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CONTENTS

THE SEARCH FOR A UNIVERSAL COVER

BASIL C. RENNIE, James Cook University of North Queensland

The *diameter* of a plane set is the least upper bound of the distances apart of pairs of points. A *universal cover* (for plane sets of diameter 1) is a set C such that if any set F has diameter ≤ 1 then C (by suitable rotation and translation) will cover F. In 1914 Lebesgue asked what was the smallest possible area of a universal cover. The question has still not been answered but some progress has been made.

A square of unit side is a universal cover. To prove this imagine any set E of diameter one or less drawn on your paper; and think of a horizontal barrier moving up until it meets F, leave it there, then bring down another horizontal barrier from the top to unit distance from the first. Then E must be between the two barriers, and so you have covered F by a horizontal strip of width one. Now in the same way cover E with a vertical strip, and you have F inside a unit square. This gives a universal cover of area one, but we can do better. In fact the regular hexagon of side $1/\sqrt{3}$ is a universal cover.

Take any E of diameter < 1 and cover it with a horizontal strip of minimal width, then making the obvious modification to the construction, cover E with a strip of minimal width at $60°$ to the horizontal (see Figure 1). This puts F inside a parallelogram.

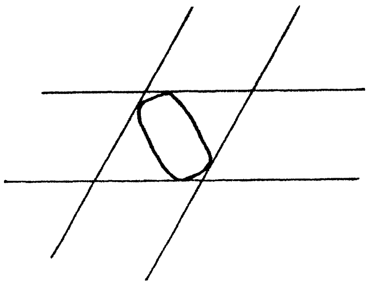


Figure 1

Now if we similarly take a third strip of minimal width at $60°$ to the other two we shall have F inside a hexagon, but this is not good enough because the middle line of the new strip may not go through the centre of the parallelogram. To get over this difficulty we slowly rotate F, translating it all the time to keep the centre of the parallelogram in the same place. If the centre of the parallelogram was initially on one side of the middle line of the third strip then after $180°$ of rotation it will be on the other side, and so at some intermediate stage the median lines of the three strips are concurrent. Now if necessary expand the strips symmetrically to unit width and E is trapped in a regular hexagon of side $1/\sqrt{3}$ and area $\frac{1}{2}\sqrt{3} = .8660$.

This is a smaller universal cover than the unit square, but we can still do a little better.

Consider the little triangles at the six corners of the hexagon cut off by lines that make angles of 30° with the sides and are at a perpendicular distance $\frac{1}{2}$ from the middle of the hexagon (see Figure 2).

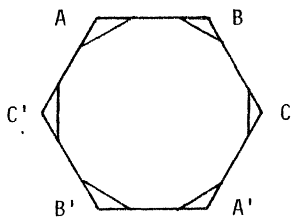


Figure 2

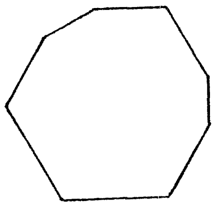


Figure 3

Every point inside the corner triangle at A is at a distance > 1 from every point in the corner triangle at A'. Consequently when the hexagon covers a set E of a diameter ≤ 1, either the triangle at A or the triangle at A' contains no points of E. A similar argument applies to the other two pairs, and so at least three of the little triangles are unoccupied. Therefore we may cut off two of the little triangles, at corners that are neither adjacent nor opposite one another. The slightly mutilated hexagon (see Figure 3) will still be a universal cover because by rotation we may keep the cut-off corners away from the occupied corners. The area of our new universal cover is $2 - 2/\sqrt{3} = .8453$. It was discovered by J. Pál in 1920 and as far as I know it is still the smallest known. The best result in the opposite direction is by the same author, who showed that every universal cover must have area at least $\pi/8 + \sqrt{3}/4 = .8257$

REFERENCES

1. Hadwiger, Debrunner and Klee, *Combinatorial Geometry in the Plane*, Holt Rinehart and Winston, 1964.

2. B. Grünbaum, "Borsuk's Problem and Related Questions", in *Convexity* (A.M.S. Symposia, VII, 1963).

* * *

MAMA-THEMATICS I

1. Mother to Leonardo Fibonacci: "So that's why you wanted those Easter bunnies."

2. Mother to Cardan: "Why can't you get along with that Tartaglia lad?"

CHARLES W. TRIGG

* * *

SOME FOURTH POWER CURIOSA

CHARLES W. TRIGG, Professor Emeritus, Los Angeles City College

In the decimal system, there are 20 fourth powers each composed of distinct digits, namely:

N	N^4	N	N^4	N	N^4	N	N^4
1	1	6	1296	14	38416	*32	1048576
*2	16	*7	2401	17	83521	38	2085136
*3	81	*8	4096	18	104976	44	3748096
*4	256	12	20736	23	279841	48	5308416
5	625	13	28561	25	390625	49	5764801

In the 6 cases indicated by an asterisk (*) the digits of N and N^4 together are distinct. In the last case (32^4), the nine digits are consecutive, although not in order of magnitude.

In four cases, 6^4, 12^4, 13^4, and 14^4, the sum of 3 of the digits of N^4 is equal to the sum of the remaining digits. This property is also exhibited by $11^4 = 14641$, which is the smallest of an infinite series of palindromic fourth powers, having the general term $(10_n 1)^4 = 10_n 40_n 60_n 40_n 1$. Here 0_n represents a sequence of n 0's, where $n = 0, 1, 2, 3, \ldots$.

Also palindromic is the first nine-digit portion of the ten-digit period, 1 6 1 6 5 6 1 6 1 0, of the repetitive sequence formed by the units' digits of successive fourth powers.

The sum of the first two digits of $31^4 = 923521$ equals the sum of the remaining digits. Save for the second 2, the digits of $22^4 = 234256$ are consecutive in increasing order of magnitude. Made to order for the current year is

$$(375)^4 = \underline{1977}5390625.$$

* * *

The Ten-Digit Man
A Bust

CHARLES W. TRIGG

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well-known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by someone else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than June 1, 1977.

221. *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Solve this cryptarithm: $CW^2 = TRI.GG$, subject to the condition that, since he is unique (the only 1 of his kind), the solution should not contain the digit 1.

222. *Proposed by Bruce McColl, St. Lawrence College, Kingston, Ontario.*

Prove that

$$\tan\frac{\pi}{11}\tan\frac{2\pi}{11}\tan\frac{3\pi}{11}\tan\frac{4\pi}{11}\tan\frac{5\pi}{11} = \sqrt{11}.$$

223. *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

Without using any table which lists Pythagorean triples, find the smallest integer which can represent the area of two noncongruent primitive Pythagorean triangles.

224. *Proposed by M.S. Klamkin, University of Alberta.*

Let P be an interior point of a given n-dimensional simplex of vertices $A_1, A_2, \ldots, A_{n+1}$. Let P_i $(i = 1,2,\ldots,n+1)$ denote points on A_iP such that $A_iP_i/P_iP = 1/n_i$. Finally, let V_i denote the volume of the simplex cut off from the given simplex by a hyperplane through P_i parallel to the face of the given simplex opposite A_i. Determine the minimum value of ΣV_i and the location of the corresponding point P.

225. *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*

C is a point on the diameter AB of a circle. A chord through C, perpendicular to AB, meets the circle at D. Two chords through B meet CD at T_1, T_2 and arc AD at U_1, U_2 respectively. It is known from Problem 220 that there are circles C_1, C_2 tangent to CD at T_1, T_2 and to arc AD at U_1, U_2 respectively. Prove that the radical axis of C_1 and C_2 passes through B.

226. *Proposed by David L. Silverman, West Los Angeles, California.*

The positive integers are divided into two disjoint sets A and B. A positive integer is an A-number if and only if it is the sum of two different A-numbers or of two different B-numbers. Find A.

227. *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

It is well-known that

$$\sqrt{a^2+1} = \langle a, \overline{2a} \rangle = a + \frac{1}{2a+}\,\frac{1}{2a+}\,\frac{1}{2a+}\cdots$$

for all positive integers a. Solve completely in positive integers each of the equations

$$\sqrt{a^2+y} = \langle a, \overline{x, 2a} \rangle \quad \text{and} \quad \sqrt{a^2+y} = \langle a, \overline{x, x, 2a} \rangle,$$

where in both cases $x \neq 2a$.

