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ON THE DISTANCES OF A POINT TO THE VERTICES OF A TRIANGLE

O. BOTTEMA

1. *Introduction.*

We consider a triangle $A_1A_2A_3$ with sides a_i and angles α_i $(i = 1,2,3)$, and a point P in its plane. Let $d_i = PA_i$. We derive in Section 2 a formula for d_3 when d_1 and d_2 are known. We then obtain in Section 3 a symmetric relation for d_1, d_2, d_3. Certain applications of this relation are given in Section 4, and alternative proofs of it appear in Section 5.

2. *A formula for d_3.*

We assume for the time being that P is an interior point of the triangle (see Figure 1). Let F be the area of triangle $A_1A_2A_3$ and F_1 that of triangle A_1A_2P. Then we have

$$\begin{cases} 2a_2a_3\cos\alpha_1 = -a_1^2 + a_2^2 + a_3^2, \\ a_2a_3\sin\alpha_1 = 2F. \end{cases} \tag{1}$$

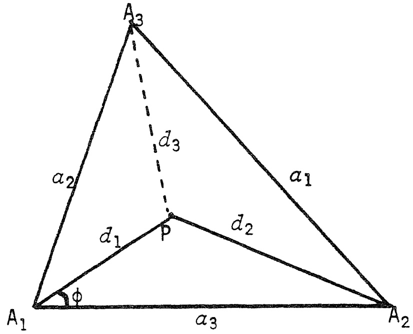

Figure 1

If $\angle PA_1A_2 = \phi$, we obtain from triangle A_1A_2P

$$\begin{cases} 2a_3d_1\cos\phi = d_1^2 - d_2^2 + a_3^2, \\ a_3d_1\sin\phi = 2F_1, \end{cases} \tag{2}$$

and, from triangle A_3A_1P,

$$d_3^2 = a_2^2 + d_1^2 - 2a_2d_1\cos(\alpha_1-\phi). \tag{3}$$

Multiplying (3) by $2a_3^2$ and then substituting from (1) and (2) result in

$$\begin{aligned} 2a_3^2d_3^2 &= 2a_2^2a_3^2 + 2a_3^2d_1^2 - 4a_2a_3^2d_1(\cos\alpha_1\cos\phi + \sin\alpha_1\sin\phi) \\ &= 2a_2^2a_3^2 + 2a_3^2d_1^2 - (-a_1^2+a_2^2+a_3^2)(d_1^2-d_2^2+a_3^2) - 16FF_1 \\ &= (a_1^2-a_2^2+a_3^2)d_1^2 + (-a_1^2+a_2^2+a_3^2)d_2^2 + (a_1^2+a_2^2-a_3^2)a_3^2 - 16FF_1. \end{aligned}$$

Therefore

$$a_3d_3^2 = a_1d_1^2\cos\alpha_2 + a_2d_2^2\cos\alpha_1 + a_1a_2a_3\cos\alpha_3 - \frac{8FF_1}{a_3}. \tag{4}$$

If P lies outside the triangle but on the same side of A_1A_2 as vertex A_3, it is easily seen that formulas (2), (3), and (4) remain valid, for $\cos(\phi-\alpha_1) = \cos(\alpha_1-\phi)$. The situation is slightly different if P lies below A_1A_2, for we must write $\cos(\alpha_1+\phi)$ in (3), and the last term in (4) changes sign. Summing up, we have for any point P the formula

$$a_3d_3^2 = a_1d_1^2\cos\alpha_2 + a_2d_2^2\cos\alpha_1 + a_1a_2a_3\cos\alpha_3 \pm \frac{8FF_1}{a_3}. \tag{5}$$

It was to be expected that, corresponding to given distances d_1 and d_2, there would be *two* distances d_3. Indeed, for a point P and its reflection P' in the line A_1A_2, the distances d_1 and d_2 are the same but d_3 and d_3' are different. There is a unique answer if and only if $F_1 = 0$ in (5), that is, if and only if P lies on the line A_1A_2. In particular, if P lies between A_1 and A_2, then $d_1+d_2 = a_3$ and, bearing in mind that

$$a_1\cos\alpha_2 + a_2\cos\alpha_1 = a_3,$$

it follows from (5) that

$$a_3d_3^2 = a_1^2d_1 + a_2^2d_2 - a_3d_1d_2.$$

This shows that (5) can be considered as *a generalization of Stewart's theorem.*

3. *A symmetric relation for d_1,d_2,d_3.*

We multiply (5) by a_3, isolate the last term, and then square both sides to obtain

$$(-a_3^2d_3^2 + a_1a_3d_1^2\cos\alpha_2 + a_2a_3d_2^2\cos\alpha_1 + a_1a_2a_3^2\cos\alpha_3)^2 = 64F^2F_1^2.$$

If we substitute into this

$$4F^2 = a_2^2a_3^2\sin^2\alpha_1 = a_3^2a_1^2\sin^2\alpha_2 = a_1^2a_2^2\sin^2\alpha_3$$

and

$$16F_1^2 = -d_1^4 - d_2^4 - a_3^4 + 2d_1^2d_2^2 + 2a_3^2d_1^2 + 2a_3^2d_2^2,$$

we obtain, after some algebra and division by a_3^2, *our main conclusion*:

$$\begin{aligned} a_1^2d_1^4 + a_2^2d_2^4 + a_3^2d_3^4 - 2a_2a_3\cos\alpha_1\cdot d_2^2d_3^2 - 2a_3a_1\cos\alpha_2\cdot d_3^2d_1^2 - 2a_1a_2\cos\alpha_3\cdot d_1^2d_2^2 \\ - 2a_1^2a_2a_3\cos\alpha_1\cdot d_1^2 - 2a_1a_2^2a_3\cos\alpha_2\cdot d_2^2 - 2a_1a_2a_3^2\cos\alpha_3\cdot d_3^2 + a_1^2a_2^2a_3^2 = 0. \end{aligned} \tag{6}$$

This is a quadratic equation in the d_i^2. If we express the $\cos\alpha_i$ in terms of the sides of the triangle, we see that (6) is a rational relation in a_i and d_i.

Let d_1,d_2,d_3 be three positive numbers satisfying (6). If the real circles $(A_1;d_1)$ and $(A_2;d_2)$ have two real intersections S_1 and S_2, then it follows from (5) that for one of them the distance to A_3 equals d_3. The same holds if S_1 and S_2 are (conjugate) imaginary, but if one of them is a point of the (real) circle $(A_3;d_3)$ the same is true for the other. But then the three circles belong to the pencil of circles through the points S_1 and S_2, which implies that A_1,A_2,A_3 are collinear points. Our conclusion therefore reads: *if* d_1, d_2, d_3 *are three positive numbers satisfying* (6), *then there is a unique point* P *such that* $PA_i = d_i$ $(i = 1,2,3)$.

4. *Applications of* (6).

We consider some special cases.

(a) For the distances d_i of a point P to the vertices of an equilateral triangle ($a_i = a$, $\cos\alpha_i = \frac{1}{2}$, $i = 1,2,3$), (6) is equivalent to

$$\Sigma d_i^4 - \Sigma d_2^2 d_3^2 - a^2 \Sigma d_1^2 + a^4 = 0,$$

where the sums are cyclic.

(b) If with respect to an arbitrary triangle we have $d_1 = d_2 = d_3 = d$, then the coefficient of d^4 in (6) vanishes, and the result is, as expected, equivalent to

$$d^2 = \frac{a_1^2 a_2^2 a_3^2}{16F^2} = R^2,$$

where R is the circumradius of the triangle.

(c) If P is such that

$$d_1 : d_2 : d_3 = \frac{1}{a_1} : \frac{1}{a_2} : \frac{1}{a_3}, \tag{7}$$

then $d_i = \lambda / a_i$, $i = 1,2,3$, and (6) gives a quadratic in λ^2:

$$A\lambda^4 + B\lambda^2 + C = 0, \tag{8}$$

where

$$A = a_1^4 + a_2^4 + a_3^4 - a_2^2 a_3^2 - a_3^2 a_1^2 - a_1^2 a_2^2,$$

$$B = -a_1^2 a_2^2 a_3^2 (a_1^2 + a_2^2 + a_3^2),$$

$$C = a_1^4 a_2^4 a_3^4.$$

If the triangle is equilateral, then $A = 0$ and there is one point P satisfying (7), the center. For a nonequilateral triangle, we have

$$2A = (a_2^2 - a_3^2)^2 + (a_3^2 - a_1^2)^2 + (a_1^2 - a_2^2)^2 > 0,$$

and the discriminant of (8) is

$$D = B^2 - 4AC = 48 a_1^4 a_2^4 a_3^4 \cdot F^2 > 0.$$

The two roots λ^2 of (8) are therefore real and distinct, and furthermore they are both positive since $B < 0$ and $C > 0$, so there are two real positive values for λ. Our conclusion is: *there are always two real points in the plane satisfying* (7). They are the well-known isodynamic points of the triangle. The roots λ^2 of (8) are seen to be

$$\frac{a_1^2 a_2^2 a_3^2 (a_1^2 + a_2^2 + a_3^2 \pm 4\sqrt{3}F)}{2A},$$

and, as they are both positive, we have by the way proved Weitzenböck's inequality

$$a_1^2 + a_2^2 + a_3^2 \geq 4\sqrt{3}F.$$

(d) We ask if there is a point P such that

$$d_1^2 : d_2^2 : d_3^2 = \frac{1}{a_1} : \frac{1}{a_2} : \frac{1}{a_3}.$$

Here we have $a_i d_i^2 = \lambda$, $i = 1,2,3$, and it follows from

$$\cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3 = \frac{R + r}{R}$$

and (6) that λ satisfies

$$(R-2r)\lambda^2 - 2a_1 a_2 a_3 (R+r)\lambda + a_1^2 a_2^2 a_3^2 R = 0. \qquad (9)$$

For a nonequilateral triangle we have $R - 2r > 0$, and the discriminant of (9) is $4a_1^2 a_2^2 a_3^2 r(4R+r) > 0$. Hence in such a triangle *there are always two real points with the property that the squares of their distances to the vertices are inversely proportional to the opposite sides.*

5. *Alternative proofs of* (6).

