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CONTENTS

For Vol. 4 (1978), the support of Algonquin College, the Samuel Beatty Fund, and Carleton University is gratefully acknowledged.

R. ROBINSON ROWE

1896 - 1978

Readers of this journal will be saddened to learn of the death of R. Robinson Rowe on May 4, 1978, in his eighty-second year, after a long illness.

He had been a friend of, and a contributor to, CRUX MATHEMATICORUM from its early days, and he contributed regularly to many other journals as well, notably to the *Journal of Recreational Mathematics* and to the *Pi Mu Epsilon Journal*. His interest in mathematics continued unabated until very near the end. This Editor received from him a letter dated April 11, 1978, which began as follows: "I have been in misery for nearly 3 months—15 days in hospital—27 days home—mostly on my back—a few minutes at a time sitting up—like now. But will try to brief some recreations." This was followed by solutions to half a dozen CRUX MATH. problems. But mathematics claimed his attention even beyond this date. After he made his last trip to the hospital, never to return, his son, Edwin R. Rowe, found in his typewriter an unfinished letter, dated May 1, 1978, addressed to the Editor of the *Journal of Recreational Mathematics*, which contained solutions to several *JRM* problems.

"Bob" Rowe has already told us much about his involvement with mathematics in his early life in his *Reminiscences* [1977: 184, 248; 1978: 6, 63, 129]. He had completed a few more instalments, at this Editor's request. These will be published in the months to come.

On behalf of the entire CRUX MATHEMATICORUM community, the Editor extends his deepest sympathy to the bereaved family of R. Robinson Rowe.

MORE ON THE THEOREMS OF CEVA AND MENELAUS

ALFRED AEPPLI

The article "The Theorems of Ceva and Menelaus" by Dan Pedoe [1977: 2] stimulated the following remarks. Menelaus and Ceva can be understood most naturally in the framework of projective geometry.

1. The cross ratio

$$(ab,\ cd) = (AB,\ CD) = \frac{AC}{CB}\bigg/\frac{AD}{DB} = (AB)_C/(AB)_D$$

is introduced (Figure 1) and proved to be a projective invariant (e.g. via Pappus).

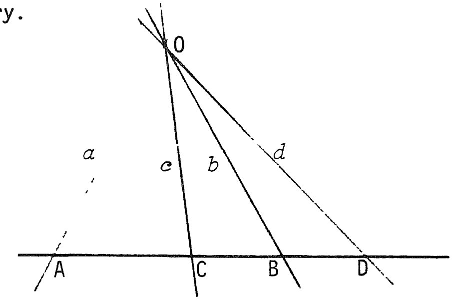

Figure 1

2. The theorem of Menelaus is proved: for the triangle Δ = ABC and the line l (Figure 2), the quantity

$$\mu = (BC)_L(CA)_M(AB)_N$$

is an invariant constant, independent of l and Δ, e.g.

$$(BC)_L(CA)_M(AB)_N = (BC)_{L'}(CA)_M(AB)_{N'},$$

by projective invariance of the cross ratio. A special example shows $\mu = -1$.

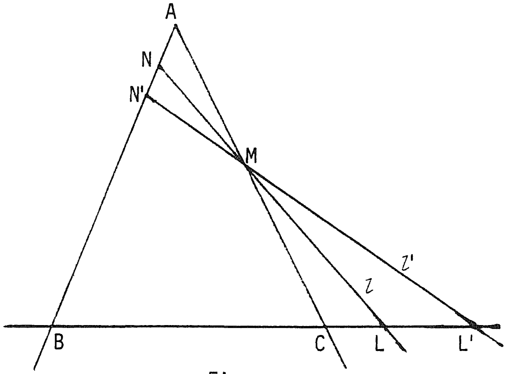

Figure 2

3. Similarly, the theorem of Ceva is proved: for Δ = ABC and the point O (Figure 3), the quantity

$$\gamma = (BC)_D(CA)_E(AB)_F$$

is an invariant constant, independent of O and Δ, e.g.

$$(BC)_D(CA)_E(AB)_F = (BC)_{D'}(CA)_{E'}(AB)_{F'},$$

by projective invariance of the cross ratio. A special example shows $\gamma = 1$.

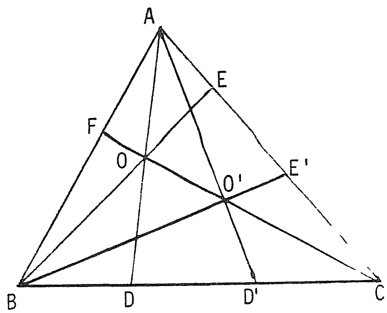

Figure 3

4. For the complete quadrangle BCEF (Figure 4),

$$(BC)_D/(BC)_{D'} = (BC, DD') = -1 \qquad (1)$$

by projective invariance of the cross ratio.

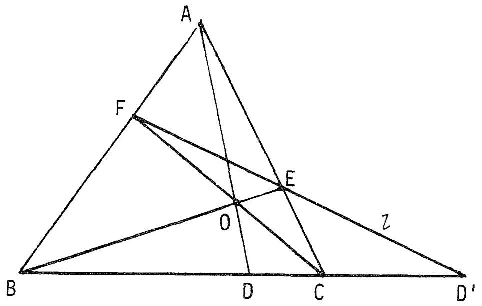

Figure 4

5. Equation (1) implies

$$(BC)_D(CA)_E(AB)_F = -(BC)_{D'}(CA)_E(AB)_F, \qquad (2)$$

where the left side is the Ceva product for the triangle Δ = ABC and the point O, and the right side is the Menelaus product for the same triangle and the line l. Thus (2) shows that Ceva $\Longleftrightarrow$ Menelaus if the theory of the cross ratio is assumed to be known. Also, (1) $\Longleftrightarrow$ (Ceva $\Longleftrightarrow$ Menelaus).

6. *Final remark.* Figure 2 in Pedoe's article [1977: 3] implies that

$$(CB, A'L) = (CB)_{A'}/(CB)_L = -1,$$
$$(AC, B'M) = (AC)_{B'}/(AC)_M = -1,$$
$$(BA, C'N) = (BA)_{C'}/(BA)_N = -1,$$

and multiplication of these three equations produces

$$(CB)_{A'}(AC)_{B'}(BA)_{C'} = -(CB)_L(AC)_M(BA)_N;$$

hence again Ceva $\Longleftrightarrow$ Menelaus on the basis of the theory of the cross ratio.

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* * *

ALGORITHMS AND POCKET CALCULATORS: SQUARE ROOTS: II

CLAYTON W. DODGE

In a first article [1978: 96] we developed the long division square root algorithm. Here we look at a second well-known technique. Newton's method, the divide-and-average method, requires the assumption of a first approximation r_1 for $\sqrt{n}$; then

$$r_2 = \frac{n/r_1 + r_1}{2}$$

is a second (better) approximation. We show that, when r_1 is at all reasonably chosen, then r_2 has at least twice the accuracy of r_1. Thus, if we guess that $\sqrt{18468.81}$ is about 130, then we take $r_1 = 130$ and, using a pocket calculator, we compute that

$$r_2 = \frac{1}{2}\left(\frac{18468.81}{130} + 130\right) = 136.0338846.$$

Let us repeat the algorithm using r_2 as our initial approximation to find the next approximation r_3. We have

$$r_3 = \frac{1}{2}\left(\frac{18468.81}{136.0338846} + 136.0338846\right) = 135.9000659.$$

One further application of the algorithm gives

$$r_4 = 135.9,$$

the exact square root of 18468.81.

Now $r_1 = 130$ does not have 2-place accuracy, r_2 has 3-place accuracy, r_3 has 6-place accuracy, and r_4 has 10-place accuracy, the maximum obtainable with a 10-place calculator. In each case we see that the accuracy has doubled from one approximation to the next. In general, the accuracy to which r_k and r_{k+1} agree is the accuracy of the approximation r_k.

We give two proofs of Newton's method, the first by calculus. If f has two continuous derivatives on the interval $(x, x+h)$ with $h > 0$ then, by the mean value theorem,

$$f(x+h) = f(x) + \frac{f(x+h) - f(x)}{h} \cdot h = f(x) + f'(x+j) \cdot h$$

for some j in the interval $(0,h)$. Applying this same formula to $f'(x+j)$, we obtain

$$f(x+h) = f(x) + f'(x) \cdot h + f''(x+k) \cdot jh$$

where $k \in (0,j)$. Hence we have

$$f(x+h) \approx f(x) + f'(x) \cdot h$$

with error less than $f''(x+k) \cdot h^2$. If we further assume that h (and therefore k) is small enough so that $f''(x) \approx f''(x+k)$, then the error will be less than

$$f''(x) \cdot h^2.$$

It is easy to remove the restriction $h > 0$, so we assume it has been done.

