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Loopy Catenaries

A.W.F. EDWARDS, *Gonville and Caius College, Cambridge*

The author is Reader in Mathematical Biology at Cambridge and a Fellow of Gonville and Caius College. His books include *Likelihood* (1972) and *Pascal's Arithmetical Triangle* (1986). In his spare time he flies gliders (he and wife are both holders of the international gold badge for gliding), and he has contributed extensively to the mathematical theory of cross-country soaring. He is an atrocious skier.

1. Introduction

Our family skiing party consisted of seven people and could boast computer science and mechanical engineering as well as mathematics amongst its skills. One evening the conversation turned to the question of what happened to the wires of the 'gondola' lifts after the gondola cars had been removed for the night. Why, some of us asked, did the slack in the wire as it passed over the pylons not all feed itself into one inter-pylon gap in a great loop, especially as the pylons were on a sloping hillside? Others could see no reason why we expected this to happen, and a lively debate ensued. All were soon agreed on a basis for argument: level equally-spaced pylons supporting a heavy flexible wire on frictionless bearings. But we could not agree on the stable configurations of such a system, for while some had visions of one big loop acquiring nearly all the slack, others imagined a sequence of identical catenary loops with the slack evenly distributed.

Finding myself unable to ski for a few days through injury I resorted to a little A-level mathematics recalled from more than thirty years ago, which led to the happy discovery that both opinions were correct, as the following account shows. Thus was family strife averted.

2. The catenary

First I had to rediscover the catenary. Consider the forces acting on a length l of catenary stretching from the origin O, where it is tangential to the

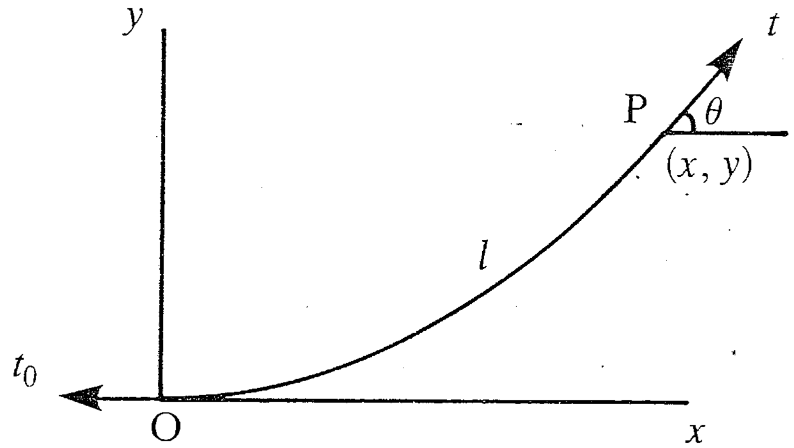

Figure 1. The catenary curve

x-axis, to the point P at (x, y) (figure 1). Let the inclination of the tangent to the curve at P be θ, the tension at P equal to t and the tension at O equal to t_0. It is assumed that the wire is of unit weight per unit length.

Resolving horizontally

$$t_0 = t\cos\theta \tag{1}$$

and vertically

$$l = t\sin\theta. \tag{2}$$

Clearly l and θ are intrinsic coordinates and

$$l = t_0\tan\theta. \tag{3}$$

Transforming to x- and y-coordinates,

$$\mathrm{d}x = t_0\sec\theta\,\mathrm{d}\theta \tag{4}$$

and

$$\mathrm{d}y = t_0\tan\theta\sec\theta\,\mathrm{d}\theta. \tag{5}$$

A momentary diversion allows us to remark that since, from (1),

$$\mathrm{d}t = t_0\tan\theta\sec\theta\,\mathrm{d}\theta, \tag{6}$$

another form of (5) is

$$\mathrm{d}t = \mathrm{d}y, \tag{7}$$

informing us that the tension in a catenary increases linearly with height; indeed,

$$t = t_0 + y. \tag{8}$$

Integrating (4) and (5) we obtain

$$x = t_0\ln(\sec\theta + \tan\theta) \tag{9}$$

and

$$y = t_0(\sec\theta - 1) \tag{10}$$

as the parametric equations of the catenary, or, eliminating θ,

$$\frac{x}{t_0} = \ln\left[1 + \frac{y}{t_0} + \left\{\left(\frac{y}{t_0}\right)^2 + \frac{2y}{t_0}\right\}^{\frac{1}{2}}\right], \tag{11}$$

from which we can see that t_0 is just a scaling factor which has to be obtained from (11) if the catenary is to pass through a particular point (X, Y).

3. The equilibrium configurations

The most obvious characteristic of an equilibrium configuration for the wire over the pylons is that the tension in the wire immediately on each side of each pylon must be the same. One equilibrium configuration is thus when all the catenaries are identical, but we need to find out if there are others. However, it turns out to be impossible to express the end-tension as a simple function of the length of the catenary between two pylons a known distance apart, so the required equation cannot be constructed. Without a knowledge of the equilibria it seems that we can hardly discuss their stability, but, oddly enough, thinking first about stability leads to a solution.

It is obvious that for a given distance between two pylons (henceforth assumed to be two units) there must be a length of the catenary between them which minimizes the end-tension. For as the catenary is made longer indefinitely it becomes heavier and its ends become ever closer to the vertical, thus ensuring an indefinitely-large tension. Conversely, as it is made shorter and approaches two units in length the tension again increases without limit. Somewhere inbetween there will be a length corresponding to a minimum end-tension (it is barely credible that there could be more than one such length).

Consider now the equilibrium configuration in which all the catenaries are the same and less than this length. If one catenary tries to steal a little wire from another its tension will decrease whilst that of the other will increase—and it will steal the wire back! Thus the all-catenaries-identical configuration is stable provided the length of each catenary is less than the critical length which gives the minimum end-tension. What *some* members of the family maintained is at least *sometimes* true!

Now consider the same equilibrium configuration when all the catenaries are greater than the critical length. If one catenary steals from its neighbour its end-tension will increase and that of its neighbour decrease, and it will go on devouring its neighbour until a new equilibrium is reached in which the neighbour has passed through the critical length and into the region of increasing end-tension again. The all-catenaries-identical configuration is thus unstable if each catenary exceeds the critical length. Moreover, the analysis reveals the likely existence of a stable equilibrium with one large loop under these conditions—what the *other* members of the family maintained is also *sometimes* true!

That there can never be more than one large (that is, longer than the critical length) loop is obvious from these considerations, because the largest of any number of large loops will always devour the others, however many pylons away they are.

To render this analysis formal we need to discover the critical catenary and then study the graph of end-tension versus length.

4. The catenary with minimum end-tension

We have assumed that the pylons are two units apart, at $x = \pm 1$. Let $y = Y$, $l = L$, $t = T$ and $\theta = \Theta$ at $x = \pm 1$. We wish to minimise T with respect to variation in the half-length L or, what is the same thing, with respect to variation in Θ. From (9)

$$t_0 = \frac{1}{\ln(\sec\Theta + \tan\Theta)} \tag{12}$$

and (1) thus yields

$$T = \frac{\sec\Theta}{\ln(\sec\Theta + \tan\Theta)}. \tag{13}$$

Setting the differential coefficient of this equal to zero leads to

$$\sin\Theta \ln(\sec\Theta + \tan\Theta) = 1 \tag{14}$$

which has to be solved numerically and gives

$$\Theta = 56.465\,84°.$$

The corresponding values for the half-length L and sag Y are

$$L = 1.257\,736 \quad \text{and} \quad Y = 0.675\,323.$$

Figure 2 shows the minimum end-tension catenary.

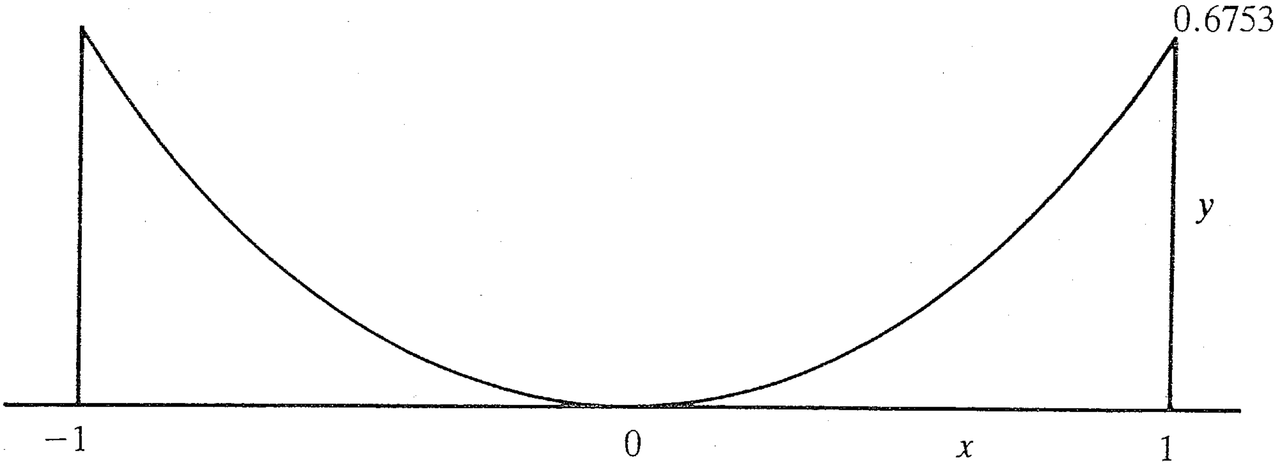

Figure 2. The catenary with minimum end-tension

5. The graph of end-tension *versus* length

Equation (13) gives T as a function of Θ, whilst (3) and (12) give

$$L = \frac{\tan\Theta}{\ln(\sec\Theta + \tan\Theta)}. \tag{15}$$

Equations (13) and (15) together enable figure 3 to be drawn, from which it will be seen that the graph is simple in form with asymptotes $L = 1$ and $T = L$, as was to be expected.