228. *Proposed by Charles W. Trigg, San Diego, California.*

(a) Find four consecutive primes having digit sums that, in some order, are consecutive primes.

(b) Find five consecutive primes having digit sums that are distinct primes.

229. *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

On an examination, one question asked for the largest angle of the triangle with sides 21, 41, and 50. A student obtained the correct answer as follows: Let C denote the desired angle; then $\sin C = 50/41 = 1 + 9/41$. But $\sin 90° = 1$ and $9/41 = \sin 12° 40' 49''$. Thus

$$C = 90° + 12° 40' 49'' = 102° 40' 49'',$$

which is correct. Find the triangle of least area having integral sides and possessing this property.

230. *Proposed by R. Robinson Rowe, Sacramento, California.*

Find the least integer N which satisfies

$$N = a^{ma+nb} = b^{mb+na}$$

with m and n positive and $1 < a < b$. (This generalizes Problem 219.)

* * *

3. Madame de Fermat to Pierre: "If the margin is too small, why not use the fly-leaf?"

4. Mrs. Shanks to son William: "You should have used a computer."

CHARLES W. TRIGG

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

145. [1976: 94, 181, 224; 1977: 16] *Proposed by Walter Bluger, Department of National Health and Welfare, Ottawa, Ont.*

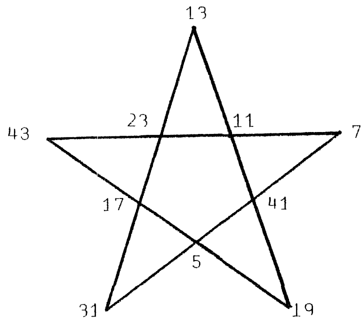
A *pentagram* is a set of 10 points consisting of the vertices and the intersections of the diagonals of a regular pentagon with an integer assigned to each point. The pentagram is said to be *magic* if the sums of all sets of 4 colinear points are equal.

Construct a magic pentagram with the ten smallest possible positive primes.

IV. *Solution by Harry L. Nelson , Livermore, California.*

If 1 is allowed as a prime, there are, for line sum 72, just the two solutions published in [1976: 225] and confirmed by Trigg in [1977· 16].

An exhaustive computer search of all line sums ≤ 100 (personal time spent 2–3 hours, machine time ½ minute) shows that, if 1 is not allowed as a prime, the smallest line sum for which any solution exists is 84, for which the set {5,7,11,13,17,19,23,31,41,43}, arranged, for example, as in the adjoining figure, is the unique solution set. As shown by Trigg in *Ten Elements on a Pentagram* [1977: 5], these numbers can be arranged to form 12 different magic prime pentagrams all with line sum 84.



For line sum 96, there are four solution sets:

{3,5,7,13,17,23,29,31,53,59},
{3,5,7,11,13,29,31,41,47,53},
{3,7,11,13,17,23,29,31,47,59},
{7,11,13,17,19,23,29,31,43,47}.

I'll leave it to the readers to discover how the numbers in these sets can be arranged to form magic prime pentagrams.

Editor's comment.

Harry L. Nelson, whom we welcome to our pages, is the Executive Editor of the *Journal of Recreational Mathematics.*

* * *

173. [1976: 171; 1977: 47] (Corrected) *Proposed by Dan Eustice, The Ohio State University.*

For each choice of n points on the unit circle ($n \geq 2$), there exists a point on the unit circle such that the product of the distances to the chosen points is ≥ 2. Moreover, the product is ≤ 2 for all points on the unit circle if and only if the n points are the vertices of a regular polygon.

Solution by the proposer.

If we set

$$P(z) = (z - z_1)\ldots(z - z_n) = z^n + a_{n-1} z^{n-1} + \ldots + a_0,$$

where the z_i are the chosen points, it follows that $|P(0)| = |a_0| = 1$. Without loss of generality, we can assume that $a_0 = 1$. (Replace z by ηz, where η is an nth root of a_0, and divide by a_0 for the equivalent simpler problem.) Thus

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0, \quad \text{with } a_n = a_0 = 1.$$

Let $\omega = \exp(2\pi i/n)$, a primitive nth root of unity, and consider

$$A = \frac{1}{n} \sum_{k=0}^{n-1} P(\omega^k), \tag{1}$$

so that

$$A = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{n} a_j (\omega^k)^j = \frac{1}{n} \sum_{j=0}^{n} a_j \sum_{k=0}^{n-1} \omega^{kj}.$$

For $j \neq 0$ or n, we have

$$\sum_{k=0}^{n-1} (\omega^j)^k = \frac{(\omega^j)^n - 1}{\omega^j - 1} = 0,$$

and since $\sum_{k=0}^{n-1} 1 = n$ and $\sum_{k=0}^{n-1} (\omega^n)^k = n$, it follows that $A = 2$. Since the average of the $P(\omega^k)$ is 2, it must happen that $|P(\omega^k)| > 2$ for some k unless $P(\omega^k) = 2$ for all k. For suppose $|P(\omega^k)| \leq 2$ for all k. Since $A = 2$ it follows from (1) that

$$\frac{1}{n} \sum_{k=0}^{n-1} \text{Re}\,(P(\omega^k)) = 2.$$

But then

$$2 = \frac{1}{n} \sum_{k=0}^{n-1} \text{Re}\,(P(\omega^k) \leq \frac{1}{n} \sum_{k=0}^{n-1} |P(\omega^k)| \leq \frac{1}{n} \cdot 2n = 2,$$

and the resulting string of equalities implies that, for all k,

$$\text{Re}\,(P(\omega^k)) = |P(\omega^k)| = P(\omega^k) = 2. \qquad (2)$$

Now the last equality in (2) yields the linear system

$$a_{n-1}(\omega^k)^{n-1} + \ldots + a_1\omega^k = 0 \qquad \text{for } k = 1, \ldots, n-1.$$

This system of equations in $a_{n-1}, \ldots, a_1$ has only the trivial all zero solution, for the determinant of the coefficient matrix is a Vandermonde determinant which is nonzero since the ω^k are all distinct. Thus the assumption that $|P(z)| \le 2$ for all z on the unit circle is equivalent to $P(z) = z^n + 1$.

Editor's comment.

The beautiful solution given here is quite different and, in the editor's opinion, much simpler than the one given in *The American Mathematical Monthly* (see Reference [1] in [1977: 48]).

* * *

176. [1976: 171; 1977: 29] *Proposé par Hippolyte Charles, Waterloo, Québec.*

Soit $f{:}R \to R$ une fonction différentiable paire. Montrer que sa dérivée f' n'est pas paire, à moins que f ne soit une fonction constante.

III. *Commentaire de L.F. Meyers, The Ohio State University.*

La solution II [1977: 30] n'est pas valide. Il existe des fonctions (même paires) partout différentiables mais dont la dérivée n'est intégrable sur aucun intervalle fermé. Voir [1].

RÉFÉRENCE

1. Bernard R. Gelbaum and J. M. H. Olmsted, *Counterexamples in Analysis*, Holden-Day, San Francisco, 1964, pp. 37, 43, 107-108.

* * *

183. [1976: 193] *Proposed by Viktors Linis, University of Ottawa.*

If $x + y = 1$, show that

$$x^{m+1}\sum_{j=0}^{n} y^j C^j_{m+j} + y^{n+1}\sum_{i=0}^{m} x^i C^i_{n+i} = 1$$

holds for all $m, n = 0, 1, 2, \ldots$.

This problem is taken from the list submitted for the 1975 Canadian Mathematical Olympiad (but not used on the actual exam).

Solution by Mark Kleiman, Stuyvesant H.S., Staten Island, N.Y.

