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Crux Mathematicorum

Crux Mathematicorum with Mathematical Mayhem

EDITORIAL

Standard approaches quickly become, well, standard. When you perform a task often, you develop efficient ways to approach it. It's effective (in that it gets done), but an opportunity to be creative gets lost. Consider this problem as an example:

$$\text{Solve } \sqrt{x} + \sqrt{x+1} = 5$$

I have seen this type of problem so many times that I don't have to even think about it. I know exactly what my plan is and I know exactly how it will play out: square both sides, isolate the root, square again, expand and solve, check that the solution is in the domain. Maybe there are some trivial simplification steps that can occur in the middle, but the process is standard:

$$\begin{aligned}
&(\sqrt{x} + \sqrt{x+1})^2 = 25, \\
&x + x + 1 + 2\sqrt{x(x+1)} = 25, \\
&\sqrt{x(x+1)} = 12 - x, \qquad (\star) \\
&x(x+1) = (12-x)^2, \\
&x^2 + x = 144 - 24x + x^2, \\
&25x = 144, \\
&x = \frac{144}{25}
\end{aligned}$$

Students also tend to follow this approach, or at least try to. Some of them will make the obvious mistake in expanding the square and forgetting the $2\sqrt{x(x+1)}$ term. Some will get stuck at the next step and wonder if square roots will ever go away. This is where I saw some ingenuity last week: to avoid squaring both sides again, at step $(\star)$ the student made a substitution using the original equation: $\sqrt{x+1} = 5 - \sqrt{x}$:

$$\begin{aligned}
&\sqrt{x}(5 - \sqrt{x}) = 12 - x, \\
&5\sqrt{x} - x = 12 - x, \\
&25x = 144, \\
&x = \frac{144}{25}.
\end{aligned}$$

What about other ways of doing it? Here's a suggestion from our Facebook group. First, multiply both sides of the equation by the conjugate $\frac{1}{5}\left(\sqrt{x} - \sqrt{x+1}\right)$:

$$\frac{1}{5}\left(\sqrt{x} - \sqrt{x+1}\right)\left(\sqrt{x} + \sqrt{x+1}\right) = \left(\sqrt{x} - \sqrt{x+1}\right).$$

Simplifying, we get

$$\sqrt{x} - \sqrt{x+1} = \frac{1}{5}.$$

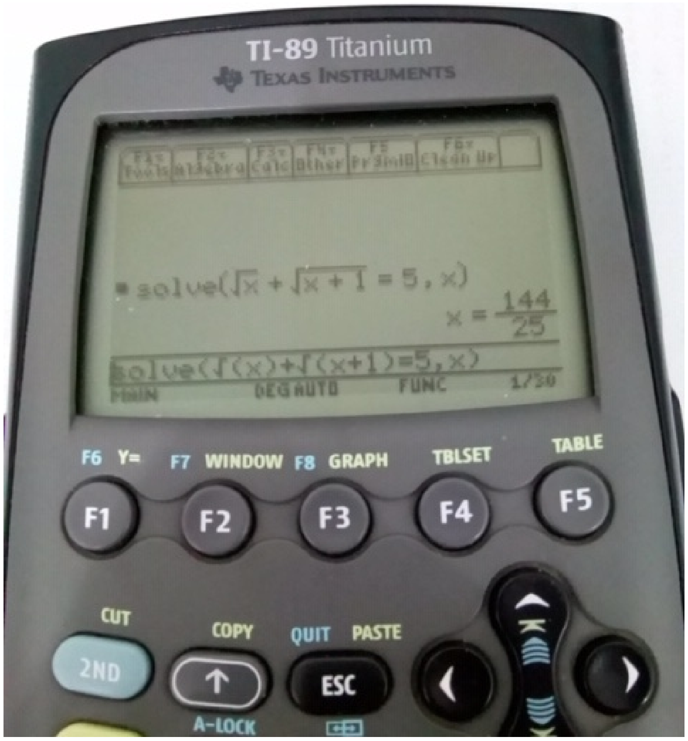
Adding the original equation yields

$$2\sqrt{x} = \frac{24}{5},$$

so $x = \frac{144}{25}$. I have to admit that I generally wouldn't have given this equation a second thought. But it's a nice way to emphasize that even with most standard and seemingly routine problems, there's room for creativity.

Kseniya Garaschuk

P.S. And there's always this (photo courtesy of Adelmárcio André):



MathemAttic

No. 8

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **November 30, 2019**.

MA36. Let A and B be sets with the property that there are exactly 144 sets which are subsets of at least one of A or B. How many elements does the union of A and B have?

MA37. Both 4 and 52 can be expressed as the sum of two squares as well as exceeding another square by 3:

$$\begin{aligned} 4 = 0^2 + 2^2 \quad &\text{and} \quad 4 - 3 = 1^2, \\ 52 = 4^2 + 6^2 \quad &\text{and} \quad 52 - 3 = 7^2. \end{aligned}$$

Show that there are an infinite number of such numbers that have these two characteristics.

MA38. Consider a 12×12 chessboard consisting of 144 1×1 squares. If three of the four corner squares are removed, can the remaining area be covered by placing 47 1×3 tiles?

MA39. Point E is selected on side AB of triangle ABC in such a way that $AE : EB = 1 : 3$ and point D is selected on side BC so that $CD : DB = 1 : 2$. The point of intersection of AD and CE is F. Determine the value of $\dfrac{EF}{FC} + \dfrac{AF}{FD}$.

MA40. In racing over a given distance d at uniform speeds, A can beat B by 20 yards, B can beat C by 10 yards, and A can beat C by 28 yards. Determine the distance d in yards.

. . . .

Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 novembre 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA36. Soient A et B des ensembles tels qu'il existe exactement 144 ensembles qui sont sous ensembles d'au moins un de A et B. Combien d'éléments la réunion de A et B a-t-elle ?

MA37. Les nombres 3 et 52 peuvent s'exprimer comme somme de deux carrés et aussi comme 3 plus un carré :

$$\begin{aligned} 4 &= 0^2 + 2^2 \quad \text{and} \quad 4 - 3 = 1^2, \\ 52 &= 4^2 + 6^2 \quad \text{and} \quad 52 - 3 = 7^2. \end{aligned}$$

Démontrer qu'il y a un nombre infini de nombres qui, comme 3 et 52, peuvent s'exprimer de ces deux façons.

MA38. Soit un échiquier 12×12, comprenant 144 carrés de taille 1×1. Si trois des quatre coins sont enlevés, la surface restante peut-elle être couverte par 47 tuiles de taille 1×3?

MA39. Le point E se situe sur le côté AB du triangle ABC de façon à ce que $AE : EB = 1 : 3$, tandis que le point D se trouve sur le côté BC de façon à ce que $CD : DB = 1 : 2$. Le point d'intersection de AD et CE est dénoté F. Déterminer la valeur de $\dfrac{EF}{FC} + \dfrac{AF}{FD}$.

MA40. Dans une course sur une distance donnée d et à des vitesses uniformes, A défait B par 20 verges et B défait C par 10 verges, tandis que A défait C par 28 verges. Déterminer la distance d, en verges.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2018: 45(3), p. 107–108.

MA11. Let $f(x) = 375x^5 - 131x^4 + 15x^2 - 435x - 2$. Find the remainder when $f(97)$ is divided by 11.

Proposed by Don Rideout.

We received two correct solutions and one incorrect solution. We present the common approach taken by both solvers, Richard Hess and Konstantine Zelator.

With

$$x = 97 \equiv -2 \pmod{11},$$

we have that

$$\begin{aligned} f(97) &\equiv (-2)^5 - (-1)(-2)^4 + 4(-2)^2 - 6(-2) - 2 \\ &\equiv 1 + 5 + 5 + 1 - 2 \\ &\equiv -1 \pmod{11}. \end{aligned}$$

The required remainder is 10.

MA12. Ten straight lines are drawn on a two-dimensional plane. Given that three of these lines are parallel to one another, what is the maximum possible number of intersection points formed by the lines?

Originally Problem 16 of 2016 Problem Solving for Irish Second level Mathematicians.

There was one correct solution, submitted by Richard Hess.

Since each pair of lines gives rise to at most one intersection, the maximum number of intersections for ten lines would be

$$\binom{10}{2} = 45.$$

However, in this case, the three parallel lines do not intersect, reducing the number of intersections by 3. Therefore, the maximum number of intersection points is 42.

Editor's comment. An alternative argument notes that the seven lines (apart from the parallel ones) are responsible for at most

$$\binom{7}{2} = 21$$

intersection points. There are 21 additional possible intersection points when these seven lines intersect the three parallel lines, for a total of 42.

MA13. How many ways can the letters of the word LETTERKENNY be arranged in a row if the R must stay in the middle position and the letters L,R,K and Y must remain in their current order LRKY? (An example of an arrangement that meets the requirements is ELTTERENKYN.)

Originally Problem 18 of 2016 Problem Solving for Irish Second level Mathematicians.

We received 3 solutions, 2 of which were correct. We present the solution by Jason Smith of Richland Community College, Decatur, Illinois.

There are 5 positions available for L. There are $4 + 3 + 2 + 1 = 10$ arrangements available for the KY combination, depending on whether K appears in the seventh, eight, ninth, or tenth position. Seven positions remain for 3 E's, 2 T's, and 2 N's. The number of arrangements for these equals the trinomial coefficient

$$\frac{7!}{3!2!2!} = 210.$$

Therefore, $5 \cdot 10 \cdot 210 = 10500$ arrangements are possible that meet the given conditions.

MA14. In $\triangle ABC$, the side AB has length 20 and $\angle ABC = 90^\circ$. If the lengths of the other sides must be positive integers, how many such triangles are possible?

Originally Problem 17 of 2016 Problem Solving for Irish Second level Mathematicians.

We received 8 solutions, 7 of which were correct. We present the solution by Richard Hess.

One leg of the Pythagorean triangle is 20. Let the other leg be n and the hypotenuse be m. Then

$$m^2 - n^2 = 400, \quad \text{or} \quad (m+n)(m-n) = 400.$$

Define $d = m - n$ as the smaller divisor of 400. Then

$$m = 200/d + d/2$$

and

$$n = 200/d - d/2.$$

The divisors of 400 up to 20 are 1, 2, 4, 5, 8, 10, and 20. Of these, only 2, 4, 8, and 10 give integer values for m and n. There are four triangles:

$$(15, 20, 25),\ (20, 21, 29),\ (20, 48, 52),\ \text{and}\ (20, 99, 101).$$

MA15. Prove that $43^{43} - 17^{17}$ is divisible by 10. (Do not use Fermat's Little Theorem.)

Originally problem 3 of grade 6 1963 Leningrad Math Olympiad.

We received ten submissions, all are correct. We present 2 solutions.

Solution 1, by Ismael El Yassini.

We have:

$3^{60} - 1 \equiv 9^{20} - 1 \equiv (-1)^{20} - 1 \equiv 1 - 1 \equiv 0 \pmod{10}$.

We note that 7 is the inverse of 3 modulo 10 because

$$7 \times 3 = 21 \equiv 1 \pmod{10}.$$

Therefore

$$43^{43} \equiv 3^{43} \equiv 3^{43} \times 1 \equiv 3^{43} \times (3 \times 7)^{17} \equiv 3^{60} \times 7^{17} \equiv 1 \times 7^{17} \equiv 17^{17} \pmod{10}.$$

Solution 2, by Henry Richardo.

We have

$$\begin{aligned} 43^{43} - 17^{17} &= \left(43^{43} + 17^{43}\right) - \left(17^{43} + 17^{17}\right) \\ &= \left(43^{43} + 17^{43}\right) - 17^{17}\left((17^2)^{13} + 1^{13}\right). \end{aligned}$$

It is known that $(x + y)|(x^n + y^n)$ for positive odd integers n. Therefore, the first term is a multiple of $43 + 17 = 60$, and the second term is a multiple of $17^2 + 1 = 290$, which implies that the original expression ends in a zero.

