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CONTENTS

THE OLYMPIAD CORNER: 72

M.S. KLAMKIN

All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada, T6G 2G1.

I start out with the nine problems of the 1985 Austria-Poland Mathematical Competition held in Hollabrunn, Austria. I am grateful to Walther Janous for their translation and transmission. As usual, I solicit from readers elegant solutions to these problems.

1st day, June 25, 1985 -- time 4 1/2 hours.

1. If a, b, c are distinct real numbers whose sum is zero, prove that $[(b-c)/a + (c-a)/b + (a-b)/c][a/(b-c) + (b/(c-a) + c/(a-b)] = 9.$

2. A given graph has $n \geq 8$ vertices. Is it possible for the vertices to have the respective valences 4, 5, 6, ..., n-4, n-3, n-2, n-2, n-2, n-1, n-1, n-1?

3. Prove that for any convex quadrilateral of unit area, the sum of the lengths of its sides and diagonals is $\geq 4 + \sqrt{8}$.

2nd day, June 26, 1985 -- time 4 1/2 hours.

4. Determine all real solutions x, y of the system

$$x^4 + y^2 - xy^3 - 9x/8 = 0,$$
$$y^4 + x^2 - yx^3 - 9y/8 = 0.$$

5. We are given a certain number of identical sets of four integral weights in grams. It is assumed that using all of these weights, one can weigh all integral weights from 1 to 1985 grams inclusively. In how many ways can such identical sets of weights be chosen if their total weight is to be a minimum? *Editorial note*: *There will be two answers depending on whether or not one allows the (weighing) weights to be used on both sides of the balance.*

6. If P is an interior point of a tetrahedron $ABCD$, show that the volume of the tetrahedron whose vertices are the centroids of the four tetrahedra $PABC$, $PBCD$, $PCDA$, $PDAB$ equals (vol. $ABCD$)/64.

3rd day, June 27, 1985 -- time 4 hours.

Team Competition

7. Determine an upper bound for

$$(xy + 2yz + zw)/(x^2 + y^2 + z^2 + w^2)$$

which is valid for all real quadruples $(x,y,z,w) \neq (0,0,0,0)$. (The smaller your upper bound is, the more points you are awarded.)

8. The consecutive vertices of a given convex n-gon are $A_0, A_1, \ldots, A_{n-1}$. The n-gon is partitioned into $n - 2$ triangles by diagonals which are non-intersecting (except possibly at the vertices). Show that there exists an enumeration $\Delta_1, \Delta_2, \ldots, \Delta_{n-2}$ of these triangles such that A_i is a vertex of Δ_i for $1 \leq i \leq n - 2$. How many enumerations of this kind exist?

9. $P_1, P_2, \ldots, P_n$ are consecutive vertices of a given convex n-gon.

Show that there exists an interior point Q of the n-gon and three vertices P_i, P_j, P_k such that the angles QP_iP_{i+1}, QP_jP_{j+1} and QP_kP_{k+1} are acute $(P_{n+1} = P_1)$. *Editorial note: I have edited this problem to remove an ambiguity in the translation I received. It is quite possible that the original version is a different problem. A related problem is to determine the maximum number of angles* QP_iP_{i+1}, $i = 1,2,\ldots,n$ *which can be acute for an interior point* Q.

*

Now through the courtesy of P.J. O'Halloran, I give the official solutions of the 1983 Australian Olympiad [1983: 173].

March 15, 1983 -- time 4 hours.

1. Each positive rational number occupies an infinite number of positions in the following pattern:

$$\frac{1}{1}; \frac{2}{1}, \frac{1}{2}; \frac{3}{1}, \frac{2}{2}, \frac{1}{3}; \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}; \frac{5}{1}, \frac{4}{2}, \frac{3}{3}, \frac{2}{4}, \frac{1}{5}; \ldots .$$

For instance the number $\frac{2}{3}$ occupies positions 9, 42, ...

(a) Find the first five positions occupied by the number $\frac{1}{2}$.

(b) Find an expression for the nth occurrence of the number $\frac{1}{2}$.

(c) Find an expression for the first occurrence of the number $\frac{p}{q}$, where p and q are relatively prime and $p < q$.

Solution.

$$\frac{1}{1};\ \frac{2}{1},\ \boxed{\frac{1}{2}};\ \frac{3}{1},\ \frac{2}{2},\ \frac{1}{3};\ \frac{4}{1},\ \frac{3}{2},\ \frac{2}{3},\ \frac{1}{4};\ \frac{5}{1},\ \frac{4}{2},\ \frac{3}{3},\ \boxed{\frac{2}{4}},\ \frac{1}{5};\ \frac{6}{1},$$

$$\frac{5}{2},\ \frac{4}{3},\ \frac{3}{4},\ \frac{2}{5},\ \frac{1}{6};\ \frac{7}{1},\ \frac{6}{2},\ \frac{5}{3},\ \frac{4}{4},\ \frac{3}{5},\ \frac{2}{6},\ \frac{1}{7};\ \frac{8}{1},\ \frac{7}{2},\ \frac{6}{3},\ \frac{5}{4},\ \frac{4}{5},\ \boxed{\frac{3}{6}},$$

$$\frac{2}{7},\ \frac{1}{8};\ \frac{9}{1},\ \frac{8}{2},\ \frac{7}{3},\ \frac{6}{4},\ \frac{5}{5},\ \frac{4}{6},\ \frac{3}{7},\ \frac{2}{8},\ \frac{1}{9};\ \frac{10}{1},\ \frac{9}{2},\ \frac{8}{3},\ \frac{7}{4},\ \frac{6}{5},\ \frac{5}{6},\ \frac{4}{7},\ \frac{3}{8},$$