Let

$$f(m,n) = x^{m+1}\sum_{j=0}^{n} y^j C^j_{m+j} + y^{n+1}\sum_{i=0}^{m} x^i C^i_{n+i}.$$

Since $x = 1-y$, we have

$$f(m,n) = x^m \sum_{j=0}^{n} (y^j - y^{j+1})\, C^j_{m+j} + y^{n+1} \sum_{i=0}^{m} x^i C^i_{n+i}$$

$$= x^m \sum_{j=0}^{n} y^j (C^j_{m+j} - C^{j-1}_{m+j-1}) - x^m y^{n+1} C^n_{m+n} + y^{n+1} \sum_{i=0}^{m-1} x^i C^i_{n+i} + y^{n+1} x^m C^m_{m+n}$$

$$= x^m \sum_{j=0}^{n} y^j C^j_{m+j-1} + y^{n+1} \sum_{i=0}^{m-1} x^i C^i_{n+1}$$

$$= f(m-1,n),$$

and similarly $f(m,n) = f(m,n-1)$. An easy induction now shows that, for all $m,n = 0,1,2,\ \ldots,$

$$f(m,n) = f(0,0) = x+y = 1$$

* * *

184. [1976: 193] *Proposé par Hippolyte Charles, Waterloo, Québec.*

Si $I = \{x \in R \mid a \le x \le b\}$ et si la fonction $f : I \to I$ est continue, montrer que l'équation $f(x) = x$ admet au moins une solution dans I.

Solution de Bernard Vanbrugghe, Université de Moncton.

Nous supposerons $f(a) \ne a$ et $f(b) \ne b$, car autrement le théorème est vérifié. Si l'on pose $g(x) = f(x) - x$, alors la fonction g est continue et

$$g(a) = f(a) - a > 0, \qquad g(b) = f(b) - b < 0.$$

Le théorème des valeurs intermédiaires nous assure qu'il existe un $x_0 \in I$ tel que $g(x_0) = 0$, c'est-à-dire tel que $f(x_0) = x_0$.

Also solved in the same way by CLAYTON W. DODGE, *University of Maine at Orono; and the proposer.*

* * *

185. [1976: 194] *Proposed by H.G. Dworschak, Algonquin College.*

Prove that, for any positive integer $n > 1$, the equation

$$1 + 2x + 3x^2 + \ldots + nx^{n-1} = n^2$$

has a rational root between 1 and 2.

I. *Solution by T.J. Griffiths, A.B. Lucas S.S., London, Ontario.*

We have, for $x > 1$,

$$1 + 2x + 3x^2 + \ldots + nx^{n-1} = \frac{d}{dx}\left(x + x^2 + \ldots + x^n\right) = \frac{d}{dx}\left(\frac{x - x^{n+1}}{1-x}\right)$$

$$= \frac{1 - nx^n\left(1 + \frac{1}{n} - x\right)}{(1-x)^2}$$

and this expression equals n^2 when $x = 1 + \frac{1}{n}$.

II. *Solution by R. Robinson Rowe, Sacramento, California.*

The given equation can be written

$$1 - n^2 + 2x + 3x^2 + \ldots + nx^{n-1} = 0,$$

and the left member can be factored as follows:

$$[nx - (n+1)][x^{n-2} + 2x^{n-3} + \ldots + (n-2)x + (n-1)] = 0.$$

The first factor vanishes for $x = 1 + \frac{1}{n}$, which is the required rational root.

Also solved by HIPPOLYTE CHARLES, *Waterloo, Québec;* RADFORD DE PEIZA, *Woburn C.I., Scarborough, Ont. (partial solution);* G.D. KAYE, *Department of National Defence, Ottawa (partial solution);* F.G.B. MASKELL, *Algonquin College, Ottawa;* L.F. MEYERS, *The Ohio State University;* DAVID R. STONE, *Georgia Southern College, Statesboro, Georgia;* KENNETH M. WILKE, *Topeka, Kansas; and the proposer. One incorrect solution was received.*

Editor's comment.

This problem was reprinted, with permission, in *James Cook Mathematical Notes No.7* (January 1977), edited by Basil C. Rennie, James Cook University of North Queensland, Australia, and a solution by J.B. Parker appeared in *JCMN No.8* (February 1977).

* * *

186. [1976: 194] *Proposed by Leroy F. Meyers, The Ohio State University.*

Let A, B, C, and D be the subsets of the plane R^2 having, respectively, both coordinates rational, both coordinates irrational, exactly one coordinate rational, and both coordinates or neither rational. Which of these sets is/are connected? (A subset of the plane is connected just when it cannot be expressed as the union of two disjoint nonempty sets neither of which contains a boundary point of the other.)

Solution by the proposer.

If there is a line in the plane which separates a set into two nonempty parts but does not intersect the set, then the set must be unconnected. Since A, B, and C are obviously so separated by the lines with equations $x = \sqrt{2}$, $x = 0$, and $x = y$, respectively, these sets cannot be connected.

On the other hand, suppose S and T are disjoint nonempty sets whose union is D. Then one of them, say S, must include the connected set U consisting of all points (x,y) for which $x + y$ or $x - y$ is rational. Since the boundary of U is the entire plane, the boundary of S must contain some point of T. Hence D is connected.

* * *

187. [1976: 194] *Proposé par André Bourbeau, École Secondaire Garneau.*

Si $m = 2^n \cdot 3 \cdot p$, où n est un entier positif et p un nombre premier impair, trouver toutes les valeurs de m pour lesquelles $\sigma(m) = 3m$, $\sigma(m)$ étant la somme de tous les diviseurs de m.

Solution de L.F. Meyers, The Ohio State University.

On a

$$\sigma(m) = \begin{cases} (2^{n+1} - 1)\cdot 13 & \text{si } p = 3, \\ (2^{n+1} - 1)\cdot 4 \cdot (p+1) & \text{si } p > 3. \end{cases}$$

Mais $p = 3$ est impossible, car alors $\sigma(m)$ et $3m$ ont des parités opposées. Il nous reste l'équation

$$(2^{n+1} - 1) \cdot 4 \cdot (p+1) = 2^n \cdot 9 \cdot p,$$

où $p > 3$. Les nombres 2^n et $2^{n+1} - 1$ étant premiers entre eux, il existe un entier positif r tel que $4(p+1) = 2^n r$, de sorte que $(2^{n+1} - 1)r = 9p$ et, enfin, $r = 8 - p$. Les seules possibilités sont donc $p = 5$ et 7, qui donnent

$$m = 2^3 \cdot 3 \cdot 5 = 120 \text{ et } m = 2^5 \cdot 3 \cdot 7 = 672.$$

Réciproquement, on peut facilement s'assurer que ces deux nombres vérifient bien les conditions du problème.

Also solved by RADFORD DE PEIZA, *Woburn C.I., Scarborough, Ontario;* CLAYTON W. DODGE, *University of Maine at Orono;* MARK KLEIMAN, *Stuyvesant H.S., Staten Island, N.Y.;* R. ROBINSON ROWE, *Sacramento, California;* DAVID R. STONE, *Georgia Southern College, Statesboro, Georgia;* KENNETH M. WILKE, *Topeka, Kansas; and the proposer. One incorrect solution was received.*

Editor's comment.

Natural numbers m such that $\sigma(m) = 3m$ are called *triply perfect numbers* (*TPN*) (ordinary perfect numbers satisfy $\sigma(m) = 2m$). So this problem shows that the only even triply perfect numbers of the form $2^n \cdot 3 \cdot p$, where p is an odd prime, are 120 and 672. Rowe, quoting information found in [2], says that this result has been known since Fermat (1636), was rediscovered by Desboves (1878), and perhaps by many others up to and including our own proposer (1976). Other even *TPN* are known. Rowe quotes the following from [2]:

$$523776 = 2^9 \cdot 3 \cdot 11 \cdot 31$$
$$459818240 = 2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$$
$$1476304896 = 2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127$$
$$51001180160 = 2^{14} \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151$$

It is not known if odd *TPN* exist. The latest information I've been able to uncover about them is due to Claude W. Anderson, University of California, Berkeley [1]. He has shown that an odd *TPN* has the form $m = (p^{3k+1} \cdot n)^2$, where $p \equiv 1 \pmod 6$ is prime, n is odd, and $(p,n) = 1$; and that, if such a number exists, it must exceed 10^{20}.

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1. Claude W. Anderson, Problem P.229, *Canadian Mathematical Bulletin*, Vol. 19 (1976), p. 122.

2. Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, 1966, Vol. I, pp. 33 - 37.

* * *

188. [1976: 194] *Proposed by Daniel Rokhsar, Susan Wagner H.S., Staten Island, N.Y.*

Show that the only positive integer solution of the equation $a^b = b^a$, $a < b$, is $a = 2$, $b = 4$.

