Crux Mathematicorum

VOLUME 38, NO. 2

FEBRUARY / FÉVRIER 2012

IN THIS ISSUE / DANS CE NUMÉRO

EDITORIAL

Shawn Godin

The call goes out for nominations for problems editors. With our change of format the frequency of issues has increased and we have added new problem features, so we need a few more people on board to help deal with the work.

We are also going to be starting some new regular features. We are searching for individuals to help edit these new features. Recommendations or inquiries can be sent to the Editor-in-Chief.

Returning in this issue is Chris Fisher's *Recurring Crux Configurations*. This is a 9-part series of short articles on geometric themes that have appeared in *Crux* repeatedly over the years. The article this issue is on cyclic orthodiagonal quadrilaterals, I hope you enjoy it!

At this point I would like to acknowledge a few solutions that slipped through the cracks: Roy Barbara, Lebanese University, Fanar, Lebanon (problem 3536); Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India (problem 3540); Oliver Geupel, Brühl, NRW, Germany (problems 3576, 3577, 3578, 3579, 3580, 3582, 3584, 3585, 3586, and 3587(a)); Matti Lehtinen, National Defence College, Helsinki, Finland (problems 3358 and 3365); Edmund Swylan, Riga, Latvia (problems 3576, 3577 and 3582); and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA (problems M464, M466, M467, and M468). These are my errors, and I apologize. Please let us know of any errors or omissions.

Finally, this issue features another solution to an unsolved *Crux* problem by Tomasz Cieśla. Last issue we featured his solution to problem 1580 and this issue we give his solution to problem 478 from 1979! Recall, in the December 2010 issue of *Crux* [2010 : 545–547], we published a list of unsolved problems from the journal. Remember, we are always happy to publish solutions to unsolved problems or new solutions, insights or generalizations to previous problems. Submit your solutions to the editor for consideration.

Shawn Godin

MAYHEM SOLUTIONS

Mathematical Mayhem *is being reformatted as a stand-alone mathematics journal for high school students. Solutions to problems that appeared in the last volume of* **Crux** *will appear in this volume, after which time* **Mathematical Mayhem** *will be discontinued in* **Crux**. *New* **Mayhem** *problems will appear when the journal is relaunched in 2013.*
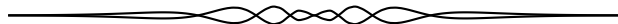
M482. *Proposed by the Mayhem Staff.*

Using four sticks with lengths of 1 cm, 2 cm, 3 cm, and 5 cm, respectively, you can measure any integral length from 1 cm to 10 cm. Note that a stick may only be used once in a particular measurement, so the 1 cm, 2 cm, and 3 cm sticks could be used to measure 6 cm, but not the 3 cm stick twice.

(a) Find a set of ten stick lengths that can be used to represent any integral length from 1 cm to 100 cm.

(b) What is the fewest number of sticks that are needed to represent any integral length from 1 cm to 100 cm?

Solution by Florencio Cano Vargas, Inca, Spain.

(a) We can apply a similar reasoning to the one given in the problem to measure the tens: with sticks of 10, 20, 30 and 50 cm we can measure those lengths from 10 to 100 which are multiples of 10. Therefore, if we combine these sticks with the set of 1, 2, 3 and 5 cm (to measure any length from 1 to 10) it is clear that we cover all the integral distances between 1 and 100 cm. To meet the condition stated in the problem we can add two more sticks of, say 40 and 60 cm. Then the required set of sticks is (in cm units):

$$\{1, 2, 3, 5, 10, 20, 30, 40, 50, 60\}$$

(b) We will show that the minimal number of sticks is seven and we generalize the method to any length. Let l_i represent the length (in cm) of the ith stick and we order them such that $l_i < l_j$ if $i < j$. Clearly we start with $l_1 = 1$ and $l_2 = 2$; with which, we can measure any length up to 3. Then we choose $l_3 = l_1 + l_2 + 1 = 4$. With l_1, l_2 and l_3 we can measure up to length 7, so that we choose $l_4 = 8$. In general, we choose

$$l_{n+1} = \sum_{k=1}^{n} l_k + 1 \quad l_1 = 1$$

and with the set $\{l_1, ..., l_n\}$ will be able to measure lengths up to $L = \sum_{k=1}^{n} l_k$. It is easy to show by induction that an expression for the length of the sticks is

$l_i = 2^{i-1}$ from which we can calculate the lengths we can measure with n sticks, that is:

$$\sum_{k=1}^{n} l_k = 2^n - 1.$$

By imposing that $2^n - 1 > 100$ we have that the minimal value for n is 7. Therefore, the required set is:

$$\{1, 2, 4, 8, 16, 32, 64\}$$

As a matter of fact, we can measure lengths up to $2^7 - 1 = 127$ cm.

Also solved by GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; THARIQ SURYA GUMELAR, student, SMPN 8, Yogyakarta, Indonesia; JUZ'AN NARI HAIFA, student, SMPN 8, Yogyakarta, Indonesia; HASNA NADILA, student, SMPN 8, Yogyakarta, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; and DEXTER WEI and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON. Five incorrect solutions were submitted.

M483. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Triangle ABC has $\angle BAC = 90°$. The feet of the perpendiculars from A to the internal bisectors of $\angle ABC$ and $\angle ACB$ are P and Q, respectively. Determine the measure of $\angle PAQ$.

Solution by Gusnadi Wiyoga, student, SMPN 8, Yogyakarta, Indonesia.

Let $\angle ABC = 2\beta$ and $\angle ACB = 2\alpha$. It is clear that $2\beta + 2\alpha = 90°$, and hence $\beta + \alpha = 45°$. Now consider ΔABP. Since $\angle ABP = \frac{1}{2}\angle ABC = \beta$, then $\angle PAB = 90° - \beta$. Similarly, by considering ΔACQ we get $\angle CAQ = 90° - \alpha$. We now obtain:

$$\begin{aligned}
\angle PAQ &= \angle PAB - \angle QAB \\
&= \angle PAB - [\angle CAB - \angle CAQ] \\
&= (90° - \beta) - [90° - (90° - \alpha)] \\
&= 90° - (\beta + \alpha) \\
&= 90° - 45° \\
&= 45°
\end{aligned}$$
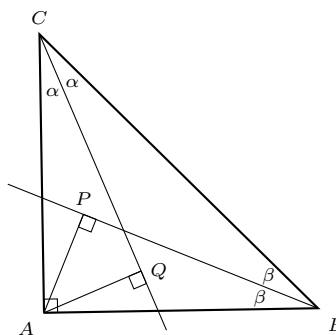
The measure of $\angle PAQ$ is therefore $45°$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; FLORENCIO CANO VARGAS, Inca, Spain; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; MUHAMMAD HAFIZ FARIZI, student, SMPN 8, Yogyakarta, Indonesia; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; VIJAYA PRASAD NULLURI, Rajahmundry, India; RICARD

PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON.

M484. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Solve the equation

$$x^2 + 4\left(\frac{x}{x-2}\right)^2 = 45\,.$$

Solution by Cássio dos Santos Sousa, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil.

Rewriting the equation, we obtain:

$$x^2 + \left(\frac{2x}{x-2}\right)^2 = 45.$$

Completing the square on the left hand side yields:

$$x^2 + \left(\frac{2x}{x-2}\right)^2 + \frac{4x^2}{x-2} = 45 + \frac{4x^2}{x-2}$$
$$\left(x + \frac{2x}{x-2}\right)^2 = 45 + \frac{4x^2}{x-2}$$
$$\left(\frac{x^2}{x-2}\right)^2 = 45 + 4\left(\frac{x^2}{x-2}\right).$$

We can take this expression, rearrange it and complete the square again to get:

$$\left(\frac{x^2}{x-2}\right)^2 - 4\left(\frac{x^2}{x-2}\right) + 4 = 45 + 4$$
$$\left(\frac{x^2}{x-2} - 2\right)^2 = 49,$$

hence, as $x \neq 2$,

$$\frac{x^2}{x-2} - 2 = 7 \Rightarrow x^2 - 9x + 18 = 0, \tag{1}$$

$$\frac{x^2}{x-2} - 2 = -7 \Rightarrow x^2 + 5x - 10 = 0. \tag{2}$$

Finally, from (1) we get the first two roots $x_1 = 6$, $x_2 = 3$ and from (2) we get the last two roots $x_3 = \dfrac{-5 + \sqrt{65}}{2}$ and $x_4 = \dfrac{-5 - \sqrt{65}}{2}$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SCOTT BROWN, Auburn University, Montgomery, AL, USA; FLORENCIO CANO VARGAS, Inca, Spain; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; MUHAMMAD HAFIZ FARIZI, student, SMPN 8, Yogyakarta, Indonesia; IAN JUNE L. GARCES, Ateneo

With the exception of Mariana and the proposer, all other solvers expanded and rearranged the equation to get $x^4 - 4x^3 - 37x^2 + 180x - 180 = 0$, which factors to $(x-3)(x-6)(x^2 + 5x - 10) = 0$, from which the roots are easily extracted.

M485. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Prove that

$$\prod_{k=1}^{n} \binom{n}{k} = \frac{1}{n!} \prod_{k=1}^{n} \frac{k^k}{(n-k)!}$$

for all $n \in \mathbb{N}$.

Solution by Geneviève Lalonde, Massey, ON.

By direct computation we get

$$\prod_{k=1}^{n} \binom{n}{k} = \prod_{k=1}^{n} \frac{n!}{k!(n-k)!}$$

but since these are all products, we can rewrite this as

$$\prod_{k=1}^{n} \binom{n}{k} = \frac{\prod_{k=1}^{n} n!}{\left(\prod_{k=1}^{n} k!\right)\left(\prod_{k=1}^{n} (n-k)!\right)}. \tag{1}$$

Working with the product in the numerator, can rewrite it as

$$\prod_{k=1}^{n} n! = \prod_{k=1}^{n} \prod_{j=1}^{n} j.$$

We can think of this product as if we are considering a number of lattice points and taking the product of their ordinates. From the given product, we travel through the points as indicated in the first diagram below (with the example $n = 5$). Since we are multiplying all these numbers together and multiplication is commutative, it doesn't matter what order we do the multiplication in. We will choose the order in the second diagram below because it will simplify to a more useful form.
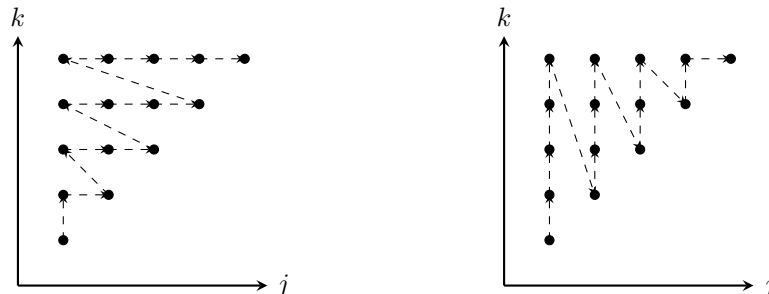
Now the product can be simplified to

$$\prod_{k=1}^{n} n! = \prod_{k=1}^{n}\prod_{j=1}^{n} j = \prod_{j=1}^{n}\prod_{k=1}^{n} j = \prod_{j=1}^{n} j^n.$$

Similarly, working with one of the bottom products we get

$$\prod_{k=1}^{n} k! = \prod_{k=1}^{n}\prod_{j=1}^{k} j = \prod_{j=1}^{n}\prod_{k=j}^{n} j = \prod_{j=1}^{n} j^{n+1-j}.$$

Hence

$$\frac{\displaystyle\prod_{k=1}^{n} n!}{\displaystyle\prod_{k=1}^{n} k!} = \frac{\displaystyle\prod_{j=1}^{n} j^n}{\displaystyle\prod_{j=1}^{n} j^{n+1-j}} = \prod_{j=1}^{n} \frac{j^n}{j^{n+1-j}} = \prod_{j=1}^{n} \frac{j^j}{j} = \frac{\displaystyle\prod_{j=1}^{n} j^j}{\displaystyle\prod_{j=1}^{n} j} = \frac{\displaystyle\prod_{j=1}^{n} j^j}{n!},$$

which, when substituted into (1) yields the result.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; FLORENCIO CANO VARGAS, Inca, Spain; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; CÁSSIO DOS SANTOS SOUSA, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and the proposer. One incorrect solution was received.

