Crux Mathematicorum

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

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ISSN 0705 - 0348

CRUX MATHEMATICORUM

Vol. 9, No. 1

January 1983

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
Publié par le Collège Algonquin, Ottawa
Printed at Carleton University

The assistance of the publisher and the support of the Canadian Mathematical Olympiad Committee, the Carleton University Department of Mathematics and Statistics, the University of Ottawa Department of Mathematics, and the endorsement of the Ottawa Valley Education Liaison Council are gratefully acknowledged.

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CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is $22 in Canada, US$20 elsewhere. Back issues: $2 each. Bound volumes with index: Vols. 1&2 (combined) and each of Vols. 3-7, $17 in Canada and US$15 elsewhere. Cheques and money orders, payable to CRUX MATHEMATICORUM, should be sent to the managing editor.

All communications about the content (articles, problems, solutions, etc.) should be sent to the editor. All changes of address and inquiries about subscriptions and back issues should be sent to the managing editor.

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CONTENTS

A MIXTILINEAR ADVENTURE

LEON BANKOFF

Triangles in the Euclidean plane may be classified as rectilinear, mixtilinear, or curvilinear depending on whether all, some, or none of the bounding lines are straight. Our interest here will first lie in the consideration of circles inscribed in a special kind of mixtilinear triangle, one bounded by two sides of a rectilinear triangle and an arc of its circumcircle. We will then consider circles inscribed in another type of mixtilinear triangle. We will consistently use the convenient notation (M)n (or sometimes simply (M)) for a circle with center M and radius n. We start with a relatively simple problem.

Let (O)R and (I)r be the circumcircle and incircle, respectively, of a given triangle ABC with sides a,b,c in the usual order. Our objective is to learn what we can about the circle (W)p inscribed in the mixtilinear triangle bounded by the sides AC and BC and the arc AB of the circumcircle (O). As shown in Figure 1, circle (W) touches AC at D and BC at E. To find the radius p, we proceed as follows.

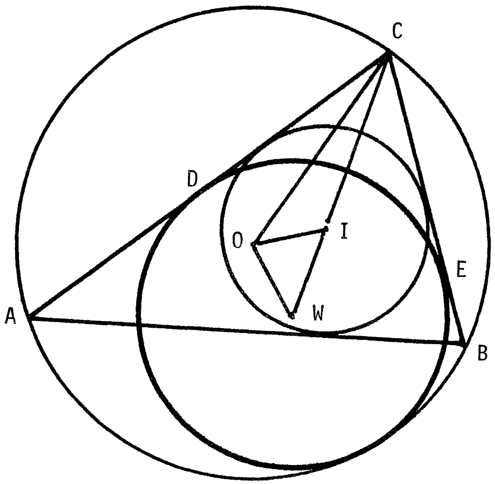

Figure 1

We apply the cosine law to triangle COW, in which

$$CO = R, \qquad OW = R-p, \qquad CW = p\csc\frac{C}{2}, \qquad \angle OCW = \frac{|A-B|}{2},$$

obtaining

$$(R-p)^2 = R^2 + p^2\csc^2\frac{C}{2} - 2Rp\csc\frac{C}{2}\cos\frac{A-B}{2},$$

which is equivalent to

$$p^2\csc^2\frac{C}{2} - p^2 = 2Rp\csc\frac{C}{2}\cos\frac{A-B}{2} - 2Rp.$$

Dividing throughout by $p\csc^2\frac{C}{2}$, we get

$$p - p\sin^2\frac{C}{2} = 2R\sin\frac{C}{2}\cos\frac{A-B}{2} - 2R\sin^2\frac{C}{2},$$

whereupon

$$\begin{aligned} p\cos^2\frac{C}{2} &= 2R\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right) \\ &= 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\ &= r, \end{aligned}$$

and we have the desired result

$$p = r\sec^2\frac{C}{2}. \tag{1}$$

Another simple proof of (1) results from the application of Stewart's Theorem to the point O and the three points C,I,W (which are collinear on the internal bisector of angle C). With

$$OW = R-p,\quad OI^2 = R^2-2Rr,\quad CI = r\csc\frac{C}{2},\quad CW = p\csc\frac{C}{2},\quad OC = R,\quad IW = (p-r)\csc\frac{C}{2}, \tag{2}$$

Stewart's relation

$$OW^2\cdot CI + OC^2\cdot IW = OI^2\cdot CW + CI\cdot CW\cdot IW$$

becomes

$$r(R-p)^2\csc\frac{C}{2} + R^2(p-r)\csc\frac{C}{2} = p(R^2-2Rr)\csc\frac{C}{2} + pr(p-r)\csc^3\frac{C}{2}.$$

This reduces to $r = p\cos^2(C/2)$, and (1) follows.

A special case of interest arises when C is a right angle. Then $\sec^2(C/2) = 2$, and the radius p is twice the inradius r. For any right triangle with hypotenuse c, the inradius is $r = \frac{1}{2}(a+b-c)$, and so

$$p = a + b - c. \tag{3}$$

The special case leading to (3) was proposed by the author in 1954 [1]. Of the two solutions later published [2], one stemmed from the use of the nine-point circle, and the other from a straightforward use of analytic geometry. A still more special case (with a,b,c = 3,4,5) was proposed recently in this journal (Crux 703 [1982: 14]). [A solution appears later in this issue. (Editor)]

An interesting sidelight of the general case illustrated in Figure 1 is that I is the midpoint of chord DE. To show this, we note from (2) and (1) that

$$\frac{\mathrm{IW}}{\mathrm{CW}} = \frac{p-r}{p} = 1 - \frac{r}{p} = 1 - \cos^2\frac{C}{2} = \sin^2\frac{C}{2} = \frac{p^2}{\mathrm{CW}^2},$$

and so $\mathrm{IW}\cdot\mathrm{CW} = p^2$. This means that CW is perpendicular to DE and intersects it at I, which is therefore the midpoint of DE. In particular, when C = 90°, then CDWE is a square with center I.

There are various ways of expressing the length of the equal segments CD and CE in Figure 1; for example, by the chain of equalities

$$\mathrm{CD} = p\cot\frac{C}{2} = r\sec^2\frac{C}{2}\cot\frac{C}{2} = \frac{2r}{\sin C} = \frac{4Rr}{c} = \frac{ab}{s}.$$

This chain provides us with another way of expressing p, one not involving r:

$$p = \frac{ab}{s}\tan\frac{C}{2}. \tag{4}$$

Radford [3] finds the value of CD by use of an inversion centered at C with constant of inversion $\sqrt{\mathrm{CD}\cdot\mathrm{CD'}}$, where D' is the point of contact of the escribed circle opposite C with the side AC (see Figure 2). Then, since the circle (W) inverts into the escribed circle and the circumcircle into the line A'B' (where A',B' are the points inverse to A,B), the line A'B' must touch the escribed circle. Also, since CB·CB' = CA·CA', the triangles CAB and CB'A' are similar and equal in all respects. Hence CB' = CA and we have CA·CB = CD·CD'. Since CD' = s, it follows that CD = ab/s.

Let us now go on to satisfy our curiosity about the circle $(\mathrm{N})q$ inscribed in the mixtilinear triangle bounded by side AC and the arcs of circles (O) and (W), as shown in Figure 3. Since this is a challenge far more daunting than those already grappled with, we might as well recognize that any attempt to find the radius q by ordinary synthetic procedures will only lead to frustration. The solution of this problem, however, is a splendid example of how the principles of inversion can come to the rescue. Inversion enables us to transform a given figure into one more amenable to interpretation and calculation, and to transfer the information so obtained back to the original configuration by means of certain ratios.

