

Mathematicorum

Crux

Published by the Canadian Mathematical Society.



<http://crux.math.ca/>

The Back Files

The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Volume 16 #3

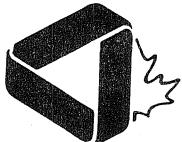
March / Mars

1990

CONTENTS / TABLE DES MATIÈRES

<i>The Ochoa curve</i>	<i>Richard K. Guy</i>	65
<i>The Olympiad Corner: No. 113</i>	<i>R.E. Woodrow</i>	69
<i>Problems 1521-1530</i>		74
<i>Solutions 1258, 1294, 1317, 1321, 1355, 1403-1413</i>		76
<i>Editorship of Crux Mathematicorum</i>		

Canadian Mathematical Society



Société mathématique du Canada

Founding Editors: Léopold Sauvé, Frederick G.B. Maskell
Editor: G.W. Sands
Managing Editor: G.P. Wright

GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

G.W. Sands
Department of Mathematics & Statistics
University of Calgary
Calgary, Alberta
Canada, T2N 1N4

SUBSCRIPTION INFORMATION

Crux is published monthly (except July and August). The 1989 subscription rate for ten issues is \$17.50 for members of the Canadian Mathematical Society and \$35.00 for non-members. Back issues: \$3.50 each. Bound volumes with index: volumes 1 & 2 (combined) and each of volumes 3, 7, 8, 9 and 10: \$10.00. (Volumes 4, 5 & 6 are out-of-print). All prices quoted are in Canadian dollars. Cheques and money orders, payable to the CANADIAN MATHEMATICAL SOCIETY, should be sent to the Managing Editor:

Graham P. Wright
Canadian Mathematical Society
577 King Edward
Ottawa, Ontario
Canada K1N 6N5

ACKNOWLEDGEMENT

The support of the Departments of Mathematics and Statistics of the University of Calgary and Carleton University, and of the Department of Mathematics of the University of Ottawa, is gratefully acknowledged.

© Canadian Mathematical Society, 1990

Published by the Canadian Mathematical Society
Printed at Carleton University

THE OCHOA CURVE

Richard K. Guy

In *Crux* [1987: 248] appears a problem that was "proposed but not used" at the 28th I.M.O. in Havana:

Spain 1. *Determine, with justification, the integer solutions of the equation*

$$3z^2 = 2x^3 + 385x^2 + 256x - 58195. \quad (1)$$

In *Crux* [1989: 41] a partial solution was given by Gillian Nonay and Edward T.H. Wang: twelve pairs of points found by computer, and some modular arithmetic which showed that the computer was not really necessary.

I was intrigued how such a problem came to be asked. Correspondence elicited a document with the original proposal:

"1. Hallar, razonadamente, soluciones de la ecuación diofántica..."

Passage through French and English*, through an informant and an editor, imported a definite article and a small but significant semantic change, so that, even if the original was a suitable proposal, the printed version certainly was not.

First, we give the proposer's solution, from which we get a glimpse of why the question might have been suggested for an Olympiad: "completing the square" is a fundamental idea, which should occur to all contestants.

Multiply by 3 and put $3z = y$:

$$y^2 = 6x^3 + 1155x^2 + 768x - 174585.$$

Notice that the coefficients of the odd powers of x are 2·3 and 2·3·128, so that these terms appear when we square $y = x^2 + 3x + 128$, and, on substitution, cancel out, leaving the quadratic in x^2 :

$$x^4 - 890x^2 + 190969 = 0$$

$$(x^2 - 19^2)(x^2 - 23^2) = 0$$

leading to some (namely, all that was originally asked for) solutions $(x, \pm z)$ where (x, z) equals

5. $(-23, 196)$ 6. $(-19, 144)$ 7. $(19, 182)$ 8. $(23, 242)$

and bold labels have been given for future reference. By way of attribution I quote the paper [1].

Had this problem actually confronted an I.M.O. contestant, how might she have attacked it? The roots of the cubic are approximately -191.03, -13.10 and

*Traduttore, traditore, as my Spanish correspondent observes.

11.63, so a search only needs to be made with $-191 \leq x \leq -14$ and $12 \leq z$. As observed by Nonay & Wang, x must be odd, and is not a multiple of 3. Indeed, calculations modulo 9 show that $x \equiv \pm 1, -2$ or ± 4 , leaving only 5 possibilities to be examined out of each 18.

Similarly, it can be shown that $x \not\equiv \pm 1 \pmod{7}$, that $x \not\equiv 0, \pm 2$ or $5 \pmod{11}$ and $x \not\equiv 0, 2$ or $\pm 5 \pmod{13}$. Without having to leave the realm of the multiplication tables that we used to learn at school, we are soon left with only the possibilities

$$\begin{array}{l} \text{1 } x = -191, -157, -89, -67, -49, -23, -19, 19, 23, 61, 91, 95, 103, 149, \dots ? \end{array}$$

To test these, surely we need a calculator? Or not?

$$\begin{aligned} x = -191? (\times 2) - 382 (+ 385) 3 (\times -191) - 573 (+ 256) - 317 \\ (\times -191) 63400 - 3170 + 317 = 60547 (- 58195) 2352 (\div 3) 784 (\sqrt) 28. \end{aligned}$$

In this way, we find that all the bold entries, including the proposer's $x = \pm 19, \pm 23$, give solutions:

$$\begin{array}{lll} \mathbf{1} (-191, 28) & \mathbf{2} (-157, 742) & \mathbf{3} (-67, 592) \\ \mathbf{4} (-49, 454) & \mathbf{9} (61, 784) & \mathbf{10} (103, 1442) \end{array}$$

together with the corresponding solutions with $z < 0$, which we label $\bar{1}, \bar{2}, \dots, \bar{10}$.

Could our contestant find more? The fundamental fact about elliptic curves, such as we have here, is that the straight line joining two points with rational coordinates meets the curve in a third point with rational coordinates. And, if we draw the tangent at a rational point, it meets the curve again at a rational point. For example, the slope of the present curve, by elementary calculus, is

$$\frac{3x^2 + 385x + 128}{3z} = \frac{(3x + 1)(x + 128)}{3z},$$

which, at $1 (-191, 28)$ is $(-572)(-63)/(3 \times 28) = 429$ and the equation to the tangent is $z - 28 = 429(x + 191)$. If we substitute $429x + 81967$ for z in the original equation, we have

$$3(429x + 81967)^2 = 2x^3 + \dots - 58195,$$

which we know has to simplify to

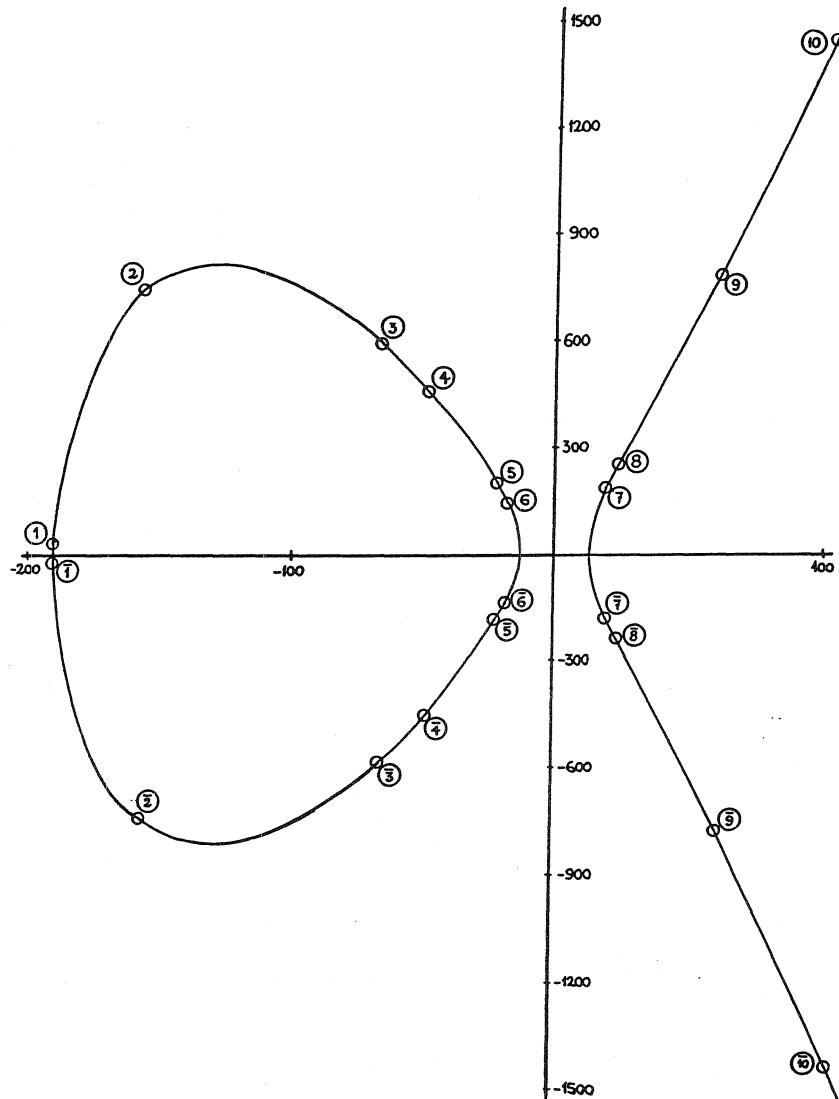
$$2(x + 191)^2(x - t) = 0$$

where $t = (3 \times 81967^2 + 58195)/2 \times 191^2$ has got to be a whole number. This check enables us to find $t = 276251$, even though our calculator (and I claim that this is the first time we really need one) may not be able to handle all of the eleven digits that arise on the way. So we've found

$$15. (276251, 118593646)!$$

Now join $\bar{7}$ to 9 to find $11 (521, 11364)$,
 1 to 2 to find $12 (817, 21196)$,
 $\bar{9}$ to 10 to find $13 (3857, 200404)$,

and $\bar{5}$ to 6 to find 14 (10687, 910154),
the first two of which were already found by Nonay & Wang.