M486. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

How many distinct numbers are in the list

$$\frac{1^2 - 1 + 4}{1^2 + 1}, \ \frac{2^2 - 2 + 4}{2^2 + 1}, \ \frac{3^2 - 3 + 4}{3^2 + 1}, \ \ldots, \ \frac{2011^2 - 2011 + 4}{2011^2 + 1} \ ?$$

Solution by Florencio Cano Vargas, Inca, Spain.

We can rewrite the general term of this sequence as

$$a_n = \frac{n^2 - n + 4}{n^2 + 1} = \frac{n^2 + 1 - n + 3}{n^2 + 1} = 1 - \frac{n-3}{n^2 + 1}.$$

If two terms, a_n and a_m, are equal then

$$1 - \frac{n-3}{n^2+1} = 1 - \frac{m-3}{m^2+1}$$

which yields

$$\frac{m-3}{m^2+1} = \frac{n-3}{n^2+1}$$
$$mn^2 - 3n^2 + m - 3 = m^2 n - 3m^2 + n - 3$$
$$3m^2 - 3n^2 + m - n = m^2 n - mn^2$$
$$(3m + 3n + 1)(m - n) = mn(m - n).$$

Since $m = n$ leads to the trivial case $a_n = a_n$, we will assume that $m \neq n$, hence

$$mn = 3m + 3n + 1$$
$$m(n - 3) = 3n + 1.$$

Clearly $n \neq 3$, since m and n are integers and $n = 3 \Rightarrow 3n + 1 = 0$, a contradiction. Thus

$$m = \frac{3n+1}{n-3} = \frac{3n-9+10}{n-3} = 3 + \frac{10}{n-3}.$$

Since the denominator, $n - 3$, has to divide 10 we have only four possibilities:

$$n - 3 = 1 \Rightarrow n = 4, m = 13;$$
$$n - 3 = 2 \Rightarrow n = 5, m = 8;$$
$$n - 3 = 5 \Rightarrow n = 8, m = 5;$$
$$n - 3 = 10 \Rightarrow n = 13, m = 4.$$

So there are only two pairs of terms that are equal: $a_4 = \frac{16}{17} = a_{13}$ and $a_5 = \frac{12}{13} = a_8$. Therefore there are 2009 distinct numbers in the list.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; CÁSSIO DOS SANTOS SOUSA, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer. Three incorrect solutions were received.

Note, the equation $mn = 3m + 3n + 1$ could be rewritten as $mn - 3m - 3n + 9 = 10$ which could be factored as $(m-3)(n-3) = 10$. Then, using the fact that both $m - 3$ and $n - 3$ must be factors of 10, we come to the same conclusion as the featured solution. Also note that if we looked at the infinite sequence $\left\{ \frac{n^2-n+4}{n^2+1} \right\}_{n=1}^{\infty}$, the featured solutions shows that there are only two repeated values in the whole sequence.

M487. *Proposed by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.*

Let m be a positive integer. Find all real solutions to the equation

$$m + \sqrt{m + \sqrt{m + \cdots \sqrt{m + \sqrt{m + \sqrt{x}}}}} = x,$$

in which the integer m occurs n times.

Solution by Florencio Cano Vargas, Inca, Spain.

Let us consider first the equation $m + \sqrt{x} = x$. To solve it, let us denote $t = \sqrt{x} > 0$ and then $x = t^2$ and the equation to solve is $t^2 - t - m = 0$ which has solutions $t = \dfrac{1 \pm \sqrt{1 + 4m}}{2}$. Since $1 - \sqrt{1 + 4m} < 1 - \sqrt{1} = 0$, we will keep only $t = \dfrac{1 + \sqrt{1 + 4m}}{2}$ as t must be positive.

In order to solve the equation given in the problem we define the function $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = m + \sqrt{x}$. Thus we see that

$$m + \sqrt{x} = x \iff x = f(x),$$
$$m + \sqrt{m + \sqrt{x}} = x \iff x = (f \circ f)(x) = f^2(x),$$
$$m + \sqrt{m + \sqrt{m + \sqrt{x}}} = x \iff x = (f \circ f \circ f)(x) = f^3(x),$$
$$\vdots$$
$$m + \sqrt{m + \sqrt{m + \cdots \sqrt{m + \sqrt{m + \sqrt{x}}}}} = x \iff x = (f \circ f \circ f \circ \cdots f \circ f)(x) = f^n(x),$$

where, in the last line, the integer m appears n times.

We can easily see that if $x = f(x)$ then $f^2(x) = (f \circ f)(x) = f(x) = x$ and hence $f^n(x) = x$ for all $n \geq 1$. Thus, a solution to the equation $x = f(x)$ is also a solution to all the equations of the form $f^n(x) = x$. If we let x_0 be the solution to $x = f(x), x > 0$, then

$$\sqrt{x_0} = \frac{1 + \sqrt{1 + 4m}}{2} \Rightarrow x_0 = \frac{1 + 2m + \sqrt{1 + 4m}}{2}.$$
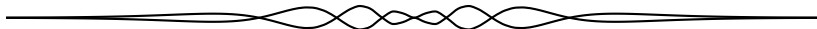
Note that $f(x) - x$ is negative on $(0, x_0)$ and positive on (x_0, ∞). Thus we can prove, by induction, that:

- if $x \in (0, t)$ then $f^n(x) < x$ and

- If $x \in (t, \infty)$ then $f^n(x) > x$.

Hence x_0 is the *only* solution to the equation $f^n(x) = x$ for all $n \geq 1$.

Also solved by by the proposer. One incorrect solution was received.

THE CONTEST CORNER
No. 2

Shawn Godin

*The Contest Corner est une nouvelle rubrique offerte pas **Crux Mathematicorum**, comblant ainsi le vide suite à la mutation en 2013 de* Mathematical Mayhem *et* Skoliad *vers une nouvelle revue en ligne. Il s'agira d'un amalgame de* Skoliad*,* The Olympiad Corner *et l'ancien* Academy Corner *d'il y a plusieurs années. Les problèmes en vedette seront tirés de concours destinés aux écoles secondaires et au premier cycle universitaire ; les lecteurs seront invités à soumettre leurs solutions ; ces solutions commenceront à paraître au prochain numéro.*

Les solutions peuvent être envoyées à :

Shawn Godin
Cairine Wilson S.S.
975 Orleans Blvd.
Orleans, ON, CANADA
K1C 2Z5

ou par courriel à

crux-contest@cms.math.ca.

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 août 2013***.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

CC6. Déterminer des entiers positifs a et b tels que

$$3^{x+a} + 2^{x+a} + 2^x = 2^{x+b} + 3^x$$

soit satisfaite par un certain entier x.

CC7. Soit $U = \{(x,y) : x^2 + y^2 < 1\}$, le disque unitaire ouvert dans le plan $\mathbb{R}^2$. Une corde dans U est définie tout naturellement comme étant une corde du cercle unitaire, moins ses extrémités. Prouver vrai ou prouver faux : il existe une bijection $f : \mathbb{R}^2 \to U$ telle que toute droite dans $\mathbb{R}^2$ corresponde à une corde dans U.

CC8. Voici un jeu simple qui déterminera si A ou B paiera pour une pizza. On brasse un jeu de cartes, puis A et B y pigent une carte en alternance. Le premier à piger un as va payer pour la pizza. Si A est le premier à piger une carte, quelle est la probabilité qu'il paye ? (Fournir votre réponse sous forme de fraction sans facteur commun.)

CC9. Soit $k \geq 3$, entier. Posons $n = \frac{k(k+1)}{2}$. Soit $S \subset \mathbb{Z}_n$ tel que $\|S\| = k$. Démontrer que $S + S \neq \mathbb{Z}_n$. Noter que $\|S\|$ représente la cardinalité de S et que $S + S = \{x + y \mid x \in S, y \in S\}$.

CC10. Soit m un entier positif et soit $d(m)$ le nombre de diviseurs entiers positifs de m. Déterminer tous les entiers positifs n tels que $d(n) + d(n + 1) = 5$.

. . . .

CC6. Determine all pairs of positive integers a and b for which

$$3^{x+a} + 2^{x+a} + 2^x = 2^{x+b} + 3^x$$

is satisfied for some integer x.

CC7. Let $U = \{(x, y) : x^2 + y^2 < 1\}$ be the open unit disc in the plane $\mathbb{R}^2$. A chord of U is naturally defined to be a chord of the unit circle with its distinct endpoints removed. Prove or disprove: there is a bijection $f : \mathbb{R}^2 \to U$ such that every straight line in $\mathbb{R}^2$ is mapped by f onto a chord of U.

CC8. To see who pays for a pizza, A and B play the following simple game. They shuffle a deck of cards, and then in turns draw cards. The first person to draw an ace pays for the pizza. If A draws first, what is the probability that he buys? (Express your answer as a fraction in lowest terms.)

CC9. Let $k \geq 3$ be an integer. Let $n = \frac{k(k+1)}{2}$. Let $S \subset \mathbb{Z}_n$ with $\|S\| = k$. Show that $S + S \neq \mathbb{Z}_n$. Note that $\|S\|$ denotes the cardinality of S and $S + S = \{x + y \mid x \in S, y \in S\}$.

CC10. Given a positive integer m, let $d(m)$ be the number of positive divisors of m. Determine all positive integers n such that $d(n) + d(n + 1) = 5$.

If you know of a mathematics contest at the high school or undergraduate level whose problems you would like to see in *Contest Corner*, please send information about the contest to `crux-contest@cms.math.ca`.

THE OLYMPIAD CORNER
No. 300

Nicolae Strungaru

*Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 août 2013**.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

OC66. Soit $n \geq 2$ un entier positif. Trouver toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ de sorte que
$$f(x - f(y)) = f(x + y^n) + f(f(y) + y^n), \qquad \forall x, y \in \mathbb{R}.$$

OC67. Un 2011-gone convexe est dessiné au tableau. Pierre est occupé à dessiner ses diagonales de sorte que chaque nouvelle diagonale ne coupe pas plus qu'une seule des diagonales déjà dessinées. Quel est le plus grand nombre de diagonales que Pierre peut dessiner ?

OC68. Trouver tous les entiers x, y de sorte que
$$x^3 + x^2 + x = y^2 + y.$$

OC69. Soit n un entier positif et soit $P(x, y) = x^n + xy + y^n$. Montrer qu'on ne peut pas trouver deux polynômes $G(x, y)$ et $H(x, y)$ à coefficients réels tels que
$$P(x, y) = G(x, y) \cdot H(x, y).$$

OC70. $\triangle ABC$ est un triangle tel que $\angle C$ and $\angle B$ sont acutangles. Soit D un point variable sur BC tel que $D \neq B, C$ et AD n'est pas perpendiculaire à BC. Soit d la droite passant par D et perpendiculaire à BC. Supposons que $d \cap AB = E, d \cap AC = F$. Soit M, N, P les centres des cercles inscrits de $\triangle AEF, \triangle BDE, \triangle CDF$. Montrer que A, M, N, P sont cocycliques si et seulement d passe par le centre du cercle inscrit de $\triangle ABC$.