In the problem we are considering here, we invert the figure by using (C)CD as the circle of inversion. Its radius, as we now know, is CD = ab/s. As shown in Figure 3, the circle of inversion cuts the circumcircle (O) at J and K, the extremities of their common chord. In turn, the chord JK cuts CA and CB at A' and B',

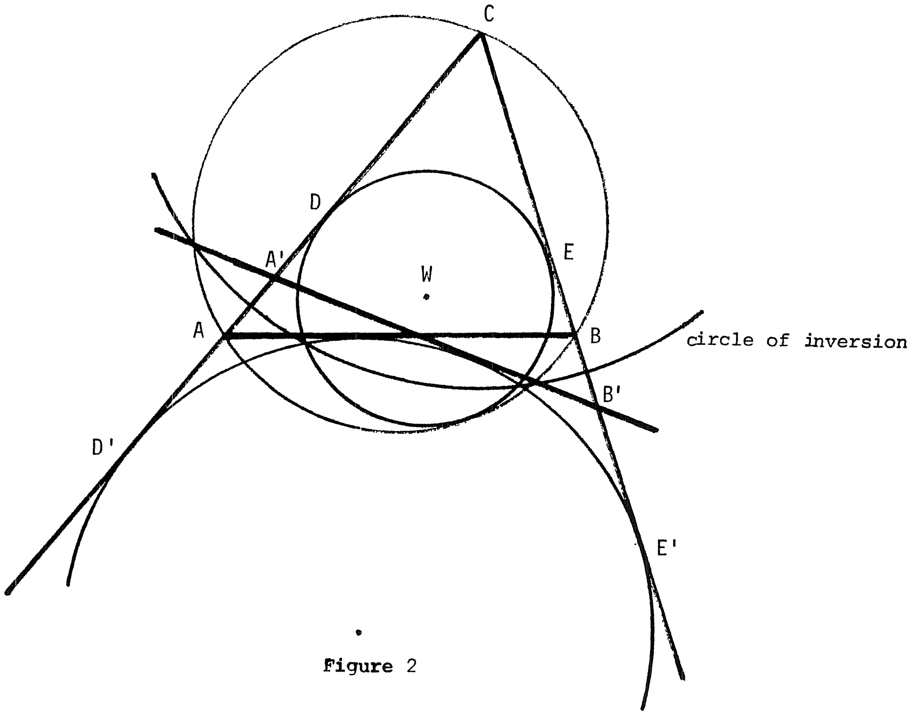

Figure 2

the points inverse to A and B. The circle (W) is self-inverse since it is cut orthogonally by the circle of inversion. Since (W) is tangent to the arc AB, it now becomes tangent to A'B', the segment of the line JK corresponding by inversion to the arc AB.

It is instructive to see what happens when the circle $(N)q$ is inverted into the circle $(S)x$. It is seen that circles (N) and (S) both remain tangent to the self-inverse line CA, which passes through C, the center of inversion. Furthermore, circle (S), like its inverse circle (N), remains tangent to the self-inverse circle (W). Then the tangency of circle (N) with arc AB is transformed into tangency of circle (S) with the line A'B'. We now find that the circle (S), like its tangent circle (W), is tangent to CA and JK. The common internal tangent of circles (S) and (W) is perpendicular to the internal bisector of angle AA'B' and, as shown in Figure 3, cuts the sides of this angle in U and V.

We now focus our attention on the circle (S) inscribed in the isosceles triangle UA'V. By reference to intercepted arcs on circle (O), it is apparent that $\angle CA'B' = \angle CBA$ and $\angle CB'A' = \angle CAB$, and it follows that $\angle A'UV = \angle A'VU = \frac{1}{2}\angle CBA$. It is easily confirmed that the ratio of inradius ρ to exradius ρ_m of any isosceles triangle with base m and

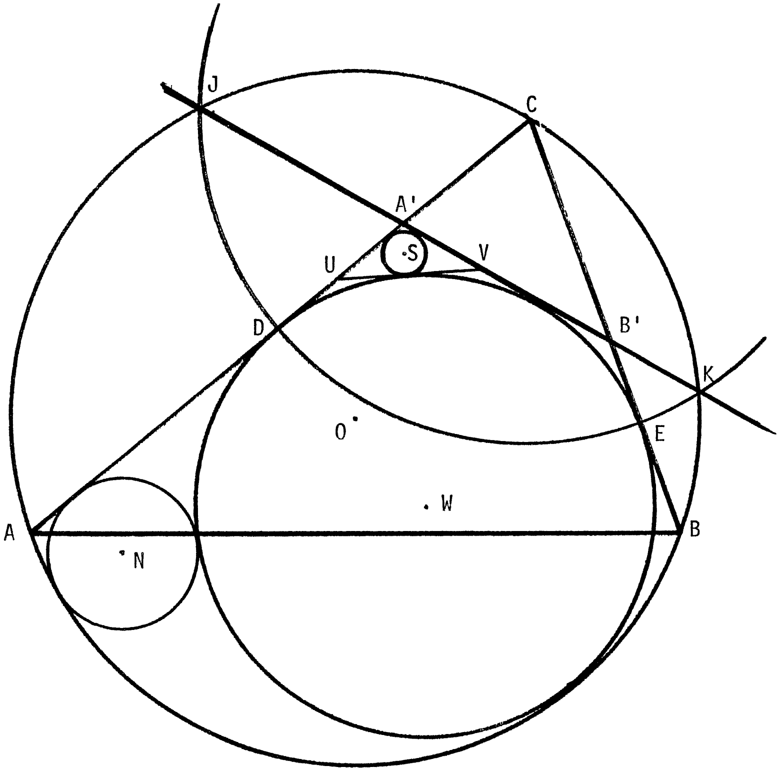

Figure 3

base angles u is $\rho/\rho_m = \tan^2(u/2)$. Consequently, with $u = B/2$, we may write

$$\frac{x}{p} = \tan^2\frac{B}{4}, \tag{5}$$

thus establishing the relationship between circles (W)p and (S)x.

Now comes the denouement of our inversion drama. We have used information regarding circle (W) to discover certain characteristics of circle (S). Our final step is to relate circle (S)x to circle (N)q. We accomplish this by utilizing another one of the principles of inversion: *the ratio of x to q is equal to the ratio of the square of the tangent from* C *to circle* (S) *to the square of the radius of inversion.* The latter is $(ab/s)^2$, and the former is the square of the difference between ab/s and the length of the common external tangent of circles (S) and (W), the last of which is known to be $2\sqrt{px}$, or $2p\tan(B/4)$. Therefore

$$\frac{x}{q} = \frac{(ab/s - 2p\tan(B/4))^2}{(ab/s)^2}.$$

With x given by (5) and p by (1) or (4), the only unknown in this relation is q. With very few lines of paper work, we arrive at the result

$$q = \frac{r\sec^2(C/2)}{(\cot(B/4) - 2\tan(C/2))^2}.$$

Another way of looking at this relation is

$$\frac{p}{q} = (\cot\frac{B}{4} - 2\tan\frac{C}{2})^2. \qquad (6)$$

At first glance this relation may appear erroneous, since we would not normally expect the angle B/4 to be related to the radius of the circle tucked away in the corner near angle A. However, the formula is correct. Mathematical reality is more to be trusted than intuition, which often turns out to be deceptive.

Finally, we apply (4) and (6) to a few special cases. For the Pythagorean triangle considered earlier, in which $a = 3$, $b = 4$, $c = 5$, we find that $p = 2$ and $p/q = 5$, a rather neat ratio. For the Pythagorean triangle in which $a = 7$, $b = 24$, $c = 25$, we find $p = q = 6$: the circles (W) and (N) are equal! In the triangle considered in the article "An Heronian Oddity" [1982: 206], in which $a = 15$, $b = 14$, $c = 13$, we had found that certain strategically placed circles had radii 2, 3, and 4. We now find that (4) yields $p = 5$: the sequence marches on, and the Heronian oddity has become an Heronian odyssey! However, digit delvers will draw cold comfort from the value of p/q in this case.

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1. Leon Bankoff, (proposer of) Problem E 1141, *American Mathematical Monthly*, 61 (1954) 711.

2. Solutions to Problem E 1141, *American Mathematical Monthly*, 62 (1955) 444.

3. Rev. E.M. Radford, *Solutions to Mathematical Problem Papers*, Cambridge, 1925, pp. 160, 418.

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* * *

THE PUZZLE CORNER

Answer to Puzzle No. 26 [1982: 328]: NOW = 12 4 11, WE = 11 16, ARE = 5 13 16, SIX = 18 9 1, ONE = 4 12 16, TWO = 2 11 4.

Answer to Puzzle No. 27 [1982: 328]: STEAK = 20753, COW = 841, I = 9, EAT = 750, MEAT = 6750.

Answer to Puzzle No. 28 [1982: 328]: SQUARE = 164025, RUBIK = 24389, IRIS = 8281.

A 1983 SALMAGUNDI

CHARLES W. TRIGG

The results of delving into the digits of some previous years have been reported in *Crux Mathematicorum* in "1980 Kaleidoscope" [1980: 200-203], "A 1981 Gallimaufry" [1981: 6-9], and "A 1982 Potpourri" [1982: 6-10]. Following in their wake, we have concocted a salmagundi for 1983, seeking a pleasing combination of substance and flavor. The ingredients were dropped into the pot in no particular order and then thoroughly stirred, so dip into it at random. *Bon appétit*!

