Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

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Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor

A New Largest Prime

On 23 August 2008, a new largest prime was discovered,

$$2^{43\,112\,609} - 1,$$

a number with 12 978 189 digits. Its discoverer, Edson Smith of the University of California, thereby wins a \$100 000 prize from the Electronic Frontier Foundation, a civil liberties group dedicated to protecting internet freedoms, which gives a clue to the use of large primes, even if not as huge as this, in cryptography to protect our privacy when our details are transmitted electronically. The prize was offered for the first prime number to be discovered with more than 10 million digits. Edson Smith employed software from the Great Internet Merseene Prime Search (GIMPS for short), used by around 100 000 volunteers worldwide, who allow their computers to gurgle away happily all through the night. This largest prime is a *Mersenne prime*, named after a 17th century French monk Father Marin Mersenne who considered primes of the form $2^n - 1$. At the time of writing, this is the 46th Mersenne prime known.

So you too could join GIMPS in the search for ever larger primes. As an incentive, the Electronic Frontier Foundation is offering a \$150 000 prize for the first person to discover a prime with 100 million digits and \$250 000 for the first to find one with a billion digits. So get searching!

To think that over 2000 years ago in the writings of Euclid is a proof that there are infinitely many primes. So these primes are out there somewhere.

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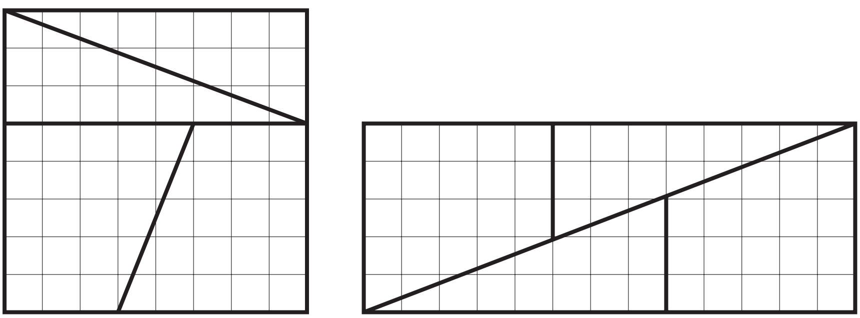
64 = 65!

Cut an 8×8 square into two triangles and two trapeziums and reassemble them into a 5×13 rectangle as shown. So $64 = 65$!



C/o A. A. Khan, Regional Office, Indian Overseas Bank, Ashok Marg, Lucknow, India **M. A. Khan**

Sumlines

JONNY GRIFFITHS

This simple question occurred to me the other day: what happens if you take three nonparallel nonvertical lines in the form $y = mx + c$ and 'add' them? For example,

$$y = 2x - 1, \quad y = 3x + 2, \quad y = x + 2 \qquad \text{add to} \qquad 3y = 6x + 3 \quad \text{or} \quad y = 2x + 1.$$

A name was needed, and naturally enough I called the sum of the lines the *sumline*. Plotting all four lines, I found that the sumline crossed the triangle formed by the original three lines. (See figure 1 for another example.) Would this always happen? I mentioned this problem to the Editor, who came up with the following elegant proof.

Firstly, note that the sumline is well-behaved under translation. Suppose that we carry out the transformation

$$x' = x + a, \qquad y' = y + b.$$

Then the trio of lines $y = m_1 x + c_1$, $y = m_2 x + c_2$, and $y = m_3 x + c_3$ (which have sumline $y = \frac{1}{3}(m_1 + m_2 + m_3)x + \frac{1}{3}(c_1 + c_2 + c_3)$) become

$$y' - b = m_1(x' - a) + c_1, \qquad y' - b = m_2(x' - a) + c_2, \qquad y' - b = m_3(x' - a) + c_3,$$

and these have sumline $y' - b = \frac{1}{3}(m_1 + m_2 + m_3)(x' - a) + \frac{1}{3}(c_1 + c_2 + c_3)$, which is simply the translation applied to the original sumline.

If the three lines are concurrent, then clearly their sumline must pass through the point of intersection. In all other cases, the three lines will form a triangle. Let us say that this triangle has vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3), where $x_1 < x_2 < x_3$ (no side can be vertical). Now translate the axes to make (x_2, y_2) the origin. Thus, the equations of the sides become (without loss of generality)

$$y = px, \qquad y = qx, \qquad y = rx + s.$$

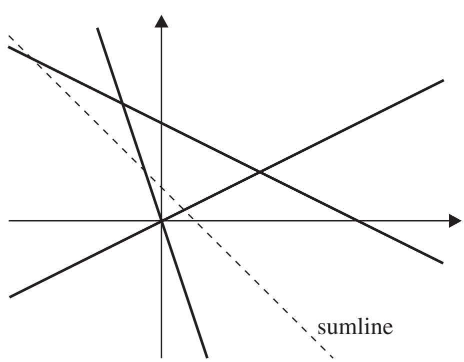


Figure 1

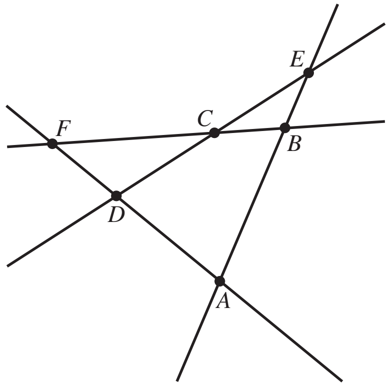


Figure 2

Therefore, the sumline is $y = \frac{1}{3}(p+q+r)x + \frac{1}{3}s$, and it is clear by considering the y-intercept of this line that it must cross the triangle.

Do the lines have to be in the form $y = mx + c$? For example, what happens if you add the following three lines:

$$2x+3y+5=0, \quad x+6y-5=0, \quad -2x+y+1=0 \qquad \text{adding to} \qquad x+10y+1=0?$$

We are in effect taking a weighted average of the lines this time. Will this sumline always cross the triangle as before? The Editor came to my aid again, this time with the following counterexample

$$9y+x=0, \quad y+9x=0, \quad -9y-5x-9=0 \qquad \text{give the sumline} \qquad y+5x-9=0,$$

which does not cross the triangle.

However, if we write our lines in the form $y = mx + c$, and if we weight our lines with positive constants, then the resulting sumline does cross the triangle. As before, translate the origin of coordinates to a vertex so that the y-axis crosses the triangle. We get

$$\begin{aligned} & k_1 y = m_1 k_1 x, \quad k_2 y = m_2 k_2 x, \quad k_3 y = m_3 k_3 x + c_3 k_3 \\ \text{add to} \qquad & y = \frac{m_1 k_1 + m_2 k_2 + m_3 k_3}{k_1 + k_2 + k_3} x + c_3 \frac{k_3}{k_1 + k_2 + k_3}. \end{aligned}$$

The point $(0, c_3 k_3/(k_1+k_2+k_3))$ is on the line and inside the triangle, since $k_3/(k_1+k_2+k_3) < 1$.

The next natural question is: will four lines that define a quadrilateral have a sumline that crosses it? The answer is, it depends what you mean by a quadrilateral. Consider figure 2. $ABCD$ is what we might call the *conventional quadrilateral* formed by the four lines, while the figure $ADFCEB$ is called the *complete quadrilateral*. The sumline of the four lines that border a conventional quadrilateral need not cross it (see figure 3).

However, the sumline *will* cross the complete quadrilateral formed by the four lines. There is a neat, if informal, argument that shows this. First, consider the sumline of two nonparallel nonvertical lines. Clearly, their sumline will go through their point of intersection. Moreover, the sumline is *vertical-averse*, i.e. it will ‘avoid’ the regions containing the vertical line through

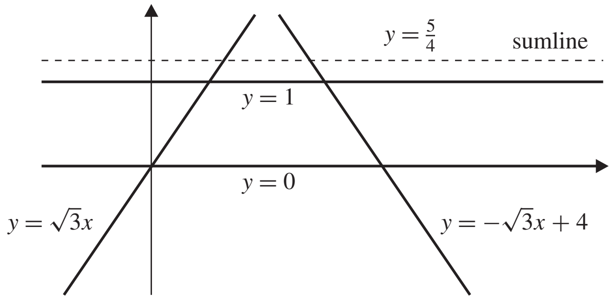


Figure 3

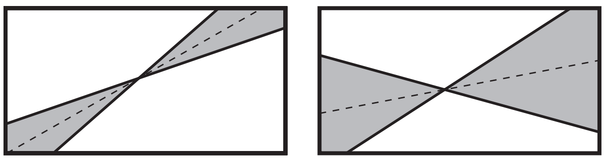


Figure 4

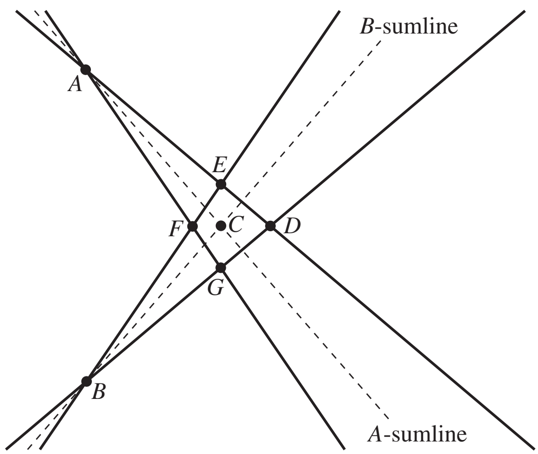


Figure 5

the point of intersection. This is because it must have a gradient that lies between the gradients of the two lines, which means it must lie in the shaded zones in figure 4.

The next thing to note is that the sumline of four lines is the sumline of (the sumline of two of the lines) and (the sumline of the other two).

Now the complete quadrilateral will be made up of a conventional quadrilateral $DEFG$ with two adjacent triangles AEF and BGF, as in figure 5.

Define the A-sumline as the sumline of the two lines through A. In figure 5 the A-sumline crosses the B-sumline within the conventional quadrilateral at C, so the sumline of all four

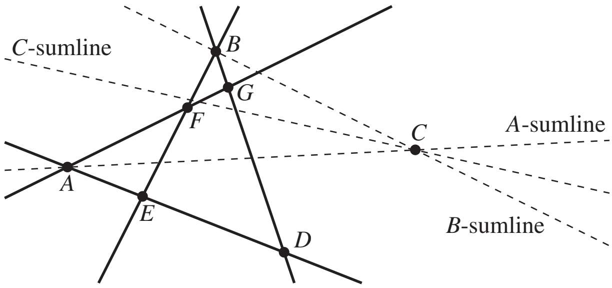


Figure 6

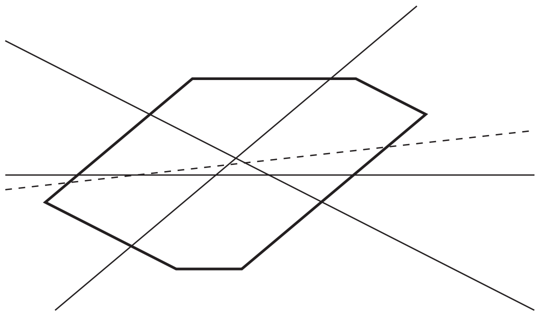


Figure 7

lines must go through C and thus cross the complete quadrilateral. In figure 6, the A-sumline and the B-sumline meet outside the conventional quadrilateral at C, but the C-sumline (the sumline of all four lines) must now have a gradient between that of AC and BC, and so will cross the complete quadrilateral. It is easy to satisfy yourself that this will happen for all other cases, including those in which there are parallel sides.

There are some conventional quadrilaterals that will always be crossed by their sumline, however. Take any parallelogram. Translate it so that its centre is at the origin. The four sides must now be of the form

$$y = m_1 x \pm c_1 \quad \text{and} \quad y = m_2 x \pm c_2,$$

and so their sum will be a line through the origin. This simple argument extends to any $2n$-sided shape that has opposite sides equal and parallel. (Does this kind of shape have a name? I am tempted to call any $2n$-sided shape with opposite sides parallel a *paragon* and if in addition the opposite sides are equal, to call this a *paragon of virtue* (!)) This will of course include all regular $2n$-agons. Indeed, one conjecture is that the sumline will always cross a general paragon: it will surely cross a hexagonal paragon, as figure 7 shows. Adding the parallel lines in pairs gives three sumlines that cross in a triangle within the hexagonal paragon. The sumline of these three sumlines is the sumline of the whole figure, and it must cross this triangle, by our earlier work.

So what happens for regular $(2n + 1)$-agons? Take the equilateral triangle: surely the sumline passes through its centre? Perhaps surprisingly, the answer is no. Take the triangle

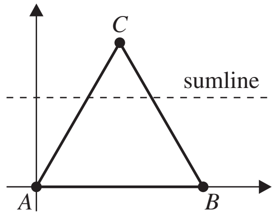


Figure 8

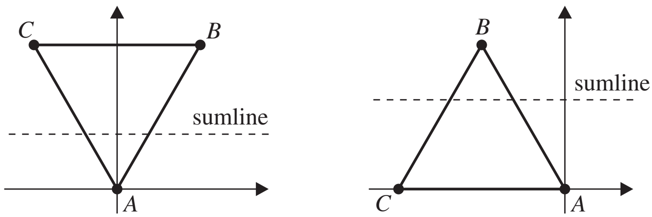


Figure 9

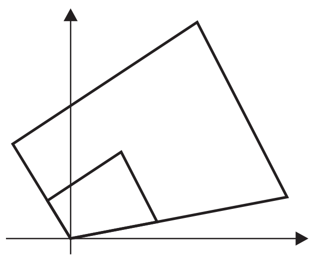


Figure 10

whose sides are formed by $y = 0$, $y = \sqrt{3}x$, and $y = -\sqrt{3}x + 2\sqrt{3}$. The sumline is $y = 2\sqrt{3}/3$, which does not pass through the triangle's centre at $(1, \sqrt{3}/3)$; see figure 8.

I might point out at this point that a triangle's sumline is not well-behaved under rotation, i.e. the sumline of a rotated triangle is not the sumline of the original when rotated. Take the triangle in figure 8 and rotate it about A so that BC and then CA are horizontal (see figure 9).

However, a sumline is well-behaved under enlargement. Given two similar shapes oriented in the same way, translate them so that the enlargement becomes one centred at the origin, as in figure 10. Clearly all gradients are unaffected by the enlargement, and the y-intercepts are simply multiplied by the scale factor of the enlargement, so the sumline of the enlarged shape is in the same proportion to its host as the original sumline is to the original shape.

So a triangle has no centre through which its sumlines pass. However, figures 8 and 9 suggest an alternative conjecture. All three sumlines shown are tangents to the equilateral triangle's incircle. Could this be true however the equilateral triangle is aligned?