We give two other proofs for (6). In each case we assume that P is an interior point of the triangle (the argument must be modified if P is an exterior point). Referring to Figure 2, we have $\phi_1+\phi_2+\phi_3 = 2\pi$. As

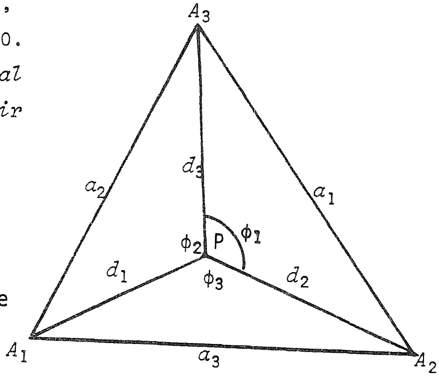

Figure 2

$$2\cos\phi_1 \cos\phi_2 \cos\phi_3 - \cos^2\phi_1 - \cos^2\phi_2 - \cos^2\phi_3 + 1 = 0$$

and

$$\cos\phi_1 = \frac{-a_1^2 + d_2^2 + d_3^2}{2d_2 d_3}, \quad \text{etc.},$$

we obtain

$$(-a_1^2+d_2^2+d_3^2)(-a_2^2+d_3^2+d_1^2)(-a_3^2+d_1^2+d_2^2) - (-a_1^2+d_2^2+d_3^2)^2 d_1^2$$
$$- (-a_2^2+d_3^2+d_1^2)^2 d_2^2 - (-a_3^2+d_1^2+d_2^2)^2 d_3^2 + 4d_1^2 d_2^2 d_3^2 = 0. \qquad (10)$$

It is easy to verify that the coefficients of cubic terms of d_i^2, such as $d_1^2 d_2^2 d_3^2$ and $d_1^4 d_2^2$, vanish and that (10) is equivalent to (6).

A third proof, less elementary than the first two, starts from a formula for the volume V of a tetrahedron $A_1A_2A_3A_4$ in which the edge A_iA_j is denoted by a_{ij}:

$$288V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a_{12}^2 & a_{13}^2 & a_{14}^2 \\ 1 & a_{21}^2 & 0 & a_{23}^2 & a_{24}^2 \\ 1 & a_{31}^2 & a_{32}^2 & 0 & a_{34}^2 \\ 1 & a_{41}^2 & a_{42}^2 & a_{43}^2 & 0 \end{vmatrix}.$$

If our point P coincides with A_4, then we have $a_{23} = a_1$, $a_{31} = a_2$, $a_{12} = a_3$, $a_{14} = d_1$, $a_{24} = d_2$, $a_{34} = d_3$, and $V = 0$. Hence

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a_3^2 & a_2^2 & d_1^2 \\ 1 & a_3^2 & 0 & a_1^2 & d_2^2 \\ 1 & a_2^2 & a_1^2 & 0 & d_3^2 \\ 1 & d_1^2 & d_2^2 & d_3^2 & 0 \end{vmatrix} = 0,$$

where our condition is given by means of a symmetric determinant of order five. If we subtract the fifth row from the second, third, and fourth, and then subtract from the resulting three rows the first row multiplied by d_1^2, d_2^2, d_3^2, respectively, we get

$$\begin{vmatrix} 2d_1^2 & -a_3^2+d_1^2+d_2^2 & -a_2^2+d_3^2+d_1^2 \\ -a_3^2+d_1^2+d_2^2 & 2d_2^2 & -a_1^2+d_2^2+d_3^2 \\ -a_2^2+d_3^2+d_1^2 & -a_1^2+d_2^2+d_3^2 & 2d_3^2 \end{vmatrix} = 0,$$

and this relation is equivalent to (10).

Charlotte de Bourbonstraat 2, 2628 BN Delft, The Netherlands.

*　　　*　　　*

MATHEMATICAL CLERIHEWS

Georg Mohr
Gave before
Lorenzo Mascheroni
For compasses only
A set of instructions
For Euclidean contructions.

Muhammad ibn Musa Al-Khwarizmi
Was told by an Arab, "Kindly don't quiz me,
Your *Algoritmus de numero Indorum*
Is inflicted on students to bore 'em."

ALAN WAYNE, Holiday, Florida

*　　　*　　　*

ANOTHER PROOF OF THE MATRIX INVEPSION FORMULA

EDWARD L. COHEN

Let $A = [a_{ij}]$ be an invertible matrix of order n with inverse $A^{-1} = [b_{ij}]$. The inversion formula

$$A^{-1} = \frac{\operatorname{adj} A}{\det A} \tag{1}$$

is proved here by a simple method. The proof avoids the technique of linear algebra textbooks that use one of the relations

$$a_{i1}A_{k1} + a_{i2}A_{k2} + \dots + a_{in}A_{kn} = \delta_{ik}\det A \tag{2}$$

or

$$a_{1j}A_{1k} + a_{2j}A_{2k} + \dots + a_{nj}A_{nk} = \delta_{jk}\det A. \tag{3}$$

This proof is to appear in [1]. We need only know that

$$\det AB = (\det A)(\det B), \tag{4}$$

which is proved earlier in [1]. It is interesting to note that a proof of Cramer's Rule by a similar method [2] also makes the use of (2) and (3) unnecessary.

We find $A^{-1} = [b_{ij}]$ by considering two matrix equations and solving for each b_{ij} separately. For the sake of clarity, we consider only the special case where A is a 4×4 matrix and show how to obtain the typical entry b_{23} of the 4×4 matrix A^{-1}. The other entries b_{ij} are found in the same way, and the generalization to the $n\times n$ case is obvious.

Keeping our sights on b_{23}, the result

$$A\cdot\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \\ b_{43} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

which is a consequence of the identity $AA^{-1} = I$, is equivalent to

$$A\cdot\begin{bmatrix} 0 & b_{13} & 0 & 0 \\ 0 & b_{23} & 0 & 0 \\ 0 & b_{33} & 0 & 0 \\ 0 & b_{43} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

It is easy to verify that

$$A\cdot\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}a_{11} & 0 & a_{13} & a_{14}\\ a_{21} & 0 & a_{23} & a_{24}\\ a_{31} & 0 & a_{33} & a_{34}\\ a_{41} & 0 & a_{43} & a_{44}\end{bmatrix}, \qquad (6)$$

where all the entries in the second column of I (on the left) and of A (on the right) are replaced by zeros. Adding the matrix equations (5) and (6), we obtain, by the distributive law,

$$A\cdot\begin{bmatrix}1 & b_{13} & 0 & 0\\ 0 & b_{23} & 0 & 0\\ 0 & b_{33} & 1 & 0\\ 0 & b_{43} & 0 & 1\end{bmatrix} = \begin{bmatrix}a_{11} & 0 & a_{13} & a_{14}\\ a_{21} & 0 & a_{23} & a_{24}\\ a_{31} & 1 & a_{33} & a_{34}\\ a_{41} & 0 & a_{43} & a_{44}\end{bmatrix}. \qquad (7)$$

Taking determinants on both sides of (7) and using (4), we obtain finally

$$(\det A)b_{23} = A_{32}.$$

Similarly $(\det A)b_{ij} = A_{ji}$, and (1) follows.

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* * *

THE PUZZLE CORNER

Puzzle No. 58: Transposal (7, 2 5)

With ONE and delta one may show
A curve continuous, if so.
Derivatives may sometimes do
To demonstrate the changes TWO.

Puzzle No. 59: Charade (8)

Inversion's useful, you'll agree;
*ONE also TWOtimes like to THREE
A length preserving map, or KEY.

ALAN WAYNE, Holiday, Florida

* * *

THE OLYMPIAD CORNER: 58

M.S. KLAMKIN

In this survey, I give the results of this year's International Mathematical Olympiad (I.M.O.) as well as some personal reflections on the I.M.O. since the U.S.A. entered into this competition in 1974. Only some of these reflections are shared by the team leaders and deputy leaders of other countries. As is to be expected, one important reason for this is that their basic philosophies concerning the I.M.O. are not all the same. In view of this and of the increasing number of new or relatively new countries participating in the I.M.O., it would be very helpful if these philosophies were discussed at the very beginning of the next I.M.O. (in Finland).

At the end of each I.M.O., the host country is warmly thanked (and deservedly so) for having held the I.M.O. with its myriad of requirements and costs, and for its gracious hospitality. There has not been a single exception to this in my experience from our first I.M.O. in East Germany in 1974 to the latest one in Czechoslovakia in 1984, and presumably the same happened in the earlier I.M.O.s. So, at the conclusion of an I.M.O., everyone is loath to criticize publicly any of the technical aspects of running the competition. Consequently, certain complaints are voiced privately and repeatedly from year to year. In view of this, I strongly believe it would be very beneficial if there were a special Jury session at the end of each I.M.O. charged with coming up with constructive ideas to improve the technical aspects of running the subsequent competition. I will return to this point later in this column.

The Twenty-Fifth International Mathematical Olympiad was held this year in Prague, Czechoslovakia from June 29 to July 10. Teams from 34 countries took part in the competition. This was again a record number of participating countries, up from last year's record of 32 countries. The following table gives the corresponding results for all past I.M.O.s.