1		1
9	$\longrightarrow$	$3 + 3 + 3$
8		$2 \cdot 2 \cdot 2$
3		$1 + 1 + 1$

★

1	1
9 8	17
3 1 9	13
8 3 1 9	21
8 3 1 9 8	29
3 1 9 8 3 1	25
9 8 3 1 9 8 3	41
1 9 8 3 1 9 8 3	42
1 9 8 3 1 9 8 3 1	43
9 8 3 1 9 8 3 1 9 8	59
3 1 9 8 3 1 9 8 3 1 9	55
8 3 1 9 8 3 1 9 8 3 1 9	63
8 3 1 9 8 3 1 9 8 3 1 9 8	71
3 1 9 8 3 1 9 8 3 1 9 8 3 1	67
9 8 3 1 9 8 3 1 9 8 3 1 9 8 3	83
1 9 8 3 1 9 8 3 1 9 8 3 1 9 8 3	84
1 9 8 3 1 9 8 3 1 9 8 3 1 9 8 3 1	85

Seven 1983's are written in sequence from the left in seven rows to form a triangular array. Four of the row sums are prime, two of the sums are squares, and the remaining sum, the middle one, is a multiple of 7 and the arithmetic mean of the other six. When the repetitive distribution is continued, certain patterns appear in the triangular array. For $k = 1, 2, 3, \ldots,$

(a) 1983 begins the $8k$th and $8(k + 1)$th row, and terminates the $8k$th and $8(k - 1)$th row;

(b) the sums of the elements in the $(8k - 1)$th, $8k$th, and $8(k + 1)$th rows are the consecutive integers $42k - 1$, $42k$, and $42k + 1$;

(c) beginning with the central column, alternate columns contain like digits, which in successive columns are 1, 9, 8, 3, ...;

(d) the other alternate columns contain like pairs repeated, which in successive columns are 93, 81, ...;

(e) rows $4k - 1$, $4k$, and $4k + 1$ together contain the digits 1, 9, 8, 3, ... in order as triads of like digits forming like angles.

Five 1983's are written in sequence from the left in five rows to form the truncated triangular array on the right. The odd middle row sum is flanked by pairs of consecutive even row sums of which the middle row sum is the arithmetic mean—the same mean as in the previous seven-row triangular array.

		1	9				10
	8	3	1				12
	9	8	3	1			21
9	8	3	1	9			30
8	3	1	9	8	3		32

★

1983 is one of the sequence of years from 1980 to 1986, inclusive, each of which is divisible by its digital root. The reverses of 1980, 1981, and 1983 also are divisible by their digital roots. Thus $1983/3 = 661$, the 121st prime, and $121 = 11^2$; and $3891/3 = 1297$, the 211th prime, 211 is the 47th prime, 47 is also prime, and $4 + 7 = 11$. Permutations of the same digit set, both 121 and 211 have the digital root 2^2.

We have just seen how instructive it can be to dip into the prime life of our years. Dipping further, we find that 1983 begins 9 six-digit primes and 64 seven-digit primes. Both frequencies are squares. It also begins 3 nine-digit palindromic primes, namely

$$198343891, \quad 198353891, \quad \text{and} \quad 198383891.$$

The sum of the middle digits of these primes is 17, the 7th prime, and 7 is prime.

$1983 = 2^2 + 3^2 + 11^2 + 43^2 = 2^2 + 3^2 + 17^2 + 41^2$, the sum of the squares of primes.

★

Applying Ducci's routine [1982: 262] to 1983, we have

$$1983 \rightarrow 8152 \rightarrow 7436 \rightarrow 3131 \rightarrow 2222 \rightarrow 0000$$

where the year vanishes in five operations, the second and third integers in the sequence together contain eight consecutive digits, the other two decimal digits occurring in the extreme members of the sequence.

1983 is the 27th term of the sequence generated from

$$F(1) = 0, \quad F(2) = 2, \quad F(3) = 3, \quad F(n) = F(n-2) + F(n-3).$$

Thus 0, $\underline{2}$, $\underline{3}$, $\underline{2}$, $\underline{5}$, $\underline{5}$, $\underline{7}$, 10, 12, $\underline{17}$, 22, $\underline{29}$, 39, 51, 68, 90, 119, 158, 209, $\underline{277}$, $\underline{367}$, 486, 644, $\underline{853}$, 1130, 1497, 1983. Among these terms there is a prime number, 11, of primes (underlined).

★

We now represent by the digits 1, 9, 8, 3 in order and standard algebraic symbols all the primes less than 100, except 59 which proved refractory[1]. Recourse at times is made to subfactorials, $!x$, where $!1 = 0$, $!2 = 1$, $!3 = 2$, $!4 = 9$, $!5 = 44$, and $!6 = 265$.

$2 = (1 + 9)/(8 - 3)$	$41 = -\sqrt{1 \cdot 9} + !(8 - 3)$
$3 = 1 - 9 + 8 + 3$	$43 = 19 + 8 \cdot 3$
$5 = 1 + 9 - 8 + 3$	$47 = \sqrt{1 \cdot 9} + !(8 - 3)$
$7 = -1 - \sqrt{9} + 8 + 3$	$53 = 1 \cdot 9 + !(8 - 3)$
$11 = 1 - !(!\sqrt{9}) + 8 + 3$	$61 = (-1 + 9) \cdot 8 - 3$
$13 = -1 + \sqrt{9} + 8 + 3$	$67 = 19 + 8(3!)$
$17 = 19 - \sqrt{8/(!3)}$	$71 = 1 + 9 \cdot 8 - !3$
$19 = 1 + 9\sqrt[3]{8}$	$73 = -1 - 9 + 83$
$23 = 19 + 8/(!3)$	$79 = 1 + 9 \cdot 8 + 3!$
$29 = 19 + 8 + !3$	$83 = (!1) \cdot 9 + 83$
$31 = (1 + \sqrt{9})! + 8 - !(!3)$	$89 = (\sqrt{1 \cdot 9})! + 83$
$37 = 1 + 9 \cdot 8/(!3)$	$97 = 1 + 98 - !3$

★

None of the permutations of 1983 is prime. However, 11 smaller primes are formable from the digits of the year, namely: 3, 13, 19, 31, 83, 89, 139, 193, 389, 839, and 983. Note that 19 is the 8th prime, 83 is the 23rd prime, and $8 + 23 = 31$, the 11th prime. Also, $83 - 19 = 64 = 2^6$, so 19 and 83 are the extreme terms of five arithmetic progressions of n terms and common difference d, where $(n, d) = (33, 2)$, $(17, 4)$, $(9, 8)$, $(5, 16)$, and $(3, 32)$.

For example:

19 27 35 43 51 59 67 75 83

---

[1]Will some reader please show Trigg how to do 59? Also 19, where the order of the digits is questionable. (Editor)

in which the sum of the <u>five</u> prime terms is 271, a prime with a digit sum of <u>5</u>•2; and the sum of the <u>four</u> composite terms is the composite 188 = <u>4</u>•47. Using the "trade 'n' twist" routine [*Mathematics Magazine*, 46 (March 1973) 99], a third-order magic square with a magic constant of 153 can be formed from the aforesaid arithmetic progression. Thus

$$\begin{array}{ccc} \underline{83} & 75 & 67 \\ 59 & 51 & 43 \\ 35 & 27 & \underline{19} \end{array} \rightarrow \begin{array}{ccc} \underline{19} & 75 & 35 \\ 59 & 51 & 43 \\ 67 & 27 & \underline{83} \end{array} \rightarrow \begin{array}{ccc} 75 & 35 & 43 \\ \underline{19} & 51 & \underline{83} \\ 59 & 67 & 27 \end{array}$$

★

$1983 = 19 \cdot 83 + 1 \cdot 98 \cdot 3 + 1 \cdot 9 \cdot 8 \cdot 3 - 1 - 98 - 3 + 1 \cdot 9 - 8 - 3.$

$1 \cdot 983 + 198 \cdot 3 = 19 \cdot 83 = (-1 + 9 + 8 + 3)\{(1+9) \cdot 8 + 3\}.$

1, $\sqrt[\sqrt{9}]{8}$, 3 are consecutive integers.

$(1 + 9 + 8)/3 = 6$ and $198/3 = 66$.

9 and $1 + 8 \cdot 3$ are squares.

$-1 - 9 + 8 + 3 = 1,$

$1 + \sqrt{9} + 8 - 3 = 9,$

$1 \cdot \sqrt{9} + 8 - 3 = 8,$

$-1 + 9 - 8 + 3 = 3.$

$1983 = 3\{(2 \cdot 3)^2 + (2+3)^{2+2}\}.$

$1983_{ten} = 2201110_{three} = 11110111111_{two}$. In bases two and three, all digits in each base appear in the year.

★

The first ten square numbers represented by the digits 1, 9, 8, 3 in order and algebraic symbols are:

$1 = -1 - 9 + 8 + 3,$	$36 = (1 + 9 + 8) \cdot (!3),$
$4 = 1 \cdot 9 - 8 + 3,$	$49 = -1 + !\sqrt{9} + 8 \cdot (3!),$
$9 = (19 + 8)/3,$	$64 = -19 + 83,$
$16 = -1 + \sqrt{9} + 8 + 3!,$	$81 = (1 + 9) \cdot 8 + !(!3),$
$25 = 1 + 9 \cdot 8/3,$	$100 = -1 + 98 + 3.$

★

The roots of $x^2 - 102x + 1577 = 0$ are 19 and 83. In the equation

$$x^4 - 21x^3 + 143x^2 - 339x + 216 = 0,$$

the absolute values of three coefficients are the products of two primes, while the constant term, the product of the roots, is $1 \cdot 9 \cdot 8 \cdot 3 = 216 = 6^3$.

