Crux Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

The Back Files

The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

C R U X   M A T H E M A T I C O R U M

Vol. 13, No. 6
June 1987

Published by the Canadian Mathematical Society/
Publié par la Société Mathématique du Canada

The support of the Departments of Mathematics and Statistics of the University of Calgary and Carleton University, and the Department of Mathematics of The University of Ottawa, is gratefully acknowledged.

*   *   *

CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is $22.50 for members of the Canadian Mathematical Society and $25 for nonmembers. Back issues: $2.75 each. Bound volumes with index: Vols. 1 & 2 (combined) and each of Vols. 3-10: $20. All prices quoted are in Canadian dollars. Cheques and money orders, payable to CRUX MATHEMATICORUM, should be sent to the Managing Editor.

All communications about the content of the journal should be sent to the Editor. All changes of address and inquiries about subscriptions and back issues should be sent to the Managing Editor.

Founding Editors: Léo Sauvé, Frederick G.B. Maskell.

Editor: G.W. Sands, Department of Mathematics and Statistics, University of Calgary, 2500 University Drive N.W., Calgary, Alberta, Canada, T2N 1N4.

Managing Editor: Dr. Kenneth S. Williams, Canadian Mathematical Society, 577 King Edward Avenue, Ottawa, Ontario, Canada, K1N 6N5.

ISSN 0705 - 0348.

Second Class Mail Registration No. 5432. Return Postage Guaranteed.

*   *   *

CONTENTS

THE OLYMPIAD CORNER: 86

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this column with the problems from the *Canadian Mathematical Olympiad, 1987* which we reproduce with the permission of The Canadian Mathematical Society. The official solutions will be discussed in a later issue, but we welcome your "nice" solutions as well.

1. Find all solutions of $a^2 + b^2 = n!$ for positive integers a, b and n with $a \le b$ and $n < 14$.

2. The number 1987 can be written as a three-digit number xyz in some base b. If $x + y + z = 1 + 9 + 8 + 7$, determine all possible values of x, y, z and b.

3. Suppose $ABCD$ is a parallelogram and E is a point between B and C on the line BC. If the triangles DEC, BED, and BAD are isosceles, what are the possible values for the angle DAB?

4. On a large flat field, n people are positioned so that for each person, the distances to all the other people are different. Each person holds a water pistol and at a given signal fires and hits the person who is closest. When n is odd, show that there is at least one person left dry. Is this always true when n is even?

5. For every positive integer n, show that

$$[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+1}] = [\sqrt{4n+2}] = [\sqrt{4n+3}]$$

where $[x]$ is the greatest integer less than or equal to x (for example, $[2.3] = 2$, $[\pi] = 3$, and $[5] = 5$).

*　　　　*　　　　*

The next set of problems we pose are from the *Sixteenth U.S.A. Mathematical Olympiad (1987)*. These problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems, may be obtained for a nominal fee from

Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A., 68588-0322. As always, we welcome your original "nice" solutions for use in future columns.

1. Determine all solutions in non-zero integers a and b of the equation

$$(a^2 + b)(a + b^2) = (a - b)^2.$$

2. AD, BE and CF are the bisectors of the interior angles of triangle ABC, with D, E and F lying on the perimeter. If angle EDF is 90°, determine all possible values of angle BAC.

3. Construct a set S of polynomials inductively by the rules:

(i) $x \in S$;

(ii) If $f(x) \in S$ then $xf(x) \in S$ and $x + (1 - x)f(x) \in S$. Prove that there are no two distinct polynomials in S whose graphs intersect within the region $\{0 < x < 1\}$.

4. Three circles C_i are given in the plane. C_1 has diameter $AB = 1$. C_2 is concentric and has diameter k where $1 < k < 3$. C_3 has center A and diameter $2k$. We regard k as fixed. Now consider all straight line segments XY which have one endpoint X on C_2, one endpoint Y on C_3 and contain the point B. For what ratio XB/BY will the segment XY have minimum length?

5. Given a sequence $X_1, X_2, \ldots, X_n$ of 0's and 1's, let A be the number of triples (X_i, X_j, X_k) with $i < j < k$ such that (X_i, X_j, X_k) equals $(0,1,0)$ or $(1,0,1)$. For $1 \le i \le n$, let d_i denote the number of j for which either $j < i$ and $X_j = X_i$ or else $j > i$ and $X_j \ne X_i$.

(a) Prove that $A = \binom{n}{3} - \binom{d_1}{2} - \binom{d_2}{2} - \ldots - \binom{d_n}{2}$.

(b) Given an odd number n, what is the maximum possible value of A?

*   *   *

We now return to the problems posed by students at the 1985 U.S.A.M.O. training sessions. The two solutions we present here are to problems that were starred in the November 1985 issue of the Corner. This must be because no solution was available when they were proposed, and as you will appreciate, the problem can then be rather more involved than at first anticipated.

8*. [1985: 271] *Proposed by Randall Rose, Plainview, N.Y.*

The Heronian mean $(a + \sqrt{ab} + b)/3$ of two distinct positive numbers is certainly less than the arithmetic mean and greater than the geometric mean. Determine the largest $r \geq 0$ and the smallest $s \leq 1$ for which

$$\left[\frac{a^r + b^r}{2}\right]^{1/r} \leq \frac{a + \sqrt{ab} + b}{3} \leq \left[\frac{a^s + b^s}{2}\right]^{1/s}$$

are valid inequalities.

Solutions independently by Dr. Horst Alzer, Waldbröl, Federal Republic of Germany; and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let $M_t(a,b) = \left[\frac{a^t + b^t}{2}\right]^{1/t}$ and $F(a,b) = \frac{a + \sqrt{ab} + b}{3}$. We show that

$$M_{\frac{\ln 2}{\ln 3}}(a,b) \leq F(a,b) \leq M_{2/3}(a,b) \qquad (1)$$

holds for all positive numbers a and b, but that (1) is not true for all a and b if $\frac{\ln 2}{\ln 3}$ is replaced by a larger number, or if 2/3 is replaced by a smaller number. Equality holds in (1) just in case $a = b$.

If $a = b$ then equality holds in (1). Without loss of generality we may assume that $b > a$. We shall prove

$$M_{\frac{\ln 2}{\ln 3}}(t,1) < F(t,1) < M_{2/3}(t,1) \qquad \text{for} \quad 0 < t < 1. \qquad (2)$$

If we replace t by a/b in (2) and multiply by b then we obtain

$$M_{\frac{\ln 2}{\ln 3}}(a,b) < F(a,b) < M_{2/3}(a,b) \qquad \text{for} \quad 0 < a < b.$$

For $t > 0$ define

$$p(t) = \tfrac{3}{2}\ln(t^{2/3} + 1) - \ln(t + 1 + t^{1/2}) + \ln 3 - \tfrac{3}{2}\ln 2.$$

Then

$$2t^{1/6}(1 + t + t^{1/2})(t + t^{1/3})p'(t) = (t^{1/6} - 1)^3(t^{1/6} + 1)$$

and we find $p'(t) < 0$ for $0 < t < 1$ and

$$p(t) > p(1) \qquad \text{for} \quad 0 < t < 1$$

which implies

$$F(t,1) < M_{2/3}(t,1) \qquad \text{for} \quad 0 < t < 1.$$

Next we show that the value 2/3 in (2) cannot be replaced by a smaller number. Let $s < 2/3$. For $t > 0$, define

$$q_s(t) = M_s(t,1) - F(t,1)$$

(where, using L'Hôpital's Rule we can set $M_0(t,1) = t^{1/2}$). A calculation gives

$$q_s'(t) \;=\; s\cdot\frac{t^{s-1}}{2}\cdot\frac{1}{s}\left[\frac{t^s+1}{2}\right]^{\frac{1}{s}-1} - \frac{1}{3} - \frac{1}{6}t^{-1/2}$$

so

$$q_s'(1) = 0.$$

Further calculation gives

$$q_s''(t) \;=\; \frac{s-1}{2}\cdot t^{s-2}\left[\frac{t^s+1}{2}\right]^{\frac{1}{s}-1} + s\cdot\frac{t^{s-1}}{2}\cdot\frac{t^{s-1}}{2}\left(\frac{1}{s}-1\right)\left[\frac{t^s+1}{2}\right]^{\frac{1}{s}-2} + \frac{1}{12}t^{-3/2}.$$

So

$$q_s''(1) \;=\; \frac{s-1}{2} + \frac{1}{2}(1-s)\cdot\frac{1}{2} + \frac{1}{12} = \frac{1}{4}\left(s - \frac{2}{3}\right) \;<\; 0.$$

Therefore there is a positive number δ such that $q_s(t) < 0$ for $1 < t < 1 + \delta$. We conclude that

$$q_s(t) < q_s(1) = 0$$

for $1 < t < 1 + \delta$. But $q_s(t) < 0$ implies $M_s(t,1) < F(t,1)$ so that (1) fails for $(t,1)$.

We next turn to establishing the left-hand inequality of (2).

Let $r = \ln 2/\ln 3$. For $t > 0$ we define

$$f(t) \;=\; \ln(1 + t + t^{1/2}) - \frac{1}{r}\ln(t^r + 1).$$

We have to show that $f(t) > 0$ for t in $(0,1)$.

For a contradiction, we assume that there is $t_0 \in (0,1)$ such that $f(t_0) = 0$. Noting that $f(0) = 0 = f(1)$, we see that this means that f has (at least) 3 zeros on $[0,1]$, and thus $f'(t)$ has (at least) 2 zeros on $(0,1)$. Now,

$$2t^{1/2}(1 + t + t^{1/2})(t^r + 1)f'(t) \;=\; 2t^{1/2} + 1 - t^r - 2t^{r-1/2}.$$

Since $f'(1) = 0$ we conclude that f' and

$$g(t) = 2t^{1/2} + 1 - t^r - 2t^{r-1/2}$$

each have at least *three* zeros on $(0,1]$.

Then we find that

$$h(t) \;=\; t^{-r+3/2}g'(t) \;=\; -rt^{1/2} + t^{1-r} - 2r + 1$$

vanishes twice on $(0,1)$. But this is impossible. First notice that

$$\frac{4}{r}t^{r+1}h''(t) \;=\; t^{r-1/2} - 4(1-r) \;<\; 1 - 4(1-r) \;<\; 0,$$

as $1 - 4(1-r) < 0$ iff $4\frac{\ln 2}{\ln 3} < 3$.

