Mathematical Spectrum

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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

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From the Editor

It's About Time

I confess that I have never been able to get my head round the theory of relativity. A newly-published book by N. David Mermin is what I have needed. He taught the subject for many years to nonspecialist students at Cornell University in the USA. This volume, entitled *It's About Time*, with the subtitle *Understanding Einstein's Relativity*, was published in 2005, the centenary year of the publication of Einstein's theory of special relativity.

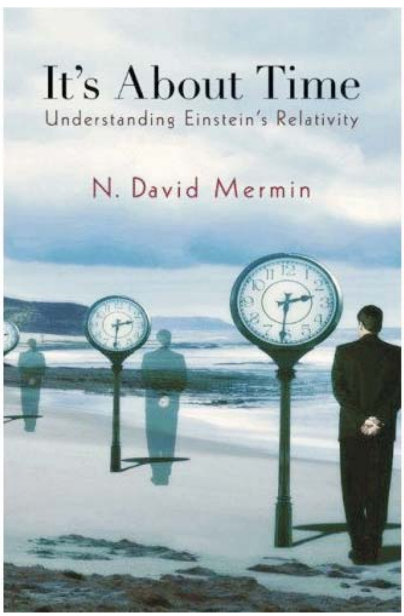
I could never grasp the underlying assumptions before reading this book. I am more at home in axiomatic mathematics like Euclidean geometry or group theory, which begins with a set of axioms or postulates and deduces things from them called theorems. What are the axioms of relativity theory? Mermin gives two.

The principle of relativity says that no phenomena have properties corresponding to the concept of absolute rest. In other words, all other things being equal, it doesn't matter how fast you are going if you are moving with fixed speed along a straight line.

The principle of the constancy of the velocity of light says that light in empty space moves with a velocity that is independent of the velocity of the body that emitted that light.

Using these, he brilliantly deduces, in a way even I can understand, how the rule for adding speeds u and v along the same straight line is not $w = u + v$ but

$$w = \frac{u + v}{1 + uv/c^2},$$



where c is the velocity of light. This makes it impossible to exceed the speed of light. He goes on to explain how two events can be simultaneous in one frame of reference, such as on a moving train, but not in another, such as standing on the track, and obtains the formula

$$T = \frac{Dv}{c^2}$$

for the difference in time between the two events in the track frame, where D is the distance along the track between the two events, v is the speed of the train, and c is our old friend the speed of light. Or, to turn it round, if two clocks are synchronized and separated by a distance D when at rest then, when they are moving at a speed v along the line joining them, the reading of the clock at the front lags behind the reading of the clock at the rear by this amount. The slowing-down factor of a clock moving with speed v is

$$\sqrt{1 - v^2/c^2}.$$

This is also the factor by which a stick moving at speed v in the direction of the line of the stick will shrink. As Einstein is famously quoted as saying 'At last it came to me that time was suspect'. It explains why sub-atomic particles decay much more slowly when accelerated in a particle accelerator; in their time, they are still decaying at the same rate.

One mind-blowing fact follows another. There is a *Doppler effect* in relation to time. For example, if a clock is moving towards you at speed v emitting a flash every second, then you will receive the signal every

$$\sqrt{\frac{1 - v/c}{1 + v/c}}$$

seconds; if it is moving away from you at speed v, then you will receive it every

$$\sqrt{\frac{1 + v/c}{1 - v/c}}$$

seconds.

My little grey cells were stretched beyond limit when I reached space–time geometry, $E = mc^2$, and the brief introduction to Einstein's theory of general relativity, but I am grateful to Mermin for showing me that relativity theory is not the closed book I thought it was.

Reference

1 N. D. Mermin, *It's About Time: Understanding Einstein's Relativity* (Princeton University Press, 2005).

What is

$$\sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}} \quad ?$$

10 Shahid Azam Lane,
Makki Abad Avenue, Sirjan, Iran

Abbas Roohol Amini

Construction of a Golden Rectangle with a Given Area

LYNDON O. BARTON

Introduction

There are many textbooks in geometry where ruler-and-compass constructions are described to convert one polygon to another of equal area, such as a rectangle to a square, or a pentagon to a quadrilateral, and so on. However, in the case of the conversion of a square to a golden rectangle, this construction has not been encountered in textbooks and other literature to date. The *golden rectangle* is a rectangle whose length to width ratio is $(1+\sqrt{5})/2$ to 1. We shall describe such a construction in this article. The construction will enable us to convert any given rectangle into a golden rectangle of equal area by first converting the given rectangle to a square (see reference 1), and in turn converting the square to the required golden rectangle.

Procedure

We begin with the square ABCD in figure 1. We want to construct, using a ruler and compass, a golden rectangle whose area is equal to that of the square. We shall say that the square has unit area.

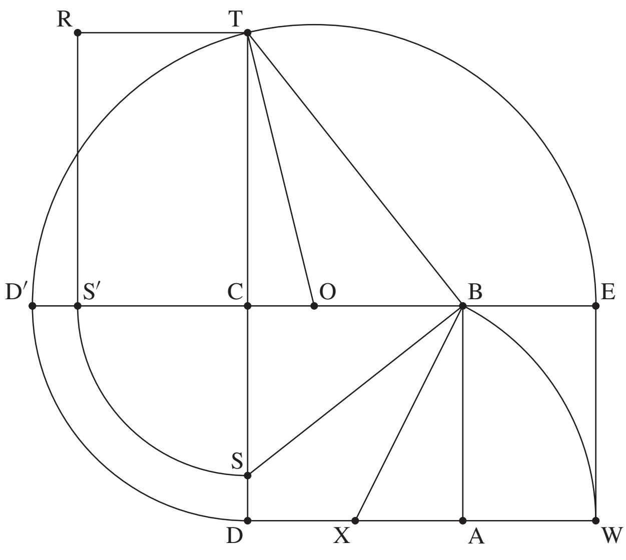


Figure 1

Step 1 Find a length equal to $(1+\sqrt{5})/2$ units.

(a) Locate the midpoint of segment DA and call this point X.

(b) With X as the centre and XB as the radius, describe an arc from B to meet DA extended at a point W.

The distance XW is $\sqrt{5}/2$ units. Consequently, the distance DW is $(1+\sqrt{5})/2$ units.

Step 2 From W, erect a perpendicular to DW to meet CB extended at a point E and complete the rectangle WECD.

Step 3 Find the length of the golden rectangle.

(a) With C the centre and CD the radius, describe an arc from D to meet EC extended at a point D'.

(b) Locate the midpoint of $\mathrm{D}'\mathrm{E}$ and call this point O.

(c) With O as the centre and OD' or OE as the radius, describe a semicircle on $\mathrm{D}'\mathrm{E}$ and extend DC to meet this semicircle at a point T.

The segment CT thus formed will be the longer side of the required golden rectangle.

Step 4 Find the width of the golden rectangle.

(a) Join T to B with a segment TB, and from B draw a perpendicular to TB to meet CD at S and form the right-angled triangle TBS.

(b) With the centre at C and the radius CS, describe an arc from S to meet CD' at S'.

The segment CS' thus formed is the width of the required golden rectangle.

Step 5 Finally, erect perpendiculars from S' on CD' and from T on CT to meet at a point R and complete the required golden rectangle CTRS'.

Analysis

Referring to figure 1, let points O and T be joined with segment OT and let $x = (1+\sqrt{5})/2$. From this construction, we obtain

$$\begin{aligned}\mathrm{OT} &= \frac{\mathrm{CE}+\mathrm{CD}'}{2}\\ &= \frac{1+x}{2}\end{aligned}$$

and

$$\mathrm{CT}^2 = \mathrm{OT}^2 - \mathrm{OC}^2,$$

where

$$\mathrm{OC} = \mathrm{CE} - \mathrm{OT} = x - \mathrm{OT}.$$

Then we have

$$\begin{aligned}\mathrm{CT}^2 &= (\mathrm{OT} + x - \mathrm{OT})(\mathrm{OT} - x + \mathrm{OT})\\ &= x(2\mathrm{OT} - x)\\ &= x\left(2\frac{1+x}{2} - x\right)\\ &= x(1+x) - x^2\\ &= x + x^2 - x^2\\ &= x,\end{aligned}$$

so that

$$\mathrm{CT} = \sqrt{x}.$$

Also,

$$\begin{aligned}\mathrm{BT}^2 &= \mathrm{CB}^2 + \mathrm{CT}^2 = 1 + x,\\ \mathrm{CS}' &= \mathrm{CS} = \mathrm{ST} - \mathrm{CT},\end{aligned}$$

where

$$\begin{aligned}\mathrm{ST} &= \mathrm{BT}\sec \mathrm{BTC}\\ &= \mathrm{BT}\frac{\mathrm{BT}}{\mathrm{CT}}\\ &= \frac{\mathrm{BT}^2}{\mathrm{CT}}\\ &= \frac{1+x}{\sqrt{x}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{CS}' &= \frac{1+x}{\sqrt{x}} - \sqrt{x}\\ &= \frac{1}{\sqrt{x}}.\end{aligned}$$

Hence,

$$\frac{\mathrm{CT}}{\mathrm{CS}'} = x \qquad \text{or} \qquad \frac{1+\sqrt{5}}{2} \quad \text{(the golden ratio)},$$

so that CTRS' is a golden rectangle with the same area as the given square ABCD.

Reference

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Lyndon O. Barton *teaches mathematics at Delaware State University, Dover, DE, USA. He is the author of the textbook 'Mechanism Analysis', as well as various articles on simplified graphical and analytical techniques for solving problems in mechanical engineering and mathematics. Apart from his mathematics and engineering pursuits, Mr Barton is an artist and has held one-man exhibitions of his work on Landmarks of Newark, Delaware, as well as Landmarks of Buxton, Guyana.*

Cluster Analysis and the Planetary Status of Pluto

A. TAN, R. RAMACHANDRAN and T. X. ZHANG

1. Introduction

Since its discovery in 1930 until fairly recently, Pluto had enjoyed the status of a planet. It was demoted from that status at the International Astronomical Union (IAU) conference in August 2006. In its final resolution, the IAU redefined a planet in the Solar System as a celestial body that is in orbit around the Sun, has sufficient mass so that it assumes a hydrostatic equilibrium (i.e. nearly round) shape, and has cleared the neighbourhood around its orbit (see reference 1). The first criterion eliminates all satellites from contention, regardless of their size, and the second criterion eliminates the smaller objects orbiting the Sun, mostly in the asteroid belt. It is the new third criterion that drove the final nail in the coffin of Pluto's membership in the planetary society.

Pluto's demise seems to have been forthcoming for quite some time. For many, Pluto was just too small to be a planet. It has less than 4% of the mass of the next smallest planet (Mercury) and less than 7 millionth of that of the largest planet (Jupiter). Others believed that Pluto was once a satellite of Neptune. The discovery of Pluto's own satellites seemed to contradict that. However, Pluto's ultimate downfall began with the advent of charge-coupled device (CCD) cameras in the early 1990s, when other objects in the vicinity of Pluto's orbit were discovered, some of whose sizes rivalled that of Pluto. The discovery of new members of this family, called *Kuiper belt objects*, (to which Pluto is now believed to belong) continues today, and one object (2003UB313) is actually reported to be larger than Pluto itself. It was this state of affairs that prompted the IAU to reclassify Pluto.

While the IAU decision seemed to be fair and justifiable, the redefinition of a planet, by which the IAU executed Pluto's demotion, still lacks any quantitative or mathematical basis. In this article, we compare the individual planets using cluster analysis to arrive at a quantitative assessment regarding the planetary status of Pluto.

2. Cluster analysis

Cluster analysis consists of the classification of a group of objects according to certain similarities and criteria (see references 2–4). Cluster analysis has been applied to all fields of empirical science and has led to important discoveries. For example, classification of animals by Darwin resulted in his theory of evolution. Similarly, classification of elements by Mendeleyev gave rise to the periodic table and the prediction and discovery of missing elements. More recently, the classification of stars by the Russell–Hertzsprung diagram led to the theory of stellar evolution.