I. *Solution by W.J. Blundon, Memorial University of Newfoundland.*

Let $b = a(1+t)$, where $t(>0)$ is rational. The given equation then reduces to

$$a^t = 1 + t < e^t < 3^t,$$

so that $a < 3$. Since $a = 1$ gives $t = 0$, we must have $a = 2$, and it follows easily that $b = 4$.

II. *Comment by L.F. Meyers, The Ohio State University.*

The beautiful solution paraphrased below can be found in [7].

Since the given equation is equivalent to $\sqrt[a]{a} = \sqrt[b]{b}$ and the function $\sqrt[x]{x}$ is strictly decreasing for $x > e$ and tends to 1 as $x \to \infty$, it follows that

$$\sqrt[3]{3} > \sqrt[4]{4} = \sqrt[2]{2} > \sqrt[5]{5} > \sqrt[6]{6} > \ldots > \sqrt[1]{1},$$

and so we must have $a = 2$, $b = 4$.

Also solved by LEON BANKOFF, *Los Angeles, California;* STEVEN R. CONRAD, *Benjamin N. Cardozo H.S., Bayside, N.Y.;* RADFORD DE PEIZA, *Woburn C.I., Scarborough, Ontario;* CLAYTON W. DODGE, *University of Maine at Orono;* G.D. KAYE, *Department of National Defence, Ottawa;* R. ROBINSON ROWE, *Sacramento, California;* DAVID R. STONE, *Georgia Southern College, Statesboro, Georgia;* CHARLES W. TRIGG, *San Diego, California;* BERNARD VANBRUGGHE, *Université de Moncton, Nouveau Brunswick;* KENNETH M. WILKE, *Topeka, Kansas; and the proposer.* MURRAY S. KLAMKIN, *University of Alberta, sent in reference* [2].

Editor's comment.

This problem is indeed an old chestnut, but even old chestnuts, when properly roasted, can be tasty. Some of the references given below were sent in by several solvers.

It has been known at least since Euler (see [2]) that all the positive rational solutions are given by

$$a=(1+\frac{1}{n})^n, \quad b=(1+\frac{1}{n})^{n+1}, \quad n=1,2,\ldots .$$

Recent discussions of this generalization can be found in [3], [5], and [6].

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3. Man-Keung Siu, "An interesting exponential equation", *The Mathematical Gazette,* Vol.60, 1976, pp. 213 - 215.

4. Trygve Nagell, *Introduction to Number Theory,* Chelsea, 1964, p.125, Problem 52.

5. R. Robinson Rowe, "The Mutuabola", *Journal of Recreational Mathematics,* pp.176 - 178. (The volume number and year of issue do not appear on my reprint of the article.)

6. Shklarsky, Chentzov, and Yaglom, *The USSR Olympiad Problem Book,* W.H. Freeman and Co., 1962, pp. 29, 218 - 220.

7. Wacław Sierpiński, *Elementary Theory of Numbers,* Warsaw, 1964, pp. 106 - 107.

8. John T. Varner III, "Comparing a^b and b^a using elementary calculus", *The Two Year College Mathematics Journal,* Vol.7, 1976, p. 46.

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189. [1976: 194] *Proposed by Kenneth S. Williams, Carleton University.*

If a quadrilateral circumscribes an ellipse, prove that the line through the midpoints of its diagonals passes through the centre of the ellipse.

I. *Comment by Sahib Ram Mandan, Indian Institute of Technology, Kharagpur, India.*

This theorem is true not only for an ellipse but for any central conic S inscribed in a quadrilateral q. The conics like S are said to form a *range* r such that the pairs of opposite vertices of q form its *degenerate* members.

The theorem then states that *the centres of the conics of r are collinear.* For a proof of this, see, for example, Durell [3].

II. *Comment by Leon Bankoff, Los Angeles, California.*

This theorem is due to Newton [6]. A solution by analytic geometry can be found in Smith [7], and a synthetic geometrical solution appears in Durell [2, p. 196].

By an affine transformation, it is sufficient to consider the theorem for a circle. Newton's own solution for this special case is given in Catalan [1].

III. *Comment by Murray S. Klamkin, University of Alberta.*

A proof valid for any central conic can be found in Starr [8], but by an affine transformation it suffices to consider a circle.

IV. *Comment by Dan Pedoe, University of Minnesota.*

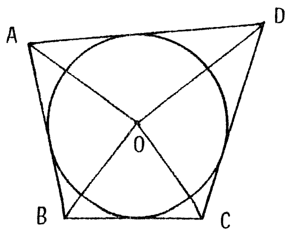
This problem, like Problem 180 [1977: 56], can be considered as an exercise on Desargues' Involution Theorem and its dual, for which see Durell [2, p. 184]. The problem is stated and proved in Durell [2, p. 196].

Editor's comment

A proof of this theorem valid for any central conic can also be found in *Mathesis* [5], a reference I found in F.G.-M.[4]. As far as I can tell, none of the proofs in the references adduced thus far can really be called elementary. But Bankoff and Klamkin have noted that it is sufficient to prove the theorem holds for a circle, and a really elementary proof of this special case can be found in [4], where it is a nearly trivial consequence of the following theorem of Léon Anne (ca. 1850), which is proved in [4] with only the rudiments of high school geometry:

THEOREM (Léon Anne). Let ABCD *be a quadrilateral. Then the line joining the midpoints of the diagonals* AC *and* BD *is the locus of the points* O *such that the sum of the areas of* Δ*s* OAD, OBC *is equal to the sum of the areas of* Δ*s* OAB, OCD.

Our own problem now follows easily. For if quadrilateral ABCD circumscribes circle (O), as shown in the figure, then the four triangles OAD, OBC, OAB, OCD have the same altitude from O (the radius of the circle), and since AD + BC = AB + CD, it follows that the sum of the areas of the first two triangles equals the sum of the areas of the last two.



Now, by the theorem of Léon Anne, O must lie on the line through the midpoints of diagonals AC and BD.

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1. Eugène Catalan, *Théorèmes et problèmes de géométrie élémentaire,* sixième édition, 1879, p. 127, Théorème LIX.

2. C.V. Durell, *Projective Geometry,* Macmillan, London, 1945.

3. C.V. Durell, *Projective Geometry,* Macmillan, London, 1952, Theorem 125(3), p. 187.

4. F.G.-M., *Exercices de Géométrie,* Mame et Fils, Tours, quatrième édition, 1907, pp. 749 - 750.

5. *Mathesis,* 1898, p. 194, No. 19.

6. Isaac Newton, *Principia,* Book I, Lemma XXV, Corollary 3.

7. Charles Smith, *Conic Sections,* Macmillan and Co., 1948, p. 174, Ex. 5; p. 229, section 219; p. 337, section 244.

8. A.T. Starr, *Scholarship Mathematics,* Sir Isaac Pitman & Sons, London, Vol.II, pp. 156 - 158.

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190. [1976: 194] *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

Find all integral values of m for which the polynomial

$$P(x) = x^3 - mx^2 - mx - (m^2 + 1)$$

has an integral zero.

Solution by David R. Stone, Georgia Southern College, Statesboro, Georgia.

For integral m, suppose $P(n) = 0$, where n is an integer; then

$$m^2 + n(n+1)m - (n^3 - 1) = 0$$

and

$$m = -\frac{n(n+1)}{2} \pm \sqrt{\frac{n^4 + 6n^3 + n^2 - 4}{4}}\,. \tag{1}$$

Since $-n(n+1)/2$ is an integer, the radicand in (1) must be an integer and a perfect square which we will call $T_n^{\,2}$. If we let $A_n = (n^2 + 3n - 4)/2$, which is an integer for any parity of n, then

$$T_n^{\,2} = (A_n - 1)^2 + (n+10)(n-1) = A_n^{\,2} + 6n - 5 = (A_n + 1)^2 - (n-1)(n-2). \tag{2}$$

The values of n for which

$$A_n^{\,2} < T_n^{\,2} < (A_n + 1)^2 \tag{3}$$

and those for which

$$(A_n - 1)^2 < T_n^{\,2} < A_n^{\,2} \qquad (4)$$

can surely be excluded, for $T_n^{\,2}$ cannot lie between consecutive squares. It follows easily from (2) that (3) holds whenever $n \geq 3$ and (4) holds whenever $n \leq -11$. So the only values of n that need to be tested in (1) are $n = 2, 1, 0, \ldots, -10$. It will be found that only $n = 2, 1, -10$ yield integral values of m. These are, respectively, 1, -7; 0, -2; -13, -77.

