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CONTENTS

THE EXTENDED ERDÖS-MORDELL INEQUALITY

CLAYTON W. DODGE

Ten years ago *The American Mathematical Monthly* published the following Problem E 2462 [81 (1974) 281], which is an extension of the earlier Problem 3740 proposed by Paul Erdös [42 (1935) 396] and first solved by L.J. Mordell [44 (1937) 252-254]:

"E 2462. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let P be a point interior to the triangle $A_1A_2A_3$. Denote by R_i the distance from P to the vertex A_i, and denote by r_i the distance from P to the side a_i opposite to A_i. The Erdös-Mordell inequality asserts that

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3).$$

Prove that the above inequality holds for every point P in the plane of $A_1A_2A_3$ when we make the interpretation $R_i \geq 0$ always and r_i is positive or negative depending on whether P and A_i are on the same side of a_i or on opposite sides."

It was my pleasure in 1974 to referee the solutions to this problem. Curiously, each of the solvers started with the solution to the original Erdös inequality given by Kazarinoff [1] and modified it for the case where r_1, r_2, or r_3 is negative. Each made the same error, invalidating the proof. Curiously, Kazarinoff stated that his proof "holds even if P lies outside the triangle, provided it remains inside the circumcircle", but the Elementary Problem Department editors could not see that such an extension of the proof was possible without commiting the same error the other solvers had made. We outline Kazarinoff's proof and describe the error. Since we shall rely heavily on this proof, our outline is quite complete. It is interesting to note that, if Kazarinoff's statement could have been verified then, a proof would have been published in 1975.

In Demir's notation, Kazarinoff let P lie within angle A_1 and then he reflected triangle $A_1A_2A_3$ in the bisector A_1T of angle A_1 into triangle $A_1A_2'A_3'$, as shown in Figure 1. Noting that the bisector of angle A_1 also bisects the angle between the altitude A_1D and the circumradius OA_1, he used a theorem of Pappus which states that the area of the parallelogram whose adjacent sides are A_1A_2' and A_1P plus the area of the parallelogram whose adjacent sides are A_1P and A_1A_3' is equal to the area of the parallelogram erected on $A_2'A_3'$ whose sides emanating from A_2' and A_3' are equal, as vectors, to $\vec{A_1P}$. Since $A_1P = R_1$, Kazarinoff obtained the first of equalities (1), and the other two are obtained in the same way when P lies within angles A_2 and A_3:

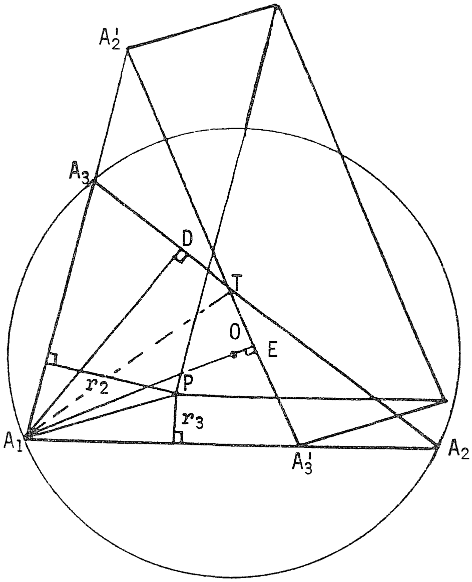

Figure 1

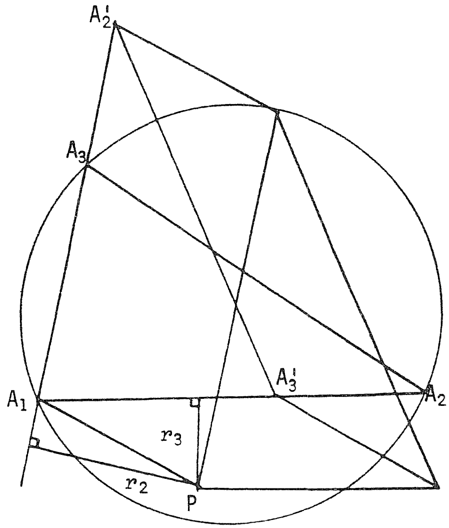

Figure 2

$$\begin{cases} a_1R_1 \cos(\mathrm{OA_1P}) = a_2r_3 + a_3r_2 \\ a_2R_2 \cos(\mathrm{OA_2P}) = a_3r_1 + a_1r_3 \\ a_3P_3 \cos(\mathrm{OA_3P}) = a_1r_2 + a_2r_1. \end{cases} \tag{1}$$

From this follows, when P is an interior point of the triangle,

$$\begin{cases} R_1 + R_2 + R_3 \geq R_1 \cos(\mathrm{OA_1P}) + R_2 \cos(\mathrm{OA_2P}) + R_3 \cos(\mathrm{OA_3P}) \\ \qquad = \left(\frac{a_2}{a_3}+\frac{a_3}{a_2}\right)r_1 + \left(\frac{a_3}{a_1}+\frac{a_1}{a_3}\right)r_2 + \left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)r_3 \\ \qquad \geq 2(r_1 + r_2 + r_3) \end{cases} \tag{2}$$

because $x + 1/x \geq 2$ when $x > 0$.

Equations (1) hold for all locations of P, provided Demir's sign convention is observed. Then also the first two lines of (2) hold. We shall make use of this in our proofs later in this paper, so a proof is presented.

Let P lie outside angle A_1 and outside angle A_2 but inside angle A_3 and inside the circumcircle of triangle $A_1A_2A_3$, as shown in Figure 2. Using the notation of Figure 1, we see that the parallelogram on side $A_2'A_3'$ now is the difference between those on sides A_1A_2' and A_1A_3'. Accordingly,

$$a_1R_1 \cos(\mathrm{OA_1P}) = -a_2r_3 + a_3r_2,$$

where we take the r_i all nonnegative; similarly,

$$a_2R_2 \cos(\mathrm{OA_2P}) = a_3r_1 - a_1r_3,$$

and we have as before

$$a_3R_3 \cos(\mathrm{OA_3P}) = a_1r_2 + a_2r_1,$$

since P lies within angle A_3. Thus equations (1) are true for this case if we observe Demir's sign convention. That they also hold in other cases is not needed here. Since the cosines of the angles $\mathrm{OA}_i\mathrm{P}$ are all still positive because P lies inside the circumcircle, the first two lines of (2) both still hold. Only the third line of (2) is in doubt. In fact, Kazarinoff's argument fails at this point, as explained in the next paragraph.

The error in the submitted solutions to Problem E 2462 occurred when one of the r_i, say r_3, is negative. Then we still have

$$\frac{a_1}{a_2} + \frac{a_2}{a_1} \geq 2,$$

but, since $r_3 < 0$, the inequality reverses to give

$$\left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right)r_3 \leq 2r_3,$$

rendering the argument inconclusive. The editors could find no simple remedy for this flaw since the extended theorem requires that either one or two of the distances r_i be negative. We wrote to those who had submitted solutions, and Leon Bankoff and I corresponded for perhaps a year in attempting to put together a satisfactory proof. Over the next nine years I returned to the problem from time to time, fascinated by its challenge.

Two cases were disposed of almost immediately.

Case 1. Point P lies inside the angle vertical to a vertex angle.

For example, let P lie inside the angle vertical to A_1, as shown in Figure 3. Then r_2 and r_3 are to be taken negative and we must prove that

$$R_1 + R_2 + R_3 \geq 2(r_1 - r_2 - r_3),$$

where the r_i have all been taken nonnegative and we have inserted the appropriate negative signs. Because R_2 and r_1 are hypotenuse and leg of a right triangle, we have

$$R_2 \geq r_1, \qquad \text{and similarly} \qquad R_3 \geq r_1.$$

Thus

$$R_1 + R_2 + R_3 \geq R_2 + R_3 \geq r_1 + r_1 \geq 2(r_1 - r_2 - r_3). \quad \square$$

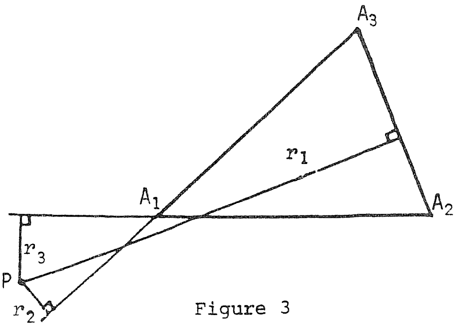

Figure 3

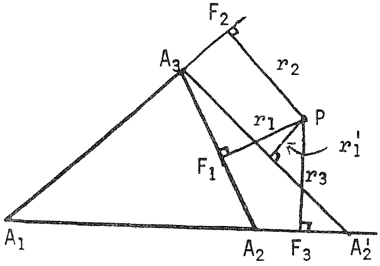

Figure 4

Case 2. Point P is interior to an angle of the triangle, but far enough outside the triangle so that a foot F_i of a distance r_i lies outside the triangle.

As shown in Figure 4, we take P lying within angle A_1 and far enough outside the triangle so that, say, the foot F_3 of distance r_3 lies outside the triangle. Then r_1 is taken negative. Choose point A_2' so that F_3 is the midpoint of segment A_2A_2'. Then $PA_2 = PA_2'$ and the three distances R_i for triangle $A_1A_2A_3$ are the same as those for triangle $A_1A_2'A_3$. Also r_2 and r_3 remain unchanged, and only r_1 changes to r_1'. If, as pictured, P lies outside triangle $A_1A_2'A_3$, then $|r_1| > |r_1'|$ and $-r_1 < -r_1'$ since they both must be taken negative. If P lies inside triangle $A_1A_2'A_3$, we get $-r_1 < 0 < r_1'$. So in either case, using the appropriate sign, we have

$$\pm r_1' + r_2 + r_3 \geq -r_1 + r_2 + r_3.$$

It therefore suffices to prove the extended theorem in the case where all three feet F_i of the distances r_i lie inside the triangle's sides or at its vertices, and when this occurs P lies inside the circumcircle. □

A comprehensive computer run showed the theorem apparently true for all points inside the circumcircle, so all that remained was to prove the theorem when the point P lies outside the triangle and inside the circumcircle. Moreover, Case 2 eliminates a portion of even that region (when, say, P lies inside triangle $A_1A_2'A_3$, for the original Erdös inequality applies to that triangle). Let D be diametrically opposite vertex A_1 on the circumcircle of triangle $A_1A_2A_3$, as shown in Figure 5. Without loss of generality, we must prove the theorem whenever P lies within or on triangle A_2A_3D. We may assume that $A_2 < 90^\circ$ and $A_3 < 90^\circ$ since otherwise the indi-

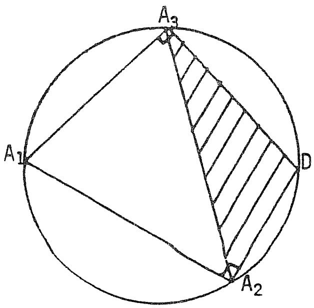

Figure 5

cated region is empty. In this region, since r_1 is to be given a negative sign, we must prove

$$R_1 + R_2 + R_3 + 2r_1 \geq 2r_2 + r_3.$$

If Kazarinoff's statement that his proof holds whenever P lies inside the circumcircle had been substantiated, then the proof of Problem E 2462 would have been complete at this point. The following cases, all developed in the past year, do complete the desired proof.