★

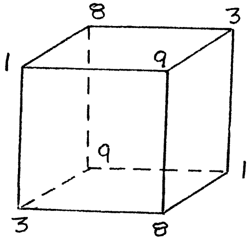

Figure 1

Figure 1 shows the only way to place the digits of 1983 on the vertices of a cube so that each digit appears on the perimeter of each face. The clockwise (or counter-clockwise) orders around the faces constitute the six cyclic permutations of four distinct digits. Digits at the extremities of space diagonals are like. Orders on opposite faces are in opposite senses.

Figure 2 shows the two ways that the digits of 1983 can be distributed on the midpoints of the edges of a cube so that each digit appears on the perimeter of each face. Like digits fall on edges having different directions and no common vertices. Orders on opposite faces are in opposite senses, and the six cyclic permutations of the four digits appear.

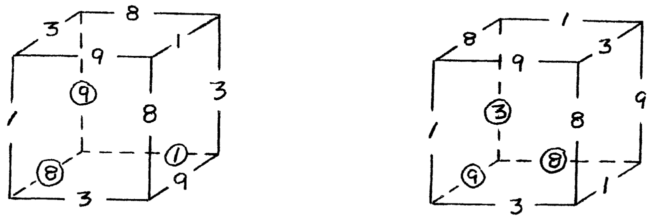

Figure 2

★

In the reiterative operation wherein an integer and its reverse are added, let N_0 be the original integer and N_k the sum in the kth addition. If N_0 = 1983, the first palindrome encountered is N_4. Thus

N_0 =	1983	5874	10659	106260
	3891	4785	95601	062601
N_1 =	5874	10659	106260	168861 = N_4.

Other palindromes in the sequence are N_8 = 1320231, N_9 = 2640462, and N_{13} = 99577599. The smallest versum in this sequence that contains all the decimal digits is N_{28} = 114279288286350.

In the repetitive Kaprekar routine, the digits of a four-digit integer are arranged in ascending order. This is then subtracted from its reverse. Thus, beginning with 1983,

9831	8442	9954	5553	9981	8820	8532
1389	2448	4599	3555	1899	0288	2358.
8442	5994	5355	1998	8082	8532	6174

Seven operations to reach Kaprekar's constant, the self-replicating 6174.

★

The 24 second-order determinants that can be formed with the digits of 1983 fall into three groups of eight that have the same absolute values. The values of representatives from each group sum to a prime. For example,

$$\begin{vmatrix} 9 & 1 \\ 3 & 8 \end{vmatrix} + \begin{vmatrix} 8 & 1 \\ 9 & 3 \end{vmatrix} + \begin{vmatrix} 9 & 1 \\ 8 & 3 \end{vmatrix} = 103.$$

The digits of 1983 and their squares are represented below by nine-digit determinants in which the central digit is the same as the value of the determinant (or its square root).

$$\begin{vmatrix} 7 & 4 & 6 \\ 2 & \underline{1} & 3 \\ 9 & 5 & 8 \end{vmatrix} = 1, \quad \begin{vmatrix} 1 & 2 & 3 \\ 7 & \underline{9} & 8 \\ 4 & 6 & 5 \end{vmatrix} = 9, \quad \begin{vmatrix} 2 & 1 & 3 \\ 4 & \underline{8} & 5 \\ 6 & 7 & 9 \end{vmatrix} = 8, \quad \begin{vmatrix} 4 & 7 & 6 \\ 1 & \underline{3} & 2 \\ 8 & 5 & 9 \end{vmatrix} = 3;$$

$$\begin{vmatrix} 2 & 6 & 3 \\ 4 & \underline{1} & 7 \\ 5 & 9 & 8 \end{vmatrix} = 1^2, \quad \begin{vmatrix} 3 & 1 & 7 \\ 4 & \underline{9} & 6 \\ 2 & 8 & 5 \end{vmatrix} = 9^2, \quad \begin{vmatrix} 1 & 4 & 2 \\ 5 & \underline{8} & 6 \\ 9 & 7 & 3 \end{vmatrix} = 8^2, \quad \begin{vmatrix} 1 & 2 & 5 \\ 8 & \underline{3} & 7 \\ 6 & 4 & 9 \end{vmatrix} = 3^2.$$

★

1983 is part of a 1560-digit additive bracelet wherein each element is the units' digit of the sum of the four preceding digits, namely

$\underline{1983}1138359521755853 1767\ldots$.

The complete bracelet is included in "A Digital Bracelet for 1967", *The Fibonacci Quarterly*, 5 (December 1967) 477-480.

1983 is a charm attached to the ten-member fourth-order multiplicative bracelet wherein each digit is the units' digit of the product of the four preceding digits, thus:

1983'6642846488'6642... .

★

Rome was founded in 753 B.C. So, as we enter into the new year, the number of years that have elapsed *ab urbe condita* is 753 + 1983 - 1 = 2735 (there was no year zero). And the first four digits of the cube root of 2735 form a permutation of the digits of 1983.

The equinoctial year 1983 lasts for 365 days, 5 hours, 48 minutes, and 46 seconds, a total of 31556926 seconds; and may you enjoy every one of them! If that is not enough, the sidereal year 1983 equals the equinoctial year plus 20 minutes: time for one more coffee.

2404 Loring Street, San Diego, California 92109.

* * *

THE OLYMPIAD CORNER: 41

M.S. KLAMKIN

I have in the past given Practice Sets consisting of three Olympiad-type problems. Solutions to the problems in the last set, Practice Set 15, appeared in [1980: 276-282]. I give below another Practice Set, for which solutions will appear here next month.

PRACTICE SET 16

16-1. Solve the equation $(x^2 - 4)(x^2 - 2x) = 2$.

16-2. Given the face angles of a trihedral angle, determine the locus of the points of contact of its faces with its inscribed spheres.

16-3. A length L of wire is cut into two pieces which are bent into a circle and a square. Determine the minimum and the maximum of the sum of the two areas formed.

*

For Problem 2 of the 1982 International Mathematical Olympiad, repeated below, a reader has sent me several interesting extensions, which I give below without proof.

2. [1982: 225] A scalene triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite A_i). For all $i = 1,2,3$, M_i is the midpoint of side a_i, T_i is the point where the incircle touches side a_i, and the reflection of T_i in the interior bisector of A_i yields the point S_i. Prove that the lines M_1S_1, M_2S_2, and M_3S_3 are concurrent.

Extensions by J.T. Groenman, Arnhem, The Netherlands.

(a) The point P of concurrency of the M_iS_i lies on the incircle of triangle $A_1A_2A_3$.

(b) If Q is antipodal to P with respect to the incircle, then Q and the point at infinity of the Euler line of triangle $T_1T_2T_3$ are isogonal conjugates with respect to triangle $T_1T_2T_3$.

(c) The Wallace-Simpson line of S_3 with respect to triangle $T_1T_2T_3$ is perpendicular to A_1A_2.

*

I now present solutions to the problems of the 1981 Hungarian Mathematical Olympiad (Second Round) [1981: 267-268]. Solutions to Problem 1 of the First

Version and to Problem 1 of the Third Version have already appeared here ([1982: 46] and [1982: 166]).

2. (First Version) [1981: 267] The real numbers x and y satisfy $0 < x < 1$ and $x+y = 1$. Determine the maximum and minimum values of the expression

$$x\cdot\frac{1+x^2}{1+x} + y\cdot\frac{1+y^2}{1+y}.$$

Solution.

Since

$$\frac{x+x^3}{1+x} = x^2 - x + 2 - \frac{2}{x+1},$$

with a similar expression in y, the given expression is equal to

$$\begin{aligned} f(x,y) &\equiv (x+y)^2 - (x+y) + 4 - 2xy - 2\left\{\frac{x+y+2}{xy+x+y+1}\right\} \\ &= 2\left\{4 - (xy+2) - \frac{3}{xy+2}\right\}. \end{aligned}$$

The problem therefore reduces to finding the extremes of the function

$$g(t) \equiv t + \frac{3}{t}$$

for $2 < t \le 2 + \max(xy)$. Now g is convex and attains its minimum value for $t = \sqrt{3} < 2$ (by the A.M.-G.M. inequality). Hence f attains its minimum value $5/6$ when $xy = t - 2$ has its maximum value $\frac{1}{4}$ (for $x = y = \frac{1}{2}$). The values of the restricted g function have greatest lower bound $7/2$, corresponding to $t = 2$, and this gives least upper bound 1 for the values of f. However, this upper bound is never attained since it would require the excluded values $\{x,y\} = \{0,1\}$. We therefore have

$$\frac{5}{6} \le f(x,y) < 1.$$

3. (First Version) [1981: 267] A frustum of a certain triangular pyramid has lower base of area A, upper base of area B (where $B < A$), and the sum of the areas of its lateral faces is P. The frustum is such that it can be divided, by a plane parallel to the bases, into two smaller frustums in each of which a sphere can be inscribed. Prove that

$$P = (\sqrt{A} + \sqrt{B})(\sqrt[4]{A} + \sqrt[4]{B})^2.$$

Solution.