From this we conclude that h is strictly concave down on $[0,1]$ and thus can have at most one zero on $(0,1)$ since

$$h(0) = -2\frac{\ln 2}{\ln 3} + 1 < 0$$

and

$$h(1) = -3\frac{\ln 2}{\ln 3} + 2 > 0.$$

This gives the desired contradiction and establishes that f does not vanish on $(0,1)$. Now f is continuous on $(0,1)$. Also $f(\frac{1}{2}) > 0$ because

$$f(\tfrac{1}{2}) > 0 \iff 1 + \frac{1}{2} + \frac{1}{\sqrt{2}} > \left[\frac{1}{2^r} + 1\right]^r .$$

But

$$\left[\frac{1}{2^r} + 1\right]^r < \frac{1}{2^r} + 1 < \frac{1}{2^{1/2}} + 1 < 1 + \frac{1}{2} + \frac{1}{\sqrt{2}}$$

since

$$\frac{1}{2} < r < 1.$$

We conclude that $f(t) > 0$ for $t \in (0,1)$ which gives the left-hand inequality for (2).

If we replace b by $a + 1$ in

$$M_k(a,b) \le F(a,b) \qquad k > 0$$

and if we let $a \longrightarrow 0$, then we obtain $k \le \frac{\ln 2}{\ln 3}$ as follows:

$$\lim_{a\to 0^+} F(a,a+1) = \lim_{a\to 0^+} \frac{a + \sqrt{a(a+1)} + a + 1}{3} = \frac{1}{3}$$

$$\lim_{a\to 0^+} M_k(a,a+1) = \lim_{a\to 0^+} \left[\frac{a^k + (a+1)^k}{2}\right]^{1/k} = \frac{1}{2^{1/k}} .$$

From $\frac{1}{2^{1/k}} \le \frac{1}{3}$ we read off $k \le \frac{\ln 2}{\ln 3}$. This means that the left hand inequality of (1) cannot be sharpened.

The solution presented above is essentially that submitted by Dr. Horst Alzer. The solution by Walther Janous is similar but works instead with $x = \sqrt{b/a} \ge 1$ *and limits as* $x \longrightarrow +\infty$. *He adds the following application and remarks.*

Application by Walther Janous, Innsbruck, Austria.

An inequality for triangles. Let $p = \frac{\ln 2}{\ln 3}$ and $q = 2/3$. Then in any acute-angled triangle one has

$$\frac{a^2+b^2-c^2}{2}\left\{3\left[\frac{a^p+b^p}{2}\right]^{1/p}+c-2s\right\} \geq \cos C$$

$$\geq \frac{a^2+b^2-c^2}{2}\left\{3\left[\frac{a^q+b^q}{2}\right]^{1/q}+c-2s\right\}^{-2},$$

where s is the semiperimeter. For C an obtuse angle, the inequalities are reversed. In this p cannot be increased and q cannot be decreased.

Remarks by Walther Janous, Innsbruck, Austria.

1. Let $M_r = M_r(a,b) = \begin{cases} \left[\dfrac{a^r+b^r}{2}\right]^{1/r}, & r > 0 \\ \sqrt{ab}, & r = 0 \end{cases}$.

The problem solved above is a special case of the more general problem (with parameters λ, μ):

Find r_{max} and s_{min} such that

$$M_r \leq \lambda M_0 + \mu M_1 \leq M_s \quad \text{where} \quad 0 \leq \lambda,\ \mu \text{ and } \lambda + \mu = 1.$$

2. This may in turn be generalized:

Find r_{max} and s_{min} such that $M_r \leq M_q(M_{i_1},\ldots,M_{i_k},p_1,\ldots,p_k) \leq M_s$

where $0 \leq i_1 < \ldots < i_k$, $p_1,\ldots,p_k$, q are given with $\sum_{i=1}^{k} p_i = 1$, $p_i > 0$ and

$$M_q(X_1,\ldots,X_k,p_1,\ldots,p_k) = \left[\sum_{i=1}^{k} p_i\, X_i^q\right]^{1/q}.$$

3. The original problem and the generalizations in 1 and 2 may be of interest if we allow

$$M_r(a_1,\ldots,a_n) = \begin{cases} \left[\dfrac{\Sigma a_i^r}{n}\right]^{1/r}, & r \neq 0 \\ (\Pi a_i)^{1/n}, & r = 0 \end{cases}.$$

What is known about the problems cited in Remarks 1, 2 and 3?

4. Interested readers should note in this context the article of Horst Alzer, Ungleichungen für $\left[\frac{e}{a}\right]^a\left[\frac{b}{e}\right]^b$ in *Elem. Math.* 40 (1985) pp.120-123.

There he proves that

$$M_r^{b-a} < \left[\frac{e}{a}\right]^a\left[\frac{b}{e}\right]^b < M_s^{b-a}$$

where $b > a$ implies $r_{max} = 2/3$ and $s_{min} = \ell n\ 2$.

9.* [1985: 272] *Proposed by Joseph Keane, Pittsburgh, Pennsylvania.*
For $0 < x < 1$, it is conjectured that
$$(1 + x)^{1+x} + (1 + x)^{1-x} - (1 - x)^{1+x} - (1 - x)^{1-x} > 4x.$$
Prove or disprove.

Solution by Harvey Abbott, The University of Alberta, Edmonton, Alberta.

By the Binomial Theorem,

$$\text{Left side} = \sum_{n=0}^{\infty} \left\{ \begin{bmatrix} 1+x \\ n \end{bmatrix} + \begin{bmatrix} 1-x \\ n \end{bmatrix} - (-1)^n \begin{bmatrix} 1+x \\ n \end{bmatrix} - (-1)^n \begin{bmatrix} 1-x \\ n \end{bmatrix} \right\} x^n$$

$$= 2 \sum_{n=0}^{\infty} \left\{ \begin{bmatrix} 1+x \\ 2n+1 \end{bmatrix} + \begin{bmatrix} 1-x \\ 2n+1 \end{bmatrix} \right\} x^{2n+1}$$

$$= 2 \sum_{n=0}^{\infty} A_n(x) x^{2n+1} \qquad \text{where } A_n(x) = \begin{bmatrix} 1+x \\ 2n+1 \end{bmatrix} + \begin{bmatrix} 1-x \\ 2n+1 \end{bmatrix}.$$

In the above $\begin{bmatrix} y \\ k \end{bmatrix}$ where y is any real number and k is a positive integer is defined to be $\dfrac{y(y-1)\dots(y-k+1)}{k!}$. Also, $\begin{bmatrix} y \\ 0 \end{bmatrix} = 1$.

One finds that

$$A_0(x) = \frac{1+x}{1} + \frac{1-x}{1} = 2$$

and

$$A_1(x) = \frac{(1+x)x(x-1)}{3!} + \frac{(1-x)(-x)(-x-1)}{3!} = 0.$$

Also, for $n > 1$,

$$A_n(x) = \frac{(1-x^2)x}{(2n+1)!} \{(2+x)(3+x)\dots(2n-1+x) - (2-x)\dots(2n-1-x)\}.$$

Since $A_n(x) > 0$ for $n > 1$ and $0 < x < 1$, the conjecture holds.

* * *

The reader may complain that both problems discussed this month involve solutions that use extra machinery, whether the calculus as in 8 or the generalized Binomial Theorem as in 9. It would be interesting to know what can be said without recourse to these extra tools. One problem, 10, from the student proposals, remains unsolved in this column. We also await solutions to the five problems from the 1985 Bulgarian Winter Olympiad posed in the November 1985 Olympiad Corner. Send in your Olympiad problems and solutions!

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before January 1, 1988, although solutions received after that date will also be considered until the time when a solution is published.

1251. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.* (Dedicated to Léo Sauvé.)

(a) Find all integral n for which there exists a regular n-simplex with integer edge and integer volume.

(b)* Which such n-simplex has the smallest volume?

1252. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle and M an interior point with barycentric coordinates λ_1, λ_2, λ_3. We denote the pedal triangle and the Cevian triangle of M by DEF and $A'B'C'$ respectively. Prove that

$$\frac{[DEF]}{[A'B'C']} \ge 4\lambda_1\lambda_2\lambda_3(s/R)^2,$$

where s is the semiperimeter and R the circumradius of ΔABC, and $[X]$ denotes the area of figure X.

1253. *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

Player A starts with \$3 and player B starts with \$10. On each turn a fair coin is tossed, with the outcome that either B pays A \$3 or A pays B \$2. Play continues until one player wins by having won all the other player's money. Which player is more likely to win?

1254. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let ABC be a triangle and $n \ge 1$ a natural number. Show that

$$\left|\sum \sin n(B-C)\right| \begin{cases} < 1 & \text{if } n = 1, \\ < 3\sqrt{3}/2 & \text{if } n = 2, \\ \le 3\sqrt{3}/2 & \text{if } n \ge 3, \end{cases}$$

where the sum is cyclic.

1255. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

(a) Find all positive integers n such that $2^{13} + 2^{10} + 2^n$ is the square of an integer.

(b)* Find all positive integers n such that $2^{14} + 2^{10} + 2^n$ is the square of an integer.

1256. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Let ABC be a triangle with sides satisfying $a^3 = b^3 + c^3$. Determine the range of angle A.

1257. *Proposed by Jordan Stoyanov, Bulgarian Academy of Sciences, Sofia, Bulgaria.*

Find all rational x such that $3x^2 - 5x + 4$ is the square of a rational number.

1258. *Proposed by Ian Witten, University of Calgary, Calgary, Alberta.*

Think of a picture as an $m \times n$ matrix A of real numbers between 0 and 1 inclusive, where a_{ij} represents the brightness of the picture at the point (i,j). To reproduce the picture on a computer we wish to approximate it by an $m \times n$ matrix B of 0's and 1's, such that every "part" of the original picture is "close" to the corresponding part of the reproduction. These are the ideas behind the following definitions:

A *subrectangle* of an $m \times n$ grid is a set of positions of the form

$$\{(i,j) \mid r_1 \le i \le r_2\ ,\ s_1 \le j \le s_2\}$$

where $1 \le r_1 \le r_2 \le m$ and $1 \le s_1 \le s_2 \le n$ are constants. For any subrectangle R, let

$$d(R) = \left| \sum_{(i,j)\in R} (a_{ij} - b_{ij}) \right| ,$$

where A and B are as given above, and define

$$d(A,B) = \max d(R),$$

the maximum taken over all subrectangles R.