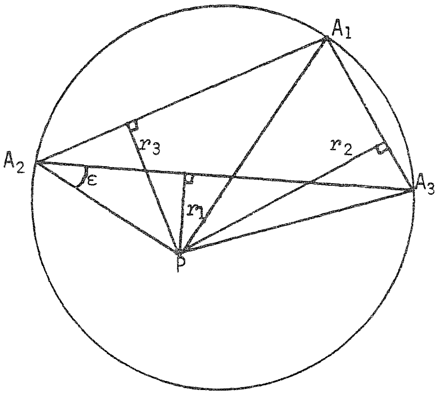

Figure 6

Case 3. Point P lies in triangle A_2A_3D and at least one of angles A_2 and A_3 does not exceed 30°.

Referring to Figure 6, let $A_2 \leq 30^\circ$, so that $\sin A_2 \leq 1/2$. If $\angle A_3A_2P = \varepsilon$, then

$$r_3 = R_2 \sin(A_2+\varepsilon), \qquad r_1 = R_2 \sin\varepsilon,$$

and

$$\begin{aligned}\sin(A_2+\varepsilon) - \sin\varepsilon &= \sin A_2 \cos\varepsilon + \cos A_2 \sin\varepsilon - \sin\varepsilon \\ &= \sin A_2 \cos\varepsilon + \sin\varepsilon(\cos A_2 - 1) \\ &\leq \sin A_2 \leq \tfrac{1}{2}.\end{aligned}$$

Now

$$R_2 = PA_2 > PA_2\{2\sin(A_2+\varepsilon) - 2\sin\varepsilon\} = 2r_3 - 2r_1.$$

Hence

$$R_1 + R_2 + R_3 + 2r_1 \geq r_2 + (2r_3 - 2r_1) + r_2 + 2r_1 = 2r_2 + 2r_3. \quad \square$$

Case 4. Point P lies inside the largest angle of the triangle.

Let P lie in triangle A_2A_3D and suppose $A_1 \geq A_2 \geq A_3$. Then we have $a_1 \geq a_2 \geq a_3$ and $r_1 \leq r_2$, and also

$$2 \leq U \equiv \frac{a_2}{a_3} + \frac{a_3}{a_2} \leq V \equiv \frac{a_3}{a_1} + \frac{a_1}{a_3}.$$

Hence, if for some number N we have $Ur_1 = Vr_2 + N$, then

$$Ur_1 \geq Ur_2 + N$$

and, since also

$$(U-2)r_1 \leq (U-2)r_2,$$

we may subtract to get

$$2r_1 \geq 2r_2 + N.$$

Therefore, since we have, by the first two lines of (2),

$$R_1 + R_2 + R_3 + (\frac{a_2}{a_3}+\frac{a_3}{a_2})r_1 \geq (\frac{a_3}{a_1}+\frac{a_1}{a_3})r_2 + (\frac{a_1}{a_2}+\frac{a_2}{a_1})r_3,$$

it follows that

$$R_1 + R_2 + R_3 + 2r_1 \geq 2r_2 + (\frac{a_1}{a_2}+\frac{a_2}{a_1})r_3 \geq 2r_2 + 2r_3. \quad \square$$

Now only one case remains to be settled, but first we prove a pair of lemmas.

LEMMA 1. The function

$$f(x) = 1 - \cos x - \frac{1}{2}\sin(x - 15^\circ)$$

has a minimum at approximately $(29^\circ, 0.044)$, so $f(x) > 0$ for all x in the interval $[15^\circ, 90^\circ]$. (See Figure 7.)

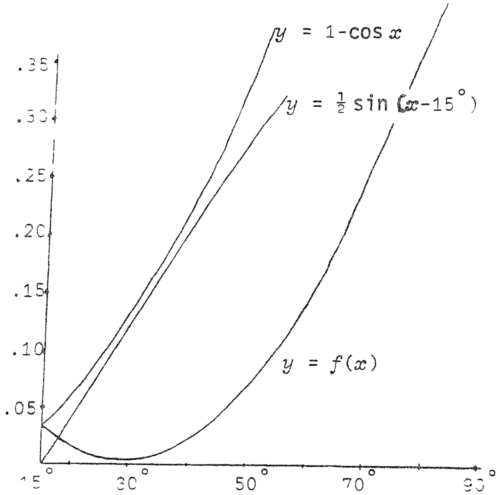

Figure 7

We have

$$f'(x) = \sin x - \frac{1}{2}\cos(x-15^\circ),$$

which vanishes when

$$\tan x = \frac{\cos 15^\circ}{2 - \sin 15^\circ},$$

that is, when $x \approx 29.019466^\circ$, at which point $f(x) \approx 0.004419 > 0$. Since

$$f''(x) = \cos x + \frac{1}{2}\sin(x-15^\circ) > 0,$$

the critical point is a minimum. $\square$

LEMMA 2. If $1 \leq x \leq 2$, then $g(x) = x + 1/x \leq 2.5$.

Clearly $g'(x) \geq 0$ in the given interval, so $g(x) \leq g(2) = 2.5$. $\square$

Our last case takes P inside the triangle A_2A_3D of Figure 5, where $A_2 > 30^\circ$ and $A_3 > 30^\circ$ (by Case 3), and A_1 is not the largest angle of the triangle (by Case 4). We may without loss of generality assume that A_2 is the largest angle. Since we need not consider $A_2 \geq 90^\circ$ (by Figure 5), we have the following case:

Case 5. Point P lies inside triangle A_2A_3D, and $30^\circ < A_3 \leq A_2$ and $A_1 \leq A_2 < 90^\circ$. (See Figure 8.)

Let $\delta = \angle OA_3P$. Since $A_3 > 30^\circ$, we have $A_1 + A_2 < 150^\circ$ and $A_1 < 75^\circ$. So $\angle A_3OA_2 < 150^\circ$ and $\angle OA_3A_2 > 15^\circ$. From

$$\frac{a_2}{a_3} = \frac{\sin A_2}{\sin A_3} \quad \text{and} \quad 30^\circ < A_3 \leq A_2 < 90^\circ,$$

we get

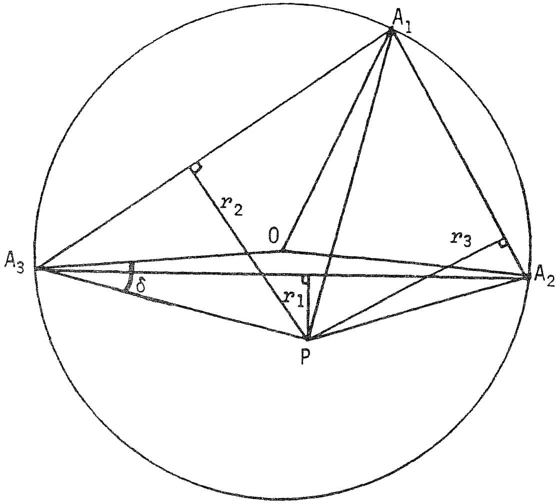

Figure 8

$$1 \leq \frac{a_2}{a_3} < \frac{\sin 90^\circ}{\sin 30^\circ} = 2,$$

and so, by Lemma 2,

$$\frac{a_2}{a_3} + \frac{a_3}{a_2} < 2.5.$$

Now $r_1 \leq R_3 \sin(\delta - 15^\circ)$, so, by Lemma 1,

$$R_3(1 - \cos\delta) \geq \tfrac{1}{2} R_3 \sin(\delta - 15^\circ) \geq \tfrac{1}{2} r_1.$$

Then

$$\begin{aligned} R_1 + R_2 + R_3 + 2r_1 &= R_1 + R_2 + R_3\cos\delta + R_3(1-\cos\delta) + 2r_1 \\ &\geq R_1 + R_2 + R_3\cos\delta + 2.5r_1 \\ &\geq R_1\cos(OA_1P) + R_2\cos(OA_2P) + R_3\cos\delta + \left(\frac{a_2}{a_3}+\frac{a_3}{a_2}\right)r_1 \\ &= \left(\frac{a_3}{a_1}+\frac{a_1}{a_3}\right)r_2 + \left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)r_3 \qquad \text{by (2)} \\ &\geq 2(r_2 + r_3), \end{aligned}$$

and we are at last finished. The proof of Problem E 2462 is complete.

Finally, we use the extended Erdös-Mordell inequality for triangles to get, as a corollary, a corresponding result for convex quadrilaterals.

Let $A_1A_2A_3A_4$ be a convex quadrilateral, and let P be any point in its plane.

