SKOLIAD No. 82

Shawn Godin

Please send your solutions to the problems in this edition by ***1 March, 2005***. A copy of **MATHEMATICAL MAYHEM Vol. 7** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk ($*$) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

We are gradually shortening the deadline for submitting solutions to Skoliad problems. Note in particular that the deadline for the problems in this issue is the same date as for the problems in the previous issue.

Our items this issue come from Game # 4 of the 1993–1994 Newfoundland and Labrador Teachers Association Senior Mathematics League. My thanks go to Bruce Shawyer of Memorial University of Newfoundland for forwarding the material to me.

The contest is completed by teams of four students. The questions in the first part have a time limit. If a team agrees on the answer and it is correct, the team receives 5 points. If a team cannot agree, then each member puts down an answer and the team receives 1 point for each answer which is correct.

The students work cooperatively on all the questions in the relay. When they have completed all four questions, they show their answer sheet to their proctor, who answers either "Yes" or "No". Of course, "Yes" means that all four are correct, whereas "No" means that at least one is incorrect. The proctor returns the sheet to the students who can, if there is time available, work further. The proctor gives no indication of where any error may lie. There are 5 points for a correct relay answer set. Otherwise, 3 points for questions 1, 2, and 3 correct, 2 points for questions 1 and 2 correct, and 1 point for question 1 correct. Any other combination gets 0 points.

1993–1994 Newfoundland and Labrador Teachers Association Senior Mathematics League
Game #4

1. ($*$) If n is a positive integer then $n!$ (read "n factorial") is defined to be

$$n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots 3 \cdot 2 \cdot 1\,.$$

For example $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ and $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

Determine the positive integer m such that the number of seconds in a year is between $m!$ and $(m+1)!$

2. ($*$) A movie showing was attended by 500 people. Adults paid \$10 each and children \$4 each. The total amount taken for tickets was \$4,160. How many children attended?

3. ($*$) Assume that the earth is a sphere with circumference 40,250 km and that a belt is placed around the equator, one metre above the earth's surface at all points. How much greater than the circumference of the earth would the length of the belt be? Would this difference be:

(a) 2π metres,

(b) 40,250 metres,

(c) $40{,}250\pi$ metres,

(d) 40,250 kilometres, or

(e) none of the above?

4. ($*$) Let a, b and c be integers. You are given that $a \star b$ is defined to be $ab - 2a - 2b + 6$. Compute

$$(a \star b) \star c - a \star (b \star c) \,.$$

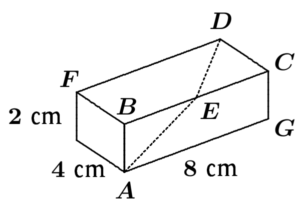
5. ($*$) A cube with sides of length 3 cm is painted red and then cut into $3 \times 3 \times 3 = 27$ cubes with sides of length 1 cm. If a denotes the number of small cubes (that is, 1 cm $\times$ 1 cm $\times$ 1 cm cubes) that are not painted at all, b the number painted on one side, c the number painted on two sides, and d the number painted on three sides, determine $a - b - c + d$.

6. For which value or values of k, if any, is $x^2 + k$ a factor of

$$x^4 - 3x^3 + 6x^2 - 3kx + 8\,?$$

7. An ant wishes to travel from A to D on the surface of a small wooden block with dimensions 2 cm by 4 cm by 8 cm, as shown on the right. The shortest such route involves crossing the edge BC at a point E.

Find the distance BE.



8. A typical large hamburger has 427 calories, 48% of them from fat. The same hamburger with cheese has 31 grams of fat, 53% of its calories coming from fat. Regular French fries have 220 calories in total and 12 grams of fat. A popular sundae has 360 calories, and 28% of these are fat calories.

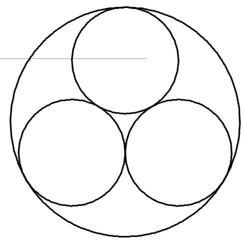
A high school student has a meal consisting of this hamburger with double cheese, an order of regular French fries and the popular sundae. What percentage of calories in the meal are fat calories?

You need to know that 1 gram of fat has 9 calories. Give your answer to the nearest whole number percentage.

9. A bag contains 2 red cabbages and 3 green cabbages. Tracy, who is blindfolded, randomly selects one of the cabbages and places it in an empty pan. Then she randomly selects a second cabbage from those remaining in the bag and also places that in the pan. What is the percentage likelihood that, of the two cabbages that are now in the pan, one is red and the other is green?

10. Three small circles each of radius 1 cm and one larger circle are located as indicated on the right. Determine the area of the larger circle.

Your answer should be expressed in the form $\left(\frac{a+b\sqrt{3}}{c}\right)\pi$, where a, b, and c are integers.



Relay

R1. The sum of five consecutive numbers is 130. Call the smallest of these numbers A.

Write the value of A in box #1 of the relay answer sheet.

R2. A triangle has vertices at $(0,0)$, $(A/6,0)$, and $(0,5)$. How many points with integer coordinates lie inside the triangle?

Write your answer, B, in box #2 of the relay answer sheet.

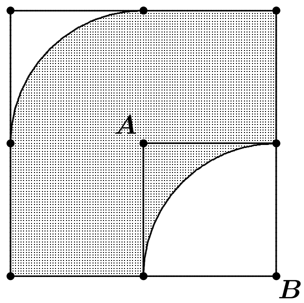
R3. Determine c if

$$\begin{aligned} (B+18)c+7d &= 4\,, \\ d+e &= 20\,, \\ e+f &= 36\,, \\ f+5c &= 15\,. \end{aligned}$$

Write the value of c in box #3 of the relay answer sheet.

R4. The sides of the large square in the diagram are twice the length of the sides of the small square. The two arcs are portions of circles with radii equal to the length of the sides of the small square and with centres at the points A and B. If the area of the hatched region is c^2, determine the length of the sides of the small square.

Write this value in box #4 of the relay answer sheet.



Collections of past problems of the Newfoundland and Labrador Teachers Association Senior Mathematics League appear in the CMS series ATOM for which Bruce Shawyer is the current Editor-in-Chief.

Next we give the solutions to the 2003 Croatian Mathematical Society competitions [2004 : 65–66] and [2004 : 129–130].

Croatian Mathematical Society County-Wide Competition Junior Level (Grade 1), April 4, 2003

1. The lengths of the sides of a triangle ABC are $a = b - \frac{r}{4}$, b, $c = b + \frac{r}{4}$, where r is the radius of the inscribed circle. Determine the lengths of the sides of this triangle as a function of r only.

[*Ed.* In [2004 : 65], there was an error in the statement of the problem. There it was stated that $c = b - \frac{r}{4}$. This has been corrected above. We apologize for the error.]

Official solution.

Let $s = \frac{1}{2}(a+b+c)$. From the formulas for the area of a triangle, we get $sr = \sqrt{s(s-a)(s-b)(s-c)}$. It follows that

$$sr^2 \;=\; (s-a)(s-b)(s-c)\,.$$

By inserting the expressions for a, b, and c, we get

$$\frac{3}{2}br^2 \;=\; \left(\frac{b}{2}+\frac{1}{4}r\right)\frac{b}{2}\left(\frac{b}{2}-\frac{1}{4}r\right)\,,$$

which implies that $b^2 \;=\; \dfrac{49}{4}r^2$; that is, $b = \dfrac{7}{2}r$. Then $a = \dfrac{13}{4}r$ and $c = \dfrac{15}{4}r$.

2. If $a > 0$, determine which points (x, y) in the Cartesian plane satisfy the inequality

$$\big||x+a| - |y-a|\big| \;<\; a\,.$$

Official solution, modified by the editors.

We can rewrite the inequality as

$$-a \;<\; |x+a| - |y-a| \;<\; a\,.$$

We have to consider four cases:

Case 1: $x + a \geq 0$ and $y - a \geq 0$.

Now we have $-a < x - y + 2a < a$, from which we get

$$x + a \;<\; y \;<\; x + 3a\,.$$

These inequalities represent the points between the lines $y = x + a$ and $y = x + 3a$. But we only want the points for which $x \geq -a$ and $y \geq a$, since we have assumed that $x + a \geq 0$ and $y - a \geq 0$.

Case 2: $x + a < 0$ and $y - a < 0$.

Now we have $-a < y - x - 2a < a$, from which we get

$$x + a \;<\; y \;<\; x + 3a\,.$$

These inequalities are the same as in Case 1, but now we want the points for which $x < -a$ and $y < a$.

Case 3: $x + a \geq 0$ and $y - a < 0$.

Now we have $-a < x + y < a$, from which we get

$$-x - a \;<\; y \;<\; -x + a\,.$$

These inequalities represent the points between the lines $y = -x - a$ and $y = -x + a$. But we only want the points for which $x \geq -a$ and $y < a$, since we have assumed that $x + a \geq 0$ and $y - a < 0$.

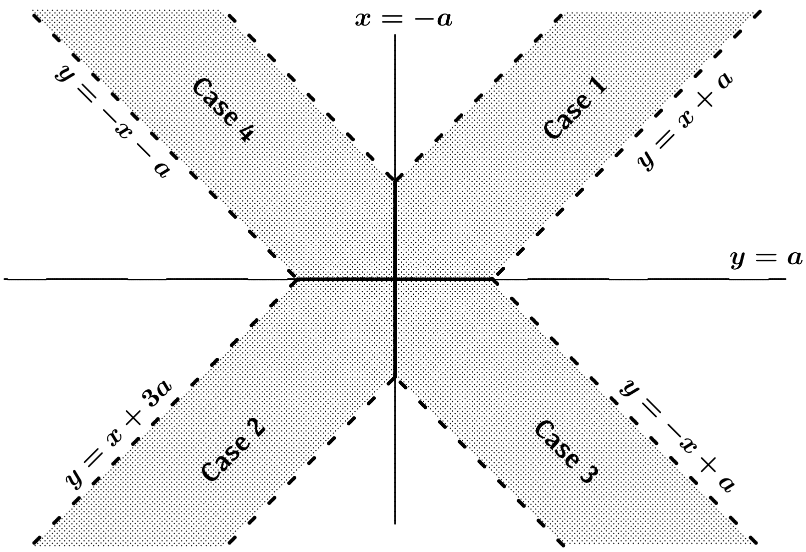
Case 4: $x + a < 0$ and $y - a \geq 0$.

Now we have $-a < -x - y < a$, from which we get

$$-x - a \;<\; y \;<\; -x + a\,.$$

These inequalities are the same as in Case 3, but now we want the points for which $x < -a$ and $y \geq a$.

These four cases come together as shown in the figure below. The shaded region represents the complete solution to the original inequalities. The boundary of the region is not included as part of the solution set. The thick solid lines are included in the solution set; they simply separate the four cases outlined above.



3. Find all integer solutions to the equation

$$4x + y + 4\sqrt{xy} - 28\sqrt{x} - 14\sqrt{y} + 48 \;=\; 0\,.$$

Solution by Luyun Zhong-Qiao, teacher, Columbia International College, Hamilton, ON.

From the given equation, we get

$$\begin{aligned}\left(2\sqrt{x}\right)^2 + 2(2\sqrt{x})(\sqrt{y}) + (\sqrt{y})^2 - 28\sqrt{x} - 14\sqrt{y} + 48 &= 0\,,\\ \left(2\sqrt{x} + \sqrt{y}\right)^2 - 14(2\sqrt{x} + \sqrt{y}) + 48 &= 0\,,\\ \left(2\sqrt{x} + \sqrt{y} - 6\right)\left(2\sqrt{x} + \sqrt{y} - 8\right) &= 0\,.\end{aligned}$$

Thus, $2\sqrt{x} + \sqrt{y} = 6$ or $2\sqrt{x} + \sqrt{y} = 8$. The first of these equations is satisfied by the integer pairs $(x,y) \in \{(0,36),\ (1,16),\ (4,4),\ (9,0)\}$, and the second is satisfied by $(x,y) \in \{(0,64),\ (1,36),\ (4,16),\ (9,4),\ (16,0)\}$. Thus, the complete solution set is

$$\left\{(0,36),\ (1,16),\ (4,4),\ (9,0),\ (0,64),(1,36),\ (4,16),\ (9,4),\ (16,0)\right\}.$$

4. How many four-digit positive integers divisible by 7 have the property that, when the first and last digits are interchanged, the result is a (not necessarily four-digit) positive integer divisible by 7?

Solution by Geneviève Lalonde, Massey, ON.

Suppose $abcd = 1000a + 100b + 10c + d$ is a four-digit number that is divisible by 7. If we interchange the first and last digits, we get the number $dbca = 1000d{+}100b{+}10c{+}a$. This new number is supposed to be divisible by 7. Therefore, the difference $abcd - dbca = 999a - 999d = 999(a-d)$ must also be divisible by 7. Since 999 is not divisible by 7, we must have $7 \mid (a-d)$; that is, $a \equiv d \pmod 7$.

Since $7 \mid abcd$, we must have

$$\begin{aligned}1000a + 100b + 10c + d &\equiv 0 \pmod 7\,,\\ -a + 10(10b + c) + d &\equiv 0 \pmod 7\,,\\ 10b + c &\equiv 0 \pmod 7\,.\end{aligned}$$

The number $10b + c$ can vary from 0 to 99, giving 15 pairs b, c such that $10b + c \equiv 0 \pmod 7$. And there are 14 pairs a, d such that $a \equiv d \pmod 7$ with $1 \le a \le 9$ and $0 \le d \le 9$ (since there are 9 pairs with $a = d$, 3 pairs with $a = d + 7$, and 2 pairs with $d = a + 7$). Therefore, we have a total of $(15)(14) = 210$ numbers with the desired property.

Croatian Mathematical Society National Competition Junior Level (Grade 1), May 7-10, 2003

1. Consider a triangle ABC whose sides have lengths which are prime numbers. Prove that the area of the triangle cannot be an integer.

Official solution.

By Heron's Formula, the square of the area of the triangle with sides a, b, c is $A^2 = s(s-a)(s-b)(s-c)$, where $s = \frac{1}{2}(a+b+c)$. If we let $p = a+b+c$, this can be written as

$$16A^2 \;=\; p(p-2a)(p-2b)(p-2c)\,.$$

Since the left side is even, p must be even. There are two possibilities:

Case 1: All the numbers a, b, c are even.

Because a, b, c are even and prime, we must have $a = b = c = 2$. But this gives $A = \sqrt{3}$, which is not an integer.

Case 2: One of the numbers a, b, c is even, and the other two are odd.

Without loss of generality, we may assume that $a = 2$ and that b and c are odd. If $b \neq c$, we can take $b < c$. Then $c-b \geq 2$; that is, $c \geq b+2 = b+a$. But this is a contradiction of the Triangle Inequality. Hence, $b = c$. Now $p = 2+2b$, and we have:

$$16A^2 \;=\; p(p-4)(p-2b)^2 \;=\; (2+2b)(2b-2)\cdot 4\,,$$

which yields $b^2 - A^2 = 1$; that is, $(b-A)(b+A) = 1$. But because $b+A \geq 2$, this is not possible.

2. The product of the positive real numbers x, y, and z is equal to 1. If

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \;\geq\; x+y+z\,,$$

prove that

$$\frac{1}{x^k}+\frac{1}{y^k}+\frac{1}{z^k} \;\geq\; x^k+y^k+z^k$$

for every positive integer k.

Official solution, modified by the editors.

Since $xyz = 1$, we have $\frac{1}{x} = yz$, $\frac{1}{y} = xz$, and $\frac{1}{z} = xy$. Therefore, the given inequality is equivalent to

$$\begin{aligned}
yz+xz+xy &\;\geq\; x+y+z\,,\\
-x-y-z+yz+xz+xy &\;\geq\; 0\,,\\
1-x-y-z+yz+xz+xy+xyz &\;\geq\; 0\,,\\
(1-x)(1-y)(1-z) &\;\geq\; 0\,. \qquad (1)
\end{aligned}$$

Similarly, for any positive integer k, the inequality that is to be proved is equivalent to

$$(1 - x^k)(1 - y^k)(1 - z^k) \;\geq\; 0\,. \qquad (2)$$

We must show that (1) implies (2). This is obvious if $x = 1$, $y = 1$, or $z = 1$. Therefore, we will assume that $x \neq 1$, $y \neq 1$, and $z \neq 1$. Then, since $xyz = 1$, at least one of x, y, and z must be less than 1, and at least one must be greater than 1. Without loss of generality, assume that $x < 1$ and $y > 1$. Then we must have $z > 1$ to satisfy (1).

Since $x = \dfrac{1}{yz}$, we can write (2) equivalently as

$$\begin{aligned}
\left(1 - \frac{1}{y^k z^k}\right)(1 - y^k)(1 - z^k) \;&\geq\; 0\,, \\
(1 - y^k)(1 - z^k) \;&\geq\; \frac{1}{y^k z^k}(1 - y^k)(1 - z^k)\,, \\
(y^k - 1)(z^k - 1) \;&\geq\; \left(1 - \frac{1}{y^k}\right)\left(1 - \frac{1}{z^k}\right)\,. \qquad (3)
\end{aligned}$$

All the factors appearing in (3) are positive, because $y > 1$ and $z > 1$. Therefore, (3) will follow if we can prove that

$$y^k - 1 \;\geq\; 1 - \frac{1}{y^k} \qquad \text{and} \qquad z^k - 1 \;\geq\; 1 - \frac{1}{z^k}\,. \qquad (4)$$

We claim that if t is any positive real number, then $t - 1 \geq 1 - \dfrac{1}{t}$. To prove this, note that $(t-1)^2 \geq 0$; that is, $t^2 - 2t + 1 \geq 0$. Hence,

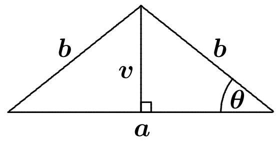
$$\begin{aligned}
t^2 - t \;&\geq\; t - 1\,, \\
t - 1 \;&\geq\; 1 - \frac{1}{t}\,. \qquad (5)
\end{aligned}$$

In (5), we let $t = y^k$ and $t = z^k$, respectively, to obtain (4).

3. Consider an isosceles triangle ABC with base length a whose two equal sides are of length b and whose altitude is of length v. If $\frac{a}{2} + v \geq b\sqrt{2}$, determine the angles of the triangle. Furthermore, if $b = 8\sqrt{2}$, calculate the area of the triangle.

Solution by Geneviève Lalonde, Massey, ON.

In the right triangle with sides $\frac{a}{2}$, v, and b, and with angle θ as shown on the right, we have $v = b\sin\theta$ and $\frac{a}{2} = b\cos\theta$. Hence, $\dfrac{a}{2} + v = b(\sin\theta + \cos\theta)$.



If $\frac{a}{2} + v \geq b\sqrt{2}$, then

$$\begin{aligned} \sin\theta + \cos\theta &\geq \sqrt{2}\,, \\ (\sin\theta + \cos\theta)^2 &\geq 2\,, \\ \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta &\geq 2\,, \\ 1 + \sin(2\theta) &\geq 2\,, \\ \sin(2\theta) &\geq 1\,, \end{aligned}$$

which implies that $\theta = 45^\circ$. Thus, the triangle is an isosceles right triangle.

If $b = 8\sqrt{2}$, the area will be $\frac{1}{2}b^2 = 64$.

4. How many divisors of the number 30^{2003} are not divisors of 20^{2000}?

Solution by Geneviève Lalonde, Massey, ON.

Note that $30^{2003} = 2^{2003} \cdot 3^{2003} \cdot 5^{2003}$ and $20^{2000} = 2^{4000} \cdot 5^{2000}$. Any divisor of 30^{2003} which also divides 20^{2000} must be a divisor of $2^{2003} \cdot 5^{2000}$. There are

$$(2003 + 1) \times (2000 + 1) \;=\; 4{,}010{,}004$$

such divisors. Altogether there are $(2003 + 1)^3 = 8{,}048{,}096{,}064$ divisors of 30^{2003}. Therefore, there must be $8{,}044{,}086{,}060$ divisors of 30^{2003} which do not divide 20^{2000}.

Croatian Mathematical Society City-Level Competition
Junior Level (Grade 1), March 7, 2003

1. A road construction unit is made up of a certain number of workers and a certain amount of equipment. Three units have paved 20 km of a road in 10 days. How many additional units are needed if the remaining 50 km of the road must be paved in 15 days?

Official solution.

Let d be the length of the section of road to be paved; let x be the total number of units used to do this paving; let y be the number of days needed; and let a be the constant of proportionality. Then $d = axy$. During the first 10 days of paving, we have $d = 20$, $x = 3$, and $y = 10$, implying that

$$a \;=\; \frac{d}{xy} \;=\; \frac{2}{3}\,.$$

For the remaining section of road, we have $d = 50$ and $y = 15$. Therefore,

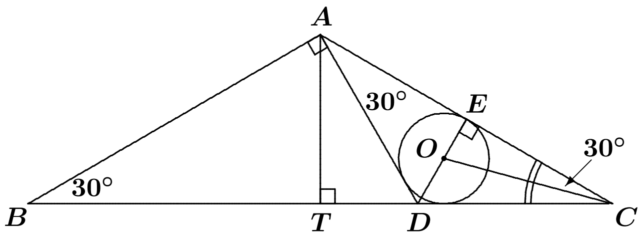
$$x \;=\; \frac{d}{ay} \;=\; \frac{50}{\frac{2}{3} \cdot 15} \;=\; 5\,.$$

Thus, the number of additional units that are needed is $5 - 3 = 2$.

2. Let $\triangle ABC$ be an isosceles triangle whose angle at vertex A equals 120°. The line passing through this vertex and perpendicular to one of the adjacent sides of the triangle divides the triangle into two triangles, one of which is obtuse and has an inscribed circle with radius equal to 1. Determine the area of $\triangle ABC$.

Solution by Luyun Zhong-Qiao, teacher, Columbia International College, Hamilton, ON.

Let D be the point on BC such that $AD \perp AB$; and let E be on AC so that $AC \perp DE$. Since $\angle BAC = 120^\circ$ and $\angle BAD = 90^\circ$, we have $\angle B = \angle C = 30^\circ$, $\angle DAC = 30^\circ$, and $\angle ADC = 120^\circ$. Since $\triangle ACD$ is isosceles, the segment DE passes through O, the centre of the inscribed circle of $\triangle ACD$. Therefore, $OE = 1$ (the radius of the circle).



Since $OE = 1$, we see that $EC = \tan\angle EOC = \tan 75^\circ = 2 + \sqrt{3}$. Thus, $AC = 2(2+\sqrt{3})$. Then

$$\begin{aligned} CT &= AC\cos 30^\circ = 2(2+\sqrt{3})\tfrac{\sqrt{3}}{2} = \sqrt{3}(2+\sqrt{3}), \\ \text{and} \quad AT &= AC\sin 30^\circ = 2(2+\sqrt{3})\tfrac{1}{2} = 2+\sqrt{3}. \end{aligned}$$

Now the area of $\triangle ABC$ is

$$CT \cdot AT = \sqrt{3}(2+\sqrt{3})^2 = 12 + 7\sqrt{3}.$$

3. Calculate the sum

$$\frac{2}{2\cdot 5} + \frac{2}{5\cdot 8} + \cdots + \frac{2}{1997\cdot 2000} + \frac{2}{2000\cdot 2003}.$$

Solution and generalization by Luyun Zhong-Qiao, teacher, Columbia International College, Hamilton, ON.

In general, consider the sum

$$S = \frac{k}{t_1t_2} + \frac{k}{t_2t_3} + \cdots + \frac{k}{t_{n-1}t_n},$$

where t_1, t_2, …, t_n is an arithmetic sequence with common difference d.

Then

$$\begin{aligned} S &= \frac{k}{d}\left[\left(\frac{1}{t_1}-\frac{1}{t_2}\right)+\left(\frac{1}{t_2}-\frac{1}{t_3}\right)+\cdots+\left(\frac{1}{t_{n-1}}-\frac{1}{t_n}\right)\right] \\ &= \frac{k}{d}\left(\frac{1}{t_1}-\frac{1}{t_n}\right) = \frac{k}{d}\left(\frac{t_n-t_1}{t_1t_n}\right) = \frac{k}{d}\left(\frac{t_1+(n-1)d-t_1}{t_1t_n}\right) \\ &= \frac{k(n-1)}{t_1t_n}. \end{aligned}$$

In the given problem we have $k = 2$, $t_1 = 2$, $t_n = 2003$, and $d = 3$. Therefore, the sum is

$$S = \frac{k}{d}\left(\frac{1}{t_1}-\frac{1}{t_n}\right) = \frac{2}{3}\left(\frac{1}{2}-\frac{1}{2003}\right) = \frac{667}{2003}.$$

4. If the real numbers a, b, c satisfy

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} = 1,$$

prove that

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} = 0.$$

Solution by Luyun Zhong-Qiao, teacher, Columbia International College, Hamilton, ON.

If $a+b+c=0$, then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} = \frac{a}{-a}+\frac{b}{-b}+\frac{c}{-c} = -3.$$

Thus, $a+b+c\neq 0$. Therefore,

$$\begin{aligned} (a+b+c)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) &= a+b+c, \\ \frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}+\frac{a(b+c)}{b+c}+\frac{b(c+a)}{c+a}+\frac{c(a+b)}{a+b} &= a+b+c, \\ \frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}+a+b+c &= a+b+c, \\ \frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} &= 0. \end{aligned}$$

That brings us to the end of another year. This month's winner of a past Volume of Mayhem is Alex Wice. Congratulations, Alex! Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). ~~The Assistant Mayhem Editor is John Grant McLoughlin~~ (University of New Brunswick). The other staff members are Larry Rice (University of Waterloo), Dan MacKinnon (Ottawa Carleton District School Board), and Ian VanderBurgh (University of Waterloo).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier mai 2005**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M151. (Reconsidéré) *Proposé par Babis Stergiou, Chalkida, Grèce.*

Soit a, b et c des nombres réels avec $abc = 1$. Montrer que

$$a^3 + b^3 + c^3 + (ab)^3 + (bc)^3 + (ca)^3 \;\geq\; 2(a^2b + b^2c + c^2a)\,.$$

[*Ed*. Vedula N. Murty, Dover, PA, USA a noté que l'inégalité n'est pas correcte. Son contre-exemple est $a = 2$, $b = -1/2$, $c = -1$. Pour établir cette inégalité, il faut supposer de plus que a, b et c sont positifs.]

M169. *Proposé par Équipe de Mayhem.*

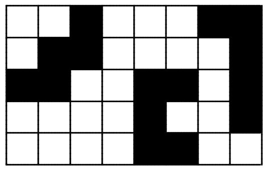
Montrer que

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2003} - \frac{1}{2004} \;=\; \frac{1}{1003} + \frac{1}{1004} + \frac{1}{1005} + \cdots + \frac{1}{2003} + \frac{1}{2004}\,.$$

M170. *Proposé par Équipe de Mayhem.*

Evaluer $\cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \cdots + \cos^2 90^\circ$.

M171. *Proposé par Neven Jurič, Zagreb, Croatie.*



Il y a 12 pentominos distincts (non-congruents), dont 3 apparaissent dans la figure à droite. Chaque pentomino couvre une surface de 5 carrés unités. (À noter que *Pentominoes* est une marque enregistrée de Solomon W. Golomb.)

1. Trouver les 9 pentominos restants.
2. Répartir les 12 pentominos sur les 60 cases numérotées du diagramme à droite, de sorte que chaque pentomino couvre des chiffres dont la somme soit égale à 10.