PROBLEM SOLVING VIGNETTES

No. 8

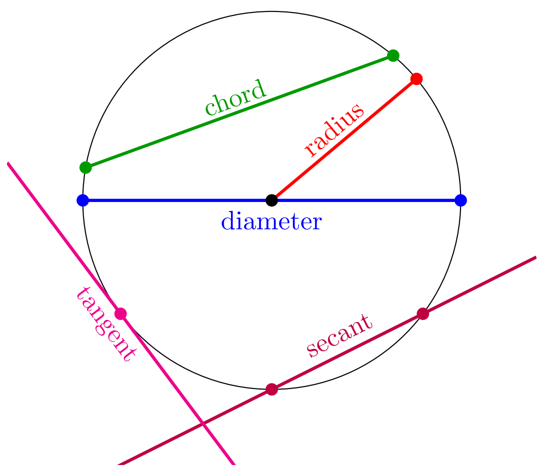
Shawn Godin

Going in Circles

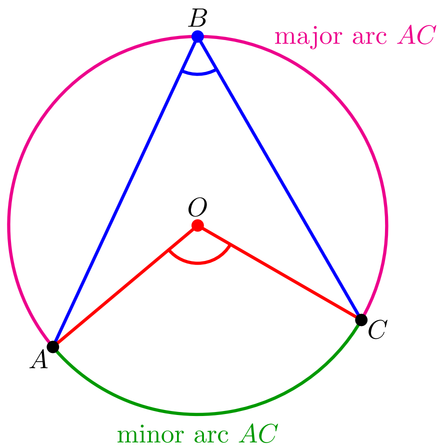
In this issue we will examine some theorems from Euclidean geometry that deal with circles.

First we will need some definitions. The following are some line segments and lines related to a circle:

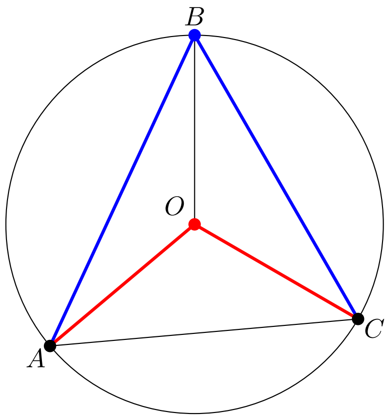
- a *chord* is a line segment joining any two points on the circumference of a circle;
- a *radius* is a line segment joining the centre of a circle to any point on its circumference;
- a *diameter* is a chord that passes through the centre of a circle;
- a *secant* is a line that passes through two points on a circle's circumference; and
- a *tangent* is a line that passes through only one point of a circle's circumference.



Two points on the circumference of a circle partition the circumference into two *arcs*. If the arcs are equal in length, they are called *semicircles*. If the arcs are different lengths, the larger one is called the *major arc* and the smaller one is called the *minor arc*. Using three points on the circumference of a circle one can create an *inscribed angle*. The vertex of an inscribed angle lies on one of the arcs created by the end points of the angle. We say the angle is *subtended* by the other arc. In the diagram below, $\angle ABC$ is an inscribed angle and is subtended by the minor arc AC. The indicated angle $\angle AOC$ is called the *central angle* that is subtended by the same arc as $\angle ABC$.



There is a relationship between central angles and inscribed angles that are subtended by the same arc. If we add two line segments to the last diagram we get the diagram below.



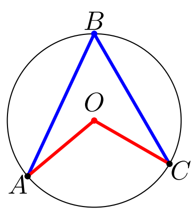
Since OA, OB and OC are radii, ΔOAB and ΔOBC are isosceles triangles, as is triangle ΔOAC. Therefore, in ΔOAB, we have $\angle OAB = \angle OBA$ and hence $\angle AOB = 180^\circ - 2\angle OBA$. Similarly, $\angle COB = 180^\circ - 2\angle OBC$ and so

$$\begin{aligned}\angle AOC &= 360^\circ - \angle AOB - \angle COB\\ &= 360^\circ - (180^\circ - 2\angle OBA) - (180^\circ - 2\angle OBC)\\ &= 2(\angle OBA + \angle OBC)\\ &= 2\angle ABC.\end{aligned}$$

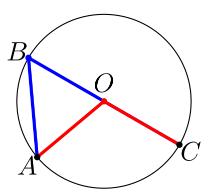
We have just demonstrated the following theorem:

Central angle theorem *The central angle subtended by an arc is twice the measure of any inscribed angle subtended by the same arc.*

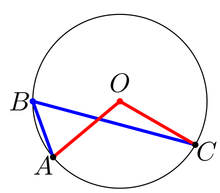
We have to be careful proving this theorem. Our diagram was only one of three possible configurations of the central and inscribed angles. It is possible for the centre to lie on one of the arms of $\angle ABC$ as shown in the case 2 diagram. It is also possible for an arm of $\angle ABC$ to intersect an arm of $\angle AOC$ at some point not on the circumference as in the case 3 diagram. In case 4 we get another situation entirely. This special case will be revisited in the problem section.



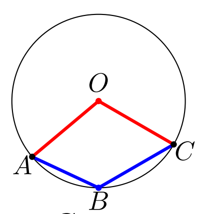
Case 1



Case 2

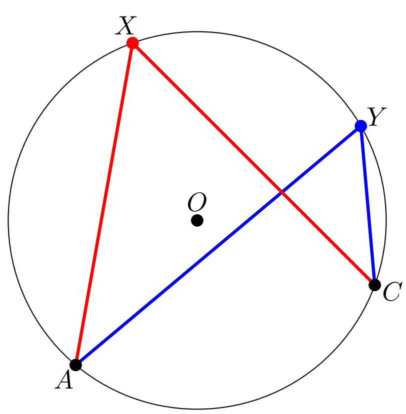


Case 3



Case 4

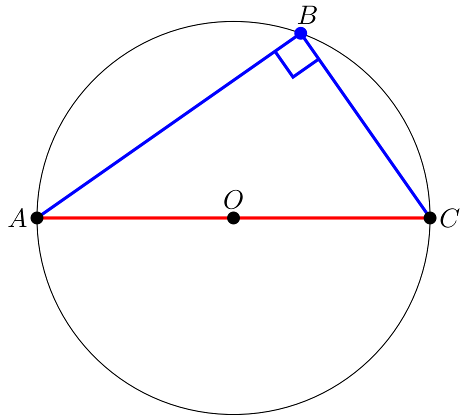
This simple theorem has some interesting consequences. Since the theorem said "The central angle subtended by an arc is twice the measure of **any** inscribed angle subtended by the same arc" it follows that any two inscribed angles subtended by the same arc are equal. Thus in the diagram below $\angle AXC = \angle AYC$.



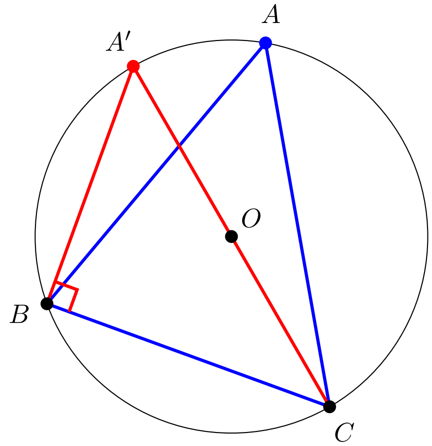
If AC is a diameter of a circle, then $\angle AOC = 180^\circ$, which means that if B is any point on the circumference then

$$\angle ABC = \frac{1}{2}\angle AOC = 90^\circ.$$

Thus the angle in a semicircle is a right angle.



Next, we will derive the extended law of sines. Suppose we have an acute angled ΔABC and we inscribe it in a circle. From the previous results, we know that if we pick a point A' on the same arc as A then $\Delta CA'B = \Delta CAB$. We will choose A' such that CA' is a diameter of the circle and hence $\angle A'BC$ is a right angle.



Thus,

$$\sin(\angle BAC) = \sin(\angle BA'C) = \frac{a}{2R}$$

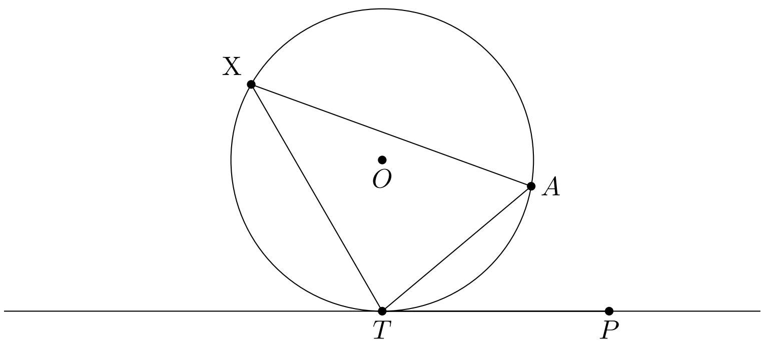
where a is the length of the side opposite to $\angle A$ and R is the radius of the circumcircle. This can be rewritten as $\dfrac{a}{\sin A} = 2R$. We can repeat the process for the other angles and their opposite sides b and c to get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

which is the usual law of sines presented in school with the extra piece of information relating the ratios to the radius (diameter) of the circumcircle.

Here are a couple of problems involving circles to practice using these theorems.

1. Show that configurations as in diagrams of case 2 and case 3 yield the same relationship between the inscribed and central angle.
2. Determine the relationship between the central angle and the inscribed angle as shown in the case 4 diagram.
3. Quadrilateral $ABCD$ has its four vertices on a circle (called a *cyclic quadrilateral*). The vertices appear in the order A, B, C, D as we move around the circle. Show that $\angle A + \angle C = \angle B + \angle D = 180°$.
4. Show that the extended law of sines holds for obtuse angled triangles.
5. Prove the **Tangent chord theorem**: *Given TA is any chord of a circle and PT is tangent to the circle at T. If X is a point on the arc TA that is outside the region bounded by $\angle PTA$, then $\angle TXA = \angle PTA$.*



6. Given a circle with perpendicular chords AC and BD that meet in a point P, prove that the line through P that is perpendicular to AB passes through the midpoint of CD.
7. Given four points A, B, C, D on a circle whose centre is O, define P to be the intersection of the lines AD and BC. Prove that:
   (a) $\angle APB = \frac{1}{2}(\angle AOB + \angle DOC)$ when P is inside the circle, and
   (b) $\angle APB = \frac{1}{2}|\angle AOB - \angle DOC|$ when P is outside.

Another theorem that may be of use in the practice problems is the fact that a tangent to a circle is perpendicular to the radius through the point of tangency. We will explore more Euclidean geometry and more circle properties in future columns.

Thank you to Chris Fisher for his comments and problem suggestions (6 and 7).

TEACHING PROBLEMS

No.5

Richard Hoshino

The Club Scheduling Problem

> Several students have formed various clubs, based on academic subjects that most interest them. The clubs consist of the following students:
>
> Astronomy Club: Michael, Breanna, Joe
>
> Biology Club: Breanna, Bonnie
>
> Calculus Club: Joe, Caitlin
>
> Dance Club: Joe, Gillian, Patrick
>
> Economics Club: Caitlin, Michael, Bonnie
>
> Food Studies Club: Bethany, Gillian, Michael
>
> Geology Club: Bethany, Patrick
>
> Each of the seven clubs wants to have an hour-long meeting Friday afternoon; each person in the club must be present for the meeting. Class ends at 12:00, and the eight students want to get their club meetings over with as soon as possible. What is the earliest possible time at which all eight students can complete each of their hour-long meetings?

The most inefficient solution is 19:00, by having each of the seven clubs meeting in hour-long slots, one right after the other. We can save time by combining slots where no conflict occurs: for example, having the Calculus and Geology clubs meeting at the same time, since no student belongs to both clubs. This produces a solution whose answer is 18:00.

Whenever I share this problem in a classroom, at least one student is able to find the correct answer of 15:00, using a trial and error approach to show that the seven clubs can meet in three hour-long slots. For example, here is one possible 3-hour solution:

12:00 to 13:00: Astronomy, Geology

13:00 to 14:00: Economics, Dance

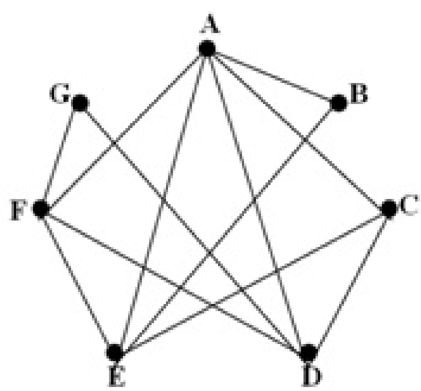
14:00 to 15:00: Food Studies, Biology, Calculus

A common approach is to make an 8×7 table with students' names in the rows and clubs in the columns, to see where the possible conflicts arise. Through such a table, students determine which clubs can meet together and which ones cannot, and find a solution such as the one above.

Once students find a solution for three hours, they are asked whether there exists a solution for two hours. They quickly see that no 2-hour solution exists, since Michael and Joe each belong to three different clubs: since each of them needs at least three hours to complete their meetings, the entire group needs at least three hours as well.

Despite satisfaction with solving the problem, students remark that making a 56-element table is tedious. They realize that all is required is to determine which clubs have conflicts – e.g. Astronomy and Biology cannot meet at the same time because one individual belongs to both clubs: what matters is that there is an individual belonging to both clubs, not who that individual is.