· ·

OC66. Let $n \geq 2$ be a positive integer. Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that
$$f(x - f(y)) = f(x + y^n) + f(f(y) + y^n), \qquad \forall x, y \in \mathbb{R}.$$

OC67. A convex 2011-gon is drawn on the board. Peter keeps drawing its diagonals in such a way that each newly drawn diagonal intersects no more than one of the already drawn diagonals. What is the greatest number of diagonals that Peter can draw?

OC68. Find all integers x, y so that

$$x^3 + x^2 + x = y^2 + y\,.$$

OC69. Let n be a positive integer and let $P(x,y) = x^n + xy + y^n$. Prove that we cannot find two non-constant polynomials $G(x,y)$ and $H(x,y)$ with real coefficients such that

$$P(x,y) = G(x,y) \cdot H(x,y)\,.$$

OC70. $\triangle ABC$ is a triangle such that $\angle C$ and $\angle B$ are acute. Let D be a variable point on BC such that $D \neq B, C$ and AD is not perpendicular to BC. Let d be the line passing through D and perpendicular to BC. Assume $d \cap AB = E$, $d \cap AC = F$. Let M, N, P be the incentres of $\triangle AEF$, $\triangle BDE$, $\triangle CDF$. Prove that A, M, N, P are concyclic if and only if d passes through the incentre of $\triangle ABC$.

OLYMPIAD SOLUTIONS

OC6. In the diagram, $ABCD$ is a square, with U and V interior points of the sides AB and CD respectively. Determine all the possible ways of selecting U and V so as to maximize the area of the quadrilateral $PUQV$. (*Originally question # 3 from the 1992 Canadian Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, France; Florencio Cano Vargas, Inca, Spain; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Bataille.
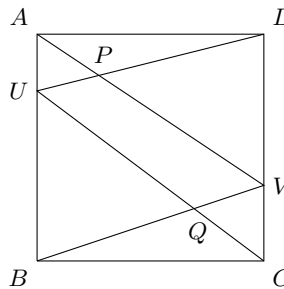
We will use $[\cdot]$ to denote the area. Let a be the length of the sides of the square. Since

$$[AVB] + [DUC] = \frac{a^2}{2} + \frac{a^2}{2} = [ABCD]\,, \tag{1}$$

we get

$$[PUQV] = [APD] + [QBC]\,.$$

Let $u = AU$ and $v = CV$, and let P', Q' be the orthogonal projections of P onto AD, respectively Q onto BC. Then

$$\frac{AP'}{AD} = \frac{PP'}{DV} \Rightarrow AP' = \frac{AD}{DV}PP' = \frac{a}{a-v}PP'\,.$$

Similarly we get $DP' = \frac{a}{u}PP'$.

Then

$$a = AP' + DP' = \frac{a}{a-v}PP' + \frac{a}{u}PP'\,,$$

which yields

$$PP' = \frac{u(a-v)}{a+u-v}.$$

In a similar way, we obtain

$$QQ' = \frac{v(a-u)}{a+v-u}\,.$$

Since $[APD] = \frac{1}{2}a \cdot PP'$ and $[BQC] = \frac{1}{2}a \cdot QQ'$, (1) yields

$$[PUQV] = \frac{a}{2}\left(\frac{u(a-v)}{a+u-v} + \frac{v(a-u)}{a+v-u}\right).$$

Now, by AM-GM, $u(a-v) \leq \frac{(a+u-v)^2}{4}$ and $v(a-u) \leq \frac{(a+v-u)^2}{4}$, with equality if and only if $u + v = a$.

Thus

$$[PUQV] \leq \frac{a}{8}(a + u - v + a + v - u) = \frac{a^2}{4}.$$

In conclusion, $[PUQV] \leq \frac{a^2}{4}$ and the maximum value $\frac{a^2}{4}$ is attained exactly when $AU + VC = a$.

OC7. Let n be a natural number such that $n \geq 2$. Show that

$$\frac{1}{n+1}\left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right).$$

(*Originally question # 3 from the 1998 Canadian Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Manes.

Our inequality is equivalent to

$$n\left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) > (n+1)\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right).$$

Using the well known Catalan's inequality

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n},$$

we get

$$1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} = \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) + \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right).$$

Thus the inequality to prove becomes

$$n\left(\frac{1}{2} + \frac{1}{4} + \cdots \frac{1}{2n}\right) + n\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) > (n+1)\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right),$$

or equivalently

$$n\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) > \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right). \tag{1}$$

Since $n \geq 2$, for all $1 \leq j \leq n$ we have $n + j < 2nj$ and hence:

$$\frac{n}{n+j} > \frac{1}{2j}.$$

Adding these inequalities yields (1) and completes the proof.

OC8. For each real number r let T_r be the transformation of the plane that takes the point (x, y) into the point $(2^r x, r2^r x + 2^r y)$. Let F be the family of all such transformations *i.e.* $F = \{T_r : r \in \mathbb{R}\}$. Find all curves $y = f(x)$ whose graphs remain unchanged by every transformation in F.
(*Originally question # 2 from the 1983 Canadian Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Oliver Geupel, Brühl, NRW, Germany. We give the solution of Bataille.

It is readily checked that T_r is a bijective transformation with $T_r^{-1} = T_{-r}$. Thus, it suffices to find all curves $\mathcal{C}_f : y = f(x)$ such that $T_r(\mathcal{C}_f) \subset \mathcal{C}_f$ for all real r.
We show that the solutions are the curves $y = f_{n,p}(x)$ where $f_{n,p}$ is defined by

$$f_{n,p}(x) = \begin{cases} x(\log_2(x) + p) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ x(\log_2(|x|) + n) & \text{if } x < 0 \end{cases}$$

where n, p are real constants.
First, let $f_{n,p}$ be such a function, $M(x, f_{n,p}(x))$ be any point on $\mathcal{C}_{f_{n,p}}$ and r any real number. Then $T_r(M)$ has coordinates $x' = 2^r x$, $y' = r2^r x + 2^r f_{n,p}(x)$.
If $x = 0$, then $y' = 2^r f_{n,p}(0) = 0 = f_{n,p}(x')$.
If $x < 0$, then

$$f_{n,p}(x') = 2^r x(\log_2(|2^r x|) + n) = 2^r x(r + \log_2(|x|) + n) = r2^r x + 2^r f_{n,p}(x) = y'$$

hence $y' = f_{n,p}(x')$ and $T_r(M)$ is on $\mathcal{C}_{f_{n,p}}$.
The case $x > 0$ is easily treated in a similar way. Thus,

$$T_r(\mathcal{C}_{f_{n,p}}) \subset \mathcal{C}_{f_{n,p}} .$$

Conversely, let f be such that $T_r(\mathcal{C}_f) \subset \mathcal{C}_f$ for all r. This means that for all r, x, we have

$$f(2^r x) = r2^r x + 2^r f(x) . \tag{1}$$

Taking $r = 1, x = 0$ in (1) yields $f(0) = 2f(0)$, hence we must have $f(0) = 0$.
Let $\alpha > 0$. Setting $r = \log_2(\alpha)$ and $x = 1$ in (1), we get

$$f(\alpha) = \alpha \log_2(\alpha) + \alpha f(1) = \alpha(\log_2(\alpha) + p) ,$$

where $p = f(1)$.
Now setting $r = \log_2(\alpha)$ and $x = -1$ in (1), we get

$$f(-\alpha) = -\alpha \log_2(\alpha) + \alpha f(-1) = (-\alpha)(\log_2(|-\alpha|) + n) ,$$

where $n = -f(-1)$. It follows that $f = f_{n,p}$.

OC9. A deck of $2n + 1$ cards consists of a joker and, for each number between 1 and n inclusive, two cards marked with that number. The $2n + 1$ cards are placed in a row, with the joker in the middle. For each k with $1 \le k \le n$, the two cards numbered k have exactly $k - 1$ cards between them. Determine all the values of n not exceeding 10 for which this arrangement is possible. For which values of n is it impossible?
(*Originally question # 5 from the 1992 Canadian Mathematical Olympiad.*)

Solution by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

We prove that such an arrangement is possible if and only if

$$n \in \{3, 4, 7, 8\}. \tag{1}$$

Let's number the places from left to right $1, 2, 3, ..., 2n+1$. The joker is placed on $n+1$. For each $1 \le k \le n$, let the two cards with number k be on position a_k and $a_k + k$. Then we have

$$\sum_{k=1}^{n}(a_k + a_k + k) = \left(\sum_{k=1}^{2n+1} k\right) - (n+1).$$

Thus

$$\sum_{k=1}^{n} a_k = \frac{3n(n+1)}{4}.$$

Thus, either n or $n+1$ is divisible by 4, which implies (1).

To complete the proof, we need to show that these n work. Indeed we have the following:

$$
\begin{array}{lll}
n = 3 & : & 2, 3, 2, J, 3, 1, 1 \\
n = 4 & : & 2, 4, 2, 3, J, 4, 3, 1, 1 \\
n = 7 & : & 5, 3, 4, 7, 3, 5, 4, J, 6, 7, 1, 1, 6 \\
n = 8 & : & 6, 8, 5, 7, 1, 1, 6, 5, J, 8, 7, 4, 2, 3, 2, 4, 3
\end{array}
$$

OC10. The number 1987 can be written as a three digit number xyz in some base b. If $x + y + z = 1 + 9 + 8 + 7$, determine all possible values of x, y, z, b.
(*Originally question # 2 from the 1987 Canadian Mathematical Olympiad.*)

Solved by Florencio Cano Vargas, Inca, Spain; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Cano Vargas.

Let $xyz_{(b)} = 1987$.
Thus

$$xb^2 + yb + z = 1987. \tag{1}$$

Since x, y, z are digits in base b it follows immediately that $b > 10$ and $b^2 < 1987$.

From the problem we also have

$$x + y + z = 25. \tag{2}$$

Subtracting (2) from (1) we get

$$(b-1)(xb + x + y) = 1962.$$

And hence

$$(b-1)|2 \cdot 3^2 \cdot 109.$$

Since $b^2 < 1987$ we have $109 \nmid (b-1)$ and hence $(b-1)$ is a divisor of 18. Hence, using $b > 10$, we get $b - 1 = 18$. Thus $b = 19$.

We showed that $b = 19$ is the only possible solution. We see now that this works. Indeed, changing 1987 into base 19 we get

$$1987_{(10)} = 59E_{(19)},$$

where E is the digit eleven in base 19, and this satisfies the requirements of the problem.

BOOK REVIEWS

Amar Sodhi

The Mathematical Mechanic: Using Physical Reasoning to Solve Problems
by Mark Levi
Princeton University Press, 2009
ISBN: 978-0-691-14020-9, Hardcover, 186 + viii pages, US$19.95
Reviewed by **Nora Franzova**, *Langara College, Vancouver, BC*

> *A mathematician and a physicist are sitting on a bench and notice a lovely
> lady sitting on a bench across the street.*
> *Physicist suggests: "Let's walk over and introduce ourselves." Then he
> immediately starts crossing the street.*
> *Mathematician remains seated.*
> *Physicist turns back with a question: "Why are you not coming?"*
> *Mathematician replies: "Well, there is no point to it. We first have to cross
> half the distance between us and her, then half of what is left and then half
> of what is left again ... We will never get there."*
> *To this the physicist replies: "Close enough for me."*
>
> *Anonymous*

Often we ask ourselves whether explanations of mathematical truths that are given
via physical approach are "close enough". Is physics just giving the ideas to math and
math just giving the tools to physics? Is one of them either "above" the other or more
needed than the other?