We set $PA_i = R_i > 0$ and denote the signed distance between P and line A_iA_j by r_{ij}, the sign being determined by Demir's convention for any triangle of which A_iA_j is a side. Thus (see Figure 9), r_{12} is associated with triangles $A_1A_2A_3$ and $A_1A_2A_4$ and has the same sign for both triangles regardless of the location of point P; and similar statements can be made about r_{23}, r_{34}, and r_{41}. The distance $|r_{13}|$, on the other hand, is associated with triangles $A_1A_2A_3$ and $A_1A_3A_4$; and if r_{13} is the signed distance associated with triangle $A_1A_2A_3$, then $-r_{13}$ is the signed distance associated with triangle $A_1A_3A_4$. Similarly, if r_{24} corresponds to triangle $A_1A_2A_4$, then $-r_{24}$ corresponds to triangle $A_2A_3A_4$. Our inequality extended to quadrilaterals reads as follows:

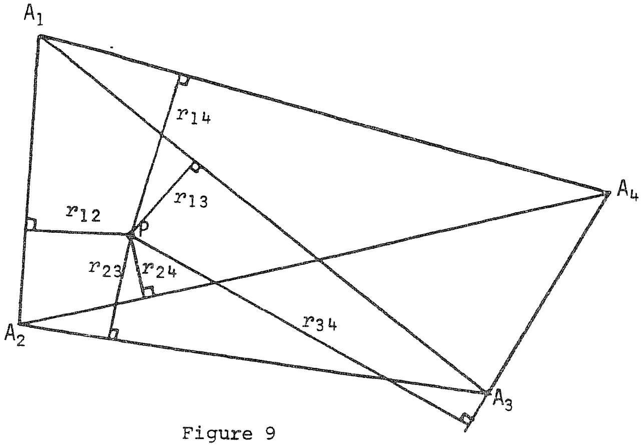

Figure 9

COROLLARY. If $A_1A_2A_3A_4$ is a convex quadrilateral, P is any point in its plane, and the distances R_i and r_{ij} are as defined above, then

$$3(R_1 + R_2 + R_3 + R_4) \geq 4(r_{12} + r_{23} + r_{34} + r_{41}). \tag{3}$$

Proof. We apply the extended Erdös-Mordell inequality successively to triangles $A_1A_2A_3$, $A_1A_2A_4$, $A_1A_3A_4$, and $A_2A_3A_4$:

$$R_1 + R_2 + R_3 \geq 2(r_{12} + r_{23} + r_{13}),$$

$$R_1 + R_2 + R_4 \geq 2(r_{12} + r_{24} + r_{41}),$$

$$R_1 + R_3 + R_4 \geq 2(r_{34} + r_{41} - r_{13}),$$

$$R_2 + R_3 + R_4 \geq 2(r_{23} + r_{34} - r_{24}),$$

and adding these four inequalities yields (3).

REFERENCE

1. D.K. Kazarinoff, "A Simple Proof of the Erdös-Mordell Inequality for Triangles", *Michigan Math. J.*, 4 (1957) 97-98.

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* * *

THE OLYMPIAD CORNER: 59

M.S. KLAMKIN

I present three new problem sets this month. The first two, which I obtained through the courtesy of Dimitris Vathis, contain the problems set at two stages of the annual High School Competition of the Greek Mathematical Society. The third contains a few interesting problems from the April 1984 issue of the Russian journal *Kvant*. I solicit from all readers elegant solutions to all of these problems.

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1984 ANNUAL GREEK HIGH SCHOOL COMPETITION

2nd Class High School - March 10, 1984

1. (a) Let $A_1A_2A_3A_4A_5A_6$ be a convex hexagon having its opposite sides parallel. Prove that triangles $A_1A_3A_5$ and $A_2A_4A_6$ have equal areas.

(b) Consider a convex octagon in which all the angles are equal and the length of each side is a rational number. Prove that its opposite sides are equal and parallel.

2. An equilateral triangle ABC and an interior point P are given such that PA = 5, PB = 4, and PC = 3. Find the length of a side of triangle ABC.

3. In a given triangle ABC, $\angle A = 5\pi/8$, $\angle B = \pi/8$, and $\angle C = \pi/4$. Prove that its angle bisector CZ, the median BE, and the altitude AD are concurrent.

4. (a) Find the real roots of the equation

$$(x^2-x-2)^4 + (2x+1)^4 = (x^2+x-1)^4.$$

(b) Find the roots of the equation

$$x^3 - 2ax^2 + (a^2+1)x - 2a + 2 = 0$$

for all real a.

(c) Find the range of the function

$$f(x)= \frac{\sqrt{x^2+1} + x - 1}{\sqrt{x^2+1} + x + 1}, \qquad -\infty < x < \infty,$$

and show that f is an odd function.

3rd Class High School - March 10, 1984

1. Prove or disprove that there exists in space a pentagon all of whose sides are equal and all of whose angles are 90°.

2. (a) Find the maximum and the minimum of $8x + 6y - 5z$, where x,y,z are real and $x^2 + y^2 + z^2 = 5$.

(b) Express the vector $\vec{x}$ in terms of the vectors $\vec{a},\vec{b}$ if

$$\vec{a}\cdot(\vec{x} + \vec{b}) = \vec{a}^2 \quad \text{and} \quad \vec{b}\cdot\vec{x} = 0.$$

It is assumed that $\vec{a} \neq \vec{0}$ and that the vectors $\vec{a},\vec{b},\vec{x}$ are coplanar.

3. If G is a multiplicative group and a,b,c are elements of G, prove:

(a) If $b^{-1}ab = ac$, $ac = ca$, and $bc = cb$, then

$$a^nb = ba^nc^n \quad \text{and} \quad (ab)^n = b^na^nc^{n(n+1)/2}$$

for all $n \in N$.

(b) If $b^{-1}ab = a^k$, where $k \in N$, then

$$b^{-1}a^nb^l = a^{nk^l}$$

for all $l,n \in N$.

4. The elements $a = (1,1,0,0)$, $b = (1,0,-1,0)$, $c = (0,0,0,1)$, $d = (-1,0,1,0)$, $e = (0,1,0,1)$, and $f = (0,0,1,0)$ of R^4 are given. Let $V = \langle a,b,c\rangle$ and $W = \langle d,e,f\rangle$ be the subspaces of R^4 generated by the vectors a,b,c and d,e,f, respectively.

(a) Prove that the set

$$V + W = \{x_1+x_2 \mid x_1 \in V,\ x_2 \in W\}$$

is a subspace of R^4.

(b) Find a basis of the space $V \cap W$ (and denote it by A).

(c) Find a basis (denoted by B) of the V containing the A, and a basis C of the W containing the A as well.

(d) Prove that the set $B \cup C$ is a basis of the space $V+W$ and that $V+W = R^4$.

(e) Prove that $\dim (V+W) + \dim (V \cap W) = \dim V + \dim W$.

*

PROBLEMS FROM *KVANT*, April 1984

M856. *Proposed by I.Z. Titovich.*

(a) Construct a quadrilateral knowing the lengths of its sides and that of the segment joining the midpoints of the diagonals.

(b) Under what conditions does the problem have a solution?

M857. *Proposed by S. Stadnichenko.*

Among the first 1984 positive integers (from 1 to 1984) we underline those which may be represented as the sum of five nonnegative integer powers of 2 (i.e., of five not necessarily different numbers 1, 2, 4, 8, ...). Is the set of underlined numbers larger than that of the nonunderlined ones?

M858. *Proposed by P.B. Gusyatnikov.*

The angles α, β, and γ of a triangle satisfy the relation

$$\sin^2\alpha + \sin^2\beta = \sin\gamma.$$

(a) Find α,β,γ if the triangle is isosceles (consider all possible cases).

(b) Can the triangle have only acute angles?

(c) What values can the largest angle of the triangle assume?

M859. *Proposed by V.P. Pikulin.*

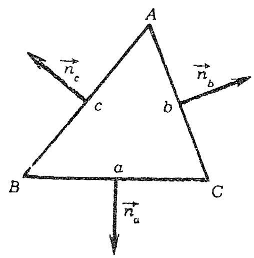
Find the least positive number a such that any quadratic trinomial $f(x)$ for which $|f(x)| \le 1$ whenever $0 \le x \le 1$ satisfies $|f'(1)| \le a$.

M860. *Proposed by Chan Quang.*

(a) Let R and O be the radius and centre of the circumcircle of triangle ABC, r and I the radius and centre of its incircle, and K the common point of the medians of the triangle whose vertices are the points where the incircle touches the sides of ABC. Prove that I lies on the segment OK and that

$$|\mathrm{OI}| : |\mathrm{IK}| = 3R : r.$$

(b) Let a,b,c be the lengths of the sides of triangle ABC, and $\vec{n}_a,\vec{n}_b,\vec{n}_c$ unit vectors perpendicular to the corresponding sides of the triangle and directed outward (see figure). Prove that



$$a^3\vec{n}_a + b^3\vec{n}_b + c^3\vec{n}_c = 12S\cdot\vec{\mathrm{GO}},$$

where S is the area of triangle ABC and G is the intersection point of its medians.

*

I now present solutions to several problems proposed earlier.

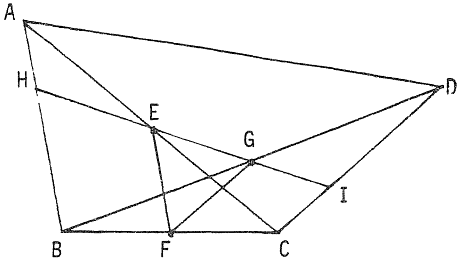
J-25. [1981: 143; 1984: 79] (Corrected) *From a list of Russian "Jewish" problems.*

In a convex quadrilateral ABCD, the sides AB and CD are congruent and the midpoint of diagonals AC and BD are distinct. Prove that the straight line through these two midpoints makes equal angles with AB and CD.

II. *Composite of the (essentially similar) solutions by Chris Fisher (Regina), Leroy F. Meyers (Ohio), D.J. Smeenk (The Netherlands), Dan Sokolowsky (Los Angeles), and Esther Szekeres (Australia).*

Solution I by vectors was easy and direct, but not very elegant. The following synthetic solution is better.

Let E, G, and F be the respective midpoints of AC, BD, and BC, and let line EG intersect AB and CD in H and I, respectively, as shown in the figure. Then EF || AB, FG || CD, and



$$EF = \frac{AB}{2} = \frac{CD}{2} = FG.$$

Therefore $\angle BHE = \angle FEG = \angle FGE = \angle CIG$. □

The same proof applies, with very minor modifications, if ABCD is not convex.

*

8. [1982: 301; 1984: 87] *Proposed by John Steinke.*

Determine all six-digit integers n such that n is a perfect square and the number formed by the last three digit of n exceeds the number formed by the first three digits of n by 1. (n might look like 123124.)

