Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

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- The Statistics of Squash
- Gym Membership
- Identifying a Rogue Ball
- Reversing Digits
- Cup and Saucer Derangements

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor

2012 Transylvanian Mathematical Competition

This was held in Gyergyoszentmiklos in Romania 3-5 February. The organizer, Mihaly Bencze, has sent us the problems set, which are reproduced below for interested readers to try. There are problems at various levels. They might suggest other problems to you. Please do not send us your solutions or requests for solutions!

Transylvanian Hungarian Mathematical Competition,
22nd edition, Gyergyószentmiklós, Romania,
3rd to 5th February 2012

Problems for the 9th form

Problem 1 Find that numbers $x, y \in \mathbb{N}$ for which relation $x + 2y + 3x/y = 2012$ holds.

Béla Kovács

Problem 2 Let $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R} \setminus \{0\}$ with $a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2$. Prove that at least one of equations $a_1x^2 + 2c_2x + b_1 = 0$, $b_1x^2 + 2a_2x + c_1 = 0$, and $c_1x^2 + 2b_2x + a_1 = 0$ has real solutions.

Mihály Bencze

Problem 3 Solve equation $2^{[x]} = 1 + 2x$ with $x \in \mathbb{R}$, where $[x]$ denotes the integer part of x.

Anna-Mária Darvas

Problem 4 Prove that for every acute angled and not isosceles triangle with the half of the segment determined by a vertex and the orthocenter, with the median from the same vertex, and with the circumradius of the triangle we can construct a triangle.

Ferenc Olosz

Problem 5 The measures two angles of a triangle are of $45°$ and $30°$. Find the ratio of the longest side of the triangle and the median from the vertex of the angle of $45°$.

Ferenc Olosz

Problem 6 Prove that among every seven vertexes of a regular 12-gon there exist three which are vertexes of a right-angled triangle! Is it also true that among every seven vertexes of a regular 12-gon there exist three which are vertexes of a right-angled and isosceles triangle?

Zoltán Bíró

Problems for the 10th form

Problem 1 Solve in $\mathbb{R}$ equation $[\log_2 x] = \sqrt{x} - 2$, where $[x]$ denotes the integer part of x.

Ferenc Kacsó

Problem 2 Find all real solutions of equation $7^{\log_5(x^2+4/x^2)} + 2(x + 2/x)^2 = 25$.

Mihály Bencze

Problem 3 (a) Prove that for each $z \in \mathbb{C}$ the following inequality holds:

$$|z^2 + 2z + 2| + |z - 1| + |z^2 + z| \geq 3.$$

(b) When does the equality hold?

Béla Bíró

Problem 4 In triangle ABC with $AB = AC$ let I denote the incenter of the triangle. Line BI meets the circumcircle secondly in point D. Find the measures of the angles of the triangle if $BC = ID$.

Géza Dávid

Problem 5 (a) Show that an interior point M of a triangle ABC belongs to the median from A if and only if $area[MAB] = area[MAC]$.

(b) Determine that interior point M of the triangle ABC, for which

$$\frac{MA}{\sin(\widehat{BMC})} = \frac{MB}{\sin(\widehat{CMA})} = \frac{MC}{\sin(\widehat{AMB})}.$$

Lajos Longáver

Problem 6 Find the 73th digit from the end of $\underbrace{111\ldots1}_{2012 \text{ digits}}{}^2$.

Anna-Mária Darvas

Problems for the 11th form

Problem 1 Find the necessary and sufficient condition for numbers $a \in \mathbb{Z} \setminus \{-1, 0, 1\}$, $b, c \in \mathbb{Z} \setminus \{0\}$, and $d \in \mathbb{N} \setminus \{0, 1\}$ for which $a^n + bn + c$ is divisible by d for each natural number n.

József Kolumbán Jr.

Problem 2 Let n be a positive integer. Find the number of those numbers of $2n$ digits in the binary system for which the sum of digits from the odd places is equal to the sum of digits from the even places.

Zoltán Bíró

Problem 3 For a positive integer n let $a_n = \sum_{k=1}^{n} (-1)^{\sum_{i=1}^{k} i} \cdot k$. Find the rank of matrix $A \in \mathcal{M}_{4,4n}(\mathbb{R})$, where

$$A = \begin{pmatrix} a_1 & a_2 & \ldots & a_{4n} \\ a_{4n+1} & a_{4n+2} & \ldots & a_{8n} \\ a_{8n+1} & a_{8n+2} & \ldots & a_{12n} \\ a_{12n+1} & a_{12n+2} & \ldots & a_{16n} \end{pmatrix}.$$

Ágnes Mikó

Problem 4 Sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ satisfy the following relations: $x_1 = 2$, $y_1 = 4$, and $x_{n+1} = 2 + y_1 + y_2 + \cdots + y_n$, $y_{n+1} = 4 + 2(x_1 + x_2 + \cdots + x_n)$, for all $n \in \mathbb{Z}^+$. Prove that sequence $(x_n\sqrt{2} + y_n)_{n\geq 1}$ is a geometric progression and find its general term.

Ferenc Kacsó

Problem 5 (a) ABM, BCN, and CDP are equilateral triangles with $AB = a$, $BC = b$, and $CD = c$. Points A, B, C, D belong to a line d in this order, and points M, N, P are situated in the same side of d. Show that the following inequality holds:

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + b^2 + c^2 + ab - ac + bc}.$$

(b) Prove that, for positive real numbers a_0, a_1, ..., a_n,

$$\sum_{k=0}^{n-1} \sqrt{a_k^2 - a_k a_{k+1} + a_{k+1}^2} \geq \sqrt{a_0^2 + a_n^2 + \left(\sum_{k=1}^{n-1}\right)^2 - a_0 a_n + (a_0 + a_n) \sum_{k=1}^{n-1} a_k}.$$

Lajos Longáver

Problem 6 Show that there exist infinitely many non-similar triangles such that the side lengths are positive integers and the areas of squares constructed on their sides are in arithmetic progression.

Ferenc Olosz

Problems for the 12th form

Problem 1 Solve in $\mathbb{Z}$ the following equation: $3/\sqrt{x} + 2/\sqrt{y} = 1/\sqrt{2}$.

Ferenc Kacsó

Problem 2 Let's consider set $M = \{a^2 - 2ab + 2b^2 \mid a, b \in \mathbb{Z}\}$. Show that $2012 \notin M$. Prove that M is a closed subset of $\mathbb{N}$ in respect of the multiplication of integers.

Béla Bíró

Problem 3 Solve in $\mathbb{R}$ equation $5x^3 - 18x^2 + 43x - 6 = 3 \cdot 2^{x+2}$.

Béla Kovács

Problem 4 In the not isosceles triangle ABC we have $m(\widehat{BAC}) = 90°$, AD, AE, AO are altitude, angle-bisector, and median, respectively ($D, E, O \in (BC)$). Prove that if $OE = 2DE$, then $AB^2 + AC^2 = 4AB \cdot AC$.

Lajos Longáver

Problem 5 Uncle John has taken blood pressure drops for a long time according to the following rule: 1 drop for one day, 2 drops daily for two days, ..., 10 drops daily for ten days, 9 drops daily for nine days, ..., 2 drops daily for two days, 1 drop for one day, 2 drops daily for two days, ... One day he forgot how many drops he should take, finally he took 5 drops. What is the probability that he guessed right the daily dose? Later he remembered taking 5 drops previous day, so he calmed down that he guessed the dose correctly with high probability. What is this newer probability?

Ágnes Mikó

Problem 6 (a) At least how many elements must be selected from the group $(\mathbb{Z}_{2k}, +)$ such that among the selected elements surely there exist three (not necessarily distinct) with sum $\hat{0}$?

(b) The same question for $(\mathbb{Z}_{15}, +)$.

Szilárd András

A Stochastic Random-Walk Analysis of the Sport of Squash

DEAN G. HATHOUT

The sport of squash has recently adopted a new international scoring system. In this paper we use a simple constrained random-walk probabilistic model to analyze the probability of winning a squash match given the probability of winning a single point. The advantage magnification of the current scoring system is illustrated by the derived function. We then undertake an efficacy analysis of the deuce rule, assessing whether it has any significant effect on the likelihood of winning a match given the probability of winning a single point.

Introduction

I have a good friend, whom I will call Stephen (because that is his name), and we play squash together a few times a week. I will admit that Stephen is a little better than me. When we play without keeping score, he wins rallies a touch more often that I do. The annoying thing, though, is that when we play matches, Stephen essentially always wins. I began to ponder the matter; was it psychological? did I imperceptibly freeze up a bit when a match was on the line, leading to a lower level of play? did Stephen elevate his level when a match was at stake? As my annoyance mounted, I began to think rigorously about the issue, and finally decided to do a probabilistic, random-walk analysis of squash to try to understand the situation.

This approach has been applied to various sports, such as tennis (see references 1 and 2). However, relatively little work has been done on squash. Franks and McGarry (see reference 3) used a Markov chain model which analyzes future performance based on past play. This model is optimal when restricted to data already available on players, because the Markov chain predicts future performance based on past performance, and this varies across different opponents.

In this paper we seek to implement a simpler random-walk probabilistic approach for squash, focusing on the newly implemented international scoring system, where games are played to 11 points and must be won by two clear points if the score reaches 10-10 (deuce), and a match is won by the first player to win three games. In the old scoring system, a player could win points only on his own serve, but in the new scoring system, it does not matter who is serving in terms of awarding points (like tennis). Unlike tennis, though, the serve does not offer a significant advantage to the server—it just gets the ball in play.

This new scoring system makes it fairly straightforward to model the probability of winning a match as a function of p, the probability of winning a single point, using a random-walk probability analysis. The rationale for the new scoring system was to make the matches move along more quickly. It would, therefore, be interesting to analyze whether the deuce rule in effect for games makes a significant difference in the likelihood of winning matches for a given p. That is because the deuce rule often leads to very long games, i.e. games which could have ended at 11-10 often go to much higher scores until one player wins by two points. This, in effect, is an analysis of the efficacy of the current scoring system in squash. Such analyses have been performed for sports such as tennis (see reference 4), but relatively few efficacy analyses address the new squash scoring system.

Analysis

I do not want to make this personal, so let us call our players A and B, and assume that A can consistently win a point with probability p, while B can win a point with probability q (which, of course, is just $1 - p$). We first wish to analyze the likelihood of A winning a single game as a function of p.

We can begin the analysis by considering the situation to be a constrained random walk on a two-dimensional score grid, where each entry represents the number of possible ways that a given score could be arrived at. This approach was used in an analysis of tennis scoring by the mathematician Ian Stewart, and we follow his lead here (see reference 5). Let us express scores as an ordered pair (m, n), where A's score is indexed by m, and can be represented as moving to the right in columns across a score grid, while B's score is indexed by n, and is then represented by the rows of the score grid. Then, at any point in the game (before its end or a deuce situation), a score of (m, n) can only be arrived at in one of two ways (see figure 1). Either the score had been $(m - 1, n)$ and A won the point, or the score had been $(m, n - 1)$ and B won the point. Therefore, the number of ways to get to a score of (m, n) is equal to the sum of the number of ways to get to $(m - 1, n)$ and the number of ways to get to $(m, n - 1)$, setting up a recursive definition. This means that someone not familiar with combinatorics can fill in a probability grid by noting that each cell (excluding for the moment a win or a deuce situation) is just the sum of the cell above and the cell to the left (see figure 2).

The first row and first column of the grid in figure 2 can be filled with 1s, since there is only one way to achieve a score of $(k, 0)$ and one way to achieve a score of $(0, k)$. Given this, the grid can be progressively filled in cell by cell using the addition rule above.

Of course, for those who are familiar with combinatorics, we can express the ways of achieving a score of (m, n) as just $\binom{m+n}{n}$, and the recursive definition amounts to Pascal's formula for the binomial coefficients:

$$\binom{m+n}{n} = \binom{m+n-1}{n} + \binom{m+n-1}{n-1}.$$

Excluding the deuce situation for the moment, player A will win the game when he gets 11 points. This is shown as the highlighted right-hand column in figure 2. In this case, there is

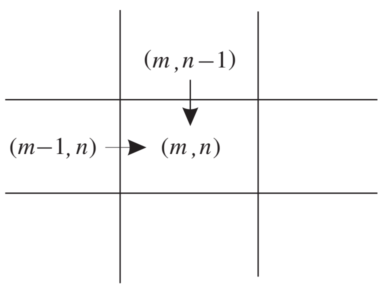

Figure 1 A score of (m, n), in the middle of the game, can only be achieved as a step from two prior positions, $(m - 1, n)$ or $(m, n - 1)$.