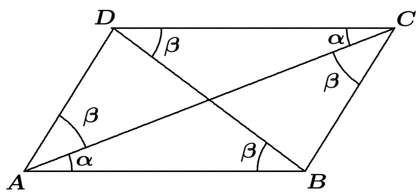
2	1	1	3	1	1		
3	1	1	3	5	2		
3	1	1	2	1	1	4	1
2	1	5	2	1	2	2	1
1	1	5	4	3	1	4	1
3	4	1	5	1	1	1	1
1	4	1	1	2	1	1	3
3	1	1	1	1	1	2	5

M172. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit I le centre du cercle inscrit d'un triangle ABC. Montrer que si l'un des triangles AIB, BIC, ou CIA est semblable au triangle ABC, alors les angles du triangle ABC sont en progression géométrique.

M173. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

On suppose que les diagonales AC et BD d'un parallélogramme $ABCD$ déterminent les angles α et β comme indiqué dans la figure ci-dessous.



1. Montrer qu'un tel arrangement des angles est possible si et seulement si les diagonales sont proportionnelles aux côtés.

2. Utiliser la trigonométrie pour exprimer β en fonction de α.

M174. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

On désigne par x la mesure d'un angle d'un triangle non dégénéré. Déterminer x, sachant que

$$\frac{1}{\sin x} = \frac{1}{\sin 2x} + \frac{1}{\sin 3x}.$$

M175. *Proposé par l'Equipe de Mayhem.*

Un ensemble S est formé de cinq entiers positifs. Montrer qu'il est toujours possible de trouver un sous-ensemble de S, contenant trois éléments, de telle sorte que la somme des éléments de ce sous-ensemble soit un multiple de 3.

M151. (Revisited) *Proposed by Babis Stergiou, Chalkida, Greece.*

Let a, b, c be real numbers with $abc = 1$. Prove that

$$a^3 + b^3 + c^3 + (ab)^3 + (bc)^3 + (ca)^3 \;\geq\; 2(a^2b + b^2c + c^2a)\,.$$

[*Ed.* Vedula N. Murty, Dover, PA, USA has observed that the inequality is incorrect. His counter-example is $a = 2$, $b = -1/2$, $c = -1$. To prove the inequality, it must be assumed that a, b, and c are positive.]

M169. *Proposed by the Mayhem Staff.*

Prove that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2003}-\frac{1}{2004} \;=\; \frac{1}{1003}+\frac{1}{1004}+\frac{1}{1005}+\cdots+\frac{1}{2003}+\frac{1}{2004}\,.$$

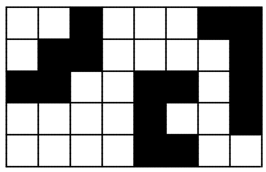
M170. *Proposed by the Mayhem Staff.*

Evaluate $\cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \cdots + \cos^2 90^\circ$.

M171. *Proposed by Neven Jurič, Zagreb, Croatia.*

There are 12 distinct (non-congruent) pentominoes, 3 of which are shown to the right. Each pentomino covers an area of 5 square units. (Note: *Pentominoes* is a registered trademark of Solomon W. Golomb.)

1. Find the remaining 9 pentominoes.
2. Arrange all 12 pentominoes on the 60 numbered cells in the diagram to the right, so that each pentomino covers numbers whose sum is 10.



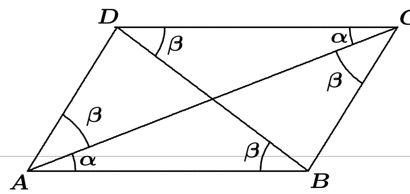
2	1	1	3	1	1		
3	1	1	3	5	2		
3	1	1	2	1	1	4	1
2	1	5	2	1	2	2	1
1	1	5	4	3	1	4	1
3	4	1	5	1	1	1	1
1	4	1	1	2	1	1	3
3	1	1	1	1	1	2	5

M172. *Proposed by Mihály Bencze, Brasov, Romania.*

Let I denote the centre of the inscribed circle in triangle ABC. Prove that if one of the triangles AIB, BIC, or CIA is similar to triangle ABC, then the angles of triangle ABC are in geometric progression.

M173. *Proposed by K.R.S. Sastry, Bangalore, India.*

Suppose that the diagonals AC and BD of a parallelogram $ABCD$ determine angles α and β as shown in the diagram below.



1. Prove that such an arrangement of angles is possible if and only if the diagonals are proportional to the sides.
2. Use trigonometry to express β in terms of α.

M174. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let x denote the measure of an angle of a non-degenerate triangle. Determine x, given that

$$\frac{1}{\sin x} = \frac{1}{\sin 2x} + \frac{1}{\sin 3x}.$$

M175. *Proposed by the Mayhem Staff.*

A set S consists of five positive integers. Show that it is always possible to find a subset of S containing three elements such that the sum of the elements in the subset is a multiple of 3.

Mayhem Solutions

M106. *Proposed by the Mayhem Staff.*

A 4 by 4 square has an area of 16 square units and a perimeter of 16 units. That is, the area and perimeter are numerically equivalent (ignoring units of measurement). Are there any other rectangles with integral dimensions that share this property? If possible, show that you have found all such examples.

Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

If a rectangle with sides x and y has the given property, then we must have $2(x+y) = xy$; that is,

$$y = \frac{2x}{x-2} = 2 + \frac{4}{x-2}.$$

Therefore, the only positive integers x for which y is a positive integer are 3, 4, and 6. It follows that a 3×6 rectangle is the only other one with the given property.

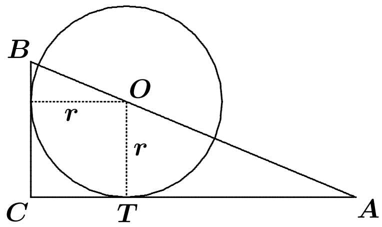
Also solved by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia; Robert Bilinski, Outremont, QC; and Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

M107. *Proposed by the Mayhem Staff.*

A right-angled triangle has legs of length a and b. A circle of radius r touches the two legs and has its centre on the hypotenuse. Show that

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{r}.$$

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.



In $\triangle ABC$, let a and b denote the sides BC and AC, respectively. Let T be the point where the circle touches the side AC. Since $\triangle ABC$ and $\triangle AOT$ are similar, we have $\frac{a}{b} = \frac{r}{b-r}$. Then

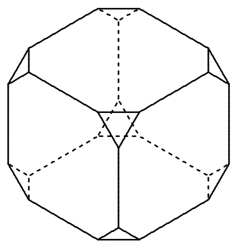
$$\begin{aligned} ab - ar &= rb \\ ab &= r(b+a) \\ \frac{1}{r} &= \frac{b+a}{ab} = \frac{1}{a} + \frac{1}{b}. \end{aligned}$$

Also solved by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia.

M108. *Proposed by the Mayhem Team.*

Given a cube with its eight corners cut off by planes, how many diagonals joining the 24 new 'corners' lie completely inside the cube?

Solution by Geneviève Lalonde, Massey, ON.



A diagonal will lie inside the new figure if it joins two vertices that are not on the same face.

Each vertex is on 3 faces—a triangular face where the corner of the cube used to be, and two octagonal faces which are remnants of the original square faces. Thus, each vertex is on the same face as 7 vertices from one of the octagonal faces and 6 additional vertices from the other octagonal face (since the two octagonal faces share a side and 2 vertices). The vertices from the triangular face have already been counted among the vertices from the two octagonal faces. Therefore, each vertex is connected by internal diagonals to $24-7-6-1 = 10$ other vertices.

Hence, the total number of internal diagonals of the new figure is $\frac{1}{2}(24 \cdot 10) = 120$.

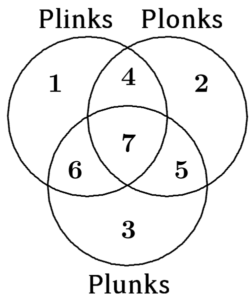
M109. *Proposed by the Mayhem Staff.*

If all plinks are plonks and some plunks are plinks, which of the statements X, Y, Z must be true?

X: All plinks are plunks.

Y: Some plonks are plunks.

Z: Some plinks are not plunks.



Solution by Geneviève Lalonde, Massey, ON.

Consider the most general way that these three sets can be related in the diagram on the right. We will use $n(k)$ to represent the number of members in region k.

Since all plinks are plonks, we have $n(1) = n(6) = 0$. Similarly, since some plunks are plinks, then $n(6) + n(7) \neq 0$. Putting this together with the first condition, we get $n(7) \neq 0$. Now we look at each of the statements X, Y, Z.

Statement X is equivalent to $n(1) = n(4) = 0$, which may or may not be true since we have no information on region 4.

Statement Y is equivalent to $n(5) + n(7) \neq 0$. Since we know that $n(7) \neq 0$, this statement is true.

Statement Z is equivalent to $n(1) + n(4) \neq 0$. We know that $n(1) = 0$, but we have no information on region 4. This means that Z may or may not be true.

Thus, the only statement that must be true is Y.

Also solved by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia; and Robert Bilinski, Outremont, QC.

M110. *Proposed by the Mayhem Staff.*

Given any starting number (other than 1), get a new number by dividing the number 1 larger than your starting number by the number 1 smaller than your starting number. Then do the same with this new number. What happens? Explain!

Solution by Gabriel Krimker, grade 9 student, Buenos Aires, Argentina.

We obtain the starting number.

Indeed, let $x \neq 1$ be the starting number. In the second step we get the number

$$\frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} = \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} = \frac{2x}{2} = x\,,$$

as claimed.

Also solved by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia; Robert Bilinski, Outremont, QC; and Laura Steil, student, Samford University, Birmingham, Alabama, USA.

M111. *Proposed by the Mayhem Staff.*

A crossnumber is like a crossword except that the answers are numbers with one digit in each square. What is the sum of all the digits in the solution to this crossnumber?

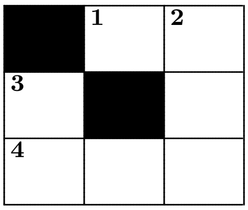
CLUES

Across

1. See 3 Down
3. A cube
4. Five times 3 Down

Down

2. A Square
3. Four times 1 Across



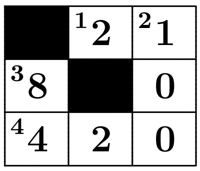
Solution by Laura Steil, student, Samford University, Birmingham, Alabama, USA.

Because 1–across, 3–down, and 4–across are related, we can write expressions for these values using a variable x. Thus, if we let 3–down be equal to x, then we know that 1–across is $x/4$ and 4–across is $5x$. This also tells us that x must be divisible by 4, because 1–across must be an integer.

Since 3–across is a cube and is only one digit, it must be 1 or 8. Also, since 3–down is divisible by 4, the only possibilities are 12, 16, 84, or 88. But we know that 4–across must be 5 times whatever value we use for 3–down. The only case that fits is 84, which makes 4–across equal to 420. Also, since we now know that 3–down is 84, we also know that 1–across is $\frac{1}{4}(84) = 21$.

Now the only value we need is 2–down, and we know that it must be a three-digit square, starting with 1 and ending with 0. The only value that will work is 100.

In summary, we get the solution on the right.



Thus, the sum of the digits that solve the crossnumber is

$$2 + 1 + 8 + 4 + 2 + 0 + 0 = 17 .$$

[*Ed:* Note that all numbers in the puzzle could be zero—a much less interesting solution!]

Also solved by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia; and Robert Bilinski, Outremont, QC.

M112. *Proposed by the Mayhem Staff.*

Given that $ABCDEF$ is a regular hexagon and G is the mid-point of AB, determine the ratio of the total area of hexagon $ABCDEF$ to the area of triangle GDE.

I. *Solution by Gabriel Krimker, grade 9 student, Buenos Aires, Argentina.*

Let r and a be the radius and the apothem of $ABCDEF$ respectively. Applying the Theorem of Pythagoras, we obtain $a = \frac{\sqrt{3}}{2}r$. Then the area of $ABCDEF$ is

$$[ABCDEF] = \frac{6r \cdot a}{2} = \frac{3\sqrt{3}\,r^2}{2}. \qquad (1)$$

Since the altitude of $\triangle GDE$ is $2a$, the area of $\triangle GDE$ is

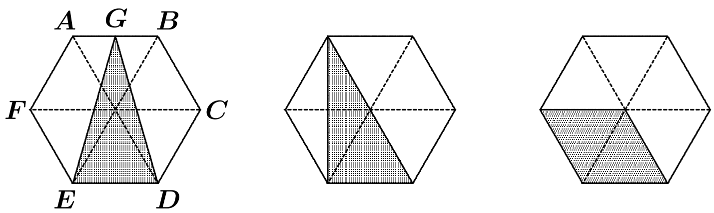
$$[GDE] = \frac{r \cdot 2a}{2} = \frac{\sqrt{3}}{2}r^2. \qquad (2)$$

From (1) and (2), we have

$$\frac{[ABCDEF]}{[GDE]} = \frac{\frac{3\sqrt{3}\,r^2}{2}}{\frac{\sqrt{3}}{2}r^2} = 3.$$

II. *Solution by Robert Bilinski, Outremont, QC.*

We give a proof without words showing that $\frac{[ABCDEF]}{[GDE]} = 3$. (Shaded areas in the diagrams are equal.)



Also solved by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia.

M113. *Proposed by Neven Jurič, Zagreb, Croatia.*

The king is on an open $m \times n$ chessboard. On each of its mn cells the total number of possible moves by the king from that cell is written. Find the sum of all these mn numbers.

Solution by Geneviève Lalonde, Massey, ON.

A king can move one space in any direction (horizontally, vertically, or diagonally). We get 4 cases: if the king is in the interior of the board, he has 8 possible moves (a move to each of the 8 surrounding squares); if he is on an edge but not in a corner, he has 5 possible moves; and if he is in a corner, he has 3 possible moves. Thus, if we place the number of possible moves from each position into that position, our board looks like this:

n columns

m rows

3	5	...	5	3
5	8	...	8	5
⋮	⋮	⋱	⋮	⋮
5	8	...	8	5
3	5	...	5	3

The total of all the numbers on the board is

$$8(m-2)(n-2)+5\Big(2(m-2)+2(n-2)\Big)+3(4) \;=\; 8mn-6m-6n+4\,.$$

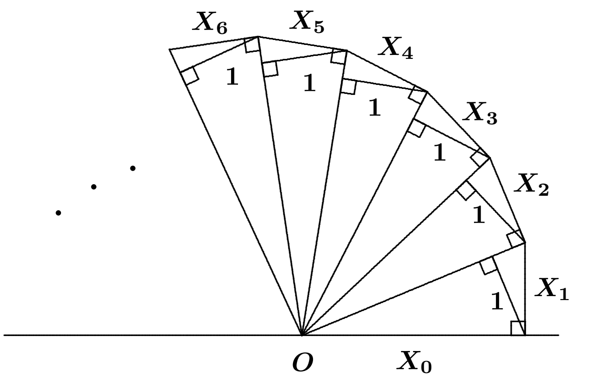
One incorrect solution was received.

M114. *Proposed by Seyamack Jafari, Bandar Imam, Khozestan, Iran.*

In the spiral below prove that

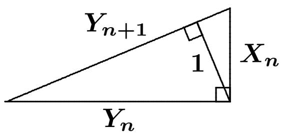
$$X_0^2+X_1^2+X_2^2+\cdots+X_n^2 \;=\; X_0^2\cdot X_1^2\cdot X_2^2\cdots X_n^2\,,$$

where the height of each triangle indicated in the diagram is 1 unit.



Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina, modified by the editors.

For $n = 1, 2, 3, \ldots$, the n^{th} triangle in the spiral is a right triangle in which one leg has length X_n. Let Y_n be the length of the other leg. The hypotenuse then has length Y_{n+1}. Note that $Y_1 = X_0$.



By the Pythagorean Theorem,

$$Y_{n+1}^2 \;=\; X_n^2+Y_n^2\,. \tag{1}$$

On the other hand, by writing the area of the triangle in two different ways (using two different bases and corresponding heights), we get

$$Y_{n+1} \;=\; X_n Y_n \,. \qquad (2)$$

Applying mathematical induction to (1), with $Y_1 = X_0$, it can be shown that

$$Y_{n+1}^2 \;=\; X_0^2 + X_1^2 + X_2^2 + \cdots + X_n^2 \,.$$

Similarly, applying induction to (2), we get

$$Y_{n+1} \;=\; X_0 \cdot X_1 \cdot X_2 \cdots X_n \,.$$

Therefore,

$$X_0^2 + X_1^2 + \cdots + X_n^2 \;=\; Y_{n+1}^2 \;=\; X_0^2 \cdot X_1^2 \cdot X_2^2 \cdots X_n^2 \,.$$

Also solved by Robert Bilinski, Outremont, QC.

M115. *Proposed by the Mayhem Staff.*

The twenty-third term of an arithmetic sequence is three times the value of the fifth term. Find the ratio of the twenty-third term to the first term of the sequence. Express the ratio in the form $p : q$ where p and q are integers.

Solution by Gabriel Krimker, grade 9 student, Buenos Aires, Argentina.

Let $\{a_n\}$ be the sequence in the problem, where $a_n = a_1 + (n-1)d$. Since $a_{23} = 3 \cdot a_5$, we have

$$\begin{aligned} 22d + a_1 &= 3(4d + a_1) \;=\; 12d + 3a_1 \,, \\ 10d &= 2a_1 \,, \\ 5d &= a_1 \,. \end{aligned}$$

Then

$$\frac{a_{23}}{a_1} \;=\; \frac{22d + a_1}{a_1} \;=\; \frac{22d + 5d}{5d} \;=\; \frac{27}{5} \,.$$

Also solved by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia; and Robert Bilinski, Outremont, QC.

M116. *Proposed by the Mayhem Staff.*

A polynomial $f(x)$ satisfies the condition that $f(5 - x) = f(5 + x)$ for all real numbers x. If $f(x) = 0$ has 4 distinct real roots, find the sum of these roots.

Solution by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia.

Since $f(x) = 0$ has 4 distinct real roots and we are only looking at the sum of these roots, we may assume that $f(x)$ is a quartic polynomial. Thus,

$$f(x) \;=\; ax^4 + bx^3 + cx^2 + dx + e \,,$$

and the sum of the roots of $f(x)$ is $-b/a$. Now

$$\begin{aligned} f(5-x) &= ax^4 - x^3(20a+b) + x^2(150a+15b+c) \\ &\quad - x(500a+75b+10c+d) + 625a+125b+25c+5d+e, \\ f(5+x) &= ax^4 + x^3(20a+b) + x^2(150a+15b+c) \\ &\quad + x(500a+75b+10c+d) + 625a+125b+25c+5d+e. \end{aligned}$$

Hence, $20a + b = 0$; that is, $b/a = -20$. Therefore, the sum of the roots of $f(x)$ is 20.

[*Ed.* Note that f is symmetric about $x = 5$; therefore, the sum of the four real roots must be $4 \times 5 = 20$.]

Also solved by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

M117. *Proposed by the Mayhem Staff.*

A person cashes a cheque at the bank. By mistake, the teller pays the number of cents as dollars and the number of dollars as cents. The person spends $\$3.50$ before noticing the mistake, then, on counting the money, finds that the remaining money is exactly double the amount of the cheque. For what amount was the cheque made out?

Solution by Geneviève Lalonde, Massey, ON.

Let d and c represent the number of dollars and cents on the original cheque. Thus, the amount of the cheque, in cents, is $100d + c$. Since we always have less than 100 cents on a cheque, we must have $c < 100$. The information in the problem yields

$$\begin{aligned} 100c + d - 350 &= 2(100d + c)\,, \\ 98c - 350 &= 199d\,, \\ 14(7c - 25) &= 199d\,. \end{aligned}$$

Since 199 is prime, we must have $d = 14k$ for some positive integer k. Then

$$7c - 25 = 199k\,.$$

If $k = 1$, we get $c = 32$ and $d = 14$. When we try $k = 2$ and $k = 3$, we do not get an integer solution for c. Higher values of k will force $c > 100$, which is not allowed. Thus, the amount on the cheque was $\$14.32$.

M118. *Proposed by Andrew Mao, Grade 12 student, A.B. Lucas Secondary School, London, ON.*

You are given a sheet of paper of size 2003×2004. You are allowed to cut it either horizontally or vertically (that is, parallel to an edge). You wish to obtain 2003×2004 unit squares. You are not allowed to fold or stack pieces of the paper. Determine the minimum number of cuts required.

Ed.: No solutions have been received. The problem remains open.

Problem of the Month

Ian VanderBurgh, University of Waterloo

This month, we have a "two-for-one" holiday special—two related problems together in one article! Both problems come from the 2002 Australian Mathematics Competition, and both deal with the sum

$$1 + 11 + 111 + \cdots + \underbrace{111\ldots111}_{2002 \text{ digits}}.$$

Let S be the number obtained by performing this summation.

Problem 1. What are the last five digits of S?

We can answer this question with some careful accounting work.

Solution. If we had a very large sheet of paper on which to write down all 2002 numbers in the above sum, we could do the addition just like we were taught in elementary school. In fact, we can do this calculation without the large sheet of paper. We start by adding up the units column. Since this column consists of 2002 digits all of which are 1, our result is 2002. We write down the 2 and carry 200 to the tens column. Adding up the tens column, we get $200 + 2001 = 2201$. We write down the 1 and carry 220 to the hundreds column. We could continue in this way, but let's turn instead to a more interesting method.

We start by splitting up each term in the sum into powers of 10:

$$S \;=\; 1 + (1 + 10) + (1 + 10 + 100) + \cdots + (1 + 10 + \cdots + 10^{2001}) .$$

Now we note that each term in this sum includes a 1, each term after the first includes a 10, each term after the second includes a 100, and so on. Collecting like powers of 10, we get

$$S \;=\; 2002(1) + 2001(10) + 2000(100) + \cdots + 2(10^{2000}) + 1(10^{2001}) .$$

Since we are interested in determining only the last five digits of the sum, we can ignore terms that end in at least five zeroes. This leaves us with only four terms:

$$2002(1) + 2001(10) + 1999(1000) + 1998(10000) .$$

We could calculate this directly, but we can simplify our task further. We can replace the term 1999(1000) by just 99(1000), since the difference between these is 1900(1000), which ends in five zeroes. Similarly, we can replace 1998(10000) by 8(10000). Thus, the last five digits of S are the same as the last five digits of

$$\begin{aligned} &2002(1) + 2001(10) + 99(1000) + 8(10000) \\ &\qquad = \; 2002 + 20010 + 99000 + 80000 \;=\; 201012 . \end{aligned}$$

Therefore, the last five digits of S are 01012.

That wasn't so bad. However, if we had been asked for the last 100 digits of S, neither of the above methods would have been very appealing (unless we were stuck in a blizzard with nothing to do).

Let us now look at the second problem.

Problem 2. How many times does the digit 1 occur in S?

Here we must try to be a bit more clever.

Sometimes, when a number consisting of a sequence of 1s appears, it is useful to recognize that the number is one-ninth of a number consisting of a sequence of 9s. Well, that doesn't seem totally useful until we recognize that a number consisting of a sequence of 9s is 1 less than a power of 10...

Solution. Using the above idea, we have

$$\begin{aligned} S &= 1 + 11 + 111 + \cdots + \underbrace{111\ldots111}_{\text{2002 digits}} \\ &= \tfrac{1}{9}\left(10-1\right) + \tfrac{1}{9}\left(10^2-1\right) + \tfrac{1}{9}\left(10^3-1\right) + \cdots + \tfrac{1}{9}\left(10^{2002}-1\right) \\ &= \tfrac{1}{9}\Big((10+10^2+10^3+\cdots+10^{2002}) - 2002\Big) \\ &= \tfrac{1}{9}\Big(\underbrace{111\ldots111}_{\text{2002 digits}}0 - 2002\Big) \;=\; \tfrac{1}{9}\Big(\underbrace{111\ldots111}_{\text{1998 digits}}00000 + 11110 - 2002\Big) \\ &= \tfrac{1}{9}\Big(\underbrace{111\ldots111}_{\text{1998 digits}}00000 + 9108\Big) \;=\; \tfrac{1}{9}\Big(\underbrace{111\ldots111}_{\text{1998 digits}}09108\Big) . \end{aligned}$$

We have reduced the problem so that now we just have to divide a very large number by 9. (At this stage, it is worth checking that this very large number is actually divisible by 9. The sum of its digits is $1998 + 9 + 1 + 8 = 2016$, which is divisible by 9. Therefore, the number itself is divisible by 9. That's a relief!)

We could start doing long division and hope to find a pattern, or we could notice that the integer 111111111 is divisible by 9. Using a calculator (or a napkin), we get $111111111 = 9(12345679)$. In our very large number above, we group the 1998 leading 1s into blocks of nine 1s. Thus, we get

$$\begin{aligned} S &= \tfrac{1}{9}\Big(111111111(10^5 + 10^{14} + \cdots + 10^{1994}) + 9108\Big) \\ &= \tfrac{1}{9}(111111111)(10^5 + 10^{14} + \cdots + 10^{1994}) + 1012 \\ &= 12345679(10^5 + 10^{14} + \cdots + 10^{1994}) + 1012 \\ &= 12345679012345679 \cdots 01234567901012 , \end{aligned}$$

where the block "12345679" occurs 222 times (once without a leading 0, 221 times with a leading 0). Therefore, the digit 1 occurs 224 times in S.

Pólya's Paragon

Triangular Tidbits (Part 2)

Shawn Godin

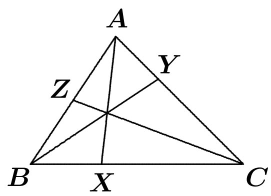
When last we met, we reviewed the definitions of the trigonometric ratios *sine*, *cosine*, and *tangent*, we looked at the Law of Sines, and we saw how the radius of the circumcircle of a triangle is related to the sides and angles of the triangle. In this issue we will continue looking at triangles, but we will save the trigonometry for the homework.

Sometimes in mathematical problem-solving, a technique or concept turns out to be useful even though it seems totally unrelated to the problem at hand. We will see an example of this. But first we need a definition.

Definition In a triangle, a line segment drawn from a vertex to any point on the opposite side is called a *cevian*. (Thus, for example, medians are just special cevians.)

Ceva's Theorem In $\triangle ABC$, if the three cevians AX, BY, and CZ are concurrent, then

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = 1\,.$$



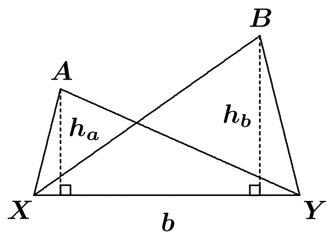
Now let's prove Ceva's Theorem. Since it involves all kinds of ratios, one might be tempted to attack the problem with vectors. You may do that if you wish, for fun, but we will attack it by bringing in an unexpected concept, namely *area*. Before that, however, we need a couple of lemmas. (These are essentially theorems, but we call them lemmas to indicate that they are just steps on the way to our main theorem.)

We will write $[ABC]$ to denote the area of a triangle ABC.

Lemma 1 If two triangles have the same base, the ratio of their areas is equal to the ratio of their respective heights.

Proof: Consider the two triangles AXY and BXY in the diagram. We have $[AXY] = \frac{1}{2}bh_a$ and $[BXY] = \frac{1}{2}bh_b$. Thus,

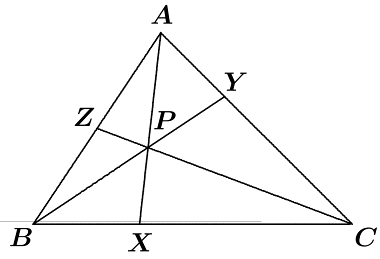
$$\frac{[AXY]}{[BXY]} = \frac{\frac{1}{2}bh_a}{\frac{1}{2}bh_b} = \frac{h_a}{h_b}\,. \qquad \blacksquare$$



Lemma 2 If two triangles have the same height, the ratio of their areas is equal to the ratio of their respective bases.

The proof of this is left as homework.

Now we will prove Ceva's Theorem. Let the point of concurrency of the three cevians be P, as in the diagram on the right.



