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THE OLYMPIAD CORNER: 65

M.S. KLAMKIN

Through the courtesy of G.J. Butler, I give the problems of the 17th Canadian Mathematics Olympiad, which took place on May 1, 1985. I shall publish the official solutions to these problems in a later column. The committee responsible for organizing this Olympiad consisted of:

C.M. Reis (Chairman), University of Western Ontario
D. Borwein, University of Western Ontario
G.J. Butler, University of Alberta
S.Z. Ditor, University of Western Ontario
T.J. Griffiths, University of Western Ontario
M. Grundland, Memorial University of Newfoundland
N.S. Mendelsohn, University of Manitoba
B.L.R. Shawyer, University of Western Ontario

17TH CANADIAN MATHEMATICS OLYMPIAD

May 1, 1985 - Time: 3 hours

1. The lengths of the sides of a triangle are 6, 8, and 10 units. Prove that there is exactly one straight line which simultaneously bisects the area and perimeter of the triangle.
2. Prove that there does not exist an integer which is doubled when the initial digit is transferred to the end.
3. Suppose that a given circle has circumference c . Let x denote the length of the perimeter of a regular polygon of 1985 sides circumscribed about the given circle. Let y denote the perimeter of a regular polygon of 1985 sides inscribed in the given circle. Prove that $x + y \geq 2c$.
4. Prove that 2^{n-1} divides $n!$ if and only if $n = 2^{k-1}$ for some positive integer k .
5. Let $1 < x_1 < 2$ and, for $n = 1, 2, 3, \dots$, define

$$x_{n+1} = 1 + x_n - \frac{1}{2}x_n^2.$$

Prove that $|x_n - \sqrt{2}| < 2^{-n}$ for $n \geq 3$.

**

Next I give the problems of the 3rd Annual American Invitational Mathematics Examination (AIME'), which took place in March 1985. The answers (only) are given at the end of this column. Students or teachers with questions or comments about this examination may write to the AIME Chairman, Professor George Berzsenyi,

Department of Mathematics, Lamar University, Beaumont, Texas 77710. For copies of the exam and detailed solutions, write to Professor Walter E. Mientka, Department of Mathematics, University of Nebraska, Lincoln, Nebraska 68588-0322.

3RD ANNUAL AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

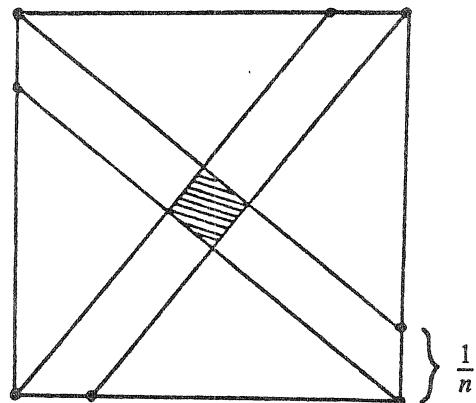
March 1985 - Time: 3 hours

1. Let $x_1 = 97$, and for $n > 1$ let $x_n = n/x_{n-1}$. Calculate the product $x_1x_2\dots x_8$.
2. When a right triangle is rotated about one leg, the volume of the cone produced is $800\pi \text{ cm}^3$. When the triangle is rotated about the other leg, the volume of the cone produced is $1920\pi \text{ cm}^3$. What is the length (in cm) of the hypotenuse of the triangle?
3. Find c if a , b , and c are positive integers which satisfy

$$c = (a + bi)^3 - 107i,$$

where $i^2 = -1$.

4. A small square is constructed inside a square of area 1 by dividing each side of the unit square into n equal parts, and then connecting the vertices to the division points closest to the opposite vertices, as shown in the figure on the right. Find the value of n if the area of the small square (shaded in the figure) is exactly $1/1985$.



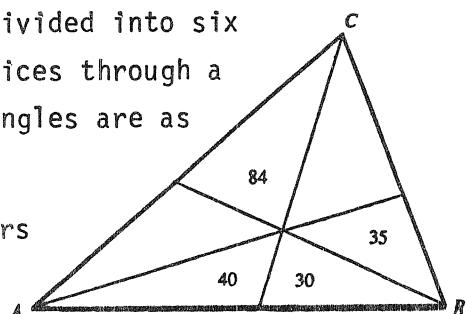
5. A sequence of integers a_1, a_2, a_3, \dots is chosen so that

$$a_n = a_{n-1} - a_{n-2}$$

for each $n \geq 3$. What is the sum of the first 2001 terms of this sequence if the sum of the first 1492 terms is 1985, and the sum of the first 1985 terms is 1492?

6. As shown in the figure on the right, $\triangle ABC$ is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of $\triangle ABC$.

7. Assume that a , b , c , and d are positive integers such that



$$a^5 = b^4, \quad c^3 = d^2, \quad \text{and} \quad c - a = 19.$$

Determine $d - b$.

8. The sum of the following seven numbers is exactly 19:

$$\alpha_1 = 2.56, \alpha_2 = 2.61, \alpha_3 = 2.65, \alpha_4 = 2.71, \alpha_5 = 2.79, \alpha_6 = 2.82, \alpha_7 = 2.86.$$

It is desired to replace each α_i by an integer approximation A_i , $1 \leq i \leq 7$, so that the sum of the A_i 's is also 19, and so that M , the maximum of the "errors" $|A_i - \alpha_i|$, is as small as possible. For this minimum M , what is $100M$?

9. In a circle, parallel chords of lengths 2, 3, and 4 determine central angles of α , β , and $\alpha+\beta$ radians, respectively, where $\alpha+\beta < \pi$. If $\cos \alpha$, which is a positive rational number, is expressed as a fraction in lowest terms, what is the sum of its numerator and denominator?

10. How many of the first 1000 positive integers can be expressed in the form

$$[2x] + [4x] + [6x] + [8x],$$

where x is a real number, and $[z]$ denotes the greatest integer less than or equal to z ?

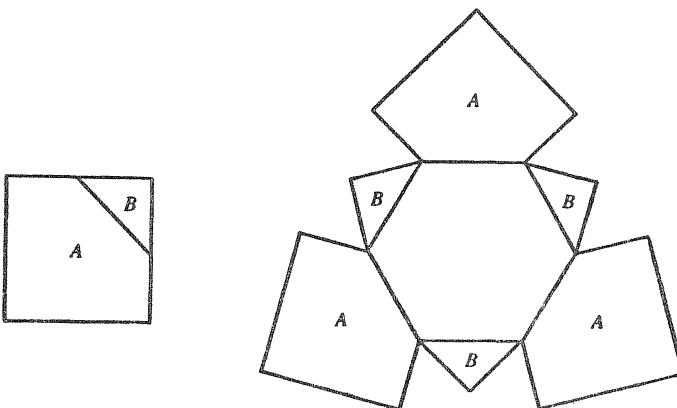
11. An ellipse has foci at $(9, 20)$ and $(49, 55)$ in the xy -plane and is tangent to the x -axis. What is the length of its major axis?

12. Let A, B, C, and D be the vertices of a regular tetrahedron, each of whose edges measures 1 meter. A bug, starting from vertex A, observes the following rule: at each vertex it chooses one of the three edges meeting at that vertex, each edge being equally likely to be chosen, and crawls along that edge to the vertex at its opposite end. Let $p = n/729$ be the probability that the bug is at vertex A when it has crawled exactly 7 meters. Find the value of n .

13. The numbers in the sequence 101, 104, 109, 116, ... are of the form $\alpha_n = 100 + n^2$, where $n = 1, 2, 3, \dots$. For each n , let d_n be the greatest common divisor of α_n and α_{n+1} . Find the maximum value of d_n as n ranges through the positive integers.

14. In a tournament each player played exactly one game against each of the other players. In each game the winner was awarded 1 point, the loser got 0 points, and each of the two players earned $1/2$ point if the game was a tie. After the completion of the tournament, it was found that exactly half of the points earned by each player were earned in games against the ten players with the least number of points. (In particular, each of the ten lowest scoring players earned half of her/his points against the other nine of the ten.) What was the total number of players in the tournament?

15. Three $12 \text{ cm} \times 12 \text{ cm}$ squares are each cut into two pieces A and B , as shown in the first figure below, by joining the midpoints of two adjacent sides. These six pieces are then attached to a regular hexagon, as shown in the second figure, so as to fold into a polyhedron. What is the volume (in cm^3) of this polyhedron?



*

For the final set of problems for this month, I give the 14th U.S.A. Mathematical Olympiad, which took place on April 23, 1985. These problems were set by the Examination Subcommittee of the U.S.A.M.O., consisting of:

M.S. Klamkin (Chairman), University of Alberta
J.D.E. Konhauser, Macalester College
Andy Liu, University of Alberta
C.C. Rousseau, Memphis State University

(I will be resigning from this committee shortly, after having served on it since it started in 1971.) Solutions to these problems, along with those of the 1985 International Mathematical Olympiad (due to take place in June-July 1985), can be obtained later this year by writing to Professor Walter E. Mientka at the address given earlier in this column.

14TH U.S.A. MATHEMATICAL OLYMPIAD

April 23, 1985 - Time: $3\frac{1}{2}$ hours

1. Determine whether or not there are any positive integral solutions of the simultaneous Diophantine equations

$$x_1^2 + x_2^2 + \dots + x_{1985}^2 = y^3,$$
$$x_1^3 + x_2^3 + \dots + x_{1985}^3 = z^2,$$

such that $x_i \neq x_j$ for all $i \neq j$.