Let C be the area of the section formed by the indicated plane parallel to the bases, and let T_a denote the triangular pyramid from which the given frustum is obtained. The planes of B and C divide T_a into two other pyramids, T_b and T_c, whose bases have areas B and C, respectively. Let h_b, h_c, h_a be the altitudes, and r_b, r_c, r_a the radii of the inscribed spheres, of pyramids T_b, T_c, T_a, respectively. Since the three pyramids are homothetic, we have, for some constants k and l,

$$\frac{r_a}{\sqrt{A}} = \frac{r_b}{\sqrt{B}} = \frac{r_c}{\sqrt{C}} = k \qquad \text{and} \qquad \frac{h_a}{\sqrt{A}} = \frac{h_b}{\sqrt{B}} = \frac{h_c}{\sqrt{C}} = l.$$

The volumes of T_a and T_b are given by

$$V(T_a) = \tfrac{1}{3}h_a A = \tfrac{1}{3}r_a(A + L_a)$$

and

$$V(T_b) = \tfrac{1}{3}h_b B = \tfrac{1}{3}r_b(B + L_b),$$

where L_a and L_b are the lateral areas of T_a and T_b, respectively. Thus

$$lA = k(A + L_a) \qquad \text{and} \qquad lB = k(B + L_b),$$

which implies that

$$P = L_a - L_b = \left(\frac{l}{k} - 1\right)(A - B). \tag{1}$$

Since a sphere can be inscribed in each of the two smaller frustums, we must have

$$h_b + 2r_c = h_c \qquad \text{and} \qquad h_c + 2r_a = h_a,$$

from which we get

$$l\sqrt{B} + 2k\sqrt{C} = l\sqrt{C} \qquad \text{and} \qquad l\sqrt{C} + 2k\sqrt{A} = l\sqrt{A},$$

and eliminating $\sqrt{C}$ from these two equations yields

$$\frac{l}{k} = \frac{2\sqrt[4]{A}}{\sqrt[4]{A} - \sqrt[4]{B}}.$$

Hence, from (1),

$$P = \left(\frac{l}{k} - 1\right)(\sqrt{A} + \sqrt{B})(\sqrt[4]{A} + \sqrt[4]{B})(\sqrt[4]{A} - \sqrt[4]{B})$$

$$= (\sqrt{A} + \sqrt{B})(\sqrt[4]{A} + \sqrt[4]{B})^2.$$

1. (Second Version) [1981: 267] Make pairs from the medians of the faces of a tetrahedron in such a way that medians starting from the same midpoint of an edge form a pair. Suppose that in each pair the two medians have equal lengths. How many different lengths of these medians can there be?

Solution by Nelson M. Ferrer, student, Universidad Nacional de Colombia.

Let the tetrahedron be A-BCD, and let E,F,G be the midpoints of BC,CD,DB, respectively. Then AE = ED, AG = GC, and AF = FB. Thus point A belongs simultaneously to the spheres with centers E,F,G and radii ED,FB,GC, respectively. This implies that if there is such a tetrahedron with base BCD, then it is unique (to within a reflection). On the other hand, it follows easily that such a tetrahedron exists having all faces congruent to BCD, i.e., the tetrahedron must be isosceles. Hence there are at most three different lengths for the medians of the faces.

Comment by M.S.K.

R.R. Rottenberg, The Technion, Haifa, Israel, gave a synthetic proof based on the facts that the three bimedians concur at the centroid and then that the centroid coincides with the circumcenter. These results can also be obtained very directly by vectors.

2. (Second Version) [1981: 268] Let

$$f(x) = \begin{cases} \sin \pi x, & \text{if } x < 0 \\ f(x-1)+1, & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \cos \pi x, & \text{if } x < \frac{1}{2} \\ g(x-1)+1, & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Solve the equation $f(x) = g(x)$.

Solution by Nelson M. Ferrer, Universidad Nacional de Colombia.

For $x < 0$, f and g are periodic with period 2, and the equation has the solutions $-3/4$ and $-7/4$ in the interval $(-2, 0)$. We therefore have the solutions

$$x = -(n + \tfrac{3}{4}), \qquad n = 0,1,2,\ldots \quad . \tag{1}$$

Also, 0 is a solution and there are no solutions in the interval $(0, \frac{1}{2})$. For $x \geq \frac{1}{2}$, the two functions satisfy

$$f(x) = g(x) \iff f(x+1) = g(x+1).$$

As the unique solution in the interval $(1/2, 3/2)$ is $x = 1$, another set of solutions is

$$x = n, \qquad n = 0,1,2,\ldots \quad . \tag{2}$$

All solutions are included in (1) and (2).

3. (Second Version) [1981: 268] Denote by $f(k)$ the number of zeros in the decimal representation of the natural number k. Compute

$$S_n = \sum_{k=1}^{n} 2^{f(k)},$$

where $n = 10^{10}-1$.

Solution.

We solve the following more general problem:

Let a be a nonzero real number, and let b and m be natural numbers. Denote by $f(k)$ the number of zeros in the base $b+1$ representation of the natural number k. Compute

$$S_n = \sum_{k=1}^{n} a^{f(k)},$$

where $n = (b+1)^m - 1$.

For each k, we have $0 \le f(k) \le m-1$. For $i = 0,1,\ldots,m-1$, let $N(a^i)$ be the number of summands a^i in S_n. Then we have

$$N(a^0) = b + b^2 + \ldots + b^m,$$

$$N(a^1) = \binom{1}{1}b + \binom{2}{1}b^2 + \ldots + \binom{m-1}{1}b^{m-1},$$

$$N(a^2) = \binom{2}{2}b + \binom{3}{2}b^2 + \ldots + \binom{m-1}{2}b^{m-2},$$

$$\vdots$$

$$N(a^{m-1}) = \binom{m-1}{m-1}b^{m-(m-1)}.$$

Consequently,

$$\begin{aligned} S_n = {} & b\left\{\binom{0}{0}1 + \binom{1}{1}a + \ldots + \binom{m-1}{m-1}a^{m-1}\right\} \\ & + b^2\left\{\binom{1}{0}1 + \binom{2}{1}a + \ldots + \binom{m-1}{m-2}a^{m-2}\right\} + \\ & \vdots \\ & + b^m\left\{\binom{m-1}{0}1\right\}. \end{aligned}$$

Dividing both sides by b and collecting terms of the same degree in a and b, we get

$$\begin{aligned} S_n/b &= 1 + \left\{\binom{1}{0}b + \binom{1}{1}a\right\} + \left\{\binom{2}{0}b^2 + \binom{2}{1}ba + \binom{2}{2}a^2\right\} \\ & \quad + \ldots\ldots + \left\{\binom{m-1}{0}b^{m-1} + \binom{m-1}{1}b^{m-2}a + \ldots + \binom{m-1}{m-1}a^{m-1}\right\} \\ &= 1 + (b+a) + (b+a)^2 + \ldots + (b+a)^{m-1}, \end{aligned}$$

and so

$$S_n = \begin{cases} bm, & \text{if } b+a = 1, \\ b\cdot\dfrac{(b+a)^m-1}{b+a-1}, & \text{if } b+a \ne 1. \end{cases} \quad \square$$

For the given problem, where $a = 2$, $b+1 = 10$, and $m = 10$, we get

$$S_n = \frac{9(11^{10} - 1)}{10}.$$

2. (Third Version) [1981: 268] Let n be a positive integer, and let $f(n)$ denote the number of triplets consisting of three different positive integers the sum of which is exactly n. (Two triplets are considered to be identical if they differ only by the order of their elements.) For which n is $f(n)$ an even number?

Solution.

The determination of $f(n)$ is given in N.Y. Vilenkin, *Combinatorics*, Academic Press, New York, 1971, page 225.