A →

B ↓	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	1	1	1	1	1	1	1	1	1	1	1			
1	1	2	2	3	4	5	6	7	8	9	10	11			
2	1	3	6	10	15	21	28	36	45	55	66	66			
3	1	4	10	20	35	56	84	120	165	220	286	286			
4	1	5	15	35	70	126	210	330	495	715	1001	1001			
5	1	6	21	56	126	252	462	792	1287	2002	3003	3003			
6	1	7	28	84	210	462	924	1716	3003	5005	8008	8008			
7	1	8	36	120	330	792	1716	3432	6435	11 440	19 448	19 448			
8	1	9	45	165	495	1287	3003	6435	12 870	24 310	48 620	48 620			
9	1	10	55	220	715	2002	5005	11 440	24 310	48 620	92 378	92 378			
10	1	11	66	286	1001	3003	8008	19 448	48 620	92 378	184 756	184 756	184 756		
11	1	11	66	286	1001	3003	8008	19 448	48 620	92 378	184 756	369 512	369 512	369 512	
12											184 756	369 512	739 024	739 024	739 024
13												369 512			

B wins · Advantage B · Deuce · Advantage A · A wins

Figure 2 Combinatorics of a squash game. Each cell in the grid gives the number of ways that a certain score (m, n) can be achieved.

only one way to get to any cell in this column. For example, for player A to win by a score of $(11, 4)$, the score must have been $(10, 4)$, with player A then winning the next point. The score could not have been $(11, 3)$ with B then winning a point, as the game would have already been over. Therefore, the number of ways to get to $(11, 4)$ is just the same as the number of ways to get to $(10, 4)$, and the highlighted cells in column 11 are calculated only from the cell to the left, and not the one above. The same logic applies to row 11, which shows the wins for player B. Thus, excluding deuce, each score from $(11, k)$, where $0 \leq k < 10$, can be achieved with a probability of $\binom{10+k}{k} p^{11} q^k$. Therefore, the probability P of A winning a game is

$$P = \sum_{k=0}^{9} \binom{10+k}{k} p^{11} q^k + \text{the probability of winning a deuce.}$$

To calculate the probability of winning a deuce, we look at the highlighted diagonal cells in figure 2 that represent wins for player A, and we see that starting at cell $(12, 10)$, the entries are 184 756, 369 512, and 739 024, corresponding to scores of $(12, 10)$, $(13, 11)$, and $(14, 12)$, respectively. We can see that these entries represent a progressive doubling of 92 378, which is the entry in cell $(11, 9)$, which itself came from cell $(10, 9)$. This makes sense, since if A wins by a score of $(12, 10)$, this means that the previous position had to be $(11, 10)$, and the one before that had to be $(10, 10)$. Therefore, these three entries are identical. A score of $(10, 10)$ is the sum of the cells above and to the left i.e. those corresponding to $(10, 9)$ and to $(9, 10)$. Both of these are 92 378, which is just $\binom{19}{9} = \binom{19}{10}$. Therefore, the entry in cell $(12, 10)$ is just twice the entry in cell $(11, 9)$. The same logic continues down the diagonal, and we see that the probability of winning a deuce is just

$$184\,756 p^{12} q^{10} + 369\,512 p^{13} q^{11} + 739\,024 p^{14} q^{12} + \cdots,$$

which continues as an infinite series. Now we can say that the probability of winning a game, P, is given by

$$\sum_{k=0}^{9} \binom{10+k}{k} p^{11} q^k + 184\,756 p^{12} q^{10} + 369\,512 p^{13} q^{11} + 739\,024 p^{14} q^{12} + \cdots.$$

Let us rewrite this sum as

$$\sum_{k=0}^{8} \binom{10+k}{k} p^{11} q^k + 92\,378 p^{11} q^9 + 184\,756 p^{12} q^{10} + 369\,512 p^{13} q^{11} + 739\,024 p^{14} q^{12} + \cdots.$$

Now we can factor $92\,378 p^{11} q^9$ out of the infinite series portion and get

$$P = \sum_{k=0}^{8} \binom{10+k}{k} p^{11} q^k + 92\,378 p^{11} q^9 (1 + 2pq + 4p^2 q^2 + 8p^3 q^3 + \cdots).$$

We see that the brackets contain a simple geometric series, with 1 as the first term and common ratio $2pq$. Therefore, we can now write the probability of winning a game as

$$P = \sum_{k=0}^{8} \binom{10+k}{k} p^{11} q^k + \frac{92\,378 p^{11} q^9}{1 - 2pq}.$$

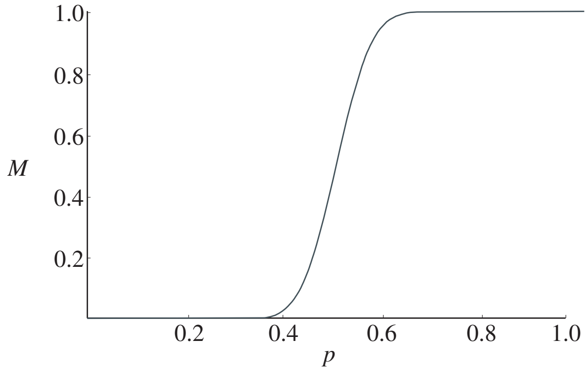

Figure 3 Probability of winning a match, M, as a function of the probability of winning a single point, p.

If we substitute $q = 1 - p$ and expand, we get

$$P = 75\,582p^{11} - 554\,268p^{12} + 1\,790\,712p^{13} - 3\,325\,608p^{14} + 3\,879\,876p^{15} - 2\,909\,907p^{16} + 1\,369\,368p^{17} - 369\,512p^{18} + 43\,758p^{19} + \frac{92\,378p^{11}(1-p)^9}{1-2p+2p^2}.$$

Now that we have the probability P of player A winning a single game as a function of p, we can easily calculate the probability of A winning a match. We know that the probability of A losing a game is $1 - P$. Player A can win a match by winning the third game after being up 2-0 in games (which can happen in only one way), or after being up 2-1 in games, which can happen in $\binom{3}{1} = 3$ ways, or after the score is 2-2, which can happen in $\binom{4}{2} = 6$ ways. Therefore, the probability $M(P)$ of player A winning a match is

$$M = P^3 + 3P^3(1-P) + 6P^3(1-P)^2, \tag{1}$$

where P is a function of p. The expansion of this function in terms of p is too long to write down, but is a 95th-degree polynomial. However, we can now plot M as a function of p (see figure 3). By tabulating (see table 1) this function for certain values of p, rounded to six significant figures, we find the explanation to the mystery I had been grappling with.

We see that if a player can win 30% of the points, he has only a 0.01% chance of winning a match. In my case, I would estimate that Stephen is somewhat better than me, able to win points with a likelihood of 0.6. This translates into a 96.6% chance of winning a match, and explains why I hardly ever seem to beat Stephen. The scoring system, of course, magnifies tiny advantages in skill if the players are relatively well matched. In looking at the world of professional squash, this would seem to explain the dominance by a single player across an era of time almost unparalleled in any other major sport (compare, for example, golf or car racing). All a player has to do is become just a tiny bit more skilled than his opponents—just slightly more likely to win any given point—and he will have a major advantage in terms of match play.

To give a flavour of this, let us look at the championship record for the British Squash Open and the World Squash Open, the sport's two largest annual tournaments. In an intensely competitive sport, the British Open was won by Geoff Hunt from 1976–1981, by Jahangir Khan from 1982–1991, and by Jansher Khan from 1992–1997. The World Squash Open was won by Geoff Hunt from 1976–1980, by Jahangir Khan from 1981–1985 and in 1988 (runner up in

Table 1

Probability of winning a point, p	Probability of winning a match, M
0.0	0.0
0.1	0.0
0.2	0.0
0.3	0.000 101 983
0.4	0.033 725 4
0.5	0.5
0.6	0.966 275
0.7	0.999 898
0.8	1.0
0.9	1.0
1.0	1.0

1986), and by Jansher Khan in 1987, 1989, 1990, and 1992–1996 (runner up in 1988). This sort of dominance is consistent with the above analysis.

Another issue of interest is the deuce rule for games; does it have any significant effect on the shape of the above curve? For example, does it blunt the advantage amplification that the current scoring system provides, and, if so, does it do so enough to justify the significant lengthening of games which it sometimes imposes?

From the above analysis, we see that the formula for the probability P_{nd} of winning a game where there is no deuce rule (i.e. first player to 11) is simply given by

$$P_{nd} = \sum_{k=0}^{10} \binom{10+k}{k} p^{11} q^k,$$

which can be written explicitly as

$$\begin{aligned} P_{nd} = {} & 35\,2716 p^{11} - 3\,233\,230 p^{12} + 13\,430\,340 p^{13} - 33\,256\,080 p^{14} + 54\,318\,264 p^{15} \\ & - 61\,108\,047 p^{16} + 47\,927\,880 p^{17} - 25\,865\,840 p^{18} + 9\,189\,180 p^{19} \\ & - 1\,939\,938 p^{20} + 184\,756 p^{21}. \end{aligned}$$

The probability of winning a match can then be calculated precisely the same as before, using P_{nd} instead of P in (1). From figure 4 we see that the two resultant curves are essentially identical.

We see that the deuce rule produces almost no difference in the curves, and does not offer any significant mitigating advantage to the less-skilled player. For $p = 0.3$, the no-deuce scoring gives a likelihood of winning a match as 0.000 176 589 versus 0.000 101 983 for deuce scoring, and, with $p = 0.4$, the results are 0.040 122 2 versus 0.033 725 4, respectively. Therefore, we see that the deuce rule has no significant effect on the curve shape, and that eliminating deuce scoring would not only shorten matches, but would also minimally blunt the advantage amplification which favours the better player, both desirable outcomes. Therefore, mathematically speaking, a strong case can be made to eliminate deuce scoring from squash.

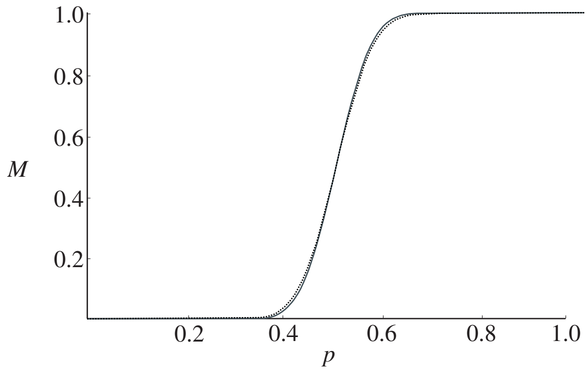

Figure 4 Two overlapping curves for the probability of winning a match M versus p, employing the deuce (*solid line*) and no-deuce (*dotted line*) rules in games.

Conclusions

The squash scoring system provides significant advantage magnification in match play—being very slightly better than an opponent on a per-point basis has huge rewards. Even for those quite seasoned in probability, it possibly defies intuition that if a player is able to win 30% of the points, he can expect to win only 0.01% of the matches! To be able to win 30% of the matches, a player needs to be able to win, on average, just over 47% of the points, i.e. to be nearly evenly matched with his opponent.

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Dean Hathout *is a high school student in California, taking advanced mathematics classes through his school and Stanford University's Education Programme for Gifted Youth, where he studied calculus while in eighth grade. He is a gold medal winner in the Los Angeles County Science Fair in mathematics, and has participated in national math contests in the US since the age of ten. He is also an avid junior squash player who enjoys the game's physical and mental aspects.*

Gym Membership: an Application of the Weibull Distribution

JOHN C. B. COOPER

The Swedish physicist, Waloddi Weibull, is renowned for formulating a family of probability distributions for modelling the variability of nonnegative, quantitative, random phenomena (see reference 3). The distribution was originally designed to model the breaking strength of steel (Weibull was scientific advisor to Bofors, the Swedish steel company) but, since then, Weibull analysis has evolved to provide a concise, but flexible, framework for describing and interpreting survival-type data. Indeed, it has been extensively applied in various disciplines ranging from engineering (see reference 1) to ecology (see reference 2) despite Weibull's own modest understatement that '... it may sometimes render good service.' The illustrative example in this article is a business application.