2. Determine each real root of

$$x^4 - (2 \cdot 10^{10} + 1)x^2 - x + 10^{20} + 10^{10} - 1 = 0$$

correct to four decimal places.

3. Let A, B, C, and D denote any four points in space such that at most one of the distances AB, AC, AD, BC, BD, and CD is greater than 1. Determine the maximum value of the sum of the six distances.

4. There are n people at a party. Prove that there are two people such that, of the remaining $n-2$ people, there are at least $[n/2] - 1$ of them, each of whom knows both or else knows neither of the two. Assume that "knowing" is a symmetric relation, and that $[x]$ denotes the greatest integer less than or equal to x .

5. Let $\alpha_1, \alpha_2, \alpha_3, \dots$ be a nondecreasing sequence of positive integers. For $m \geq 1$, define

$$b_m = \min\{n : \alpha_n \geq m\},$$

that is, b_m is the minimum value of n such that $\alpha_n \geq m$. If $\alpha_{19} = 85$, determine the maximum value of

$$\alpha_1 + \alpha_2 + \dots + \alpha_{19} + b_1 + b_2 + \dots + b_{85}.$$

*

I now give solutions to some problems proposed in earlier columns. Please note that readers who submit solutions should *clearly* identify themselves by including their names, school (student or teacher), address, and also to *clearly* identify the problems by giving their numbers, as well as the year and page number of the issue where they appeared.

4. [1981: 46] From the 1978 Rumanian Mathematical Olympiad, Final Round, 11th class.

Let $f: R \rightarrow R$ be the function defined by

$$f(x) = x|x-\alpha_1| + |x-\alpha_2| + \dots + |x-\alpha_n|, \quad (1)$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are fixed real numbers. Find a condition for f to be everywhere differentiable.

Solution by M.S.K.

The function $|x - \alpha|$ is differentiable everywhere except at $x = \alpha$. Consequently, (1) will not be everywhere differentiable if there are at least two distinct α_i 's, and so a necessary condition for (1) to be everywhere differentiable is that

$$\alpha_1 = \alpha_2 = \dots = \alpha_n = \alpha, \text{ say,}$$

and then (1) becomes

$$f(x) = |x - \alpha|(x + n - 1),$$

which is differentiable everywhere except possibly at $x = \alpha$. Now $f'(\alpha)$ exists if and only if

$$\lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} \quad \text{and} \quad \lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a} \quad (2)$$

both exist and are equal, and then $f'(a)$ is their common value. The limits (2) are respectively

$$a + n - 1 \quad \text{and} \quad -(a + n - 1),$$

and they are equal if and only if each is zero, and so $a = 1 - n$. A necessary and sufficient condition for (1) to be everywhere differentiable is therefore that

$$\alpha_1 = \alpha_2 = \dots = \alpha_n = 1 - n.$$

*

1. [1981: 46] From the 1978 Rumanian Mathematical Olympiad, Final Round, 12th class.

Let $f: R \rightarrow R$ be a continuous function and let

$$g(x) = f(x) \int_0^x f(t) dt.$$

Prove that, if g is decreasing, then $f \equiv 0$ on R .

Solution by M.S.K.

The word "decreasing" in the proposal should be understood in the less restrictive sense of "nonincreasing".

First of all, $g(0) = 0$.

Suppose $f \not\equiv 0$ for $x \geq 0$. Then, by the continuity of f , there is a number $\alpha \geq 0$ such that $f(x) \equiv 0$ for $0 \leq x \leq \alpha$ and $f(x) > 0$ (or < 0) in some small neighborhood $(\alpha, \alpha+\varepsilon)$. But then

$$g(\alpha+\varepsilon) = f(\alpha+\varepsilon) \int_0^{\alpha+\varepsilon} f(t) dt > 0 = g(0),$$

contradicting the nonincreasing property of g .

Suppose $f \not\equiv 0$ for $x \leq 0$. Then there is a number $b \leq 0$ such that $f(x) \equiv 0$ for $b \leq x \leq 0$ and $f(x) > 0$ (or < 0) in some small neighborhood $(b-\varepsilon, b)$. But then

$$g(b-\varepsilon) = f(b-\varepsilon) \int_{b-\varepsilon}^0 f(t) dt = -f(b-\varepsilon) \int_{b-\varepsilon}^0 f(t) dt < 0 = g(0),$$

again contradicting the nonincreasing property of g .

Therefore $f \equiv 0$ on R .

*

3. [1981: 47] From the 1978 Rumanian Mathematical Olympiad, Final Round, 12th class.

Let F be the set of all continuous functions $f: [0,1] \rightarrow [0,2]$ such that

$$\int_0^1 f(x) dx = 1.$$

(a) As f ranges over F , find all the possible values of

$$\int_0^1 x f(x) dx.$$

(b) Show that, if both f and f^2 belong to F , then f is a constant function.

Solution by M.S.K.

(a) Geometrically, we must find all the possible values of the abscissa \bar{x} of the centroid of a region of area 1 bounded by the graph of a function $f \in F$ and the lines $x = 0$, $x = 1$, and $y = 0$. The greatest lower bound is clearly $\bar{x} = 1/4$, obtained for the function f_1 defined by $f_1(x) = 2$ for $0 \leq x \leq 1/2$ and $f_1(x) = 0$ for $1/2 < x \leq 1$; and the least upper bound is $\bar{x} = 3/4$, obtained for the function f_2 defined by $f_2(x) = 0$ for $0 \leq x < 1/2$ and $f_2(x) = 2$ for $1/2 \leq x \leq 1$. These bounds are never attained for any $f \in F$ because f_1 and f_2 are discontinuous at $x = 1/2$.

For any α such that $1/4 < \alpha < 3/4$, we can obtain $\bar{x} = \alpha$ for the piecewise linear function $f \in F$ whose graph encloses an isosceles trapezoid of height 2, upper and lower bases of lengths $1/2 - \varepsilon$ and $1/2 + \varepsilon$, respectively, and having $x = \alpha$ as axis of symmetry.

The required answer is therefore $\{\bar{x}: 1/4 < \bar{x} < 3/4\}$.

(b) If f and f^2 both belong to F , then

$$\int_0^1 \{f^2(x) - 2f(x) + 1\} dx = \int_0^1 \{f(x) - 1\}^2 dx = 0,$$

and therefore $f(x) \equiv 1$.

*

3. [1983: 303] (Corrected) From the 1980 Leningrad High School Olympiad, Third Round.

Three people are playing table tennis, and after every match the loser gives up his place to the person not playing. At the end, the first player has played 10 games and the second has played 21. How many games has the third player played?

Solution by Andy Liu, University of Alberta.

Note that in the proposal we have replaced the verb "won" by the verb "played"

in three places. As originally formulated, the problem is trivial and does not have a unique solution.

Let n be the number of games played between the first and second players. Then $n \leq 10$. Now $21-n$ games have been played between the second and third players, and between any two such games, in $21-n-1$ instances, must have intervened at least one game played by the first player. Hence $21-n-1 \leq 10$, or $n \geq 10$. Thus $n = 10$, and it follows easily that the third player has played 11 games.

*

6. [1983: 304] From the 1980 Leningrad High School Olympiad, Third Round.

For natural numbers m and n , what is the smallest value of n such that in its decimal representation the fraction m/n has the sequence ...501... after the decimal point?

Solution by Andy Liu, University of Alberta.

If the decimal expansion of m/n contains a particular sequence S of digits, then, for some $h < n$, the decimal expansion of h/n begins with S . In fact, h is the remainder that appears, in the long division of m by n , at the step just before S appears in the quotient. So, for our problem, it suffices to find the smallest value of n for which $m/n = 0.501\dots$ for some $m < n$.

Now the sequences $\{k/(2k-1)\}$ and $\{(k+1)/2k\}$ both decrease as k increases.

Since

$$\frac{125}{249} = 0.502\dots, \quad \frac{126}{250} = 0.504, \quad \text{and} \quad \frac{126}{251} = 0.501\dots,$$

the desired minimum value is $n = 251$.

*

7. [1983: 304] From the 1980 Leningrad High School Olympiad, Third Round.

Among nine coins are two counterfeits. The real coins weigh 10 g and the false ones weigh 11 g. How can the false coins be identified in five weighings on a single-pan balance if the counterweights are in units of 1 g?

Solution by Mark Rabenstein, student, McKerran Junior High School, Edmonton.

Label the coins A, B, C, D, E, F, G, H , and I .

First weighing: (A, B, C) .

Second weighing: (D, E, F) .

From these, the total weight of the group \dots can be deduced.

Case 1: One of the three groups say (A, B, C) has total weight 32 g. Then two of these coins are false.

Third weighing: A .

Fourth weighing (if necessary): I .

From these, both false coins can be identified.

Case 2: Two of the three groups, say (A,B,C) and (D,E,F) , each have a total weight of 31 g.

Third weighing: (A,D) .

Case 2a: The total weight of (A,D) is 22 g. Then A and D are the false coins.

Case 2b: The total weight of (A,D) is 21 g. Then one of these coins is false.

Fourth weighing: A .

From this, one false coin can be identified, say A . Then the other false coin is in (E,F) .

Fifth weighing: E .

From this, the other false coin can be identified.

Case 2c: The total weight of (A,D) is 20 g. Then there is a false coin in each of (B,C) and (E,F) .

Fourth weighing: B .

Fifth weighing: E .

From these, both false coins can be identified.

*

9. [1983: 304] (Corrected) *From the 1980 Leningrad High School Olympiad,
Third Round.*

Three people are playing table tennis, and after every match the loser gives up his place to the person not playing. At the end, it turns out that the first player has played 10 games, the second 15, and the third 17. Who lost the second game?

