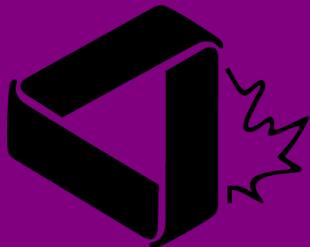


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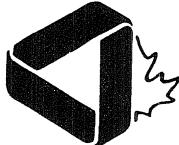
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THE OLYMPIAD CORNER

No. 130

R.E. WOODROW

*All communications about this column should be sent to Professor R.E. Woodrow,
Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta,
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This month I have selected contests from two countries whose contests have not before appeared in the Corner. I would like to thank Professor Florentin Smarandache for sending me a version in French of the first of these, from Turkey. My guess is that the exam is from 1988, although Prof. Smarandache gives the following reference [Bilim ve Teknik, Ankara, Turkey, No. 251, pp. 38–9, October 1989 and No. 252, pp. 51–2, November 1989].

TURKISH MATHEMATICAL OLYMPIAD

1. Two concentric circles are given in the plane with radii r and R ($r < R$). Let P be a fixed point on the smaller circle, and let B be a point which varies on the larger circle. The line BP cuts the large circle again at C . The line ℓ passes through P and is perpendicular to BP , and cuts the small circle again at A . (In the case where ℓ is tangent to the circle at P , set $A = P$.) Consider the expression $BC^2 + CA^2 + AB^2$. Give the set of values of the expression as B varies. Find the locus of midpoints of AB .

2. For n a positive integer, $A_1, A_2, \dots, A_{2n+1}$ are subsets of a given set B . Suppose that the following conditions hold:

- (a) each set A_i has $2n$ elements;
- (b) for $1 \leq i < j \leq 2n + 1$, $A_i < A_j$;
- (c) each element of B belongs to at least two of the A_i .

We wish to label each element of B with 0 or 1. For what values of n can we label so that each A_i contains exactly n elements labelled 0?

[*Editor's query.* Part (b) doesn't make much sense or seem to enter into the problem. How should the problem read?]

3. Let f be a function from the natural numbers to itself. For each positive integer n we have the following:

$$\begin{aligned} f(1) &= 1, \quad f(3) = 3, \quad f(2n) = n, \\ f(4n+1) &= 2f(2n+1) - f(n), \quad f(4n+3) = 3f(2n+1) - 2f(n). \end{aligned}$$

Find the number of positive integers up to 1988 which satisfy $f(n) = n$.

4. Consider

$$S = \left\{ x : \sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4} \right\}.$$

Show that S is a union of several intervals the sum of whose lengths is 1988.

5. Let ABC be a right triangle, with $\angle A = 90^\circ$. Let D be a point on BC . Let the line ℓ , passing through the centers of the inscribed circles for ABD and ACD , cut AB

and AC in K and L , respectively. Show $[ABC] \geq 2[AKL]$, where $[XYZ]$ is the area of triangle XYZ .

6. Let a and b be distinct positive integers such that the quotient on division of $a^2 + b^2$ by $ab + 1$ is an integer. Show that $(a^2 + b^2)/(ab + 1)$ is the square of a positive integer.

[*Editor's note.* This was problem 6 of the 29th I.M.O. [1988: 197] for which we gave a solution last month [1991: 262].]

*

The second set of problems are from the 1991 Japanese selection test. They were translated by Hidetosi Fukagawa and Leung Yu Kiang, and it is Willie Yang of Singapore whom we thank for forwarding them to us.

1991 JAPANESE I.M.O.
 Second Selection Test — 15 February 1991
 Time: 3 hours

1. Let P , Q and R be three points on the sides BC , CA and AB of a triangle ABC respectively, such that

$$BP : PC = CQ : QA = AR : RB = t : 1 - t$$

for some real number t . Show that the three line segments AP , BQ and CR will form a triangle, Δ say. Find the ratio of the area of triangle ABC to the area of Δ in terms of t .

2. Let \mathbb{N} be the set of positive integers and let p , q be mappings from \mathbb{N} to \mathbb{N} given by:

$$p(1) = 2, p(2) = 3, p(3) = 4, p(4) = 1, p(n) = n \text{ for } n \geq 5,$$

$$q(1) = 3, q(2) = 4, q(3) = 2, q(4) = 1, q(n) = n \text{ for } n \geq 5.$$

(i) Find the mapping f from \mathbb{N} to \mathbb{N} such that $f(f(n)) = p(n) + 2$, $n \geq 1$.

(ii) Show that there is no mapping g from \mathbb{N} to \mathbb{N} such that $g(g(n)) = q(n) + 2$, $n \geq 1$.

3. Suppose A is a positive 16-digit integer. Show that we can find some consecutive digits of A such that the product of these digits is a perfect square.

4. Consider the 10×14 matrix $(a_{i,j})$, where $a_{i,j} = 0$ or 1 for $1 \leq i \leq 10$, $1 \leq j \leq 14$, and such that each column contains an odd number of ones and each row also contains an odd number of ones. Show that among the numbers $a_{i,j}$ such that $i + j$ is even, there is precisely an even number of ones.

5. Let S be a set of n distinct points in a plane where $n \geq 2$. Show that there are two distinct points P_i and P_j of S such that the circle with diameter $P_i P_j$ contains at least $[n/3]$ distinct points of S , where $[m]$ is the integer part of m .

* * *

Now we turn to solutions from the readers, continuing with solutions to problems of the XIV “*ALL UNION Mathematical Olympiad*” (U.S.S.R.) [1990: 33–34, 70–72]. Solutions to the first five appeared last month. The acronym JACL is for the “team” consisting of two Edmonton students, Jason A. Colwell of Old Scona School and Calvin Li of Archbishop MacDonald School, and Andy Liu of The University of Alberta.

6. Given point E on diameter AC of a circle, construct chord BD passing through E and such that the quadrilateral $ABCD$ has the largest possible area.

Solutions by JACL; by Murray S. Klamkin, University of Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. The solution we give is that of JACL.

For a polygon P , let $[P]$ denote its area. Let O be the centre of the circle. If $E = O$ then obviously $BD \perp AC$ gives the answer, so suppose $E \neq O$. Then

$$\frac{[ACD]}{[ODE]} = \frac{AC}{OE} = \frac{[ABC]}{[OBE]}.$$

Hence we have

$$\frac{[ABCD]}{[OBD]} = \frac{AC}{OE} = \text{constant},$$

so that $[ABCD]$ is maximum iff $[OBD]$ is. Now OBD is a triangle with two sides equal to r , the radius of the circle. Hence $[OBD]$ is maximum iff $\sin BOD$ is. Let PQ be the chord through E perpendicular to AC . Now $PE \cdot EQ = BE \cdot ED$. By the Arithmetic-Mean Geometric-Mean inequality,

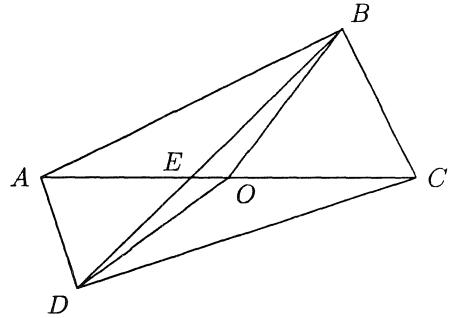
$$BD = BE + ED \geq 2\sqrt{BE \cdot ED} = 2\sqrt{PE \cdot EQ} = PQ.$$

If $BD \neq PQ$ then $BE \neq ED$, so $BD > PQ$ and $\angle BOD > \angle POQ$. If $\angle POQ \geq \pi/2$ then $\sin POQ > \sin BOD$ and the maximum area is generated by the chord PQ . If $\angle POQ < \pi/2$, we want BD so that $\angle BOD = \pi/2$. Draw a circle with centre O and radius $r/\sqrt{2}$. A tangent from E to this circle yields the desired chord of the original circle.

7. Several points are located along the shore of a circular lake. Some of these are linked by ferry service. Points A and B are linked by a ferry if and only if the next two adjacent points A' and B' (proceeding clockwise around the lake) are not linked. Show that one can go by ferry from any point to any other point, changing boats not more than twice.

Solution by JACL.

First suppose A and B are separated by at least two points. Let A'' precede A and A' follow A in clockwise order. Also let B'' precede B and B' follow B . Suppose A and B are not linked. By the hypothesis A' and B' are linked. Also A'' and B'' are linked. Now if A'' and A are not linked, then A and A' are. By symmetry we may assume A and A' are linked. If B and B' are linked, we can get from A via A' and B' to B . If B and B' are



not linked, then B'' and B must be. If B'' is linked to A , we can go from A via B'' to B . If B'' is not linked to A , then B is linked to A' and we can get from A via A' to B .

Suppose next that we have $CADBE$ as five consecutive points. If A and B are not linked C and D must be, as must D and E . If C is linked to A , A is not linked to D and D is linked to B . We can go from A via C and D to B . If C is not linked to A , A is linked to D and B to E , and we can go from A via D and E to B . Note that the argument remains the same if C and E coincide.

Finally suppose we have $CABD$ as four consecutive points. If A and B are not linked, C and A must be linked, as must B and D . If C and B are linked, we can go from A via C to B . If C and B are not linked, A and D must be, and we can go from A via D to B . Note that we cannot have here that C and D coincide.

8. A number is written (in base 10 notation) using six distinct non-zero digits, and it is divisible by 37. Show that by permuting the digits one can obtain at least 23 different new numbers, each divisible by 37.

Solution by JACL, and by Stewart Metchette, Culver City, California.

Suppose $x = 10^5a + 10^4b + 10^3c + 10^2d + 10e + f$ is divisible by 37. Let $y = 10^5d + 10^4b + 10^3c + 10^2a + 10e + f$ be obtained from x by switching digits a and d . Then $x - y = 10^2(10^3 - 1)(a - d)$. Since 37 divides 999, 37 divides $x - y$, and since it divides x , it must divide y . Similarly we get other multiples of 37 by switching the digits b and e or the digits c and f . This yields 4 desired permutations (including the identity). To obtain 24, we now prove that a cyclic permutation of the digits does not affect divisibility by 37. Suppose $z = 10^4b + 10^3c + 10^2d + 10e + f$ so that $x = 10^5a + z$. The number obtained by moving the first digit to the end is $y = 10^5b + 10^4c + 10^3d + 10^2e + 10f + a = 10z + a$. Now $10x - y = (10^3 + 1)(10^3 - 1)a$ is divisible by 37. Since 37 divides $10x$, it divides y . It is easy to see that the 24 permutations are distinct. In passing, we note that 37 *does* have a multiple consisting of 6 different digits, 456728 being an example.