We are interested in the function f defined by

$$f(x) = \sqrt{x}, \qquad x > 0,$$

so that

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \text{and} \quad f''(x) = -\frac{1}{4\sqrt{x^3}}.$$

Here we have

$$\sqrt{x+h} \approx \sqrt{x} + \frac{h}{2\sqrt{x}}$$

with maximum error $h^2/4\sqrt{x^3}$. Letting

$$x + h = n \quad \text{and} \quad \sqrt{x} = r_1,$$

we get $h = n - r_1^2$ and

$$\sqrt{n} \approx r_2 = r_1 + \frac{n - r_1^2}{2r_1} = r_1 + \frac{n}{2r_1} - \frac{r_1}{2} = \frac{1}{2}\left(\frac{n}{r_1} + r_1\right)$$

with

$$|\sqrt{n} - r_2| \le \frac{(n - r_1^2)^2}{4\sqrt{n^3}}.$$

Since

$$|\sqrt{n} - r_1| \le \frac{n - r_1^2}{2\sqrt{n}},$$

the relative errors for r_1 and r_2 satisfy

$$\frac{|\sqrt{n} - r_1|}{\sqrt{n}} \le \frac{n - r_1^2}{2n} \quad \text{and} \quad \frac{|\sqrt{n} - r_2|}{\sqrt{n}} \le \frac{(n - r_1^2)^2}{4n^2};$$

hence, at each iteration of Newton's method, the maximum relative error is squared, at least doubling the accuracy.

Again, the assumption that $h > 0$ is not necessary to the theorem; we can have $h < 0$, and the proof is readily modified for that case.

Next we give an algebraic proof, more suitable for presentation in high school classes.

Suppose $ab = n$ with $a > 0$, $b > 0$, and $a < \sqrt{n}$. Then, of course, we must have $b > \sqrt{n}$, and so

$$a < \sqrt{n} < b = \frac{n}{a}.$$

(If $a > \sqrt{n}$, then the two inequality signs are reversed.) We now seek the linear combination of a and n/a that will best approximate $\sqrt{n}$. To that end, suppose

$$r_1 = a = \sqrt{n}(1 + \varepsilon_1),$$

with $|\varepsilon_1| < 1$. (Generally we will have $|\varepsilon_1| < 0.1$, but this is not a requirement.) Then $|\varepsilon_1|$ is the relative error in r_1 as an approximation to $\sqrt{n}$. We search for constants p and q such that

$$r_2 = \frac{pn}{r_1} + qr_1$$

is the best approximation to $\sqrt{n}$. We have

$$\begin{aligned} r_2 &= \frac{pn}{\sqrt{n}(1+\varepsilon_1)} + q\sqrt{n}(1+\varepsilon_1) \\ &= \sqrt{n}\left(\frac{p}{1+\varepsilon_1} + q(1+\varepsilon_1)\right) \\ &= \sqrt{n}\left((p+q) + \frac{(q-p)\varepsilon_1}{1+\varepsilon_1} + \frac{q\varepsilon_1^2}{1+\varepsilon_1}\right). \end{aligned}$$

To make r_2 as close to $\sqrt{n}$ as possible for all permissible ε_1, we take $p+q = 1$ and $q-p = 0$, that is, $p=q=\frac{1}{2}$. Then

$$r_2 = \sqrt{n}\left(1 + \frac{\varepsilon_1^2}{2(1+\varepsilon_1)}\right).$$

For $-1 < \varepsilon_1 < 1$, the relative error ε_2 in r_2 as an approximation to $\sqrt{n}$ is given by

$$\varepsilon_2 = \frac{\varepsilon_1^2}{2(1+\varepsilon_1)} > 0,$$

so r_2 and all succeeding approximations are greater than $\sqrt{n}$. When $r_1 > \sqrt{n}$, so that $0 < \varepsilon_1 < 1$, then

$$\varepsilon_2 < \frac{\varepsilon_1^2}{2} < \frac{\varepsilon_1}{2},$$

and r_2 is a better approximation than r_1. When $|\varepsilon_1| < 0.1$, so that r_1 has an accuracy of 1-place or better, then

$$\varepsilon_2 \approx \frac{\varepsilon_1^2}{2}.$$

When, in fact, $\varepsilon_1 > 0$, then $\varepsilon_2 < \varepsilon_1^2/2$, and the accuracy of r_2 is at least double that of r_1. This again proves the assertion of the divide-and-average method.

The amount of work required to calculate a 10-digit approximation to $\sqrt{n}$ seems much greater by the long division method than by the divide-and-average method. On the other hand, the latter method requires a reasonable first approximation r_1. But observe that the former method suffers from the same drawback, too; a first approximation to $\sqrt{n}$ must be obtained.

In the next article, we shall see how some computers and some calculators compute square roots.

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* * *

THINKING OF THE FUNDAMENTAL THEOREM OF ARITHMETIC? THINK AGAIN!

RICHARD A. GIBBS

Richard S. Field recently [1977: 152] had the courage to break from tradition and challenge the widely-held belief that the odd-even pattern of positive integers continues indefinitely. He used an ingenious probabilistic approach to produce a convincing argument against this popular bit of mathematical folklore.

Given confidence by Field's bold move, and with a deeper appreciation of probabilistic logic, I would like to announce publicly at this time (having concealed my discovery these many years for fear of ridicule) that there appear to be serious doubts as to the validity of the Fundamental Theorem of Arithmetic.

Imagine, if you will, a (large) container holding strips of paper on each of which is written a prime number, and that there is an unlimited supply of strips for each prime. Now draw out a handful of strips and record the product of their primes. Do this indefinitely. The Fundamental Theorem of Arithmetic (I used to abbreviate this "FTA" until a student of mine informed me that those initials have an entirely different interpretation in the armed services) implies that the list of products will contain each natural number (except 1) an equal number of times. This means that, on the average, the prime p will appear in only $1/p$ of the handfuls of strips. But this is ridiculous! It is absurd to expect that, for instance, a strip marked 5 will be selected more than twice as often as a strip marked 11. And is a strip marked 2 going to be selected almost 2^{19936} times as often as a strip marked $(2^{19937}-1)$? The answer is obvious. Clearly, any strip is as likely to be selected as any other. One can only conclude that the Fundamental Theorem of Arithmetic, so easily verified for small numbers, must eventually fail. And so we have another example of intuition gone astray.

Whatever may be the consequences of the breakdown in the odd-even pattern for positive integers, the ramifications of the failure of the Fundamental Theorem of Arithmetic are ominous indeed. How are we to reduce fractions? How can we find common denominators? (Praise be for the Metric System where these skills are no longer needed!) What of the Sylow Theorems? Is $\sqrt[k]{n}$ really irrational for $n \neq m^k$? Are there really infinitely many primes? The list goes on and on (the list of ramifications, I mean, not necessarily the list of primes).

As we have seen, the methods of probabilistic logic are extremely powerful.

I'm sure that further results will be forthcoming which will reveal other popular "truths" for the misconceptions they really are. The consequences are too grim to ponder.

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* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before November 1, 1978, although solutions received after that date will also be considered until the time when a solution is published.

351. *Proposed by Sidney Kravitz, Dover, N.J.*

Solve the following base ten alphametic addition:

$$\begin{array}{r} \text{GRAPE} \\ \underline{\text{APPLE}} \\ \text{CHERRY} \end{array}.$$

352. *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*

Let

$$x^{(0)} = 1; \qquad x^{(n)} = \prod_{k=1}^{n} \{x + (k-1)c\}, \quad c \text{ constant}, \quad n = 1,2,\ldots .$$

Prove that

$$(a+b)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} a^{(n-k)} b^{(k)}, \quad n = 0,1,2,\ldots .$$

353. *Proposed by Orlando Ramos, Instituto Politécnico José Antonio Echevarría, Habana, Cuba.*

Prove that, if a triangle is self-polar with respect to a parabola, its nine-point circle passes through the focus.