Clearly, $\triangle ABX$ and $\triangle AXC$ have a common height. Therefore,

$$\frac{[ABX]}{[AXC]} = \frac{BX}{XC}.$$

Similarly, since $\triangle PBX$ and $\triangle PXC$ have a common height, we get $\frac{[PBX]}{[PXC]} = \frac{BX}{XC}$. Since we have common ratios, the ratio of the differences is also the same; that is,

$$\frac{[ABX] - [PBX]}{[AXC] - [PXC]} = \frac{BX}{XC}.$$

When we look at the diagram, we also see that $[ABX] - [PBX] = [ABP]$ and $[AXC] - [PXC] = [APC]$. Hence,

$$\frac{[ABP]}{[APC]} = \frac{BX}{XC}. \tag{1}$$

By similar arguments, we get

$$\frac{[BCP]}{[ABP]} = \frac{CY}{YA} \qquad \text{and} \qquad \frac{[APC]}{[BCP]} = \frac{AZ}{ZB}. \tag{2}$$

Using (1) and (2), we have

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{[ABP]}{[APC]} \cdot \frac{[BCP]}{[ABP]} \cdot \frac{[APC]}{[BCP]} = 1. \qquad \blacksquare$$

The converse of Ceva's Theorem is also true; that is, if three cevians AX, BY, CZ satisfy the equation

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1,$$

then the cevians are concurrent. Note that this equation is certainly satisfied when the three cevians are medians, because then $\frac{BX}{XC} = \frac{CY}{YA} = \frac{AZ}{ZB} = 1$. Thus, we have an easy proof that the medians of a triangle are concurrent.

For homework, use Ceva's Theorem to show that the three altitudes of a triangle are concurrent and that the three angle bisectors are concurrent (don't forget your trigonometry!)

Mayhem Year End Wrap Up

Shawn Godin

December already! How the time has flown. We have continued to see increased contribution to ***Mayhem*** by the readers, and for that I say a heartfelt "Thank you". ***Mayhem*** is really a function of its readers, and their contribution is invaluable to making the journal what it is. We hope that in 2005 we will continue to gain more readers (and contributors), and still hear from our regulars.

At this point I must thank the people without whom I would have had a nervous breakdown long ago! First, and foremost, I would like to thank Mayhem Assistant Editor, JOHN GRANT McLOUGHLIN. John has continued to supply us with very interesting problems and has provided me with guidance and support at every turn.

Next I must thank those who are leaving us and won't be returning in 2005. The first is PAUL OTTAWAY, who started the feature we call *Pólya's Paragon*. His informal articles were a great addition to ***Mayhem***. Paul is continuing his graduate studies. Paul, your contributions will be missed. Also leaving us is LARRY RICE, who assisted with the ***Mayhem*** solutions. Thanks, Larry. We wish all the best to Paul and Larry in their future endeavours.

We have a new face: IAN VANDERBURGH joined us in September and has resurrected the Problem of the Month. Ian's columns have really added to ***Mayhem***. I look forward to more of his columns and to working with him in 2005.

I also need to thank those people who have been so helpful behind the scenes: RICHARD HOSHINO, DAN MACKINNON, BRUCE SHAWYER, and GRAHAM WRIGHT. They were always there when I was in need, and they always came through. Thanks everyone!

All the best of the season to all our readers and contributors! I hope that you have a great year in 2005 and that you will help us continue to make ***CRUX with MAYHEM*** grow and improve. Happy problem solving! We'll see you in 2005.

THE OLYMPIAD CORNER

No. 242

R.E. Woodrow

As a first set, we give the problems of the Thirteenth Irish Mathematical Olympiad. Thanks go to Chris Small, Canadian Team Leader to the 42^{nd} IMO, for collecting the problems.

13^{th} IRISH MATHEMATICAL OLYMPIAD

May 6, 2000

Time: 6 hours

1. Let S be the set of all numbers of the form $a(n) = n^2 + n + 1$, where n is a natural number. Prove that the product $a(n)a(n+1)$ is in S for all natural numbers n. Give, with proof, an example of a pair of elements s, $t \in S$ such that $st \notin S$.

2. Let $ABCDE$ be a regular pentagon with its sides of length one. Let F be the mid-point of AB, and let G and H be points on the sides CD and DE, respectively, such that $\angle GFD = \angle HFD = 30^\circ$. Prove that the triangle GFH is equilateral. A square is inscribed in the triangle GFH with one side of the square along GH. Prove that FG has length

$$t = \frac{2\cos 18^\circ(\cos 36^\circ)^2}{\cos 6^\circ},$$

and that the square has side length $\dfrac{t\sqrt{3}}{2+\sqrt{3}}$.

3. Let $f(x) = 5x^{13} + 13x^5 + 9ax$. Find the least positive integer a such that 65 divides $f(x)$ for every integer x.

4. Let $a_1 < a_2 < a_3 < \cdots < a_M$ be real numbers. The sequence $\{a_1, a_2, \ldots, a_M\}$ is called a *weak arithmetic progression* of length M if there exist real numbers $x_0, x_1, x_2, \ldots, x_M$ and d such that

$$x_0 \le a_1 < x_1 \le a_2 < x_2 \le a_3 < x_3 \le \cdots \le a_M < x_M$$

and $x_{i+1} - x_i = d$ for $i = 0, 1, 2, \ldots, M-1$ (that is, $\{x_0, x_1, x_2, \ldots, x_M\}$ is an arithmetic progression).

(a) Prove that if $a_1 < a_2 < a_3$, then $\{a_1, a_2, a_3\}$ is a weak arithmetic progression of length 3.

(b) Let A be a subset of $\{0, 1, 2, 3, \ldots, 999\}$ with at least 730 members. Prove that A contains a weak arithmetic progression of length 10.

5. Consider all parabolas of the form $y = x^2 + 2px + q$ (for real p, q) which intersect the x– and y–axes in three distinct points. For such a pair p, q, let $C_{p,q}$ be the circle through the points of intersection of the parabola $y = x^2 + 2px + q$ with the axes. Prove that all the circles $C_{p,q}$ have a point in common.

6. Let $x \geq 0$, $y \geq 0$ be real numbers with $x + y = 2$. Prove that

$$x^2y^2(x^2+y^2) \;\leq\; 2\,.$$

7. Let $ABCD$ be a cyclic quadrilateral and R the radius of the circumcircle. Let a, b, c, d be the lengths of the sides of $ABCD$, and let Q be its area. Prove that

$$R^2 \;=\; \frac{(ab+cd)(ac+bd)(ad+bc)}{16Q^2}\,.$$

Deduce that $R \geq \dfrac{(abcd)^{\frac{3}{4}}}{Q\sqrt{2}}$, with equality if and only if $ABCD$ is a square.

8. For each positive integer n, determine, with proof, all positive integers m such that there exist positive integers $x_1 < x_2 < \cdots < x_n$ which satisfy $\dfrac{1}{x_1} + \dfrac{2}{x_2} + \dfrac{3}{x_3} + \cdots + \dfrac{n}{x_n} = m$.

9. Prove that in each set of ten consecutive integers there is one which is coprime with each of the other integers. For example, taking 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, the numbers 119 and 121 are each coprime with all the others. [Two integers a, b are coprime if their greatest common divisor is 1.]

10. Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with non-negative real coefficients. Suppose that $p(4) = 2$ and $p(16) = 8$. Prove that $p(8) \leq 4$, and find, with proof, all such polynomials with $p(8) = 4$.

As a final set for this issue, we give the problems of the Third Hong Kong (China) Mathematical Olympiad. Thanks go to Chris Small, Canadian Team Leader to the 42nd IMO, for collecting these problems for us.

THIRD HONG KONG (CHINA) MATHEMATICAL OLYMPIAD

December 2, 2000

Time: 3 hours

1. Let O be the circumcentre of $\triangle ABC$. Suppose $AB > AC > BC$. Let D be a point on the minor arc BC of the circumcircle. Let E and F be points on AD such that $AB \perp OE$ and $AC \perp OF$. Let P be the intersection of BE and CF. If $PB = PC + PO$, prove that $\angle BAC = 30^\circ$.

2. Let $a_1 = 1$, $a_{n+1} = \frac{a_n}{n} + \frac{n}{a_n}$ for $n = 1, 2, 3, \ldots$. Find the greatest integer less than or equal to a_{2000}. Be sure to prove your claim.

3. Find all prime numbers p and q such that $\frac{(7^p - 2^p)(7^q - 2^q)}{pq}$ is an integer.

4. In the coordinate plane, a *lattice point* is a point with integer coordinates. Find all positive integers $n \geq 3$ such that there exists an n–sided polygon with lattice points as vertices and all sides of equal length.

We turn to the file of readers' solutions for the October 2002 number of the *Corner*. The first group are to problems of the Ukranian Mathematical Olympiad 1999 given [2002 : 353–354].

2. (8th grade). Let us consider the "sunflower" figure (see Figure 1). The cells A, B, C in the "sunflower" are marked. The marker is situated in cell A. Each move of the marker may be one of the moves demonstrated in Figure 2. In how many different ways can the marker move from A to B if the marker cannot visit C?

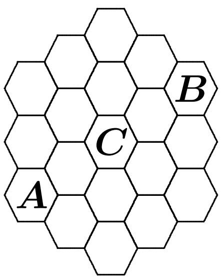


Figure 1

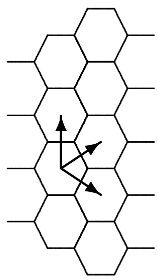
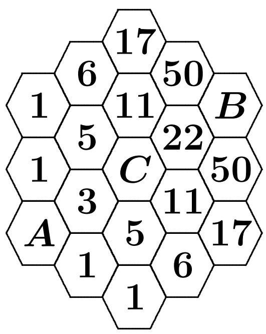


Figure 2

Solution by Robert Bilinski, Outremont, QC, adapted by the editor.

In each cell, we place a number which is the number of ways to reach that cell from cell A. To obtain these values, we first place 1 in hexagon A and 0 in hexagon C. Then, in the cells which neighbour cell A, we place the number which is the sum of the numbers in its neighbouring cells that connect to it by moves that are allowed. We continue in this manner with the result as shown.



The number in hexagon B is

$$50 + 50 + 22 = 122 .$$

This represents the number of paths from A to B avoiding C.

3. (9th grade) Prove that the number $9999999 + 1999000$ is composite.

Solved by Robert Bilinski, Outremont, QC; and Bruce Crofoot, University College of the Cariboo, Kamloops, BC. We give the solution of Crofoot.

Observe that $9999999 = 10^7 - 1$ and $1999000 = 2 \cdot 10^6 - 10^3$. Therefore,

$$\begin{aligned} 9999999 + 1999000 &= 10^7 + 2 \cdot 10^6 - 10^3 - 1 \;=\; 12 \cdot 10^6 - 10^3 - 1 \\ &= (3 \cdot 10^3 - 1)(4 \cdot 10^3 + 1) \;=\; 2999 \cdot 4001 \,. \end{aligned}$$

4. (9th grade) The sequence of positive integers a_1, a_2, ..., a_n, ... is such that $a_{a_n} + a_n = 2n$ for all $n \geq 1$. Prove that $a_n = n$ for all n.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Yuming Chen and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the write-up of Chen and Wang.

We proceed by induction. For $n = 1$, we get $a_{a_1} + a_1 = 2$. Since a_1 is a positive integer, we must have $a_1 = 1$. Assume that for some $n \geq 1$, we have $a_k = k$ for $k = 1, 2, \ldots, n$. Let $a_{n+1} = \ell$. Then

$$a_\ell + \ell \;=\; a_{a_{n+1}} + a_{n+1} \;=\; 2(n+1)\,.$$

If $\ell \leq n$, then $a_\ell = \ell$ (by the induction hypothesis), and hence,

$$2(n+1) \;=\; a_\ell + \ell \;=\; \ell + \ell \;\leq\; 2n \;<\; 2(n+1)\,,$$

a contradiction.

If $\ell > n + 1$, then, since $a_\ell + \ell = 2(n+1)$, we must have $a_\ell < n + 1$. By the induction hypothesis, $a_{a_\ell} = a_\ell$, which implies that

$$2\ell \;=\; a_{a_\ell} + a_\ell \;=\; 2a_\ell \;<\; 2(n+1) \;<\; 2\ell\,,$$

a contradiction again.

Thus, $\ell = n + 1$. That is, $a_{n+1} = n + 1$, completing the induction.

5. (10th grade) Let $P(x)$ be a polynomial with integer coefficients. The sequence of integers x_1, x_2, ..., x_n, ... satisfies the conditions $x_1 = x_{2000} = 1999$, $x_{n+1} = P(x_n)$, $n \geq 1$. Find the value of

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{1999}}{x_{2000}}\,.$$

Solved by Pierre Bornsztein, Maisons-Laffitte, France and Moubinool Omarjee, Paris, France by similar methods. We give Bornsztein's write-up.

Subscripts are considered modulo 1999. For any positive integer n, let $y_n = x_n - x_{n-1}$. Then

$$\sum_{i=1}^{1999} y_i \;=\; \sum_{i=1}^{1999} (x_i - x_{i-1}) \;=\; x_{1999} - x_0 \;=\; 0\,. \tag{1}$$

Suppose that for all n, we have $y_n \neq 0$. Since $P(x)$ has integer coefficients, it is well known that, for any integers $a \neq b$, the integer $a - b$ divides $P(a) - P(b)$. It follows that y_n divides y_{n+1}, for all n. Then the numbers $|y_1|$, $|y_2|$, ..., $|y_n|$, ... form a non-decreasing sequence. Since $|y_1| = |x_1 - x_{1999}| = |x_{2000} - x_{1999}| = |y_{2000}|$, we deduce that $|y_1| = |y_2| = \cdots = |y_{2000}|$. Let $a \neq 0$ be this common value.

Let k be the number of terms among y_1, y_2, ..., y_{1999} which have the value a. Then the remaining $1999 - k$ terms have the value $-a$. Hence,

$$\sum_{i=1}^{1999} y_i = a(2k - 1999) \neq 0,$$

contradicting (1).

It follows that there is some n for which $y_n = 0$; that is, $x_n = x_{n-1}$. An easy induction leads to $x_n = x_1 = 1999$ for all n. Then

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{1999}}{x_{2000}} = 1 + 1 + \cdots + 1 = 1999.$$

6. (10th grade) For real numbers x_1, x_2, ..., $x_6 \in [0, 1]$ prove the inequality

$$\frac{x_1^3}{x_2^5 + x_3^5 + x_4^5 + x_5^5 + x_6^5 + 5} + \frac{x_2^3}{x_1^5 + x_3^5 + x_4^5 + x_5^5 + x_6^5 + 5} + \cdots + \frac{x_6^3}{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + 5} \leq \frac{3}{5}.$$

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Club de mathématiques du lyceé Henri IV, Paris, France; and Moubinool Omarjee, Paris, France. Bornsztein and Omarjee used similar methods. We give Omarjee's write-up.

Since $x_1, \ldots, x_6$ are in the interval $[0, 1]$,

$$x_2^5 + x_3^5 + x_4^5 + x_5^5 + x_6^5 + 5 \geq x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + x_6^5 + 4.$$

By permuting the subscripts, we see that the left side of the inequality in the problem is at most

$$\sum_{i=1}^{6} \frac{x_i^3}{x_1^5 + x_2^5 + \cdots + x_6^5 + 4} = \frac{\sum\limits_{i=1}^{6} x_i^3}{\sum\limits_{i=1}^{6} x_i^5 + 4}.$$

For any $y \geq 0$, the AM–GM Inequality gives us

$$\frac{y^5 + y^5 + y^5 + 1 + 1}{5} \geq \sqrt[5]{y^5 \cdot y^5 \cdot y^5} = y^3;$$

that is, $3y^5 + 2 \geq 5y^3$. Thus,

$$5\sum_{i=1}^{6} x_i^3 \;\leq\; \sum_{i=1}^{6}(3x_i^5 + 2) \;=\; 3\left(\sum_{i=1}^{6} x_i^5 + 4\right) \;;$$

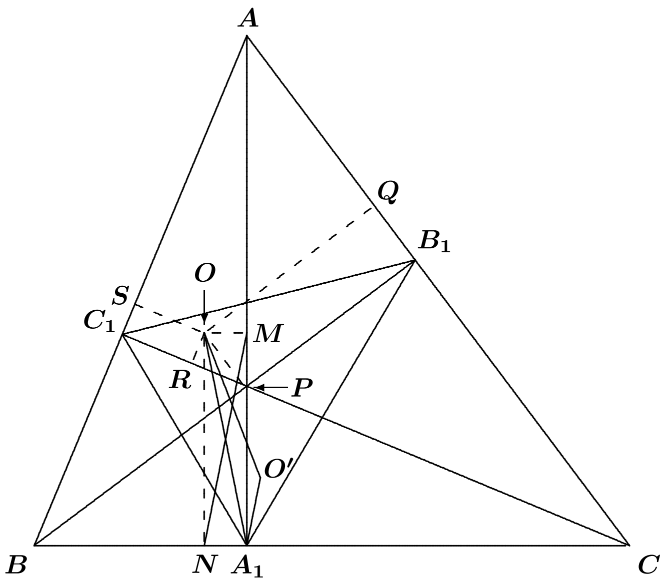
that is,

$$\frac{\sum_{i=1}^{6} x_i^3}{\sum_{i=1}^{6} x_i^5 + 4} \;\leq\; \frac{3}{5}\,.$$

This, together with our initial observations, completes the proof.

8. (11th grade) Let AA_1, BB_1, CC_1 be the altitudes of acute triangle ABC, let O be an arbitrary point inside the triangle $A_1B_1C_1$. Let us denote by M and N the bases of perpendiculars drawn from O to lines AA_1 and BC, respectively, by P and Q — ones from O to lines BB_1 and CA, respectively, by R and S — ones from O to lines CC_1 and AB, respectively. Prove that the lines MN, PQ, RS are concurrent.

Solution by Toshio Seimiya, Kawasaki, Japan.



Since $\angle BB_1C = \angle BC_1C = 90^\circ$, the points B, C, B_1, C_1 are concyclic. Similarly, C, A, C_1, A_1 are concyclic, and A, B, A_1, B_1 are concyclic. Hence, $\angle B_1A_1A = \angle B_1BA = \angle ACC_1 = \angle AA_1C_1$.

Let O' be the isogonal conjugate of O with respect to $\triangle A_1B_1C_1$. Then

$$\begin{aligned} \angle AA_1O &= \angle AA_1B_1 - \angle OA_1B_1 \\ &= \angle C_1A_1A - \angle C_1A_1O' \;=\; \angle O'A_1A\,. \end{aligned} \qquad (1)$$

Since OMA_1N is a rectangle, we get from (1)

$$\angle A_1MN = \angle OA_1M = \angle OA_1A = \angle O'A_1A.$$

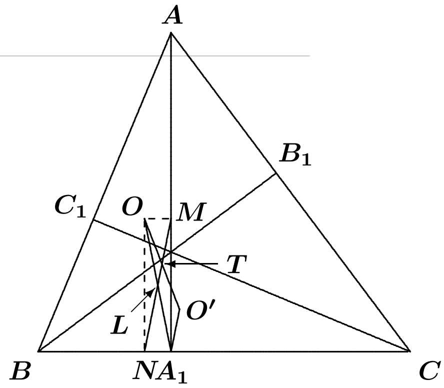
Therefore, $MN \parallel A_1O'$.

Let L be the intersection of OA_1 and MN. Then L is the mid-point of OA_1. Let T be the intersection of OO' with MN. Since $MN \parallel A_1O'$, we have

$$OT : TO' = OL : LA_1 = 1 : 1.$$

Thus, $OT = TO'$, and MN passes through the mid-point of OO'.

Similarly, PQ and RS pass through the mid-point of OO'. Hence, MN, PQ, and RS are concurrent at the mid-point of OO'.



Now we turn to solutions for problems of the XLIII Mathematical Olympiad of Moldova, 10th Form, given [2002 : 354–355].

1. Let the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 2ax - a^2 - \frac{3}{4}$, be considered. Find the values a for which the inequality $|f(x)| \le 1$ is true for every $x \in [0,1]$.

Solved by Robert Bilinski, Outremont, QC; Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's solution, modified by the editors.

We will prove that the values of a for which the inequality $|f(x)| \le 1$ is true for every $x \in [0,1]$ are those such that

$$-\tfrac{1}{2} \le a \le \tfrac{1}{2\sqrt{2}}. \qquad (1)$$

We will need the following function values:

$$f(0) = -a^2 - \tfrac{3}{4}, \qquad f(1) = -a^2 - 2a + \tfrac{1}{4}, \qquad f(a) = -2a^2 - \tfrac{3}{4}.$$

If $|a| > \frac{1}{2}$, then $|f(0)| = a^2 + \frac{3}{4} > \left(\frac{1}{2}\right)^2 + \frac{3}{4} = 1$, and therefore it is not true that $|f(x)| \le 1$ for all $x \in [0,1]$.

If $\frac{1}{2\sqrt{2}} < a \le 1$, then $a \in [0,1]$, and

$$|f(a)| = 2a^2 + \tfrac{3}{4} > 2\left(\tfrac{1}{2\sqrt{2}}\right)^2 + \tfrac{3}{4} = 1.$$

Thus, once again, it is not true that $|f(x)| \le 1$ for all $x \in [0,1]$.

The only values of a remaining to be considered are those satisfying (1). We will prove that, for all such a, we have $|f(x)| \leq 1$ for all $x \in [0,1]$.

Note that $f(x) = (x-a)^2 + f(a)$. The graph of f is a parabola with vertex at $\big(a, f(a)\big)$, opening upward. It follows that the maximum and minimum values of $f(x)$ on the interval can occur only where $x = 0$ or $x = 1$, or where $x = a$ if $a \in [0,1]$. Therefore, the maximum value of $|f(x)|$ on $[0,1]$ can occur only at these points.

Consider any a satisfying (1). Then, since $|a| \leq \frac{1}{2}$, we have

$$|f(0)| \;=\; a^2 + \tfrac{3}{4} \;\leq\; \big(\tfrac{1}{2}\big)^2 + \tfrac{3}{4} \;=\; 1\,.$$

Since $a \geq -\frac{1}{2}$, we have

$$1 - f(1) \;=\; a^2 + 2a + \tfrac{3}{4} \;=\; \big(a + \tfrac{3}{2}\big)\big(a + \tfrac{1}{2}\big) \;\geq\; 0\,,$$

and therefore $f(1) \leq 1$. Also, since $-\frac{5}{2} < a < \frac{1}{2}$, we get

$$f(1) - (-1) \;=\; -a^2 - 2a + \tfrac{5}{4} \;=\; \big(\tfrac{5}{2} + a\big)\big(\tfrac{1}{2} - a\big) \;>\; 0\,,$$

and therefore, $f(1) > -1$. Thus, $|f(1)| \leq 1$.

Finally, if $a \in [0,1]$ and a satisfies (1), then

$$|f(a)| \;=\; 2a^2 + \tfrac{3}{4} \;\leq\; 2\big(\tfrac{1}{2\sqrt{2}}\big)^2 + \tfrac{3}{4} \;=\; 1\,.$$

We conclude that $|f(x)| \leq 1$ for all $x \in [0,1]$.

2. Let n be a natural number such that the number $2n^2$ has 28 distinct divisors and the number $3n^2$ has 30 distinct divisors. How many distinct divisors has the number $6n^2$.
[*Ed:* we suspect that this problem has no solution.]

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We will prove that the statement of the problem is incorrect, as the editor suspected. Let $d(k)$ denote the number of positive divisors of the positive integer k.

(a) We will show that there is no natural number n that satisfies both $d(2n^2) = 28$ and $d(3n^2) = 30$. This will show that the problem is incorrect if the divisors in the problem are required to be positive.

Suppose, for the purpose of contradiction, that the natural number n satisfies $d(2n^2) = 28$ and $d(3n^2) = 30$.

Let $n = \prod\limits_{i=1}^{k} p_i^{\alpha_i}$ be the prime decomposition of n, where the integers p_i are in increasing order. If n is odd, then $d(2n^2) = 2\prod\limits_{i=1}^{k}(2\alpha_i + 1) = 28$; that is,

$$\prod_{i=1}^{k}(2\alpha_i + 1) \;=\; 14\,.$$

This is impossible, since the left side is odd and the right side is even.

Therefore, n must be even. Then $d(2n^2) = (2\alpha_1+2)\prod_{i=2}^{k}(2\alpha_i+1) = 28$ (a product indexed by the empty set is equal to 1); that is,

$$(\alpha_1+1)\prod_{i=2}^{k}(2\alpha_i+1) \;=\; 14 \;=\; 2\times 7\,.$$

Using parity, we deduce that either $\alpha_1 + 1 = 14$ (and $k = 1$), or $\alpha_1 + 1 = 2$ and $2\alpha_2 + 1 = 7$ (and $k = 2$). In the first case, $n = 2^{13}$. Then $d(3n^2) = d(3 \times 2^{26}) = 2 \times 27 \neq 30$. In the second case, $n = 2p^3$, for some odd prime p. If $p \neq 3$, then $d(3n^2) = d(2^2 \times 3 \times p^6) = 3 \times 2 \times 7 \neq 30$; if $p = 3$, then $d(3n^2) = d(2 \times 3^7) = 3 \times 8 \neq 30$. In each case, we have a contradiction.

(b) We will show that there is no natural number n that satisfies both $d(2n^2) = 14$ and $d(3n^2) = 15$. This will show that the problem is incorrect if the divisors in the problem are not required to be positive.

Suppose, for the purpose of contradiction, that the natural number n satisfies $d(2n^2) = 14$ and $d(3n^2) = 15$. If n is odd, then the condition $d(2n^2) = 14$ leads to $\prod_{i=1}^{k}(2\alpha_i+1) = 7$; that is, $n = p^3$ for some odd prime number p. If $p \neq 3$, we have $d(3n^2) = 2 \times 7 \neq 15$; if $p = 3$, we have $d(3n^2) = 8 \neq 15$. In each case we have a contradiction.

Therefore, n must be even. Then the condition $d(2n^2) = 14$ leads to $(\alpha_1+1)\prod_{i=2}^{k}(2\alpha_i+1) = 7$; that is, $n = 2^6$. Now $d(3n^2) = 2 \times 13 \neq 15$, which is a contradiction.

3. All the natural numbers from 1 to 100 are arranged arbitrarily along a circle. The sum of every three consecutively arranged numbers is calculated. Prove that there exist two such sums, with the difference between them being greater than 2.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

First, omit 1 and divide all the remaining numbers into 33 pairwise disjoint groups of three consecutive numbers along the circle. The total sum of these 33 groups is $\sum_{i=2}^{100} i = 5049$, and the average sum is $\dfrac{5049}{33} = 153$. It follows that there is at least one sum, say S, such that $S \geq 153$.

Now, omit 100 and divide all the remaining numbers into 33 pairwise disjoint groups of three consecutive numbers along the circle. The total sum of these 33 groups is $\sum_{i=1}^{99} i = 4950$, and the average sum is $\dfrac{4950}{33} = 150$. It follows that there is at least one sum, say S', such that $S' \leq 150$.

Then $S - S' \geq 3$, and we are done.

5. Find all the functions $f : \mathbb{R} \to \mathbb{R}$, which satisfy the relation

$$x \cdot f(x) = \lfloor x \rfloor \cdot f\big(\{x\}\big) + \{x\} \cdot f\big(\lfloor x \rfloor\big), \quad \forall\, x \in \mathbb{R},$$

where $\lfloor \cdot \rfloor$ and $\{\cdot\}$ denote the integral part and fractional part functions, respectively.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's solution.

The solutions are the constant functions.