In the introductory chapter, Mark Levi states that: "The two subjects are so
intimately intertwined that both suffer if separated." Levi goes on to say that: "In this
book physics is put to work for mathematics, proving to be a very efficient servant (with
apologies to physicists)".

Physical ideas implemented in very creative ways to solve math problems are the
main feature of this book. The author recalls that Archimedes already used this approach
in proving his famous integral calculus theorem on the volumes of a cylinder, a sphere,
and a cone using an imagined balancing scale. How unfortunate it is that not many
Calculus books mention this!

Following the introduction, there are 10 chapters that gradually build up the dif-
ficulty of the problems and an appendix in which the necessary physics background is
presented. The reader will discover some very imaginative physical solutions to famous
topics like: Pythagorean Theorem, Max and Min problems — including some beautiful
calculus classics like: The Cheapest Can, The Best Spot in a Drive-in Theatre, Maneuver-
ing a Ladder (through perpendicular hallways), Lifeguard Saving a Swimmer. Topics also
go beyond usual undergraduate math classes to higher-level topics like Euler-Lagrange
Equations and Gauss-Bonnet Theorem, passing through finding trigonometric derivatives
and integrals, and Green's Theorem.

The author's imagination brings in "physical incarnations" of these interesting
problems. Some problems invite more than one such presentation. Pythagorean The-
orem offers itself in several different physical ideas — maybe because it is a topic that
occurs so naturally. So we witness a proof by using a prism shaped fish tank, we "put"
Pythagoras on ice and use kinetic energy as we push off the x–axis and y–axis. And
then, there is a much more "explosive" proof, with a very compressed spring that is cut

and literally releases another proof of Pythagorean Theorem. My favorite is the proof done by sweeping. Levi uses electric shorting to prove inequalities between arithmetic and geometric mean. One can find centre of mass by conservation of energy, and can compute some integrals by lifting weights.

Physical illustrations are chosen from our everyday life — like spokes on a bike wheel, the path traced by a wheel of a shopping cart, even a person's nose becomes a vector. When imagination has to be stretched, illustrations are provided to describe some imaginary physical set up. It is almost tempting to go beyond just illustrations and write some java program that shows the process how a vacuum filled piston with two rings attached at its ends becomes the ladder one maneuvers through perpendicular hallways. There are a few physical incarnations though, that require almost a cartoon style imagination. The author obviously found many of these images inspiring in solving a variety of problems and the reader can feel the excitement as each topic is extended to more and more possibilities.

Have you ever wondered why negative times negative is positive? There is a thoughtful argument at the beginning of Chapter 11. This ambitious chapter prides itself in making Complex Variables Simple(r). Here Mark Levi links complex functions with idealized fluid flow in a plane and offers inspiration to those of us who need to find yet another explanation to justify to the students how imaginary numbers help us solve real world problems.

There are additional problems introduced at the end of sections, inviting the reader to use presented strategies and apply them to solve new or old favorites. Just for a teaser show that the orthogonality of eigenvectors of a symmetric matrix is a consequence of the non-existence of a perpetual motion machine. Hints, but not full solutions, are provided to many of the problems.

At the beginning of the book Levi admits that his main focus is on presenting a concept and not complete detailed proofs. Yet, only in a few places does one feel like they are really missing details. The book is not a textbook, but can serve as a supplement to instruction in providing a different approach to the material. It would, however, work only with students who have sufficient physics background. Even though the appendix claims to provide all one needs from physics, the reviewer feels that some background in calculus-based physics would definitely make the book a better read.

The book is a collection of playful ideas that present math as a subject developed in the world where physics is the axiomatic basis to sciences, to paraphrase the author. If we think of physics and mathematics as two different languages, this book confirms a well known, but underappreciated truth: it is great to be bilingual.

RECURRING CRUX CONFIGURATIONS 5

J. Chris Fisher

Cyclic Orthodiagonal Quadrilaterals

A quadrilateral $ABCD$ is called *cyclic* if its vertices are arranged around its circumcircle in the same order as they appear in the name; it is *orthodiagonal* if $AC \perp BD$. Standard references such as [1, pages 136-139] list some of their familiar properties; other references that have been suggested by **Crux** readers are [1] and [3]. A familiar property that was generalized in a "Klamkin Quickie" is quick enough to reproduce here in full:

[2003 : 375, 377] For any four points A, B, C, D in Euclidean 3-space, AC and BD are orthogonal if and only if

$$AB^2 + CD^2 = BC^2 + DA^2. \tag{1}$$

Proof. Denote the vectors from A to B, C, and D by $\vec{B}, \vec{C}$, and $\vec{D}$, respectively. Then we have the identity

$$
\begin{aligned}
AB^2 + CD^2 - BC^2 - DA^2 &= \vec{B}^2 + (\vec{C} - \vec{D})^2 - (\vec{B} - \vec{C})^2 - \vec{D}^2 \\
&= 2\vec{C} \cdot (\vec{B} - \vec{D}).
\end{aligned}
$$

Thus $AB^2 + CD^2 = BC^2 + DA^2$ if and only if $\vec{C} \cdot (\vec{B} - \vec{D}) = 0$, which is true if and only if AC and BD are orthogonal.

Moreover, if these four points are the vertices of an orthodiagonal quadrilateral inscribed in a circle of radius R, and the diagonals intersect in the point P, we have

$$PA^2 + PB^2 + PC^2 + PD^2 = AB^2 + CD^2 = BC^2 + DA^2 = 8R^2. \tag{2}$$

Problem 3 of the 1991 British Mathematical Olympiad [1993 : 5; 1994 : 69-70] asked for a proof of (2) and inquired about its converse; a nonsquare rectangle serves to show that the equality

$$PA^2 + PB^2 + PC^2 + PD^2 = \frac{1}{2}\left(AB^2 + CD^2 + BC^2 + DA^2\right)$$

fails to imply that a cyclic quadrilateral be orthodiagonal.

An interesting result that solver Joe Howard found to be an easy consequence of equation (1) appeared in **Crux** as Problem 3402 [2009 : 42, 44; 2010 : 50-51] (proposed by Mihály Bencze): *If D and E are the midpoints of sides AB and AC of triangle ABC, then $CD \perp BE$ if and only if $5BC^2 = AC^2 + AB^2$.*

Among other familiar properties of the cyclic orthodiagonal quadrilateral $ABCD$ found in [1] we have

- (*Brahmagupta's theorem*) Any line through the intersection point P of the diagonals of an orthodiagonal quadrilateral that is perpendicular to a side bisects the opposite side.

- The midpoints of the four sides and the projections of P (where the diagonals intersect) onto the sides all lie on a circle whose centre is the midpoint between P and the circumcentre of $ABCD$.

You might recognize the second item from our previous configurations column: *The projections of the point P onto the sides of a cyclic orthodiagonal quadrilateral form the vertices of a bicentric quadrilateral*. This is Problem 2209 [1997 : 47; 1998 : 112-113], which comes with a proof, numerous references, and a converse: *If $QRST$ is a bicentric quadrilateral, then the lines that are perpendicular at $Q, R, S,$ and T to the lines from the incentre P form the sides of a cyclic orthodiagonal quadrilateral whose diagonals meet at P.*
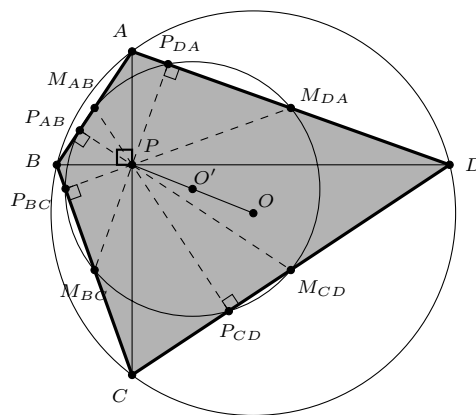

Figure 1: For the cyclic quadrilateral $ABCD$, $AC \perp BD$ implies that the midpoints of the four sides and the projections of P are cyclic; lines through P perpendicular to one side pass through the midpoint of the opposite side. Here P_{XY} and M_{XY} represent the projection of P and the midpoint of side XY, respectively.

Problem 1062 [1985 : 219; 1987 : 17-19] (Proposed by Murray S. Klamkin). If a convex quadrilateral is inscribed in a circle with centre O, then the distance from O to any side is half the length of the opposite side if and only if the diagonals are orthogonal.

Curiously, on page 138 of [1] Theorem 278 claims only that the orthodiagonal property of a cyclic quadrilateral implies the half-length property (and fails to mention the converse). The featured solution by Klamkin to part (b) of his problem proved, furthermore, that *given an oval which is centrosymmetric with centre O, should all convex orthodiagonal quadrilaterals inscribed in the oval have the property that the distance of any side from O is half the length of the opposite side, then that oval must be a circle.* It remains an open question whether an oval must be a circle if the half-length property of an inscribed convex quadrilateral always implies the orthodiagonal property.

Problem 1836 [1993 : 113; 1994 : 84-85] (proposed by Jisho Kotani and reworded here). If $ABCD$ is a cyclic quadrilateral, then the sum of the areas

of the four crescent-shaped regions outside the circumcircle and inside the circles with diameters AB, BC, CD, DA equals the area of $ABCD$ if and only if its diagonals are orthogonal.

Problem 2338 [1998 : 234; 1999 : 243-245] (proposed by Toshio Seimiya, extended by Peter Y. Woo). When a convex quadrilateral is subdivided into four triangles by its two diagonals, then the incentres of the four triangles are the vertices of a cyclic orthodiagonal quadrilateral if and only if the initial quadrilateral has an incircle.

Problem 2978 [2004 : 429, 432; 2005 : 470-472] (proposed by Christopher J. Bradley). If $QRST$ is a cyclic quadrilateral whose adjacent interior angle bisectors intersect in the points A, B, C, and D, then $ABCD$ is a cyclic orthodiagonal quadrangle. Furthermore, the point where the diagonals AC and BD intersect is collinear with the circumcentres of the two quadrilaterals.

References

[1] Agnis Andžāns, On the Inscribed Orthodiagonal Quadrilaterals, *Mathematics and Informatics Quarterly*, **3**:1 (1993) 6-8.

[2] Nathan Altshiller Court, *College Geometry: An Introduction to the Modern Geometry of the Triangle and the Circle*, 2nd ed. Barnes & Noble, 1965.

[3] Jordan Tabov, Simple Properties of Orthodiagonal Quadrilaterals, *Mathematics and Informatics Quarterly*, **1**:1 (1991) 1-5.
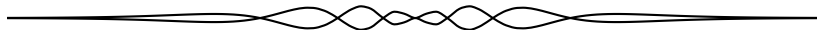
PROBLEMS

*Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 août 2013**. Une étoile ($\star$) après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7, et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8, et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal et Rolland Gaudet, de Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.
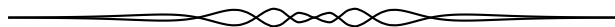
3711. *Proposé par Mehmet Sahin, Ankara, Turquie.*

Soit ABC un triangle rectangle avec $\angle A = 90°$. Soit AD une hauteur et supposons que la bissectrice de $\angle B$ coupe AD en K. Si $\angle ACK = 2\angle DCK$ montrer que $KC = 2AD$.