II. *Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.*

There is an old Chinese proverb that when two men are arguing, the one who first resorts to physical violence is the one who first runs out of ideas. Similarly, when a mathematician runs out of ideas he resorts to a computer. Mr. Prielipp's computer took 7 seconds to solve this problem, but it took a good deal longer to write the program and debug it. It takes no longer to solve the problem mathematically, and it gives a lot more satisfaction.

a	b	N	n
7	143	428	183184
11	91	727	528529
13	77	846	715716
77	13	155	24025
91	11	274	75076
143	7	573	328329

Let $N^2 = n$; then $N^2 - 1 = 1001z$, where z is a three-digit integer. Let $ab = 1001 = 7\cdot11\cdot13$ and $z = xy$. Then $(N-1)(N+1) = abxy$, and we may take $N-1 = ax$ and $N+1 = by$. Then $N = ax+1 = by-1$, from which

$$by = ax + 2.$$

This equation may be solved for positive integers x and y when a and b are given. When $a = 7$ and $b = 143$, we obtain

$$x = 143w - 82, \qquad y = 7w - 4, \qquad \text{whence} \qquad N = 1001w - 573.$$

Choosing w so that $0 < N < 1000$ gives $N = 428$, $n = 183184$, as shown in the first line of the table. The rest of the table gives the results corresponding to the other possible choices for a and b. If initial zeros are not allowed, we have only the four solutions n = 183184, 328329, 528529, 715716.

[A similar solution was received from J.T. Groenman, Arnhem, The Netherlands.]

*

1. [1983: 303] *From the 1980 Leningrad High School Olympiad, Third Round.* If $a,b,c,d > 0$ and $a+b+c+d = 1$, prove that

$$\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} + \sqrt{4d+1} < 6.$$

(Grade 8)

I. *Solution by Mark Kantrowitz, student, Maimonides School, Brookline, Massachusetts.*

We weaken the hypothesis to $a,b,c,d \geq 0$ and $a+b+c+d = 1$. By the A.M.-G.M. inequality,

$$\sqrt{4a+1} \leq \frac{(4a+1) + 1}{2} = 2a + 1,$$

with equality if and only if $a = 0$. From this and three similar relations,

$$\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} + \sqrt{4d+1} \leq 2(a+b+c+d) + 4 = 6,$$

and the inequality must be strict since a,b,c,d cannot all be zero.

II. *Solution by M.S.K.*

By the power mean inequality

$$\frac{\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} + \sqrt{4d+1}}{4} \leq \sqrt{\frac{(4a+1)+(4b+1)+(4c+1)+(4d+1)}{4}} = \sqrt{2}.$$

Thus

$$\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} + \sqrt{4d+1} \leq 4\sqrt{2} < 6,$$

with equality if and only if $a = b = c = d = 1/4$.

2. [1983: 303] *From the 1980 Leningrad High School Olympiad, Third Round.* On the sides AC and BC of a triangle ABC, points M and K, respectively, are chosen such that

$$\mathrm{BK}\cdot\mathrm{AB} = \mathrm{IB}^2 \qquad \text{and} \qquad \mathrm{AM}\cdot\mathrm{AB} = \mathrm{IA}^2,$$

where I is the incenter of the triangle. Prove that the points M, I, and K are collinear. (Grade 8)

Solution by Paul Wagner, Chicago, Illinois.

The direct similarity of triangles AIB and IKB follows from the first part of the hypothesis, and the direct similarity of triangles AIB and AMI follows from the second part. Therefore

$$\angle MIA + \angle AIB + \angle BIK = \angle IBA + \angle AIB + \angle BAI = \pi,$$

and the points M, I, and K are collinear.

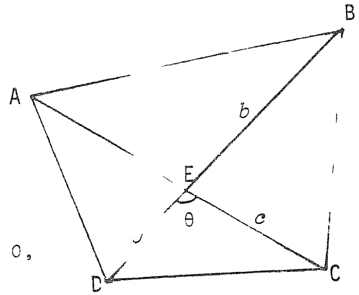
*

5. [1983: 304] *From the 1980 Leningrad High School Olympiad, Third Round.*

A convex quadrilateral is divided by its diagonals into four triangles. The sum of the squares of the areas of the triangles adjacent to opposite sides is the same. Show that at least one of the diagonals is bisected by the point of intersection. (Grade 8)

Solution by Paul Wagner, Chicago, Illinois.

Referring to the figure, we have, with square brackets denoting area,



$$[AEB] = \tfrac{1}{2}ab\sin\theta, \qquad [BEC] = \tfrac{1}{2}bc\sin\theta,$$

$$[CED] = \tfrac{1}{2}cd\sin\theta, \qquad [DEA] = \tfrac{1}{2}da\sin\theta.$$

The hypothesis therefore gives

$$a^2b^2 + c^2d^2 = b^2c^2 + d^2a^2, \text{ or } (a^2 - c^2)(b^2 - d^2) = 0,$$

and so $a = c$ or $b = d$.

*

8. [1983: 304] *From the 1980 Leningrad High School Olympiad, Third Round.*

A triangle ABC has sides a,b,c in the usual order. If angle A is twice angle B, show that $a^2 = b(b+c)$. (Grade 9)

Solution by Gali Salvatore, Perkins, Québec.

Since A = 2B, we have $\sin A = 2\sin B\cos B$, and so

$$a = 2b\cdot\frac{c^2+a^2-b^2}{2ca},$$

which is equivalent to $\{a^2 - b(b+c)\}(b-c) = 0$. If $b \neq c$, then $a^2 = b(b+c)$. If $b = c$, then A = 90° and B = C = 45°, so $a^2 = 2b^2 = b(b+c)$ again.

The converse, if $a^2 = b(b+c)$ then A = 2B, is also true. See solutions IV and V to Problem 102 in this journal [1976: 74].

*

10. [1983: 304] (Corrected) *From the 1980 Leningrad High School Olympiad, Third Round.*

Show that if, for any value of x in the interval [0,1], the inequality $|ax^2+bx+c| \leq 1$ is satisfied, then $|a|+|b|+|c| \leq 17$. (Grade 9)

[In the original version 17 had been replaced by 1.]

Solution by Paul Wagner, Chicago, Illinois.

For $x = 0,\ 1/2,\ 1$, we obtain respectively

$$|c| \le 1, \qquad |a+2b+4c| \le 4, \qquad |a+b+c| \le 1.$$

If we let $a + 2b + 4c = m$ and $a + b + c = n$, then

$$a = -m + 2n + 2c \quad \text{and} \quad b = m - n - 3c.$$

Thus

$$|a| = |-m+2n+2c| \le |m| + 2|n| + 2|c| \le 8,$$

$$|b| = |m-n-3c| \le |m| + |n| + 3|c| \le 8,$$

and so $|a| + |b| + |c| \le 17$.

The maximum value 17 is attained for $8x^2 - 8x + 1$, which satisfies

$$-1 \le 8x^2 - 8x + 1 \le 1 \quad \text{when} \quad 0 \le x \le 1.$$

*

11. [1983: 304] *From the 1980 Leningrad High School Olympiad, Third Round.*

Find two different natural numbers whose arithmetic and geometric means are two-digit numbers one of which is obtained from the other by interchanging the digits. (Grade 9)

Solution by Daniel Ropp, student, Stillman Valley High School, Illinois.

Let a and b be the required natural numbers. Then there are nonzero decimal digits c and d such that

$$\frac{a+b}{2} = 10c + d \quad \text{and} \quad \sqrt{ab} = 10d + c,$$

and $(a+b)/2 \ge \sqrt{ab}$ implies that $c \ge d$. Thus $a+b = 2(10c+d)$, $ab = (10d+c)^2$, and a and b are the roots of

$$x^2 - 2(10c+d)x + (10d+c)^2 = 0,$$

that is,

$$\{a,\ b\} = \{10c + d \pm 3\sqrt{11(c^2-d^2)}\}. \tag{1}$$

Now $(c+d)(c-d) = 11$ or 44 since $c^2-d^2 \le 9^2-1^2$; and since $c+d$ and $c-d$ have the same parity, we must have

$$\begin{cases} c + d = 11 \\ c - d = \ \ 1 \end{cases} \quad \text{or} \quad \begin{cases} c + d = 22 \\ c - d = \ \ 2. \end{cases}$$

Only the first system has a solution in digits c and d: $(c,d) = (6,5)$, and then (1) gives the unique solution $\{a,\ b\} = \{32,\ 98\}$.

*

12. [1983: 304] *From the 1980 Leningrad High School Olympiad, Third Round.*

We shall call a segment in a convex quadrilateral a *midline* if it joins the midpoints of opposite sides. Show that if the sum of the midlines of a quadrilateral is equal to its semiperimeter, then the quadrilateral is a parallelogram. (Grades 9, 10)

Solution by Daniel Ropp, student, Stillman Valley High School, Illinois.

Let ABCD be a convex quadrilateral with the stated midline-semiperimeter property. We use A as origin of vectors and use the notation $\vec{AX} = \vec{x}$ for all points X. The sum of the midlines is

$$\frac{|\vec{b} + \vec{c} - \vec{d}|}{2} + \frac{|\vec{d} + \vec{c} - \vec{b}|}{2}$$

and the semiperimeter is

$$\frac{|\vec{b}| + |\vec{c}-\vec{d}| + |\vec{d}| + |\vec{c}-\vec{b}|}{2}.$$

Thus

$$|\vec{b}+\vec{c}-\vec{d}| + |\vec{d}+\vec{c}-\vec{b}| = |\vec{b}| + |\vec{c}-\vec{d}| + |\vec{d}| + |\vec{c}-\vec{b}|. \tag{1}$$

Now, by the triangle inequality,

$$|\vec{b}| + |\vec{c}-\vec{d}| \geq |\vec{b}+\vec{c}-\vec{d}|,$$

with equality if and only if $\vec{b} = m(\vec{c}-\vec{d})$, or AB || CD: and similarly

$$|\vec{d}| + |\vec{c}-\vec{b}| \geq |\vec{d}+\vec{c}-\vec{b}|,$$

with equality if and only if $\vec{d} = n(\vec{c}-\vec{b})$, or AD || BC. Therefore (1) holds just when ABCD is a parallelogram.

*

13. [1983: 304] *From the 1980 Leningrad High School Olympiad, Third Round.*

Are there real numbers a and b such that the function $f(x) = ax + b$ satisfies the inequality

$$\{f(x)\}^2 - \cos x \cdot f(x) < \tfrac{1}{4}\sin^2 x$$

for all $x \in [0,2\pi]$? (Grade 10)

Solution by Daniel Ropp, student, Stillman Valley High School, Illinois.