The figure shows the first ten pairs of integer points. Have we found them all? We believe so, but this is a very hard question. By investigating other collinearities you can find other pairs of rational points (but evidently no more integer ones):

$$H \left(\frac{33239}{2}, \frac{1759471}{2} \right)$$

$$V \left(-\frac{277}{2}, 812 \right)$$

$$X \left(\frac{107}{3}, \frac{3788}{9} \right)$$

$$T \left(\frac{35}{3}, \frac{100}{9} \right)$$

$$S \left(-\frac{565}{3}, \frac{2260}{9} \right)$$

$$P \left(-\frac{79}{3}, \frac{2114}{9} \right)$$

$$Y \left(\frac{1477}{6}, \frac{37889}{9} \right)$$

$$U \left(\frac{145}{6}, \frac{2330}{9} \right)$$

$$W \left(-\frac{1465}{8}, \frac{6685}{16} \right)$$

$$R \left(-\frac{1267}{27}, \frac{106006}{243} \right)$$

If we make the transformation $X = 6x + 385$, $Y = 18z$, then the first eight of these pairs become integer points on the new curve

$$Y^2 = X^3 - 3 \cdot 383^2 X + 2 \cdot 5 \cdot 73 \cdot 145307 . \quad (2)$$

The collinearities include

1.2.12	$\bar{4}.5.12$	$\bar{7}.8.H$	2.3.X	7.12. \bar{T}	4. $\bar{8}.P$	3.6. \bar{U}	$\bar{3}.13.R$	5.X. \bar{W}
$\bar{1}.3.10$	5.6. $\bar{1}\bar{0}$	$\bar{1}\bar{1}.12.H$	4.6. \bar{X}	8.11. \bar{T}	6.13. \bar{P}	$\bar{7}.14.U$	$\bar{4}.15.R$	V.Y.W
1.4.9	$\bar{5}.6.14$	$\bar{1}\bar{4}.15.H$	7.9.X	9.10.T	3.6.Y	$\bar{8}.15.U$	6.11. \bar{R}	X. $\bar{T}.Y$
1.5.8	7.8.10	$\bar{1}.11.V$	$\bar{8}.13.X$	1.14.S	$\bar{7}.10.Y$	1.13.W	$\bar{1}.H.S$	X. $\bar{S}.P$
$\bar{1}.6.7$	$\bar{7}.9.11$	$\bar{2}.14.V$	10.12. \bar{X}	2.11.S	11.13. \bar{Y}	4.10. \bar{W}	5.V.T	$\bar{U}.R.P$
$\bar{2}.5.9$	$\bar{8}.9.12$	4.7.V	2.6.T	5.7.S	$\bar{1}.5.U$	$\bar{1}.9.R$	12.V. \bar{S}	
3.5. $\bar{7}$	$\bar{9}.10.13$	6.9. \bar{V}	3.4. \bar{T}	6.8. \bar{S}	2.4.U	2.8.R	10. $\bar{V}.P$	

- a configuration that might have interested Sylvester: 25 pairs of points, and the point at infinity, making 51 in all; 61 lines, their reflexions in the x -axis (swap bars over the labels), and 25 lines through the point at infinity, 147 in all, each with 3 points. All this linear dependence can be exhibited by expressing each point in terms of four suitably selected generators, say 1, 2, 3, 4:

1	1	0	0	0	6	0	-1	-1	-1	11	2	1	1	2	H	3	2	1	2	P	-2	-1	0	-2
2	0	1	0	0	7	1	1	1	1	12	-1	-1	0	0	V	-1	-1	-1	-2	Y	0	1	2	1
3	0	0	1	0	8	-2	-1	0	-1	13	-2	0	1	-1	X	0	-1	-1	0	U	0	-1	0	-1
4	0	0	0	1	9	-1	0	0	-1	14	1	2	1	2	T	0	0	1	1	W	1	0	-1	1
5	1	1	0	1	10	1	0	-1	0	15	-2	0	0	0	S	-2	-2	-1	-2	R	2	0	0	1

I.e. the rank of the curve is 4, and the points 1, 2, 3, 4, will serve as generators, although to establish this rigorously would require "height" calculations.

To return to the problem of finding *all* integer solutions, there's a theorem of Siegel which states that if $f(x)$ is an integer polynomial of degree at least 3 with distinct roots, then the equation $y^2 = f(x)$ has only a finite number of integer solutions. The condition of distinctness of roots is essential. For example, line $y = mx$, through the singular point $(0,0)$, meets the curve $y^2 = x^2(x+5)$ in the point $(m^2-5, m(m^2-5))$. If m is rational, so is the point. If m is an integer, the point has integer coordinates and there are infinitely many of them. Notice also that even in the non-singular case, the number of integer points is unbounded. We have seen that (1) contains (at least) 15 pairs of integer points, and that (2) contains 23 such. This number can be further increased, at the expense of the size of the coefficients, by further scaling. If we multiply equation (2) by $(n!)^6$ and put $\mathcal{X} = (n!)^2 X$, $\mathcal{Y} = (n!)^3 Y$, then the result will contain, as integer points, all points

corresponding to rational points on (2) where the denominators of X were divisors of $(n!)^2$.

Siegel's theorem is not effective. A large step forward was Alan Baker's work on linear forms of logarithms, but the bounds that this gives are far beyond computer reach. However, new algorithmic techniques now bring the problem within reach, at least for curves whose discriminant is less than a million, say. A remarkable paper appeared earlier this year [2]. Unfortunately, the curve (2) has discriminant $-4(-3 \cdot 383^2)^3 - 27(2 \cdot 5 \cdot 73 \cdot 145307)^2 = 2^{10} 3^7 7^2 79 \cdot 311 \cdot 13759$, which may be too large and too composite, even for the new methods. On the other hand, recent correspondence with de Weger suggests that even in this case the situation is far from hopeless, and their methods may extend even to discriminants of this size.

Ochoa's construction is evidently a particular case of a construction described by Barry Mazur as having been used by Néron in his thesis. Nine points in the projective plane determine a cubic curve, and if these are chosen sufficiently independently, then a curve of rank 9 (or even a little higher, but not much!) is produced.

References:

- [1] Juan Ochoa Melida, La ecuacion diofántica $b_0y^3 - b_1y^2 + b_2y - b_3 = z^2$, *Gaceta Matematica*, 1978, 139–141.
- [2] N. Tzanakis & B.M.M. de Weger, On the practical solution of the Thue equation, *J. Number Theory* 31 (1989) 99–132.

University of Calgary
Calgary, Alberta, Canada

THE OLYMPIAD CORNER
No. 113
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

This month's column will appear short to readers. This is partly to allow space for the article by Professor R.K. Guy which had its genesis in one of the

problems published here in the Corner, but which does not appear as a solution in the column as such.

We begin with the remaining questions from the XIV "All Union" Mathematical Olympiad of the U.S.S.R. Thanks again go to Willie Yong of Singapore who sent in these problems which were translated by Mark Saul. The first ten problems were given last month.

XIV "ALL UNION" MATHEMATICAL OLYMPIAD (U.S.S.R.)

11. Let us denote the sum of the digits of the natural number n by $s(n)$.

(a) Does there exist a natural number n such that $n + s(n) = 1980$?

(b) Show that at least one of any two consecutive natural numbers can be expressed as $n + s(n)$ for some third natural number n .

12. Some of the squares on an infinite piece of (Cartesian) graph paper are coloured red, and the remainder are coloured blue. This is done in such a way that any 2×3 rectangle made up of six squares contains exactly two red squares. How many red squares may be included in a 9×11 rectangle made up of 99 squares?

13. An epidemic hit the land of the Munchkins. One day several Munchkins fell ill. Any healthy Munchkin who was not immunized and who visited a sick Munchkin fell ill on the day after the visit. Each ill Munchkin was sick for exactly one day, and after recovery, was immune from further infection for at least one day; on these days (s)he was healthy and could not fall ill (each Munchkin might have this immunization for a different length of time). Despite the epidemic, each healthy Munchkin visited his sick friends every day. As soon as the epidemic began, the Munchkins forgot about any possible vaccines and did not use them. Show that

(a) If some of the Munchkins were vaccinated *before* the epidemic, and were immune on the first day, then the epidemic could continue for an arbitrarily long time.

(b) If no Munchkin was immune on the first day, the epidemic must eventually end.

14. Let us denote by $p(n)$ the product of the decimal digits of the number n . The sequence $\langle n_k \rangle$ is defined recursively by $n_{k+1} = n_k + p(n_k)$, and by the choice of any natural number as n_1 . Can the sequence $\langle n_k \rangle$ be

unbounded?

15. A line parallel to side AC of equilateral triangle ABC intersects AB and BC in points M and P , respectively. Point D is the center (of symmetry) of triangle PMB , and E is the midpoint of segment AP . Find the measures of the angles of triangle DEC .

16. The lengths of the edges of a rectangular parallelepiped are x , y and z units, where $x < y < z$. If

$$p = 4(x + y + z), \\ s = 2(xy + yz + zx),$$

and

$$d = \sqrt{x^2 + y^2 + z^2}$$

respectively are the perimeter, surface area, and diagonal of the parallelepiped, show that

$$x < \frac{1}{3}\left(\frac{p}{4} - \sqrt{d^2 - s/2}\right) \quad \text{and} \quad z > \frac{1}{3}\left(\frac{p}{4} + \sqrt{d^2 - s/2}\right).$$

17. The set M consists of integers, the smallest of which is 1 and the largest 100. Each element of M , except for 1, is equal to the sum of two (possibly identical) numbers in M . Of all such sets, find one with the smallest possible number of elements.

18. Show that there are infinitely many numbers a for which the equation
- $$[x^{3/2}] + [y^{3/2}] = a$$

has at least 1980 solutions in natural numbers x , y . Here $[z]$ stands for the integer part of z .

19. In tetrahedron $ABCD$, $AC \perp BC$ and $AD \perp BD$. Show that the cosine of the angle between lines AC and BD is less than CD/AB .

20. The number x , $0 \leq x < 1$, is written as an infinite decimal. If we permute at random the first five digits to the right of the decimal point, we will obtain a new infinite decimal, corresponding to some new number x_1 . If we then permute the second through the sixth digit in the decimal representation of x_1 , we obtain a decimal representation of some new number x_2 . In general, we can start with a number x_k and obtain x_{k+1} by permuting the digits in places $(k+1)$ through $(k+5)$ after the decimal point.

(a) Show that no matter how the digits are permuted at each step, the sequence $\langle x_n \rangle$ will always have a limit. Let us call that limit y .

(b) Can one start with a *rational* x and by the process described above obtain an *irrational* y ?

(c) Find a number x such that starting with x , the above process always produces an irrational y .

*

*

*

The two solutions we include this month are those received for problems published in the May 1988 number of the Corner.

2. [1988: 131] *Final round of the 23rd Spanish Mathematical Olympiad.*

Show that for each natural number $n > 1$

$$1 \cdot \sqrt{\binom{n}{1}} + 2 \cdot \sqrt{\binom{n}{2}} + \cdots + n \cdot \sqrt{\binom{n}{n}} < \sqrt{2^{n-1} n^3}.$$

(Here $\binom{n}{k}$ is, of course, the binomial coefficient.)

Solutions by Bob Prielipp, University of Wisconsin-Oshkosh; by M.A. Selby, Department of Mathematics and Statistics, The University of Windsor, Ontario; and by Edward T.H. Wang, Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario.