$$\frac{2}{9},\ \frac{1}{10};\ \frac{11}{1},\ \frac{10}{2},\ \frac{9}{3},\ \frac{8}{4},\ \frac{7}{5},\ \frac{6}{6},\ \frac{5}{7},\ \boxed{\frac{4}{8}},\ \frac{3}{9},\ \frac{2}{10},\ \frac{1}{11};\ \frac{12}{1},\ \frac{11}{2},\ \frac{10}{3},$$

$$\frac{9}{4},\ \frac{8}{5},\ \frac{7}{6},\ \frac{6}{7},\ \frac{5}{8},\ \frac{4}{9},\ \frac{3}{10},\ \frac{2}{11},\ \frac{1}{12};\ \frac{13}{1},\ \ldots;\ \frac{14}{1},\ \frac{13}{2},\ \frac{12}{3},\ \frac{11}{4},\ \frac{10}{5},\ \frac{9}{6},$$

$$\frac{8}{7},\ \frac{7}{8},\ \frac{6}{9},\ \boxed{\frac{5}{10}},\ \frac{4}{11},\ \frac{3}{12},\ \frac{2}{13},\ \frac{1}{14}.$$

(a) The first 5 occurrences of 1/2 are at positions

3, 14, 34, 63, 101.

(b) Using finite differences

1	3		
		11	
2	14		9
		20	
3	34		9
		29	
4	63		9
		38	
5	101		

suggests the quadratic function

$$\begin{aligned} f(n) &= 3 + 11(n - 1) + \frac{9}{2}(n - 1)(n - 2) \\ &= \frac{1}{2}(9n^2 - 5n + 2). \end{aligned}$$

The nth occurrence of $\frac{1}{2}$ is $\frac{n}{2n}$. This occurs in the section

$$\frac{3n-1}{1},\ \frac{3n-2}{2}\ \ldots,\ \frac{n}{2n},\ \ldots .$$

There are $3n - 2$ sections preceding this one containing 1, 2, ..., $3n - 2$ terms respectively, so the number of terms preceding this section is $\frac{1}{2}(3n - 2)(3n - 1)$. Therefore $\frac{n}{2n}$ first occurs in position

$$\frac{1}{2}(3n - 2)(3n - 1) + 2n = \frac{1}{2}(9n \quad - 5n + 2).$$

(c) Similarly, the first occurrence of p/q $(p < q)$ occurs in the section

$$\frac{p+q-1}{1},\ \ldots,\ \frac{p}{q},\ \ldots$$

There are $p + q - 2$ sections preceding this one so the total number of terms preceding this section is

$$\frac{1}{2}(p + q - 2)(p + q - 1).$$

Thus the position in which p/q $(p < q)$ first occurs is

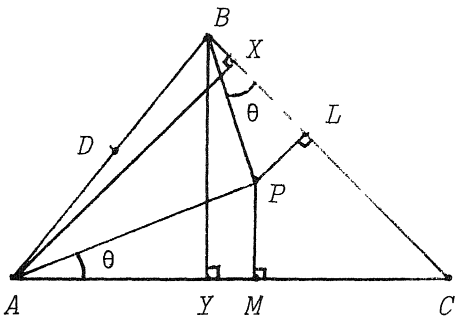
$$\tfrac{1}{2}(p + q - 2)(p + q - 1) + q.$$

Subsequent occurrences of p/q are of the form $\frac{kp}{kq}$ $(k > 1)$ which is in position kq in the string beginning

$$\frac{k(p + q) - 1}{1}, \ldots .$$

As $k > 1$, this string occurs after the stated first occurrence.

2. ABC is a triangle and P is a point inside it. Angle PAC=angle PBC. The perpendiculars from P to BC, CA meet these sides at L, M respectively, and D is the midpoint of AB. Prove that $DL = DM$.



Solution.

Draw altitudes AX, BY. D is the centre of circle $AYXB$, with radius $r = AB/2$. So

$$LD^2 - r^2 = LX \cdot LB$$
$$MD^2 - r^2 = MY \cdot MA$$

(see for example Corollary page 29 of *Advanced Euclidean Geometry* by R.A. Johnson) so we have to prove $LX \cdot LB = MY \cdot MA$. Now $\angle YBX = \angle YAX$, so subtracting the given equal angles we obtain $\angle YBP = \angle PAX = \theta$ (say) and thus

$$\frac{MY}{BP} = \frac{LX}{AP} = \sin\theta.$$

Since ΔBPL and ΔAPM are similar,

$$\frac{MY}{LX} = \frac{BP}{AP} = \frac{BL}{AM}.$$

Therefore

$$LX \cdot LB = MY \cdot MA.$$

3. A box contains p white balls and q black balls. Beside the box there is a pile of black balls. Two balls are taken out from the box. If they are of the same colour, a black ball from the pile is put into the box. If they are of different colours, the white ball is put back into the box. This procedure is repeated until the last pair of balls are removed from the box and one last ball is put in. What is the probability that this last ball is white?

Solution.

If the two balls taken out are both white, then the number of white balls decreases by two, in the other cases it remains unchanged. Hence the parity of the number of white balls does not change during the procedure. Therefore if p is even, the last ball cannot be white - the probability is 0. If p is odd, the last ball has to be white - the probability is 1.

Editorial note: See also #30 *in Olympiad Corner* 63 [1985: 71].

4. Find all pairs of natural numbers (n,k) for which

$$(n + 1)^k - 1 = n!$$

Solution.

For $n = 1$, the equality $2^k - 1 = 1$ must be satisfied, whence $k = 1$, so one pair is (1,1).

For $n = 2$, the equality $3^k - 1 = 2$ must be satisfied, whence $k = 1$, and (2,1) is a required pair.