The inequality of the proposal is equivalent to

$$\left(f(x) - \frac{1+\cos x}{2}\right)\left(f(x) + \frac{1-\cos x}{2}\right) < 0,$$

and this implies

$$-\frac{1 - \cos x}{2} < f(x) < \frac{1 + \cos x}{2}.$$

Setting $x = 0, \pi, 2\pi$ successively, we obtain

$$0 < b < 1, \qquad -1 < a\pi+b < 0, \qquad 0 < 2a\pi+b < 1.$$

If we sum the first and third of these relations, we get $0 < 2(a\pi+b) < 2$, and this contradicts the second. So there are no real numbers a and b with the desired property.

*

15. [1983: 305] *From the 1980 Leningrad High School Olympiad, Third Round.*

How many different numbers appear in the sequence

$$\left[\frac{1^2}{1980}\right], \left[\frac{2^2}{1980}\right], \ldots, \left[\frac{1980^2}{1980}\right],$$

where the square brackets denote the greatest integer function? (Grade 10)

Solution by Daniel Ropp, student, Stillman Valley High School, Illinois.

Let S_1 and S_2 be the subsequences corresponding $k = 1,2,\ldots,990$ and $k = 991,992,\ldots,1980$, respectively, of the sequence

$$S = \{[k^2/1980] \mid k = 1,2,\ldots, 1980\}.$$

Every integer from 0 to $[990^2/1980] = 495$ appears in S_1. For if not, let n be a missing integer. Then $n < 495$ and, for some k,

$$\frac{k^2}{1980} < n < n+1 \le \frac{(k+1)^2}{1980}.$$

This implies that $(2k+1)/1980 > 1$, so $k \ge 990$ and $n \ge 495$, a contradiction.

No integer appears twice in S_2. For suppose m appears twice. Then there is a positive integer $k \ge 991$ such that

$$m \le \frac{k^2}{1980} < \frac{(k+1)^2}{1980} < m+1.$$

This implies that $(2k+1)/1980 < 1$ and $k < 990$, a contradiction.

Finally, observing that $[990^2/1980] < [991^2/1980]$, we conclude that the subsequences S_1 and S_2 are disjoint, and therefore the number of different numbers in S is

$$1 + 495 + 990 = 1486.$$

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

THE PUZZLE CORNER

Answer to Puzzle No. 58 [1984: 248]: Epsilon, in slope.

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly hand-written on signed, separate sheets, should preferably be mailed to the editor before April 1, 1985, although solutions received after that date will also be considered until the time when a solution is published.

981. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Solve the doubly true multiplication

$$5\cdot\text{ELEVEN} = \text{FIFTY5}$$

where FIVE is, of course, prime.

982. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let P and Q be interior points of triangle $A_1A_2A_3$. For $i = 1,2,3$, let $PA_i = x_i$, $QA_i = y_i$, and let the distances from P and Q to the side opposite A_i be p_i and q_i, respectively. Prove that

$$\sqrt{x_1y_1} + \sqrt{x_2y_2} + \sqrt{x_3y_3} \geq 2(\sqrt{p_1q_1} + \sqrt{p_2q_2} + \sqrt{p_3q_3}).$$

When P = Q, this reduces to the well-known Erdös-Mordell inequality. (See the article by Clayton W. Dodge in this journal [1984: 274-281].)

983*. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Let $A_0A_1\ldots A_n$ be an n-simplex in E^n.

(a) If m_k is the median through A_k, prove that

$$n^2m_k^2 = nS_k - T_k,$$

where S_k is the sum of the squares of all the edges meeting in A_k, and T_k is the sum of the squares of all the edges not passing through A_k.

(b) Deduce from (a), or otherwise, that if the medians of the simplex are all equal, then the sum of the squares of all the edges meeting in a vertex is the same for all vertices. Is the converse also true?

984. *Proposed by J.C. Fisher and H.N. Gupta, University of Regina.*

For which $k \geq 3$ is $\binom{k}{2} - 1$ a prime power p^n?

985. *Proposed by John J. Martinez, Gonzaga High School, Washington, D.C.*

Let ABC be a triangle with sides a,b,c, and let AP,BQ,CR be concurrent cevians terminating in the opposite sides at P,Q,R. We use square brackets to denote the area of a triangle.

(a) If AP,BQ,CR are the internal angle bisectors of the triangle, prove that [PQR] = [BPR] if and only if a,b,c, in some order, are in arithmetic progression.

(b) If AP,BQ,CR are the altitudes of the triangle and p,q,r are the sides of triangle PQR, prove that

$$\frac{[PQR]}{[ABC]} = \frac{2pqr}{abc}.$$

986. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Let

$$x = \sqrt[3]{p + \sqrt{r}} + \sqrt[3]{q - \sqrt{r}},$$

where p,q,r are integers and $r \ge 0$ is not a perfect square. If x is rational, prove that $p = q$ and x is integral.

987*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

If triangle ABC is acute-angled, prove or disprove that

(a) $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \ge \frac{4}{3}\{1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\}$,

(b) $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \ge \frac{4}{\sqrt{3}}\{1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\}$.

988. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Prove that

$$\sum_{k=1}^{5} \sec\frac{2\pi k}{11} + \sum_{k=1}^{6} \sec\frac{2\pi k}{13} = 0.$$

989. *Proposed by Kurt Schiffler, Schorndorf, Federal Republic of Germany.*

Let H be the orthocenter of triangle ABC. Prove that the Euler lines of triangles ABC, BCH, CAH, and ABH are all concurrent. In what remarkable point of triangle ABC do they concur?

990. *Proposed by Bob Prielipp, University of Wisconsin-Oshkosh.*

Find all pairs (u,v) of positive integers such that

$$u^3 + (u+1)^3 = v^2.$$

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THE PUZZLE CORNER

Answer to Puzzle No. 59 [1984: 248]: Isometry (I-some-try).

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

93. [1975: 97; 1976: 45, 111] *Proposed by H.G. Dworschak, Algonquin College.*

Is there a convex polyhedron having exactly seven edges?

IV. *Second solution by Charles W. Trigg, San Diego, California.*

This proposition has been considered by Starke [1], and solutions by Maskell, Sauvé, and Trigg have appeared in this journal. An approach differing from these solutions follows:

If there is a polyhedron having exactly 7 edges, its faces cannot all be triangular, for each edge of a polyhedron joins two faces, and $7\cdot 2/3$ is not an integer. And for every polyhedron having an n-gonal face, where $n \geq 4$, the number of edges is at least $2n \geq 8$. For, in addition to the n sides of the n-gonal face, the polyhedron has at least n additional edges, there being at least one emanating from each vertex of the n-gon and terminating at a polyhedral vertex not in the plane of the n-gon. So there is no polyhedron having exactly 7 edges.

Editor's comment.

This problem has also appeared in [2] with an outline of solution based on Euler's formula $v-e+f = 2$. A question now naturally arises: For which n is there a convex polyhedron having exactly n edges? *This* problem has also appeared (without solution) in [2] and in this journal (Problem 121 [1976: 113]) with solutions by Meyers, Trigg, and Sauvé.

REFERENCES

1. E.P. Starke, "Possible Number of Edges for a Polyhedron" (Solution of Problem E 923), *American Mathematical Monthly*, 58 (March 1951) 190.

2. Anatole Beck, Michael N. Bleicher, and Donald W. Crowe, *Excursions Into Mathematics* (Experimental Edition), Worth Publishers, New York, 1967, p. 29.

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346. [1978: 134; 1979: 26] *Proposed by Leroy F. Meyers, The Ohio State University.*

It has been conjectured by Erdös that every rational number of the form $4/n$, where n is an integer greater than 1, can be expressed as the sum of three or fewer unit fractions (reciprocals of positive integers, also called Egyptian fractions), not necessarily distinct. As a partial verification of the conjecture, show that at least 23/24 of such numbers have the required expansions.

III. *Comment by J.L. Brenner, Palo Alto, California.*

The proposal, as clarified in Wilke's solution I [1979: 26-29], in effect asks for a proof that $\delta = 1/24 \approx 0.0417$ is an upper bound to the density of integers $n \geq 3$ such that $4/n$ *cannot* be expressed as the sum of three distinct unit fractions (Egyptian fractions), that is, such that, for integers x,y,z with $1 \leq x < y < z$, the equation

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \tag{1}$$

has no solution. In fact, Wilke found the better upper bound $\delta = 1/140 \approx 0.0071$, and Maskell (see the editor's comment [1979: 30]) improved this still further to

$$\delta = \frac{1}{140}\cdot\frac{9}{11}\cdot\frac{10}{13} \approx 0.0045.$$

It is easy to improve even this result by extensions of the elementary methods used by Wilke and Maskell, but to do so would be pointless because it is already known that the least upper bound is $\delta = 0$, that is, almost every n is representable in the form (1). This (and more) was proved in 1970 by Vaughan [4]. Moreover, in a personal communication Koichi Yamamoto stated that he has verified that (1) has a solution for every n such that $3 \leq n \leq 10^8$.

The list of references given below augments that given earlier [1979: 30].

REFERENCES

1. Paul Erdös and Sherman Stein, "Sums of distinct unit fractions", *Proc. Amer. Math. Soc.*, 14 (1963) 126-131. MR 26, 71.

2. B.M. Stewart and W.A. Webb, "Sums of fractions with bounded numerators", *Can. J. Math.*, 18 (1966) 999-1003. MR 33, 7297.

3. William A. Webb, "On $4/n = 1/x + 1/y + 1/z$", *Proc. Amer. Math. Soc.*, 25 (1970) 578-584. MR 41, 1639.

4. R.C. Vaughan, "On a problem of Erdös, Straus and Schinzel", *Mathematika*, 17 (1970) 193-198. MR 44, 6600.

5. Li Delang, "On the equation $4/n = 1/x + 1/y + 1/z$", *J. of Number Theory*, 13 (1981) 485-494.

6. Xun Qian Yang, "A note on $4/n = 1/x + 1/y + 1/z$", *Proc. Amer. Math. Soc.*, 83 (1982) 496-498.

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851. [1983: 177] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Solve the doubly true decimal alphametic

תשעים = עשר - עשר × עשר

which means TEN × TEN - TEN = NINETY in Hebrew. (*Goyim* readers will find it convenient to rewrite the alphametic as ABC × ABC - ABC = DECBF.)

This same Hebrew alphametic appeared in the *Journal of Recreational Mathematics* 15 (1982-1983) 136, proposed by Meir Feder, who asked for a solution in base 7.

Solution by the proposer (revised by the editor).