(a) Show that there exist matrices A such that $d(A,B) > 1$ for every 0 - 1 matrix B of the same size.

(b)* Is there a constant c such that for every matrix A of any size, there is some 0 - 1 matrix B of the same size such that $d(A,B) < c$?

1259. *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

If $x,y,z \geq 0$, disprove the inequality

$$(yz + zx + xy)^2(x + y + z) \geq 9xyz(x^2 + y^2 + z^2).$$

Determine the largest constant one can replace the 9 with to obtain a valid inequality.

1260. *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokaisi, Aichi, Japan.*

Let ABC be a triangle with angles B and C acute, and let H be the foot of the perpendicular from A to BC. Let O_1 be the circle internally tangent to the circumcircle O of ΔABC and touching the segments AH and BH. Let O_3 be the circle similarly tangent to O, AH and CH. Finally let O_2 be the incircle of ΔABC, and denote the radii of O_1, O_2, O_3 by r_1, r_2, r_3, respectively.

(a) Show that $r_2 = \dfrac{r_1 + r_3}{2}$.

(b) Show that the centers of O_1, O_2, O_3 are collinear.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1111. [1986: 26] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let α, β, γ be the angles of an acute triangle and let

$$f(\alpha,\beta,\gamma) = \cos\alpha/2 \cos\beta/2 + \cos\beta/2 \cos\gamma/2 + \cos\gamma/2 \cos\alpha/2.$$

(a) Prove that $f(\alpha,\beta,\gamma) > \frac{3}{2}\sqrt[3]{2}$.

(b)* Prove or disprove that $f(\alpha,\beta,\gamma) > 1/2 + \sqrt{2}$.

I. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For (a), the arithmetic-geometric mean-inequality, and item 2.28 of the Bottema bible *Geometric Inequalities*, immediately yield

$$f(\alpha,\beta,\gamma) \geq 3(\cos\alpha/2 \cos\beta/2 \cos\gamma/2)^{2/3} > 3(1/2)^{2/3} = 3\sqrt[3]{2}/2.$$

In fact, the stronger inequality (b) holds. We assume without loss of generality that $\pi/2 > \alpha \geq \beta \geq \gamma$, and thus $\pi/4 \leq \beta < \pi/2$. Then

$$\cos\alpha/2 \leq \cos\beta/2 \leq \cos\gamma/2.$$

Since

$$(\cos\gamma/2 - \cos\beta/2)(\cos\beta/2 - \cos\alpha/2) \geq 0$$

we obtain

$$\cos \alpha/2 \cos \beta/2 + \cos \beta/2 \cos \gamma/2 \ \geq \ \cos \alpha/2 \cos \gamma/2 + \cos^2\beta/2$$

and so

$$\begin{aligned} f(\alpha,\beta,\gamma) &\geq 2 \cos \alpha/2 \cos \gamma/2 + \cos^2\beta/2 \\ &= \sin \beta/2 + \cos \frac{\alpha - \gamma}{2} + \cos^2\beta/2 \\ &= 1 + \sin \beta/2 - \sin^2\beta/2 + \sin(\beta/2 + \gamma). \end{aligned}$$

The conjectured inequality $f(\alpha,\beta,\gamma) > 1/2 + \sqrt{2}$ thus follows if we show

$$\sin \beta/2 - \sin^2\beta/2 + \sin(\beta/2 + \gamma) > \sqrt{2} - 1/2. \tag{1}$$

Let β, $\pi/4 \leq \beta < \pi/2$, be fixed and put

$$g(\gamma) = \sin \beta/2 - \sin^2\beta/2 + \sin(\beta/2 + \gamma).$$

Then, from $\gamma \leq \alpha$, $\beta/2 + \gamma \leq \pi/2$ and thus g increases. Now

$$\gamma = \pi - \alpha - \beta > \pi/2 - \beta,$$

and thus

$$g(\gamma) > g(\pi/2 - \beta) = \sin \beta/2 + \cos \beta/2 - \sin^2\beta/2.$$

Hence (1) follows if we show

$$\sin \beta/2 + \cos \beta/2 - \sin^2\beta/2 > \sqrt{2} - 1/2$$

for $\pi/4 \leq \beta < \pi/2$. Putting $\sin \beta/2 = t$, this is equivalent to showing

$$h(t) = t + \sqrt{1 - t^2} - t^2 > \sqrt{2} - 1/2$$

where

$$\frac{\sqrt{2 - \sqrt{2}}}{2} \leq t < \frac{\sqrt{2}}{2} .$$

But

$$h'(t) = 1 - 2t - \frac{t}{(1 - t^2)^{1/2}}$$

and

$$h''(t) = -2 - (1 - t^2)^{-3/2} < 0,$$

and as

$$h'\left[\frac{\sqrt{2 - \sqrt{2}}}{2}\right] < 0,$$

we get that

$$h(t) > h(\sqrt{2}/2) = \sqrt{2} - 1/2,$$

as was to be shown.

II. *Solution by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

It suffices to prove (b). We use Schur-convexity and the Majorization Inequality (see page 57 of A.W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic Press, New York, 1979). We first give the necessary definitions and results.

$F(x,y,z)$ is Schur-convex on a convex region D if

(i) $F(x,y,z)$ is symmetric in x, y, z in D, continuous on D, and continuously differentiable in the interior of D, and

(ii) $(x - y)\left[\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right] \geq 0$ for all $x,y \in D$.

It follows that if

$$(x,y,z) \succ (x',y',z')$$

(i.e. $x \geq x'$, $x + y \geq x' + y'$, $x + y + z = x' + y' + z'$ where $x \geq y \geq z$ and $x' \geq y' \geq z'$), then

$$F(x,y,z) \geq F(x',y',z').$$

Now, since

$$-\frac{\partial f}{\partial \alpha} = 1/2 \sin \alpha/2(\cos \beta/2 + \cos \gamma/2), \text{ etc.},$$

we get that

$$(\alpha-\beta)\left[-\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right] = 1/2(\alpha-\beta)\left[\sin\left[\frac{\alpha - \beta}{2}\right] + \cos \gamma/2(\sin \alpha/2 - \sin \beta/2)\right], \text{ etc.};$$

it follows easily that $-f$ is Schur-convex in the region

$$D: \quad \{(\alpha,\beta,\gamma) \mid 0 \leq \alpha,\beta,\gamma \leq \pi/2 , \alpha + \beta + \gamma = \pi\}.$$

Also,

$$(\pi/2,\pi/2,0) \succ (\alpha,\beta,\gamma) \succ (\pi/3,\pi/3,\pi/3).$$

Hence,

$$9/4 = f(\pi/3,\pi/3,\pi/3) \geq f(\alpha,\beta,\gamma) \geq f(\pi/2,\pi/2,0) = 1/2 + \sqrt{2}.$$

There is equality on the right hand side only for the degenerate triangle of angles $\pi/2$, $\pi/2$, 0, and equality on the left hand side for the equilateral triangle.

More generally, it follows in the same way that

$$3 \cos^2 k\pi/6 \geq f(k\alpha,k\beta,k\gamma) \geq \cos^2 k\pi/4 + 2 \cos k\pi/4$$

where k is any constant in $[0,1]$. Also, if the triangle were not restricted to being acute, the right hand side inequality would be replaced by

$$f(k\alpha,k\beta,k\gamma) \geq 1 + 2 \cos k\pi/2$$

since now $(\pi,0,0) \succ (\alpha,\beta,\gamma)$.

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; VEDULA N. MURTY ((a) and part of (b)), Pennsylvania State University, Middletown, Pennsylvania; PETER WATSON-HURTHIG, Columbia College, Burnaby, B.C.; and (part (a)) the proposer.

* * *

<u>1112</u>. [1986: 26] *Proposed by Allan W. Johnson Jr., Washington, D.C.*

Solve the synonymical base 10 addition

$$\begin{array}{r} \text{LITHE} \\ \underline{\text{PLIANT}} \\ \text{SUPPLE} \end{array}.$$

Solution.

$$\begin{array}{r} 76094 \\ \underline{276180} \\ 352274 \end{array} \quad \text{or} \quad \begin{array}{r} 76084 \\ \underline{276190} \\ 352274 \end{array}.$$

Found by RICHARD I. HESS, Rancho Palos Verdes, California; GLEN E. MILLS, Valencia Jr. College, Orlando, Florida; J. SUCK, Essen, Federal Republic of Germany; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, Federal Republic of Germany; and the proposer.

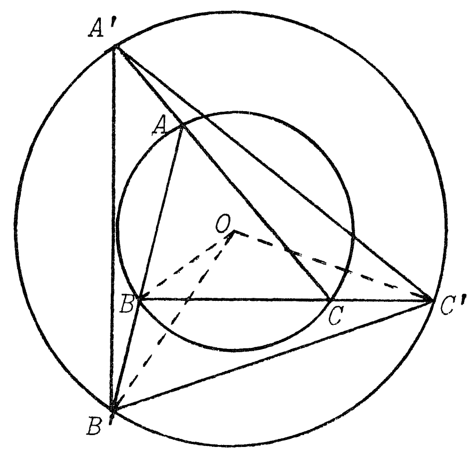
* * *

1113*. [1986: 26] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Consider two concentric circles with radii r and $2r$, and a triangle ABC inscribed in the inner circle. Points A', B', C' on the outer circle are determined by extending AB to B', BC to C', and CA to A'. Prove that the perimeter of triangle $A'B'C'$ is at least twice the perimeter of ABC. Equality is attained when ABC is equilateral.

Solution by C. Festraets-Hamoir, Brussels, Belgium.

Let O be the centre of the circles, and denote the sides of triangles ABC and $A'B'C'$ by a, b, c and a', b', c', respectively.



In quadrilateral $OBB'C'$, we have by Ptolemy's theorem that

$$B'C'\cdot OB + BB'\cdot OC' \geq OB'\cdot BC'$$

or

$$a'r + BB'\cdot 2r \geq 2r(a + CC')$$

from which

$$a' \geq 2a + 2CC' - 2BB'.$$

Similarly

$$b' \geq 2b + 2AA' - 2CC',$$

$$c' \geq 2c + 2BB' - 2AA'.$$

Adding, we have

$$a' + b' + c' \geq 2a + 2b + 2c \tag{1}$$

as required.