3712. *Proposé par Johan Gunardi, étudiant, SMPK 4 BPK PENABUR, Jakarta, Indonésie.*

Montrer que pour trois nombres réels positifs arbitraires a, b, c, on a

$$\sqrt{\frac{a(a^2+bc)}{b+c}} + \sqrt{\frac{b(b^2+ca)}{c+a}} + \sqrt{\frac{c(c^2+ab)}{a+b}} \geq a+b+c.$$

3713. *Proposé par D. M. Bătineţu, Giurgiu, Bucharest and Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Calculer

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!!c_n}} - \frac{n^2}{\sqrt[n]{(2n-1)!!e_n}}\right),$$

où $e_n = \left(1+\dfrac{1}{n}\right)^n$ et $c_n = -\ln n + \displaystyle\sum_{k=1}^{n}\frac{1}{k}$, pour tout entier positif n.

3714. *Proposé par Panagiote Ligouras, École Secondaire Léonard de Vinci, Noci, Italie.*

Dans un triangle donné ABC on choisit respectivement les points B_1 sur le côté AB et C_1 sur le côté AC tels que $B_1C_1 \parallel BC$, les points C_2 et A_2 sur les côtés BC et BA de sorte que $C_2A_2 \parallel CA$, les points A_3 et B_3 sur CA et CB de sorte que $A_3B_3 \parallel AB$. De plus, notons A_i', B_i', C_i' A_i, B_i, C_i les projections de A_i, B_i, C_i sur les côtés parallèles correspondants du triangle (pour former trois rectangles, tels $B_1B_1'C_1'C_1$).

(a) Montrer que si les rapports des aires de chacun des triangles définis à celles de son rectangle adjacent sont égaux, à savoir

$$\frac{[AB_1C_1]}{[B_1B_1'C_1'C_1]} = \frac{[BC_2A_2]}{[C_2C_2'A_2'A_2]} = \frac{[CA_3B_3]}{[A_3A_3'B_3'B_3]},$$

alors les rayons des cercles inscrits de ces trois triangles sont aussi égaux.

b) Trouver le rapport du rayon du cercle inscrit du triangle formé par les droites C_1B_1, A_2C_2, B_3A_3 au rayon du cercle inscrit du ΔABC.

3715. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Soit h_a, h_b, h_c les hauteurs d'un triangle, r_a, r_b, r_c les rayons des cercles extérieurement tangents, r le rayon du cercle inscrit et R celui du cercle circonscrit. Montrer que

$$\frac{h_a^2}{r_a} + \frac{h_b^2}{r_b} + \frac{h_c^2}{r_c} \geq 4r \left(2 - \frac{r}{R}\right)^2.$$

3716. *Proposé par Michel Bataille, Rouen, France.*

Etant donné trois entiers positifs a, u_1, u_2, on définit u_n via la récursion

$$u_{n+2} = 2au_{n+1} + u_n, \ n \in \mathbb{N}.$$

Montrer qu'il existe un nombre réel positif r tel que

$$u_{n+2^{k+1}} = \lfloor 1 + r^{2^k} \rfloor \cdot u_{n+2^k} - u_n$$

pour tous les entiers positifs n, k.

3717. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Déterminer toutes les solutions réelles $(x_1, x_2, \ldots, x_n)$ au système d'équations

$$x_1 = \exp\left[\sin\left(x_2 - \sqrt{1 - \ln^2 x_1}\right)\right]$$
$$x_2 = \exp\left[\sin\left(x_3 - \sqrt{1 - \ln^2 x_2}\right)\right]$$
$$\vdots$$
$$x_n = \exp\left[\sin\left(x_1 - \sqrt{1 - \ln^2 x_n}\right)\right].$$

3718. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soient a, b et c des nombres réels tels que $a > b > c$ et $b + c = 503$.

(i) Déterminer la valeur minimale de

$$A = \frac{a^2}{a-b} + \frac{b^2}{b-c}.$$

(ii) Déterminer des valeurs de a, b, c telles que A atteint sa valeur minimale.

3719. *Proposé par Johan Gunardi, étudiant, SMPK 4 BPK PENABUR, Jakarta, Indonésie.*

Démontrer que pour tous nombres réels a, b, c, l'inégalité suivante tient

$$\sqrt{\frac{a(a^2+bc)}{b+c}} + \sqrt{\frac{b(b^2+ca)}{c+a}} + \sqrt{\frac{c(c^2+ab)}{a+b}} \geq a + b + c.$$

3720. *Proposé par Michel Bataille, Rouen, France.*

Soit ABC un triangle tel que $\angle BAC = 90°$, soit O le mipoint de BC et soit H le pied de l'altitude émanant de A. Enfin, soit K sur le segment AH, tel que $\angle BKC = 135°$ et soit L tel que $AHCL$ est un rectangle. Démontrer que $OL = OB + KH$.

. . . .

3711. *Proposed by Mehmet Sahin, Ankara, Turkey.*

Let ABC be a right triangle with $\angle A = 90°$. Let AD be an altitude, and let the angle bisector of $\angle B$ meet AD in K. If $\angle ACK = 2\angle DCK$ then prove that $KC = 2AD$.

3712. *Proposed by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.*

Prove that for any positive real numbers a, b, c

$$\sqrt{\frac{a(a^2+bc)}{b+c}} + \sqrt{\frac{b(b^2+ca)}{c+a}} + \sqrt{\frac{c(c^2+ab)}{a+b}} \geq a + b + c.$$

3713. *Proposed by D. M. Bătinețu, Giurgiu, Bucharest and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Compute

$$\lim_{n \to \infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!!c_n}} - \frac{n^2}{\sqrt[n]{(2n-1)!!e_n}} \right),$$

where $e_n = \left(1 + \frac{1}{n}\right)^n$ and $c_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$, for any positive integer n.

3714. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Given a triangle ABC define points B_1 on side AB and C_1 on AC so that $B_1C_1 \parallel BC$; similarly take C_2 and A_2 on sides BC and BA with $C_2A_2 \parallel CA$, and A_3, B_3 on CA, CB with $A_3B_3 \parallel AB$. Furthermore, denote A_i', B_i', C_i' the projections of A_i, B_i, C_i onto the corresponding parallel sides of the given triangle (to form three rectangles such as $B_1B_1'C_1'C_1$).

(a) Prove that if the ratios of the areas of each defined triangle to that of its adjacent rectangle are equal, namely

$$\frac{[AB_1C_1]}{[B_1B_1'C_1'C_1]} = \frac{[BC_2A_2]}{[C_2C_2'A_2'A_2]} = \frac{[CA_3B_3]}{[A_3A_3'B_3'B_3]},$$

then the inradii of those three triangles are also equal.

(b) Determine the ratio of the inradius of the triangle formed by the lines C_1B_1, A_2C_2, B_3A_3 to the inradius of ΔABC.

3715. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let h_a, h_b, h_c be the altitudes, r_a, r_b, r_c the exradii, r the inradius and R the circumradius of a triangle. Prove that

$$\frac{h_a^2}{r_a} + \frac{h_b^2}{r_b} + \frac{h_c^2}{r_c} \geq 4r\left(2 - \frac{r}{R}\right)^2.$$

3716. *Proposed by Michel Bataille, Rouen, France.*

Given positive integers a, u_1, u_2 let u_n be defined by the recursion

$$u_{n+2} = 2au_{n+1} + u_n, \ n \in \mathbb{N}.$$

Show that there exists a positive real number r such that

$$u_{n+2^{k+1}} = \lfloor 1 + r^{2^k} \rfloor \cdot u_{n+2^k} - u_n$$

for all positive integers n, k.

3717. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Find all real solutions $(x_1, x_2, \ldots, x_n)$ of the system of equations

$$x_1 = \exp\left[\sin\left(x_2 - \sqrt{1 - \ln^2 x_1}\right)\right]$$
$$x_2 = \exp\left[\sin\left(x_3 - \sqrt{1 - \ln^2 x_2}\right)\right]$$
$$\vdots$$
$$x_n = \exp\left[\sin\left(x_1 - \sqrt{1 - \ln^2 x_n}\right)\right].$$

3718. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b and c be real number such that $a > b > c$ and $b + c = 503$.

(i) Find the minimum value of the expression

$$A = \frac{a^2}{a-b} + \frac{b^2}{b-c}.$$

(ii) Determine values of a, b, c for which A attains its minimum value.

3719. *Proposed by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.*

Prove that for any positive real numbers a, b, c,

$$\sqrt{\frac{a(a^2+bc)}{b+c}} + \sqrt{\frac{b(b^2+ca)}{c+a}} + \sqrt{\frac{c(c^2+ab)}{a+b}} \geq a + b + c.$$

3720. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle with $\angle BAC = 90°$, O be the midpoint of BC and H be the foot of the altitude from A. Let K, on segment AH, be such that $\angle BKC = 135°$ and L be such that $AHCL$ is a rectangle. Show that $OL = OB + KH$.
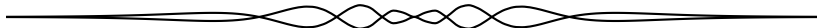
SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

478 ⋆ . [1979 : 229; 1980 : 219-220; 1985 : 189-190; 1987: 151-152] *Proposed by Murray S. Klamkin, University of Alberta.*

Consider the following theorem: *If the circumcircles of the four faces of a tetrahedron are congruent, then the circumcentre O of the tetrahedron and its incentre I coincide.* An editor's comment following *Crux* 330 [1978: 264] claims that the proof of this theorem is "easy." Prove it.

V. Solution by Tomasz Cieśla, student, University of Warsaw, Poland with help from Dominik Burek, student, Jagiellonian University, Cracow, Poland.

We begin with a lemma. (See [1, Article 263], where an alternative proof can be found.)

Lemma *The points F where the face ABC of tetrahedron $ABCD$ touches the insphere, and G where it touches the exsphere opposite D, are isogonal conjugates in triangle ABC.*

Proof of the Lemma. Consider the infinite cone with vertex D that is tangent to the faces DAB, DBC, DCA. Let e be the intersection of the face ABC with this cone; thus e is an ellipse inscribed in triangle ABC. There is a nice pictorial proof by Dandelin (in 1822) that F and G are the foci of e. (See, for example, [2, Section 17.3, page 227] or Wikipedia's article on "Dandelin Spheres." Indeed, consider any point P on e. Let DP be tangent to the insphere at X and to the exsphere at Y. Since segments PF and PX are tangent to the insphere, we have $PF = PX$. Likewise, $PG = PY$, whence $FP + PG = PX + PY = XY$, which does not depend on the choice of P on e.) Because F and G are foci of an ellipse inscribed in ΔABC, they are isogonal conjugates with respect to that triangle. (See, for example, [3, Paragraph 2323, pages 1109-1110].)

For the proof of the theorem, observe that the four finite cones having the circumcircles as base and O as vertex are congruent, whence O must be equidistant from each face of the tetrahedron; here are the details: Denote the orthogonal projections of O into faces BCD, CDA, DAB, ABC of the tetrahedron $ABCD$ by O_A, O_B, O_C, O_D, respectively. Let $R = OA = OB = OC = OD$. By Pythagoras we have $R^2 - OO_A^2 = O_A B^2 = O_A C^2 = O_A D^2$, thus O_A is the circumcentre of BCD. Analogously, O_B, O_C, O_D are the circumcentres of CDA, DAB, ABC. Let the common radius of the congruent circumcircles of these faces equal r. Then $R^2 - r^2 = OO_A^2 = OO_B^2 = OO_C^2 = OO_D^2$, whence O is equidistant from faces of tetrahedron $ABCD$; that is, O coincides with either the incentre or an excentre of tetrahedron $ABCD$. A tetrahedron can have two types of excentres. We must show that it is not possible for the circumcentre of $ABCD$ to coincide with either type of excentre.