For typographical convenience we solve the "goyim" transliteration

$$ABC \times ABC - ABC = DECBF. \tag{1}$$

We note at the outset that $A \in \{1,2,3\}$, for DECBF ≤ 98765 implies ABC < 315; and that (1) implies

$$100B^2 + 100A(2C-1) - 100C + 20B(C-1) + C(C-1) - F \equiv 0 \pmod{10^3}. \tag{2}$$

It follows from (2) that $C(C-1) - F \equiv 0 \pmod{20}$, which is possible only if

$$(C,F) \in \{(1,0),\ (3,6),\ (5,0),\ (7,2)\}; \tag{3}$$

and that

$$B^2 + A(2C-1) - C + \frac{B(C-1)}{5} + \frac{C(C-1)-F}{100} \equiv 0 \pmod{10}. \tag{4}$$

For $(C,F) = (1,0)$, relation (4) reduces to $B^2 + A - 1 \equiv 0 \pmod{10}$. If $A = 3$, there is no value for B; and if $A = 2$, then $B = 3$ or 7, but neither ABC = 231 nor 271 satisfies (1). Proceeding likewise with the other possible values of (C,F) in (2), we obtain the following results:

$$(C,F) = (3,6) \implies B \in \{0,5\} \implies \text{no value for } A,$$

$$(C,F) = (5,0) \implies B \in \{1,6\} \implies \text{no value for } A,$$

$$(C,F) = (7,2) \implies \begin{cases} B = 3 \implies \text{no value for } A, \\ B = 8 \implies A = 1. \end{cases}$$

We find that ABC = 187 satisfies (1), and the unique solution is

$$187 \times 187 - 187 = 34782.$$

Also solved by CLAYTON W. DODGE, University of Maine at Orono; MEIR FEDER, Haifa, Israel; RICHARD I. HESS, Rancho Palos Verdes, California; J.A. McCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Pensacola Junior College, Florida; KENNETH M. WILKE, Topeka, Kansas; and ANNELIESE ZIMMERMANN, Bonn, West Germany. Comments were received from PETER GILBERT and STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; and STEVEN KAHAN, Hollis Hills, N.Y.

Editor's comment.

Rabinowitz stated that, in addition to bases 7 and 10, the Hebrew alphametic has a unique solution in bases 13 and 14, but not in bases 6, 8, 9, 12, 15, or 16; and Feder stated [1] that it has two solutions in base 11. Mills proved that the corresponding English alphametic

$$\text{TEN} \times \text{TEN} - \text{TEN} = \text{NINETY} \tag{5}$$

has no solution in base 10 but does have one in base 11, and Feder stated [1] that the *next* base in which (5) has a solution is 66.

REFERENCE

1. Solution to Alphametics 1177 and 1178 (proposed by Meir Feder), *Journal of Recreational Mathematics*, 16 (1983-84) 134.

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852. [1983: 177] *Proposed by Jordi Dou, Barcelona, Spain.*

Given are three distinct points A,B,C on a circle. A point P in the plane has the property that if the lines PA,PB,PC meet the circle again in A',B',C', respectively, then A'B' = A'C'.

Find the locus of P.

Solution by Jordan B. Tabov, Sofia, Bulgaria.

Let Γ be the given circle. It is clear that Γ is part of the required locus, for $P \in \Gamma$ implies that A' = B' = C' = P. We now assume that $P \notin \Gamma$. Triangles ABP and ACP are similar to triangles B'A'P and C'A'P, respectively, and so

$$\frac{BA}{PB} = \frac{A'B'}{PA'}, \qquad \frac{CA}{PC} = \frac{A'C'}{PA'}.$$

The condition A'B' = A'C' is therefore equivalent to

$$\frac{PB}{PC} = \frac{AB}{AC}. \tag{1}$$

We now consider two cases.

Case 1. AB = AC. Then (1) is satisfied if and only if $P \in l$, where l is the line bisecting $\angle BAC$. The complete locus in this case is $\Gamma \cup l$.

Case 2. $AB \neq AC$. Then (1) is satisfied if and only if $P \in \Phi$, where Φ is the circle of Apollonius relative to side BC [1]. This is the circle on diameter MN, where M and N are the intersections with line BC of the internal and external bisectors of angle A. The complete locus in this case is $\Gamma \cup \Phi$.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

REFERENCE

1. Roger A. Johnson, *Advanced Euclidean Geometry (Modern Geometry)*, Dover, New York, 1960, p. 295, Theorem 491.

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853. [1983: 178] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Let f be a real-valued function, defined for all $x \geq 0$, such that

$$\lim_{x\to\infty} f(x) \qquad \text{and} \qquad \int_0^r \frac{f(x)}{x}\,dx \tag{1}$$

both exist, the second for all $r > 0$. Evaluate

$$I_1 = \int_0^\infty \frac{f(ax) - f(bx)}{x}\,dx,$$

where $a > b > 0$, and deduce the value of

$$I_2 = \int_0^\infty \frac{e^{ax} - e^{bx}}{x(e^{ax}+1)(e^{bx}+1)}\,dx.$$

(This problem was suggested by Problem A-3 on the 1982 William Lowell Putnam Mathematical Competition.)

I. *Solution by Leroy F. Meyers, The Ohio State University.*

By the integral property in (1), the integrals

$$\int_0^{at} \frac{f(y)}{y}\,dy = \int_0^t \frac{f(ax)}{x}\,dx \qquad \text{and} \qquad \int_0^{bt} \frac{f(y)}{y}\,dy = \int_0^t \frac{f(bx)}{x}\,dx$$

both exist for $t > 0$, and their difference is

$$I_t = \int_0^t \frac{f(ax) - f(bx)}{x}\,dx = \int_{bt}^{at} \frac{f(y)}{y}\,dy.$$

Let $f(\infty)$ denote the limit in (1). Given $\varepsilon > 0$, there exists a real number x_0 such that

$$x > x_0 \implies f(\infty) - \varepsilon < f(x) < f(\infty) + \varepsilon.$$

If $t > x_0/b$, then $at > bt > x_0$, and so

$$(f(\infty)-\varepsilon)\cdot\ln\frac{a}{b} = \int_{bt}^{at} \frac{f(\infty)-\varepsilon}{y}\,dy < \int_{bt}^{at} \frac{f(y)}{y}\,dy < \int_{bt}^{at} \frac{f(\infty)+\varepsilon}{y}\,dy = (f(\infty)+\varepsilon)\cdot\ln\frac{a}{b}.$$

Since ε is arbitrary, we obtain

$$I_1 = \lim_{t\to\infty} I_t = f(\infty)\cdot\ln\frac{a}{b}. \tag{2}$$

In particular, if

$$f(x) = \frac{1 - e^{-x}}{2(1 + e^{-x})} = \frac{e^x}{e^x + 1} - \frac{1}{2}, \qquad x \geq 0,$$

then $f(\infty) = 1/2$ and, for every $r > 0$, the integral in (1) is a *proper* integral, for $\lim_{x\downarrow 0} f(x)/x = 1/4$ by l'Hôpital's rule. For this f, the integral I_1 becomes I_2. Therefore

$$I_2 = f(\infty)\cdot\ln\frac{a}{b} = \frac{1}{2}\cdot\ln\frac{a}{b}.$$

It may be noted that (2) is valid also if $b \geq a > 0$.

II. *Solution by M.S. Klamkin, University of Alberta.*

Instead of (1), we assume only that $f(\infty)$ exists and that $f'(x)$ is continuous for $x \geq 0$. For such a function f, we have immediately, by Elliott's extension of Frullani's Theorem [1],

$$I_1 = (f(\infty) - f(0))\cdot\ln\frac{a}{b}.$$

In particular, I_1 becomes I_2 for $f(x) = -1/(1+e^x)$, and therefore

$$I_2 = (f(\infty) - f(0))\cdot\ln\frac{a}{b} = (0 + \frac{1}{2})\cdot\ln\frac{a}{b} = \frac{1}{2}\cdot\ln\frac{a}{b}.$$

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer. One incorrect solution was received.

Editor's comment.

References [2] and [3] were provided by Meyers. In [2], Problems 132 and 134 give (without attribution) Frullani's Theorem and Elliott's extension thereof, respectively. In [3] Elliott's extension is mentioned and misleadingly called Frullani's Theorem. The most thorough (and accessible) discussion of the Frullani and Elliott results is probably that in Edwards [4].

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2. R. Courant, *Differential and Integral Calculus* (translated by E.J. McShane), Interscience, New York, 1937, Vol. I, p. 561.

3. Bruce C. Berndt, "The Quarterly Reports of S. Ramanujan", *American Mathematical Monthly*, 90 (1983) 505-516, esp. p. 511.

4. Joseph Edwards, *A Treatise on the Integral Calculus*, Chelsea, New York, 1955, Vol. II, pp. 337-341, esp. p. 339.

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854. [1983: 178] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

For $x,y,z > 0$, let

$$A = \frac{yz}{(y+z)^2} + \frac{zx}{(z+x)^2} + \frac{xy}{(x+y)^2}$$

and

$$B = \frac{yz}{(y+x)(z+x)} + \frac{zx}{(z+y)(x+y)} + \frac{xy}{(x+z)(y+z)}.$$

It is easy to show that $A \le \frac{3}{4} \le B$, with equality if and only if $x = y = z$.

(a) Show that the inequality $A \le 3/4$ is "weaker" than $3B \ge 9/4$ in the sense that

$$A + 3B \ge \frac{3}{4} + \frac{9}{4} = 3.$$

When does equality occur?

(b) Show that the inequality $4A \le 3$ is "stronger" than $8B \ge 6$ in the sense that

$$4A + 8B \le 3 + 6 = 9.$$

When does equality occur?

I. *Solution by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

We will use the following easily established results:

$$1 - B = \frac{2xyz}{(y+z)(z+x)(x+y)},$$

from which follows

$$2 - 8(1-B) = \frac{2\{x(y-z)^2 + y(z-x)^2 + z(x-y)^2\}}{(y+z)(z+x)(x+y)}, \qquad (1)$$

and

$$3 - 4A = \frac{(y-z)^2}{(y+z)^2} + \frac{(z-x)^2}{(z+x)^2} + \frac{(x-y)^2}{(x+y)^2}. \qquad (2)$$

Observe that $A \le 3/4 \le B$, mentioned in the proposal, follows from (1) and (2).