Solution by Andy Liu, University of Alberta.

Note that in the proposal we have replaced the verb "won" by the verb "played". As originally formulated, the problem is trivial and does not have a unique solution.

Let n be the number of games played between the first and second players. Then $15-n$ games were played between the second and third players, and $10-n$ games were played between the first and third players. From $(15-n) + (10-n) = 17$ follows $n = 4$. Thus 11 games were played between the second and third players. Now any two such games must be separated by at least one game played by the first player; and since the first player has played 10 games, he must have played in and lost all the even-numbered games. Hence the first player lost the second game.

*

14. [1983: 304] *From the 1980 Leningrad High School Olympiad, Third Round.*

Three people are playing table tennis, and after every match the loser gives up his place to the person not playing. At the end, it turns out that the first player has won 10 games, the second 12, and the third 14. How many games did each person play?

Solution by Andy Liu, University of Alberta.

For each player, a loss (except the final one) must be followed by a bye while a bye (except the initial one) is preceded by a loss. Hence the losses and byes come in tandem except for the initial bye and the final loss. Now the number of wins overall is even, and the number of wins for each player is also even. Hence the player who has the initial bye also takes the final loss. It follows easily that the numbers of games played by the respective players are

$$10 + \frac{12+14}{2} = 23, \quad 12 + \frac{10+14}{2} = 24, \quad 14 + \frac{10+12}{2} = 25.$$

*

16. [1983: 305] *From the 1980 Leningrad High School Olympiad, Third Round.*

At the intersections of an $n \times n$ square array real numbers are located. We decide to write in the places of any two numbers their arithmetic mean. Find all natural numbers n for which, from any initial arrangement of these numbers in the array, this operation can end up with a single number appearing everywhere in the array.

Solution by Andy Liu, University of Alberta.

We first point out that the structure of the array plays no part in the solution. We may simply deal with sequences of real numbers. Consider a sequence of m terms all of which are 0's except for one 1. The averaging process produces only fractions whose denominators are powers of 2. On the other hand, if a constant sequence is obtained, then every term equals $1/m$. This is impossible unless m is a power of 2.

We now use induction on k to prove that, for all $m = 2^k$, the averaging process can be applied to any sequence of m terms to produce a constant sequence. The result holds trivially for $k = 0$. Now divide a sequence of 2^{k+1} terms into two subsequences of 2^k terms each, which can be made identical by averaging the corresponding terms. By the induction hypothesis, each subsequence can be transformed into a constant sequence. It follows easily that the whole sequence has also turned into a constant sequence, completing the induction.

For $m = n^2$, as in the problem, n must also be a power of 2.

Query.

To utilize the structure of the array, suppose only adjacent terms may be averaged. What would be the answer then?

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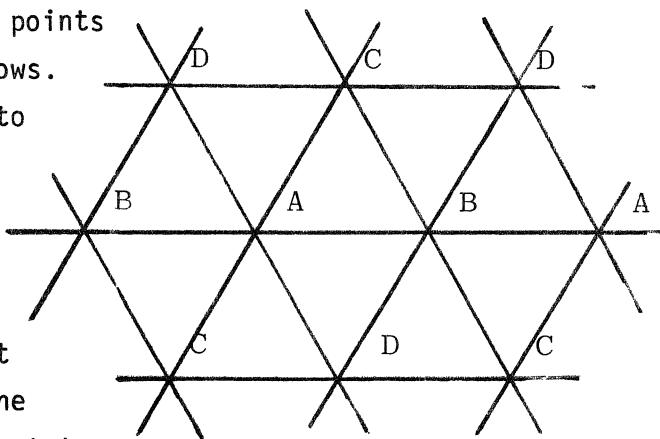
17. [1983: 305] *From the 1980 Leningrad High School Olympiad, Third Round.*

Show that any two points on the surface of a regular tetrahedron of edge 1 cm can be joined by a broken line passing along the surface of the tetrahedron whose length does not exceed $2/\sqrt{3}$ cm.

Solution by Andy Liu, University of Alberta.

Map the points of the plane onto the points on the surface of the tetrahedron as follows.

Roll the tetrahedron ABCD over the plane to generate a tessellation with labelled equilateral triangles, as shown in the figure. The points of each triangle are mapped canonically onto the points on the corresponding face of the tetrahedron. It is easy to see that the points of the plane that are mapped into the same point on the tetrahedron form an isometric lattice of distance 2.



For any point P on the tetrahedron, find one corresponding point Q of the plane. The circle with centre Q and radius $2/\sqrt{3}$ encloses an equilateral triangle T of side 2. For any point R on the tetrahedron, there exists a corresponding point S within T. The image of the line segment QS on the tetrahedron yields the desired broken line joining P and R.

*

18. [1983: 305] *From the 1980 Leningrad High School Olympiad, Third Round.*

Two spiders sit along the sides of a convex polygon. Simultaneously they begin to run along the polygon in the same sense and with the same speed. For what initial arrangement of the spiders will the shortest distance between them during the motion be the greatest?

Solution by Andy Liu, University of Alberta.

More generally, we consider any convex set. The optimal arrangement places the spiders where the chord they determine bisects the perimeter of the convex set. Then clearly, for $0^\circ \leq \theta < 180^\circ$, there exists a unique chord $c(\theta)$ determined by the spiders at some time which makes an angle of θ with a fixed axis. If the initial

arrangement is other than the above, then the spiders determine two chords $p(\theta)$ and $q(\theta)$, parallel to and one on each side of $c(\theta)$. By convexity,

$$\min \{ \overline{p(\theta)}, \overline{q(\theta)} \} \leq \overline{c(\theta)},$$

where the bars denote length. It follows easily that

$$\min \{ \min \{ \overline{p(\theta)}, \overline{q(\theta)} \}: 0^\circ \leq \theta < 180^\circ \} \leq \min \{ \overline{c(\theta)}: 0^\circ \leq \theta < 180^\circ \}.$$

*

The answers to the 15 problems of the 1985 AIME are as follows:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
384	26	198	32	986	315	757	61	49	600	85	182	401	25	864

Editor's note. All communications about this column should be sent directly to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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SOURCES IN RECREATIONAL MATHEMATICS

I have embarked on a project to find the sources of classical problems in recreational mathematics. Some of these problems are quite new - e.g. the twelve coins problem of 1945 or Rubik's Cube. Others are much more ancient than I expected - e.g. the Monkey and the Coconuts problem goes back about a thousand years more than I initially knew.

The initial objective was to produce a book of sources, translated into English with some annotation, for the Series in Recreational Mathematics that I am editing for Oxford University Press. However, it now appears that first priority must be to compile an annotated bibliography of the material. I am putting this material into a computer file, arranged by subject (some 160 subjects so far) and chronologically within each subject. This is presently 88 pages long. I also have a file of Queries relating to this material, presently 17 pages long, and I have a paper outlining the project and some of the topics. I would be delighted to hear from anyone interested in this project.

I am also compiling a list of mathematical monuments and have a computer file of a draft article on this. I would be happy to hear from anyone who knows of an interesting mathematical monument.

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MATHEMATICAL SWIFTIES

"The transformation is conformable," Tom replied harmoniously.

"Those lines are neither perpendicular nor parallel," Tom answered obliquely.

M.S. KLAMKIN

PROBLEMS -- PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before December 1, 1985, although solutions received after that date will also be considered until the time when a solution is published.

1041. Proposed by Allan Wm. Johnson Jr., Washington, D.C.

The deepest mine in the world is Western Deep Levels near Carletonville, Transvaal, South Africa. It is both

$$\begin{array}{r} \text{A} \\ \text{GOLD} \\ \hline \text{**} \\ \text{*4} \\ \text{**} \\ \text{*} \\ \hline \text{LODE} \end{array} \quad \text{and} \quad \begin{array}{r} \text{A} \\ \text{GOLD} \\ \hline \text{**} \\ \text{*4} \\ \text{**} \\ \text{*} \\ \hline \text{LOAD} . \end{array}$$

Solve these homophonic decimal multiplications independently.

1042. Proposed by Clark Kimberling, University of Evansville, Indiana.

Let P be a point in the plane of a given triangle ABC; let A'B'C' be the cevian triangle of the point P for the triangle ABC (with A' on line BC, etc.); and let the circumcircle of triangle A'B'C' meet the lines BC, CA, AB again in A'', B'', C'', respectively. Prove that the lines AA'', BB'', CC'' are concurrent.

1043. Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.

Prove that the projections of a point P on the faces of a tetrahedron T are coplanar if and only if P lies on a particular cubic surface which passes through the edges of T.

(This is an extension to three dimensions of the Wallace-Simson theorem, which states that the projections of a point P on the sides of a triangle are collinear if and only if P lies on the circumcircle of the triangle.)

1044. Proposed by Peter Messer, M.D., Mequon, Wisconsin.

Find a simple expression for the positive root of the equation

$$x^3 - 3x^2 - x - \sqrt{2} = 0.$$

1045. Proposed by George Tsintsifas, Thessaloniki, Greece.

Let P be an interior point of triangle ABC; let x, y, z be the distances of P from vertices A, B, C, respectively; and let u, v, w be the distances of P from sides BC, CA, AB, respectively. The well-known Erdős-Mordell inequality states that

$$x + y + z \geq 2(u + v + w).$$

Prove the following related inequalities:

$$(a) \frac{x}{vw} + \frac{y}{wu} + \frac{z}{uv} \geq 12,$$

$$(b) \frac{x}{v+w} + \frac{y}{w+u} + \frac{z}{u+v} \geq 3,$$

$$(c) \frac{x}{\sqrt{vw}} + \frac{y}{\sqrt{wu}} + \frac{z}{\sqrt{uv}} \geq 6.$$

1046. Proposed by Jordan B. Tabov, Sofia, Bulgaria.

The Wallace point W of any four points A_1, A_2, A_3, A_4 on a circle with center O may be defined by the vector equation

$$\vec{OW} = \frac{1}{2}(\vec{OA}_1 + \vec{OA}_2 + \vec{OA}_3 + \vec{OA}_4)$$

(see the article by Bottema and Groenman in this journal [1982: 126]).