354. *Proposed by Sidney Penner, Bronx Community College, Bronx, N.Y.*

Along a circular road there are n identical parked automobiles. The total amount of gas in all of the vehicles is enough for only one of them to travel the whole circular road. Prove that at least one of these cars could travel the entire road, taking on gas along the way from the other $n-1$ vehicles.

355. *Proposed by James Gary Propp, Great Neck, N.Y.*

Given a finite sequence $A=(a_n)$ of positive integers, we define the family of sequences

$$A_0 = A; \quad A_i = (b_r), \quad i = 1,2,3,\ldots ,$$

where b_r is the number of times that the rth lowest term of A_{i-1} occurs in A_{i-1}.

For example, if $A = A_0 = (2,4,2,2,4,5)$, then $A_1 = (3,2,1)$, $A_2 = (1,1,1)$, $A_3 = (3)$, and $A_4 = (1) = A_5 = A_6 = \ldots$.

The *degree* of a sequence A is the smallest i such that $A_i = (1)$.

(a) Prove that every sequence considered has a degree.

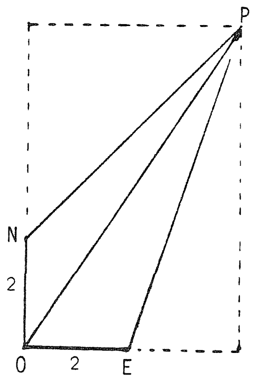
(b) Find an algorithm that will yield, for all integers $d \geq 2$, a shortest sequence of degree d.

(c)* Let $\Lambda(d)$ be the length of the shortest sequence of degree d. Find a formula, recurrence relation, or asymptotic approximation for $\Lambda(d)$.

(d)* Given sequences A and B, define C as the concatenation of A and B. Find sharp upper and lower bounds on the degree of C in terms of the degrees of A and B.

356. *Proposed by the late R. Robinson Rowe, Sacramento, California.*

Jogging daily to a landmark windmill P on the northeasterly horizon (see figure), Joe wondered how far it was. Directly (path OP), his time was 25 minutes; jogging first 2 miles due North (path ONP) took 30 minutes, and jogging first 2 miles due East (path OEP) took 35 minutes. How far was Joe's jog (path OP)?



357. *Proposed by Leroy F. Meyers, The Ohio State University.*

In a certain multiple-choice test, one of the questions was illegible, but the choice of answers was clearly printed. Determine the true answer(s).

(a) All of the below.

(b) None of the below.

(c) All of the above.

(d) One of the above.

(e) None of the above.

(f) None of the above.

358. *Proposed by Murray S. Klamkin, University of Alberta.*

Determine the maximum of x^2y, subject to the constraints

$$x + y + \sqrt{2x^2 + 2xy + 3y^2} = k \text{ (constant)}, \quad x,y \geq 0.$$

359. *Proposed by Charles W. Trigg, San Diego, California.*

Construct a third-order additive magic square that contains three prime elements and has a magic constant of 37.

360. *Proposé par Hippolyte Charles, Waterloo, Québec.*

Montrer directement (c'est-à-dire sans faire appel à un théorème plus général) que, pour $n = 1,2,3,\ldots$, la mantisse de $\sqrt{n}$,

$$\{\sqrt{n}\} = \sqrt{n} - [\sqrt{n}],$$

est dense dans l'intervalle $(0,1)$.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

256. [1977: 155; 1978: 53, 102] Late solution: DAVID KARR, student, Bronx High School of Science, Bronx, N.Y.

294. [1977: 297] *Proposed by Harry D. Ruderman, Hunter College, New York.*

Prove that there are infinitely many integers that cannot be expressed in the form $3ab + a + b$, where a and b are nonzero integers.

Solution de F.G.B. Maskell, Collège Algonquin, Ottawa.

Montrons, plus généralement, qu'il existe une infinité d'entiers positifs n qui ne peuvent être exprimés sous la forme

$$\lambda ab + \mu a + \nu b, \tag{1}$$

où a, b, λ, μ, ν sont des entiers positifs et $(\lambda,\mu\nu) = 1$.

Si $n = \lambda ab + \mu a + \nu b$, le nombre

$$\lambda n + \mu\nu = (\lambda a + \nu)(\lambda b + \mu)$$

est composé; donc si $\lambda n + \mu\nu$ est premier, le nombre n n'est pas exprimable sous la forme (1). Or, d'après un théorème bien connu de Dirichlet (voir [1] ou [3], par exemple), la suite arithmétique

$$\lambda n + \mu\nu, \quad n = 1,2,3,\ldots$$

contient une infinité de nombres premiers. Notre assertion du début est donc justifiée.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; J.D. DIXON, Haliburton Highlands Secondary School, Haliburton, Ontario; CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montréal, Québec; VIKTORS LINIS, University of Ottawa; BASIL C. RENNIE, James Cook University of North Queensland, Australia; the late R. ROBINSON ROWE, Sacramento, California; KENNETH M. WILKE, Topeka, Kansas; and KENNETH S. WILLIAMS, Carleton University, Ottawa.

Editor's comment.

The following application of the generalization proved in our featured solution occurs in [2], a reference sent by Wilke. Setting $\lambda = 2$, $\mu = \nu = 1$ in (1) yields an unexpected sieve for odd primes. Specifically, let E be the set of all numbers

$$E(a,b) = 2ab + a + b;$$

then, for $k > 0$, an odd number $p = 2k + 1$ is a prime if and only if k is not in the set E.

REFERENCES

1. William J. LeVeque, *Topics in Number Theory*, Addison-Wesley, 1956, Vol. I, p. 76.

2. B.M. Stewart, *Theory of Numbers*, Second Edition, Macmillan, New York, 1964, p. 50, Problem 9.12.

3. J.V. Uspensky and M.A. Heaslet, *Elementary Number Theory*, McGraw-Hill, New York, 1939, p. 93.

* * *

295. [1977: 297] *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

If $0 < b \le a$, prove that

$$a + b - 2\sqrt{ab} \ge \frac{1}{2}\,\frac{(a-b)^2}{a+b}\,.$$

I. *Solution by W.J. Blundon, Memorial University of Newfoundland.*

If the two sides of $2(a+b) = (\sqrt{a}+\sqrt{b})^2 + (\sqrt{a}-\sqrt{b})^2$ are multiplied by $(\sqrt{a}-\sqrt{b})^2$, we obtain

$$2(a+b)(a+b-2\sqrt{ab}) = (a-b)^2 + (\sqrt{a}-\sqrt{b})^4 \ge (a-b)^2,$$

with equality if and only if $a = b$. Division by $2(a+b)$ gives the required result.

II. *Solution by Kenneth M. Wilke, Washburn University, Topeka, Kansas.*

Let

$$A = \frac{a+b}{2}, \qquad G = \sqrt{ab}, \qquad H = \frac{2ab}{a+b}.$$

Applying the well-known arithmetic mean - geometric mean inequality to A and H, we have

$$A + H \geq 2\sqrt{AH} = 2G;$$

hence

$$2A - 2G \geq A - H,$$

that is,

$$a + b - 2\sqrt{ab} \geq \frac{a+b}{2} - \frac{2ab}{a+b} = \frac{(a-b)^2}{2(a+b)},$$

equality holding if and only if $a = b$.

III. *Solution by Kenneth S. Williams, Carleton University, Ottawa.*

We expand the left side of $(\sqrt{a} - \sqrt{b})^4 \geq 0$ and rearrange to obtain

$$a^2 + 6ab + b^2 \geq 4(a+b)\sqrt{ab},$$

from which we get successively

$$2(a+b)^2 - (a-b)^2 \geq 4(a+b)\sqrt{ab}$$

and

$$2(a+b)^2 - 4(a+b)\sqrt{ab} \geq (a-b)^2.$$

Finally, dividing both sides by $2(a+b)$ yields the desired inequality, in which it is clear that equality holds if and only if $a = b$.