When $n = 2$ the given inequality is equivalent to $\sqrt{2} < 2$ and when $n = 3$ it is equivalent to $1 < \sqrt{3}$. Hence in the remainder of the solution we assume that $n > 3$.

By Cauchy's inequality the left side of the given inequality is less than or equal to

$$\begin{aligned} (1^2 + 2^2 + \cdots + n^2)^{1/2} &\left[\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \right]^{1/2} \\ &= \left[\frac{n(n+1)(2n+1)}{6} \right]^{1/2} (2^n - 1)^{1/2}. \end{aligned}$$

Thus to complete the solution it suffices to establish that

$$\frac{n(n+1)(2n+1)}{6} (2^n - 1) < 2^{n-1} n^3 \quad (\text{for } n > 3). \quad (*)$$

But $(*)$ is equivalent to

$$(2n^2 + 3n + 1)(2^n - 1) < 3n^2 2^n.$$

Since $n > 3$, $n^2 > 3n$, so $n^2 \geq 3n + 1$. It follows that $3n^2 \geq 2n^2 + 3n + 1$. Because $2n^2 + 3n + 1 \leq 3n^2$ and $2^n - 1 < 2^n$, $(*)$ holds, completing the solution.

6. [1988: 132] *Final Round of the 23rd Spanish Mathematical Olympiad.*

For all natural numbers n , define the polynomial

$$P_n(x) = x^{n+2} - 2x + 1.$$

(a) Show that the equation $P_n(x) = 0$ has one and only one root c_n in the open interval $(0,1)$.

(b) Find $\lim_{n \rightarrow \infty} c_n$.

Solution by David Vaughan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

(a) Since $P'_n(x) = (n+2)x^{n+1} - 2$, the only critical value of $P_n(x)$ on $(0,1)$ is $k_n = [2/(n+2)]^{1/n+1}$. Furthermore, $P_n(k_n)$ is the absolute minimum of $P_n(x)$ on $[0,1]$ since $P_n(x)$ is decreasing on $(0, k_n)$ and increasing on $(k_n, 1)$. But $P_n(1) = 0$ and thus we must have $P_n(k_n) < 0$. Since $P_n(0) = 1$, the Intermediate Value Theorem implies that there exists $c_n \in (0, k_n)$ such that $P_n(c_n) = 0$. That this c_n is unique follows from the above analysis regarding the increasing and decreasing of $P_n(x)$ on $[0,1]$.

(b) Let α be arbitrary such that $0 < \alpha < 1$. Then

$$\lim_{n \rightarrow \infty} P_n\left(\frac{1+\alpha}{2}\right) = \lim_{n \rightarrow \infty} \left(\frac{1+\alpha}{2}\right)^{n+2} - \alpha = -\alpha < 0.$$

Since $P_n(1/2) = (1/2)^{n+2} > 0$ it must be true that $1/2 < c_n < (1+\alpha)/2$ for all sufficiently large n . By letting $\alpha \rightarrow 0^+$ we conclude that $\lim_{n \rightarrow \infty} c_n = 1/2$.

Remark: Using exactly the same argument one can show that for any positive integer $k \geq 2$ the polynomial $P_n(x) = x^{n+k} - kx + (k-1)$ has a unique root c_n in $(0,1)$ and $\lim_{n \rightarrow \infty} c_n = 1/k$.

[Editor's note. A slightly different solution was submitted by M.A. Selby, Department of Mathematics and Statistics, The University of Windsor, Ontario. Having established that P_n is decreasing on $(0, t_n)$, with $P_n(t_n) < 0$, where t_n is its critical point, he notes that $P_{n+1}(x) \leq P_n(x)$ gives that $P_{n+1}(c_{n+1}) = 0 \leq P_n(c_{n+1})$. Since $P_n(c_n) = 0$ this means that $c_{n+1} \leq c_n$. As $0 < c_n$ for all n , $\lim_{n \rightarrow \infty} c_n = L$

exists. Also, since $c_n \leq c_1 < 1$, $c_n^{n+2} \rightarrow 0$ as $n \rightarrow \infty$ so

$$\lim_{n \rightarrow \infty} P_n(c_n) = 0 = \lim_{n \rightarrow \infty} (c_n^{n+2} - 2c_n + 1) = -2L + 1,$$

and therefore $L = 1/2$.]

*

*

*

Please send me your contests and nice solutions.

*

*

*

PROBLEMS

Problem proposals and solutions should be sent to the editor, B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before October 1, 1990, although solutions received after that date will also be considered until the time when a solution is published.

1521. Proposed by J.T. Groenman, Arnhem, The Netherlands.

Triangle ABC has angles α, β, γ , circumcenter O , incenter I , and orthocenter H . Suppose that the points A, H, I, O, B are concyclic.

- (a) Find γ .
- (b) Prove $HI = IO$.
- (c) If $AH = HI$, find α and β .

1522. Proposed by M.S. Klamkin, University of Alberta.

Show that if $a, b, c, d, x, y > 0$ and

$$xy = ac + bd, \quad \frac{x}{y} = \frac{ad + bc}{ab + cd},$$

then

$$\frac{abx}{a + b + x} + \frac{cdx}{c + d + x} = \frac{ady}{a + d + y} + \frac{bcy}{b + c + y}.$$

1523. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let $0 < t \leq 1/2$ be fixed. Show that

$$\sum \cos tA \geq 2 + \sqrt{2} \cos(t + 1/4)\pi + \sum \sin tA,$$

where the sums are cyclic over the angles A, B, C of a triangle. [This generalizes Murray Klamkin's problem E3180 in the Amer. Math. Monthly (solution p. 771, October 1988).]

1524. Proposed by George Tsintsifas, Thessaloniki, Greece.

ABC is a triangle with sides a, b, c and area F , and P is an interior point. Put $R_1 = AP, R_2 = BP, R_3 = CP$. Prove that the triangle with sides aR_1, bR_2, cR_3 has circumradius at least $4F/(3\sqrt{3})$.

1525. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Let m, n be given positive integers and d be their greatest common divisor. Let $x = 2^m - 1, y = 2^n + 1$.

- (a) If m/d is odd, prove that x and y are coprime.
- (b) Determine the greatest common divisor of x and y when m/d is even.

1526. *Proposed by Sydney Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let n and q denote positive integers. The identity

$$\sum_{k=1}^n k \binom{n}{k} q^{n-k} = n(q+1)^{n-1}$$

can be proved easily from the Binomial Theorem. Establish this identity by a combinatorial argument.

1527. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

In quadrilateral $ABCD$ the midpoints of AB, BC, CD and DA are P, Q, R and S respectively. T is the intersection point of AC and BD , M that of PR and QS . G is the centre of gravity of $ABCD$. Show that T, M and G are collinear, and that $\overline{TM}:\overline{MG} = 3:1$.

1528^{*} *Proposed by Ji Chen, Ningbo University, China.*

If a, b, c, d are positive real numbers such that $a + b + c + d = 2$, prove or disprove that

$$\frac{a^2}{(a^2 + 1)^2} + \frac{b^2}{(b^2 + 1)^2} + \frac{c^2}{(c^2 + 1)^2} + \frac{d^2}{(d^2 + 1)^2} \leq \frac{16}{25}.$$

1529. *Proposed by Jordi Dou, Barcelona, Spain.*

Given points A, B, C on line l and A', B', C' on line l' , construct the points $P \neq l \cap l'$ such that the three angles APA', BPB', CPC' have the same pair of bisectors.

1530^{*} *Proposed by D.S. Mitrinović, University of Belgrade, and J.E. Pečarić, University of Zagreb.*

Let

$$I_k = \frac{\int_0^{\pi/2} \sin^{2k} x \, dx}{\int_0^{\pi/2} \sin^{2k+1} x \, dx}$$

where k is a natural number. Prove that

$$1 \leq I_k \leq 1 + \frac{1}{2k}.$$

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1258. [1987: 180; 1988: 220] *Proposed by Ian Witten, University of Calgary, Alberta.*

Think of a picture as an $m \times n$ matrix A of real numbers between 0 and 1 inclusive, where a_{ij} represents the brightness of the picture at the point (i,j) . To reproduce the picture on a computer we wish to approximate it by an $m \times n$ matrix B of 0's and 1's, such that every "part" of the original picture is "close" to the corresponding part of the reproduction. These are the ideas behind the following definitions.

A *subrectangle* of an $m \times n$ grid is a set of positions of the form

$$\{(i,j) \mid r_1 \leq i \leq r_2, s_1 \leq j \leq s_2\}$$

where $1 \leq r_1 \leq r_2 \leq m$ and $1 \leq s_1 \leq s_2 \leq n$ are constants. For any subrectangle R , let

$$d(R) = \left| \sum_{(i,j) \in R} (a_{ij} - b_{ij}) \right|,$$

where A and B are as given above, and define

$$d(A,B) = \max d(R),$$

the maximum taken over all subrectangles R .

(a) Show that there exist matrices A such that $d(A,B) > 1$ for every 0-1 matrix B of the same size.

(b) Is there a constant c such that for every matrix A of any size, there is some 0-1 matrix B of the same size such that $d(A,B) < c$?

III. *Solution by Vojtech Rodl and Peter Winkler, Emory University.*

It turns out that this problem (in particular for constant matrices, i.e., matrices all of whose entries are equal) is closely related to the study of irregularities of distributions of points in Euclidean space. This subject is about fifty years old and has seen many deep results by such figures as Klaus Roth and Wolfgang Schmidt; an excellent source is the recent book [1] of Beck and Chen.

Here we will show using a theorem of Gábor Halász [3] that the answer to part (b) is NO, even if the matrices A are restricted to be square and constant, and the error is measured using only square subrectangles.

Let P be a collection of N points in the unit square $[0,1]^2$, and let

$$S = [x, x + s] \times [y, y + s]$$

be a subsquare. Define

$$D(P, S) = |P \cap S| - Ns^2 \quad (1)$$

[this is the difference between how many points of P are in S and how many "should" be], and let $D(P)$, the *discrepancy* of the point distribution P , be the maximum of $|D(P, S)|$ over all subsquares S . Halász's Theorem (Theorem 3C, p.6 of [1]) implies that there is a constant $k > 0$ such that $D(P) > k \log N$ for any distribution P of N points in the unit square.

Let $\sigma(A)$ be the sum of the entries of a matrix A , and let $E(A)$ be the minimum of $d(A, B)$ over all B . We show that for any $k' < k/2$, and for sufficiently large $\sigma(A)$, there are square constant matrices A such that

$$E(A) > k' \log \sigma(A), \quad (2)$$

i.e. the constant c of part (b) cannot exist.