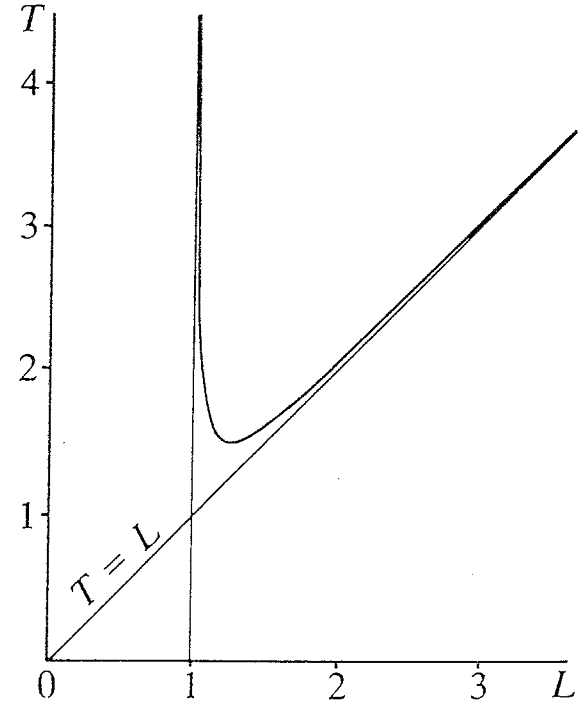

Figure 3. End-tension T as a function of half-length L for a unit half-span

6. Graphical analysis of the equilibria

With the aid of figure 3 the behaviour of two adjacent catenaries (and hence of the whole sequence) can be analysed. Consider three pylons with the wire fixed at the outer two and free to slide over the middle one. Whilst figure 3 describes the situation in the left-hand catenary (say), the situation in the right-hand catenary can be described by turning figure 3 over and placing it on the original figure 3 with the x-axes coincident and the y-axes separated by an amount equal to the sum of the half-lengths of the wire in the two catenaries. There will be three kinds of result depending on whether the total half-length of wire exceeds the critical total ($2\times1.257\,736$), is equal to it, or is less than it (figures 4 (a), (b) and (c) respectively).

The trick of inverting and superimposing figure 3 provides a complete analysis. In figure 4 (a) there are three equilibria, A, B and C, or points at which the end-tensions are the same in both catenaries. The symmetrical one, C, is unstable by the argument already given, which is here made obvious graphically. A and B, which are mirror-images of each other, are, however, stable, because shortening the shorter catenary evidently increases its tension faster than that of the longer catenary (at both A and B the gradients are both of the same sign, but the curve corresponding to the shorter catenary is steeper), leading it to recover the bit it has lost, whilst lengthening it has the reverse effect.

In figure 4 (b), the critical case where each catenary is of the critical length, there is only one equilibrium, just stable by the preceding argument

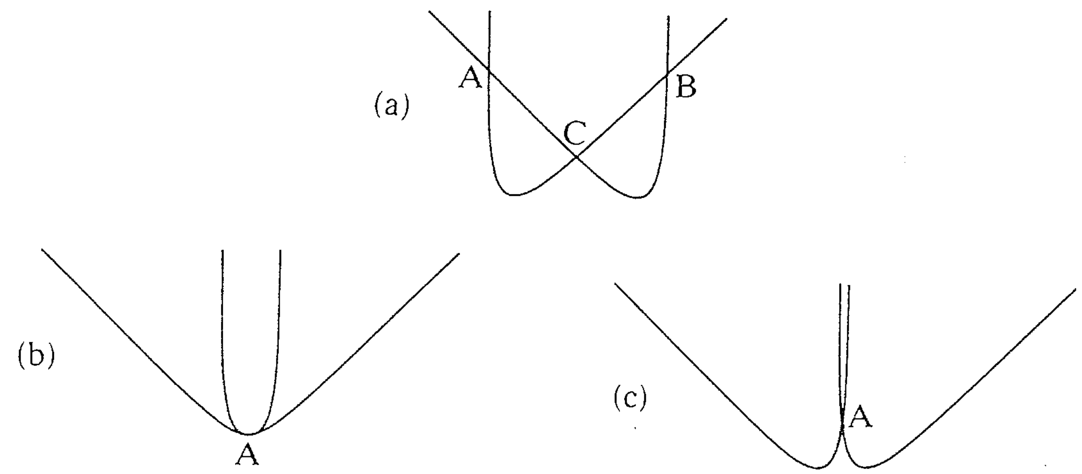

Figure 4. (a) The wire exceeds the critical length; (b) the wire equals the critical length; (c) the wire is less than the critical length

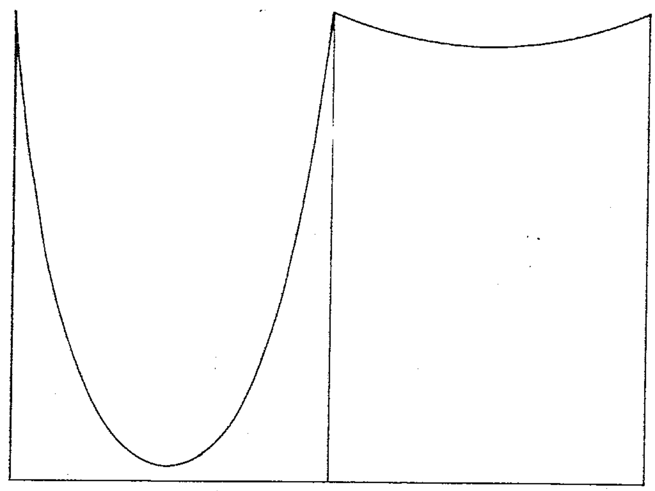

Figure 5. An example of an asymmetric equilibrium; the wire is 90% longer than the distance between the outer pylons

(assuming, as we shall without proof, that the left branch in figure 3 is always steeper than the right at the same value of T), whilst in figure 4 (c) the stable equilibrium already discussed in section 3 is depicted.

Figure 5 shows an example of an asymmetric equilibrium.

7. The complete solution

As explained in section 3, there can never be more than one catenary longer than the critical length. The complete solution to our ski-holiday problem is, therefore, that given a wire suspended over a horizontal line of equally-spaced pylons and fixed to the two end ones, the following two types of stable configuration are possible.

Case I. The length of the wire exceeds the distance between the end pylons by more than the factor L ($L \approx 1.257\,736$). There is only one kind of stable equilibrium, in which one catenary (it might be any one) is long and all the others are equal and shorter than the critical length. Moreover, all initial configurations will lead to this stable equilibrium, for one loop will initially be the largest and will start consuming the others.

Case II. The length of the wire exceeds the distance between the end pylons by less than or equal to the factor L. There is then a stable equilibrium with each catenary the same. However, if there are four or more pylons an equilibrium of the former type (Case I) may also be possible for lengths of wire close to factor L, because an increase in the length of the largest loop can be taken equally from each of the two or more other loops. The system will converge to the equal-catenary solution if initially the longest loop is less then the critical length, but if it is longer than the critical length which solution is reached will depend on its precise length and that of the other loops.

With only three pylons (and two catenaries) the appropriate version of figure 4 will give the graphical solution to Case I, but with more pylons recourse will have to be had to the numerical solutions of the end-tension equations mentioned in section 3. But at least our analysis has led to a qualitative description of the possible stable equilibria.

Now consider the case of the sloping hillside!

Smith numbers

Malcolm Smithers, a student of the Open University, has sent us a cutting from the *Daily Mail* of 3 September 1986, about a mathematician in the USA whose brother-in-law has the telephone number 493 7775.

He factorized this number into primes:

$$4937\,775 = 3\times5\times5\times65\,837$$

and noticed that the sum of the digits of these factors is the same as the sum of the digits of the original number: they are both 42. The brother-in-law is called Smith, so now you've got a *Smith number*.

L. J. Upton, of Mississauga, Ontario, Canada, read a similar report in a Canadian newspaper. He tells us that the mathematician who originated Smith numbers is Professor Albert Wilansky of LeHigh University, Bethlehem, Pennsylvania. Our correspondent asks two questions about Smith numbers:

1. Clearly every prime number is a Smith number. Now 4 is the smallest non-prime Smith number. Is there a largest such, or are there infinitely many non-prime Smith numbers?

2. A perfect number is a number which is the sum of its positive divisors, excluding the number itself (e.g. $6 = 1+2+3$ is perfect). Is there a perfect Smith number?

Ratio Derivatives

ANTHONY QUAS, *Gordano Comprehensive School, Bristol*

The author had this idea and wrote the article when he was a sixth former at Gordano Comprehensive School, Bristol. He is now reading mathematics at King's College, Cambridge.

Many common functions are differentiable, i.e. the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

is defined. This definition uses subtraction and division. We define here another limit $f^*(x)$ using instead division and powers, i.e.

$$f^*(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{1/h}.$$

We shall write $\rho_x(f(x))$ for $f^*(x)$, analogous to $(\mathrm{d}/\mathrm{d}x)(f(x))$. Then, if λ is a constant, it is easy to see that

$$\rho_x(\lambda f(x)) = \rho_x(f(x)).$$

This corresponds to

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)+\lambda) = \frac{\mathrm{d}}{\mathrm{d}x}f(x).$$

The formula for differentiating a product is actually easier for this 'ratio derivative' than for the usual derivative. For

$$\rho_x(f(x)g(x)) = \lim_{h\to 0}\left(\frac{f(x+h)g(x+h)}{f(x)g(x)}\right)^{1/h} = \rho_x(f(x))\rho_x(g(x)).$$

As an example,

$$\rho_x(\mathrm{e}^x) = \lim_{h\to 0}\left(\frac{\mathrm{e}^{x+h}}{\mathrm{e}^x}\right)^{1/h} = \lim_{h\to 0}(\mathrm{e}^h)^{1/h} = \mathrm{e}.$$

The well-known limit

$$\lim_{h\to 0}(1+h)^{1/h} = \mathrm{e}$$

enables us to evaluate various ratio derivatives. For example,

$$\rho_x(x) = \lim_{h\to 0}\left(\frac{x+h}{x}\right)^{1/h} = \lim_{h\to 0}\left[\left(1+\frac{h}{x}\right)^{x/h}\right]^{1/x} = \mathrm{e}^{1/x},$$

whence, from the product formula,

$$\rho_x(x^n) = e^{n/x},$$

when n is a positive integer.