The basic scheme of cluster analysis consists of the following steps.

Step 1 Suppose that there are n *objects* which are to be grouped according to m *properties*, which can be *quantitative* or *qualitative* in nature. For the time being, we restrict ourselves to quantitative properties. The quantitative values of the properties for each object are assigned and an $n \times m$ data matrix is constructed.

Step 2 The *similarity* or *dissimilarity* between any two objects is determined by some predetermined schemes. Numerous different schemes have been devised (see reference 5). For instance, the similarity between two objects can be determined by any of the *resemblance coefficients* found in the literature. Likewise, the dissimilarity between the objects can be determined by any of the *distance coefficients* that have been widely used (see reference 5). One of the most common distance coefficients between two objects denoted by the variables x and y is the *Euclidean distance* in m dimensions:

$$d = \sqrt{\sum_{i=1}^{m}(x_i - y_i)^2}.$$

The smaller the value of d, the greater the similarity between the two objects. For the resemblance coefficients, we use *Pearson's correlation coefficient*, where

$$r = \frac{\sum_{i=1}^{m}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{m}(x_i - \bar{x})^2 \sum_{i=1}^{m}(y_i - \bar{y})^2}},$$

with

$$\bar{x} = \frac{\sum_{i=1}^{m} x_i}{m} \quad \text{and} \quad \bar{y} = \frac{\sum_{i=1}^{m} y_i}{m}.$$

Here, the larger the value of r, the greater the similarity between the two objects.

Neither the Euclidean distance nor Pearson's correlation coefficient give satisfactory results when the range of the variables is large. For variables having nonnegative values, a particularly simple resemblance coefficient is given by

$$r = \frac{1}{m}\sum_{i=1}^{m}\frac{\min(x_i, y_i)}{\max(x_i, y_i)}. \tag{1}$$

The value of r ranges from 0 to a maximum value of 1 (for total resemblance). The complement of r from (1) is the *Wave–Hedges coefficient*

$$d = 1 - \frac{1}{m}\sum_{i=1}^{m}\frac{\min(x_i, y_i)}{\max(x_i, y_i)}, \tag{2}$$

which serves as the distance coefficient (see reference 5).

Step 3 The resemblance and distance coefficients are calculated between every two objects in the group. The two objects having the greatest value of r or the least value of d are then deemed the *most similar*. They are then joined to form a single *new object* in the group, whereby themselves disappearing in the process. The number of objects in the group is thereby reduced by one. The new object acquires a name and assumes some mean values for the variables.

Amongst the various means between two variables x_i and y_i, the three most common means are the *arithmetic mean* (*AM*), the *geometric mean* (*GM*), and the *harmonic mean* (*HM*), which are given by

$$
\begin{aligned}
\mathrm{AM} &= \frac{x_i + y_i}{2}, \\
\mathrm{GM} &= \sqrt{x_i y_i}, \\
\mathrm{HM} &= \frac{2x_i y_i}{x_i + y_i}.
\end{aligned} \tag{3}
$$

Two other means, known as the *quadratic mean* (*QM*), also called the *root mean square*, and the *harmonic root mean* (*HRM*) are also used (see reference 6), where

$$
\begin{aligned}
\mathrm{QM} &= \sqrt{\frac{x_i^2 + y_i^2}{2}}, \\
\mathrm{HRM} &= \sqrt{\frac{2x_i^2 y_i^2}{x_i^2 + y_i^2}}.
\end{aligned}
$$

For variables having a wide range of values, the GM is the most appropriate. It is skewed towards the smaller variable from the middle point, i.e. the AM. The GM lies between the AM and HM; it also lies between the QM and HRM. Also,

$$\mathrm{GM} = \sqrt{\mathrm{AM} \times \mathrm{HM}} = \sqrt{\mathrm{QM} \times \mathrm{HRM}}.$$

Step 4 Steps 2 and 3 are repeated, and each time the number of objects is reduced by one. The process is continued until all the objects have been clustered into one. A tree diagram called a *dendrogram* is constructed to illustrate the entire clustering process.

3. Physical properties of the Planets

We consider for the quantitative variables the following physical properties of the planets.

Volumetric radius Under hydrostatic equilibrium, a rotating planet acquires the shape of an oblate spheroid with volume

$$V = \tfrac{4}{3}\pi r_{\mathrm{e}}^2 r_{\mathrm{p}}, \tag{4}$$

where r_{e} and r_{p} are, respectively, the equatorial and polar radii of the planet. The volumetric radius, r_{v}, of the planet is the radius of a sphere of the same volume as that of the oblate spheroid, i.e.

$$V = \tfrac{4}{3}\pi r_{\mathrm{v}}^3. \tag{5}$$

Equating (4) and (5), we obtain

$$r_{\mathrm{v}} = \sqrt[3]{r_{\mathrm{e}}^2 r_{\mathrm{p}}}.$$

Density The mean density of the planet is its mass divided by its volume, i.e.

$$\rho = \frac{M}{V}.$$

Table 1 Physical properties of the planets.

	Volumetic radius (km)	Density (g/cc)	Mass (10^{24}kg)	Equatorial gravity (m/s^2)	Equatorial escape velocity (km/s)
Mercury	2 440	5.43	0.33	3.70	4.4
Venus	6 052	5.20	4.87	8.87	10.4
Earth	6 371	5.51	5.97	9.78	11.2
Mars	3 390	3.93	0.642	3.69	5.0
Jupiter	69 910	1.33	1 900	23.12	59.5
Saturn	58 230	0.69	569	8.96	35.5
Uranus	25 360	1.32	86.8	8.69	21.3
Neptune	24 620	1.64	102	11.00	23.5
Pluto	1 137	2.10	0.013	0.66	1.1

Mass The mass of the planet is the total quantity of matter in it. The presence of a satellite furnishes an accurate determination of the mass of the planet. If P is the period of the satellite and a its semi major axis then, by Kepler's third law, we obtain

$$M = \frac{4\pi^2}{GP^2} a^3,$$

where G is the universal gravitational constant.

Equatorial gravity The acceleration of gravity at the equator of a planet is given by Newton's law of gravitation:

$$g_e = \frac{GM}{r_e^2}.$$

Equatorial escape velocity The velocity of escape from the equator of the planet can be obtained from energy considerations at the surface and at infinity as follows:

$$v_e = \sqrt{\frac{2GM}{r_e}}.$$

The values of the five physical quantities of each planet (including Pluto) are taken from references 7 and 8 and entered in table 1.

4. Results

We begin with the nine classical planets, including Pluto. The resemblance coefficient, r, and the distance coefficient, d, between all planets are calculated using (1) and (2) respectively. Upon the union of the two planets having the greatest value of r or the smallest value of d, the new planet assumes the mean values of the variables as given by (3). The results, as represented by a dendrogram (see figure 1), are summarized below.

- Among all the planets, Venus and Earth are the most similar, having a resemblance coefficient r of 0.909 and distance coefficient d of only 0.091. They are the first two planets to combine to form Venus–Earth.

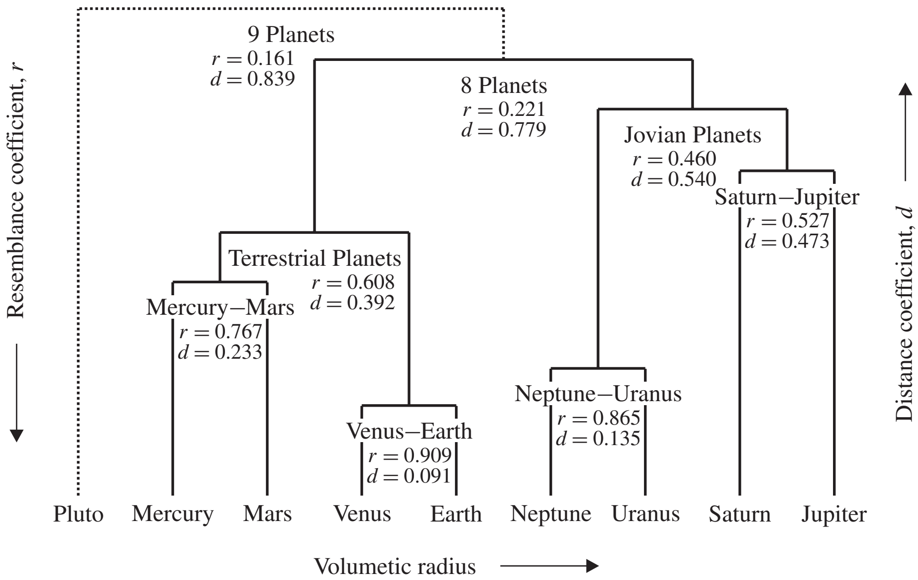

Figure 1 Dendrogram of the planets.

- In the second round, Neptune and Uranus are the most similar planets, having $r = 0.865$ and $d = 0.135$. They join to form Neptune–Uranus.
- In the third round, Mercury and Mars (with $r = 0.767$ and $d = 0.233$) join to form Mercury–Mars.
- In the fourth round, Mercury–Mars joins Venus–Earth (with $r = 0.608$ and $d = 0.392$) to form the *Terrestrial Planets*.
- In the fifth round, Saturn joins Jupiter (with $r = 0.527$ and $d = 0.473$) to form Saturn–Jupiter.
- In the sixth round, Neptune–Uranus and Saturn–Jupiter (with $r = 0.460$ and $d = 0.540$) join to form the *Jovian Planets*. This is the first occasion where d exceeds r. In other words, the dissimilarity exceeds the resemblance.
- In the next round, the Terrestrial Planets and the Jovian Planets are made to join. But r drops to 0.221 while d rises to 0.779. At this stage, the eight legitimate planets have joined to form the *Eight Planets* cluster, leaving only Pluto out.
- In the final round, Pluto is allowed to join the Eight Planets to form the nine classical planets. However, the value of r drops to 0.161, while the value of d rises to 0.839.

We also make the following observations.

- There are four pairs of similar planets (Venus and Earth, Neptune and Uranus, Mercury and Mars, and Saturn and Jupiter); Pluto is the only odd planet out. Among the four pairs, Venus and Earth are the most similar, followed by Neptune and Uranus. Saturn and Jupiter are the least similar, but they still have a resemblance coefficient greater than 0.5.

- There are two major groups of planets, each containing four members: the Terrestrial Planets are relatively small having high densities and the Jovian Planets are large having low densities. The two groups are, in fact, more different than similar (the resemblance coefficient is only 0.221).

- Pluto fails to cluster with any other planet in the Solar System. Its resemblance coefficient with the rest of the Eight Planets was only 0.161. Thus, the IAU decision to demote Pluto from its planetary status is not unjustified from a quantitative point of view.

5. Qualitative attributes and similarity

The discussion of cluster analysis remains incomplete unless we mention that, besides quantitative variables, there are qualitative attributes on which the similarity or dissimilarity between two objects can be based. Once again, many different schemes have been devised to study this problem (see reference 5). In this study, we have devised our own scheme to find the *Similarity coefficient for qualitative attributes(s)*. We give two examples to illustrate this scheme.

Example 1 In table 2, the similarity coefficient, s, is evaluated for qualitative attributes between Venus and Earth. There are two types of attributes. The first seven attributes are *mutual attributes* (if the answer is *Yes* then score 1, if the answer is *No* then score 0). The last six attributes are *individual attributes* (if the answers are *Yes* and *Yes* then score 1, if the answers are either *Yes* and *No* or *No* and *Yes* then score 0). We divide the total score by the total number of attributes to obtain the similarity coefficient, s. If the answers are *No* and *No* then that attribute is not applicable and therefore not considered. Interestingly, the similarity coefficient, s, between Venus and Earth is only 0.387, compared to their resemblance coefficient of 0.909. Thus, Venus and Earth are quite dissimilar as far as qualitative attributes are concerned.