Also solved by LEON BANKOFF, Los Angeles, California; PAUL J. CAMPBELL for the Beloit College Solvers, Beloit, Wisconsin; RADFORD de PEIZA, L'Amoreaux Collegiate Institute, Agincourt, Ontario; CLAYTON W. DODGE, University of Maine at Orono; JACK GARFUNKEL, Forest hills H.S., Flushing, N.Y.; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; G.C. GIRI, Research Scholar, Indian Institute of Technology, Kharagpur, India; ALLAN Wm. JOHNSON, Jr., Washington, D.C.; ROBERT S. JOHNSON, Montréal, Québec; J.D.E. KONHAUSER, Macalester College, Saint Paul, Minnesota; N. KRISHNASWAMY, student, Indian Institute of Technology, Kharagpur, India; JACK LeSAGE, Eastview Secondary School, Barrie, Ontario; VIKTORS LINIS, University of Ottawa; BOB MAHONEY, Algonquin College, Ottawa; F.G.B. MASKELL, Collège Algonquin, Ottawa; LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; the late R. ROBINSON ROWE, Sacramento, California; HARRY D. RUDERMAN, Hunter College, New York; DONALD P. SKOW, Edinburg, Texas; BERNARD VANBRUGGHE, Université de Moncton; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer.

Editor's comment.

The condition $b \leq a$ was not used in any of the above solutions, so the proposed inequality holds equally well if $0 < a \leq b$, a fact mentioned by only three solvers, LeSage, Meyers, and Wang.

The proposer mentioned that the problem was suggested to him by a double inequality from Mitrinović's *Analytic Inequalities* which is given in Problem 247

[1977: 131; 1978: 23, 37]. Mitrinović's inequality (one-half of it, anyway) is, in fact, a *consequence* of our present problem, but only if $0 < b \le a$.

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296. [1977: 297] *Proposé par F.G.B. Maskell, Collège Algonquin, Ottawa.*

Soit p un nombre premier. Montrer que $p^4 - 20p^2 + 4$ n'est pas un nombre premier.

Solution by Bob Prielipp, The University of Wisconsin-Oshkosh.

We will show that $f(x) = x^4 - 20x^2 + 4$ is composite when x is any integer (not merely a prime). Since

$$\begin{aligned} f(x) &= (x^2 - 2)^2 - 16x^2 \\ &= (x^2 - 4x - 2)(x^2 + 4x - 2), \end{aligned}$$

$f(x)$ is a prime only if

$$x^2 - 4x - 2 = \pm 1 \quad \text{or} \quad x^2 + 4x - 2 = \pm 1,$$

that is, only if one of the equations

$$x^2 \pm 4x - 3 = 0, \qquad x^2 \pm 4x - 1 = 0$$

has an integral root. But the discriminants of these equations, 28 and 20, are not perfect squares, so none has an integral root, and we conclude that $f(x)$ is composite for all integers x.

Also solved by LEON BANKOFF, Los Angeles, California; W.J. BLUNDON, Memorial University of Newfoundland; LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; PAUL J. CAMPBELL for the Beloit College Solvers, Beloit, Wisconsin; CECILE M. COHEN, John F. Kennedy H.S., New York; STEVEN R. CONRAD, Benjamin N. Cardozo H.S., Bayside, N.Y.; RADFORD de PEIZA, L'Amoreaux Collegiate Institute, Agincourt, Ontario; CLAYTON W. DODGE, University of Maine at Orono; JACK GARFUNKEL, Forest Hills H.S., Flushing, N.Y.; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; ROBERT S. JOHNSON, Montréal, Québec; VIKTORS LINIS, University of Ottawa; LAI LANE LUEY, Willowdale, Ontario; BOB MAHONEY, Algonquin College, Ottawa; LEROY F. MEYERS, The Ohio State University, Columbus; HERMAN NYON, Paramaribo, Surinam; the late R. ROBINSON ROWE, Sacramento, California; HARRY D. RUDERMAN, Hunter College, New York; DAVID R. STONE, University of Kentucky, Lexington; BERNARD VANBRUGGHE, Université de Moncton; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer.

* * *

297. [1977: 298] *Proposed by Kenneth M. Wilke, Washburn University, Topeka, Kansas.*

A young lady went to the store to purchase four items. In computing her bill, the nervous clerk multiplied the four amounts together and announced that the bill was \$6.75. Since the young lady had added the four amounts mentally and obtained the same total, she paid her bill and left. Assuming that the prices for each item are distinct, what are the individual prices?

Solution by the proposer.

Let the distinct individual prices, in *cents*, be a, b, c, d. Then we must solve in distinct positive integers the system

$$abcd = 675000000 = 2^6 \cdot 3^3 \cdot 5^8, \tag{1}$$

$$a+b+c+d = 675. \tag{2}$$

Observe from (2) that each price is less than 675. If the eight factors 5 in (1) were allocated to only one or two of the prices, then one price would be greater than 675, or else two of the distinct prices would each be equal to 5^4. Thus at least three of the prices (hence also the fourth) must be divisible by 5. Since the sum of the four prices ends in 5, their last digits must be, in some order,

$$(0,\ 0,\ 0,\ 5) \quad \text{or} \quad (5,\ 5,\ 5,\ 0). \tag{3}$$

The second possibility in (3) requires that the sole even price, call it d, be

$$d = 2^6 \cdot 5 = 320, \tag{4}$$

and so $abc = 3^3 \cdot 5^7$. But then the A.M.-G.M. inequality implies that

$$a+b+c > 3\sqrt[3]{abc} = 3\sqrt[3]{3^3 \cdot 5^7} > 384; \tag{5}$$

and (4) and (5) are incompatible with (2).

The first possibility in (3) must therefore hold. Suppose the last digits of (a, b, c, d), in that order, are $(0, 0, 0, 5)$. If we divide each price by 5 in (1) and (2), we obtain the new system

$$a_1b_1c_1d_1 = 2^6 \cdot 3^3 \cdot 5^4, \tag{6}$$

$$a_1+b_1+c_1+d_1 = 135, \tag{7}$$

where the first three of a_1, b_1, c_1, d_1 are even and all are less than 135. We assume, without loss of generality, that $a_1 > b_1 > c_1$.

Suppose $(d_1, 5) = 1$. The four factors 5 in (6) cannot be allocated to only one or two of the even numbers a_1, b_1, c_1; for then one of them would be greater than 135, or else two of them would each be equal to 50. So all three of a_1, b_1, c_1 are divisible by 5, and so is d_1 by (7), a contradiction.

Suppose $d_1 = 125$. Then $a_1b_1c_1 = 2^6 \cdot 3^3 \cdot 5$ and, as before,

$$a_1+b_1+c_1 > 3\sqrt[3]{a_1b_1c_1} = 3\sqrt[3]{2^6 \cdot 3^3 \cdot 5} > 61,$$

which is incompatible with (7).

Two cases are left to be considered.

Case 1. $(d_1, 25) = 5$. Here $d_1 = 5$, 15, or 45. If $d_1 = 45$, then

$$a_1 + b_1 + c_1 = 90, \qquad a_1 b_1 c_1 = 2^6 \cdot 3 \cdot 5^3,$$

and dividing each of the even numbers a_1, b_1, c_1 by 2 leaves the system

$$a_2 b_2 c_2 = 2^3 \cdot 3 \cdot 5^3, \tag{8}$$

$$a_2 + b_2 + c_2 = 45. \tag{9}$$

Allocating all three of the factors 5 from (8) to only one of a_2, b_2, c_2 would violate (9), so at least two (and hence all three) are divisible by 5. Now dividing each by 5 leaves the system

$$a_3 b_3 c_3 = 2^3 \cdot 3, \qquad a_3 + b_3 + c_3 = 9,$$

whose only solution is clearly $(a_3, b_3, c_3) = (4, 3, 2)$. Reverting back to a, b, c, d gives as one solution to our problem:

$$(a, b, c, d) = (200, 150, 100, 225).$$

A similar analysis if $d_1 = 5$ or $d_1 = 15$ yields no further solutions.

Case 2. $(d_1, 25) = 25$. Here $d_1 = 25$ or 75. If $d_1 = 25$, then

$$a_1 + b_1 + c_1 = 110, \qquad a_1 b_1 c_1 = 2^6 \cdot 3^3 \cdot 5^2,$$

and dividing each number by 2 gives

$$a_2 + b_2 + c_2 = 55, \qquad a_2 b_2 c_2 = 2^3 \cdot 3^3 \cdot 5^2.$$

Now consideration of the possible distribution of the factors 2, 3, and 5 shows that we must have $(a_2, b_2, c_2) = (25, 18, 12)$. (If this is not convincing, consider that $55 = a_2 + b_2 + c_2 < 3a_2$, so $18 < a_2 < 55$. If $a_2 = 25$, then b_2 and c_2 are the roots of

$$x^2 - (b_2 + c_2)x + b_2 c_2 = x^2 - 30x + 216 = (x - 18)(x - 12) = 0,$$

so $b_2 = 18$, $c_2 = 12$; and no other possible value of a_2 yields integral values for b_2 and c_2.) Reverting back to a, b, c, d gives another solution to our problem:

$$(a, b, c, d) = (250, 180, 120, 125).$$

A similar analysis if $d_1 = 75$ yields no further solutions.