For suppose otherwise and choose M large enough so that

$$2k' \log M < k \log (M - k' \log M). \quad (3)$$

For each positive integer $n \geq \sqrt{M}$, let A_n be the $n \times n$ matrix each of whose entries is M/n^2 (so that $\sigma(A_n) = M$), and let B_n be an approximating $\{0,1\}$ -matrix which minimizes $d(A_n, B_n)$. Using the entire matrix as the subsquare, we have by our assumption that

$$\begin{aligned} |\sigma(B_n) - M| &= |\sigma(B_n) - \sigma(A_n)| \leq d(A_n, B_n) = E(A_n) \\ &\leq k' \log \sigma(A_n) = k' \log M, \end{aligned}$$

so

$$M - k' \log M \leq \sigma(B_n) \leq M + k' \log M. \quad (4)$$

Thus for some integer N in this range, infinitely many of the B_n 's have the same entry sum $\sigma(B_n) = N$, that is, they all have exactly N 1's.

For each such B_n let $(i_1, j_1), \dots, (i_N, j_N)$ be the index pairs corresponding to the positions of the 1's in B_n , in any order, and to B_n associate the ordered point distribution

$$P_n = \left\{ \left(\frac{i_1}{n}, \frac{j_1}{n} \right), \dots, \left(\frac{i_N}{n}, \frac{j_N}{n} \right) \right\}$$

in the unit square.

Regarding each P_n as a point in the $2N$ -dimensional unit hypercube $[0,1]^2$, which is compact, we can find a limit point P in $[0,1]^2$ which we can then interpret back as N points in the unit square. Thus for each $\delta > 0$ there are infinitely many values of n such that each point in P_n is within δ of the corresponding point in P .

Applying Halász's Theorem to P yields a subsquare

$$S = [x, x+s] \times [y, y+s]$$

for which

$$|D(P, S)| > k \log N. \quad (5)$$

Assuming (via a small adjustment of x and y if necessary) that no point of P falls within some fixed $\epsilon > 0$ of the boundary of S , we have for infinitely many values of n that a point of P_n is inside S if and only if the corresponding point of P is. Furthermore, by letting $\delta = \epsilon/2$ above, we may assume that for each such n , no point of P_n is within $\epsilon/2$ of the square's boundary.

For each such n which also satisfies $n > 2/\epsilon$, expand S to the (almost?) square submatrix S_n of B_n consisting of all entries whose indices (i,j) satisfy

$$[nx] \leq i \leq [n(x+s)], \quad [ny] \leq j \leq [n(y+s)].$$

Note that if a point $(i/n, j/n)$ of P_n lies in S then

$$x \leq \frac{i}{n} \leq x+s,$$

which means (since no point is within $\epsilon/2$ of the boundary of S) that

$$x + \frac{\epsilon}{2} \leq \frac{i}{n} \leq x + s - \frac{\epsilon}{2}$$

or

$$[nx] < nx + 1 \leq nx + \frac{n\epsilon}{2} \leq i \leq nx + ns - \frac{n\epsilon}{2} \leq nx + ns - 1 < [n(x+s)], \quad (6)$$

and thus (using the same calculation for j) that the (i,j) entry of B_n (which is a 1) lies in S_n .

Also note that, because of the strict inequalities in (6), S_n could be shrunk by as much as 2 in its horizontal or vertical dimension without losing any 1's and without significantly affecting its area. Thus we may assume S_n is square.

Similarly, if $(i/n, j/n)$ lies outside S then

$$\frac{i}{n} \leq x - \frac{\epsilon}{2} \quad \text{or} \quad \frac{i}{n} \geq x + s + \frac{\epsilon}{2},$$

so

$$i \leq nx - \frac{n\epsilon}{2} < [nx] \quad \text{or} \quad i \geq nx + ns + \frac{n\epsilon}{2} > [n(x+s)],$$

and we get that the 1 in the (i,j) position of B_n lies outside S_n . Hence S_n contains precisely $|P \cap S|$ 1's, so the sum of the entries of B_n in S_n is $|P \cap S|$. The sum of the entries in the corresponding subsquare of A_n is

$$([n(x+s)] - [nx])([n(y+s)] - [ny]) \cdot M/n^2$$

which approaches Ms^2 as n gets large. By (5) let

$$\epsilon' = |D(P, S)| - k \log N > 0. \quad (7)$$

Then for infinitely many n ,

$$\begin{aligned}
 E(A_n) &= d(A_n, B_n) > ||P \cap S| - Ms^2| - \epsilon' \\
 &\geq ||P \cap S| - Ns^2| - |(M - N)s^2| - \epsilon' \\
 &\geq |D(P, S)| - |M - N| - \epsilon' \quad (\text{by (1)}) \\
 &= k \log N - |M - N| \quad (\text{by (7)}) \\
 &\geq k \log(M - k' \log M) - k' \log M \quad (\text{by (4)}) \\
 &> k' \log M \quad (\text{by (3)})
 \end{aligned}$$

showing (2).

Note: As a result of recent work of Géza Bohus [2], one can show that $E(A_n)$ has an upper bound of constant $\cdot (\log n)^3$.

References:

- [1] József Beck and William Chen, *Irregularities of Distribution*, Cambridge University Press, 1987.
- [2] G. Bohus, On the discrepancy of 3 permutations, *Random Structures and Algorithms* 1 (1990) 215–220.
- [3] Gabor Halász, On Roth's method in the theory of irregularities of point distributions, *Recent Progress in Analytic Number Theory*, Vol. 2, pp. 79–94, Academic Press, London, 1981.

*

*

*

1294. [1987: 320; 1989: 17, 235] *Proposed by P. Penning, Delft, The Netherlands.*

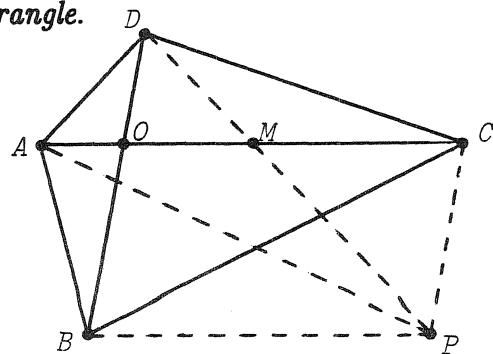
Find a necessary and sufficient condition on a convex quadrangle $ABCD$ in order that there exists a point P (in the same plane as $ABCD$) such that the areas of the triangles PAB , PBC , PCD , PDA are equal.

III. Comment by the proposer.

The second family of solutions, given on [1989: 235], can be described by the condition: *the area of one of the four triangles generated by the diagonals of the quadrangle is equal to half the area of the quadrangle.*

Assuming that the condition of the problem holds for the figure at right, we show that $[OBC] = \frac{1}{2}[ABCD]$ (where $[X]$ denotes the area of region X).

Given the three vertices A , B , C and the direction of diagonal BD , the points P and D satisfying the problem will first be constructed. Now $[ABP] = [BCP]$ implies that A and C are the same distance from



BP , so BP is parallel to AC . Similarly $[BCP] = [CDP]$ implies that PC is parallel to BD . Thus point P is found. Finally $[CDP] = [DAP]$ implies that PD must pass through the midpoint M of diagonal AC , so point D is found. Note that $[ABP] = [DAP]$ means that PA passes through the midpoint of the other diagonal BD . We may now prove that $[OBC] = \frac{1}{2}[ABCD]$ by simply counting areas and taking into account that $[BCP] = [OBC]$.

[Editor's note. It can also be shown that Penning's condition is sufficient for the quadrangle to satisfy the condition of the problem. Combined with the first published solution [1989: 17], we get quite an interesting characterization of quadrangles satisfying the problem: they are precisely those quadrangles so that either one or both of the diagonals generate a triangle of area half the area of the quadrangle.]

The same solution has also been sent in by TOSHIO SEIMIYA, Kawasaki, Japan.

*

*

*

1317. [1988: 45; 1989: 91] Proposed by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen.

Crux 1133 [1987: 225] suggests the following problem. In a triangle ABC the excircle touching side AB touches lines BC and AC at points D and E respectively. If $AD = BE$, must the triangle be isosceles?

V. Comment by Toshio Seimiya, Kawasaki, Japan.

Answering a question of the editor [1989: 96], I show that equations $R = r_c$ and $\cos C = \cos A + \cos B$ are equivalent. By the known formula

$$r_c = 4R \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2},$$

we get

$$R = r_c \Leftrightarrow 1 = 4 \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2}.$$

Now the required equivalence follows from

$$\begin{aligned} \cos C - (\cos A + \cos B) &= 1 - 2 \sin^2 \frac{C}{2} - 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \\ &= 1 - 2 \sin \frac{C}{2} \left(\cos \frac{A+B}{2} + \cos \frac{A-B}{2} \right) \\ &= 1 - 4 \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2}. \end{aligned}$$

I have found the following problem in Shiko Iwata, *Encyclopedia of Geometry* (3) (1976), quoted from problem 2091 of *Mathesis* 37 (1923):

If ABC be a triangle such that $\cos A = \cos B + \cos C$, prove that

(i) $4s(s - b)(s - c) = abc$,

(ii) $r_a = R$,

(iii) circumcenter O lies on W_2W_3 , where W_2 and W_3 are intersections of the bisectors of $\angle B$ and $\angle C$ with the opposite sides,

(iv) the foot of the altitude from A to BC lies on OI , where I is the incenter of ΔABC . (Proposed by V. Thébault.)

A proof of the equivalence of the above two conditions has also been sent in by K.R.S. SASTRY, Addis Ababa, Ethiopia.

*

*

*

1321. [1988: 76; 1989: 116] *Proposed by Jordi Dou, Barcelona, Spain.*

(Dedicated in memoriam to Léo Sauvé.)

The circumcircle of a triangle is orthogonal to an excircle. Find the ratio of their radii.

II. *Solution by Shailesh Shirali, Rishi Valley School, India.*

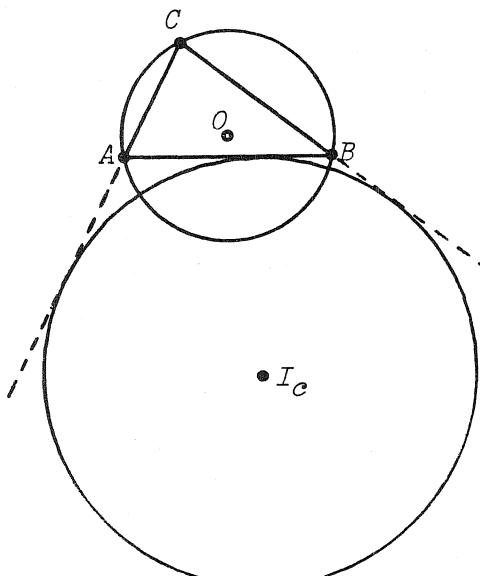
Orthogonality invites inversion,

so invert in circle (I_c, r_c) . The lines CA , CB , AB all "fold" into circles passing through I_c and internally tangent to (I_c, r_c) , therefore with radius $r_c/2$. Circumcircle (O, R) , being orthogonal to (I_c, r_c) , inverts into itself, so the maps A' , B' , C' of A , B , C lie on (O, R) . Now by a known theorem, when three equal circles pass through a point, the circumcircle of their other three points of intersection is congruent to the original three circles. Invoking this we see that circle $A'B'C'$ has radius $r_c/2$, in other words $R = r_c/2$.