Conversely,

$$\begin{aligned}
\text{for } m &= 1, & \text{we have } P(x) &= (x-2)(x^2+x+1),\\
m &= 0, & P(x) &= (x-1)(x^2+x+1),\\
m &= -2, & P(x) &= (x-1)(x^2+3x+5),\\
m &= -7, & P(x) &= (x-2)(x^2+9x+25),\\
m &= -13, & P(x) &= (x+10)(x^2+3x-17),\\
m &= -77, & P(x) &= (x+10)(x^2+67x-593).
\end{aligned}$$

This problem has the most puzzling collection of answers I've ever seen!

Also solved by R. ROBINSON ROWE, *Sacramento, California; and the proposer.*

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191. [1976: 219] *Proposed by R. Robinson Rowe, Sacramento, California.*

In the November 1976 *Scientific American*, p. 134, Martin Gardner recited an algorithm equivalent to

$$N_n N_{n-2} = N_{n-1} + 1$$

and demonstrated that, for any N_0 and N_1, the algorithm led to $N_5 = N_0$. Considering the more general relation

$$N_n N_{n-2} = N_{n-1} + e$$

(a) find sets of square integers N_0 and N_1 for which $N_5 = N_0$ when $e = 2$;

(b) find the general relation between N_0 and N_1 for any value of e.

I. *Solution of part (b) by Leroy F. Meyers, The Ohio State University.*

If $e = 1$, Gardner's algorithm shows that $N_5 = N_0$ holds for arbitrary N_0 and N_1, so it is clear we must assume $e \neq 1$.

Suppose $N_5 = N_0$ and assume, at the outset, that $N_0 \neq 0$. The relations

$$N_0N_2 = N_1 + e, \quad N_1N_3 = N_2 + e, \quad N_2N_4 = N_3 + e, \quad N_0N_3 = N_4 + e$$

imply

$$N_1(N_4 + e) = N_0N_1N_3 = N_0(N_2 + e) = N_1 + e + N_0e$$

and

$$(N_1 + e)N_4 = N_0 N_2 N_4 = N_0 (N_3 + e) = N_4 + e + N_0 e.$$

Subtraction yields $(N_1 - N_4)e = N_1 - N_4$, and so $N_1 = N_4$ since $e \neq 1$. But then

$$N_0 (N_2 - N_3) = (N_1 + e) - (N_4 + e) = 0,$$

and $N_2 = N_3$ since $N_0 \neq 0$. Now we have

$$N_1 (N_1 + e) = N_0 N_1 N_2 = N_0 N_1 N_3 = N_0 (N_2 + e) = N_1 + e + N_0 e,$$

and so

$$N_0 e = (N_1 - 1)(N_1 + e). \tag{1}$$

Finally, from $N_2 N_0 = N_1 + e$, we see that $N_0 = 0$ implies $N_1 + e = 0$, and so (1) holds even when $N_0 = 0$.

Hence (1) is the desired relation between N_0 and N_1 which must hold whenever $N_5 = N_0$ and $e \neq 1$.

II. *Solution of part (a) by the proposer.*

If we substitute $e = 2$, $N_0 = 9v^2$, and $N_1 = \frac{1}{2}(3u - 1)$ in (1), we obtain the Pell equation

$$u^2 - 8v^2 = 1,$$

whose general solution in positive integers (see [1], for example) is (u_n, v_n), where

$$u_n + v_n \sqrt{8} = (3 + \sqrt{8})^n, \qquad n = 0, 1, 2, \ldots .$$

For $n = 0$, we get	$(u_0, v_0) = (1,0)$	and	$(N_0, N_1) = (0^2, 1^2)$,
$n = 1$,	$(u_1, v_1) = (3,1)$		$(N_0, N_1) = (3^2, 2^2)$,
$n = 2$,	$(u_2, v_2) = (17,6)$		$(N_0, N_1) = (18^2, 5^2)$.

This method will, of course, yield a square N_0 for every n, but for $n = 3, 4, \ldots, 14$, we find

$$N_1 = 148,\ 865,\ 5044,\ 29401,\ 171364,\ 998785,$$
$$5821348,\ 33929305,\ 197754484,\ 1152597601,$$
$$6717831124,\ 39154389145,$$

none of which is a square.

Whether this method will yield larger square values of N_1, or whether some other method will yield additional "small" square values of N_0 and N_1, is still to be determined.

Also solved by LEROY F. MEYERS *(part (a) as well);* KENNETH M. WILKE, *Topeka, Kansas; and the proposer (part (b) as well).*

REFERENCE

1. Trygve Nagell, *Introduction to Number Theory*, Chelsea, New York, 1964, pp. 197 - 198.

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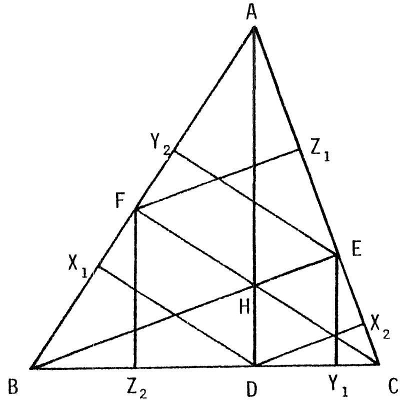
192. [1976: 219] *Proposed by Ross Honsberger, University of Waterloo.*

Let D, E, F denote the feet of the altitudes of ΔABC, and let (X_1,X_2), (Y_1,Y_2), (Z_1,Z_2) denote the feet of perpendiculars from D, E, F, respectively, upon the other two sides of the triangle. Prove that the six points X_1, X_2, Y_1, Y_2, Z_1, Z_2 lie on a circle.

Adapted from the solution of Clayton W. Dodge, University of Maine at Orono; and a comment from Dan Pedoe, University of Minnesota.

Let H be the orthocenter of ΔABC, and let Z_2 and Y_1 lie on BC, X_2 and Z_1 on CA, and Y_2 and X_1 on AB, as shown in the figure.



From the similarities

$$\Delta BDX_1 \sim \Delta BFZ_2, \quad \Delta BEY_2 \sim \Delta BHF, \quad \Delta BEY_1 \sim \Delta BHD,$$

we get

$$\frac{BX_1}{BZ_2} = \frac{BD}{BF}, \quad \frac{BY_2}{BE} = \frac{BF}{BH}, \quad \frac{BY_1}{BE} = \frac{BD}{BH},$$

so that

$$\frac{BY_1}{BY_2} = \frac{BD}{BF} = \frac{BX_1}{BZ_2} \quad \text{and} \quad BZ_2 \cdot BY_1 = BX_1 \cdot BY_2.$$

Hence Y_2, X_1, Z_2, Y_1 lie on a circle γ_1; and similarly Z_2, Y_1, X_2, Z_1 lie on a circle γ_2; and X_2, Z_1, Y_2, X_1 lie on a circle γ_3.

If γ_1 and γ_2, for example, coincide, our theorem is proved; and if they are distinct their radical axis is the line BC. Thus, if our theorem is not true, we have three distinct circles γ_1, γ_2, and γ_3, whose radical axes, taking the circles in pairs, form the sides of ΔABC, which contradicts the theorem that such radical axes are either concurrent or parallel (see [13], for example).

Also solved by HARRY D. RUDERMAN, *Hunter College, New York; and* DAN SOKOLOWSKY, *Yellow Springs, Ohio (two solutions). One incorrect solution was received.*

Editor's comment.

The circle in this problem is known in the literature as *The Taylor Circle,* after H.M. Taylor (1842-1927) who discussed it in [14] in 1882. Since it seems to be little-known today, we are grateful to the proposer for having resurrected it for the enjoyment of mathematical afissionados of the nuclear age.

In spite of its name, this circle did not originate with Taylor. It was discovered earlier, like so many other good things in life, by the French. It was mentioned by Catalan [3] in 1879, and was apparently first proposed in 1877 by the French mathematician known as Eutaris [6]. Other references given below chronicle the story of its reappearance over the years. J.D.E. Konhauser, Macalester College, Saint Paul, Minnesota, sent in reference [5]; Sahib Ram Mandan, Indian Institute of Technology, Kharagpur, India, sent in references [4],[5], and [14]; and the proposer gave reference [11].