Let γ be a cyclic quadrilateral the Wallace point of whose vertices lies inside γ . Let a_i ($i = 1, 2, 3, 4$) be the sides of γ , and let G_i be the midpoint of the side opposite to a_i . Find the minimum value of

$$f(X) \equiv a_1 \cdot G_1 X + a_2 \cdot G_2 X + a_3 \cdot G_3 X + a_4 \cdot G_4 X,$$

where X ranges over all the points of the plane of γ .

1047*. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let p, q, r be three different natural numbers, not all even. Prove or disprove that $(x, y, z) = (1, 1, 1)$ is the only real solution of the system

$$\begin{cases} x^q + y^r + z^p = 3 \\ x^r + y^p + z^q = 3 \\ x^p + y^q + z^r = 3. \end{cases}$$

Generalize.

1048. Proposed by J.T. Groenman, Arnhem, The Netherlands.

In base ten, $361 = 19^2$. Find at least three other bases in which 361 is a perfect square.

1049*. Proposed by Jack Garfunkel, Flushing, N.Y.

Let ABC and A'B'C' be two nonequilateral triangles such that $A \geq B \geq C$ and $A' \geq B' \geq C'$. Prove that

$$A - C > A' - C' \iff \frac{s}{r} > \frac{s'}{r'},$$

where s, r and s', r' are the semiperimeter and inradius of triangles ABC and A'B'C', respectively.

1050. Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.

In the plane, you are given the curve known as the folium of Descartes. Show how to construct the asymptote to this curve using straightedge and compasses only.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

922. [1984: 88] Proposed by A.W. Goodman, University of South Florida.

Let

$$S_n(z) = \frac{n(n-1)}{2} + \sum_{k=1}^{n-1} (n-k)^2 z^k, \quad (1)$$

where $z = e^{i\theta}$. Prove that, for all real θ ,

$$\operatorname{Re}(S_n(z)) = \frac{\sin \theta}{2(1 - \cos \theta)^2} (n \sin \theta - \sin n\theta) \geq 0. \quad (2)$$

Solution by M.S. Klamkin, University of Alberta.

The function (1) is periodic with period 2π , so it suffices to show that (2) holds for all $\theta \in [0, 2\pi)$. From (1),

$$\operatorname{Re}(S_n(z)) = \frac{n(n-1)}{2} + n^2 \sum_{k=1}^{n-1} \cos k\theta - 2n \sum_{k=1}^{n-1} k \cos k\theta + \sum_{k=1}^{n-1} k^2 \cos k\theta. \quad (3)$$

We assume for now that $\theta \neq 0$. The following sums are well known [1]:

$$\sum_{k=1}^{n-1} \cos k\theta = \frac{\cos(n\theta/2) \sin((n-1)\theta/2)}{\sin(\theta/2)} = \frac{\sin((2n-1)\theta/2)}{2 \sin(\theta/2)} - \frac{1}{2}, \quad (4)$$

$$\sum_{k=1}^{n-1} k \cos k\theta = \frac{n \sin((2n-1)\theta/2)}{2 \sin(\theta/2)} - \frac{1 - \cos n\theta}{4 \sin^2(\theta/2)}, \quad (5)$$

$$\sum_{k=1}^{n-1} k \sin k\theta = \frac{\sin n\theta}{4 \sin^2(\theta/2)} - \frac{n \cos((2n-1)\theta/2)}{2 \sin(\theta/2)}; \quad (6)$$

and differentiating (6) with respect to θ yields

$$\sum_{k=1}^{n-1} k^2 \cos k\theta = \frac{n \cos n\theta}{4 \sin^2(\theta/2)} - \frac{\sin n\theta \cos(\theta/2)}{4 \sin^3(\theta/2)} + \frac{n(2n-1) \sin((2n-1)\theta/2)}{4 \sin(\theta/2)} + \frac{n \cos((2n-1)\theta/2) \cos(\theta/2)}{4 \sin^2(\theta/2)}. \quad (7)$$

Now substituting (4), (5), and (7) into (3) yields, after trigonometric simplifications,

$$\operatorname{Re}(S_n(z)) = \frac{\sin \theta}{2(1 - \cos \theta)^2} (n \sin \theta - \sin n\theta), \quad (8)$$

and we must show that

$$\sin \theta (n \sin \theta - \sin n\theta) \geq 0, \quad 0 < \theta < 2\pi. \quad (9)$$

Equality holds in (9) for $\theta = \pi$. For $\theta = m\pi + \phi$, where $m = 0$ or 1 and $0 < \phi < \pi$, (9) becomes

$$(-1)^m \sin \phi ((-1)^m n \sin \phi - (-1)^{mn} \sin n\phi) \geq 0,$$

and this is equivalent to

$$n \sin \phi \geq (-1)^{m(n-1)} \sin n\phi. \quad (10)$$

We now appeal to inequality 3.4.4 in [2]: "If $0 < x_k < \pi$ for $k = 1, 2, \dots$, and if $n > 1$ is an integer, then

$$\sum_{k=1}^n \sin x_k > \left| \sin \sum_{k=1}^n x_k \right|. \quad (11)$$

(This result follows from the subadditivity of the sine function.) Now (10) follows by setting $x_1 = x_2 = \dots = x_n = \phi$ in (11).

Finally, we must verify that (8) is nonnegative for $\theta = 0$. Since this is an indeterminate form of the type $0/0$ and $\operatorname{Re}(S_n(z))$ is continuous at $\theta = 0$, we find the limiting value of (8) as $\theta \rightarrow 0$. This limit is most conveniently found by setting $\theta = 0$ in (3). The result is $(n-1)n(n+1)/3 \geq 0$.

An incomplete solution was received from WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.

REFERENCES

1. L.B.W. Jolley, *Summation of Series*, Dover, New York, 1961, pp. 78, 80 (Series Nos. 418, 427, 428).
2. D.S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Heidelberg, 1970, p. 236.

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923. [1984: 88] Proposed by Clark Kimberling, University of Evansville, Indiana.

Let the ordered triple (a,b,c) denote the triangle whose side lengths are a,b,c . Similarity being an equivalence relation on the set of all triangles, let the ordered ratios $a:b:c$ (which we will call a *triclass*) denote the equivalence class of all triangles (a',b',c') such that

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'};$$

and let T be the set of all triclasses. A multiplication \circ on T is defined by

$$a:b:c \circ \alpha:\beta:\gamma = a\alpha + (c-b)(\gamma-\beta) : b\beta + (c-a)(\gamma-\alpha) : c\gamma + (b-a)(\beta-\alpha).$$

(a) Prove that (T,\circ) is a group.

(b) If \hat{T} is the set of all $a:b:c$ in T such that a,b,c are integers, prove that every triclass in \hat{T} is a unique product (to within order of factors) of "prime" triclasses (relative to the multiplication induced on \hat{T}).

Solution by the proposer.

(a) Let

$$u = \frac{b+c-a}{2}, \quad v = \frac{c+a-b}{2}, \quad w = \frac{a+b-c}{2},$$

so that

$$v+w=a, \quad w+u=b, \quad u+v=c.$$

Then (a,b,c) is a triangle if and only if $u > 0$, $v > 0$, and $w > 0$. Let

$$P = \{u:v:w \mid u > 0, v > 0, w > 0\}.$$

We define a multiplication " \cdot " on P as follows:

$$u:v:w \cdot u':v':w' = uu':vv':ww'.$$

Then it is easily verified that (P,\cdot) is an Abelian group with identity 1:1:1.

Let the transformation $\phi: P \rightarrow T$ be defined by

$$\phi(u:v:w) = (v+w):(w+u):(u+v) = a:b:c. \quad (1)$$

Then ϕ is clearly a bijection from P onto T , with

$$\phi^{-1}(a:b:c) = (b+c-a):(c+a-b):(a+b-c) = u:v:w, \quad (2)$$

and a bit of simple algebra shows that

$$\phi(u:v:w \cdot u':v':w') = \phi(u:v:w) \circ \phi(u':v':w').$$

Thus ϕ is an isomorphism between (P,\cdot) and (T,\circ) , and therefore (T,\circ) is an Abelian group with identity $\phi(1:1:1) = 1:1:1$.

(b) Let \hat{P} be the set of all $u:v:w$ in P such that u,v,w are integers. It follows from (1) and (2) that the restriction of ϕ to \hat{P} maps \hat{P} bijectively onto \hat{T} , so our search for "prime" triclasses can be carried out in \hat{P} .

In \hat{P} , $u:v:w = u:v:v \cdot v:v:w$, so each triclass is a product of two isosceles triclasses. If $u = lm$ and $v = pq$, then

$$u:v:v = l:p:p \cdot m:q:q.$$

This shows that u and v must be prime integers or 1 in order for $u:v:v$ to be a "prime" triclass. In summary, the "prime" triclasses in \hat{P} are isosceles triclasses of the form $u:v:v$ or $v:v:w$, where u,v,w are prime integers or 1, and every $u:v:w$ can be factored all the way down to "prime" triclasses. That this factorization is unique except for the order of multiplication follows from the uniqueness of factorization of integers and from the fact that the multiplication induced in \hat{P} is commutative.