*

*

*



1355. [1988: 175; 1989: 240] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle and I its incenter. The perpendicular to AI at I intersects the line BC at the point A' . Analogously we define B' , C' . Prove that A' , B' , C' lie in a straight line.

II. *Comment by Dan Pedoe, Minneapolis, Minnesota.*

The generalization by Hut and Janous of this problem says: Let ABC be a triangle and P a point in its plane, and let A' , B' and C' lie on BC , CA and AB , respectively, where $PA' \perp PA$, $PB' \perp PB$ and $PC' \perp PC$. Then A' , B' and C' are collinear. [See [1989: 240] for Walther Janous's proof.]

Why the perpendiculars? Is there an underlying structure? Can we paint in some background to this bleak-looking but fascinating theorem so that it becomes obvious?

The generalization follows from standard results of classical projective geometry. (See, for example, Dan Pedoe, *Geometry: A Comprehensive Course*, Dover Publications, 1989.) The essential result is that given a quadrangle $ABCP$, there is exactly one conic having P as a focus and ABC as self-polar triangle. (Projectively, a focus is the point of intersection of conjugate tangents from the two circular points of infinity.) Recall that perpendicular lines through the focus are conjugate, each containing the pole of the other, and that the poles of the lines through a point are collinear. Then A' is the pole of the line PA , B' is the pole of PB , and C' is the pole of PC , and since the three lines PA , PB , PC are concurrent in the point P , the line which is the polar of P contains the points A' , B' and C' , and this line is the directrix of the conic.

*

*

*

1403* [1989: 12] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For $n \geq 2$, prove or disprove that

$$1 < \frac{x_1 + \cdots + x_n}{n} \leq 2$$

for all natural numbers x_1, x_2, \dots, x_n satisfying

$$x_1 + x_2 + \cdots + x_n = x_1 \cdot x_2 \cdot \cdots \cdot x_n.$$

Solution by Robert B. Israel, University of British Columbia.

The assertion is true. Moreover, the second inequality is strict except in the cases

$$x + 2 + \overbrace{1 + \cdots + 1}^{x-2 \text{ times}} = x \cdot 2 \cdot 1 \cdot \cdots \cdot 1. \quad (1)$$

Rearrange the numbers so that $x_j \geq 2$ for $j \leq l$ and $x_j = 1$ for $j > l$. Note that $l \geq 2$. Let

$$P = x_1 \cdots x_l \quad \text{and} \quad S = x_1 + \cdots + x_l.$$

Then $n - l = P - S$. The first inequality of the problem being trivial, we must

show that

$$P \leq 2n = 2(l + P - S),$$

i.e.

$$P \geq 2(S - l).$$

Let $x_j = 2 + y_j$. Then

$$P = \prod_{j=1}^l (2 + y_j) \geq 2^l + 2^{l-1} \sum_{j=1}^l y_j \quad (2)$$

$$\geq 2l + 2 \sum_{j=1}^l y_j \quad (3)$$

$$\begin{aligned} &= 2 \sum_{j=1}^l (1 + y_j) = 2 \sum_{j=1}^l (x_j - 1) \\ &= 2(S - l). \end{aligned}$$

Note that equality in (3) requires $l = 2$, in which case equality in (2) requires $y_1 y_2 = 0$, which produces the cases (1).

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MARCIN E. KUCZMA, Warszawa, Poland; and C. WILDHAGEN, Breda, The Netherlands.

EDWARD T.H. WANG, Wilfrid Laurier University, points out that the problem was proposed by him as problem E2447 in the American Math. Monthly, with a solution by C. Hurd appearing in Vol. 82 (1975) pp. 78–80. Wang nevertheless was delighted to see the problem reappear in Crux, and (especially since Israel's solution seems to be a bit simpler than Hurd's) the editor agrees.

The published solution of the Monthly problem also included an observation by D.M. Bloom that the lower bound of 1 can be approached arbitrarily closely by the natural numbers

$$1, \dots, 1, s+2, \frac{n+s}{s+1},$$

where s is a large natural number and $n \equiv 1 \pmod{s+1}$ is then chosen sufficiently large.

*

*

*

1404. [1989: 13] *Proposed by J.T. Groenman, Arnhem, The Netherlands, and D.J. Smeenk, Zaltbommel, The Netherlands.*

Let ABC be a triangle with circumradius R and inradius ρ . A theorem of Poncelet states that there are an infinity of triangles having the same

circumcircle and the same incircle as ΔABC .

- (a) Show that the orthocenters of these triangles lie on a circle.
(b) If $R = 4\rho$, what can be said about the locus of the centers of the nine-point circles of these triangles?

Solution by Dan Sokolowsky, Williamsburg, Virginia.

(a) Let Δ denote any triangle having the same circumcircle, $O(R)$, and incircle, $I(\rho)$, as ΔABC . Let H denote the orthocenter of Δ . Then the following facts are well known (e.g. [1], [2]):

(i) the nine-point circle K of Δ has as center the midpoint N of OH , and radius $R/2$;

(ii) K touches the incircle $I(\rho)$ (by the Feuerbach theorem).

This implies that $IN = R/2 - \rho$. Extend OI to J with $OI = IJ$. Then I, N are the respective midpoints of OJ and OH , so

$$JH = 2IN = R - 2\rho .$$

Hence the locus of H is the circle $J(R - 2\rho)$.

(b) If $R = 4\rho$, then

$$IN = R/2 - \rho = 2\rho - \rho = \rho .$$

Hence the locus of N is the incircle $I(\rho)$ of ΔABC .

References:

- [1] D. Pedoe, *A Course of Geometry for Colleges and Universities*, Cambridge University Press, 1970, pp. 56–58, 87. (Also reprinted by Dover in 1988 as *Geometry: A Comprehensive Course*.)
[2] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960, pp. 195, 200.

Also solved by the proposers, who ask whether the loci found in (a) and (b) are in fact the entire circles given.

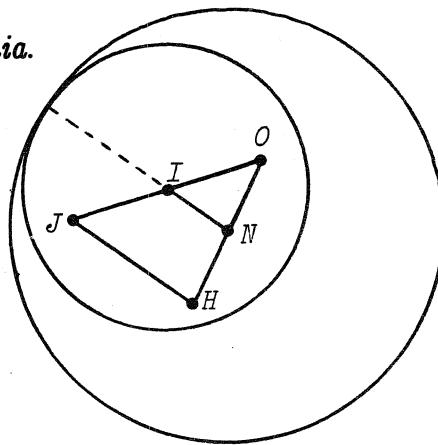
*

*

*

1405. [1989: 13] *Proposed by Murray S. Klamkin and Andy Liu, University of Alberta.*

Two distinct congruent n -gons P and P' are inscribed in a noncircular ellipse E . Prove or disprove that if $n > 4$, P' must be obtainable from P by a reflection across the axes or center of E . (For the cases $n = 3$ and 4 see [1988: 131, 139].)



Solution by Jordi Dou, Barcelona, Spain.

Let κ be the congruence such that $\kappa(P) = P'$. Then it holds that $\kappa(E) = E$, since $\kappa(E)$ and E have five common points (namely the vertices of P'). If κ conserves the sense of rotation, κ is a rotation of 180° , with centre the centre of E . If κ does not conserve the sense of rotation, κ is a symmetry with respect to an axis of E .

Also solved (the same way) by the proposers.

*

*

*

1406. [1989: 13] *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville.*

If $0 < \theta < \pi$, prove without calculus that

$$\cot \frac{\theta}{4} - \cot \theta > 2.$$

Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.

Let $\theta = 2\alpha$. Then

$$\begin{aligned}\cot \frac{\theta}{4} - \cot \theta &= \cot \frac{\alpha}{2} - \cot 2\alpha \\ &= \frac{1 + \cos \alpha}{\sin \alpha} - \frac{2 \cos^2 \alpha - 1}{2 \sin \alpha \cos \alpha} \\ &= \frac{2 \cos \alpha + 1}{2 \sin \alpha \cos \alpha} \\ &= \frac{1}{\sin \alpha} + \frac{1}{\sin 2\alpha} \geq 2.\end{aligned}$$

But since $\sin \alpha$ and $\sin 2\alpha$ have their maxima (within the region $0 \leq \alpha \leq \pi/2$) at different points, we are justified in changing \geq to $>$. Thus

$$\cot \frac{\theta}{4} - \cot \theta > 2.$$

Also solved by HAYO AHLBURG, Benidorm, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Federal Republic of Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Flushing, N.Y.; DOUGLASS L. GRANT, University College of Cape Breton; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; J. WALTER LYNCH, Georgia Southern College, Statesboro; J.A. MCCALLUM, Medicine Hat, Alberta; V.H. MURTY, Penn State University; BOB PRIELIPP, University of Wisconsin-Oshkosh; TOSHIO SEIMIYA, Kawasaki, Japan; M.A. SELBY, University of Windsor; C. WILDHAGEN, Breda, The Netherlands; KENNETH S. WILLIAMS, Carleton University; and the proposer. One other reader

sent in a solution using calculus.

Ahlburg and Hut observe that if $0 < \theta < \pi/2$ then in fact

$$\cot \theta/4 - \cot \theta > 1 + \sqrt{2}$$

(as can be seen from the above proof). Hut goes on to calculate the best lower bound for the problem. He finds it to be ≈ 2.28 , occurring at $\theta \approx 111^\circ$.

Murty points out that the problem appears with solution on p. 88 of E.W. Hobson's Plane Trigonometry, Cambridge University Press (republished as A Treatise on Plane and Advanced Trigonometry by Dover).

*

*

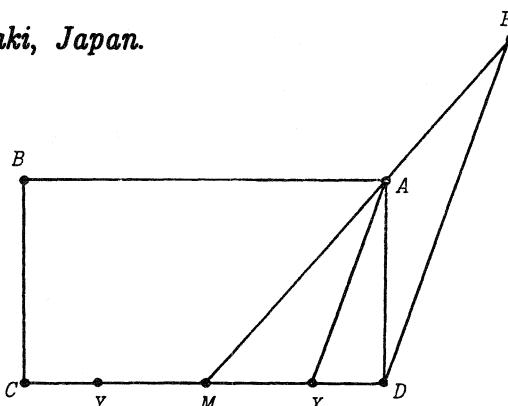
*

1407. [1989: 13] Proposed by G.R. Veldkamp, De Bilt, The Netherlands.