I have examined seven or eight different proofs of this theorem, both new and old, and in my opinion the proof given here is one of the shortest, simplest, as well as the most elementary of all those I have seen. One of the worst was Taylor's own in [14]. The proof that comes closest to the one given here is that of Clarke in [11]. He uses radical axes to prove the coincidence of circles γ_1, γ_2, γ_3, just as in our own proof, but he proves that Y_2, X_1, Z_2, Y_1 are concyclic by the theory of antiparallels, which would not be familiar to some readers today.

It is most appropriate to see this theorem reappear in our pages exactly one hundred years after its first appearance in 1877.

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4. Julian Lowell Coolidge, *A Treatise on the Circle and the Sphere,* Chelsea, New York, 1971, p. 50, Theorem 66; pp. 71-74, especially p. 72, Theorem 111.

5. Nathan Altshiller Court, *College Geometry,* Second Edition, Barnes and Noble, New York, 1952, pp. 286-287, No. 689.

6. Eutaris (nom de plume (anagramme) de Restiau, du Collège Chaptal, Paris), *Journal de Mathématiques élémentaires* de M. Vuibert, Vol. 2, 1877, pp. 30, 43, No. 60.

7. F.G.-M., *Exercices de Géométrie,* quatrième édition, Mame et Fils, Tours, 1907, pp. 1102, 1140-1141.

8. *Intérmediaire des Mathématiciens,* 1895, *p.* 166, *No.*154.

9. R.A. Johnson, *Modern Geometry*, Houghton Mifflin, Boston, 1929, p. 277.

10. *Mathesis*, 1889, p. 250.

11. *National Mathematics Magazine*, Vol. 18, 1943 - 1944, pp. 40 -41, Problem 485 proposed by Paul D. Thomas, Sherburne, N.Y. and solved by Walter B. Clarke, San Jose, California.

12. *Nouvelle Correspondance Mathématique*, Vol. VI, 1880, p. 183.

13. D. Pedoe, *A Course of Geometry for Colleges and Universities*, Cambridge University Press, London, 1970, p. 110, Theorem 28.1.

14. H.M. Taylor, "On a six-point circle connected with a triangle", *The Messenger of Mathematics*, new series, Vol. XI, 1881 - 1882, pp. 177-179.

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193. [1976: 219] *Proposed by L.F. Meyers, The Ohio State University.*

A river with a steady current flows into a still-water lake at Q. A swimmer swims down the river from P to Q, and then across the lake to R, in a total of 3 hours. If the swimmer had gone from R to O to P, the trip would have taken 6 hours. If there had been a current in the lake equal to that in the river, then the downstream trip PQR would have taken $2\frac{1}{4}$ hours. How long would the upstream trip RQP have taken under the same circumstances? (This is a reconstruction of a problem that I could not solve while participating in a high school contest.)

Solution by Kenneth M. Wilke, Topeka, Kansas.

Let s be the speed of the swimmer in still water, c the speed of the current, and PQ, QR the distances from P to Q and Q to R; then we have the equations

$$\frac{\mathrm{PQ}}{s+c}+\frac{\mathrm{QR}}{s}=3 \quad \text{or} \quad s(\mathrm{PQ}+\mathrm{QR})+c\mathrm{QR}=3s^2+3sc, \tag{1}$$

$$\frac{\mathrm{PQ}}{s-c}+\frac{\mathrm{QR}}{s}=6 \quad \text{or} \quad s(\mathrm{PQ}+\mathrm{QR})-c\mathrm{QR}=6s^2-6sc, \tag{2}$$

$$\frac{\mathrm{PQ}+\mathrm{QR}}{s+c}=\frac{9}{4} \quad \text{or} \quad 4(\mathrm{PQ}+\mathrm{QR})=9(s+c), \tag{3}$$

and the answer sought is the value of $\dfrac{\mathrm{PQ}+\mathrm{QR}}{s-c}$.

Adding (1) and (2) gives $2(\mathrm{PQ}+\mathrm{QR})=9s-3c$ which, when combined with (3), yields

$$\frac{s}{c}=\frac{5}{3}, \qquad \frac{s+c}{s-c}=\frac{8}{2}=4.$$

Finally, substitution of $s+c=4(s-c)$ in (3) gives

$$\frac{\mathrm{PQ}+\mathrm{QR}}{s-c}=9,$$

and the required time is 9 hours.

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* G.D. KAYE, *Department of National Defence, Ottawa;* MURRAY S. KLAMKIN, *University of Alberta;* R. ROBINSON ROWE, *Sacramento, California;* CHARLES W. TRIGG, *San Diego, California;* *and the proposer. One incorrect solution was received.*

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194. [1976: 219] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

A sequence $\{a_n\}$ is defined by

$$a_1 = X, \qquad a_n = X^{a_{n-1}}, \qquad n = 2, 3, \ldots$$

where $X = \left(\frac{4}{3}\right)^{3/4}$. Discuss the convergence of the sequence and find the value of the limit if any.

Solution and comment by David R. Stone, Georgia Southern College, Statesboro, Georgia.

If $\lim_{n\to\infty} a_n$ exists and equals A then, by the continuity of the exponential function,

$$A = \lim_{n\to\infty} a_n = \lim_{n\to\infty} X^{a_{n-1}} = X^A,$$

and so A satisfies the equation $t = X^t$. Graphically, this last equation has two solutions: $t_1 \approx 1.3$ and $t_2 \approx 11.2$. Clearly $t_1 = 4/3$, since $4/3 = X^{4/3}$.

Now the sequence $\{a_n\}$ is strictly increasing, since $a_1 = X > 1$ and

$$a_2 = X^{a_1} = a_1^{a_1} > a_1, \qquad a_n > a_{n-1} \Rightarrow a_{n+1} = X^{a_n} > X^{a_{n-1}} = a_n.$$

The sequence is also bounded above by $4/3$, since

$$a_1 = X \approx 1.24 < \frac{4}{3}, \qquad a_n < \frac{4}{3} \Rightarrow a_{n+1} = X^{a_n} < X^{4/3} = \frac{4}{3}.$$

It now follows from the opening paragraph that the desired limit is $A = t_1 = 4/3$.

In trying to solve the equation $t = g(t)$, that is, to find a fixed point of the function g, one useful method is that of iteration, which is described in the following theorem (see [5]):

THEOREM. Let $t = A$ be a solution of $t = g(t)$ and suppose g has a continuous derivative in some interval J containing A. Then if $|g'(t)| \le c < 1$ in J, the iteration process defined by

$$a_{n+1} = g(a_n), \qquad n = 1, 2, \ldots$$

converges to A for any $a_1 \in J$.

Attempting to solve $t = X^t$ by this method with $a_1 = X$ would give the sequence of the problem, which must thus converge to $4/3$. But the boundedness condition on the derivative is merely sufficient, not necessary, and I think we could start with a_1

equal to any value from $-\infty$ to $t_2 \approx 11.1951$ and still have the sequence $\{a_n\}$ converge to 4/3. But no choice of a_1 would initiate a sequence converging to t_2, except for $a_1 = t_2$ itself.

Also solved by R. MAUDE, *The University, Leeds, U.K.;* DANIEL ROKHSAR, *Susan Wagner H.S., Staten Island, N.Y.;* R. ROBINSON ROWE, Sacramento, California; and the proposer.

Editor's comment.

Problems similar to this one have been frequently discussed in the literature in recent years. The proposer gave reference [1]; Murray S. Klamkin, University of Alberta, gave reference [2]; and Harry Schor, Greenbrae, California, sent in a copy of his paper [6]. Our solver, David R. Stone, gave additional references [3] and [4] for the iteration process.

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2. T.J.I'a Bromwich, *Introduction to the Theory of Infinite Series,* Macmillan, London, 1947, p. 23, Ex. 11.