I.M.O. No.	Year	Host Country	No. of Participating Countries
1	1959	Romania	7
2	1960	Romania	5
3	1961	Hungary	6
4	1962	Czechoslovakia	7
5	1963	Poland	8
6	1964	Soviet Union	9
7	1965	East Germany	10
8	1966	Bulgaria	9
9	1967	Yugoslavia	13
10	1968	Soviet Union	11
11	1969	Romania	13
12	1970	Hungary	13
13	1971	Czechoslovakia	14
14	1972	Poland	13
15	1973	Soviet Union	14
16	1974	East Germany	18
17	1975	Bulgaria	17
18	1976	Austria	18
19	1977	Yugoslavia	21
20	1978	Romania	17
21	1979	Great Britain	22
	1980	(No I.M.O. in 1980)	

22	1981	U.S.A.	27
23	1982	Hungary	30
24	1983	France	32
25	1984	Czechoslovakia	34
26	1985	Finland	?

The team size this year was six students from each country (the same as last year), except for Luxembourg and Norway which sent only one student each, and Algeria which sent four, resulting in 192 participating students, tying last year's record. Cyprus and Norway participated for the first time; Mongolia was back again with its best showing ever (10th place), presumably due to its new personable team leader. Israel was unfortunately not invited this year, presumably because Czechoslovakia does not maintain formal diplomatic relations with Israel.

Continuing with the last three years' break with tradition (see my last three I.M.O. reports in [1981: 220], [1982: 223], and [1983: 205]), the six problems of the competition were assigned equal weights of 7 points each, for a maximum possible score of 42. I believe that this year's competition was easier than the previous two, as evidenced by 24 students having scores of at least 35, and 8 having perfect scores, 4 more than last year. The perfect scores were achieved by

D.B. Mihov,	Bulgaria	K. Ignatiev,	Soviet Union
K. Groger,	East Germany	L. Orydoroga,	Soviet Union
D. Tataru,	Romania	D. Moews,	United States
A. Astrelin,	Soviet Union	T. Son Dan,	Vietnam.

Eleven of the contestants were girls, including K. Groger above. It would be nice to see an increase in this number and in the number of first prizes for girls at the next I.M.O.

The results of the competition are announced officially only for individual team members. However, team standings are usually compiled unofficially by adding up the scores of individual team members. This year we were provided with such a list (see table). Congratulations to the Soviet Union team, which was first by a wide margin. Congratulations also to the runner-up, the Bulgarian team, which achieved what I think is their best showing ever.

Rank	Country	Score (max 252)	Prizes 1st	2nd	3rd	Total Prizes
1	Soviet Union	235	5	1	0	6
2	Bulgaria	203	2	3	1	6
3	Romania	199	2	2	2	6
4,5	Hungary	195	1	4	1	6
4,5	U.S.A.	195	1	4	1	6
6	Great Britain	169	1	3	1	5
7	Vietnam	162	1	2	3	6
8	East Germany	161	1	2	3	6
9	West Germany	150	-	2	4	6
10	Mongolia	146	-	3	2	5
11	Poland	140	-	1	5	6
12	France	126	-	2	2	4
13	Czechoslovakia	125	-	2	2	4
14	Yugoslavia	105	-	-	4	4
15	Australia	103	-	1	2	3
16	Austria	97	-	1	2	3
17	Netherlands	93	-	1	2	3
18	Brazil	92	-	-	3	3
19	Greece	88	-	1	-	1
20	Canada	83	-	-	1	1

21	Colombia	80	-	-	2	2
22	Cuba	67	-	-	1	1
23,24	Belgium	56	-	-	1	1
23,24	Morocco	56	-	-	1	1
25	Sweden	53	-	-	-	-
26	Cyprus	47	-	-	1	1
27	Spain	43	-	-	-	-
28	Algeria	36	-	-	-	-
29	Finland	31	-	-	-	-
30	Tunisia	29	-	-	-	-
31	Norway	24	-	-	1	1
32	Luxembourg	22	-	-	1	1
33	Kuwait	9	-	-	-	-
34	Italy	0	-	-	-	-

The members, scores, and leaders of the Canadian and U.S.A. teams were respectively as follows:

M. Bradley	15	Leon Bowden, University of Victoria
F. D'Ippolito	12	Edward Barbeau, University of Toronto
M. Molloy	16	
M. Piotte	21 (3rd prize)	
T. Vo Minh	15	
L. Yen	4	
D. Davidson	24 (3rd prize)	M.S. Klamkin, University of Alberta
D. Grabiner	35 (2nd prize)	Andy Liu, University of Alberta
J. Kahn	30 (2nd prize)	
D. Moews	42 (1st prize)	
S. Newman	30 (2nd prize)	
M. Reid	34 (2nd prize)	

The problems of this year's competition are given below. Solutions to these problems, along with those of the 13th U.S.A. Mathematical Olympiad, will appear in a booklet, *Olympiads for 1984*, obtainable (for a small charge) from

Dr. W.E. Mientka, Executive Director
M.A.A. Committee on H.S. Contests
917 Oldfather Hall
University of Nebraska
Lincoln, Nebraska 68588

The 1985 and 1986 I.M.O.s are expected to be held in Finland and Poland, respectively.

25th INTERNATIONAL MATHEMATICAL OLYMPIAD

First day, 4 July 1984. Time: 4½ hours

1. Prove that

$$0 \le yz + zx + xy - 2xyz \le \frac{7}{27},$$

where x,y,z are nonnegative real numbers for which $x+y+z = 1$.

2. Find one pair of positive integers a,b such that:
(1) $ab(a+b)$ is not divisible by 7,
(2) $(a+b)^7 - a^7 - b^7$ is divisible by 7^7.
Justify your answer.

3. In the plane two different points O and A are given. For each point X of the plane, other than O, denote by $\alpha(X)$ the measure of the angle between OA and OX in radians, counterclockwise from OA ($0 \le \alpha(X) < 2\pi$). Let $C(X)$ be the circle with centre O and radius of length OX + $\alpha(X)$/OX. Each point of the plane is colored by one of a finite number of colors. Prove that there exists a point Y for which $\alpha(Y) > 0$ such that its color appears on the circumference of the circle $C(Y)$.

Second day, 5 July 1984. Time: 4½ hours.

4. Let ABCD be a convex quadrilateral such that the line CD is tangent to the circle on AB as diameter. Prove that the line AB is tangent to the circle on CD as diameter if and only if the lines BC and AD are parallel.

5. Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices ($n > 3$), and let p be its perimeter. Prove that

$$n - 3 < \frac{2d}{p} < \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] - 2,$$

where $[x]$ denotes the greatest integer not exceeding x.

6. Let a, b, c, d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if

$$a + d = 2^k, \quad b + c = 2^m$$

for some integers k and m, then $a = 1$.

*

Since I continually get requests for detailed information regarding the operation of the I.M.O., and also since it will provide background information for my further comments on the actual technical running of the I.M.O., I now give the complete set of instructions which were sent by Czechoslovakia to each of the invited countries. Except for dates, these instructions are essentially the same from year to year.

25th International Mathematical Olympiad

Regulations

I. General Provisions

1. The 25th International Mathematical Olympiad (hereinafter I.M.O.) is an international contest for secondary school students in the solution of mathematical problems. It will consist of the contest proper and of related events concerning the problem of detecting and developing pupils with a gift for mathematics.

2. The 25th I.M.O. will be held in the Czechoslovak Socialist Republic from June 29 to July 10, 1984.

3. The related events will be prepared by the I.M.O. organizers. They will include a Symposium and an Exhibition related to the problem area of work with pupils with a talent for mathematics.

4. The contest consists of two papers which will be written in the morning of July 4 and of July 5, 1984. In each of these papers the participants solve three problems in 4-4½ hours of working time.

5. Each country which accepts the invitation to participate in the 25th I.M.O. may send a delegation consisting of the leader of the delegation, his deputy and six contestants.

6. The I.M.O. is open only to secondary school students or to pupils of schools on an equal level, born after July 5, 1964.

7. The Head (leader) of the delegation and his deputy should be mathematicians or mathematics teachers. They must be able to express themselves exactly and clearly on mathematical and technical aspects of the contest in at least one of the official languages of the I.M.O., i.e., in English, French, German, or Russian.

8. Each country which accepts the invitation to participate in the 25th I.M.O. is requested to send to the organizers 3 to 5 problem proposals with solutions, formulated in one of the official languages of the Olympiad, by April 15. The problems should come from various areas of mathematics, such as are included in math curricula at secondary schools. The solution of these problems should, however, require exceptional mathematical ability and excellent mathematical knowledge on the part of the contestants. It is assumed that the problems will be original and not yet published anywhere. Czechoslovakia, as the organizing country, does not submit any problems for the contest.

9. The Symposium will discuss "Forms and methods of identification and further specialist guidance of pupils with mathematical talent, and pupils' mathematical contests". The Symposium will be attended by the leaders of the delegations and by selected Czechoslovak specialists. The participants are kindly requested to send in their papers and communications in writing to the organizers, in any one of the official languages of the Olympiad, by May 15.

10. The content of the Exhibition will be information on pupils' contests in the solution of mathematical problems (mathematical olympiads, etc.) organized in the countries taking part in the 25th I.M.O., examples of contest problems, and publications intended for the participants in these contests and aimed at widening their mathematical knowledge and developing their talent. Countries taking part in the 25th I.M.O. are requested to send their contributions to the Exhibition to the organizers by May 15.

11. The 25th I.M.O. is managed by the Organizing Committee. The pupils' contest itself is directed by the International Jury of the contest. The Symposium and the Exhibition are organized by the respective commissions of the Organizing Committee of the 25th I.M.O.