The connection between the ratio derivative and the usual derivative is given by

$$\begin{aligned}
\ln f^*(x) &= \ln \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{1/h} \\
&= \lim_{h\to 0} \ln\left(\frac{f(x+h)}{f(x)}\right)^{1/h} \\
&= \lim_{h\to 0} \frac{\ln f(x+h) - \ln f(x)}{h} \\
&= (\ln f(x))' \\
&= \frac{f'(x)}{f(x)},
\end{aligned}$$

so that

$$f'(x) = f(x)\ln f^*(x), \qquad f^*(x) = e^{f'(x)/f(x)}.$$

Thus, for example, we can recover the formula for $(\mathrm{d}/\mathrm{d}x)(x^n)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = x^n \ln \rho_x(x^n) = x^n\frac{n}{x} = nx^{n-1},$$

and the formula for differentiating a product follows from the ratio derivative of a product:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) &= f(x)g(x)\ln(f^*(x)g^*(x)) \\
&= f(x)g(x)\ln f^*(x) + f(x)g(x)\ln g^*(x) \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

We can define the analogue of integration by writing

$$P_x(f^*(x)) = f(x).$$

The connection between P_x and the usual integration comes from

$$\begin{aligned}
& & \ln f^*(x) &= \frac{f'(x)}{f(x)} \\
&\Rightarrow & \int \ln f^*(x)\,\mathrm{d}x &= \int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln f(x) \\
&\Rightarrow & f(x) &= \exp\left(\int \ln f^*(x)\,\mathrm{d}x\right).
\end{aligned}$$

Thus

$$P_x(u(x)) = \exp\left(\int \ln u(x)\ \mathrm{d}x\right).$$

We note that, since the ordinary indefinite integral involves an arbitrary additive constant, the indefinite ratio integral has an arbitrary multiplicative constant.

It is reasonable to define the definite ratio integral by

$${}_a^bP_x(u(x)) = \frac{[P_x(u(x))]_b}{[P_x(u(x))]_a},$$

so that

$${}_a^bP_x(u(x)) = \frac{\left[\exp\left(\int \ln u(x)\ \mathrm{d}x\right)\right]_b}{\left[\exp\left(\int \ln u(x)\ \mathrm{d}x\right)\right]_a} = \exp\left(\left[\int \ln u(x)\ \mathrm{d}x\right]_a^b\right),$$

i.e.

$${}_a^bP_x(u(x)) = \exp\left(\int_a^b \ln u(x)\ \mathrm{d}x\right).$$

This can be used to obtain an approximation for $n!$. We divide the interval from a to b into equal intervals of width h. Then, for small h,

$$\int_a^b \ln f(x)\ \mathrm{d}x \simeq h[\ln f(a)+\ln f(a+h)+\ln f(a+2h)+\ldots+\ln f(b-h)]$$

$$= \ln[f(a)f(a+h)f(a+2h)\ldots f(b-h)]^h.$$

Thus

$${}_a^bP_x(f(x)) \simeq [f(a)f(a+h)f(a+2h)\ldots f(b-h)]^h.$$

Now put $f(x) = x$, $a = 1/n$, $b = 1$ and $h = 1/n$. Then

$${}_{1/n}^{1}P_x(x) = \exp\left(\int_{1/n}^1 \ln x\ \mathrm{d}x\right)$$

$$= \exp([x\ln x - x]_{1/n}^1)$$

$$= \exp\left(-1-\frac{1}{n}\ln\frac{1}{n}+\frac{1}{n}\right)$$

$$= \exp\left(-1+\frac{1}{n}\right)n^{1/n},$$

so that, for large n,

$$\exp\left(-1+\frac{1}{n}\right)n^{1/n} \simeq \left(\frac{1}{n}\times\frac{2}{n}\times\frac{3}{n}\times\ldots\times\frac{n-1}{n}\right)^{1/n}.$$

Thus

$$e^{-n+1}n \simeq \frac{(n-1)!}{n^{n-1}},$$

or

$$(n-1)! \simeq \frac{n^n}{e^{n-1}},$$

or

$$n! \simeq \frac{(n+1)^{n+1}}{e^n}.$$

If we compare values, this approximation seems unimpressive. For example, $5! \simeq 314$ (in fact $5! = 120$) and $10! \simeq 12\,950\,000$ (instead of $3\,528\,800$). However, if the graphs of the logarithms of $n!$ and the approximation are plotted, a very close correspondence is immediately noticeable. The two graphs have almost exactly the same form. And our approximation bears a certain resemblance to a famous approximation to $n!$, namely *Stirling's formula*, which is

$$n! \simeq \sqrt{2\pi}\,\frac{n^{n+\frac{1}{2}}}{e^n}.$$

The usual derivative has a geometrical interpretation in terms of the slope of the graph and a physical interpretation in terms of rates of change (for example, ds/dt gives speed, where s is distance and t is time). It would be interesting to know if there are corresponding geometrical and physical interpretations of the ratio derivative. Perhaps readers could help.

We leave readers with an exercise. Produce a formula for the ratio derivative of the sum of two functions.

The biggest prime in the world

Joseph Mclean, now Research Assistant in the Department of Computing Science at the University of Strathclyde, writes:

'It has come to my attention, and my sources are many and reliable, that the new largest known prime is the Mersenne prime

$$2^{216091}-1$$

with an exponent roughly double that of the previous largest (see Volume 19 No. 2, page 46).'

Waiting time for a run of wins

A. V. BOYD, *University of the Witwatersrand, Johannesburg*

The author studied at the University of the Witwatersrand and, from 1951 to 1953, at Fitzwilliam House, Cambridge, and later spent a year at the University of Aberdeen. He has lectured on mathematics and statistics for over 30 years and, after a spell as Head of Department of Statistics, is now giving statistics courses for actuarial science students.

1. Statement of the problem

Statistics textbooks discuss the usefulness of the techniques of probability generating functions and moment generating functions for tackling problems on distributions but do not often give guidance on when one of these is to be preferred, or stress the ease with which one can convert from one generating function to the other when this leads to a desirable simplification. Some advantages of these methods will be illustrated by discussing the problem considered by D. O. Forfar and T. W. Keogh (reference 2), of finding the waiting time for the first occurrence of a run of m wins by either player when two players engage in a series of independent games, and each has probability $\frac{1}{2}$ of winning a game. Forfar and Keogh used only first and second moments of certain distributions; use of generating functions will provide higher-order moments should they be needed for considering skewness and kurtosis of the distributions concerned.

We shall use the following definitions. For a discrete variate X taking the value n with probability $P(n)$, $(n = 0, 1, 2, \ldots)$, the probability generating function is

$$p(t) = E(t^X) = \sum_{x=0}^{\infty} t^x P(x),$$

which converges for $|t| < 1$, and the moment generating function will be taken as

$$M(t) = E(\mathrm{e}^{tX}) = \sum_{x=0}^{\infty} \mathrm{e}^{tx} P(x)$$

which converges for $t < 0$. The relations between these are seen to be

$$M(t) = p(\mathrm{e}^t) \quad \text{and, conversely,} \quad p(t) = M(\log t). \tag{1}$$

Expanding e^{tx} in a Maclaurin series leads formally to

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{x=0}^{\infty} x^n P(x) = \sum_{n=0}^{\infty} \mu_n' \frac{t^n}{n!}, \tag{2}$$

where

$$\mu_n' = \sum_{x=0}^{\infty} x^n P(x)$$

is the nth moment of X about the origin. It is known that the mean of X is given by $p'(1)$ or μ_1', and the variance by either $p''(1)+p'(1)-\{p'(1)\}^2$ or $\mu_2'-(\mu_1')^2$, the form to be used depending on whether it is easier to differentiate a closed expression for $p(t)$ or to expand $M(t)$ in powers of t and use equation (2) to identify μ_1' and μ_2' as coefficients in the expansion.

2. The first two moments in the cases $m = 2$ and $m = 3$

Forfar and Keogh defined $u(n)$ to be the number of different series terminating with a win on the nth game, and $P_n^{(m)}$ to be the probability that the series ends on the nth game, so that $P_n^{(m)} = u(n)/2^n$. If

$$S_r = \sum_{n=1}^{\infty} \frac{n^r u(n)}{2^n} = \sum_{n=1}^{\infty} n^r P_n^{(m)},$$

which is the rth moment μ_r' of the set of probabilities $P_n^{(m)}$, they showed that

$$S_0 = 1,$$

$$\frac{m}{2^{m-1}} = S_1 - \tfrac{1}{2}(S_1+S_0) - \tfrac{1}{4}(S_1+2S_0) - \ldots - \frac{1}{2^{m-1}}[S_1+(m-1)S_0],$$

and

$$\frac{m^2}{2^{m-1}} = S_2 - \tfrac{1}{2}(S_2+2S_1+S_0) - \tfrac{1}{4}(S_2+4S_1+4S_0) - \ldots$$
$$- \frac{1}{2^{m-1}}[S_2+2(m-1)S_1+(m-1)^2 S_0].$$