Given a rectangle $ABCD$ with $AB = CD > AD = BC$, construct points X, Y on CD between C and D such that $AX = XY = YB$.

I. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Construction. Let M be the midpoint of the side CD . Take a point P on the ray MA such that $DP = DC$. Take a point X on the segment DM such that $AX \parallel PD$. Take a point Y on the segment MC such that $MX = MY$. Then X and Y are the required points.



Proof. Since $AX \parallel PD$ and $DP = DC$,

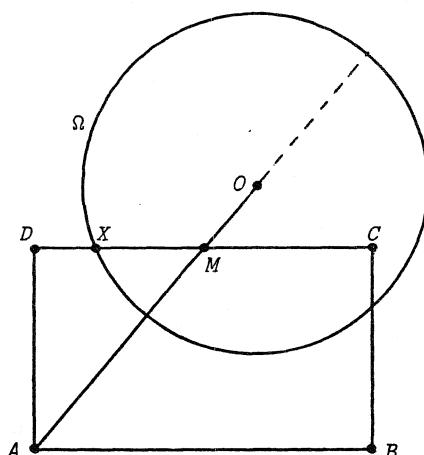
$$AX : XM = PD : DM = CD : DM = 2 : 1 .$$

Therefore $AX = 2XM = XY$, and by symmetry $AX = BY$.

II. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

Let M be the midpoint of CD .

Extend segment AM beyond M to the point O such that $AO = (4/3)AM$. The circle Ω of center O and radius $(2/3)AM$ is the locus of points whose distance from A equals twice the distance from M (the Apollonius circle). Since $AD < CD = 2MD$, point D lies outside Ω . Hence Ω intersects segment MD ; the point of intersection is the desired X . Y shall lie symmetrically to X with respect to M .



Also solved by JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Federal Republic of Germany; HERITA T. FREITAG, Roanoke, Virginia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; ANTONIO LEONARDO P. PASTOR, Universidade de Sao Paulo, Brazil; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

Solution I was also given by Pastor and the proposer; solution II by Engelhaupt, Freitag, and Lau.

Regrettably, Professor Veldkamp passed away last September, at the age of 82.
(The editor thanks Dr. L.J. Hut for this information.)

*

*

*

1408. [1989: 13] Proposed by Jordi Dou, Barcelona, Spain.

Given the equilateral triangle ABC , find all positive real numbers r for which there is a point $P(r)$ such that

$$\frac{PA}{1} = \frac{PB}{r} = \frac{PC}{r^2},$$

and describe the locus of $P(r)$.

Solution by Marcin E. Kuczma, Warszawa, Poland.

I choose the coordinate system so that $A = (2, \sqrt{3})$, $B = (-1, 0)$, $C = (2, -\sqrt{3})$.

Given $r > 0$, let

$$\Omega_1 = \left\{ P : \frac{PB}{PA} = r \right\}, \quad \Omega_2 = \left\{ P : \frac{PB}{PC} = \frac{1}{r} \right\}.$$

Thus $P(r)$ exists if and only if $\Omega_1 \cap \Omega_2 \neq \emptyset$.

Evidently, $P(1)$ exists; it is the center of the triangle: $P(1) = (1, 0)$.

For $r \neq 1$, Ω_1 is the Apollonius circle for the pair of points A, B and ratio r . Formulas expressing its center and radius are well-known (or easy to derive). The same concerns Ω_2 . And so:

$$\Omega_1 \text{ has center } \left(\frac{1 + 2r^2}{r^2 - 1}, \frac{\sqrt{3}r^2}{r^2 - 1} \right) \text{ and radius } \frac{2\sqrt{3}r}{|r^2 - 1|},$$

$$\Omega_2 \text{ has center } \left(\frac{r^2 + 2}{1 - r^2}, \frac{\sqrt{3}}{r^2 - 1} \right) \text{ and radius } \frac{2\sqrt{3}r}{|r^2 - 1|}.$$

Now, for $r \neq 1$ the two circles intersect if and only if the distance between the centers does not exceed the sum of the radii. This leads to the biquadratic inequality $r^4 - 3r^2 + 1 \leq 0$, whose solution in positive numbers is the interval

$$\frac{\sqrt{5} - 1}{2} \leq r \leq \frac{\sqrt{5} + 1}{2}. \quad (1)$$

The "exceptional" value $r = 1$ is here again included, so that (1) is a necessary and sufficient condition for the existence of $P(r)$.

Note that $P(r)$ is not unique in general; it is unique for $r = 1$ and $r = (\sqrt{5} \pm 1)/2$; for other values of r we have two points $P(r)$.

To determine the locus of $P(r)$, it suffices to observe that a point $P = (x, y)$ is equal to $P(r)$ for some r (i.e., belongs to the locus) if and only if PB is the geometric mean of PA and PC :

$$((x+1)^2 + y^2)^2 = ((x-2)^2 + (y-\sqrt{3})^2)((x-2)^2 + (y+\sqrt{3})^2).$$

Equivalently,

$$((x+1)^2 + y^2)^2 = ((x+1)^2 + y^2 + 6(1-x))^2 - 12y^2,$$

i.e.,

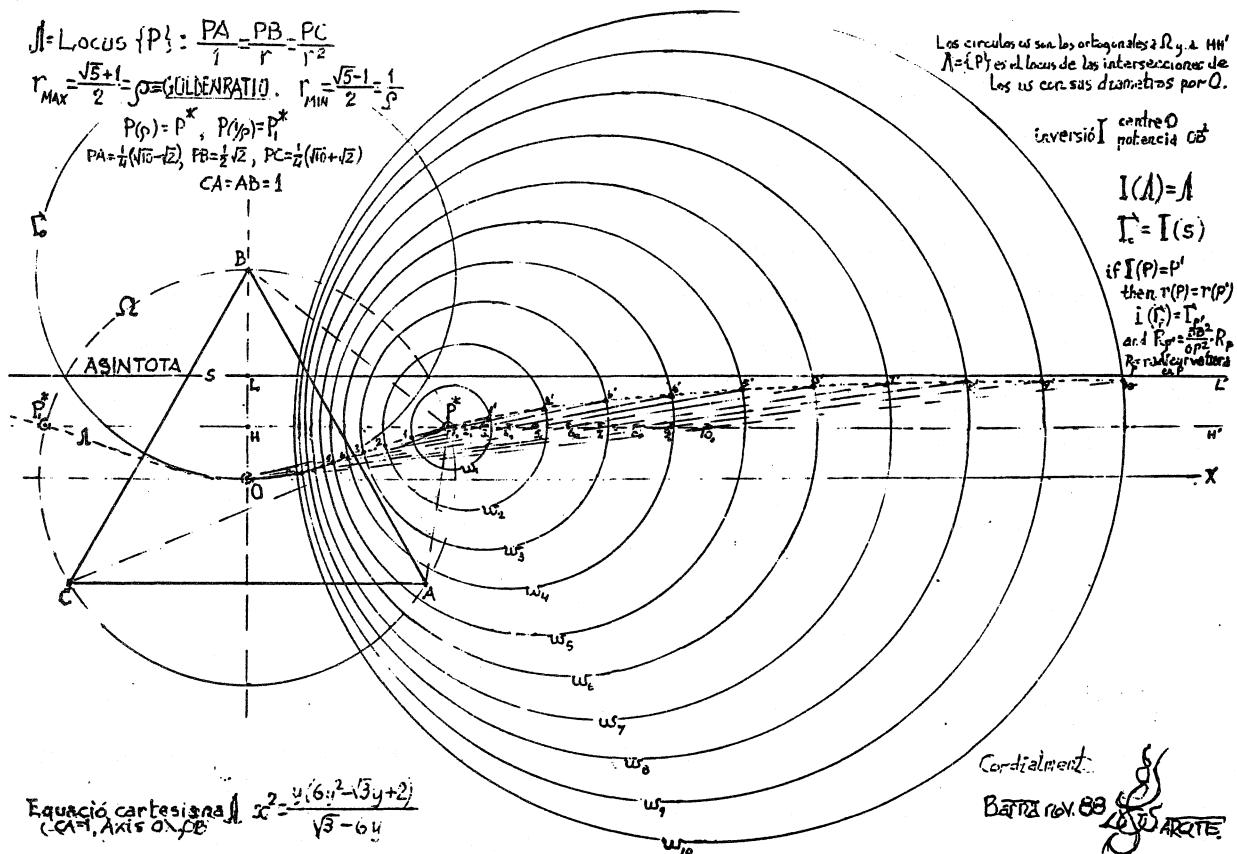
$$12y^2 = 12(1-x)((x+1)^2 + y^2) + 36(1-x)^2,$$

and finally

$$xy^2 = (1-x)(x^2 - x + 4). \quad (2)$$

Equation (2) describes a third order algebraic curve, symmetric with respect to the x -axis. This is the desired locus of $P(r)$.

[Editor's note: As an extra treat for *Crux* readers, here is the proposer's fantastic drawing of the locus!]



Also solved by MATHEW ENGLANDER, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

*

*

*

1409. [1989: 13] Proposed by Shailesh Shirali, Rishi Valley School, India.

Show that

$$\frac{n}{n+1} + \frac{2n(n-1)}{(n+1)(n+2)} + \frac{3n(n-1)(n-2)}{(n+1)(n+2)(n+3)} + \cdots = \frac{n}{2}$$

for all positive integers n . What if $n \geq 0$ is not an integer?

I. Solution by Kenneth S. Williams, Carleton University.

Let n be a positive integer and let x be a real number $\neq -1, -2, \dots, -n$.

Then

$$\begin{aligned} & \frac{x}{x+1} + \frac{2x(x-1)}{(x+1)(x+2)} + \frac{3x(x-1)(x-2)}{(x+1)(x+2)(x+3)} + \cdots + \frac{nx(x-1)\cdots(x-n+1)}{(x+1)(x+2)\cdots(x+n)} \\ &= \frac{x}{x+1} + \sum_{k=1}^{n-1} \frac{(k+1)x(x-1)\cdots(x-k)}{(x+1)(x+2)\cdots(x+k+1)} \quad | \quad 2 \cdot \frac{k+1}{x+k+1} = \frac{2k+2}{x+k+1} = \frac{x+k+1}{x+k+1} \\ &= \frac{x}{x+1} + \frac{1}{2} \sum_{k=1}^{n-1} \left[\frac{x(x-1)\cdots(x-k)}{(x+1)(x+2)\cdots(x+k)} - \frac{x(x-1)\cdots(x-k-1)}{(x+1)(x+2)\cdots(x+k+1)} \right] \\ &= \frac{x}{x+1} + \frac{1}{2} \left[\frac{x(x-1)}{x+1} - \frac{x(x-1)\cdots(x-n)}{(x+1)(x+2)\cdots(x+n)} \right] \\ &= \frac{x}{2} - \frac{x(x-1)\cdots(x-n)}{2(x+1)(x+2)\cdots(x+n)}. \end{aligned}$$

Taking $x = n$ we obtain

$$\frac{n}{n+1} + \frac{2n(n-1)}{(n+1)(n+2)} + \frac{3n(n-1)(n-2)}{(n+1)(n+2)(n+3)} + \cdots = \frac{n}{2}.$$

II. Solution by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.

We generalize the result for n a positive integer.