9. Solve simultaneously:

$$\sin x + 2 \sin(x + y + z) = 0,$$

$$\sin y + 3 \sin(x + y + z) = 0,$$

$$\sin z + 4 \sin(x + y + z) = 0.$$

Solution by JACL.

The equations may be rewritten as

$$\frac{\sin x}{2} = \frac{\sin y}{3} = \frac{\sin z}{4} = \sin x \sin y \sin z - \sin x \cos y \cos z - \cos x \sin y \cos z - \cos x \cos y \sin z,$$

which by putting $\sin x = \frac{2}{3} \sin y$ and $\sin z = \frac{4}{3} \sin y$ implies

$$\frac{\sin y}{3} = \frac{8}{9} \sin^3 y - \frac{2}{3} \sin y \cos y \cos z - \sin y \cos z \cos x - \frac{4}{3} \sin y \cos x \cos y.$$

Now, if $\sin y = 0$, then $\sin x = 0 = \sin z$ and each of x, y and z is an integral multiple of π . Suppose $\sin y \neq 0$. Then the above equation can be rewritten in the form

$$6 \cos y \cos z + 12 \cos x \cos y = 5 - 9 \cos z \cos x - 8 \cos^2 y.$$

Squaring both sides and simplifying, using $4 \cos^2 y = 9 \cos^2 x - 5$ and $16 \cos^2 y = 9 \cos^2 z + 7$, we have

$$8 \cos^2 y + 1 = -9 \cos x \cos z.$$

Squaring again and simplifying, we have $\cos^2 y = 1$. Hence $\cos y = \pm 1$ and $\sin y = 0$ after all. It follows that each of x, y and z is an integral multiple of π .

10. A set of 1980 vectors are given in the plane, not all of which are collinear. It is known that the sum of any 1979 of these vectors is collinear with the one vector not included in the sum. Prove that the sum of all 1980 given vectors is equal to the vector $\mathbf{0}$.

Solution by JACL; and by Murray S. Klamkin, University of Alberta.

We can just as well have n vectors in space with the above properties. Let \mathbf{S} denote the sum of the n vectors $\mathbf{V}_1, \dots, \mathbf{V}_n$. Then by hypothesis $\mathbf{S} - \mathbf{V}_i = \lambda_i \mathbf{V}_i$ for each i , where the λ_i are real numbers. Hence $\mathbf{S} = (1 + \lambda_1)\mathbf{V}_1 = (1 + \lambda_2)\mathbf{V}_2 = \dots = (1 + \lambda_n)\mathbf{V}_n$. Since the \mathbf{V}_i 's are not all collinear, at least one $1 + \lambda_i = 0$. Hence $\mathbf{S} = \mathbf{0}$.

11. Let us denote the sum of the digits of the natural number n by $s(n)$.

(a) Does there exist a natural number n such that $n + s(n) = 1980$?

(b) Show that at least one of any two consecutive natural numbers can be expressed as $n + s(n)$ for some third natural number n .

Solution by Curtis Cooper, Central Missouri State University, Warrensburg.

(a) We want to find a natural number n such that $n + s(n) = 1980$. Since $s(n) \geq 1$, $n \leq 1979$. Now since $n \leq 1979 \leq 1999$, $s(n) \leq 28$ and so $n \geq 1952$. Searching n from 1952 to 1979 we find that $1962 + s(1962) = 1980$.

(b) Take two consecutive natural numbers and let m be the larger of the two. Let $k \geq 0$ be the largest integer such that $10^k + 1 \leq m$. Let $m_k = m$, and let $d_k \geq 1$ be the largest integer such that

$$d_k(10^k + 1) \leq m_k.$$

Let $m_{k-1} = m_k - d_k(10^k + 1)$ and let $d_{k-1} \geq 0$ be the largest integer such that

$$d_{k-1}(10^{k-1} + 1) \leq m_{k-1}.$$

Continuing this process for $i = k, \dots, 1$ we let $m_{i-1} = m_i - d_i(10^i + 1)$ and let $d_{i-1} \geq 0$ be the largest integer such that

$$d_{i-1}(10^{i-1} + 1) \leq m_{i-1}.$$

Finally let $m_{-1} = m_0 - 2d_0$. By our process, d_k is a non-zero decimal digit, and d_i is a decimal digit for $i = 0, \dots, k-1$. In addition m_{-1} is 0 or 1. Now let $n = \sum_{i=0}^k d_i 10^i$. It

follows that

$$\begin{aligned} n + s(n) &= \sum_{i=0}^k d_i(10^i + 1) = \sum_{i=0}^k (m_i - m_{i-1}) = m_k - m_{-1} \\ &= \begin{cases} m & \text{if } m_{-1} = 0, \\ m - 1 & \text{if } m_{-1} = 1. \end{cases} \end{aligned}$$

This is what we set out to prove.

12. Some of the squares on an infinite piece of (Cartesian) graph paper are coloured red, and the remainder are coloured blue. This is done in such a way that any 2×3 rectangle made up of six squares contains exactly two red squares. How many red squares may be included in a 9×11 rectangle made up of 99 squares?

Solutions by Duane M. Broline, Eastern Illinois University; by JACL; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

[Editor's note. The solutions are different, but interesting. First we give the solution of JACL.]

B	B	B
R	R	B
B	B	B

B	B	B
R	B	R
B	B	B

Suppose we have two red squares adjacent to each other. Without loss we may suppose they are in a row, and we consider the 3×3 rectangle in which they are the first two entries in the second row, as in the first diagram. Then the remaining entries must be B by considering the upper and lower 2×3 matrices. But now the right 3×2 matrix contains only one red square, a contradiction. Similarly we cannot have two red squares with exactly one blue square in between, as shown in the second diagram. Finally, if we have three blue squares in a row, consider any three squares that complete a 2×3 rectangle. We must have two red squares, which are adjacent or separated by 1. Neither is possible. It follows that exactly one of every three squares in a row is red, so that in a 9×11 rectangle the number of red squares is $3 \times 11 = 33$. A possible configuration is indicated in the figure below.

R			R			R			R	
	R			R			R			R
		R			R			R		
R			R			R			R	
	R			R			R			R
		R			R			R		
R			R			R			R	
	R			R			R			R
		R			R			R		

[Editor's note. The next solution, by Wang, is not as constructive.]

Let R denote any 9×11 rectangle and k the number of red squares in R . We show $k = 33$. First it is easy to check that any 3×3 square must contain at least 3 red squares. Since R can be partitioned into the disjoint union of nine 3×3 "squares" and three 3×2 rectangles, $k \geq 27 + 6 = 33$. On the other hand if we put two copies of R side by side to obtain a 9×22 rectangle R' then clearly R' can be partitioned into the disjoint union of 33 3×2 rectangles. Thus the number of red squares that R' contains is exactly 66 giving that $k > 33$ is impossible. Thus $k = 33$.

[Editor's note. Both solutions make no distinction between 2×3 and 3×2 rectangles when counting red squares. What is the answer if there is only a restriction on 2×3 rectangles?]

13. An epidemic hit the land of the Munchkins. One day several Munchkins fell ill. Any healthy Munchkin who was not immunized and who visited a sick Munchkin fell ill on the day after the visit. Each ill Munchkin was sick for exactly one day, and after recovery, was immune from further infection for at least one day; on these days (s)he was healthy and could not fall ill (each Munchkin might have this immunization for a different length of time). Despite the epidemic, each healthy Munchkin visited his sick friends every day. As soon as the epidemic began, the Munchkins forgot about any possible vaccines and did not use them. Show that

- (a) If some of the Munchkins were vaccinated *before* the epidemic, and were immune on the first day, then the epidemic could continue for an arbitrarily long time.
- (b) If no Munchkin was immune on the first day, the epidemic must eventually end.

Solutions by JACL and by Duane M. Broline, Eastern Illinois University.

Let S_i be the set of Munchkins sick on day i , I_i the set of those immune on day i , and H_i the set of Munchkins who are healthy but not immune on day i .

(a) Suppose that immunity lasts exactly one day and that each Munchkin visits every other Munchkin each day. Then $I_{i+1} = S_i$, $S_{i+1} = H_i$, and $H_{i+1} = I_i$. If each group is initially non-empty the three remain non-empty forever.

(b) Suppose $I_1 = \emptyset$. We can construct a graph, the friendship graph, whose "vertices" are the Munchkins, and for which there is an edge between two Munchkins just if they are friends. Then for each Munchkin M , let $d(M)$ be the shortest distance in this graph between M and some member of S_1 . Thus $d(M) = 0$ iff $M \in S_1$; $d(M) = 1$ if $M \in H_1$ but M has a friend in S_1 ; $d(M) = 2$ if $M \in H_1$, no friend of M is in S_1 , but some friend of M has a friend in S_1 ; and so on. We claim that $S_n = \{M : d(M) = n - 1\}$, for each $n \geq 1$, and for $n \geq 2$ $I_n = S_{n-1}$. The claim follows by induction. For $n = 2$, S_2 is the set of friends of members of S_1 since $I_1 = \emptyset$, and $I_2 = S_1$. The inductive step follows since the friends of a Munchkin M with $d(M) = n - 1$ are those N with $d(N) = n - 2$ or with $d(M) = n$, and the first group are immune that day. Since there are only finitely many Munchkins the result follows easily from the claim since $\{M : d(M) = n\}$ is empty for large enough n .

14. Let us denote by $p(n)$ the product of the decimal digits of the number n . The sequence $\langle n_k \rangle$ is defined recursively by $n_{k+1} = n_k + p(n_k)$, and by the choice of any

natural number as n_1 . Can the sequence $\langle n_k \rangle$ be unbounded?

[*Editor's note.* Solutions were received from JACL, as well as from Curtis Cooper, Central Missouri State University, Warrensburg. In the interim a solution was published to question 4 of the Austro-Polish Mathematics Competition 1982. This problem is identical and the solution is found on [1991: 4–5].]