Assume first that O coincides with the excentre opposite vertex D. Then O_D is the point where that exsphere is tangent to face ABC. Denote the points where the insphere touches faces BCD, CDA, DAB, and ABC by P_A, P_B, P_C, and P_D, respectively. By the lemma, P_D and O_D are isogonal conjugates in triangle ABC; consequently, because O_D is its circumcentre, so P_D is its orthocentre.

In the circle BCD we have $\sin \angle BDC = \frac{BC}{2r} = \sin \angle BAC$; thus either $\angle BDC = \angle BAC$ or $\angle BDC = \pi - \angle BAC$. Since the planes through BC that are tangent at P_A and P_D to the insphere of $ABCD$ are symmetric about the plane that bisects the dihedral angle, we have $\angle BP_AC = \angle BP_DC$. But P_A lies inside triangle BCD, whence $\angle BDC < \angle BP_AC = \angle BP_DC = \pi - \angle BAC$. (The last equality is a consequence of P_D being the orthocentre.) It follows that $\angle BDC = \angle BAC$. Analogously we get $\angle CDA = \angle CBA$ and $\angle ADB = \angle ACB$.

Keeping in mind these pairs of equal angles, we now look at the net of our tetrahedron.
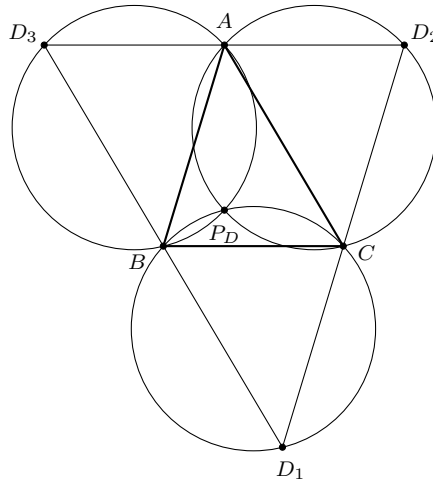
We deduce that circumcircles of BCD_1, CAD_2, ABD_3 pass through P_D. From $AD_2 = AD_3$ we get $\angle D_2P_DA = \angle AP_DD_3$ or $\angle D_2P_DA = \pi - \angle AP_DD_3$, with analogous possibilities for the pairs $\angle D_3P_DB$, $\angle BP_DD_1$ and $\angle D_1P_DC$, $\angle CP_DD_2$. Since at most one of these three pairs can form a straight angle (because D_1 cannot lie on D_2D_3), we may assume (without loss of generality), that $\angle D_3P_DB = \angle BP_DD_1$ and $\angle D_1P_DC = \angle CP_DD_2$. Now two sides, an angle, and circumradius of ΔBP_DD_3 equal the corresponding elements of ΔBP_DD_1, whence they are congruent. Similarly, triangles CP_DD_2 and CP_DD_1 are also congruent. This implies that $P_DD_3 = P_DD_1 = P_DD_2$ and, therefore, triangles AP_DD_2 and AP_DD_3 are congruent by SSS. So we have $\angle P_DAD_2 = \pi - \angle D_2CP_D = \pi - \angle P_DCD_1 = \angle P_DBD_1 = \angle D_3BP_D = \pi - \angle D_3AP_D$, which implies that points D_2, A, D_3 are collinear; furthermore, the equality $D_3A = AD_2$ implies that A is midpoint of D_2D_3. Analogously B, C are midpoints of D_3D_1 and D_1D_2, whence the faces of $ABCD$ are congruent.

To summarize the previous paragraph, the assumption that O coincides with the excentre opposite vertex D has led to the conclusion that the tetrahedron $ABCD$ is isosceles. But the automorphism group of an isosceles tetrahedron contains the three involutions that interchange the endpoints of opposite edges, whence by symmetry, O must also coincide with other excentres. This contradiction shows that O cannot coincide with the centre of an exsphere that is tangent to a face of $ABCD$. But in general, a tetrahedron will also have three exspheres that are tangent to the sides of the dihedral angles determined by pairs of opposite edges; thus, it remains to show that O cannot coincide with the centre of this type of exsphere.

To that end we assume, to the contrary, that the circumcentre coincides with the centre of the exsphere tangent to the interior side of the face planes through the edge AB, namely ABD and ABC, and to the exterior of the face planes through CD, namely CDA and CDB. As before, the point of tangency of the exsphere with plane DAB is the circumcenter of ΔDAB. Of course O_C lies inside the circumcircle. On the other hand, O_C is separated from A by BD (in the circle DAB on the exterior side of the plane CDB) and separated from B by AD (in the circle DAB on the exterior side of the plane CDA). This contradiction completes the proof.

References

[1] Nathan Altshiller-Court, *Modern Pure Solid Geometry*, Chelsea Publishing, Bronx, N.Y., 1964.

[2] Marcel Berger, *Geometry*, Vol. 2, Springer-Verlag, 1987.

[3] F. G.-M., *Exercices de géométrie—comprenant l'exposé des méthodes géométriques et 2000 questions résolues*, quatrième édition, Mame et Fils, Tours, 1907.

No other complete solutions have ever been received.

The failed solution I to Problem 478 claimed, incorrectly, that the proof is implicit in Altshiller-Court's solid-geometry text [1]. Klamkin and Liu refuted the claim [1987 : 151-152], thereby moving the problem to our unsolved problem list. Note an immediate consequence of the theorem: For a tetrahedron to be isosceles, it is sufficient (as well as necessary) that the circumcircles of the faces be congruent. This joins the list of better-known necessary and sufficient conditions, most of which can be found in [1, pages 103-112]; specifically, if a tetrahedron satisfies one of the following seven properties, it is isosceles and it satisfies them all.

1. *The opposite edges are congruent. (This is the standard definition.)*
2. *The faces are congruent.*
3. *The three face angles at each vertex sum to $180°$.*
4. *The incentre coincides with the circumcentre.*
5. *The faces have the same perimeter.*
6. *The faces have the same area.*
7. *There exist exactly four exspheres.*

On the other hand, Problem 330 [1978: 263-264]—mentioned above in the statement of the current problem—asserts that the congruence of the incircles of the four faces does not *imply that the tetrahedron be isosceles.*

3611. [2011 : 48, 51] *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Given x, y, and z are positive integers such that

$$\frac{x(y+1)}{x-1}, \frac{y(z+1)}{y-1}, \text{ and, } \frac{z(x+1)}{z-1}$$

are positive integers. Find the smallest positive integer N such that $xyz \leq N$.

Remark from the editor.

We make three preliminary observations: (1) If $(x, y, z) = (a, b, c)$ satisfies the conditions, then so also does (b, c, a) and (c, a, b); (2) each of the integers x, y, z must be not less than 2; (3) since $(x - 1, x)$, $(y - 1, y)$ and $(z - 1, z)$ are coprime pairs, it follows that $x - 1$ divides $y + 1$, $y - 1$ divides $z + 1$ and $z - 1$ divides $x + 1$ for any valid triple (x, y, z). The first solution answers the question of the problem, while the second indicates how to obtain the set of all valid (x, y, z).

I. Composite of solutions by Joseph DiMuro, Biola University, La Mirada, CA, USA; and Oliver Geupel, Brühl, NRW, Germany.

We must have that $x - 1 \leq y + 1$, $y - 1 \leq z + 1$ and $z - 1 \leq x + 1$. Equality cannot hold in all three cases, so we may suppose with no loss of generality that $2(z - 1) \leq x + 1$.

Then $2(z - 1) \leq x + 1 \leq y + 3 \leq z + 5$, whence $z \leq 7$, $y \leq 9$ and $x \leq 11$, so that $xyz \leq 7 \times 9 \times 11 = 693$. On the other hand, $(x, y, z) = (11, 9, 7)$ satisfies the conditions of the problem, so the smallest value of N is 693.

II. Solution by Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India, modified by the editor.

Suppose, first, that two of x, y, z are equal, say $x = y$. Since $x - 1$ must divide $y + 1 = x + 1 = (x - 1) + 2$, $x - 1$ must divide 2.

If $x = 2$, then $z - 1$ must divide 3, so that $z = 2$ or $z = 4$. If $x = 3$, then $z - 1$ must divide 4, so that $z = 2$, $z = 3$ or $z = 5$. However, $(x, y, z) \neq (3, 3, 2)$ since $y(z + 1) = 9$ is not a multiple of $y - 1 = 2$. However $(x, y, z) = (2, 2, 2), (2, 2, 4), (3, 3, 3), (3, 3, 5)$ are all valid.

Suppose now that x, y, z are all distinct and that z is the largest. Since $z - 1$ divides $x + 1$, $x < z \leq x + 2$, so that $z = x + 1$ or $z = x + 2$. The first option is excluded since $z \geq 4$ and $z - 1$ does not divide $z(x + 1) = z^2 = (z - 1)(z + 1) + 1$. Hence $z = x + 2$.

If $x - 1 = y + 1$, then $y = x - 2$, and $x - 3$ divides $x + 3 = (x - 3) + 6$. Therefore $x - 3$ must be one of the numbers 1, 2, 3, 6. These lead to the valid triples $(4, 2, 6)$, $(5, 3, 7)$, $(6, 4, 8)$ and $(9, 7, 11)$.

On the other hand, let $x - 1 < y + 1$. Since $x - 1 = z - 3$ divides $y + 1$ and $y + 1 \leq z$, then $z - 3$ must have two multiples, namely $z - 3$ and $y + 1$, between $z - 3$ and z inclusive. Hence $2(z - 3) \leq z$, so that $z \leq 6$. Checking possible values of y leads to the triples $(2, 2, 4), (3, 3, 5)$.

Therefore, up to a cyclic permutation, the valid triples are $(2, 2, 2)$, $(2, 2, 4)$, $(3, 3, 3)$, $(3, 3, 5)$, $(4, 2, 6)$, $(5, 3, 7)$, $(6, 4, 8)$, $(9, 7, 11)$. It follows that the minimum value of N is $9 \times 7 \times 11 = 693$.

Also solved by DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; GERHARDT HINKLE, Student, Central High School, Springfield, MO, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; SHAUNDA SAWYER, California State University, Fresno, CA, USA; DIGBY SMITH, Mount Royal University, Calgary, AB;

ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; and the proposer. One incomplete solution was received.

Some of the solutions were quite long and inefficient. Besides De, Bailey, Malikić, Sawyer, Stadler and Swylan gave the correct set of triples satisfying the conditions of the problem. Bailey, Bataille and Stadler solved the system of equations $y + 1 = m(x - 1)$, $z + 1 = n(y - 1)$, $x + 1 = k(z - 1)$ to obtain

$$x = 1 + \frac{2[k(n+1)+1]}{mnk - 1};$$
$$y = 1 + \frac{2[m(k+1)+1]}{mnk - 1};$$
$$z = 1 + \frac{2[n(m+1)+1]}{mnk - 1}.$$

Bailey showed that the minimum of m, n, k does not exceed 3 and worked through the cases. Stadler began with the observation that

$$x = \frac{1 + 2kn + 2k + mnk}{mnk - 1} \leq \frac{5mnk + 1}{mnk - 1}$$
$$= 5 + \frac{6}{mnk - 1} \leq 11,$$

with equality if and only $(m, n, k) = (1, 1, 2)$. Since also y and z do not exceed 11, it is straightforward to go through the cases.

