Mathematical Spectrum

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- J. J. Sylvester
- Catalan numbers
- Playing cards with Buffon

A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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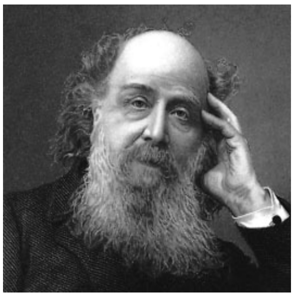
Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems and to mathematics in the classroom, as well as a computer column. The copyright of all published material is vested in the Applied Probability Trust.

A Mathematical Centenary
J. J. Sylvester 1814–1897

ROGER COOK

J. J. Sylvester died in 1897 so this may be a fitting time to look back at the career of someone who was a leading mathematician of Victorian England and who also played a significant part in stimulating mathematical research in America.

JJS was born James Joseph on 3 September 1814, adopting the surname Sylvester later in life — following the lead of his eldest brother. His mathematical skills were evident early and by the age of fourteen he was a student of Augustus de Morgan at University College London, which had only just opened in 1828. From 1829 to 1831 he studied at the Royal Institution at Liverpool, where the examiners' reports placed him in a special class of his own. In 1831, at the age of 17, Sylvester went to St John's College Cambridge but, due to health problems, did not take the mathematical tripos until 1837. At that time the candidates were placed in order of merit by the examiners, a practice that continued until early this century. Sylvester was placed second in the list; the man who came first never did any more mathematics.



In 1838 JJS was appointed Professor of Natural Philosophy (essentially, physics) at University College London, where his former teacher de Morgan was a colleague. The following year, at the age of 25, he was elected to the Royal Society. At this point his future career appeared well set up, but his skills were mathematical rather than scientific and he found the teaching of scientific courses irksome. He resigned from his post at UCL and took up the post of Professor of Mathematics at the University of Virginia in 1841. He spent only three months in this post before resigning. Newman (p. 340 of reference 7) summarises the events, which took place in one of the southern states of America some 20 years before the American Civil War, as follows:

> 'One day a young member of the chivalry whose classroom recitation he had criticized prepared an ambush and fell upon Sylvester with a heavy walking stick. He (JJS) speared the student with a sword cane; the damage was slight, but the professor found it advisable to leave his post and take the earliest possible passage for England.'

His American experiences had disillusioned him on the prospects of teaching and he took up employment as an actuary for a life insurance company. In 1846 he set out upon a legal career, entering the Inner Temple at the age of 32, and was called to the Bar in 1850. During these wilderness years JJS had remained mathematically active by taking private pupils including, rather surprisingly, Florence Nightingale.

In 1852 JJS made the acquaintance of another mathematician who had stumbled into the wrong profession, law. This was Arthur Cayley, remembered for the Cayley–Hamilton theorem, amongst other things. The two men had a common interest in developing the theory of algebraic invariants which arose from the earlier work of Boole in 1841. (A simple example of an algebraic invariant is the discriminant $d = b^2 - 4ac$ of the quadratic polynomial $ax^2 + bx + c$.)

Two years later, JJS applied, unsuccessfully, for the professorship of mathematics at the Royal Military College, Woolwich. He was appointed to the post the following year after the death of the successful candidate, and remained there until 1870 when he was compelled to retire at the age of 56 in somewhat acrimonious circumstances. During this tenure he was elected to the French Academy of Sciences in 1863. On his retirement he pursued his many interests which included reading the classics, playing chess and versifying. In 1870 he published a pamphlet *The Laws of Verse*.

The Johns Hopkins University was founded, at Baltimore, Maryland, in 1875. H. F. Baker (reference 3) comments:

> 'The authorities seem to have felt that a Professor of Mathematics and a Professor of Classics could inaugurate the work of an (*sic*) University without expensive buildings or elaborate apparatus (*plus ça change... RJC*). It was finally agreed that Sylvester should go, securing, besides his travelling expenses, an annual stipend of 5000 dollars "paid in gold".'

This stipend represented a considerable sum at that time and JJS remained at Johns Hopkins until 1883, revitalized as a mathematician. The *American Journal of Mathematics* was founded by JJS in 1878 and it remains one of the world's leading research journals. In its early years it did much to stimulate mathematical research in the United States.

The eminent number theorist H. J. S. Smith died in 1883, at the age of 57. Oxford University invited Sylvester, by then aged 70, to take the vacant post of Savilian Professor of Geometry. Thus Sylvester returned to England and delivered his inaugural lecture, at the age of 71, on 12 December 1885. In 1869, in a presidential address to the British Association, he had declared 'The mathematician lives long and lives young'. He was living proof of this, continuing in post at Oxford until 1894 when failing eyesight forced him into retirement, some 24 years after he was compelled to retire from the post at Woolwich. In March 1897 he suffered from a stroke and died a few days later on 15 March.