Now if $n \geq 2$, then $n!$ is even; therefore the pair (n,k) for $n \geq 2$ satisfies the given equality only if n is even.

For $n = 4$, the equality $5^k - 1 = 24$ must be satisfied, so another pair is (4,2).

For $n > 4$, an even number, take $n = 2m$ and a pair (n,k) satisfying the given equality. Then $n! = (2m)! = 2m(2m - 1)!$, and $(2m - 1)!$ contains both factors 2 and m where $m \neq 2$. Therefore n^2 divides $n!$ and must also divide $(n + 1)^k - 1$. But $(n + 1)^k = n^k + \ldots + \binom{k}{2}n^2 + kn + 1$, hence n must divide k. That means $k \geq n$ is true, which leads to $(n + 1)^k \geq (n + 1)^n > n! + 1$. This contradicts the assumption that the pair (n,k) satisfies the given equality. So there is no solution for $n > 4$, and the only solutions are $(n,k) = (1,1)$, $(2,1)$, and $(4,2)$.

5. (a) Find the rearrangement $\{a_1,a_2,\ldots,a_n\}$ of $\{1,2,\ldots,n\}$ which maximises

$$a_1a_2 + a_2a_3 + \ldots + a_na_1 = Q.$$

(b) Find the rearrangement that minimises Q.

Solution.

(a) Suppose we have a rearrangement $\{a_1,a_2,\ldots,a_n\}$. Then if $1 \leq i < i + 2 \leq j < n$ and $(a_i - a_{j+1})(a_j - a_{i+1}) > 0$, we can increase Q with the rearrangement

$$\{a_1,\ldots,a_i,a_j,a_{j-1},\ldots,a_{i+2},a_{i+1},a_{j+1},a_{j+2},\ldots,a_n\}.$$

Indeed, by this alteration we increase Q by the quantity

$$a_ia_j + a_{i+1}a_{j+1} - a_ia_{i+1} - a_ja_{j+1} = (a_i - a_{j+1})(a_j - a_{i+1}) > 0.$$

Let A be a rearrangement that maximises Q (since the number of rearrangements is finite, A exists). Using a cyclic permutation (which does not alter the value of Q) we can suppose that A is of the form $\{1 = a_1, a_2, \ldots, a_n\}$. Let $2 = a_i$. Then, since $(1 - a_{i-1})(2 - a_2) \geq 0$, we must have $i = 2$ or $i = n$ because otherwise we could increase Q. We can suppose $i = n$ for otherwise we use a cyclic permutation and a reflexion (which does not alter Q either). If $3 = a_j$, then $j = 2$ since otherwise, from $(1 - a_{j+1})(3 - a_2) > 0$, we could improve A again. Applying a reflexion, we get the rearrangement $\{2, b_2, \ldots, 3, 1\}$.

In the same way as above we can prove that $b_2 = 4$. Continuing this argument we obtain that

$$A = \{1,3,5,\ldots,2\left[\frac{n-1}{2}\right] + 1,\ 2\left[\frac{n}{2}\right],\ldots,4,2\}$$

(or a cyclic permutation and/or reflexion of this).

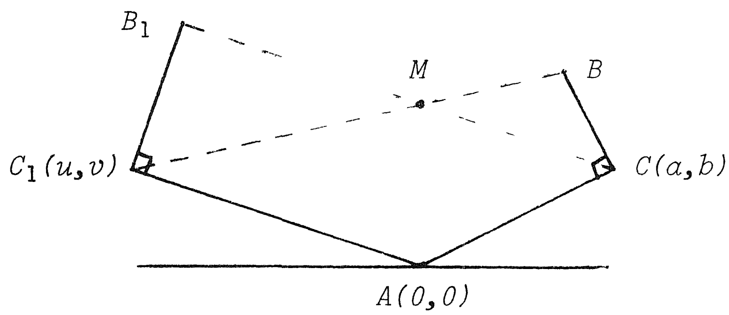
(b) A similar argument leads to the minimising rearrangement

$$\{1,n,2,n-1,\ldots,\left[\frac{n}{2}\right] + 1\}.$$

Editorial note: See also #34 in Olympiad Corner 63 [1985: 71], *and Problem* 1059 (*Cyclic extrema*) *by D.E. Daykin, Math. Mag.* 53 (1980) 115.

6. The right triangles ABC and AB_1C_1 are similar and have opposite orientation. The right angles are at C and C_1, and $\angle CAB = \angle C_1AB_1$. M is the point of intersection of the lines BC_1 and CB_1. Prove that if the lines AM and CC_1 exist, then they are perpendicular.

Solution.



Take A to be the origin for a system of Cartesian co-ordinates. Let the co-ordinates of C be (a,b) and the co-ordinates of C_1 be (u,v).

If the ratio $AC:CB = 1:k$, then the co-ordinates of B are

$$k(-b,a) + (a,b) = (-kb + a, ka + b),$$

and those of B_1 are

$$k(v,-u) + (u,v) = (kv + u, -ku + v).$$

The equations of the lines BC_1 and B_1C are

$$((ka + b) - v)x - ((-kb + a) - u)y = (ka + b)u - (-kb + a)v$$

and

$$(b - (-ku + v))x - (a - (kv + u))y = b(kv + u) - a(-ku + v).$$

Rearranging these we have

$$kax + kby = kau + kbv + bu - av - (b - v)x + (a - u)y$$

$$kux + kvy = kau + kbv + bu - av - (b - v)x + (a - u)y.$$

The co-ordinates (s,t) of the point M satisfy both these equations, so

$$kas + kbt = kus + kvt,$$

that is,

$$\frac{s}{t} = -\frac{b - v}{a - u}$$

which means the lines AM and CC_1 are perpendicular.