12. All expenses related to the stay of the delegations of the participating countries in accordance with the Programme of the 25th I.M.O. will be covered by the organizing country. The participating countries will cover the costs of travel of the members of their delegations to Prague and back, and the costs of their possible stay in Czechoslovakia before the set day of arrival and after the set day of departure. The organizing country does not cover the costs related to the stay of any other persons.

II. The International Jury of the 25th I.M.O. and its tasks.

1. The International Jury of the 25th I.M.O. consists of a Chairman who is appointed from the ranks of Czechoslovak specialists by the Ministry of Education of the Czech Socialist Republic in agreement with the Ministry of Education of the Slovak Socialist Republic, and of the leaders of the delegations of the countries participating in the Olympiad. All abovesaid persons have a vote on the Jury's deliberations. Taking part in the work of the Jury is the Deputy Chairman who, in case of the absence of the Chairman, presides over the Jury and takes the vote in his place. After the problems of the second day have been set to the contestants, the members of the Jury are joined in their work by the Deputy Heads of their dele-

gations who only have the right to vote in case of the absence of the leader of their delegation. Also partaking of the work of the Jury, without a vote, is the Chairman of the Problem Selection Committee of the Organizing Committee of the 25th I.M.O., the Main Coordinators (see Article III.7), and possibly other specialists invited by the Chairman in case the situation requires their presence.

2. The Jury may only take decisions on questions that are related to the preparation and realization of the contest proper and to the evaluation of its results in accordance with the provisions of these Regulations. The quorum of the Jury is the presence of at least 50% of the members with the right of vote. The decisions of the Jury are adopted by simple majority. In case of a parity of votes, the vote is decided by the vote of the Chairman.

3. The Jury deliberations are held in the official languages of the Olympiad. In case of necessity, interpreters will attend the meeting without the right of vote.

4. The Jury will start its work at the set date well in advance of the start of the contest in such a manner as to be able to carry out the following preparatory work:

(a) From the preliminary broader selection of problems prepared by the Problem Selection Committee, select 6 problems for the contest.

(b) Determine the sequence of the problems for the contest and their division under the two days of the event.

(c) With regard to the difficulty of the contest problems, determine the duration of the working time and the number of points to be gained for the complete solution of the individual contest problems in such a manner as to make the sum of the points for the complete solution of all 6 problems a total of 40 points.

(d) Prepare and approve the formulation of the texts of the selected contest problems for the 25th I.M.O. in all official languages of the Olympiad.

(e) Decide on possible objections to the translations of the contest problems made by the leaders of the delegations into the working languages of the pupils. The responsibility for the correctness of the translation rests with the delegation leader.

(f) Determine the layout of the sheets with the texts of the contest problems to be distributed to the contestants.

5. Before the start of the contest and during the contest, the Jury will fulfill the following tasks:

(a) On the arrival of the contestants, verify that they satisfy the conditions set for participation under Article I.6. The Jury is entitled to exclude from the contest any pupil who does not meet these conditions.

(b) Decide on possible objections as to the regularity of the contest.

(c) In the course of the contest, decide in each individual case on the answer to the written queries of the contestants related to the text of the contest problems (see Article III.6).

6. Concerning the marking of the solutions to the contest problems, the Jury fulfills the following tasks:

(a) Prior to the coordination of the marking of the solutions, meet with the coordinators to determine the principles for the marking of the solutions of the individual contest problems.

(b) Decide who of the members of the Jury will coordinate the marking of the solutions presented by the Czechoslovak contestants (see Article III.8).

(c) Decide on the marking of the solution of the problems in those cases where agreement could not be reached between the coordinators and the delegation leader.

(d) Agree on the final approval of the marking of the solutions and of the results of the contest.

7. At its final meeting, determine the number of points needed to win the awarded prizes, and decide on the possible awarding of special awards for outstanding original solutions to the individual contest problems.

8. The Chairman of the Jury has to direct the deliberations of the Jury so that the Jury takes an unambiguous decision on every question related to the contest. In case it is necessary to take a vote on any question, the Chairman must make sure, prior to the vote, that the subject of the vote is clear to all members of the Jury.

III. Contest problems, written work, and marking of solutions.

1. From the draft problems submitted by the participating countries before the given deadline (see Article I.8), the Problem Selection Committee will prepare two variants of 6 contest problems each, and 8 substitute problems. Their texts in the official languages of the Olympiad will be handed over to the members of the Jury by the Chairman of the Jury together with the Programme of the 25th I.M.O. on the first day of the Olympiad (June 29).

2. The Chairman of the Jury directs the deliberations of the Jury on the selection of the contest problems in such a manner as to ensure, if possible, agreement of all members of the Jury.

3. The translation of the texts of contest problems from the texts officially approved by the Jury into the working languages of the pupils taking part in the contest is made by the leader of the respective delegation; he is also responsible for the preparation of an adequate number of copies of the texts for the pupils.

4. All persons who know the contest problems are obliged to keep them secret up to the termination of the respective part of the contest.

5. Each contestant writes the solution of the contest problems in his language on a sheet of paper provided by the Organizing Committee of the 25th I.M.O. All other aids—writing utensils, rulers, compasses—the contestants bring with them. The use of other aids outside of those given above is prohibited.

6. The leaders of delegations may answer pupils' queries related to the texts of the contest problems only with the approval of the Jury. The question should be submitted in writing within 30 minutes after the start of the session and the answer will be given to the contestant in writing.

7. The first evaluation of the solutions of the contest problems will be made by the leaders of delegations and their deputies. The coordination of the marking of the solutions of the individual problems is the task of groups of coordinators appointed by the Organizing Committee of the 25th I.M.O. One Main Coordinator and at least two further Coordinators are appointed for each contest problem. They are Czechoslovak mathematicians with appropriate experience in the evaluation of the solutions of mathematical problems. In each group there is, for each official language, at least one Coordinator who speaks that language.

8. The marking of the solutions submitted by the Czechoslovak participants will be coordinated by the delegation leaders appointed by the Jury (see Article II.6b) in the presence of the main coordinator for the respective problem.

9. The coordination of the marking will follow a time schedule announced by the Chairman of the Jury not later than one day before the start of the coordination.

10. At the coordination of the marking of the solutions of the problems, the leader of the respective delegation or his deputy will, at the request of the Coordinators, translate parts of the solutions or the entire solutions submitted by his contestants into one of the official languages of the Olympiad. The translation may be oral or in writing.

11. The responsibility for the objective evaluation of the problem solutions rests with the Main Coordinator for the respective problem. A protocol will be

made on the results of the coordination of the marking of each problem. The protocol will be signed by the leader of the respective delegation and the Main Coordinator for the respective problem.

IV. Awarding of prizes and conclusion of the 25th I.M.O.

1. Each participant will receive a Diploma certifying to his participation in the Olympiad.

2. The most successful contestants will be awarded 1st, 2nd, and 3rd prizes. Special awards may be presented for outstanding original solutions.

3. The total number of awarded prizes will not exceed half of the number of all contestants. The number of 1st, 2nd, and 3rd prizes awarded will if possible be in the ratio 1:2:3.

4. Proposals for special awards for outstanding original solutions of individual problems are submitted to the final meeting of the Jury by the Main Coordinators of the individual problems.

5. The prizes and special awards are not linked with the right to any financial remuneration or material award.

6. The results of the 25th I.M.O. are announced, and the prizes and special awards are handed over, at a solemn public assembly. The diplomas are handed to the participants by representatives of the Ministries of Education of the Czech Socialist Republic and of the Slovak Socialist Republic and by the Chairman of the Jury.

V. Written materials of the 25th I.M.O.

Prior to their departure from the venue of the Olympiad, the leaders of delegations and their deputies will receive the draft of selected contest problems, the programme (see Article III.1), the results of the contest as approved by the Jury, a list of contestants who have won prizes and awards at the 25th I.M.O., and the drafts of all problems and their solutions sent in by the participating countries to the organizers of the contest within the set deadline (see Article I.8).

*

In the socialist countries, the Mathematical Olympiads, as well as the Physics and Chemistry Olympiads, are well supported by their respective Ministries of Education. As an indication of this support, I now give the English translation of the speech given by the First Deputy Minister of Education of the ČSR, Professor Ing. Václav Císař, CSc., at the opening of the 25th I.M.O.

Dear guests, dear young friends,

I greet you all most cordially here in the capital of Socialist Czechoslovakia, in the historical premises of the venerable Carolinum where we have met at this festive opening of the 25th International Mathematical Olympiad. We are pleased that this jubilee event of the oldest of international pupils' contests is taking place in our country which has thus been offering, for the third time already, a hospitable reception to young secondary school students—mathematicians from various countries of all continents. I sincerely welcome all the delegations who are present here; I especially welcome the delegations of Norway and Cyprus who are taking part in the Olympiad for the first time.

All of us who are responsible for the education of the young generation for a creative life in—as we may say without exaggerating—the third millennium, are fully aware of the fact that mathematics is one of the effective means for forming the intellectual profile of Man and we highly appreciate its educational function. Mathematical education induces young people to precision, perseverance, purposefulness, and makes them acquire a number of other qualities necessarily needed for a rich, active life and for creative work. Mathematics is the basis of natural and technical sciences and by means of an ever-growing use of automatic computers it penetrates more and more into chemical, medical, economic and social sciences as well. It is but natural, therefore, that the importance given to it in all the countries who try to make the scientific and technical development more rapid and better has been growing. These countries include the Czechoslovak Socialist Republic where our people, under the guidance of the Communist Party of Czechoslovakia, are building a developed Socialist society in which industrial production requires an all-round development of each individual and, at the same time, the rising of the intellectual and scientific potential especially of the younger generation.