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924. [1984: 89] *Proposed by Charles W. Trigg, San Diego, California.*

In a 3-by-3 array, when the sums, S , of the elements in the four 2-by-2 subarrays are the same, the large square is said to be gnomon-magic.

Find all third-order gnomon-magic squares in which the elements are consecutive digits, and the digits in one of the 2-by-2 arrays form an arithmetic progression.

Solution by the proposer.

In the gnomon-magic square

$$\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & k \end{array}$$

$a + d = c + f$ and $d + g = f + k$. Subtracting, $a - g = c - k$, whereupon $a + e + k = c + e + g$. Also, $a + b = g + h$ and $b + c = h + k$. The element e appears in four subarrays, whereas no other element appears in more than two subarrays.

Every gnomon-magic square with the nine nonzero digits as elements has a complementary square obtained by subtracting each digit in the square from 10. In the same sense, four-digit arithmetic progressions exist in complementary pairs:

$$\begin{array}{lll} (1, 2, 3, 4) & \text{and} & (9, 8, 7, 6), \\ (2, 3, 4, 5) & \text{and} & (8, 7, 6, 5), \\ (3, 4, 5, 6) & \text{and} & (7, 6, 5, 4), \\ (1, 3, 5, 7) & \text{and} & (9, 7, 5, 3). \end{array}$$

The only progression not accounted for so far, (2,4,6,8), is self-complementary. Consequently, only five progressions need be tested in the a,b,d,e positions in 3-by-3 arrays.

The digits of (1,2,3,4) sum to $S = 10$. In a gnomon-magic square there must be four different sums of distinct digits equal to S . There is only one partitioning of 10 into distinct digits. Hence this progression cannot appear in a 2-by-2 array in a gnomon-magic square.

Placing the digits of (2,4,6,8) in a small square leaves all other positions to be occupied by odd digits. Thus one diagonal of the large square will contain one odd and two even digits, whereas the other diagonal will contain one even and two odd digits. Since the diagonals cannot have equal sums, the large square will not be gnomon-magic.

$2 + 3 + 4 + 5 = 14$. The other four partitions of 14 into distinct digits are

$$1 + 2 + 3 + 8, \quad 1 + 2 + 4 + 7, \quad 1 + 2 + 5 + 6, \quad 1 + 3 + 4 + 6.$$

Of the digits in the progression, only 2 appears in four partitions, so $e = 2$. But then 1 appears in three partitions, so (2,3,4,5) cannot be the progression sought.

The digits of (3,4,5,6), which sum to $S = 18$, can appear in the a,b,d,e positions in twelve essentially different orders, namely:

$$\begin{array}{cccccccccccc} 4 & 5 & 5 & 4 & 5 & 3 & 6 & 3 & 6 & 3 & 6 & 4 \\ 6 & 3 & 6 & 3 & 6 & 4 & 5 & 4 & 4 & 5 & 5 & 3 \\ 3 & 5 & 3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 & 5 & 3 \\ 6 & 4 & 6 & 5 & 6 & 5 & 5 & 6 & 5 & 6 & 4 & 6 \end{array}$$

In these twelve respective cases, we must have

$$c + f = 10, 11, 11, 11, 10, 11, 9, 9, 10, 8, 9, 9$$

and

$$g + h = 9, 9, 8, 9, 9, 10, 8, 7, 7, 7, 7, 8.$$

In all but the first five cases, no combination of the unused digits {1,2,7,8,9} gives sums $c + f$ and $g + h$ that are both satisfactory. For the first five cases, we have the unique respective possibilities

$$c + f = 9+1, 9+2, 9+2, 9+2, 9+1;$$

$$g + h = 7+2, 8+1, 7+1, 8+1, 7+2.$$

The value of k is then uniquely determined, it being the last unused digit, namely

4 5	5 4	5 3	6 3	6 3
6 3	6 3	6 4	5 4	4 5
8	7	8	7	8

Thus the sums of the principal diagonals are 15, 15, 17, 17, and 19, respectively. But in no case can the other diagonal be completed to have an equal sum from the available c, f and g, h pairs.

We have left only the progression (1,3,5,7). Here $S = 1 + 3 + 5 + 7 = 16$. The other partitions of 16 into four distinct digits are

$$1+2+4+9, \quad 1+2+5+8, \quad 1+2+6+7, \quad 1+3+4+8, \quad 1+4+5+6, \quad 2+3+4+7, \quad 2+3+5+6.$$

Only three of these partitions contain 7, so $e \neq 7$. In the four partitions containing 5, the digit 1 appears three times, so $e \neq 5$. Thus there are only six essentially different possible distributions in the a, b, d, e positions, namely:

5 3	3 5	7 3	1 5	5 1	7 1
7 1	7 1	5 1	7 3	7 3	5 3

which, for conversion into gnomon-magic squares, require respectively

$$c + f = 12, 10, 12, 8, 12, 12$$

and

$$g + h = 8, 8, 10, 6, 6, 8.$$

The only sums that can be formed from the set of unused digits {2,4,6,8,9} to meet these requirements are

$$c + f = 4 + 8 \quad \text{and} \quad g + h = 2 + 6,$$

to be applied to the first and last squares. It follows that $k = 9$, and the diagonal sums in the last square must be 19, but no two available digits sum to 16 to complete the other diagonal. Hence, to within rotations and reflections, the only third-order gnomon-magic squares formed from the nine nonzero digits that have a subsquare with digits in arithmetic progression are

$$5 \ 3 \ 8$$

7 1 4 and its complement

$$6 \ 2 \ 9$$

$$S = 16$$

$$5 \ 7 \ 2$$

$$3 \ 9 \ 6$$

$$4 \ 8 \ 1$$

$$S = 24$$

When 1 is subtracted from each digit in these squares, two more gnomon-magic squares with consecutive digit elements and a 2-by-2 array with elements in arithmetic progression are formed, namely:

$$\begin{array}{ccc}
 4 & 2 & 7 \\
 6 & 0 & 3 \\
 5 & 1 & 8
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 4 & 6 & 1 \\
 2 & 8 & 5 \\
 3 & 7 & 0
 \end{array}$$

$$S = 12 \qquad S = 20$$

One incorrect solution was received.

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925, [1984: 89] Proposed by J.T. Groenman, Arnhem, The Netherlands.

The points A_i , $i = 1, 2, 3$, are the vertices of a triangle with sides a_i and median lines m_i . Through a point P in the plane, the lines parallel to m_i intersect a_i in S_i . Find the locus of P if the three points S_i are collinear.

Solution by Dan Pedoe, University of Minnesota.

By an orthogonal projection the triangle $A_1A_2A_3$ can be mapped onto an equilateral triangle, and since parallel lines map into parallel lines, and midpoints into midpoints, the mapped locus is that of a point P' such that the feet of the perpendiculars from P' onto the sides of the triangle are collinear. This locus is the circumcircle of the triangle, the line being the Wallace-Simson line, so that the original locus is an ellipse which passes through the vertices of the given triangle.

It is, in fact, a special ellipse, the Steiner ellipse, in which the tangents at the vertices are parallel to the opposite sides. Taking the given triangle as triangle of reference, and using areal coordinates, the equation of the ellipse is

$$yz + zx + xy = 0.$$

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; JORDI DOU, Barcelona, Spain; JORDAN B. TABOV, Sofia, Bulgaria; and the proposer.

Editor's comment.

The problem, as the editor belatedly discovered, is not new. It appears in the following form in [1], where it is credited to E. Cesaro (*Mathesis*, 1893, p.70):

Les parallèles menées par un point de l'ellipse de Steiner aux médianes d'un triangle, rencontrent les côtés opposés sur une droite.

Dou stated without proof that the locus of P remains an ellipse circumscribed to the triangle if the medians m_i are replaced by three arbitrary concurrent cevians.

The proposer noted that the problem was suggested to him by the following extension of O. Bottema [2]:

A tetrahedron $A_1A_2A_3A_4$ is given, a_i is the face opposite to A_i , and m_i is the median through A_i . The line through a point P parallel to m_i intersects a_i at S_i ($i = 1, 2, 3, 4$). Determine the 'locus c.' P if the four points S_i are coplanar.

REFERENCES

1. F.G.-M., *Exercices de Géométrie*, Quatrième édition, Mame & Fils, Tours, 1907, p. 1064.

2. Problem 652, (proposed by) O. Bottema, *Nieuw Archief voor Wiskunde*, July 1983, p. 249.

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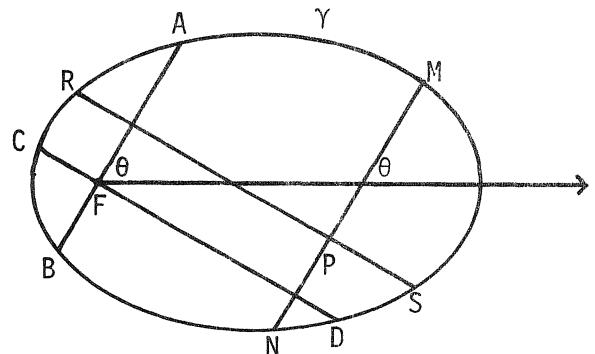
926, [1984: 89] Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.