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THE EUGENE STRENS MEMORIAL CONFERENCE ON INTUITIVE
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PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before September 1, 1986, although solutions received after that date will also be considered until the time when a solution is published.

1111. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let α, β, γ be the angles of an acute triangle and let

$$f(\alpha,\beta,\gamma) = \cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \cos\frac{\beta}{2}\cos\frac{\gamma}{2} + \cos\frac{\gamma}{2}\cos\frac{\alpha}{2}.$$

(a) Prove that $f(\alpha,\beta,\gamma) > \frac{3}{2}\sqrt[3]{2}$.

(b)* Prove or disprove that $f(\alpha,\beta,\gamma) > \frac{1}{2} + \sqrt{2}$.

1112. *Proposed by Allan W. Johnson Jr., Washington, D.C.*

Solve the synonymical base 10 addition

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1113.* *Proposed by Jack Garfunkel, Flushing, N.Y.*

Consider two concentric circles with radii r and $2r$, and a triangle ABC inscribed in the inner circle. Points A', B', C' on the outer circle are determined by extending AB to B', BC to C', and CA to A'. Prove that the perimeter of triangle $A'B'C'$ is at least twice the perimeter of ABC. Equality is attained when ABC is equilateral.

1114. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC, $A'B'C'$ be two triangles with sides a,b,c, a',b',c' and areas F, F' respectively. Show that

$$aa' + bb' + cc' \ge 4\sqrt{3}\sqrt{FF'}.$$

1115. *Proposed by Helen Sturtevant and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Determine all positive integers n such that an equilateral triangle can be dissected into exactly n equilateral triangles.

1116. *Proposed by David Grabiner, Claremont High School, Claremont, California.*

(a) Let $f(n)$ be the smallest positive integer which is not a factor of n. Continue the series $f(n)$, $f(f(n))$, $f(f(f(n)))$, ... until you reach 2. What is the maximum length of the series?

(b) Let $g(n)$ be the second smallest positive integer which is not a factor of n. Continue the series $g(n)$, $g(g(n))$, $g(g(g(n)))$, ... until you reach 2. What is the maximum length of the series?

1117. *Proposed by Jordi Dou, Barcelona, Spain.*

Let $ABCD$ be an isosceles trapezoid with bases $AB > DC$, and let M and N be points on AD and BC respectively so that MN is parallel to AB and DC. Let D' be the projection of D on AB, let $E = DD' \cap BM$, $F = BD \cap AE$, and $P = NF \cap DC$. Prove that PA is perpendicular to AB.

1118. *Proposed by P. Erdős, Hungarian Academy of Sciences.*

Let $a_1, a_2, a_3, \ldots$ be a sequence of numbers such that $\lim_{i\to\infty} (a_{i+1} - a_i) = \infty$. Construct an infinite sequence $b_1 < b_2 < \ldots$ so that none of the sums $\sum_{i=1}^{\infty} \epsilon_i b_i$ (where $\epsilon_i = 0$ or 1 for each i and all but finitely many are 0) equals any of the a_i's. See also problem #85 [1976: 29].

1119. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

The following problem, for which I have been unable to locate the source, has been circulating around DEC. A rectangle is partitioned into smaller rectangles. If each of the smaller rectangles has the property that one of its sides has integral length, prove that the original rectangle also has this property.

1120*. *Proposed by D.S. Mitrinovic, University of Belgrade, Belgrade, Yugoslavia.*

(a) Determine a positive number λ so that

$$(a + b + c)^2(abc) \ge \lambda(bc + ca + ab)(b + c - a)(c + a - b)(a + b - c)$$

holds for all real numbers a, b, c.

(b) As above, but a, b, c are assumed to be positive.

(c) As above, but a, b, c are assumed to satisfy

$$b + c - a > 0\ ,\ c + a - b > 0\ ,\ a + b - c > 0.$$

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

982. [1984: 291] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let P and Q be interior points of triangle $A_1A_2A_3$. For $i = 1,2,3$, let $PA_i = x_i$, $QA_i = y_i$, and let the distances from P and Q to the side opposite A_i be p_i and q_i, respectively. Prove that

$$\sqrt{x_1y_1} + \sqrt{x_2y_2} + \sqrt{x_3y_3} \ge 2(\sqrt{p_1q_1} + \sqrt{p_2q_2} + \sqrt{p_3q_3}).$$

When $P = Q$, this reduces to the well-known Erdős-Mordell inequality. (See the article by Clayton W. Dodge in this journal [1984: 274-281].)

Solution by M.S. Klamkin, University of Alberta.

We prove a generalization of the Erdős-Mordell inequality which includes the inequality proposed here as a special case.

Let P_j $(j = 1,2,\ldots,n)$ denote any set of n points lying in the interior or on the boundary of a given triangle $A_1A_2A_3$ with respective sides a_1, a_2, a_3. Let R_{ij} and r_{ij} $(i = 1,2,3)$ denote the distances from P_j to the vertices A_i and to the sides a_i, respectively. It is known that

$$R_{1j} \ge r_{2j}(a_3/a_1) + r_{3j}(a_2/a_1),$$
$$R_{2j} \ge r_{3j}(a_1/a_2) + r_{1j}(a_3/a_2),$$
$$R_{3j} \ge r_{1j}(a_2/a_3) + r_{2j}(a_1/a_3),$$

with equality in all three inequalities if and only if P_j is the circumcenter of $A_1A_2A_3$. (Incidentally, these inequalities can be used to provide an elegant proof of the Erdős-Mordell inequality. Just add the three inequalities and note that $(a_2/a_3) + (a_3/a_2) \ge 2$, etc. See O. Bottema et al, *Geometric Inequalities*, 12.13 and 12.16.)