Bell describes Sylvester as short and stocky, with a fiery personality and broad intellectual interests. He read the classics and French, German and Italian literature in the original, illuminating many of his papers with apt quotations. His interest in language led him to introduce much of the terminology in mathematics, in particular he is credited with having introduced the term *matrix* into mathematical terminology (see p. 40 of reference 8).

His interests in mathematics were also wide ranging. Much of the work he did on matrices is still relevant to current research in linear programming (see reference 8). Nearly a century and a half later, one part of Sylvester's work, dating from 1852, still survives in many undergraduate mathematics courses. Despite its name, *Sylvester's law of inertia* is a theorem in linear algebra, or geometry, rather than mechanics. When a quadratic form of rank r is reduced by two real non-singular linear transformations to the diagonal forms

$$a_1x_1^2 + \cdots + a_rx_r^2, \qquad b_1x_1^2 + \cdots + b_rx_r^2,$$

the number of positive a's is the same as the number of positive b's (see for example p. 377 of reference 6).

In number theory, a positive integer n is called *perfect* when the sum of its divisors is $2n$. The first few perfect numbers are 6, 28, 496, 8128, $\ldots$ and it is known that an even number is perfect if and only if it is of the form

$$2^{p-1}(2^p - 1),$$

where $2^p - 1$ (and therefore also p) is prime. It is still not known whether there are any odd perfect numbers. In 1888 Sylvester showed that any odd perfect number must have at least five different prime divisors. All we know along these lines, over a century later and with the advantage of massive computing power, is that such numbers must have at least eight different prime divisors.

Andrews (reference 1) draws attention to Sylvester's work on partitions, a subject first considered systematically by Euler. A partition of a positive integer n is an expression of n as a sum of positive integers. For example, $7 = 4 + 3$ or $7 = 3 + 2 + 1 + 1$, and such partitions can be exhibited graphically by the *Ferrers graph*. For example,

$$\begin{array}{llll} \times & \times & \times & \times \\ \times & \times & \times & \end{array}$$

illustrates the partition $7 = 4+3$. Counting points vertically we also have the *conjugate* partition $7 = 2 + 2 + 2 + 1$. A partition is called *self-conjugate* when it has the same Ferrers graph as its conjugate partition; for example, $6 = 3 + 2 + 1$ is self-conjugate.

It is not too difficult, if you are Euler, to see that the number of partitions $p(n)$ of n can be generated by the formula

$$\sum_{n=1}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-1}$$

since $(1 - x^n)^{-1} = 1 + x^n + x^{2n} + x^{3n} + \ldots$. (If you lack Euler's insight see ch. 14 of reference 2 or ch. XIX of reference 5.) In 1750 Euler considered the reciprocal of this generating function

$$\prod_{n=1}^{\infty} (1 - x^n)$$

and obtained a series expansion for it which is called *Euler's pentagonal number theorem* (see reference 2 or 5 for the gory details; we need not go into specifics here). By considering certain self-conjugate partitions Sylvester was able to generalize Euler's pentagonal number theorem. This led F. Franklin to develop a purely combinatorial proof of the pentagonal number theorem, arising so naturally from Sylvester's work that it is referred to as the Franklin–Sylvester method. This proof is described as 'remarkable' by Apostol, 'very beautiful' by Hardy and Wright and 'probably the first major discovery in American mathematics' by Andrews. A more detailed biography is provided by Bell (reference 4), which I have used extensively in preparing this article.

References

1. G. E. Andrews, 'J. J. Sylvester, Johns Hopkins and Partitions'. In *A Century of Mathematics in America* ed. P. Duren (American Mathematical Society, Providence, RI, 1988) pages 21–40.
2. T. M. Apostol, *Introduction to Analytic Number Theory* (Springer, New York, 1976).
3. H. F. Baker, Biographical Notice of J. J. Sylvester (pages xv–xxxvii of reference 9).
4. E. T. Bell, *Men of Mathematics* vol. 2 (Penguin, London, 1953).
5. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Clarendon, Oxford, 1938).
6. L. Mirsky, *An Introduction to Linear Algebra* (Clarendon, Oxford, 1955).
7. J. R. Newman, *The World of Mathematics* vol. 1 (Simon and Schuster, New York, 1956).
8. A. Schrijver, *Theory of Linear and Integer Programming* (Wiley, Chichester, 1986).
9. J. J. Sylvester, *Collected Papers* (Cambridge University Press, Cambridge, 1912). (Reprinted by Chelsea, New York, 1973). □

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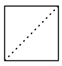
Catalan Numbers

ISAAC VUN and PAUL BELCHER

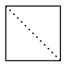
How many ways are there to divide up a field into triangular pieces? To bracket objects into pairs? To never be behind when votes are counted in an election? The Catalan numbers help to describe all of these problems!