Let P be a fixed point inside an ellipse, L a variable chord through P , and L' the chord through P that is perpendicular to L . If P divides L into two segments of lengths m and n , and if P divides L' into two segments of lengths r and s , prove that $1/mn + 1/rs$ is a constant.

I. Solution by Gali Salvatore, Perkins, Québec.

More generally, we show that the desired result holds for any point P in the plane of any conic γ . We introduce a polar coordinate system (ρ, ϕ) with the pole at a focus F and polar axis along the focal axis. If, as in the figure, the directrix lies to the left of the pole, then the equation of γ is

$$\rho = \frac{\ell}{1 - e \cos \phi},$$



where ℓ (semi-latus rectum) and e (eccentricity) are constants.

Let P be any point in the plane of γ , and suppose that two perpendicular chords through P meet γ in M, N and R, S , as shown in the figure, with the chord MN making an angle θ with the polar axis. We set

$$PM = m, \quad PN = n, \quad PR = r, \quad PS = s,$$

and we will show later that $1/mn + 1/rs$ is a constant independent of θ .

Through the pole F , draw chords AB and CD parallel to MN and RS , respectively, and set

$$FA = a, \quad FB = b, \quad FC = c, \quad FD = d.$$

Then we have

$$a = \frac{\ell}{1 - e \cos \theta},$$

$$b = \frac{\ell}{1 - e \cos(\theta + \pi)} = \frac{\ell}{1 + e \cos \theta},$$

$$c = \frac{\ell}{1 - e \cos(\theta + \pi/2)} = \frac{\ell}{1 + e \sin \theta},$$

$$d = \frac{\ell}{1 - e \cos(\theta + 3\pi/2)} = \frac{\ell}{1 - e \sin \theta}.$$

Now

$$\frac{1}{ab} + \frac{1}{cd} = \frac{1 - e^2 \cos^2 \theta}{\ell^2} + \frac{1 - e^2 \sin^2 \theta}{\ell^2} = \frac{2 - e^2}{\ell^2},$$

which is a constant independent of θ .

We now appeal to the following theorem (translated from [1]):

THEOREM OF NEWTON. Through a point P in the plane of an ellipse two arbitrary chords are drawn. The ratio of the product of the segments determined by the curve on one of these chords to the product of the segments of the other chord is constant for every point P , provided the direction of the chords is invariant.

F.G.-M. proves the theorem, and then remarks that the theorem of Newton holds for an arbitrary algebraic curve, the particular case relative to conics being due to Apollonius (*Nouvelles Annales mathématiques*, 1844, p. 510).

We can now complete our proof. It follows from the theorem of Newton that $ab/cd = mn/rs$, from which

$$\frac{1}{mn} = \frac{k}{ab} \quad \text{and} \quad \frac{1}{rs} = \frac{k}{cd},$$

where k is a constant, and therefore

$$\frac{1}{mn} + \frac{1}{rs} = k\left(\frac{1}{ab} + \frac{1}{cd}\right) = \frac{k(2 - e^2)}{\ell^2},$$

and this is a constant independent of θ .

II. Comment by M.S. Klamkin, University of Alberta.

A problem [equivalent to the general result proved in solution I above] appears in Loney [2]. Analogous results hold in n dimensions. In 3-dimensional space, for example, we have:

Through a fixed point O in space, three mutually perpendicular chords are drawn to meet a given quadric surface in P, P' ; Q, Q' ; and R, R' . Then

$$\frac{1}{PO \cdot OP'} + \frac{1}{QO \cdot OQ'} + \frac{1}{RO \cdot OR'} = \text{constant}.$$

Also solved by W.J. BLUNDON, Memorial University of Newfoundland (partial solution); J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; DAN PEDOE, University of Minnesota; JORDAN B. TABOV, Sofia, Bulgaria; and the proposer. A comment was received from WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.

REFERENCES

1. F.G.-M., *Exercices de Géométrie*, Quatrième édition, Mame & Fils, Tours, 1907, pp. 993-994.
2. S.L. Loney, *The Elements of Coordinate Geometry*, Macmillan, London, 1953, p. 377.

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927. [1984: 89] Proposed by J.L. Brenner, Palo Alto, California, and Lorraine L. Foster, California State University, Northridge, California.

Find all sets of four integers x, y, z, w such that

$$3^x + 3^y = 2^z + 2^w.$$

I. Solution by Kenneth M. Wilke, Topeka, Kansas.

The equation is symmetric in x and y , as well as in z and w , and clearly no exponent can be negative; so we may assume without loss of generality that $x \geq y \geq 0$ and $z \geq w \geq 0$. We will show that there are exactly seven solutions:

$$(x, y, z, w) = (0, 0, 0, 0), \quad (1)$$

$$(1, 0, 1, 1), \quad (2)$$

$$(2, 0, 3, 1), \quad (3)$$

$$(1, 1, 2, 1), \quad (4)$$

$$(2, 2, 4, 1), \quad (5)$$

$$(2, 1, 3, 2), \quad (6)$$

$$(3, 2, 5, 2). \quad (7)$$

We will several times refer to Problem 250 [1978: 39], which states that $|3^m - 2^n| = 1$ for positive integers m and n if and only if $(m, n) = (1, 1), (1, 2)$, or $(2, 3)$.

The given equation can be rewritten as

$$3^y(3^{x-y} + 1) = 2^w(2^{z-w} + 1). \quad (8)$$

If $w = 0$, then the two sides of (8) have different parities unless also $z = 0$, and this leads immediately to the trivial solution (1). We now assume that $w > 0$. It is easy to show that $3^m \equiv 1$ or $3 \pmod{8}$ according as m is even or odd. Accordingly,

$$3^{x-y} + 1 \equiv \begin{cases} 2, & \text{if } x-y \text{ is even,} \\ 4, & \text{if } x-y \text{ is odd,} \end{cases} \pmod{8} \quad (9)$$

and, from this and (8), $w = 1$ or 2 .

Case 1: $w = 1$. If $y = 0$, then (8) is equivalent to $3^x - 2^z = 1$, where $x > 0$

and $z > 0$, and, by Problem 250, $(x,z) = (1,1)$ or $(2,3)$, yielding solutions (2) and (3). If $y > 0$, then, with (8) reading

$$3^y(3^{x-y} + 1) = 2(2^{z-1} + 1), \quad (10)$$

considerations of parity show that $z > 1$ (for $z = 1$ implies $y = 0$) and hence that $3^{x-y} + 1 = 2$, so $x = y$. Now (10) is equivalent to $3^y - 2^{z-1} = 1$, and Problem 250 gives $(y,z) = (1,2)$ or $(2,4)$. Solutions (4) and (5) follow.

Case 2: $w = 2$. Equation (8) now reads

$$3^y(3^{x-y} + 1) = 4(2^{z-2} + 1).$$

Since 3^y is odd and $z > 2$ (for $z = 2$ would contradict (9)), we must have

$$3^{x-y} + 1 = 4 \quad \text{and} \quad 3^y - 2^{z-2} = 1.$$

The first of these gives $x-y = 1$ and, from Problem 250, the second gives $(y,z) = (1,3)$ or $(2,5)$. Solutions (6) and (7) follow.

II. Comment by Curtis Cooper, Central Missouri State University at Warrensburg.

In [1], Pillai found all the solutions to the Diophantine equations

$$2^x - 3^y = 3^Y - 2^X, \quad 2^x - 3^y = 2^X + 3^Y, \quad 3^y - 2^x = 2^X + 3^Y,$$

the first of which is equivalent to our own.

III. Comment by the proposers.

Readers who wish more information on exponential Diophantine equations (*eDe's*) like this one will find a considerable literature, much of it referenced in [2]. The equation $2^x + 3^y = 2^z + 3^w$ has only a finite number of nonobvious solutions. But if $x = z$ and $y = w$ there are infinitely many obvious solutions. Schlickewei [3] showed that $\sum \pm p_i^{x_i} = 0$, where the p_i are distinct primes, has at most finitely many solutions. He does not show how to find them. Many *eDe's* are solved in [2]. It is shown in [4] that no algorithm can be given for solving an *eDe*. We do not know how to recognize when an *eDe* can be solved using modular arithmetic, though some progress has been made along these lines. The two equations

$$1 + 3^x = 2^y + 2^z 3^w \quad \text{and} \quad 1 + 3^x = 5^y + 3^z 5^w$$

seem to be similar. The first can be solved using modular arithmetic; the second cannot be solved by that method.

Incomplete solutions were submitted by CLAYTON W. DODGE, University of Maine at Orono; JACK LESAGE, Eastview Secondary School, Barrie, Ontario; and the proposers.

Editor's comment.

Our other solvers found only 4, 5, and 6, respectively, of the 7 solutions. And *none* of them found solution (3). So let's check to make sure. Is it true that

$$3^2 + 3^0 = 2^3 + 2^1?$$

You bet it is.

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928.* [1984: 89] *Proposed by Kaidy Tan, Fukien Teachers' University, Foochow, Fukien, People's Republic of China.*

Through a given point in space, construct a plane that bisects the total surface area and volume of a given tetrahedron.

Editor's comment.

No solution was received for this problem, which therefore remains open.

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929. [1984: 89] *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

Given a triangle ABC, find all interior points P such that, if AP,BP,CP meet the circumcircle again in A_1,B_1,C_1 , respectively, then triangles ABC and $A_1B_1C_1$ are congruent.