We now employ the following case of Hölder's inequality:

$$(u_1 + v_1)^{1/n}(u_2 + v_2)^{1/n} \cdots (u_n + v_n)^{1/n} \ge (u_1u_2\ldots u_n)^{1/n} + (v_1v_2\ldots v_n)^{1/n},$$

where all u_i, $v_i \ge 0$. With all products running from $j = 1$ to $j = n$, we then have

$$\Pi R_{1j}^{1/n} \ge \Pi(r_{2j}(a_3/a_1) + r_{3j}(a_2/a_1))^{1/n} \ge (a_3/a_1)\Pi r_{2j}^{1/n} + (a_2/a_1)\Pi r_{3j}^{1/n},$$

$$\Pi R_{2j}^{1/n} \ge \Pi(r_{3j}(a_1/a_2) + r_{1j}(a_3/a_2))^{1/n} \ge (a_1/a_2)\Pi r_{3j}^{1/n} + (a_3/a_2)\Pi r_{1j}^{1/n},$$

$$\Pi R_{3j}^{1/n} \ge \Pi(r_{1j}(a_2/a_3) + r_{2j}(a_1/a_3))^{1/n} \ge (a_2/a_3)\Pi r_{1j}^{1/n} + (a_1/a_3)\Pi r_{2j}^{1/n}.$$

Since $(a_2/a_3) + (a_3/a_2) \ge 2$ with equality if and only if $a_2 = a_3$, etc., it follows by addition that

$$\sum_{i=1}^{3} \prod_{j=1}^{n} R_{ij}^{1/n} \ge 2 \sum_{i=1}^{3} \prod_{j=1}^{n} r_{ij}^{1/n}, \tag{1}$$

with equality if and only if the triangle is equilateral and all the points P_j coincide with its center.

The proposer's inequality corresponds to the special case $n = 2$. □

By using the following more general version of Hölder's inequality,

$$(u_1 + v_1)^{1/n_1}(u_2 + v_2)^{1/n_2}\ldots(u_n + v_n)^{1/n_n} \ge \prod_{j=1}^{n} u_j^{1/n_j} + \prod_{j=1}^{n} v_j^{1/n_j},$$

where $n_k > 1$ for $k = 1,2,\ldots,n$ and $(1/n_1) + (1/n_2) + \ldots + (1/n_n) = 1$, we obtain the following generalization of (1):

$$\sum_{i=1}^{3} \prod_{j=1}^{n} R_{ij}^{1/n_j} \ge 2 \sum_{i=1}^{3} \prod_{j=1}^{n} r_{ij}^{1/n_j}. \tag{2}$$

By specializing the points P_j, (2) leads to a host of triangle inequalities. For example, take $n = 2$ and let P_1 and P_2 be the circumcenter and incenter, respectively, of an acute triangle $A_1A_2A_3$. R and r being the circumradius and inradius, respectively, we then have, for $i = 1,2,3$,

$$R_{i1} = R\ , \quad R_{i2} = r \csc\,(A_i/2)\ , \quad r_{i1} = R\cos A_i\ , \quad r_{i2} = r.$$

This leads to the interesting triangle inequality

$$\sum_{i=1}^{3} \csc^{1/n_2}(A_i/2) \ge 2 \sum_{i=1}^{3} \cos^{1/n_1} A_i, \tag{3}$$

where n_1, $n_2 > 1$ and $(1/n_1) + (1/n_2) = 1$. Actually, one can show that the limiting cases $n_1 \longrightarrow \infty$ or $n_2 \longrightarrow \infty$ are valid. These lead respectively to the known inequalities

$$\sum_{i=1}^{3} \csc (A_i/2) \geq 6 \quad \text{and} \quad 3 \geq 2 \sum_{i=1}^{3} \cos A_i .$$

Using the last two inequalities, plus

$$8 \sin (A_1/2) \sin (A_2/2) \sin (A_3/2) \leq 1,$$

and then applying the power mean inequality, we obtain the following chain of inequalities (where all sums and products run from $i = 1$ to 3):

$$\frac{\Sigma \csc (A_i/2)}{3} \geq \left\{\frac{\Sigma \csc^{1/n_2}(A_i/2)}{3}\right\}^{n_2} \geq \Pi \csc^{1/3}(A_i/2) \geq 2$$

$$\geq 4 \cdot \frac{\Sigma \cos A_i}{3} \geq 4 \left\{\frac{\Sigma \cos^{1/n_1} A_i}{3}\right\}^{n_1} \geq 4 \cdot \Pi \cos^{1/3} A_i . \qquad (4)$$

Here n_1, $n_2 \geq 1$ but need not be otherwise related. The terms with the cube roots correspond to the limiting cases of the power mean inequality for which $n_2 \longrightarrow \infty$ and $n_1 \longrightarrow \infty$.

We now compare

$$\Sigma \csc^{1/n_2}(A_i/2) \geq 3 \cdot 2^{2/n_2} \left\{\frac{\Sigma \cos^{1/n_1} A_i}{3}\right\}^{n_1/n_2} \qquad (5)$$

from (4), where now $(1/n_1) + (1/n_2) = 1$, with inequality (3). We show that (3) is stronger than (5) for $n_1 > 2$, that is,

$$2 \Sigma \cos^{1/n_1} A_i \geq 3 \cdot 2^{2/n_2} \left\{\frac{\Sigma \cos^{1/n_1} A_i}{3}\right\}^{n_1/n_2} . \qquad (6)$$

When $1/n_2$ is replaced by $1 - 1/n_1$, (6) reduces to

$$3 \geq 2^{1/n_1} \Sigma \cos^{1/n_1} A_i ,$$

which follows from (4) for $n_1 > 2$. For $1 \leq n_1 < 2$, (3) is weaker than (5).