3. F.B. Hildebrand, *Introduction to Numerical Analysis,* McGraw-Hill, Second Edition, p. 568.

4. J.A. Jensen and J.H. Rowland, *Methods of Computation,* Chapter 4, from p. 132.

5. Erwin Kreyszig, *Advanced Engineering Mathematics,* Third Edition, Wiley, 1972, p. 640.

6. Harry Schor, *On Infinite Exponential Expressions,* unpublished paper.

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195. [1976: 220] *Proposed by John Karam, Coop. Student, University of Waterloo.*

The following two problems are given together since they both appear to be related to the celebrated Birthday Problem, which says that if 23 persons are in a room the odds are better than 50% that two persons in the room have the same birthday.

(a) How many persons would have to be in a room for the odds to be better than 50% that three persons in the room have the same birthday?

(b) In the Québec-based lottery *Loto Perfecta,* each entrant picks six distinct numbers from 1 to 36. If, at the draw, his six numbers come out in some order (*dans le désordre*) he wins a sum of money; if his numbers come out in order (*dans l'ordre*), he wins a larger sum of money. How many entries would there have to be for the odds to be better than 50% that two persons have picked the same numbers (i) *dans le désordre,* (ii) *dans l'ordre?*

I. *Editor's comment on part (a).*

We assume there are $N = 365$ possible birthdays, all equally likely. Suppose the room contains $n(\geq 3)$ persons; let p_n denote the probability of success, that is, that three of the n persons have the same birthday; and let $q_n = 1 - p_n$ be the probability of failure. The problem requires us to find the smallest value of n for which $p_n \geq \frac{1}{2}$ (or $q_n \leq \frac{1}{2}$).

R. Robinson Rowe, Sacramento, California, gave an elementary derivation of the formula

$$q_n = (1 - \frac{1}{N^2})(1 - \frac{3}{N^2})(1 - \frac{6}{N^2}) \;\dots\; \left(1 - \frac{\frac{1}{2}(n-2)(n-1)}{N^2}\right), \tag{1}$$

where $1, 3, 6, \dots, \frac{1}{2}(n-2)(n-1)$ are the first $n-2$ triangular numbers. Formula (1) is exact, but awkward to calculate. With a sigh of relief that we live in an age of calculators, we find

$$q_{82} = 0.511857\dots, \qquad p_{82} = 0.488142\dots,$$
$$q_{83} = 0.499098\dots, \qquad p_{83} = 0.500901\dots,$$

and we conclude that the required number is $n = 83$.

Rowe shortened his calculations for q_n by averaging 10 factors at a time, finding $q_{12}, q_{22}, \dots,$ ending with $q_{82} = 0.510827\dots$ and $q_{83} = 0.498093\dots$, so the answer was not affected.

This problem is not new, since it is a special case of the following problem investigated by Klamkin and Newman [4]: Given are N equally likely alternatives. We choose from among them repeatedly until we find that one of the alternatives has occurred k times. Our purpose is to find $E(N,k)$, the *expected* number of repetitions necessary for this success.

The answer to our problem would then be $E(365,3)$ in the unlikely event that it is an integer, or $[E(365,3)] + 1$ if it is not.

Klamkin and Newman found the following formula

$$E(N,k) = \int_0^\infty \left(S_k(\tfrac{t}{N})\right)^N e^{-t}\,dt, \tag{2}$$

where $S_k(x)$ is the k-th partial sum of e^x, that is,

$$S_k(x) = \sum_{j<k} \frac{x^j}{j!}.$$

They then used (2) to obtain the following asymptotic estimate for E: for each fixed k,

$$E(N,k) \sim \sqrt[k]{k!}\;\Gamma(1 + \tfrac{1}{k})\, N^{1-1/k} \quad \text{as } N \to \infty. \tag{3}$$

For our problem, we find from (3)

$$E(365,3) \approx \sqrt[3]{6} \cdot \Gamma(\tfrac{4}{3}) \cdot 365^{2/3} \approx 82.87,$$

which confirms that the answer is 83.

For a clear discussion of the original Birthday Problem see Mosteller [5], and for other extensions of it see Abramson and Moser [1], Austin [2], and Greenwood and Richert [3].

II. *Solution of part (b) by R. Robinson Rowe, Sacramento, California.*

This problem is the same as the classical Birthday Problem mentioned in the proposal, except that instead of $N = 365$ we have

(i) $N_1 = \dfrac{36!}{6!\,30!} = 1{,}947{,}792$ *dans le désordre*,

(ii) $N_2 = \dfrac{36!}{30!} = 1{,}402{,}410{,}240$ *dans l'ordre*.

From any standard account of the classical Birthday Problem (see [5], for example, or almost any probability text), we have, in the notation used in part (a) above,

$$Q = q_n = \frac{(N-1)!}{N^{n-1}(N-n)!}, \qquad p_n = 1 - q_n, \tag{1}$$

and we have to find the smallest value of n for which $p_n \geq \frac{1}{2}$ (or $Q \leq \frac{1}{2}$). Because of the large values of N, we must find an efficient way of estimating Q.

From Stirling's formula

$$\ell n\,(a!) \approx \ell n \sqrt{2\pi} + (a + \tfrac{1}{2})\,\ell n\, a - a,$$

we get from (1)

$$\ell n\, Q \approx (N-\tfrac{1}{2})\,\ell n(N-1) - (n-1)\,\ell n\, N - (N - n + \tfrac{1}{2})\,\ell n\,(N-n) - (N-1) + (N-n)$$

$$= (N-\tfrac{1}{2})\,\ell n \frac{N-1}{N-n} - (n-1)\,\ell n \frac{N}{N-n} - (n-1). \tag{2}$$

Now $\dfrac{N-1}{N-n} = 1 + \dfrac{n-1}{N-n}$, $\dfrac{N}{N-n} = 1 + \dfrac{n}{N-n}$ and, for small k,

$$\ell n(1+k) \approx k - \tfrac{1}{2}k^2. \tag{3}$$

Applying (3) to (2) now gives, after simplification,

$$\ell n\, Q \approx -\frac{n(n-1)}{2(N-n)} + \frac{(n-1)(3n-1)}{4(N-n)^2}, \tag{4}$$

and neglecting the relatively small second term in (4) gives

$$n^2 - n(1+2\,\ell n\, Q) + 2N\,\ell n\, Q \approx 0,$$

whence

$$n \approx \tfrac{1}{2} + \ln Q + \sqrt{(\tfrac{1}{2} + \ln Q)^2 - 2N \ln Q}. \qquad (5)$$

We can now use (5) to approximate n when $Q = q_n = \frac{1}{2}$:

$$n \approx \tfrac{1}{2} - \ln 2 + \sqrt{(\tfrac{1}{2} - \ln 2)^2 + 2N \ln 2}.$$

Since the first term in the radicand, whose value is 0.0373 ..., is not likely to affect the integral part of n, we can neglect it and use

$$n \approx \tfrac{1}{2} - \ln 2 + \sqrt{2N \ln 2}. \qquad (6)$$

In the classical Birthday Problem, $N = 365$, and (6) gives $n \approx 22.301$, as expected. For our own problem, (6) gives:

for $N = N_1 = 1{,}947{,}792$: $\quad n \approx 1643.039$;

for $N = N_2 = 1{,}402{,}410{,}240$: $\quad n \approx 44092.362$.

We can now seek confirmation from the more accurate formula (2). With $N = N_1$ and $n = 1643$ and $n = 1644$ successively, we find

$$q_n = 0.500212 \quad \text{and} \quad q_n = 0.499790,$$

so the least value sought is $n = 1644$, for which

$$p_n = 1 - q_n = 0.500210.$$

With $N = N_2$ and $n = 44092$ and $n = 44093$ successively, we find from (2)

$$q_n = 0.500013 \quad \text{and} \quad q_n = 0.499997,$$

so the least value sought in this case is $n = 44093$, for which

$$p_n = 1 - q_n = 0.500003.$$

(For $N = N_2$, the values of q_n are so close to $\frac{1}{2}$ that some of the calculations in (2) must be carried out to great accuracy, say to 12 significant digits, to determine with certainty on which side of $\frac{1}{2}$ the values of q_n lie.)

Thus in all three cases the more accurate computation with (2) confirms the approximation obtained from (6), warranting the conclusion that (6) is a very convenient way of solving the classical Birthday Problem for a "year" of N "days".