The number of ordered partitions of n into three positive summands is

$$\binom{n-1}{2} = \frac{n^2-3n+2}{2}.$$

(Just consider the generating function $(x+x^2+x^3+\ldots)^3$.) If n is even, then $(n-2)/2$ of the unordered partitions have two equal summands. If n is odd, then $(n-1)/2$ of the unordered partitions have equal summands. Also, if n is divisible by 3, then there is one partition with three equal summands. Applying the principle of inclusion and exclusion, we find that the number of ordered partitions of n into distinct summands is

$$\begin{aligned}
&\binom{n-1}{2} - \tfrac{3}{2}(n-2) + 2 = 18k^2 - 18k + 6, && \text{if } n = 6k,\\
&\binom{n-1}{2} - \tfrac{3}{2}(n-1) \qquad\;\; = 18k^2 - 12k, && \text{if } n = 6k+1,\\
&\binom{n-1}{2} - \tfrac{3}{2}(n-2) \qquad\;\; = 18k^2 - 6k, && \text{if } n = 6k+2,\\
&\binom{n-1}{2} - \tfrac{3}{2}(n-1) + 2 = 18k^2, && \text{if } n = 6k+3,\\
&\binom{n-1}{2} - \tfrac{3}{2}(n-2) \qquad\;\; = 18k^2 + 6k, && \text{if } n = 6k+4,\\
&\binom{n-1}{2} - \tfrac{3}{2}(n-1) \qquad\;\; = 18k^2 + 12k, && \text{if } n = 6k+5.
\end{aligned}$$

To find the number $f(n)$ of unordered partitions, we divide the number of ordered partitions by 6. Thus $f(n)$ is even when $n = 6k+2$ or $6k+4$ for any k, and when $n = 6k+1$ or $6k+3$ or $6k+5$ for even k.

Note that, for any n,

$$f(n) = \left[\frac{n^2 - 6n + 12}{12}\right],$$

where the brackets denote the greatest integer function.

3. (Third Version) [1981: 268] Construct (and prove your result) a polynomial $P(x)$ with integral coefficients such that

$$|P(x) - 0.5| < \frac{1}{1981}$$

for every real number x in the interval $[0.19, 0.81]$.

Solution by A. Meir, University of Alberta.

More generally, we find a polynomial $P(x)$ satisfying

$$\left|P(x) - \frac{p}{q}\right| < \varepsilon \qquad \text{for} \qquad \delta \le x \le 1-\delta,$$

where p/q is a given rational fraction (with $q > 0$), and ε and δ satisfy $0 < \varepsilon < 1$ and $0 < \delta < \frac{1}{2}$.

The answer is trivial if $p = 0$ (take the zero polynomial), so we assume that $p \neq 0$. It is easy to see that a suitable polynomial is

$$P(x) = \frac{p}{q}\{1 - [1 - qx^r(1-x)^r]^n\},$$

where $4^r > q$ and

$$n > \left|\frac{\ln(q\varepsilon/|p|)}{\ln\{1 - q\delta^r(1-\delta)^r\}}\right|.$$

For the given problem, we can take $p = 1$, $q = 2$, $r = 1$, $\varepsilon = 1/1981$, $\delta = 0.19$, and $n = 19$, resulting in the polynomial

$$P(x) = \tfrac{1}{2}\{1 - [1 - 2x(1-x)]^{19}\}. \quad \square$$

As a rider, determine the supremum of the total length of all intervals of x such that, for a suitable integral polynomial $P(x)$,

$$\left|P(x) - \frac{p}{q}\right| < \varepsilon$$

holds with given p/q $(\neq 0)$ and ε as above.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

A BORDERED MAGIC SQUARE FOR 1983

The adjoining magic square of order 6, with magic constant 1983, contains the 36 consecutive integers from 313 to 348. Removal of the outer border leaves a magic square of order 4 with magic constant 1322.

I offer it do *Crux* readers with my best wishes for 1983.

319	344	318	330	347	325
313	341	322	321	338	348
346	332	327	328	335	315
345	326	333	334	329	316
324	323	340	339	320	337
336	317	343	331	314	342

KAMALA KUMARI
Radhaur, Bihar, India

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before June 1, 1983, although solutions received after that date will also be considered until the time when a solution is published.

801. *Proposed by Sidney Kravitz, Dover, New Jersey.*

At this time (late January 1983), Canada's peripatetic Prime Minister, the apostle of the North-South dialogue, is catching his breath at home. So let us lose no time in proposing the following alphametic before he takes off again (or bows out):

$$\begin{array}{r} \text{PIERRE} \\ \underline{\text{ELLIOTT}} \\ \text{TRUDEAU} \end{array}.$$

802. *Proposed by Joseph Gillis, Weizmann Institute of Science, Rehovot, Israel.*

Points P_1 and P_2 move with equal angular velocities along circles C_1 and C_2, respectively, in the plane. Prove that in general there is one point S such that $|SP_1|^2 - |SP_2|^2$ is constant (where $|SP|$ denotes the distance between S and P), and discuss the exceptional cases.

(This problem generalizes Problem 3 of the 1979 International Mathematical Olympiad [1979: 194].)

803. *Proposed by Leroy F. Meyers, The Ohio State University.*

Let f be the operation which takes a positive integer n to $\frac{1}{2}n$ if n is even, and to $3n+1$ if n is odd. It is as yet unknown whether or not every positive integer n can be reduced to 1 by successively applying f to it. (This is the substance of Problem 133 [1976: 67, 144-150, 221].) In comment I (Trigg, p. 144), it is stated that it is unnecessary to carry the sequence N, fN, ffN, $fffN$, ... for N beyond any term less than N. In particular, it is not necessary to test N if N is even or is of the form $4k+1$.

Is this argument justified?

804*. *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus.*

Let $P_n(x)$ be a polynomial of degree $n \geq 2$ with real coefficients, leading coefficeient $a \neq 0$, and n real zeros x_i with

$$x_1 \leq x_2 \leq \ldots \leq x_n.$$

It is easily verified that

$$\int_{x_1}^{x_2} |P_2(x)|\ dx = \frac{|a|}{6}(x_2-x_1)^3$$

and (more tediously) that

$$\int_{x_1}^{x_3} |P_3(x)|\ dx = \frac{|a|}{12}\{(x_2-x_1)^3(3x_3-\Sigma x_i) + (x_3-x_2)^3(\Sigma x_i-3x_1)\},$$

where the indicated sum is for $i = 1,2,3$.

Find a "nice" compact formula for

$$\int_{x_1}^{x_n} |P_n(x)|\ dx.$$

805. *Proposed by M.S. Klamkin, University of Alberta.*

If $x,y,z > 0$, prove that

$$\frac{x+y+z}{3\sqrt{3}} \geq \frac{yz+zx+xy}{\sqrt{y^2+yz+z^2}+\sqrt{z^2+zx+x^2}+\sqrt{x^2+xy+y^2}},$$

with equality if and only if $x = y = z$.

806. *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

Let LMN be the cevian triangle of the point S for the triangle ABC (i.e., the lines AS,BS,CS meet BC,CA,AB in L,M,N, respectively). It is trivially true that

S is the centroid of ΔABC $\Longrightarrow$ S is the centroid of ΔLMN.

Prove the converse.

807. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Three balls are taken at random from an urn that contains w white balls and r red balls. The probability that the three balls are white is p. If the urn had contained one more white ball, the probability of three white balls would have been $4p/3$. Find all possible values of the pair (w,r).

808*. *Proposed by Stanley Rabinowitz, Digital Equipemnt Corp., Merrimack, New Hampshire.*

Find the length of the largest circular arc contained within the right triangle with sides $a \leq b < c$.

809. *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

Let A and B be two angles of a triangle, with opposite sides a and b, respectively. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n}(\cos 2nA - \cos 2nB) = \ln \frac{b}{a}.$$

810. *Proposed by Charles W. Trigg, San Diego, California.*

Place different integers on the vertices of a triangle, chosen so that the sums of integers on the extremities of each side of the triangle will be squares.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

584. [1980: 283; 1981: 290; 1982: 16, 51, 107] An analytic solution was received from J.T. GROENMAN, Arnhem, The Netherlands.

* * *

682. [1981: 274; 1982: 287] *Proposed by Robert C. Lyness, Southwold, Suffolk, England.*

Triangle ABC is acute-angled and Δ_1 is its orthic triangle (its vertices are the feet of the altitudes of triangle ABC). Δ_2 is the triangular hull of the three excircles of triangle ABC (that is, its sides are the external common tangents of the three pairs of excircles that are not sides of triangle ABC).

Prove that the area of triangle Δ_2 is at least 100 times the area of triangle Δ_1.

III. *Comment by G.R. Veldkamp, De Bilt, The Netherlands.*

As the editor requested [1982: 288], I give a noncalculus proof of the inequality

$$\frac{1 + \cos A + \cos B + \cos C}{2 \cos A \cos B \cos C} \geq 10 \tag{1}$$

for acute-angled triangles. From the inequality

$$\cos A \cos B \cos C \leq \frac{1}{8}$$

(for which a reference was given on page 288) and the A.M.-G.M. inequality, we have

$$\begin{aligned}\frac{1 + \cos A + \cos B + \cos C}{\cos A \cos B \cos C} &\geq 8 + \sec B \sec C + \sec C \sec A + \sec A \sec B \\ &\geq 8 + 3\sqrt[3]{\sec^2 A \sec^2 B \sec^2 C} \\ &\geq 8 + 3\sqrt[3]{64} \\ &= 20,\end{aligned}$$

and (1) follows, with equality just when the triangle is equilateral. For acute-angled triangles, (1) is equivalent to

$$1 + \cos A + \cos B + \cos C \geq 20 \cos A \cos B \cos C. \qquad (2)$$

Since its left member is positive for all triangles [1], it is clear that (2) holds for right- and obtuse-angled triangles as well, hence for all triangles.