Eight years ago, with the backing of the top Party and State organs, we started gradually to put into effect a project of a further development of the Czechoslovak educational system the core of which is the reorganization of the content of education at all levels of the school system. One of the characteristic features of this reorganization is exactly the extraordinary stress put on mathematics at all levels of our schools.

An effective means for the development of our young people's gifts for mathematics is also the Mathematical Olympiad which has grown out of the rich tradition of mathematical contests of secondary school students since the seventies of last century. This year the 33rd Mathematical Olympiad took place, with over 6,000 secondary school students and nearly 25,000 pupils of the top grades of primary schools participating. Tens and hundreds of successful contestants of these Olympiads have become eminent scientists and successful creative workers, not only in mathematics, physics, or in technical fields, but also in medicine and in other nontechnical fields.

For you, dear young friends, mathematics and the successes you achieve in it are also becoming an important means how to make new friendship ties many of which will last the whole of your lives, for they are bound by a noble interest in a branch of science which for many of you will probably become a decisive part of your life and your vocation. Mathematics also gives you the opportunity to see foreign countries and meet their inhabitants. We are very pleased that you will spend a few days in our Socialist Czechoslovakia, in a country of rich cultural traditions, in our capital, Prague, where more than six hundred years ago the first university in Central Europe, Charles University, was founded. Many distinguished scientists who contributed considerably to the enrichment of human knowledge worked there as teachers. This country also gave birth to the great teacher of nations John Amos Comenius, prominent scholars of world fame worked here, such as Johannes Kepler, Tycho de Brahe, Bernard Bolzano and others.

Although you have come primarily to compete in solving exacting mathematical problems, I firmly believe you will have plenty of opportunity also to see the sights of our capital and to get acquainted with the rich life of its inhabitants. I am sure you will see that the inhabitants of this country are anxious for peace to be preserved as the basic precondition of life on Earth. In you, too, we welcome the messengers of Peace and international co-operation. I am sure this International Mathematical Olympiad will also contribute to the understanding among nations and stress the importance of the fight for preserving and consolidating world peace.

To you, dear friends, I wish much success in the forthcoming contest. I wish the jubilee 25th International Mathematical Olympiad much sucess, the best possible results in the solution of the contest problems to all the contestants, and the best of impressions from their stay in our country to all foreign participants—the contestants, the delegation heads as well as all other guests.

And now, dear friends, allow me to fulfil my pleasant duty. I state that to all intents and purposes the contest has been prepared. I take due notice that the contest problems have been prepared and that the organization of the contest has been ensured.

It is with great pleasure, therefore, that in accordance with the set rules I now pass the running of the contest into the hands of the International Jury present here. I have no doubt that its members will use all their expert and pedagogical knowledge and experience to ensure the contest to be stimulating and fair and to proceed in a dignified way. I wish them much success in this important work and herewith declare the contest opened.

*

My first I.M.O. should have been the one held at Erfurt, East Germany in 1974. At the time, I was employed by the Scientific Laboratories of the Ford Motor Co. as a Principal Research Scientist. Also, I had been recently transferred from the Physics Department to one of the engineering departments. As a member of the Physics Department, I would have been allowed to participate in the U.S.A. training session and then in the I.M.O., a period of approximately five weeks, without any problems. However, in my new department, I was informed that I could participate only if I used up my three weeks of vacation time and got a leave of absence without pay for the remaining time. I declined to go under these conditions. This led to my leaving the company four months later to join the Applied Mathematics Department of the University of Waterloo. It turned out that I was very glad to return to academia.

In 1975 the I.M.O. was held in Burgas, Bulgaria. This is the one that holds the greatest nostalgia for me because of many incidents that occurred, mostly good but some not so good. For a short amusing description of this Olympiad, see the article of J.H. Durran, the British deputy leader, in *Mathematical Spectrum*, 8 (1975-76) 37-39. Sam Greitzer and I trained the 8-man U.S.A. team plus 16 other non-senior students at Rutgers University just prior to the competition. The team members at this session were very good mathematically, were hard working and hard playing, a credit to their families and to their country. This has also been the case for all subsequent teams through 1981. Since then, and particularly in 1982 and 1984, we have hada few "ugly Americans" representing the U.S.A. abroad. I now would like to see some mechanism put in place for weeding out such students from the I.M.O. team, even if it turns out that these are the best students mathematically.

At Burgas, both the leader and the deputy leader of each team participated in the problem selection, etc. Since there were some 34 of us, this led to lots of interaction and noise. Nevertheless it led to a fairly good selection of problems except for No. 1. This problem followed immediately from an elementary rearrangement inequality. Although a number of us tried to eliminate this problem, we did not succeed. Now, because of the large number of participating countries, only the team leaders take part in the problem selection. This has led to a neglect in the treatment of the deputy leaders. This year they did not even see the first set of problems until the students finished the first part and asked questions about them. The deputy leaders are housed with the students and kept separate from the leaders until after the start of the second part of the test. In the future, the deputy leaders should receive copies of the first set of problems with their solutions after the students have started to write the first part, and similarly for the second set. In this way, they will be prepared to discuss the problems with the students and not feel left out.

A great deal of the responsibility for an I.M.O. falls on the shoulders of the Chairman of the Jury. This is a very difficult job to do well. In my view, the extremes in the Chairmen's attitudes took place in Austria in 1976, where he more or less took a dictatorial stance, and in Czechoslovakia in 1984, where he bent over backward in attempting to appease members of the Jury.

In the last few years there have been many discussions about the cutoff scores for awarding 1st, 2nd, and 3rd place honors. The longest such discussion ever was the one in Paris in 1983. The shortest one was in Austria in 1976, where we were presented with a *fait accompli*. The Chairman simply asserted that the requisite number of prizes had already been obtained. This year in Prague, I was informed it had taken approximately two hours of discussion before coming to an agreement. Feeling very tired, I had left (for the first time) this part of the Jury deliberation at 10:30 p.m. (the session had started at 5 p.m. with a short break for dinner in the same building). Also, I knew that there would be considerable pressure to have low cutoff scores so that quite a number of leaders could return to their countries with honors of some kind for their students. Personally, I feel that the awards have been cheapened by giving out so many because of cutoff scores as low as 17. This year 98 out of 192 students received honors. The breakdown is as follows:

1st prizes: 14 (score $\geq$ 40),
2nd prizes: 35 (39 $\geq$ score $\geq$ 26),
3rd prizes: 49 (25 $\geq$ score $\geq$ 17).

For countries whose scores were appreciable but not near the top, having newer countries with weak Olympiad experience increases their chances of obtaining more prizes. Something should be done about the number of prizes awarded. One possibility is not to use such a low bottom cutoff score. Another possibility is to have A and B groups, as in World Football. However, before considering any changes, there should be discussion and then agreement on the basic philosophy of the I.M.O., as mentioned previously.

I brought up a matter of disturbances on the second day of the competition caused by the intrusion of television cameras and lights in some of the examination rooms for a period of approximately 20-30 minutes. This occurred in at least five rooms and affected five U.S.A. team members. One of these students, nominally our best one, was highly disturbed by the unwarranted and unexpected interference. Although an apology was given, I trust that such intrusions in the examination rooms will never happen again.

I now turn to the awarding of special prizes for elegant solutions and/or non-trivial generalizations. I strongly feel that the procedure used this year and previously is not adequate for such decisions, especially if one is serious about these prizes. I have voted very sparingly for these prizes in the past. I cannot say the same about some of my colleagues. No doubt this is again due to a difference in basic philosophy concerning prizes.

During the grading process, the coordinators are always on the lookout for solutions to be considered for special prizes. These are then reviewed by the Chief Coordinator who usually eliminates some of them. At the Jury meeting, I was informed by the Chief Coordinator that one of the two solutions of my students being considered was eliminated but was replaced by another one. Each leader goes over these solutions in the presence of the Jury. I thought that the first one, a solution to No. 2 by student Jeremy Kahn, was a reasonable candidate. (It will appear in the booklet *Olympiads for 1984* mentioned earlier.) It turned out that a Russian student also had an equivalent solution. In the light of that and the ensuing discussion of the Jury, I decided to vote against awarding it a special prize. The vote was against it even without my vote. I declined to even discuss the other candidate's solution since it was quite long, 4 pages. The official solution was very much better and much shorter. I strongly felt it was a waste of time to discuss it and was sure the Jury would correctly vote against it. However, this led to one of the few instances where the Chairman was forceful and determined. He demanded that the solution be considered. I handed over the solution which was projected onto a screen producing a very hard to read image. The Chief Coordinator started going over the solution and after a very short time gave up on it, to the relief of the Jury. The one solution which was awarded a special prize was a Russian one for No. 5. While it is true the student first proved that it suffices to consider all the points on a line, his resulting calculations were no simpler than if he had not used this lemma. His solution also took 4 pages. I personally did not think it deserved a special prize. If there had been a valid extension, even with a long proof, then I would have voted for a special prize. There were presumably 18 leaders voting for it, just one more than half the Jury number. This was one of the very few instances in which the negative vote and abstentions were not considered. In a crowded room with new leaders and deputy leaders, sometimes a deputy leader will inadvertently vote by raising his hand, even though he has no right to vote. I recommend strongly that in the future, if any solution is to be considered for a special prize, it should first be duplicated and given to all the leaders so that they can discuss it among themselves for a reasonable time period. Then they can vote on it with a sufficient knowledge of the solution.

The most important task of the Jury is to come up with a good challenging selection of six problems. Even though this is the first order of business for the Jury, I have left it for the end in my survey because of its importance.