This conjecture is not true for triangles in general. Consider the $3:4:5$ triangle in figure 11 (which has an incircle radius 1). The sumline will be $y = \frac{7}{36}x + \frac{23}{36}$, which crosses the incircle

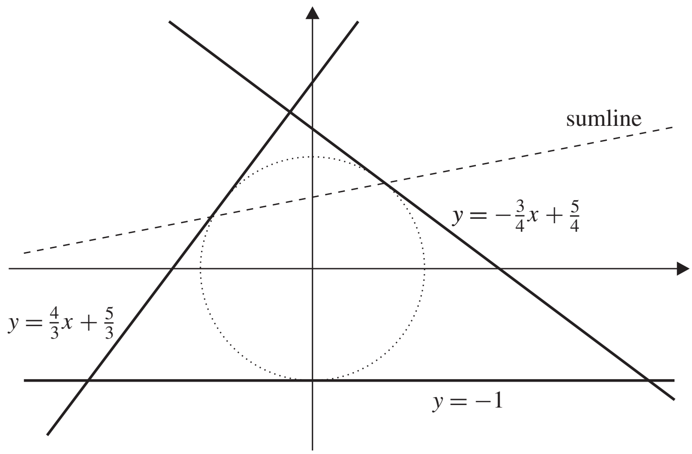


Figure 11

(radius 1, centre the origin) in two places. (Please note, the sumline here does *not* go through the points where the sides touch the incircle!) However, might our conjecture remain true for equilateral triangles?

Take two points on the unit circle with centre at the origin, and call them $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$. The straight line joining them has equation

$$y = \frac{\sin\theta - \sin\phi}{\cos\theta - \cos\phi}x + \sin\theta - \cos\theta\frac{\sin\theta - \sin\phi}{\cos\theta - \cos\phi},$$

which simplifies to

$$y = \frac{-1}{\tan((\theta+\phi)/2)}x + \frac{\cos((\theta-\phi)/2)}{\sin((\theta+\phi)/2)}.$$

So let us take three points on the circumference of this circle that are the vertices of an equilateral triangle, say $(\cos\theta, \sin\theta)$, $(\cos(\theta+120), \sin(\theta+120))$, and $(\cos(\theta+240), \sin(\theta+240))$. The sumline of the three lines joining them must be

$$\begin{aligned}3y = &-\left(\frac{1}{\tan(\theta+60)} + \frac{1}{\tan(\theta+180)} + \frac{1}{\tan(\theta+300)}\right)x \\ &+ \frac{1}{2}\left(\frac{1}{\sin(\theta+60)} + \frac{1}{\sin(\theta+180)} + \frac{1}{\sin(\theta+300)}\right).\end{aligned}$$

After simplification using some standard trigonometry, this becomes

$$y = \cot\theta\cot(\theta-60)\cot(\theta+60)x + \tfrac{1}{8}\operatorname{cosec}\theta\operatorname{cosec}(\theta-60)\operatorname{cosec}(\theta+60).$$

Let us call this $y = Px + Q$. The incircle is centred at the origin, radius $\frac{1}{2}$. So where does the sumline cross this circle, i.e. $x^2 + y^2 = \frac{1}{4}$? Solving the two equations simultaneously, we obtain

$$x^2 + (Px+Q)^2 = \tfrac{1}{4} \qquad \Longrightarrow \qquad x^2(1+P^2) + x(2PQ) + Q^2 - \tfrac{1}{4} = 0.$$

Thus, the sumline will be a tangent to the circle if and only if '$b^2 = 4ac$', i.e.

$$4P^2Q^2 = 4(1+P^2)\big(Q^2 - \tfrac{1}{4}\big),$$

which gives

$$4Q^2 = 1 + P^2.$$

So we need

$$\tfrac{1}{16}\operatorname{cosec}^2\theta \operatorname{cosec}^2(\theta-60)\operatorname{cosec}^2(\theta+60) = 1 + \cot^2\theta\cot^2(\theta-60)\cot^2(\theta+60)$$

or

$$\tfrac{1}{16} = \sin^2\theta\sin^2(\theta-60)\sin^2(\theta+60) + \cos^2\theta\cos^2(\theta-60)\cos^2(\theta+60).$$

Happily, it is easy to show by expanding using standard trigonometry that this is indeed an identity. So the sumline will always touch the incircle of the equilateral triangle, whatever θ may be.

How could we extend this result? The obvious thought is that the sumline might always touch the incircle of a regular $(2n+1)$-sided polygon. I have tested this for a regular pentagon and a regular heptagon, and it would appear to be true. The trigonometry for $5, 7, 9\ldots$ sides would seem to extend naturally from the case for the equilateral triangle.

If true, this is quite a result. Given a regular 1000-agon, we know the sumline will go through its centre. Tweak this slightly into a regular 1001-agon, and the sumline still crosses the figure, but only just, traversing the tiny annulus between the shape's inscribed and circumscribed circles. Now if the figure is changed slightly once more into a regular 1002-agon, the sumline jumps back to passing through the centre again. Adding one side can make a big difference, it seems!

This notion of adding the equations of several curves to generate a *sumcurve* can be taken further. What happens if you add the equations of three circles to get a *sumcircle*? If the three circles overlap, it is clear that the sumcircle must enclose the common area, because

$$\begin{aligned}&(x-a_1)^2+(y-b_1)^2<r_1^2,\quad (x-a_2)^2+(y-b_2)^2<r_2^2,\quad (x-a_3)^2+(y-b_3)^2<r_3^2\\ &\implies\quad (x-a_1)^2+(y-b_1)^2+(x-a_2)^2+(y-b_2)^2+(x-a_3)^2+(y-b_3)^2\\ &\qquad\qquad < r_1^2+r_2^2+r_3^2.\end{aligned}$$

It is fun to put the equations of the three circles and the sumcircle into a graphing program such as Autograph and watch the way the sumcircle surrounds the intersection as the parameters are varied. Sometimes the most enjoyable results are the simple ones!

Jonny Griffiths *teaches at Paston College in Norfolk, where he has been for the last thirteen years. He has also taught at St Dominic's Sixth Form College in Harrow-on-the-Hill and at the Islington Sixth Form Centre. Possible claims to fame include being an ex-member of Harvey and the Wallbangers, a popular band in the 1980s, and playing the character Stringfellow on the childrens' television programme Playdays. He was a Gatsby Teacher Fellow for the year 2005–2006.*

Curvature, Not Second Derivative

SONJA HUBER and MICHAEL HANKE

1. Introduction

Minimum, maximum, and inflection points – every high school calculus course includes these basic concepts. The geometric interpretation of the first derivative as a gradient is well known. In contrast, the exact geometric meaning of the second derivative is more elusive.

Here, we will shed some light on a widespread misinterpretation of the absolute value of the second derivative as a measure of curvature in a geometric sense. Imagine the graph of the function in question as a road, and you are driving along this road by car at a constant speed. The larger the angle of the steering wheel relative to its null position (12 o'clock), the larger is the curvature of the road. Formulae for calculating curvature can be found in books on differential geometry (see, e.g. reference 1). In this article, we devise a simple, straightforward measure of curvature for functions of single variables which can be used directly to find points of maximum or minimum curvature.

Faced with the task of finding the point of largest curvature on a graph, we may intuitively look for the maximum value of the second derivative. A justification might go something like this.

- At an inflection point, the second derivative of the function is zero (no curvature).
- At a maximum, the second derivative is negative (the graph is curved to the right) and at a minimum it is positive (curved to the left).
- Thus, the larger the (absolute) value of the second derivative, the larger the curvature of the function.

The correct part of this argument is that a positive second derivative indicates a convex function and a negative value indicates a concave one. However, this classification depends only on the sign, not on the magnitude of the second derivative. Finding the maximum or mimimum value of curvature in the sense described above is not usually discussed. We will try to rectify this omission.

2. Why not simply use the second derivative as a measure of curvature?

Locating the maximum of the second derivative is not the route to finding the point of greatest curvature. This can easily be seen using the unit circle as an example:

$$x^2 + y^2 = 1.$$

Defining $y = f(x)$ as the function describing the lower semi-circle, the second derivative is given by

$$f''(x) = y'' = \frac{1}{(1-x^2)^{3/2}}.$$

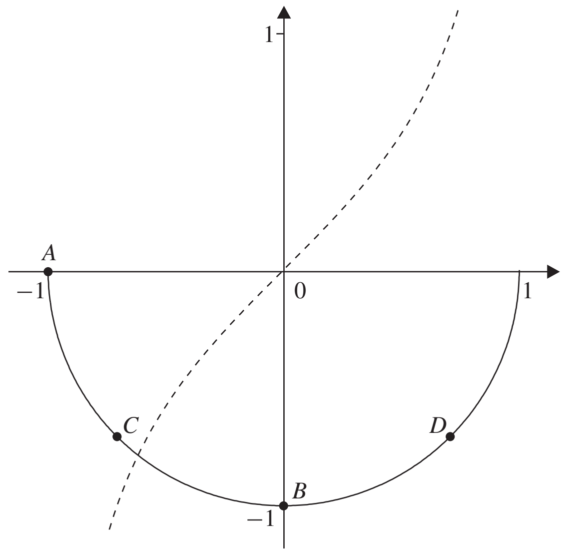


Figure 1 The (lower) unit (semi-) circle (solid line) and its first derivative (dashed line).

This reaches its maximum for $x \to \pm 1$. Driving along the circle at constant speed, however, you do not have to change the angle of the steering wheel at all once you have found the correct position. The main problem is that curvature in our geometric sense should be independent of the orientation of the graph in the plane whereas the second derivative depends on this orientation. This is illustrated in figure 1. For this semi-circle, the change in slopes of the tangents, i.e. the second derivative, is small in the neighbourhood of 0 and increases for $x \to \pm 1$. Furthermore, due to its dependence on the orientation of the graph in the plane, identical changes in tangent angle lead to different values for the second derivative, depending on the initial slope of the curve. The smaller the (absolute value of the) initial slope, the smaller the second derivative will be for identical changes in tangent angle. The solution to this problem is obvious: use the change in tangent angle directly.

3. A simple measure of curvature

Apart from independence of the exact orientation of a curve in the plane, what are the desired properties of a better measure of curvature? It is quite obvious that a linear curve should have zero curvature (the steering wheel in the 12 o'clock position). Furthermore, a circle should have constant curvature over its whole domain (since, as described above, there is no change in the steering wheel's angle when following the circle by car).

Let α be the slope of the curve at a given point and s the arc length along the curve to the given point from some fixed point of the curve. For $f(x)$ twice continuously differentiable, we define $\breve{f}(x)$, the curvature of f, measured at x, by

$$\breve{f}(x) = \left.\frac{\mathrm{d}\alpha}{\mathrm{d}s}\right|_x .$$

Table 1 summarizes the differences between $\breve{f}(x)$ and $f''(x)$.

Table 1 Differences between the second derivative and our curvature measure.

	$f''(x)$	$\breve{f}(x)$
‘numerator’	change in slope	change in tangent angle
‘denominator’	change in argument x	change in arc length s

To compute the new curvature measure, we proceed as follows:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}s} = \frac{\mathrm{d}\alpha}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}\alpha}{\mathrm{d}x}\frac{1}{\mathrm{d}s/\mathrm{d}x}.$$

An analytical expression for the arc length s between points A and B is given by

$$s = \int_A^B (1 + f'(x)^2)^{1/2}\,\mathrm{d}x.$$

The angle α is given by

$$\alpha = \arctan(f'(x)).$$

Thus,

$$\frac{\mathrm{d}\alpha}{\mathrm{d}x} = \frac{1}{1 + f'(x)^2} f''(x),$$
$$\frac{\mathrm{d}s}{\mathrm{d}x} = (1 + f'(x)^2)^{1/2},$$

and finally

$$\frac{\mathrm{d}\alpha}{\mathrm{d}s} = \frac{f''(x)}{(1 + f'(x)^2)^{3/2}}. \tag{1}$$

In contrast to the second derivative, we can use this new measure $\breve{f}$ directly to find maxima and minima of geometric curvature. In figure 2, the application of our curvature measure is presented using several well-known functions as examples. The solid lines show these functions, the (absolute values of) curvatures are dashed lines, and the second derivatives are dotted lines. In all of our examples, the maximum value of curvature is attained where it should be expected from visual inspection.

Going back to our introductory problem, we want to calculate the curvature of a circle with radius 1 by plugging the functional and its derivatives into (1) as follows:

$$\begin{aligned}
\frac{\mathrm{d}\alpha}{\mathrm{d}s} &= \frac{f''(x)}{(1 + f'(x)^2)^{3/2}} \\
&= \frac{1}{(1 - x^2)^{3/2}} \frac{1}{(1 + x^2/(1 - x^2))^{3/2}} \\
&= \frac{(1 - x^2)^{3/2}}{(1 - x^2)^{3/2}} \\
&= 1.
\end{aligned}$$

As expected, our measure is constant over the whole support.

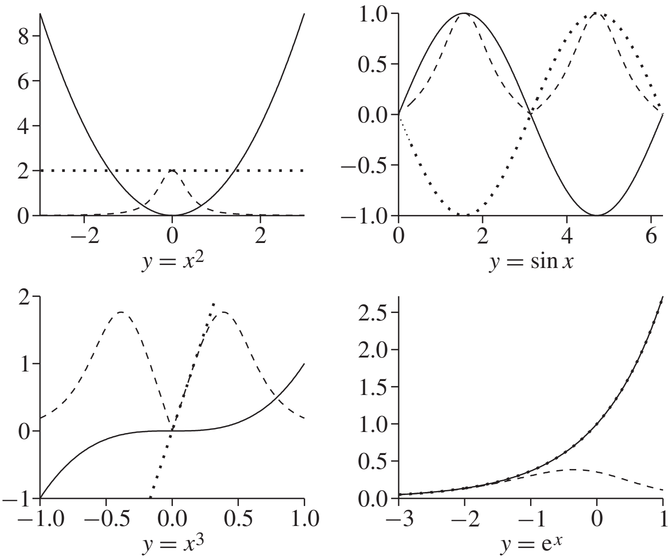


Figure 2 Four well-known functions (solid lines), together with their curvatures (dashed lines), and second derivatives (dotted lines).

A nice aspect of our curvature measure is its relation to the convention of expressing the curvature of roads via the radius of the osculating circle (see, e.g. reference 1, p. 35). Retracing our steps, we find that a circle of radius r has curvature $1/r$. Since $\breve{f}$ directly measures curvature in this sense, $\breve{f}(x)$ can be used to calculate the radius of the osculating circle to f at x, which is given by

$$r_{\text{oscc}}(f(x)) = \frac{1}{\breve{f}(x)}.$$

Reference

1 E. Kreyszig, *Differential Geometry* (Dover, New York, 1991).

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Sonja Huber *is a research assistant at the same department. She is currently pursuing a PhD in Finance, working on extreme value theory.*

Two spheres, 14 cm and 8 cm in diameter, are in contact and rest on a level surface. How high is the point of contact above the surface?