Table 2 Similarity coefficient between Venus and Earth.

Qualitative attribute	Venus	Earth	Score	Similarity coefficient
Whether neighbours	Yes		1	
Similar rotation periods	No		0	
Similar axial tilt	No		0	
Similar orbital plane	Yes		1	
Similar atmospheric pressure	No		0	
Similar atmospheric composition	No		0	
Similar cloud composition	No		0	0.387
Whether terrestrial planet	Yes	Yes	1	
Whether rocky planet	Yes	Yes	1	
Has oceans	No	Yes	0	
Has clouds	Yes	Yes	1	
Has magnetic field	No	Yes	0	
Has natural satellite	No	Yes	0	

Table 3 Similarity coefficient between Neptune and Uranus.

Qualitative attribute	Neptune	Uranus	Score	Similarity coefficient
Whether neighbours	Yes		1	
Similar rotation periods	Yes		1	
Similar axial tilt	No		0	
Similar orbital plane	Yes		1	
Similar oblateness	Yes		1	
Similar atmospheric composition	Yes		1	
Discovered through telescope	Yes	Yes	1	
Whether Jovian Planet	Yes	Yes	1	0.867
Whether Gas Planet	Yes	Yes	1	
Has strong magnetic field	Yes	Yes	1	
Has smooth appearance	Yes	Yes	1	
Has great dark spot	Yes	No	0	
Has many satellites	Yes	Yes	1	
Has few rings	Yes	Yes	1	
Large tilt between magnetic and rotational axes	Yes	Yes	1	

Example 2 The similarity coefficient between Neptune and Uranus is evaluated in table 3. Interestingly, the two planets are very similar, with a similarity coefficient of $s = 0.867$, which is more than twice that between Venus and Earth.

In summary, whereas Venus and Earth are the most similar when quantitative attributes are considered, Neptune and Uranus are the most similar when qualitative attributes are considered.

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A. Tan *is a Professor of Physics at Alabama Agricultural and Mechanical University. He has a special interest in Applied Mathematics and has published numerous articles in Mathematical Spectrum.*

R. Ramachandran *is a Research Scientist at The University of Alabama in Huntsville, who specializes in Statistical Methods.*

T. X. Zhang *is an Assistant Professor of Physics at Alabama Agricultural and Mechanical University, with specialization in Astrophysics and Space Science.*

Truncatable Semi-Primes

SHYAM SUNDER GUPTA

Introduction

Truncatable semi-primes can be categorised as right-truncatable semi-primes and left-truncatable semi-primes in a similar way as truncatable primes are defined in reference 1. A number N is defined to be a *right-truncatable semi-prime* if N and all numbers obtained by successively deleting its right-most digits are *semi-prime* (i.e. numbers with only two, possibly equal, prime factors). For example, the semi-prime 933 593 is a right-truncatable semi-prime because 933 593 and each of its right-truncations, 93 359, 9 335, 933, 93, and 9, are all semi-primes.

Similarly, a *left-truncatable semi-prime* is defined as a zero-free number N such that N and all numbers obtained by successively deleting the left-most digit are semi-prime. For example, the semi-prime 2 641 339 is a left-truncatable semi-prime because 2 641 339 and each of its left truncations, 641 339, 41 339, 1 339, 339, 39, and 9, are all semi-primes.

Right-truncatable semi-primes

If N is a d-digit right-truncatable semi-prime, then the left-most $d-1, d-2, d-3, \ldots, d-(d-1)$ digits of N also form right-truncatable semi-primes. Starting with the single-digit semi-primes, i.e. 4, 6, and 9, digits can be added onto the right-hand side of these and then tested to see whether it is a semi-prime. In this way, larger and larger right-truncatable semi-primes can be obtained.

The first few right-truncatable semi-primes are

4, 6, 9, 46, 49, 62, 65, 69, 91, 93, 94, 95, 466, 469,
493, 497, 622, 623, 629, 655, 694, 695, 697, 698, 699,
913, 914, 917, 933, 934, 939, 943, 949, 951, 955, 958, 959.

Based on computer computations of all right-truncatable semi-primes, the largest right-truncatable semi-prime consists of 38 digits and is

95 861 957 783 594 714 393 831 931 415 189 937 897.

Two more 38-digit right-truncatable semi-primes are

93 359 393 537 779 942 973 989 331 953 313 839 313

and

95 861 957 783 594 714 393 831 931 415 118 711 938.

Table 1 The number, R_d, of right-truncatable semi-primes with d digits.

d	R_d	d	R_d	d	R_d	d	R_d	d	R_d
1	3	9	1 253	17	4 683	25	1 083	33	49
2	9	10	1 773	18	4 369	26	787	34	26
3	26	11	2 396	19	4 017	27	542	35	17
4	49	12	3 044	20	3 554	28	393	36	13
5	120	13	3 677	21	2 837	29	268	37	7
6	248	14	4 243	22	2 324	30	161	38	3
7	449	15	4 667	23	1 858	31	110		
8	794	16	4 726	24	1 430	32	68		

There are no other right-truncatable semi-primes with 38 or more digits. If the number of right-truncatable semi-primes with d digits is denoted by R_d, then $R_d = 0$ for $d > 38$. The values of R_d for $d = 1, 2, \dots, 38$ are listed in table 1. Other important observations are as follows.

(i) The total number of right-truncatable semi-primes including single-digit semi-primes, i.e. 4, 6, and 9, is 56 076. Out of these, the numbers of right-truncatable semi-primes starting with 4, 6, and 9 are 13 343, 19 773, and 22 960 respectively.

(ii) As can be seen from table 1, the 16-digit right-truncatable semi-primes, 4 726 in all, are the most numerous as compared to any other digit right-truncatable semi-primes.

It can also be noted that the number of right-truncatable semi-primes increases from $d = 1$ to $d = 15$, reaches a maximum for $d = 16$, and then decreases from $d = 16$ to $d = 38$.

Left-truncatable semi-primes

If N is a d-digit left-truncatable semi-prime, then the right-most $d-1, d-2, d-3, \dots, d-(d-1)$ digits of N also form left-truncatable semi-primes. Starting with the single-digit semi-primes, 4, 6, and 9, digits can be added onto the left-hand side of these and then tested to see whether it is a semi-prime. This process is repeated recursively to obtain larger and larger left-truncatable semi-primes.

The first few left-truncatable semi-primes are

$$4, 6, 9, 14, 26, 34, 39, 46, 49, 69, 74, 86, 94, 134, 146,$$
$$169, 194, 214, 226, 249, 274, 314, 326, 334, 339, 346, 386, 394.$$

Based on computer computations, we find that there are large numbers of left-truncatable semi-primes, in fact much more than right-truncatable semi-primes. The largest right-truncatable semi-prime has 38 digits, whereas it is not yet certain as to how many digits the largest

Table 2 The number, L_d, of left-truncatable semi-primes with d digits.

d	L_d	d	L_d	d	L_d
1	3	7	2 168	13	421 719
2	10	8	5 642	14	893 161
3	40	9	14 575	15	1 764 898
4	129	10	36 391	16	3 336 037
5	335	11	87 579	17	6 057 204
6	837	12	199 073	18	10 562 466

Table 3 Numbers of odd and even semi-primes with d digits.

d	Number of even semi-primes	Number of odd semi-primes
1	2	1
2	13	18
3	80	185
4	574	1 752
5	4 464	16 289
6	36 405	150 252
7	306 975	1 387 314
8	2 652 621	12 870 313
9	23 354 733	120 006 545
10	208 598 356	1 124 389 551
11	1 884 700 355	10 581 513 544
12	17 188 481 564	99 977 545 272

left-truncatable semi-prime will have, if indeed there is a largest. An example of a large left-truncatable semi-prime is

$$95\,289\,564\,275\,756\,333\,917\,136\,664\,351\,291\,247\,738\,715\,994\,261\,177\,867\,513\,112\,112\,641\,339,$$

which has 68 digits.

Let us denote the number of left-truncatable semi-primes with d digits by L_d. The values of L_d for $d = 1, 2, \dots, 18$ are listed in table 2. It can be seen from table 2 that L_d increases up to $d = 18$. Other observations are as follows.

(i) The total number of left-truncatable semi-primes ending in 4 and 6 are, respectively, 2 680 and 1 866. The largest left-truncatable semi-primes ending in 4 and 6 are, respectively,

$$16\,249\,539\,484\,927\,563\,367\,274 \quad \text{(with 23 digits)}$$

and

$$984\,248\,439\,959\,634\,986\,762\,446 \quad \text{(with 24 digits).}$$

The total number of left-truncatable semi-primes is not yet known.

(ii) Left-truncatable semi-primes ending in 4 and 6 are much fewer than those ending in 9. The obvious reason for this is the fact that left-truncatable semi-primes ending in 4 or 6 are even semi-primes whereas those ending in 9 are odd semi-primes. The numbers of even semi-primes are much fewer than the number of odd semi-primes, as can be seen from table 3, where the number of even and odd semi-primes with d digits are given for $d = 1, \ldots, 12$.

(iii) On comparing the numbers of right- and left-truncatable semi-primes, we find that $R_d < L_d$, for $d > 1$. The main reason for this is the fact that, to obtain a new left-truncatable semi-prime, all the possibilities of adding any digit from 1 to 9 on the left-hand side are valid, whereas for right-truncatable semi-primes, the possibilities of adding 2, 4, 5, 6, and 8 are limited as follows.

(a) Addition of the digit 5 will require that the number, after dividing by 5, must yield a prime. So this possibility is limited.

(b) If the right-most digit is even then digits 4 and 8 cannot be added to make it a right-truncatable semi-prime, as it will have more than two prime factors.

(c) If the right-most digit is odd, then digits 2 and 6 cannot be added for the same reason as in (b).

Conjecture 1 *The number of left-truncatable semi-primes is finite.*

It is an interesting challenge to compute all left-truncatable semi-primes and find the largest left-truncatable semi-prime.

Conjecture 2 *The number, L_d, of left-truncatable semi-primes with d digits increases as d increases up to a maximum value, $d_{\max}$, and then decreases as d increases further.*

We have already seen that the corresponding results for right-truncatable semi-primes is true, where $d_{\max} = 16$.

It is already known that there are 83 right-truncatable primes and 4 260 left-truncatable primes (see reference 1). Also, $d_{\max} = 4$ for right-truncatable primes and $d_{\max} = 9$ for left-truncatable primes (see reference 1).

Reference

1 E. W. Weisstein, Truncatable Prime, From *Mathworld* – A Wolfram Web Resource (http://mathworld.wolfram.com/TruncatablePrime.html).

Shyam Sunder Gupta *joined the Indian Railways in 1983 and presently works as Chief Bridge Engineer of the North Central Railway in Allahabad, India. His main hobby is the theory of numbers and many of his contributions have been published and are also available at http://www.shyamsundergupta.com/.*

Prove that the product of four consecutive integers plus 1 is a perfect square.

10 Shahid Azam Lane, **Abbas Roohol Amini**
Makki Abad Avenue, Sirjan, Iran

The First Indeterminate Diophantine Problem

DIMITRIS SARDELIS and THEODOROS VALAHAS

Introduction

This article deals with the first indeterminate problem encountered in the work *Arithmetica*, which was written by Diophantus in around 250 AD. We present Diophantus's particular solution to one of the problems from *Arithmetica*, where one of the two unknowns is set beforehand as an example. We then give the general solution. We find that the indeterminate character of this problem derives itself from the prime factorization of natural numbers.

Arithmetica, Book I, Problem 14

It is required to find two numbers so that their product and their sum form a given ratio.