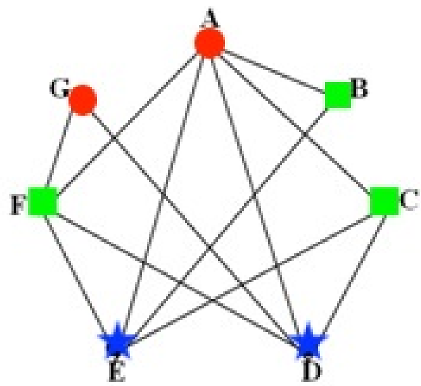
Through this process, we motivate the key idea of solving scheduling problems using Graph Theory, to show that the above scheduling question can be solved by creating a "conflict graph" on seven vertices (labelling the seven clubs using the first letter of their names – A, B, C, D, E, F, G), where two vertices are joined by an edge if and only if some individual belongs to both clubs, and would therefore have a conflict if both clubs scheduled their meetings at the same time. For this problem, the conflict graph has seven vertices and twelve edges, and looks like this:



Now we colour the vertices, with each colour representing a time slot. For example, if we colour vertex A red, then we see that we cannot assign red to vertices B, C, D, E, F, since each of those five clubs has a conflict with vertex A. Thus, we must use a different colour.

What is the fewest number of colours we need to ensure that no edge is joined by two vertices of the same colour? Do you see how the answer to this question must be the same as the fewest number of time slots needed to schedule all meetings of the student clubs?

We show that only three colours are required. To do this, we let A be red, E be blue, and F be green. Then, B and C must be green since they are both adjacent to the red vertex A and the blue vertex E, which in turn forces D to be blue and G to be red. (We say that vertices are adjacent when they are connected by an edge.) Therefore, we have found a valid 3-colouring, with red circles, green squares, and blue stars:



In the above picture, we see that no edge connects two vertices of the same colour (for emphasis, we will use different shapes to distinguish the different colours). Thus, we are guaranteed a solution to the seven-club scheduling problem by simply assigning time slots to the three colours in any order such as: Red = 12:00-13:00, Blue = 13:00-14:00, and Green = 14:00-15:00. Indeed, we can quickly verify that this is the exact same solution as what was given earlier.

A natural question is whether two colours suffice. To see why this is impossible, note that AEF is a triangle, representing the three different clubs Michael is in. And so each of these three points must be assigned different colours; thus we require at least three colours, i.e., at least three time slots.

As an interesting side note, I worked with a Grade 12 student named Sam to optimally schedule final exams to days at his Vancouver high school, so that no student would be taking two exams on the same day. In this context, the academic clubs were just the courses to which each student belonged. Sam's conflict graph with 475 students and 29 courses required a total of 7 colours, and so the optimal exam schedule required 7 days. This was a significant improvement from the previous year's exam schedule, which was constructed by hand, and lasted 11 days.

Many mathematical problems can only be solved in routine and mundane ways. However, if students see "Teaching Problems" that can be solved in both a routine way and a surprising innovative way, then many unexpected benefits arise: a greater confidence in doing mathematics; a deeper appreciation for the beauty of mathematics; a development in one's creativity; as well as the opportunity to engage in applied problem-solving. These skills and opportunities would help our students in so many ways, and serve them well for their future.

We end with three questions for consideration. Communications, including solutions, concerning these questions are welcomed via email to `richard.hoshino@questu.ca`

Question #1

You are given four colours: A, B, C and D. Your task is to colour the 17 westernmost states of the continental United States, with each state being assigned one of these four colours. You choose the colour of each state.

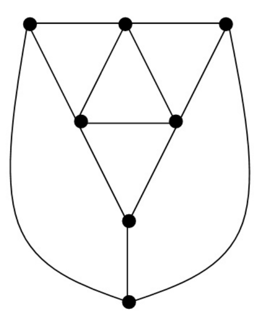
There's only one rule: if two states share a common border, they cannot receive the same colour. (For example, Utah and New Mexico are allowed to have the same colour, but each must have a colour different from Arizona's.)

You are given 1 penalty point for every state whose colour is D. In other words, your goal is to colour this map using only the colours A, B, and C.

Is it possible to colour this map without getting any penalty points? If not, why?

Question #2

a) We say that a graph is 3-colourable if each of its vertices can be coloured with one of three colours (red, blue, green) so that there is no edge connecting two vertices of the same colour. Carefully explain why the graph below, with 7 vertices and 12 edges, is not 3-colourable.



b) Alejandro, Beatrice, Chase, Donald, Elianne, Farley, Gina are seven students who all decide to take one summer course. There are exactly three course options for the summer, all offered at the same time. Below is the list of courses these students took in the most recent semester.

Alejandro	Algebra	Organic Chemistry	Physiology	Neuroscience
Beatrice	Algebra	Algorithm Design	English	Discrete Math
Chase	Biology	Algorithm Design	Art History	Statistics
Donald	Economics	Calculus	Physiology	Discrete Math
Elianne	Biology	Calculus	English	Sociology
Farley	Economics	Physics	Nutrition	Sociology
Gina	Programming	Physics	Art History	Neuroscience

The seven students are best of friends; however, they want to maximize their experience by having classes with as many different people as possible. Specifically, if two of them have shared a class at anytime, they refuse to take the same summer course. (For example, Alejandro and Beatrice won't sign up for the same class, as they took Algebra together.)

Is it possible for the seven students to select their summer courses this way, or must some pair of students be forced to take another summer course together?

Hint: how might this summer course scheduling problem relate to part a)?

Question #3

This powerful technique of graph colouring has numerous applications, including airplane flight scheduling, mobile radio frequency assignments, and colouring maps.

Did you know that every Sudoku puzzle is just a graph colouring problem in disguise?

Let G be the 81-vertex Sudoku graph, where each cell is a vertex, and we join two vertices if and only if two vertices share a common row, column, or 3 by 3 box.

Explain why solving a Sudoku puzzle is completely identical to colouring the vertices of the Sudoku graph G.

Further Readings

R. Hoshino (2018). *Supporting mathematical creativity through problem solving.* In Kajander, A., Holm, J., & Chernoff, E.J. (Eds.), Teaching and Learning Secondary School Mathematics: Canadians' Perspective in an International Context. New York: Springer.

P. Liljedahl (2015). *Building thinking classrooms: Conditions for problem solving.* Proceedings of the 2015 Annual Meeting of the Canadian Mathematics Education Study Group (CMESG), 131-138, Moncton, New Brunswick.

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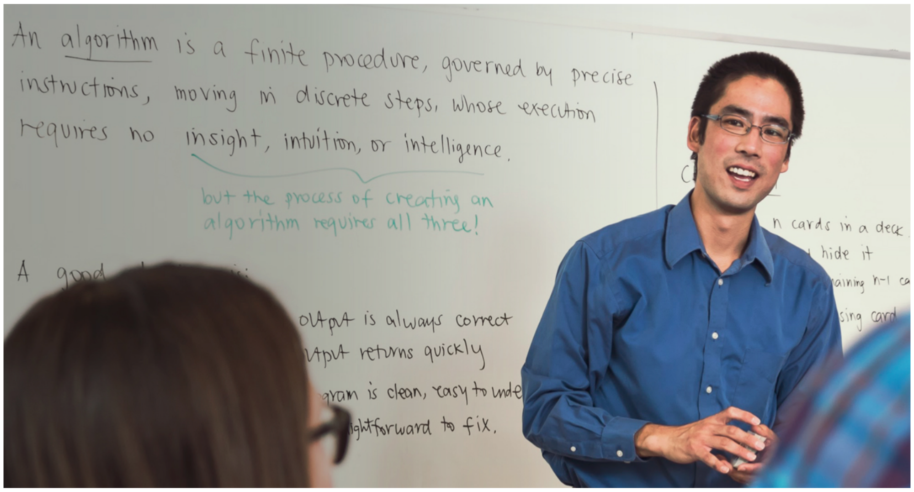
Note: Submissions for consideration in Teaching Problems are welcomed. Please feel free to send along a contribution concerning a valuable teaching example from your experience. It is also appreciated if you can include some related problems for consideration as has been done here. Our readers welcome opportunities to solve problems.



Richard Hoshino

Quest University Canada

OLYMPIAD CORNER

No. 376

The problems in this section appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **November 30, 2019**.

OC446. Given the numbers $2, 3, \ldots, 2017$ and the natural number $n \leq 2014$, Ivan and Peter play the following game: Ivan selects n numbers from the given ones, then Peter selects 2 numbers from the remaining numbers, then all the selected $n+2$ numbers are ranked in value:

$$a_1 < a_2 < \ldots < a_{n+2}.$$

If there exists i, $1 \leq i \leq n+1$ for which a_i divides a_{i+1}, then Peter wins, otherwise Ivan wins. Find all n for which Ivan has a winning strategy.

OC447. Let $m > 1$ be an integer and let $N = m^{2017} + 1$. Positive numbers $N, N-m, N-2m, \ldots, m+1, 1$ are written in a row. At each step, the leftmost number and all of its divisors (if any) are erased. This process continues until all the numbers are erased. What are the numbers deleted at the last step?

OC448. Let $x_1 \leq x_2 \leq \ldots \leq x_{2n-1}$ be real numbers whose arithmetic mean is equal to A. Prove that

$$2\sum_{i=1}^{2n-1}(x_i - A)^2 \geq \sum_{i=1}^{2n-1}(x_i - x_n)^2.$$

OC449. A sequence $(a_1, a_2, \ldots, a_k)$ consisting of pairwise distinct squares of an $n \times n$ chessboard is called a *cycle* if $k \geq 4$ and the squares a_i and a_{i+1} have a common side for all $i = 1, 2, \ldots, k$, where $a_{k+1} = a_1$. Subset X of this chessboard's squares is *mischievous* if each cycle on it contains at least one square in X. Determine all real numbers C with the following property: for each integer $n \geq 2$, on an $n \times n$ chessboard there exists a mischievous subset consisting of at most Cn^2 squares.

OC450. Find all pairs (x, y) of real numbers satisfying the system of equations

$$x \cdot \sqrt{1-y^2} = \frac{1}{4}\left(\sqrt{3}+1\right),$$
$$y \cdot \sqrt{1-x^2} = \frac{1}{4}\left(\sqrt{3}-1\right).$$

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 novembre 2019**.

La rédaction souhaite remercier Valérie Lapointe, Carignan, QC, d'avoir traduit les problèmes.

OC446. Soit les nombres $2, 3, \ldots, 2017$ et le nombre naturel $n \leq 2014$. Ivan et Peter jouent au jeu suivant : Ivan sélectionne n nombres parmi ceux donnés, puis Peter selectionne 2 nombres parmi ceux qui restent, puis tous les $n+2$ nombres sélectionnés sont ordonnés en ordre croissant :

$$a_1 < a_2 < \ldots < a_{n+2}.$$

S'il existe i, $1 \leq i \leq n+1$ pour lequel a_i divise a_{i+1}, Peter gagne, sinon Ivan gagne. Trouvez toutes les valeurs de n pour lesquelles Ivan a une stratégie gagnante.

OC447. Soit $m > 1$ un entier et soit $N = m^{2017} + 1$. Les nombres positifs $N, N-m, N-2m, \ldots, m+1, 1$ sont écrits sur une rangée. À chaque étape, le nombre le plus à gauche et tous ses diviseurs (s'il en possède) sont effacés. Le processus continue jusqu'à ce que tous les nombres soient effacés. Quels sont les nombres effacés à la dernière étape ?

OC448. Soit $x_1 \leq x_2 \leq \ldots \leq x_{2n-1}$ des nombres réels dont la moyenne arithmétique est égale à A. Prouvez que

$$2\sum_{i=1}^{2n-1}(x_i - A)^2 \geq \sum_{i=1}^{2n-1}(x_i - x_n)^2.$$

OC449. Une suite $(a_1, a_2, \ldots, a_k)$ constituée de paires distinctes d'un échiquier $n \times n$ est appelée un *cycle* si $k \geq 4$ et les carrés a_i et a_{i+1} ont un côté commun pour tout $i = 1, 2, \ldots, k$, où $a_{k+1} = a_1$. Le sous-ensemble X des carrés de cet échiquier est *mischievous* si chaque cycle contient au moins un carré de X. Trouvez tous les nombres réels C ayant la propriété suivante : pour tout entier $n \geq 2$ sur un échiquier $n \times n$, il existe un mischievous sous-ensemble consistant en au plus Cn^2 carrés.