Their technique will also give the result

$$\frac{m^3}{2^{m-1}} = S_3 - \frac{1}{2}\left\{S_3 + \binom{3}{1}S_2 + \binom{3}{2}S_1 + \binom{3}{3}S_0\right\}$$
$$- \frac{1}{4}\left\{S_3 + \binom{3}{1}S_2 + \binom{3}{2}2^2 S_1 + \binom{3}{3}2^3 S_0\right\} - \ldots$$
$$- \frac{1}{2^{m-1}}\left\{S_3 + \binom{3}{1}(m-1)S_2 + \binom{3}{2}(m-1)^2 S_1 + \binom{3}{3}(m-1)^3 S_0\right\}$$

and so on, in an obvious way. They examined S_1 and S_2 in order to get the mean and the variance $S_2 - S_1^2$ of the set of probabilities, but it is worth noting that for specific values of m one can do appreciably more than this by elementary means. For example, in the case $m = 2$ the above equations become

$$1 = S_1 - \tfrac{1}{2}(S_1 + S_0), \qquad 2 = S_2 - \tfrac{1}{2}(S_2 + 2S_1 + S_0),$$
$$4 = S_3 - \tfrac{1}{2}(S_3 + 3S_2 + 3S_1 + S_0)$$

and, more generally,

$$2^{k-1} = S_k - \frac{1}{2}\left\{\binom{k}{0}S_k + \binom{k}{1}S_{k-1} + \binom{k}{2}S_{k-2} + \ldots + \binom{k}{k}S_0\right\}$$

for $k = 1, 2, \ldots$. If we put $S_0 = 2x_0 - 1$ and $S_n = 4x_n - 1$ for $n = 1, 2, \ldots$ and use $S_0 = 1$, some straightforward algebraic manipulation leads to $x_0 = 1$ and

$$x_k = \binom{k}{1}x_{k-1} + \binom{k}{2}x_{k-2} + \ldots + \binom{k}{k-1}x_1 + \binom{k}{k}x_0$$

for $k = 1, 2, \ldots$. Then $x_0 = 1$, $x_1 = 1$, $x_2 = 3$, $x_3 = 13$, $x_4 = 75$, etc. are the numbers introduced by Cayley in 1859 to discuss trees with n knots.

These numbers x_n were also considered by Gross in 1962 and shown to represent the numbers of possible outcomes in a race where ties are allowed. Thus for 3 runners the possible outcomes of a race are the 6 permutations of A, B and C together with

(i) (ABC), where all 3 tie for first place;
(ii) $(AB)C$, $(AC)B$ and $(BC)A$, where 2 tie for first place;
(iii) $C(AB)$, $B(AC)$ and $A(BC)$, where 2 tie for second place.

Interesting formulae and recurrence methods for calculating the x_n were given by E. Mendelson (reference 3). He shows that if

$$f(t) = \sum_{n=0}^{\infty} \frac{x_n}{n!}t^n \quad \text{then} \quad f(t) = \frac{1}{2 - e^t},$$

and so

$$M(t) = \sum_{n=0}^{\infty} S_n \frac{t^n}{n!} = 4\sum_{n=0}^{\infty} x_n \frac{t^n}{n!} - 2x_0 - \sum_{n=0}^{\infty} \frac{t^n}{n!},$$

which simplifies to $e^{2t}/(2 - e^t)$. This $M(t)$ is the moment generating function of the set of probabilities $P_n^{(2)}$ and so could be expanded in a Maclaurin series to give moments for their distribution, but if only the mean and variance are required then it is easier to exploit the simple relationship given by equation (1) and to write

$$p(t) = M(\ln t) = \frac{t^2}{2-t} = -t - 2 + \frac{4}{2-t}$$

from which we get the mean as $p'(1) = 3$ and the variance as $p''(1) + p'(1) - \{p'(1)\}^2 = 2$. Higher-order moments can, of course, also be found from $p(t)$.

In the case $m = 3$ the formulae of Forfar and Keogh, and their extensions, give rise to

$$\tfrac{3}{4} = S_1 - \tfrac{1}{2}(S_1 + S_0) - \tfrac{1}{4}(S_1 + 2S_0),$$
$$\tfrac{9}{4} = S_2 - \tfrac{1}{2}(S_2 + 2S_1 + S_0) - \tfrac{1}{4}(S_2 + 4S_1 + 4S_0),$$

and

$$\frac{3^k}{4} = S_k - \frac{1}{2}\left\{\binom{k}{0}S_k + \binom{k}{1}S_{k-1} + \binom{k}{2}S_{k-2} + \ldots + \binom{k}{k}S_0\right\}$$
$$- \frac{1}{4}\left\{\binom{k}{0}S_k + \binom{k}{1}2S_{k-1} + \binom{k}{2}2^2S_{k-2} + \ldots + \binom{k}{k}2^kS_0\right\} \quad (3)$$

for $k = 1, 2, \ldots$. Now expressing e^t, e^{2t} and $M(t)$ as power series in t, multiplying out and rearranging terms, we get

$$e^tM(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!}\left\{\binom{n}{0}S_0 + \binom{n}{1}S_1 + \ldots + \binom{n}{n}S_n\right\}$$

and

$$e^{2t}M(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!}\left\{\binom{n}{0}2^nS_0 + \binom{n}{1}2^{n-1}S_1 + \ldots + \binom{n}{n}2^0S_n\right\}.$$

Hence by using equation (3) we get

$$(\tfrac{1}{2}e^t + \tfrac{1}{4}e^{2t})M(t) = \tfrac{1}{2} + \tfrac{1}{4} + \sum_{n=1}^{\infty} \frac{t^n}{n!}\left\{S_n - \frac{3^n}{4}\right\}$$
$$= \tfrac{3}{4} + M(t) - 1 - \tfrac{1}{4}(e^{3t} - 1)$$

and so

$$M(t) = \frac{e^{3t}}{4 - 2e^t - e^{2t}}$$

and

$$p(t) = M(\ln t) = \frac{t^3}{4 - 2t - t^2} = -t + 2 + \frac{8(t-1)}{4 - 2t - t^2}.$$

This leads to the mean number of trials $= p'(1) = 7$ and var(number of trials) $= 22 \simeq (4.69)^2$. Higher-order moments can again be found, and larger values of m can be tackled in a similar way. An alternative approach, which avoids the use of the moments S_r but is based on the theory of recurrent events, can be developed as in reference 1, Chapter XIII.

3. The probabilities in the cases $m = 2$ and $m = 3$

In the cases $m = 2$ and 3 it is worth noting that $p(t)$ is a rational function with denominator of degree 1 or 2 respectively, and so it is easily resolved into partial fractions and then expanded to yield the probabilities $P_n^{(m)}$. Thus

$$p(t) = \frac{t^2}{2-t} = \sum_{n=2}^{\infty} \frac{t^n}{2^{n-1}}, \quad \text{giving} \quad P_n^{(2)} = \frac{1}{2^{n-1}} \quad \text{for } n \geq 2,$$

which could easily be obtained by considering the 2 sequences ending with successes for one of the two players on the $(n-1)$th and nth games, i.e. sequences $\ldots ABABAA$ and $\ldots BABABB$; and, in the case $m = 3$,

$$p(t) = \frac{t^3}{4-2t-t^2} = \frac{t^3}{2\sqrt{5}}\left\{\frac{1}{t+1+\sqrt{5}} - \frac{1}{t+1-\sqrt{5}}\right\}$$

$$= \sum_{n=3}^{\infty} \frac{(1+\sqrt{5})^{n-2}-(1-\sqrt{5})^{n-2}}{2^{2n-3}\sqrt{5}} t^n,$$

leading to the solution published to Problem 17.8 in Volume 18, Number 2 of *Mathematical Spectrum*.

It will be found, however, that the probability generating function obtained either by extending the above method or by using a result derived by Feller (equation (8.2) in Chapter XIII of reference 1) is

$$p(t) = \frac{(2-t)t^m}{t^m+2^m(1-t)}$$

and is not readily expanded in powers of t when m exceeds 3.

References

1. W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. I, 3rd edn (Wiley, New York, 1968).
2. D. O. Forfar and T. W. Keogh, Independence and the length of a run of wins, *Mathematical Spectrum* **18** (1985/6), 70–75.
3. E. Mendelson, Races with ties, *Mathematical Magazine* **55** (1982), 170–175.

Autocash

A simple cash dispenser has n keys labelled from 1 to n. To obtain money, a customer inserts his card and keys in two digits in order. If the wrong digits are keyed in, the machine does nothing. It allows an unlimited number of tries, and reads the last two numbers keyed in. If you have forgotten your number (or have someone else's card and wish to relieve the owner of cash!), what sequence of numbers would you key into the dispenser?

Liquid Measurements Using Two Jugs

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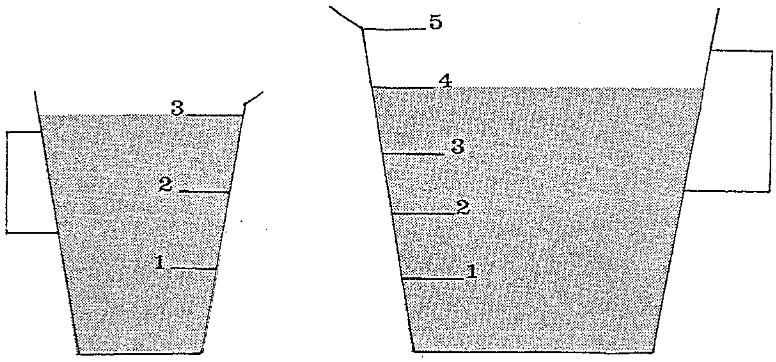

A well-known problem asks how to obtain exactly 4 litres of water using only a 5-litre jug and a 3-litre jug. Table 1 provides a concise description of the sequence of pourings required to achieve this.

TABLE 1

5-litre jug	3-litre jug
5	0
2	3
2	0
0	2
5	2
4	3

This problem suggests the following generalization.

Given jugs of capacities L and S, where $L > S$, which volumes of water can be obtained exactly?

We shall restrict our discussion to relatively prime pairs L and S ($L > S$) and shall show that *all* of the volumes $1, 2, \ldots, L$ are exactly obtainable.