Theory

The probability density function (PDF) and the cumulative distribution function (CDF) of the two-parameter Weibull distribution may be written respectively as

$$f(x) = \alpha\beta x^{\beta-1}\exp\{-\alpha x^{\beta}\}$$

and

$$F(x) = 1 - \exp\{-\alpha x^{\beta}\}, \tag{1}$$

where $x \leq 0$ and $\alpha, \beta > 0$. Here, α, the scale parameter, governs the 'peakedness' of the distribution while β, the shape parameter, gives the distribution its flexibility. By changing β, the Weibull distribution can model a wide variety of data. Thus, for example, when $\beta = 1$, the Weibull reduces to the exponential distribution; when $\beta = 2$, it reduces to the Rayleigh distribution; and when $\beta = 3.5$ and $\alpha = 1$, it closely approximates the normal distribution.

Occasionally, it is necessary to include a third parameter, γ, the so-called location, shift, or threshold parameter. In this case, we would have the three-parameter Weibull distribution with PDF

$$f(x) = \alpha\beta(x-\gamma)^{\beta-1}\exp\{-\alpha(x-\gamma)^{\beta}\}.$$

In the two-parameter version, $\gamma = 0$, and the distribution starts at the origin. When $\gamma > 0$, the distribution starts at the location, γ, to the right of the origin. Graphically, the effect is to slide the PDF to the right along the abscissa. As the empirical application to follow uses the two-parameter Weibull, we shall not explore the three-parameter version any further.

A very useful feature of the CDF is that it may be readily linearised. Thus, from (1),

$$\begin{aligned} 1 - F(x) &= \exp\{-\alpha x^{\beta}\}, \\ -\ln[1 - F(x)] &= \alpha x^{\beta}, \\ -\ln[-\ln[1 - F(x)]] &= \ln\alpha + \beta\ln x. \end{aligned} \tag{2}$$

Thus, if $\ln[-\ln[1 - F(x)]]$ is plotted against $\ln x$, we have a straight line with slope β and intercept $\ln\alpha$, each of which may be estimated using ordinary least squares.

Application

A certain leisure centre offers various grades of membership of its gym facilities for a monthly fee. A sample of 20 male clients who purchased the same grade of membership in January 1998 was identified, and the time to cancellation of their contract was noted. These times to cancellation in years, x, are listed in ascending order in the first column of table 1; ranks (the equivalent of cumulative cancellations in this particular study) are listed in the second column; while the natural logarithm of the cancellation time, $\ln x$, is listed in the fifth column.

In order to estimate the parameters, α and β, of the Weibull distribution, we require an estimate of $F(x)$ for all 20 data points. These estimates, shown in column three, are obtained by calculating Benard's approximation of the median rank (MR) defined as

$$F(x) = \mathrm{MR}(x) = \frac{\mathrm{Rank}(x) - 0.3}{n + 0.4},$$

where n denotes the sample size (further explanation of the median rank is provided in the appendix). The double log transformation of $1 - \mathrm{MR}(x)$, namely $\ln[-\ln[1 - \mathrm{MR}(x)]]$, is listed in the fourth column. We may now estimate (2) by regressing column four onto column five ($\ln x$) to obtain

$$\hat{\alpha} = 0.0419, \qquad \hat{\beta} = 1.7972, \qquad \text{and} \qquad R^2 = 0.97.$$

At this point, it is worth mentioning that, had the leisure centre stipulated a minimum initial period during which membership could not be cancelled, as some do, then the three-parameter

Table 1

Cancellation time (x)	Rank	Median rank (MR)	$\ln[-\ln[1 - \mathrm{MR}]]$	$\ln(x)$	Expected cumulative failures
1.05	1	0.03	−3.49	0.05	1
1.72	2	0.08	−2.48	0.54	2
1.87	3	0.13	−1.97	0.63	2
1.96	4	0.18	−1.62	0.67	3
2.38	5	0.23	−1.34	0.87	4
2.79	6	0.28	−1.11	1.03	5
3.91	7	0.33	−0.92	1.36	8
3.99	8	0.38	−0.74	1.38	8
4.33	9	0.43	−0.58	1.47	9
4.89	10	0.48	−0.42	1.59	10
5.00	11	0.52	−0.31	1.61	11
5.12	12	0.57	−0.17	1.63	11
5.23	13	0.62	−0.03	1.65	11
5.58	14	0.67	0.10	1.72	12
6.19	15	0.72	0.24	1.82	13
6.34	16	0.77	0.39	1.85	14
8.62	17	0.82	0.54	2.15	17
9.42	18	0.87	0.71	2.24	18
11.29	19	0.92	0.93	2.42	19
11.94	20	0.97	1.25	2.48	19

Weibull distribution may have been more appropriate. The third parameter γ would have been interpreted as the earliest possible time after which cancellation could occur. For example, if a client were not allowed to cancel in the first six months, γ would be set at 0.5 and this value would be subtracted from each cancellation time, x, prior to the analysis.

We have identified a coherent sample of individuals who have opened their membership at the same point in time and who use the facilities to which they are entitled in the same way. If each individual has a cumulative cancellation distribution given by (1), then the expected number of failures by each time x is

$$N(x) = (1 - \exp\{-\alpha x^{\beta}\})n.$$

For example, the expected number of cancellations by year five is calculated as

$$(1 - \exp\{-0.0149 \times 5^{1.7972}\}) \times 20 = 10.61 \approx 11.$$

All expected cancellations, rounded to the nearest integer, are shown in the final column of table 1 while, of course, actual cancellations are given in the second column. Clearly, the Weibull distribution provides a good fit.

Appendix: Median Rank

Suppose that we have a sample of only one client and he cancels his membership after, say, one year. We cannot deduce from this that all clients will cancel after one year. That is, we cannot deduce that the cumulative probability of cancellation after one year is 100% because $F(x) = 1 - \exp\{-\alpha x^{\beta}\}$ only approaches 1, or 100%, as time, x, tends to ∞. We can say, however, that there is a 50% probability that the cancellation will occur prior to the median time to cancellation, x_m, and a 50% probability that it will occur after x_m. Thus, the proportion of cancellations at x_m is 0.5. This is known as the *median rank* or MR.

Suppose now that we have a sample of three clients who cancel their memberships after one, two, and three years, respectively. The probability that at least one cancellation has occurred by x_m is obtained by solving the binomial equation:

$$\binom{3}{1}p^1(1-p)^2 + \binom{3}{2}p^2(1-p)^1 + \binom{3}{3}p^3(1-p)^0 = 0.5,$$

from which $p = 0.2063$. Thus, $\text{MR}_1 = 0.2063$, which implies that 20.63% of clients have cancelled by within one year.

Similarly, the probability that at least two cancellations have occurred by x_m is obtained by solving

$$\binom{3}{2}p^2(1-p)^1 + \binom{3}{3}p^3(1-p)^0 = 0.5,$$

from which $p = 0.5$. Thus, $\text{MR}_2 = 0.5$, which implies that 50% of clients have cancelled within two years.

Finally, the probability that three cancellations have occurred by x_m is obtained by solving

$$\binom{3}{3}p^3(1-p)^0 = 0.7937$$

from which $p = 0.7937$. Thus, $\text{MR}_3 = 0.7937$, which implies that 79.37% of clients have cancelled within three years.

We may calculate reasonably accurate estimates of these median ranks more easily, using Benard's approximation, to obtain $\mathrm{MR}_1 = 0.2059$, $\mathrm{MR}_2 = 0.5$, and $\mathrm{MR}_3 = 0.7941$. Indeed, reliability analysis software will calculate these for the user as a matter of course.

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Cubes and triangular numbers

$$1^3 = 1^2 - 0^2, \qquad 2^3 = 3^2 - 1^2,$$
$$3^3 = 6^2 - 3^2, \qquad 4^3 = 10^2 - 6^2, \qquad 5^3 = 15^2 - 10^2.$$

The sequence (T_n) of triangular numbers begins 1, 3, 6, 10, 15, with $\mathrm{T}_n = \frac{1}{2}n(n+1)$. Generally,

$$n^3 = \mathrm{T}_n^2 - \mathrm{T}_{n-1}^2,$$

so that every cube is a difference of the squares of consecutive triangular numbers.

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How good is your algebra?

$$a = (2p^2 - 1)x^2 + 2pqx + 2pr,$$
$$b = 2pqx^2 + (2q^2 - 1)x + 2qr,$$
$$c = 2rpx^2 + 2rqx + (2r^2 - 1),$$

and $p^2 + q^2 + r^2 = 1$. What is $a^2 + b^2 + c^2$?

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Strategy for Identifying a Rogue Ball. I

MERVYN HUGHES

Among a set of balls, which are all alike, is one differing by its weight only, which we call a rogue. We are allowed to only use simple scales, that is, we can determine only whether a ball placed in the pan on one side of the scales is lighter, heavier, or the same weight as a ball placed in the other pan. In this article we determine the minimum number of weighings in order to guarantee that this rogue ball will be found, among a set of up to 12 balls. Moreover, in the case when its relative weight is unknown, we give a strategy to locate it and determine whether it is lighter or heavier than the others.

In this article we investigate the problem of identifying a different ball among many balls which are alike. All balls are similar in shape and colour and all have the same density except one, which may be lighter or heavier than the rest. We shall call this a rogue ball. There are three distinct problems.

Problem 1. Identify the rogue ball when it is known to be lighter than the others.

Problem 2. Identify the rogue ball when it is known to be heavier than the others.

Problem 3. Identify the rogue ball when its relative weight is unknown, i.e. it could be either lighter or heavier than the others.

The rogue ball problem came to mind at Christmas when my son had 12 crackers and said that one was different from all the others and was either lighter or heavier than the others but would not tell me which. How could we find out which one was the rogue? We decided to change the problem into balls with a toy inside (or possibly no toys or two toys). We were allowed only to use simple scales, that is, one with two pans. Thus, we could distinguish only whether one pan was lighter or heavier than the other. We clearly wanted to minimize the number of weighings, but we wanted to guarantee that we could find out which one was the rogue ball and not only that we wanted to know whether it was heavier or lighter than the others. Is there a strategy to use?

In Section 1 we deal with problem 1, in Section 2 we deal with problem 2, and in Section 3 we deal with problem 3. In this article we shall only deal with a set of up to 12 balls, and in a followup article we will consider a set of up to 120 balls and look at different strategies when we relax our certainty condition as well as the probabilities and mathematical expectations.

For n balls, we find the number of weighings required to *guarantee* that the rogue ball can be identified in all possible cases. Let $f(n)$ be the minimum of all these values. Suppose that $n = 6$. We can find the rogue ball after just one weighing if we know whether it is lighter or heavier than the others, but we clearly need more to guarantee that we find it. We can continue to weigh balls until we find the rogue ball, by systematically weighing the first ball against the others in turn. Thus, we can guarantee finding the rogue ball upon the fifth weighing. However, a little thought tells us that we do not need the fifth weighing as, if the rogue has not been identified after the fourth weighing, then the sixth ball must be the rogue ball. This gives $f(6) \leq 4$, but there may be other ways to reduce this bound. Indeed, we shall see that $f(6) = 2$.

1. Problem 1

1.1. Notation

Consider n balls, labelled $B_1, B_2 \ldots$. Let $S_1 = \{B_1, B_2, \ldots\}$ denote the set of balls. Let $W(S)$ and $W(B)$ be the weights of set S and ball B, and, for simplicity, let W_i be the weight of ball B_i. We shall compare the weights of two sets consisting of an equal number of balls. If we compare the weights of sets S_1 and S_2, say, using simple scales, then either the pans of the scales balance or one side is lighter than the other. Without loss of generality we shall always consider the lighter side to be the left side. If $W(S_1) = W(S_2)$ then neither set contains the rogue ball, and if $W(S_1) < W(S_2)$ then the rogue ball is on the left.

1.2. Results

We now establish the results for up to 12 balls.

Case 1: $n = 2$. If we weigh B_1 against B_2 and find that $W_1 < W_2$, then B_1 is the rogue ball and $f(2) = 1$.

Case 2: $n = 3$. Weigh B_1 against B_2 and if

- $W_1 = W_2$ then both balls are 'good' and B_3 is the rogue ball,
- $W_1 < W_2$ then B_1 is necessarily the rogue ball.

Thus, $f(3) = 1$.

Case 3: $n = 4$. There are many ways of weighing the balls against each other in this case, but we shall consider weighing two balls against the other two, say B_1 with B_2 against B_3 with B_4. If $W_1 + W_2 < W_3 + W_4$, we then weigh B_1 against B_2 and the lighter of these is the rogue. Thus, $f(4) = 2$.