Diophantus's solution

One of the numbers must be greater than the given ratio. Let their product and their sum have a ratio of 3. Let also one of the numbers be x and the other, according to the restriction, greater than 3, say 12. Their product will then be $12x$ and their sum $x+12$. Therefore, $12x$ is three times $x+12$, i.e. $3(x+12)=12x$. Hence, $x=4$. So one of the numbers is 4, the other is 12, and the problem is solved. (Translation from the original; see reference 1.)

The general solution

Let the positive integers x, y, and n satisfy the equation

$$\frac{xy}{x+y}=n, \tag{1}$$

where n is given. If $x=y$ then $x=y=2n$ is the solution to (1), so we suppose that $x<y$. Suppose that x and y differ from n by d_1 and d_2 respectively, i.e.

$$x=n+d_1 \quad \text{and} \quad y=n+d_2. \tag{2}$$

Then we obtain

$$(n+d_1)y=n(x+y) \quad \text{and} \quad (n+d_2)x=n(x+y),$$

so that

$$d_1y=nx \quad \text{and} \quad d_2x=ny.$$

Hence, both d_1 and d_2 must be positive, i.e. each of x and y must be greater than n. Also, $n^2=d_1d_2$. Therefore, the number of distinct solutions is the same as the number of all the possible representations of n^2 as a product of two positive integers, d_1 and d_2 with $d_1 < d_2$. More specifically, let n have the prime factorization form

$$n = p^a q^b r^c \cdots ,$$

where $p, q, r, \ldots$ are distinct primes and $a, b, c, \ldots$ are positive integers. Then the number of divisors of n^2 is

$$d(n^2) = (2a + 1)(2b + 1)(2c + 1) \cdots ,$$

and there are $\frac{1}{2}(d(n^2) - 1)$ ways of writing n^2 as a product of two positive integers, d_1 and d_2 with $d_1 < d_2$, and as many solutions to (1) given by (2). In particular, if n is prime then $d_1 = 1$ and $d_2 = n^2$, and we obtain the following unique solution:

$$x = n + 1 \quad \text{and} \quad y = n + n^2.$$

Examples

In table 1, we present some numerical examples.

A closing remark

It is a remarkable fact that the multiplicity of solutions of this simple problem depends on the concept of the prime factorization of positive integers. The first clear statement of this was presented by Euclid (*Elements*, Proposition 14, Book IX), and its proof was given by Gauss some twenty centuries after Diophantus.

Reference

1 P. Tannery, *Diophanti Alexandrini Opera omnia cum graecis commentariis* (Leipzig, 1893–1895).

Dimitris Sardelis *is Professor of Mathematics and Physics at The American College of Greece.*

Theodoros Valahas *is the head of the Mathematics and Natural Sciences Department of The American College of Greece.*

Table 1

n	$n^2 = d_1 d_2$	$x = n + d_1$	$y = n + d_2$
2	$1 \cdot 2^2$	$2 + 1 = 3$	$2 + 2^2 = 6$
3	$1 \cdot 3^2$	$3 + 1 = 4$	$3 + 3^2 = 12$
4	$1 \cdot 2^4$	$4 + 1 = 5$	$4 + 2^2 = 20$
	$1 \cdot 2^3$	$4 + 2 = 6$	$4 + 2^3 = 12$
6	$1 \cdot (2 \cdot 3)^2$	$6 + 1 = 7$	$6 + 2^2 \cdot 3^2 = 42$
	$2 \cdot (2 \cdot 3^2)$	$6 + 2 = 8$	$6 + 2 \cdot 3^2 = 42$
	$3 \cdot (2^2 \cdot 3)$	$6 + 3 = 9$	$6 + 2^2 \cdot 3 = 18$
	$2^2 \cdot 3^2$	$6 + 2^2 = 10$	$6 + 3^2 = 15$
8	$1 \cdot 2^6$	$8 + 1 = 9$	$8 + 2^6 = 72$
	$2 \cdot 2^5$	$8 + 2 = 10$	$8 + 2^5 = 40$
	$2^2 \cdot 2^4$	$8 + 2^2 = 12$	$8 + 2^4 = 24$

The Quadratic Equation as Solved by Persian Mathematicans of the Middle Ages

MOHAMED TEYMOUR and THOMAS J. OSLER

Today we can write down a quadratic equation in the form

$$ax^2 + bx + c = 0,$$

and solve it using the technique of *completing the square*, i.e.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The algebraic techniques we take for granted today are the result of a long struggle of over 2000 years of thought beginning with the Babylonian, Greek, and Arab mathematicians. We will look at the solution to the quadratic equation as viewed by two Persian mathematicians, Al-Khwarazmi (777–850) and Omar Khayyam (1043–1123).

Before showing this solution, we need to understand the restrictions under which these mathematicians worked. First, the condensed mathematical symbolism as seen in the equation $x^2 = 3x + 4$ was unknown to them. They would have expressed this equation in literal terms as follows.

> If three times an unknown added to 4 is equal to the square of that unknown, what is the value of the unknown?

It is remarkable to reflect that the invention of our condensed symbols is only 500 years old. Before the 16th century, mathematicians struggled with these literal expressions. Like the invention of the wheel, shorthand symbolism seems obvious today, but history proves otherwise. Even the ancient Greeks, for all their sophistication, missed this essential ‘wheel’ on which all modern mathematics moves. A second restriction was the failure to recognize negative and complex numbers. To mathematicians of the middle ages, all numbers were zero or positive.

In the spirit of Al-Khwarazmi and Omar Khayyam, we now solve the equation $x^2 = bx + c$, in which x, b, and c are assumed to be positive. We make the following constructions (see figure 1).

1. Construct the square ABCD with side length x. (Since x is unknown, imagine any quantity greater than b.)

2. Find point E on BC such that EC has length b.

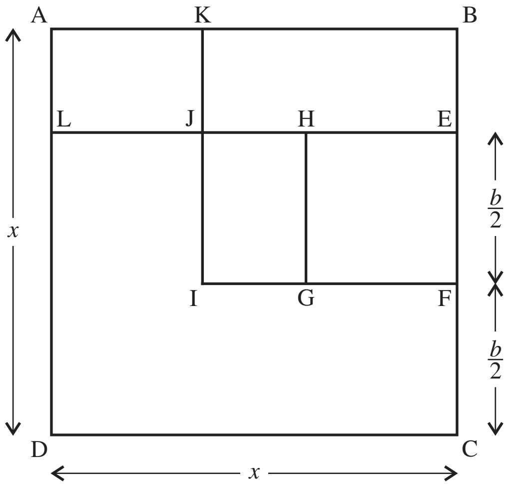

Figure 1

3. Construct EL parallel to AB. Notice that CDLE has area bx and, since $x^2 = bx + c$, ABEL must have area c.
4. Bisect EC at F and construct the square EFGH with side length $b/2$.
5. Extend FG to I so that $\text{IG} = \text{BE}$.
6. Notice that $\text{LJ} = \text{HE} = b/2$.
7. It is now clear that KBFI is a square.
8. The rectangles AKJL and JHGI are congruent.
9. Thus, the area of KBEHGI is equal to the area of ABEL, which is equal to c.
10. The square KBFI has area $(b/2)^2 + c$ (we have added the area of EFGH to the area of KBEHGI).
11. The side of this square KBFI is $\text{BC} - \text{FC} = x - b/2$.
12. Thus,

$$x - \frac{b}{2} = \sqrt{\frac{b^2}{4} + c},$$

and we have the solution

$$x = \frac{b + \sqrt{b^2 + 4c}}{2}.$$

Al-Khwarazmi used the geometrical method described above to find an algebraic solution to the specific equation $x^2 = 3x + 4$, while Omar Khayyam solved the general equation $x^2 = bx + c$. Al-Khwarazmi (also known as Muhammad Ben Musa Al-Khoarazmi) was born

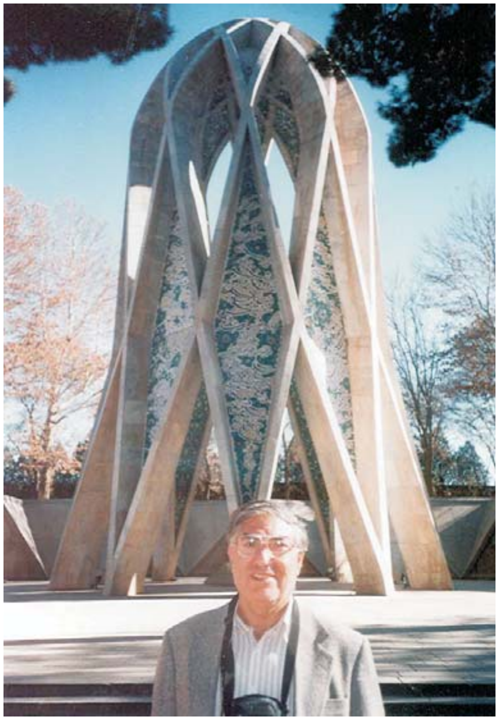

Figure 2 The first author at the tomb of Omar Khayyam in Nishpour, Iran.

in Khwarazm, a region in northern Persia. He was a scholar at the House of Wisdom called *Bit Al-Hikma* in Baghdad, Iraq, where there was a large library and a famous observatory. Within the walls of this institution lived some of the greatest scientists of the period. It housed translators, busy rendering into Arabic scientific classics written in Sanskrit, Pahlavi (the classical language of Persia), Syriac, and Greek (see reference 1).

Khwarazmi's most famous algebra work is called *Al-jabr Wal Moqabelah* (see reference 2). It was the first algebra book ever written. It is described in reference 1 as the best known algebra for Europeans until the French Mathematician Viete (1550–1603) extended algebra and trigonometry in the 16th century. Western Europeans learned about this book after it was translated into Latin in the 12th century. The book is over 1200 years old and the word *algebra* is originally taken from the name of this book. The word *Jabr* means transposing (when moving a negative term and making it positive) and completing. The word *Moqabelah* means cancelling the same terms from both sides of an equation. Khwarazmi's book contains first- and second-degree equations with their geometric solutions, the four basic arithmetic operations, topics in areas and volumes, and problems related to inheritance.

Omar Khayyam was a celebrated Persian mathematician, astronomer, philosopher, and poet. Khayyam was born in Nishapour, Iran, and worked as an astronomer in the observatory of Isfehan, Iran. He introduced an accurate calendar called *Tarikh Jalali*. This calendar is superior to the Georgian calendar and is still used by most Near Eastern countries. He also worked on the triangular array known today as Pascal's triangle. Written in Farsi and in Arabic, Khayyam's *Jabr O Moqabeleh* or *Maqalat Fi Al Jabr Wal Moqabelah* (see reference 3) classified equations according to their degree and gave rules for solving quadratic equations which are very similar to the ones used today. He also gave a geometrical method for solving cubic equations with real roots (see reference 4).

The fame of Khayyam as a mathematician has been partially eclipsed by his famous poetry book *Rubaiyat*, translated into English by Edward Fitzgerald in the middle of the 19th century.

References

1 V. S. Varadarajan, *Algebra in Ancient and Modern Times* (American Mathematical Society, Providence, RI, 1998), pp. 45, 65.
2 M. B. M. Al-Khwarazmi, *Kitab Al-Jabr Wa'l Muqabeleh* (UNESCO, Tehran, 1983).
3 O. Khayyam, *Hakim Omare Khayyam as an Algebraist*, 2nd edn. (UNESCO, Tehran, 2000).
4 G. G. Joseph, *The Crest of the Peacock* (Penguin Books, London, 1991), p. 303.