OC450. Trouvez toutes les paires (x, y) de nombres réels qui satisfont au système d'équations

$$x \cdot \sqrt{1-y^2} = \frac{1}{4}\left(\sqrt{3}+1\right),$$
$$y \cdot \sqrt{1-x^2} = \frac{1}{4}\left(\sqrt{3}-1\right).$$

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 45(3), p. 122–123.

OC421. Mim has a deck of 52 cards, stacked in a pile with their backs facing up. Mim separates the small pile consisting of the seven cards on the top of the deck, turns it upside down, and places it at the bottom of the deck. All cards are again in one pile, but not all of them face down; the seven cards at the bottom do, in fact, face up. Mim repeats this move until all cards have their backs facing up again. In total, how many moves did Mim make?

Originally Problem 3 from the 2017 Italy Math Olympiad, Final Round.

We received no solutions.

OC422. A 2017×2017 table is filled with nonzero digits. Among the 4034 numbers whose decimal expansion is formed with the rows and columns of this table, read from left to right and from top to bottom, respectively, all but one are divisible by a prime number p, and the remaining number is not divisible by p. Find all possible values of p.

Originally problem 5 from the 2017 Moscow Math Olympiad, Grade 11, Second Day, Final Round.

We received 1 *submission. We present the solution by Oliver Geupel.*

We generalise the problem to an $M \times N$ table, where $M, N > 1$, and show that the possible values of p are exactly 2 and 5.

To see that $p \in \{2, 5\}$ has the desired property, put the top-right table entry equal to 1 and all other entries equal to p.

Next, suppose $p \notin \{2, 5\}$ and let an $M \times N$ table be filled with nonzero digits, where $d_{i,j}$ is the digit in row i and column j. Then, the numbers whose decimal expansions are formed with the rows and columns of the table are

$$r_i = \sum_{j=1}^{N} 10^{N-j} d_{i,j} \quad \text{and} \quad c_j = \sum_{i=1}^{M} 10^{M-i} d_{i,j},$$

respectively. Note that

$$\sum_{i=1}^{M} \sum_{j=1}^{N} 10^{M+N-i-j} d_{i,j} = \sum_{i=1}^{M} 10^{M-i} r_i = \sum_{j=1}^{N} 10^{N-j} c_j \tag{1}$$

and that the prime p is not divisible by 10.

If exactly one of the numbers in $\{r_1, \ldots, r_M\} \cup \{c_1, \ldots c_N\}$ were not divisible by p, then exactly one of the numbers $\sum_{i=1}^{M} 10^{M-i} r_i$ and $\sum_{j=1}^{N} 10^{N-j} c_j$ were not divisible by p. By (1), this is impossible. This proves that p fails to have the required property.

OC423. There are 100 gnomes with weight $1, 2, \ldots, 100$ kg gathered on the left bank of the river. They cannot swim, but they have one boat with capacity 100 kg. Because of the current, it is hard to row back, so each gnome has enough power only for one passage from right side to left as oarsman. Can all gnomes get to the right bank?

Originally problem 3 from the 2017 Russia Math Olympiad, Grade 9, Final Round.

We received 2 *submissions. We present the solution by Oliver Geupel.*

The answer is No. Let us generalize the problem to N gnomes $G_1, \ldots, G_N$ with mass $1, \ldots, N$ kilograms, respectively, and a boat with capacity N kilograms. We show that the passage is possible if and only if N is an odd number.

First assume N is odd, say, $N = 2n + 1$. A passage is:

- step k $(1 \le k \le n)$: transfer G_k and G_{2n+1-k} to right (R), then G_k to L
- step $n + 1$: transfer G_{2n+1} to R, then G_{2n} to L
- step $n + 1 + k$ $(1 \le k < n)$: transfer G_k, G_{2n+1-k} to R, then G_{2n-k} to L
- step $2n + 1$: transfer G_n and G_{n+1} to R.

Next, assume N is even. We prove by contradiction that there is no passage.

Suppose there exists a passage with t transfers to R. There are $t - 1$ transfers to L with total mass at least $1 + 2 + \cdots + (t - 1)$ kilograms. Hence, the total mass transferred to R is at least

$$(1 + 2 + \cdots + N) + (1 + 2 + \cdots + (t - 1))$$

kilograms and at most Nt kilograms by the given capacity. Thus,

$$N(N + 1) + (t - 1)t \le 2Nt,$$

which we rewrite in the form

$$(2N + 1 - 2t)^2 \le 1.$$

We obtain $t \in \{N, N+1\}$, and in all transfers to R the boat is full. If $t = N+1$ then every gnome paddles twice to R, while in the case $t = N$ the gnome G_N crosses the river only once. Since the boat is full in all transfers to R, G_{N-1} shares the boat with G_1 twice. Also G_{N-2} shares the boat with G_2 twice. Continuing this way, there remains no partner for $G_{N/2}$. This is the desired contradiction.

OC424. Let n be a nonzero natural number, let $a_1 < a_2 < \ldots < a_n$ be real numbers and let $b_1, b_2, \ldots, b_n$ be real numbers. Prove that:

(a) if all the numbers b_i are positive, then there exists a polynomial f with real coefficients and having no real roots such that $f(a_i) = b_i$ for $i = 1, 2 \ldots, n$;

(b) there exists a polynomial f of degree at least 1 having all real roots and such that $f(a_i) = b_i$ for $i = 1, 2 \ldots, n$.

Originally problem 2 from the 2017 Russia Math Olympiad, Grade 12, Final Round.

We received 2 *submissions. We present the solution by the Missouri State University Problem Solving Group.*

(a) Let

$$\begin{aligned} h_i(x) &= \prod_{j=1}^{i-1}(x-a_j)^2 \prod_{j=i+1}^{n}(x-a_j)^2, \\ g_i(x) &= \frac{h_i(x)}{h_i(a_i)}. \end{aligned}$$

Then

$$\begin{aligned} g_i(a_i) &= 1, \\ g_i(a_j) &= 0 \text{ for } i \neq j \\ g_i(x) &> 0 \text{ for } x \neq a_j. \end{aligned}$$

Take

$$f(x) = \sum_{i=1}^{n} b_i g_i(x).$$

Note that $f(a_i) = b_i, i = 1, \ldots, n$, as desired.

Since $b_i > 0, f(x) > 0$ for $x \neq a_i, i = 1, \ldots, n$ and $f(a_i) = b_i > 0$. Therefore $f(x) > 0$ for all x and hence f has no real roots.

(b) If $b_i = 0$ for all i, let

$$f(x) = \prod_{i=1}^{m}(x-a_i).$$

Otherwise, if necessary, insert additional data points so that the non-zero y-coordinates alternate in sign making sure that all the data points with y-coordinate 0 lie between two data points whose y-coordinates have opposite signs. Next, if necessary, insert additional data points with y-coordinate 0, so that the number of such points between two points with non-zero y-coordinates is even. Denote the

x-coordinates of the points with non-zero y-coordinate by $a_i^{'}, i = 1, \ldots, m$, with $a_1^{'} < \ldots < a_m^{'}$ and denote the number of data points with y-coordinate 0 whose x-coordinates lie between $a_i^{'}$ and $a_{i+1}^{'}$ by $2k_i$.

Let f be the interpolating polynomial through this augmented set of data points having minimal degree. There are $m + \sum_{i=1}^{m-1} 2k_i$ data points, so

$$\deg f \leq m - 1 + \sum_{i=1}^{m-1} 2k_i.$$

Now for any polynomial, the sum of the multiplicities of its zeros between a point where the polynomial takes a positive value and one where it takes a negative value must be odd. In our case, for each interval $[a_i^{'}, a_{i+1}^{'})$, the sum of the multiplicities of the zeros in that interval must be at least $2k_i + 1$. Therefore the sum of the multiplicities of the zeros of f must be at least

$$\sum_{i=1}^{m-1}(2k_i + 1) = m - 1 + \sum_{i=1}^{m-1} 2k_i.$$

Therefore

$$\deg f = m - 1 + \sum_{i=1}^{m-1} 2k_i$$

and all of its roots are real, as desired.

For example, suppose $a_1 = 0$, $a_2 = 2$, $a_3 = 4$, $a_4 = 6$ and $b_1 = 5$, $b_2 = 0$, $b_3 = 0$, $b_4 = 3$. We could insert the data point $(1, -2)$ and obtain $a_1^{'} = 0, a_2^{'} = 1, a_3^{'} = 6$ and $2k_1 = 0$, $2k_2 = 2$. On the other hand, if we inserted the data point $(3, -4)$, we would need to add two more zeros, say at $x = 1$ and $x = 5$ to obtain $a_1^{'} = 0$, $a_2^{'} = 3$, $a_3^{'} = 6$ and $2k_1 = 2$, $2k_2 = 2$.

OC425. Consider a triangle ABC with $\angle A < \angle C$. Point E is on the internal angle bisector of $\angle B$ such that $\angle EAB = \angle ACB$. Let D be a point on line BC such that $B \in CD$ and $BD = AB$. Prove that the midpoint M of the segment AC is on the line DE.

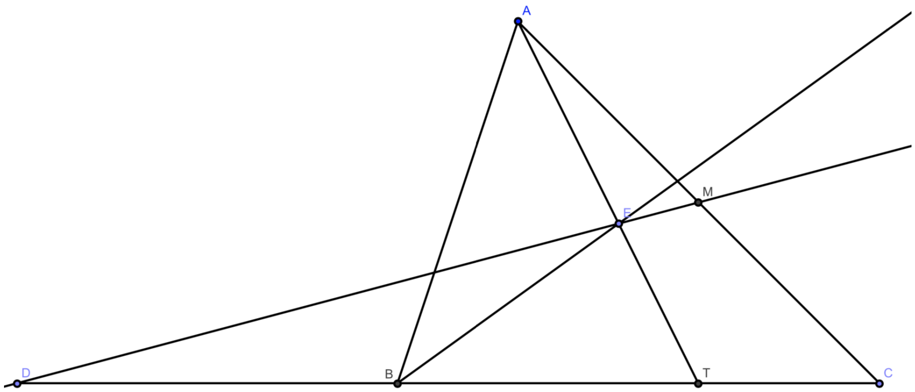
Originally problem 4 from the 2017 Romania Math Olympiad, Grade 7, District Round.

We received 7 *submissions. We present the solution by Sushanth Sathish Kumar.*

Define $T = \overline{BC} \cap \overline{AE}$. The key is to apply Menelaus's theorem to triangle TCA. We want to show that

$$\frac{AM}{MC} \cdot \frac{TE}{EA} \cdot \frac{CD}{DT} = -1,$$

where lengths are taken to be directed.



Clearly $AM/MC = 1$. By the angle bisector theorem, we have $TE/EA = BT/BA$. However, triangles TBA and ABC are similar since $\angle BAT = \angle BCA$ and $\angle B$ is shared. It follows that

$$TE/EA = BA/BC = c/a.$$

To compute the last ratio, note that

$$CD = CB + BD = a + c$$

and

$$DT = DB + BT = c + BT.$$

But by the similar triangles found above,

$$BT = BA^2/BC = c^2/a.$$

Hence,

$$\frac{CD}{DT} = -\frac{a+c}{c + c^2/a} = -\frac{a}{c}.$$

Taking the product of the three ratios in question, we get -1, as desired.

Up in the Air:
The Mathematics of Juggling

Aaron Berk and Ethan White

Captivating audiences for more than 4000 years, juggling has been practiced in countries such as Ancient Egypt, Greece, China and the Polynesian Islands (to name a few) [1]. In Medieval Europe, juggling was at best buffoonery (no help from the pejorative collective noun, "a neverthriving of jugglers"). At worst, it was considered a form of witchcraft [2]! Today, jugglers comprise an active community the world over, a disproportionate number of whom possess an appreciation for mathematics. This curious connection between jugglers and mathematicians is brought to bear by icons like Ronald Graham, who has been the president of both the American Mathematical Society and the International Jugglers Association.

Mathematics aids in the communication and creation of juggling patterns. Its tools help us to distill a system for describing "most" juggling patterns, and to determine what makes one pattern unique from another. Modular arithmetic will be a key tool for us in this investigation (for a review of modular arithmetic, see Donald Rideout's *Problem Solving Vignettes* in ***Crux*** 45(3)). In this investigation, we will study only *simple juggling patterns* — ones for which the following conditions hold:

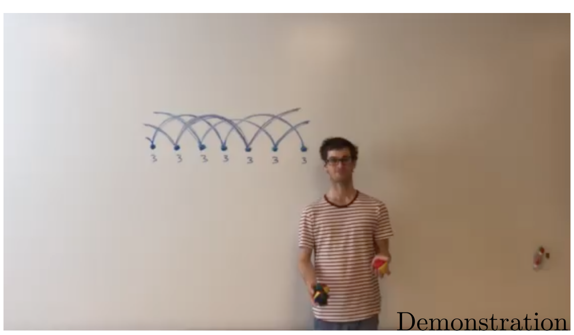
(A) Throws are made on evenly spaced 'beats', a ball is thrown as soon as it is caught;

(B) At most one ball is thrown or caught on each beat;

(C) The throws made by the juggler are a finite repeating pattern, repeating forever (forwards and backwards in time).