Modular arithmetic is an essential part of the discussion. We use the notation $[M]$ to denote the remainder obtained when the whole number M is divided by S. Thus, S divides $M-[M]$, $0 \leqslant [M] < S$ and $[M] = [N]$ if and only if S divides $M-N$.

Theorem 1. If we can obtain exactly $[X]$ litres, then we can obtain exactly $[L+X]$ litres.

Proof. Since S divides both $[X]-X$ and $L+X-[L+X]$, it also divides their sum, the positive number $L+[X]-[L+X]$. Thus,

$$L+[X]-[L+X] = kS$$

for some $k > 0$. We use the sequence of pourings given in table 2 noting that

$$L-(S-[X]) = L+[X]-S.$$

TABLE 2

L-litre jug	S-litre jug
0	$[X]$
L	$[X]$
$L+[X]-S$	S
$L+[X]-S$	0
$L+[X]-2S$	S
$L+[X]-2S$	0
$\vdots$	$\vdots$
$L+[X]-kS$	S

The result follows upon observing that

$$L+[X]-kS = [L+X].$$

Corollary. We can obtain exactly $[L]$ litres.

Theorem 2. All of the volumes $1, 2, \ldots, L$ are exactly obtainable.

Proof. Since we can obtain exactly $[L]$ litres, we can obtain exactly $[L+L] = [2L]$ litres. Similarly, from $[2L]$ we can obtain $[2L+L] = [3L]$, and so forth. Thus, each of the volumes $[L], [2L], \ldots, [(S-1)L]$ is exactly obtainable. Since L and S are relatively prime, none of these numbers $[jL]$, where $1 \leqslant j < S$, can be zero. Moreover, $[jL] \neq [kL]$ if $0 \leqslant j < k < S$ since, if $[jL] = [kL]$, then S divides $(k-j)L$, which we just observed to be impossible. Therefore, each of the numbers $1, 2, \ldots, S-1$ appears exactly once among $[L], [2L], \ldots, [(S-1)L]$, from which we conclude that each of the volumes $1, 2, \ldots, S-1$ is exactly obtainable. Finally, by adding multiples of S, we can exactly obtain any of the volumes $1, 2, \ldots, L$.

As an example, suppose $L = 21$ and $S = 8$. Then, $[L] = 5$, $[2L] = 2$, $[3L] = 7$, $[4L] = 4$, $[5L] = 1$, $[6L] = 6$ and $[7L] = 3$, thereby giving us each of the volumes 1, 2, 3, 4, 5, 6 and 7 exactly once, and hence also the volumes 8 to 15 and 16 to 21.

The Knight's Tour of a Chessboard

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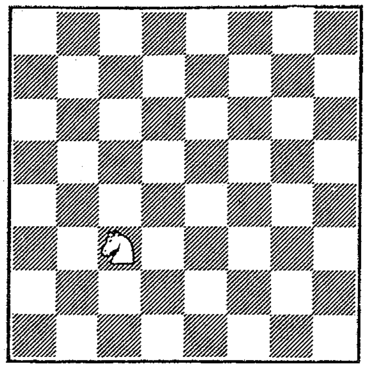

This famous problem was posed to readers in *Mathematical Spectrum* Vol. 18 (1985/86), p. 34. The problem is to find a sequence of knight's moves on a chessboard so that the knight lands on each square of the board just once. If we include the condition that the first and last squares are a knight's move apart, it is called a *re-entrant* (or *closed* or *cyclic*) tour. If, when the squares are numbered in the order in which the knight visits them, the row and column sums are the same, it is called a *magic square* as well as a knight's tour.

An example of a magic square fulfilling all these conditions is given in figure 1: a knight's tour is possible, a re-entrant tour is possible and the magic square has row and column sums equal to 260.

50	11	24	63	14	37	26	35
23	62	51	12	25	34	15	38
10	49	**64**	21	40	13	36	27
61	22	9	52	33	28	39	16
48	7	60	**1**	20	41	54	29
59	4	45	8	53	32	17	42
6	47	2	57	44	19	30	55
3	58	5	46	31	56	43	18

Figure 1

Once such a re-entrant tour is known, a simple method of producing a knight's tour starting from a given position is to renumber the known tour cyclicly. However, if such a 'closed' tour is not available or if 'open' tours are required, a method of generating tours is needed.

Background and algorithm suitable for computer programming

A good account of the history and solutions of the problem of finding a knight's tour, which dates back certainly to Euler (reference 1) is given by Rouse Ball and Coxeter (reference 3), who provide references to many methods. Most of their methods do not produce simple computer programs, whilst the use of trials with backtracking is simple but very time-consuming. However, a remarkably concise method (best considered as a tree which must be traversed until a solution is found) was given by H. C. Warnsdorff (reference 4); it goes to the second level of search which we will call LEVEL2. It is: 'if there are no possible moves STOP, otherwise move to the square from which there are fewest subsequent moves'. This algorithm provides an interesting computer-programming exercise, particularly if a solution has not been seen. Thus, to derive full pleasure from writing a program the reader is advised against turning to the program listings which follow.

Two programs written in BBC BASIC are given. The first implements Warnsdorff's rule and the second implements a refinement of the rule in an attempt to overcome a surprising failure of the rule which is as follows. Some tours which obey the algorithm do not traverse all the squares of the board before STOP. Clearly, the only difference between implementations of the rule is the choice of move when, for two or more possible moves, the numbers of subsequent moves are the same. Warnsdorff believed that the choice was arbitrary, but this is untrue. Rouse Ball and Coxeter reported that some cases of failure were constructed 'with great ingenuity'. However, the advent of computers has allowed a more rigorous investigation than was previously practicable. Most implementations of Warnsdorff's rule succeed for all starting points; occasionally a re-entrant tour is produced, and very rarely a failure occurs.

There seems to be a tendency for the programs which fail to be those which allocate the next move in a 'biased' way rather than a random or scattered way. For instance, the method of choosing the next move by looking at possible moves clockwise round the current position, only accepting a better move as the next possible position, fails when started from (4, 3). No clear reason is known for such failures, and no proof is known that there *is* an implementation of Warnsdorff's rule which succeeds for all starting points on a board of any shape for which a knight's tour exists. This remains a widely held conjecture, however. In fact, Kraitchik (reference 2) observed that the rule frequently succeeds even when some mistakes are made when using it.

A new algorithm which overcomes the known failures of Warnsdorff's rule

The purpose of this article is to provide an algorithm which succeeds regardless of arbitrary choices made when the next move is not defined uniquely.

Program A

```
10 REM PROGRAM A
20 DIM B%(12,12)
30 PRINT"INPUT TWO INTEGERS"
40 PRINT"WHICH ARE THE STARTING POSITION"
50 PRINT"GIVEN AS ROW NUMBER THEN COLUMN NUMBER"
60 PRINT
70 PRINT"KNIGHT'S TOUR STARTING FROM:"
80 READ I%,J%
90 PRINT ;I%"  "J%;
100 PRINT
110 PRINT
120 I%=I%+2
130 J%=J%+2
140 PROCknight(I%,J%)
150 PROCprint(B%)
160 END
170
180 DEF PROCprint(B%)
190 REM PROC PRINTS THE MATRIX B%
200 FOR COL%=3 TO 10
210 PRINT
220   FOR ROW%=3 TO 10
230 REM THE FOLLOWING LINE IS INCLUDED TO GET TIDY OUTPUT SIMPLY
240 IF (B%(COL%,ROW%)<10)THEN PRINT;"  ";ELSE PRINT;" ";
250   PRINT;B%(COL%,ROW%);
260   NEXT ROW%
270 NEXT COL%
280 ENDPROC
290
300 DEF PROCknight(I%,J%)
310 REM MAIN PROCEDURE TO FIND TOUR
320 REM FROM GIVEN STARTING POINT
330 FOR K%=1 TO 12
340   FOR L%=1 TO 12
350     B%(K%,L%)=99
360   NEXTL%
370 NEXTK%
380 FOR K%=3 TO 10
390   FOR L%= 3 TO 10
400     B%(K%,L%)=0
410   NEXTL%
420 NEXTK%
430 FOR P%=1 TO 64
440   B%(I%,J%)=P%
450   MIN%=8
460 REM PROGRAM RANGES OVER 8 POSSIBLE MOVES
470   FOR K%=-2 TO 2
480     FOR L%=-2 TO 2
490     IF ABS(K%*L%)<>2 THEN 600
500       IP%=I%+K%
510       JP%=J%+L%
520       IF B%(IP%,JP%)<>0 THEN 600
530         C%=0
540         FOR KD%=-2 TO 2
550           FOR LD%=-2 TO 2
560 IF ABS(KD%*LD%)=2 AND B%(IP%+KD%,JP%+LD%)=0 THEN C%=C%+1
570           NEXTLD%
580         NEXTKD%
590 IF C%<MIN% THEN IBEST%=IP%:JBEST%=JP%:MIN%=C%
600     NEXTL%
610   NEXTK%
620 I%=IBEST%
630 J%=JBEST%
640 NEXTP%
650 ENDPROC
660 REM   END OF PROGRAM A
670 DATA 3,3
```