Mohamed Teymour *is a graduate student in mathematics at Rowan University and teaches mathematics at the Community College of Philadelphia. Mohamed is bilingual and conducts research in ancient mathematical topics. He is also a Persian poet and enjoys cultural activities and journalism. He has written a number of cultural newsletters in Farsi called 'Payke Shabahang'.*

Tom Osler *is professor of mathematics at Rowan University. He is the author of 73 mathematical papers. In addition to teaching university mathematics for the past 44 years, Tom has a passion for long-distance running. He has been competing for the past 51 consecutive years. Included in his over 1800 races are wins in three national championships in the late 1960s at distances from 25 kilometres to 50 miles. He is the author of two books on running.*

Mathematical Spectrum Awards for Volume 38

Prizes have been awarded to the following student readers for contributions in Volume 38:

Linda Ettlin

for the article 'But I Only Want to Half Differentiate!' (with Paul Belcher);

Matthew Oster

for the article 'A Spiral of Triangles Related to the Great Pyramid' (with Thomas J. Osler).

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £50 for letters, solutions to problems and other items.

Circles of Best Fit

GUIDO LASTERS

The problem we consider is how to find a circle of best fit at a given point of a given curve. Consider, for example, the parabola with equation $y = x^2$. We aim to find the circle which best fits this parabola at the origin; see figure 1. The normal to the parabola at $P(0, 0)$ is the y-axis, with equation $x = 0$. The parabola at the neighbouring point $Q \neq P$, with coordinates (h, h^2), has slope $2h$, so the normal at Q has slope $-1/2h$ and equation

$$y - h^2 = -\frac{1}{2h}(x - h).$$

The normals at P and Q meet when $x = 0$ in this equation, and so when

$$y = h^2 + \tfrac{1}{2},$$

i.e. they meet at the point $(0, h^2 + \frac{1}{2})$. As $Q \to P$, $h \to 0$ and this point tends to $(0, \frac{1}{2})$. This will be the centre of the circle of best fit at the origin, so the circle of best fit has equation

$$x^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{4} \qquad \text{or} \qquad y = x^2 + y^2.$$

A similar calculation can be carried out by taking P as the point $(1, 1)$, again with $Q(h, h^2)$ as a neighbouring point (not P). The normal to the parabola at P has equation

$$y - 1 = -\tfrac{1}{2}(x - 1) \qquad \text{or} \qquad y = -\tfrac{1}{2}x + \tfrac{3}{2},$$

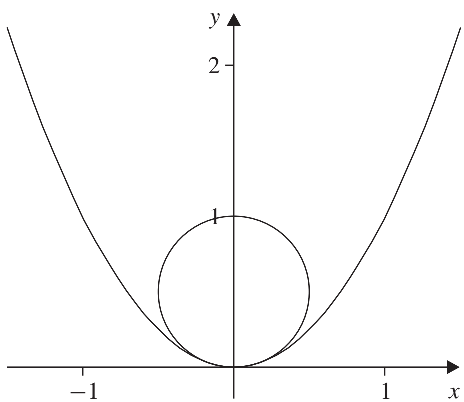

Figure 1

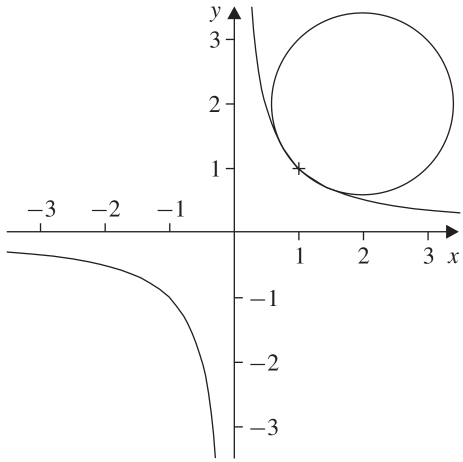

Figure 2

and this meets the normal at Q when

$$-\frac{1}{2}x+\frac{3}{2}-h^2=-\frac{1}{2h}(x-h),$$

i.e. when

$$\frac{x}{2h}(1-h)=h^2-1 \qquad \text{or} \qquad x=-2h(h+1).$$

(Note that $h \neq 1$ because $Q \neq P$.) As $Q \to P$, $h \to 1$ and $x \to -4$, $y \to \frac{7}{2}$. Thus, the limiting point of intersection of the two normals is $(-4, \frac{7}{2})$ and the circle of best fit at the point $(1, 1)$ has centre $(-4, \frac{7}{2})$ and radius

$$\sqrt{25+\tfrac{25}{4}}=\tfrac{1}{2}5\sqrt{5},$$

and so has equation

$$(x+4)^2+\left(y-\tfrac{7}{2}\right)^2=\tfrac{125}{4}.$$

As a third example, consider the point $P(1, 1)$ on the rectangular hyperbola with equation $xy = 1$; see figure 2. The normal at P is $y = x$. The hyperbola at the neighbouring point $Q(h, 1/h) \neq P$ has slope $-1/h^2$, so its normal at Q has slope h^2 and equation

$$y-\frac{1}{h}=h^2(x-h).$$

This meets the normal at P when

$$x-\frac{1}{h}=h^2(x-h) \qquad \text{or} \qquad x(h^2-1)=h^3-\frac{1}{h} \qquad \text{or} \qquad x=\frac{h^2+1}{h}.$$

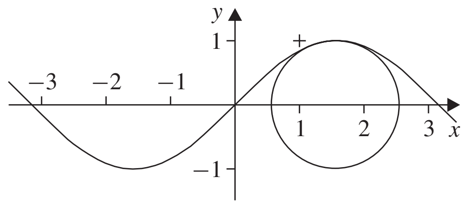

Figure 3

As $Q \to P$, $h \to 1$ and the point of intersection of the two normals tends to $(2, 2)$, which will be the centre of the circle of best fit at P, so this circle has equation

$$(x-2)^2 + (y-2)^2 = 2.$$

As a fourth example, consider the point $P(\pi/2, 1)$ on the curve $y = \sin x$; see figure 3. The normal to the curve at P is the line $x = \pi/2$. The normal to the curve at the neighbouring point $Q(h, \sin h) \neq P$ has equation

$$y - \sin h = -\frac{1}{\cos h}(x - h),$$

and the normals meet when

$$\begin{aligned} y &= \sin h - \frac{1}{\cos h}\left(\frac{\pi}{2} - h\right) \\ &= \sin h - \frac{\pi/2 - h}{\sin(\pi/2 - h)}. \end{aligned}$$

As $Q \to P$, $h \to \pi/2$ and $y \to 1 - 1 = 0$. Thus, the point of intersection of the two normals tends to $(\pi/2, 0)$ and the circle of best fit at P has equation

$$\left(x - \frac{\pi}{2}\right)^2 + y^2 = 1.$$

We leave readers to verify that the circle of best fit to the curve $y = \tan x$ at the point $(\pi/4, 1)$ has equation

$$\left(x - \frac{\pi}{4} + \frac{5}{2}\right)^2 + \left(y - \frac{9}{4}\right)^2 = \frac{125}{16}.$$

As a final illustration, we ask what is the sphere of best fit of the surface $z = x^2 + y^2$ at the origin (see figure 4)? The normal to the surface at $P(0, 0, 0)$ is the z-axis. The normal at the neighbouring point $Q(h, k, h^2 + k^2)$, with h, $k \neq 0$, is given by the equations

$$\begin{aligned} \frac{x-h}{-2h} &= \frac{y-k}{-2k} \\ &= \frac{z - h^2 - k^2}{1}, \end{aligned}$$

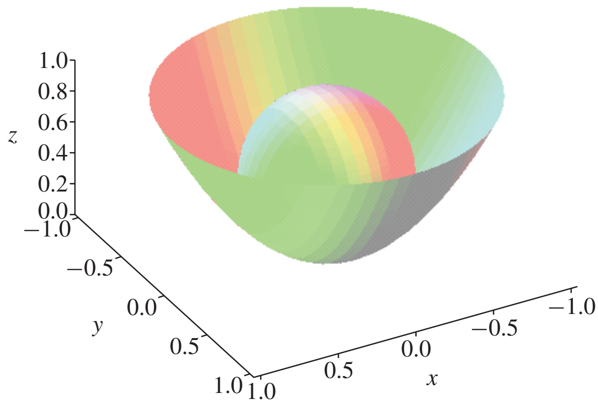

Figure 4

and the normals meet. In fact, they meet at the point $(0, 0, h^2+k^2+\frac{1}{2})$. As $Q \to P$, $h, k \to 0$ and the limit of this point of intersection is the point $(0, 0, \frac{1}{2})$. The sphere of best fit thus has centre $(0, 0, \frac{1}{2})$ and radius $\frac{1}{2}$, and so has equation

$$x^2 + y^2 + \left(z - \tfrac{1}{2}\right)^2 = \tfrac{1}{4} \qquad \text{or} \qquad z = x^2 + y^2 + z^2.$$

Guido Lasters *is a teacher in a secondary school in Leuven, Belgium. As a teacher, he tries to do a lot of mathematics by simple means. He is grateful to his colleague, Rudi Peetermans, who created figure 4 using MAPLE®.*

From 2006 to 666

Using only factorials, multiplication, square roots, and concatenation (so that 1, 1, 1 concatenated gives 111), we have

$$\left(\left(\sqrt{\sqrt{\sqrt{\cdots 2}}}\,\right) 0!\; 0!\right) \times 6 = 666.$$

Repton School
Derbyshire, UK

Alexei Gelbutovski

On Sums of Squares of Multivariate Polynomials

J. A. SCOTT

Introduction

In reference 1, we read that in the late nineteenth century Hilbert demonstrated the existence of a class C of nonnegative polynomials which cannot be expressed as a sum of squares of polynomials. However, it was not until the 1960s that the simple example

$$f(x, y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1$$

was found by Motzkin. In 1925 (whilst addressing Hilbert's 17th problem), Artin showed that polynomials belonging to C may be expressed as a finite sum of squares of rational functions, but at the beginning of the present millennium such a representation for $f(x, y)$ was apparently still unknown.

In this article we confirm that $f(x, y) \in C$ and provide a sum of squared rational functions.

Motzkin's polynomial

To establish nonnegativity, we use the arithmetic-geometric mean inequality for the three positive numbers x^4y^2, x^2y^4, and 1. Thus,

$$\begin{aligned} A &= \tfrac{1}{3}(x^4y^2 + x^2y^4 + 1) \\ &\geq G \\ &= (x^6y^6)^{1/3} \\ &= x^2y^2; \end{aligned}$$

whence, $f(x, y) \geq 0$ with equality at the four points given by $x^2 = y^2 = 1$. Now, if $f(x, y)$ is a sum of squares of polynomials, then each square must have the form

$$(a_i + b_ixy + c_ix^2y + d_ixy^2)^2,$$

for some constants a_i, b_i, c_i, and d_i. We therefore immediately have the *reductio ad absurdum*

$$\sum b_i^2 = -3.$$

Next, writing $p = x^2 + y^2$, we have

$$\begin{aligned} (x^2 - y^2)^2 + (p+1)(p-2)^2x^2y^2 &= p^2 - 4x^2y^2 + (p^3 - 3p^2 + 4)x^2y^2 \\ &= p^2((p-3)x^2y^2 + 1); \end{aligned}$$

whence,

$$\begin{aligned} f(x, y) &= (p-3)x^2y^2 + 1 \\ &= \left[\frac{x^2-y^2}{p}\right]^2 + \left[\frac{xy(p-2)}{p}\right]^2 + \left[\frac{x^2y(p-2)}{p}\right]^2 + \left[\frac{xy^2(p-2)}{p}\right]^2, \end{aligned}$$

i.e. a sum of squares of four rational functions in x and y.