As mentioned by the editor [1982: 289], it is known that the orthic triangle Δ_1 is homothetic to the tangential triangle (call it Δ_3) of Δ. It is also known [2] that the homothetic centre of Δ_1 and Δ_3 lies on the Euler line of Δ. However, the homothetic centres of the pairs $\{\Delta_1,\Delta_2\}$ and $\{\Delta_1,\Delta_3\}$ are, in general, different.

IV. *Comment by the proposer.*

I reiterate my suggestion that the homothetic centre of Δ_1 and Δ_2 be called the *Crucial Point* of the triangle. It won't do to call it the Lyness Point, as the editor kindly suggested [1982: 289], because there is already a geographical Lynas Point in Anglesey.

Noncalculus proofs of (1) were also submitted by JACK GARFUNKEL, Flushing, N.Y.; M.S. KLAMKIN, University of Alberta; and BOB PRIELIPP, University of Wisconsin-Oshkosh.

Editor's comment.

Let Δ_4 be the triangle whose sides are the tangents to the nine-point circle of Δ at the midpoints of the sides of Δ. It is known [3] that Δ_4 is homothetic to Δ_1, that their homothetic centre lies on the Euler line of Δ, and that their homothetic ratio is $4 \cos A \cos B \cos C$.

So the triangles Δ_1, Δ_2, Δ_3, Δ_4 (and there may be more!) are homothetic in pairs. An interesting article could result from an exhaustive investigation of the homothetic centres and homothetic ratios of each pair of triangles.

Edith Orr, the poetess who routinely sanitizes and adds a bit of sparkle to this French-speaking editor's English, paraphrased the last paragraph as follows: "Our gay little orthic triangle sure has a lot of homo(thetic) friends! It would be interesting to know who does what to whom when they all get together."

REFERENCES

1. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969, p. 22, No. 2.16.

2. Nathan Altshiller Court, *College Geometry*, Barnes & Noble, New York, 1952, p. 102, Art. 205.

3. Solutions to Problem E 1350 (proposed by N.A. Court), *American Mathematical Monthly*, 66 (1959) 594-595.

* * *

686. [1981: 275; 1982: 294] *Proposed by Charles W. Trigg, San Diego, California.*

Without using calculus, analytic geometry, or trigonometry, find the area of the region which is common to the four quadrants that have the vertices of a square as centers and a side of the square as a common radius.

[A solution using analytic geometry appears in *School Science and Mathematics*, 78 (April 1978) 355.]

II. *Comment by E. Frederick Lang, M.D., Grosse Pointe, Michigan.*

The solution to this problem referred to in the editor's comment [1982: 294, 278] is indeed by Samuel I. Jones. But it is not in his book *Mathematical Wrinkles* (the editor's crystal ball must have been clouded). Rather, it can be found in his book *Mathematical Nuts*, published in 1932. It is Problem No. 9 on page 86, and the solution appears on page 301.

* * *

700. [1981: 302] *Proposed by Jordi Dou, Barcelona, Spain.*

Construct the centre of the ellipse of minimum eccentricity circumscribed to a given convex quadrilateral.

I. *Solution by the proposer (translated from Spanish by Dan Pedoe, University of Minnesota).*

Let $A_1A_2A_3A_4$ be the given convex quadrilateral. Draw an auxiliary circle ϕ through $V = A_3A_4 \cap A_2A_1$, as shown in Figure 1, and let C be its centre. Project from V onto ϕ the Desargues involution cut by the sides of the quadrilateral on the line at infinity, by drawing the lines V4 parallel to A_4A_1, V2 parallel to A_2A_3, V1 along A_1A_2, and V3 along A_3A_4. The intersection C_1 of the lines 24 and 13 is the centre of the involution as represented on ϕ . Draw the polar d of C_1, the points 5 and 6 on d being the points of contact of tangents from C_1 to ϕ.

The involutions on the line at infinity by the conics which pass through the A_i being harmonic (permutable) with regard to the Desargues involution, their centres lie on d. The centre which corresponds to the minimum eccentricity is C_2, the point on d nearest to the centre C of ϕ. The points $1' = \phi \cap 1C_2$ and $2' = \phi \cap 2C_2$ give the directions V1' and V2' of the diameters conjugate to A_1A_2, A_2A_3 which pass through the respective midpoints M_1, M_2. The centre O of the required ellipse is the intersection of these conjugate diameters.

II. *Comment by the translator.*

There are two involutions used in this solution. The first is the Desargues involution cut by circumconics of the A_i on the line at infinity, pairs of the involution being given by the sides of the quadrilateral. Choosing V at the intersection of A_3A_4 and A_2A_1 simplifies a little the representation of this involution on the auxiliary circle ϕ. This method and some remarks on involutions are to be found in my article "Pascal Redivivus: II" in this journal [1979: 281] and in my book [1], the theory of involutions being an important part of projective geometry.

Pairs of the involution are represented by points cut on ϕ by lines through C_1. The double (fixed) points of the involution are represented by the points of contact 5 and 6 of tangents through C_1 to ϕ.

The second involution used in the solution is that cut, for a given conic, by conjugate diameters on the line at infinity. The midpoints of a set of chords of the conic in a given direction lie on a diameter, and this diameter and the one in the given direction are conjugate diameters. On the line at infinity conjugate diameters cut pairs of points which are harmonic with respect to the intersections of the given conic and the line at infinity.

Since the double points of an involution are harmonic with respect to any pair of the involution, we see that the points 5 and 6 on ϕ represent a pair of every involution cut on the line at infinity by conjugate diameters of conics through the A_i. Hence the directions V5 and V6 are directions for a pair of conjugate diameters of all circumconics, and the centre of the involution of conjugate diameters for any circumconic in the ϕ representation must lie on d.

In Dörrie [2], the existence of the most nearly circular ellipse circumscribing a quadrilateral is discussed algebraically. We have proved *Auxiliary Theorem I* [2, p. 233] synthetically. The problem was posed in the seventeenth volume of Gergonne's *Annales de Mathématiques* (1826) and solved by J. Steiner in *Crelle's Journal*, vol. II (1827). (See also Steiner, *Gesammelte Werke*, vol. I.)

But eccentricity is a metrical concept, and we need the algebraic proof that for a given ellipse the minimum acute angle between a pair of conjugate diameters occurs when they are equi-conjugate diameters, and that

$$\tan\left(\tfrac{1}{2}\,\text{minimum angle}\right) = \sqrt{1 - e^2},$$

where e is the eccentricity of the ellipse. It follows that the most nearly circular ellipse through the A_i, that with least eccentricity, is the one which has its equi-conjugate diameters along the directions for conjugate diameters common to all the circumscribing ellipses.

A separate diagram (Figure 2) will help to explain the final step in the proposer's construction. Let C^* be the centre of the involution of conjugate diameters in the ϕ representation corresponding to a given circumellipse. If P and P' are the intersections with ϕ of a line through C^*, then VP and VP' give the directions of a pair of conjugate diameters of this ellipse. To obtain the principal axes of