Any constant function satisfies the given relation, since $x = \lfloor x \rfloor + \{x\}$. Conversely, consider a function f for which the the given relation holds, and let $C = f(0)$. Substituting an arbitrary non-zero integer k for x in the given relation, we get $kf(k) = k \cdot C + 0 \cdot f(k) = k \cdot C$. Hence, $f(k) = C$. Similarly, for all $x \in (0,1)$, we have $xf(x) = 0 \cdot f(x) + x \cdot f(0) = C \cdot x$. Thus, $f(x) = C$.

Now let x be an arbitrary non-zero real number. Observing that $\lfloor x \rfloor \in \mathbb{Z}$ and $\{x\} \in [0,1)$, we get

$$xf(x) = \lfloor x \rfloor \cdot C + \{x\} \cdot C = C \cdot \big(\lfloor x \rfloor + \{x\}\big) = C \cdot x,$$

and $f(x) = C$ follows. In conclusion, $f(x) = C$ for all real numbers x.,

6. Find a polynomial of degree 3 with real coefficients such that each of its roots is equal to the square of one root of the polynomial $P(X) = X^3 + 9X^2 + 9X + 9$.

Solved by Jean-Claude Andrieux, Beaune, France; Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's solutions.

Let u, v, w be the (complex) roots of $P(X)$. We can obtain the required polynomial by two methods:

Solution I. Since $u + v + w = -9$, $uv + vw + wu = 9$, and $uvw = -9$,

$$\begin{aligned} u^2 + v^2 + w^2 &= (u+v+w)^2 - 2(uv+vw+wu) = 63, \\ u^2v^2 + v^2w^2 + w^2u^2 &= (uv+vw+wu)^2 - 2uvw(u+v+w) = -81, \\ u^2v^2w^2 &= (uvw)^2 = 81. \end{aligned}$$

Thus, the required polynomial is

$$Q(X) = (X - u^2)(X - v^2)(X - w^2) = X^3 - 63X^2 - 81X - 81.$$

Solution II. The polynomial $Q(X) = (X - u^2)(X - v^2)(X - w^2)$ satisfies

$$\begin{aligned} Q(X^2) &= (X^2 - u^2)(X^2 - v^2)(X^2 - w^2) \\ &= (X-u)(X-v)(X-w)(X+u)(X+v)(X+w) \\ &= P(X) \cdot \big(-P(-X)\big) \\ &= (X^3 + 9X)^2 - (9X^2 + 9)^2 = X^6 - 63X^4 - 81X^2 - 81. \end{aligned}$$

Hence, $Q(X) = X^3 - 63X^2 - 81X - 81$.

7. Prove that for all strictly positive numbers a, b, and c the inequality

$$(a+b+x)^{-1} + (b+c+x)^{-1} + (c+a+x)^{-1} \;\le\; x^{-1}\,,$$

holds, where $x = \sqrt[3]{abc}$.

Solution by Michel Bataille, Rouen, France.

Let $(a+b+x)^{-1} + (b+c+x)^{-1} + (c+a+x)^{-1} = N/D$. An easy calculation yields

$$\begin{aligned} N &= 3x^2 + 4x(a+b+c) + a^2 + b^2 + c^2 + 3(ab+bc+ca)\,, \\ D &= x^3 + 2x^2(a+b+c) + x\Big(a^2 + b^2 + c^2 + 3(ab+bc+ca)\Big) \\ &\qquad + a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 2abc\,. \end{aligned}$$

Observing that

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 \;=\; (a+b+c)(ab+bc+ca) - 3abc$$

and $x^3 = abc$, we have

$$\begin{aligned} D &= 2x^2(a+b+c) + x\Big(a^2 + b^2 + c^2 + 3(ab+bc+ca)\Big) \\ &\qquad + (a+b+c)(ab+bc+ca)\,. \end{aligned}$$

Hence, $D - xN = (a+b+c)\big((ab+bc+ca) - 2x^2\big) - 3x^3$.

By the AM–GM Inequality, we have $a+b+c \ge 3\sqrt[3]{abc} = 3x$ and $ab+bc+ca \ge 3\sqrt[3]{a^2b^2c^2} = 3x^2$. Thus,

$$D - xN \;\ge\; 3x(3x^2 - 2x^2) - 3x^3 \;=\; 0\,.$$

It follows that $\dfrac{N}{D} \le \dfrac{1}{x}$, as required.

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

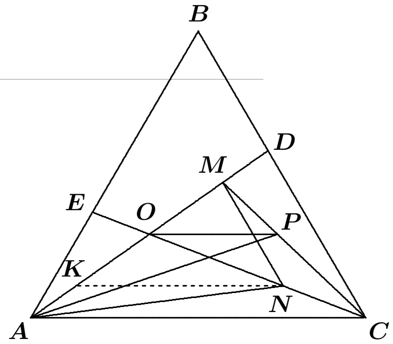
Letting $\alpha = a^3$, $\beta = b^3$ and $\gamma = c^3$, this problem is similar to problem #5 in the USAMO 1997, a solution of which appears in *Math. Magazine*, Vol. 71, no. 3 (June 1998), p. 237.

8. On the sides BC and AB of the equilateral triangle ABC the points D and E, respectively, are taken such that $CD : DB = BE : EA = (\sqrt{5}+1)/2$. The straight lines AD and CE intersect in the point O. The points M and N are interior points of the segments OD and OC, respectively, such that $MN \parallel BC$ and $AN = 2\,OM$. The parallel to the straight line AC, drawn through the point O, intersects the segment MC in the point P. Prove that the half-line AP is the bisectrix of the angle MAN.

Comments by Toshio Seimiya, Kawasaki, Japan.

The conditions of the problem statement cannot all be true. We will show that the condition $AN = 2OM$ cannot be true if the other conditions are true.

Since $CD : DB = (\sqrt{5}+1) : 2$, we get $CD : CB = (\sqrt{5}+1) : (\sqrt{5}+3)$. By Menelaus' Theorem for $\triangle ABD$, we have



$$\frac{BE}{EA}\cdot\frac{AO}{OD}\cdot\frac{DC}{CB} \;=\; 1\,;$$

that is, $\dfrac{\sqrt{5}+1}{2}\cdot\dfrac{AO}{OD}\cdot\dfrac{\sqrt{5}+1}{\sqrt{5}+3} = 1$. Thus,

$$\frac{AO}{OD} \;=\; \frac{2(\sqrt{5}+3)}{(\sqrt{5}+1)^2} \;=\; 1\,;$$

whence, $AO = OD$.

We are given that M and N are points on the segments OD and OC, respectively, such that $MN \parallel DC$. Let K be the point on OA such that $KN \parallel AC$. Then $OK : OA = ON : OC = OM : OD$. Since $OA = OD$, we see that $OK = OM$. Since $\angle KMN = \angle ADC > \angle ABC = 60^\circ$ and $\angle KNM = \angle ACD = 60^\circ$, we have $\angle KMN > \angle KNM$. Therefore, $KN > KM = 2OM$. Since

$$\angle AKN \;=\; 180^\circ - \angle DAC > 180^\circ - \angle BAC \;=\; 120^\circ\,,$$

it follows that $AN > KN$. Therefore, $AN > 2OM$.

Thus, it is not true that $AN = 2OM$.

Now we turn to solutions to problems of the XLIII Mathematical Olympiad of Moldova, 11th–12th Forms, given [2002 : 355–356].

1. Grandfather distributes n sweets among n grandchildren arranged along a circle: first of all he gives one sweet to some grandchild, then he gives one sweet to the next grandchild, then one sweet skipping one grandchild, then one sweet skipping two grandchildren and so on. The distribution is executed in the same direction. For what values of n does every grandchild get a sweet?

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

This problem is equivalent to problem #2 of the Tournament of the Towns 1985 (Spring, Main Version). A solution appears in *International Mathematics Tournament of Towns 1984–1989, questions and solutions*, edited by P.J. Taylor, published by Australian International Centre for Mathematics Enrichment, Australia.

2. Let the number $n \in \mathbb{N}^*$ be given. Denote by M the set of all real numbers x for which there exists a finite sequence (a_p), $p = 1, \ldots, n$, with $a_p \in \{0, 1\}$, $p = 1, \ldots, n$, such that

$$x = 2^{-1} \cdot a_1 + 2^{-2} \cdot a_2 + \cdots + 2^{-n} \cdot a_n .$$

(a) Determine the set M, and prove that for every number $x \in M$ there exists a unique finite sequence (a_p), $p = 1, \ldots, n$, with the mentioned property.

(b) Find the function $f : M \to \mathbb{R}$ such that if (a_p) is the sequence defined above by the number x, then

$$f(x) = 2^{-1} \cdot 2000^{a_1} + 2^{-2} \cdot 2000^{a_2} + \cdots + 2^{-n} \cdot 2000^{a_n} , \quad \forall x \in M .$$

Solution by Pierre Bornsztein, Maisons-Laffitte, France, modified by the editors.

(a) Let $M' = \{2^n x : x \in M\}$. Then M' consists of all real numbers y that can be represented in the form

$$y = a_n + 2a_{n-1} + \cdots + 2^{n-1} a_1 ,$$

where $a_p \in \{0, 1\}$ for $p = 1, 2, \ldots, n$. The above representation for y is simply the binary expansion of y. Thus, $M' = \{0, 1, \ldots, 2^n - 1\}$ and therefore, $M = \{2^{-n} y : y \in M'\}$. The uniqueness of the representation for each $x \in M$ follows from the uniqueness of the binary expansion of each $y \in M'$.

(b) Let $x = 2^{-1}a_1 + 2^{-2}a_2 + \cdots + 2^{-n}a_n \in M$. Let $I = \{p \mid a_p \neq 0\}$. Then $x = \sum_{p \in I} 2^{-p}$. It follows that

$$\sum_{p \notin I} 2^{-p} = \sum_{p=1}^{n} 2^{-p} - \sum_{p \in I} 2^{-p} = 1 - 2^{-n} - x .$$

We deduce that

$$\begin{aligned} f(x) &= \sum_{p \in I} \frac{2000}{2^p} + \sum_{p \notin I} \frac{1}{2^p} = 2000x + 1 - 2^{-n} - x \\ &= 1999x + 1 - 2^{-n} . \end{aligned}$$

5. Find all the integer values of m, for which the equation

$$\left\lfloor \frac{m^2 x - 13}{1999} \right\rfloor = \frac{x - 12}{2000}$$

has 1999 distinct real solutions ($\lfloor \cdot \rfloor$ denotes the integral part function).

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

If x is a solution of the given equation, then $x = 2000k + 12$ for some integer k. Moreover, the given equation may be rewritten as $k = m^2k + \left\lfloor \dfrac{m^2k + 12m^2 - 13}{1999} \right\rfloor$; that is, $f(k) = 0$, where

$$f(k) \;=\; (m^2 - 1)k + \left\lfloor \frac{m^2k + 12m^2 - 13}{1999} \right\rfloor .$$

Suppose that $m^2 > 1$. Since f is the sum of an increasing function and a non-decreasing function on $\mathbb{Z}$, the function f is increasing on $\mathbb{Z}$. It follows that the equation $f(k) = 0$ has at most one solution. Therefore, m is not a solution of the problem.

If $m = 0$, then the given equation becomes $-1 = \dfrac{x - 12}{2000}$, which does not have 1999 solutions. Thus, $m = 0$ is not a solution of the problem.

Finally, suppose that $m^2 = 1$. Then $f(k) = \left\lfloor \dfrac{k-1}{1999} \right\rfloor = 0$, and the equation $f(k) = 0$ has solutions $k = 1, 2, \ldots, 1999$.

Thus, the desired values of m are $m = 1$ and $m = -1$.

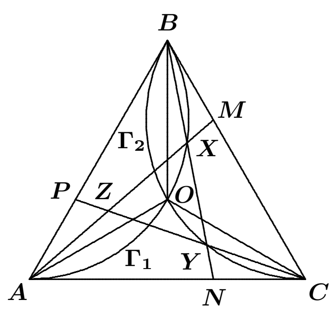
7. Prove that the number $a = \dfrac{m^{n+1} + n^{n+1}}{m^m + n^n}$ satisfies the relation $a^m + a^n \geq m^m + n^n$ for non-zero natural numbers m and n.

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

This problem is #4 of the USAMO 1991. A solution appears in the booklet of Dr. Walter Mientka, published by the MAA.

8. On the sides BC, AC and AB of the equilateral triangle ABC the points M, N and P, respectively, are considered such that $AP : PB = BM : MC = CN : NA = \lambda$. Find all the values λ for which the circle with the diameter AC covers the triangle bounded by the straight lines AM, BN and CP. (In the case of concurrent straight lines, the mentioned triangle degenerates into a point.)

Solution by Toshio Seimiya, Kawasaki, Japan.



Let the intersections of AM and BN, BN and CP, and CP and AM be X, Y, and Z, respectively, and let O be the circumcentre of $\triangle ABC$.

Since $AP = BM = CN$, we see that $\triangle PAC \equiv \triangle MBA \equiv \triangle NCB$, from which we have

$$\angle PCA \;=\; \angle MAB \;=\; \angle NBC .$$

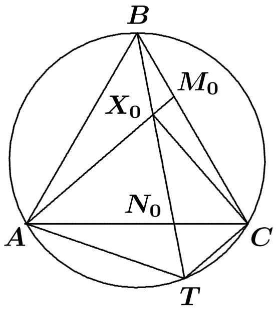
It follows that $\angle AXB = \angle BYC = \angle CZA = 120^\circ$.

We denote the circumcircles of $\triangle OAB$ and $\triangle OBC$ by Γ_1 and Γ_2, respectively. Since $\angle AOB = \angle BOC = 120°$, we have $\angle AXB = \angle AOB$ and $\angle BYC = \angle BOC$. Therefore, X lies on the minor arc AOB of Γ_1, and Y lies on the minor arc BOC of Γ_2.

If M varies on the side BC from B to C, then X moves on the arc BOA from B to A, and if N varies on the side CA from C to A, then Y moves on the arc BOC from C to B.

If $\lambda = 1$, then X, Y, and Z coincide with O. If $\lambda < 1$, then X lies on the minor arc BO of Γ_1 and Y lies on the minor arc CO of Γ_2. If $\lambda > 1$, then X lies on the minor arc AO of Γ_1 and Y lies on the minor arc BO of Γ_2.

Now we consider the case $\lambda = \frac{1}{2}$. In this case we denote M, N, and X by M_0, N_0, and X_0, respectively.



Let T be the second intersection of BN_0 with the circumcircle of $\triangle ABC$. Then

$$\begin{aligned} \angle ATN_0 &= \angle ATB = \angle ACB = 60° \\ \text{and} \quad \angle CTN_0 &= \angle CTB = \angle CAB = 60° . \end{aligned}$$

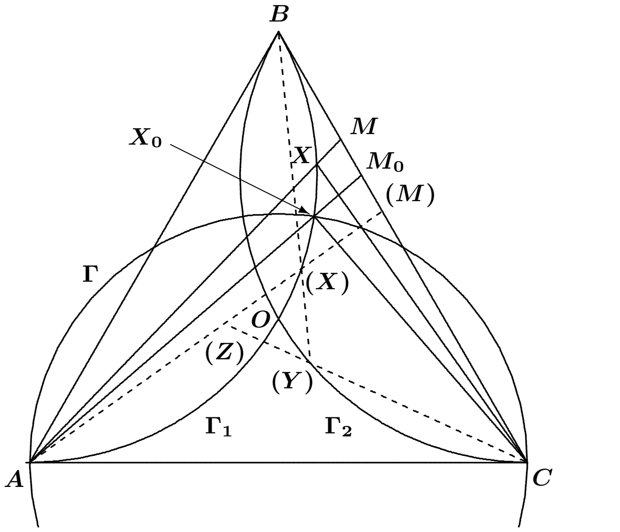
Thus, $\angle ATN_0 = \angle CTN_0$; whence,

$$AT : TC = AN_0 : N_0C = 1 : \lambda = 2 : 1 ,$$

which implies that $AT = 2CT$.

Since $\angle AX_0T = 60°$ and $\angle ATX_0 = 60°$, we see that $\triangle ATX_0$ is equilateral. Hence, $X_0T = AT = 2CT$. Since $\angle X_0TC = 60°$, it follows that $\angle X_0CT = 90°$. Hence, $\angle TX_0C = 30°$. Therefore,

$$\angle AX_0C = \angle AX_0T + \angle TX_0C = 60° + 30° = 90° .$$



We denote the circle with diameter AC by Γ.

If $\lambda < \frac{1}{2}$, then X lies on the minor arc BX_0 of Γ_1, which means that $\angle AXC < \angle AX_0C = 90^\circ$. Thus, X is an exterior point of Γ, from which we see that $\triangle XYZ$ is not contained in Γ.

If $\frac{1}{2} \leq \lambda \leq 1$, then X lies on the minor arc X_0O of Γ_1. Thus,

$$\begin{array}{rrcl} & \angle AXC & \geq & \angle AX_0C \;=\; 90^\circ\,, \\ & \angle AYC & \geq & \angle AXC \;\geq\; 90^\circ\,, \\ \text{and} & \angle AZC & \geq & \angle AXC \;\geq\; 90^\circ\,. \end{array}$$

Hence, X, Y, and Z are contained in Γ. Thus, $\triangle XYZ$ is contained in Γ.

Finally, suppose that $\lambda > 1$. Let $\mu = 1/\lambda$. Then $0 < \mu < 1$, and $AN : NC = CM : MB = BP : PA = \mu$. The above argument shows that for $\frac{1}{2} \leq \mu \leq 1$ the triangle XYZ is contained in Γ, and for $\mu < \frac{1}{2}$, this is not the case. Therefore, $\triangle XYZ$ is contained in Γ if and only if $\frac{1}{2} \leq \lambda \leq 2$.

Next we turn to solutions to problems of the Team Selection Contest, Cortona, Italy, 1999 given [2002 : 356–357].

1. Prove that for each prime number p the equation

$$2^p + 3^p \;=\; a^n$$

has no solutions (a, n), with a and n integers > 1.

Solved by Club de mathématiques du lycée Henri IV, Paris France; Moubinool Omarjee, Paris, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's write-up.

Let $f(p) = 2^p + 3^p$. Note first that $f(2) = 13$ and $f(5) = 275$, neither of which is an n^{th} power for $n > 1$. Now we assume that $f(p) = a^n$ for some integers a and $n > 1$, where p is a prime such that $p \neq 2$ and $p \neq 5$. Since p is odd, we have $f(p) = (2+3)(2^{p-1} - 2^{p-2} \cdot 3 + \cdots + 3^{p-1}) = 5A$ where

$$A \;=\; \sum_{k=0}^{p-1} 2^{p-k-1}(-3)^k\,.$$

In particular, $5 \mid f(p)$, and hence, $5 \mid a$. Since $f(p) = a^n$ and $n > 1$, we have $25 \mid f(p)$, and thus, $5 \mid A$. However, since $-3 \equiv 2 \pmod 5$, we get

$$A \;\equiv\; \sum_{k=0}^{p-1} 2^{p-k-1} \cdot 2^k \;\equiv\; p \cdot 2^{p-1} \pmod 5\,,$$

which is not divisible by 5, since $5 \nmid p$. This is a contradiction.

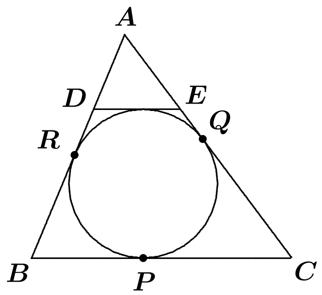
Comment by Pierre Bornsztein, Maisons-Laffitte, France.

This problem is similar to #8 of the Ninth Irish Mathematical Olympiad, for which a solution appeared [2001 : 184].

2. Points D and E are given on the sides AB and AC of $\triangle ABC$ in such a way that DE is parallel to BC and tangent to the incircle of $\triangle ABC$. Prove that

$$DE \;\le\; \tfrac{1}{8}(AB + BC + CA)\,.$$

Solved by Michel Bataille, Rouen, France; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.



We set $BC = a$, $CA = b$, $AB = c$, and $2s = a + b + c$. Let the incircle touch BC, CA, AB at P, Q, R, respectively.

Since DE is parallel to BC, we have $\triangle ADE \sim \triangle ABC$. Thus,

$$\frac{AD + DE + AE}{AB + BC + AC} \;=\; \frac{DE}{BC} \;=\; \frac{DE}{a}\,.$$

Since $AD + DE + AE = AR + AQ = b + c - a$, we have

$$\frac{b + c - a}{a + b + c} \;=\; \frac{DE}{a}\,;$$

whence, $DE = \dfrac{a(b + c - a)}{a + b + c}$. Then

$$\begin{aligned}
&\tfrac{1}{8}(AB + BC + CA) - DE \\
&\quad= \frac{a+b+c}{8} - \frac{a(b+c-a)}{a+b+c} \;=\; \frac{(a+b+c)^2 - 8a(b+c-a)}{8(a+b+c)} \\
&\quad= \frac{(b+c)^2 - 6a(b+c) + 9a^2}{8(a+b+c)} \;=\; \frac{(b+c-3a)^2}{8(a+b+c)} \;\ge\; 0\,.
\end{aligned}$$

Thus, $\frac{1}{8}(AB + BC + CA) \ge DE$.

3. (a) Determine all the strictly monotone functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(x + (f(y)\big) \;=\; f(x) + y\,, \quad \forall\, x, y \in \mathbb{R}\,.$$

(b) Prove that for every integer $n > 1$ there do not exist strictly monotone functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(x + f(y)\big) \;=\; f(x) + y^n\,, \quad \forall\, x, y \in \mathbb{R}\,.$$

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Club de mathématiques du lycée Henri IV, Paris, France; and Moubinool Omarjee, Paris, France. We give Bataille's solution.

(a) Clearly, the functions $x \mapsto x$ and $x \mapsto -x$ meet the required conditions. We will show that there are no other solutions. We will use the notation f^2 for $f \circ f$, the composition of f with itself.

Let $f : \mathbb{R} \to \mathbb{R}$ be strictly monotone and have the property that, for all $x, y \in \mathbb{R}$,

$$f\big(x + f(y)\big) \;=\; f(x) + y\,. \tag{1}$$

With $x = y = 0$, equation (1) yields $f\big(f(0)\big) = f(0)$. Then, $f(0) = 0$, since f is injective.

First, suppose that f is strictly increasing. Note that the following property holds: if $f^2(a) = a$, then $f(a) = a$ (since $f(a) < a$ leads to the contradiction $a = f^2(a) < f(a)$ and, similarly, $f(a) > a$ is impossible).

Now, for arbitrary real numbers x, y, let $a = x + f(y)$. Using (1) twice,

$$f^2(a) \;=\; f\big(y + f(x)\big) \;=\; f(y) + x \;=\; a\,,$$

from which $f(a) = a$ follows. Using (1) again, this gives $f(x)+y = x+f(y)$, or $f(x) - x = f(y) - y$. Thus, the function $x \mapsto f(x) - x$ is constant. The constant must be 0, since $f(0) = 0$, and finally $f(x) = x$ for all x.

Next, suppose that f is strictly decreasing. For all x, y, the relation (1) provides

$$f\big(x + f^2(y)\big) \;=\; f(x) + f(y)\,. \tag{2}$$

Hence, $f^2\big(x + f^2(y)\big) = f\big(f(x) + f(y)\big) = f^2(x) + y$. We deduce that the function f^2 satisfies condition (1), for all $x, y \in \mathbb{R}$. In addition, this function is strictly increasing. Thus, by the previous case, $f^2(x) = x$ for all x.

Returning to (2), we get the usual Cauchy functional equation

$$f(x + y) \;=\; f(x) + f(y)\,, \quad \text{for all } x,\, y\,.$$

A well-known method gives first $f(k) = kf(1)$ for all integers k, next $f(r) = rf(1)$ for all rational r, and finally, $f(x) = xf(1)$ for all real x (using two adjacent rational sequences converging to x). In particular, for $x = f(1)$, we have $f^2(1) = \big(f(1)\big)^2$. Substituting $x = 0$ and $y = 1$ into (1), we obtain $f^2(1) = f(0) + 1 = 1$. Then $\big(f(1)\big)^2 = 1$. Observe that $f(1)$ is negative, since f is strictly decreasing and $f(0) = 0$. Therefore, $f(1) = -1$, and $f(x) = -x$ for all x.

(b) Suppose that f is strictly monotone and satisfies

$$f\big(x + f(y)\big) \;=\; f(x) + y^n\,,$$

for all $x, y \in \mathbb{R}$, where $n > 1$ is an integer. Since $f(0) = 0$ still holds, we have $f^2(y) = y^n$ for all y. Hence, for all y,

$$f\big(f^2(y)\big) \;=\; f^2\big(f(y)\big) \;=\; \big(f(y)\big)^n\,.$$

For arbitrary x, y, we have, on the one hand,

$$f^2\big(x + f(y)\big) \;=\; \big(x + f(y)\big)^n$$

and, on the other hand,

$$\begin{aligned} f^2\big(x + f(y)\big) \;&=\; f\big(f(x) + y^n\big) \;=\; f(y^n) + x^n \\ &=\; f\big(f^2(y)\big) + x^n \;=\; \big(f(y)\big)^n + x^n\,. \end{aligned}$$

Thus, for all x and y,

$$\left(x + f(y)\right)^n \;=\; x^x + \left(f(y)\right)^n . \tag{3}$$

Since f is strictly increasing and $f(0) = 0$, there is some y such that $f(y) > 0$. Choose some such y and choose $x = 1$ in (3). Letting $b = f(y)$, we have $(1 + b)^n = 1 + b^n$, with $b > 0$. This is impossible for $n > 1$. The required result follows.

4. Let X be a set with n elements, and let $A_1, \ldots, A_m$ be subsets of X such that

(i) $|A_i| = 3$ for every $i = 1, \ldots, m$

(ii) $|A_i \cap A_j| \le 1$ for every $i \ne j$ $(i, j \in \{1, \ldots, m\})$.

Prove that there exists a subset of X with at least $\lfloor\sqrt{2n}\rfloor$ elements, which does not contain A_i for $i = 1, \ldots, m$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Moubinool Omarjee, Paris, France. We give Bornsztein's write-up.

Let E be a subset of X which does not contain any of the sets A_i and which has maximal cardinality, say $|E| = p$. Consider any fixed $x \in X\backslash E$. By the maximality of p, the set $E \cup \{x\}$ must contain at least one set A_i. Choose some such set A_i, and denote it by $A(x)$. Thus, we have $A(x) \subseteq E \cup \{x\}$ and $A(x) \not\subseteq E$. It follows that $x \in A(x)$. Let $B(x) = A(x) \setminus \{x\}$. Then $B(x) \subseteq E$ and $|B(x)| = 2$.

Now let $x, y \in X \setminus E$, with $x \ne y$. Then

$$\begin{aligned} A(x) \cap A(y) &= \left(B(x) \cup \{x\}\right) \cap \left(B(y) \cup \{y\}\right) \\ &= B(x) \cap B(y) . \end{aligned}$$

If $B(x) = B(y)$, then $A(x) \cap A(y) = B(x) = B(y)$, which implies that $|A(x) \cap A(y)| = 2$, contradicting condition (ii) in the problem. Therefore, $B(x) \ne B(y)$.

Thus, the number of subsets of E containing exactly two elements is at least as great as the number of elements of X which do not belong to E. That is, $\binom{p}{2} \ge n - p$. Hence, $p(p+1) \ge 2n$; that is, $\left(p + \frac{1}{2}\right)^2 \ge 2n + \frac{1}{4}$. Then

$$p + \tfrac{1}{2} \;\ge\; \sqrt{2n + \tfrac{1}{4}} \;>\; \sqrt{2n} .$$

Since p is an integer, this leads to $p \ge \lfloor\sqrt{2n}\rfloor$, as desired.

That completes the *Corner* for this issue. Send me your nice solutions to recent problems for use in upcoming issues as well as Olympiad Contests.