There is equality in (1) if and only if each of the quadrilaterals $OBB'C'$, $OCC'A'$, $OAA'B'$ is cyclic. $OBB'C'$ is cyclic if and only if

$$\angle OBC' = \angle OB'C' = 1/2(180^\circ - \angle B'OC')$$
$$= 1/2(180^\circ - \angle B'BC') = 1/2 \angle CBA,$$

that is, OB is the interior bisector of angle B of ΔABC. Using analogous results for OA and OC, we see that equality holds in (1) if and only if O is the incentre of ΔABC, that is, if and only if ΔABC is equilateral.

There was one partial solution submitted.

* * *

<u>1114</u>. [1986: 26] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC, $A'B'C'$ be two triangles with sides a, b, c, a', b', c' and areas F, F' respectively. Show that

$$aa' + bb' + cc' \geq 4\sqrt{3}\sqrt{FF'}.$$

I. *Solution and generalization by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We will extend the proposed inequality to the following n-triangle-inequality:

Let a_i, b_i, c_i and F_i be the sides and area of triangle Δ_i $(i = 1,2,\ldots,n)$. Then

$$a_1\ldots a_n + b_1\ldots b_n + c_1\ldots c_n \geq 3\left[\frac{4}{\sqrt{3}}\right]^{n/2}\sqrt{F_1\ldots F_n}. \tag{1}$$

Indeed, by [1], item 4.13, it follows

$$4\sqrt{3}F_i \leq \frac{9a_ib_ic_i}{a_i + b_i + c_i}$$

for $i = 1,\ldots,n$. Multiplication of these n inequalities leads to

$$(4\sqrt{3})^n \prod_{i=1}^{n} F_i \leq 9^n \prod_{i=1}^{n} \frac{a_ib_ic_i}{a_i + b_i + c_i}.$$

In order to show the validity of (1) we thus have to prove

$$9^n \prod_{i=1}^{n} \frac{a_ib_ic_i}{a_i + b_i + c_i} \leq \frac{3^n(a_1\ldots a_n + b_1\ldots b_n + c_1\ldots c_n)^2}{9},$$

that is,

$$\prod_{i=1}^{n} a_ib_ic_i \leq \left[\frac{a_1\ldots a_n + b_1\ldots b_n + c_1\ldots c_n}{3}\right]^2 \prod_{i=1}^{n} \frac{a_i + b_i + c_i}{3}. \tag{2}$$

But by the arithmetic-geometric mean-inequality,

$$\frac{a_1\ldots a_n + b_1\ldots b_n + c_1\ldots c_n}{3} \geq \sqrt[3]{a_1\ldots a_nb_1\ldots b_nc_1\ldots c_n}$$

and

$$\frac{a_i + b_i + c_i}{3} \geq \sqrt[3]{a_i b_i c_i} ,$$

and (2) follows.

We will now employ the following result of Oppenheim (cf. [2]):

If a, b, c, F are the sides and area of a triangle, then, for $0 < t \leq 1$, a^t, b^t, c^t are the sides of a triangle of area F_i, where

$$F_i \geq \left[\frac{\sqrt{3}}{4}\right]^{1-t} F^t. \tag{3}$$

Inequalities (1) and (3) imply the following:

For each $i = 1,\ldots,n$ let $0 < t_i \leq 1$ be a real number and a_i, b_i, c_i and F_i be the sides and area of a triangle Δ_i. Then

$$a_1^{t_1}\ldots a_n^{t_n} + b_1^{t_1}\ldots b_n^{t_n} + c_1^{t_1}\ldots c_n^{t_n} \geq 3\left[\frac{4}{\sqrt{3}}\right]^{\left[\sum_{i=1}^{n} t_i/2\right]} \prod_{i=1}^{n} F_i^{t_i/2}. \tag{4}$$

In fact we show that (4) holds for *all* positive real numbers $t_1,\ldots,t_n$. We may and do assume that $t_n = \max\{t_1,\ldots,t_n\}$. Then, by (4) and the general mean inequality we infer

$$a_1^{t_1}\ldots a_n^{t_n} + b_1^{t_1}\ldots b_n^{t_n} + c_1^{t_1}\ldots c_n^{t_n} \geq 3\left[\frac{\Sigma\, a_1^{t_1/t_n}\ldots a_{n-1}^{t_{n-1}/t_n} a_n}{3}\right]^{t_n}$$

$$\geq 3\left[\frac{4}{\sqrt{3}}\right]^{\left[\sum_{i=1}^{n} t_i/2\right]} \prod_{i=1}^{n} F_i^{t_i/2}.$$

We will now use the well-known median-duality, which says: If $I(a,b,c,F) \geq 0$ is a valid triangle-inequality, then so is

$$I(m_a,m_b,m_c,3F/4) \geq 0$$

where m_a, m_b, m_c are the medians. (This was first noted by M.S. Klamkin and follows simply from the fact that m_a, m_b, m_c form a triangle of area $3F/4$.)

Applying the median-duality to (4) we obtain that if $0 \leq k \leq n$ then

$$\Sigma\, a_1^{t_1}\ldots a_k^{t_k}\cdot m_{a_{k+1}}^{t_{k+1}}\ldots m_{a_n}^{t_n} \geq 3\left[\frac{4}{\sqrt{3}}\right]^{\left[\sum_{i=1}^{k} t_i/2\right]} (\sqrt{3})^{\left[\sum_{i=k+1}^{n} t_i/2\right]} \prod_{i=1}^{n} F_i^{t_i/2}. \tag{5}$$

Of course, using other transformations (e.g. the ones investigated in [3] or [4]) we could get very many consequences of (4), sometimes asymmetric and complicated ones.

Putting in (4) the values $n = 2$, $t_1 = s$, $t_2 = t$ $(s,t > 0)$, and

$$a_1 = a_2 = a, \quad b_1 = b_2 = b, \quad c_1 = c_2 = c,$$

or

$$a_1 = c_2 = a, \quad b_1 = a_2 = b, \quad c_1 = b_2 = c,$$

or

$$a_1 = c_2 = a, \quad b_1 = b_2 = b, \quad c_1 = a_2 = c,$$

we get the respective inequalities

$$a^{s+t} + b^{s+t} + c^{s+t} \geq 3\left[\frac{4F}{\sqrt{3}}\right]^{(s+t)/2}, \tag{6}$$

$$a^s b^t + b^s c^t + c^s a^t \geq 3\left[\frac{4F}{\sqrt{3}}\right]^{(s+t)/2}, \tag{7}$$

and the asymmetric

$$a^s c^t + a^t c^s + b^{s+t} \geq 3\left[\frac{4F}{\sqrt{3}}\right]^{(s+t)/2}. \tag{8}$$

(6), (7) and (8) generalize many items from Chapter 4 of [1]. Similarly we get from (5), for $n = 2$,

$$a^s m_a^t + b^s m_b^t + c^s m_c^t \geq 3\cdot 2^s(\sqrt{3})^{(t-s)/2}F^{(s+t)/2}, \text{ etc.} \tag{9}$$

Finally, we will use (4), in case $n = 2$, to give a restricted generalization of the Neuberg-Pedoe inequality (cf. [1], item 10.8).

Let s, t be positive real numbers, and $a \geq b \geq c$ and $a' \leq b' \leq c'$ be the sides of two triangles of areas F and F', respectively. Then

$$\Sigma\, a'^t(-a^s + b^s + c^s) \geq 3\left[\frac{4}{\sqrt{3}}\right]^{(s+t)/2} F^{s/2}F'^{t/2}. \tag{10}$$

By (4), in order to show (10) we only have to prove

$$\Sigma\, a'^t(-a^s + b^s + c^s) \geq \Sigma\, a^s a'^t. \tag{11}$$

But (11) reads equivalently

$$\Sigma\, a^s \cdot \Sigma\, a'^t \geq 3\, \Sigma\, a^s a'^t,$$

which follows from Tchebyshev's inequality.

As $a \geq b \geq c$ implies $m_a \leq m_b \leq m_c$ we can put $a' = m_a$ etc. in (10). This leads to

$$\Sigma(-a^s + b^s + c^s)m_a^t \geq 3\cdot 2^s(\sqrt{3})^{(t-s)/2}F^{(s+t)/2}$$

for $s,t > 0$.

References:

[1] O. Bottema et al, *Geometric Inequalities*, Groningen, 1968.

[2] A. Oppenheim, Inequalities involving elements of triangles, quadrilaterals or tetrahedra, *Univ. Belgrade Publ. El. Fak. Ser. Mat. Fiz.*, No.461-497 (1974), 257-263.

[3] Sv. Bilchev, E. Velikova, Some asymmetric triangle-inequalities (Bulgarian), *Matematika* (Sofija) 25 (4) (1986), 13-17.

[4] Sv. Bilchev, E. Velikova, On one generalization of the parallelogram transformation (Bulgarian), *Mat. i mat. obras.* (Sofija) (1986), 525-531.

II. *Applications by the proposer.*

The triangle with sides m_a, m_b, m_c has area $3F/4$. Therefore we can easily see that

$$am_b + bm_c + cm_a \geq 6F.$$

In the same way we have

$$am_a + bm_b + cm_c \geq 6F$$

and

$$am_c + bm_a + cm_b \geq 6F.$$

Adding, we have

$$2s(m_a + m_b + m_c) \geq 18F$$

or

$$m_a + m_b + m_c \geq \frac{9F}{s},$$

where s is the semiperimeter.

Now we put $a' = \sqrt{a}$, $b' = \sqrt{b}$, $c' = \sqrt{c}$. For the triangle $A'B'C'$ with sides a', b', c' it is known that

$$4F'^2 \geq \sqrt{3}F$$

(see 10.3 of [1]). Therefore our basic inequality gives

$$a^{3/2} + b^{3/2} + c^{3/2} \geq 2^{3/2}3^{5/8}F^{3/4}.$$

For another application, we take

$$a' = \sqrt{b^2 + c^2}\ ,\ b' = \sqrt{a^2 + c^2}\ ,\ c' = \sqrt{a^2 + b^2}\ .$$

From 10.12 of [1] we have $F' \geq 2F$, therefore

$$a\sqrt{b^2 + c^2} + b\sqrt{a^2 + c^2} + c\sqrt{a^2 + b^2} \geq 4F\sqrt{6}.$$

Similarly

$$a\sqrt{a^2 + c^2} + b\sqrt{b^2 + a^2} + c\sqrt{c^2 + b^2} \geq 4F\sqrt{6},$$
$$a\sqrt{a^2 + b^2} + b\sqrt{b^2 + c^2} + c\sqrt{c^2 + a^2} \geq 4F\sqrt{6}.$$

Adding, we obtain

$$\sqrt{a^2 + b^2} + \sqrt{b^2 + c^2} + \sqrt{c^2 + a^2} \geq 6r\sqrt{6}$$

where r is the inradius.