Each participating country can submit up to 5 problems. For the 1984 I.M.O., 72 problems were submitted. The host country usually filters down this list to approximately 20 problems encompassing different fields of mathematics. Ostensibly, this first filtration is based on some of the problems being too well known, too easy, too hard, or in some other way inappropriate. According to Article III.1 of the Instructions, the Problem Selection Committee (of the host country) is supposed to prepare two sets of 6 contest problems each and 8 substitute problems (for a total of 20 problems). Unfortunately this was not done. Actually, the Jury received only a narrow selection of 15 problems, including 5 number theory problems and 5 from geometry or combinatorial geometry. Four problems from this list were discarded as being too well known. Two other reasonably nice Canadian problems were discarded since, due to a misunderstanding, the Canadian leader had given them to his team for practice. I objected to the resulting short narrow list of remaining problems. So the next day we received an additional list of 5 problems, one of which was discarded as being known. Two more were then discarded after the Problem Selection Committee Chairman declared they were too easy. In retrospect, I should have asked at the time why they were even included for consideration. In my view, the new geometry problem was no easier than the geometry problem No. 4 used, and the new inequality problem was more challenging than inequality problem No. 1 used. It turned out, to the chagrin of some members of the Jury including myself, that many students solved No. 1 in a routine way using multivariate calculus. Also, before No. 1 was selected, I had moved that the left inequality be deleted since it followed immediately from the well-known stronger inequality

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9.$$

Even more elementarily, it follows immediately from

$$yz, zx, xy \geq xyz.$$

Nevertheless, it was voted overwhelmingly to be kept in. Incidentally, this part of the solution was worth 2 points out of the 7 points for the problem.

After the 6 problems to be used were finally accepted, I made a comment which I still stand by. I had no objections to any one of the problems, except those noted above, but taken altogether the test was too narrow. Note the final composition of the problems: two from number theory, one from plane geometry, one from combinatorial geometry, one combinatorial geometric inequality, and one inequality. The net result was a test easier than the previous two, as evidenced by 8 perfect scores and 24 scores of at least 35 (out of 42). The 1981 I.M.O. in the U.S.A. was unfortunately too easy, with a record number of 26 perfect scores. Even quite a few students (some with perfect scores) complained that the exam was not up to the usual I.M.O. standards. For this 1981 I.M.O., there was initially a good set of problems. Unfortunately, many of the "weaker" and newer countries with enough of the older countries voted to eliminate the more challenging problems. However, I feel sure that the Jury as a whole at that time did not expect that there would be such an avalanche of perfect scores.

Although I do not think that the 1984 Problem Selection Committee did a very good job, the ultimate responsibility for the problem selection rests with the Jury. The Jury could have insisted to see more problems from the original 72-problem pool. (A recurring comment from a number of long time leaders is that the best problems of this initial pool always seem to be left out.) However, the Jury has only a limited time to do its selection. Perhaps in future I.M.O.s an extra day can be provided for this. A possible way to get extra time, and also to reduce costs, is to reduce the number of team members to 4 or 5, as was done in Hungary in 1982. This should also make it much easier for a country with a relatively small population to come up with a better team.

Finally, I hope that the I.M.O. Jury in Finland next year will address itself to some of the criticisms raised here, and in so doing maintain the very high regard in which the I.M.O. is held around the world. As for myself, it is time for a younger person to replace me in the I.M.O. and it is time for me to be doing other things, even though I shall miss the annual companionship of many of the leaders, deputy leaders, and their spouses.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before March 1, 1985, although solutions received after that date will also be considered until the time when a solution is published.

971. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Solve the piscine alphametic

$$\text{FISH} + \text{FISH} + \text{FISH} + \ldots + \text{FISH} = \text{SHOAL}.$$

This SHOAL has 73 FISH.

972.* *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

(a) Prove that two equilateral triangles of unit side cannot be placed inside a unit square without overlapping.

(b) What is the maximum number of regular tetrahedra of unit side that can be packed without overlapping inside a unit cube?

(c) Generalize to higher dimensions.

973. *Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.*

Evaluate $\lim_{n\to\infty} P_n$ if $P_1 = 4$ and

$$P_{n+1} = 2^{n+1}\sqrt{2}\sqrt{1-\sqrt{1-\left(\frac{P_n}{2^{n+1}}\right)^2}}, \qquad n = 1,2,3,\ldots\ .$$

974. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Consider the following double inequality, where A,B,C are the angles of any triangle:

$$\cos A \cos B \cos C \le 8 \sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2} \le \frac{1}{8}.$$

The inequality involving the first and third members and that involving the second and third members are both well known. Prove the inequality involving the first and second members.

975. *Proposed by Herta T. Freitag, Roanoke, Virginia.*

Parabolas $y^2 = 2px$ and $x^2 = 2qy$ are given. A triangle with vertices $P_i(x_i,y_i)$, $i = 1,2,3$, is inscribed in $y^2 = 2px$ and the lines containing two of its sides are tangent to $x^2 = 2qy$.

(a) Prove that $2p^2q = y_1y_2y_3$.

(b) Deduce from (a), or otherwise, that the line containing the third side of the triangle is also tangent to $x^2 = 2qy$.

976*. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

(a) For all possible sets of n distinct points in a plane, let $T(n)$ be the maximum number of equilateral triangles having their vertices among the n points. Evaluate $T(n)$ explicitly in terms of n, or (at least) find a good upper bound for $T(n)$.

(b) If $a_n = T(n)/n$, prove or disprove that the sequence $\{a_n\}$ is monotonically increasing.

(c) Prove or disprove that $\lim_{n\to\infty} a_n = \infty$.

977. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $f(n) = 2n^2 + 14n + 25$. It is easy to verify that $f(17) = 29^2$. Find two more positive integers n such that $f(n)$ is a perfect square.

978. *Proposed by Andy Liu, University of Alberta.*

Determine the smallest positive integer m such that

$$529^n + m \cdot 132^n$$

is divisible by 262417 for all odd positive integers n.

979. *Proposed by R.B. Killgrove, Alhambra, California.*

(a) Find a ring of smallest possible order which has nonzero products but in which all squares are zero.

(b) Characterize all rings R with the property that, for all $x,y \in R$, $x \neq y$ implies $xy = 0$.

980. *Proposed by Leon Bankoff, Los Angeles, California.*

Show that

$$\frac{\Pi \sin A}{\Sigma \sin A} + \Sigma \sin^2\frac{A}{2} = 1,$$

where the sums and product are cyclic over the angles A,B,C of a triangle.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

783. [1982: 277; 1984: 30, 90, 218] *Proposed by R.C. Lyness, Southwold, Suffolk, England.*

Let n be a fixed natural number. We are interested in finding an infinite sequence $(v_0,v_1,v_2,\ldots)$ of strictly increasing positive integers, and a finite sequence $(u_0,u_1,\ldots,u_n)$ of nonzero integers such that, for all integers $m \geq n$,

$$u_0^2v_m + u_1^2v_{m-1} + \ldots + u_n^2v_{m-n} = u_0v_m^2 + u_1v_{m-1}^2 + \ldots + u_nv_{m-n}^2. \qquad (1)$$

(a) Prove that (1) holds if

$$u_r = \text{coefficient of } x^r \text{ in } (1-x)^n$$

and

$$v_r = \text{coefficient of } x^r \text{ in } (1-x)^{-n-1}.$$

(b) Find other sequences (u_r) and (v_r) for which (1) holds.

IV. *Solution to part (b) by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

There is indeed an example different from the sequences of part (a), at least *if n is odd.* Take

$$u_r = (-1)^r, \quad r = 0,1,2,\ldots,n \qquad \text{and} \qquad v_r = r+1, \quad r = 0,1,2,\ldots .$$

Then, for all $m \geq n$,

$$\begin{aligned}\sum_{r=0}^{n} u_r^2 v_{m-r} &= (m+1) + m + (m-1) + (m-2) + \ldots + (m-n+2) + (m-n+1)\\ &= (2m+1) + \{2(m-2)+1\} + \{2(m-4)+1\} + \ldots + \{2(m-n+1)+1\}\\ &= (m+1)^2 - m^2 + (m-1)^2 - (m-2)^2 + \ldots + (m-n+2)^2 - (m-n+1)^2\\ &= \sum_{r=0}^{n} u_r v_{m-r}^2.\end{aligned}$$

Editor's comment.

Readers will note that in the above example

$$u_r = \text{coefficient of } x^r \text{ in } \frac{1 - x^{n+1}}{1 + x}, \qquad n \text{ odd}$$

and

$$v_r = \text{coefficient of } x^r \text{ in } \frac{1}{(1 - x)^2}.$$

A characterization of all sequences for which (1) holds would wrap up this problem very nicely.

* * *

844. [1983: 143] *Proposed by Peter M. Gibson, University of Alabama in Huntsville, and Michael H. Rodgers, student at the same university.*

(a) A triangle $A_0B_0C_0$ with centroid G_0 is inscribed in a circle Γ with center O. The lines A_0G_0, B_0G_0, C_0G_0 meet Γ again in A_1, B_1, C_1, respectively, and G_1 is the centroid of triangle $A_1B_1C_1$. A triangle $A_2B_2C_2$ with centroid G_2 is obtained in the same way from $A_1B_1C_1$, and the procedure is repeated indefinitely, producing triangles with centroids G_3, G_4, ...

If $g_n = OG_n$, prove that the sequence $\{g_0, g_1, g_2, \ldots\}$ is decreasing and converges to zero.

(b)* Prove or disprove that a result similar to (a) holds for a tetrahedron inscribed in a sphere, or, more generally, for an n-simplex inscribed in an n-sphere.