Case 4: $n = 5$. Split the five balls into three sets: $S_1 = \{B_1, B_2\}$, $S_2 = \{B_3, B_4\}$, and $S_3 = \{B_5\}$. Weigh S_1 against S_2 and if

- $W_1 + W_2 = W_3 + W_4$ then B_5 is the rogue,
- $W_1 + W_2 < W_3 + W_4$ then the lighter of B_1 and B_2 is the rogue.

Thus, $f(5) = 2$.

Case 5: $n = 6$. Split the six balls into three sets: $S_1 = \{B_1, B_2\}$, $S_2 = \{B_3, B_4\}$, and $S_3 = \{B_5, B_6\}$. Weigh S_1 against S_2 and if

- $W_1 + W_2 = W_3 + W_4$ then the rogue is in the third set and so one more weighing is needed,
- $W_1 + W_2 < W_3 + W_4$ then the lighter of B_1 and B_2 is the rogue.

Thus, $f(6) = 2$.

The results for $n = 7$ and $n = 8$ are very similar, and it can easily be established that $f(7) = f(8) = 2$. However, we shall demonstrate that $f(9) = 2$.

Case 6: $n = 9$. Split the nine balls into three sets: $S_1 = \{B_1, B_2, B_3\}$, $S_2 = \{B_4, B_5, B_6\}$, and $S_3 = \{B_7, B_8, B_9\}$. Weigh S_1 against S_2 and if

- $W_1 + W_2 + W_3 = W_4 + W_5 + W_6$ then the rogue is in the third set and so one more weighing is needed,

- $W_1 + W_2 + W_3 < W_4 + W_5 + W_6$ then the lightest of B_1, B_2, and B_3 is the rogue and again one more weighing suffices.

Thus, $f(9) = 2$.

The results for $n = 10$ and $n = 11$ balls are again similar, but we will establish that $f(10) = 3$.

Case 7: $n = 10$. Let $S_1 = \{B_1, B_2, B_3\}$, $S_2 = \{B_4, B_5, B_6\}$, and $S_3 = \{B_7, B_8, B_9, B_{10}\}$. Weigh S_1 against S_2 and if

- $W_1 + W_2 + W_3 = W_4 + W_5 + W_6$ then the rogue is in the third set and so two more weighings are needed,
- $W_1 + W_2 + W_3 < W_4 + W_5 + W_6$ then the lightest of B_1, B_2, and B_3 is the rogue and again one more weighing suffices.

Thus, $f(10) = 3$.

Case 8: $n = 11$. Split the 11 balls into sets of size 4, 4, and 3. Proceed similarly as in case 7.

Case 9: $n = 12$. Split the balls into three sets of four. Weigh the first set against the second set and if

- $W_1 + W_2 + W_3 + W_4 = W_5 + W_6 + W_7 + W_8$ then the rogue is in the third set and so two more weighings are needed,
- $W_1 + W_2 + W_3 + W_4 < W_5 + W_6 + W_7 + W_8$ then the lightest in the first set is the rogue and again one more weighing suffices.

Thus, $f(12) = 3$.

2. Problem 2

In this section we consider a rogue ball which is known to be heavier than the rest. We simply change all the less-than inequality signs to greater-than inequality signs in the arguments above. Thus, if we know that the rogue ball is lighter or heavier than the rest then we obtain the same results for $f(n)$ as above; see table 1.

Table 1

n	$f(n)$
2	1
3	1
4	2
5	2
6	2
7	2
8	2
9	2
10	3
11	3
12	3

3. Problem 3

In this section the relative weight of the rogue ball is unknown. As above, we find the number of weighings required to guarantee that it is found and classify it as lighter or heavier. We let $g(n)$ be the least value of all the possible number of weighings. At first, it may seem that $g(n) = f(n) + 1$, but we shall see that this is not so. However, it is clear that

$$f(n) \leq g(n) \leq f(n) + 1.$$

The strategy will be the same as before. Split the balls into three sets and weigh the first set against the second; however, this time there are three possible outcomes.

- $W(S_1) = W(S_2)$ and the rogue ball is in S_3.
- $W(S_1) < W(S_2)$ and the light rogue is in S_1.
- $W(S_1) < W(S_2)$ and the heavy rogue is in S_2.

3.1. Control balls and x-ball plants

We shall call a correctly weighted ball a *control* ball, which will become useful in this section. We shall also be swapping balls in the pans and use the following definition of a *ball plant*, borrowed from the game of snooker. Take a control ball and place it in the left pan and remove a ball from this left pan and place it in the right pan, removing a ball from the right pan (see figure 1). Since we are always considering the left pan as lighter than the right pan, then either the left pan originally held the light rogue or the right pan held the heavy rogue. By performing a ball plant, the light ball is either still in the left pan or has moved to the right pan, or the heavy rogue is either still in the right pan or has been removed from it.

As we shall be planting many balls, we shall extend the above definition.

Definition Let there be y balls in each pan and x control balls available such that $x < y$, $x, y > 0$.

An *x-ball plant* is when we place x control balls in the left pan whilst removing x balls from the left pan. We then place these balls in the right pan whilst removing x balls from the right pan. Note that we can only perform an x-ball plant if we have x control balls to use and, therefore, we must have at least $3x + 2$ balls in total.

We now illustrate how a one-ball plant works. We need at least five balls in total. Let two balls be in the left pan and two balls be in the right pan. After weighing, as usual, we make the left pan the lighter pan. Thus, $W_1 + W_2 < W_3 + W_4$ with B_5 the correctly weighted ball. We perform a one-ball plant by replacing B_2 with B_5 and then replacing B_4 with the displaced B_2.

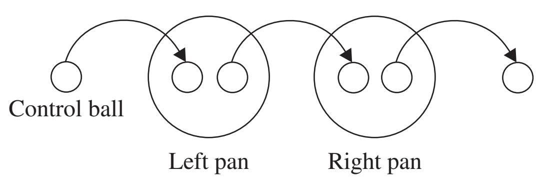

Figure 1

A second weighing then has three possible outcomes:

- $W_1 + W_5 = W_3 + W_2$, in which case the heavy rogue is B_4,
- $W_1 + W_5 > W_3 + W_2$, in which case the light rogue has moved pans and is B_2,
- $W_1 + W_5 < W_3 + W_2$, in which case we still cannot identify and classify the rogue ball; it is either B_1 and has remained in the left-hand pan and is light, or it is B_3 and has remained in the right-hand pan and is heavy. A further weighing against a control ball is necessary.

Let us consider the cases in which $n = 2, 3,$ and 4. For $n = 2$ balls, we will never know whether one is a light rogue or the other a heavy rogue. Thus, we must start with at least three balls. If we weigh one against another, they will either balance, in which case the remaining ball is rogue and must be weighed against a control ball, or they do not balance, in which case we must weigh one of them against a control ball. In either case we need two weighings. Thus, $g(3) = 2$.

For $n = 4$ balls, there is little use weighing two balls against the other two balls as we will still not know which side contains the rogue and more weighings will be needed. Instead, a better strategy is to weigh one ball against another, leaving two balls. If $W_1 < W_2$ then the remaining balls are control balls and can be used to test these. However, if $W_1 = W_2$ then both of these are control balls, but we will still need two extra weighings to determine the rogue out of B_3 and B_4, and whether it is light or heavy. Thus, $g(4) = 3$.

For $n = 5$ balls, split the balls into three sets: $S_1 = \{B_1, B_2\}$, $S_2 = \{B_3, B_4\}$, and $S_3 = \{B_5\}$. Weigh S_1 against S_2 and if

- $W_1 + W_2 = W_3 + W_4$ then B_5 is the rogue and one more weighing is needed,
- $W_1 + W_2 < W_3 + W_4$ then we employ the one-ball plant to establish which is the rogue, requiring two more weighings.

Thus, $g(5) = 3$.

Let us now consider the cases in which $n = 6, 7, \ldots, 12$. The strategy is very similar for each case, i.e. split the balls into three sets and employ ball plants. We illustrate the $n = 12$ case. Split the balls into three sets of size four and weigh the first set against the second.

- If $W_1 + W_2 + W_3 + W_4 = W_5 + W_6 + W_7 + W_8$ then the rogue is in the third set, but, although $g(4) = 3$, we can reduce this number by using three of the control balls. We weigh three control balls against $B_9 + B_{10} + B_{11}$. If they balance then the rogue ball is B_{12} and is weighed against a control, B_1 say; if they do not balance then we know whether the rogue is heavier or lighter, so we require one more weighing as $f(3) = 1$.
- If $W_1 + W_2 + W_3 + W_4 < W_5 + W_6 + W_7 + W_8$ then we perform a three-ball plant using B_9, B_{10}, and B_{11} with three possible outcomes.
  - If $W_1 + W_9 + W_{10} + W_{11} = W_5 + W_2 + W_3 + W_4$ then the heavy rogue is one of $\{B_5, B_6, B_7\}$.
  - If $W_1 + W_9 + W_{10} + W_{11} > W_5 + W_2 + W_3 + W_4$ then the light rogue is one of $\{B_2, B_3, B_4\}$.
  - If $W_1 + W_9 + W_{10} + W_{11} < W_5 + W_2 + W_3 + W_4$ then either B_1 is a light rogue or B_5 is a heavy rogue. Testing against a control determines which statement holds.

Thus, $g(12) = 3$.

Table 2

n	$f(n)$	$g(n)$
2	1	–
3	1	2
4	2	3
5	2	3
6	2	3
7	2	3
8	2	3
9	2	3
10	3	3
11	3	3
12	3	3

We summarize the results for problem 3 in table 2.

3.2. Strategy

- Consider n balls, where $n = 3x - 1,\ 3x,$ or $3x + 1$.
- Decompose the balls into three sets of size $(x, x, x - 1),\ (x, x, x),$ or $(x, x, x + 1)$.
- Weigh S_1 against S_2.
- Without loss of generality, we always consider the lighter side of the scales to be the left side.
  - If $W(S_1) = W(S_2)$ then neither set contains the rogue ball and it must be in S_3.
  - If $W(S_1) < W(S_2)$ then the rogue ball is in S_1 if it is known to be light, or in either S_1 or S_2 if we do not know its relative weight, in which case we need to perform ball plants to discover its location and classification.

We found in the previous sections that it takes three weighings to identify a rogue ball of known weight among 12 balls, yet to identify and classify a rogue ball of unknown weight among 12 balls it also takes three weighings.

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Divisibility of a number and its reverse by 13 and 73

32 178 146 and 64 187 123 are both divisible by 13, while 364 546 451 and 154 645 463 are both divisible by 73. Such numbers seem difficult to find. Can readers find other examples?

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Reversing Digits: Divisibility by 27, 81, and 121

DAVID SHARPE and ROGER WEBSTER

We find conditions under which reversing digits preserves divisibilty by 27, 81, and 121. This article answers a question posed by M. A. Khan in his letter in Volume 45, Number 1, p.41.

Readers may well be familiar with the following tests for divisibility by 3 (or 9): a number is divisible by 3 (or 9) if and only if the sum of its digits is divisible by 3 (or 9). Take 294 591, for example. The sum of its digits is 30, which *is* divisible by 3, but *not* by 9, whence the same is true for 294 591 itself. For some numbers, we may need to repeat the test, until it becomes clear at sight whether or not the number reached is divisible by 3 (or 9). The reasoning behind these tests derives from the congruences

$$a_n \ldots a_0 = a_n(10)^n + a_{n-1}(10)^{n-1} + \cdots + a_0 \equiv a_n + a_{n-1} + \cdots + a_0 \pmod{3 \text{ or mod } 9},$$

where $a_n \ldots a_0$ denotes the decimal integer with digits $a_n, \ldots, a_0$. Thus, $a_n \ldots a_0$ is divisible by 3 (or 9) if and only if $a_n + \cdots + a_0$ is divisible by 3 (or 9). It follows at once that, if a number is divisible by 3 (or 9), so is its *reverse*, i.e. the number obtained by reversing its digits.