Proof of (a). This inequality is equivalent to $A \ge 3(1-B)$, that is, to

$$\frac{yz}{(y+z)^2} + \frac{zx}{(z+x)^2} + \frac{xy}{(x+y)^2} \ge \frac{6xyz}{(y+z)(z+x)(x+y)},$$

or to

$$\frac{yz(z+x)(x+y)}{y+z} + \frac{zx(x+y)(y+z)}{z+x} + \frac{xy(y+z)(z+x)}{x+y} \ge 6xyz. \qquad (3)$$

If we denote by a,b,c the terms on the left side of (3), then

$$abc = x^2y^2z^2(y+z)(z+x)(x+y) \ge x^2y^2z^2(2\sqrt{yz})(2\sqrt{zx})(2\sqrt{xy}) = 8x^3y^3z^3$$

and

$$a + b + c \ge 3\sqrt[3]{abc} \ge 6xyz.$$

Thus (3) is established, with equality if and only if $x = y = z$.

Proof of (b). This inequality is equivalent to $(3-4A) - \{2-8(1-B)\} \geq 0$, that is, to

$$\frac{(y-z)^2}{(y+z)^2} + \frac{(z-x)^2}{(z+x)^2} + \frac{(x-y)^2}{(x+y)^2} - \frac{2\{x(y-z)^2 + y(z-x)^2 + z(x-y)^2\}}{(y+z)(z+x)(x+y)} \geq 0. \tag{4}$$

If we multiply this by $(y+z)^2(z+x)^2(x+y)^2 \geq 0$, we obtain an equivalent inequality $f(x,y,z) \geq 0$. After some tedious algebra, we find that

$$f(x,y,z) = (y-z)^2(z-x)^2(x-y)^2 \geq 0.$$

Thus (4) is established, with equality if and only if at least two of x,y,z are equal. (Most of the tedious algebra can be avoided by observing that f is a homogeneous and symmetric polynomial of degree 6 in x,y,z, and that both f and $\partial f/\partial x$ vanish when $x = y$.)

II. *Comment by M.S. Klamkin, University of Alberta.*

Let

$$g(x,y,z,\lambda) = A + \lambda(B-\tfrac{3}{4}) - \tfrac{3}{4}.$$

We show that $g \geq 0$ if $\lambda \geq 3$ and $g \leq 0$ if $\lambda \leq 2$. If $\lambda \geq 3$, then

$$A + \lambda(B-\tfrac{3}{4}) - \tfrac{3}{4} \geq A + 3(B-\tfrac{3}{4}) - \tfrac{3}{4} = A + 3B - 3 \geq 0$$

by part (a). If $\lambda \leq 2$, then

$$A + \lambda(B-\tfrac{3}{4}) - \tfrac{3}{4} \leq A + 2(B-\tfrac{3}{4}) - \tfrac{3}{4} = A + 2B - \tfrac{9}{4} \leq 0$$

by part (b). Either $g > 0$ or $g < 0$ may occur if $2 < \lambda < 3$. For example, if $2 < \lambda < 361/180$, then $g(2,1,1,\lambda) > 0$ and $g(1,2,3,\lambda) < 0$.

Putting it another way, we have shown the following: Of the two inequalities

$$A + \lambda B \geq \tfrac{3}{4}(1 + \lambda) \qquad \text{and} \qquad A + \lambda B \leq \tfrac{3}{4}(1 + \lambda),$$

the first holds for all (x,y,z) if $\lambda \geq 3$, the second holds for all (x,y,z) if $\lambda \leq 2$, and either one may hold if $2 < \lambda < 3$.

Also solved by the COPS of Ottawa; J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus (second solution); and the proposer.

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855. [1983: 178] *Proposed by Christian Friesen, student, University of New Brunswick.*

Let N be the set of natural numbers. For each $n \in N$, prove the existence of a polynomial $f_n(x_1,x_2,\ldots,x_n)$ such that the mapping $f_n : N^n \to N$ is a bijection.

Solutions were received from MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; F.D. HAMMER, Palo Alto, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LEROY F. MEYERS, The Ohio State University; and KENNETH S. WILLIAMS, Carleton University, Ottawa. A joint comment was received from M.S. KLAMKIN and A. LIU, both from University of Alberta, and GREGG PATRUNO, student, Princeton University.

Editor's comment.

This problem has a long history, going back to Cauchy (1821), and there has been much activity about it and related problems in the more recent literature, as evidenced by the list of references given below, most of which were found through information provided by readers.

In response to an anonymous query [1], Lew [2] gave a brief outline of the history of the problem, and he referred to his note [3] where a much fuller account is given, with an abundance of references that augments the list given below. The information in the next paragraph is taken from [3].

Relative to the polynomial bijections $f_n: N^n \to N$ with which we are concerned, Chowla [4] has displayed one such polynomial for each positive integer n, and Lew [5] has constructed $c(n)$ essentially distinct such polynomials (i.e., distinct to within a permutation of the variables), where $c(1) = 1$,

$$(c(2), c(3), c(4), c(5), c(6), c(7), \ldots) = (1, 3, 11, 45, 197, 903, \ldots)$$

is Sequence 1163 in Sloane [6], and, according to Knuth [7, pp. 239, 534, Ex. 12], as $n \to \infty$,

$$c(n) \sim k(3+2\sqrt{2})^n \cdot n^{-3/2}, \text{ where } k = \tfrac{1}{2}\sqrt{(3\sqrt{2}-4)/\pi} \approx 0.139.$$

The constructed polynomials have least and greatest degrees n and 2^{n-1}.

Of all our solvers, only Meyers showed how to find a polynomial of least degree n. He proved that $g_n: N_0^n \to N_0$, where N_0 is the set of *nonnegative* integers and g_n is defined by

$$g_n(x_1,x_2,\ldots,x_n) = \sum_{j=1}^{n} \binom{x_1+x_2+\ldots+x_j+j-1}{j},$$

is a polynomial bijection of degree n. The nontrivial part of the proof is equivalent to a well-known problem which can be found in Comtet [8], Knuth [7, pp. 72, 488, Ex. 56], and (with solution) in Liu [9]. Meyers then concluded that $f_n: N^n \to N$ is a polynomial bijection of degree n, where

$$f_n(x_1,x_2,\ldots,x_n) = 1 + g_n(x_1-1,x_2-1,\ldots,x_n-1). \qquad (1)$$

For $n = 1$ and $n = 2$, (1) becomes

$$f_1(x_1) = x_1 \qquad (2)$$

and

$$f_2(x_1,x_2) = x_1 + \frac{(x_1+x_2-1)(x_1+x_2-2)}{2}, \qquad (3)$$

respectively. Formula (3), which Lew [3] credits to Cantor (1878), is given in the more recent references Barnard & Child [10], Mathews [11], and Pólya-Szegö [12]. It was conjectured in 1923 by Fueter and Pólya (reference 6 in [3]) that (3) is essentially the *only* polynomial bijection $N^2 \to N$. It is known that (3) is essentially the only *quadratic* polynomial with this property and that, if there is another such polynomial, it must be at least of fifth degree [2].

We now consider polynomial bijections $f_n: N^n \to N$ of highest degree 2^{n-1}. For $n = 1$ and $n = 2$, such polynomials are given by (2) and (3), respectively. For $n > 2$, Meyers (and all other solvers) argued that the required polynomials can be defined recursively by

$$f_n(x_1,x_2,\ldots,x_n) = f_2(f_{n-1}(x_1,x_2,\ldots,x_{n-1}),\ x_n).$$

For if $f_{n-1}: N^{n-1} \to N$ is a polynomial bijection of degree 2^{n-2}, then $f_n: N^n \to N$ is a polynomial bijection of degree 2^{n-1}.

Now a look at some related problems.

Meyers showed how to find a bijection $f_\infty: N^\infty \to N$, where N^∞ is the set of all finite sequences (tuples) of positive integers, but he proved that such a bijection cannot be a polynomial (of finite degree).

Hammer [13] asked if there is a polynomial in two variables with integral coefficients which is a bijection from Z^2 onto Z, where Z is the set of all integers, and if so, how many there are. His questions remain unanswered to this day.

Finally, Lew [3] posed the following hybrid problem: to find a polynomial $f(x_1,\ldots,x_n)$ which maps Z^n *surjectively* onto N. He writes that this is clearly impossible when $n = 1$, and it is always possible when $n \geq 3$, by the famous theorem of Legendre and Gauss: each positive integer is the sum of three triangular numbers. Hence the two-dimensional case is the sole undecided one.

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3. _________, "Polynomials in two variables taking distinct integer values at lattice-points", *Amer. Math. Monthly*, 88 (1981) 344-346.

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7. Donald E. Knuth, *The Art of Computer Programming, Vol. 1: Fundamental Algorithms*, Second Edition, Addison-Wesley, Reading, 1973.

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10. S. Barnard & J.M. Child, *Advanced Algebra*, Macmillan, London, 1939, p. 373, Ex. 109.

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13. F.D. Hammer, (proposer of) Problem 6028*, *Amer. Math. Monthly*, 82 (1975) 410.

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856. [1983: 179] *Proposed by Jack Garfunkel, Flushing, N.Y.*

For a triangle ABC with circumradius R and inradius r, let $M = (R-2r)/2R$. An inequality $P \geq Q$ involving elements of triangle ABC will be called *strong* or *weak*, respectively, according as

$$P - Q \leq M \quad \text{or} \quad P - Q \geq M.$$

(a) Prove that the following inequality is strong:

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} \geq \frac{3}{4}.$$

(b) Prove that the following inequality is weak:

$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} \geq \sin B\sin C + \sin C\sin A + \sin A\sin B.$$

Solution by W.J. Blundon, Memorial University of Newfoundland.

We will use the following well-known relations, where all sums are cyclic over A,B,C:

$$R - 2r \geq 0, \qquad \Sigma\sin^2\frac{A}{2} = \frac{2R-r}{2R}, \qquad \Sigma\cos^2\frac{A}{2} = \frac{4R+r}{2R}, \qquad (1)\text{-}(2)\text{-}(3)$$

and $s^2 \leq 4R^2 + 4Rr + 3r^2$ (with equality just when $R = 2r$), from which follows

$$\Sigma \sin B \sin C = \frac{s^2 + 4Rr + r^2}{4R^2} \le \frac{(R+r)^2}{R^2}. \tag{4}$$

For each of the proposed inequalities, it will be seen that equality holds just when $R = 2r$, that is, just when the triangle is equilateral.

(a) From (1) and (2),

$$P - Q = \Sigma \sin^2 \frac{A}{2} - \frac{3}{4} = \frac{2R-r}{2R} - \frac{3}{4} = \frac{R-2r}{4R} \le \frac{R-2r}{2R} = M,$$

and the given inequality is strong.