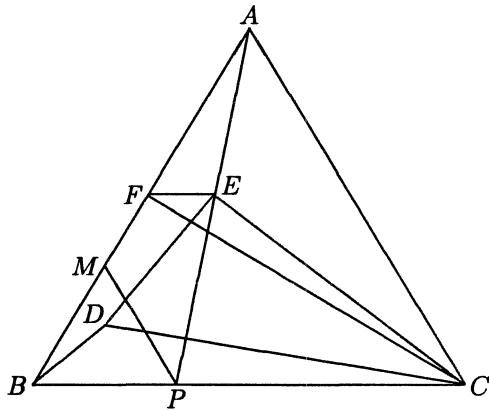
15. A line parallel to side AC of equilateral triangle ABC intersects AB and BC in points M and P , respectively. Point D is the center (of symmetry) of triangle PMB , and E is the midpoint of segment AP . Find the measures of the angles of triangle DEC .

Solutions by Leon Bankoff, Los Angeles, California, and by JACL. We give Bankoff's solution.

Let F be the midpoint of AB . It is the foot of the altitude from C . Since $\angle EFC = \angle DBC = 30^\circ$, and

$$\frac{BC}{FC} = \frac{2}{\sqrt{3}} = \frac{BD}{(BP/2)},$$

it follows that $BC/FC = BD/FE$ and triangles EFC and CDB are similar. Then $\angle ECF = \angle DCB$ and $\angle ECD = \angle FCB$. Since $EC/DC = FC/BC$, triangle DEC is similar to the 60–90–30 degree triangle BFC .



16. The lengths of the edges of a rectangular parallelepiped are x , y and z units, where $x < y < z$. If

$$p = 4(x + y + z),$$

$$s = 2(xy + yz + zx),$$

and

$$d = \sqrt{x^2 + y^2 + z^2}$$

respectively are the perimeter, surface area, and diagonal of the parallelepiped, show that

$$x < \frac{1}{3} \left(\frac{p}{4} - \sqrt{d^2 - s/2} \right) \quad \text{and} \quad z > \frac{1}{3} \left(\frac{p}{4} + \sqrt{d^2 - s/2} \right).$$

Solutions by JACL and by Bob Prielipp, University of Wisconsin-Oshkosh.

We have $3(z - x)(y - x) > 0$ since $z > y > x$. Thus

$$4x^2 + y^2 + z^2 + 2yz - 4zx - 4xy > x^2 + y^2 + z^2 - yz - zx - xy.$$

Taking square roots yields

$$y + z - 2x > \sqrt{d^2 - s/2}.$$

Hence

$$x + y + z - \sqrt{d^2 - s/2} > 3x$$

so

$$x < \frac{1}{3} \left(\frac{p}{4} - \sqrt{d^2 - s/2} \right).$$

Similarly, we have $3(z-x)(z-y) > 0$, so that

$$x^2 + y^2 + 4z^2 - 4yz - 4zx + 2xy > x^2 + y^2 + z^2 - yz - zx - xy.$$

Taking square roots we get

$$2z - x - y > \sqrt{d^2 - s/2}.$$

Hence

$$3z > x + y + z + \sqrt{d^2 - s/2}$$

and

$$z > \frac{1}{3} \left(\frac{p}{4} + \sqrt{d^2 - s/2} \right).$$

17. The set M consists of integers, the smallest of which is 1 and the largest 100. Each element of M , except for 1, is equal to the sum of two (possibly identical) numbers in M . Of all such sets, find one with the smallest possible number of elements.

Solutions by JACL and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $1 = a_1 < a_2 < \dots < a_n = 100$ be the chosen integers. Then $a_k \leq 2^{k-1}$ for all k . In particular $a_7 \leq 64$, so that $n \geq 8$. Suppose $a_8 = 100$. Since $a_6 + a_7 \leq 32 + 64 < 100$, we must have $a_8 = a_7 + a_7$ or $a_7 = 50$. Similarly $a_5 + a_6 \leq 16 + 32 - 48 < 50$, so $a_6 = 25$. Now $a_4 + a_4 \leq 8 + 8 = 16 < 25$. Since a_6 is odd, $a_6 = a_5 + a_4$. Since $a_6 \leq 16$, $a_4 \geq 9$ contradicting $a_4 \leq 8$. Hence $n \geq 9$. For $n = 9$, we may have $a_1 = 1, a_2 = 2, a_4 = 8, a_5 = 9, a_6 = 16, a_7 = 25, a_8 = 50$, and $a_9 = 100$. Other examples are $\{1, 2, 4, 8, 12, 13, 25, 50, 100\}$ and $\{1, 2, 3, 5, 10, 20, 40, 60, 100\}$.

18. Show that there are infinitely many numbers a for which the equation

$$[x^{3/2}] + [y^{3/2}] = a$$

has at least 1980 solutions in natural numbers x, y . Here $[z]$ stands for the integer part of z .

Solution by JACL.

For any integer n , define

$$b_k = [(n^2 + 4k)^{3/2}] \quad \text{and} \quad c_k = [(4n^2 + 2k)^{3/2}],$$

for $1 \leq k \leq 1980$. Choose $n > 6 \cdot 1980^2$ so that $n > 6k^2$. Note that

$$(n^2 + 4k)^3 - (n^3 + 6kn)^2 = 12n^2k^2 + 64k^3 > 0.$$

On the other hand

$$(n^3 + 6kn + 1)^2 - (n^2 + 4k)^3 = 2n^3 + 12kn + 1 - 12k^2n^2 - 64k^3 > 0,$$

because $2n^3 > 2n^2 \cdot 6k^2 = 12k^2n^2$ and $12kn > 12k \cdot 6k^2 = 72k^3$. Hence $b_k = n^3 + 6kn$, and the sequence $\{b_1, b_2, \dots, b_{1980}\}$ is an arithmetic progression with common difference $6n$.

Similarly

$$(4n^2 + 2k)^3 - (8n^3 + 6kn)^2 = 12k^2n^2 + 8k^3 > 0$$

and

$$(8n^3 + 6kn + 1)^2 - (4n^2 + 2k)^3 = 16n^3 + 12kn + 1 - 12k^2n^2 - 8k^3 > 0.$$

Hence $c_n = 8n^3 + 6kn$ and the sequence $\{c_k\}_{k=1}^{1980}$ is also arithmetic with common difference $6n$. It follows that if $a = b_1 + c_{1980} = b_2 + c_{1979} = \dots = b_{1980} + c_1$, then the equation $[x^{3/2}] + [y^{3/2}] = a$ has at least 1980 solutions in the positive integers x and y . Since the only requirement on n is $n > 6(1980)^2$, we can find infinitely many numbers a with the desired property.

19. In tetrahedron $ABCD$, $AC \perp BC$ and $AD \perp BD$. Show that the cosine of the angle between lines AC and BD is less than CD/AB .

Solution by JACL.

Note that the midpoint O of AB is the circumcenter of the tetrahedron. Set up a coordinate system with O as origin, so that the coordinates of A and B are $(-k, 1, 0)$ and $(k, -1, 0)$ respectively where k is some real number. We may take the coordinates of D to be $(-k, -1, 0)$ and those of C to be (x, y, z) where $x^2 + y^2 + z^2 = k^2 + 1$ and $z \neq 0$. Let C' be $(k, 1, 0)$. Then

$$\overrightarrow{AB} = \langle 2k, -2, 0 \rangle, \quad |\overrightarrow{AB}| = 2\sqrt{k^2 + 1},$$

$$\overrightarrow{DB} = \overrightarrow{AC'} = \langle 2k, 0, 0 \rangle, \quad |\overrightarrow{AC'}| = 2k,$$

$$\overrightarrow{AC} = \langle x + k, y - 1, z \rangle, \quad |\overrightarrow{AC}| = \sqrt{2(k^2 + 1 + xk - y)},$$

$$\overrightarrow{DC} = \langle x + k, y + 1, z \rangle, \quad |\overrightarrow{DC}| = \sqrt{2(k^2 + 1 + xk + y)},$$

and

$$\overrightarrow{AC'} \cdot \overrightarrow{AC} = 2k(x + k).$$

The angle in question is the angle between $\overrightarrow{AC'}$ and \overrightarrow{AC} , and its cosine is given by

$$\frac{2k(x + k)}{2k\sqrt{2(k^2 + 1 + xk - y)}}.$$

We wish to show that this is less than

$$\frac{\sqrt{2(k^2 + 1 + xk + y)}}{2\sqrt{k^2 + 1}}.$$

Equivalently, we want $(x + k)^2(k^2 + 1) < (k^2 + 1 + xk)^2 - y^2$. This follows from

$$\begin{aligned}
 (k^2 + 1 + xk)^2 - y^2 - (x + k)^2(k^2 + 1) &= (k^2 + 1)^2 - (k^2 + 1) - x^2 - y^2 \\
 &= k^2(k^2 + 1) - x^2 - y^2 \geq k^2 + 1 - x^2 - y^2 = z^2 > 0,
 \end{aligned}$$

as desired.

20. The number x , $0 \leq x < 1$, is written as an infinite decimal. If we permute at random the first five digits to the right of the decimal point, we will obtain a new infinite decimal, corresponding to some new number x_1 . If we then permute the second through the sixth digit in the decimal representation of x_1 , we obtain a decimal representation of some new number x_2 . In general, we can start with a number x_k and obtain x_{k+1} by permuting the digits in places $(k+1)$ through $(k+5)$ after the decimal point.

- (a) Show that no matter how the digits are permuted at each step, the sequence $\langle x_n \rangle$ will always have a limit. Let us call that limit y .
- (b) Can one start with a *rational* x and by the process described above obtain an *irrational* y ?
- (c) Find a number x such that starting with x , the above process always produces an irrational y .

Solution by JACL.

- (a) Notice that the digits of x_n up to the n th are unchanged in x_{n+1} . This means that the first n digits of the limit y are then determined and the sequence converges.
- (b) This is possible. Start with $x = 0.010101\dots$, with repeating block 01, so that x is rational. We interchange the digits 0 and 1 within the n th block, if and only if $n = 10, 10^2, 10^3, \dots$. The resulting decimal expansion cannot be periodic and thus an irrational number y is obtained.

- (c) Let x be such that its decimal expansion consists only of blocks of five 0's and blocks of five 1's, with the 1's in the n th block if and only if $n = 10, 10^2, 10^3, \dots$. For any y obtained by this process, suppose its decimal expansion ends with repeating blocks of length m , beginning at the n th decimal place, say. Let k be the number of 1's in the repeating block. Note that $k \geq 1$. Consider a long grouping of blocks of five 0's of x occurring after the n th decimal place, with at least $10m$ zero's. (There are blocks of five 0's used for the $9 \times 10^\ell - 1$ positions from between the block of five 1's at 10^ℓ and the next block at $10^{\ell+1}$ of x). In y this grouping must account for at least 9 of the m -blocks and so contain at least nine 1's. However, the permuting process can introduce at most $4 + 4 = 8$ 1's into this stretch of 0's, a contradiction.