We stop here with the remark that two-triangle inequalities are very productive!

[1] Bottema et al, *Geometric Inequalities*.

Also solved by GEORGE EVAGELOPOULOS, *Athens, Greece;* C. FESTRAETS-HAMOIR, *Brussels, Belgium;* J.T. GROENMAN, *Arnhem, The Netherlands;* M.S. KLAMKIN, *University of Alberta, Edmonton, Alberta;* BOB PRIELIPP, *University of Wisconsin, Oshkosh, Wisconsin; and the proposer.*

Janous' generalization (4) was also obtained by Klamkin.

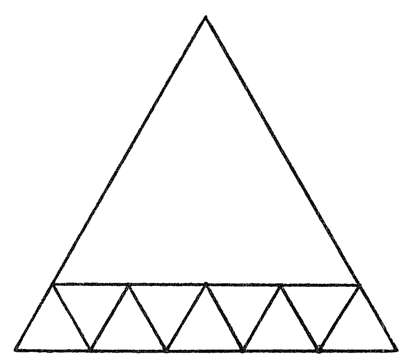
*　　　　　*　　　　　*

<u>1115</u>. [1986: 27] *Proposed by Helen Sturtevant and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Determine all positive integers n such that an equilateral triangle can be dissected into exactly n equilateral triangles.

Solution by Leroy F. Meyers, The Ohio State University, Columbus, Ohio.

Such a dissection is possible for all positive integers except 2, 3, and 5. The dissection is trivial if $n = 1$. By dividing one side of the given equilateral triangle of side 1 into k equal segments, where $k \geq 2$, and erecting a strip of $2k - 1$ equilateral triangles of side $1/k$ on that side, we are left with one equilateral triangle of side $\frac{k-1}{k}$, for a total of $2k$ equilateral triangles in the dissection. If we divide one of these $2k$ triangles into four congruent equilateral triangles, we have a dissection into $2k + 3$ equilateral triangles. Hence any equilateral triangle can be dissected into n equilateral triangles, except possibly when $k \leq 1$, i.e., when n is 2, 3, or 5. Now a dissection into more than one piece requires a triangle for each vertex of the original triangle, so that we cannot have a dissection into two pieces; but three pieces, one at each vertex, would leave a space in the middle (a triangle, trapezoid, pentagon, or hexagon), and this space cannot be dissected into exactly two equilateral triangles. See also *Crux* 256 [1977: 155; 1978: 53, 102].



Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; and the proposers. One other respondent misinterpreted the problem and counted the number of equilateral triangles contained in a tesselation of an equilateral triangle into congruent equilateral triangles.

* * *

<u>1116</u>. [1986: 27, 77 (corrected)] *Proposed by David Grabiner, Claremont High School, Claremont, California.*

(a) Let $f(n)$ be the smallest positive integer which is not a factor of n. Continue the series $f(n), f(f(n)), f(f(f(n))), \ldots$ until you reach 2. What is the maximum length of the series?

(b) Let $g(n)$ be the second smallest positive integer which is not a factor of n. Continue the series $g(n), g(g(n)), g(g(g(n))), \ldots$ until you reach 3. What is the maximum length of the series?

Solution by Leroy F. Meyers, The Ohio State University, Columbus, Ohio.

More generally, let $f_r(n)$ be the rth smallest positive integer which is not a divisor of n, and let f, g, and h abbreviate f_1, f_2, and f_3, respectively. We show that $f_r(n)$ is always of the form jp^k for some prime p and positive integers j and k, where $j \le r$. In particular, $f(n)$ is always a prime power.

Suppose that $f(n) = q_1^{a_1} \ldots q_s^{a_s}$, where $q_1, \ldots, q_s$ are distinct primes and $a_1, \ldots, a_s$ are positive integers. Obviously, $f(n) > 1$, and, more generally, $f_r(n) > r$. If $s > 1$, then $q_i^{a_i} < f(n)$ for $1 \le i \le s$, and so $q_i^{a_i} | n$ by the definition of $f(n)$. Hence $f(n)$ divides n, contrary to the definition of $f(n)$. Therefore $s = 1$, and so $f(n)$ is a prime power for all n.

Since $f(n) \nmid n$, no multiple of $f(n)$ divides n. Hence the rth smallest nondivisor of n is at most r times the smallest prime power not dividing n, and must contain as a factor a prime power not dividing n, and so is j times some prime power not dividing n, where $j \le r$. In particular, $g(n)$ is either a prime power or twice a prime power for each n.

The following tables show the results of successive applications of f, g, and h. Superscripts on function names denote repeated composition, so that, for example, $f^3(n) = f(f(f(n)))$. Exponents are ≥ 1 unless otherwise indicated, and p is a prime.

$f(n)$	$f^2(n)$	$f^3(n)$
2^α	3	2
p^α , $p \geq 3$	2	

$g(n)$	$g^2(n)$	$g^3(n)$	$g^4(n)$
2^α , $\alpha \geq 2$	5	3	
3^α	4	5	3
p^α , $p \geq 5$	3		
$2 \cdot 3^\alpha$	5	3	
$2 \cdot p^\alpha$, $p \geq 5$	4	5	3

$h(n)$	$h^2(n)$	$h^3(n)$	$h^4(n)$	$h^5(n)$
2^α , $\alpha \geq 2$	6	7	4	
3^α , $\alpha \geq 2$	5	4		
p^α , $p \geq 5$	4			
$2 \cdot 3^\alpha$	7	4		
$2 \cdot 5^\alpha$	6	7	4	
$2 \cdot p^\alpha$, $p \geq 7$	5	4		
12	8	6	7	4
$3 \cdot 2^\alpha$, $\alpha \geq 3$	9	5	4	
$3 \cdot 5^\alpha$	6	7	4	
$3 \cdot p^\alpha$, $p \geq 7$	5	4		

Thus, for all n:

$f^m(n) = 2$ for some m not exceeding 3;

$g^m(n) = 3$ for some m not exceeding 4;

$h^m(n) = 4$ for some m not exceeding 5.

It seems likely that for any n, $f_k^m(n)$ will equal $k + 1$ for some m not exceeding some fixed number depending only on k; possibly that fixed number will not be larger than $k + 2$.

Also solved by RICHARD I. HESS (part (b) only), Rancho Palos Verdes, California; DAN SOKOLOWSKY, Williamsburg, Virginia; LAWRENCE SOMER, Washington, D.C.; PETER WATSON-HURTHIG, Columbia College, Vancouver, B.C.; KENNETH M. WILKE (part (a) only), Topeka, Kansas; and the proposer. There was one faulty proof submitted.

Meyers' interesting results and conjecture cry out for further investigation; any takers?

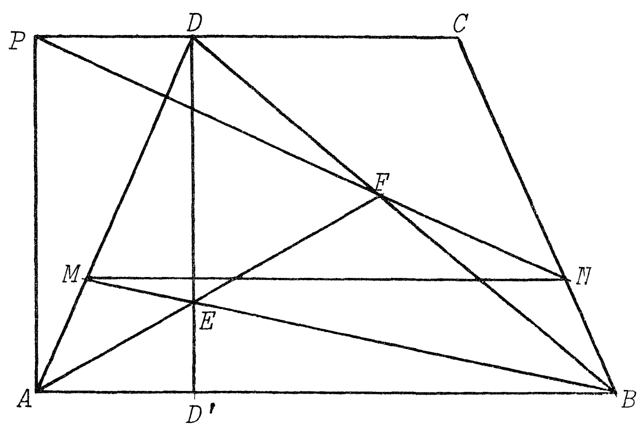
Watson-Hurthig compliments Mr. Grabiner for his proposal, saying that "a couple of other teachers including the principal enjoyed the problem and it provided as well a nice programming assignment for my computer science class". I appreciate efforts like this to acquaint more people with Crux. This seems like a good place to remind you readers out there that new problem proposals are always needed. I would especially invite problems suitable to be worked on by students.

* * *

<u>1117</u>. [1986: 27] *Proposed by Jordi Dou, Barcelona, Spain.*

Let $ABCD$ be an isosceles trapezoid with bases $AB > DC$, and let M and N be points on AD and BC respectively so that MN is parallel to AB and DC. Let D' be the projection of D on AB, let $E = DD' \cap BM$, $F = BD \cap AE$, and $P = NF \cap DC$. Prove that PA is perpendicular to AB.

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.



From Ceva's theorem applied to ΔABD,

$$\frac{D'A}{D'B}\cdot\frac{FB}{FD}\cdot\frac{MD}{MA} = 1. \qquad (1)$$

From Menelaus' theorem applied to ΔBCD,

$$\frac{PC}{PD}\cdot\frac{FD}{FB}\cdot\frac{NB}{NC} = 1. \qquad (2)$$

Since MN is parallel to AB and DC,

$$\frac{MA}{MD}\cdot\frac{NC}{NB} = 1. \qquad (3)$$

Multiplying (1), (2), and (3) we obtain

$$\frac{D'A}{D'B}\cdot\frac{PC}{PD} = 1. \qquad (4)$$

Put $AB = a$ and $CD = c$. Then $D'A = \frac{a - c}{2}$, $D'B = \frac{a + c}{2}$, and $PC = PD + c$, and from (4),

$$\frac{a - c}{a + c}\cdot\frac{PD + c}{PD} = 1$$

and so

$$PD = \frac{a - c}{2} = AD'.$$

Thus $PDD'A$ is a rectangle, and $PA \perp AB$.

Also solved by C. FESTRAETS-HAMOIR, *Brussels, Belgium;* J.T. GROENMAN, *Arnhem, The Netherlands;* WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria;* KEE-WAI LAU, *Hong Kong;* DAN SOKOLOWSKY, *Williamsburg, Virginia; and the proposer.*

* * *

1118. [1986: 27] *Proposed by P. Erdös, Hungarian Academy of Sciences.*

Let $a_1,a_2,a_3,...$ be a sequence of numbers such that $\lim_{i\to\infty}(a_{i+1} - a_i) = \infty$. Construct an infinite sequence $b_1 < b_2 < ...$ so that none of the sums $\sum_{i=1}^{\infty} \epsilon_i b_i$ (where $\epsilon_i = 0$ or 1 for each i and all but finitely many are 0) equals any of the a_i's. See also Problem #85 [1976: 29].