For positive integers n, m and any real number r , define

$$\begin{aligned} F(n, m, r) &= \frac{n}{m} \cdot r + \frac{n(n-1)}{m(m+1)} \cdot (r+1) + \frac{n(n-1)(n-2)}{m(m+1)(m+2)} \cdot (r+2) + \cdots \\ &\quad + \frac{n(n-1)\cdots 2 \cdot 1}{m(m+1)\cdots(m+n-1)} \cdot (r+n-1) \\ &= \sum_{i=1}^n \binom{n}{m+i-1} (r+i-1). \end{aligned}$$

Since

$$F\left(1, m, \frac{m}{2}\right) = \frac{1}{2}$$

and

$$F\left(k+1, m, \frac{m-k}{2}\right) = \frac{k+1}{m} \left(\frac{m-k}{2} + F\left(k, m+1, \frac{m-k+2}{2}\right) \right),$$

it follows by induction on n that, for all positive integers n and m ,

$$F\left(n, m, \frac{m-n+1}{2}\right) = \frac{n}{2}. \quad (1)$$

Setting $m = n + 1$ in (1) yields the desired identity for n a positive integer.

Note that doubling both sides of (1) gives the interesting identity

$$\sum_{i=1}^n \frac{\binom{n}{i}}{\binom{m+i-1}{i}} (m+2i-n-1) = n,$$

which holds for all positive integers n and m .

III. Solution by Robert B. Israel, University of British Columbia.

The series converges to $n/2$ if $n = 0$ or if n is any complex number with $\Re n > 0$, and diverges otherwise. (Of course, it is not defined if n is a negative integer.)

Let

$$a_k = k \prod_{j=1}^k \frac{n-j+1}{n+j} \quad \text{and} \quad S_m = \sum_{k=1}^m a_k,$$

so that the given sum is equivalent to $\lim_{m \rightarrow \infty} S_m = n/2$. Let

$$R_k = \frac{n}{2} \prod_{j=1}^k \frac{n-j}{n+j}.$$

Then

$$\begin{aligned} a_{m+1} + R_{m+1} &= \frac{n}{2} \prod_{j=1}^m \frac{n-j}{n+j} \left(\frac{2(m+1)}{n+m+1} + \frac{n-m-1}{n+m+1} \right) \\ &= \frac{n}{2} \prod_{j=1}^m \frac{n-j}{n+j} \cdot 1 = R_m. \end{aligned}$$

Noting that

$$S_1 + R_1 = a_1 + R_1 = R_0 = \frac{n}{2}$$

and

$$S_m + R_m = S_{m-1} + a_m + R_m = S_{m-1} + R_{m-1},$$

by induction we obtain $S_m + R_m = n/2$ for all $m \geq 1$. Thus it suffices to show that $R_m \rightarrow 0$ as $m \rightarrow \infty$ if $n = 0$ or $\Re n > 0$, but not otherwise. Of course, if n is a

nonnegative integer, $R_m = 0$ for $m \geq n$; this establishes the proposal. If n is not an integer,

$$\ln|R_m| = \ln\left|\frac{n}{2}\right| + \sum_{j=1}^n \ln\left|\frac{1-n/j}{1+n/j}\right|.$$

Since as $j \rightarrow \infty$,

$$\ln\left|\frac{1-n/j}{1+n/j}\right| = \frac{-2\operatorname{Re} n}{j} + O\left(\frac{1}{j^2}\right), \quad (2)$$

we find that $\ln|R_m| \rightarrow -\infty$ if and only if $\operatorname{Re} n > 0$, and we are done.

[Editor's note. Expert colleague Len Bos offers the following derivation of (2). For any sufficiently small complex number z ,

$$\begin{aligned} \ln\left|\frac{1-z}{1+z}\right| &= \frac{1}{2}\ln\left|\frac{1-z}{1+z}\right|^2 = \frac{1}{2}\ln\left(\frac{(1-z)(1-\bar{z})}{(1+z)(1+\bar{z})}\right) \\ &= \frac{1}{2}[\ln(1-z) + \ln(1-\bar{z}) - \ln(1+z) - \ln(1+\bar{z})] \\ &= \frac{1}{2}\left[-\left(z+\frac{z^2}{2}+\dots\right) - \left(\bar{z}+\frac{\bar{z}^2}{2}+\dots\right) - \left(z-\frac{z^2}{2}+\dots\right) - \left(\bar{z}-\frac{\bar{z}^2}{2}+\dots\right)\right] \\ &= -(z+\bar{z}) + O(z^2) + O(\bar{z}^2) \\ &= -2\operatorname{Re} z + O(z^2) + O(\bar{z}^2), \end{aligned}$$

and putting $z = n/j$ yields (2).]

Also solved by NICOS D. DIAMANTIS, student, University of Patras, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEWAI LAU, Hong Kong; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; and the proposer.

The problem was solved for all positive real values of n by Klamkin, Kuczma, Lau, and the proposer. The proposer's solution was similar to Israel's solution (III).

*

*

*

1410*. [1989: 13] Proposed by Svetoslav Bilchev and Emilia Velikova, Technical University, Russe, Bulgaria.

Given is a triangle with circumcentre O and circumradius R . Interior points P, P' are isogonal conjugates, and r_1, r_2, r_3 are the distances from P to the sides of the triangle. Prove that

$$(R^2 - \overline{OP}^2)^2(R^2 - \overline{OP'}^2) = 8r_1r_2r_3R^3.$$

Solution by Murray S. Klamkin, University of Alberta.

Letting r_1' , r_2' , r_3' be the distances from P' to the sides of the triangle, then as is known [1],

$$r_1 r_1' = r_2 r_2' = r_3 r_3' = \lambda^2. \quad (1)$$

The proportionality constant λ^2 is gotten from

$$\sum a_1 r_1 = \sum a_1 r_1' = 2F$$

(the sums here and subsequently being cyclic over 1,2,3), so that

$$\lambda^2 = \frac{2F}{\sum \frac{a_1}{r_1}} = \frac{2F}{\sum \frac{a_1}{r_1'}}. \quad (2)$$

It is also known [1987: 260], [2] that

$$R^2 - \overline{OP}^2 = \frac{r_1 r_2 r_3 R}{F} \sum \frac{a_1}{r_1},$$

so that from (2),

$$R^2 - \overline{OP'}^2 = \frac{r_1' r_2' r_3' R}{F} \sum \frac{a_1}{r_1'} = \frac{r_1' r_2' r_3' R}{F} \sum \frac{a_1}{r_1}.$$

Hence

$$(R^2 - \overline{OP}^2)^2 (R^2 - \overline{OP'}^2) = (r_1 r_2 r_3)^2 \left(\frac{R}{F}\right)^3 \left(\sum \frac{a_1}{r_1}\right)^3 r_1' r_2' r_3'.$$

Finally, since (by (1) and (2))

$$r_1' r_2' r_3' = \frac{\lambda^6}{r_1 r_2 r_3} = \frac{8F^3}{r_1 r_2 r_3 \left(\sum \frac{a_1}{r_1}\right)^3},$$

we get the desired result.

References:

- [1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, New York, 1960, p. 155.
- [2] M.S. Klamkin, An identity for simplexes and related inequalities, *Simon Stevin* 48 (1974/75) 57–64.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposers.

*

*

*

1411. [1989: 47] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

$\triangle ABC$ is acute angled with sides a , b , c and has circumcircle Γ with centre O . The inner bisector of $\angle A$ intersects Γ for the second time in A_1 . D is the projection on AB of A_1 . L and M are the midpoints of CA and AB respectively. Show that

- (i) $AD = \frac{1}{2}(b + c)$;
(ii) $A_1D = OM + OL$.

Solution by Wilson da Costa Areias, Rio de Janeiro, Brazil.

Drawing the radii OA , OA_1 and letting J be on A_1D such that $OJ \perp A_1D$, we form the isosceles triangle OAA_1 and the rectangle $OMDJ$. Then

$$\begin{aligned}\angle JA_1O + \angle OAL &= \angle DA_1A + \angle A_1AO + \angle OAL \\ &= \angle DA_1A + \angle A_1AL \\ &= \angle DA_1A + \angle DAA_1,\end{aligned}$$

which means

$$\triangle OJA_1 \cong \triangle OAL,$$

and so

$$MD = OJ = AL = b/2$$

and

$$A_1J = OL.$$

Therefore

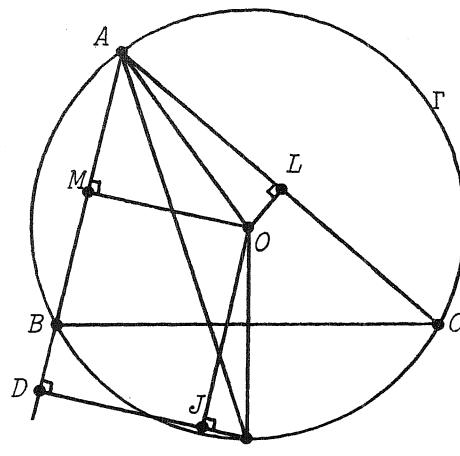
$$AD = AM + MD = \frac{c}{2} + \frac{b}{2} = \frac{1}{2}(b + c)$$

and

$$A_1D = JD + A_1J = OM + OL.$$

[*Editor's note.* The above diagram applies as long as $\angle B > \angle C$, which can be assumed because (despite appearances) B and C are symmetric in the problem; the lengths AD and A_1D are unchanged if D is chosen as the projection of A_1 on AC rather than AB . However, it appears to the editor that the result remains true for nonacute triangles. Maybe some reader can find a unified proof.]

Also solved by RAUL F.W. AGOSTINO, Rio de Janeiro, Brazil; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; GRAHAM DENHAM, student, University of Alberta; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Federal Republic of Germany; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; P. PENNING, Delft, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer. Several of the proofs were similar to the one above.



1412. [1989: 47] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find all positive integers n such that

$$(n - 36)(n - 144) = 4964$$

is the square of an integer.

Solution by Graham Denham, student, University of Alberta.