So there are two solutions to our problem: the prices are

$$\$2.00,\ \$1.50,\ \$1.00,\ \$2.25$$

or

$$\$2.50,\ \$1.80,\ \$1.20,\ \$1.25.$$

Partial solutions were submitted by LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; PAUL J. CAMPBELL for the Beloit College Solvers, Beloit, Wisconsin; CECILE M. COHEN, John F. Kennedy H.S., New York; ALLAN Wm. JOHNSON, Jr., Washington, D.C.; ROBERT S. JOHNSON, Montréal, Québec; the following students in the class of JACK LeSAGE, Eastview Secondary School, Barrie, Ontario: DAVE GOURLEY, ROB HIGGINSON, MIKE HODGINS, GEORGE JEWELL, KAROL KOMAR, JANE KOREVAAR, SUSAN McARTHUR, MARK McCUAIG, PATTI MENEZES, CARYL ONYSCHUK, SUSAN REYNOLDS, JIM ROBB, LAURIE ROBILLARD, PAT WALSH, and NANCY WHETHAM; BOB MAHONEY and F.G.B. MASKELL, Algonquin College, Ottawa; HERMAN NYON, Paramaribo, Surinam; the late R. ROBINSON ROWE, Sacramento, California; and HARRY D. RUDERMAN, Hunter College, New York.

Editor's comment.

Believe it or not, *each* of the twenty-five other solvers gave as the unique answer: \$1.00, \$1.50, \$2.00, \$2.25. This was a rare display of unanimity in a lost cause, for it is certainly true that also

$$1.20 + 1.25 + 1.80 + 2.50 = 1.20 \times 1.25 \times 1.80 \times 2.50 = 6.75.$$

I will not speculate as to the reason for this collective blindness to the logical gaps in their seat-of-the-pants solutions (when there *was* a solution: three of them sent in *only* an answer, a pointless exercise if ever there was one).

The proposer pointed out that this problem is a variant of Problem 65 in I.A. Graham's *Ingenious Mathematical Problems and Methods*, Dover, 1959, pp. 39, 201.

* * *

298. [1977: 298] *Proposed by Clayton W. Dodge, University of Maine at Orono.*

The equation $x^2 - 9x + 18 = 4$ has the property that, if the left side is factored, so that $(x-3)(x-6) = 4$, then one of the roots, $x = 7$, is found by illegally setting one of the factors equal to the constant on the right, $x - 3 = 4$. Unfortunately, the second root cannot be similarly found; it is not $x - 6 = 4$. Find all such quadratic equations in which both roots can be obtained by equating each factor in turn to the nonzero constant on the right.

I first heard of this problem in a lecture by Howard Eves some years ago.

Solution by Gali Salvatore, Ottawa, Ontario.

Any quadratic equation has two roots, real or complex, distinct or not, and any two quadratic equations are *equivalent* if they have the same roots. Any given equation has infinitely many equivalent forms which can be obtained by adding the same terms to both sides and/or multiplying both sides by a nonzero constant. These elementary considerations disposed of, we affirm that *any* given quadratic equation, without exception, can be completely and correctly solved by the *pseudosolution* described in the proposal, and in infinitely many ways, in the sense that it has infinitely many equivalent forms to which the pseudosolution applies.

If n is any nonzero number (not necessarily an integer or even real), it is easily verified that the equation in the proposal, for example, is equivalent to

$$(1 + 2n - nx)(nx + 1 - 7n) = 1 - 5n \tag{1}$$

and to

$$(1 + 7n - nx)(nx + 1 - 2n) = 1 + 5n. \tag{2}$$

Now, with the sole restriction $n \neq \pm 1/5$, so that the right sides are nonzero, the pseudosolution applied to (1) and (2) will correctly yield *both* roots $x = 2$ and $x = 7$ in each case. (The restriction $n \neq \pm 1/5$ can even be lifted. If we select $n = -1/5$ we use only (1), and if we select $n = 1/5$ we use only (2).)

In general, suppose we have given a quadratic equation whose roots are a and b. One of its equivalent forms is obviously

$$(x - a)(x - b) = 0 \tag{3}$$

and, for any nonzero n, two other equivalent forms are

$$(1 + nb - nx)(nx + 1 - na) = 1 - n(a - b) \tag{4}$$

and

$$(1 + na - nx)(nx + 1 - nb) = 1 - n(b - a), \tag{5}$$

since each of these is equivalent to (3), a matter which is easily verified. If $a = b$, the pseudosolution applied to (4) and (5) will yield the correct roots in both cases. If $a \neq b$, and we impose the sole restriction $n \neq \pm 1/(a - b)$ (a restriction which can be lifted, as explained earlier), the pseudosolution will again yield the correct roots $x = a$ and $x = b$ in each case.

There is a fly in the ointment, of course, which limits the usefulness of the method: the roots a and b must be known before a given equation can be recast in one of the equivalent forms (4) or (5). And this disadvantage cannot be obviated, for we show that every equivalent form of a quadratic equation for which the pseudosolution works must be of the form (4) or (5).

Consider an equation whose roots are a and b, and let

$$(mx + p)(nx + q) = r, \qquad m,n,r \neq 0 \tag{6}$$

be one of its equivalent forms for which the pseudosolution works, so that the roots are

$$a = \frac{r - p}{m}, \qquad b = \frac{r - q}{n}. \tag{7}$$

The sum and product of the roots in (6) are

$$-\frac{pn+qm}{mn} \quad \text{and} \quad \frac{pq-r}{mn},$$

so we must have

$$\frac{r-p}{m}+\frac{r-q}{n}=-\frac{pn+qm}{mn} \tag{8}$$

and

$$\frac{(r-p)(r-q)}{mn}=\frac{pq-r}{mn}. \tag{9}$$

From (8) and (9), we get $m+n = 0$ and $r-p-q = -1$; so $m=-n$, $r = p+q-1$, and (6) becomes

$$(p-nx)(nx+q) = p+q-1. \tag{10}$$

Now, from (7),

$$-na = r-p = q-1, \qquad nb = r-q = p-1,$$

so

$$p = 1+nb, \qquad q = 1-na,$$

and (10) becomes identical to (4). Similarly, interchanging a and b in (7) would make (10) identical to (5).

I have heard it said that Euclide Paracelso Bombasto Umbugio [1977: 118-128] considers that our pseudosolution is the *only* way to solve a quadratic equation, and that in the textbook he uses at the University of Guayazuela (a book he wrote himself) *every* quadratic equation is of the form (4) or (5).

Also solved by PAUL J. CAMPBELL for the Beloit College Solvers, Beloit, Wisconsin; CECILE M. COHEN, John F. Kennedy H.S., New York; STEVEN R. CONRAD, Benjamin N. Cardozo H.S., Bayside, N.Y.; MICHAEL W. ECKER, City University of N.Y.; G.C. GIRI, Research Scholar, Indian Institute of Technology, Kharagpur, India; ROSS HONSBERGER, University of Waterloo; TIMOTHY S. HOUSTON, Rensselaer Polytechnic Institute Class of 1981, Troy, N.Y.; ALLAN Wm. JOHNSON, Jr., Washington, D.C.; JACK LeSAGE, Eastview Secondary School, Barrie, Ontario; PETER A. LINDSTROM, Genesee Community College, Batavia, N.Y.; BOB MAHONEY, Algonquin College, Ottawa; F.G.B. MASKELL, Collège Algonquin, Ottawa; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; the late R. ROBINSON ROWE, Sacramento, California; HARRY D. RUDERMAN, Hunter College, New York; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer.

Editor's comment.

This problem has been floating around for quite some time, as can be seen from the references given below, most of which were sent in by readers. But, as far as I can tell, nowhere else is it treated with the same generality as in our own solution. In all the references I've seen, only particular numerical examples are given or, at best, the general case is treated but only for $n=1$.

So the pseudosolution is only a mathematical curiosity of limited usefulness except, perhaps, as a party game. Astound your friends (those who know about quadratic equations) with a few carefully prepared examples. Better still, get a copy of Umbugio's book and ask your friends to dip into it at random.