I. *Solution by Jordi Dou, Barcelona, Spain.*

Let Γ be the circumcircle and O its center. We assume that "interior points" means "points inside the circumcircle". Let $T \equiv ABC$ have sides a,b,c in the usual order. Suppose an interior point $P = P_i$ generates, in the manner described in the proposal, a triangle $T_i \equiv A_iB_iC_i$, with sides a_i,b_i,c_i in the usual order, which is congruent to T . This can happen in six different ways:

$$\begin{aligned}
 P_1 &\rightarrow \alpha_1 = a, b_1 = b, c_1 = c; \\
 P_2 &\rightarrow \alpha_2 = b, b_2 = c, c_2 = a; \\
 P_3 &\rightarrow \alpha_3 = c, b_3 = a, c_3 = b; \\
 P_4 &\rightarrow \alpha_4 = a, b_4 = c, c_4 = b; \\
 P_5 &\rightarrow \alpha_5 = c, b_5 = b, c_5 = a; \\
 P_6 &\rightarrow \alpha_6 = b, b_6 = a, c_6 = c.
 \end{aligned}$$

It is easy to show that

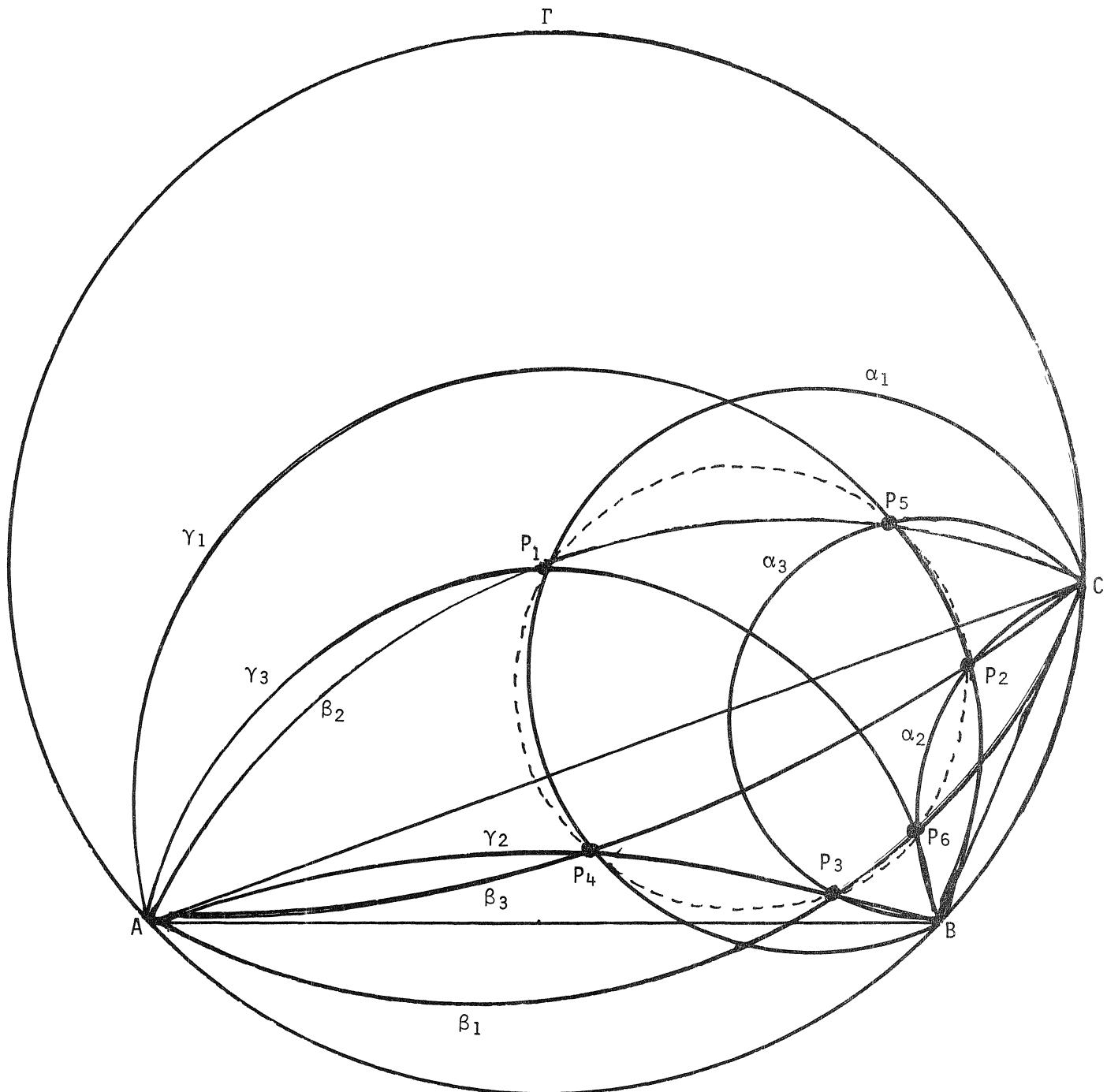
$$\begin{aligned}
 \{P \mid \alpha_1 = a\} &= \text{circular arc } \alpha_1 \text{ through } O, B, C; \\
 \{P \mid \alpha_2 = b\} &= \text{circular arc } \alpha_2 \text{ through } B, C, \text{ tangent to } b; \\
 \{P \mid \alpha_3 = c\} &= \text{circular arc } \alpha_3 \text{ through } B, C, \text{ tangent to } c; \\
 \{P \mid b_1 = a\} &= \text{circular arc } \beta_1 \text{ through } C, A, \text{ tangent to } a; \\
 \{P \mid b_2 = b\} &= \text{circular arc } \beta_2 \text{ through } O, C, A; \\
 \{P \mid b_3 = c\} &= \text{circular arc } \beta_3 \text{ through } C, A, \text{ tangent to } c; \\
 \{P \mid c_1 = a\} &= \text{circular arc } \gamma_1 \text{ through } A, B, \text{ tangent to } a; \\
 \{P \mid c_2 = b\} &= \text{circular arc } \gamma_2 \text{ through } A, B, \text{ tangent to } b; \\
 \{P \mid c_3 = c\} &= \text{circular arc } \gamma_3 \text{ through } O, A, B.
 \end{aligned}$$

It now follows easily that

$$\begin{aligned}
 P_1 &= \alpha_1 \cap \beta_2 \cap \gamma_3, & P_4 &= \alpha_1 \cap \beta_3 \cap \gamma_2, \\
 P_2 &= \alpha_2 \cap \beta_3 \cap \gamma_1, & P_5 &= \alpha_3 \cap \beta_2 \cap \gamma_1, \\
 P_3 &= \alpha_3 \cap \beta_1 \cap \gamma_2, & P_6 &= \alpha_2 \cap \beta_1 \cap \gamma_3.
 \end{aligned}$$

The required point set is $\{P_1, P_2, \dots, P_6\}$, and some of these points may coincide in special triangles. It is clear that P_1 is the circumcenter O , and it follows from Theorem 446 in [1] that P_2 and P_3 are the positive and negative Brocard points Ω and Ω' , respectively, of triangle ABC . \square

The configuration yields an unexpected bonus: the six points P_i are all concyclic. The points P_1, P_4, P_6, P_3 are the intersections (distinct from B, A) of circles α_1, β_1 with circles γ_3, γ_2 . An inversion I_C with center C transforms circles γ_3, γ_2 into circles γ'_3, γ'_2 which intersect in the inverses A', B' of A, B . Circle α_1 transforms into the straight line α'_1 which passes through B , and circle β_1 into the straight line β'_1 which passes through A . The circumcircle Γ transforms into the straight line $A'B'$. As circles α_1 and β_1 each form with circumcircle Γ an angle



equal to A , the straight lines α'_1 and β'_1 each form with the common chord $B'A'$ of γ'_3 and γ'_2 an angle equal to A . Thus the points P'_1, P'_4, P'_6, P'_3 are the vertices of an isosceles trapezoid and so are concyclic. Therefore P_1, P_4, P_6, P_3 are concyclic. It can be shown in the same way that P_1, P_5, P_4, P_3 are concyclic, and so are P_1, P_4, P_5, P_2 . Therefore all six points P_i are concyclic.

II. Comment by Jordan B. Tabov, Sofia, Bulgaria.

The following extension of our problem was given at the 1984 All-Union Mathematical Olympiad in the U.S.S.R.:

3. A triangle ABC is given. Through a point P in the plane, straight lines PA, PB, PC are drawn to meet the circumcircle of the triangle in points A₁, B₁, C₁ different from the vertices of the triangle. It happens that triangle A₁B₁C₁ is congruent to triangle ABC. Prove that there are at most 8 points P with this property. (Igor Sharygin)

In addition to the (at most) 6 points P inside the circumcircle, there is (assuming A ≤ B ≤ C) one point P in the region outside the circumcircle bounded by the rays AB and AC, and one point P in the region outside the circumcircle bounded by the rays AC and BC.

In a comment [2] on this Problem 3, the Editor of the Problem Department of *Kvant* in effect proved the following:

Two (not necessarily congruent) triangles ABC and A₁B₁C₁ are inscribed in a circle Γ. Through a point P in the plane straight lines PA, PB, PC are drawn to meet Γ in points A₂, B₂, C₂, respectively, distinct from the vertices A, B, C. Prove that there exist at most 12 positions of P such that triangles A₁B₁C₁ and A₂B₂C₂ are congruent.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; DAN PEDOE, University of Minnesota; MALCOLM A. SMITH, Georgia Southern College, Statesboro; and the proposer.

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930. [1984: 89] Proposed by the Cops of Ottawa.

Does there exist a tetrahedron such that all its edge lengths, all its face areas, and its volume are integers? If so, give a numerical example.