BOOK REVIEW

John Grant McLoughlin

La Magie du Carré
par René Descombes, édité chez Vuibert, 2004
ISBN 2-7117-5325-5, couverture molle, 608 pages, 60 Euros.
Critique de revue par ***Steve Mazerolle****, candidat au doctorat, Université de Montréal, Montréal, QC.*

Alors que la mathématique était en pleine ébullition au temps des Grecs, Euclide définissait le carré comme étant un quadrilatère équilatéral ayant des angles droits (Roger Cooke, *The History of Mathematics : A Brief Course*, John Wiley and Sons, 1997, p. 96). De plus, il y avait un certain Pythagore, pour qui le carré d'un nombre était la base de son célèbre théorème. Déjà-là, il était possible d'entrevoir l'importance et la multitude d'applications que l'on peut donner au terme "carré". Bien que le carré n'ait rien de magique, *La Magie du Carré* fait suite à un livre précédent écrit par René Descombes et ayant pour titre *Les Carrés Magiques*. *La Magie du Carré*, dépassant les 600 pages, est intéressant par la quantité de connaissances véhiculant les différents concepts mathématiques du carré et décevant par le manque de profondeur de ces concepts abordés.

Descombes nous présente plus de 250 problèmes tirés à même l'histoire des mathématiques qui portent sur le concept de carré. Le carré est évidemment vu sous plusieurs facettes mathématiques telles que la géométrie, la théorie des nombres et l'algèbre. Plusieurs de ces concepts sont présentés à l'aide de jeux, de grilles, de labyrinthes et de damiers. Abondamment illustré, les problèmes sont accompagnés de formidables images clarifiant des concepts et un vocabulaire parfois difficile à saisir. Ce travail volumineux donne des pistes de départ, stimulant ainsi la curiosité du lecteur. Parce que l'auteur présente des objets sur le carré, mais explicite rarement sur l'objet, l'ouvrage demeure un travail de référence. Ce document est sûrement un excellent point de départ didactique pour tout professionnel de l'enseignement qui désire parler de carrés à ses étudiants. Bien que le lecteur puisse consulter une table des matières élaborée, il devra s'en remettre à une lecture rapide s'il cherche un élément particulier tant l'index est dépourvu d'information. Voilà un livre qui devrait se retrouver dans plusieurs bibliothèques où la mathématique et l'enseignement de la mathématique sont parmi les intérêts de leurs utilisateurs.

International Mathematical Olympiads 1986–1999
by Marcin E. Kuczma, published by the Math. Association of America, 2003
ISBN 0-88385-811-8, paperbound, 208 pages, US$34.95.
Reviewed by ***Bill Sands****, University of Calgary, Calgary, AB.*

Well, let's face it, you all know what this book is about. If you have been involved with the IMO and/or reading Crux the last 15 years or more, you also know about Marcin Kuczma's superb abilities when it comes to solving problems and writing clear and correct solutions. So you won't need any urging to get your own copy of this book.

But, for the record, let's mention that this is the third book published by the MAA which contains problems from the IMO. The previous two were compiled by Samuel Greitzer and Murray Klamkin respectively, and together covered the years from the beginning of the IMO (1959) up to and including 1985. Hence, we are overdue for an update. Let's also mention that the problems and solutions are listed by year, and that there is often more than one solution given. These comprise the bulk of the book, naturally. But after that comes the IMO (team) results for the years 1986 to 1999, followed by a list of symbols, glossary of terms and frequently used theorems, and a shortish subject index. I might also mention that, in my opinion, the book (in particular its cover) is quite attractively designed.

That's it! Now go get your copy.

Mathematical Treks: From Surreal Numbers to Magic Circles
by Ivars Peterson, published by the Math. Association of America, 2002
ISBN 0-88385-537-2, paperbound, 170+x pages, US$26.95.
Reviewed by ***John Grant McLoughlin****, University of New Brunswick, Fredericton, NB.*

Mathematical Treks features 33 journeys into the world of mathematics. The style of presentation is familiar to readers of books by Martin Gardner, to whom this book is dedicated. The 33 chapters represent distinct themes, each offering updated versions of Ivars Peterson's columns from 1996 and 1997 issues of *Science News*. A sampling of the chapter titles provides a sense of the diversity of topics: *Calculation and the Chess Master*; *A Passion for Pi*; *DNA Adds Up*; *Waring Experiments*; *Beyond the Ellipse*; *Prime Theorems*; *More than Magic Squares*; and *Fair Play and Dreidel*.

The book offers at least 33 entry points for any reader. This makes for a wonderful resource, particularly for those who like to simply delve into an idea, or as a source of stimulation for students who will surely find topics of interest. The chapters average about 5 pages in length. The inclusion of a bibliography at the end of each chapter is an asset to the book's value as a resource, particularly to those who may wish to examine a topic in more depth. In fact, the cumulative result is a rich resource for entertainment and mathematical development.

The Triangle: A Parametric Description

K.R.S. Sastry

1. Introduction

A family of triangles can be described in various ways. For example, look at the family of right triangles. Naming the vertices A, B, C so that the right angle is at A, and using the customary notation $a = BC$, $b = CA$, and $c = AB$, we can describe this family completely in terms of two positive real parameters u and v as follows: $b = u$, $c = v$, and $a = \sqrt{u^2 + v^2}$. This same family may also be described in terms of m_a, the length of the median to the hypotenuse BC. It is well known and easily established that m_a is half the length of BC. In fact, the relation $m_a = \frac{1}{2}a$ characterizes right triangles; that is, $\triangle ABC$ has a right angle at A if and only if $m_a = \frac{1}{2}a$. We can put it this way: *The family of triangles in which $m_a = \frac{1}{2}a$ is the family whose sides have the form $a = \sqrt{u^2 + v^2}$, $b = u$, $c = v$.*

This motivates us to describe the complete family of triangles as the union of families for which $m_a = \frac{1}{2}\lambda a$, where $\lambda > 0$. The triangles which are right-angled at A will then be the triangles for which $\lambda = 1$. After obtaining this description, we derive the constraints on λ that characterize a triangle ABC in which the Euler line is parallel to the side BC.

We assume familiarity with basic trigonometric results. The equation $4m_a^2 = 2b^2 + 2c^2 - a^2$ is also assumed to be known.

2. Description of triangles with a given ratio m_a/a

Theorem 1 Let triangle ABC be given, and let λ be a positive real number. The following statements are equivalent:

(A) $m_a = \frac{1}{2}\lambda a$;

(B) The sides a, b, c have the form

$$\begin{aligned} a &= \sqrt{u^2 + v^2}\,, \\ b &= \sqrt{\left(\frac{\lambda+1}{2}\right)^2 u^2 + \left(\frac{\lambda-1}{2}\right)^2 v^2}\,, \\ c &= \sqrt{\left(\frac{\lambda-1}{2}\right)^2 u^2 + \left(\frac{\lambda+1}{2}\right)^2 v^2}\,, \end{aligned} \tag{1}$$

where u and v are positive real parameters.

(C) The sides a, b, c are related by the equation

$$b^2 + c^2 = \left(\frac{\lambda^2+1}{2}\right) a^2\,. \tag{2}$$

Proof. We refer to Figure 1. Without loss of generality, assume that $b \geq c$. Let D be the mid-point of BC, and let $\theta = \angle ADB$. Then $0 < \theta \leq \pi/2$. Choose A' on AD (on the same side of D as A) such that $DA' = DB = DC$. Depending on whether $\angle BAC$ is acute, right, or obtuse, the point A' lies in the interior of the segment AD, coincides with A, or lies on AD extended. (Figure 1 shows all three cases.) Let $u = CA'$ and $v = BA'$. Then $u \geq v$ because $b \geq c$.

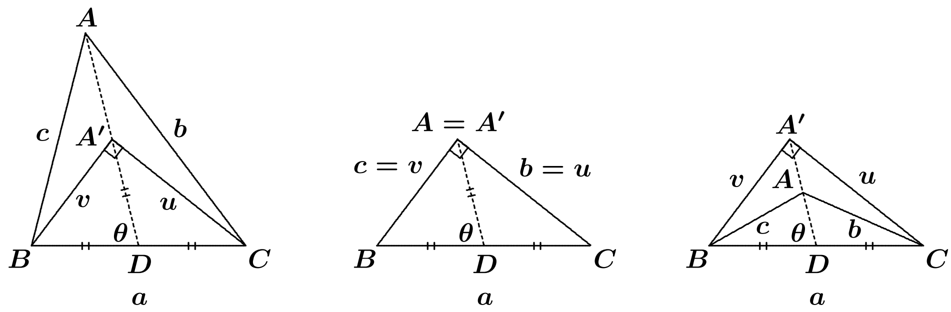


Figure 1

We note that $a = \sqrt{u^2 + v^2}$,

$$\cos\frac{\theta}{2} = \frac{u}{a} = \frac{u}{\sqrt{u^2+v^2}}, \quad \text{and} \quad \sin\frac{\theta}{2} = \frac{v}{a} = \frac{v}{\sqrt{u^2+v^2}}.$$

Hence,

$$\cos\theta = \frac{u^2 - v^2}{u^2 + v^2} \quad \text{and} \quad \sin\theta = \frac{2uv}{u^2 + v^2}. \tag{3}$$

Now suppose that $m_a = \frac{1}{2}\lambda a$ (statement (A) in the theorem). Then

$$AD = m_a = \tfrac{1}{2}\lambda\sqrt{u^2 + v^2}. \tag{4}$$

We now derive expressions for the sides b and c by applying the Cosine Law twice, once in $\triangle ADC$ and once in $\triangle ADB$. Routine simplification leads to the expressions in (1). Thus, statement (A) implies statement (B).

By substituting the expressions in (1) into (2), we can easily prove that (B) implies (C). Finally, to show that (C) implies (A), we just substitute from (2) into the equation $4m_a^2 = 2b^2 + 2c^2 - a^2$. ∎

[*Editor's comment:* Theorem 1 can be interpreted as a theorem about parallelograms. The given triangle ABC is half of a parallelogram in which one of the diagonals is BC and the sides have length b and c. The parameter λ is just the ratio of the diagonals in this parallelogram, and equation (2) is essentially the parallelogram law—the sum of the squares of the diagonals in a parallelogram equals the sum of the squares of the four sides. The reader may wish to explore this alternative point of view.]

The expressions for a, b, and c in (1) are homogeneous in u and v. Consequently, multiplying u and v by the same positive real number has the effect of multiplying all three sides a, b, and c by this same factor, thereby producing a triangle which is similar to the original. Conversely, triangles which are similar have the same value of λ and the same parameter ratio u/v. Note that $u = v$ if and only if $\triangle ABC$ is isosceles with $b = c$.

The sides a, b, c in (1) may appear irrational, but integer-sided triangles are not excluded. For example, set $u = 4\sqrt{65/11}$, $v = 3\sqrt{91/11}$, and $\lambda = 11/13$ to get the triangle with $a = 13$, $b = 9$, $c = 8$, and $m_a = 11/2$.

Next, we deduce some corollaries from Theorem 1. For Corollary 2, we need to recall that the *medial triangle* of a given triangle ABC is the triangle whose sides have lengths m_a, m_b, m_c.

Corollary 1 In triangle ABC, if $m_a = \frac{1}{2}\lambda a$, and if u and v satisfy equations (1), then the area of $\triangle ABC$ is $\frac{1}{2}\lambda uv$.

Proof: The result follows from (4) and the fact that the area of $\triangle ABC$ is $\frac{1}{2}(BC)(AD)\sin\theta$. ∎

Corollary 2 Let triangle ABC be given, and let λ be such that $m_a = \frac{1}{2}\lambda a$, where the value a is between the values of b and c (allowing equality). Then triangle ABC is similar to its medial triangle if and only if $\lambda = \sqrt{3}$.

Proof: Since a is between b and c, we have m_a between m_b and m_c (because the lengths of the medians have the opposite order to the lengths of the corresponding sides).

Suppose that triangle ABC is similar to its medial triangle. Then

$$\frac{m_b}{c} = \frac{m_c}{b} = \frac{m_a}{a} = \frac{\lambda}{2}.$$

We have

$$\begin{aligned} 2b^2 + 2c^2 - a^2 &= 4m_a^2, \\ 2c^2 + 2a^2 - b^2 &= 4m_b^2, \\ 2a^2 + 2b^2 - c^2 &= 4m_c^2. \end{aligned}$$

Adding these three equations gives

$$3(a^2 + b^2 + c^2) = 4(m_a^2 + m_b^2 + m_c^2) = \lambda^2(a^2 + b^2 + c^2),$$

which means that $\lambda = \sqrt{3}$.

Conversely, suppose that $\lambda = \sqrt{3}$. Then (2) becomes $b^2 + c^2 = 2a^2$. Using this in the relations $2c^2 + 2a^2 - b^2 = 4m_b^2$ and $2a^2 + 2b^2 - c^2 = 4m_c^2$, we obtain $m_b = \frac{1}{2}\sqrt{3}c$ and $m_c = \frac{1}{2}\sqrt{3}b$. Therefore, triangle ABC is similar to its medial triangle. ∎

When $\lambda = \sqrt{3}$, we have $b^2 + c^2 = 2a^2$ (as noted in the proof above). Then we can express a, b, and c in the following alternative form:

$$a = \sqrt{p^2 + q^2}\,, \quad b = p + q\,, \quad c = p - q\,,$$

where p, q are real numbers with $p > q > 0$. We deduce that if $\triangle ABC$ has integer sides and is similar to its medial triangle, then

$$a = m^2 + n^2\,, \quad b = m^2 - n^2 + 2mn\,, \quad c = |m^2 - n^2 - 2mn|\,,$$

where m and n are integers with $m > n$.

3. The parallelism of the Euler line with a side

The Euler line of a triangle ABC is a line containing the circumcentre, centroid, nine-point centre, and orthocentre of the triangle. The Euler line is parallel to the side BC if and only if $\tan B \tan C = 3$ and the angles B and C are not equal (see [1], [2], or [3]). If $\tan B \tan C = 3$ and the angles B and C are equal, then $\triangle ABC$ is equilateral and its Euler line is just a point.

Theorem 2 Let $\lambda > 0$ be given. There exists a triangle ABC with $m_a = \frac{1}{2}\lambda a$ and with its Euler line parallel to the side BC if and only if $1 < \lambda < \sqrt{3}$. When such a triangle exists, it is uniquely determined by λ up to similarity.

Proof. We refer to Figure 2. The altitude from A meets the side BC at P, and the mid-point of BC is D. The circumcentre and orthocentre of $\triangle ABC$ are O and H, respectively. The Euler line of $\triangle ABC$ is then OH.

The Euler line is parallel to BC if and only if $\tan B \tan C = 3$ and the angles B and C are not equal (as noted above). The condition $\tan B \cdot \tan C = 3$ is equivalent to $AP^2 = 3BP \cdot PC$; that is,

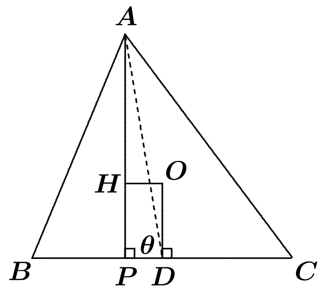


Figure 2

$$\begin{aligned} AP^2 &= 3\left(\frac{a}{2} - PD\right)\left(\frac{a}{2} + PD\right) \\ &= \frac{3}{4}(a^2 - 4PD^2)\,. \end{aligned}$$

Using (3) and (4) from the proof of Theorem 1, we have

$$\begin{aligned} AP &= m_a \sin\theta = \frac{\lambda uv}{\sqrt{u^2 + v^2}}\,, \\ PD &= m_a \cos\theta = \frac{\lambda(u^2 - v^2)}{2\sqrt{u^2 + v^2}}\,. \end{aligned}$$

Thus, the condition $\tan B \tan C = 3$ becomes

$$\frac{\lambda^2 u^2 v^2}{u^2+v^2} \;=\; \frac{3}{4}\left(u^2+v^2 - \frac{\lambda^2(u^2-v^2)^2}{u^2+v^2}\right).$$

We simplify the above equation and write it as a quadratic in u^2/v^2:

$$3(\lambda^2-1)\left(\frac{u^2}{v^2}\right)^2 - 2(\lambda^2+3)\left(\frac{u^2}{v^2}\right) + 3(\lambda^2-1) \;=\; 0\,.$$

The quadratic formula then yields

$$\frac{u^2}{v^2} \;=\; \frac{\lambda^2+3\pm 2\lambda\sqrt{2(3-\lambda^2)}}{3(\lambda^2-1)} \;=\; \frac{(\sqrt{3-\lambda^2}\pm\sqrt{2}\lambda)^2}{3(\lambda^2-1)}\,. \tag{5}$$

It is clear that (5) gives real values for u/v if and only if $1 < \lambda \leq \sqrt{3}$. If $\lambda = \sqrt{3}$, then (5) yields $u/v = 1$. But then the angles B and C are equal. We conclude that the Euler line of $\triangle ABC$ is parallel to the side BC if and only if $1 < \lambda < \sqrt{3}$ and u/v is given by (5).

When $1 < \lambda < \sqrt{3}$, equation (5) gives two values for u/v. But these values are reciprocals of one another. Without loss of generality, we can assume that $u > v$. Then the value of u/v corresponding to the negative sign in (5) has to be discarded. Any fixed value of $\lambda \in (1, \sqrt{3})$ then determines a unique ratio $u/v > 1$ such that triangles defined by (1) have Euler lines parallel to BC. Since the ratio u/v determines u and v up to a constant multiple, it also determines the triangle ABC up to similarity. ∎

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PROBLEMS

Solutions to problems in this issue should arrive no later than **1 June 2005**. *An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

2988★. *Proposé par Faruk Zejnulahi et Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Soit x, y et z des nombres réels non-négatifs satisfaisant $x+y+z=1$. Montrer ou réfuter que :

(a) $xy^2+yz^2+zx^2 \;\geq\; \frac{1}{3}(xy+yz+zx)$;

(b) $xy^2+yz^2+zx^2 \;\geq\; xy+yz+zx-\frac{2}{9}$.

Comment les membres de droite de (a) et (b) se comparent-ils ?

2989. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Montrer que si $0<a<b<d<\pi$ et $a<c<d$ satisfont $a+d=b+c$, alors

$$\frac{\cos(a-d)-\cos(b+c)}{\cos(b-c)-\cos(a+d)} \;<\; \frac{ad}{bc}\,.$$

2990. *Proposé par Václav Konečný, Big Rapids, MI, USA.*

On donne une ellipse par son centre O et un foyer F, une droite ℓ et un point P. Avec la règle seulement, construire la droite par P perpendiculaire à ℓ. (Si l'on donne un cercle avec son centre au lieu d'une ellipse, alors la construction est donnée par le Théorème de Poncelet–Steiner bien connu.)

2991. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit n un entier, $n\geq 3$. Montrer que pour tout n-uplet de nombres complexes $z_1,z_2,\ldots,z_n$, on a

$$\begin{aligned}(n-1)&\left|\sum_{i=1}^{n} z_i^3 \;-\; 3\sum_{1\leq i<j<k\leq n} z_iz_jz_k\right| \\ &\qquad\leq\; \left|\sum_{i=1}^{n} z_i\right| \sum_{1\leq i<j\leq n}\Big(|z_i-z_j|^2+(n-3)|z_i+z_j|\Big)\,.\end{aligned}$$

2992. *Proposé par Pham Van Thuan, Hanoi, Viêt Nam.*

Soit Q un point intérieur au triangle ABC. Soit M, N et P des points sur les côtés BC, CA et AB, respectivement, de telle sorte que $MN \parallel AQ$, $NP \parallel BQ$, et $PM \parallel CQ$. Montrer que

$$[MNP] \;\le\; \tfrac{1}{3}[ABC]\,,$$

où $[XYZ]$ désigne l'aire du triangle XYZ.

2993★. *Proposé par Faruk Zejnulahi et Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Soit x, y et z des nombres réels non-négatifs satisfaisant $x+y+z=1$. Montrer ou réfuter que :

(a) $\dfrac{x}{xy+1}+\dfrac{y}{yz+1}+\dfrac{z}{zx+1} \;\ge\; \dfrac{9}{10}$;

(b) $\dfrac{x}{y^2+1}+\dfrac{y}{z^2+1}+\dfrac{z}{x^2+1} \;\ge\; \dfrac{9}{10}$.

Comment les membres de gauche de (a) et (b) se comparent-ils ?

2994. *Proposé par Faruk Zejnulahi et Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Soit a, b et c des nombres réels non-négatifs satisfaisant $a+b+c=3$. Montrer que

(a) $\dfrac{a^2}{b+1}+\dfrac{b^2}{c+1}+\dfrac{c^2}{a+1} \;\ge\; \dfrac{3}{2}$;

(b) $\dfrac{a}{b+1}+\dfrac{b}{c+1}+\dfrac{c}{a+1} \;\ge\; \dfrac{3}{2}$;

(c) $\dfrac{a^2}{b^2+1}+\dfrac{b^2}{c^2+1}+\dfrac{c^2}{a^2+1} \;\ge\; \dfrac{3}{2}$;

(d) $\dfrac{a}{b^2+1}+\dfrac{b}{c^2+1}+\dfrac{c}{a^2+1} \;\ge\; \dfrac{3}{2}$.

2995. *Proposé par Christopher J. Bradley, Bristol, GB.*

Soit $ABCD$ un quadrilatère cyclique dans lequel les diagonales AC et BD se coupent à angle droit en E. Soit O le centre de son cercle circonscrit. Soit P le point d'intersection des tangentes en A et B. Soit Q, R et S définis de manière analogue pour les paires respectives B et C, C et D, D et A. C'est un fait connu que $PQRS$ est un quadrilatère cyclique.

Soit T, U, V et W les orthocentres respectifs des triangles AOB, BOC, COD, DOA.

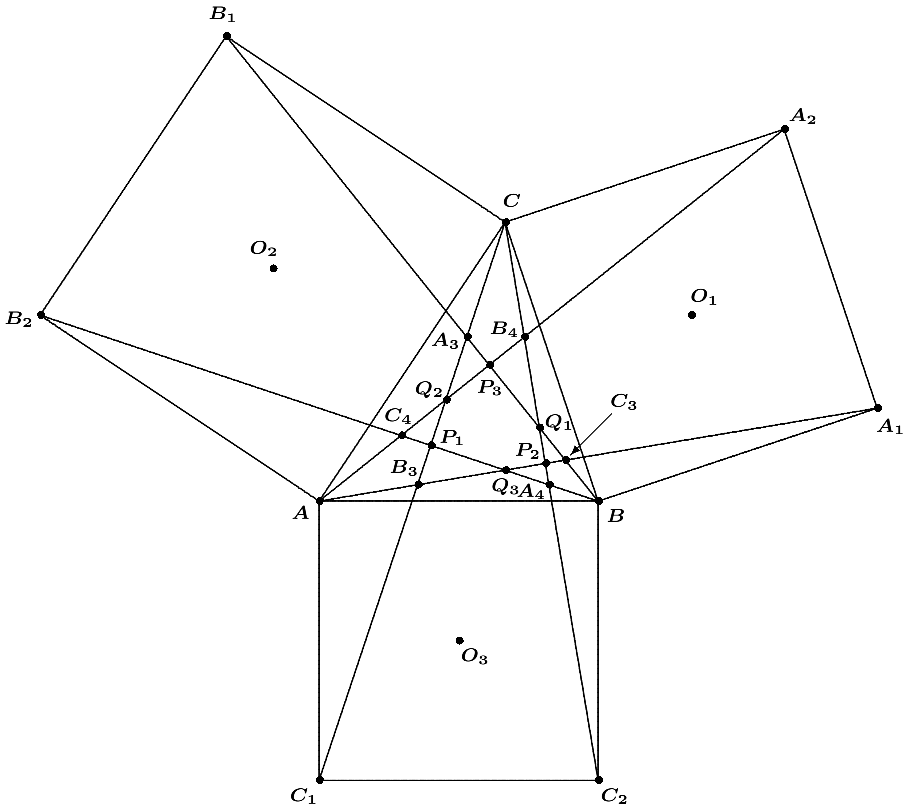
Soit F, G, H et K les orthocentres respectifs des triangles POQ, QOR, ROS, SOP.

Montrer que $TUVW$ et $FGHK$ sont des droites se coupant à angle droit en E.

2996. *Proposé par Roland Eddy et Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Etant donné un triangle ABC, on dessine, comme indiqué dans la figure, des carrés extérieurs AC_1C_2B, BA_1A_2C et CB_1B_2A, de centres respectifs O_3, O_1 et O_2. Soit

$$\begin{aligned}
A_3 &= BB_1 \cap CC_1\,, & B_3 &= CC_1 \cap AA_1\,, & C_3 &= AA_1 \cap BB_1\,,\\
A_4 &= BB_2 \cap CC_2\,, & B_4 &= CC_2 \cap AA_2\,, & C_4 &= AA_2 \cap BB_2\,,\\
P_1 &= BB_2 \cap CC_1\,, & P_2 &= CC_2 \cap AA_1\,, & P_3 &= AA_2 \cap BB_1\,,\\
Q_1 &= BB_1 \cap CC_2\,, & Q_2 &= CC_1 \cap AA_2\,, & Q_3 &= AA_1 \cap BB_2\,.
\end{aligned}$$



Montrer que :

(a) $AC \parallel A_3C_4 \parallel A_4C_3$;

(b) AQ_1, BQ_2, CQ_3 sont concourantes ;

(c) $AA_1 \perp CC_2$;

(d) AP_1 est la bissectrice de $\angle C_1P_1B_2$;

(e) A, P_1, O_1 sont colinéaires ;

(f) AP_1, BP_2, CP_3 sont concourantes.

Les proposeurs peuvent démontrer tous ces résultats. Mais tous, excepté c), utilisent la géométrie analytique. Ils seraient heureux de voir des démonstrations en géométrie synthétique.

2997. *Proposé par Christopher J. Bradley, Bristol, GB.*

Soit ABC un triangle de cercle inscrit Γ. On suppose que Γ touche les côtés BC, CA et AB en X, Y et Z, respectivement. On suppose aussi que YZ et BC se coupent en X' ; que ZX et CA se coupent en Y' et que XY et AB se coupent en Z'. Soit P un point quelconque sur la droite $X'Y'$. On suppose que AP et BC se coupent L, que BP et CA se coupent en M, et que CP et AB se coupent en N. Soit U le point d'intersection de MN et BC, V celui de NL et CA, et W celui de LM et AB.

Montrer que UVW est une droite, tangente à Γ.

[*Ed :* Bradley ajoute : "Ce problème n'est pas original. On le trouve dans un livre de problèmes par Wolstenholme (St. John's College, Cambridge) daté du 19^e siècle, où le problème porte en fait sur toute conique touchant les côtés."]

2998. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit n un entier positif qui n'est pas un multiple de 3 et soit A, B, C des matrices réelles $n \times n$ satisfaisant

$$A^2 + B^2 + C^2 \;=\; AB + BC + CA\,.$$

Montrer que

$$\det\big((AB - BA) + (BC - CB) + (CA - AC)\big) \;=\; 0\,.$$

2999. *Proposé par José Luis Díaz-Barrero et Juan José Egozcue, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit m et n deux entiers positifs. Montrer que

$$\left(\frac{m+1}{m}\sum_{k=1}^{n}\frac{k}{n^{m+2}}(n^m - k^m)\right)^m \;<\; \frac{1}{m+1}\,.$$

3000. *Proposé par Paul Dayao, Université Ateneo de Manille, Philippines.*

Soit f une fonction continue, non-négative et deux fois différentiable sur $[0,\infty)$. On suppose que $xf''(x) + f'(x)$ est non nulle et ne change pas de signe sur $[0,\infty)$. Si x_1, x_2, …, x_n sont des nombres réels non-négatifs de moyenne géométrique c, montrer que

$$f(x_1) + f(x_2) + \cdots + f(x_n) \;\geq\; nf(c)\,,$$

avec égalité si et seulement si $x_1 = x_2 = \cdots = x_n$.