12 Pinewood Road, Midsomer Norton, Bath BA3 2RG, UK

Bob Bertuello

Room One

FRED WITZGALL

I admit it. My inspiration came from that tragic but compelling story by Nancy Casey: *Hotel Infinity*. It's amazing how many tired travellers you can accommodate when you have an infinite number of rooms to rent. But that's her strange story. Here's mine.

I wasn't renting. I was looking to buy. I needed a spacious townhouse with *lots* of interesting rooms. I guess I get bored easily. When I discovered that cryptic ad – '*sequential suite sale: infinite satisfaction within your limits*' – in an obscure mathematical journal, I sensed my days of boredom might be over. Unfortunately I was right.

Later that evening I pulled up to One Convergence Drive and rang an exquisitely carved epsilon-shaped doorbell. If you put ten mathematical geeks in a line-up and said 'identify Mr Wizard', that's who answered the door. Frightfully dishevelled, it was hard to tell where his eyebrows ended and his beard began. What he wore would have sent shudders through Generation X. It turns out that he was a world-renowned mathematician – chaos theory, no doubt – but he wasn't much of a poet as I'd soon discover. His greeting was bizarre, 'I did it all without mirrors'.

Surprisingly, the townhouse was lovely and spacious, and it had many interesting rooms. Did I say interesting? They were fascinating rooms, the kind of rooms that could ambush your attention for a lifetime – or longer. But I'm getting ahead of myself. I suppressed the disquieting feeling that the townhouse had looked much smaller from the outside, and continued my tour. I was quickly sold on its expansive beauty.

I figured I could afford it. I'd invested in Hotel Infinity, and had liquidated my stock before the fire occurred. 'What do you want for this place?' I asked.

With the manic eyes of a desert prophet who had eaten a bad locust, he replied, 'An answer'. He paused dramatically, which is like saying an iceberg got colder, and added, 'Fathom its mystery, and owner you'll be'.

Well, you can't beat *that* price. 'Mind if I look around on my own?'

During the harrowing months that followed, I often wished I had asked him more questions. I wished I had listened to my inner doubts. Above all, I wished I'd paid more attention during my Analysis class at Augsburg College.

He nodded grimly and escorted me to a stairway. 'Down one flight and jaunt to the right'. I glanced once at him – he seemed strangely calmer in a way that reminded me of Attila the Hun sipping his morning coffee out of a human skull – and I descended.

At the bottom of the long winding stairs, a sign on the door read 'ROOM ONE: THIS GLORIOUS CONUNDRUM WILL LEAVE YOUR MIND NUMB'. For the second time, I swallowed my apprehensions and entered.

I was simultaneously relieved and disappointed. This room was exactly like the one above it, in size, shape, and decor. The only difference was another quirky sign – a gigantic red arrow on the opposite wall pointing to the right, and neatly painted words: 'FOLLOW TO ROOM ONE – IT'S NOT TOO LATE TO RUN!'. Feeling more ridiculous than alarmed, I followed the sign to another door which was on the right-hand side of the room. There was another sign, of course: 'WELCOME TO ROOM ONE. YOUR DILEMMA HAS BEGUN'.

I went in, and immediately felt a wave of pity. What pathetic mind would conceive of two adjacent rooms one floor below a third room – *and all three exactly the same?* I'd had enough. The poor fool could keep his townhouse. I left 'Room One' and went back into 'Room One'.

And suddenly I was whimpering like a lost child. My knees buckled and I collapsed on the floor. My brain tried to scramble out of one ear and slap some sense into me. If reality depends upon our perception, then I was getting very bad reception.

Room One had — *shrunk!* It was but a fraction of its previous size. I leaped up and whirled to face the door of the adjacent Room One. But the door had disappeared. In a frantic sprint I covered the shortened distance to the stairway exit. But this door too had vanished. I was trapped in Room One.

I put the next hour to good use, screaming and slamming my body futilely against the walls of the diminished Room One. Bruised and exhausted, I decided to change strategies. I surveyed my environment more carefully. That's when I noticed another door. It was on the opposite side of Room One from the entrance-that-no-longer-existed, and I swore by my teetering sanity that it hadn't been there before.

This door also had a sign. I wanted to rip it to shreds, and that's just what I did after reading it: 'NO DECEPTION BETRAYS YOUR EYES. THIS ROOM IS 0.5 THE ORIGINAL SIZE. THROUGH THIS DOOR AND DOWN THE STAIRS, AWAITS ROOM ONE AND MORE NIGHTMARES'.

Down the stairway I scrambled, and barged into another 'Room One'. It seemed to be the same size as the original. I didn't even need to read the sign guiding me to the other Room One on my right. I'm a quick learner. I rushed to the door of the adjacent Room One, stepped inside just long enough to make sure it was exactly like a good Room One should be, and backed out. And yes, the first Room One was now significantly smaller, while the second Room One had disappeared. The only thing different was the sign on the new stairway door: 'WE TRULY HOPE YOU'RE HAVING FUN – 0.666 666 66 THE SIZE OF ROOM ONE'.

The pattern went on and on, down level after level. The only difference was the size of each reduced replica of Room One. After 0.666 666 66 came 0.6, then 0.625, 0.615 384 6, 0.619 047 6, 0.617 647, and so on. I won't weary you with all the sizes I encountered. But you can easily see from these first few that they were oscillating above and below a certain point, while getting closer and closer to each other. I vaguely remembered the phrase Cauchy Sequence from my Analysis class, and deduced that the numbers were converging. But *why*? Never mind the impossibility of the entire situation! *What was the nature of the pattern?*

'WHY DO THE NUMBERS NARROW? LEARN, AND ESCAPE YOUR PERIL.'

I carefully recorded each number on my note pad, which I had brought along with me as a potential home buyer. I also had a calculator that I hoped would prove useful. But after many levels, I had many questions and no answers.

By now I should mention that each Room One was generously stocked: loaded refrigerator, microwave, toilet, and shower, even a bed to sleep on. My host was coldly eccentric, but he wasn't insensitive. I would have everything I needed for a very long visit.

Days passed, weeks, and months. At one point I took a break and spent seven days sitting in a corner of Room One, laughing hysterically. Then I went back to work. But after eleven months, I was no closer to the solution than when I started. It seemed that I was cursed to finish out my miserable life in Room One.

Thank God I found the diary!

A previous buyer had apparently left it on a glass coffee table in Room One – thousands of levels below Room One. The very last entry brought me to crocodile tears.

I just can't believe I've wasted a whole afternoon figuring this place out! Wouldn't you know it? As soon as I started to get the answer, the stairway turned into an elevator. So I went down a hundred thousand flights to be sure I was right. Just as I thought, the sequence is

$$A_n = \frac{1}{1 + A_{n-1}},$$

with $A_0 = 1$. It's definitely a Cauchy Sequence, as the terms get closer and closer to each other. And it oscillates like an old rocking chair. It converges of course – to a value of around 0.618 033 9, for the lay person, commonly known as the Golden Ratio.

I started to suspect this sequence when I realized how the 'Room Ones' were set up: One over One plus One over One plus One over One plus one....

It's rather interesting visually. When you look at it, begin at a given level and compute back upwards to determine the value of A_n. For instance, computing backwards from the seventh level down, you get $A_7 = \frac{21}{34}$, which is about 0.617 647 in decimal terms.

$$\begin{array}{lll}
1 & = \frac{1}{1} & = 1.0 \\
1+1 & = \frac{1}{2} & = 0.500 \\
\quad 1+1 & = \frac{2}{3} & = 0.666\,666 \\
\qquad 1+1 & = \frac{3}{5} & = 0.600 \\
\qquad\quad 1+1 & = \frac{5}{8} & = 0.625 \\
\qquad\qquad 1+1 & = \frac{8}{13} & = 0.615\,384\,6 \\
\qquad\qquad\quad 1+1 & = \frac{13}{21} & = 0.619\,047\,6 \\
\qquad\qquad\qquad 1+1 & = \frac{21}{34} & = 0.617\,647\,1
\end{array}$$

Another fascinating aspect is that both the numerators and denominators of the fractions increase in a Fibonacci manner: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... Each number is the sum of the two previous numbers.... Oh well, I don't really want this place. It's almost 3:00 and I need to beat rush hour.

In feverish ecstasy I began to shout out what I had learned from the diary. I yelled so loud that you could have heard me all the way up in Room One. For added effect, I clapped my bare heels together (my shoes had worn out) and shouted over and over, 'there's no place like Room One'. Suddenly, I was back in Room One. Only this time, the door opened to the outside world.

Now that I'm back, I pledge to pay more attention in my Analysis class. And oh yes, I need to give my mathematics professor back her diary.

The author writes: after graduating with a mathematics degree from Bethel University, Saint Paul, Minnesota, mathematics and I went our separate ways. But in mid-life I fell passionately in love again with the beauty and mystery of mathematics. I attained a mathematics education degree at Augsburg College in Minneapolis, and am now an Honors teacher at Paschal High School in Forth Worth, Texas. I am also an avid unicyclist.

A Tribute to Joseph Liouville: 2009 Marks the Bicentenary Anniversary of his Birth

SCOTT H. BROWN

During the 19th century, Joseph Liouville was one of the most highly regarded mathematicians in France. He made significant contributions in many of the branches of pure and applied mathematics. His name is associated with several major theorems in mathematics. In addition to having a brilliant mathematical career, he established and was editor for almost 40 years of the *Journal de Mathématiques Pures et Appliquées* (Liouville's Journal).

Liouville was born on 24th March 1809 in Saint-Omer, Pas de Calais, in northern France. He was the son of Claude-Joseph Liouville (an Army Captain) and Theresa Balland, both of whom came from rather notable families of Lorraine. While his father served during the final years of the Napoleonic War, he and his older brother lived in a small town near Commercy with their uncle. During his early childhood years in Commercy, he began his initial studies. However, according to one of his teachers, Liouville was too interested in playing rather than learning and 'would not go a long way'. Obviously, Liouville's mathematical genius was not apparent to this teacher.

After the Napoleonic War, his father retired and moved to Toul, where Liouville was educated in the ancient languages at the 'college'. He then entered the *Collège St. Louis* in Paris where he began his studies in mathematics. During this period, he began to study the mathematical works in Joseph Diez Gergonne's *Annales de Mathématiques Pures et Appliquées*. Based on this early research, he wrote a few articles that focused on such subjects as projective and analytical geometry, but the articles were never published.

Joseph Liouville

At the age of 16, Liouville was accepted to the *École Polytechnique*. Here he was introduced to Andre-Marie Ampère's lectures on electrodynamics, which would have a strong influence on Liouville's research in this branch of mathematics. In 1827, he decided to pursue a career as an engineer, and transferred to the *École des Ponts et Chaussées*. Though he completed his requirements to become an engineer, several circumstances such as poor health and strained relationships with his superiors in the engineering corps eventually led him to leave his career and pursue his interest in mathematics and science.

One of the first endeavours in his new profession was a manuscript based on one of Ampère's theories in electrodynamics which was a proof of a conjecture concerning the force between two conductors. The manuscript was presented to the Académie des Sciences and was published in the *Annales de Chimie et Physique* in 1829. Liouville followed this up with a second paper on electrodynamics which was published in the same journal in 1831.

By this time he had married his maternal cousin Marie-Louise Balland, so he turned to teaching to support his family. He would eventually have three daughters and a son. Throughout the next several years, he took on various teaching positions at the *Université*, the *École Polytechnique*, and the *École Centrale*. He was literally teaching at all of the institutions at the same time in order to make ends meet. However, when he was appointed as a professor at the *École Centrale des Arts et Manufacture*, Liouville made a comfortable salary. Along with monies from the other positions, this allowed him more time to conduct his research.

Now in his early twenties, Liouville had already produced several papers in electrodyamics, heat conduction, and linear partial differential equations. In 1832, he began conducting research in the field of analysis, focusing on differentiation of arbritrary order, later known as fractional calculus. This concept had been previously investigated by several mathematicians. Gottfried Wihelm Leibniz, who had intoduced the generalized differentiation notation $\mathrm{d}^n y/\mathrm{d}x^n$, began exploring the concept of a fractional index when l'Hôpital asked him the following question: 'what if $n = \frac{1}{2}$?'. His work was followed by Lacroix, Fourier, and Euler, who addressed the subject of a derivative of arbitrary order in the early 1800s. However, it was Niels Henrik Abel who considered applying fractional calculus to the solution of an integral equation. Liouville, after reading Abel's work, began to study the theory of fractional calculus. In doing so, he first observed the well-known result for derivatives of integral order

$$\frac{\mathrm{d}^n \mathrm{e}^{mx}}{\mathrm{d}x^n} = m^n \mathrm{e}^{mx},$$

which he extended to derivatives of arbitrary order as follows:

$$\frac{\mathrm{d}^u \mathrm{e}^{mx}}{\mathrm{d}x^u} = m^u \mathrm{e}^{mx}.$$

For a function of x which may be expressed in the form $\sum A_m \mathrm{e}^{mx}$, he defined the arbitrary derivative by

$$\frac{\mathrm{d}^u y}{\mathrm{d}x^u} = \sum A_m \mathrm{e}^{mx} m^u.$$

This is now known as Liouville's initial formula for a fractional derivative. It was in his first paper that he set this foundation for fractional calculus and further showed how this new calculus was applicable to finding solutions to mechanics and geometrical problems. His second paper was devoted to a more detailed analysis of the foundations of the theory of fractional calculus.

Liouville's third paper focused on solving the equation

$$(mx^2 + nx + p)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (qx + r)\frac{\mathrm{d}y}{\mathrm{d}x} + sy = 0$$

using fractional calculus. All three of these papers were published in 1832 in the *Journal de l'École Polytechnique*. Liouville had an important part in establishing the theory of differentiation of arbitrary order, mainly because he showed how the theory could be applied to problems in mathematics and physics.

As Liouville was working on the theory of fractional calculus, he developed an interest in the integration of algebraic functions in finite terms. In his first paper, published in 1833, he focused on the question of when an algebraic function has an algebraic integral. His next paper in 1834 addressed the conditions in which an algebraic function has an elementary integral. In general, Liouville was determining when a given indefinite integral could be expressed in a finite expression consisting of just algebraic, exponential, and logarithmic functions. His efforts in addressing this problem led to the following well-known theorem, named after him.

Theorem *If y is an arbitrary algebraic function of x and the integral $\int y \, \mathrm{d}x$ is an elementary integral, then*

$$\int y \, \mathrm{d}x = t + A \log u + B \log v + \cdots + C \log w,$$

where $A, B, \ldots, C$ are constants and $t, u, v, \ldots, w$ are algebraic functions of x.

Liouville extended his study to integrals of transcendental functions. In this work he developed a generalized form of his original theorem to determine the integral of a transcendental function when it is elementary. He published these results in 1835. His accomplishments in this field were important because they established a foundation for future research on the problem of indefinite integration.