Related polynomials

Let

$$\begin{aligned} g(x, y) &= x^6 + y^6 - 3x^2y^2 + 1 \\ &= \left[\frac{x^2-y^2}{p}\right]^2 + \left[\frac{xy(p-2)}{p}\right]^2 + \left[\frac{x(px^2-2y^2)}{p}\right]^2 + \left[\frac{y(py^2-2x^2)}{p}\right]^2, \end{aligned}$$

where nonnegativity and absence of polynomial squares may be treated in a similar way to the above. In contradistinction, we have the polynomial

$$h(x, y) = x^8 + y^8 - 4x^2y^2 + 2.$$

This *is* a sum of squares of polynomials, namely

$$\begin{aligned} (x^4-1)^2 + (y^4-1)^2 + 2(x^2-y^2)^2 &= (x^4-y^4)^2 + 2(x^2y^2-1)^2 \\ &\geq 0, \end{aligned}$$

with equality when $x^2 = y^2 = 1$. Finally, note the expression of $h(x, y)$ as a sum of squares of *three* rational functions as follows:

$$h(x, y) = \left[\frac{\sqrt{2}(x^4-y^4)}{q}\right]^2 + \left[\frac{x^2(qx^2-2y^2)}{q}\right]^2 + \left[\frac{y^2(qy^2-2x^2)}{q}\right]^2, \qquad q = x^4 + y^4.$$

Reference

1 M.-F. Roy, Three problems in real algebraic geometry and their descendants. In *Mathematics Unlimited: 2001 and Beyond*, eds B. Engquist and W. Schmid (Springer, Berlin, 2001), pp. 991–1002.

Since his retirement from a mathematics lecturing post at Bristol Polytechnic in 1991, the author has written many articles for mathematics education. Personal interests include classical music, reading, gardening, and tennis.

What are the last four digits of 1249^{1249}?

10 Shahid Azam Lane,
Makki Abad Avenue, Sirjan, Iran

Abbas Roohol Amini

A ballad of pi

π = 3.141 592 653 …

Anthony Summers can recite you pi
Digit by digit to two thousand and eight.
His friends prefer football, his Dad TV,
But he learns numbers early and late.

Nothing but kid's stuff, says Anthony Summers,
To remember Cup Final teams, every Top Ten.
No pattern, no end to its digits,
Pi's the challenge for us memory men.

School's a bore, he's glad to leave;
His teachers say he doesn't shine.
But he knows pi to six thousand and six,
Eight thousand's his next deadline.

It's running the mile a little bit faster,
Jumping higher than anyone before,
Testing memory as it's never been tested,
Says Anthony Summers at ten thousand and four.

His mother wishes he'd find a girl,
As Anthony Summers improves his technique,
Ten digits a time to the clock's tick-tock;
Reached twenty thousand non-stop last week.

In a Huddersfield hall the auditor's finger
Follows Anthony Summers over pages of pi,
As he recites more digits than any one before.
Thirty thousand, the record's his, first try!

The one thing intelligent beings out there
In space are sure to recognise is pi,
Says Anthony Summers as he radios digits,
Hopes some far star will send them in reply.

(Anthony Summers is imaginary but you may have met his cousin, Timothy Winters, in a poem by Charles Causley.)

***Derek Collins** is Emeritus Professor of Applied Mathematics at the University of Sheffield. His present research work is on the dynamics of synchronous electric motors. He is also a published poet though this is the first time one of his poems has appeared in a mathematics journal.*

Mathematics in the Classroom

Forgetting to integrate

In a recent Mathematics in the Classroom column, *A Magical Mistake* (*Math. Spectrum*, Volume 36, Number 2), Anand Kumar gave a number of examples of how it is possible to get the right answer to an integral by the wrong method based on specific cases of

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a),$$

i.e. where the integration step has been forgotten! There is one really rather interesting example of this kind that is included in Anand's article, but which doesn't stand out because he chooses his integrand as $\ln(1+x)$ and limits as 0 and $\mathrm{e}-1$. We would like to bring this example to the readers attention in what we feel is a more aesthetic form as follows:

$$\int_1^{\mathrm{e}} \ln x\,\mathrm{d}x = \int_1^{\mathrm{e}} \frac{1}{x}\,\mathrm{d}x. \tag{1}$$

(Both integrals in (1) are unity.) In terms of the graphs of the functions (see figure 1), (1) says that

$$\mathrm{area}_{\mathrm{I}} + \mathrm{area}_{\mathrm{II}} = \mathrm{area}_{\mathrm{II}} + \mathrm{area}_{\mathrm{IV}},$$

i.e.

$$\mathrm{area}_{\mathrm{I}} = \mathrm{area}_{\mathrm{IV}}.$$

Hence, the area between $x = 1$ and the curves $y = 1/x$ and $y = \ln x$ is the same as the area between these curves and $x = \mathrm{e}$. Can readers think of a nicer example?

Kendrick School, Reading — **Elizabeth M. Glaister**

University of Reading — **Paul Glaister**

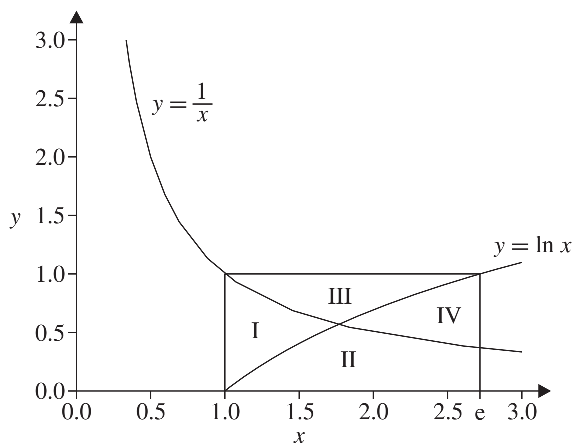

Figure 1

Letters to the Editor

Dear Editor,

The areas problem

Atara Shriki's article *The Areas Problem* (*Math. Spectrum*, Volume 39, Number 1) fulfils two very useful criteria. Firstly, it takes us through the thought processes of the writer, something rare in mathematical articles, which it might be good to see more often. Secondly, it demonstrates the usefulness of the computer in taking some of the hard graft out of mathematical investigation.

Having said this, I would like to give a simpler proof of the basic result in Shriki's article. To remind readers, I reproduce figure 1. Let P be the point (x, x^n) and Q the point $(kx, (kx)^n)$. Apply a two-way stretch with scale factors k and k^n parallel to the x- and y-axes respectively with these two axes invariant, i.e. the transformation represented by the matrix

$$\begin{bmatrix} k & 0 \\ 0 & k^n \end{bmatrix}.$$

Then P maps onto Q. The curve $y = x^n$ maps onto $y/k^n = (x/k)^n$, that is $y = x^n$, i.e. the curve maps onto itself, so Q maps onto R. Also, area A maps onto area B. Since the area scale factor for the transformation is given by the determinant of the matrix, k^{n+1}, it follows that

$$\frac{\text{area } B}{\text{area } A} = k^{n+1}.$$

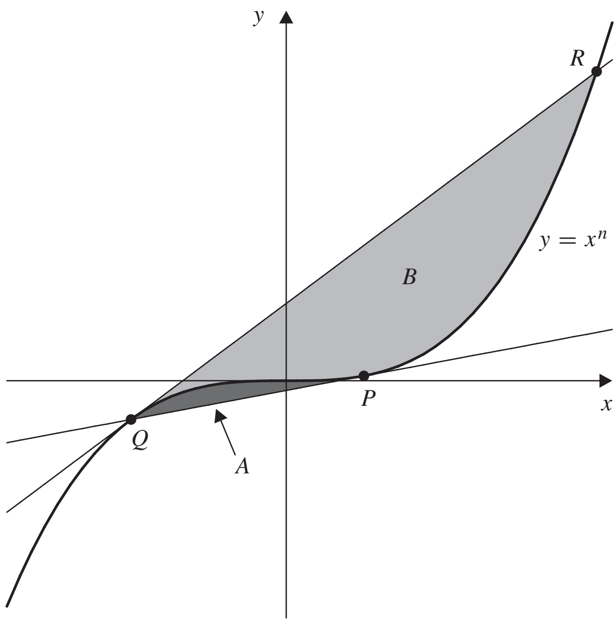

Figure 1

Furthermore, the gradient of PQ may be deduced from the coordinates of P and Q, and also from the gradient of the tangent at P. Hence, equating these two evaluations we obtain

$$\frac{(kx)^n - x^n}{kx - x} = nx^{n-1},$$

giving

$$\frac{k^n - 1}{k - 1} = n. \tag{1}$$

From the context, it is clear that this equation only has one solution. Also it is independent of x, so k is a function of n only. Values of k can then be found by Atara's method or by an equivalent numerical method.

Regarding the solutions to (1), letting $u = -k$ we obtain, for n odd,

$$u^n = n(u+1) - 1 \qquad \Longrightarrow \qquad n \ln u = \ln(n(u+1) - 1),$$

where we know that $u > 1$ from the context. Hence, using the well-known inequality $\ln u < u - 1$, we obtain

$$n(u-1) > \ln(n(u+1) - 1) > \ln(2n - 1),$$

and so

$$u > \frac{\ln(2n-1)}{n} + 1. \tag{2}$$

Atara gives the approximate formula $y = -2.76029/x - 1.00819$ or, using my variable names,

$$u = \frac{2.76029}{n} + 1.00819.$$

Comparing this with (2), we see that this is no longer a useful formula for larger n.

Regarding the linear formula Atara gives for the area factor, if we let the area factor be $z = u^{n+1}$ we get, from $u^n = n(u+1) - 1$ on multiplication by u, $z = n(u^2 + u) - u$. Also, as $n \to \infty$, it is easy to show that $u \to 1$. It follows that asymptotically

$$\text{area factor} = z \sim 2n - 1,$$

which gives some credence to Atara's formula which, in my variable names, is $z = 2.13436n + 9.58882$, but it obviously needs adjusting for larger n. *Note the '+' sign here. There is, I think, a typographical error in the original.*

Yours sincerely,

Alastair Summers
(57 Conduit Road
Stamford
Lincolnshire PE9 1QL
UK)

Dear Editor,

Symmetrical Fibonacci products

The products $f_{n-m}f_{n+m} = P_m$, where $m = 0, 1, 2, \ldots, n-1$, are symmetrical about f_n in the Fibonacci sequence $\{f_n\}$, where

$$f_1 = 1 = f_2 \qquad \text{and} \qquad f_{n+2} = f_{n+1} + f_n, \quad \text{for } n > 0.$$

From P_3 they are generated by the recurrence relation

$$P_{m+3} - P_m = 2(P_{m+1} - P_{m+2}), \tag{1}$$

which can be proved by expansion of the left-hand side. The first application depends upon $P_0 = f_n^2$, $P_1 = f_{n-1}f_{n+1} = f_n^2 + (-1)^n$ (which is well known and can be proved by induction), and $P_2 = f_n^2 - (-1)^n$ (which is proved by expanding the product). Hence,

$$P_3 = 2(P_1 - P_2) + P_0 = f_n^2 + 4(-1)^n.$$

These three products conform to the pattern

$$P_m = f_n^2 - f_m^2(-1)^{n+m}, \tag{2}$$

which includes P_0 if $f_0 = 0$. This can be verified by substitution into (1) with

$$\begin{aligned} P_{m+1} &= 2(P_{m-1} - P_m) + P_{m-2} \\ &= f_n^2 + (2(f_m^2 + f_{m-1}^2) - f_{m-2}^2)(-1)^{n+m} \\ &= f_n^2 + ((f_m + f_{m-1})^2 + (f_m - f_{m-1})^2 - f_{m-2}^2)(-1)^{n+m} \\ &= f_n^2 - f_{m+1}^2(-1)^{n+m+1}. \end{aligned}$$

From the final product, we obtain

$$\begin{aligned} f_n^2 - f_{n-1}^2(-1)^{2n-1} &= P_{n-1} \\ &= f_1 f_{2n-1} \end{aligned}$$

and

$$f_n^2 + f_{n-1}^2 = f_{2n-1}.$$

This identity can be used with (2) to give

$$P_{m+1} = P_m + f_{2m+1}(-1)^{n+m}.$$

By addition over $P_{k+1} - P_k$ from $k = 0$ to m, we obtain

$$\sum_{k=0}^{m}(-1)^k f_{2k+1} = (-1)^m f_{m+1}^2,$$

which, with $\sum_{k=0}^{m} f_{2k+1} = f_{2m+2}$, gives

$$\sum_{k=0}^{n} f_{4k+1} = \frac{f_{4n+2} + f_{2n+1}^2}{2}.$$

Alternatively, these summation formulae can be proved by induction. The formula (2) can also be found by solving the difference equation specified by (1) with the initial conditions P_0, P_1, and P_2. However, it requires the Binet formula,

$$f_m = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right),$$

and is more involved than the method we have presented.