Most often, left and right hand will alternate throwing. The number of beats a ball is airborne is called the *height* of the throw; all heights are non-negative integers. If we chronologically list the heights of all the throws made on each beat in a simple juggling pattern, we obtain a bi-infinite sequence of the form

$$\ldots, t_{n-1}, t_n, t_1, t_2, \ldots, t_{n-1}, t_n, t_1, t_2, \ldots, t_{n-1}, t_n, t_1, t_2, \ldots \tag{1}$$



Demonstration of 333 pattern.

We call any bi-infinite sequence of the form in (1) a *simple pattern*, and say that it is *generated* by the tuple $(t_1, t_2, \ldots, t_n)$.

One of the first 3-ball juggling patterns a new juggler learns is the 3-ball cascade. Every throw is a throw of height 3, and so this pattern is generated by the singleton tuple (3). The paths that the 3 balls trace through time as they are thrown is visualized in the juggling diagram above. For example, the path of one of the balls, as it is thrown from left hand to right hand to left again, is shown in bold.

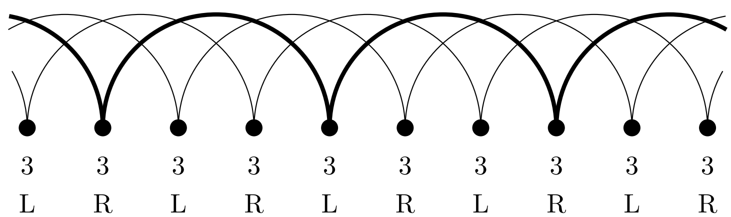


Figure 1: Juggling diagram of (3)

The tuple $(1, 5, 0)$ also generates a juggling pattern, whose sequence of throw heights is:

$$\ldots, 1, 5, 0, 1, 5, 0, 1, 5, 0, 1, 5, \ldots$$

We can verify that $(1, 5, 0)$ generates a simple juggling pattern by drawing its corresponding juggling diagram.

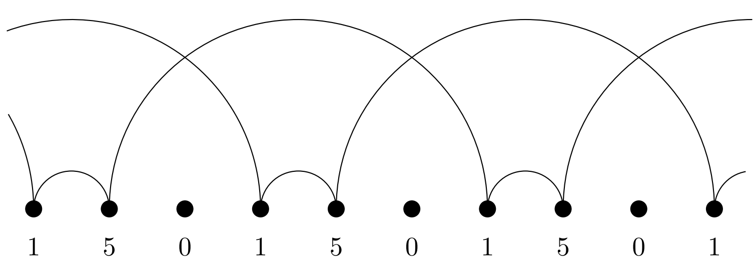


Figure 2: Juggling diagram of (1,5,0)

Demonstration of 150 pattern.

The dark dots represent beats, and the arcs represent ball throws (left hand, right hand, left hand, right hand, ...). A juggling diagram gives a simple way of checking that a tuple generates a simple juggling pattern; i.e. each dot must be incident to exactly 0 or 2 arcs in order for (B) to hold. The simple pattern generated by any tuple will satisfy (A) and (C), but not all simple patterns are simple juggling patterns. For example, $(3, 2, 1)$ generates a pattern with the following juggling diagram.

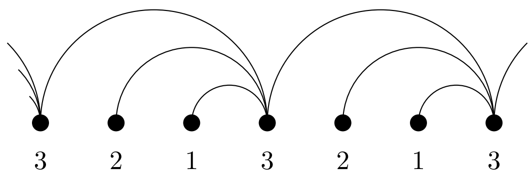


Figure 3: Juggling diagram of $(3, 2, 1)$

The diagram makes it clear that several balls land on a single beat, violating assumption (B). We say a tuple is *jugglable* if it generates a simple juggling pattern. In general, a suitably large juggling diagram can always be used to verify if a tuple is jugglable, but this can be unwieldy for longer tuples. Assuming that a throw of height t_1 is made on beat 1, the list of beats that a ball is caught in the simple juggling pattern generated by $(t_1, t_2, \ldots, t_n)$ is

$$\ldots, t_{n-1} - 1, t_n, t_1 + 1, t_2 + 2, \ldots, t_n + n, t_1 + n + 1, \ldots \tag{2}$$

The tuple $(t_1, t_2, \ldots, t_n)$ is jugglable if and only if every term in the bi-infinite sequence (2) is distinct.

Problem 1 *Prove that a tuple $t = (t_1, t_2, \ldots, t_n)$ is jugglable if and only if $t_1 + 1, t_2 + 2, \ldots, t_n + n$ are all distinct modulo n.*

Proof. Let T_n be the sequence in (1), where $T_1 = t_1$. Suppose two balls are caught simultaneously, then there exist distinct integers i, j such that $T_i + i = T_j + j$. By the division algorithm, find integers q_i, q_j such that $i = q_i n + r_i$ and $j = q_j n + r_j$ where $1 \le r_i, r_j \le n$. Notice that $T_i = t_{r_i}$ and $T_j = t_{r_j}$. We have

$$t_{r_i} + r_i \equiv t_{r_i} + i = T_i + i = T_j + j \equiv t_{r_j} + j \equiv t_{r_i} + r_j,$$

where all equivalences above are modulo n. On the other hand, if $t_i + i \equiv t_j + j \pmod{n}$ for $1 \le i \ne j \le n$, then there is an integer q such that $t_i + i = t_j + j + qn$. This implies $T_i + i = T_{j+qn} + j + qn$, and so the balls thrown on beats i and $j + qn$ land on the same beat.

□

Two tuples that differ by a circular permutation generate the same pattern; for example $(1, 5, 0)$ and $(5, 0, 1)$. On the other hand, there is an elementary operation we can apply to a simple juggling pattern that can generate a new simple juggling pattern, namely by exchanging the beats that two different balls land on. Precisely, we say $s = (s_1, s_2, \ldots, s_n)$ is a *siteswap* of $t = (t_1, t_2, \ldots, t_n)$ if $s = (t_2 + 1, t_1 - 1, t_3, \ldots, t_n)$. The ball thrown on the first beat in the pattern generated by s, lands on the same beat as the ball thrown on the second beat in the pattern generated by t. For example, $(4, 2, 3)$ is a siteswap of $(3, 3, 3)$. Applying a siteswap twice returns the original tuple.

Problem 2 *If s is a siteswap of t, show that s is jugglable if and only if t is jugglable.*

Proof. Notice that $\{t_1 + 1, t_2 + 2, \ldots, t_n + n\} = \{s_1 + 1, s_2 + 2, \ldots, s_n + n\}$. The result follows by Problem 1.

□

Constant tuples are always jugglable, and so too are any siteswap and circular permutation of constant tuples. Furthermore, any jugglable tuple can be obtained from a constant tuple via siteswaps and circular permutations.

Problem 3 *Show that for any juggling tuple t, successive siteswaps and circular permutations can be applied to t to obtain a constant tuple.*

Proof. If t is constant, then we are done. Suppose t is not constant, and by applying circular permutations, assume without loss of generality that t_1 is a largest entry in t and $t_1 > t_2$. Note $t_1 \neq t_2 + 1$, since otherwise by Problem 1, t would not be jugglable. The siteswapped tuple $(t_2 + 1, t_1 - 1, t_3, \ldots, t_n)$ now has strictly less elements with value t_1. We can repeat this procedure until we obtain a tuple t' with largest element strictly less than t_1. We can continue reducing the value of the largest entry until all entries are the same. This procedure must terminate since the largest entry will never be negative, since the sum of all entries remains constant throughout. □

The sum of the entries in a tuple does not change after a siteswap. Thus if a jugglable n-tuple $t = (t_1, t_2, \ldots, t_n)$ is obtained from a constant n-tuple $(a, \ldots, a)$, then $a = \frac{1}{n}\sum_{i=1}^{n} t_i$. The number of balls used in the pattern generated $(a, \ldots, a)$ is a. Since a siteswap does not change the number of balls used in the pattern it generates, a is also the number of balls used in the pattern generated by t. This gives a quick way of determining how many balls are required in a juggling pattern; the average of the terms!

We say a tuple is *scramblable* if any permutation of its elements results in a jugglable tuple.

Problem 4 *Show that an n-tuple $t = (t_1, t_2, \ldots, t_n)$ is scramblable if and only if there is a nonnegative constants c and q_i for $1 \leq i \leq n$ such that $t_i = c + q_i n$ for $1 \leq i \leq n$.*

Proof. By Problem 1, a tuple of the form described will be scramblable since constant tuples are scramblable. On the other hand, suppose t is scramblable. Subtract multiples of n from entries in t to obtain a new tuple t' such that all entries of t' are between 1 and n. By Problem 1, since t is scramblable, so too is t'. If t' is constant, we are done. Otherwise if $1 \leq a < b \leq n$ are two entries of t', then the n-tuple with b in the first entry, a in the $b - a + 1$ entry, and the remaining entries of t' in any positions, is a permutation of t' that is not jugglable, by Problem 1. This is a contradiction since t' is scramblable, and so t' must be constant, and t is of the form claimed.

$\square$

Problem 5 *Show that* $(1, 2, \ldots, n)$ *is never a juggling sequence if* n *is even.*

Problem 6 *Let* n *be any odd integer. Show that there exists a jugglable* n*-tuple with entries from the set* $\{1, 2, \ldots, n\}$ *each appearing once. Is there more than one such tuple?*

Juggling Resources

There is no end to excellent online resources for learning to juggle. A good tutorial for learning to juggle three balls is available at `https://www.youtube.com/watch?v=x2_j6kMg1co`. A library of juggling tricks, with companion animations is available at `http://www.libraryofjuggling.com/`. Finally, a fun juggling simulator that uses siteswap notation is available at `http://www.gunswap.co/`.

For more information on the mathematics of juggling, we refer the avid reader to [2]. This book provides a thorough covering of the mathematics of juggling, and makes reference to other valuable resources on the topic.

References

[1] Karl-Heinz Ziethen and Andrew Allen. *Juggling: the art and its artists.* Rausch & Luft, 1985.

[2] Burkard Polster and Ehrhard Behrends. *The mathematics of juggling.* The Mathematical Intelligencer, 28(2): 88–89, 2006.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **November 30, 2019**.

4471. *Proposed by Michael Diao.*

In $\triangle ABC$, let H be the orthocenter. Let M_A be the midpoint of AH and D be the foot from H onto BC, and define M_B, M_C, E and F similarly. Suppose P is a point in the plane distinct from the circumcenter of ΔABC, and suppose that P_A, P_B and P_C are points such that quadrilaterals $PABC$, P_AAEF, P_BDBF and P_CDEC are similar with vertices in that order. Show that M_AP_A, M_BP_B and M_CP_C concur on the circumcircle of $\triangle DEF$.

4472. *Proposed by Liam Keliher.*

Let n be a positive integer. Prove that n divides

$$\prod_{i=0}^{n-1}\left(2^n - 2^i\right).$$

4473. *Proposed by Nguyen Viet Hung.*

Let $\lfloor a \rfloor$ denote the greatest integer not exceeding a. For every positive integer n,

(a) find the last digit of $\lfloor (2+\sqrt{3})^n \rfloor$,

(b) find $\gcd(\lfloor (2+\sqrt{3})^{n+1} \rfloor + 1, \lfloor (2+\sqrt{3})^n \rfloor + 1)$, where $\gcd(a, b)$ denotes the greatest common divisor of a and b.

4474. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \frac{\pi}{2}$, $\angle ADB = \frac{\pi}{12}$, $\angle BDC = \frac{\pi}{6}$ and $\angle DBC = \frac{\pi}{8}$. Prove that BD passes through the midpoint of AC.