We want to find all positive integers n such that

$$(n - 36)(n - 144) = m^2$$

for some nonnegative integer m . Expanding and rearranging, this becomes

$$n^2 - 180n + 220 = m^2, \quad (1)$$

or

$$(n - 90)^2 - 7880 = m^2.$$

For convenience, set $k = n - 90$. Then we get

$$(k - m)(k + m) = k^2 - m^2 = 7880 = 2^3 \cdot 5 \cdot 197, \quad (2)$$

where 2, 5 and 197 are all prime. Therefore

$$k - m = 2^a 5^b 197^c \quad \text{and} \quad k + m = 2^{3-a} 5^{1-b} 197^{1-c} \quad (3)$$

for some a, b, c . [Editor's note. Since from (1) $n = 1$ is not a solution, we know $n \geq 2$ and thus $k \geq -88$. Since also $|k| \geq \sqrt{7880}$ from (2), we must have $k > 0$, so both factors in (3) are positive.] Solving in (3) for k , we get

$$k = \frac{2^a 5^b 197^c + 2^{3-a} 5^{1-b} 197^{1-c}}{2} = 2^{a-1} 5^b 197^c + 2^{2-a} 5^{1-b} 197^{1-c},$$

where $a = 1$ or 2, $b = 0$ or 1, $c = 0$ or 1. Discarding duplicates, one obtains

$$k = 1 + 2 \cdot 5 \cdot 197 = 1971, \quad \text{so} \quad n = k + 90 = 2061,$$

$$k = 2 + 5 \cdot 197 = 987, \quad \text{so} \quad n = 1077,$$

$$k = 5 + 2 \cdot 197 = 399, \quad \text{so} \quad n = 489,$$

$$k = 2 \cdot 5 + 197 = 207, \quad \text{so} \quad n = 297,$$

which is a complete solution set.

Also solved by HAYO AHLBURG, Benidorm, Spain; SEUNG-JIN BANG, Seoul, Republic of Korea; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Federal Republic of Germany; MATHEW ENGLANDER, Toronto, Ontario; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; BOTOND KÖSZEGI, student, Halifax West High School, Halifax, Nova Scotia; SIDNEY KRAVITZ, Dover, New Jersey; INDY LAGU, student, University of

Calgary; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; D.J. MORRISS, George Brown College, Toronto, Ontario; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; C. WILDHAGEN, Breda, The Netherlands; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer. Six other readers sent in partial solutions.

*

*

*

1413. [1989: 47] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For $0 < x, y, z < 1$ let

$$u = z(1 - y), \quad v = x(1 - z), \quad w = y(1 - x).$$

Prove that

$$(1 - u - v - w) \left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w} \right) \geq 3.$$

When does equality occur?

I. *Solution by Guo-Gang Gao, student, Université de Montréal.*

Observe that for $0 < x, y, z < 1$,

$$\begin{aligned} \frac{1-z}{y} + \frac{y}{1-z} &\geq 2 \quad \text{with equality if and only if } y = 1 - z, \\ \frac{1-x}{z} + \frac{z}{1-x} &\geq 2 \quad \text{with equality if and only if } z = 1 - x, \\ \frac{1-y}{x} + \frac{x}{1-y} &\geq 2 \quad \text{with equality if and only if } x = 1 - y. \end{aligned}$$

Adding, we have

$$\left(\frac{1-x}{z} + \frac{x}{1-y} \right) + \left(\frac{1-y}{x} + \frac{y}{1-z} \right) + \left(\frac{1-z}{y} + \frac{z}{1-x} \right) \geq 6,$$

where equality holds if and only if $x = y = z = 1/2$. By substitution, we have

$$\frac{1-v-w}{u} + \frac{1-w-u}{v} + \frac{1-u-v}{w} \geq 6,$$

where equality holds if and only if $u = v = w = 1/4$. Notice that this inequality is equivalent to the inequality to be proved.

II. *Comment by Murray S. Klamkin, University of Alberta.*

[Klamkin first solved the problem as above. - Ed.]

An equivalent inequality in a geometric setting appears in problem 4212 by Dmitry Mavlo in *School Science and Mathematics*.

Let S be the area of $\triangle KLM$ inscribed in $\triangle ABC$, where K, L, M lie on AB, BC, CA , respectively. Let S_1, S_2, S_3 be the areas of triangles AMK, BKL, CLM , respectively. Prove that

$$S \geq \frac{3}{1/S_1 + 1/S_2 + 1/S_3}, \quad (1)$$

with equality if and only if K, L, M are the midpoints of the respective sides of $\triangle ABC$.

This strengthens the Erdős–Debrunner inequality

$$S \geq \min(S_1, S_2, S_3)$$

(*Elem. der Math.* 11 (1956) 20), and is equivalent to the USA–inequality

$$(1+U)\left(1 + \frac{1}{S}\right) + (1+S)\left(1 + \frac{1}{A}\right) + (1+A)\left(1 + \frac{1}{U}\right) \geq \frac{3(U+1)(S+1)(A+1)}{USA + 1}, \quad (2)$$

U, S, A any positive reals, which appeared as problem 4200 in *School Science and Mathematics*, October 1988.

Since the ratios S/S_i are affine invariant, it suffices to prove (1) for $\triangle ABC$ being an equilateral triangle of side 1. Then letting $AK = x, BL = y, CM = z$, we have

$$S_1 = x(1-z)[ABC], \quad S_2 = y(1-x)[ABC], \quad S_3 = z(1-y)[ABC]$$

and

$$S = [1 - x(1-z) - y(1-x) - z(1-y)][ABC] = [xyz + (1-x)(1-y)(1-z)][ABC],$$

where $[ABC]$ is the area of $\triangle ABC$. (1) can now be rewritten as

$$[xyz + (1-x)(1-y)(1-z)]\left[\frac{1}{x(1-z)} + \frac{1}{y(1-x)} + \frac{1}{z(1-y)}\right] \geq 3, \quad (3)$$

which is the same as the Janous inequality. Also, (2) can be rewritten as

$$(USA + 1)\left(\frac{1}{S(A+1)} + \frac{1}{A(U+1)} + \frac{1}{U(S+1)}\right) \geq 3, \quad (4)$$

and (3) and (4) can be converted into each other by the transformations

$$\begin{aligned} x &= \frac{U}{U+1}, & y &= \frac{S}{S+1}, & z &= \frac{A}{A+1}, \\ U &= \frac{x}{1-x}, & S &= \frac{y}{1-y}, & A &= \frac{z}{1-z}. \end{aligned}$$

Inequality (2) with different letters appeared as problem 642 in the *Pi Mu Epsilon Journal*, and in the SOCLE (the problem section of the *Canadian Mathematical Society Notes*), again proposed by Mavlo.

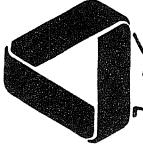
Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; and the proposer.

The proposer asks for a generalization. (To n variables?)

*

*

*



Canadian Mathematical Society
Société Mathématique du Canada

577 KING EDWARD, OTTAWA, ONT. CANADA K1N 6N5

TEL. (613) 564-2223

EDITORSHIP OF CRUX MATHEMATICORUM
CALL FOR NOMINATIONS

The Publication Committee of the Canadian Mathematical Society invites applications for the editorship of **Crux Mathematicorum** for a five-year period commencing January 1, 1991.

An application should consist of a letter from the candidate which includes an expression of views of **Crux** and indicates whether any changes in direction of policy are contemplated, a *curriculum vitae* and a letter of support from the head of the potential host department.

Applications should be sent to:

Prof. Michael Barr, Chairman
CMS Publications Committee
Department of Mathematics
McGill University
805 Sherbrooke St. W.
Montreal, Quebec
H3A 2K6

The absolute deadline for receipt of applications is August 31, 1990.

CRUX MATHEMATICORUM
APPELS AUX NOMINATIONS

Le Comité des publications de la Société mathématique du Canada invite les mises en candidatures pour le poste de rédacteur de **Crux Mathematicorum** pour un mandat de cinq ans débutant le 1 janvier 1991.

Une mise en candidature devrait inclure une lettre du candidat (de la candidate) décrivant sa vision de **Crux** et qui indique si des changements de direction ou de politiques sont envisagés, un *curriculum vitae* et une lettre d'appui du département du candidat (de la candidate).

Les mises en candidature devraient être acheminées à:

Prof. Michael Barr, Président
Comité des publications de la SMC
Department of Mathematics
McGill University
805 Sherbrooke St. W.
Montréal, Québec
H3A 2K6

La date limite pour la réception des mises en candidature est le 30 août 1990.

!!!! BOUND VOLUMES !!!!

THE FOLLOWING BOUND VOLUMES OF CRUX MATHEMATICORUM
ARE AVAILABLE AT \$ 10.00 PER VOLUME

1 & 2 (combined), 3, 7, 8, 9 and 10

PLEASE SEND CHEQUES MADE PAYABLE TO
THE CANADIAN MATHEMATICAL SOCIETY

The Canadian Mathematical Society

577 King Edward

Ottawa, Ontario

Canada K1N 6N5

Volume Numbers _____

Mailing:
Address _____

____ volumes X \$10.00 = \$ _____

!!!! VOLUMES RELIÉS !!!!

CHACUN DES VOLUMES RELIÉS SUIVANTS À 10\$:

1 & 2 (ensemble), 3, 7, 8, 9 et 10

S.V.P. COMPLÉTER ET RETOURNER, AVEC VOTRE REMISE LIBELLÉE
AU NOM DE LA SOCIÉTÉ MATHÉMATIQUE DU CANADA, À L'ADRESSE
SUIVANTE:

Société mathématique du Canada

577 King Edward

Ottawa, Ontario

Canada K1N 6N5

Volumes: _____

Adresse: _____

____ volumes X 10\$ = \$ _____

PUBLICATIONS

The Canadian Mathematical Society
577 King Edward, Ottawa, Ontario K1N 6N5
is pleased to announce the availability of the following publications:

1001 Problems in High School Mathematics

Collected and edited by E.J. Barbeau, M.S. Klamkin and W.O.J. Moser.

Book I:	Problems 1-100 and Solutions 1-50	58 pages	(\$5.00)
Book II:	Problems 51-200 and Solutions 51-150	85 pages	(\$5.00)
Book III:	Problems 151-300 and Solutions 151-350	95 pages	(\$5.00)
Book IV:	Problems 251-400 and Solutions 251-350	115 pages	(\$5.00)
Book V:	Problems 351-500 and Solutions 351-450	86 pages	(\$5.00)

The Canadian Mathematics Olympiads (1968-1978)

Problems set in the first ten Olympiads (1969-1978) together with suggested solutions. Edited by E.J. Barbeau and W.O.J. Moser. 89 pages (\$5.00)

The Canadian Mathematics Olympiads (1979-1985)

Problems set in the Olympiads (1979-1985) together with suggested solutions. Edited by C.M. Reis and S.Z. Ditor. 84 pages (\$5.00)

Prices are in Canadian dollars and include handling charges.
Information on other CMS publications can be obtained by writing
to the Executive Director at the address given above.