Solution by the proposer.

By choosing the b_i's large enough we can assume that the a_i's are increasing. Let b_1 be a sufficiently large positive number distinct from all the a_i's. If $b_1,b_2,...,b_k$ have already been constructed, let $b_{k+1} = a_\ell + 1$ where ℓ is chosen such that $a_{\ell+1} - a_\ell > b_1 + ... + b_k + 1$. By induction we assume that all sums of elements from $\{b_1,b_2,...,b_k)$ are distinct from the a_i's. Thus we need only consider sums of elements from $\{b_1,b_2,...,b_{k+1}\}$ that involve b_{k+1}, and since $b_{k+1} > a_\ell$ and

$$b_1 + b_2 + ... + b_{k+1} = b_1 + b_2 + ... + b_k + a_\ell + 1 < a_{\ell+1} ,$$

no such sum can equal any of the a_i's.

* * *

1120*. [1986: 27] *Proposed by D.S. Mitrinovic, University of Belgrade, Belgrade, Yugoslavia.*

(a) Determine a positive number λ so that

$$(a + b + c)^2(abc) \geq \lambda(bc + ca + ab)(b + c - a)(c + a - b)(a + b - c)$$

holds for all real numbers a, b, c.

(b) As above, but a, b, c are assumed to be positive.

(c) As above, but a, b, c are assumed to satisfy

$$b + c - a > 0 \ , \ c + a - b > 0 \ , \ a + b - c > 0.$$

Solution by Jordan B. Tabov, Sofia, Bulgaria.

(a) For $a = 2$, $b = c = -1$, the proposed inequality reduces to $0 \geq 48\lambda$, and therefore there is no λ with the required property.

(b) and (c) We shall show that $\lambda = 3$ is the largest suitable real number. Taking $a = b = c = 1$, we get $3 \geq \lambda$, and it remains to prove that

$$(a + b + c)^2abc \geq 3(bc + ca + ab)(b + c - a)(c + a - b)(a + b - c) \qquad (1)$$

where a, b, c satisfy either the condition in (b) or that in (c); the first follows from the second.

In order to prove (1), we note that

$$(a + b + c)^2 \geq 3(bc + ca + ab) \qquad (2)$$

and that the substitution

$$b + c - a = 2u \ , \ c + a - b = 2v \ , \ a + b - c = 2w$$

reduces the inequality

$$abc \geq (b + c - a)(c + a - b)(a + b - c) \qquad (3)$$

to

$$(v + w)(w + u)(u + v) \geq 8uvw. \qquad (4)$$

If we establish (4), then (2) and (3) together give (1), since the right side of (2) is positive.

When $u,v,w > 0$ (i.e. case (c) holds), we can do this using the A.M.-G.M. inequality:

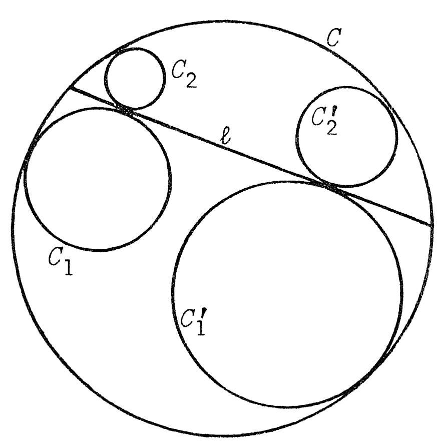
$$(v + w)(w + u)(u + v) \geq 2\sqrt{vw}\cdot 2\sqrt{wu}\cdot 2\sqrt{uv} = 8uvw.$$

Suppose that $a,b,c > 0$ but that not all of u, v, w are positive. The left side of (4) is equal to abc and therefore is positive. But by virtue of $a = v + w$ etc. we conclude that exactly one of u, v, w is non-positive, and hence the right side of (4) is non-positive. Consequently (4) is true also in case (b) as well as in case (c), and the stated result is proved.

Also solved by WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria; and* M.S. KLAMKIN, *University of Alberta, Edmonton, Alberta.*

* * *

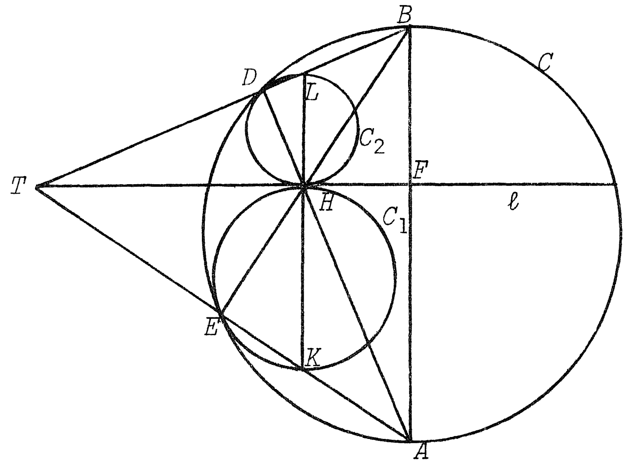
<u>1121</u>. [1986: 50] *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokai-City, Aichi, Japan.*



Let ℓ be a chord of a circle C. Let C_1 and C_2 be circles of radii r_1 and r_2 respectively, interior to C and tangent to C, and on opposite sides of ℓ and tangent to ℓ at a common point. Let C_1' and C_2' be another such pair of circles, of radii r_1' and r_2' respectively, with C_1 and C_1' on the same side of ℓ. Show that $\dfrac{r_1}{r_2} = \dfrac{r_1'}{r_2'}$.

I. *Solution by Dan Sokolowsky, Williamsburg, Virginia.*

It will suffice to show $r_1/r_2 = k$, a constant.



Thus let C_1, C_2 touch C at D, E respectively, and ℓ at H. Let the line through H perpendicular to L meet C_1 at K, C_2 at L. Then HK, HL are respective diameters of C_1, C_2.

Since C_1 touches C at E, EK and EH meet C at the respective endpoints A, B of the diameter AB of C parallel to HK, so $AB \perp \ell$. Likewise, since C_2 touches C at D, DH and DL also pass through A, B respectively. Let AB meet ℓ at F, and let AE, BD meet at T. Then, since $AD \perp BT$ and $BE \perp AT$, H is the orthocenter of ΔABT, so $TH \perp AB$. Since $HF \perp AB$, T, H, and F are collinear. We then have

$$\frac{HL}{FB} = \frac{TH}{TF} = \frac{HK}{FA},$$

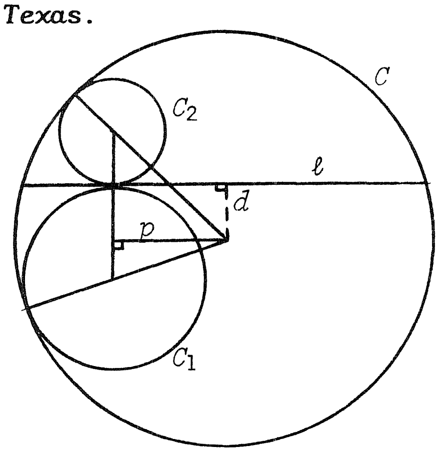
so

$$\frac{r_1}{r_2} = \frac{HK}{HL} = \frac{FA}{FB},$$

and FA/FB is the required constant k.

II. *Solution by Sam Baethge, San Antonio, Texas.*

The problem is equivalent to showing that r_1/r_2 is a constant for a given circle and chord.



Let R be the radius of C, let d be ℓ's distance from the center of C and let p be the distance from the center of C to the line of centers of C_1 and C_2. Then we have

$$p^2 + (r_1 - d)^2 = (R - r_1)^2$$

and

$$p^2 + (r_2 + d)^2 = (R - r_2)^2,$$

which yield

$$r_1 = \frac{R^2 - p^2 - d^2}{2(R - d)}$$

and

$$r_2 = \frac{R^2 - p^2 - d^2}{2(R + d)},$$

and thus

$$\frac{r_1}{r_2} = \frac{R + d}{R - d},$$

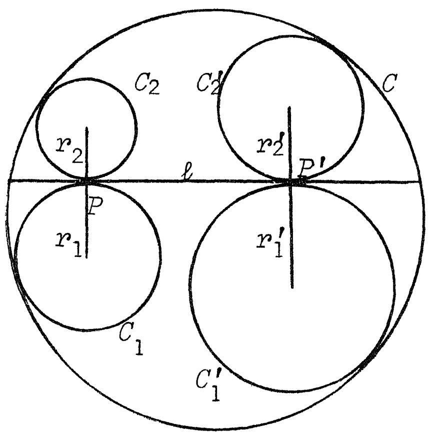
a constant. Note that if $r_1 < d$ the result is unchanged.

III. *Solution by J. Chris Fisher, University of Regina, Regina, Saskatchewan.*

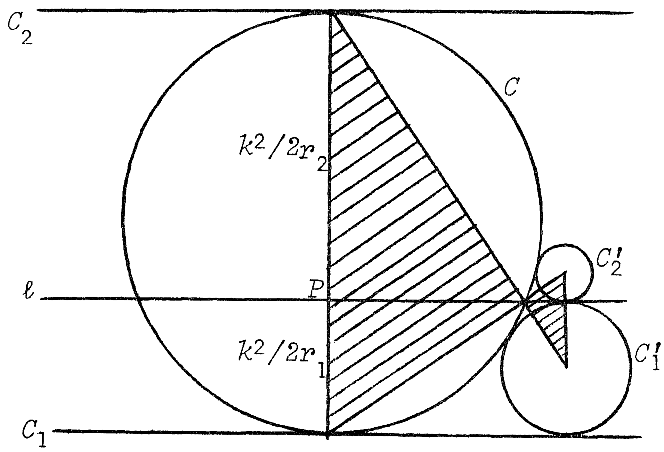
Let P and P' be the points where ℓ is tangent to the given circles. Inversion in the circle centered at P with radius $k = PP'$ fixes ℓ and the primed circles (since they are perpendicular to the circle of inversion) and takes C_1 and C_2 to lines parallel to ℓ at distances of $k^2/2r_1$ and $k^2/2r_2$ respectively. Since the shaded triangles in the "after" figure below are similar (the given radii of C and the primed circles are perpendicular to their tangents ℓ and the lines parallel to ℓ), and ℓ is their common altitude,

$$r_1' : k^2/2r_2 = r_2' : k^2/2r_1 .$$

That is, $r_1' : r_2' = r_1 : r_2$ as desired.