4475. *Proposed by Michel Bataille.*

Let a, b be real numbers with $a, b, a+b, a-b \neq 0$. Prove the inequality

$$\frac{\sinh(2(a+b))}{a+b} + \frac{\sinh(2(a-b))}{a-b} \geq 4\left(\frac{\sinh^2(a)}{a} + \frac{\sinh^2(b)}{b}\right).$$

4476. *Proposed by Leonard Giugiuc.*

Prove that for any real number a, b and c, we have

$$3\sqrt{6}(ab(a-b)+bc(b-c)+ca(c-a)) \leq ((a-b)^2+(a-c)^2+(b-c)^2)^{3/2}.$$

4477. *Proposed by Warut Suksompong.*

Given a positive integer n, let $a_1 \geq \cdots \geq a_n \geq 0$ and $b_1 \geq \cdots \geq b_n \geq 0$ be integers such that

1. $a_1 + \cdots + a_i \geq b_1 + \cdots + b_i$ for all $i = 1, \ldots, n-1$;
2. $a_1 + \cdots + a_n = b_1 + \cdots + b_n$.

Assume that there are n boxes, with box i containing a_i balls. In each move, Alice is allowed to take two boxes with an unequal number of balls, and move one ball from the box with more balls to the other box. Prove that Alice can perform a finite number of moves after which each box i contains b_i balls.

4478. *Proposed by Florin Stanescu.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a^2) - f(b^2) \leq (f(a)+b)(a - f(b))$$

for all $a, b \in \mathbb{R}$.

4479. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with $\angle A = 90°$ and let H be the foot of the altitude from A. Prove that

$$\frac{6}{(AB+AC)^2} - \frac{1}{2 \cdot AH^2} \leq \frac{1}{BC^2}.$$

4480. *Proposed by Leonard Giugiuc.*

Find all the solutions to the system

$$\begin{cases} a+b+c+d = 4, \\ a^2+b^2+c^2+d^2 = 6, \\ a^3+b^3+c^3+d^3 = \frac{94}{9}, \end{cases}$$

in $[0,2]^4$.

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 novembre 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4471. *Proposé par Michael Diao.*

Soit H l'orthocentre de $\triangle ABC$. Aussi, soit MA le mi point de AH et D le pied de l'altitude de H vers BC ; MB, MC, E et F sont définis de façon similaire. Supposons que P est un point dans le plan, autre que le centre du cercle circonscrit de $\triangle ABC$, puis que P_A, P_B et P_C sont des points tels que les quadrilatères $PABC$, P_AAEF, P_BDBF et P_CDEC sont similaires, avec sommets dans l'ordre donné. Démontrer que M_AP_A, M_BP_B et M_CP_C sont concourantes, sur le cercle circonscrit de $\triangle DEF$.

4472. *Proposé par Liam Keliher.*

Soit n un entier positif. Démontrer que n divise

$$\prod_{i=0}^{n-1} \left(2^n - 2^i\right).$$

4473. *Proposé par Nguyen Viet Hung.*

Le terme $\lfloor a \rfloor$ dénote le plus grand entier ne dépassant pas a. Pour tout entier positif n,

(a) Déterminer le dernier chiffre de $\lfloor (2+\sqrt{3})^n \rfloor$,

(b) Déterminer pgcd$(\lfloor (2+\sqrt{3})^{n+1} \rfloor + 1, \lfloor (2+\sqrt{3})^n \rfloor + 1)$, où pgcd$(a, b)$ dénote le plus grand commun diviseur de a et b.

4474. *Proposé par Kadir Altintas et Leonard Giugiuc.*

Soit $ABCD$ un quadrilatère convexe tel que $\angle ABC = \frac{\pi}{2}$, $\angle ADB = \frac{\pi}{12}$, $\angle BDC = \frac{\pi}{6}$ et $\angle DBC = \frac{\pi}{8}$ Démontrer que BD passe par le mi point de AC.

4475. *Proposé par Michel Bataille.*

Soient a, b des nombres réels tels que $a, b, a+b, a-b \neq 0$. Démontrer l'inégalité

$$\frac{\sinh(2(a+b))}{a+b} + \frac{\sinh(2(a-b))}{a-b} \geq 4\left(\frac{\sinh^2(a)}{a} + \frac{\sinh^2(b)}{b}\right).$$

4476. *Proposé par Leonard Giugiuc.*

Démontrer que, pour tous a, b et c nombres réels, l'inégalité suivante tient

$$3\sqrt{6}(ab(a-b) + bc(b-c) + ca(c-a)) \leq ((a-b)^2 + (a-c)^2 + (b-c)^2)^{3/2}.$$

4477. *Proposé par Warut Suksompong.*

Pour un entier positif n donné, soient $a_1 \geq \cdots \geq a_n \geq 0$ et $b_1 \geq \cdots \geq b_n \geq 0$ des entiers tels que

1. $a_1 + \cdots + a_i \geq b_1 + \cdots + b_i$ pour tous $i = 1, \ldots, n-1$;
2. $a_1 + \cdots + a_n = b_1 + \cdots + b_n$.

Supposer qu'Alice possède n boîtes, la boîte i contenant a_i billes. À chaque étape du jeu, Alice choisit deux boîtes avec nombres différents de billes et déplace une bille de la boîte la mieux nantie vers l'autre boîte. Démontrer qu'après un nombre fini d'étapes, Alice peut parvenir à une situation où toute boîte i contient b_i billes.

4478. *Proposé par Florin Stanescu.*

Déterminer toute fonction $f : \mathbb{R} \to \mathbb{R}$ telle que

$$f(a^2) - f(b^2) \leq (f(a) + b)(a - f(b))$$

pour tous $a, b \in \mathbb{R}$.

4479. *Proposé par George Apostolopoulos.*

Soit ABC un triangle tel que $\angle A = 90^\circ$ et soit H le pied de l'altitude émanant de A. Démontrer que

$$\frac{6}{(AB+AC)^2} - \frac{1}{2 \cdot AH^2} \leq \frac{1}{BC^2}.$$

4480. *Proposé par Leonard Giugiuc.*

Déterminer toute solution de

$$\begin{cases} a + b + c + d = 4, \\ a^2 + b^2 + c^2 + d^2 = 6, \\ a^3 + b^3 + c^3 + d^3 = \frac{94}{9}, \end{cases}$$

dans $[0, 2]^4$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2018: 45(3), p. 144–147.

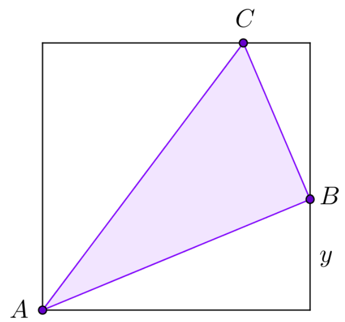
4421. *Proposed by Peter Y. Woo. (Correction.)*

Show that the area of the largest $30^\circ - 60^\circ - 90^\circ$ triangle that fits inside the unit square is greater than $1/3$.

We originally published a duplicate of problem 4372 under this number. We received 3 more correct submissions to 4372 by Andrea Fanchini, Oliver Geupel and Christóbal Sánchez-Rubio (done independently).

For this problem, we received 14 submissions, 12 of which were correct. We present the solution by Richard Hess.

Consider a $30^\circ - 60^\circ - 90^\circ$ triangle positioned in the unit square as shown below.



Let $B = (1, y)$. Then $AB = \sqrt{1+y^2}$ and $BC = \sqrt{(1+y^2)/3}$. From this we find that the y-coordinate of C is $1/\sqrt{3} + y$. The triangle will fit inside the square if $1/\sqrt{3} + y \le 1$. If the y-coordinate of C is equal to 1, then $y = 1 - 1/\sqrt{3}$ and the area of the triangle is

$$\frac{1}{2}BC \cdot AB = \frac{1+y^2}{\sqrt{12}} = \frac{7\sqrt{3}}{18} - \frac{1}{3} = 0.3402\cdots > \frac{1}{3}.$$

4422. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Let ABC be a scalene triangle with incenter I and nine-point center N. Find $\angle A$ given that A, N and I are collinear.

We present two of the 15 solutions we received, all of which were correct.

Solution 1, by Ricardo Barroso Campos, with numerous details supplied by the editor.

We shall see that $\angle A = 60^\circ$. First observe that for any point N on the bisector of $\angle BAC$ and for points C' between A and B and B' between A and C, if $NC' = NB'$ then either $AC' = AB'$ or $AC'NB'$ is a cyclic quadrangle. Here we are given that N is the center of the nine-point circle of ΔABC, and that circle contains the midpoints C' of side AB and B' of side AC; that is, $NB' = NC'$. We are also given that N is on the bisector of $\angle BAC$ (because it lies on the line AI) and, because the triangle is scalene, we have $AB' \neq AC'$. It follows that $AC'NB'$ is a cyclic quadrangle. The midpoint A' of BC is also on the nine-point circle. We also have that $\angle B'A'C' = \angle A$ (corresponding sides are parallel). Because the angle at the center equals twice the angle on the circumference, $\angle B'NC' = 2\angle B'A'C' = 2\angle A$, whence in the isosceles triangle $C'NB'$ we have

$$\angle C'B'N = 90^\circ - \angle A.$$

Finally in the circle $AC'NB'$

$$\angle A/2 = \angle C'AN = \angle C'B'N = 90^\circ - \angle A,$$

so that $\angle A = 60^\circ$, as claimed.

Solution 2, by Ivko Dimitrić.

This solution uses barycentric coordinates with respect to triangle ABC and presupposes knowledge of the homogeneous barycentric coordinates of the nine-point center, namely

$$N = (a\cos(B-C) : b\cos(C-A) : c\cos(A-B)),$$

where A, B, C are the angle measures at the corresponding vertices. [This follows from the homogeneous coordinates of $O = (\sin 2A : \sin 2B : \sin 2C)$, those of the centroid $G = (1:1:1)$, and the fact that $ON : NG = 3 : (-1)$; see Paul Yiu, *Introduction to Triangle Geometry*, p. 28. N is the point $X(5)$ in Clark Kimberling's *Encyclopedia of Triangle Centers*.]

The line through $A = (1:0:0)$ and $I = (a:b:c)$ has an equation $cy - bz = 0$. If A, I and N are collinear, the coordinates of N satisfy the above equation of the line AI, so that

$$c\,b\cos(C-A) = b\,c\cos(A-B),$$

from where we have $(A-B) = \pm(C-A)$. Finally, since the triangle is scalene it follows that

$$A - B = C - A \quad\Longrightarrow\quad B + C = 2A \quad\Longrightarrow\quad 180^\circ = A + B + C = 3A,$$

which gives $\angle A = 60^\circ$.

Editor's comments. This problem has made several appearances in ***Crux*** ; see, for example, the article "Recurring Crux Configurations 3: Triangles Whose Angles

Satisfy $2B = C + A$" [**35**:7 (November 2011) 449-453], properties 5 and 6 (pages 450-451). For three other proofs see problem 2855 [2004: 308-309] and problem 5 on the 2007 Indian Team Selection Test [2011: 370-371].

4423. *Proposed by Mihaela Berindeanu.*

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a twice differential function such that

$$f(x) + f''(x) = -x \cdot c^x \cdot f'(x)$$

for all real values of x and an arbitrary constant c. Find $\lim_{x\to 0} x \cdot f(x)$.

We received 2 submissions, both correct. We present the solution by Michel Bataille.

Being twice differentiable, f certainly is continuous on $\mathbb{R}$; in particular, we have $\lim_{x\to 0} f(x) = f(0)$. It follows that

$$\lim_{x\to 0} x \cdot f(x) = 0.$$

4424. *Proposed by Marius Drăgan and Neculai Stanciu.*

Let $k \in \mathbb{N}$ such that $9k + 9, 9k + 10$ and $9k + 13$ are not perfect squares. Prove that

$$[\sqrt{k+x} + \sqrt{k+x+1} + \sqrt{k+x+2}] = [\sqrt{9k+7}]$$

for all $x \in [0, 7/9]$, where $[a]$ denotes the integer part of number a.

We received 7 submissions, all correct. We present the solution by Oliver Geupel.