Finally, it should be noted that inequality (2) holds more generally as an n triangle inequality. Consider n triangles $A_{1j}A_{2j}A_{3j}$ of sides a_{1j}, a_{2j}, a_{3j} ($j = 1,\ldots,n$), and n points $P_1,\ldots,P_n$ where P_j is an interior or boundary point of triangle $A_{1j}A_{2j}A_{3j}$ for each j. Then if R_{ij} and r_{ij} denote the distances from P_j to the vertex A_{ij} and the side a_{ij}, respectively, inequality (2) is still valid. The same proof using Hölder's inequality goes through, the only change being that instead of coefficients of the form $a_1/a_2 + a_2/a_1$, we get

$$\prod_{j=1}^{n}\left[\frac{a_{1j}}{a_{2j}}\right]^{1/n_j} + \prod_{j=1}^{n}\left[\frac{a_{2j}}{a_{1j}}\right]^{1/n_j}$$

which are still ≥ 2 since the product of the two products is 1. There is equality in the generalized inequality if and only if each of the n triangles is equilateral and each P_j coincides with the center of its triangle.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

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983*. [1984: 291] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Let $A_0A_1 \ldots A_n$ be an n-simplex in $\mathbb{R}^n$.

(a) If m_k is the median through A_k, prove that

$$n^2m_k^2 = nS_k - T_k ,$$

where S_k is the sum of the squares of all the edges meeting in A_k , and T_k is the sum of the squares of all the edges not passing through A_k .

(b) Deduce from (a), or otherwise, that if the medians of the simplex are all equal, then the sum of the squares of all the edges meeting in a vertex is the same for all vertices. Is the converse also true?

Solution by G.P. Henderson, Campbellcroft, Ontario.

(a) Let A_0 be the origin and let the coordinates of A_k be $(x_{1k},\ldots,x_{nk})$, $k = 1,2,\ldots,n$. Then m_0 is the distance from the origin to the point whose i^{th} coordinate is $\sum_k x_{ik}/n$. Hence

$$n^2m_0^2 = \sum_i(\sum_k x_{ik})^2 = \sum_i\sum_j\sum_k x_{ij}x_{ik}.$$

We have

$$T_0 = \frac{1}{2}\sum_j\sum_k\sum_i(x_{ij} - x_{ik})^2 = \frac{1}{2}\sum_i\sum_j\sum_k(x_{ij}^2 - 2x_{ij}x_{ik} + x_{ik}^2)$$

$$= n\sum_k\sum_i x_{ik}^2 - \sum_i\sum_j\sum_k x_{ij}x_{ik}.$$

Hence (a) is true for $k = 0$ and similarly for $k = 1,2,\ldots,n$.

(b) Let E be the sum of the squares of all the edges of the simplex. Then $S_k + T_k = E$ for all k, and

$$n^2m_k^2 = (n + 1)S_k - E.$$

Therefore m_k is independent of k if and only if S_k is.

Also solved by O. BOTTEMA, Delft, The Netherlands; ROGER CUCULIERE, Paris, France; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; BASIL RENNIE, James Cook University, Australia; and GEORGE TSINTSIFAS, Thessaloniki, Greece. The proposer noted that part (a) *in the case* $n = 3$ *is known (see Theorem* 187 *in N. Altshiller-Court, Modern Pure Solid Geometry, Chelsea,* 1964). *Theorem* 295 *of the same book gives a generalization of part* (b) *in the case* $n = 3$.

* * *

<u>986</u>. [1984: 292] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Let

$$x = \sqrt[3]{p + \sqrt{r}} + \sqrt[3]{q - \sqrt{r}}\ ,$$

where p, q, r are integers and $r \geq 0$ is not a perfect square. If x is rational, prove that $p = q$ and x is integral.

Comment and solution by Leroy F. Meyers, Ohio State University, Columbus, Ohio.

If $x = 0$ then p and q need not be equal. Indeed, $p = -q$.

Assume $x \neq 0$, and let $a = \sqrt[3]{p + \sqrt{r}}$ and $b = \sqrt[3]{q - \sqrt{r}}$, so that $x = a + b$. If x is rational, then so is

$$x^3 = a^3 + b^3 + 3ab(a + b) = p + q + 3abx.$$

Hence ab is rational, and so is

$$a^3b^3 = pq - r + (q - p)\sqrt{r}.$$

But since $\sqrt{r}$ is irrational, it must be that $q = p$, in which case a^3b^3 is the integer $p^2 - r$, and x is a root of the polynomial equation

$$(x^3 - 2p)^3 = 27a^3b^3x^3$$

with integer coefficients and leading coefficient 1. Every rational root of such an equation must be an integer.

Also solved (for $x \neq 0$) *by ANDREW CUSUMANO, Great Neck, N.Y.; KARL DILCHER, Dalhousie University, Halifax, Nova Scotia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARK KANTROWITZ, Maimonides School, Brookline, Massachusetts; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; ROBERT C. LYNESS, Suffolk, England; J.A. McCALLUM, Medicine Hat, Alberta; KENNETH M. WILKE, Topeka, Kansas; and the proposer.*

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<u>987</u>*. [1984: 292] *Proposed by Jack Garfunkel, Flushing, N.Y.*

If triangle ABC is acute-angled, prove or disprove that

(a) $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \geq \frac{4}{3}\{1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\}$

(b) $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \geq \frac{4}{\sqrt{3}}\{1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\}$.

Solution to part (a) by Basil Rennie, James Cook University, Australia.

Let $\beta = \sin^{-1}\left[\frac{\sqrt{3}-1}{2}\right]$. We prove a result slightly stronger than the one suggested, namely: if the largest angle of triangle ABC is $\leq \pi - 4\beta$ (approximately 94.12°), then inequality (a) holds.