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This completes the solutions to problems from the “All Union” Mathematical Olympiad. Send in your nice solutions and contest problem sets!

* * * * *

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

Problem Solving Series, by D. A. Holton, published by the University of Otago, Dunedin, New Zealand, 1988-90, ISBN # 0-908903-AB-C (where AB ranges from 01 to 15 and C satisfies $11 \mid 5 + 3A + 2B + C$), softcover, each between 30 and 52 pages, US\$2.30 each. *Reviewed by Andy Liu, University of Alberta.*

Professor Holton, the leader of the New Zealand I.M.O. team, has written fifteen booklets in a Problem Solving Series for high school students. The titles are

- | | | |
|------------------------|-------------------------|------------------------|
| 01, How To; | 02, Combinatorics I; | 03, Graph Theory; |
| 04, Number Theory I; | 05, Geometry I; | 06, Proof; |
| 07, Geometry II; | 08, I.M.O. Problems I; | 09, Combinatorics II; |
| 10, Geometry III; | 11, Number Theory II; | 12, Inequalities; |
| 13, Combinatorics III; | 14, I.M.O. Problems II; | 15, Creating Problems. |

The topics cover the most popular areas of mathematics in the I.M.O. Besides the two booklets (8 and 14) featuring actual I.M.O. problems, there are three (1, 6 and 15) which discuss problem-solving techniques in general.

Each booklet consists of a large number of exercises interspersed with a minimum amount of text, followed by a solution section. The style is informal, and much of the small amount of text is devoted to talking to, but not down to, the students.

The levels of the problems vary from the simplest to open problems, but not necessarily in ascending order of difficulty. It is not until checking the back of the booklet that a student discovers that she or he has been working on an unsolved problem. Had this been known in advance, the student might have passed up the challenge. This way, an unsuspecting youngster may some day surprise the experts.

This series is highly recommended, both for individual study and club meetings. This is especially so in North America, since the topics covered are largely ignored in the regular curriculum.

The author is negotiating with some possible Canadian distributors or publishers. For now, the booklets can be ordered directly from

Prof. Derek Holton,
Department of Mathematics and Statistics,
University of Otago,
Dunedin, New Zealand.

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PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before July 1, 1992, although solutions received after that date will also be considered until the time when a solution is published.

1691*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $n \geq 2$. Determine the best upper bound of

$$\frac{x_1}{x_2 x_3 \dots x_n + 1} + \frac{x_2}{x_1 x_3 \dots x_n + 1} + \dots + \frac{x_n}{x_1 x_2 \dots x_{n-1} + 1},$$

over all x_1, \dots, x_n with $0 \leq x_i \leq 1$ for $i = 1, 2, \dots, n$.

1692. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with circumcenter O and with $\overline{AB} < \overline{AC}$. Let P be a point on BA produced beyond A such that $\overline{BP} = \overline{AC}$, and let Q be a point on side AC such that $\overline{CQ} = \overline{AB}$. Suppose that P, Q, O are collinear. Find $\angle BAC$.

1693. *Proposed by Murray S. Klamkin, University of Alberta.*

If $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are four distinct unit vectors in space such that

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{A} = \mathbf{D} \cdot \mathbf{B} = \mathbf{D} \cdot \mathbf{C} = 1/2,$$

determine $\mathbf{A} \cdot \mathbf{D}$.

1694. *Proposed by Jordi Dou, Barcelona, Spain.*

Let ℓ be a line through the orthocentre H of ΔABC . Let A', B', C' be the feet of the perpendiculars to ℓ from A, B, C , and let A'', B'', C'' be chosen on ℓ so that A', B', C' are the midpoints of HA'', HB'', HC'' , respectively. Prove that AA'', BB'', CC'' meet in a point L , and determine the locus of L as ℓ rotates about H .

1695. *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*

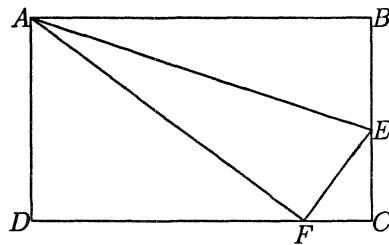
Let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$ with $a_0 > 0$ and

$$a_0 + \frac{a_0 + a_2}{3} + \frac{a_2 + a_4}{5} + \frac{a_4}{7} < 0.$$

Prove that there exists at least one zero of $p(x)$ in the interval $(-1, 1)$.

1696. *Proposed by Ed Barbeau, University of Toronto.*

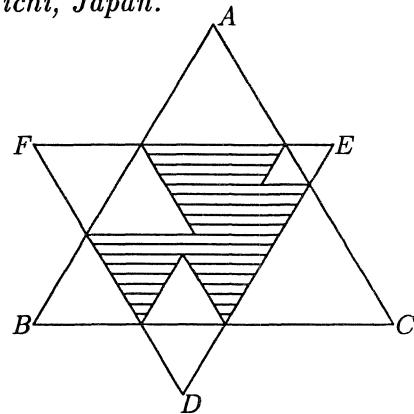
An $8\frac{1}{2}$ by 11 sheet of paper is folded along a line AE through the corner A so that the adjacent corner B on the longer side lands on the opposite longer side CD at F . Determine, with a minimum of measurement or computation, whether triangle AEF covers more than half the quadrilateral $AECD$.

**1697.** *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Suppose that P, Q, R are lattice points (i.e., points with integer coordinates) in the plane such that Q and R lie on the parabola $y = x^2$ and PQ, PR are tangents to the parabola. Let n be the total number of lattice points on the sides of triangle PQR , including the vertices. Find the area of ΔPQR .

1698. *Proposed by Hidetosi Fukagawa, Aichi, Japan.*

ABC is an equilateral triangle of area 1. DEF is an equilateral triangle of variable size, placed so that the two triangles overlap, with $DE \parallel AB$, $EF \parallel BC$, $FD \parallel CA$, and D, E, F not in ΔABC , as shown. The corners of ΔDEF sticking outside ΔABC are then folded over. Find the maximum possible area of the uncovered (shaded) part of ΔDEF .

**1699.** *Proposed by Xue-Zhi Yang and Ji Chen, Ningbo University, China.*

Let $R, r, h_a, h_b, h_c, r_a, r_b, r_c$ be the circumradius, inradius, altitudes, and exradii of a triangle. Prove that

$$\sqrt{\frac{2R}{r} + 5} \leq \sqrt{\frac{r_a}{h_a}} + \sqrt{\frac{r_b}{h_b}} + \sqrt{\frac{r_c}{h_c}} \leq \sqrt{\frac{4R}{r} + 1}.$$

1700. *Proposed by Sydney Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Suppose a and b are two elements of a group satisfying $ba = ab^2$, $b \neq 1$ and $a^{31} = 1$. Determine the order of b .

* * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1148. [1986: 108; 1988: 83] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Find the triangle of smallest area that has integral sides and integral altitudes.

II. Solution and comments by Sam Maltby, student, University of Calgary.

Let ABC be a triangle with integral sides and altitudes. We first note that at least one of the altitudes meets the opposite side between its two vertices. Let D be the foot of the altitude from A to BC with D between B and C . Then $DC = \sqrt{AC^2 - AD^2}$ is the square root of an integer, as is BD . But $BD + DC = BC$ which is an integer, so we conclude that DC and BD are also integers. Thus (AD, DC, AC) and (DB, AD, AB) are Pythagorean triples.

Case (i): $\angle BAC = 90^\circ$. Then $\Delta ABD \sim \Delta CBA \sim \Delta CAD$, so we have $BD = k_1r$, $AD = k_1s$, $AB = k_1t$ and $AD = k_2r$, $DC = k_2s$, $AC = k_2t$ where (r, s, t) is a primitive Pythagorean triple and k_1, k_2 are positive integers. Then $k_1s = k_2r$ and $\gcd(r, s) = 1$, so $k_1 = nr$ and $k_2 = ns$ for some positive integer n . Then the area of ΔABC is

$$\frac{AD \cdot BC}{2} = \frac{nrs(nr^2 + ns^2)}{2} = \frac{n^2rs(r^2 + s^2)}{2}.$$

This is clearly minimal for $n = 1$, and the three sides AB , AC , BC and three altitudes AB , AC , AD all have integral lengths for $n = 1$. In this case the area is $rs(r^2 + s^2)/2$, which is clearly minimal over Pythagorean triples for $r = 3, s = 4$ (or $r = 4, s = 3$). This gives us $BD = 9$, $DC = 16$, $BC = 25$, $AD = 12$, $AB = 15$ and $AC = 20$, with area 150.

Case (ii): $\angle BAC \neq 90^\circ$. Again AD is a leg of two integral-sided right triangles, and if the area of ΔBAC is to be no more than 150, one of them must have area at most 75, and each must have area less than 150. The only triangles this leaves (for ΔABD and ΔADC) are those with sides

$$(3, 4, 5), (6, 8, 10), (9, 12, 15), (12, 16, 20), (5, 12, 13), (10, 24, 26), (7, 24, 25), (8, 15, 17).$$

If ΔABC is isosceles then we use the same triple twice, but none of these yields integral altitudes other than AD . If the two triples are distinct, then they must share a common leg. The pair $(7, 24, 25)$ and $(10, 24, 26)$ is discarded because neither has area ≤ 75 . $(6, 8, 10)$ with $(8, 15, 17)$, and $(5, 12, 13)$ with either $(9, 12, 15)$ or $(12, 16, 20)$, do not give us three integral altitudes. Finally $(9, 12, 15)$ with $(12, 16, 20)$ gives us our triangle from Case (i). Thus Case (ii) does not give us a smaller triangle than Case (i).

Therefore the smallest triangle with integral sides and altitudes is the 15-20-25 triangle, as stated on [1988: 83].

Note. The smallest non-right triangles with integral sides and altitudes are the 25-25-30 (acute) and 25-25-40 (obtuse) triangles with areas of 300 each and altitudes 24, 24, 20 and 24, 24, 15 respectively.