Let $f(x) = \sqrt{k+x} + \sqrt{k+x+1} + \sqrt{k+x+2}$ for $x \in [0, 7/9]$. By the AM-GM inequality, we have

$$\begin{aligned}\left(\sqrt{k} + \sqrt{k+1} + \sqrt{k+2}\right)^6 - (9k+7)^3 &> 3^6 k(k+1)(k+2) - (9k+7)^3 \\ &= 486k^2 + 135k - 343 \\ &> 0,\end{aligned}$$

so

$$f(0) = \sqrt{k} + \sqrt{k+1} + \sqrt{k+2} > \sqrt{9k+7} \tag{1}$$

Since the square root function is concave, we have by Jensen's inequality that

$$\begin{aligned}f\left(\frac{7}{9}\right) &= \sqrt{k+\frac{7}{9}} + \sqrt{k+\frac{16}{9}} + \sqrt{k+\frac{25}{9}} \\ &\leq \sqrt{\frac{(9k+7)+(9k+16)+(9k+25)}{3}} \\ &= \sqrt{9k+16}.\end{aligned}$$

Because $f(x)$ is an increasing function, it follows from (1) that

$$\sqrt{9k+7} < f(0) \leq f(x) \leq f\left(\frac{7}{9}\right) < \sqrt{9k+16}. \tag{2}$$

It is well known that the quadratic residues modulo 9 are 0, 1, 4, and 7. Thus, the numbers $9k+8$, $9k+11$, $9k+12$, $9k+14$, and $9k+15$ are not perfect squares. Moreover, by assumptions, $9k+9$, $9k+10$ and $9k+13$ are not perfect squares either. Hence, if there exists $m \in \mathbb{N}$ such that

$$[\sqrt{9k+7}] < m < f(x) < [\sqrt{9k+16}],$$

then by (2) we get

$$9k+7 < m^2 < (f(x))^2 < 9k+16,$$

a contradiction. Hence $[f(x)] = \left[\sqrt{9k+7}\right]$ for all $x \in [0, 7/9]$, completing the proof.

4425. *Proposed by Nguyen Viet Hung.*

Prove the following identities

(a) $\tan^3\theta + \tan^3(\theta - 60^\circ) + \tan^3(\theta + 60^\circ) = 27\tan^3 3\theta + 24\tan 3\theta$,

(b) $\dfrac{1}{1+\tan\theta} + \dfrac{1}{1+\tan(\theta - 60^\circ)} + \dfrac{1}{1+\tan(\theta+60^\circ)} = \dfrac{3\tan 3\theta}{\tan 3\theta - 1}$.

We received 13 submissions, all of which were correct, and we feature the solution by Aram Tangboonduangjit.

From the triple-angle identity,

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta},$$

we have

$$\tan^3\theta - (3\tan 3\theta)\tan^2\theta - 3\tan\theta + \tan 3\theta = 0.$$

This implies that $x = \tan\theta$ is a zero of the polynomial $P(x)$ defined by

$$P(x) = x^3 - (3\tan 3\theta)x^2 - 3x + \tan 3\theta,$$

where θ is a real number for which $\tan\theta$ and $\tan 3\theta$ are both defined. But since

$$\tan 3(\theta \pm 60^\circ) = \tan(3\theta \pm 180^\circ) = \tan 3\theta,$$

it follows that $\tan(\theta - 60^\circ)$ and $\tan(\theta + 60^\circ)$ are zeros of $P(x)$ as well. Because they come from translates of a strictly increasing function, the three numbers $\tan\theta, \tan(\theta \pm 60^\circ)$ are distinct; therefore, they compose all the zeros of the polynomial $P(x)$. We deduce that $P(x)$ factors as

$$P(x) = (x-p)(x-q)(x-r),$$

where $p = \tan\theta$, $q = \tan(\theta - 60^\circ)$, and $r = \tan(\theta + 60^\circ)$. Then, comparing the coefficients of the polynomial $P(x)$ yields

$$p + q + r = 3\tan 3\theta, \quad pq + qr + rp = -3, \quad \text{and} \quad pqr = -\tan 3\theta.$$

For part (a), we have

$$\begin{aligned} p^3 + q^3 + r^3 &= (p+q+r)\left((p+q+r)^2 - 3(pq+qr+rp)\right) + 3pqr \\ &= 3\tan 3\theta(9\tan^2 3\theta - 3(-3)) - 3\tan 3\theta \\ &= 27\tan^3\theta + 24\tan 3\theta, \end{aligned}$$

as desired.

For part (b), differentiating the polynomial $P(x)$ with respect to x on both sides, we obtain $P'(x) = 3x^2 - (6\tan 3\theta)x - 3$ and, from the factored form of $P(x)$,

$$\frac{P'(x)}{P(x)} = \frac{1}{x-p} + \frac{1}{x-q} + \frac{1}{x-r}.$$

Hence,

$$\begin{aligned} \frac{1}{1+p} + \frac{1}{1+q} + \frac{1}{1+r} &= -\frac{P'(-1)}{P(-1)} \\ &= -\frac{3 + 6\tan 3\theta - 3}{-1 - 3\tan 3\theta + 3 + \tan 3\theta} = \frac{3\tan 3\theta}{\tan 3\theta - 1}, \end{aligned}$$

as desired.

4426. *Proposed by Michel Bataille.*

Let distinct points A, B, C on a rectangular hyperbola $\mathcal{H}$ be such that $\angle BAC = 90^\circ$. A point M of $\mathcal{H}$, other than A, B, C, is called *good* if the triangles MAB and MAC have the same circumradius. Show that either infinitely many M are good or a unique M is good. Characterize the triangle ABC in the former case and find M and the common circumradius in the latter one.

The two submissions we received were both correct; we will combine them into the featured solution. More precisely, the solution from Walther Janous generalized the problem, but because it relied on computer calculations, we will modify the proposer's solution to incorporate Janous' generalization.

Choosing the asymptotes of $\mathcal{H}$ as the coordinate axes, we take the equation of $\mathcal{H}$ to be $xy = 1$. We begin with A, B, C, M four arbitrarily chosen points of $\mathcal{H}$, and let a, b, c, m denote, respectively, their abscissas. At the end we will investigate what happens when $\angle BAC$ is a right angle (as the proposer intended). From the law of Sines, we see that the triangles MAB and MAC have the same circumradius if and only if $\sin(\angle MBA) = \sin(\angle MCA)$; that is, if and only if $\cos^2(\angle MBA) = \cos^2(\angle MCA)$, which itself is equivalent to

$$\left(\frac{MB^2 + AB^2 - MA^2}{2MB \cdot AB}\right)^2 = \left(\frac{MC^2 + AC^2 - MA^2}{2MC \cdot AC}\right)^2 \tag{1}$$

By easy calculations, we find

$$\begin{aligned}
&MB^2 + AB^2 - MA^2\\
&= (m-b)^2 + \left(\frac{1}{m} - \frac{1}{b}\right)^2 + (b-a)^2 + \left(\frac{1}{b} - \frac{1}{a}\right)^2 - (m-a)^2 - \left(\frac{1}{m} - \frac{1}{a}\right)^2\\
&= 2(b-a)(b-m)\left(1 + \frac{1}{ab^2m}\right)\\
&= 2(b-a)(b-m)\left(\frac{ab^2m+1}{ab^2m}\right),
\end{aligned}$$

and

$$\begin{aligned}
MB^2 \cdot AB^2 &= \left((b-m)^2 + \left(\frac{1}{b} - \frac{1}{m}\right)^2\right)\left((b-a)^2 + \left(\frac{1}{b} - \frac{1}{a}\right)^2\right)\\
&= (b-a)^2(b-m)^2\left(1 + \frac{1}{b^2m^2}\right)\left(1 + \frac{1}{a^2b^2}\right)\\
&= (b-a)^2(b-m)^2\left(\frac{b^2m^2+1}{b^2m^2}\right)\left(\frac{a^2b^2+1}{a^2b^2}\right).
\end{aligned}$$

Replacing b by c, we obtain $MC^2 + AC^2 - MA^2$ and $MC^2 \cdot AC^2$, and then (1), written as

$$\left(\frac{MB^2 + AB^2 - MA^2}{MC^2 + AC^2 - MA^2}\right)^2 = \frac{MB^2 \cdot AB^2}{MC^2 \cdot AC^2},$$

becomes

$$\frac{(ab^2m+1)^2}{(ac^2m+1)^2} = \frac{(b^2m^2+1)(a^2b^2+1)}{(c^2m^2+1)(a^2c^2+1)},$$

which, after expanding and arranging, rewrites as

$$\begin{aligned}
0 &= a^2(b^2-c^2) - 2am(b^2-c^2) + m^2(b^2-c^2)\\
&\qquad -a^4b^2c^2m^2(b^2-c^2) + 2a^3b^2c^2m^3(b^2-c^2) - a^2b^2c^2m^4(b^2-c^2)\\
&= (b^2-c^2)\left((a^2 - 2am + m^2) - a^2b^2c^2m^2(a^2 - 2am + m^2)\right),
\end{aligned}$$

and finally,

$$(b-c)(b+c)(a-m)^2(abcm+1)(abcm-1) = 0. \tag{2}$$

Because we assume that A, B, C, M are distinct points, neither $b-c$ nor $a-m$ can be zero, which leaves us with two cases.

- **Case 1**. When $b+c=0$, equation (2) has infinitely many solutions m with corresponding good points M. The triangles ABC are given by a arbitrary points A and B on $\mathcal{H}$ and point C the reflection of B in the origin.

- **Case 2**. For $b+c \neq 0$, (2) reduces to $(abcm+1)(abcm-1) = 0$. It follows that there will generally be exactly two good points, namely

$$\left(\frac{1}{abc},\ abc\right) \quad \text{and} \quad \left(-\frac{1}{abc},\ -abc\right).$$

Of course, should any of a, b, or c equal $\pm\frac{1}{abc}$, there will be only one candidate for a good point. The computer tells us that when $m = \pm\frac{1}{abc}$ the common circumradius of triangles ABM and ACM is

$$\frac{\sqrt{(a^2b^2+1)(b^2c^2+1)(c^2a^2+1)}}{2abc}. \tag{3}$$

Solution to the original problem. We now assume that $\angle BAC = 90°$. Expressing that the dot product $\overrightarrow{AB}\cdot\overrightarrow{AC}$ vanishes, we deduce that a, b, c satisfy

$$a^2bc = -1. \tag{4}$$

After replacing abc by $-\frac{1}{a}$ and multiplying the equation by $-a^2$, equation (2) becomes

$$(b-c)(b+c)(m+a)(m-a)^3 = 0. \tag{5}$$

Thus, any $M \neq A, B, C$ on $\mathcal{H}$ is good if $b+c=0$; otherwise, only one point is good: the one with abscissa $-a$. As with an arbitrary triangle, the former case occurs if and only if B and C are symmetrical about the centre O of $\mathcal{H}$, that is, if and only if the hypotenuse is a diameter of $\mathcal{H}$ (that is, if and only if the legs of ΔABC are parallel to the axes of $\mathcal{H}$, as in figure 1). In the latter case, the only good point is the reflection of A in O. It is known that if a triangle is inscribed in a rectangular hyperbola, then the centre O is on the nine-point circle of the triangle (see for example C.V. Durell, *A Concise Geometrical Conics*, MacMillan, 1952, p. 72), hence the reflection in O of the orthocentre is on the circumcircle of the triangle. Here, the orthocentre of ABC being A, it follows that the only good point is on the circumcircle of ABC and therefore the common circumradius of ΔMAB and ΔMAC is the circumradius of ΔABC (see figure 2). Because the circumradius of a right triangle equals half the hypotenuse BC, equation (3) reduces to

$$\frac{|b-c|}{2bc}\sqrt{b^2c^2+1}$$

when $a^2bc = -1$ (and, therefore, $a^2b^2 = -\frac{b}{c}$, $a^2c^2 = -\frac{c}{b}$, and $-a^2 = \frac{1}{bc}$).

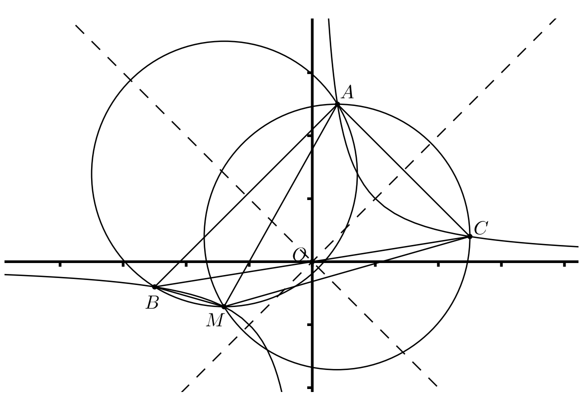


Figure 1

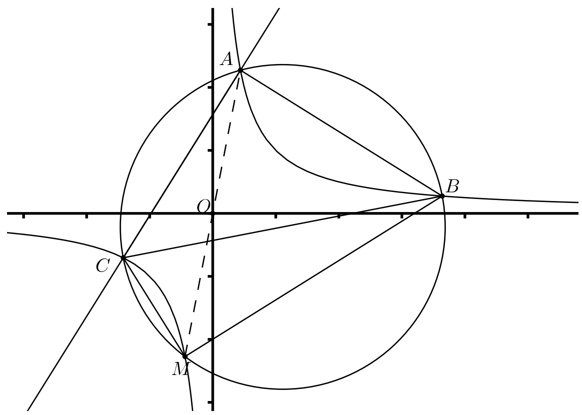


Figure 2

4427. *Proposed by Max A. Alekseyev.*

Prove that the equation

$$u^8 + v^9 + w^{14} + x^{15} + y^{16} = z^8$$

has infinitely many solutions in positive integers with $\gcd(u, v, w, x, y, z) = 1$.