During the early 1800s, few journals existed which were devoted to mathematics. Gergonne had established his *Annales de Mathématiques Pures et Appliquées* in 1810, but by 1831 his journal did not focus entirely on mathematics. The *Journal de l'École Polytechnique* also existed during this period, where Liouville submitted his two major papers on fractional calculus and integration of finite terms, which were published in the early 1830s. In 1826, the *Journal für die Reine and Angewandte Mathematik* specializing in mathematics was established by August Leopold Crelle and became widely known for publishing the papers of numerous famous mathematicians who were mainly from outside France.

Liouville started submitting a few of his papers to Crelle. This gave him the opportunity to see the influence of the mathematical powers outside France. Subsequently, he saw that a French journal devoted to mathematics was needed. In 1836, he began publishing what is perhaps his finest work in his career, the *Journal de Mathématiques Pures et Appliquées* (Liouville's Journal). Liouville considered papers, as the title suggested, from all areas in pure and applied mathematics.

In his first issue he introduced the *advertissement* which set forth his guidelines for submission of papers. His initial table of contents provided a list of several prominent mathematicians of the day, including Libri, Ampère, Jacobi, Sturm, and Lebesgue. Liouville contributed to the first volume and quite often to subsequent volumes. He continued to have numerous famous mathematicians contribute to the journal, which would make it extremely successful. As a result, his career benefited and he became famous worldwide. This was quite an achievement for a man only 26 years old whose contributions were still being evaluated.

By 1836, Liouville had made considerable contributions in several fields of mathematics and was now the editor of a scholarly journal. He had also earned a doctorate in mathematical physics in 1836. His next significant contribution to mathematics during this time would be a paper co-authored with Charles-François Sturm. This paper on heat conduction and boundary value problems laid the foundation for what is known today as Sturm–Liouville theory. Liouville continued to conduct research in the field of higher-order Sturm–Liouville theory and published a paper in 1838. However, he began to focus on other areas of research as well as on his teaching and administrative duties in academia.

Liouville continued to hold several teaching positions while he was making significant contributions in mathematics. He was chosen in 1837 to assume the duties as Chair of General Physics and Mathematics at the *Collège de France*. In 1838 he was selected as Professor of Analysis and Mechanics at the *École Polytechnique*. He reached perhaps the highest point of his career when he was elected to the *Académie des Sciences* and the *Bureau des Longitudes*.

By 1840, Liouville was busily trying to balance his teaching, advising, and, most importantly his research duties. According to several of his students, he was considered to be an effective lecturer. The astronomer Hervé Faye who enrolled in several courses at the *Collège de France* noted that 'His lectures impressed me so strongly in my youth that today I still have vivid recollections of the startling clarity with which he was endowed'. However, there were conflicting reviews that suggest that Liouville's lectures were geared to his most talented mathematics students.

Liouville was interested in guiding those students with strong mathematical talent and developing their careers. He would often collaborate with students in research. Some of the students he influenced included Le Verrier, Hermite, and Serret.

During the 1840s, Liouville found there was less time to conduct research due to his teaching, administrative, and editorial duties. However, he still found time to make considerable contributions in various subjects of mathematics. One such subject was transcendental numbers which are real numbers that are not solutions of algebraic equations, unlike $\sqrt{2}$, which is a solution of the equation $x^2 - 2 = 0$. His proof showing the existence of these numbers is perhaps one of his most important results. He had initially investigated the transcendence of e and e^2, which were published in brief notes in his journal during 1840. He returned to his research on the existence of transcendental numbers in 1844. Liouville published a paper during this year in the *Comptes Rendus des l'Académie des Sciences* which showed that transcendental numbers exist.

Liouville later wrote a more detailed paper, introducing another famous Liouville theorem involving transcendental numbers, which was published in 1851 in his journal. One of his famous transcendental numbers was derived from the series

$$\frac{1}{l} + \frac{1}{l^{2!}} + \cdots + \frac{1}{l^{n!}} + \cdots, \qquad l \in \mathbb{N}.$$

He established the Liouville constant $0.110\,001\,000\,000\,000\,000\,000\,001\ldots$, which has 1 in the $n!$th decimal place for each n and 0 otherwise.

At about the same time that he was working on transcendental numbers, his interest in elliptic functions was being reignited. He had previously studied elliptic integrals of the first and second kind based on the work of Henrik Abel and Carl Jacobi, who focused on the inverse functions of the elliptic integral. A paper written by Charles Hermite on Abelian functions in 1843 was probably the driving force behind Liouville to begin his investigation of elliptic functions. He took a different approach from Abel and Jacobi in that he determined the properties of elliptic functions from their double periodicity. In 1844, he presented a brief note to the *Académie* containing the following famous theorem.

Theorem *A well-determined doubly periodic function which never becomes infinite is reduced to a constant.*

Liouville's proof was not only satisfied for two real-valued periods, he further showed in his notes that the proof could be satisfied for complex periods. While he published little on this subject, his work on elliptic functions became an important part of the development of complex functions.

Liouville made another important contribution in the 1840s by introducing to the mathematics community the work of Évariste Galois. In 1842, he received Galois' papers from Auguste Chevalier. He announced the importance of Galois' methods to the *Académie* in 1843 and briefly highlighted Galois' work in the *advertissement* of the 1843 volume of his journal. Further plans to publish Galois' collected works in their entirety and write a commentary providing an in-depth analysis of these works never materialized. However, Liouville was influential in the future development of algebra as a result of his efforts to spread the ideas of Galois' work.

Liouville was a committed republican and throughout the 1840s supported various political actions to oppose the French government. However, the revolution of 1848 would have a personal effect on him. As a result of the revolution a new provisional government was established and had announced an election to fill positions in the constituting assembly. Supported by some of the government members, he became a candidate and was elected to the assembly. His tenure in this position was then impacted by further changes in the leadership of the French government. In 1849, he was not re-elected to the assembly, which left him bitter and despondent. His mathematical productivity was noticeably impacted, since he published only a few articles during this period. Soon thereafter, Liouville was elected to the Chair of Mathematics at the *Collège de France*. He then resigned from the *École Polytechnique* due to the conditions placed on the teachers, after which he began to lecture at the *Collège de France* in 1851 on such subjects as differential forms, geometry, and the theory of elliptic functions.

While in his new position, Liouville continued to give inspiring lectures which would have an impact on many of his students. He continued to guide the gifted students who included Pfanuti Chebyshev. Perhaps most importantly to him during the early 1850s was seeing his only son, Ernest, pursue a career as an astronomer. He published several of his son's articles in his journal. However, his son would eventually turn to a career in law.

In the mid 1850s there was a rejuvenation of Liouville's interest in research and an increased productivity in mathematics, particularly in the field of mechanics. Earlier in his life, he had made significant contributions in applied mathematics, including celestial mechanics. He had also made brief contributions in 1846 in rational or analytical mechanics. Liouville's more recent research in rational mechanics was a result of his lectures at the *Collège de France* during 1852–1853. His discoveries in the field had been based on the work of William Hamilton and Carl Jacobi. In 1855 Liouville published the last theorem named after him, which in summary can be stated as follows.

Theorem *The knowledge of n independent integrals in involution is enough to solve Hamilton's equation of motion in $2n$-dimensional phase space by quadrature.*

Following this theorem, Liouville would publish only a few more papers devoted to rational mechanics.

Liouville's mathematical productivity was at its peak during this period. He began to publish a series of articles in number theory which became of special interest to him. During the period 1858–1865, he published his famous series of eighteen articles in his journal, focusing on analogous functions. He continued to mainly conduct research in this branch of mathematics throughout the last years of his life.

By the beginning of the 1860s, Liouville had already been appointed as the chair of mechanics at the *Faculté des Science*. Combined with his teaching load at the *Collège de France*, he found little time for research. Also, his health was starting to deteriorate, which was another reason for his decline in creativity and productivity. Perhaps one of his final mathematical efforts was the more than two hundred short notes on number theory, based on his lectures at the *Collège de France* entitled *Théorie des Nombres*. He continued lecturing at the *Collège de France* up until his death. He taught his final course at the *Faculté des Sciences* during the 1870–1871 academic year. Then, with his health declining rapidly, he turned his journal over to Henri Resal in 1875. He was enjoying his family life during his latter years until he lost his wife in an accident in 1880, and his son died soon thereafter. Liouville died on 8th September 1882, in Paris. He left behind a substantial legacy of mathematical work. Most important were the 39 volumes of his journal. He published approximately 400 pages devoted to both pure and applied mathematics. He wrote 340 notebooks containing information regarding his published and unpublished material.

References

1 W. O. Amrein, A. M. Hinz and D. Pearson (eds), *Sturm–Liouville Theory: Past and Present* (Birkhäuser, Boston, MA, 2005).

2 G. Chrystal, Joseph Liouville, *Proc. R. Soc. Edinburgh* **14** (1884), pp. 83–91.

3 G. Loria, J. Liouville and his work, *Scripta Math.* **4** (1936), pp. 147–154, 257–262, 301–306.

4 T. Kasper, Integration in finite terms: the Liouville theory, *Math. Magazine* **53** (1980), pp. 195–201.

5 J. Lutzen, Sturm and Liouville's work on ordinary linear differential equations: the emergence of Sturm–Liouville theory, *Archive History Exact Sci.* **29** (1984), pp. 309–376.

6 J. Lutzen, *Joseph Liouville 1809–1882: Master of Pure and Applied Mathematics* (Springer, New York, 1990).

7 J. Ritt, *Integration in Finite Terms* (Columbia University Press, New York, 1948).

8 B. Ross, Fractional calculus, *Math. Magazine* **50** (1977), pp. 115–117.

9 R. Taton, Joseph Liouville. In *Biographical Dictionary of Mathematics: Reference Biographies from the Dictionary of Mathematics* (Charles Scribner, New York, 1991), pp. 1594–1600.

10 D. P. Watson, Fractional calculus and its application (www.umw.edu/cas/math/students/documents/damian1.pdf).

11 E. W. Weisstein, Liouville's constant (http://mathworld.wolfram.com/LiouvillesConstant.html).

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Radii of Touching Circles: A Trigonometric Solution

PRITHWIJIT DE

The problem of finding the radii of circles touching three circles which touch each other externally in the plane is a well-known problem in plane geometry (see reference 1, pp. 13–14). In this article we give a simple trigonometric proof of the problem. As a corollary we deduce Descartes' or Soddy's result on the curvatures of four touching circles. Here is the problem.

> Three coplanar circles of radii a, b, and c touch each other externally. Find the radii of the circles touching the three given circles.

Let A, B, and C be the centres of the circles with radii a, b, and c (see figure 1). In triangle ABC, $AB = a + b$, $BC = b + c$, and $CA = c + a$. Suppose that O is the centre of the circle touching the three given circles externally and r its radius. Join OA, OB, and OC. Let $\angle AOB = \theta_1$, $\angle BOC = \theta_2$, and $\angle COA = \theta_3$. Observe that $OA = a+r$, $OB = b+r$, and $OC = c + r$. From this we easily derive, using the cosine formula, the following set of identities:

$$\begin{aligned}
\cos(\theta_1) &= 1 - \frac{2ab}{(a+r)(b+r)} &&\Longrightarrow && \cos\left(\frac{\theta_1}{2}\right) = \sqrt{\frac{r(a+b+r)}{(a+r)(b+r)}},\\
\cos(\theta_2) &= 1 - \frac{2bc}{(b+r)(c+r)} &&\Longrightarrow && \cos\left(\frac{\theta_2}{2}\right) = \sqrt{\frac{r(b+c+r)}{(b+r)(c+r)}},\\
\cos(\theta_3) &= 1 - \frac{2ca}{(c+r)(a+r)} &&\Longrightarrow && \cos\left(\frac{\theta_3}{2}\right) = \sqrt{\frac{r(c+a+r)}{(c+r)(a+r)}}.
\end{aligned} \tag{1}$$

Now we are going to introduce a trigonometric identity that will form the crux of the proof. If $\theta_1 + \theta_2 + \theta_3 = 2\pi$ then

$$1 + \cos(\theta_1) + \cos(\theta_2) + \cos(\theta_3) = -4\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_3}{2}\right). \tag{2}$$

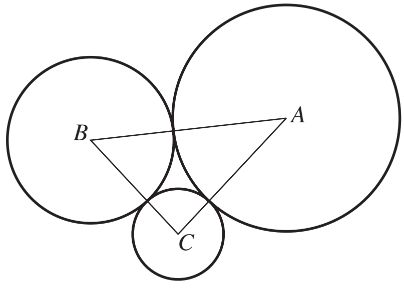

Figure 1 Three circles in the plane touching each other externally.

The left-hand side can be written as

$$
\begin{aligned}
1 &+ 2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) + 2\cos^2\left(\frac{\theta_3}{2}\right) - 1 \\
&= 2\cos\left(\pi - \frac{\theta_3}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) + 2\cos^2\left(\frac{\theta_3}{2}\right) \\
&= -2\cos\left(\frac{\theta_3}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) + 2\cos^2\left(\frac{\theta_3}{2}\right) \\
&= 2\cos\left(\frac{\theta_3}{2}\right)\left(-\cos\left(\frac{\theta_1-\theta_2}{2}\right) + \cos\left(\frac{\theta_3}{2}\right)\right) \\
&= -2\cos\left(\frac{\theta_3}{2}\right)\left(\cos\left(\frac{\theta_1-\theta_2}{2}\right) + \cos\left(\frac{\theta_1+\theta_2}{2}\right)\right) \\
&= -4\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_3}{2}\right).
\end{aligned}
$$

Substituting (1) into (2) we obtain

$$
\begin{aligned}
4 - \frac{2(3abc + (ab+bc+ca)r)}{(a+r)(b+r)(c+r)} &= -4\frac{\sqrt{r^3(a+b+r)(b+c+r)(c+a+r)}}{(a+r)(b+r)(c+r)} \\
\implies\ & 2(a+r)(b+r)(c+r) - (3abc + (ab+bc+ca)r) \\
& = -2\sqrt{r^3(a+b+r)(b+c+r)(c+a+r)} \\
\implies\ & (2r^3 + 2(a+b+c)r^2 + (ab+bc+ca)r - abc)^2 \\
& = 4r^3(a+b+r)(b+c+r)(c+a+r).
\end{aligned}
$$

Put $x = 1/r$, $a+b+c = \lambda_1$, $ab+bc+ca = \lambda_2$, and $abc = \lambda_3$ in the above expressions to give

$$(2 + 2\lambda_1 x + \lambda_2 x^2 - \lambda_3 x^3)^2 = 4(1 + 2\lambda_1 x + (\lambda_1^2 + \lambda_2)x^2 + (\lambda_1\lambda_2 - \lambda_3)x^3).$$

Upon simplification this reduces to

$$\lambda_3^2 x^2 - 2\lambda_2\lambda_3 x + (\lambda_2^2 - 4\lambda_3\lambda_1) = 0. \tag{3}$$

The roots of (3) are $(\lambda_2 + 2\sqrt{\lambda_1\lambda_3})/\lambda_3$ and $(\lambda_2 - 2\sqrt{\lambda_1\lambda_3})/\lambda_3$. Therefore,

$$r = \frac{\lambda_3}{\lambda_2 \pm 2\sqrt{\lambda_2\lambda_3}} = \frac{abc}{ab+bc+ca \pm 2\sqrt{abc(a+b+c)}}. \tag{4}$$

We decide on the sign shortly.