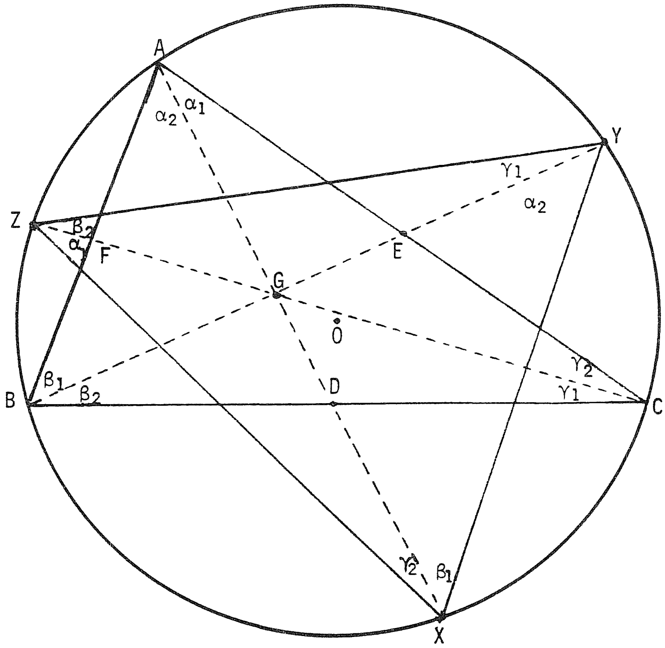
Joint solution to part (a) by R.B. Killgrove, Alhambra, California; and Dan Sokolowsky, California State University at Los Angeles.

We first show that the sequence $\{g_0, g_1, g_2, \ldots\}$ is monotonically decreasing. It suffices to show that $g_0 \geq g_1$, for then, by the same argument, $g_1 \geq g_2$, $g_2 \geq g_3$, ... We will use the well-known facts [1] that

$$OH_n = 3OG_n \qquad \text{and} \qquad OH_n^2 = 9R^2 - a_n^2 - b_n^2 - c_n^2, \tag{1}$$

where R is the radius of Γ, and H_n, a_n, b_n, c_n are the orthocenter and sides of triangle $A_nB_nC_n$. But for typographical convenience we denote triangle $A_0B_0C_0$ by ABC and its sides by a, b, c, and triangle $A_1B_1C_1$ by XYZ and its sides by x, y, z. It then follows from (1) that $g_0 \geq g_1$ is equivalent to

$$x^2 + y^2 + z^2 \geq a^2 + b^2 + c^2. \tag{2}$$



We find an expression for x in terms of sides a, b, c and medians d = AD, e = BE, f = CF (see figure). From

$$x = 2R\sin X = 2R\sin(\beta_1+\gamma_2) = 2R(\sin\beta_1\cos\gamma_2 + \cos\beta_1\sin\gamma_2)$$

and

$$\sin\beta_1 = \frac{b\sin A}{2e}, \ \sin\gamma_2 = \frac{c\sin A}{2f}, \ \cos\beta_1 = \frac{c^2+e^2-b^2/4}{2ce}, \ \cos\gamma_2 = \frac{b^2+f^2-c^2/4}{2bf},$$

we obtain (bearing in mind that $2R\sin A = a$)

$$x = \frac{a}{16ef}(3b^2 + 3c^2 + 4e^2 + 4f^2).$$

From the known relations

$$4d^2 = 2b^2+2c^2-a^2, \qquad 4e^2 = 2c^2+2a^2-b^2, \qquad 4f^2 = 2a^2+2b^2-c^2, \tag{3}$$

we now get

$$x = \frac{a(a^2+b^2+c^2)}{4ef}.$$

From this and two similar relations for y and z, we obtain

$$x^2 + y^2 + z^2 = \frac{(a^2+b^2+c^2)^2}{16d^2e^2f^2}(a^2d^2+b^2e^2+c^2f^2),$$

and (2) is equivalent to

$$(a^2+b^2+c^2)(a^2d^2+b^2e^2+c^2f^2) \geq 16d^2e^2f^2,$$

or, using (3) again, to

$$a^6 + b^6 + c^6 - \{b^2c^2(b^2+c^2) + c^2a^2(c^2+a^2) + a^2b^2(a^2+b^2)\} + 3a^2b^2c^2 \geq 0. \tag{4}$$

If we set $k^2 = c^2/(a^2+b^2)$, so $c^2 = k^2(a^2+b^2)$, then (4) is found to be equivalent to

$$k^2(2k^2-1)^2a^2b^2(a^2+b^2) + (k^2-1)^2(k^2+1)(a^2+b^2)(a^2-b^2)^2 \geq 0, \tag{5}$$

and this is clearly true. Equality holds in (5) if and only if $k^2 = \frac{1}{2}$ and $a^2 = b^2$, hence if and only if $a^2 = b^2 = c^2$ and triangle ABC is equilateral. This completes the proof that the sequence $\{g_0, g_1, g_2, \ldots\}$ is monotonically decreasing.

The sequence is also bounded below by zero, so it converges. To show that it converges to zero, it suffices to note that [2]

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \frac{\pi}{3}.$$

REFERENCES

1. H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, New Mathematical Library, No. 19, Mathematical Association of America, 1967, pp. 19-20.

2. Problem 913 (proposed by J. Garfunkel), *Mathematics Magazine*, 48 (1975) 246-247.

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845. [1983: 143] *Proposed by B.M. Saler, Agincourt, Ontario.*

Let r_1, r_2, r_3 be the focal radii (all from the same focus F) of the points P_1, P_2, P_3, respectively, on the ellipse $b^2x^2+a^2y^2 = a^2b^2$. A circle with centre F and radius $r = \sqrt[3]{r_1r_2r_3}$ intersects the focal radii r_1, r_2, r_3 in P_1', P_2', P_3', respectively. Find the ratio of the areas of triangles $P_1P_2P_3$ and $P_1'P_2'P_3'$.

(This is Theorema Elegantissimum from *Acta Eruditorum*, A.D. 1771, page 131, by an unknown author.)

Solution by G.P. Henderson, Campbellcroft, Ontario.

We can take F to be $(ae,0)$ where e is the eccentricity. We use polar coordinates (ρ,θ) with the pole at F and the initial ray along the positive x-axis. The polar equation of the ellipse is then

$$\rho = \frac{s}{1+e\cos\theta}, \tag{1}$$

where $s = b^2/a$ is the semi-latus rectum. Let $P_i = (\rho_i,\theta_i)$, where $\rho_i = r_i$ and $0 \le \theta_i < 2\pi$, $i = 1,2,3$. With square brackets denotina *signed* area, we have

$$\begin{aligned}
A &\equiv [P_1P_2P_3]\\
&= [FP_2P_3] + [FP_3P_1] + [FP_1P_2]\\
&= \frac{1}{2}\{r_2r_3\sin(\theta_3-\theta_2) + r_3r_1\sin(\theta_1-\theta_3) + r_1r_2\sin(\theta_2-\theta_1)\}\\
&= \frac{r^3}{2}\{\frac{\sin(\theta_3-\theta_2)}{r_1} + \frac{\sin(\theta_1-\theta_3)}{r_2} + \frac{\sin(\theta_2-\theta_1)}{r_3}\}\\
&= \frac{r^3}{2s}\{(1+e\cos\theta_1)\sin(\theta_3-\theta_2) + (1+e\cos\theta_2)\sin(\theta_1-\theta_3) + (1+e\cos\theta_3)\sin(\theta_2-\theta_1)\}.
\end{aligned}$$

Since

$$\cos\theta_1\sin(\theta_3-\theta_2) + \cos\theta_2\sin(\theta_1-\theta_3) + \cos\theta_3\sin(\theta_2-\theta_1) = 0,$$

we therefore have

$$A = \frac{r^3}{2s}\{\sin(\theta_3-\theta_2) + \sin(\theta_1-\theta_3) + \sin(\theta_2-\theta_1)\}.$$

Now

$$A' \equiv [P_1'P_2'P_3'] = \frac{r^2}{2}\{\sin(\theta_3-\theta_2) + \sin(\theta_1-\theta_3) + \sin(\theta_2-\theta_1)\},$$

so the required ratio is

$$\frac{A}{A'} = \frac{r}{s} = \frac{ra}{b^2}.$$

Also solved by the COPS of Ottawa; HOWARD EVES, University of Maine; M.S. KLAMKIN, University of Alberta; LEROY F. MEYERS, The Ohio State University; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and JORDAN B. TABOV, Sofia, Bulgaria.

Editor's comment.

Every nondegenerate conic has an equation of the form (1). So it follows from the above solution that the ratio $A/A' = r/s$ holds for every nondegenerate conic. This fact was brought out in the solution of the Cops of Ottawa.

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846. [1983: 143] *Proposed by Jack Garfunkel, Flushing, N.Y. and George Tsintsifas, Thessaloniki, Greece.*

Given is a triangle ABC with sides a,b,c and medians m_a, m_b, m_c in the usual order, circumradius R, and inradius r. Prove that

(a)
$$\frac{m_a m_b m_c}{m_a^2+m_b^2+m_c^2} \geq r;$$

(b)
$$12Rm_a m_b m_c \geq a(b+c)m_a^2 + b(c+a)m_b^2 + c(a+b)m_c^2;$$

(c)
$$4R(am_a + bm_b + cm_a) \geq bc(b+c) + ca(c+a) + ab(a+b);$$

(d)
$$2R\left(\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}\right) \geq \frac{m_a}{m_b m_c} + \frac{m_b}{m_c m_a} + \frac{m_c}{m_a m_b}.$$

Solution by M.S. Klamkin, University of Alberta.