I. Solution by Jordan B. Tabov, Sofia, Bulgaria.

The answer is yes. The following example is known (sorry, I am not able to give a reference): the tetrahedron with edges

896, 990, 1073, 1073, 1073, 1073,

the first two being opposite edges, has integral face areas and integral volume.

It is known also that a tetrahedron with edges

$$6, 7, 8, 9, 10, 11$$

has integral volume, but not, unfortunately, all integral face areas.

II. Comment by Richard K. Guy, University of Calgary.

I have a note that reads: "John Leech observes that four copies of a Heronian triangle will fit together to make such a tetrahedron provided that the volume is made rational, and this is not difficult. E.g., three pairs of opposite edges of lengths 148, 195, 203. Is there a smaller example?"

The Leech references [3]-[5] contain interesting information about related problems.

III. Comment by M.S. Klamkin, University of Alberta.

It is known that such tetrahedra exist. I quote from Dickson [1, p. 224], in which the superscript references are identified:

"R. Güntsche¹⁵² made use of F. Bessel's¹⁵³ relations between the face and trihedral angles and reduced the problem of the rational tetrahedron to a diophantine equation quadratic in q and quadratic in r with coefficients involving an arbitrary parameter p . Euler's¹⁴⁴ process of Ch. XXII is used to find solutions q, r rational in p , so that the six edges, the surface areas and volume are expressed rationally in p .

Güntsche¹⁵⁴ considered tetrahedra whose edges, surface areas and volume are all rational and having all faces congruent. He reduced the problem to the solution of

$$\psi\theta(\psi\theta + \psi + \theta - 1)(\psi\theta - \psi - \theta - 1) = h^2,$$

but did not solve it in general. But seven particular sets of solutions involving an arbitrary parameter are found.¹⁵⁵ The tetrahedra of Hoppe¹⁴⁹ are all of the type here considered."

An alternative set of Diophantine equations to solve for isosceles tetrahedra is

$$16A^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c) \quad (\text{area condition}),$$

$$72V^2 = (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2) \quad (\text{volume condition}).$$

IV. Solution by the proposers.

We consider only one type of tetrahedron, with three mutually orthogonal not all concurrent edges a, b, c , as shown in Figure 1, and we show that, even for this single type, there are infinitely many solutions.

The six edge lengths are

$$a, b, c, \sqrt{a^2+b^2}, \sqrt{b^2+c^2}, \sqrt{a^2+b^2+c^2};$$

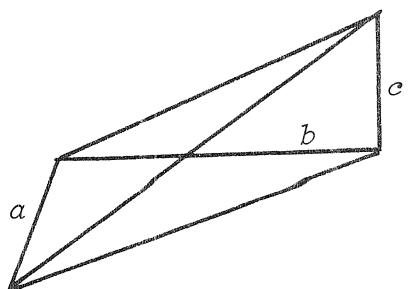


Figure 1

the four face areas are

$$\frac{1}{2}ab, \quad \frac{1}{2}bc, \quad \frac{1}{2}a\sqrt{b^2+c^2}, \quad \frac{1}{2}c\sqrt{a^2+b^2};$$

and the volume is

$$\frac{1}{6}abc.$$

We must choose integers a, b, c such that a^2+b^2 , b^2+c^2 , and $a^2+b^2+c^2$ are all squares. The first two conditions are met by choosing

$$a = |(pq)^2 - (rs)^2|, \quad b = |2pqrs|, \quad c = |(pr)^2 - (qs)^2|, \quad (1)$$

and then we find that

$$a^2 + b^2 + c^2 = (p^4 + s^4)(q^4 + r^4).$$

This expression is a square if $p^4 + s^4 = q^4 + r^4$, an equation first solved by Euler [2]. The simplest parametric solution known is

$$\begin{aligned} p &= x^7 + x^5y^2 - 2x^3y^4 + 3x^2y^5 + xy^6, \\ q &= x^7 + x^5y^2 - 2x^3y^4 - 3x^2y^5 + xy^6, \\ r &= x^6y + 3x^5y^2 - 2x^4y^3 + x^2y^5 + y^7, \\ s &= x^6y - 3x^5y^2 - 2x^4y^3 + x^2y^5 + y^7, \end{aligned}$$

from which we may obtain numerical examples. The smallest known solution is obtained from $x = 1$ and $y = 2$, which yield $p = 133$, $q = -59$, $r = 158$, $s = 134$. The resulting

$$133^4 + 134^4 = 59^4 + 158^4$$

is well known.

With the above values of p, q, r, s , we obtain from (1) the edges

$$\begin{aligned} a &= 386\ 678\ 175, & \sqrt{a^2+b^2} &= 509\ 828\ 993, \\ b &= 332\ 273\ 368, & \sqrt{b^2+c^2} &= 504\ 093\ 032, \\ c &= 379\ 083\ 360, & \sqrt{a^2+b^2+c^2} &= 635\ 318\ 657; \end{aligned}$$

the face areas

$$\begin{aligned} \frac{1}{2}ab &= 64\ 241\ 429\ 769\ 671\ 700, \\ \frac{1}{2}bc &= 62\ 979\ 652\ 389\ 978\ 240, \\ \frac{1}{2}a\sqrt{b^2+c^2} &= 97\ 460\ 886\ 821\ 988\ 300, \\ \frac{1}{2}c\sqrt{a^2+b^2} &= 96\ 633\ 843\ 845\ 928\ 240; \end{aligned}$$

and the volume

$$\frac{1}{6}abc = 8\ 117\ 619\ 016\ 097\ 058\ 044\ 304\ 000.$$

V. *Comment by Gali Salvatore, Perkins, Québec.*

[As can be seen from the above solutions and comments], there are infinitely many tetrahedra with integral edges, face areas, and volume. But, as far as we know, none of them is of the type illustrated in Figure 2, where the trihedral angle at vertex O is trirectangular. For such a tetrahedron, we have the following relation among the face areas:

$$(OBC)^2 + (OCA)^2 + (OAB)^2 = (ABC)^2.$$

This extension of the Pythagorean theorem to three dimensions is due to Gua de Malves (1712-1785).

In such a tetrahedron, the edges are

$$a, b, c, \sqrt{b^2+c^2}, \sqrt{c^2+a^2}, \sqrt{a^2+b^2};$$

the face areas are (the last by the theorem of Gua de Malves)

$$\frac{1}{2}bc, \frac{1}{2}ca, \frac{1}{2}ab, \frac{1}{2}\sqrt{b^2c^2 + c^2a^2 + a^2b^2};$$

and the volume is

$$\frac{1}{6}abc.$$

It is possible to make the four face areas and the volume integral, but then the edges are in general not all integral. Dickson [1, p. 502] gives a complicated set of parametric equations for such a solution, but a much simpler one is

$$a = 6m, \quad b = 6n, \quad c = 6(m+n),$$

which yields face areas

$$18n(m+n), \quad 18m(m+n), \quad 18mn, \quad 18(m^2+mn+n^2),$$

and volume $36mn(m+n)$.

It is possible to make the six edges and the volume integral, but then the face areas are in general not all integral. For this, we need three integers a, b, c such that $b^2+c^2, c^2+a^2, a^2+b^2$ are all squares. Dickson [1, p. 497] gives parametric equations from which infinitely many such triples can be obtained, the simplest of which is $a = 44, b = 240, c = 117$. The remaining edges are then 267, 125, 244, and the volume is 205920.

Editor's comment.

With respect to solution I, we have calculated that the first tetrahedron mentioned has surface areas 436800, 436800, 471240, 471240, and volume 124185600. As

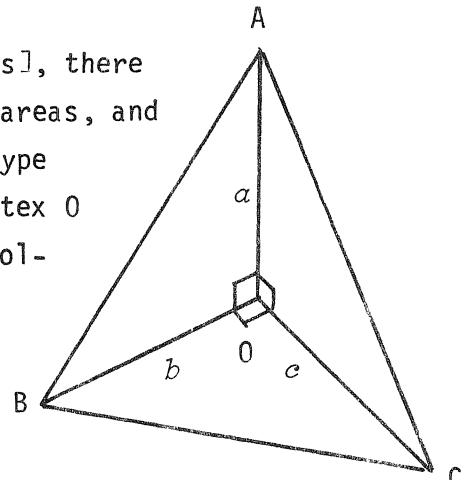


Figure 2

for the second tetrahedron mentioned, with the remarkably small consecutive integral edge lengths, it would have been more helpful if Tabov had stated how these edges are to be arranged. We did not have the time or the patience to try all possible arrangements to verify if one of them in fact leads to an integral volume. Perhaps some reader, or Tabov himself, can tell us which arrangement is satisfactory.

With respect to comment II, we have calculated that Leech's tetrahedron, consisting of four copies of a Heronian triangle with sides 148, 195, and 203, has four face areas 13650 and volume 611520.

With respect to the proposers' solution IV, we have a note to the effect that the calculations for this solution were "all done with paper and pencil" by the late and regretted Fred Maskell, who founded the problem-solving group known as *The Cops of Ottawa*, the proposers of our problem.

In the Spring 1985 issue of the *Pi Mu Epsilon Journal*, on pages 119-120, appears a series of puzzles. One of them, by Editor J.D.E. Konhauser, reads: *Are there three positive integers such that the sum of the squares of any two of them is a perfect square?* An answer to this puzzle appears in our comment V.

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MATHEMATICAL CLERIHEWS

Pierre-Simon Laplace
Escaped each *coup de grâce*
From commoner or royalty
By simply switching loyalty.

William Jones
Glory owns:
He's the guy
Called it " π ".

ALAN WAYNE
Holiday, Florida

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