Lemma 1. If $0 \leq v \leq u \leq \frac{\pi}{6}$, then $4\cos v\cos 2u - 3\sin u > 0$.

Proof. Let $g(u,v) = 4\cos v\cos 2u - 3\sin u$. Then for $0 \leq u \leq \frac{\pi}{6}$ and for fixed v, $g(u,v)$ decreases as u increases. Thus

$$g(u,v) \geq g(\tfrac{\pi}{6},v) = 2\cos v - \frac{3}{2} \geq 2\cos\frac{\pi}{6} - \frac{3}{2} = \sqrt{3} - \frac{3}{2} > 0. \qquad \square$$

Lemma 2. If $\beta \leq u \leq \frac{\pi}{6}$ and $s = \sin u$ then $2s^2 + 2s - 1 \geq 0$.

Proof. The positive root of the quadratic $2x^2 + 2x - 1 = 0$ is at $x = \frac{\sqrt{3}-1}{2} = \sin\beta \leq \sin u = s.$ $\square$

Now assume $0 \leq A \leq B \leq C \leq \pi - 4\beta$. Then we can write $A = 2u - 2v$, $B = 2u + 2v$, and $C = \pi - 4u$, where $0 \leq v \leq u$ and $u \geq \beta$. Furthermore, from $C \geq \frac{\pi}{3}$ we see that $u \leq \frac{\pi}{6}$. Using elementary trigonometry,

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 2\sin u\cos v + \cos 2u$$

and

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{1}{2}(\cos 2v - \cos 2u)\cos 2u.$$

Let

$$\begin{aligned} F(u,v) &= 3(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}) - 4 - 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\ &= 6\sin u\cos v + 3\cos 2u - 4 - 2(\cos 2v - \cos 2u)\cos 2u. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial F}{\partial v} &= -6\sin u\sin v + 4\sin 2v\cos 2u \\ &= 2\sin v(4\cos v\cos 2u - 3\sin u) \\ &\geq 0 \quad \text{by Lemma 1.} \end{aligned}$$

Thus

$$F(u,v) \geq F(u,0) = 2\cos^2 2u + \cos 2u + 6\sin u - 4.$$

Now put $s = \sin u$; then $\cos 2u = 1 - 2s^2$, so

$$\begin{aligned} F(u,0) &= 2(1-2s^2)^2 + 1 - 2s^2 + 6s - 4 \\ &= 8s^4 - 10s^2 + 6s - 1 \\ &= (2s-1)^2(2s^2+2s-1) \\ &\geq 0 \quad \text{by Lemma 2.} \end{aligned}$$

Hence $F(u,v) \geq 0$, and (a) follows.

Solution to part (b) by G.P. Henderson, Campbellcroft, Ontario; proof adapted by the editor from Rennie's proof of part (a).

We prove that (b) holds if all angles of the triangle are $\leq \pi - 4\cos^{-1}(\sqrt{3}t)$, where t is the unique real root of the polynomial

$$12x^3 + 12x^2 - 3x - 4.$$

Let $0 \leq A \leq B \leq C$, and write $A = 2u - 2v$, $B = 2u + 2v$, $C = \pi - 4u$, so that $0 \leq v \leq u \leq \frac{\pi}{6}$ as before. Then

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 2\cos u\cos v + \sin 2u$$

and again

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{1}{2}(\cos 2v - \cos 2u)\cos 2u.$$

Let

$$\begin{aligned} F(u,v) &= \sqrt{3}\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right) - 4 - 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\ &= 2\sqrt{3}\cos u\cos v + \sqrt{3}\sin 2u - 4 - 2(\cos 2v - \cos 2u)\cos 2u. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial F}{\partial v} &= -2\sqrt{3}\cos u\sin v + 4\sin 2v\cos 2u \\ &= 2\sin v(4\cos v\cos 2u - \sqrt{3}\cos u). \end{aligned}$$

Letting $g(u,v) = 4\cos v\cos 2u - \sqrt{3}\cos u$, we see that g decreases as a function of v, and thus for fixed u,

$$g(u,v) \geq g(u,u) = \cos u(4\cos 2u - \sqrt{3}).$$

But

$$h(u) = 4\cos 2u - \sqrt{3}$$

is positive for $0 \leq u \leq \frac{\pi}{6}$, since h decreases and $h(\frac{\pi}{6}) = 2 - \sqrt{3} > 0$. Thus $g(u,v) \geq 0$ for $0 \leq v \leq u \leq \frac{\pi}{6}$, so $F(u,v)$ increases as a function of v. Hence

$$F(u,v) \geq F(u,0) = \cos u(2\sqrt{3} + 2\sqrt{3}\sin u - 12\cos u + 8\cos^3 u).$$

Putting $\sin u = \sqrt{1 - \cos^2 u}$ and rationalizing, we get that

$$2\sqrt{3} + 2\sqrt{3}\sin u - 12\cos u + 8\cos^3 u \geq 0$$

provided that

$$16\cos^5 u - 48\cos^3 u + 8\sqrt{3}\cos^2 u + 39\cos u - 12\sqrt{3} \leq 0.$$

Set $\cos u = \sqrt{3}t$; then this is equivalent to

$$0 \geq 48t^5 - 48t^3 + 8t^2 + 13t - 4 = (2t-1)^2(12t^3 + 12t^2 - 3t - 4)$$

which will hold provided t is less than or equal to the unique real root of the polynomial

$$12x^3 + 12x^2 - 3x - 4,$$

approximately $x = 0.551$. Thus if $\frac{\pi}{6} \geq u \geq \cos^{-1}(\sqrt{3}x) \approx 17.34^\circ$, $F(u,v)$ will be ≥ 0. This amounts to insisting that $C \leq 110.64^\circ$.