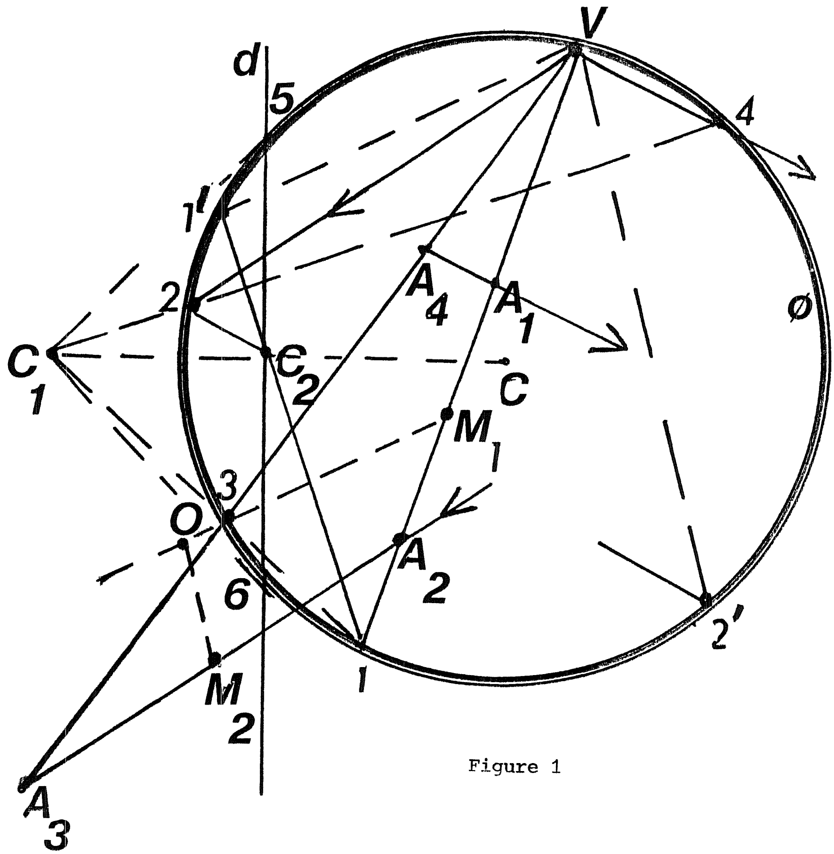

Figure 1

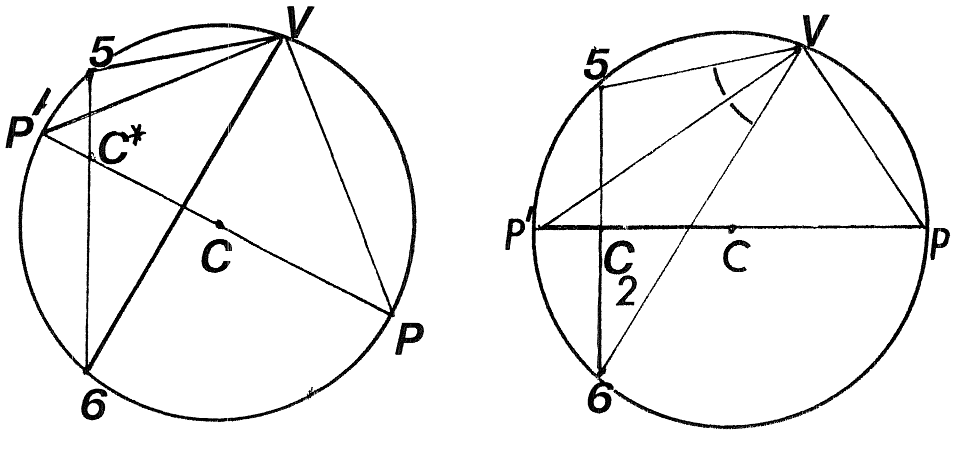

Figure 2

the ellipse, we want VP to be perpendicular to VP'. If we join C* to the centre C of ϕ, then VP and VP' will give the directions of the principal axes. If V5 and V6 are to be the directions of equi-conjugate diameters of this ellipse, we want V5 and V6 to make equal angles with VP'. If C* is the foot of the perpendicular from C onto d, the arcs 6P' and P'5 are equal, and so the angles they subtend at V are equal. This explains the location of C_2 in the construction, and other explanations are possible. The final determination of the centre O of the ellipse is based on the geometric property of conjugate diameters given earlier. Much of the above can be found in my *Crux* paper referred to at the beginning of this comment.

Editor's comment.

Reference [2] was supplied to the translator by the editor, who himself received it from Thomas J. Banchoff, Brown University, via H.S.M. Coxeter, University of Toronto.

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1. D. Pedoe, *A Course of Geometry for Colleges and Universities*, Cambridge University Press, New York, 1970.

2. Heinrich Dörrie, *100 Great Problems of Elementary Mathematics*, Dover, New York, 1965, pp. 231-236.

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703. [1982: 14] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

A right triangle ABC has legs AB = 3 and AC = 4. A circle γ with center G is drawn tangent to the two legs and tangent internally to the circumcircle of the triangle, touching that circumcircle in H. Find the radius of γ and prove that GH is parallel to AB.

I. *Comment by John A. Winterink, Albuquerque Technical Vocational Institute, Albuquerque, New Mexico.*

This problem is a special case of the fifth problem pertaining to the Apollonius Tangency Problem. As described by Vieta in *Apollonius Gallus*, page 329, it reads as follows:

Problema V. Dato circulo, & duabus lineis describere circulum quem datus circulus, & datæ duæ lineæ rectæ contingant.

II. *Solution by Leon Bankoff, Los Angeles, California.*

The corresponding problem for an arbitrary triangle ABC right-angled at A was proposed by me in 1954 [1]. The two simple solutions later published [2], one of which was synthetic and the other analytic, showed that the radius of γ is $b+c-a$.

For the present problem, where $b = 4$, $c = 3$, and $a = 5$, the radius of γ is $4+3-5 = 2 = b/2$. Thus G and the circumcenter O are equidistant from AB. Hence OG (and therefore GH) is parallel to AB. (The notation OGH is traditionally reserved for the Euler line of a triangle, so the choice of notation in this problem was not a particularly happy one.) □

For an arbitrary triangle ABC, the radius of the circle γ tangent to AB and AC, and tangent internally to the circumcircle, is given by either of the two equal expressions

$$r\sec^2\frac{A}{2} \qquad \text{and} \qquad \frac{bc}{s}\tan\frac{A}{2},$$

where r is the inradius and s the semiperimeter. For further details, see my article "A Mixtilinear Adventure" [pages 2-7 in this issue].

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; SAM BAETHGE, Southwest High School, San Antonio, Texas; E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; MILTON P. EISNER, Mount Vernon College, Washington, D.C.; J.T. GROENMAN, Arnhem, The Netherlands; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; CHARLES W. TRIGG, San Diego, California; JOHN A. WINTERINK, Albuquerque Technical Vocational Institute, Albuquerque, New Mexico; and the proposer.

REFERENCES

1. Leon Bankoff, (proposer of) Problem E 1141, *American Mathematical Monthly*, 61 (1954) 711.

2. Solutions to Problem E 1141, American Mathematical Monthly, 62 (1955) 444.

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706. [1982: 15] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $F(x) = 7x^{11} + 11x^7 + 10ax$, where x ranges over the set of all integers. Find the smallest positive integer a such that $77|F(x)$ for every x.

Solution by Kenneth S Williams, Carleton University, Ottawa.

Let $P = p_1p_2\ldots p_n$, where the factors p_i are n given distinct primes, and let $k_1, k_2, \ldots, k_{n+1}$ be $n+1$ given integers. Set

$$F_a(x) = k_1\frac{P}{p_1}x^{p_1} + k_2\frac{P}{p_2}x^{p_2} + \ldots + k_n\frac{P}{p_n}x^{p_n} + k_{n+1}ax,$$

where x ranges over the set of all integers and a is an integer (to be determined) such that

$$P|F_a(x) \quad \text{for all integers } x. \tag{1}$$

Since, for any integer x,

$$\frac{P}{p_j}x^{p_j} \equiv \begin{cases} 0, & \text{if } j \neq i, \\ \frac{P}{p_i}x, & \text{if } j = i, \end{cases} \pmod{p_i}$$

we have, for $i = 1,2,\ldots,n$,

$$F_a(x) \equiv \left(k_i\frac{P}{p_i} + k_{n+1}a\right)x \pmod{p_i}.$$

Hence (1) holds if and only if a is a solution of the n simultaneous congruences

$$k_i\frac{P}{p_i} + k_{n+1}a \equiv 0 \pmod{p_i}, \qquad i = 1,2,\ldots,n. \tag{2}$$

The solutions a of (2) can now be found routinely by the Chinese Remainder Theorem.

For the present problem, we have $n = 2$, $p_1 = 7$, $p_2 = 11$, $k_1 = k_2 = 1$, and $k_3 = 10$, so the congruences (2) are

$$11 + 10a \equiv 0 \pmod 7 \qquad \text{and} \qquad 7 + 10a \equiv 0 \pmod{11},$$

with solutions $a \equiv 29 \pmod{77}$. The smallest positive solution is $a = 29$, and for this value of a the function in the proposal can be written

$$F(x) = 7(x^{11} - x) + 11(x^7 - x) + 4\cdot 7\cdot 11x.$$

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; SAM BAETHGE, Southwest High School, San Antonio, Texas; E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; CLAYTON W. DODGE, University of Maine at Orono; MILTON P. EISNER, Mount Vernon College, Washington, D.C.; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; W.C. IGIPS, Danbury, Connecticut; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; VIKTORS LINIS, University of Ottawa; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Mathematicians best earn their keep when they generalize. So it was with real surprise that the editor found that the other fifteen solvers all proved (and, for the most part, proved very well) only the very special case of the proposal. They are worthy disciples of William Blake [1982: 60].

* * *

MATHEMATICAL CLERIHEWS

Leonardo Fibonacci
Left a sequence, not botchy,
As you readily may spy
In his *Liber Abaci*.

Henri Poincaré
"N'était point carré!"—
Indeed, well rounded,
In theorems abounded.

ALAN WAYNE
Holiday, Florida