In order to find the radius of the circle containing these three circles and touching them, the initial conditions have to be altered in the following fashion.

Case 1 Assume that the centre O of the circle lies inside ΔABC. If this is the case then the previous trigonometric identity is applicable but the algebraic forms of the terms in the identity have to be recalculated as follows:

$$\cos(\theta_1) = 1 - \frac{2ab}{(r-a)(r-b)},$$
$$\cos(\theta_2) = 1 - \frac{2bc}{(r-b)(r-c)},$$
$$\cos(\theta_3) = 1 - \frac{2ca}{(r-c)(r-a)}.$$

These are just the formulae in (1) with the signs of a, b, and c changed so we can change the signs of a, b, and c in (4) to give that this circle has radius

$$r = \frac{-abc}{ab+bc+ca \pm 2\sqrt{abc(a+b+c)}}.$$

To make this positive, we take the negative sign to give

$$r = \frac{abc}{2\sqrt{abc(a+b+c)} - (ab+bc+ca)}. \tag{5}$$

Case 2 Assume that the centre O of the circle lies outside ΔABC. In this case the trigonometric identity introduced earlier is no longer valid because the angles AOB, BOC, and COA do not add up to 2π; but, depending on the position of O, any two of the angles add up to the third. For instance, if OB lies between OA and OC then $\angle AOC = \angle AOB + \angle BOC$ and $\theta_3 = \theta_1 + \theta_2$ (see figure 2). Thus, $\theta_1 + \theta_2 + (2\pi - \theta_3) = 2\pi$, and if we replace θ_3 by $2\pi - \theta_3$ in (2) we get

$$1 + \cos(\theta_1) + \cos(\theta_2) + \cos(\theta_3) = 4\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_3}{2}\right).$$

Since we squared (2), this will still give (5). The cases $\theta_1 = \theta_2 + \theta_3$ and $\theta_2 = \theta_1 + \theta_3$ can be dealt with in a similar manner. Since (5) is positive we now see that we need a positive sign to make (4) positive. Thus, the radius of the circle touching the given circles externally is given by

$$r = \frac{abc}{ab+bc+ca+2\sqrt{abc(a+b+c)}}.$$

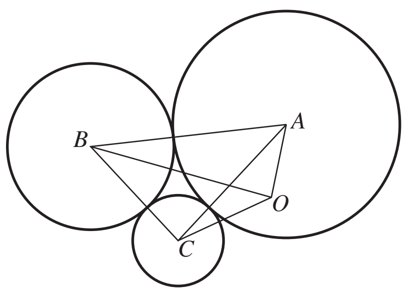

Figure 2

Now, as a corollary, we undertake the derivation of Soddy's equation in two-dimensions. Another important aspect of this exercise is its connection with Soddy's equation on the curvature of four circles touching each other in the plane. The curvature of a circle is the reciprocal of its radius. Soddy's equation in two-dimensions is

$$2\left(\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2+\left(\frac{1}{r}\right)^2\right)=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\pm\frac{1}{r}\right)^2.$$

The positive sign corresponds to the inner Soddy circle and the negative sign corresponds to the outer Soddy circle.

Our calculation yields

$$\frac{1}{r}=2\sqrt{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}\pm\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Rearranging and squaring both sides, we obtain

$$\left(\frac{1}{r^2}+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\mu\frac{2}{r}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right).$$

Adding $1/r^2+1/a^2+1/b^2+1/c^2$ to both sides and simplifying, we get

$$2\left(\frac{1}{r^2}+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\pm\frac{1}{r}\right)^2.$$

Reference

1 H. S. M. Coxeter, *Introduction to Geometry*, 2nd edn. (John Wiley, New York, 1969).

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$$37\times(3+7)=3^3+7^3,$$
$$37+(3\times 7)=3^2+7^2,$$
$$37\times 3\times 7=777,$$

$259=37\times 7,$	$185=37\times 5,$	$296=37\times 8,$
$592=37\times 16,$	$518=37\times 14,$	$629=37\times 17,$
$925=37\times 25,$	$851=37\times 23,$	$962=37\times 26.$

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Degrees of Latitude

A. W. F. EDWARDS

Whilst reading Bill Bryson's *A Short History of Nearly Everything*, I was fascinated by his account of Newton's theory of the oblateness of Earth. 'Specifically, the length [of a degree of latitude] would shorten as you moved away from the poles'.

In the *Principia* Newton calculated the oblateness of Earth arising from its rotation. With a spherical Earth each degree of latitude, taken along a meridian – a line of longitude – correponds to an equal distance on the surface. But if Earth is a Newtonian oblate ellipsoid this distance will decrease away from the poles and Newton gave a table of these distances which he had calculated. To see if he was right, in the early eighteenth century 'geometers' – Earth measurers – marched all over the place with their instruments to measure the length of a degree as close to the North Pole and as close to the Equator as they could get. (We cannot say 'conveniently get' for they endured much discomfort and danger.)

But suppose that they had found that the length of a degree was the same everywhere. Turning the problem inside-out, could they have concluded that Earth was spherical? Is the sphere the only figure for which this is true? Let's find out.

Of course we shall assume that Earth is 'rotationally' symmetrical, i.e. every line of latitude is a circle centred on the axis of rotation. Then we only have to consider its figure along a meridian: is a circle the only figure with the property that an arc of one degree subtended at a fixed point is the same length for all arcs? Equivalently, if s is arc length and θ the angle subtended, is the circle centred on the fixed point the only figure for which $\mathrm{d}s/\mathrm{d}\theta$ is constant?

Using polar coordinates with r for the radius from the fixed point as origin, a standard result (see figure 1) is

$$\mathrm{d}s = [(\mathrm{d}r)^2 + r^2(\mathrm{d}\theta)^2]^{1/2}.$$

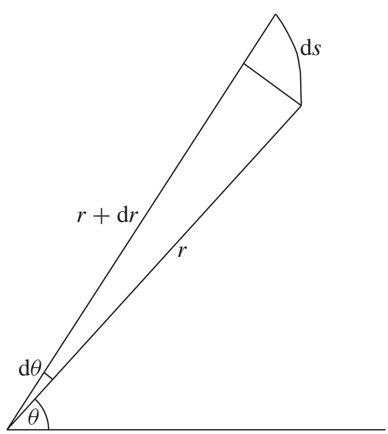

Figure 1

Therefore,

$$\frac{\mathrm{d}s}{\mathrm{d}\theta} = \left[\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2\right]^{1/2},$$

which we set equal to a constant, 1 say. Then $r = 1$ is obviously a solution. But is there another? Yes; we have

$$\begin{aligned}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2 &= 1,\\ \frac{\mathrm{d}r}{\mathrm{d}\theta} &= \sqrt{1-r^2},\\ \mathrm{d}\theta &= \frac{\mathrm{d}r}{\sqrt{1-r^2}},\end{aligned}$$

with solution

$$\theta = \int \frac{\mathrm{d}r}{\sqrt{1-r^2}} + c.$$

Put $r = \sin\phi$ and $1 - r^2 = \cos^2\phi$, then $\mathrm{d}r = \cos\phi\,\mathrm{d}\phi$ and $\theta = \phi + c$. The constant c simply rotates any figure and may be set equal to zero; whence, the required figure is $r = \sin\theta$, a circle of radius $\frac{1}{2}$ but this time centred on $(r,\theta) = (\frac{1}{2}, \pi/2)$. This result is in fact the ancient definition of sine in terms of the half-chord and the angle it subtends at the centre of the circle,

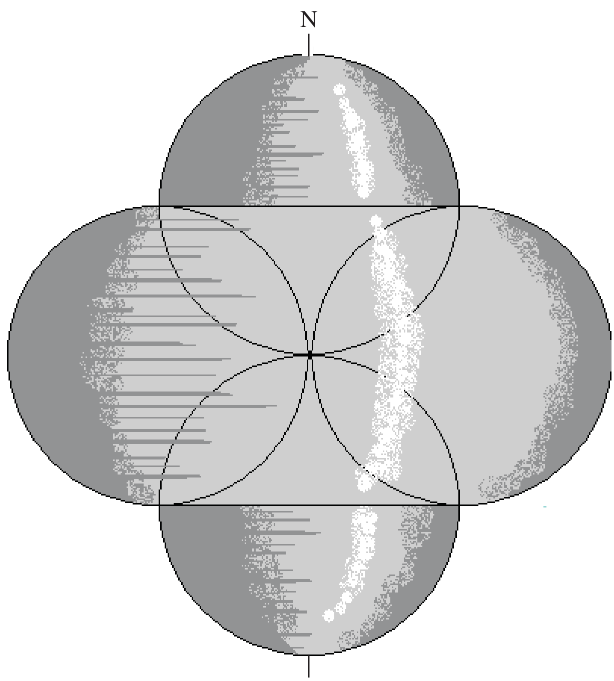

Figure 2 A variant ‘Earth’ in which a degree of latitude is a constant length on the surface.

but it may also be derived by transforming to rectangular coordinates

$$x = r\cos\theta = \sin\theta\cos\theta,$$
$$y = r\sin\theta = \sin^2\theta;$$

whence,

$$x^2 = \sin^2\theta(1-\sin^2\theta) = y(1-y) = -\big(y-\tfrac{1}{2}\big)^2 + \tfrac{1}{4} \quad \text{and} \quad x^2 + \big(y-\tfrac{1}{2}\big)^2 = \big(\tfrac{1}{2}\big)^2,$$

a circle with radius $\frac{1}{2}$ centred on $(0, \frac{1}{2})$, as required.

We should have guessed this result! For all it says is that if from a point on a circle two lines are drawn inclined to each other at angle θ, the arc in which they cut the circle again has a fixed length. Which is another way of saying that the angle subtended by an arc of a circle to any other point on the circle is of fixed size, a very elementary theorem.

Surprisingly, therefore, a figure consisting of arcs of circles of unit radius all of which pass through a fixed point possesses the desired property. Rotate it about any line in the plane passing through the fixed point and we have a nonspherical Earth with equal arcs for each degree of latitude. Figure 2 shows a simple example. More complex examples will look increasingly like a Michelin man – or at least his head!

The author is a retired Professor of Biometry at the University of Cambridge, and a previous contributor to Mathematical Spectrum, especially on Pascal's triangle (see Volume 20 Number 2 and Volume 23 Number 4). His book 'Pascal's Arithmetical Triangle' originally appeared in 1987, followed by 'Cogwheels of the Mind – The Story of Venn Diagrams' in 2004, both now published by Johns Hopkins University Press.

$$\begin{aligned}
1^2 &= 1,\\
11^2 &= 121,\\
111^2 &= 12321,\\
1111^2 &= 1234321,\\
11111^2 &= 123454321,\\
111111^2 &= 12345654321,\\
1111111^2 &= 1234567654321,\\
11111111^2 &= 123456787654321,\\
111111111^2 &= 12345678987654321.
\end{aligned}$$

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The Purchasing Power of Earnings Worldwide: An Application of the Lognormal Distribution

JOHN C. B. COOPER

Introduction

The probability density function of the normal distribution is given by

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right), \qquad \text{where } -\infty < y < \infty.$$

It is a symmetrical, continuous distribution defined by two parameters, its mean μ and standard deviation σ. This means that there is an infinite number of possible normal distributions, each uniquely governed by a particular combination of μ and σ. Therefore, to facilitate implementation, any given normal distribution is transformed to the standard normal distribution with $\mu = 0$ and $\sigma = 1$. This is easily accomplished by making the substitution $z = (y-\mu)/\sigma$, whereupon the probability density function becomes

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right), \qquad \text{where } -\infty < z < \infty.$$

Thereafter, special published tables are readily available to provide appropriate probabilities.

Closely related to the normal distribution is the lognormal distribution. A random variable X is lognormally distributed when $Y = \ln(X)$ is normally distributed; this facilitates the fitting of the lognormal.

The probability density function of the lognormal distribution is given by

$$f(x) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2\right), \qquad \text{where } 0 < x < \infty.$$

This distribution is also continuous but positively skewed and shares the same two parameters as the normal distribution. However, the mean and variance of the lognormal distribution are given by

$$\mathrm{E}(X) = \exp\left(\mu + \tfrac{1}{2}\sigma^2\right) \quad \text{and} \quad \mathrm{var}(X) = \exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1)$$

respectively. Further brief details are provided in the Appendix.

Table 1 Prices, earnings, and purchasing power in 70 cities worldwide. (Source: reference 1 and author's calculations.)

City	Net income ($US)	Price of basket ($US)	Baskets	City	Net income ($US)	Price of basket ($US)	Baskets
Amsterdam	24 200	1 581	15.31	Luxembourg	44 400	1 599	27.77
Athens	14 300	1 508	9.48	Madrid	27 500	1 398	19.67
Auckland	15 600	1 270	12.28	Manama	15 900	1 351	11.77
Bangkok	6 000	937	6.4	Manila	2 700	752	3.59
Barcelona	24 600	1 287	19.11	Mexico City	6 300	1 249	5.04
Basel	54 600	1 993	27.4	Miami	30 200	1 524	19.82
Berlin	25 400	1 541	16.48	Milan	20 100	1 521	13.21
Bogota	15 600	778	20.05	Montreal	27 500	1 342	20.49
Bratislava	6 600	782	8.44	Moscow	9 500	1 097	8.66
Brussels	24 300	1 619	15.01	Mumbai	3 300	587	5.62
Bucharest	4 000	678	5.9	Nairobi	2 800	1 095	2.56
Budapest	12 700	1 143	11.11	New York	41 800	2 136	19.57
Buenos Aires	11 900	625	19.04	Oslo	36 300	2 409	15.07
Caracas	8 300	972	8.54	Paris	51 500	1 826	28.2
Chicago	47 400	1 987	23.86	Prague	6 600	828	7.97
Copenhagen	25 100	2 021	12.42	Riga	10 500	887	11.84
Dubai	31 000	1 330	23.31	Rio de Janeiro	4 700	781	6.02
Dublin	24 700	1 694	14.58	Rome	19 300	1 501	12.86
Frankfurt	34 800	1 604	21.7	Santiago	22 900	847	27.04
Geneva	55 200	1 954	28.25	Sao Paulo	4 800	852	5.63
Helsinki	19 900	1 759	11.31	Seoul	27 500	1 564	17.58
Hong Kong	17 600	2 211	7.96	Shanghai	16 900	1 425	11.86
Istanbul	11 100	1 121	9.9	Singapore	13 000	1 473	8.83
Jakarta	3 200	1 029	3.11	Sofia	2 200	723	3.04
Johannesburg	7 800	918	8.5	Stockholm	20 200	1 863	10.84
Karachi	2 200	670	3.28	Sydney	15 000	1 351	11.1
Kiev	2 600	665	3.91	Taipei	16 100	1 495	10.77
Kuala Lumpur	6 300	878	7.18	Talinn	7 400	1 022	7.24
Lagos	1 400	1 215	1.15	Tel Aviv	15 400	1 435	10.73
Lima	9 200	929	9.9	Tokyo	66 300	2 181	30.4
Lisbon	10 900	1 331	8.19	Toronto	19 500	1 362	14.32
Ljubljana	7 200	1 124	6.41	Vienna	23 400	1 721	13.6
London	39 200	1 995	19.65	Vilnius	8 300	999	8.31
Los Angeles	46 700	1 723	27.1	Warsaw	5 500	1 036	5.31
Lugano	47 100	1 919	24.54	Zürich	62 200	2 044	30.43

Empirical application

In 2003, the Union Bank of Switzerland conducted a survey of prices and earnings in 70 cities worldwide (see reference 1). Prices were computed from a standardized basket of goods and services consisting of some 140 items. Data on gross earnings, denominated in US dollars, were collected for several different professions and these were converted to net earnings by deducting social security contributions, any mandatory superannuation payments, and income tax.