2988★. *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let x, y, z be non-negative real numbers satisfying $x+y+z=1$. Prove or disprove:

(a) $xy^2+yz^2+zx^2 \;\geq\; \frac{1}{3}(xy+yz+zx)$;

(b) $xy^2+yz^2+zx^2 \;\geq\; xy+yz+zx-\frac{2}{9}$.

How do the right sides of (a) and (b) compare?

2989. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that if $0<a<b<d<\pi$ and $a<c<d$ satisfy $a+d=b+c$, then

$$\frac{\cos(a-d)-\cos(b+c)}{\cos(b-c)-\cos(a+d)} \;<\; \frac{ad}{bc}\,.$$

2990. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Given are an ellipse with centre O and a focus F, a line ℓ, and a point P. Construct with straightedge alone the line passing through the point P perpendicular to the line ℓ. (If a circle with its centre is given instead of an ellipse, then the construction is given by the well-known Poncelet–Steiner Construction Theorem.)

2991. *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be an integer, $n\geq 3$. For all $z_i\in\mathbb{C}$, $i=1,2,\ldots,n$, prove

$$(n-1)\left|\sum_{i=1}^{n} z_i^3 \;-\;3\sum_{1\leq i<j<k\leq n} z_iz_jz_k\right| \leq \left|\sum_{i=1}^{n} z_i\right|\sum_{1\leq i<j\leq n}\Big(|z_i-z_j|^2+(n-3)|z_i+z_j|\Big)\,.$$

2992. *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*

Let Q be a point interior to $\triangle ABC$. Let M, N, P be points on the sides BC, CA, AB, respectively, such that $MN \parallel AQ$, $NP \parallel BQ$, and $PM \parallel CQ$. Prove that

$$[MNP] \;\leq\; \tfrac{1}{3}[ABC]\,,$$

where $[XYZ]$ denotes the area of triangle XYZ.

2993★. *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let x, y, z be non-negative real numbers satisfying $x + y + z = 1$. Prove or disprove:

(a) $\dfrac{x}{xy+1} + \dfrac{y}{yz+1} + \dfrac{z}{zx+1} \geq \dfrac{9}{10}$;

(b) $\dfrac{x}{y^2+1} + \dfrac{y}{z^2+1} + \dfrac{z}{x^2+1} \geq \dfrac{9}{10}$.

How do the left sides of (a) and (b) compare?

2994. *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let a, b, c be non-negative real numbers satisfying $a+b+c = 3$. Show that

(a) $\dfrac{a^2}{b+1} + \dfrac{b^2}{c+1} + \dfrac{c^2}{a+1} \geq \dfrac{3}{2}$;

(b) $\dfrac{a}{b+1} + \dfrac{b}{c+1} + \dfrac{c}{a+1} \geq \dfrac{3}{2}$;

(c) $\dfrac{a^2}{b^2+1} + \dfrac{b^2}{c^2+1} + \dfrac{c^2}{a^2+1} \geq \dfrac{3}{2}$;

(d) $\dfrac{a}{b^2+1} + \dfrac{b}{c^2+1} + \dfrac{c}{a^2+1} \geq \dfrac{3}{2}$.

2995. *Proposed by Christopher J. Bradley, Bristol, UK.*

Let $ABCD$ be a cyclic quadrilateral in which the diagonals AC and BD intersect at right angles at E. Let O be the centre of its circumscribing circle. Let P be the point of intersection of the tangent lines at A and B. Let Q, R, S be similarly defined for the pairs B and C, C and D, D and A, respectively. It is known that $PQRS$ is a cyclic quadrilateral.

Let T, U, V, W be the orthocentres of $\triangle AOB$, $\triangle BOC$, $\triangle COD$, $\triangle DOA$, respectively.

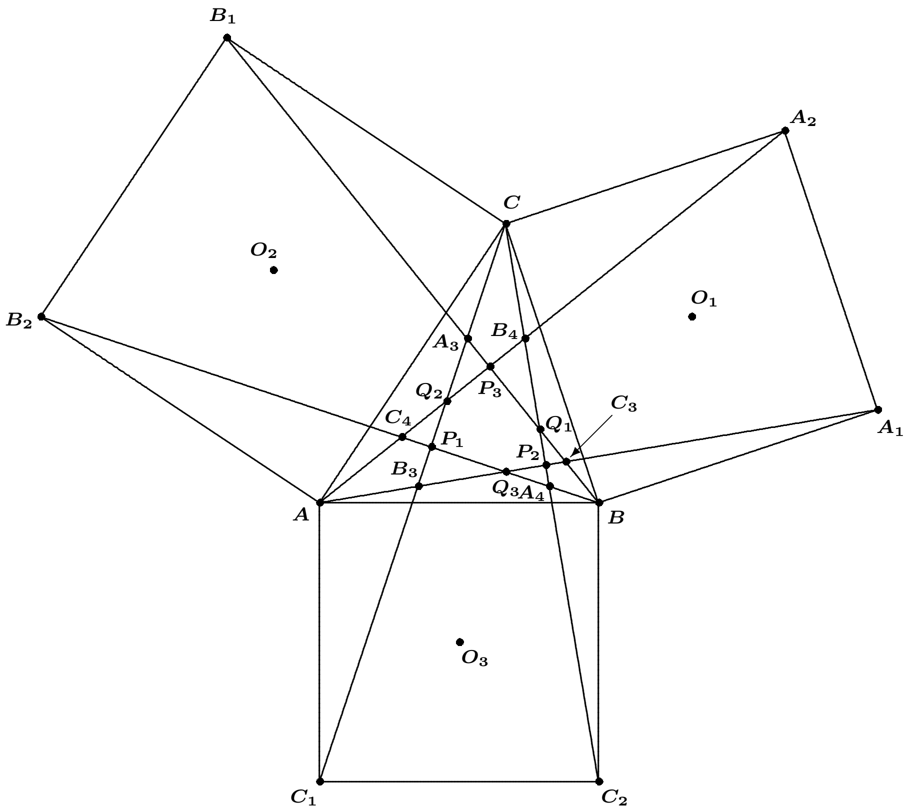
Let F, G, H, K be the orthocentres of $\triangle POQ$, $\triangle QOR$, $\triangle ROS$, $\triangle SOP$, respectively.

Prove that $TUVW$ and $FGHK$ are straight lines intersecting at right angles at E.

2996. *Proposed by Roland Eddy and Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Given $\triangle ABC$, draw squares AC_1C_2B, BA_1A_2C, CB_1B_2A outwards as indicated, with centres O_3, O_1, O_2, respectively. Let

$$\begin{aligned}
A_3 &= BB_1 \cap CC_1\,, & B_3 &= CC_1 \cap AA_1\,, & C_3 &= AA_1 \cap BB_1\,,\\
A_4 &= BB_2 \cap CC_2\,, & B_4 &= CC_2 \cap AA_2\,, & C_4 &= AA_2 \cap BB_2\,,\\
P_1 &= BB_2 \cap CC_1\,, & P_2 &= CC_2 \cap AA_1\,, & P_3 &= AA_2 \cap BB_1\,,\\
Q_1 &= BB_1 \cap CC_2\,, & Q_2 &= CC_1 \cap AA_2\,, & Q_3 &= AA_1 \cap BB_2\,.
\end{aligned}$$



Prove that:

(a) $AC \parallel A_3C_4 \parallel A_4C_3$;

(b) AQ_1, BQ_2, and CQ_3 are concurrent;

(c) $AA_1 \perp CC_2$;

(d) AP_1 bisects $\angle C_1P_1B_2$;

(e) A, P_1, and O_1 are collinear;

(f) AP_1, BP_2, and CP_3 are concurrent.

The proposers have proofs of all these results, but, except for (c), all are done using coordinate geometry. They would like to see nice synthetic proofs.

2997. *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle with incircle Γ. Suppose that Γ touches the sides BC, CA, AB at X, Y, Z, respectively. Let YZ meet BC at X'; let ZX meet CA at Y'; and let XY meet AB at Z'. Let P be any point on the line $X'Y'$. Suppose that AP meets BC at L, that BP meets CA at M, and that CP meets AB at N. Now let MN meet BC at U; let NL meet CA at V; and let LM meet AB at W.

Prove that UVW is a straight line, and that it is tangent to Γ.

[*Ed:* Bradley adds: "This problem is not original. It comes from a book of problems by Wolstenholme (St. John's College, Cambridge) dated in the 19^{th} Century, where the problem actually involves any conic touching the sides."]

2998. *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be a positive integer which is not a multiple of 3, and let A, B, C be $n \times n$ matrices with real entries that satisfy

$$A^2 + B^2 + C^2 \;=\; AB + BC + CA\,.$$

Prove that

$$\det\big((AB - BA) + (BC - CB) + (CA - AC)\big) \;=\; 0\,.$$

2999. *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let m, n be positive integers. Prove that

$$\left(\frac{m+1}{m}\sum_{k=1}^{n}\frac{k}{n^{m+2}}(n^m - k^m)\right)^m \;<\; \frac{1}{m+1}\,.$$

3000. *Proposed by Paul Dayao, Ateneo de Manila University, The Philippines.*

Let f be a continuous, non-negative, and twice-differentiable function on $[0,\infty)$. Suppose that $xf''(x) + f'(x)$ is non-zero and does not change sign on $[0,\infty)$. If x_1, x_2, $\ldots$, x_n are non-negative real numbers and c is their geometric mean, show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \;\geq\; nf(c)\,,$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2845. [2003 : 241; 2004 : 252–253] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let Q be a square of side length 1, and let S be a set consisting of a finite number of squares such that the sum of their areas is $\frac{1}{2}$.

Prove that the set S can be packed inside the square Q.

Editor: The theorem in the featured solution in [2004 : 252–253] is flawed. We present here a different solution.

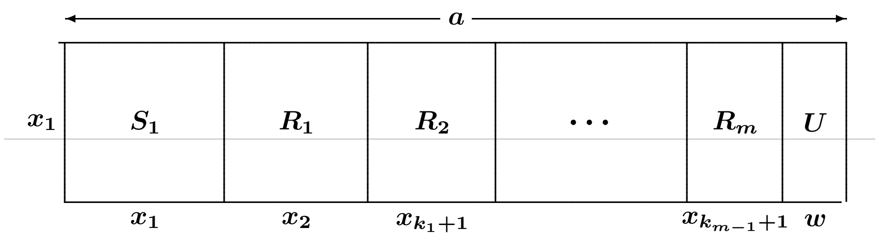
Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA, adapted by the editors.

Lemma. *Let $S = \{S_i\}_{i=1}^{n}$ be a set of squares with respective side lengths $x_1 \geq x_2 \geq \cdots \geq x_n$. Let R be an $a \times x_1$ rectangle such that $x_1 \leq a$. If S cannot be packed inside R, then there exists an integer k with $1 \leq k < n$ such that $\{S_i\}_{i=1}^{k}$ can be packed inside R to cover at least half the area of R.*

Proof. The proof is by induction on the number of squares n. If $n = 1$, then $S = \{S_1\}$ can be packed inside R and the lemma is vacuously true. Fix $n \geq 2$, and assume that the lemma is true whenever the number of squares is less than n. Let $S = \{S_i\}_{i=1}^{n}$ be a set of squares satisfying the hypothesis of the lemma. Pack S_1 into one end of R (as shown in the figure on the next page).

Now we give a recursive construction that defines a positive integer k. Let $k_0 = 1$. If $x_2 > a - x_1$, then we take $k = k_0$. Otherwise, we consider the rectangle R_1 of dimensions $x_2 \times x_1$ that is immediately beside S_1 in R. By the induction hypothesis, there exists an integer k_1 with $2 \leq k_1 < n$ such that $\{S_i\}_{i=2}^{k_1}$ can be packed inside R_1 to cover at least half the area of R_1. If $x_{k_1+1} > a - (x_1 + x_2)$, then we take $k = k_1$. Otherwise, we consider the rectangle R_2 of dimensions $x_{k_1+1} \times x_1$ that is uncovered and immediately beside R_1 in R. By the induction hypothesis, there exists an integer k_2 with $k_1 + 1 \leq k_2 < n$ such that $\{S_i\}_{i=k_1+1}^{k_2}$ can be packed inside R_2 to cover at least half the area of R_2.

We continue in this manner until the process terminates, which must happen after a finite number of iterations, say m iterations. We then have $k = k_m$, and the set of squares $\{S_i\}_{i=1}^{k}$ is packed inside R. We claim that these squares cover at least half the area of R.



R

Let U be the uncovered rectangle that is immediately beside R_m in R, and let w be the width of U (as shown in the figure). By our construction, $w < x_{k_m+1}$, and hence, $w < x_1$. Therefore, the area of square S_1 is greater than the area of U. Moreover, by our construction, each of the rectangles $R_1, R_2, \ldots, R_m$ is at least half covered by squares. It follows that our claim is true, and the lemma is proved. ■

Theorem. *Let $S = \{S_i\}_{i=1}^n$ be a set of squares with respective side lengths $x_1 \geq x_2 \geq \cdots \geq x_n$. Let Q be an $a \times b$ rectangle such that $x_1 \leq a \leq b$ and $\sum\limits_{i=1}^{n} x_i^2 \leq \frac{1}{2}ab$. Then S can be packed inside Q.*

Proof. The proof is by induction on the number of squares n. The theorem is clearly true for $n = 1$. Fix $n \geq 2$, and assume that the theorem is true whenever the number of squares is less than n. Let $S = \{S_i\}_{i=1}^n$ be a set of squares satisfying the hypothesis of the theorem. Embed Q in the xy–plane as $[0,a] \times [0,b]$. If S can be packed inside the rectangle $R = [0,a] \times [0,x_1]$ (a subset of Q), then we are done. Otherwise, the lemma yields an integer k with $1 \leq k < n$ such that $\{S_i\}_{i=1}^k$ can be packed inside R to cover at least half the area of R. Hence, $\sum\limits_{i=k+1}^{n} x_i^2 \leq \frac{1}{2}a(b - x_1)$. Also,

$$(x_1 + x_{k+1})^2 \;\leq\; 2(x_1^2 + x_{k+1}^2) \;\leq\; ab \;\leq\; b^2\,.$$

Thus, $x_{k+1} \leq b - x_1$. Also, $x_{k+1} \leq x_1 \leq a$. By the induction hypothesis, we see that $\{S_i\}_{i=k+1}^n$ can be packed inside $[0,a] \times [x_1, b]$, completing the proof. ■

2881. [2003 : 466] *Proposed by Christopher J. Bradley, Bristol, UK.*

A set of four non-negative integers a, b, c, d are said to have the property $\mathcal{P}$ if all of $bc + cd + db$, $ac + cd + da$, $ab + bd + da$, $ab + bc + ca$ are perfect squares.

The sequence $\{u_n\}$ is defined by $u_1 = 0$, $u_2 = 1$, $u_3 = 1$, $u_4 = 4$ and, for $n \geq 1$,

$$u_{n+4} \;=\; 2u_{n+3} + 2u_{n+2} + 2u_{n+1} - u_n\,.$$

Prove that the set $\{u_n,\, u_{n+1},\, u_{n+2},\, u_{n+3}\}$ has the property $\mathcal{P}$ for all $n \geq 1$.

Solution by Kathleen E. Lewis, SUNY Oswego, Oswego, NY, USA.

This problem can be reduced to Problem 2802 [2003 : 44; 2004 : 50] by showing that the sequence defined here also satisfies the recurrence relation

$$u_{n+3} \;=\; u_n + u_{n+1} + u_{n+2} + 2\sqrt{u_n u_{n+1} + u_n u_{n+2} + u_{n+1} u_{n+2}}$$

for all $n \geq 1$. In this case, u_n, u_{n+1}, u_{n+2}, and u_{n+3} play the roles of the integers p, q, r, and s, respectively, of Problem 2802.

We will show by induction that the sequence satisfies this alternative recurrence relation. First note that

$$u_4 \;=\; 4 \;=\; 0+1+1+2\sqrt{0+0+1}\,;$$

Thus, the relation is satisfied at the first stage. Now suppose that

$$u_{k+3} \;=\; u_k + u_{k+1} + u_{k+2} + 2\sqrt{u_k u_{k+1} + u_k u_{k+2} + u_{k+1} u_{k+2}}\,,$$

for some $k \geq 1$. In the notation of problem 2802, we will use m for the quantity $\sqrt{u_k u_{k+1} + u_k u_{k+2} + u_{k+1} u_{k+2}}$. We want to show that

$$\begin{aligned} u_{k+4} \;&=\; u_{k+1} + u_{k+2} + u_{k+3} \\ &\qquad + 2\sqrt{u_{k+1}u_{k+2} + u_{k+1}u_{k+3} + u_{k+2}u_{k+3}}\,. \end{aligned} \tag{1}$$

Consider the quantity under the radical. By the induction hypothesis,

$$\begin{aligned} & u_{k+1}u_{k+2} + u_{k+1}u_{k+3} + u_{k+2}u_{k+3} \\ &= u_{k+1}u_{k+2} + (u_{k+1}+u_{k+2})(u_k + u_{k+1} + u_{k+2} + 2m) \\ &= u_{k+1}u_{k+2} + u_k u_{k+1} + u_k u_{k+2} \\ &\qquad + (u_{k+1}+u_{k+2})^2 + 2m(u_{k+1}+u_{k+2}) \\ &= m^2 + 2m(u_{k+1}+u_{k+2}) + (u_{k+1}+u_{k+2})^2 \\ &= (m + u_{k+1} + u_{k+2})^2\,. \end{aligned}$$

Therefore, the quantity on the right side of (1) is equal to

$$u_{k+1}+u_{k+2}+u_{k+3}+2(m+u_{k+1}+u_{k+2}) \;=\; u_{k+3}+3u_{k+2}+3u_{k+1}+2m\,.$$

But, by the induction hypothesis, $2m = u_{k+3} - u_{k+2} - u_{k+1} - u_k$; whence, the right side of (1) is equal to $2u_{k+3} + 2u_{k+2} + 2u_{k+1} - u_k$, which is the recursive definition of u_{k+4}.

This shows that the sequence satisfies the alternative recursive relation of problem 2802 for all $n \geq 1$; hence, the set $\{u_n, u_{n+1}, u_{n+2}, u_{n+3}\}$ has the property $\mathcal{P}$ for all $n \geq 1$.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Bataille points out that "a solution can be found in the interesting article by the [proposer] of the problem: Heron Triangles and Touching Circles, Math. Gazette **87**(2003) No 508, pp 36–41."

2882. [2003 : 466] *Proposed by Mihály Bencze, Brasov, Romania.*

If $x \in \left(0, \frac{\pi}{2}\right)$, $0 \le a \le b$, and $0 \le c \le 1$, prove that

$$\left(\frac{c+\cos x}{c+1}\right)^b < \left(\frac{\sin x}{x}\right)^a .$$

Solution by Michel Bataille, Rouen, France.

We add the hypothesis $(a,b) \neq (0,0)$, since the inequality is false for $a = b = 0$. If $a = 0$, then $b > 0$, and the function $f(x) = t^b$ is strictly increasing on $(0,\infty)$. Since $0 < \dfrac{c+\cos x}{c+1} < 1$, we have

$$\left(\frac{c+\cos x}{c+1}\right)^b < 1 = \left(\frac{\sin x}{x}\right)^0 .$$

Suppose now that $0 < a \le b$. Letting $r = \dfrac{b}{a}$, the proposed inequality becomes $\left(\dfrac{c+\cos x}{c+1}\right)^r < \dfrac{\sin x}{x}$. Since $r \ge 1$, we see that

$$\left(\frac{c+\cos x}{c+1}\right)^r \le \frac{c+\cos x}{c+1} .$$

Hence, it suffices to prove that $\dfrac{c+\cos x}{c+1} < \dfrac{\sin x}{x}$. But for a fixed $\alpha \in (0,1)$, the function $f(t) = \dfrac{t+\alpha}{t+1} = 1 - \dfrac{1-\alpha}{t+1}$ is clearly increasing on $[0,1]$. Thus, it suffices to show that $\dfrac{1+\cos x}{2} < \dfrac{\sin x}{x}$, which is equivalent to

$$x\cos^2\left(\frac{x}{2}\right) < 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) , \qquad \text{or} \qquad \frac{x}{2} < \tan\left(\frac{x}{2}\right) .$$

The last inequality is well known to be true for $x \in (0,\pi)$ and our proof is complete.

Also solved by CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Besides Bataille, four other solvers noticed and stated that the proposed inequality is not valid without additional constraints on a and b. Guersenzvaig assumed that $b \neq 0$. Hess and Janous excluded the case $a = b = 0$. Zhou assumed that $a > 0$ or $a < b$. It is easy to see that all these hypotheses are equivalent to that used by Bataille.

2883. [2003 : 466] *Proposed by Šefket Arslanagić and Faruk Zejnulahi, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that x, y, $z \in [0,1)$ and that $x+y+z = 1$. Prove that

$$\sqrt{\frac{xy}{z+xy}} + \sqrt{\frac{yz}{x+yz}} + \sqrt{\frac{zx}{y+zx}} \le \frac{3}{2} .$$

Essentially the same solution by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Vasile Cîrtoaje, University of Ploiesti, Romania; Titu Zvonaru, Bucharest, Romania; and the proposers.

Since $z + xy = z(x + y + z) + xy = (x + z)(y + z)$, the AM–GM Inequality yields

$$\sqrt{\frac{xy}{z+xy}} = \sqrt{\frac{xy}{(x+z)(y+z)}} \le \frac{1}{2}\left(\frac{x}{x+z} + \frac{y}{y+z}\right).$$

Hence,

$$\begin{aligned}\sum_{\text{cyclic}} \sqrt{\frac{xy}{z+xy}} &\le \frac{1}{2}\sum_{\text{cyclic}}\left(\frac{x}{x+z} + \frac{y}{y+z}\right) \\ &= \frac{1}{2}\left(\frac{x}{x+z} + \frac{y}{y+z} + \frac{y}{y+x} + \frac{z}{z+x} + \frac{z}{z+y} + \frac{x}{x+y}\right) \\ &= \frac{1}{2}\left(\frac{x+y}{x+y} + \frac{y+z}{y+z} + \frac{z+x}{z+x}\right) = \frac{3}{2}.\end{aligned}$$

Also solved by CHRISTOPHER J. BRADLEY, Bristol, UK; VASILE CÎRTOAJE, University of Ploiesti, Romania (a second solution); CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA (two solutions); ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers (a second solution).

Though it is easy to show that the equality holds if and only if $x = y = z = \frac{1}{3}$, only Bornsztein, Specht, Woo, Zvonaru, and the proposer explicitly mentioned this, with Zvonaru being the only one who actually gave a full proof.

Zhao commented that if we replace x, y, z with $\frac{1}{bc}$, $\frac{1}{ca}$, and $\frac{1}{ab}$, respectively (assuming x, y, $z > 0$ since the inequality is trivial if any of x, y, z is zero), then the constraint becomes $a + b + c = abc$ and the inequality becomes

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \le \frac{3}{2},$$

which is well known and appeared in the 1998 Korean Math Olympiad.

2884. [2003 : 467] *Proposed by Niels Bejlegaard, Copenhagen, Denmark.*

Suppose that a, b, c are the sides of a non-obtuse triangle. Give a geometric proof and hence, a geometric interpretation of the inequality

$$a + b + c \ge \sum_{\text{cyclic}} \sqrt{a^2 + b^2 - c^2}.$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Lemma. Suppose that A and B are two points outside a circle centred at O such that AB intersects the circle. If X and Y are two points on the circle such that AX and BY are tangents, then $AB \ge AX + BY$.

Proof: Since the two tangents from a point outside the circle have the same length, we may assume that the points X and Y are in the half-plane (with respect to the line AB) which does not contain the point O (the point O can lie on AB). Let the lines OX and OY intersect the line AB at points P and Q, respectively. Then the points A, P, Q, and B are on the line AB, in that order. We know that the length of any side of a right triangle does not exceed the length of the hypotenuse. Since AXP and BYQ are right triangles, we have

$$AX + BY \;\leq\; AP + BQ \;\leq\; AB\,.$$

Equality occurs if and only if the line AB is tangent to the circle. This completes the proof of the lemma. ∎

Now, let ABC be an acute triangle, let M be the mid-point of BC, and let D be the foot of the altitude from the vertex A. Let Y and Z be points on the nine-point circle such that BY and CZ are tangents. Using the Cosine Law, we obtain

$$\begin{aligned}
&\sqrt{a^2+b^2-c^2}+\sqrt{a^2-b^2+c^2}\\
&= \sqrt{2ab\cos C}+\sqrt{2ac\cos B} \;=\; \sqrt{2\cdot CB\cdot CD}+\sqrt{2\cdot BC\cdot BD}\\
&= 2\sqrt{CM\cdot CD}+2\sqrt{BD\cdot BM} \;=\; 2\cdot CZ+2\cdot BY \;\leq\; 2a\,,
\end{aligned}$$

where the inequality follows from our lemma. Thus,

$$\sqrt{a^2+b^2-c^2}+\sqrt{a^2-b^2+c^2} \;\leq\; 2a\,.$$

Similarly,

$$\begin{aligned}
\sqrt{b^2+c^2-a^2}+\sqrt{b^2-c^2+a^2} &\;\leq\; 2b\\
\text{and}\qquad \sqrt{c^2+a^2-b^2}+\sqrt{c^2-a^2+b^2} &\;\leq\; 2c\,.
\end{aligned}$$

The proposed inequality follows by adding the last three inequalities. Equality holds if and only if the triangle is equilateral.

We note that the proposed inequality is also true for a right triangle. If, say, $c^2 = a^2 + b^2$, then the inequality becomes $a^2 + b^2 \geq (\sqrt{2}-1)ab$, which is true, since $a^2 + b^2 \geq 2ab > (\sqrt{2}-1)ab$.

A geometric interpretation is as follows: the perimeter of a non-obtuse triangle is always greater than or equal to the sum of the lengths of all six tangents from the vertices of the triangle to its nine-point circle, with equality if and only if the triangle is equilateral.

The same geometric interpretation was found by MICHEL BATAILLE, Rouen, France; and the proposer. PETER Y. WOO, Biola University, La Mirada, CA, USA found a different geometric interpretation (it is lengthier and requires additional constructions; thus, we do not include it here). Each of these three solvers has also submitted a proof of the inequality. The following solvers have only proved the inequality: ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CON AMORE PROBLEM GROUP, The Danish University of Education Copenhagen, Denmark; PANOS E. TSAOUSSOGLOU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU, Bucharest, Romania (3 solutions).

The moderator of this problem does not believe that there is a good definition of "geometric proof"; hence, the list of solvers includes those who have submitted any valid proof.

2885. [2003 : 467] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let O and I be the circumcentre and the incentre, respectively, of triangle ABC. Denote the cevians through O by AA', BB', and CC', and those through I by AD, BE, and CF. The sides of the triangle are a, b, and c.

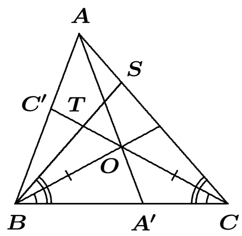
1. If $\dfrac{AA'}{a} = \dfrac{BB'}{b} = \dfrac{CC'}{c}$, prove that $\triangle ABC$ is equilateral.

2. If $\dfrac{AD}{a} = \dfrac{BE}{b} = \dfrac{CF}{c}$, prove that $\triangle ABC$ is equilateral.

3. Give an answer to Sastry's question [1998 : 280]: For an internal point P and its corresponding cevians AD, BE, CF, with $\dfrac{AD}{a} = \dfrac{BE}{b} = \dfrac{CF}{c}$, prove or disprove that $\triangle ABC$ is equilateral.