Before inversion



After inversion

Also solved by AAGE BONDESEN, *Royal Danish School of Educational Studies, Copenhagen, Denmark;* K. CAPELL, *University of Queensland, St. Lucia, Australia;* JORDI DOU, *Barcelona, Spain;* J.T. GROENMAN, *Arnhem, The Netherlands;* RICHARD I. HESS, *Rancho Palos Verdes, California;* WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria;* DAN SOKOLOWSKY *(a second solution); and the proposer.*

The problem is from the 1854 Japanese book Sanpo Tusyo *("mathematical book"), with a solution by Hirosi Hasegawa (1810-1887). It appears on p.43 of the proposer's Japanese book* Study of Sangaku.

* * *

1122. [1986: 50] *Proposed by Richard K. Guy, University of Calgary, Calgary, Alberta.*

Find a dissection of a $6 \times 6 \times 6$ cube into a small number of connected pieces which can be reassembled to form cubes of sides 3, 4, and 5, thus demonstrating that $3^3 + 4^3 + 5^3 = 6^3$. One could ask this in at least four forms:

(a) the pieces must be bricks, with integer dimensions;

(b) the pieces must be unions of $1 \times 1 \times 1$ cells of the cube;

(c) the pieces must be polyhedral;

(d) no restriction.

Editor's comment. The proposer has since located this problem (in form (b)) in *Eureka* 12 (1949), page 6. The published solution (*Eureka* 13 (1950) 23), due to R.F. Wheeler, gives a dissection of the $6 \times 6 \times 6$ cube into eight pieces of type (b) which can be reassembled as required. Here is the dissection given layer by layer, with the eight pieces labelled *A* to *H*.

A	A	A	H	H	H
A	A	A	H	H	H
A	A	A	H	H	H
F	F	F	H	H	H
F	F	F	H	H	H
F	F	F	F	F	C

layer 1

A	A	A	H	H	H
A	A	A	H	H	H
A	A	A	H	H	H
F	F	F	H	H	H
F	F	F	H	H	H
F	F	F	F	F	C

layer 2

A	A	A	H	H	H
A	A	A	H	H	H
A	A	A	H	B	B
F	F	B	B	B	B
F	F	B	B	B	B
F	F	B	B	B	B

layer 3

E	E	E	E	H	H
E	E	E	E	H	H
E	E	B	B	B	B
F	F	B	B	B	B
F	F	B	B	B	B
F	F	B	B	B	B

layer 4

E	E	E	E	D	D
E	E	E	E	D	D
E	E	B	B	B	B
F	F	B	B	B	B
F	F	B	B	B	B
F	F	B	B	B	B

layer 5

E	E	E	E	D	D
E	E	E	E	D	D
E	E	B	B	B	B
G	G	B	B	B	B
G	G	B	B	B	B
G	G	B	B	B	B

layer 6

A forms the 3-cube, *B* and C the 4-cube, and the 5-cube is formed from *D*, E, *F*, G, *H* as follows:

G	G	*H*	*H*	*H*
G	G	*H*	*H*	*H*
G	G	*H*	*H*	*H*
D	*D*	*H*	*H*	*H*
D	*D*	*H*	*H*	*H*

layer 1

E	E	*H*	*H*	*H*
E	E	*H*	*H*	*H*
E	E	*H*	*H*	*H*
D	*D*	*H*	*H*	*H*
D	*D*	*H*	*H*	*H*

layer 2

E	E	*H*	*H*	*H*
E	E	*H*	*H*	*H*
E	E	*H*	*F*	*F*
F	*F*	*F*	*F*	*F*
F	*F*	*F*	*F*	*F*

layer 3

E	E	E	*H*	*H*
E	E	E	*H*	*H*
E	E	E	*F*	*F*
F	*F*	*F*	*F*	*F*
F	*F*	*F*	*F*	*F*

layer 4

E	E	E	*F*	*F*
E	E	E	*F*	*F*
E	E	E	*F*	*F*
F	*F*	*F*	*F*	*F*
F	*F*	*F*	*F*	*F*

layer 5

To quote from the *Eureka* solution, "it is apparent that the dissection cannot be effected in less than eight pieces using only planes parallel to the faces, since then each vertex of the six-cube must be on a separate piece". Thus the best solution for (b) is eight pieces. Eight pieces is surely optimal for (c) and (d) also, although the above argument doesn't quite go through. Can anyone supply a proof?

For (a), CHARLES H. JEPSON, Grinnell College, Grinnell, Iowa, writes that he has found several 11-brick solutions, with the 4-cube being one brick, the 3-cube cut into two, and the 5-cube cut into eight. Can any reader find a smaller solution, or show that there are none with 8, 9, or 10 bricks?

Non-optimal solutions were also received from RICHARD I. HESS, Rancho Palos Verdes, California; and the proposer. One reader must have misinterpreted the problem, as he sent in a dissection of the 6-cube into seven bricks which can be reassembled to form three pieces of *volumes* 3^3, 4^3, and 5^3.

* * *

<u>1123</u>. [1986: 51] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with sides a, b, c and angles α, β, γ such that $a \neq b$. Let the interior bisectors of α and β intersect the opposite sides at D and E respectively, and find D_1 and E_1 on AC and BC respectively such that $AD_1 = AD$ and $BE_1 = BE$. Suppose that $D_1E_1 \parallel AB$. Find γ.

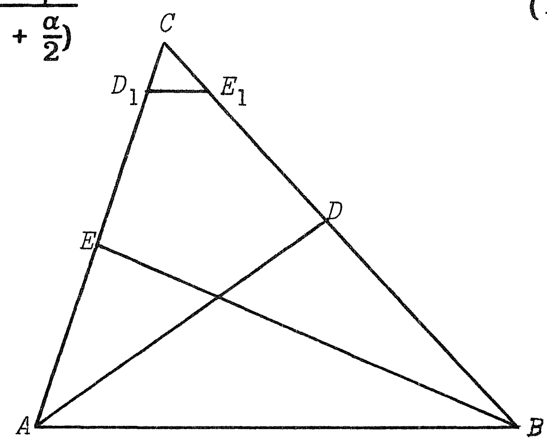
Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

In ΔABD,

$$AD:AB = \sin\beta:\sin(\beta + \tfrac{\alpha}{2})$$

so that

$$AD = \frac{c\sin\beta}{\sin(\beta + \frac{\alpha}{2})} \tag{1}$$



and similarly

$$BE = \frac{c\sin\alpha}{\sin(\alpha + \frac{\beta}{2})} . \tag{2}$$

Since $D_1E_1 \parallel AB$, $AD_1:BE_1 = CA:CB$ or

$$AD:BE = \sin\beta:\sin\alpha. \tag{3}$$

From (1), (2), and (3),

$$\sin(\beta + \tfrac{\alpha}{2}) = \sin(\alpha + \tfrac{\beta}{2}).$$

Since $a \neq b$, $\alpha \neq \beta$, and so

$$\beta + \frac{\alpha}{2} + \alpha + \frac{\beta}{2} = \pi$$

$$\alpha + \beta = \frac{2\pi}{3}$$

and thus

$$\gamma = \frac{\pi}{3} .$$

Also solved by JORDI DOU, *Barcelona, Spain;* WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria;* DAN SOKOLOWSKY, *Williamsburg, Virginia; and the proposer.*

Smeenk and Sokolowsky observe that the converse also holds, i.e., if $\gamma = \pi/3$ and $a \neq b$ then $D_1E_1 \parallel AB$. Dou states that $DD_1 \perp EE_1$. (Proof?)

* * *

<u>1124</u>. [1986: 51] *Proposed by Stanley Rabinowitz and Peter Gilbert, Digital Equipment Corp., Nashua, New Hampshire.*

If $1 < a < 2$ and k is an integer, prove that

$$[a[k/(2-a)] + a/2] = [ak/(2-a)]$$

where $[x]$ denotes the greatest integer not larger than x.

Solution by Richard I. Hess, Rancho Palos Verdes, California.

Letting
$$A = \left[a\left[\frac{k}{2-a}\right] + \frac{a}{2}\right]$$
and
$$B = \left[\frac{ak}{2-a}\right] ,$$
we wish to prove $A = B$. Put
$$\frac{k}{2-a} = N + x \qquad (1)$$
where
$$\left[\frac{k}{2-a}\right] = N , \qquad 0 \le x < 1.$$
Note that since $1 < a < 2$, $0 < \frac{k}{N+x} < 1$. From (1),
$$a = 2 - \frac{k}{N+x}$$
or
$$\frac{a}{2} = 1 - \frac{k}{2(N+x)} .$$
Then
$$\begin{aligned} A &= \left[(2 - \frac{k}{N+x})N + 1 - \frac{k}{2(N+x)}\right] \\ &= \left[2N - \frac{kN}{N+x} + 1 - \frac{k}{2(N+x)}\right] \\ &= 2N - k + \left[1 + \frac{(x-1/2)k}{N+x}\right] \end{aligned}$$
and
$$B = [(2 - \frac{k}{N+x})(N+x)] = 2N - k + [2x].$$
Now if $0 \le x < 1/2$, then
$$A = 2N - k = B,$$
while if $1/2 \le x < 1$, then
$$A = 2N - k + 1 = B.$$

Also solved by the proposers.

* * *

<u>1125</u>*. [1986: 51] *Proposed by Jack Garfunkel, Flushing, N.Y.*

If A, B, C are the angles of an acute triangle ABC, prove that
$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} \le \frac{3}{2}(\csc 2A + \csc 2B + \csc 2C)$$
with equality when triangle ABC is equilateral.

Solution by Bob Prielipp, University of Wisconsin, Oshkosh, Wisconsin.