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299. [1977: 298] *Proposed by M.S. Klamkin, University of Alberta.*

If

$$F_1 = (-r^2+s^2-2t^2)(x^2-y^2-2xy) - 2rs(x^2-y^2+2xy) + 4rt(x^2+y^2),$$

$$F_2 = -2rs(x^2-y^2-2xy) + (r^2-s^2-2t^2)(x^2-y^2+2xy) + 4st(x^2+y^2),$$

$$F_3 = -2rt(x^2-y^2-2xy) - 2st(x^2-y^2+2xy) + (r^2+s^2+2t^2)(x^2+y^2),$$

show that F_1, F_2 and F_3 are functionally dependent and find their functional relationship. Also, reduce the five-parameter representation of F_1, F_2 and F_3 to one of two parameters.

Solution de F.G.B. Maskell, Collège Algonquin, Ottawa.

Posons

$$A = x(r-t) + y(s+t),$$

$$B = x(s-t) - y(r+t).$$

On vérifie facilement, avec un peu de patience, que

$$\begin{cases} F_3 + F_2 = 2A(A - B), & F_3 - F_2 = 2B(A + B), \\ F_3 - F_1 = 2A(A + B), & F_3 + F_1 = -2B(A - B). \end{cases} \tag{1}$$

On a donc

$$(F_3 + F_2)(F_3 - F_2) + (F_3 + F_1)(F_3 - F_1) = 0,$$

ce qui amène

$$F_1^2 + F_2^2 = 2F_3^2. \tag{2}$$

C'est la relation fonctionnelle recherchée.

On verra que les deux paramètres A et B suffisent pour représenter les trois fonctions données. En effet, de

$$2F_3 - F_1 + F_2 = 4A^2 \quad \text{et} \quad 2F_3 + F_1 - F_2 = 4B^2,$$

on obtient

$$-F_1 + F_2 = 2(A^2 - B^2) \tag{3}$$

et

$$F_3 = A^2 + B^2. \tag{4}$$

Portons maintenant dans (2) les valeurs de F_2 et F_3 de (3) et (4); il résulte

$$F_1^2 + 2(A^2 - B^2)F_1 + A^4 - 6A^2B^2 + B^4 = 0,$$

d'où $F_1 = -A^2 \pm 2AB + B^2$. Ici il faut prendre

$$F_1 = -A^2 - 2AB + B^2,$$

car l'autre choix, avec la valeur de F_2 qui découle alors de (3), ne vérifie aucune des relations (1). La représentation recherchée est donc

$$F_1 = -A^2 - 2AB + B^2, \qquad F_2 = A^2 - 2AB - B^2, \qquad F_3 = A^2 + B^2.$$

Partial solutions were submitted by N. Krishnaswamy, student, Indian Institute of Technology, Kharagpur, India; and by the proposer.

Editor's comment.

The proposer mentioned that the problem arose in applying the method of Desboves in obtaining the general solution of the Diophantine equation $F_1^2 + F_2^2 = 2F_3^2$ from the knowledge of one particular solution. An analogous but more complicated set of equations would arise if we started with the general homogeneous quadratic Diophantine equation in n variables and one particular solution.

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300. [1977: 298] *Proposed by Léo Sauvé, Algonquin College (editor).*

The sine and cosine are known as transcendental functions, so one would expect that $\sin x$ and $\cos x$ would be transcendental numbers for most values of x. Does there exist a *dense* subset E of the reals such that $\sin x$ and $\cos x$ are both algebraic for every x in E?

I. *Solution by Michael W. Ecker, City University of N.Y.*

In view of the periodicity of the functions, it is sufficient to find a dense subset E of the interval $I = [0, 2\pi]$ with the desired property. Furthermore, we need only investigate the cosine function, since the identity $\sin^2 x + \cos^2 x = 1$ shows that $\cos x$ is algebraic if and only if $\sin x$ is algebraic.

We will show that the set

$$E = \left\{\frac{m}{n}\cdot\frac{\pi}{2}\;\middle|\; n = 1,2,3,\ldots;\quad m = 0,1,2,\ldots,4n\right\},$$

which is clearly dense in I, has the desired property.

The proof is based on De Moivre's Theorem:

$$(\cos x + i\sin x)^k = \cos kx + i\sin kx.$$

Taking real parts and noting that only even powers of sines occur, we have

$$\cos kx = P_k(\cos x), \tag{1}$$

where $P_k(\cos x)$ is a polynomial in $\cos x$ with integral coefficients. If we put $k = n$ and $x = \pi/2n$ in (1), we get

$$\cos\frac{\pi}{2} = 0 = P_n\left(\cos\frac{\pi}{2n}\right), \tag{2}$$

and so $\cos(\pi/2n)$ is an algebraic number. Finally, putting $k = m$ and $x = \pi/2n$ in (1) gives

$$\cos\left(\frac{m}{n}\cdot\frac{\pi}{2}\right) = P_m\left(\cos\frac{\pi}{2n}\right), \tag{3}$$

which shows that $\cos\left(\frac{m}{n}\cdot\frac{\pi}{2}\right)$ is algebraic.

II. *Solution de Jean Dhombres, Conseiller scientifique, Ambassade de France au Canada.*

Grâce aux formules trigonométriques d'addition et de soustraction des arcs, il est clair que l'ensemble E des réels pour lesquels $\cos x$ et $\sin x$ sont algébriques constitue un sous-groupe de la droite réelle. Sous-groupe E qui ne peut être discret puisque π/n en est un élément pour tout entier $n \geq 1$. Donc E est dense selon un résultat connu.

Ce même raisonnement s'adapte aux familles de fonctions transcendantes pour lesquelles on dispose de formules algébriques d'addition et la même conclusion prévaut... donc la possibilité de formuler des problèmes analogues avec d'autres fonctions. A remarquer d'ailleurs que le sous-groupe E en question est de mesure nulle au sens de Lebesgue, car dénombrable.

Bien sûr, un argument topologique direct et plus général est le suivant. Si $f: R \to R$ est telle que l'image, lorsque non vide, de tout ouvert contienne un ouvert non vide, alors l'ensemble des points où cette fonction prend une valeur algébrique (voire même rationnelle) est dense dans la droite réelle. Tel est le cas pour toute fonction presque partout localement inversible et donc le résultat est acquis pour $x \to \cos x$ et $x \to \sin x$. Avec la formule $\cos^2 x + \sin^2 x = 1$ on termine la démonstration.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; VIKTORS LINIS, University of Ottawa; J. WALTER LYNCH, Georgia Southern College, Statesboro, Georgia; F.G.B. MASKELL, Collège Algonquin, Ottawa; LEROY F. MEYERS, The Ohio State University (three solutions); the late R. ROBINSON ROWE, Sacramento, California; HARRY D. RUDERMAN, Hunter College, New York; and the proposer.

Editor's comment.

Show me a dense set of algebraic numbers and I'll show you close encounters of the surd kind.

When explaining the concept of transcendental numbers to students, most teachers of course begin by pointing out that e and π are transcendental. When asked to give further examples, many teachers refer the students to their trigonometric tables where, they are told, they will find a veritable army of transcendental numbers in battle array. Of course, the teacher hastens to add, the tables contain only rational approximations, but if the numbers could be written out in full, the vast majority of them would be transcendental numbers.

It may come as a surprise to some to learn that in every trigonometric table with sexagesimal arguments, *every* entry, without a single exception, would be an algebraic number if written out in full. There is not one transcendental number in the lot.

The proof of this startling statement follows easily from solution I. For let a, b, c be integers such that $0 \le a$, $0 \le b < 60$, $0 \le c < 60$. If wet set $n = 90 \cdot 60^2$ in (2) and (3), and $m = a \cdot 60^2 + b \cdot 60 + c$ in (3), it follows from (2) that $\cos 1''$ is algebraic, and from (3) that $\cos (a°b'c'')$ is algebraic. So $\sin (a°b'c'')$ is also algebraic and (since the real numbers algebraic over the rationals form a field) so are all the other trigonometric functions of $a°b'c''$.

Does this mean that the trigonometric functions seldom take on transcendental values? Hardly. It is known that e^{iu} is transcendental whenever u is a nonzero algebraic number; hence so is

$$\sin u = \frac{e^{iu} - e^{-iu}}{2i}$$

and so are all the other trigonometric functions of u. There is no contradiction between this and the previous statement about $a°b'c"$, since $1" = \pi/(2 \cdot 90 \cdot 60^2)$ is transcendental.