Looked at in isolation, of course, such prices and earnings figures are misleading, especially for comparative purposes. What is more meaningful is the purchasing power of net earnings and this can be determined very simply by calculating the number of baskets of goods that net earnings will buy. For the purpose of this study, we use the purchasing power of the net earnings of a hypothetical, married, 35-year-old bank clerk with ten years experience.

As shown in table 1, the 70 observations on purchasing power vary markedly from country to country and range from a minimum of 1.15 baskets in Lagos, Nigeria, to a maximum of 30.43 baskets in Zürich, Switzerland.

Table 2 Distribution of purchasing power.

Baskets (X)	Observed frequency	Observed percentage	Theoretical probability	Expected frequency
1–5	7	0.100	0.122 8	9
5–10	22	0.314	0.317 4	22
10–15	16	0.229	0.236 8	17
15–20	11	0.157	0.133 4	9
20–25	6	0.086	0.076 3	5
> 25	8	0.114	0.113 1	8
Total	70	1.000	0.999 8	70

The mean purchasing power is 13.40 baskets with a standard deviation of 7.78. Furthermore, a glance at the first three columns of table 2 suggests that the value of net earnings measured by baskets (X) is positively skewed. Accordingly, it is appropriate to attempt to fit a lognormal distribution to this data. This, in turn, is accomplished by fitting a normal distribution to the natural logarithm of the data, whose mean and standard deviation were calculated to be 2.40 and 0.68 respectively.

As an example, the probability that purchasing power lies between, say, 15 and 20 baskets is obtained as follows:

$$\begin{aligned} \mathrm{P}(15 < x < 20) &= \mathrm{P}(2.71 < \ln x < 3.00) \\ &= \mathrm{P}\left(\frac{2.71 - 2.40}{0.68} < z < \frac{3.00 - 2.40}{0.68}\right) \\ &= \mathrm{P}(0.46 < z < 0.88) \\ &= 0.1334. \end{aligned}$$

The expected number of cities where the purchasing power of a bank teller's earnings lies between 15 and 20 is therefore $70 \times 0.1334 = 9$, compared with the actual number of cities, which is 11. A comparison of observed frequency of cities to theoretical frequency of cities in table 2 suggests that the lognormal distribution provides a very good fit.

Appendix

Let X and Y be two random variables; Y is normally distributed and $X = \exp(Y)$ is lognormally distributed. Thus,

$$\int_x^\infty f(x)\,\mathrm{d}x = \int_y^\infty g(y)\,\mathrm{d}y,$$

so that

$$f(x) = g(y)\frac{\partial y}{\partial x}.$$

Also,

$$\mathrm{E}(X) = \int_0^\infty x f(x)\,\mathrm{d}x = \int_{-\infty}^\infty \exp(y) g(y)\,\mathrm{d}y.$$

After some manipulation, it can be shown that this reduces to $\exp(\mu + \sigma^2/2)$. Now,

$$\mathrm{E}(X^2) = \int_0^\infty x^2 f(x)\,\mathrm{d}x = \int_{-\infty}^\infty \exp(2y)g(y)\,\mathrm{d}y,$$

which can be shown to be equal to $\exp(2\mu + 2\sigma^2)$. Thus,

$$\operatorname{var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2) = \exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1).$$

For full details of the mathematical manipulations required to achieve these end results together with diagrams of both the normal and lognormal distributions, see reference 2, pp. 157–159.

References

1 Union Bank of Switzerland, *Prices and Earnings* (Zürich, 2003).
2 T. J. Watsham and K. Parramore, *Quantitative Methods in Finance*. (Thomson Learning, London, 1997).

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Mathematical Spectrum Awards for Volume 40

Prizes have been awarded to the following student readers for contributions in Volume 40:

Nathan Olson and **Christopher Phillips**
for the article 'Two Derivations of a Higher-Order Newton-Type Method' (with Jennifer Switkes);

Kalyaan M. Rao
for the article 'The Mathematics Behind a Certain Card Trick' (with K. P. S. Bhaskara Rao and M. Bhaskara Rao);

Daniel Schultz
for the letter 'Regular polygons' and submitting a problem (number 40.5)

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £50 for letters, solutions to problems and other items.

Geometric and Harmonic Variations of the Fibonacci Sequence

ANNE J. SHIU and CARL R. YERGER

Introduction

First told by Fibonacci himself, the story that often accompanies an initial encounter with the sequence $1, 1, 2, 3, 5, 8, \ldots$ describes the size of a population of rabbits. The original question concerns the number of pairs of rabbits there are in a population; for simplicity, we consider individual rabbits rather than pairs. In general, a rabbit is born in one season, grows up in the next, and in each successive season gives birth to one baby rabbit. Here, the sequence $\{f_n\}$ that enumerates the number of births in each season is given by $f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$, with $f_1 = f_2 = 1$, which coincides precisely with the Fibonacci sequence. Also, recall that the asymptotic exponential growth rate of the Fibonacci numbers is equal to the golden ratio, $\frac{1}{2}(1+\sqrt{5})$. Further discussion of this golden ratio can be found in reference 1. In addition, there is a large literature on the Fibonacci sequence, including *The Fibonacci Quarterly*, a journal entirely devoted to the Fibonacci sequence and its extensions.

In this article, we consider similar recurrences and examine their asymptotic properties. One way this has been previously studied is by defining a new sequence,

$$G_{n+r} = \alpha_1 G_{n+r-1} + \alpha_2 G_{n+r-2} + \cdots + \alpha_r G_n \quad \text{for } n \geq 1,$$

and giving a set of initial conditions $\{G_1, G_2, \ldots, G_r\}$. Other modifications include a nondeterministic version that allows for randomness in the values of the terms of the sequence, while still having successive terms depend on the previous two: one such recurrence is given by $t_{n+2} = \alpha_{n+2} t_{n+1} + \beta_{n+2} t_n$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of random variables distributed over some subset of the real numbers. In the case when $\{\alpha_n\}$ and $\{\beta_n\}$ are independent Rademachers (symmetric Bernoullis), i.e. each taking values ± 1 with equal probability, Divakar Viswanath showed that although the terms of $\{t_n\}$ are random, asymptotically the sequence experiences exponential growth almost surely; $\sqrt[n]{|t_n|}$ approaches a constant, $1.1319\ldots$, as $n \to \infty$ (see reference 2). Building from this result, Mark Embree and Lloyd Trefethen determined the asymptotic growth rate when α_n and β_n take the form of other random variables (see reference 3). In this article, we determine the growth rates of other variations of the Fibonacci sequence, specifically those which we call the geometric and harmonic Fibonacci sequences.

The geometric and harmonic Fibonacci sequences

There has been significant study of Fibonacci-like sequences that are linear, i.e. recurrence relations of the form given by $\{G_n\}$ defined above. In this article, though, we will consider two *nonlinear* Fibonacci recurrences. First, note that we can view the Fibonacci sequence as a recurrence in which each term is twice the *arithmetic* mean of the two previous terms.

Table 1 The first eight terms of each Fibonacci sequence.

Term number	Fibonacci sequence	Geometric Fibonacci sequence	Harmonic Fibonacci sequence
1	1	$1 = 2^0$	1
2	1	$1 = 2^0$	1
3	2	$2 = 2^1$	2
4	3	$2.828\ldots = 2^{3/2}$	$2.666\ldots = \frac{8}{3}$
5	5	$4.756\ldots = 2^{9/4}$	$4.571\ldots = \frac{32}{7}$
6	8	$7.336\ldots = 2^{23/8}$	$6.736\ldots = \frac{128}{19}$
7	13	$11.814\ldots = 2^{57/16}$	$10.893\ldots = \frac{512}{47}$
8	21	$18.619\ldots = 2^{135/32}$	$16.650\ldots = \frac{2048}{123}$

In this light, we introduce the *geometric Fibonacci sequence* $\{g_n\}$ and the *harmonic Fibonacci sequence* $\{h_n\}$, in which each successive term is twice the geometric or harmonic mean respectively of the previous two terms in the sequence. That is, we define

$$g_{n+2} = 2\sqrt{g_{n+1}g_n} \quad \text{for } n \geq 1, \qquad \text{with } g_1 = g_2 = 1,$$

and

$$h_{n+2} = \frac{4}{1/h_{n+1} + 1/h_n} \quad \text{for } n \geq 1, \qquad \text{with } h_1 = h_2 = 1.$$

We motivate the study of the geometric and harmonic sequences by a desire to examine properties associated with the triumvirate of the arithmetic, geometric, and harmonic means.

Arithmetic–geometric–harmonic mean relations

The first historical reference to the arithmetic, geometric, and harmonic means is attributed to the school of Pythagoras, where it was applied to both mathematics and music. Initially dubbed the subcontrary mean, the harmonic mean acquired its current name because it relates to

> the ‘geometrical harmony’ of the cube, which has 12 edges, 8 vertices, and 6 faces, and 8 is the mean between 12 and 6 in the theory of harmonics

(see reference 4, pp.85–86). Today, the harmonic mean has direct applications in such fields as physics, where it is used in circuits and in optics (through the well-known lens-makers’ formula).

We also know that the following hierarchy always holds: the arithmetic mean of two numbers is always at least as great as their geometric mean, which in turn is at least as great as their harmonic mean. That is, given two positive numbers a and b,

$$\frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2}{1/a + 1/b}.$$

As a result of the arithmetic–geometric–harmonic mean inequalities, the terms of the corresponding sequences we defined satisfy the inequality $f_n \geq g_n \geq h_n$ for all n. Next, we will see that the asymptotic *growth rates* of the Fibonacci sequence, along with those of our geometric and harmonic variations of the sequence, exist and also satisfy this inequality.

Calculating the growth rates for the geometric and harmonic Fibonacci sequences

In order to solve the difference equations for $\{g_n\}$ and $\{h_n\}$, we will proceed in the same manner as solving a nonhomogeneous differential equation. First, we will define a characteristic equation for the recurrence from which we can obtain a homogeneous solution. Then, using the roots of the characteristic equation, we will apply the method of undetermined coefficients to obtain a particular solution (if necessary) which, when combined with the homogeneous solution and the initial conditions, yields a solution to the difference equation.

As a first example, we will derive the growth rate for the Fibonacci sequence in this manner. Our characteristic equation of the recursive sequence $\{f_n\}$ defined by $f_{n+2} = f_{n+1} + f_n$, is $x^2 - x - 1 = 0$. This has solutions of $x = \frac{1}{2}(1 \pm \sqrt{5})$ So, our homogeneous solution is

$$f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Using our two initial conditions of the Fibonacci sequence, namely $f_1 = 1$ and $f_2 = 1$, we see that $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$. This gives a general form (Binet's formula) for the nth Fibonacci number as

$$f_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

Thus, $f_{n+1}/f_n \to \frac{1}{2}(1+\sqrt{5})$ as n approaches infinity. We say that the Fibonacci sequence $\{f_n\}$ has asymptotic bound $\frac{1}{2}(1+\sqrt{5})$, the golden ratio.

Next, we consider our *geometric* Fibonacci sequence $\{g_n\}$ as defined above and proceed to determine its growth rate. (Note, however, that it is not entirely clear that an asymptotic growth rate exists by inspection.) A naive way to guess what this rate is results from the following steps. If we assume that this asymptotic growth rate exists, we can determine the limit of the ratio of successive terms in the geometric mean recurrence directly from the recurrence relations. Let R_g be the asymptotic growth rate, i.e. $R_g = \lim_{n\to\infty}(g_{n+1}/g_n)$. Next, we solve for R_g as follows:

$$\begin{aligned}
& & g_{n+2} &= 2\sqrt{g_{n+1}g_n} \\
&\Longrightarrow & (g_{n+2})^2 &= 4g_{n+1}g_n \\
&\Longrightarrow & \lim_{n\to\infty}\frac{{g_{n+2}}^2}{{g_{n+1}}^2} &= 4\lim_{n\to\infty}\frac{g_n}{g_{n+1}} \\
&\Longrightarrow & {R_g}^2 &= 4\frac{1}{R_g} \\
&\Longrightarrow & R_g &= 4^{1/3}.
\end{aligned}$$

From this calculation the surprising result emerges that the asymptotic growth rate of our geometric Fibonacci sequence is likely to be the cube root of four.

To obtain this result in a more rigorous manner, we instead solve for a closed-form expression; from this expression, the growth rate is shown to exist and is indeed equal to $4^{1/3}$. The most common method for solving this form of recursive relation is by using generating functions; the asymptotic growth rate of the regular Fibonacci sequence can be found in this way. Here we use a different technique – the one described above – that, in this case, simplifies calculations. Recall that we have the following relation for our geometric Fibonacci sequence: $g_{n+2} = 2\sqrt{g_{n+1}g_n}$. Squaring both sides, we obtain $(g_{n+2})^2 = 4g_{n+1}g_n$. By making the substitution

$$b_n = \log(g_n),$$

we obtain a nonhomogeneous linear recurrence, $2b_{n+2} = \log 4 + b_{n+1} + b_n$, whose solution is computed here, using a method which is analogous to that of solving a similar differential equation (such as $f(x) = 17 + f'(x) + f''(x)$). To begin, we identify the characteristic polynomial as $2x^2 - x - 1 = (2x+1)(x-1)$, which has roots $x = -\frac{1}{2}$ and $x = 1$. Thus, the homogeneous solution is $b_n = c_1(-\frac{1}{2})^n + c_2(1)^n$. To obtain the particular solution, we try $b_n = An\log(4)$. If we substitute this into $2b_{n+2} = \log(4) + b_{n+1} + b_n$, we obtain $A = \frac{1}{3}$.