As for the triangle of smallest area having integral sides, area, circumradius and inradius (given as an open problem on [1988: 83]), it is the 6-8-10 triangle with area 24, circumradius 5 and inradius 2. This follows from the formulae $p = \Delta/s$, $R = abc/(4\Delta)$. The theorem (L.E. Dickson, *History of the Theory of Numbers* Vol II, p. 200) that a triangle with integral sides and area must have its area divisible by 6 is then used, the values $\Delta = 6, 12$ and 18 yielding no solutions but $\Delta = 24$ giving the triangle above. If we add the restriction that the altitudes must also be integral, we get the 30-40-50 triangle.

By the way, the quote from p. 200 of Dickson given on [1988: 83] contains an error (in Dickson): the triangle should have sides 4, 13 and 15.

* * * *

1251. [1987: 179; 1988: 210] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts. (Dedicated to Léo Sauvé.)*

(a) Find all integral n for which there exists a regular n -simplex with integer edge and integer volume.

(b)* Which such n -simplex has the smallest volume?

II. *Solution to part (b) by Sam Maltby, student, University of Calgary.*

It was originally shown on [1988: 210] that the only n for which an n -simplex with integer edge and integer volume exists is when $n = 4k^2 + 4k$ or $n = 2k^2 - 1$ for some integer $k > 1$. The smallest of these is 7, so from now on assume $n \geq 7$.

It was conjectured that the smallest n -simplex with integer edge and volume is the 7-simplex with edge 210 and volume 893397093750. This is correct.

We shall use the following results.

LEMMA 1. Assume $\sqrt{n+1}$ or $\sqrt{(n+1)/2}$ is an integer ℓ . Then if p is prime and $p|n!$, $p|(n!/\ell)$.

Proof. Since $2\ell < n$ for $n \geq 7$, $n! = \ell \cdot 2\ell \cdot m$ for some integer m , so $p|2m\ell = n!/\ell$. \square

LEMMA 2. n is less than the product of the primes not larger than n .

Proof. For $n = 7$ it is trivial. For $n \geq 8$, by Bertrand's Postulate there is a prime p such that $n/2 \leq p \leq n$, and $n/2 \geq 4$ so $p \neq 2$ or 3. Thus the product of the primes not greater than n is at least $2 \cdot 3 \cdot p \geq 6n/2 > n$. \square

LEMMA 3. The function $f : \mathbf{N} \rightarrow \mathbf{R}$ defined by

$$f(n) = \frac{n^n \sqrt{n+1}}{\sqrt{2^n} n!}$$

is increasing.

Proof. The result follows from

$$\begin{aligned} \frac{f(n+1)}{f(n)} &= \frac{(n+1)^{n+1} \sqrt{n+2}}{\sqrt{2^{n+1}} (n+1)!} \cdot \frac{\sqrt{2^n} n!}{n^n \sqrt{n+1}} \\ &= \left(\frac{n+1}{n}\right)^n \cdot \sqrt{\frac{n+2}{n+1}} \cdot \frac{1}{\sqrt{2}} > 2 \cdot 1 \cdot \frac{1}{\sqrt{2}} > 1. \quad \square \end{aligned}$$

On with the proof. The volume of an n -simplex with edge a is

$$V = \frac{a^n \sqrt{n+1}}{n! \sqrt{2^n}} = \frac{a^n}{2^{\lfloor n/2 \rfloor} n! / \ell},$$

where ℓ is as in Lemma 1. Suppose p is a prime which divides $n!$. By Lemma 1 p divides the denominator of V , and V is an integer, so we must have $p|a^n$. But then $p|a$, and all primes not greater than n divide $n!$, so they all divide a . Thus by Lemma 2 $a > n$, so

$$V > \frac{n^n \sqrt{n+1}}{n! \sqrt{2^n}}.$$

For $n = 48$ this is greater than the volume given above for the 7-simplex, so by Lemma 3 we need only examine $n < 48$. Since $n = 2k^2 - 1$ or $n = 4k^2 + 4k$ this gives us $n \in \{7, 8, 17, 24, 31\}$. By Lemma 1 and the above, since 2, 3, 5 and 7 are all divisors of a for each of these, $a \geq 210$; it is then easy to check that for each of these values of n , $V \geq 893397093750$ with equality at $a = 210$, $n = 7$.

* * * *

1400. [1988: 302; 1989: 177; 1990: 59] *Proposed by Robert E. Shafer, Berkeley, California.*

In a recent issue of the *American Mathematical Monthly* (June–July 1988, page 551), G. Klambauer showed that if $x^s e^{-x} = y^s e^{-y}$ ($x, y, s > 0$, $x \neq y$) then $x + y > 2s$. Show that if $x^s e^{-x} = y^s e^{-y}$ where $x \neq y$ and $x, y, s > 0$ then $xy(x + y) < 2s^3$.

III. *Comment by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

In what follows we extend an inequality quoted in the editor's footnote at the end of the solutions to this problem [1990: 62]. That is, we improve the inequality

$$M_0(x, \frac{x+y}{2}, y) < s < M_1(x, \frac{x+y}{2}, y)$$

to

$$M_0(x, \frac{x+y}{2}, y) < s < M_{1/3}(x, \frac{x+y}{2}, y), \quad (1)$$

where x, y, s are as in the problem and

$$M_t(u, v, w) = \begin{cases} \left(\frac{u^t+v^t+w^t}{3}\right)^{1/t} & \text{if } t \neq 0, \\ (uvw)^{1/3} & \text{if } t = 0. \end{cases}$$

Due to

$$\frac{1}{s} = \frac{\log y - \log x}{y - x},$$

we show for the right-hand inequality of (1) that

$$\frac{\log y - \log x}{y - x} > \left(\frac{3}{x^{1/3} + \left(\frac{x+y}{2}\right)^{1/3} + y^{1/3}} \right)^3.$$

Putting here $y = tx$ (with $t > 1$; since (1) is symmetric in x and y , we may and do assume $y > x$) we get

$$\frac{\log t}{t-1} > \left(\frac{3}{1 + \left(\frac{t+1}{2}\right)^{1/3} + t^{1/3}} \right)^3. \quad (2)$$

Taking logarithms of (2) we get

$$g(t) := \log \log t - \log(t-1) + 3 \log \left(1 + \sqrt[3]{\frac{1+t}{2}} + \sqrt[3]{t} \right) - \log 27 > 0.$$

We now show $g'(t) > 0$ for $t > 1$. From

$$g'(t) = \frac{1}{t \log t} - \frac{1}{t-1} + \frac{\frac{1}{2} \left(\frac{1+t}{2}\right)^{-2/3} + t^{-2/3}}{1 + \sqrt[3]{\frac{1+t}{2}} + \sqrt[3]{t}} ,$$

a short manipulation yields that $g'(t) > 0$ is equivalent to

$$\frac{\log t}{t-1} < \frac{1 + \sqrt[3]{t} + \sqrt[3]{\frac{t+1}{2}}}{t + \sqrt[3]{t} + t \left(\frac{t+1}{2}\right)^{-2/3}} (= A). \quad (3)$$

In D.S. Mitrinović, *Analytic Inequalities*, the following is given as item 3.6.16:

$$\frac{\log t}{t-1} < \frac{1 + \sqrt[3]{t}}{t + \sqrt[3]{t}} (= B), \quad t \neq 1.$$

In order to prove (3), it's thus enough to check $B \leq A$, i.e.,

$$(1 + \sqrt[3]{t}) \left(t^{2/3} + 1 + t^{2/3} \left(\frac{t+1}{2}\right)^{-2/3} \right) \leq (t^{2/3} + 1) \left(1 + \sqrt[3]{t} + \sqrt[3]{\frac{t+1}{2}} \right) ,$$

$$(1 + \sqrt[3]{t}) t^{2/3} \leq (t^{2/3} + 1) \left(\frac{t+1}{2}\right) ,$$

and finally

$$0 \leq t^{2/3}t - t^{2/3} - t + 1 = (t^{2/3} - 1)(t - 1),$$

which is true. Thus (3) is shown. Since for t decreasing to 1 both sides of (2) approach 0, we get the validity of (2) and thus (1).

We now show that the left-hand inequality in (1) is best possible. Assume that $M_r(x, \frac{x+y}{2}, y) < s$ for $r > 0$, i.e. (as before)

$$\frac{\log t}{t-1} < \left(\frac{3}{1 + t^r + \left(\frac{t+1}{2}\right)^r} \right)^{1/r} < \frac{3^{1/r}}{t} ;$$

then this is a contradiction if $t \rightarrow \infty$.

Finally, I would like to state the

Conjecture: the right-hand inequality in (1) is also best possible, i.e. for $\lambda < 1/3$,

$$\frac{\log t}{t-1} > \left(\frac{3}{1 + t^\lambda + \left(\frac{t+1}{2}\right)^\lambda} \right)^{1/\lambda}$$

doesn't hold for all $t > 1$.

Remark. Inequality (1) can also be understood as follows. Let $f(x) = \log x$, and consider values of p and q such that

$$M_p \left(f'(x), f' \left(\frac{x+y}{2} \right), f'(y) \right) < \frac{f(y) - f(x)}{y-x} < M_q \left(f'(x), f' \left(\frac{x+y}{2} \right), f'(y) \right). \quad (4)$$

Then $p_{\max} \geq -1/3$ and $q_{\min} = 0$. This tells us a lot about how well we can approximate the difference quotient of f by the derivatives.

(4) also gives rise to the following question. Let $p < q$ be given. Does there exist a differentiable function $f : I \rightarrow \mathbb{R}$ satisfying (4) such that $p_{\max} = p$ and $q_{\min} = q$? (Here I denotes a finite or infinite subinterval of \mathbb{R} .)

* * * *

1579. [1990: 240] *Proposed by S. Kotani, Akita, Japan, and H. Fukagawa, Aichi, Japan.*

Given a triangle ABC , we erect three similar rectangles $ABDE, CAFG, BCHI$ outside ABC . Let M_a and N_a be the midpoints of BC and EF respectively. Define M_b, N_b, M_c, N_c analogously. Prove that the lines M_aN_a, M_bN_b, M_cN_c are concurrent.

Solution by P. Penning, Delft, The Netherlands.