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301. [1978: 11] *Proposed by Herman Nyon, Paramaribo, Surinam.*

The following cryptarithmic decimal subtraction is dedicated to two of the outstanding digit-delvers of our day, J.A.H. Hunter and Charles W. Trigg:

HUNTER - TRIGG = DIGITS.

There are, as one would expect, exactly two solutions. That our two protagonists tower equally in the field of digital recreations is shown by the fact that the sums of the digits of HUNTER and TRIGG in one solution are equal, respectively, to the sums of the digits of TRIGG and HUNTER in the other solution.

Solution de Hippolyte Charles, Waterloo, Québec.

L'analyse du problème sera facilitée si l'on transforme la soustraction en addition:

$$\begin{array}{r} \text{TRIGG} \\ \underline{\text{DIGITS}} \\ \text{HUNTER} \end{array}.$$

En partant de la droite, nous avons

$$\mathrm{G} + \mathrm{S} = \mathrm{R} + 10c_1, \tag{1}$$

$$\mathrm{G} + \mathrm{T} + c_1 = \mathrm{E} + 10c_2, \tag{2}$$

$$2\mathrm{I} + c_2 = \mathrm{T} + 10c_3, \tag{3}$$

$$\mathrm{R} + \mathrm{G} + c_3 = \mathrm{N} + 10c_4, \tag{4}$$

$$\mathrm{T} + \mathrm{I} + c_4 = \mathrm{U} + 10, \tag{5}$$

$$\mathrm{D} + 1 = \mathrm{H}, \tag{6}$$

où tous les c_i sont 0 ou 1.

Notons en partant que les lettres initiales des trois mots ne sont pas nulles, ce qui entraîne, avec (1) et (5), que le produit TDHGSI $\neq 0$. Éliminons T entre (3) et (5); cela donne

$$3I = U + 10 - c_2 + 10c_3 - c_4. \qquad (7)$$

Le tableau qui suit exhibe, selon les valeurs du triplet (c_2,c_3,c_4), les valeurs possibles du triplet (U,I,T), les valeurs de (U,I) provenant de (7) et celles de T de (5).

	(c_2,c_3,c_4)	(U,I,T)
(a)	(0,0,0)	(2,4,8)
(b)	(0,0,1)	(0,3,6)
(c)		(3,4,8)
(d)	(0,1,0)	(1,7,4)
(e)		(4,8,6)
(f)		(7,9,8)
(g)	(0,1,1)	(2,7,4)
(h)		(5,8,6)

	(c_2,c_3,c_4)	(U,I,T)
(i)	(1,0,0)	(0,3,7)
(j)		(3,4,9)
(k)	(1,0,1)	(1,3,7)
(l)	(1,1,0)	(2,7,5)
(m)		(5,8,7)
(n)	(1,1,1)	(0,6,3)
(o)		(3,7,5)
(p)		(6,8,7)

Supposons d'abord $c_1 = 0$; alors $R \geq 3$ d'après (1) et 0 doit être représenté par une des lettres U, N, E. Il faut maintenant s'attaquer à la tâche facile mais fastidieuse de reporter un par un tous les cas (a)-(p) dans (1)-(6). On verra à la longue que, pour des raisons diverses, tous les cas (a)-(p) sont éliminés sauf (b), et que celui-ci donne la solution

HUNTER - TRIGG = DIGITS

501689 - 69322 = 432367.

On verrait de même que, pour $c_1 = 1$, tous les cas (a)-(p) sont éliminés sauf (p), qui donne la deuxième et dernière solution

HUNTER - TRIGG = DIGITS

260734 - 74855 = 185879.

Also solved by RICHARD BURNS and KRISTIN DIETSCHE, East Longmeadow H.S., East Longmeadow, Massachusetts (jointly); CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer.

Editor's comment.

Let ΣH_i and ΣT_i denote the sums of the digits in HUNTER and TRIGG in the two solutions. We have indeed, as mentioned in the proposal,

$$\Sigma H_1 = \Sigma T_2 = 29 \quad \text{and} \quad \Sigma H_2 = \Sigma T_1 = 22, \qquad (8)$$

a most cozy relationship.

Trigg started his solution by converting the subtraction to an addition, as in our featured solution. But he added that this creates the paradox that TRIGG is on top but HUNTER is the greater. He also noted that the sums of the digits of the sums of the digits of DIGITS are the two lucky primes, 7 and 11. Finally, he observed that the difference of the two values of TRIGG is 5533, that the difference of the two values of HUNTER is 240955, and that the digital root of each of these two differences is 7.

Burns and Dietsche discovered that in each solution the sums of the digits of HUNTER and TRIGG is 51, but this is a consequence of (8).

The proposer noted that in the first solution HUNTER is the product of two primes differing by 1000:

$$501689 = 367 \cdot 1367.$$

Not to be outdone, the Editor found out that in the second solution HUNTER is also a product of two primes differing, this time, by 130365:

$$260734 = 2 \cdot 130367.$$

The two TRIGGs, on the other hand, have most uninteresting factorizations. But Trigg can take some consolation from the fact that in each reincarnation he is divisible by the lucky prime 11.

So this problem pays a well-earned tribute to "two of the outstanding digit-delvers of our day," and many pretty features were concealed in the answers. In spite of this, the Editor finds it necessary to say that this problem is not a good example of the *genre*. It does not respond readily to a logical attack, and too much tedious calculation is necessary to arrive at the answers. It is a type of problem that could, and should, be handled most efficiently by a computer, and as such, it is not entirely appropriate for this journal. Readers should try to submit problems of this type that can be comfortably solved by other readers, not by computers. Why should computers have all the fun with our problems? They don't even subscribe to our journal.

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302. [1978: 11] *Proposed by Leroy F. Meyers, The Ohio State University.*

Show that if p is a prime, then $p^2 + 5$ is not a prime. (I first heard of this problem from H.J. Ryser.)

Solution by Hyman Rosen, Yeshiva University High School, Brooklyn, N.Y.

If p is odd (prime or not), then p^2 is odd and $p^2 + 5$ is even and greater than 2, hence not a prime. If $p = 2$, then $p^2 + 5 = 9$, also not a prime.

Also solved by LEON BANKOFF, Los Angeles, California; RICHARD BURNS and KRISTIN DIETSCHE, East Longmeadow H.S., East Longmeadow, Massachusetts (jointly); LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; HIPPOLYTE CHARLES, Waterloo, Québec; CECILE M. COHEN, John F. Kennedy H.S., New York; RADFORD de PEIZA, L'Amoreaux Collegiate Institute, Agincourt, Ontario; CLAYTON W. DODGE, University of Maine at Orono; ANDREJS DUNKELS, University of Luleå, Sweden; MICHAEL W. ECKER, City University of N.Y.; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; V.G. HOBBES, Westmount, Québec; J.A.H. HUNTER, Toronto, Ontario; ROBERT S. JOHNSON, Montréal, Québec; F.G.B. MASKELL, Collège Algonquin, Ottawa; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; FREDERICK NEIL ROTHSTEIN, New Jersey Department of Transportation, Trenton, N.J.; the late R. ROBINSON ROWE, Sacramento, California; GALI SALVATORE, Ottawa, Ontario; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DONALD P. SKOW, McAllen Senior High School, "Home of the Battlin' Bulldogs," McAllen, Texas; CHARLES W. TRIGG, San Diego, California; BERNARD VANBRUGGHE, Université de Moncton; KENNETH M. WILKE, Washburn University, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

Editor's comment.

The problem was not incorrectly stated, as many readers thought. It was, and was meant to be, trivial (although a few of the solutions submitted were pretty complicated). The proposer submitted it in the hope that it could be used to fulfill the Editor's promise [1976: 230] to encourage more readers to submit solutions by slipping in one day a problem that *everybody* could solve.

The experiment was a failure: 91.64% of readers did not respond.

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303. [1978: 11] *Proposed by Viktors Linis, University of Ottawa.*

Huygens' inequality $2\sin\alpha + \tan\alpha \geq 3\alpha$ was proved in Problem 115. Prove the following hyperbolic analogue:

$$2\sinh x + \tanh x \geq 3x, \qquad x \geq 0.$$

Solution by Allan Wm. Johnson, Washington, D.C.