Program B

```
10 REM PROGRAM B
20 DIM B%(12,12)
25 DIM X%(7),Y%(7)
30 PRINT"INPUT TWO INTEGERS"
40 PRINT"WHICH ARE THE STARTING POSITION"
50 PRINT"GIVEN AS ROW NUMBER THEN COLUMN NUMBER"
60 PRINT
70 PRINT"KNIGHT'S TOUR STARTING FROM:"
80 READ I%,J%
90 PRINT ;I%"  "J%;
100 PRINT
110 PRINT
120 I%=I%+2
130 J%=J%+2
140 PROCknight(I%,J%)
150 PROCprint(B%)
160 END
170
```

continued overleaf

Program B continued

```
180 DEF PROCprint(B%)
190 REM PROC PRINTS THE MATRIX B%
200 FOR COL%=3 TO 10
210 PRINT
220   FOR ROW%=3 TO 10
230 REM THE FOLLOWING LINE IS INCLUDED TO GET TIDY OUTPUT SIMPLY
240 IF (B%(COL%,ROW%)<10)THEN PRINT;"  ";ELSE PRINT;" ";
250   PRINT;B%(COL%,ROW%);
260   NEXT ROW%
270 NEXT COL%
280 ENDPROC
290
300 DEF PROCknight(I%,J%)
310 REM MAIN PROCEDURE TO FIND TOUR
320 REM FROM GIVEN STARTING POINT
330 FOR K%=1 TO 12
340   FOR L%=1 TO 12
350     B%(K%,L%)=99
360   NEXTL%
370 NEXTK%
380 FOR K%=3 TO 10
390   FOR L%= 3 TO 10
400     B%(K%,L%)=0
410   NEXTL%
420 NEXTK%
430 FOR P%=1 TO 64
440   B%(I%,J%)=P%
450   MIN%=8
460 REM PROGRAM RANGES OVER 8 POSSIBLE MOVES
465 REM MAKING A LIST OF THOSE WITH LEAST SUBSEQUENT MOVES
470   FOR K%=-2 TO 2
480     FOR L%=-2 TO 2
490     IF ABS(K%*L%)<>2 THEN 600
500       IP%=I%+K%
510       JP%=J%+L%
520       IF B%(IP%,JP%)<>0 THEN 600
530         C%=0
540         FOR KD%=-2 TO 2
550           FOR LD%=-2 TO 2
560 IF ABS(KD%*LD%)=2 AND B%(IP%+KD%,JP%+LD%)=0 THEN C%=C%+1
570           NEXTLD%
580         NEXTKD%
590 IF C%<MIN% THEN PROCstart ELSE IF C%=MIN% THEN PROCadd
600     NEXTL%
610   NEXTK%
620 REM LIST CONTAINS ONE OR MORE POSSIBLE MOVES
630 IF LISN%=1 THEN I%=X%(1):J%=Y%(1) ELSE PROClevel2
640 NEXTP%
650 ENDPROC
660 REM    END OF PROGRAM B
670
680 DEF PROCstart
690 REM STORE FIRST ON LIST
700 MIN%=C%
710 X%(1)=IP%
720 Y%(1)=JP%
730 LISN%=1
740 ENDPROC
760 DEF PROCadd
770 REM PROC ADDS A POINT TO THE LIST
780 LISN%=LISN%+1
790 X%(LISN%)=IP%
800 Y%(LISN%)=JP%
810 ENDPROC
820
830 DEF PROClevel2
840 REM PROC RANGES ROUND THE POINTS
850 REM WHICH HAVE EQUAL NUMBERS OF SUBSEQUENT MOVES
860 MIN%=8
870 FOR IP%=1 TO LISN%
880   C%=0
890   FOR K%=-2 TO 2
900     FOR L%=-2 TO 2
910 IF ABS(K%*L%)=2 AND B%(X%(IP%)+K%,Y%(IP%)+L%)=0 THEN PROClevel3
920     NEXTL%
930   NEXTK%
940 IF C%<MIN% THEN I%=X%(IP%):J%=Y%(IP%):MIN%=C%
950 NEXTIP%
960 ENDPROC
970
980 DEF PROClevel3
990 REM PROC TO CONSIDER NEXT LEVEL DOWN TO TRY TO SPLIT TIES
1000 C%=0
1010 FOR KD%=-2 TO 2
1020   FOR LD%=-2 TO 2
1030 IF ABS(KD%*LD%)=2 AND B%(X%(IP%)+K%+KD%,Y%(IP%)+L%+LD%)=0 THEN C%=C%+1
1040   NEXTLD%
1050 NEXTKD%
1060 ENDPROC
1070 DATA 3,3
```

Since Warnsdorff's rule is successful for a large number of cases it seems sensible to use it as a starting point for an improved algorithm. A backtracking version of the rule would certainly succeed since it would always eventually find the same move as one of the implementations which produce a tour, and the existence of a tour from any starting point is proved by the fact that there are cyclic tours which have been produced by the rule. This is not in the spirit of Warnsdorff's rule, which works forward defining each move as it goes.

The following algorithm is one which has succeeded for all the tested implementations which previously failed.

Proposed algorithm which searches to LEVEL3

Use Warnsdorff's algorithm but, when there are several equal numbers of possible subsequent moves, find for each of those moves the number of subsequent moves; then move to the one for which one of its subsequent moves allows the least number of subsequent moves.

In other words, move to the square from which the number of moves is minimal and for equal minima make the number of *subsequent* moves minimal.

The algorithm suffers from the same type of defect as the original rule in not stating how a choice is made between points with equal numbers where the word 'least' is used. One could again proceed in the same way, successively evaluating the numbers of subsequent moves to a greater and greater depth, knowing that for a finite chessboard the number would ultimately be reduced to zero and the problem would be solved (assuming that the algorithm were successful). One suspects that this might lead to an unnecessary amount of computation. We should like to prove that a depth of just one further than the Warnsdorff rule is sufficient always to produce a tour. However, although this appears to be the case, the proof that it will succeed for *any* selection amongst the arbitrary choices is elusive.

Program A implements Warnsdorff's LEVEL2 rule, the underlying program which tests to level 2 for the fewest possible subsequent moves. Program B implements the LEVEL3 algorithm which succeeds for all starting positions.

No attempt has been made to make this a comprehensive program. There are few comments. Unstructured GOTOs have been used where convenient. There is no testing for validity of data (that is, checking for whether they are greater than 0 and less than 9). The program is not interactive, so allowing several data sets, the array subscripts are used from 1 rather than 0 for convenience and it does not draw a board and an icon of a knight. These restrictions are not central to the problem and attempts to overcome them could obscure the program structure.

Line numbers are common to the two programs to ease comparison. From the program given, it should be easy to produce a version in your favourite programming language. However, the language chosen does have some effect on readability. The programs were run on a BBC 'B' microcomputer using (3, 3) as starting position. The results are shown in figure 2. As it happened, the LEVEL3 algorithm produced a re-entrant tour (but it is not a magic square).

23	2	19	34	41	38	17	36
20	33	22	51	18	35	44	39
3	24	**1**	42	59	40	37	16
32	21	50	29	52	43	58	45
25	4	31	60	49	62	15	56
10	7	28	53	30	57	46	63
5	26	9	12	61	48	55	14
8	11	6	27	54	13	**64**	47

A

33	2	19	**64**	31	50	17	62
20	57	32	51	18	63	30	49
3	34	**1**	56	53	48	61	16
58	21	52	47	60	43	54	29
35	4	59	22	55	46	15	42
10	7	36	39	44	41	28	25
5	38	9	12	23	26	45	14
8	11	6	37	40	13	24	27

B

Figure 2

Three and more dimensions

The concept of a knight's move and hence a knight's tour generalises easily to higher dimensions but there then are several possible choices for a knight's move. For instance, the move corresponding to steps (or increments) (1, 2) or (2, 1) which are those of normal chess in two dimensions might be taken to be any permutation of (0, 1, 2) or (1, 2, 3), for instance, in three dimensions.

Warnsdorff's rule is unaltered by a change of dimension. In fact, any implementation of Warnsdorff's rule appears to be satisfactory for the first choice of increments, (0, 1, 2), but, as yet, no successful implementation has been found for the second choice. Similarly, all Warnsdorff-based methods which have been attempted using the triples (1, 1, 2), (1, 2, 2), (1, 1, 3) and (1, 3, 3) have failed, but it seems too early to say that no such method exists, and it would certainly be unwise to rule out the possibility of a tour with one of these sets of increments. These tests were performed on a VAX 11/750 computer with a FORTRAN77 version of the program.

What is clear is the power, robustness and usefulness of Warnsdorff's rule. Kraitchik states that the rule 'is a rule of common sense'. Possibly it is, for example when stated as 'if you start by painting the most difficult places, you avoid painting yourself into a corner'. Stated in that way, perhaps it has more general application than just finding tours of a chessboard.

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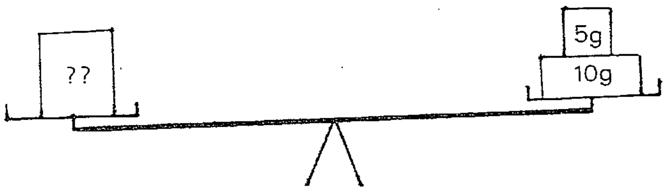
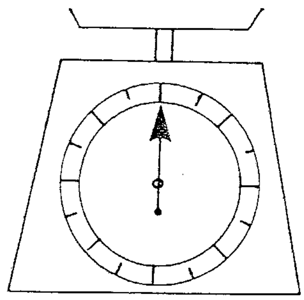
2. M. Kraitchik. *Mathematical Recreations* (Allen and Unwin, London, 1943), pp.257–266.
3. W. W. Rouse Ball and H. S. M. Coxeter. *Mathematical Recreations and Essays* 11th edn (Macmillan, London, 1939), pp.174–185.
4. H. C. Warnsdorff. *Des Rösselsprunges einfachste und allgemeinste Lösung* (Schmalkalden, 1823).

Postscript L. J. Upton of Ontario has raised the question of whether it is possible to obtain a 'perfect' magic square (where the rows, columns *and diagonals* all add up to the same number) from a re-entrant knight's tour. He has sent us an example in which the row and column sums are 260 but the diagonal sums are 256 and 264. The first example in the above article (figure 1) has diagonal sums 328 and 192. Our reader understands that it is extremely doubtful whether such a perfect magic square exists. Curiously, in both these cases the diagonal sums have mean 260.

Weighing Problems

sent in by Arthur Pounder, Manchester.