The procedure to construct the points N_a, N_b, N_c can be made much simpler. For example, CN_c is perpendicular to AB and away from AB , and its length is $\lambda \cdot AB$, where

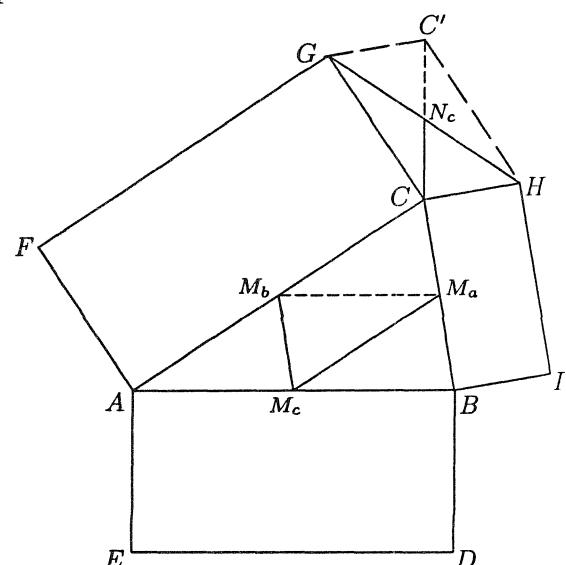
$$\frac{AF}{AC} = \frac{CH}{BC} = \frac{BD}{BA} = 2\lambda.$$

The proof is easy by rotating $\triangle GCH$ 90° ; CN_c is then parallel to M_aM_b . Furthermore $\triangle GCH$ is similar to $\triangle CM_bM_c$ because of the similarity of the rectangles.

Editor's note. This is really the same nice result that Jordi Dou pointed out in his solution of *Crux* 1493 [1991: 53]. Parallelograms $GCHC'$ and $CM_bM_cM_a$ are similar, and are oriented at 90° to each other, so the corresponding diagonals CC' and M_bM_a are perpendicular. Their lengths are in the ratio

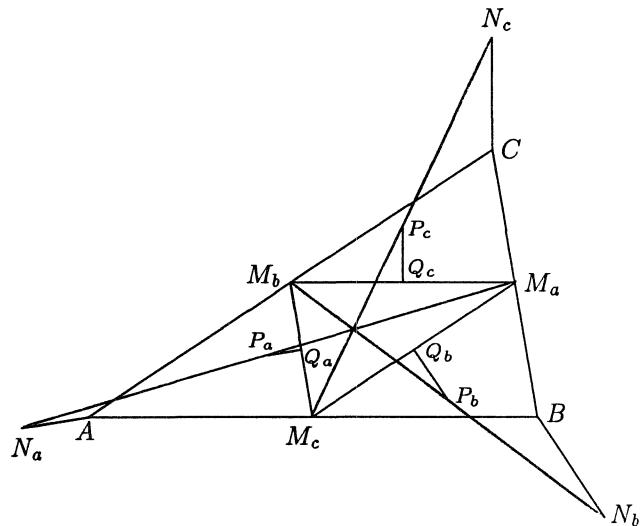
$$\frac{CG}{CM_b} = \frac{AF}{AC/2} = 4\lambda,$$

so $CN_c/AB = \lambda.$]



Let P_c be the midpoint of N_cM_c , and define similar points P_b and P_a halfway on N_bM_b and N_aM_a respectively. I shall show that P_cM_c , P_bM_b and P_aM_a are concurrent (which will of course solve the problem).

Draw P_cQ_c parallel to N_cC , with Q_c the intersection with M_cC . ΔN_cCM_c is similar to $\Delta P_cQ_cM_c$, the ratio being $1/2$. So $M_cQ_c = Q_cC$ and Q_c lies on M_aM_b also, in fact is the midpoint thereof. So P_cQ_c is the perpendicular bisector of M_aM_b and its length is proportional to M_aM_b . Similar results are obtained for P_bQ_b and P_aQ_a . [That is, P_a , P_b , P_c are the apexes of three similar isosceles triangles erected outwardly on the sides of $\Delta M_aM_bM_c$.] It is known that in this situation P_cM_c , P_bM_b and P_aM_a are concurrent. [For example, see Theorem 357, p. 223 of R.A. Johnson, *Advanced Euclidean Geometry*.]



Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; JOHN RAUSEN, New York; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposers.

For the above result (Theorem 357 of Johnson) Rausen quoted problem 53, page 35 of Aubert & Papelier, Exercices de Géometrie Analytique, Tome 1, 10ième Édition, Paris (Vuibert), 1957.

* * * * *

1580*. [1990: 240] *Proposed by Ji Chen, Ningbo University, China.*

For every convex n -gon, if one circle with centre O and radius R contains it and another circle with centre I and radius r is contained in it, prove or disprove that

$$R^2 \geq r^2 \sec^2 \frac{\pi}{n} + \overline{IO}^2.$$

Editor's comment.

No solutions to this problem have been received.

* * * * *

1581*. [1990: 266] *Proposed by Murray S. Klamkin and Andy Liu, University of Alberta.*

If T_1 and T_2 are two triangles with equal circumradii, it is easy to show that if the angles of T_2 majorize the angles of T_1 , then the area and perimeter of T_2 is not greater than

the area and perimeter, respectively, of T_1 . (One uses the concavity of $\sin x$ and $\log \sin x$ in $(0, \pi)$.) If T_1 and T_2 are two tetrahedra with equal circumradii, and the solid angles of T_2 majorize the solid angles of T_1 , is it true that the volume, the surface area, and the total edge length of T_2 are not larger than the corresponding quantities for T_1 ?

Editor's Note.

No solutions to this problem have been received, either!

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1582. [1990: 267] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

A nonzero polynomial $P(x) = a_0 + a_1x + \cdots + a_nx^n$ is called *n-palindromic* if $a_j = a_{n-j}$ for $j = 0, 1, \dots, n$. (Note that a_n may equal 0, so that $P(x)$ need not have degree n .) It is plain that if P and Q are *n*-palindromic then $F = P/Q$ satisfies

$$(*) \quad F(1/x) = F(x) \text{ for all } x \text{ such that } F(x), F(1/x) \text{ are defined.}$$

Prove or disprove that, conversely, every rational function $F \not\equiv 0$ satisfying $(*)$ is the quotient of two *n*-palindromic polynomials for some n .

Solution by Chris Wildhagen, Rotterdam, The Netherlands.

Suppose that $F(x) = F(1/x)$, for all but a finite number of $x \in \mathbf{C}$, where $F(x) = P(x)/Q(x) \not\equiv 0$, and $P(x), Q(x)$ are relatively prime polynomials with complex coefficients. We can assume that

$$P(x) = a_0 + a_1x + \cdots + a_nx^n, \quad Q(x) = b_0 + b_1x + \cdots + b_nx^n,$$

with $(a_n, b_n) \neq (0, 0)$. Let

$$P^*(x) := a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = x^n P(1/x),$$

$$Q^*(x) := b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n = x^n Q(1/x).$$

In the following, $x \in \mathbf{C}$ is chosen in such a way that the resulting expressions in x are well-defined. Note

$$\begin{aligned} F(x) = F(1/x) &\iff \frac{P(x)}{Q(x)} = \frac{P(1/x)}{Q(1/x)} = \frac{x^n P(1/x)}{x^n Q(1/x)} = \frac{P^*(x)}{Q^*(x)} \\ &\iff P(x)Q^*(x) = Q(x)P^*(x). \end{aligned} \tag{1}$$

Suppose that $a_n \neq 0$ (the case $b_n \neq 0$ is similarly dealt with). From (1) it follows that $P(x)|P^*(x)$. Since $\deg P(x) \geq \deg P^*(x)$ this means that

$$P^*(x) = a \cdot P(x) \tag{2}$$

for some $a \in \mathbf{C}$. Substitution of this relation in (1) yields $Q^*(x) = a \cdot Q(x)$. (2) implies $a_0 = aa_n$ and $a_n = aa_0$, hence $a_n = a^2a_n$, from which it follows that $a = \pm 1$. If $a = -1$ then $P(1) = P^*(1) = -1 \cdot P(1)$ (by (2)) and $Q(1) = Q^*(1) = -1 \cdot Q(1)$, so $P(1) = Q(1) = 0$, meaning that both $P(x)$ and $Q(x)$ are divisible by $x - 1$. This is however impossible, since $P(x)$ and $Q(x)$ are relatively prime. Therefore $a = 1$, hence $P^*(x) = P(x)$ and $Q^*(x) = Q(x)$. Conclusion: $P(x)$ and $Q(x)$ are *n*-palindromic.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; and the proposer.

* * * *

1583. [1990: 267] Proposed by Toshio Seimiya, Kawasaki, Japan.

ABC is a triangle with circumcenter O . Let P, Q be points on the sides AB, AC respectively, such that

$$\overline{BP} : \overline{PQ} : \overline{QC} = \overline{AC} : \overline{CB} : \overline{BA}.$$

Prove that A, P, Q and O are concyclic.

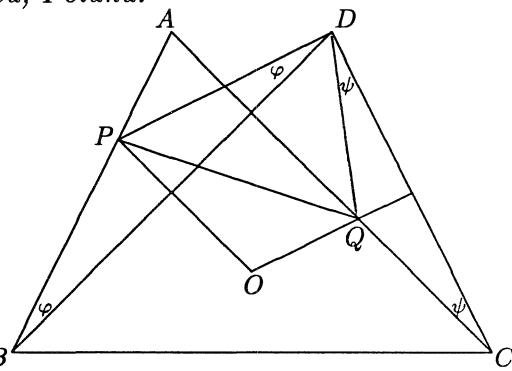
Solution by Marcin E. Kuczma, Warszawa, Poland.

Let α, β, γ be the angles of triangle ABC ; assume $\beta \geq \gamma$, without loss. Erect triangle DPQ similar to ACB (and exterior to quadrilateral $QPBC$), so that

$$DP : PQ : QD = AC : CB : BA.$$

By the condition of the problem,

$$PD = PB \quad \text{and} \quad QD = QC. \quad (1)$$



Let the isosceles triangles BDP and CDQ have angles $\varphi, \varphi, 180^\circ - 2\varphi$ and $\psi, \psi, 180^\circ - 2\psi$ respectively. In quadrangle $BCDP$,

$$\begin{aligned} 360^\circ &= \angle PBC + \angle BCD + \angle CDP + \angle DPB \\ &= \beta + (\gamma + \psi) + (\psi + \alpha) + (180^\circ - 2\varphi) \\ &= 360^\circ + 2\psi - 2\varphi. \end{aligned}$$

Hence $\varphi = \psi$ and

$$\angle CDB = \angle QDP - \varphi + \psi = \angle QDP = \alpha,$$

showing that triangles ABC and DBC have a common circumcircle. Thus, in view of (1), lines OP and OQ are the perpendicular bisectors of DB and CD respectively. Therefore $\angle POQ = 180^\circ - \angle CDB = 180^\circ - \alpha$, and this is just enough to conclude that P, O, Q and A are concyclic.