There was only one solution, by the proposer. However, the problem remains open in case a different family of solutions can be found.

Let $z = s + t$ and $u = |s - t|$ for integers s and t. Then

$$z^8 - u^8 = 2^4 st^7 + 2^4 \cdot 7s^3t^5 + 2^4 \cdot 7s^5t^3 + 2^4 s^7 t.$$

We make the choice $s = 2^a$ and $t = 7^b$ for suitable choices of a and b to arrange that

$$\begin{aligned} v^9 &= 2^4 st^7 = 2^{4+a} \cdot 7^{7b}; \\ w^{14} &= 2^4 \cdot 7s^3t^5 = 2^{4+3a} \cdot 7^{1+5b}; \\ x^{15} &= 2^4 s^7 t = 2^{4+7a} \cdot 7^b; \\ y^{16} &= 2^4 \cdot 7s^5t^3 = 2^{4+5a} \cdot 7^{1+3b}. \end{aligned}$$

To obtain integer values of the variables, we require that a be congruent to 5 (mod 9), 8 (mod 14), 8 (mod 15), 12 (mod 16), and that b be congruent to 0 (mod 9), 11 (mod 14), 0 (mod 15) and 5 (mod 16). Therefore, modulo 5040,

$$a \equiv 428 \qquad \text{and} \qquad b \equiv 1845.$$

For arbitrary nonnegative integers m and n, we let

$$a = 5040m + 428 \quad \text{and} \quad b = 5040n + 1845.$$

This yields the family of solutions given by

$$\begin{aligned}
u &= \left|2^{5040m+428} - 7^{5040n+1845}\right| \\
v &= 2^{560m+48} \cdot 7^{3920n+1435} \\
w &= 2^{1080m+92} \cdot 7^{1800n+659} \\
x &= 2^{2352m+200} \cdot 7^{336n+123} \\
y &= 2^{1575m+134} \cdot 7^{945n+346} \\
z &= 2^{5040m+428} + 7^{5040n+1845}.
\end{aligned}$$

Any common divisor of u and z must divide $u \pm z$ as well, and so is a common divisor of a power of 2 and a power of 7. It follows that the greatest common divisor of u and z, and so of all the variables, is 1.

4428. *Proposed by Leonard Giugiuc.*

Let ABC be a triangle and let O be an arbitrary point in the same plane. Let A', B' and C' be the reflections of A, B and C in O. Prove that

$$\frac{AB' \cdot B'C}{AB \cdot BC} + \frac{BC' \cdot C'A}{BC \cdot CA} + \frac{CA' \cdot A'B}{CA \cdot AB} \geq 1.$$

There were 4 correct solution, all of which used complex numbers and the remaining one used vectors. One additional solution was submitted, but relied on an unproven inequality that seems as difficult as the proposed problem. We present two solutions.

Solution 1, by Michel Bataille and Sushanth Sathish Kumar (independently).

Locate the points O, A, B, C, A', B', C', respectively, in the complex plane at 0, a, b, c, $-a$, $-b$, $-c$. It is required to show that

$$\left|\frac{(a+b)(b+c)}{(a-b)(b-c)}\right| + \left|\frac{(b+c)(c+a)}{(b-c)(c-a)}\right| + \left|\frac{(c+a)(a+b)}{(c-a)(a-b)}\right| \geq 1.$$

Observe that

$$\begin{aligned}
&|(a+b)(b+c)(c-a)| + |(b+c)(c+a)(a-b)| + |(c+a)(a+b)(b-c)| \\
&\geq |(a+b)(b+c)(c-a) + (b+c)(c+a)(a-b) + (c+a)(a+b)(b-c)| \\
&= |(ab+bc+ca)[(c-a)+(a-b)+(b-c)] + b^2(c-a) + c^2(a-b) + a^2(b-c)| \\
&= |0 - (a-b)(b-c)(c-a)| \\
&= |(a-b)(b-c)(c-a)|.
\end{aligned}$$

Dividing by $|(a-b)(b-c)(c-a)|$ yields the desired inequality.

Solution 2, by the proposer.

Locate the points in the complex plane as in the previous solution. Let

$$u = \frac{b+c}{b-c}; \qquad v = \frac{c+a}{c-a}; \qquad w = \frac{a+b}{a-b}.$$

The homogeneous system

$$\begin{aligned} (1-u)y + (1+u)z &= 0 \\ (1+v)x \qquad\qquad + (1-v)z &= 0 \\ (1-w)x + (1+w)y \qquad\qquad &= 0 \end{aligned}$$

has a nontrivial solution $(x, y, z) = (a, b, c)$. Therefore, the determinant of its coefficients, $2(1 + (wu + uv + vw))$, is zero. Hence

$$\begin{aligned} \left|\frac{(a+b)(b+c)}{(a-b)(b-c)}\right| + \left|\frac{(b+c)(c+a)}{(b-c)(c-a)}\right| + \left|\frac{(c+a)(a+b)}{(c-a)(a-b)}\right| &= |wu| + |uv| + |vw| \\ &\geq |wu + uv + vw| \\ &= |-1| \\ &= 1. \end{aligned}$$

4429. *Proposed by Lorian Saceanu.*

Let a, b, c be positive real numbers. Prove that

$$\sqrt{\frac{a^2+b^2+c^2}{2(ab+bc+ca)}} \geq \frac{a+b+c}{\sqrt{a(b+c)} + \sqrt{b(a+c)} + \sqrt{c(a+b)}}.$$

We received 4 solutions, of which one was incorrect and another is incomplete. We present the proof by Vasile Cirtoaje, modified and enhanced by the editor.

The proposed inequality is equivalent to

$$\sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)} \geq (a+b+c)\sqrt{\frac{2(ab+bc+ca)}{a^2+b^2+c^2}}$$

or

$$\left(\sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)}\right)^2 \geq \frac{2(a+b+c)^2(ab+bc+ca)}{a^2+b^2+c^2} \qquad (1)$$

By AM-GM inequality, we have

$$\begin{aligned} &b + c \geq 2\sqrt{bc} \Longrightarrow \\ &ab + ac \geq 2a\sqrt{bc} \Longrightarrow \\ &(a+b)(a+c) \geq a^2 + 2a\sqrt{bc} + bc \Longrightarrow \\ &\sqrt{(a+b)(a+c)} \geq a + \sqrt{bc}. \end{aligned}$$

Similarly, $\sqrt{(b+c)(b+a)} \geq b + \sqrt{ca}$ and $\sqrt{(c+a)(c+b)} \geq c + \sqrt{ab}$. Hence,

$$\begin{aligned}
&\left(\sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)}\right)^2 \\
&= 2(ab+bc+ca) + 2\sum_{\text{cyc}} \sqrt{bc(a+b)(a+c)} \\
&\geq 2(ab+bc+ca) + 2\sum_{\text{cyc}} \sqrt{bc}(a+\sqrt{bc}) \\
&= 4(ab+bc+ca) + 2\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c}). \qquad (2)
\end{aligned}$$

From (1) and (2), we see that it suffices to show that

$$2(ab+bc+ca) + \sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c}) \geq \frac{(a+b+c)^2(ab+bc+ca)}{a^2+b^2+c^2},$$

which is equivalent to

$$\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})(a^2+b^2+c^2) \geq (ab+bc+ca)(2(ab+bc+ca)-a^2-b^2-c^2). \quad (3)$$

Since

$$\begin{aligned}
&(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})(-\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c}) \\
&\quad = ((\sqrt{a}+\sqrt{b})^2 - c)(c - (\sqrt{a}-\sqrt{b})^2) \\
&\quad = (\sqrt{a}+\sqrt{b})^2 c + (\sqrt{a}-\sqrt{b})^2 - c^2 - (a-b)^2 \\
&\quad = 2(a+b)c - c^2 - (a-b)^2 \\
&\quad = 2(ab+bc+ca) - a^2 - b^2 - c^2,
\end{aligned}$$

we see from (3) that it now suffices to prove that

$$\sqrt{abc}(a^2+b^2+c^2) \geq (ab+bc+ca)(\sqrt{a}+\sqrt{b}-\sqrt{c})(-\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c}),$$

which follows from the trivial fact that

$$a^2+b^2+c^2 \geq ab+bc+ca$$

and the known inequality that

$$\sqrt{abc} \geq (\sqrt{a}+\sqrt{b}-\sqrt{c})(-\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c}). \qquad (4)$$

Finally, equality holds if $a = b = c$.

Editor's Comment. The proposer in private communication pointed out that (4) is known as Schur's inequality of first degree and could be proved as follows. Assume without loss of generality that $a \leq b \leq c$ and consider a, b and c as the sides of a triangle. Then using the transformation

$$x = \frac{1}{2}(b+c-a), \quad y = \frac{1}{2}(c+a-b), \quad z = \frac{1}{2}(a+b-c),$$

and applying the inequality

$$(x+y)(y+z)(z+x) \geq 8xyz,$$

(which is an immediate consequence of the AM-GM inequality), we obtain

$$abc \geq (a+b-c)(b+c-a)(c+a-b),$$

so (4) follows.

4430. *Proposed by Leonard Giugiuc.*

Let $s \geq \dfrac{28}{3}$ be a fixed real number. Consider the real numbers a, b, c and d such that

$$a+b+c+d=4 \quad \text{and} \quad a^2+b^2+c^2+d^2=s.$$

Find the maximum value of the product $abcd$.

We received 7 solutions, 3 of which were correct. We present the solution by Walther Janous, modified by the editor.

We shall use Lagrange multipliers to determine the desired maximum.

At the boundary of the set

$$B = \{(a,b,c,d) : a+b+c+d=4 \text{ and } a^2+b^2+c^2+d^2=s\},$$

at least one of the variables a, b, c, d is zero, whence $abcd = 0$.

We now set

$$F = abcd - \lambda(a+b+c+d-4) - \mu\left(a^2+b^2+c^2+d^2-s\right),$$

From $\dfrac{d}{da}F = 0$, it follows that

$$-2a\mu + bcd - \lambda = 0.$$

Setting

$$f(t) = 2\mu t^2 + \lambda t,$$

we have

$$f(a) = abcd.$$

Similarly, we get $f(b) = f(c) = f(d) = abcd$. Since $f(t)$ is quadratic in t, we infer that among the four numbers a, b, c, d, there are at most four values. We consider three cases.

Case 1: $a=b=c=d$. Here, $a=b=c=d=1$, implying that

$$a^2+b^2+c^2+d^2 = 4 < \frac{28}{3},$$

a contradiction.

Case 2: $a = b = c = A$ and $d = B$, with $A \neq B$. Solving the system

$$\begin{cases} 3A + B & = 4 \\ 3A^2 + B^2 & = s \end{cases}$$

gives

$$\begin{cases} A & = \dfrac{\sqrt{3}\left(2\sqrt{3} - \sqrt{s-4}\right)}{6} \\ B & = \dfrac{2 + \sqrt{3} \cdot \sqrt{s-4}}{2} \end{cases}$$

or

$$\begin{cases} A & = \dfrac{\sqrt{3}\left(2\sqrt{3} + \sqrt{s-4}\right)}{6} \\ B & = \dfrac{2 - \sqrt{3} \cdot \sqrt{s-4}}{2} \end{cases}.$$

We obtain

$$abcd = A^3B = \frac{-16\sqrt{3}(s-4)^{3/2} - 3s^2 - 48s + 384}{144}.$$

Case 3: $a = b = A$ and $c = d = B$, with $A \neq B$. Solving the system

$$\begin{cases} 2A + 2B & = 4 \\ 2A^2 + 2B^2 & = s \end{cases}$$

gives

$$\begin{cases} A & = \dfrac{2 - \sqrt{s-4}}{2} \\ B & = \dfrac{2 + \sqrt{s-4}}{2} \end{cases}$$

or

$$\begin{cases} A & = \dfrac{2 + \sqrt{s-4}}{2} \\ B & = \dfrac{2 - \sqrt{s-4}}{2} \end{cases}.$$

This gives

$$abcd = A^2B^2 = \left(\frac{s-8}{4}\right)^2.$$

It remains to determine which of the two obtained maxima is the greater one. Straightforward calculation shows that the maximum in Case 3 is greater than or equal to the maximum in Case 2 if and only if $s \geq \dfrac{28}{3}$. Hence, the maximum for $s \geq \dfrac{28}{3}$ is $\left(\dfrac{s-8}{4}\right)^2$.