Yours sincerely,

Robert J. Clarke
(44 Webb Court
Drury Lane
Stourbridge DY8 1BN
UK)

Dear Editor,

A modified Newton–Raphson procedure

I am writing concerning Bob Bertuello's letter *The rational mean* (*Math. Spectrum*, Volume 39, Number 2). In this, Bertuello developed a new iterative approach which, for certain problems (for example, computing the nth root of a number P), converges more rapidly than the standard Newton–Raphson (NR) technique. The purpose of this letter is to point out that the formulation of the NR procedure is not unique and that it may be re-stated in a way which converges more rapidly and which, in computing $P^{1/n}$, yields exactly the same result as that given by the 'rational mean' approach. It should be noted that this modified NR technique (equation (6), below) is not new, and although it converges in fewer steps than the standard form of NR, it may be less efficient since it is more cumbersome and involves calculation of second derivatives.

Now, the standard NR iterative formula for solving $f(x) = 0$ takes the form

$$x_{p+1} = x_p - \frac{f(x_p)}{f'(x_p)}, \tag{1}$$

where x_p is the pth iterate. This is readily shown to exhibit second-order convergence, since if X satisfies $f(X) = 0$ and $x_p = X + \varepsilon_p$, then (1) yields

$$\begin{aligned}\varepsilon_{p+1} &= \varepsilon_p - \frac{f(X+\varepsilon_p)}{f'(X+\varepsilon_p)} \\ &= \varepsilon_p - \frac{f(X) + \varepsilon_p f'(X) + \frac{1}{2}\varepsilon_p^2 f''(X) + \cdots}{f'(X) + \varepsilon_p f''(X) + \cdots} \\ &= k\varepsilon_p^2,\end{aligned} \tag{2}$$

where

$$k = \frac{f''(X)}{2f'(X)} \tag{3}$$

on retaining only the leading term on the right-hand side of (2). The essential modification that we now wish to introduce is based on realising that our equation $f(x) = 0$ can equivalently be expressed as $g(x)f(x) = 0$, where $g(x)$ is any function for which $g(X)$ is finite. The above NR analysis applied to this latter equation then yields results (1) and (3), but with $f(x)$ replaced by $f(x)g(x)$. Thus, these results respectively become

$$x_{p+1} = x_p - \frac{f(x_p)}{f'(x_p) + f(x_p)g'(x_p)/g(x_p)} \tag{4}$$

and

$$k = \frac{[f(x)g(x)]''_{x=X}}{2[f(x)g(x)]'_{x=X}} = \frac{2f'(X)g'(X) + f''(X)g(X)}{2[f(X)g'(X) + f'(X)g(X)]} \tag{5}$$

(since $f(X) = 0$), valid for all $g(x)$, $g(X) < \infty$. We can now improve the convergence of the NR iteration by choosing g so that $k = 0$, so then the technique will exhibit third-order convergence, corresponding to the leading term in $x_{p+1} - x_p$ being proportional to ε_p^3. It is clear from (5) that this will be so if

$$\frac{g'(x)}{g(x)} = -\frac{f''(x)}{2f'(x)},$$

corresponding to

$$g(x) = \text{constant} \times x[f'(x)]^{-1/2}.$$

Equation (4) then takes the form

$$x_{p+1} = x_p - \frac{2f(x_p)f'(x_p)}{2[f'(x_p)]^2 - f(x_p)f''(x_p)}. \tag{6}$$

Equation (6) is our modified NR iteration formula, and although it will generally converge in fewer steps than the standard formula (1), this does not necessarily mean that it is more efficient since, as mentioned earlier, it is more cumbersome and involves computation of the second derivative. However, if we now apply it to the computation of $P^{1/n}$, by taking $f(x) = x^n - P$, the calculation of derivatives becomes trivial, and it is then readily shown that the iterative formula obtained is identical to that given by the 'rational mean' approach as stated at the bottom of page 81 of Bertuello's letter.

Yours sincerely,
Stuart Simons
(170 Holmleigh Road
London N16 5PY
UK)

Dear Editor,

Square roots by continued fraction convergents

In M. A. Khan's article *Rational Approximation to Square Roots of Integers* (*Math. Spectrum*, Volume 39, Number 2), he shows us how to obtain rational approximations to square roots of integers by a matrix algorithm. The same results can be obtained by continued fractions.

For those unfamiliar with the latter, the following example will give the general idea. Suppose that we want to find the square root of 11. This (irrational) number lies between 3 and 4 so the integer part is 3, i.e.

$$\sqrt{11} = 3 + (\sqrt{11} - 3).$$

Now, $\sqrt{11}-3$ is less than unity so we look for a proper fraction approximation. Using the identity $(a+b)(a-b)=a^2-b^2$, we obtain

$$\frac{1}{\sqrt{11}-3}=\frac{\sqrt{11}+3}{2}=3+\frac{\sqrt{11}-3}{2}.$$

Hence,

$$\sqrt{11}-3=\cfrac{1}{3+\cfrac{\sqrt{11}-3}{2}}.$$

The first rational approximations to $\sqrt{11}$ are therefore 3 and $\frac{10}{3}$. Now,

$$\frac{2}{\sqrt{11}-3}=\frac{2(\sqrt{11}+3)}{2}=\sqrt{11}+3.$$

Once again, we convert the right-hand side into an integer plus a number less than unity as follows:

$$\frac{2}{\sqrt{11}-3}=6+(\sqrt{11}-3).$$

Thus,

$$\frac{\sqrt{11}-3}{2}=\frac{1}{6+(\sqrt{11}-3)}.$$

From now on, the integers repeat since we have got the same noninteger part, $\sqrt{11}-3$. This turns out always to be the case for a square root turned into a continued fraction. In continued fraction form we have

$$\sqrt{11}=3+\cfrac{1}{3+\cfrac{1}{6+\cfrac{1}{3+\cdots}}}.$$

Expressing this more succinctly, we have $\sqrt{11}=[3;\overline{3,6}]$, where we have a nonrepeating initial integer and then a repeating part. The so-called *convergents*, p_n/q_n, are obtained by taking the infinite periodic continued fraction so far and then cutting it off. In this case they are

$$\tfrac{3}{1},\ \tfrac{10}{3},\ \tfrac{63}{19},\ \tfrac{199}{60},\ \ldots.$$

This gives a sequence of evermore accurate rational approximations to $\sqrt{11}$.

But what has this to do with the equation used by Khan, $y_0^2-nx_0^2=1$, the so-called Pell's equation? It is a remarkable property of continued fractions that a whole set of convergents satisfy

$$p_n^2-1=nq_n^2.$$

For example, in the above, $10^2-1=11\times 3^2$. The rule can be expressed as follows. If there is an even number, n, of digits in the period, then (p_{kn-1},q_{kn-1}) for $k=1,2,3,\ldots$ satisfy Pell's equation; if n is odd, then (p_{kn-1},q_{kn-1}) will give -1 and $+1$ alternately.

Readers interested in continued fractions should consult the relevant chapters in any textbook on elementary number theory; see, for example, reference 1.

Reference

1 David M. Burton, *Elementary Number Theory* (McGraw-Hill, New York, 2001).

Yours sincerely,

Sebastian Hayes
(12 Castle Hill Close
Shaftesbury
Dorset SP7 8LQ
UK)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

39.9 Prove that the sum of two consecutive odd primes numbers is the product of at least three primes.

(Submitted by Bob Bertuello, Midsomer Norton, Bath, UK)

39.10 For real numbers a, b, and c such that $0 \leq a, b, c \leq 1$, prove that

$$a^{17} - a^{10}b^7 + b^{17} - b^{10}c^7 + c^{17} - c^{10}a^7 \leq 1.$$

(Submitted by Alexei Gelbutovski, Repton School, Derbyshire, UK)

39.11 For a triangle ABC, draw through A the line perpendicular to CA, through B the line perpendicular to AB, and through C the line perpendicular to BC, to form a triangle $A'B'C'$. Prove that the ratio of the areas of triangles $A'B'C'$ and ABC is

$$(\cot A + \cot B + \cot C)^2.$$

(Submitted by Konstantine Zelator, University of Toledo, Ohio, USA)

39.12 The function $f : \mathbb{N} \to \mathbb{N}$ has the properties

1. $f(1) = 1$,
2. $3f(n)f(2n+1) = f(2n)(1 + 3f(n))$ for all $n \in \mathbb{N}$,
3. $f(2n) < 6f(n)$ for all $n \in \mathbb{N}$.

Find all positive integers k and m such that

$$f(k) + f(m) = 2007.$$

(Submitted by Farshid Arjomandi, San Diego, California, USA)

Solutions to Problems in Volume 39 Number 1

39.1 For which natural numbers n can the set $\{1, 2, \ldots, n\}$ be partitioned into two subsets, A and B, such that the sum of the numbers in A is equal to the sum of the numbers in B?

Solution by Henry Ricardo, Medgar Evers College, New York, USA

A necessary condition for such a partition to exist is that the sum of the numbers in each subset must be half the sum of the numbers $1, 2, \ldots, n$. Thus, $\frac{1}{2}n(n+1)$ must be an even integer, so that either n or $n+1$ must be divisible by 4. This condition is also sufficient. First note that

$$\{m, m+1, m+2, m+3\} = \{m, m+3\} \cup \{m+1, m+2\}$$

and the numbers in the two subsets have the same sum. Now if $n = 4k$ for some $k \in \mathbb{N}$, we can take

$$A = \{1, 4, 5, 8, \ldots, 4k-3, 4k\},$$
$$B = \{2, 3, 6, 7, \ldots, 4k-2, 4k-1\};$$

if $n = 4k - 1$ for some $k \in \mathbb{N}$, we can take

$$A = \{1, 2, 4, 7, 8, 11, \ldots, 4k-4, 4k-1\},$$
$$B = \{3, 5, 6, 9, 10, \ldots, 4k-3, 4k-2\}.$$

39.2 Solve the following equation:

$$2\log_3 \cot x = \log_2 \cos x.$$

Solution by Bor-Yann Chen, University of California, Irvine, USA

Denote the given expression by y. Then

$$3^y = \cot^2 x \quad \text{and} \quad 2^y = \cos x, \tag{1}$$

so that

$$3^y = \frac{\cos^2 x}{\sin^2 x} = \frac{\cos^2 x}{1 - \cos^2 x} = \frac{4^y}{1 - 4^y}.$$

Hence, $3^y - 12^y = 4^y$. Write

$$f(y) = 4^y + 12^y - 3^y.$$

Then

$$f'(y) = 4^y \ln 4 + 12^y \ln 12 - 3^y \ln 3 > 0$$

when $y > 0$ and $f(0) = 1$. Hence, there is no $y > 0$ satisfying (1). Now write

$$g(y) = 4^{-y} - 3^{-y} - 1.$$

Then

$$g'(y) = -4^{-y} \ln 4 + 3^{-y} \ln 3 < 0$$

when $y < 0$ and $g(-1) = 0$, so $y = -1$ is the only solution to (1), so

$$\cos x = \frac{1}{2} \quad \text{and} \quad \cot x = \pm\frac{1}{\sqrt{3}},$$

giving $x = \pm\frac{1}{3}\pi + 2n\pi$ $(n \in \mathbb{Z})$ as the solutions.

39.3 Two different natural numbers lie strictly between the same successive perfect squares. Prove that their product is not a perfect square.