As a comment on the Stirling formula used in this solution, I note that for $N = N_2$ the numerator in (1) is a number of more than 12 *billion* digits! Praise be to Stirling!

Editor's comment on part (b).

With the Klamkin-Newman asymptotic formula (3) given in my comment on part (a), we find

$$E(N_1,2) \approx \sqrt{2}\,\Gamma\left(\frac{3}{2}\right)\sqrt{N_1} \approx 1749.2,$$

$$E(N_2,2) \approx \sqrt{2}\,\Gamma\left(\frac{3}{2}\right)\sqrt{N_2} \approx 46935.1,$$

which differ greatly from the values 1643.039 and 44092.362 obtained by Rowe. Thus, unless there is some undetected error in Rowe's solution to part (b), it would seem that the Klamkin-Newman formula substantially overestimates the value of $E(N,k)$, at least when $k=2$. The overestimation would seem to be confirmed by

$$E(365,2) \approx \sqrt{2}\,\Gamma\left(\frac{3}{2}\right)\sqrt{365} \approx 23.94,$$

which implies that the answer to the classical Birthday Problem is 24, whereas in fact it is known to be 23.

Of course, the Stirling formula used by Rowe is itself asymptotic, but it is unlikely that using the more accurate

$$\ln(a!) \approx \ln\sqrt{2\pi} + (a+\tfrac{1}{2})\ln a - a + \frac{1}{12a} - \frac{1}{360a^3} + \cdots$$

would affect Rowe's results, especially when $N = N_1$ or $N = N_2$.

Rowe pointed out, however, that the error due to neglect of the second term of (4) increases with n. Thus for $Q = 0.03$, that is, for a 97% probability of matching birthdays when $N = 365$, (5) gives $n = 48$ as the number of persons required, whereas the better figure from (2) is $n = 50$.

REFERENCES

1. Morton Abramson and W.O.J. Moser, "More birthday surprises," *American Mathematical Monthly*, Vol. 77 (1970), pp. 856-858.

2. Joe Dan Austin, "The Birthday Problem revisited," *The Two-Year College Mathematics Journal*, Vol. 7 (1976), pp. 39-42.

3. Robert E. Greenwood and Arthur Richert, Jr., "A birthday holiday problem," *Journal of Combinatorial Theory*, Vol. 5 (1968), pp. 422-424.

4. M.S. Klamkin and D.J. Newman, "Extensions of the birthday surprise," *Journal of Combinatorial Theory*, Vol. 3 (1967), pp. 279-282.

5. F. Mosteller, "Understanding the Birthday Problem," *The Mathematics Teacher*, Vol. 55 (1962), pp. 322-325.

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ANNOUNCEMENTS

1. At the Fifth U.S.A. Mathematical Olympiad held on May 4, 1976, three students submitted perfect papers. One of them was Mark Kleiman, Stuyvesant H.S., Staten Island, N.Y., who is a EUREKA subscriber and contributor (see his solution to Problem 183 in this issue). Our congratulations go to Mr. Kleiman. We are proud to be able to present in EUREKA one of the early manifestations of his talent.

2. Dr. Dagmar R. Henney, Department of Mathematics, The George Washington University, Washington, D.C. 20006, is preparing a new edition of her book *Open Questions in Mathematics (II)*, to be published by Princeton University Press. She invites EUREKA readers to send her their favourite unsolved problems for possible inclusion in her book. To be considered for publication, contributions should preferably be no more than two pages in length, and should be written in such a way that they can be reproduced exactly as they are received.

3. Don Baker, 647 Northern Ave., Mill Valley, California 94941, was the problemist who prepared the questions for the San Francisco Junior High Contest, held on March 5, 1977. He would like to exchange contest questionnaires with other EUREKA readers who are similarly engaged. Steven R. Conrad, 39 Arrow St., Selden, N.Y. 11784, who is president of the New York City Interscholastic Mathematics League, made a similar request last year [1976: 62]. Mr. Conrad would also like to hear from EUREKA readers who might like to try their hand at writing contest questionnaires.

4. The Department of Mathematics and Statistics of Miami University, Oxford, Ohio 45056, announces a fifth Annual Conference for Sept. 30 - Oct. 1, 1977. The topic this year will be: *Number Theory - Pure and Simple*. Among the featured speakers: Ivan Niven (University of Oregon), George Andrews (Pennsylvania State University), Ron Graham (Bell Laboratories), Paul Erdös (Hungarian Academy of Science), and Johnny Hill (Miami University). Authors intending to contribute a paper must submit an abstract before July 1, 1977, to Dr. Stanley E. Payne or Dr. David Kullman, both of Miami University.

5. Professor Miriam Grosof, Yeshiva University, New York, N.Y., recently saw, she did not say where, the following advertisement:

> LOWER YOUR ERDÖS NUMBER
>
> Mathematician with an Erdös No. 1 is available for joint authorship. Requirement: already completed paper of substantial merit. Fee: negotiable.

EUREKA readers interested in availing themselves of this service should get in touch with Professor Grosof, who may be able to establish contact with the other interested party. Readers who wish to know what is an Erdös Number and how many boxtops are needed to get one can find the information in:

1. Casper Goffman, "And what is your Erdös Number," *The American Mathematical Monthly*, Vol. 76 (1969), p. 791.
2. P. Erdös, "On the fundamental problem of mathematics," *The American Mathematical Monthly*, Vol. 79 (1972), pp.149 - 150.

6. An advertisement on page 90 of this issue gives detailed information about a remarkable collection of booklets, published by the Canadian Mathematical Congress, which should greatly interest high school students and teachers, and

EUREKA readers in general. They are edited by three of the best, and best-known, Canadian problemists: E.J. Barbeau (University of Toronto), M.S. Klamkin (University of Alberta), and W.O.J. Moser (McGill University). They contain problems and solutions that are unmatched as "setting-up exercises" before tackling EUREKA problems, or before participating in a high school mathematics contest. And at $2.30 and $1.80 each in dèpreciated Canadian dollars, they are inexpensive enough. Organizers of mathematics contests may well wish to purchase a few dozen copies to give away as prizes to their winning students (or as consolation prizes to the losing students who need them more). High school and college teachers will find that these problem collections constitute a superbly enriching mathematical diet for their better students.

* * *

REAL IMAGE

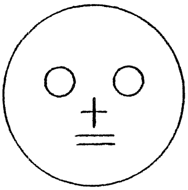
A typical mathematician,
Whom everyone thinks a magician,
  Can solve with precision
  All problems in fission,
But stumbles on simple addition.

CATHERINE A. CALLAGHAN
Ohio State University

* * *

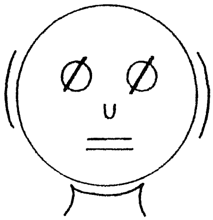
VARIATIONS ON A THEME BY BANKOFF III

The theme by Leon Bankoff



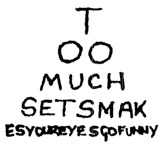
$0 + 0 = 0$

Variation No. 6 by B.C. Rennie

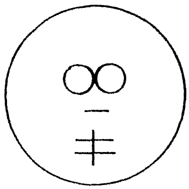


$\mu(\emptyset \cup \emptyset) = 0$

or

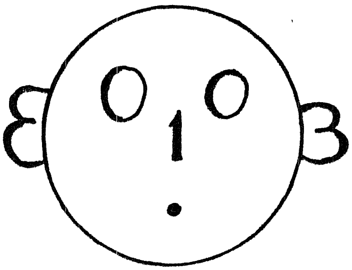


Variation No. 7 by C.W. Trigg



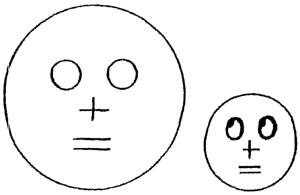
That's going to extremes!

Variation No. 5 by A. Dunkels



All people have in their faces
Log 2 to five decimal places.

Variation No. 8 by Edith Orr



Daddy, why are those people
always making fun of us?

* * *

5. Madame de Buffon to her son: "What has suddenly made you so clumsy, dropping needles all over the place?"

6. Madame Bourbaki to herself: "Why did I take those fertility pills?"

7. Mother to Lobachevsky: "If parallel lines diverge, why haven't there been more wrecks on the Trans-Siberian railroad?"

CHARLES W. TRIGG