Henderson found the strong solutions for both (a) *and* (b) *using noncalculus techniques. He also pointed out that if* $C > 2\sin^{-1}\left[\frac{\sqrt{2}+4}{6}\right] \approx 128.94^\circ$, *then* (a) *will fail for all* A *and* B *(assuming* $A \leq B \leq C$*), and that if* $94.12^\circ < C < 128.94^\circ$ *then* (a) *will hold provided that* $B - A$ *is small enough. One partial solution was also received.*

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988. [1984: 292] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Prove that

$$\sum_{k=1}^{5} \sec\frac{2\pi k}{11} + \sum_{k=1}^{6} \sec\frac{2\pi k}{13} = 0.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We show more generally that

$$\sum_{k=1}^{n} \sec\frac{2\pi k}{4n-1} + \sum_{k=1}^{n+1} \sec\frac{2\pi k}{4n+1} = 0$$

for all $n > 0$. To do this we compute $S_n = \sum_{k=1}^{n} \sec\frac{2\pi k}{2n+1}$ (a known series).

From the identity

$$\cos(2n+1)x = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} \cos^{2n+1-2k}x \sin^{2k}x$$

(provable with DeMoivre's theorem) it follows that $\cos\frac{2\pi t}{2n+1}$, $t = 0,1,\ldots,2n$, are all the roots of the polynomial

$$1 = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} x^{2n+1-2k} (1-x^2)^k.$$

Thus $\sec\frac{2\pi t}{2n+1}$, $t = 0,1,\ldots,2n$, are all the roots of the polynomial

$$1 = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} \left(\frac{1}{x}\right)^{2n+1-2k} \left(1 - \frac{1}{x^2}\right)^k,$$

which simplifies to

$$x^{2n+1} = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} 2n+1 \\ 2k \end{bmatrix} (x^2 - 1)^k$$

or

$$x^{2n+1} - (-1)^n(2n + 1)(x^2 - 1)^n - \sum_{k=0}^{n-1} (-1)^k \begin{bmatrix} 2n+1 \\ 2k \end{bmatrix} (x^2 - 1)^k = 0.$$

The negative of the coefficient of x^{2n} in this polynomial will be the sum of its roots, and hence

$$\sum_{k=0}^{2n} \sec \frac{2\pi k}{2n+1} = (-1)^n(2n + 1).$$

Since $\sec \frac{2\pi k}{2n+1} = \sec \frac{2\pi(2n+1-k)}{2n+1}$, we get that

$$\sum_{k=0}^{2n} \sec \frac{2\pi k}{2n+1} = \sec 0 + 2 \sum_{k=0}^{n} \sec \frac{2\pi k}{2n+1} = 1 + 2S_n ,$$

so $S_n = \frac{1}{2}\left[(-1)^n(2n + 1) - 1\right] = \begin{cases} n & \text{if } n \text{ is even} \\ -n-1 & \text{if } n \text{ is odd} \end{cases}$. Hence $S_{2n-1} + S_{2n} = 0$, as claimed.

Also proved by CURTIS COOPER, Central Missouri State University, Warrensburg, Missouri; KARL DILCHER, Dalhousie University, Halifax, Nova Scotia; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; ROBERT LYNESS, Suffolk, England; LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; VEDULA N. MURTY, Pennsylvania State University, Middletown, Pennsylvania; and the proposer. Most solvers did the general case. Meyers comments that there is no such simple formula for $\sum_{k=1}^{m} \csc \frac{2\pi k}{2m+1}$ *, as may be seen from the sums* $\frac{2}{\sqrt{3}}$ *and* $\frac{2\sqrt{2 + \sqrt{5}}}{\sqrt[4]{5}}$ *for* $m = 1$ *and* $m = 2$*, respectively.*

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989. [1984: 292] *Proposed by Kurt Schiffler, Schorndorf, Federal Republic of Germany.*

Let H be the orthocenter of triangle ABC. Prove that the Euler lines of triangles ABC, BCH, CAH, and ABH are all concurrent. In what remarkable point of triangle ABC do they concur?

Solution by Basil Rennie, James Cook University, Australia.

For any triangle there is a "pedal triangle" formed by the feet of the perpendiculars from each vertex to the opposite side, and a "nine-point" circle through these three feet. The four triangles mentioned all have the same pedal triangle and so they share the same nine-point circle. Since the Euler line of any triangle contains the center of the nine-point circle, all four Euler lines meet in this point.

Also solved by JORDI DOU, Barcelona, Spain; ROLAND EDDY, Memorial University, St. John's, Newfoundland; H. FUKAGAWA, Yokosuka High School, Aichi, Japan; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; ROBERT C. LYNESS, Suffolk, England; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Brooklyn, N.Y.; and GEORGE TSINTSIFAS, Thessaloniki, Greece. Klamkin observed that the result is a simple consequence of the theorem "The four triangles of an orthocentric system have a common nine-point circle" (see R.A. Johnson, Advanced Euclidean Geometry, Dover, 1960, *p.*197). *Most proofs received were along these lines; the most succinct is reproduced above.*

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A MESSAGE FROM THE EDITOR

With this issue the editorship of *Crux Mathematicorum* changes. I would like to echo Ken Williams' words of last month in wishing Léo Sauvé a quick recovery and a long and happy retirement. And may we not have seen the last of him in these pages!

In preparing this first issue, I have been struck by the immense amount of knowledge and effort Léo has brought to *Crux*. Both are difficult to replace, but knowledge the more so. Undoubtedly there will occur, for a while at least, dubious problems, missed references, and other assorted booboos. I hope you will patiently, but firmly, let me know of my errors.

As it has these past eleven years, *Crux* will continue to increase in interest and value to you, its readers.

Bill Sands