Solution by Anand Kumar, who proposed the problem

Let n, a, and b be natural numbers such that

$$n^2 < a < b < (n+1)^2.$$

Then

$$n < \sqrt{a} < \sqrt{b} < n+1,$$

so that

$$0 < \sqrt{b} - \sqrt{a} < 1,$$

whence

$$0 < b + a - 2\sqrt{ab} < 1.$$

Since there is no integer strictly between 0 and 1, this means that $\sqrt{ab}$ cannot be an integer, so that ab cannot be a perfect square.

39.4 Let z_1, z_2, z_3, and z_4 be complex numbers whose sum is zero. Prove that

$$(z_1^3 + z_2^3 + z_3^3 + z_4^3)^2 = 9(z_2z_3 - z_1z_4)(z_1z_3 - z_2z_4)(z_1z_2 - z_3z_4) \tag{2}$$

and that

$$8|z_1^3 + z_2^3 + z_3^3 + z_4^3|^2 \le 9(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)^3. \tag{3}$$

Solution by Mihály Bencze, who proposed the problem

Let $x^4 + ax^2 + bx + c$ be the polynomial with leading coefficient 1 whose roots are z_1, z_2, z_3, and z_4. (Note that the coefficient of x^3 is zero because the roots have sum zero.) Suppose initially that no z_i is zero. Then

$$z_1^3 + az_1 + b + \frac{c}{z_1} = 0,$$

and similarly for z_2, z_3, and z_4. If we sum these equations we get

$$\sum z_1^3 + a\sum z_1 + 4b + c\sum \frac{1}{z_1} = 0,$$

so that

$$\begin{aligned}\sum z_1^3 &= -4b - c\sum \frac{1}{z_1}\\ &= 4\sum z_1z_2z_3 - c\sum \frac{1}{z_1}\\ &= 4c\sum \frac{1}{z_1} - c\sum \frac{1}{z_1}\\ &= 3c\sum \frac{1}{z_1},\end{aligned}$$

and the left-hand side of (2) is equal to

$$9c^2\Big(\sum \frac{1}{z_1}\Big)^2.$$

The right-hand side of (2) is equal to

$$\begin{aligned}&9(z_1^2z_2^2z_3^2 - z_1z_2z_3^3z_4 - z_1z_2^3z_3z_4 + z_2^2z_3^2z_4^2 - z_1^3z_2z_3z_4 + z_1^2z_3^2z_4^2 + z_1^2z_2^2z_4^2 - z_1z_2z_3z_4^3)\\ &\quad = -9z_1z_2z_3z_4(z_1^2+z_2^2+z_3^2+z_4^2) + 9\sum z_1^2z_2^2z_3^2\\ &\quad = -9c\Big(\Big(\sum z_1\Big)^2 - 2\sum z_1z_2\Big) + 9c^2\sum \frac{1}{z_1^2}\\ &\quad = 18ca + 9c^2\Big(\Big(\sum \frac{1}{z_1}\Big)^2 - 2\sum \frac{1}{z_1z_2}\Big)\\ &\quad = 18ca + 9c^2\Big(\sum \frac{1}{z_1}\Big)^2 - 18c^2\frac{1}{c}\sum z_1z_2,\end{aligned}$$

which is equal to the left-hand side of (2).

Now suppose that $z_4 = 0$. Then z_1, z_2, and z_3 are the roots of the polynomial $x^3 + ax + b$ and we have

$$\sum z_1^3 + a\sum z_1 + 3b = 0,$$

so that

$$\sum z_1^3 = -3b.$$

The right-hand side of (2) is $9z_1^2z_2^2z_3^2 = 9b^2$, which is equal to the left-hand side of (2).

From (2) we obtain

$$\begin{aligned}\Big|\sum z_1^3\Big|^2 &= 9|z_2z_3 - z_1z_4||z_1z_3 - z_2z_4||z_1z_2 - z_3z_4|\\ &\le 9(|z_2||z_3| + |z_1||z_4|)(|z_1||z_3| + |z_2||z_4|)(|z_1||z_2| + |z_3||z_4|)\\ &\le 9 \times \tfrac{1}{2}(|z_2|^2 + |z_3|^2 + |z_1|^2 + |z_4|^2) \times \tfrac{1}{2}(|z_1|^2 + |z_3|^2 + |z_2|^2 + |z_4|^2)\\ &\quad \times \tfrac{1}{2}(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2),\end{aligned}$$

from which (3) follows.

Reviews

Understanding Uncertainty. By Dennis V. Lindley. John Wiley, Chichester, 2006. Hardback, 250 pages, £35.50 (ISBN 0-470-04383-0).

When this book arrived for me to review, a nonmathematician friend was visiting. He very quickly picked it up to browse and became quite absorbed, so much so that it was the next day before I could retrieve it to see what was causing all the fascination. Consequently, a very auspicious start for a reviewer; it soon became clear to me why, as this book certainly has that hard-to-put-down quality.

Following a working life spent almost entirely as an academic statistician, the author focuses on the importance of discoveries made in academe relating to uncertainty, something we all face every day, and how the results may be used to benefit our lives. The Preface explains that the objective is to describe the work that has been done in the 20th century about this phenomenon, equipping the reader with enough skill to act sensibly in the face of uncertainty, measuring and using it in all aspects of his or her decision making. It is described as a book for the layperson and carefully explains that part of the language of mathematics that it uses, but some mathematical maturity is definitely helpful. It applies the ideas about uncertainty to various disciplines including law, science, economics, and politics, and encouragingly promises and delivers some surprises along the way.

Chapter 1 presents a collection of examples, many familiar, where uncertainty impinges on our lives, Chapter 2 contains the mathematics tutorial and other stylistic issues, whilst the remaining eleven chapters are devoted to those areas of probability necessary for any study of uncertainty, namely the rules governing single and multiple events, probability, variation, decision analysis, and scientific method. Throughout the exposition there are interesting illustrative examples. For instance, in the discussion of three-event theory, Simpson's paradox (in the context of clinical trials where a treatment appears to be good for men, good for women, but bad for all of us!) is used to demonstrate that, as Lindley remarks, 'our uncertain world does not always behave in the common-sense way that we might expect'.

This is without doubt an interesting, readable book that you can dip into and, tantalisingly, you are left wanting more. One small irritation: for a book written by someone living in Minehead, UK, that uses examples relating to the likelihood of a white Christmas in the UK, and the probability that Richard III was responsible for the murder of the princes in the Tower, why is it necessary to express any currency referred to in dollars?

Nonetheless, this is a delightful book providing an excellent and accessible overview of progress in understanding uncertainty – definitely recommended reading.

Carol Nixon

Aha! A Two-Volume Collection. Aha! Insight and Aha! Gotcha. By Martin Gardner. MAA, Washington, DC, 2006. Hardback, 376 pages, $47.50 (ISBN 0-88385-551-5).

This is another first-rate book from the Mathematical Association of America, by the author of *Scientific American* fame, Martin Gardner. It will prove to be very suitable as a resource for teachers of mathematics in schools, and I believe will be of wider interest to nonmathematicians as well. It often strays outside the realms of traditional school mathematics, and includes articles that are not strictly mathematical.

The book is a combination of two volumes entitled *Aha! Gotcha: Paradoxes to Puzzle and Delight* and *Aha! Insight*. Both cover a very wide range of different branches of mathematics, and some are not really what you would call mathematics at all! The level of mathematics required to understand most of the articles is not high, never beyond AS level, and very little as advanced as that. On the other hand, some of the extension ideas are very mathematically sophisticated. Many articles would make ideal discussion material for school classes, or for bright school students to read for themselves. Each of the 145 paradoxes and puzzles is introduced by a very imaginatively drawn cartoon, making the book so much more appealing to students especially, but I think to anyone else too! They certainly added to my enjoyment. On the opposite page, for each item, there follows a lucid explanation. Many articles introduce an idea and then extend it, or add extra problems to solve, for which solutions are in the back of the book. You will also find a bibliography there.

The first volume is on paradoxes. Some of them made me laugh more than I have done for years! Many are quite profound. The areas of mathematics covered are logic, number, geometry, probability, statistics, and time. Here is just one of the more difficult to whet your appetite.

> A worm is at one end of a rope. The rope is 1km long. The worm crawls along the rope at a steady pace of 1cm/s. After the first second, the rope stretches like a rubber band to 2km. After the next second, it stretches to 3km, and so on. Will the worm ever reach the end of the rope?

The second volume deals with problems that require a special moment of insight to solve. There are six chapters: Combinatorial aha!, Geometry aha!, Number aha!, Logic aha!, Procedural aha!, and Word aha! We meet Professor Quibble and Dr Wordle, among others. Here are two of Dr Wordle's puzzle questions which start off one of the articles.

> What is the opposite of 'Not in'?
> What 11-letter word do all Yale graduates spell incorrectly?

Just a little bit of simple lateral thinking is required to answer the second question!

The reference to Yale graduates will remind you this is an American book, but this will not spoil the enjoyment or relevance of the articles for other readers. I thoroughly recommend it. It will give any mathematically inclined reader many hours of enjoyment. I think many of the articles will also interest nonmathematicians who like puzzles.

Alastair Summers

The Mathematics of Games and Gambling. By Edward Packel. MAA, Washington, DC, 2nd edn., 2006. Hardback, 192 pages, $44.00 (ISBN 0-88385-646-8).

This is the second edition of a book that first appeared in 1981 and went to eight reprints. Greeted with enthusiasm as a useful exposition of the important but elementary mathematics needed for the analysis of various gambling and game activities, it covered standard casino games (e.g. roulette, craps, blackjack, and keno), some social games (e.g. backgammon, poker, and bridge), and various other chance-related activities (e.g. state lotteries and horse racing). The areas of probability, expectation, binomial coefficients, and ideas of elementary game theory were explored in a way that was intended to engender a fascination for the mathematics behind the games.

Although this mathematics has not changed dramatically in the 25 years since the first edition appeared, the author acknowledges that changes have occurred in the games themselves and the way in which they are played. Online gaming has developed and slot machines have become increasingly sophisticated. It is changes such as these that have persuaded the author to update the first edition with the addition of some 33 pages of new material on sports betting, game theory applied to bluffing in poker, and the Nash equilibrium concept that was brought to general attention in the recent film *A Beautiful Mind*. Internet links to games and Java applets for playing certain games against the computer have been included as part of the modernisation of this text.

We are frequently advised that students will be more actively involved in their own learning with mathematics that is presented in an interesting, relevant, and up-to-date context. That being the case, this book will no doubt continue to fascinate and inform just as its predecessor has done.

Carol Nixon

Other books received

The Oxford Dictionary of Statistical Terms. Edited by Yadolah Dodge. Oxford University Press, 2006. Paperback, 512 pages, £12.99 (ISBN 0-19-920613-9).

Smoothing and Decay Estimates for Nonlinear Diffusion Equations: Equations of Porous Medium Type. By Juan Luis Vazquez. Oxford University Press, 2006. Hardback, 248 pages, £45.00 (ISBN 0-19-920297-4).

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LONDON MATHEMATICAL SOCIETY

POPULAR LECTURES 2007

This year's lectures will take place
in London on Thursday 12 July and
in Birmingham during September.

The Lecturers will be:

Dr Stephen Huggett
(University of Plymouth)
'Knots'

Dr Hinke Osinga
(University of Bristol)
'Chaos and Crochet'

LONDON, Institute of Education, 12 July. Commences at 7.00 pm, refreshments at 8.00 pm, ends at 9.30 pm. Admission is free, with ticket. Apply by 6 July to Lee-Anne Parker, London Mathematical Society, De Morgan House, 57-58 Russell Square, London WC1B 4HS (email: parker@lms.ac.uk). A stamped addressed envelope would be appreciated.

BIRMINGHAM. Once details are finalised, they will appear on the LMS website (www.lms.ac.uk).

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