Thus, $b_n = \frac{1}{3}n\log(4) + c_1(-\frac{1}{2})^n + c_2(1)^n$. By substituting $b_1 = b_2 = 0$ as initial conditions, we can solve for c_1 and c_2 as follows:

$$\begin{aligned}0 &= \tfrac{1}{3}\log(4) + c_1(-\tfrac{1}{2}) + c_2,\\ 0 &= \tfrac{2}{3}\log(4) + c_1(\tfrac{1}{4}) + c_2.\end{aligned}$$

Solving for c_1 and c_2 yields $c_1 = -\frac{4}{9}\log(4)$ and $c_2 = -\frac{5}{9}\log(4)$. So, the solution to our recurrence relation is

$$b_n = \log(4)\left(\frac{n}{3} - \frac{4}{9}\left(-\frac{1}{2}\right)^n - \frac{5}{9}\right).$$

Thus, for $n \geq 1$, we have the following closed-form expression for our geometric Fibonacci sequence:

$$g_n = \exp(b_n) = 2^{(2n/3-(8/9)(-1/2)^n-10/9)}.$$

As predicted by the simple calculation performed above, the asymptotic growth rate is indeed the cube root of four: $R_{gr} = \lim_{n\to\infty}(g_{n+1}/g_n) = 4^{1/3} = 1.5874\ldots$. Note that this rate of growth is close to that of the arithmetic (that is, the usual) Fibonacci sequence which we noted above as being the golden ratio, $1.6180\ldots$, but is less than the golden ratio. Of course, however, just as we know that, in the long-term, slight differences in interest rates result in large differences in bank account balances, for the same reason, the small difference in the growth rate with time results in quite large differences between the terms of the regular Fibonacci sequence and those of our geometric Fibonacci sequence.

Finally, we analyze our *harmonic* Fibonacci sequence $\{h_n\}$, whose recurrence relation we recall is given by $h_{n+2} = 4/(1/h_n + 1/h_{n+1})$. Again, it is not intuitively clear what type of growth this sequence undergoes, but we find that it too experiences exponential growth. By employing a heuristic procedure similar to the derivation of the geometric Fibonacci sequence, here we determine the limiting ratio $R_h = \lim_{n\to\infty}(h_{n+1}/h_n)$. Rearranging the recurrence relation yields $h_{n+2}h_{n+1} + h_{n+2}h_n = 4h_nh_{n+1}$, so that $h_{n+2}h_{n+1}/h_{n+1}h_n + h_{n+2}/h_{n+1} = 4$. Thus, assuming that the limit R_h exists, we have $R_h^2 + R_h = 4$, and by the quadratic formula, we obtain roots $\frac{1}{2}(-1 \pm \sqrt{17})$. Finally, our growth rate is known to be positive, so $R_h = \frac{1}{2}(-1 \pm \sqrt{17}) = 1.5615\ldots$.

Another way we can prove this is by the method presented for the calculation of the growth rate of the geometric Fibonacci sequence. We notice from table 1 that each of the terms of h_n for $n \geq 3$ is of the form $2^{2n-5} = j_n$, where $j_3 = 1$, $j_4 = 3$, $j_5 = 7$, and $j_{n+2} = j_{n+1} + 4j_n$ for $n \geq 5$. We can solve this recurrence relation by the methods described above, which gives the following closed-form expression for $n \geq 3$:

$$j_n = \frac{51 + 5\sqrt{17}}{1088}\left(\frac{1+\sqrt{17}}{2}\right)^n + \frac{51 - 5\sqrt{17}}{1088}\left(\frac{1-\sqrt{17}}{2}\right)^n.$$

When using the relation between h_n and j_n, namely that $h_n = 2^{2n-5}/j_n$, we obtain an explicit expression for h_n. This gives us an asymptotic growth rate of

$$\frac{4}{(1+\sqrt{17})/2} = \frac{-1+\sqrt{17}}{2},$$

as desired.

Thus, we have constructed the arithmetic–geometric–harmonic inequality for the growth rates

$$\frac{1+\sqrt{5}}{2} \geq 4^{1/3} \geq \frac{-1+\sqrt{17}}{2},$$

with corresponding decimal approximations

$$1.6180\ldots \geq 1.5874\ldots \geq 1.5615\ldots,$$

where the three terms correspond to the asymptotic growth rates we determined for the arithmetic (i.e. the usual), geometric, and harmonic Fibonacci sequences.

Appendix. Integer-valued versions of the geometric and harmonic Fibonacci sequences

It is interesting to note that, although the growth rate of the Fibonacci sequence is an irrational number, namely the golden ratio, each term of the sequence is an integer. Note, however, that neither the geometric nor harmonic Fibonacci sequence is a sequence of integers. So we now define sequences whose recurrences are given by rounding up to the nearest integer twice the geometric or harmonic mean of the previous two terms. That is, consider, for example, the following *rounded-up* version of the geometric Fibonacci sequence, which we denote by $\{g_n^u\}$:

$$g_{n+2}^u = \left\lceil 2\sqrt{g_{n+1}^u g_n^u}\right\rceil \quad \text{with } g_1^u = g_2^u = 1.$$

By bounding this sequence above and below, we can show that it has the same growth rate as that of the regular geometric Fibonacci sequence $\{g_n\}$. Similarly, a rounded-down version of $\{g_n\}$ or a rounded-up or rounded-down version of the harmonic Fibonacci sequence $\{h_n\}$ can be shown to have the same growth rates as the corresponding nonrounded versions.

Note that it is initially unclear whether rounded-down versions of these sequences are even increasing. For example, consider the sequence given by the recurrence $d_{n+2} = 2.5d_{n+1} - d_n$, with $d_1 = 20$ and $d_2 = 10$. While this sequence approaches zero, in fact the corresponding rounded-down version is decreasing for all $n \geq 1$ $(20, 10, 5, 2, 0, -2, -5, -11, \ldots)$ and negative for $n > 5$. The absolute value of the terms of this sequence grows exponentially. When we consider the rounded-up version we see that, for $n \geq 6$, the nth term is $(20/256)2^n$. (The first few terms of this sequence are $20, 10, 5, 3, 3, 5, 10, 20, 40, 80, \ldots$.) From this example, we see that rounded-up and rounded-down sequences may differ vastly from the original sequence. The above example is adapted from one mentioned by past NCTM President Johnny Lott in a recent plenary address to the Tennessee Math Teachers Association in Memphis. See reference 5, pp. 297–300 for a comprehensive theory of rounding.

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References

1 R. A. Dunlap, *The Golden Ratio and Fibonacci Numbers* (World Scientific, River Edge, NJ, 1997).

2 D. Viswanath, Random Fibonacci sequences and the number $1.13198824\ldots$, Math. Comp. **69** (2000), pp. 1131–1155.

3 M. Embree and L. N. Trefethen, Growth and decay of random Fibonacci sequences, *R. Soc. London Proc. Ser. A Math. Phys. Eng. Sci.* **455** (1999), pp. 2471–2485.

4 T. Heath, *A History of Greek Mathematics* (Dover, New York, 1981).

5 R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd edn. (Addison-Wesley, Reading, MA, 1994).

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Mathematics in the Classroom

The most fundamental inequality

We have all used the most fundamental inequality $x^2 \geq 0$ for all real x, and equality holds for $x = 0$. The inequality is very simple but in this column, we will see its vital role in the solution of challenging problems from various competitions around the world.

Problem 1 Find the minimum value of the expression $x^8 + x^6 - x^4 - 2x^3 - x^2 - 2x + 9$.

Solution At first this looks like a calculus problem, but calculus is not necessary. The expression can be rewritten as $(x^4 - 1)^2 + (x^3 - 1)^2 + (x^2 - 1)^2 + (x - 1)^2 + 5$. Here it is clearly seen that the minimum value of the expression is 5, and it occurs when $x = 1$.

Problem 2 If a, b, and c are real numbers such that $a^2 + 2b = 7$, $b^2 - 4c = -7$, and $c^2 + 6a = -14$, then find the value of $a^2 + b^2 + c^2$.

Solution We can try to solve this by obtaining an equation in one unknown a, say, but such an equation is difficult to solve. However, by using the most fundamental inequality, we arrive quickly at the solution. Adding all three equations, we get

$$a^2 + 6a + b^2 + 2b + c^2 - 4c + 14 = 0.$$

After completing squares, it becomes

$$(a+3)^2 + (b+1)^2 + (c-2)^2 = 0.$$

Here each square is zero, i.e. $a = -3$, $b = -1$, and $c = 2$. Thus, $a^2 + b^2 + c^2 = 14$.

Problem 3 For real positive numbers a, b, and c, prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Solution This is a famous problem and its solution appears in various books and magazines. But again, we use the most fundamental inequality. For any positive real numbers a, b, and c, we have $(a/(b+c) - \frac{1}{2})^2 \geq 0$. After simplification, this fact can be rewritten as

$$\frac{a}{b+c} \geq \frac{1}{4}\frac{8a/(b+c) - 1}{a/(b+c) + 1} = \frac{8a - b - c}{4(a+b+c)}.$$

Similarly,

$$\frac{b}{c+a} \geq \frac{8b - c - a}{4(a+b+c)} \quad \text{and} \quad \frac{c}{a+b} \geq \frac{8c - a - b}{4(a+b+c)}.$$

Summing these inequalities, we get

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{8a-b-c}{4(a+b+c)} + \frac{8b-c-a}{4(a+b+c)} + \frac{8c-a-b}{4(a+b+c)} = \frac{3}{2}.$$

Problem 4 In any triangle ABC, prove that $\cos A + \cos B + \cos C \leq \frac{3}{2}$.

Solution Despite numerous possible methods of solving this trigonometrical problem, here we are interested in a solution using the most fundamental inequality. By using the trigonometric transformation and submultiples angle formula, the inequality can be written as

$$2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2} \leq \frac{3}{2}.$$

We have

$$\sin^2\frac{A-B}{2} + \cos^2\frac{A-B}{2} = 1$$

and, in any triangle ABC,

$$\cos\frac{A+B}{2} = \sin\frac{C}{2}.$$

By using these results, the inequality can be rewritten as

$$\sin^2\frac{A-B}{2} + \cos^2\frac{A-B}{2} - 2\cos\frac{A-B}{2}\left(2\sin\frac{C}{2}\right) + \left(2\sin\frac{C}{2}\right)^2 \geq 0,$$

or

$$\sin^2 \frac{A-B}{2} + \left(\cos \frac{A-B}{2} - 2\sin \frac{C}{2}\right)^2 \geq 0,$$

which is clearly true. For equality,

$$\sin^2 \frac{A-B}{2} = 0 \quad \text{and} \quad \cos \frac{A-B}{2} = 2\sin \frac{C}{2}$$

must hold simultaneously and this is only possible if the triangle is equilateral.

Problem 5 For any triangle ABC with sides a, b, and c and area Δ, prove that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta.$$

(International Mathematical Olympiad, 1961.)

Solution This problem is also trigonometrical and can be solved in various ways, but we are again interested in a solution by using the most fundamental inequality. The inequality is equivalent to

$$(a^2 + b^2 + c^2)^2 \geq 48s(s-a)(s-b)(s-c)$$

(where s is the semi-perimeter of the triangle ABC), or

$$(a^2 + b^2 + c^2)^2 \geq 3(a+b+c)(-a+b+c)(a-b+c)(a+b-c),$$

or

$$(a^2 + b^2 + c^2)^2 \geq 3(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4),$$

or

$$(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 \geq 0,$$

which is obvious. Also, equality holds when $a = b = c$, i.e. when the triangle is equilateral.

Problem 6 Determine all one-to-one functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the condition $f(x^2) - f^2(x) \geq \frac{1}{4}$.

Solution At first this looks like a calculus problem, but it can also be solved with the aid of the most fundamental inequality. Setting $x = 0$ and completing the square yields $(f(0) - \frac{1}{2})^2 \leq 0$, from which it follows that $f(0) = \frac{1}{2}$. Likewise, by setting $x = 1$, we find that $f(1) = \frac{1}{2}$. Hence, $f(0) = f(1) = \frac{1}{2}$, so that no one-to-one function can be found which will satisfy the given relation.

References

1 I. Reiman, *International Mathematical Olympiad 1959–1999* (Anthem Press, London, 2001).
2 T. Andreescu and R. Gelca, *Mathematical Olympiad Challenge* (Birkhäuser, Boston, MA, 2000).
3 AHSME Solution Pamphlet, American Mathematics Competitions.

Patna, India **Anand Kumar**

Letters to the Editor

Dear Editor,

The last five digits of 1249^{1249}

With reference to Bob Bertuello's letter in Volume 40, Number 3, p. 138, following on from the question I posed in Volume 39, Number 3, p. 124, we have

$$\begin{aligned}
1249^2 &= 1\,560\,001 \equiv 60\,001 \pmod{10^5},\\
1249^{2^2} &\equiv 20\,001 \pmod{10^5},\\
1249^{2^3} &\equiv 40\,001 \pmod{10^5},\\
1249^{2^4} &\equiv 80\,001 \pmod{10^5},\\
1249^{2^5} &\equiv 60\,001 \pmod{10^5},\\
1249^{2^6} &\equiv 20\,001 \pmod{10^5},\\
1249^{2^7} &\equiv 40\,001 \pmod{10^5},\\
1249^{2^8} &\equiv 80\,001 \pmod{10^5},\\
1249^{2^9} &\equiv 60\,001 \pmod{10^5},\\
1249^{2^{10}} &\equiv 20\,001 \pmod{10^5},\\
1249 &= 2^{10} + 2^7 + 2^6 + 2^5 + 2^0,\\
1249^{1249} &= 1249^{2^{10}} \times 1249^{2^7} \times 1249^{2^6} \times 1249^{2^5} \times 1249^{2^0}\\
&\equiv 20\,001 \times 40\,001 \times 20\,001 \times 60\,001 \times 1249 \pmod{10^5}\\
&\equiv 40\,001 \times 1249 \pmod{10^5}\\
&\equiv 61\,249 \pmod{10^5}.
\end{aligned}$$

Yours sincerely,

Abbas Roohol Amini
(10 Shahid Azam Lane
Makki Abad Avenue
Sirjan
Iran)

[Or, $1249^2 \equiv 60\,001 \pmod{10^5}$, $1249^8 \equiv 40\,001 \pmod{10^5}$, $1249^{10} \equiv 1 \pmod{10^5}$ so $1249^{1249} = (1249^{10})^{124} \times 1249^8 \times 1249 \equiv 1^{124} \times 40\,001 \times 1249 \equiv 61\,249 \pmod{10^5}$. – Ed.]

Minimum difference

Arrange the digits 0 to 9 in two numbers of five digits so that their positive difference is a minimum.