Let
$$I(a,\ b,\ c,\ m_a,\ m_b,\ m_c,\ F,\ R,\ r) \geq 0 \tag{1}$$

be any inequality involving the stated elements of triangle ABC, where F is the area of the triangle. It is shown in [1] that (1) is equivalent to the following inequality, called the *median dual* of (1):

$$I\left(m_a,\ m_b,\ m_c,\ \tfrac{3}{4}a,\ \tfrac{3}{4}b,\ \tfrac{3}{4}c,\ \tfrac{3}{4}F,\ \frac{m_a m_b m_c}{3F},\ \frac{3F}{2(m_a+m_b+m_c)}\right) \geq 0.$$

(a) The median dual of (a) is

$$\frac{abc}{a^2+b^2+c^2} \geq \frac{2F}{m_a+m_b+m_c},$$

or, equivalently,

$$2R(m_a + m_b + m_c) \geq a^2 + b^2 + c^2,$$

and this inequality was established in the solution to Crux 733 [1983: 121].

(b) We start from

$$4Rm_a \geq b^2 + c^2, \tag{2}$$

also established in the solution to Crux 733 [1983: 122]. Its median dual is $am_a m_b m_c/F \geq m_b^2 + m_c^2$, or, equivalently,

$$4Rm_a m_b m_c \geq bc(m_b^2 + m_c^2).$$

From this and two similar relations, we obtain

$$\begin{aligned} 12Rm_a m_b m_c &\geq bc(m_b^2+m_c^2) + ca(m_c^2+m_a^2) + ab(m_a^2+m_b^2) \\ &= a(b+c)m_a^2 + b(c+a)m_b^2 + c(a+b)m_c^2. \end{aligned}$$

(c) From (2) and two similar relations,

$$4R(am_a + bm_b + cm_c) \geq a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2)$$

$$= bc(b+c) + ca(c+a) + ab(a+b).$$

(d) This inequality is equivalent to

$$\frac{m_a m_b m_c}{m_a^2+m_b^2+m_c^2} \geq \frac{abc}{2R(a+b+c)} = r,$$

that is, equivalent to (a).

Also solved by J.T. GROENMAN, Arnhem, The Netherlands (parts (c) and (d) only); V.N. MURTY, Pennsylvania State University, Capitol Campus; and the proposers.

REFERENCE

1. Aufgabe 677, *Elemente der Mathematik*, 28 (1973) 129-130.

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847. [1983: 143] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Prove that

$$\sum_{j=0}^{n} \frac{\binom{n}{2j-n-1}}{5^j} = \tfrac{1}{2}(0.4)^n F_n,$$

where F_n is the nth Fibonacci number. (Here we make the usual assumption that $\binom{a}{b} = 0$ if $b < 0$ or $b > a$.)

Solution by M.S. Klamkin, University of Alberta.

Letting $j = n-k$, the proposed equality is equivalent to

$$\frac{1}{2^{n-1}} \sum_{k=0}^{n} \binom{n}{2k+1} 5^k = F_n,$$

and the truth of this follows immediately by applying the binomial theorem to the Binet formula

$$F_n = \frac{1}{\sqrt{5}}\{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n\}.$$

Also solved by CURTIS COOPER, Central Missouri State University at Warrensburg; UNDERWOOD DUDLEY, DePauw University; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; BOB PRIELIPP, University of Wisconsin-Oshkosh; DAVID STONE, Georgia Southern College; and the proposer.

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848. [1983: 144] *Proposed by Charles W. Trigg, San Diego, California.*

Tetrahedral numbers have the form $T(n) = n(n+1)(n+2)/6$ and triangular numbers have the form $n(n+1)/2$. The third tetrahedral number, 10, is also the fourth triangular number. Show that at least one-third of the tetrahedral numbers are also polygonal numbers.

Solution by David Stone, Georgia Southern College, Statesboro, Georgia.

We show that *more than* one-third of the tetrahedral numbers are also polygonal numbers. Let $P(n,m)$ denote the nth m-gonal number. According to Hypsicles (*ca.* 175 B.C.) [1],

$$P(n,m) = \frac{n}{2}\{2 + (n-1)(m-2)\}. \qquad (1)$$

Thus $1 = P(1,m)$ for all m and $n = P(n,2)$ for all n. Hence every positive integer, and in particular *every tetrahedral number*, is a polygonal number.

If we are restricted to "nondegenerate" polygonal numbers ($m > 2$), as the proposer probably intended, that *at least one-third* of the tetrahedral numbers are also polygonal numbers follows from the relation

$$T(3k-1) = P(3k-1,\ k+3), \qquad k = 1,2,3,\ldots,$$

which is easily verified from (1) and the defining relation for $T(n)$. Since, as noted by the proposer, also

$$T(3) = 10 = P(4,3),$$

we conclude that *more than* one-third of the tetrahedral numbers are also polygonal numbers.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; BOB PRIELIPP, University of Wisconsin-Oshkosh; MALCOLM A. SMITH, Georgia Southern College; and the proposer. Comments were received from FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; and LEROY F. MEYERS, The Ohio State University.

REFERENCE

1. Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1952, Vol. II, p. 1.

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849. [1983. 144] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

The functions defined by

$$f(x) = \frac{Ax^7}{7} + \frac{Bx^5}{5} + \frac{19}{35}x$$

and

$$g(x) = \frac{Bx^7}{7} + \frac{Ax^5}{5} - \frac{1}{35}x,$$

where A and B are primes, have integral values for each integer x. Find the smallest possible values of A and B.

Solution by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.

We find the smallest possible nonnegative integral (not necessarily prime) satisfactory values of A and B. Letting $x = 1$ shows that we must have

$$5A + 7B \equiv 16 \pmod{35} \tag{1}$$

and

$$5B + 7A \equiv 1 \pmod{35}. \tag{2}$$

Adding (1) and (2) and multiplying by 3 give

$$A + B \equiv 16 \pmod{35}; \tag{3}$$

and subtracting (1) from (2) and multiplying by 18 give

$$A - B \equiv 10 \pmod{35}. \tag{4}$$

Finally, from (3) and (4), the smallest possible nonnegative values are $A = 13$ and $B = 3$, and these just *happen* to be both primes.

Conversely, for these values of A and B,

$$f(x) = 13\cdot\frac{(x^7-x)}{7} + 3\cdot\frac{(x^5-x)}{5} + 3x,$$

$$g(x) = 3\cdot\frac{(x^7-x)}{7} + 13\cdot\frac{(x^5-x)}{5} + 3x;$$

and these, by Fermat's theorem, both have integral values for each integer x.

The method is easily generalized to primes other than 5 and 7.

Also solved by SAM BAETHGE, San Antonio, Texas; CURTIS COOPER, Central Missouri State University at Warrensburg; the COPS of Ottawa; UNDERWOOD DUDLEY, DePauw University; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; EDWIN M. KLEIN, University of Wisconsin at Whitewater; LEROY F. MEYERS, The Ohio State University; GLEN E. MILLS, Pensacola Junior College, Florida; D.J. SMEENK, Zaltbommel, The Netherlands; MALCOLM A. SMITH, Georgia Southern College; DAVID STONE, Georgia Southern College; JORDAN B. TABOV, Sofia, Bulgaria; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

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850. [1983: 144] *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

Let $x = r/R$ and $y = s/R$, where r,R,s are the inradius, circumradius, and semiperimeter, respectively, of a triangle with side lengths a,b,c. Prove that

$$y \geq \sqrt{x}(\sqrt{6} + \sqrt{2-x}),$$

with equality if and only if $a = b = c$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

The following inequalities are known to hold for every triangle, with equality in each case just when the triangle is equilateral [1]:

$$2r \le R \qquad \text{and} \qquad r(16R - 5r) \le s^2.$$

In terms of x and y, these are equivalent to

$$0 < x \le \tfrac{1}{2} \qquad \text{and} \qquad y \ge \sqrt{x}\sqrt{16-5x},$$

with equality in each case just when $x = \frac{1}{2}$. To complete our solution, it therefore suffices to establish that

$$\sqrt{16 - 5x} \ge \sqrt{6} + \sqrt{2 - x},$$

with equality just when $x = \frac{1}{2}$. We first square both sides and reduce to obtain the equivalent

$$2(2 - x) \ge \sqrt{6}\sqrt{2 - x}.$$

We now divide both sides by $\sqrt{2-x} > 0$ and then square both sides again to obtain the equivalent, and true, inequality $x \le \frac{1}{2}$.

Also solved by CURTIS COOPER, Central Missouri State University at Warrensburg; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

REFERENCE

1. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969, pp. 48, 50.

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UNVANQUISHED PROBLEMS IN SCIENCE AND MATHEMATICS

Another collection of unsolved problems (originally edited by Dagmar Henney) is being readied for publication. The survey will contain mainly contributions by academicians of various countries (plus the occasional Nobel Prize winner). The book is to be published by the American Mathematical Society "Contemporary Mathematics Series". According to its executive editor R. James Milman (Stanford University), "This collection will provide a major service to the mathematical community and should be of interest to the mature as well as occasional scientist".

Already accepted contributions represent more than 24 countries and some of the following authors: Aczél, Ahlburg, Akutowicz, Auluck, Bellman, Bers, Bollabás, Bruckner, Cohen, Dimitrov, Dirac, Djokovic, Ehrhart, Erdös, Fraenkel, Goldberg, Goodman, Garfunkel, Harary, Hughes, Jahnke, Klee, Koethe, Lions, Moppert, Nieto, Nottrot, Oda, Ogilvy, Pekeris, Peretti, Porubsky, Rassias, Rojas, Rus, Saaty, Saff, Shrikhande, Singmaster, Sprindzŭk, Thorp, Ulam, Vestergaard and Eckert, Walker, Wilansky, Wunderlich.

To submit *your* favorite unsolved problem for publication in the next edition, mail it in camera-ready form to Prof. R. James Milman, Department of Mathematics, Stanford University, Stanford, CA 94305; or to Dr. Dagmar R. Henney, 6912 Prince Georges Ave., Takoma Park, MD 20912.