Solution by Toshio Seimiya, Kawasaki, Japan.

1. Assume first that $b > c$. Then $\angle ABC < \angle ACB$. Since $OB = OC$, we have $\angle OBC = \angle OCB$. Let S be a point on the side AC such that $\angle SBC = \angle ACB$, and let T be the point of intersection of BS and OC'. Since $\angle B'BC = \angle TCB$ and $\angle B'CB = \angle TBC$, we have $\triangle B'BC \cong \triangle TCB$. Thus, $BB' = CT < CC'$.

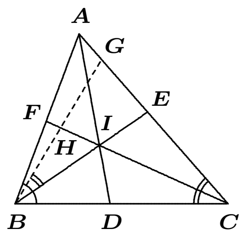


Since $AB < AC$ and $BB' < CC'$, it follows that $\dfrac{BB'}{AC} < \dfrac{CC'}{AB}$; that is, that $\dfrac{BB'}{b} < \dfrac{CC'}{c}$. This contradicts $\dfrac{BB'}{b} = \dfrac{CC'}{c}$.

If $b < c$, we have a similar contradiction. Therefore, $b = c$.

Similarly, from $\dfrac{AA'}{a} = \dfrac{BB'}{b}$, it follows that $a = b$. Hence, $a = b = c$, so that $\triangle ABC$ is equilateral.

2. Assume first that $b > c$. Then $\angle ABC > \angle ACB$. Hence, $\angle ABE = \angle EBC > \angle ACF = \angle FCB$. Let G be the point on the segment AE such that $\angle GBE = \angle ACF$, and let H be the point where BG intersects CF. Then H is a point on the segment FI, and hence, $CH < CF$. Since $\triangle GBE \sim \triangle GCH$, we have $\dfrac{BE}{CH} = \dfrac{BG}{CG}$.



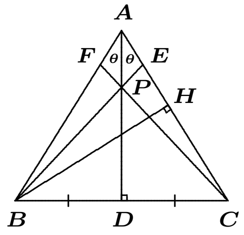
Since $\angle GBE = \angle ACH$ and $\angle EBC > \angle HCB$, it follows that $\angle GBC > \angle GCB$. Then $BG < CG$; that is $\dfrac{BG}{CG} < 1$. Hence, $\dfrac{BE}{CH} < 1$, and thus, $BE < CH < CF$.

Since $AB < AC$ and $BE < CF$, it follows that $\dfrac{BE}{AC} < \dfrac{CF}{AB}$; that is $\dfrac{BE}{b} < \dfrac{CF}{c}$. This contradicts $\dfrac{BE}{c} = \dfrac{CF}{c}$.

If $b < c$, we have a similar contradiction. Therefore, $b = c$.

Similarly, from $\frac{AD}{a} = \frac{BE}{b}$, it follows that $a = b$. Hence, $a = b = c$, which shows that $\triangle ABC$ is equilateral.

3. The conclusion that $\triangle ABC$ is equilateral is not true. We give a counterexample as follows.



Let acute angle θ be such that $\frac{1}{2} < \tan\theta < \frac{1}{\sqrt{3}}$. Then $0 < \theta < \frac{\pi}{6}$, and $\sin\theta < \frac{1}{2}$. Construct isosceles triangle ABC with $AB = AC$ and $\angle BAC = 2\theta$. We have $BC = a$, $CA = b$ and $AB = C$, where $b = c$.

Let D be the mid-point of BC. Then, $\angle BAD = \angle CAD = \theta$ and $AD \perp BC$. Thus,

$$\frac{AD}{BC} = \frac{AD}{2DC} = \frac{1}{2\tan\theta} < 1\,.$$

Hence, $AD < BC$. Since

$$2\tan\theta\sin 2\theta = \frac{2\sin\theta}{\cos\theta} \times 2\sin\theta\cos\theta = 4\sin^2\theta < 1\,,$$

we have

$$\sin 2\theta < \frac{1}{2\tan\theta} = \frac{AD}{BC} < 1\,.$$

Let H be the foot of the perpendicular from B to AC. Then

$$BH = AB\sin 2\theta < AB \times \frac{AD}{BC} < AB\,.$$

There must be a point E on the segment AH such that $BE = AB \times \frac{AD}{BC}$. Now

$$\frac{AD}{BC} = \frac{BE}{AB} = \frac{BE}{AC}\,.$$

Let P be the point of intersection of BE and AD, and let F be the point of intersection of CP with AB. Since AD is the perpendicular bisector of BC, we have $CF = BE$. Thus, $\frac{AD}{BC} = \frac{BE}{AC} = \frac{CF}{AB}$; that is,

$$\frac{AD}{a} = \frac{BE}{b} = \frac{CF}{c}\,.$$

This relation holds for the three concurrent cevians AD, BE, and CF, but $\triangle ABC$ is not equilateral.

Remark: The following theorem can easily be proved.

Theorem. Suppose that $\triangle ABC$ is equilateral and that P is an interior point. Suppose that AD, BE, and CF are three cevians through P.
If $\frac{AD}{BC} = \frac{BE}{CA} = \frac{CF}{AB}$, then P is the circumcentre (incentre) of $\triangle ABC$.

Parts 1 and 2 above are the converses of this theorem.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA (parts 1 and 2 only); and the proposer.

2886. [2003 : 468] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

If a, b, c are positive real numbers such that $abc = 1$, prove that

$$ab^2 + bc^2 + ca^2 \ \geq \ ab + bc + ca\,.$$

I. *Nearly identical solutions Chip Curtis, Missouri Southern State College, Joplin, MO, USA; Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA; Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; and Li Zhou, Polk Community College, Winter Haven, FL, USA.*

By the AM–GM Inequality, $\dfrac{ab^2 + 2bc^2}{3} \ \geq \ \sqrt[3]{(ab^2)(bc^2)^2} \ = \ bc$, and similarly, $\dfrac{bc^2 + 2ca^2}{3} \ \geq \ ca$ and $\dfrac{ca^2 + 2ab^2}{3} \ \geq \ ab$. Adding the three inequalities completes the proof.

II. *Solution by Christopher J. Bradley, Bristol, UK.*

Since $abc = 1$, the inequality is equivalent to

$$\frac{b}{c} + \frac{c}{a} + \frac{a}{b} \ \geq \ \frac{1}{c} + \frac{1}{a} + \frac{1}{b} \tag{1}$$

Applying the Cauchy–Schwarz Inequality to the vectors $\left[\sqrt{\frac{b}{c}}, \sqrt{\frac{c}{a}}, \sqrt{\frac{a}{b}}\,\right]$ and $\left[\frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{a}}\right]$, we have $\left(\frac{b}{c} + \frac{c}{a} + \frac{a}{b}\right)\left(\frac{1}{b} + \frac{1}{c} + \frac{1}{a}\right) \geq \left(\frac{1}{c} + \frac{1}{a} + \frac{1}{b}\right)^2$, from which (1) follows.

III. *Similar solutions by Arkady Alt, San Jose, CA, USA; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Joe Howard, Portales, NM, USA; and Titu Zvonaru, Bucharest, Romania.*

Since $abc = 1$, there are positive real numbers x, y, z such that $a = \dfrac{x}{y}$, $b = \dfrac{y}{z}$, and $c = \dfrac{z}{x}$. The given inequality is then equivalent to

$$\begin{aligned} \frac{xy}{z^2} + \frac{yz}{x^2} + \frac{zx}{y^2} \ &\geq \ \frac{x}{z} + \frac{y}{x} + \frac{z}{y}\,, \\ \text{or} \qquad x^3y^3 + y^3z^3 + z^3x^3 \ &\geq \ x^3y^2z + xy^3z^2 + x^2yz^3\,. \end{aligned} \tag{2}$$

Inequality (2) follows from Muirhead's Theorem on majorization since the vector $[3, 3, 0]$ majorizes the vector $[3, 2, 1]$. Note that equality holds if and only if $x = y = z$; that is, if and only if $a = b = c = 1$. Alternately, the AM–GM Inequality could be applied to obtain

$$x^3y^3 + 2y^3z^3 \ \geq \ 3\sqrt[3]{(x^3y^3)(y^3z^3)^2} \ = \ 3xy^3z^2\,.$$

Similarly, $y^3z^3 + 2z^3x^3 \geq 3x^2yz^3$ and $z^3x^3 + 2x^3y^3 \geq 3x^3y^2z$.

Adding these three inequalities, (2) follows.

Also solved by MICHEL BATAILLE, Rouen, France; VASILE CÎRTOAJE, University of Ploiesti, Romania; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA; MARCELO RUFINO de OLIVEIRA, Belem, Brazil; ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2887. [2003 : 468; corrected 2004 : 38] *Proposed by Vedula N. Murty, Dover, PA, USA.*

If a, b, c are the sides of $\triangle ABC$ in which at most one angle exceeds $\frac{\pi}{3}$, and if R is its circumradius, prove that

$$a^2+b^2+c^2 \;\le\; 6R^2\sum_{\text{cyclic}}\cos A\,.$$

Solution by Joe Howard, Portales, NM, USA.

We use the following well-known facts (see [1]):

$$\prod_{\text{cyclic}}\cos A \;=\; \frac{s^2-4R^2-4Rr-r^2}{4R^2}\,, \tag{1}$$

$$\sum_{\text{cyclic}}\cos A \;=\; \frac{R+r}{R}\,, \tag{2}$$

$$\sum_{\text{cyclic}}\cos B\cos C \;=\; \frac{r^2+s^2-4R^2}{4R^2}\,, \tag{3}$$

$$a^2+b^2+c^2 \;=\; 2s^2-2r^2-8Rr\,, \tag{4}$$

$$R \;\ge\; 2r \qquad \text{(Euler's Inequality)}\,. \tag{5}$$

Under the given condition, we must have $\sum\limits_{\text{cyclic}}(2\cos A-1)\le 0$, which expands to

$$8\prod_{\text{cyclic}}\cos A+2\sum_{\text{cyclic}}\cos A \;\le\; 1+4\sum_{\text{cyclic}}\cos B\cos C\,.$$

Using equations (1), (2) and (3), we obtain

$$8\left(\frac{s^2-4R^2-4Rr-r^2}{4R^2}\right)+2\left(\frac{R+r}{R}\right) \;\le\; 1+4\left(\frac{r^2+s^2-4R^2}{4R^2}\right)\,,$$

which simplifies to

$$\begin{aligned} s^2 &\le 3r^2+3R^2+6Rr \;=\; r^2+3R^2+7rR+r(2r-R)\\ &\le r^2+3R^2+7rR\,, \end{aligned}$$

using (5). This last inequality is easily seen to be equivalent to the proposed inequality (by using (2) and (4)).

Equality holds if and only if the triangle is equilateral.

Reference

[1] D.S. Mitrinovic, J.E. Pekaric, V. Volenec, Recent Advances in Geometric Inequalities, Kluwer, 1989.

The corrected version of the problem was also solved by VASILE CÎRTOAJE, University of Ploiesti, Romania; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. The following solvers submitted counterexamples to the original statement of the problem and suggested alternatives to correct the inequality: VASILE CÎRTOAJE, University of Ploiesti, Romania; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; and ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany. Counterexamples to the original statement of the problem only were found by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA. There were also two incorrect solutions and one incorrect counterexample submitted.

Klamkin and Specht suggested and proved the inequality

$$a^2+b^2+c^2 \;\le\; 8R^2\sum_{\text{cyclic}}\cos A\,,$$

while Cîrtoaje suggested and proved

$$18R^2\Big(-1+\sum_{\text{cyclic}}\cos A\Big) \;\le\; a^2+b^2+c^2 \;\le\; 2R^2\Big(3+\sum_{\text{cyclic}}\cos A\Big)\,,$$

instead of the original inequality. Janous gave a proof similar to Howard's proof; he also commented that triangles such as the ones considered in the present problem are referred to (in [1]) as Triangles of Bager's Type II.

2888★. [2003 : 468; corrected 2004 : 38] *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let a, b, c be the sides of $\triangle ABC$, in which at most one angle exceeds $\frac{\pi}{3}$. Give an algebraic proof of

$$8a^2b^2c^2+\prod_{\text{cyclic}}\left(b^2+c^2-a^2\right) \;\le\; 3abc\sum_{\text{cyclic}}a\left(b^2+c^2-a^2\right)\,.$$

Comment: The proposer was looking for a proof not involving trigonometry. Since this point was not stated clearly enough, all trigonometric proofs were also deemed acceptable.

I. *Solution by Chip Curtis, Missouri Southern State College, Joplin, MO, USA.*

Let $F(a,b,c)$ be the expression obtained by subtracting the left side of the proposed inequality from the ride side. Then

$$\begin{aligned} F(a,b,c) &= \sum_{\text{cyclic}} a^6 + 3abc \sum_{\text{cyclic}} (a^2b + ab^2) - 3abc \sum_{\text{cyclic}} a^3 \\ &\qquad - \sum_{\text{cyclic}} (a^4b^2 + a^2b^4) \; - \; 6a^2b^2c^2 \,. \end{aligned}$$

We need to show that $F(a,b,c) \geq 0$.

Without loss of generality, we may assume that $A \leq B \leq C$. Then $\cos C \leq \frac{1}{2} \leq \cos B \leq \cos A$. Then, applying the Law of Cosines to angle B, we have

$$c^2 + a^2 - b^2 - ca \;\geq\; 0\,. \tag{1}$$

We now make the change of variables: $x = a$, $y = b - a$, and $z = c - b$. Clearly, x, y, $z \geq 0$. Then (1) is equivalent to

$$(x+y+z)^2 + x^2 - (x+y)^2 - x(x+y+z) \;\geq\; 0\,,$$

which in turn is equivalent to

$$z(x+2y+z) \;\geq\; xy\,. \tag{2}$$

With further help from a computer algebra system, we obtain

$$\begin{aligned} F(a,b,c) &= F(x, x+y, x+y+z) \\ &= z^6 + 6xz^5 + 6yz^5 + 25xyz^4 + 14y^2z^4 + x^4yz + 10x^2z^4 \\ &\qquad + 6x^3z^3 + 16y^3z^3 + 39xy^2z^3 + 29x^2yz^3 + x^4z^2 \\ &\qquad + 8y^4z^2 + 26xy^3z^2 + 11x^3yz^2 + 27x^2y^2z^2 + x^4y^2 \\ &\qquad - (xy)^2(2y^2 + 2xy + 3xz) - (xy)(4xy^2z)\,. \end{aligned}$$

The inequality (2) implies that

$$-(xy)^2(2y^2+2xy+3xz) \;\geq\; -z^2(x+2y+z)^2(2y^2+2xy+3xz)$$

and

$$-(xy)(4xy^2z) \;\geq\; -z(x+2y+z)(4xy^2z)\,.$$

Substituting into the expression for $F(a,b,c)$ above, we obtain

$$\begin{aligned} F(a,b,c) &\geq z^6 + 6xz^5 + 6yz^5 + 25xyz^4 + 14y^2z^4 + x^4yz + 10x^2z^4 \\ &\qquad + 6x^3z^3 + 16y^3z^3 + 39xy^2z^3 + 29x^2yz^3 + x^4z^2 \\ &\qquad + 8y^4z^2 + 26xy^3z^2 + 11x^3yz^2 + 27x^2y^2z^2 + x^4y^2 \\ &\qquad - z^2(x+2y+z)^2(2y^2+2xy+3xz) \\ &\qquad - z(x+2y+z)(4xy^2z)\,. \end{aligned}$$

Expanding and combining like terms yields

$$\begin{aligned} F(a,b,c) &\geq z^6 + 3(x+2y)z^5 + (4x^2 + 11xy + 12y^2)z^4 \\ &\qquad + (3x^3 + 8y^3)z^3 + xy(13x+11y)z^3 + x(x^3+2y^3)z^2 \\ &\qquad + x^2y(9x+13y)z^2 + x^4yz + x^4y^2 \\ &\geq 0\,, \end{aligned}$$

as desired.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Dividing both sides of the proposed inequality by $a^2b^2c^2$, we see that it is equivalent to

$$8(1+\cos A\cos B\cos C) \;\leq\; 6\sum_{\text{cyclic}}\cos A\,.$$

Using $1+\cos A\cos B\cos C = \frac{1}{2}(\sin^2 A+\sin^2 B+\sin^2 C)$, the above inequality becomes

$$4(\sin^2 A+\sin^B+\sin^2 C) \;\leq\; 6\sum_{\text{cyclic}}\cos A\,.$$

This is equivalent to the inequality in problem 2887 above.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer (both of whom used trigonometry). There was one incorrect solution.

ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; and PETER Y. WOO, Biola University, La Mirada, CA, USA simply pointed out that the originally posed problem was incorrect and gave counter-examples to show this. Hess demonstrated that the original inequality would hold if the summation appearing on the right side was replaced by

$$\sum_{\text{cyclic}} a\left(b^2+c^2-a^2\right)\,.$$

Klamkin demonstrated that the original inequality would hold if the term $8a^2b^2c^2$ on the left side was replaced by $6a^2b^2c^2$.

2889. [2003 : 514] *Proposed by Vedula N. Murty, Dover, PA, USA.*

Suppose that A, B, C are the angles of $\triangle ABC$, and that r and R are its inradius and circumradius, respectively. Show that

$$4\cos(A)\cos(B)\cos(C) \;\leq\; 2\left(\frac{r}{R}\right)^2\,.$$

Solution by Michel Bataille, Rouen, France.

Let I and H be the incentre and orthocentre of $\triangle ABC$, respectively. We know that

$$IH^2 \;=\; 2r^2-4R^2\cos A\cos B\cos C$$

(see problem 2747 [2003 : 251]). Thus, $2r^2-4R^2\cos A\cos B\cos C \geq 0$, and the proposed inequality follows immediately. Equality occurs if and only if $I=H$; that is, $\triangle ABC$ is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Bristol, UK; SCOTT BROWN, Auburn University, Montgomery, AL, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; ANDREI SIMION, student, Cooper Union for Advancement of Science *and Art, New York, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.*

Bradley remarks that this problem "goes back a long way, and appears as part of Problem 517 in Wolstenholme's Mathematical Problems, *Macmillan (1867)." Solvers also cited the article by the proposer* [2003 : 82–83] *and the book* D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities,* Kluwer Academic Publishers, 1989.

2890. [2003 : 515] *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Suppose that the polynomial $A(z) = z^n + \sum\limits_{k=0}^{n-1} a_k z^k$ can be factored into $A(z) = \prod\limits_{k=1}^{n} (z - z_k)$, where the z_k are positive real numbers.

Prove that, for $k = 1, 2, \ldots, n-1$,

$$\left|\frac{a_{n-k}}{C(n,k)}\right|^{\frac{1}{k}} \geq \left|\frac{a_{n-k-1}}{C(n,k+1)}\right|^{\frac{1}{k+1}},$$

where $C(n,k)$ denotes the binomial coefficient $\binom{n}{k}$. When does equality occur?

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

As a matter of fact, the claimed inequality is nothing less than the *Maclaurin–Newton Inequality*, dating back at least to 1729, the year Maclaurin published his note [1]. Equality occurs if and only if all the roots of $A(z)$ are equal.

As a very recent reference (including also a proof), consult Chapter 12, entitled "Symmetric Sums", of the book [2]. [*Ed.* Janous recommends this book highly.]

References

[1] C. Maclaurin, A second letter to Martin Folges, Esq.; concerning the roots of equations with the demonstration of other rules in algebra, *Phil. Trans.* **36** (1729), 59–96.

[2] J.M. Steele, *The Cauchy–Schwarz Master Class (An introduction to the Art of Mathematical Inequalities).* Cambridge University Press, UK, and The Mathematical Association of America, 2004.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER BOWEN, Halandri, Greece; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers.

2891. [2003 : 515] *Proposed by Vedula N. Murty, Dover, PA, USA, adapted by the Editors.*

Two proofreaders, Chris and Pat, were asked to read a manuscript and find the errors. Let B be the number of errors which both Chris and Pat found, C the number of errors found only by Chris, and P the number found only by Pat; lastly, let N be the number of errors found by neither of them.

Prove that $\sqrt{(B+P)(C+N)(B+C)(P+N)} \;\geq\; |BN-CP|$.

I. *Solution by Kathleen E. Lewis, SUNY Oswego, Oswego, NY, USA.*

The information given about numbers B, C, P, and N actually tells us only that they are non-negative integers.

The square of the left side of the proposed inequality, namely $(B+P)(C+N)(B+C)(P+N)$, when multiplied out, is a sum of sixteen non-negative terms, two of which are B^2N^2 and C^2P^2. Therefore, this sum is greater than or equal to $B^2N^2+C^2P^2$. The square of the right side is $B^2N^2-2BNCP+C^2P^2$, which is less than or equal to $B^2N^2+C^2P^2$. Therefore, the square of the right side is less than or equal to the square of the left. Since taking the positive square roots preserves the relationship, the inequality holds.

II. *Solution by the proposer.*

Define two indicator variables X and Y, where $X=1$ if Chris catches the error (and $X=0$ otherwise), and $Y=1$ if Pat catches the error (and $Y=0$ otherwise). The correlation coefficient between X and Y is

$$r_{x,y} \;=\; \frac{BN-CP}{\sqrt{(B+P)(C+N)(B+C)(P+N)}}\,.$$

The proposed inequality is immediately obtained by observing the known inequality $-1\leq r_{x,y}\leq 1$.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER BOWEN, Halandri, Greece; CHRISTOPHER J. BRADLEY, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU, Bucharest, Romania.

2892. [2003 : 516] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Let A and B be arbitrary 2×2 matrices over $\mathbb{C}$. For all complex numbers α, β, γ, prove that

$$\det(\alpha I+\beta AB+\gamma BA) \;=\; \det(\alpha I+\gamma AB+\beta BA)\,.$$

(Here, I denotes the 2×2 identity matrix.)

(b)★ Is there a similar identity for $n\times n$ matrices?

[The proposer gives a "Machine" proof for (a). We want a *purely algebraic* proof.]

I. *Solution to* (a) *by Richard I. Hess, Rancho Palos Verdes, CA, USA* (*modified slightly by the editor*).

Let $AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $BA = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Set

$$P \;=\; \alpha I + \beta AB + \gamma BA \quad \text{and} \quad Q \;=\; \alpha I + \gamma AB + \beta BA\,.$$

Then by direct (human) computations, we have

$$\begin{aligned}
&\det(P) - \det(Q) \\
&\quad= (\alpha + \beta a + \gamma e)(\alpha + \beta d + \gamma h) - (\beta b + \gamma f)(\beta c + \gamma g) \\
&\qquad\qquad - (\alpha + \gamma a + \beta e)(\alpha + \gamma d + \beta h) + (\gamma b + \beta f)(\gamma c + \beta g) \\
&\quad= \alpha(\beta - \gamma)(a + d - e - h) + (\beta^2 - \gamma^2)(ad - bc + fg - eh) \\
&\quad= \alpha(\beta - \gamma)\big(\mathrm{tr}(AB) - \mathrm{tr}(BA)\big) + (\beta^2 - \gamma^2)\big(\det(AB) - \det(BA)\big) \\
&\quad= 0\,.
\end{aligned}$$

II. *Solution by Michel Bataille, Rouen, France.*

(a) Let $C = \beta AB + \gamma BA$ and $D = \gamma AB + \beta BA$. If $\gamma = 0$, then clearly $\det(C) = \det(D)$. For $\gamma \neq 0$, consider the polynomial

$$P(x) \;=\; \det(xAB + \gamma BA) - \det(xBA + \gamma AB)\,.$$

We readily see that $P(0) = P(\gamma) = P(-\gamma) = 0$. Since P has degree at most two and has three distinct roots, it must be the zero polynomial. It follows that $\det(C) = \det(D)$ for all β, γ.

Now, fix β and γ. Note that $\mathrm{tr}(C) = \mathrm{tr}(D) = (\beta + \gamma)\mathrm{tr}(AB)$. Hence, C and D must have the same characteristic polynomial, since they have the same determinant and the same trace. That is, $\det(xI - C) = \det(xI - D)$. With $x = -\alpha$, we obtain the required result.

(b) No, the identity in (a) does not hold if $n \geq 3$.

Let I_n denote the $n \times n$ identity matrix, and let

$$A_3 \;=\; \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B_3 \;=\; \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then straightforward calculations yield $\det(I_3 + 2A_3B_3 + B_3A_3) = 352$ and $\det(I_3 + A_3B_3 + 2B_3A_3) = 348$.

For $n > 3$, let $A_n = O_{n-3} \oplus A_3$ and $B_n = O_{n-3} \oplus B_3$, where O_{n-3} denotes the $(n-3) \times (n-3)$ zero matrix. Then, from the basic properties of direct sum of matrices, we have

$$A_nB_n \;=\; O_{n-3} \oplus A_3B_3 \quad \text{and} \quad B_nA_n \;=\; O_{n-3} \oplus B_3A_3\,.$$

Hence,

$$\begin{aligned}
\det(I_n + 2A_nB_n + B_nA_n) &= \det(I_{n-3} \oplus (I_3 + 2A_3B_3 + B_3A_3)) \\
&= \det(I_{n-3})\det(I_3 + 2A_3B_3 + B_3A_3) \\
&= 352
\end{aligned}$$

and

$$\begin{aligned}\det(I_n + A_nB_n + 2B_nA_n) &= \det(I_{n-3} \oplus (I_3 + A_3B_3 + 2B_3A_3)) \\ &= \det(I_{n-3})\det(I_3 + A_3B_3 + 2B_3A_3) \\ &= 348\,.\end{aligned}$$

Also solved (part (a) only) by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; CRISTINEL MORTICI, Valahia University of Targoviste, Romania. Both parts were also solved by LI ZHOU, Polk Community College, Winter Haven, FL, USA.

Using the Cayley-Hamilton Theorem and more complicated arguments, Guersenzvaig derived the following formula (for 2×2 matrices) from which the identity in (a) follows immediately:

$$\begin{aligned}\det(\alpha I + \beta AB + \gamma BA) &= \alpha^2 + \alpha(\beta+\gamma)\mathrm{tr}(AB) + (\beta^2+\gamma^2)\det(AB) \\ &\quad + \beta\gamma((\mathrm{tr}(AB))^2 - \mathrm{tr}(A^2B^2))\,.\end{aligned}$$

2893. [2003 : 516] *Proposed by Vedula N. Murty, Dover, PA, USA.*

In [2001 : 45–47], we find three proofs of the classical inequality

$$1 \le \sum_{\text{cyclic}} \cos(A) \le \frac{3}{2}\,.$$

In [2002 : 85–87], we find Klamkin's illustrations of the Majorization (or Karamata) Inequality.

Prove the above "classical inequality" using the Majorization Inequality.

[*Ed.* For the convenience of the reader, we review the Majorization Inequality. Let $S = (a_1, a_2, \ldots, a_n)$ and $T = (b_1, b_2, \ldots, b_n)$, where $a_1 \ge a_2 \ge \cdots \ge a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$. Suppose that $\sum\limits_{j=1}^{n} a_j = \sum\limits_{j=1}^{n} b_j$ and $\sum\limits_{j=1}^{k} a_j \ge \sum\limits_{j=1}^{k} b_j$ for each $k = 1, 2, \ldots, n-1$. Then we say that S *majorizes* T, and we write $S \succ T$. If $S \succ T$ and F is a convex function, then

$$\sum_{j=1}^{n} F(a_j) \ \ge\ \sum_{j=1}^{n} F(b_n)\,.$$

If $S \succ T$ and F is a concave function, then the above inequality is reversed.]

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

The angles in the problem are angles in a triangle ABC. Without loss of generality, we assume that $A \ge B \ge C$. We distinguish the case where the triangle is obtuse ($A > \pi/2$) from the case where it is non-obtuse ($A \le \pi/2$). In both cases, we will use the fact that the function $f(x) = \cos x$ is concave on the interval $[0, \pi/2]$.