1. There are 12 seemingly equal snooker balls. Eleven of them are of equal weights but one has weight different from the other 11. How can the odd ball be determined in three uses of a pair of scales?
2. What is the least number of weights required with a pair of scales in order to be able to weigh 1 g, 2 g, ..., 40 g?
3. There are 10 piles with 10 coins in each pile. Each coin in each pile weighs 10 g except for the coins in one pile, which weigh 9 g each. How can the odd pile be determined using just one weighing on a weighing machine?

Computer Column

MIKE PIFF

This BBC Micro program simulates, in real time, the motion of a planet round a sun under an attraction force proportional to r^p, where r is the planet's distance from the sun. You first specify a value of p, then an initial distance $r \geqslant 50$. The constant of proportionality is then chosen so that an initial velocity $v = 1$ perpendicular to r will result in circular motion. You must now specify $v \geqslant 1$, and the planet will be seen following the orbit you have chosen. Some examples are:

$p = -2$; $r = 50$; $v = 1.3$:	familiar motion round an ellipse with the sun at one focus;
$p = 1$; $r = 50$:	elliptical motion, with the sun at the centre of the ellipse;
$p = -4$; $r = 50$; $v = 1.1$:	the planet spirals into space;
$p = 10$; $r = 50$; $v = 1000$:	the planet turns regularly through an acute angle, whilst otherwise travelling virtually in a straight line.

The restriction $r \geqslant 50$ is a technical one. The numerical algorithm, chosen because of its speed, becomes inaccurate for small values of r when $p = -2$, say, as you will see if you delete line 90 and input $v = 0.5$. Also, at line 200, the time interval is so chosen that the planet initially travels at 10 units per iteration, so in fact the orbits for large values of v are really seen in slow motion.

```
10 MODE1
20 INPUT "INPUT POWER",POWER
30 X=0
40 PRINT "INPUT DISTANCE FROM SUN";
50 INPUT "(50..400)",Y
60 IF Y<50 THEN Y=50
70 G=Y^-(POWER+1)
80 VY=0
90 INPUT "GIVE SPEED OF PLANET > 1 ",VX
100 IF ABS(VX)<1 THEN VX=SGN(VX)
110 IF VX=0 THEN VX=1
120 VDU19,0,4,0,0,0
130 VDU19,1,1,0,0,0
140 VDU23,1,0;0;0;0;
150 VDU5:CLG
160 VDU29,640;512;
170 PROCBLOB(0,0)
180 MOVE X,Y
190 PROCBLOB(X,Y)
200 DT=10/ABS(VX)
210 REPEAT
220   XOLD=X:YOLD=Y
230   ACC=FNACC(X,Y)
240   AX=X*ACC:AY=Y*ACC
250   X=X+DT*(VX+DT*AX/2)
260   Y=Y+DT*(VY+DT*AY/2)
270   ACC=FNACC(X,Y)
280   AX=(AX+X*ACC)/2:AY=(AY+Y*ACC)/2
290   VX=VX+AX*DT:VY=VY+AY*DT
300   PROCBLOB(XOLD,YOLD)
310   GCOL0,1:DRAW X,Y:PROCBLOB(X,Y)
320   UNTIL FALSE
330 END
340 DEF PROCBLOB(X,Y)
350 GCOL 3,2:MOVE X-16,Y+16
360 *FX 19
370 PRINT "o":MOVE X,Y
380 ENDPROC
390 DEF FNACC(X,Y)
400 =-G*(X^2+Y^2)^((POWER-1)/2)
```

Letter to the Editor

Dear Editor,

On sums of unlike powers II

Extending my previous results (*Mathematical Spectrum* Volume 18 Number 3, page 88), we look for solutions of:

$$(1)\quad x^4+y^4 = z^2 \pm d, \qquad (2)\quad x^3+y^3 = z^3 \pm d, \qquad (3)\quad x^3+y^3 = z^2 \pm d,$$

such that $d = 1, 2$ or 3 and supposing that $x \geqslant y$. The results are presented in tabular form.

Equation (1), with $x \leqslant 5000$

Value of d	Number of solutions	Largest solution
+1	69	$4451^4+4091^4 = 25\,934\,431^2+1$
+2	4	$427^4+115^4 = 182\,808^2+2$
−2	6	$1253^4+1101^4 = 1\,983\,522^2-2$
−3	2	$825^4+116^4 = 680\,758^2-3$

Equation (2), with $x \leqslant 5000$

Value of d	Number of solutions	Largest solution
+1	12	$4528^3+3753^3 = 5262^3+1$
−1	12	$3230^3+2676^3 = 3753^3-1$
−2	9	$4373^3+486^3 = 4375^3-2$
+3	1	$4^3+4^3 = 5^3+3$

Equation (3), with $x \leqslant 2000$

Value of d	Number of solutions	Largest solution
−1	144	$1992^3+920^3 = 93\,183^2-1$
+1	25	$1863^3+595^3 = 81\,711^2+1$
−2	18	$1759^3+64^3 = 73\,775^2-2$
+2	3	$1909^3+1309^3 = 95\,916^2+2$
−3	14	$1945^3+213^3 = 85\,835^2-3$
+3	67	$1951^3+306^3 = 86\,342^2+3$

JOSEPH MCLEAN
(M.Sc. Student, University of Glasgow,
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Glasgow, G41 5DA)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

19.8. (Submitted by Gregory Economides, Form V, Royal Grammar School, Newcastle upon Tyne)
What are the last two digits of 7^{1987}?

19.9 (Submitted by A. J. Douglas and G. T. Vickers, University of Sheffield)
Show that

$$\frac{1}{\pi} = \tfrac{1}{4}\tan\tfrac{1}{4}\pi + \tfrac{1}{8}\tan\tfrac{1}{8}\pi + \tfrac{1}{16}\tan\tfrac{1}{16}\pi + \dots .$$

(Hint: $\tan\frac{1}{4}\pi = \cot\frac{1}{4}\pi$.)

19.10. (Submitted by László Cseh, a student at Babes-Bolyai University, Romania)
Prove that

$$[x] + [2x+2y] \geqslant [2x] + [y] + [x+y],$$

where $[a]$ denotes the integer part of the real number a.

19.11. (Submitted by Imre Merényi, Babes-Bolyai University, Romania)
Let a, b and c be real numbers with $0 < a,b,c < \frac{1}{2}$ and $a+b+c = 1$. Prove that

$$\sqrt{a(1-2a)} + \sqrt{b(1-2b)} > \sqrt{c(1-2c)}.$$

19.12. (Submitted by J. N. MacNeill, Royal Wolverhampton School)
For which positive integers n is the standard deviation of n consecutive integers rational?

Solutions to Problems in Volume 19 Number 1

19.1 A prime pair is a pair of prime numbers which differ by 2, e.g. 11 and 13. If p, q is a prime pair, what happens when p^2+q^2 is divided by 72 and why?

Solution by Gregory Economides (Form V, Royal Grammar School, Newcastle upon Tyne)

Every prime number greater than 3 is congruent to 1 or $-1 \pmod 6$. Thus, if $p, q > 3$ with $p < q$, then $p = 6k-1$ and $q = 6k+1$ for some positive integer k, and

$$p^2+q^2 = (6k-1)^2+(6k+1)^2 = 72k^2+2.$$

Thus the quotient when p^2+q^2 is divided by 72 is a perfect square and the remainder is 2.

Also solved by Adrian Hill (Trinity College, Cambridge), Keith Gordon (The Haberdashers' Aske's School, Elstree), Michael Boore (Wymondham College), Simon Gay (City of Stoke-on-Trent Sixth Form College), Mike Day (University of Lancaster), Jean Corriveau (McMaster University, Canada), K. W. Lau (Hong Kong), Eddie Cheng (Memorial University of Newfoundland, Canada), Chris Hillman (Cornell University, U.S.A.), Guy Willard (Corpus Christi College, Cambridge) and Amites Sarkar (aged 13, Winchester College).

19.2 A bag contains n balls, labelled 1 to n, and r balls are picked at random from the bag, one by one, without replacement. What is the probability that the numbers of the balls picked form an arithmetic progression (a) in the order in which they are picked and (b) after a possible rearrangement.

Solution 1 by Adrian Hill (Trinity College, Cambridge)

An arithmetic progression is characterized by its starting number s and by the difference d between successive terms. The number N of pairs (s,d) such that $s,d \geqslant 1$ and $s+(r-1)d \leqslant n$ is half the number of arithmetic progressions with r terms, since this number counts only increasing arithmetic progressions. We require

$$1 \leqslant d \leqslant \left[\frac{n-s}{r-1}\right],$$

where $[\alpha]$ denotes the integer part of the real number α. Hence

$$N = \sum_{s=1}^{n-r+1} \left[\frac{n-s}{r-1}\right].$$

In case (a), there are $n!/(n-r)!$ possible outcomes, so the required probability is $2N(n-r)!/n!$. In case (b), there are $n!/r!\,(n-r)!$ outcomes, but we need only count increasing arithmetic progressions, since we have a rearrangement. Hence the required probability is $Nr!\,(n-r)!/n!$.

Solution 2 by Gregory Economides (Form V, Royal Grammar School, Newcastle upon Tyne)

The number of increasing arithmetic progressions with a common difference $d > 0$ which can be formed is $n-d(r-1)$ [with first terms $1, 2, \ldots, n-d(r-1)$]. Hence the total number of arithmetic progressions which can be formed, increasing and decreasing, is

$$2\sum_{d=1}^{\left[\frac{n}{r-1}\right]} \{n-d(r-1)\} = \left[\frac{n}{r-1}\right]\left\{2n-\left(\left[\frac{n}{r-1}\right]+1\right)(r-1)\right\}.$$

Hence the probability in case (a) is

$$\frac{\left[\frac{n}{r-1}\right]\left\{2n-\left(\left[\frac{n}{r-1}\right]+1\right)(r-1)\right\}}{n(n-1)(n-2)\ldots(n-r+1)} = p \quad \text{(say)}.$$

In case (b), the probability is $\frac{1}{2}r!\,p$.