A similar test exists for divisibility by 11, but this time it is *not* the sum of the digits that is used, but their *alternating sum*, in which alternating plus and minus signs are introduced between them. Thus, to test whether 294 591 is divisible by 11, form the alternating sum $2 - 9 + 4 - 5 + 9 - 1 = 0$, which is divisible by 11. Hence, 294 591 is divisible by 11. This test follows from the congruence

$$\begin{aligned} a_n \ldots a_0 &= a_n(10)^n + a_{n-1}(10)^{n-1} + a_{n-2}(10)^{n-2} + \cdots + a_0 \\ &\equiv a_n(-1)^n + a_{n-1}(-1)^{n-1} + a_{n-2}(-1)^{n-2} + \cdots + a_0 \pmod{11} \\ &\equiv (-1)^n(a_n - a_{n-1} + a_{n-2} - \cdots + (-1)^n a_0) \pmod{11}, \end{aligned}$$

which shows that $a_n \ldots a_0$ is divisible by 11 if and only if $a_n - a_{n-1} + a_{n-2} - \cdots + (-1)^n a_0$ is. It follows that, if a number is divisible by 11, then so is its reverse.

Divisibility by 1, 3, 9, or 11 is preserved under reversal of digits, whence (since 3 and 9 are coprime to 11) so too is divisibility by 33 or 99. Indeed, these six numbers are the *only* ones known to the authors to have this property. Other prospective candidates, 27, 81, and 121, all fall at the first hurdle. For example, 72 is not divisible by 27, 18 is not divisible by 81, and 506 is not divisible by 121 even though 605 is. This brings us to the crux of this article, to answer the question: *For which numbers does reversing digits preserve divisibility by 27, 81, or 121?*

Divisibility by 81

In the argument below and elsewhere, we use the fact that, if $a \equiv b \pmod{n}$, then $ac \equiv bc \pmod{cn}$. For each positive integer r,

$$\begin{aligned} 10^r - 1 &= (10-1)((10)^{r-1} + (10)^{r-2} + \cdots + 1) \\ &\equiv 9(1^{r-1} + 1^{r-2} + \cdots + 1) \\ &\equiv 9r \pmod{81}. \end{aligned}$$

Hence,

$$\begin{aligned} a_n \dots a_0 &= a_n(10)^n + a_{n-1}(10)^{n-1} + \cdots + a_0 \\ &\equiv a_n(9n+1) + a_{n-1}(9(n-1)+1) + \cdots + a_0 \pmod{81} \\ &\equiv (a_n + \cdots + a_0) + 9(na_n + (n-1)a_{n-1} + \cdots + a_1) \pmod{81}. \end{aligned}$$

By the same token,

$$a_0 \dots a_n \equiv (a_0 + \cdots + a_n) + 9(na_0 + (n-1)a_1 + \cdots + a_{n-1}) \pmod{81}.$$

Adding these congruences shows that

$$\begin{aligned} a_n \dots a_0 + a_0 \dots a_n &\equiv 2(a_n + \cdots + a_0) + 9n(a_n + \cdots + a_0) \pmod{81} \\ &\equiv (2+9n)(a_n + \cdots + a_0) \pmod{81}. \end{aligned} \tag{1}$$

Suppose that $a_n \dots a_0$ is divisible by 81, if $a_n + \cdots + a_0$ is also divisible by 81, then from (1) the same will be true of the reverse $a_0 \dots a_n$. Conversely, if the reverse $a_0 \dots a_n$ is divisible by 81,

$$(2 + 9n)(a_n + \cdots + a_0)$$

will also by divisible by 81, as will $a_n + \cdots + a_0$, since $2 + 9n$ is coprime to 81. Thus we have proved our first result.

Result 1 *If a number is divisible by 81, then its reverse is divisible by 81 if and only if the sum of its digits is divisible by 81.*

As an example of a number divisible by 81 whose reverse is divisible by 81, how about

818 181 818 181 818 181?

Sceptical readers are invited to check for themselves that its reverse really is divisible by 81!

Divisibility by 27

The argument above works with mod 81 replaced by mod 27. Thus we have our second result.

Result 2 *If a number is divisible by 27, then its reverse is divisible by 27 if and only if the sum of its digits is divisible by 27.*

Divisibility by 121

The congruences

$$\begin{aligned} (10)^r - (-1)^r &= (10+1)((10)^{r-1} - (10)^{r-2} + (10)^{r-3} - \cdots + (-1)^{r-1}) \\ &\equiv 11((-1)^{r-1} - (-1)^{r-2} + (-1)^{r-3} - \cdots + (-1)^{r-1}) \pmod{121} \\ &\equiv 11r(-1)^{r-1} \pmod{121} \end{aligned}$$

show that

$$\begin{aligned}
a_n \dots a_0 &= a_n(10)^n + a_{n-1}(10)^{n-1} + \dots + a_0 \\
&\equiv a_n(11n(-1)^{n-1} + (-1)^n) \\
&\quad + a_{n-1}(11(n-1)(-1)^{n-2} + (-1)^{n-1}) + \dots + a_0 \pmod{121} \\
&\equiv (-1)^n(a_n - a_{n-1} + \dots + (-1)^n a_0) \\
&\quad + (-1)^{n-1}11(na_n - (n-1)a_{n-1} + \dots + (-1)^{n-1}a_1) \pmod{121}.
\end{aligned}$$

By the same token,

$$\begin{aligned}
a_0 \dots a_n &\equiv (-1)^n(a_0 - a_1 + a_2 - \dots + (-1)^n a_n) \\
&\quad + (-1)^{n-1}11(na_0 - (n-1)a_1 + \dots + (-1)^{n-1}a_{n-1}) \pmod{121}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(-1)^n(a_n \dots a_0) + a_0 \dots a_n \\
&\qquad \equiv 2(a_n - a_{n-1} + a_{n-2} - \dots + (-1)^n a_0) \\
&\qquad\quad - 11(na_n - na_{n-1} + na_{n-2} - \dots + (-1)^n na_0) \pmod{121} \\
&\qquad \equiv (2 - 11n)(a_n - a_{n-1} + a_{n-2} - \dots + (-1)^n a_0) \pmod{121}. \qquad (2)
\end{aligned}$$

Suppose that $a_n \dots a_0$ is divisible by 121. If $a_n - a_{n-1} + a_{n-2} - \dots + (-1)^n a_0$ is divisible by 121, then from (2) the same will be true of the reverse $a_0 \dots a_n$. Conversely, if the reverse $a_0 \dots a_n$ is divisible by 121,

$$(2 - 11n)(a_n - a_{n-1} + a_{n-2} - \dots + (-1)^n a_0)$$

will also be divisible by 121. Since $2 - 11n$ is coprime to 121, this means that $a_n - a_{n-1} + a_{n-2} + \dots + (-1)^n a_0$ is divisible by 121. We have thus reached our destination.

Result 3 *If a number is divisible by 121, then its reverse is divisible by 121 if and only if the alternating sum of its digits is divisible by 121.*

As an example of a number divisible by 121 whose reverse is also divisible by 121, but the alternating sum of its digits is not zero, how about the forty-three digit

6 050 605 060 506 050 605 060 506 050 605 060 506 050 605 ?

Could this be the *largest* number ever to be written out explicitly in *Mathematical Spectrum*?

David Sharpe *taught at the University of Sheffield until his final swansong in 2011 (which is not divisible by 3 or 9 or 11!). He is currently editor of* Mathematical Spectrum*.*

Roger Webster *lectures on the history of mathematics at Sheffield University. His* Mathematical Spectrum *article* On the trail of Reverse Divisors: 1089 and all that follow *(with co-author Gareth Williams) to be published in a future issue, reflects (!) his interest in digit reversal.*

Double-Angle Theorem for Triangles

HAROLD STENGEL

When does a triangle have one angle twice as big as another?

Before retiring, I taught secondary mathematics for 18 years at Boston Latin School, established in 1635, a year before Harvard. I am also an alumnus of Latin School and felt it a privilege to teach at such an illustrious and historic institution. As a student, I had many inspiring teachers, and as a teacher I worked with an excellent staff. One of my colleagues, Dan Pratt, had been teaching the law of cosines to one of his classes and he used a 4–5–6 triangle to illustrate the theorem. Remarkably this triangle has the property that the measure of the angle opposite the side of length 6 is twice the measure of the angle opposite the side of length 4. He showed me this discovery and wondered if you can classify triangles where two of the angles' measures are in the ratio of $2:1$. I looked into Dan's query and as a result proved three theorems concerning this type of triangle (see figure 1).

Theorem 1 *The measures of two angles of a triangle are in the ratio of* $2:1$ *if and only if the lengths of the sides are in the ratio of* $1:x:x^2-1,\ 1<x<2$. *Note the angle opposite the side corresponding to* x *has twice the measure of the angle opposite the side corresponding to* 1.

Proof Since one angle's measurement is twice another, the measurements can be represented by θ, 2θ, and $180-3\theta$, $0<\theta<60$. Using trigonometry formulas, we have

$$\sin(180-3\theta)=\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

Using the law of sines,

$$\begin{aligned} a:c:b &= \sin\theta : \sin 2\theta : \sin(180-3\theta)\\ &= \sin\theta : 2\sin\theta\cos\theta : 3\sin\theta - 4\sin^3\theta\\ &= 1 : 2\cos\theta : 3-4\sin^2\theta\\ &= 1 : 2\cos\theta : 3-4(1-\cos^2\theta)\\ &= 1 : 2\cos\theta : 4\cos^2\theta - 1\\ &= 1 : x : x^2-1, \qquad 1<x<2. \end{aligned}$$

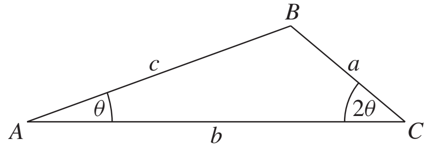

Figure 1

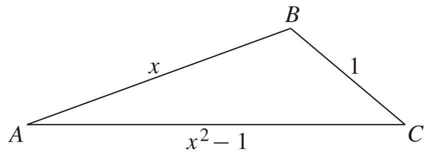

Figure 2

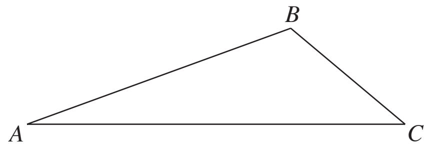

Figure 3

Proving the converse statement (see figure 2), given

$$BC : AB : AC = 1 : x : x^2 - 1,$$

with $1 < x < 2$, then by the law of cosines,

$$\begin{aligned}
\cos C &= \frac{1 + (x^2 - 1)^2 - x^2}{2(x^2 - 1)} = \frac{x^4 - 3x^2 + 2}{2(x^2 - 1)} = \frac{(x^2 - 2)(x^2 - 1)}{2(x^2 - 1)} = \frac{x^2 - 2}{2}, \\
\cos A &= \frac{x^2 + (x^2 - 1)^2 - 1}{2x(x^2 - 1)} = \frac{x^4 - x^2}{2x(x^2 - 1)} = \frac{x}{2}, \\
\cos 2A &= 2\cos^2 A - 1 = 2\left(\frac{x}{2}\right)^2 - 1 = \frac{x^2 - 2}{2} = \cos C,
\end{aligned}$$

which implies that $\angle C = 2\angle A$.

Theorem 2 *In $\triangle ABC$ (see figure 3), $\angle C = 2\angle A$ if and only if $AB^2 - BC^2 = AC \cdot BC$ and $BC < AB < 2BC$.*

Proof Suppose that $\angle C = 2\angle A$. Using theorem 1 we have

$$\frac{BC}{1} = \frac{AB}{x} = \frac{AC}{x^2 - 1} \quad \Longrightarrow \quad x = \frac{AB}{BC}$$

and

$$x^2 - 1 = \frac{AC}{BC} \quad \Longrightarrow \quad \left(\frac{AB}{BC}\right)^2 - 1 = \frac{AC}{BC} \quad \Longrightarrow \quad AB^2 - BC^2 = AC \cdot BC.$$

Note that since $AB/BC = x$ and $1 < x < 2$, then $BC < AB < 2BC$.

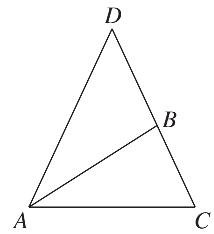

Figure 4

Proving the converse statement, if $AB^2 - BC^2 = AC \cdot BC$ and $BC < AB < 2BC$,

$$\begin{aligned} BC : AB : AC &= BC : AB : \frac{AB^2 - BC^2}{BC} \\ &= 1 : \frac{AB}{BC} : \frac{AB^2 - BC^2}{BC^2} \\ &= 1 : \frac{AB}{BC} : \left(\frac{AB}{BC}\right)^2 - 1. \end{aligned}$$

If we let $x = AB/BC$, then

$$BC : AB : AC = 1 : x : x^2 - 1, \qquad 1 < x < 2,$$

and so by theorem 1 $\angle C = 2\angle A$.