Note. It is easily seen that $ABCD$ is an isosceles trapezoid. Hence follows the method of construction of points P, Q fulfilling the condition of the problem: construct trapezoid $ABCD$ and draw the perpendicular bisectors of DB and CD ; they cut lines AB and CA at P and Q . In order that P should actually lie on segment AB , it is necessary and sufficient that $\beta \leq 2\gamma$ (in which case also Q lies on segment CA); if $2\gamma < \beta < \gamma + 90^\circ$, P lies on AB produced beyond A ; if $\beta > \gamma + 90^\circ$, P lies on AB produced beyond B ; and if $\beta = \gamma + 90^\circ$, P does not exist; in each case when P and Q exist they are concyclic with A and O .

Also solved by ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; JOSÉ YUSTY PITA, Madrid, Spain; and the proposer.

* * * *

1584. [1990: 267] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove that for $\lambda > 1$

$$\left(\frac{\ln \lambda}{\lambda - 1} \right)^3 < \frac{2}{\lambda(\lambda + 1)}.$$

Solution by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.

Considering the function

$$f(\lambda) = \sqrt[3]{2}(\lambda - 1)(\lambda^2 + \lambda)^{-1/3} - \ln \lambda,$$

we need to show $f(\lambda) > 0$ for $\lambda > 1$. We obtain

$$\begin{aligned} f'(\lambda) &= \sqrt[3]{2}(\lambda^2 + \lambda)^{-1/3} + \sqrt[3]{2}(\lambda - 1)(-1/3)(\lambda^2 + \lambda)^{-4/3}(2\lambda + 1) - \lambda^{-1} \\ &= \frac{1}{3}(\lambda^2 + \lambda)^{-4/3}[\sqrt[3]{2} \cdot 3(\lambda^2 + \lambda) - \sqrt[3]{2}(\lambda - 1)(2\lambda + 1) - 3\lambda^{1/3}(\lambda + 1)^{4/3}] \\ &= \frac{1}{3}(\lambda^2 + \lambda)^{-4/3}[\sqrt[3]{2}(\lambda^2 + 4\lambda + 1) - 3(\lambda + 1)(\lambda^2 + \lambda)^{1/3}]. \end{aligned}$$

Since $f(1) = 0$, we need only prove that $f'(\lambda) \geq 0$ for $\lambda \geq 1$, i.e., that

$$\sqrt[3]{2}(\lambda^2 + 4\lambda + 1) \geq 3(\lambda + 1)(\lambda^2 + \lambda)^{1/3}.$$

This is equivalent to

$$2(\lambda^2 + 4\lambda + 1)^3 \geq 27\lambda(\lambda + 1)^4,$$

or, dividing both sides by λ^3 , to

$$2 \left[\left(\lambda + \frac{1}{\lambda} \right) + 4 \right]^3 \geq 27 \left[\left(\lambda^2 + \frac{1}{\lambda^2} \right) + 4 \left(\lambda + \frac{1}{\lambda} \right) + 6 \right].$$

Now we set $\lambda + 1/\lambda = t \geq 2$, and obtain

$$2(t + 4)^3 \geq 27((t^2 - 2) + 4t + 6) = 27(t + 2)^2$$

which simplifies to

$$(t - 2)^2(2t + 5) = 2t^3 - 3t^2 - 12t + 20 \geq 0,$$

which is true for $t \geq 2$ with equality only for $t = 2$, i.e., $\lambda = 1$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; BEATRIZ MARGOLIS, Paris, France; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One incorrect solution was sent in.

Some solvers observed that since replacing λ by $1/\lambda$ results in an equivalent inequality, the given inequality actually holds for all $\lambda > 0, \lambda \neq 1$.

The proposer also showed that the maximum value of $q < 1$ such that

$$\left(\frac{\ln \lambda}{\lambda - 1} \right)^3 < \frac{1}{\lambda(1 - q + \lambda q)}$$

holds for all $\lambda > 1$ is $q = 1/2$, the value in the problem.

* * * *

1585. [1990: 267] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

We are given a triangle $A_1A_2A_3$ and a real number $r > 0$. Inside the triangle, inscribe a rectangle R_1 whose height is r times its base, with its base lying on side A_2A_3 . Let B_1 be the midpoint of the base of R_1 and let C_1 be the center of R_1 . In a similar manner, locate points B_2, C_2 and B_3, C_3 , using rectangles R_2 and R_3 .

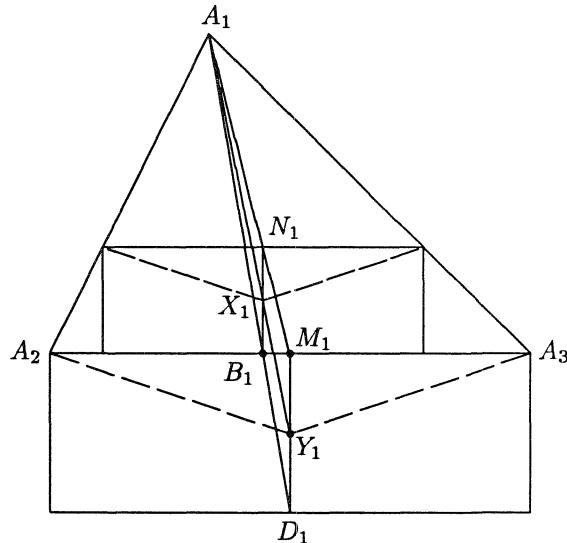
- (a) Prove that lines $A_iB_i, i = 1, 2, 3$, concur.
- (b) Prove that lines $A_iC_i, i = 1, 2, 3$, concur.

Solution by P. Penning, Delft, The Netherlands.

Multiply R_1 by such a factor with respect to A_1 that the side parallel to the base lies in A_2A_3 . The side of the rectangle then coincides with side A_2A_3 of the triangle. The perpendicular bisector N_1B_1 of R_1 transforms into the perpendicular bisector M_1D_1 of A_2A_3 . A point X_1 on N_1B_1 , determined by an arbitrary factor $\lambda = N_1X_1/N_1B_1$ ($0 \leq \lambda \leq 1$), transforms into Y_1 in the same relative position on M_1D_1 .

Note that $\Delta A_2A_3Y_1$ is isosceles with apex angle only dependent on r and λ . For the other two rectangles R_2 and R_3 similar points Y_2 and Y_3 are found, with similar isosceles triangles. It is known that the lines A_1Y_1, A_2Y_2, A_3Y_3 concur. [See the solution of *Crux* 1579, this issue! — Ed.] $\lambda = 1$ gives part (a) and $\lambda = 1/2$ part (b). But the result is true for any λ .

Also solved by JORDI DOU, Barcelona, Spain; ROLAND EDDY, Memorial University of Newfoundland, St. John's; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano,



Valladolid, Spain; JOHN RAUSEN, New York; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Several solvers, including the proposer, gave the more general result, including the “isosceles triangles” theorem mentioned above and in the solution of Crux 1579. The editor is sorry that two so similar problems were used so close together.

* * * *

1586. [1990: 267] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Let ABC be a triangle with angles $A \geq B \geq C$ and sides $a \geq b \geq c$, and let $A'B'C'$ be a triangle with sides

$$a' = a + \lambda, \quad b' = b + \lambda, \quad c' = c + \lambda$$

where λ is a positive constant. Prove that $A - C \geq A' - C'$ (i.e., $\Delta A'B'C'$ is in a sense “more equilateral” than ΔABC).

Solution by Jordi Dou, Barcelona, Spain.

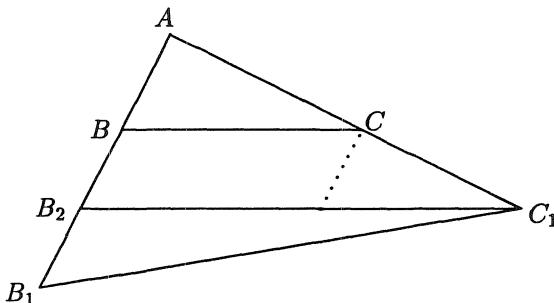


Figure 1

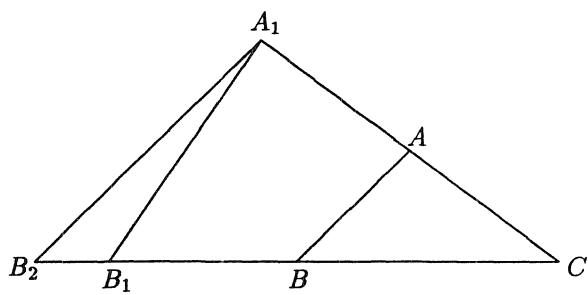


Figure 2

Let B_1 and C_1 lie on rays AB and AC respectively so that $BB_1 = CC_1 = \lambda$, and let B_2 lie on AB so that $C_1B_2 \parallel CB$. (See Figure 1.) Then

$$BB_2 = \frac{c}{b}\lambda \leq \lambda,$$

so B_2 is between B and B_1 , and

$$B_1C_1 \geq B_2C_1 = a + \frac{a}{b}\lambda \geq a + \lambda.$$

It follows that the sides $AB_1 = c + \lambda$ and $AC_1 = b + \lambda$ forming an angle equal to A determine a third side $B_1C_1 \geq a + \lambda$. Therefore the angle A' in $\Delta A'B'C'$ must be less than or equal to A . Analogously in Figure 2, if $A_1A = B_1B = \lambda$ and $A_1B_2 \parallel AB$, then B_1 will lie between B_2 and B , and

$$A_1B_1 \leq A_1B_2 = c + \frac{c}{b}\lambda \leq c + \lambda.$$

Thus the sides $B_1C = a + \lambda$ and $A_1C = b + \lambda$ forming an angle equal to C determine the third side $A_1B_1 \leq c + \lambda$, so angle C' in $\Delta A'B'C'$ must be larger than or equal to C . Therefore $A' \leq A$ and $C' \geq C$, and in particular $A' - C' \leq A - C$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

Klamkin, Seimiya, and the proposer also observed that $A' \leq A$ and $C' \geq C$ (which makes the problem much easier!).