Midsomer Norton, Bath, UK **Bob Bertuello**

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

41.5 Prove that, for all positive integers n,

$$n^{2n-1} \geq (2n-1)!$$

(Submitted by Dorin Mărghidanu, Colegiul Naţional "A. I. Cuza", Corabia, Romania)

41.6 Determine all sequences of consecutive Fibonacci numbers which are in arithmetic progression.

(Submitted by Thomas Koshy, Framingham State College, Massachusetts, USA)

41.7 A is 15km south x degrees west of B, B is 7km west x degrees north of C, and D is 9km north x degrees east of C. How far is A from D?

(Submitted by Bob Bertuello, Midsomer Norton, Bath, UK)

41.8 The vertices of a triangle lie on the rectangular hyperbola $xy = 1$. Show that the orthocentre also lies on the hyperbola.

(Submitted by J. A. Scott, Chippenham, UK)

Solutions to Problems in Volume 40 Number 3

40.9 Solve the simultaneous equations

$$2008\log[x] + \{\log y\} = 0,$$
$$2008\log[y] + \{\log z\} = 0,$$
$$2008\log[z] + \{\log x\} = 0,$$

where $[\cdot]$ and $\{\cdot\}$ denote the integer part and the fractional part respectively, and the logs are to base 10.

Solution by Bor-Yann Chen, University of California, Irvine, USA

Let (x, y, z) be a simultaneous solution. Since $0 \leq \{\log y\} < 1$,

$$2008\log[x] \leq 2008\log[x] + \{\log y\} < 2008\log[x] + 1,$$

i.e. $2008\log[x] \leq 0 < 2008\log[x] + 1$.

Hence,
$$-\tfrac{1}{2008} < \log[x] \le 0,$$
so that
$$10^{-1/2008} < [x] \le 1.$$
Hence, $[x] = 1$; whence, $\{\log y\} = 0$. Similarly, $[y] = 1$, $[z] = 1$, $\{\log z\} = 0$, and $\{\log x\} = 0$. Hence, $\log x$, $\log y$, and $\log z$ are integers, so that $x = 10^a$, $y = 10^b$, and $z = 10^c$ for some integers a, b, and c. Since $[x] = [y] = [z] = 1$, the only possibility is $x = y = z = 1$. Further, this is a solution, so it is the only solution.

40.10 Let $n > 7$ be an integer such that $n - 1$ and $n + 1$ are both prime. Show that $n^2(n^2 - 4)(n^2 - 9)$ is divisible by 2 721 600.

Solution by Abbas Roohol Amini, Sirjan, Iran

Since $n - 1$ and $n + 1$ are prime, $n = 3k$ for some $k \in \mathbb{N}$. Hence,
$$n^2(n^2 - 4)(n^2 - 9) = 3^4k^2(k - 1)(k + 1)(9k^2 - 4).$$
Since $(k - 1)k(k + 1)$ is divisible by 3, this is divisible by 3^5. Also n must be even, say $n = 2p$ for some $p \in \mathbb{N}$, so
$$n^2(n^2 - 4)(n^2 - 9) = 2^4(p - 1)pp(p + 1)(4p^2 - 9).$$
Since $(p - 1)p$ and $p(p + 1)$ are divisible by 2, this is divisible by 2^6.

Next, $n \equiv 0, 2,$ or $-2 \pmod 5$. If $n \equiv 0 \pmod 5$, then $n^2 \equiv 0 \pmod{5^2}$; if $n \equiv \pm 2 \pmod 5$, then $n^2 - 4 \equiv 0 \pmod 5$ and $n^2 - 9 \equiv 0 \pmod 5$. Hence,
$$n^2(n^2 - 4)(n^2 - 9) \equiv 0 \pmod{5^2}.$$
Finally, $n \equiv 0, \pm 2,$ or $\pm 3 \pmod 7$, so one of n, $n^2 - 4$, $n^2 - 9$ is divisible by 7, so $n^2(n^2-4)(n^2-9)$ is divisible by 7. Hence, $n^2(n^2-4)(n^2-9)$ is divisible by $2^6 \times 3^5 \times 5^2 \times 7 =$ 2 721 600.

40.11 For a positive integer n, prove the identity
$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$
(a) algebraically, (b) graphically.

Solution by Paul Levrie, who proposed the problem

(a) We have
$$\begin{aligned}\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} &= \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2n}\right) - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right)\\ &= \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)\\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}.\end{aligned}$$

Also solved by Abbas Roohol Amini, Sirjan, Iran.

(b) See figure 1.

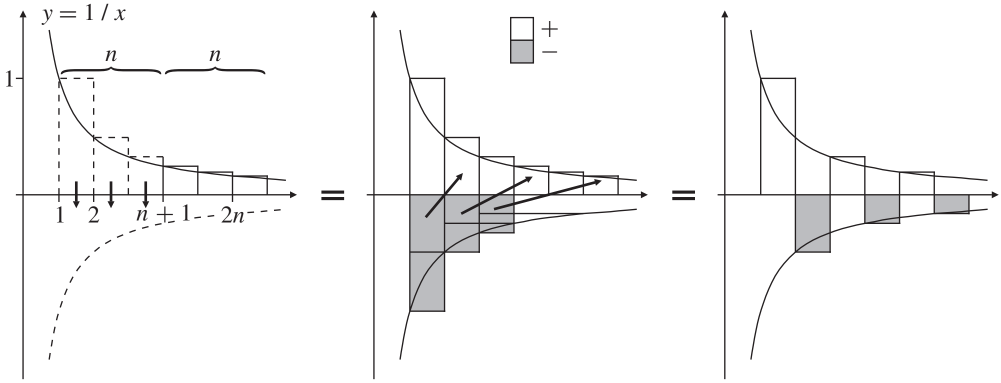

Figure 1

40.12 Show that, if a five-digit number is divisible by 41, then so is every cyclic permutation of its digits, for example

$$28\,577 = 41 \times 697, \qquad 57\,728 = 41 \times 1408.$$

Solution by Henry Ricardo, Medgar Evers College, Brooklyn, New York, USA

Write

$$N = a_4 10^4 + a_3 10^3 + a_2 10^2 + a_1 10 + a_0,$$
$$N' = a_3 10^4 + a_2 10^3 + a_1 10^2 + a_0 10 + a_4.$$

Then

$$10N - N' = a_4(10^5 - 1) = a_4 \times 2439 \times 41;$$

so, if N is divisible by 41, so is N'.

Also solved by Abbas Roohol Amini, Sirjan, Iran.

Products of consecutive numbers

The following algorithm determines two consecutive positive integers whose product is known. For instance, if 156 is the product of two consecutive integers, then write 25 after the number to get 15 625. The square root of this number is $\sqrt{15\,625} = 125$. Now, delete the last digit, 5, and you are left with 12, which is the first of two numbers, so $156 = 12 \times 13$. Can you explain this?

C/o A. A. Khan, Regional Office, Indian Overseas Bank,
Ashok Marg, Lucknow, India

M. A. Khan

Reviews

The Harmony of the World: 75 years of *Mathematics Magazine*. Edited by Gerald L. Alexanderson and Peter Ross. MAA, Washington, DC, 2007. Hardback, 304 pages, $55.95 (ISBN 0-88385-560-7).

Quite a few articles in this book assume knowledge of topics, concepts, and language not covered until the second or third year of University (and maybe beyond), so parts will be inaccessible to some readers. However, there is a considerable breadth of difficulty and some articles are straightforward, though the student may sometimes stumble over a technical term in an otherwise accessible article.

The articles are published in chronological order from 1928 to 1988. The subject matter is very wide ranging. Even where I couldn't follow everything I was stimulated to want to know more. For example, I must revisit some of the terminology of elemental topological analysis, e.g. 'Hausdofff space'! The articles are well written, and there is also a good deal of Mathematical History (hurrah!).

I'll give a flavour by describing the first five articles. The first was written by an 18-year-old student on her research into perfect numbers and is easy, but it is followed by one on the theory of manifold continuity including extensive untranslated quotes in French and German! The third article is a review of E. T. Bell's famous book, *Men of Mathematics*, in the year it was published. That was fine, but then a description of an important Mathematics Congress in Oslo in 1936 probably won't mean much to most of us today, nor the next one about Vigelan's monument to Abel in Oslo, unless you were visiting Oslo with a view to learning about the famous mathematician there.

There are very short (usually interesting and relatively easy) articles and some very long ones (some I made perfunctory attempts to follow, until I realised I would have to spend hours in textbooks to stand any chance).

I'll mention some of the very accessible articles. Firstly, there are the historical ones on 'Hypatia of Alexandria' and 'The Harmony of the World', the latter by Morris Kline just as he was bringing out his influential and popular book, *Mathematics in Western Culture*, in 1953. By chance I read this article on the Copernican revolution the day after listening to Heather Cooper's brilliant broadcast on BBC Radio 4 on the same subject, which was great as the article further illuminated the broadcast. The article on Hamilton's discovery of quaternions is very interesting as it explains the *order* in which Hamilton tackled the problem, including a very significant blind alley he could have avoided had he read a very simple proof by contradiction given by Legendre. There is a brilliant article on inequalities by one of the USA's foremost applied mathematicians, a fascinating one on base-π arithmetic by a 12-year-old, and one on 'Why your class is larger than average?'.

It would be good for aspiring young mathematicians to sharpen their intellectual teeth by reading some of the articles of intermediate difficulty. Two examples are one by Paul Erdős on 'A Property of 70', and the much longer 'Pólya's Enumeration Theorem by Example', a classic result in Combinatorics. Among the geometrical articles is a very interesting and long one on the tiling (tessellation) of the plane by pentagons.

Alastair Summers

Probably Not: Future Prediction Using Probability and Statistical Inference. By Lawrence N. Dworsky. John Wiley, Hoboken, NJ, 2008. Paperback, 310 pages, £31.95 (ISBN 0-470-18401-1).

This really is a fun-filled, easy-to-read, fascinating book. My attention was grabbed as early as the Preface where the author confesses to a life-long interest in data and sets out his intention of showing what information is needed to draw a conclusion, and what conclusions can and cannot be drawn from certain information. Surprisingly, examples abound in these introductory pages and these draw you immediately into his exploration of probability.

This is definitely not a textbook, so it does not require advanced mathematics for a good understanding – an open mind, a recall of how to read a graph, manipulate fractions, and use a basic calculator will get you happily through. Reassuringly, the author understands any mathematical reticence you may have, and does his best to make you feel comfortable about what he is imparting.

The chapter headings start routinely enough moving from Probability, Distributions, Random Walks, Life Insurance, Random Numbers, Gambling, Traffic, and Scheduling, before moving into more surprising areas such as Stock Market Portfolios, Chain Letters, Bird Counting, and Chaos, with only a single chapter devoted to statistical inference. But whatever the content, each chapter is securely couched in situations that are familiar to us all. For example, in probability we meet that intriguing Monte Hall problem again, life insurance is a gamble that we don't want to win, strategies for outsmarting traffic lights can be checked by simulation, and so can scheduling appointments in doctors' waiting rooms.

Discussion of statistical anomalies such as Benford's Law leads to the examination of patterns in data such as land area and populations of countries, and Parrando's Paradox shows somewhat counter-intuitively that it is possible to combine two losing games and produce a winning game. Simpson's Paradox is used to show that things can often be trickier than they first appear to be – don't take data at face value being the clearly delivered message.

This book is dedicated to 'grandchildren everywhere – probably the best invention of all time'. Having had some interesting discussions with my four-year-old grandson, already fascinated by the concept of zero, I have to agree with that sentiment, and also the one where the author states that probability is a tricky subject, but as he says, that's why he finds it to be so much fun. This is certainly recommended reading – take it on holiday with you for a relaxing but thought-provoking read.

Carol Nixon

Understanding Mathematics. By Keith Gregson. Nottingham University Press, 2007. Paperback, 122 pages, £19.50 (ISBN 1-904761-54-2).

This is a slim, attractively produced paperback intended for those struggling with mathematics on science courses at university, or with mathematics A-level. The author says that he is not attempting to write a comprehensive A-level text – 'there are many such books and software courses'. Indeed – which leaves me puzzled as to why there is so much routine bookwork in these pages. The majority of the space is devoted to explaining, in standard fashion, subjects such as differentiation from first principles; to my mind, the world is not crying out for another book along these lines.

What the world does need, however, is exciting and interesting pieces of mathematics that will inspire our mathematical youngsters to pursue the subject further. When Gregson turns in this direction, his book comes alive. At what speed should a fish swim relative to the current to maximise its energy efficiency? How did the young Einstein prove Pythagoras' theorem? What is the Beer–Lambert law? Much of this material was completely new to me, and deeply refreshing.

Would Gregson call himself first and foremost a scientist or a mathematician? Perhaps the former. To write the general quadratic curve as $y = a + bx + cx^2$ is unusual; but that could be a good thing. He talks at one point about why mathematics is inappropriate for discussing the arts, where we need 'a more evocative language in which we can exercise a bit of imagination'. There would be some who would claim that maths is the most evocative language there is! – and certainly one where imagination can find a full expression. But I quibble – certainly approaching the mathematics here via the sciences works extremely well. Gregson finds a friendly and comfortable style with which to address his readers, and for the excellent examples given here alone, his book deserves to win many of them.

Paston College **Jonny Griffiths**

Calculus Gems: Brief Lives and Memorable Mathematics. By George F. Simmons. MAA, Washington, DC, 2007. Hardback, 376 pages, \$48.95 (ISBN 0-88385-561-4).

This excellent book is in fact two books in one. The first, entitled *Brief Lives*, offers, biographical sketches of the great mathematicians of the past, while the second, called *Memorable Mathematics*, selects key mathematical proofs and breakthroughs that have taken the author's eye down the years. With great skill and impressively wide background knowledge, Simmons works part of Book A into Book B and vice versa. His concern throughout is to be lively and accessible.

His writings on the greats of our subject are unashamedly personal, which means there are no polite hagiographies to be found here; some mathematicians are 'in' (Gauss and Fermat get warm write-ups) while some are 'out' (Euclid and Descartes receive less fulsome accounts). The book is improved by the author's willingness to be controversial. The original sketched portraits of the subjects by Maceo Mitchell add continuity and originality to the pieces.

The choice of material within *Memorable Mathematics* is inspired. Here are many things that one might often quote in a lesson, but would you be able to lay your hands on a simple proof? Here you will find topics as varied as proving that e is irrational, investigating the cycloid, and the mathematics behind the catenary, all within a handful of pages. The level for the most part is appropriate for A-level further mathematics students and for students early on in their study of mathematics at university.

I learnt a lot from this book, which I heartily recommend. One thought in particular I take away – Dirichlet, the author tells us, carried Gauss' *Disquisitiones Arithmeticae* 'as a constant companion on his travels, like a devout man with his prayer book'. I am left feeling that it is important to read the work our mathematical heroes produced as closely as we can from the horse's mouth. Simmons has clearly spent a long time doing exactly that, and the fruits of his labour here should be of benefit to many people.

Paston College **Jonny Griffiths**

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