We can use this to prove the following result.

Theorem 3 *In figure 4, if $AD = CD$ and AB bisects $\angle CAD$, then*

$$AB^2 = \frac{BC^3}{BD} + 2 \cdot BC^2.$$

Proof We have

$$AD = CD \quad \Longrightarrow \quad \angle CAD = \angle ACB$$

and

$$AB \text{ bisects } \angle CAD \quad \Longrightarrow \quad \angle BAC = \tfrac{1}{2}\angle CAD = \tfrac{1}{2}\angle ACB.$$

By theorem 2, $AB^2 - BC^2 = AC \cdot BC$, and by the angle-bisector theorem

$$\begin{aligned} \frac{BC}{BD} = \frac{AC}{AD} \quad &\Longrightarrow \quad AC = \frac{BC \cdot AD}{BD} = \frac{BC \cdot CD}{BD} = \frac{BC(BC + BD)}{BD} = \frac{BC^2}{BD} + BC \\ &\Longrightarrow \quad AB^2 - BC^2 = BC\left(\frac{BC^2}{BD} + BC\right) \\ &\Longrightarrow \quad AB^2 - BC^2 = \frac{BC^3}{BD} + BC^2 \\ &\Longrightarrow \quad AB^2 = \frac{BC^3}{BD} + 2 \cdot BC^2. \end{aligned}$$

Definition 1 A *double-angle triangle triple* (*DATT*) is a triple of three relatively prime integers (a, b, c) whose elements are the lengths of the three sides of a triangle with the angle opposite the side of length b being twice the angle opposite the side of length a.

Theorem 4 (DATT theorem) *The triple* (a, b, c) *is a DATT only if it is of the form*

$$(q^2, pq, p^2 - q^2),$$

where p *and* q *are relatively prime positive integers and* $q < p < 2q$.

Proof This follows from theorem 1 if we write x in the form p/q where p and q are relatively prime positive integers.

Problem 1 How many DATTs have 49 as one of their elements?

Solution 1 If $q^2 = 49$ then $q = 7$, which implies $7 < p < 14$, giving six possibilities. If $p^2 - q^2 = 49$ then $p - q = 1$ and $p + q = 49$, which implies $q = 24$, $p = 25$. Therefore there are seven such DATTs.

Harold Stengel *lives in Louisville, Kentucky, USA. He is an accomplished tournament bridge player, having won three national championships, and was a member of US team A that competed in the world championships in Japan in 1991.*

Triangular numbers

The triangular numbers begin 1, 3, 6, 10, ... and generally satisfy $T_n = n(n+1)/2$, where $n \geq 1$. Did you know that there are infinitely many triangular numbers T_a and T_b such that $T_a^2 + T_b^2 = T_c$? We observe that

$$\begin{aligned} 1^2 + 3^2 &= 10 = T_4, \\ 3^2 + 6^2 &= 45 = T_9, \\ 6^2 + 10^2 &= 136 = T_{16}, \\ 10^2 + 15^2 &= 325 = T_{25}, \end{aligned}$$

suggesting that $T_n^2 + T_{n+1}^2 = T_{T_n + T_{n+1}}$, $n \geq 1$, and this pleasing fact is easily verified algebraically. Also, we can show that there are infinitely many triangular numbers satisfying $T_a + T_b = T_c^2$. The computations

$$\begin{aligned} T_2 + T_3 &= 3 + 6 = 9 = T_2^2, \\ T_5 + T_6 &= 15 + 21 = 36 = T_3^2, \\ T_9 + T_{10} &= 45 + 55 = 100 = T_4^2, \\ T_{14} + T_{15} &= 105 + 120 = 225 = T_5^2, \end{aligned}$$

suggest that $T_{T_n - 1} + T_{T_n} = T_n^2$, $n \geq 2$, again verifiable algebraically.

Bridgewater State University, USA **Tom Moore**

The Arithmetic Mean-Geometric Mean Inequality

DANIEL GOULD

For positive real numbers $a_1, \ldots, a_n$, their arithmetic mean is given by

$$F(a_1, \ldots, a_n) = \frac{a_1, \ldots, a_n}{n},$$

and their geometric mean by

$$G(a_1, \ldots, a_n) = \sqrt[n]{a_1 a_2 \ldots a_n}.$$

There are many proofs that $G(a_1, \ldots, a_n) \leq F(a_1, \ldots, a_n)$, I have discovered the following proof which I have not seen elsewhere.

We can assume, without loss of generality, that $a_1 \leq a_2 \leq \ldots \leq a_n$. We first note that the two means are equal if all the a_is are equal, since $G(a_1, \ldots, a_1) = a_1 = F(a_1, \ldots, a_1)$. Suppose that, for some i $(1 \leq i \leq n-1)$,

$$G(a_1, a_2, \ldots, a_{i-1}, a_i, a_i, \ldots, a_i) \leq F(a_1, a_2, \ldots, a_{i-1}, a_i, a_i, \ldots, a_i).$$

Consider the functions f, g of a single variable x given by

$$f(x) = F(a_1, \ldots, a_i, x, \ldots, x) \qquad g(x) = G(a_1, \ldots, a_i, x, \ldots, x)$$

so that $g(a_i) \leq f(a_i)$. Now

$$\begin{aligned} f(x) &= \frac{a_1 + \ldots + a_i + (n-i)x}{n}, \\ g(x) &= (a_1 \ldots a_i x^{n-i})^{1/n}, \end{aligned}$$

so that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{n-i}{n}$$

and

$$\begin{aligned} \frac{\mathrm{d}g}{\mathrm{d}x} &= \frac{n-i}{n} \frac{(a_1 \ldots a_i x^{n-i})^{1/n}}{x} \\ &= \frac{n-i}{nx} G(a_1, \ldots, a_i, x, \ldots, x) \\ &\leq \frac{n-i}{nx} G(x, \ldots, x, x, \ldots, x) \quad \text{(when } x \geq a_i\text{)} \\ &= \frac{n-i}{n^x} x \\ &= \frac{\mathrm{d}f}{\mathrm{d}x}. \end{aligned}$$

Hence, the slope of the graph of g is less than or equal to the slope of the graph of f at all the points $x \geq a_i$. Since $g(a_i) \leq f(a_i)$, it follows that $g(x) \leq f(x)$ for all $x \geq a_i$. Hence,

$$G(a_1, a_2, \ldots, a_i, a_{i+1}, a_{i+1}, \ldots, a_{i+1}) \leq F(a_1, a_2, \ldots, a_i, a_{i+1}, a_{i+1}, \ldots, a_{i+1}).$$

After n steps, this gives

$$G(a_1, \ldots, a_n) \leq F(a_1, \ldots, a_n)$$

as required.

__Daniel Gould__ is a 16 year old student at Ampleforth College. He hopes to study mathematics at University and enjoys more challenging areas of the subject, thinking up solutions to problems which occur to him in his studies (particularly in geometry and calculus). He likes Linux and computer graphics, both animation and rendering, and has been successful at reverse engineering competitions run by Lego. For fun he also goes to the gym, helps with Ampleforth Television (which involves filming theatrical productions on the College site) and attends Senior Mathematics Club every week. He does gardening to help fund his computer interests.

A property of number 3

If m, n are positive integers and $m+1$ is divisible by 3, then mn^2+1 is also divisible by 3 provided n is not divisible by 3.

Class 10, Eram Convent College, Indira Nagar, Lucknow, India — **Ahid Nabi**

A different way of counting

1 2 3 4 5 6 7 8 9 190 191 192 193 194 195 196 197 198 199 180 181 182...

Can you work out in what base this is? You do not even need a minus sign; you can count down without one thus: 0 19 18 17...10 29 28...

James Whitman

Cup-and-Saucer Derangements

MARTIN GRIFFITHS

In this article we make a relatively short mathematical journey, via what I shall term *cup-and-saucer derangements*, taking us from the subfactorial numbers to Latin rectangles. On the way we shall both encounter and utilise Venn diagrams, PIE, recurrence relations, and generating functions.

1. Introduction

Suppose that five distinguishable cups, represented by C_1 to C_5, are set out in a line, as follows:

$$C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5.$$

A *derangement* of these cups is an arrangement such that no cup is in its original position. Thus,

$$C_3 \quad C_1 \quad C_5 \quad C_2 \quad C_4$$

is a derangement whilst

$$C_3 \quad C_1 \quad C_5 \quad C_4 \quad C_2$$

is not. It is well known that the number of derangements d_n of n distinguishable objects is given by the formula

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \tag{1}$$

These numbers are also known as the *subfactorial* or *rencontres* numbers, and the sequence $\{d_n\}$ appears as A000166 in *The on-line encyclopedia of integer sequences* (see http://oeis.org/), where many other combinatorial interpretations can be found.

The result in (1) may be proved by using the *principle of inclusion and exclusion* (PIE). The idea behind PIE is really quite simple. Let us give an illustrative example. Suppose that 80 students took part in a survey concerning their participation, or otherwise, in Chemistry, Mathematics, and Physics, which we denote by C, M, and P, respectively. The results of the survey were that 43 took C, 49 took M, and 32 took P. Furthermore, 32 took both C and M, 25 took both C and P, and 26 took both M and P. Finally, 21 took all three subjects. The question now is: How many of the 80 students did not take any of these subjects?

One way of approaching this problem is to draw a Venn diagram; see figure 1. We could start by entering the number 21 in the central region; the place at which all three bubbles, representing each of the subjects, overlap. Then, since we know that 32 students take both M and C, it follows that the region representing M and C but not P should contain the number $32 - 21 = 11$. In this way we are able to fill all the regions inside the bubbles, as shown. Finally, the number of students not taking any of these subjects is given by

$$80 - 21 - 11 - 4 - 5 - 7 - 12 - 2 = 18.$$

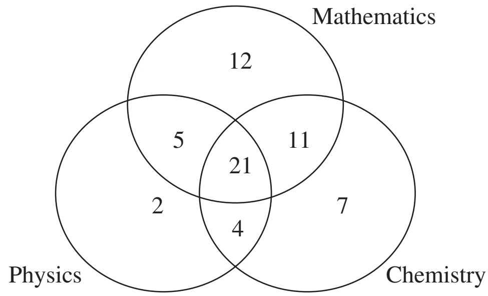

Figure 1 A Venn diagram of students' subjects.

As you can probably imagine, however, this method would become less and less attractive as the number of subjects that were included in the survey was increased. PIE gives us a way of solving such problems without the use of diagrams. Here is how it works. Using the above example once more, let us subtract from 80 the number of students taking C, to give $80-43=37$. We then in turn subtract the number taking M and the number taking P, to give $37-49=-12$ and $-12-32=-44$, respectively. This apparently ridiculous answer stems from the fact that in doing this we have actually subtracted the number doing C and M twice, and similarly for C and P, and M and P; furthermore, we have subtracted the number taking C, M, and P three times! This is counteracted first by adding back on the numbers taking C and M, C and P, and M and P; this gives $-44+32+25+26=39$. Of course, there is now one lot too many of those taking all three subjects, so we subtract this to give $39-21=18$.

In order to prove result (1) via PIE, let A be the set of all possible permutations of n cups (so that A contains $n!$ elements). For any particular $p \in B_n$, where $B_n = \{1, 2, \ldots, n\}$, note that the number of elements of A that keep cup C_p in its original place is $(n-1)!$. Similarly, for any particular pair $p, q \in \{1, 2, \ldots, n\}$ such that $p \neq q$, there are $(n-2)!$ elements of A that keep both C_p and C_q in their original positions. Then, noting that there are $\binom{n}{k}$ ways of choosing k cups from n, we have, on extending the ideas used in the above example,

$$d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!\,(n-k)!} (n-k)! = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

In table 1 we present the first few terms of the sequence $\{d_n\}$.