(b) From (1), (3), and (4),

$$P - Q - M = \Sigma \cos^2 \frac{A}{2} - \Sigma \sin B \sin C - M$$

$$\ge \frac{4R+r}{2R} - \frac{(R+r)^2}{R^2} - \frac{R-2r}{2R} = \frac{(R-2r)(R+r)}{2R^2} \ge 0,$$

and the given inequality is weak.

Editor's comment.

The proposer suggested that readers may find it interesting to pursue the idea of classifying known triangle inequalities as strong or weak, using the M given above or some other measure.

Also solved by CURTIS COOPER, Central Missouri State University at Warrensburg; the COPS of Ottawa; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

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857. [1983: 179] *Proposed by Leroy F. Meyers, The Ohio State University.*

(a) Given three positive integers, show how to determine algebraically (rather than by a search) the row (if any) of Pascal's triangle in which these integers occur as consecutive entries.

(b) Given two positive integers, can one similarly determine the row (if any) in which they occur as consecutive entries?

(c)* The positive integer k occurs in the row of Pascal's triangle beginning with $1, k, \ldots$. For which integers is this the only row in which it occurs?

Solution by the proposer.

(a) Given three positive integers a, b, c, we find necessary conditions for the existence of a positive integer n such that, for some integer k satisfying $1 \le k < n$, we have

$$\binom{n}{k-1} = a, \qquad \binom{n}{k} = b, \qquad \binom{n}{k+1} = c. \tag{1}$$

From the properties of binomial coefficients, (1) implies that

$$a(n-k+1) = bk \quad \text{and} \quad b(n-k) = c(k+1). \tag{2}$$

Our first necessary condition is $b^2 \neq ca$, for otherwise (2) implies that $n = -1$. This condition being satisfied, we solve equations (2) for n and k, obtaining

$$n = \frac{bc + 2ca + ab}{b^2 - ca}, \qquad k = \frac{a(b + c)}{b^2 - ca}. \tag{3}$$

So necessary conditions for n to be an answer to our problem are that $b^2 \neq ca$ and that n and k, as given by (3), be integers satisfying $1 \le k < n$.

These conditions, however, are not sufficient, for (2) was obtained by taking only *ratios* of the binomial coefficients (1). For example, if $a = 1$ and $b = c = 2$, then (3) yields $n = 5$ and $k = 2$, but $\binom{5}{2} = 10 \neq 2$. When the necessary conditions are satisfied, only one more calculation is needed to determine if the value found for n is satisfactory: n and the value found for k must satisfy any one of equations (1).

(b) Given two positive integers a and b, we find necessary conditions for the existence of a positive integer n such that, for some integer k satisfying $1 \le k \le n$, we have

$$\binom{n}{k-1} = a \quad \text{and} \quad \binom{n}{k} = b. \tag{4}$$

Two simple cases can be settled instantly: if $a = 1$, then $k = 1$ and $n = b$ is satisfactory; and if $b = 1$, then $k = n$ and $n = a$ is satisfactory. For arbitrary a and b, we obtain $a(n-k+1) = bk$ as in (2), and so, if $d = \gcd(a,b)$, $\alpha = a/d$, and $\beta = b/d$, we have

$$\alpha(n-k+1) = \beta k. \tag{5}$$

Since $\gcd(\alpha,\beta) = 1$, we must therefore have

$$k = \alpha t \tag{6}$$

for some positive integer t, and then, from (5),

$$n = (\alpha+\beta)t - 1. \tag{7}$$

Observe that, for any positive integer t, (6) and (7) ensure that $1 \le k \le n$. But n and k must also satisfy either one (and hence both) of equations (4). Since

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k},$$

the positive integer t must therefore satisfy

$$\binom{(\alpha+\beta)t}{\alpha t} = a + b. \tag{8}$$

Our necessary conditions are now all in place: n must be of the form (7) for some positive integer t satisfying (8).

As in part (a), these conditions are not sufficient. Having found a positive integer t satisfying (8), we must still verify if the corresponding n and k, as given by (7) and (6), satisfy one of equations (4). Fortunately, only finitely many values of t need be tested in (8) because

$$\binom{(\alpha+\beta)(t+1)}{\alpha(t+1)} = \binom{(\alpha+\beta)t}{\alpha t}\left(\prod_{j=1}^{\alpha}\frac{(\alpha+\beta)t+j}{\alpha t+j}\right)\left(\prod_{j=1}^{\beta}\frac{(\alpha+\beta)t+\alpha+j}{\beta t+j}\right) > \binom{(\alpha+\beta)t}{\alpha t}$$

shows that the function on the left side of (8) increases strictly with t.

(c)* I have no idea. Of course, a prime number can occur in only one row.

Also solved by FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio.

Editor's comment.

Kierstead's terse answer to part (c)* was the two-word statement "Almost all". We await a more detailed answer.

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858. [1983: 179] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with sides a,b,c. For $n = 0,1,2,\ldots$, let P_n be a point in the plane whose distances d_a, d_b, d_c from sides a,b,c satisfy

$$d_a : d_b : d_c = \frac{1}{a^n} : \frac{1}{b^n} : \frac{1}{c^n}.$$

(a) A point P_n being given, show how to construct P_{n+2}.

(b) Using (a), or otherwise, show how to construct the point P_n for an arbitrary given value of n.

Solution by Roland H. Eddy, Memorial University of Newfoundland.

Denote by $g(\mathrm{P})$ and $t(\mathrm{P})$ the isogonal and isotomic conjugates, respectively, of a point P in the plane of the given triangle ABC. If (α, β, γ) are the trilinear coordinates of P with respect to ABC, then [1, p. 159]

$$g(\mathrm{P}) = \left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right) \quad \text{and} \quad t(\mathrm{P}) = \left(\frac{1}{a^2\alpha}, \frac{1}{b^2\beta}, \frac{1}{c^2\gamma}\right).$$

Consequently,

$$tg(\mathrm{P}) = \left(\frac{\alpha}{a^2}, \frac{\beta}{b^2}, \frac{\gamma}{c^2}\right). \tag{1}$$

(a) It follows from (1) that $P_{n+2} = tg(P_n)$. Consequently, P_n being given, we obtain P_{n+2} by constructing the isotomic conjugate of the isogonal conjugate of P_n.

(b) For $n = 0$ and $n = 1$, we have

$$P_0 = (1, 1, 1) = I, \text{ the incentre of the triangle}$$

and

$$P_1 = (\tfrac{1}{a}, \tfrac{1}{b}, \tfrac{1}{c}) = G, \text{ the centroid of the triangle,}$$

and these points are easily constructed. For $n > 1$, we can use the method of part (a) and construct successively (and laboriously)

$$P_0, P_2, P_4, \ldots, P_n, \text{ if } n \text{ is even,}$$

or

$$P_1, P_3, P_5, \ldots, P_n, \text{ if } n \text{ is odd.}$$

A much more practical method, however, is to calculate the actual distances d_a, d_b, d_c of P_n from a, b, c using the relationships

$$a^n d_a = b^n d_b = c^n d_c = \frac{2K}{a^{1-n} + b^{1-n} + c^{1-n}},$$

where K denotes the area of triangle ABC [1, p. 157]. Any two of these distances uniquely determines P_n.

Also solved by KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

REFERENCE

1. D.M.Y. Summerville, *Analytical Conics*, G. Bell and Sons, London, 1961.

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859. [1983: 179] *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

Let ABC be an acute-angled triangle of type II, that is (see [1982: 64]), such that $A \le B \le \pi/3 \le C$, with circumradius R and inradius r. It is known [1982: 66] that for such a triangle $x \ge \frac{1}{4}$, where $x = r/R$. Prove the stronger inequality

$$x \ge \frac{\sqrt{3} - 1}{2}.$$

Solution by the proposer.

We show that the desired result holds even if angle $C = 90^\circ$. Let s be the semi-perimeter of the triangle and $y = s/R$. The following identity of Bager (p. 10 of reference 3 in [1982: 68]) is valid for every triangle:

$$4 \cos A \cos B \cos C = \frac{s^2 - (2R+r)^2}{R^2} = y^2 - (2+x)^2.$$

Since our triangle is nonobtuse, we have $y^2 - (2+x)^2 \geq 0$, and so

$$y \geq 2 + x. \tag{1}$$

The result

$$\sqrt{3}(1+x) \geq y \tag{2}$$

was established in this journal [1982: 64] for type II triangles. Finally, from (1) and (2) we obtain $\sqrt{3}(1+x) \geq 2+x$, from which follows

$$x \geq \frac{\sqrt{3} - 1}{2} > \frac{1}{4}.$$

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and GEORGE TSINTSIFAS, Thessaloniki, Greece.

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860.* [1983: 180] *Proposed by Anders Lönnberg, Mockfjärd, Sweden.*

The sequence $\{s_n\}_{n=1}^{\infty}$ is defined by

$$s_n = \sum_{m=1}^{n} (-1)^{m-1} m^{n-(m-1)},$$

so that

$$s_1 = 1^1 = 1, \quad s_2 = 1^2-2^1 = -1, \quad s_3 = 1^3-2^2+3^1 = 0, \quad s_4 = 1^4-2^3+3^2-4^1 = -2,$$

etc. Does $s_n = 0$ ever occur again for some $n > 3$?

Comment by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.

The following results suggest that $|s_{n+1}| > |s_n|$ for $n \geq 5$. If this is true, then $s_n = 0$ will never occur again for any $n > 3$.

s_5 = 1		s_{19} = -478198544		
s_6 = 5		s_{20} = -1994889946		[Oops! (Editor)]
s_7 = 20		s_{21} = -1669470783		
s_8 = 28		s_{22} = 56929813933		
s_9 = -47		s_{23} = 615188040196		
s_{10} = -525		s_{24} = 3794477505572		
s_{11} = -2056		s_{25} = 12028579019537		
s_{12} = -3902		s_{26} = -50780206473221		
s_{13} = 9633		s_{27} = -1172949397924184		
s_{14} = 129033		s_{28} = -10766410530764118		
s_{15} = 664364		s_{29} = -61183127006113951		
s_{16} = 1837904		s_{30} = -102718668475675151		
s_{17} = -2388687		s_{31} = 2573781218763700380		
s_{18} = -67004697		s_{32} = 40137777624890418072		

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