Let

$$f(x) = 2\sinh x + \tanh x - 3x;$$

then $f(0) = 0$ and, for $x > 0$,

$$f'(x) = 2\cosh x + \operatorname{sech}^2 x - 3 = (1 + 2\cosh x)(1 - \operatorname{sech} x)^2 > 0,$$

so f is increasing and $f(x) > 0$ for all $x > 0$.

Also solved by LEON BANKOFF, Los Angeles, California (two solutions); RICHARD BURNS and KRISTIN DIETSCHE, East Longmeadow H.S., East Longmeadow, Massachusetts (jointly); CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, City University of N.Y.; TIMOTHY S. HOUSTON, student, Rensselaer Polytechnic Institute, Troy, N.Y. (two solutions); LEROY F. MEYERS, The Ohio State University (two solutions); HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; HYMAN ROSEN, Yeshiva University H.S., Brooklyn, N.Y.; the late R. ROBINSON ROWE, Sacramento, California; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; KENNETH S. WILLIAMS, Carleton University, Ottawa, and the proposer.

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304. [1978: 11] *Proposed by Viktors Linis, University of Ottawa.*

Prove the following inequality:

$$\frac{\ln x}{x-1} \le \frac{1+\sqrt[3]{x}}{x+\sqrt[3]{x}}, \qquad x>0,\ x\ne 1.$$

I. *Solution by the proposer.*

Let

$$g(x) = \frac{(x-1)(1+\sqrt[3]{x})}{x+\sqrt[3]{x}} - \ln x, \qquad x>0. \tag{1}$$

Setting $x=e^{3u}$ gives

$$\begin{aligned} g(x) &= \frac{(e^{3u}-1)(1+e^{u})}{e^{3u}+e^{u}} - 3u \\ &= e^{u}-e^{-u} + \frac{e^{u}-e^{-u}}{e^{u}+e^{-u}} - 3u \\ &= 2\sinh u + \tanh u - 3u \\ &= f(u), \end{aligned}$$

where f is the function defined in the solution to the preceding problem (Crux 303).

If $x>1$, then $u>0$ and it follows from Crux 303 that $g(x)=f(u)>0$; hence

$$\frac{\ln x}{x-1} < \frac{1+\sqrt[3]{x}}{x+\sqrt[3]{x}}. \tag{2}$$

If $0<x<1$, then setting $x=e^{-3v}$ in (1) gives

$$g(x) = f(-v) = -f(v) < 0$$

since $v>0$, and this leads again to (2). Finally if, as suggested by L'Hôpital's Rule, we define the left side of (2) to be 1 when $x=1$, then

$$\frac{\ln x}{x-1} \le \frac{1+\sqrt[3]{x}}{x+\sqrt[3]{x}}$$

holds for all $x>0$.

II. *Comment extracted from a solution submitted jointly by James W. Burgmeier and Larry L. Kost, both from the University of Vermont.*

It was proved in solution I that the inequality

$$\frac{\ln x}{x-1} \le \frac{1+x^{\alpha}}{x+x^{\alpha}}, \quad x>0 \tag{3}$$

holds for $\alpha=1/3$. We show that it holds if and only if $\alpha\ge 1/3$.

Let $L(x)$ and $R(x)$ denote the left and right sides of (3), respectively, and let

$$h(x) = L(x) - R(x).$$

Then we have

$$h(1) = L(1) - R(1) = 1 - 1 = 0,$$

$$h'(1) = L'(1) - R'(1) = -\frac{1}{2} - \left(-\frac{1}{2}\right) = 0,$$

$$h''(1) = L''(1) - R''(1) = \frac{2}{3} - \frac{1}{2}(1+\alpha) = \frac{1}{2}\left(\frac{1}{3} - \alpha\right).$$

Consequently, if $\alpha < 1/3$ then $h''(1)$ is positive, $h(x)$ is positive for some $x > 1$, and so (3) does not hold. (There is in fact a deleted neighborhood of $x = 1$ in which $h(x)$ is then positive, since $h(1/x) = xh(x)$.)

Now consider R as a function of α, say

$$R_1(\alpha) = \frac{1 + x^\alpha}{x + x^\alpha}.$$

For each x such that $0 < x \neq 1$, we have

$$R_1'(\alpha) = \frac{x^{\alpha-2}(x-1)\ln x}{(1 + x^{\alpha-1})^2} > 0;$$

so R_1 is an increasing function of α, $R_1(\alpha) > R_1(1/3)$ whenever $\alpha > 1/3$, and (3) holds for all $\alpha > 1/3$.

Also solved by ALLAN Wm. JOHNSON, Jr., Washington, D.C.; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; and KENNETH S. WILLIAMS, Carleton University, Ottawa. Comments were received from LEON BANKOFF, Los Angeles, California; and W.J. BLUNDON, Memorial University of Newfoundland.

Editor's comment.

Bankoff found this inequality in Mitrinović [1], and Blundon and Williams found it in [2]. The proposer's idea was to provide a surprisingly simple proof of the Mitrinović inequality by using the much easier inequality of Crux 303, an inequality interesting in its own right because of its analogy with a well-known inequality of Huygens previously discussed in this journal [1976: 25, 98, 111, 137]. Not wanting to give too much of the game away, and yet wishing to be fair with readers, I put the two inequalities in separate problems back to back. But no other solver saw the connection between the two.

REFERENCES

1. D.S. Mitrinović, *Elementary Inequalities*, P. Noordhoff Ltd., Groningen, 1964, p. 75.

2. ________, *Analytic Inequalities*, Springer-Verlag, New York, 1970, p. 272.

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305. [1978: 11] *Proposed by Bruce McColl, St. Lawrence College, Kingston, Ontario.*

How many distinct values does $\cos(\frac{1}{3}\sin^{-1}a)$ have? What is the product of these values?

Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.

Let 3θ be any value of $\sin^{-1}a$, that is, any number such that $\sin 3\theta = a$; then the set of all values of $\sin^{-1}a$ is

$$\{n\pi + (-1)^n 3\theta \mid n \in Z\}$$

and the set of all values of $\frac{1}{3}\sin^{-1}a$ is

$$\{\phi_n \mid \phi_n = \frac{n\pi}{3} + (-1)^n\theta,\ n \in Z\}.$$

Since $\cos\phi_{n+6} = \cos(2\pi + \phi_n) = \cos\phi_n$, it follows that all the distinct values of $\cos(\frac{1}{3}\sin^{-1}a)$ are contained in the set

$$E = \{\cos\phi_n \mid n = 0,1,\ldots,5\},$$

which has at most 6 distinct elements. These are

$$\cos\phi_0 = \cos\theta, \qquad \cos\phi_3 = \cos(\pi - \theta) = -\cos\theta,$$

$$\cos\phi_1 = \cos\left(\frac{\pi}{3} - \theta\right) = -\cos\left(\frac{2\pi}{3} + \theta\right), \qquad \cos\phi_4 = \cos\left(\frac{4\pi}{3} + \theta\right) = \cos\left(\frac{2\pi}{3} - \theta\right),$$

$$\cos\phi_2 = \cos\left(\frac{2\pi}{3} + \theta\right), \qquad \cos\phi_5 = \cos\left(\frac{5\pi}{3} - \theta\right) = -\cos\left(\frac{2\pi}{3} - \theta\right).$$

The product of all the elements of E, multiplicities being counted, is thus

$$\begin{aligned}\prod_{i=0}^{5}\cos\phi_i &= -\left[\cos\theta\cos\left(\frac{2\pi}{3} - \theta\right)\cos\left(\frac{2\pi}{3} + \theta\right)\right]^2 \\ &= -\left(\frac{1}{4}\cos 3\theta\right)^2 = -\frac{1}{16}(1 - \sin^2 3\theta) = -\frac{1}{16}(1 - a^2). \qquad (1)\end{aligned}$$

If $a = 0$, then E contains exactly 4 distinct elements, ± 1 and $\pm 1/2$, whose product is $1/4$ if multiplicities are not counted. Similarly, if $a = \pm 1$, E contains exactly 3 distinct elements, 0, $\pm\sqrt{3}/2$, whose product 0 corresponds to that given by (1) whether or not multiplicities are counted. Finally, if $0 < |a| < 1$ the set E contains 6 distinct elements whose product is given by (1).

Also solved by RICHARD BURNS and KRISTIN DIETSCHE, East Longmeadow H.S., East Longmeadow, Massachusetts (jointly); VIKTORS LINIS, University of Ottawa; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam; and the late R. ROBINSON ROWE, Sacramento, California.

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