We begin by noting that if s is the semiperimeter and r is the inradius of ΔABC then

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} = \frac{s-a}{r}+\frac{s-b}{r}+\frac{s-c}{r}$$
$$= \frac{3s-(a+b+c)}{r} = \frac{s}{r}. \qquad (1)$$

Since A, B, and C are the angles of an acute triangle ABC, sin $2A$, sin $2B$, and sin $2C$ are all positive real numbers. Thus

$$\csc 2A+\csc 2B+\csc 2C = \frac{1}{\sin 2A}+\frac{1}{\sin 2B}+\frac{1}{\sin 2C}$$
$$\geq \frac{9}{\sin 2A+\sin 2B+\sin 2C}$$

(by the harmonic mean-arithmetic mean inequality)

$$= \frac{9}{2rs/R^2} = \frac{9R^2}{2rs}, \qquad (2)$$

where R is the circumradius of ΔABC.

From (1) and (2), to complete our solution it suffices to establish that

$$27R^2 \geq 4s^2, \qquad (3)$$

and this follows from 5.3 of Bottema et al, *Geometric Inequalities*.

Equality holds in (3) if and only if the triangle is equilateral. Equality holds in our application of the harmonic mean-arithmetic mean inequality if and only if sin $2A$ = sin $2B$ = sin $2C$, i.e. $A = B = C$ (since ΔABC is acute).

Also solved by R.H. EDDY, Memorial University, St. John's, Newfoundland; RICHARD I. HESS, Rancho Palos Verdes, California; EDWIN M. KLEIN, University of Wisconsin, Whitewater, Wisconsin; and VEDULA N. MURTY, Penn State University, Middletown, Pennsylvania.

* * *

<u>1126</u>. [1986: 51] *Proposed by Péter Ivady, Budapest, Hungary.*

For $0 < x \leq 1$, show that

$$\sinh x < \frac{3x}{2+\sqrt{1-x^2}} < \tan x.$$

Solution by Vedula N. Murty, Penn State University, Middletown, Pennsylvania.

We claim that

$$\tan x > \frac{3x}{3-x^2}, \qquad 0 < x \leq 1, \qquad (1)$$

$$\sinh x < \frac{3x}{3-x^2/2}, \qquad 0 < x \leq 1, \qquad (2)$$

and

$$\frac{3x}{3-x^2/2} < \frac{3x}{2+\sqrt{1-x^2}} \leq \frac{3x}{3-x^2}, \qquad 0 < x \leq 1, \qquad (3)$$

from which the required result follows.

Consider the function

$$\phi(x) = (3 - x^2)\sin x - 3x \cos x , \qquad 0 < x \le 1.$$

Then

$$\phi'(x) = x \sin x - x^2 \cos x.$$

From the known inequality $\tan x > x$, we see that $\phi'(x) > 0$ for $0 < x \le 1$. Therefore $\phi(x)$ increases on $(0,1]$, and $\phi(0) = 0$, so $\phi(x) > 0$ for $0 < x \le 1$. This proves (1).

Since

$$\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n - 1)!}$$

and

$$\frac{3x}{3 - x^2/2} = \frac{x}{1 - x^2/6} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{6^{n-1}} ,$$

and since $(2n - 1)! > 6^{n-1}$ for $n > 2$, (2) holds.

Finally, claim (3) follows from the inequalities

$$1 - \frac{x^2}{2} > \sqrt{1 - x^2}$$

and

$$1 - x^2 \le \sqrt{1 - x^2}$$

for $0 < x \le 1$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; KEE-WAI LAU, Hong Kong; ROBERT E. SHAFER, Berkeley, California; and the proposer.

*　　　　*　　　　*

<u>1127</u>*. [1986: 51] *Proposed by D.S. Mitrinovic, University of Belgrade, Belgrade, Yugoslavia.*

(a) Let a, b, c and r be real numbers > 1. Prove or disprove that

$$(\log_a bc)^r + (\log_b ca)^r + (\log_c ab)^r \ge 3\cdot 2^r.$$

(b) Find an analogous inequality for n numbers $a_1, a_2, \ldots, a_n$ rather than three numbers a, b, c.

Solution and generalization by M.S. Klamkin, University of Alberta, Edmonton, Alberta.

We claim that

$$\sum_{i=1}^{n} \left[\log_{a_i} p/a_i\right]^r \ge n(n - 1)^r \tag{1}$$

where $p = \prod_{i=1}^{n} a_i$, $a_i > 1$, $r > 0$. Part (a) corresponds to the special case $n = 3$.

In fact, we show even more generally that if G is a positive superadditive function, i.e.,

$$G(x + y) \geq G(x) + G(y)$$

for all x, y, then

$$S \equiv \sum_{i=1}^{n} \left[\frac{G(\ell n\ p/a_i)}{G(\ell n\ a_i)}\right]^r \geq n(n - 1)^r. \tag{2}$$

Since

$$\log_{a_i} p/a_i = \frac{\ell n\ p/a_i}{\ell n\ a_i},$$

inequality (1) corresponds to the special case of (2) for $G(x) = x$.

By the A.M.-G.M. inequality,

$$(S/n)^{n/r} \geq \prod_{i=1}^{n} \frac{G(\ell n\ p/a_i)}{G(\ell n\ a_i)}. \tag{3}$$

Then since G is superadditive,

$$\begin{aligned} G(\ell n\ p/a_i) &= G(\ell n\ a_1 + \ldots + \ell n\ a_n - \ell n\ a_i) \\ &\geq \sum_{j=1}^{n} G(\ell n\ a_j) - G(\ell n\ a_i) \end{aligned}$$

for each $i = 1, \ldots, n$. Thus, again by the A.M.-G.M. inequality,

$$G(\ell n\ p/a_i) \geq (n - 1)\left[\frac{\prod_{j=1}^{n} G(\ell n\ a_j)}{G(\ell n\ a_i)}\right]^{\frac{1}{n-1}}.$$

Hence

$$\begin{aligned} \prod_{i=1}^{n} G(\ell n\ p/a_i) &\geq \frac{(n - 1)^n \prod_{i=1}^{n} [G(\ell n\ a_i)]^{\frac{n}{n-1}}}{\prod_{i=1}^{n} [G(\ell n\ a_i)]^{\frac{1}{n-1}}} \\ &= (n - 1)^n \prod_{i=1}^{n} G(\ell n\ a_i). \end{aligned}$$

From (3),

$$(S/n)^{n/r} \geq (n - 1)^n$$

and (2) follows.

Also solved by WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria;* LEROY F. MEYERS, *The Ohio State University, Columbus, Ohio;* VEDULA N. MURTY, *Penn State University, Middletown, Pennsylvania; and (part (a))* ROBERT E. SHAFER, *Berkeley, California.*

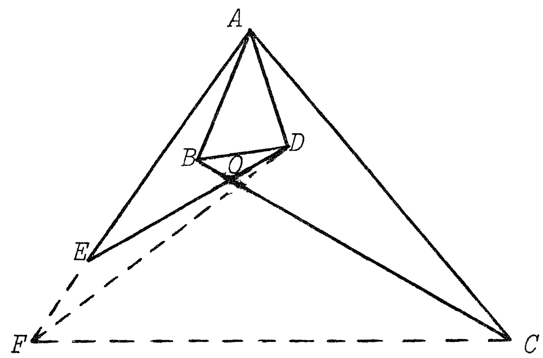
* * *

1128. [1986: 51] *Proposed by Roger Izard, Dallas, Texas.*

Triangles CBA and ADE are so placed that B and D lie inside ADE and CBA respectively. DE and CB intersect at O. Angle CAB is equal to angle DAE. AD = AB and CO = OE. Prove that triangles ADE and CBA are congruent.

Solution by Dan Sokolowsky, Williamsburg, Virginia.

To prove $\Delta ADE \equiv \Delta CBA$ it suffices to show that $AE = AC$. Suppose this is false. We can then assume $AE < AC$. Extend AE to F, where $AF = AC$. Then $\Delta AFD \equiv \Delta ACB$, so $BC = DF$ and $\angle ADF = \angle ABC$. Since $AD = AB$ implies $\angle ADB = \angle ABD$,



$$\begin{aligned}\angle BDF &= \angle ADF - \angle ADB\\ &= \angle ABC - \angle ABD = \angle DBC. \qquad (1)\end{aligned}$$

Now note that since B is inside ΔAED,

$$\angle AED < \angle ABD = \angle ADB < \angle ADE,$$

hence $\angle AED < 90^\circ$. Thus $\angle DEF > 90^\circ$, and hence $BC = DF > DE$. Then

$$BO = BC - CO > DE - OE = DO,$$

so $\angle BDE > \angle DBC$. But since $\angle BDF > \angle BDE$, this contradicts (1). Hence $AE = AC$ must hold and the conclusion follows.

Also solved by JORDI DOU, *Barcelona, Spain;* J.T. GROENMAN, *Arnhem, The Netherlands;* RICHARD I. HESS, *Rancho Palos Verdes, California; and the proposer.*

* * *

! ! ! ! ! SPECIAL OFFER ! ! ! ! !

WHILE SUPPLIES LAST, BOUND VOLUMES OF CRUX MATHEMATICORUM ARE AVAILABLE AT THE FOLLOWING REDUCED PRICES:

$10.00 per volume	(regularly $20.00)
$75.00 per complete set (volumes 1-10)	(regularly $150.00)

PLEASE SEND CHEQUES MADE PAYABLE TO THE CANADIAN MATHEMATICAL SOCIETY TO:

Canadian Mathematical Society
577 King Edward, Suite 109
Ottawa, Ontario
Canada K1N 6N5

Volume Numbers ______________________

______ volumes × $10.00 = $__________

Complete Sets (volumes 1-10)

______ sets × $75.00 = $__________

Total Payment Enclosed $__________

Mailing Address : ______________________

____

____

____

____

! ! ! ! ! GRANDE SOLDE ! ! ! ! !

CRUX MATHEMATICORUM: 10 VOLUMES RELIES EN SOLDE:

chacun des volume 1 à 10	10$	(régulier 20$)
la collection (volumes 1-10)	75$	(régulier 150$)

S.V.P. COMPLETER ET RETOURNER, AVEC VOTRE REMISE LIBELLEE AU NOM DE LA SOCIETE MATHEMATIQUE DU CANADA, A L'ADRESSE SUIVANTE:

Société mathématique du Canada
577 King Edward, Suite 109
Ottawa, Ontario
Canada K1N 6N5

volume(s) numéro(s) ________________ Adresse : ______________________________

_____ volumes × 10$ = __________$ ______________________________

____

Collection (volumes 1-10) ______________________________

_____ × 75$ = __________$ ______________________________

Total de votre remise __________$