Suppose that the n cups are removed from their positions on the shelf in order to be used, and that they are then put back at random. What is the probability P_n that the new ordering is a derangement? This is, since the number of possible arrangements is $n!$, very easy to answer. Using (1), we have

$$P_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

This gives

$$\lim_{n\to\infty} P_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{\mathrm{e}}.$$

Table 1 The number of derangements d_n of n objects, $1 \le n \le 8$.

n	d_n
1	0
2	1
3	2
4	9
5	44
6	265
7	1854
8	14 833

Note also that d_n is approximately equal to $n!/\mathrm{e}$. In fact, it can be shown that d_n is actually equal to the nearest integer to $n!/\mathrm{e}$; see reference 1. It may also be expressed as

$$d_n = \left\lfloor \frac{n! + 1}{\mathrm{e}} \right\rfloor,$$

where $\lfloor x \rfloor$ is the *floor* function, denoting the largest integer not exceeding x. Furthermore, d_n satisfies the recurrence relations

$$d_n = (n-1)\,(d_{n-1} + d_{n-2}) \quad \text{and} \quad d_n = nd_{n-1} + (-1)^n$$

for $n \ge 3$ and $n \ge 2$, respectively. Note that these results imply that d_n is divisible by all the prime factors of $n - 1$ but by none of those of n. It is reasonably straightforward to show that the ordinary and exponential generating functions for $\{d_n\}$ are

$$\sum_{k=0}^{\infty} \frac{x^k k!}{(1+x)^{k+1}} \quad \text{and} \quad \frac{1}{\mathrm{e}^x(1-x)},$$

respectively. By this we mean that d_n is equal to the coefficient of x^n in the first function and equal to the coefficient of $x^n/n!$ in the second.

2. Adding the saucers

As we shall see, the mathematics in this section and the next provides a nice link between the relatively straightforward mathematics of derangements and the somewhat more challenging work associated with Latin rectangles.

Let us now suppose that, for each of the n distinguishable cups, there is a matching saucer (the assumption is, therefore, that these n saucers are distinguishable from one another). The situation for $n = 5$ is shown below:

$$\begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \\ S_1 & S_2 & S_3 & S_4 & S_5. \end{array}$$

We consider here the problem of enumerating the number of derangements associated with this rather more complicated scenario.

There is an initial problem to be faced though in that there would seem to be several ways of defining cup-and-saucer derangements. They might enumerate the ways in which

(i) for each of the n positions at least one cup or saucer is not in its original position;

(ii) no cup or saucer is in its original position;

(iii) no pairs are together;

(iv) no pairs are together and no cup or saucer is in its original position.

For example,

$$\begin{array}{ccccc} C_3 & C_4 & C_5 & C_1 & C_2 \\ S_2 & S_4 & S_1 & S_3 & S_5. \end{array}$$

satisfies just (i),

$$\begin{array}{ccccc} C_3 & C_4 & C_5 & C_1 & C_2 \\ S_5 & S_4 & S_1 & S_3 & S_2 \end{array}$$

satisfies (i) and (ii), and

$$\begin{array}{ccccc} C_3 & C_4 & C_5 & C_1 & C_2 \\ S_2 & S_5 & S_1 & S_3 & S_4 \end{array}$$

satisfies (i)–(iv). Let us define e_n, f_n, g_n, and h_n as the number of cup-and-saucer derangements satisfying (i), (ii), (iii), and (iv), respectively. As will become clear, some of these are extremely easy to calculate while others are rather difficult.

It is certainly very straightforward to evaluate f_n. Since there are d_n ways of arranging the cups such that none of them are in their original positions, and similarly for the saucers, it follows that

$$f_n = d_n^2.$$

Furthermore, in order to calculate g_n, it suffices to note that, for each of the $n!$ arrangements of the saucers, there are d_n arrangements of the cups that will not result in a pair being together. (The roles of the cups and saucers in this argument may clearly be interchanged.) It is thus the case that

$$g_n = n!\, d_n.$$

The wording of (i) hints at a possible application of PIE, and this is indeed the case. The total number of arrangements of cups and saucers is $(n!)^2$. Let s_k denote a fixed subset of B_n (defined earlier) comprising exactly k distinct elements, $a_1, a_2, \ldots, a_k$, say. Note that the number of arrangements with $C_{a_1}, C_{a_2}, \ldots, C_{a_k}$ and $S_{a_1}, S_{a_2}, \ldots, S_{a_k}$ in their original positions is given by $((n-k)!)^2$. Then, since the number of ways of choosing k elements from n is $\binom{n}{k}$, we see, by way of PIE, that

$$\begin{aligned} e_n &= (n!)^2 \binom{n}{0} - ((n-1)!)^2 \binom{n}{1} + ((n-2)!)^2 \binom{n}{2} - \cdots + (-1)^n (0!)^2 \binom{n}{n} \\ &= n! \left(\frac{n!}{0!} - \frac{(n-1)!}{1!} + \frac{(n-2)!}{2!} - \cdots + (-1)^n \frac{0!}{n!} \right) \\ &= n! \sum_{k=0}^{n} (-1)^k \frac{(n-k)!}{k!}. \end{aligned}$$

The calculation of h_n is rather more of a challenge, and is something that is considered in the following section.

3. Latin rectangles

We start by looking at certain well-known mathematical objects called *Latin squares*. A Latin square of order n consists of an $n \times n$ array whose entries consist solely of elements from the set B_n such that each entry occurs exactly once in each column and exactly once in each row; see reference 1. Examples of 3×3 Latin squares may be seen in figure 2. As is easily checked, there are in fact 12 different Latin squares of order 3.

Of relevance to our work here, the idea of a Latin square may be extended to that of a *Latin rectangle*. A $k \times n$ Latin rectangle is a $k \times n$ array whose entries consist solely of elements from B_n in such a way that each entry occurs exactly once in each row and at most once in each column; again, see reference 1. Two examples of 3×5 Latin rectangles are given in figure 3.

Let us consider $3 \times n$ Latin rectangles in particular. It is clear that the number of such rectangles increases very rapidly with n. Thus, in counting these objects, we would like to use any labour-saving device that comes to hand. It is certainly possible to use symmetry to good effect here. We keep the bottom row fixed, and count the number of Latin rectangles under this restriction. Since the same number would be obtained whatever the arrangement of the bottom row, and there are $n!$ arrangements of n distinct objects in a line, in order to calculate the total number of $3 \times n$ Latin rectangles, it suffices to obtain the number for a particular arrangement of the bottom row and then multiply this by $n!$. A $3 \times n$ Latin rectangle with the bottom row in numerical order from 1 to n is said to be *reduced*; an example of a reduced 3×5 Latin rectangle can be seen in figure 3. A key point here is that h_n actually gives the number of reduced $3 \times n$ Latin rectangles. For example, the number of 3×3 Latin squares is 12, and so the number of reduced 3×3 Latin squares is $12/3! = 2$, giving $h_3 = 2$.

The following piece of MATHEMATICA® code, currently set at $n = 5$, allows us to calculate h_n for any value of n. It does this by comparing all possible derangements of the cups with all possible derangements of the saucers, and counting the number of times that

3	1	2
1	2	3
2	3	1

3	1	2
2	3	1
1	2	3

Figure 2 Two 3×3 Latin squares, the rightmost of which has its bottom row in numerical order (it is thus termed *reduced*).

5	2	3	1	4
1	5	4	2	3
3	4	2	5	1

2	5	1	3	4
5	1	4	2	3
1	2	3	4	5

Figure 3 Two 3×5 Latin rectangles, the rightmost of which is reduced.

condition (iv) is satisfied. Beyond $n = 7$ this process does take a considerable time to obtain h_n, and it is much more efficient to calculate it via (2) below.

```
n = 5; total = 0; j = Count[Derangements[n],_];
derangematrix = Derangements[n];
For[q = 1, q <= j,
  For[p = 1, p <= j,
    For[k = 0; i = 1, i <= n,
      If[Part[derangematrix,p,i] == Part[derangematrix,q,i], k++];
      i++];
    If[k == 0, total++]; p++]; q++]; total
```

Either way, on utilising MATHEMATICA, we obtain the results given in table 2.

It is possible, though well beyond the scope of this article, to show that the exponential generating function for the number of reduced $3 \times n$ Latin rectangles, and, hence, for h_n, is given by

$$e^{2x} \sum_{k=0}^{\infty} \frac{x^k k!}{(1+x)^{3k+3}}. \tag{2}$$

This result appears in reference 3. Going one step further, a formula for the number of $4 \times n$ Latin rectangles is given in reference 2.

As a final word, Latin rectangles are not just playthings for combinatorialists. Indeed, statisticians make use of them in the *design of experiments*, a discipline that has broad applications across both natural and social sciences.

Table 2 The number h_n of reduced $3 \times n$ Latin rectangles, $1 \le n \le 8$.

n	h_n
1	0
2	0
3	2
4	24
5	552
6	21 280
7	1 073 760
8	70 299 264

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1 P. J. Cameron, *Combinatorics: Topics, Techniques, Algorithms* (Cambridge University Press, Cambridge, 1994).
2 P. G. Doyle, The number of Latin rectangles, 2007. Preprint available at http://arxiv.org/abs/math/0703896v1.
3 I. P. Goulden and D. M. Jackson, *Combinatorial Enumeration* (Wiley, New York, 1983).

***Martin Griffiths** is a Lecturer in Mathematics Education at the University of Manchester. His mathematical interests include number theory, combinatorics and mathematical epidemiology. From an educational point of view, he is fascinated by the notions of intuition and aesthetics in mathematics. In addition to having had a wide range of mathematical papers and articles published, he is the author of a book entitled The Backbone of Pascal's Triangle which aims to introduce able students to certain aspects of combinatorics.*

Letters to the Editor

Dear Editor,

$$1^3 + 2^3 + \cdots n^3 = (1 + 2 + \cdots + n)^2$$

Readers may be interested in *Amjadi's method* of proving this result. Write

$$\begin{aligned} C_n &= 1 + 2 + \cdots + n = \tfrac{1}{2}n(n+1), \\ D_n &= 1^2 + 2^2 + \cdots + n^2, \\ E_n &= 1^3 + 2^3 + \cdots + n^3. \end{aligned}$$

Then

$$\begin{aligned} C_n^2 &= D_n + 2\sum_{i=2}^{n} iC_{i-1} \\ &= D_n + 2\sum_{i=1}^{n} i\frac{1}{2}(i-1)i \\ &= D_n + (E_n - 1) - (D_n - 1); \end{aligned}$$

whence, $E_n = C_n^2$.

Yours sincerely,

Abbas Rouholamini Gugheri
(Students' Investigation House
Shariati Avenue
Sirjan
Iran)

Dear Editor,

A trigonometric inequality

Michel Bataille in his article 'A Trip from Trig to Triangle' (Volume 44, Number 1, pp.19–23) presented the two trigonometric inequalities

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\frac{3}{2},$$
$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\le\frac{3\sqrt{3}}{2},$$

where A, B, and C are the angles of a triangle. I would like to remind readers of the following inequality:

$$\cos A+\cos B+\cos C\le\frac{3}{2}.$$

Below we give a proof of this.

It is obvious that there are at least two acute angles in any triangle. We suppose that $A, B\le\pi/2$. The function $f(x)=\cos x$ is a concave function on $[0,\pi/2]$ (i.e. concave down). Therefore the well-known Jensen's inequality holds:

$$f\left(\frac{x+y}{2}\right)\ge\frac{f(x)+f(y)}{2}\quad\text{for all } x,y\in\left[0,\frac{\pi}{2}\right].$$

Hence,

$$\cos\frac{A+B}{2}\ge\frac{\cos A+\cos B}{2}.$$

Since

$$\cos\frac{A+B}{2}=\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)=\sin\frac{C}{2},$$

we get

$$\cos A+\cos B\le 2\sin\frac{C}{2}.$$

Therefore,

$$\begin{aligned}\cos A+\cos B+\cos C&\le 2\sin\frac{C}{2}+\cos C\\&=2\sin\frac{C}{2}+1-2\sin^2\frac{C}{2}\\&=-2\left(\sin\frac{C}{2}-\frac{1}{2}\right)^2+\frac{3}{2}\\&\le\frac{3}{2},\end{aligned}$$

as claimed.

Yours sincerely,

Spiros P. Andriopoulos
(Third High School of Amaliada
Amaliada City, Eleia
Greece)

Dear Editor,

Conjectures about triangular numbers

Following from Tom Moore's item about triangular numbers in Volume 44, Number 2, p. 50, I would like to propose the following conjectures and questions:

1. The equation $T_x + T_y = z^2$ has no solutions in positive integers x, y, z except when $x = y + 1$ or $y = x + 1$. (We note that $T_x + T_{x-1} = x^2$.)

2. The only values of x for which $T_x = F_x$, where F_x is the xth Fibonacci number, are $x = 1$ and 10.

3. The only triangular numbers in the Fibonacci sequence are 1, 3, 21, 55.

4. The equation $T_x = S_y$ has only two solutions, $(x, y) = (1, 1)$ and $(10, 5)$, where $S_y = 1^2 + 2^2 + \cdots + y^2$.