First suppose that $A \le \pi/2$. Then

$$\left(\frac{\pi}{2},\frac{\pi}{2},0\right) \succ (A,B,C) \succ \left(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3}\right).$$

Applying the Majorization Inequality, we have

$$2\cos\left(\frac{\pi}{2}\right)+\cos 0 \le \cos A+\cos B+\cos C \le 3\cos\left(\frac{\pi}{3}\right).$$

Simplifying, we obtain the required inequalities.

Now suppose that $A > \pi/2$. Then

$$(\pi - A, 0) \succ (B,C) \succ \left(\frac{\pi-A}{2},\frac{\pi-A}{2}\right).$$

Applying the Majorization Inequality, we have

$$\begin{aligned} \cos(\pi-A)+\cos 0 &\le \cos B+\cos C \le 2\cos\left(\frac{\pi-A}{2}\right),\\ \text{or}\qquad -\cos A+1 &\le \cos B+\cos C \le 2\sin\left(\frac{A}{2}\right).\end{aligned}$$

Then

$$1 \le \cos A+\cos B+\cos C \le \cos A+2\sin\left(\frac{A}{2}\right).$$

Finally, we note that $\cos A+2\sin\left(\frac{A}{2}\right) \le \frac{3}{2}$, because

$$\begin{aligned}\cos A+2\sin\left(\frac{A}{2}\right)-\frac{3}{2} &= 1-2\sin^2\left(\frac{A}{2}\right)+2\sin\left(\frac{A}{2}\right)-\frac{3}{2}\\ &= -\frac{1}{2}\left(1-2\sin\left(\frac{A}{2}\right)\right)^2 \le 0.\end{aligned}$$

Also solved by LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incomplete solution.

2894. [2003 : 517] *Proposed by Vedula N. Murty, Dover, PA, USA.*

Suppose that $\triangle ABC$ is acute-angled. With the standard notation, prove that

$$4abc < \left(a^2+b^2+c^2\right)\left(a\cos A+b\cos B+c\cos C\right) \le \frac{9}{2}abc.$$

Solution by Joe Howard, Portales, NM, USA.

Let $K = (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)$. Since $\triangle ABC$ is acute, we must have $K > 0$. Let S be the area and R the circumradius of $\triangle ABC$. We will use the following well-known or easy-to-prove facts.

$$(a^2+b^2+c^2)(16S^2) = K+8(abc)^2, \tag{1}$$

$$\sum_{\text{cyclic}} a\cos A = \frac{2S}{R}, \tag{2}$$

$$4SR = abc, \tag{3}$$

$$a^2+b^2+c^2 \le 9R^2. \tag{4}$$

(The last inequality follows from $OH^2 = 9R^2 - (a^2+b^2+c^2)$, where O is the circumcentre and H the orthocentre of $\triangle ABC$.)

Using equations (1), (2), and (3), and the inequality $K > 0$, we obtain

$$\begin{aligned}(a^2+b^2+c^2)\sum_{\text{cyclic}} a\cos A &= (a^2+b^2+c^2)\cdot\frac{2S}{R} \\ &= \frac{(a^2+b^2+c^2)(16S^2)}{8SR} \\ &= \frac{K+8(abc)^2}{2abc} \;=\; \frac{K}{2abc} + 4abc \;>\; 4abc\,,\end{aligned}$$

which proves the left-hand inequality.

Using inequality (4) and equations (2) and (3), we get

$$(a^2+b^2+c^2)\sum_{\text{cyclic}} a\cos A \;\le\; 9R^2\cdot\frac{2S}{R} \;=\; \frac{9}{2}(4SR) \;=\; \frac{9}{2}(abc)\,,$$

which takes care of the right-hand inequality. Equality holds in the right-hand inequality if and only if the triangle is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; SCOTT BROWN, Auburn University, Montgomery, AL, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; MARCELO RUFINO de OLIVEIRA, Belém, Brazil; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece (2 solutions); PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2895. [2003 : 517] *Proposed by Vedula N. Murty, Dover, PA, USA.*

Suppose that A and B are two events with probabilities $P(A)$ and $P(B)$ such that $0 < P(A) < 1$ and $0 < P(B) < 1$. Let

$$K \;=\; \frac{2[P(A\cap B) - P(A)P(B)]}{P(A)+P(B)-2P(A)P(B)}\,.$$

Show that $|K| < 1$, and interpret the value $K = 0$.

Solution by Michel Bataille, Rouen, France.

We prove that $|K| \le 1$.

First, note that

$$\begin{aligned}P(A)+P(B)-2P(A)P(B) &= P(A)(1-P(B)) + P(B)(1-P(A)) \\ &> 0\,.\end{aligned}$$

Since $P(A \cap B) \leq P(A)$ and $P(A \cap B) \leq P(B)$, we have

$$2P(A \cap B) \;\leq\; P(A) + P(B)$$

and $K \leq 1$ follows.

We remark that $K = 1$ holds if and only if $P(A) = P(B) = P(A \cap B)$. This occurs, for example, if $B = A \cup N$ where $A \cap N = \emptyset$ and $P(N) = 0$.

Now, we show that $K \geq -1$, which amounts to

$$P(A) + P(B) + 2P(A \cap B) \;\geq\; 4P(A)P(B)\,. \qquad (1)$$

Equation (1) certainly holds if $P(A) + P(B) \leq 1$ since, in that case,

$$\begin{aligned} P(A) + P(B) + 2P(A \cap B) &\geq P(A) + P(B) \;\geq\; \big(P(A) + P(B)\big)^2 \\ &\geq 4P(A)P(B)\,. \end{aligned}$$

Now suppose that $P(A) + P(B) > 1$. Then at least one of $P(A)$ and $P(B)$ is greater than $\frac{1}{2}$, say $P(A) > \frac{1}{2}$. Let $P(A) = \frac{1}{2} + h$ and $P(B) = \frac{1}{2} + k$, where $0 < h < \frac{1}{2}$ and $|k| < \frac{1}{2}$. Then

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \;=\; 1 + h + k - P(A \cup B) \\ &\geq h + k\,, \end{aligned}$$

and

$$4P(A)P(B) \;=\; 1 + 2(h+k) + 4hk\,.$$

Thus, (1) will follow if $h + k - 4hk \geq 0$. This is certainly true if $k \leq 0$, since $-4hk \geq 0$ and $h + k = P(A) + P(B) - 1 > 0$. If $k > 0$, then $\sqrt{hk} \leq \frac{1}{2}$ and

$$h + k - 4hk \;\geq\; 2\sqrt{hk} - 4hk \;=\; 2\sqrt{hk}(1 - 2\sqrt{hk}) \;\geq\; 0\,.$$

Note that $K = -1$ when $P(A) = P(B) = \frac{1}{2}$ and $A \cap B = \emptyset$, and that $K = 0$ when $P(A \cap B) = P(A)P(B)$ (that is, when A and B are independent events).

Also solved by CHRISTOPHER BOWEN, Halandri, Greece; CHRISTOPHER J. BRADLEY, Bristol, UK; KEITH EKBLAW, Walla Walla, WA, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2896. [2003 : 517] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $0 < x_0 < x_1$ and that, for $n = 1, 2, 3, \ldots$,

$$\sqrt{1 + x_n}\left(1 + \sqrt{x_{n-1}x_{n+1}}\right) \;=\; \sqrt{1 + x_{n-1}}\left(1 + \sqrt{x_n x_{n+1}}\right)\,.$$

(a) Prove that the sequence $\{x_n\}$ is convergent.

(b) Find $\lim\limits_{n\to\infty} x_n$.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

The given recursive formula forces each term of the sequence $\{x_n\}$ to be non-negative. For each n, let a_n be the real number in $[0, \pi/2)$ such that $x_n = \tan^2 a_n$. The given formula becomes

$$\sec a_n(1 + \tan a_{n-1} \tan a_{n+1}) = \sec a_{n-1}(1 + \tan a_n \tan a_{n+1}) .$$

Multiplying both sides by $\cos a_n \cos a_{n-1} \cos a_{n+1}$, we obtain

$$\cos(a_{n-1} - a_{n+1}) = \cos(a_n - a_{n+1}) .$$

Suppose that $a_i = a_{i+1}$ for some $i \geq 1$. Then $\cos(a_i - a_{i+1}) = 1$, and therefore, $\cos(a_{i-1} - a_{i+1}) = 1$. Since $a_n \in [0, \pi/2)$ for all n, we must have $a_{i-1} = a_{i+1} = a_i$. Then there is no least integer $i \geq 1$ such that $a_i = a_{i+1}$, and therefore, there is no such integer i at all. Thus, no two consecutive terms of the sequence $\{a_n\}$ are equal.

Since $a_{n-1} \neq a_n$, the relation $\cos(a_{n-1} - a_{n+1}) = \cos(a_n - a_{n+1})$ implies that $2a_{n+1} = a_n + a_{n-1}$ for each n. It follows by induction that

$$a_n = \frac{a_0 + 2a_1}{3} - \frac{a_0 - a_1}{3(-2)^{n-1}} ,$$

which obviously converges to $\frac{a_0 + 2a_1}{3}$. Therefore, the sequence $\{x_n\}$ also converges, and we have

$$\lim_{n\to\infty} x_n = \tan^2\left(\frac{a_0 + 2a_1}{3}\right) = \tan^2\left(\frac{\tan^{-1}\sqrt{x_0} + 2\tan^{-1}\sqrt{x_1}}{3}\right) .$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER BOWEN, Halandri, Greece; CHRISTOPHER J. BRADLEY, Bristol, UK; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2897. [2003 : 518] *Proposed by Václav Konečný, Big Rapids, MI, USA.*

(a) Show that it is possible to divide a circular disc into four parts with the same area by means of three line segments of the same length.

(b) Does there exist a straight edge and compass construction (in the classical sense; that is, with a finite number of steps)?

Solution by Christopher Bowen, Halandri, Greece.

(a) Without loss of generality, we assume a unit radius so that the area of the circle is π. Let AB be a chord that divides the area in the ratio $1 : 3$.

Draw a parallel chord CD of equal length. Any line EF through the center O, with E on AB and F on CD, will divide the area of the circle between these two chords into two equal parts, and, therefore, the three line segments AB, CD, and EF divide the circle into four regions of equal area. It remains to insure that $EF = AB$.

Define $2\theta = \angle BOA$. Since the area of the smaller segment (of the circle) determined by AB is $(2\theta - \sin 2\theta)/2 = \pi/4$, we have

$$2\theta - \sin 2\theta \;=\; \pi/2\,. \tag{1}$$

This shows that $\theta = \dfrac{\sin 2\theta}{2} + \dfrac{\pi}{4} > \dfrac{\pi}{4}$. Hence, if H is the foot of the perpendicular to AB from O, then

$$OH \;=\; \frac{BH}{\tan\theta} \;<\; BH\,.$$

Since (by definition) AB cannot be a diameter, we see that $BH < OB$; whence, $OH < BH < OB$. As line FOE rotates about O, the length OE will vary continuously from a maximum of OB to a minimum of OH. Thus, some position of this rotating line will have length $OE = BH$, as desired.

(b) The construction of these segments must involve at least one chord of length $2\sin\theta$ such as AB, since, if another segment were to intersect this chord at a point interior to both segments, the two segments would by themselves create four regions, contrary to what is desired. One can construct AB by ruler and compass if and only if one can construct the length $k = \sin 2\theta = 2\sin\theta\cos\theta$, with θ defined by (1). However, a length is constructible by ruler and compass if and only if the corresponding number is algebraic of degree 2^n over the rationals for some positive integer n. Since

$$\cos k \;=\; \cos(\sin 2\theta) = \cos\left(2\theta - \frac{\pi}{2}\right) \;=\; \sin 2\theta = k\,,$$

we obtain

$$k - \cos k \;=\; ke^0 - \tfrac{1}{2}e^{ik} - \tfrac{1}{2}e^{-ik} \;=\; 0\,.$$

By the Hermite-Lindemann Theorem (which provided the first proof of the transcendence of π), if k were algebraic, then the above algebraic linear combination of exponentials could not be zero. The number k must therefore be transcendental, which implies that construction by ruler and compass is impossible.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2898. [2003 : 518] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that $\dfrac{(2^n)!}{1!2!4!\cdots(2^{n-1})!}$ is divisible by $\displaystyle\prod_{k=1}^{n}\left(2^{k-1}+1\right)$.

I. *Composite of essentially the same solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Michel Bataille, Rouen, France; Christopher Bowen, Halandri, Greece; Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina; Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

Since

$$\frac{(2^n)!}{1!2!4!\cdots(2^{n-1})!} = \prod_{k=1}^{n}\frac{(2^k)!}{(2^{k-1})!(2^{k-1})!} = \prod_{k=1}^{n}\binom{2^k}{2^{k-1}},$$

it suffices to show that $\binom{2^k}{2^{k-1}}$ is divisible by $2^{k-1}+1$ for all $k \in \mathbb{N}$.

Since $\binom{2^k+1}{2^{k-1}+1} = \frac{2^k+1}{2^{k-1}+1}\binom{2^k}{2^{k-1}}$ is an integer, and since the integers 2^k+1 and $2^{k-1}+1$ are relatively prime, it follows that $2^{k-1}+1$ divides $\binom{2^k}{2^{k-1}}$.

II. *Composite of similar solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Recall that the m^{th} Catalan number is defined by $C_m = \frac{1}{m+1}\binom{2m}{m}$, for $m = 1, 2, 3, \ldots$. These numbers are well known to be integers.

Let $A_n = \frac{(2^n)!}{1!2!4!\cdots(2^{n-1})!}$, and let $P_n = \prod\limits_{k=1}^{n}(2^{k-1}+1)$. Then, as in Solution I above, $A_n = \prod\limits_{k=1}^{n}\binom{2^k}{2^{k-1}}$.

Hence, $\frac{A_n}{P_n} = \prod\limits_{k=1}^{n}\frac{1}{2^{k-1}+1}\binom{2^k}{2^{k-1}} = \prod\limits_{k=0}^{n-1}C_{2^k}$, which is an integer.

Also solved by KEE-WAI LAU, Hong Kong, China; MIKE SPIVEY, Samford University, Birmingam, AL, USA; and the proposer.

There were actually several variations of the proof that $2^{k-1}+1$ *divides* $\binom{2^k}{2^{k-1}}$. *Here are some of the identities used:*

$$\begin{aligned}
\frac{1}{2^n+1}\binom{2^{n+1}}{2^n} &= \binom{2^{n+1}}{2^n} - \binom{2^{n+1}}{2^n-1}, \\
2^n\binom{2^{n+1}}{2^n} &= (2^n+1)\binom{2^{n+1}}{2^n-1}, \\
\binom{2^k}{2^{k-1}} &= \frac{2^{k-1}+1}{2^{k-1}}\binom{2^k}{2^{k-1}+1}.
\end{aligned}$$

2899. [2003 : 518] *Proposed by Hiroshi Kotera, Nara City, Japan.*

Find the maximum area of a pentagon $ABCDE$ inscribed in a unit circle such that the diagonal AC is perpendicular to the diagonal BD.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

If $ABCD$ were fixed, then since $[ABCDE] = [ABCD] + [DEA]$, the area would be maximized if E were placed as far away from line AD as possible. We may therefore assume that E is the mid-point of the arc AD not containing B. Let O be the centre of the circumcircle, and let $\alpha = \angle AOB$, and $\beta = \angle BOC$. Then $\angle COD = \pi - \alpha$ (because $AC \perp BD$ if and only if $\angle AOB + \angle COD = \pi$), and $\angle DOE = \angle EOA = (\pi - \beta)/2$. Thus,

$$\begin{aligned}
[ABCDE] &= [AOB] + [BOC] + [COD] + [DOE] + [EOA] \\
&= \frac{1}{2}\left(\sin\alpha + \sin\beta + \sin(\pi - \alpha) + 2\sin\frac{\pi - \beta}{2}\right) \\
&= \sin\alpha + \frac{1}{2}\sin\beta + \cos\frac{\beta}{2}.
\end{aligned}$$

Since $\alpha \in (0, \pi)$, we have

$$[ABCDE] \le 1 + \frac{1}{2}\sin\beta + \cos\frac{\beta}{2}.$$

The derivative, $\frac{d}{d\beta}\left(\frac{1}{2}\sin\beta + \cos\frac{\beta}{2}\right) = \frac{1}{2}\left(\cos\beta - \sin\frac{\beta}{2}\right)$, has its only zero in $(0, \pi)$ at $\beta = \pi/3$. It is easy to verify that the maximum occurs there. Thus,

$$[ABCDE] \le 1 + \frac{1}{2}\sin\beta + \cos\frac{\beta}{2} \le 1 + \frac{1}{2}\sin\frac{\pi}{3} + \cos\frac{\pi}{6} = \frac{4 + 3\sqrt{3}}{4}.$$

The desired maximum area is therefore $\frac{4 + 3\sqrt{3}}{4}$, which is attained when $\alpha = \pi/2$ and $\beta = \pi/3$.

Also solved by CHRISTOPHER J. BRADLEY, Bristol, UK; G.D. CHAKERIAN, University of California, Davis, and MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution (which made the assumption that AD was a diameter, turning the proposal into a simpler but still interesting problem).

Chakerian and Klamkin described our pentagon in terms of a regular 12–gon $P_1P_2\cdots P_{12}$ inscribed in a unit circle: $ABCDE = P_1P_4P_6P_9P_{11}$ is the pentagon of largest area inscribed in the circle having a pair of adjacent perpendicular diagonals. This pentagon has angle 120° at P_{11} $(= E)$, and angles of 105° at the other four vertices; the sides have lengths $\sqrt{2}$, 1, $\sqrt{2}$, 1, 1, (starting with $AB = P_1P_4$).

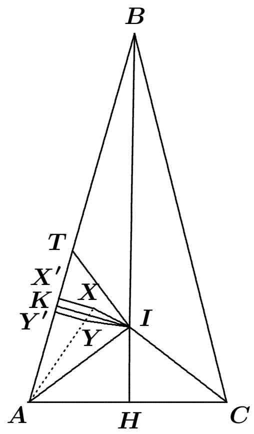
Bradley reported that in the 1980's this was training problem $X3$ of the late F.J. Budden, leader of the UK IMO team. It might well pre-date this. Woo likewise recalls having seen it before, perhaps a few years ago as a Putnam Competition problem.

2900★. [2003 : 518] *Proposed by Stanley Rabinowitz, Westford, MA, USA.*

Let I be the incentre of $\triangle ABC$, r_1 the inradius of $\triangle IAB$ and r_2 the inradius of $\triangle IAC$. Computer experiments using *Geometer's Sketchpad* suggest that $r_2 < \frac{5}{4} r_1$.

(a) Prove or disprove this conjecture.

(b) Can $\frac{5}{4}$ be replaced by a smaller constant?



I. *Solution by Toshio Seimiya, Kawasaki, Japan.*

(a) Let r be the inradius of $\triangle ABC$. Let T be the point on AB such that $\angle AIT = 90^\circ$. Let X and Y be the incentres of $\triangle IAB$ and $\triangle IAT$, respectively. Then X, Y, and A are collinear. Since $\angle AIB > 90^\circ$, we have

$$\angle AIY \;=\; \tfrac{1}{2}\angle AIT \;<\; \tfrac{1}{2}\angle AIB \;=\; \angle AIX\,.$$

Thus, Y is a point on the segment AX.

Let r_1' be the inradius of $\triangle IAT$. Let X' and Y' be the feet of the perpendiculars to AB from X and Y, respectively. Then $XX' = r_1$ and $YY' = r_1'$. Since $XX' \parallel YY'$, we have $\dfrac{YY'}{XX'} = \dfrac{AY}{AX} < 1$. That is, $\dfrac{r_1'}{r_1} < 1$, which implies that

$$r_1' \;<\; r_1\,. \tag{1}$$

Suppose that H and K are the feet of the perpendiculars from I to AC and AB, respectively. Clearly, $IK = IH = r$. In $\triangle IAC$, it is known that $IH > 2r_2$. Thus,

$$r_2 \;<\tfrac{1}{2}r\,. \tag{2}$$

In $\triangle IAT$, we see that $[IAT] = \frac{1}{2}AT \cdot IK = \frac{1}{2}AT \cdot r$, where $[IAT]$ denotes the area of $\triangle IAT$. Moreover, $[IAT] = \frac{1}{2}(AT + AI + IT)r_1'$. Therefore,

$$\begin{aligned} AT \cdot r &= (AT + AI + IT)r_1'\,, \\ \frac{r}{r_1'} &= \frac{AT + AI + IT}{AT} \;=\; 1 + \cos\alpha + \sin\alpha \\ &= 1 + \sqrt{2}\sin(\alpha + 45^\circ) \;\le\; 1 + \sqrt{2}\,, \end{aligned} \tag{3}$$

where we have set $\alpha = \angle TAI$.

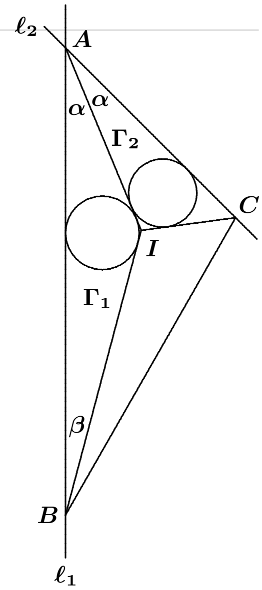
From (1), (2), and (3), we obtain

$$\frac{r_2}{r_1} \;<\; \frac{r_2}{r_1'} \;=\; \frac{\frac{1}{2}r}{r_1'} \;\le\; \frac{1+\sqrt{2}}{2} \;=\; \frac{2+\sqrt{2}}{4} \;<\; \frac{5}{4}\,.$$

Hence, $r_2 < \frac{5}{4}r_1$ is true.

(b) As shown in the above proof, $\frac{5}{4}$ can be replaced by $(1+\sqrt{2})/2$, and this is the best possible.

II. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*



We assert that the precise constant should be $(1+\sqrt{2})/2$, which is slightly less than $5/4$.

Let $\alpha = A/2$ be fixed, but $\beta = B/2$ be variable. Then $\angle BIC = 90^\circ + \alpha$ is fixed in size. Let ℓ_1 be the (fixed) line containing A and B, and let ℓ_2 be the (fixed) line containing A and C, and be fixed. Let $k = IA$, and let Γ_1 and Γ_2 be the incircles of $\triangle IAB$ and $\triangle IAC$, respectively.

Suppose that B moves along ℓ_1 away from A. Then Γ_1 grows in size, and the limit of its radius r_1 is $\frac{1}{2}k\sin\alpha$. On the other hand, if we denote the perimeter of $\triangle AIC$ by s and its area by $[AIC]$, then we see that $\angle AIC$ shrinks towards 90° as a limit, and its radius r_2 shrinks toward a limit of

$$\frac{2[AIC]}{s} = \frac{k^2\tan\alpha}{k + k\tan\alpha + k\sec\alpha} = \frac{k\sin\alpha}{\cos\alpha + \sin\alpha + 1}\,.$$

Then

$$\begin{aligned}\frac{r_1}{r_2} = \frac{1+\sin\alpha+\cos\alpha}{2} &= \tfrac{1}{2}\big(1+\sin\alpha+\sin(90^\circ-\alpha)\big)\\ &= \tfrac{1}{2} + \sin 45^\circ\cos(\alpha - 45^\circ)\,.\end{aligned}$$

The maximum occurs when $\alpha = 45^\circ$, and the limit is $(1+\sqrt{2})/2$.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; C.R. PRANESACHAR, Bangalore, India; and PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA.

YEAR END FINALE

This brings to a close the second year of my stewardship of ***CRUX with MAYHEM***. I think I have benefited from the learning that took place in my first year, but there are times when this is definitely not obvious. I continue to receive compliments from our readers about the quality of our product. Again, I insist that these compliments be shared among the many people who bring the various pieces to me for final editing and assembly. I will attempt to list them all here. If I miss anyone, it is due to faulty memory.

The first person I need to thank is BRUCE CROFOOT, my Associate Editor. In addition to acting as one of several Problems Editors, Bruce puts in vast amounts of time scrutinizing several drafts of each section. His attention to detail picks up an incredible number of typos (and out-and-out errors!) before we ever go to the proof readers.

There are many other people whom I wish to thank most sincerely for their particular contributions. These include ILIYA BLUSKOV, RICK BREWSTER, CHRIS FISHER, EDWARD WANG, and BRUCE SHAWYER for their regular and timely service in assessing the solutions; BRUCE GILLIGAN, for ensuring that ***CRUX with MAYHEM*** has quality articles; JOHN GRANT McLOUGHLIN, for ensuring that we have book reviews that are appropriate to our readership; ROBERT WOODROW for overseeing the *Olympiad Corner*; and SHAWN GODIN for doing likewise with the *Skoliad*.

For the past few years we have been posing all of our Mayhem and CRUX Problems in French, as well as English. The task of translating has again fallen on the shoulders of JEAN-MARC TERRIER and MARTIN GOLDSTEIN. I want to thank them for their efforts, and for always coming through even when I have given them very little time for turn-around. They often find ways to improve the English wording of the problems! I could not ask for two better colleagues.

Those assisting with the ***MATHEMATICAL MAYHEM*** section, our journal-within-a-journal, are thanked within that section by the Mayhem Editor, with whom they work closely. I will simply add my thanks to his.

I want to thank all the proofreaders. MOHAMMED AASSILA, DAVID FELDMAN, BRUCE KADANOFF, and ROGER COROAS assist the editors with this task. The quality of the work of all these people is a vital part of what makes ***CRUX with MAYHEM*** what it is. Thank you one and all.

Thanks also go to the University College of the Cariboo (soon to become Canada's newest University—Thompson Rivers University) and my colleagues in the Department of Mathematics and Statistics for their continued understanding and support, and for believing that my work on this journal is important enough to reduce my teaching load sufficiently to allow me to do it. Special thanks go to CAROL COSTACHE, secretary to our department, for all that she does to give me more time to edit.

Also, the LaTeX expertise of JOANNE LONGWORTH at the University of Calgary and TAO GONG at Wilfrid Laurier University, the ***MAYHEM*** staff, and all others who produce material, is much appreciated.

Thanks to GRAHAM WRIGHT, the Managing Editor, who keeps me on the right track (and adhering to deadlines!), and to the University of Toronto Press, and TAMI EHRLICH in particular, who continue to print a high-quality product.

The online version of ***CRUX with MAYHEM*** continues to grow. Thanks are due to JUDI BORWEIN at Dalhousie University for putting all the material on the Canadian Mathematical Society website.

Last but not least, I send my thanks to you, the readers. Without you, ***CRUX with MAYHEM*** would not be what it is. We receive between 150 and 200 problem proposals each year, and we publish only 100 of these in each volume. Of course, we receive hundreds of solutions, as you will see in the index that follows. Every year, we receive solutions from new readers. This is very gratifying. We hope that these new solvers will become regular solvers and proposers of new problems. Please ensure that your name and address is on EVERY problem or proposal, and that each starts on a fresh sheet of paper. Otherwise, there may be filing errors, resulting in a submitted solution or proposal being lost. We need your ARTICLES, PROPOSALS, and SOLUTIONS to keep ***CRUX with MAYHEM*** alive and well. Keep them coming!

I would like to take this time to remind our readers that we plan to have a special issue dedicated to the memory of Murray Klamkin in 2005. If you have something to contribute to such an issue, please send it, and identify it as intended for that issue. If you intend to send material to me for inclusion in that issue, please ensure that it arrives by **March 31, 2005**, to give us time to put the issue together. Thank you.

I wish everyone the compliments of the season, and a very happy, peaceful, and prosperous year 2005.

Jim Totten

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