Bataille showed that if $mnk = 2$, then (x, y, z) must be $(11, 9, 7)$ in some cyclic order. When $mnk \geq 3$, he found that

$$\frac{x + y + z}{3} = \frac{3mnk + 2(mn + nk + km) + 2(m + n + k) + 3}{3(mnk - 1)}$$
$$\leq \frac{(3 + 6 + 6)mnk + 3}{3(mnk - 1)} = 5 + \frac{6}{mnk - 1} \leq 8,$$

whence $xyz \leq ((x + y + z)/3)^3 \leq 512$.

3612. [2011 : 48, 51] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Find all nonconstant polynomials P such that $P(\{x\}) = \{P(x)\}$, for all $x \in \mathbb{R}$, where $\{a\}$ denotes the fractional part of a.

Composite of submitted solutions, modified by the editor.

Since $\{P(x + 1)\} = P(\{x + 1\}) = P(\{x\}) = \{P(x)\}$, the polynomial $P(x + 1) - P(x)$ is an integer for each real x. Since any polynomial assuming two distinct integer values assumes every intermediate value, $P(x + 1) - P(x)$ is equal to the same integer a for each x. Since $P(x)$ is not constant, it can assume each of its values only finitely often, so that $a \neq 0$.

Let $Q(x) = P(x) - ax$. Then $Q(x + 1) - Q(x) = 0$ so that $Q(x)$ is periodic with period 1, and so bounded. Hence $Q(x) = b$ for some constant b and all real x. Therefore $P(x) = ax + b$. Since $b = P(0) = \{P(0)\}$, we must have $0 \leq b < 1$. Let $x = 1/|a|$. Then

$$a \left\{ \frac{1}{|a|} \right\} + b = \{\pm 1 + b\} = b,$$

so that $a\{1/|a|\} = 0$ and $a = \pm 1$.

Suppose that $a = -1$ and $b < x < 1$. Then

$$-(x - b) = -x + b = -\{x\} + b = \{-x + b\} \geq 0,$$

a contradiction. Therefore $a = 1$. Let $x = 1 - b$. Then $0 < x \leq 1$ and

$$0 \leq b \leq \{1 - b\} + b = P(\{1 - b\}) = \{P(1 - b)\} = \{1\} = 0,$$

whence $b = 0$. Therefore the sole solution to the problem is $P(x) = x$.

Solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. There were 5 incorrect solutions received.

3613. [2011 : 112, 114] *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Solve the system of equations

$$\frac{x(y+1)}{x-1} = 7\,,\, \frac{y(z+1)}{y-1} = 5\,, \text{and } \frac{z(x+1)}{z-1} = 12,$$

where x, y, and z are positive integers.

I. Composite of essentially the same solution by Roy Barbara, Lebanese University, Fanar, Lebanon; Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; Kathleen E. Lewis, University of the Gambia, Brikama, Gambia; and Michael Parmenter, Memorial University of Newfoundland, St. John's, NL.

The only solution is $(x, y, z) = (7, 5, 3)$.

First note that $x, y, z \geq 2$. We have $x(y+1) = 7(x-1)$ so that $x \mid 7(x-1)$. Since x and $x - 1$ are coprime, $x \mid 7$ so $x = 7$ follows. Similarly, from $y(z+1) = 5(y-1)$ we deduce that $y = 5$. Finally, from $z + 1 = \frac{5(y-1)}{y} = 4$ we obtain $z = 3$.

II. Similar solutions by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; and Titu Zvonaru, Comăneşti, Romania.

Since $y + 1 = \frac{7(x-1)}{x} = 7 - \frac{7}{x}$ we have $6 - y = \frac{7}{x}$ so $x(6 - y) = 7$. Since $x > 1$ we must have $x = 7$ and $6 - y = 1$ so $y = 5$. Using either one of the other two equations, $z = 3$ follows.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; A. WIL EDIE, Missouri State University, Springfield, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; STAN WAGON, Macalester College, St. Paul, MN, USA; MENG XIONG, California State University, Fresno, CA, USA; and the proposer.
Heuver, Wagon and Zvonaru solved the given equations in real numbers and found another solution given by $(x, y, z) = \left(\frac{32}{19}, \frac{59}{32}, \frac{76}{59}\right)$.

3614. [2011 : 112, 114] *Proposed by Neven Jurič, Zagreb, Croatia.*

Taking consecutive decimal digits of $\frac{1}{7}$ the set of points $A(1, 4)$, $B(4, 2)$, $C(2, 8)$,... in the plane is obtained. Prove that all these points belong to the same ellipse. Compute the area of the ellipse.

Composite of the almost identical solutions of Roy Barbara, Lebanese University, Fanar, Lebanon; and Missouri State University Problem Solving Group, Springfield, MO, USA.

Since $\frac{1}{7} = \overline{142857}$, there are only six points: $(1, 4), (4, 2), (2, 8), (8, 5), (5, 7)$, and $(7, 1)$. Note that they are symmetric about the centroid $\left(\frac{9}{2}, \frac{9}{2}\right)$. Translate the centroid to the origin, and in the resulting coordinate system these points become

$$\left(-\frac{7}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, -\frac{5}{2}\right), \left(-\frac{5}{2}, \frac{7}{2}\right), \left(\frac{7}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{5}{2}\right), \left(\frac{5}{2}, -\frac{7}{2}\right).$$

It is a straightforward exercise to verify that these points all lie on the ellipse

$$19x^2 + 36xy + 41y^2 = 306.$$

Finally, the area of this ellipse is

$$\frac{2\pi}{\sqrt{4AC - B^2}} = \frac{306\pi}{\sqrt{455}} \approx 45.07,$$

where we used $A = \frac{19}{306}, B = \frac{36}{306}, C = \frac{41}{306}$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

*Because the given points are the vertices of a centrally symmetric hexagon (whose opposite sides are parallel), the points lie on a conic by Pascal's theorem. Because the hexagon is also convex, the conic must be an ellipse. The equation of this conic is required only to find the area it encloses. Other solvers used the conic in its original position where its equation is $19x^2 + 36xy + 41y^2 - 333x - 531y + 1638 = 0$. According to the MathWorld web page, this ellipse is known as the one-seventh ellipse; the reference given there is to David Wells, The Penguin Dictionary of Curious and Interesting Numbers (Penguin Books, 1986). For those readers with access to Zeitschrift MNU (Mathematischer und Naturwissenschaftlicher Unterricht), Milan Koman provides a modest generalization (Variationen auf die Ein-Siebtel-Ellipse, MNU **63**:4 (2010) 200-202). These references were kindly provided by Rudolf Fritsch (University of Munich).*

3615. [2011 : 112, 114] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Prove that if $x, y, z \geq 0$ and $x + y + z = 1$, then

$$\frac{xy}{\sqrt{z + xy}} + \frac{yz}{\sqrt{x + yz}} + \frac{zx}{\sqrt{y + zx}} \leq \frac{1}{2}.$$

Similar solutions by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Titu Zvonaru, Cománeşti, Romania; and the proposer.

Since $z + xy = (z + x)(z + y)$, we can use the Arithmetic-Geometric Means Inequality to obtain that

$$\frac{xy}{\sqrt{z + xy}} = \frac{xy}{\sqrt{(z + x)(z + y)}} \leq \frac{xy}{2}\left(\frac{1}{z + x} + \frac{1}{z + y}\right).$$

Combining this with similar inequalities for the other two terms shows that the left side does not exceed

$$\frac{xy}{2}\left(\frac{1}{z + x} + \frac{1}{z + y}\right) + \frac{yz}{2}\left(\frac{1}{x + y} + \frac{1}{x + z}\right) + \frac{zx}{2}\left(\frac{1}{y + x} + \frac{1}{y + z}\right) = \frac{1}{2}.$$

Equality occurs if and only if (x, y, z) is equal to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(\frac{1}{2}, \frac{1}{2}, 0)$, $(\frac{1}{2}, 0, \frac{1}{2})$ or $(0, \frac{1}{2}, \frac{1}{2})$.

An alternative approach begins with the equation $z + xy = (1 - x)(1 - y)$ and its analogues.

Also solved by MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and ALBERT STADLER, Herrliberg, Switzerland.

3616. *Proposed by Dinu Ovidiu Gabriel, Valcea, Romania.*

Compute

$$L = \lim_{n \to \infty} n^{2k} \left[\frac{\arctan(n^k)}{n^k} - \frac{\arctan(n^k + 1)}{n^k + 1} \right],$$

where $k \in \mathbb{R}$.

Comment by the editor.

All solvers noted that when $k = 0$ then, since $n^0 = 1$ for $n > 0$ the limit was $\frac{\pi}{4} - \frac{\arctan(2)}{2}$. And when $k < 0$ we have, where $K = -k > 0$,

$$\lim_{n \to \infty} n^{2k} \left[\frac{\arctan(n^k)}{n^k} - \frac{\arctan(n^k + 1)}{n^k + 1} \right]$$
$$= \lim_{n \to \infty} \left[\frac{\arctan\left(\frac{1}{n^K}\right)}{n^K} - \frac{\arctan\left(\frac{1}{n^K} + 1\right)}{n^K(n^K + 1)} \right]$$
$$= 0.$$

The rest of the solutions are for the more interesting case when $k > 0$.

I. Solution by Missouri State University Problem Solving Group, Springfield, MO, USA.

If $k > 0$, letting $N = n^k$ and using the fact that $\arctan x = \frac{\pi}{2} - \frac{1}{x} + O\left(\frac{1}{x^2}\right)$ for large x, the limit may be rewritten as

$$\lim_{N \to \infty} N^2 \left[\frac{\arctan(N)}{N} - \frac{\arctan(N + 1)}{N + 1} \right]$$
$$= \lim_{N \to \infty} N^2 \left[\frac{\frac{\pi}{2} - \frac{1}{N} + O\left(\frac{1}{N^2}\right)}{N} - \frac{\frac{\pi}{2} - \frac{1}{N+1} + O\left(\frac{1}{N^2}\right)}{N + 1} \right]$$
$$= \lim_{N \to \infty} \frac{N^2}{N(N + 1)} \left[\frac{\pi}{2} - \frac{N + 1}{N} + \frac{N}{N + 1} + O\left(\frac{1}{N}\right) \right]$$
$$= 1 \cdot \left[\frac{\pi}{2} - 1 + 1 + 0 \right]$$
$$= \frac{\pi}{2}.$$

II. Solution by Michel Bataille, Rouen, France.

Let $v_n = n^{2k} \left[\frac{\arctan(n^k)}{n^k} - \frac{\arctan(n^k + 1)}{n^k + 1} \right]$.

Suppose $k > 0$. Let $f(x) = \frac{\arctan x}{x}$. Then, $f'(x) = -\frac{\phi(x)}{x^2}$ where $\phi(x) = \arctan(x) - \frac{x}{x^2 + 1}$. Since $\phi'(x) = \frac{2x^2}{(x^2 + 1)^2} > 0$ for $x > 0$, the function ϕ is increasing

on $(0, \infty)$ and therefore $0 < \phi(x) < \frac{\pi}{2} = \lim_{x \to \infty} \phi(x)$ for positive x. Observing that

$$v_n = n^{2k} \int_{n^k}^{n^k + 1} \frac{\phi(x)}{x^2} \, dx$$

we may write for all $n \geq 1$,

$$n^{2k} \phi(n^k) \int_{n^k}^{n^k + 1} \frac{dx}{x^2} \leq v_n \leq n^{2k} \phi(n^k + 1) \int_{n^k}^{n^k + 1} \frac{dx}{x^2}$$

that is,

$$\phi(n^k) \frac{n^{2k}}{n^k(n^k + 1)} \leq v_n \leq \phi(n^k + 1) \frac{n^{2k}}{n^k(n^k + 1)}.$$

Since $\lim_{n \to \infty} \frac{n^{2k}}{n^k(n^k+1)} = 1$ and $\lim_{n \to \infty} \phi(n^k) = \lim_{n \to \infty} \phi(n^k + 1) = \frac{\pi}{2}$, we obtain $L = \frac{\pi}{2}$ by the squeeze principle.

III. Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, expanded by the editor.

Note that

$$n^{2k} \left[\frac{\arctan(n^k)}{n^k} - \frac{\arctan(n^k + 1)}{n^k + 1} \right]$$

$$= n^k \arctan(n^k) - \frac{n^{2k}}{n^k + 1} \arctan(n^k + 1)$$

$$= n^k \arctan(n^k) - \frac{(n^k + 1)^2 - 2n^k - 1}{n^k + 1} \arctan(n^k + 1)$$

$$= n^k \arctan(n^k) - (n^k + 1) \arctan(n^k + 1) + \frac{2n^k + 1}{n^k + 1} \arctan(n^k + 1). \qquad (1)$$

Let $f(x) = -x \arctan x$. Then f is continuous and differentiable on $(0, \infty)$. For each n, we have, by applying the Mean Value Theorem to the interval $[n^k, n^k + 1]$, that there exists a c_n with $f'(c_n) = n^k \arctan(n^k) - (n^k + 1) \arctan(n^k + 1)$. Thus, since

$$f'(x) = -\frac{x}{x^2 + 1} - \arctan x, \qquad (2)$$

then from (1) and (2) we get

$$L = \lim_{n \to \infty} \left(-\frac{c_n}{c_n^2 + 1} - \arctan c_n \right) + \lim_{n \to \infty} \frac{2n^k + 1}{n^k + 1} \arctan(n^k + 1)$$

$$= 0 - \frac{\pi}{2} + 2 \cdot \frac{\pi}{2} = \frac{\pi}{2}.$$

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ANASTASIOS KOTRONIS, Athens, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3617. [2011 : 112, 115] *Proposed by Michel Bataille, Rouen, France.*

Let r be a positive rational number. Show that if r^r is rational, then r is an integer.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $a, b, c,$, and d be positive integers such that

$$r = \frac{a}{b}, \ r^r = \frac{c}{d}, \ gcd(a, b) = 1.$$

It follows that

$$\frac{a^a}{b^a} = r^a = r^{rb} = \frac{c^b}{d^b}.$$

We have to prove that $b = 1$. Assume the contrary. Then, b has a prime divisor p. Let $\varepsilon(n)$ denote the exact exponent of p in the integer n, that is, the non-negative integer α such that $p^\alpha \mid n$ and $p^{\alpha+1} \nmid n$. We obtain

$$a \cdot \varepsilon(b) + b \cdot \varepsilon(c) = \varepsilon(b^a c^b) = \varepsilon(a^a d^b) = a \cdot \varepsilon(a) + b \cdot \varepsilon(d) = b \cdot \varepsilon(d).$$

Hence, $a \cdot \varepsilon(b) = b \cdot (\varepsilon(d) - \varepsilon(c))$, from which we deduce that b divides the positive number $\varepsilon(b)$. Consequently, $b < p^b \leq p^{\varepsilon(b)} \leq b$. This is a contradiction, which completes the proof.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; KAYLIN EVERETT, California State University, Fresno, CA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3618. [2011 : 113, 115] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $\alpha > 3$ be a real number. Find the value of

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n}{(n+m)^\alpha}.$$

Composite of nearly identical solutions by Anastasios Kotrononis, Athens, Greece; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

The original sum may be rewritten as

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n}{(n+m)^\alpha} = \sum_{q=2}^{\infty} \sum_{p=1}^{q-1} \frac{p}{q^\alpha} = \sum_{q=2}^{\infty} \frac{1}{2} \cdot \frac{q(q-1)}{q^\alpha} = \frac{1}{2} \left[\sum_{q=2}^{\infty} \frac{1}{q^{\alpha-2}} - \sum_{q=2}^{\infty} \frac{1}{q^{\alpha-1}} \right]$$

$$= \frac{1}{2}[(\zeta(\alpha-2) - 1) - (\zeta(\alpha-1) - 1)]$$

$$= \frac{1}{2}(\zeta(\alpha-2) - \zeta(\alpha-1) - 1),$$

where $\zeta(z)$ is the Riemann zeta function.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; ALBERT STADLER, Herrliberg, Switzerland; STAN WAGON, Macalester College, St. Paul, MN, USA; and the proposer.

Stadler and AN-anduud Problem Solving Group used symmetry to write

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n}{(n+m)^{\alpha}} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m}{(n+m)^{\alpha}} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(n+m)-m}{(n+m)^{\alpha}} = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(n+m)^{\alpha-1}},$$

from which they proceeded in a similar fashion to the featured solution. The proposer proved

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(n+m)^{p}} = \zeta(p-1) - \zeta(p),$$

from which he used a symmetry argument similar to Stadler and AN-anduud's to set up a situation where the lemma could be used. Perfetti also supplied a proof of the convergence of the double sum. Wagon pointed out that Mathematica produces the desired result.

3619. [2011 : 113, 115] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$(a^2b - c)(b^2c - a)(c^2a - b) \le 4(ab + bc + ca - 3a^2b^2c^2).$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

First we show that

$$a^2b + b^2c + c^2a + abc \le 4. \tag{1}$$

By the cyclicity of (1), we may without loss of generality suppose that either $a \le b \le c$ or $a \ge b \ge c$. In either case, the Rearrangement Inequality yields

$$a^2b + b^2c + c^2a + abc = a \cdot ab + b \cdot bc + c \cdot ac + b \cdot ac \le a \cdot ab + b \cdot ac + b \cdot ac + c \cdot bc = b(a+c)^2.$$

By the AM-GM Inequality, we have

$$b(a+c)^2 = \frac{1}{2} \cdot 2b(a+c)(a+c) \le \frac{1}{2}\left(\frac{2b + (a+c) + (a+c)}{3}\right)^3 = 4,$$

which completes the proof of (1).

As a consequence of (1),

$$\sum_{\text{cyclic}} a^3b^2 + \sum_{\text{cyclic}}(a^3bc + 2a^2b^2c) = (a^2b + b^2c + c^2a + abc)(ab + bc + ca) \le 4(ab + bc + ca).$$

Moreover, by the AM-GM Inequality, we have

$$1 = \left(\frac{a+b+c}{3}\right)^3 \ge abc \ge (abc)^3$$

and

$$12(abc)^2 \le 12(abc)^{11/6} \le \sum_{\text{cyclic}}(a^4bc^2 + a^3bc + 2a^2b^2c).$$

By expanding the expressions in the desired inequality, we finally obtain

$$(a^2b - c)(b^2c - a)(c^2a - b) - 4(ab + bc + ca - 3a^2b^2c^2)$$

$$= ((abc)^3 - abc) + 12(abc)^2 - \sum_{\text{cyclic}} a^4bc^2 + \left(\sum_{\text{cyclic}} a^3b^2 - 4(ab + bc + ca) \right)$$

$$\leq ((abc)^3 - abc) + \left(12(abc)^2 - \sum_{\text{cyclic}} (a^4bc^2 + a^3bc + 2a^2b^2c) \right)$$

$$\leq 0,$$

and we are done.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece(2 solutions with Maple); and the proposer.

3620. [2011 : 113, 115] *Proposed by John G. Heuver, Grande Prairie, AB.*

Let P be an interior point in tetrahedron $ABCD$ and let AP, BP, CP, and DP meet the corresponding opposite faces in A', B', C', and D'. Then

$$\frac{AP}{PA'} \frac{BP}{PB'} \frac{CP}{PC'} \frac{DP}{PD'} = 3 + 2\left(\frac{AP}{PA'} + \frac{BP}{PB'} + \frac{CP}{PC'} + \frac{DP}{PD'} \right)$$
$$+ \frac{AP}{PA'} \frac{BP}{PB'} + \frac{AP}{PA'} \frac{CP}{PC'} + \frac{AP}{PA'} \frac{DP}{PD'}$$
$$+ \frac{BP}{PB'} \frac{CP}{PC'} + \frac{BP}{PB'} \frac{DP}{PD'} + \frac{CP}{PC'} \frac{DP}{PD'}.$$

Solution by the proposer.

Let a, b, c, and d, respectively, be the volumes of the four tetrahedra $PBCD$, $PACD$, $PABD$, and $PABC$ that form a partition of $ABCD$. Without loss of generality we assume Volume$(ABCD) = 1$. Because $ABCD$ and $PBCD$ share the same base $\triangle BCD$ while their heights are in the ratio $\frac{AA'}{PA'}$, we have

$$\frac{AA'}{PA'} = \frac{\text{Volume}(ABCD)}{\text{Volume}(PBCD)} = \frac{a + b + c + d}{a} = \frac{1}{a}.$$

Therefore

$$\frac{AP}{PA'} = \frac{AA' - PA'}{PA'} = \frac{1}{a} - 1.$$

Similarly,

$$\frac{BP}{PB'} = \frac{1}{b} - 1, \quad \frac{CP}{PC'} = \frac{1}{c} - 1, \quad \text{and} \quad \frac{DP}{PD'} = \frac{1}{d} - 1.$$

We seek the product, Π, of these four terms. We now have

$$\Pi = \frac{AA'}{PA'} \cdot \frac{BP}{PB'} \cdot \frac{CP}{PC'} \cdot \frac{DP}{PD'}$$
$$= \left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right)\left(\frac{1}{d} - 1\right)$$
$$= \frac{1}{abcd} - \frac{1}{abc} - \frac{1}{abd} - \frac{1}{acd} - \frac{1}{bcd} + \frac{1}{ab} + \frac{1}{ac} + \frac{1}{ad}$$
$$+ \frac{1}{bc} + \frac{1}{bd} + \frac{1}{cd} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{d} + 1.$$

Since $a + b + c + d = 1$, we have

$$\frac{1}{abcd} = \frac{1}{abc} + \frac{1}{abd} + \frac{1}{acd} + \frac{1}{bcd}.$$

It follows that

$$\begin{aligned}
\Pi &= \frac{1}{ab} + \frac{1}{ac} + \frac{1}{ad} + \frac{1}{bc} + \frac{1}{bd} + \frac{1}{cd} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{d} + 1 \\
&= \sum_{\text{cyclic}} \left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right) - 6 + 2\sum_{\text{cyclic}} \left(\frac{1}{a} - 1\right) + 8 + 1 \\
&= 3 + 2\left(\frac{AP}{PA'} + \frac{BP}{PB'} + \frac{CP}{PC'} + \frac{DP}{PD'}\right) + \frac{AP}{PA'}\frac{BP}{PB'} + \frac{AP}{PA'}\frac{CP}{PC'} + \frac{AP}{PA'}\frac{DP}{PD'} \\
&\qquad\qquad + \frac{BP}{PB'}\frac{CP}{PC'} + \frac{BP}{PB'}\frac{DP}{PD'} + \frac{CP}{PC'}\frac{DP}{PD'}.
\end{aligned}$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France;and OLIVER GEUPEL, Brühl, NRW, Germany.

Geupel observed that when distances are directed, the identity holds more generally for each point P that is not in a plane determined by a face of ABCD. Heuver found the problem in Nathan Altshiller Court's Modern Pure Solid Geometry *(Chelsea, 1964), page 141 #10. He proved the analogous theorem for a simplex in four dimensions (where the product of five quotients equals an expression with 26 summands) and conjectured that his approach would extend to simplices in n-dimensional Euclidean space.*