5. I have found the following solutions to the equation $T_x T_y = T_z$:

$$(x, y, z) = (4, 6, 20), (4, 12, 39), (7, 10, 55), (26, 37, 702), (43, 61, 1891), (154, 218, 23\,870), (253, 358, 64\,261).$$

Can readers find other solutions, and what is the general solution?

Yours sincerely,
Abbas Rouholamini Gugheri
(Students' Investigation House
Shariati Avenue
Sirjan
Iran)

Dear Editor,

Consecutive numbers divisible by consecutive squares

In Volume 43, Number 3, p. 132, Abbas Rouholamini provided examples of three consecutive numbers which are divisible by squares. Here we seek examples of three consecutive integers which are divisible by *consecutive* squares. Specifically, we seek positive integers N, u, x, y, z such that

$$N = u^2 x, \quad N - 1 = (u-1)^2 y, \quad N + 1 = (u+1)^2 z.$$

We thus require

$$u^2 x - (u-1)^2 y = 1, \tag{1}$$
$$u^2 x - (u+1)^2 z = -1. \tag{2}$$

One solution of equation (1) is

$$(x, y) = ((u-2)^2, u^2 - 2u - 1);$$

One solution of (2) is

$$(x, z) = ((u^2 - 2, (u-1)^2).$$

We need the values of x to be the same in the two equations. Equation (1) has solutions

$$(x, y) = ((u-2)^2 + t(u-1)^2, (u-2u-1) + tu^2)$$

for all integers t; equation (2) has solutions

$$(x, z) = (u^2 - 2 + p(u+1)^2, (u-1)^2 + pu^2$$

for all integers p, so we require

$$(u-2)^2 + t(u-1)^2 = u^2 - 2 + p(u+1)^2.$$

This equation is satisfied by

$$t = \frac{1}{2}(2u-3)(u+2), \qquad p = \frac{1}{2}(2u-3)(u-2).$$

To ensure that t and p are integers, we need u to be even. These values of t and p give

$$\begin{aligned} x &= \frac{1}{2}(2u^4 - 3u^3 - 4u^2 + 5u + 2), \\ y &= \frac{1}{2}(2u^4 + u^3 - 4u^2 - 4u - 2), \\ z &= \frac{1}{2}(2u^4 - 7u^3 + 8u^2 - 4u + 2), \end{aligned}$$

so these values of x, y, z will give three consecutive integers divisible by three consecutive cubes for all even positive integers u. For example, $u = 2$ gives $x = 2, y = 7, z = 1,$ and

$$\begin{aligned} N - 1 = (u-1)^2 y &= 1^2 \times 7 = 7, \\ N = u^2 x &= 2^2 \times 2 = 8, \\ N + 1 = (u+1)^2 z &= 3^2 \times 1 = 9. \end{aligned}$$

The value $u = 4$ gives

$$\begin{aligned} N - 1 &= 3^2 \times 247 = 2223, \\ N &= 4^2 \times 139 = 2224, \\ N + 1 &= 5^2 \times 89 = 2225. \end{aligned}$$

The value $u = 6$ gives

$$\begin{aligned} N - 1 &= 5^2 \times 1319 = 32\,975, \\ N &= 6^2 \times 916 = 32\,976, \\ N + 1 &= 7^2 \times 673 = 32\,977. \end{aligned}$$

Yours sincerely,

M. A. Khan
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Indian Overseas Bank,
Ashok Marg,
Lucknow,
India)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st July will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

45.5 Prove that

$$\int_0^{\pi/4} \mathrm{e}^{\tan x} \cos x \, \mathrm{d}x > 1.$$

(Submitted by Spiros Andriopoulos, Third High School of Amaliada, Eleia, Greece. Dedicated to Hazel Perfect)

45.6 The trapezium $ABCD$ has $\angle ABC = \angle BAD = 90°$, and K is the orthogonal projection of B on to CD. Prove that $DK = DA$ if and only if $CD = CB$.

(Submitted by Michel Bataille, Rouen, France)

45.7 The points A, B on the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

are such that AB is perpendicular to the x-axis, and the point P with coordinates $(m, 0)$ is such that A, B and P are not collinear. The line PB meets the ellipse again at E. Show that AE passes through the point $(a^2/m, 0)$.

(Submitted by Zhang Yun, Sunshine School of Xi An Jiaotong University, China)

45.8 Find a simple expression for the infinite product

$$P = (1+x)(1+x^2)(1+x^4)\dots(1+x^{2^{n-1}})\dots,$$

where $|x| < 1$. If P_n is the nth partial product, how are $P - P_{n+1}$ and $P - P_n$ related? (Submitted by J. A. Scott, Chippenham, UK)

Solutions to Problems in Volume 44 Number 3

44.9 In figure 1, $BE = DF$. Obtain a relationship involving the sides of the quadrilaterals $ABCD$ and $AECF$, and deduce that

$$AE^2 + AF^2 = CE^2 + CF^2$$

when angles BAD and BCD are right angles.

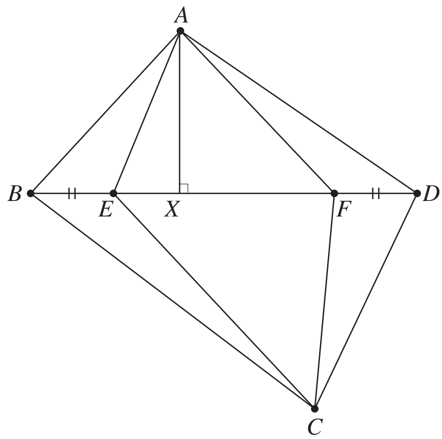


Figure 1

Solution

Drop the perpendicular from A to meet BD at X, as shown in figure 1. By Pythagoras' Theorem,

$$\begin{aligned} AB^2 &= AX^2 + BX^2 \\ &= AE^2 - EX^2 + (BE + EX)^2 \\ &= AE^2 + BE^2 + 2BE \cdot EX. \end{aligned}$$

Similarly,

$$AD^2 = AF^2 + FD^2 + 2FD \cdot FX.$$

Hence, since $BE = FD$,

$$\begin{aligned} AB^2 + AD^2 &= AE^2 + AF^2 + 2BE^2 + 2BE(EX + FX) \\ &= AE^2 + AF^2 + 2BE(BE + EX + FX) \\ &= AE^2 + AF^2 + 2BE \cdot BF. \end{aligned}$$

Similarly,

$$BC^2 + CD^2 = CE^2 + CF^2 + 2BE \cdot BF.$$

Hence,

$$AB^2 + AD^2 + CE^2 + CF^2 = BC^2 + CD^2 + AE^2 + AF^2.$$

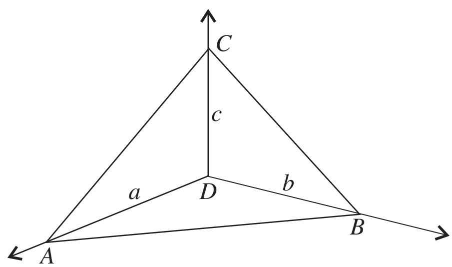
When angles BAD and BCD are right angles,

$$AB^2 + AD^2 = BC^2 + CD^2 = BD^2 \qquad \text{and} \qquad \mathrm{AE}^2 + \mathrm{AF}^2 = \mathrm{CE}^2 + \mathrm{CF}^2.$$

The second part was solved by by Abbas Rouholamini, Sirjan, Iran.

44.10 The tetrahedron $ABCD$ is such that angles ADB, BDC, CDA are right angles. Prove that
$$(\text{area } \triangle ABC)^2 = (\text{area } \triangle ABD)^2 + (\text{area } \triangle BCD)^2 + (\text{area } \triangle CAD)^2.$$

Solution by Tom Moore, who proposed the problem



Denote the areas of $\triangle$s ABC, BCD, CAD, ABD, by S, S_A, S_B, S_C, respectively. By Heron's Formula,
$$S^2 = s(s - \sqrt{b^2+c^2})(s - \sqrt{c^2+a^2})(s - \sqrt{a^2+b^2}),$$
where
$$s = \frac{1}{2}(\sqrt{b^2+c^2} + \sqrt{c^2+a^2} + \sqrt{a^2+b^2}).$$
Hence,
$$\begin{aligned}
S^2 &= \frac{1}{16}(\sqrt{b^2+c^2} + \sqrt{c^2+a^2} + \sqrt{a^2+b^2})(-\sqrt{b^2+c^2} + \sqrt{c^2+a^2} + \sqrt{a^2+b^2}) \\
&\quad \times (\sqrt{b^2+a^2} - \sqrt{c^2+a^2} + \sqrt{a^2+b^2})(\sqrt{b^2+c^2} + \sqrt{c^2+a^2} - \sqrt{a^2+b^2}) \\
&= \frac{1}{16}((\sqrt{c^2+a^2} + \sqrt{a^2+b^2})^2 - (b^2+c^2))((b^2+c^2) - (\sqrt{c^2+a^2} - \sqrt{a^2+b^2})^2) \\
&= \frac{1}{16}(2a^2 + 2\sqrt{(c^2+a^2)(a^2+b^2)})(2\sqrt{(c^2+a^2)(a^2+b^2)} - 2a^2) \\
&= \frac{1}{4}((c^2+a^2)(a^2+b^2) - a^4) \\
&= \frac{1}{4}(b^2c^2 + c^2a^2 + a^2b^2) \\
&= S_A^2 + S_B^2 + S_C^2.
\end{aligned}$$

44.11 A number of the form $2^n - 1$, where n is a positive integer, is called a *Mersenne number* after Père Marin Mersenne (1588–1648). Find all positive integer cubes which can be written as the sum of two Mersenne numbers.

Solution by Tom Moore, who proposed the problem

Suppose that $c^3 = (2^a - 1) + (2^b - 1)$. Then c is even, say $c = 2k$, and
$$8k^3 = 2^a + 2^b - 2,$$
whence,
$$4k^3 + 1 = 2^{a-1} + 2^{b-1}.$$

Since the left-hand side is odd, one of a, b must be 1, say $b = 1$. Then $4k^3 = 2^{a-1}$, whence $k^3 = 2^{a-3}$. Hence, k must be a power of 2, so c must also be a power of 2. Conversely, $(2^t)^3 = (2^1 - 1) + (2^{3t} - 1)$, so the answer is all cubes of powers of 2 greater than 1.

Also solved by Abbas Rouholamini, Sirjan, Iran, and David Christopher, The American College, Madurai, India, who generalized the problem from cubes to kth powers.

44.12 Tatyana loves track and trig. She runs one lap around a circular track with centre O and radius r. Let T be her position on the track at any time and S her starting point. Let $\theta = \angle SOT$, $0 \le \theta \le 2\pi$, and d the distance she has run along the track from S to T. For fixed r, how many values of θ are there such that $d = \sec^2\theta$?

Solution by William Gosnell, who proposed the problem

Since $d = r\theta$ with θ measured in radians, we seek the number of points of intersection of the straight line $y = r\theta$ with the curve $y = \sec^2\theta$ for $0 \le \theta \le 2\pi$. The straight line will be tangential to the curve when $r\theta = \sec^2\theta$ and $r = 2\sec^2\theta\tan\theta$, or $\tan\theta = 1/2\theta$. Solving this numerically gives two solutions, $\theta = \theta_1 \simeq 0.653$ and $\theta = \theta_2 \simeq 3.292$, with corresponding values of r, $r = r_1 \simeq 2.427$ and $r = r_2 \simeq 0.311$. From figure 2 , the number of points of intersection, and so the required number of values of θ, depends on the value of r as given in table 1.

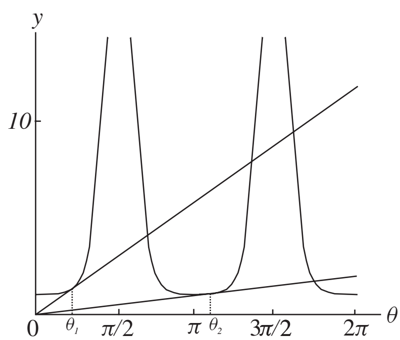


Figure 2

Table 1

Range of r	Number of intersections
$r < 1/(2\pi)$	0
$1/(2\pi) \le r < r_2$	1
$r = r_2$	2
$r_2 < r < r_1$	3
$r = r_1$	4
$r > r_1$	5

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