Klamkin notes that the areas of the two triangles are related by

$$\sqrt{[A'B'C']} \geq \sqrt{[ABC]} + \frac{\lambda\sqrt[4]{3}}{2}.$$

Janous obtains the equivalent inequality

$$\frac{s(s-b)}{ab^2c} \geq \frac{(s+3t)(s-b+t)}{(a+2t)(b+2t)^2(c+2t)}$$

for any $t \geq 0$, where s is the semiperimeter and we always assume $a \geq b \geq c$. He then gets the special cases

$$\begin{aligned} s &\geq b \left(1 + \frac{2ac}{4ac + 9(a+c)b + 18b^2} \right), \quad (t = b) \\ \frac{s^2}{3Rr} &\geq \frac{(5s-3b)b}{(s+a-b)(s+c-b)}, \quad \left(t = \frac{s-b}{2} \right) \\ \frac{1}{16Rr} &\geq \frac{2a-b+2c}{(a+2c)(2a+c)(3a+b+3c)}, \quad (t = s-b) \end{aligned}$$

where R and r are the circumradius and inradius, respectively.

* * * *

1587*. [1990: 268] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

When all the diagonals of a certain convex n -gon are drawn, it is found that no three of them are concurrent at an interior point and that they divide the interior of the n -gon into a square number of regions. Find the possible values of n .

I. *Partial solution by J.A. McCallum, Medicine Hat, Alberta.*

It is shown in many books on elementary combinatorics (e.g., Ross Honsberger, *Mathematical Gems*, MAA, 1973, pp. 99–102) that the number R of regions, formed as indicated in the statement of the problem, is given by the formula

$$R = \binom{n}{4} + \binom{n-1}{2}. \tag{1}$$

Counting the case of the triangle (no diagonals) the early instances of a square number of areas are given in the following table:

<i>n</i>	<i>R</i>
3	1
4	4
6	25
17	2500

This is about as far as it is practical to go with pencil and paper. With a small desk computer one can verify that there are no other cases with small *n*.

II. *Comment by Richard K. Guy, University of Calgary.*

Problems in which a cubic or quartic polynomial is to be made square reduce to an elliptic curve. For example, (1) above can be seen (by Mordell's *Diophantine Equations*, p. 77) to have the same *rational* solutions as

$$Y^2 = X^3 - 732X + 27056,$$

whose points $(26, \pm 160)$, $(37, \pm 225)$, $(52, \pm 360)$ and $(82, \pm 720)$ correspond to the four known solutions. A theorem of Siegel states that there is only a finite number of integer points on such a curve. In theory, Baker's method gives a bound below which such points must lie, but this has usually been so astronomical that it is not helpful in establishing that all integer points have been found. However, Tzanakis and de Weger and others have recently developed powerful algorithms that can (after much hard work and computation) determine all such points. For example, in *J. Number Theory* 31 (1989), pp. 99–132, they determine all 22 integer points on the curve $y^2 = x^3 - 4x + 1$. See also the remarks in my article *The Ochoa curve* [1990: 65–69].

For this problem, it is unlikely that there are any solutions other than those given above. However, note that the integer points $(1162, \pm 39600)$ on the curve correspond to the “solution” $n = -1$, $R = 4 = 2^2$!

The above solutions were also found by C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California (except for $n = 17$); P. PENNING, Delft, The Netherlands; and the proposer.

Penning reports that there are no further solutions for $n < 1500$. Can someone settle this problem?

* * * *

1588. [1990: 268] *Proposed by D.M. Milošević, Pranjani, Yugoslavia.*

Show that

$$\sin B \sin C \leq 1 - \frac{a^2}{(b+c)^2},$$

where a, b, c are the sides of the triangle ABC .

Solution by Seung-Jin Bang, Seoul, Republic of Korea.

The required inequality is equivalent to

$$\frac{bc}{4R^2} \leq \frac{(b+c)^2 - a^2}{(b+c)^2},$$

or

$$R^2 \geq \frac{bc(b+c)^2}{4(a+b+c)(b+c-a)}. \quad (1)$$

Since

$$R^2 = \frac{a^2 b^2 c^2}{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}$$

(e.g., see [1990: 253]), we may prove

$$\frac{a^2 bc}{(a+b-c)(a-b+c)} \geq \frac{(b+c)^2}{4},$$

or

$$4a^2 bc \geq (b+c)^2(a^2 - (b-c)^2),$$

or

$$0 \geq (b-c)^2(a^2 - (b+c)^2),$$

which is true. This completes the proof. We have equality for $b = c$.

In fact, (1) shows that the inequality is a generalization of *Crux* 1457 [1990: 252], because

$$\frac{bc(b+c)^2}{4((b+c)^2 - a^2)} \geq \frac{b^2 c^2}{(b+c)^2 - a^2}.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; PETER HURTHIG, Columbia College, Burnaby, B.C.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; LJUBOMIR LJUBENOV, Stara Zagora, Bulgaria; BOB PRIELIPP, University of Wisconsin-Oshkosh; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

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YEAR-END WRAPUP

Here we are at the end of another year and another volume of *Crux*. So, as usual, this is an appropriate time to list some comments on the last year's issues.

First an abject apology by the editor for messing up several solutions of Marcin E. Kuczma! In Kuczma's solution of **1352** [1990: 268], several occurrences of f'_r and φ'_r were printed without the primes:

[1990: 269], lines 12 and 14 from bottom	$f_r(x)$ should be $f'_r(x)$
[1990: 269], line 8 from bottom	$\varphi_r(x)$ should be $\varphi'_r(x)$
[1990: 269], line 6 from bottom, right side	$\varphi_r(0)$ should be $\varphi'_r(0)$
[1990: 269], line 5 from bottom, right side	$\varphi_r(1/2)$ should be $\varphi'_r(1/2)$
[1990: 270], lines 2 and 5	$\varphi_r(1/2)$ should be $\varphi'_r(1/2)$

Then at the end of Kuczma's solution of **1510** [1991: 92] the two displayed equations should have started with a summation:

$$\sum \cot \alpha = \dots \quad \text{and} \quad \sum \cot(A/2) = \dots .$$

In his solution of **1513** [1991: 115], the definition of point C was omitted: it should be

$$C = (1, 1, \dots, 1).$$

And near the end of his solution of **1514** [1991: 117] a $\sqrt{r^2 + t^2}$ should have been $\sqrt{R^2 + t^2}$. For these mistakes and omissions the editor could blame the convenient "computer error", but the real reason was usually too-hasty proofreading!

Regarding **1512** [1991: 93] Kuczma's value of $C(r)$ given on [1991: 96] is only *one* value which works, not the *best* one (which almost certainly has no reasonable solution). The editor's comments at the end may not have been clear on this point.

Here are a few more comments on past problems.

1520 [1991: 125]. An equivalent problem appeared as problem M748(a) in the 1982 volume of *Kvant* (issue 6, page 21).

1537 [1991: 182]. John Mason of The Open University, Milton Keynes, England, writes (in reply to a question of J.A. McCallum [1991: 184]): "Rotating the square $ABDE$ and the line DPC about the point A , through 90° , yields the usual Ladders-in-the-alley or Wires-on-poles version of the harmonic mean".

1542 [1991: 208]. Francisco Bellot Rosado of Valladolid, Spain, suggests the (only slightly related) problem 3 of the 1987 National USSR Olympiad, Class X, as what Walther Janous may have recalled [1991: 217]. Janous himself has since sent in the rather more related inequality

$$|\cos x| + |\cos 2x| + |\cos 4x| + \dots + |\cos 2^n x| \geq n/2,$$

which was a problem in the 1981/82 Polish Olympiad. He gives the Bulgarian books

[1] J.B. Tabov and K.G. Bankov, *Mathematical Competitions All Around the World*, Sofia, 1988 (p. 50, problem 3.42), and

[2] *Mathematical Olympiads for Secondary Schools—1985*, edited by the Dept. of National Education, Sofia, 1986 (problem 62) as references. Also related is problem 63 of the latter book; it reads

$$\sum_{k=0}^{n-1} \left| \cos \frac{k\pi}{n} \right| \geq \frac{n}{2} .$$

1556 [1991: 246]. The editor's comment at the end, giving a further observation of the proposer K.R.S. Sastry, should have read "...yields a Heronian triangle of sides ..." (i.e., a triangle of integer sides and area). The given sides will also, of course, have to be positive. (This shows that the editor plays no favourites when it comes to whose solutions he messes up!)

1573 [1991: 281]. In the diagram, circle Ω is missing; readers are asked to draw it in themselves. The editor had compassed the arc for Ω in pencil, but it seems that the result didn't copy very well.

A late solution to **1530** was received from Cory Pye, student, Memorial University of Newfoundland.

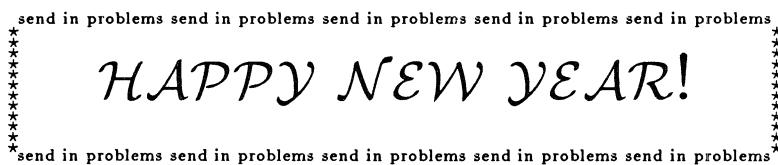
Thanks go to the following people for their help, in the form of opinions on problems, solutions, and articles, during 1991: *LEN BOS, CHARLES EDMUNDS, HIDETOSI FUKAGAWA, WALTHER JANOUS, JAMES P. JONES, CLARK KIMBERLING, MURRAY KLAMKIN, STANLEY RABINOWITZ, JONATHAN SCHAER, DAN SOKOLOWSKY, and EDWARD T.H. WANG*.

Of course this list should also include the members of *Crux*'s Editorial Board, newly formed this year. They have all been of great assistance in producing *Crux*. Still, two members might be singled out: *ANDY LIU*, who has taken over the Book Review section of *Crux*; and *CHRIS FISHER*, whose expert (and uncredited) interpretations of Jordi Dou's solutions to his problems **1486**, **1529**, and **1564** saved *Crux* readers from having to decipher the editor's versions!

Reserved for special mention is *RICHARD GUY*; not only did he take on the time-consuming task of setting up the format for producing *Crux* in LATEX, but he also lent his computer to produce the first six issues in 1991, and taught the editor how to use it!

Many thanks also to *JOANNE LONGWORTH*, who took over the task of keyboarding *Crux* into LATEX starting with the April issue, and has proven remarkably fast and reliable. Consequently *Crux* not only looks better than in past years but (it is hoped) will soon be produced somewhat closer to "on schedule" than before.

And so at the end of 1991, an unhappy or turbulent year in some countries where *Crux* is read, the editor wishes all of you a more peaceful 1992.



* * * * *

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