Also solved by Jean Corriveau (McMaster University, Canada) and Amites Sarkar (aged 13, Winchester College).

19.3 Let $0 < x \leqslant \pi$. Show that

$$\frac{2+\cos x}{3} \leqslant \frac{2(1-\cos x)}{x^2}.$$

Solution by K. W. Lau (Hong Kong)

The required inequality is equivalent to the inequality $f(x) \leqslant 0$, where

$$f(x) = (x^2+6)\cos x + 2x^2 - 6 \quad (0 < x \leqslant \pi).$$

We break the interval $(0, \pi]$ into three subintervals, $(0, \sqrt{6}]$, $(\sqrt{6}, 2.9]$ and $(2.9, \pi]$. For $x \in (0, \sqrt{6}]$, we have

$$\begin{aligned} f(x) &\leqslant (x^2+6)(1-\tfrac{1}{2}x^2+\tfrac{1}{24}x^4)+2x^2-6 \\ &= \tfrac{1}{24}x^4(x^2-6) \\ &\leqslant 0. \end{aligned}$$

Now

$$f(x) = (2+\cos x)x^2 + 6(\cos x - 1).$$

Hence, for $x \in (\sqrt{6}, 2.9]$, we have

$$f(x) < (2+\cos\sqrt{6})(2.9)^2 + 6(\cos\sqrt{6}-1) < 0.$$

Similarly, for $x \in (2.9, \pi]$, we have

$$f(x) < (2+\cos 2.9)\pi^2 + 6(\cos 2.9 - 1) < 0.$$

Also solved by Adrian Hill (Trinity College, Cambridge) and Gregory Economides (Form V, Royal Grammar School, Newcastle upon Tyne).

Book Reviews

Results and Problems in Combinatorial Geometry. By V. BOLTJANSKY and I. GOHBERG. Cambridge University Press, 1985. Pp. 108. Paperback £4.95.

This book first appeared in Russian twenty years ago. It concludes with seventeen unsolved problems, which have *remained* unsolved over that span of time. Yet the remarkable thing is that the entire book is easily accessible to sixth formers and undergraduates, even though it is an exposition of a range of current research problems from one very fascinating corner of modern mathematics. Complete and elementary proofs are given to the theorems stated, so that the beginning mathematician can feel he is on equal terms with researchers in the field—a rare achievement in books on the frontiers of mathematics!

To give us some idea of the sort of material covered, I mention two problems. Given a finite region in two- or three-dimensional space, the diameter of the region is the maximum distance between any two of its points. Now a disc cannot be chopped into two pieces, each of *smaller* diameter than the disc, but any finite plane can be chopped into three pieces, each of smaller diameter than that region. Similarly, a sphere cannot be chopped into three chunks, each of smaller diameter than the original sphere, yet any finite three-dimensional body can be chopped into four chunks of smaller diameter. Try it! It isn't as easy as it sounds! The solution to the obvious problem in four dimensions is unknown, though five pieces would appear to be sufficient.

As an example of the second sort of problem, a *reduced copy* of a region is a similar, smaller-diameter copy of it, with the same orientation, situated anywhere in the plane or in space. Any finite convex plane region can be completely covered by some collection of four reduced copies of that region. However it is not known whether eight reduced copies of it would be sufficient to cover any finite convex three-dimensional body.

A fascinating book, highly recommended to all mathematicians.

University of Sheffield — MIKE PIFF

Enterprising Mathematics, Pp. ix + 181, **Motivating A-level Mathematics,** Pp. ix + 130, **Realistic Applications in Mechanics,** Pp. viii + 76. By THE SPODE GROUP. Oxford University Press, 1986. Each £12·50.

Earlier books from the Spode Group, using mathematics at the standard of CSE and O-level, were reviewed in *Mathematical Spectrum* Vol. 16 No. 1. These three books continue in the same tradition of trying to open up real (or at least realistic) situations to mathematical modelling and analysis. They are written more for teachers than for pupils, though pupils could use them since they do describe situations fully and give suggested solutions to clear questions. The high price of the books reflects the fact that teachers are given permission to photocopy the text.

Enterprising Mathematics is aimed at less-able students in the 16+ age range. My impression is that the questions are less open and more contrived than in the previous books. There is more emphasis on arithmetic than before. For example, the first question (on football matches and attendance) is 'If the average admission charge at Liverpool is £2, how much did Liverpool earn for their five home games'. Now this is not a real question. In practice the total amount of gate money and the admission figures are known and it is only from these that the average would be calculated—and it would be most unlikely to be exactly £2. Nevertheless the book opens up the mathematics in many different areas—sport, travel, parties, money, home, house design, buying and various odds and ends—to make a useful source book to teachers who themselves could add realism to the work presented here.

Motivating A-Level Mathematics takes a number of illustrations to lead as introductions to and motivation for different techniques that occur in A-level syllabuses. The illustrations are drawn from many contexts such as gravitational force, seats in parliament, tennis, population, drugs, art forgeries, cars, advertisements and stock control. Some are fairly standard, all are brief. They are used to introduce many of the basic ideas in the pure, mechanics and statistics sections of A-level mathematics syllabuses.

Realistic Applications in Mechanics is the book most similar to the earlier books from the Spode Group. It looks in more detail, using A-level standard mathematics, at applications in mechanics. There is, for example, a fairly detailed analysis of different techniques for the high jump which concludes with an estimate of the efficiency of the Fosbury Flop in comparison with other techniques. Other examples are from rowing, downhill skiing, shot putting, golfing, ice-skating, long-jumping, speed bumps on roads, emergency stops, car braking systems, overtaking, collisions, rockets, satellites, guns, waterwheels and raindrops. The initial assumptions in the modelling are spelled out clearly and then there is a detailed analysis. All that the reader has to do is follow the analysis—the teacher might like to leave the situation more open and give less guidance to the solution.

Centre for Statistical Education, Sheffield PETER HOLMES

ALPHA—A Mathematics Handbook. By LENNART RÅDE. Studentlitteratur and Chartwell-Bratt Ltd. 1984. Pp. 199. £4·95.

This book contains useful formulae, tables (including statistical tables, but the usual logarithmic and trigonometric tables are intentionally omitted), flowcharts for programs, an outline of BASIC, sketches of curves and drawings of three-dimensional objects, a glossary of mathematical terms and some potted biographies of famous mathematicians. It is attractively produced and should prove useful to anyone from GCSE upwards.

A quick scan through revealed only one error, but that a serious one: in the section on partial fractions (p. 20) the words 'numerator' and 'denominator' should be interchanged throughout. It would also have been helpful to warn readers, on page 99, that a sum variable needs to be initialized to zero.

Royal Holloway and Bedford New College H. J. GODWIN

London Mathematical Society

An Evening of Popular Lectures

For the past five years the London Mathematical Society has held an evening of popular lectures. These lectures, in which prominent mathematicians speak on topics of current interest in a way that is accessible to a wide audience of teachers, sixth formers and people with a general interest in mathematics, have been a great success.

The 1987 lectures will be given by Professor F.C. Piper, on **Crypt and Ciphers** and Dr W.A. Hodges, on **Games that solve Problems**. The lectures will be at the University of Leeds on Friday 26 June at 7.30 pm and at Imperial College, London, on 3 July at 7.30 pm.

Admission is free, by ticket obtainable in advance. For the London lecture write to Miss Oakes, London Mathematical Society, Burlington House, Piccadilly, London W1V ONL and for the Leeds lecture write to Miss Backhouse, School of Mathematics, University of Leeds, Leeds LS2 9JT. A stamped addressed envelope would be much appreciated.

TV Maths

a series of programmes on BBC2 television on Saturday mornings at 8.55, linked to the Open University course MA290

Programmes 1 to 3 were on Greek mathematics, the development of mathematics in the Middle Ages and the birth of modern geometry. The remaining programmes in the series are outlined below.

4. **The Royal Society** 6 June

At the start of the seventeenth century Sir Francis Bacon proposed a new philosophy of experimental science. The adoption of this philosophy resulted in a revolution in scientific methodology, and the foundation of the Royal Society. Filmed at the Royal Society and the Royal Observatory, Greenwich, this programme shows how the activities of the Society led to mathematics taking its place as a tool of all the sciences. A new era of British mathematical creativity was born.

5. **Newton and Leibniz: The Invention of Calculus** 4 July

By the late seventeenth century, mathematicians and scientists had gained respectability. As a result of this respectability, the collected papers and notebooks of the famous of the time were kept and preserved. Thus we can trace the thinking processes of Newton and Leibniz as they invented the 'calculus' by studying their original hand-written notes. Jeremy Gray of the Open University visits Cambridge University Library and the Archiv Leibniz in Hanover.

6. **A New Geometry of Space** 1 August

For centuries mathematicians had thought that the geometry of Euclid's **Elements** was the only plausible model of physical space. The work of Bolyai and Lobachevski around 1830 was to completely overturn this theory and lead to a total reappraisal of what is meant by the word 'geometry'. Jeremy Gray, using studio models, illustrates the non-euclidean geometries.

7. **Paris and the New Mathematics** 29 August

Amidst the fervour of the French revolution, its greatest scientists, engineers and mathematicians came together to form the Ecole Polytechnique. Out of this educational enterprise came the spread of the mathematical text-book and the adoption of new techniques which were to change the course of mathematics in the nineteenth century. Filmed in Paris at the Ecole Polytechnique, John Fauvel of the Open University traces these developments.

8. **Hamilton and Boole: The Freeing of Algebra** 26 September

Applications as diverse as computer programming and space flights to the Moon owe their foundation to Hamilton in Dublin and Boole in Cork at the end of the last century. Until these two great mathematicians came on the scene, the subject of algebra was a mere offshoot of arithmetic. Hamilton and Boole introduced a degree of abstraction to the subject which led to mathematical systems that could solve hitherto insoluble problems. Graham Flegg visits Dublin and Cork to follow the process of their mathematical thinking.

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