Euler's problem of polygon division

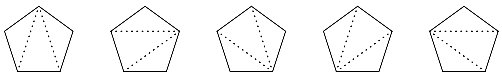
Consider a farmer who has a field in the shape of an n-sided convex polygon and wonders in how many different ways he can divide this into triangles using diagonals to join the vertices. Let E_n be the number of ways. $E_3 = 1$ since we have a triangle initially; $E_4 = 2$ as a quadrilateral can be divided up as follows:



or



$E_5 = 5$; the five different divisions are as follows:



With work, the next few values of the sequence can be found as $E_6 = 14$, $E_7 = 42$, $E_8 = 132$. Let us consider the general case as illustrated in figure 1.

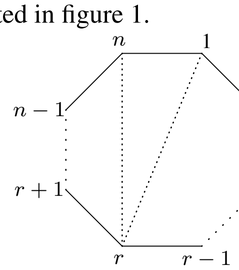


Figure 1

If we join vertices n, 1 and r, we obtain triangle $n1r$, and there are two smaller polygons on either side of this. There are two cases to consider.

Case 1: $3 \leqslant r \leqslant n-2$, which gives an r-sided polygon on one side and an $(n-r+1)$-sided one on the other side.

Case 2: $r = 2$ or $r = n-1$, which give a triangle and an $(n-1)$-sided polygon.

For convenience we will define $E_2 = 1$. Then, counting the total number of divisions as r varies, we have as a recurrence relation:

$$E_n = E_2 \times E_{n-1} + E_3 \times E_{n-2} + \dots \\ + E_r \times E_{n-r+1} \cdots + E_{n-2} \times E_3 + E_{n-1} \times E_2.$$

The number of ways of pairing n objects when working with a binary operation

Real numbers are associative under addition, that is $(a+b)+c = a+(b+c)$ for all $a, b, c \in \mathbb{R}$, and so we can just write $a+b+c$ without confusion. However, for subtraction this is not true; for instance, $(10-5)-2 \neq 10-(5-2)$. Given n objects, we are going to consider how many different ways we can put in brackets so that we only have paired objects. Here are some examples:

$n = 2$, only (a, b);
$n = 3$, $((a, b), c)$ or $(a, (b, c))$;
$n = 4$, $(((a, b), c), d)$ or $((a, (b, c)), d)$ or
$((a, b), (c, d))$ or $(a, ((b, c), d))$ or $(a, (b, (c, d)))$.

Let this sequence be denoted by $\{C_n\}$. Then we have $C_2 = 1$, $C_3 = 2$, $C_4 = 5$; and further work gives $C_5 = 14$, $C_6 = 42$, $C_7 = 132$. These are called *Catalan numbers* after the nineteenth century Belgian mathematician, Eugène Charles Catalan (1814–1894).

The German mathematician Johann von Segner (1704–1777) first solved the problem of dissecting a polygon into triangles using non-intersecting diagonals, but his solution was not as elegant as Catalan's. Euler and Binet also worked on the problem.

Let us find the recurrence relation for the sequence $\{C_n\}$. A pairing will finally be of the form (A, B), where there will be r objects combined in A and $n-r$ combined in B, with r running from 1 to $n-1$. So

$$C_n = C_1 \times C_{n-1} + C_2 \times C_{n-2} + \dots \\ + C_r \times C_{n-r} + \cdots + C_{n-2} \times C_2 + C_{n-1} \times C_1,$$

where we have defined $C_1 = 1$.

Election results

Suppose there is an election with only two candidates A and B. A total of $2n$ votes are to be cast. Candidate A is to tie with candidate B ultimately, but is leading him at every earlier stage of the election (i.e. A is always winning until the final vote is cast). In how many ways is this possible? Let us imagine a $2n \times n$ grid as shown in figure 2.

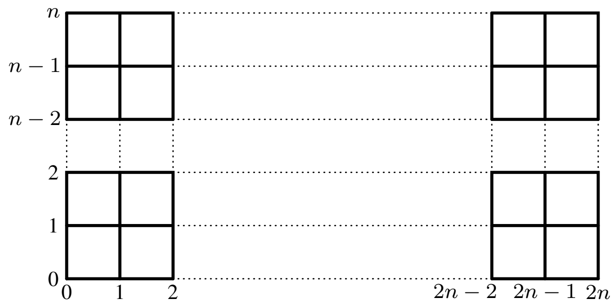


Figure 2

The situation that x votes have been cast and that A is ahead of B by y votes is represented by the point (x, y) of the grid. Thus a vote for A is denoted by a diagonally upward move and a vote for B by a diagonally downward move (see figure 3). We therefore have to move by diagonally upward and downward steps from $(0,0)$ to $(2n, 0)$ without touching the base line $y = 0$ between these two points. Denoting by $G(x)$ the number of different routes from $(0,0)$ to $(x, 0)$ which do not touch the base line between these points, we can easily check that $G(2) = 1$, $G(4) = 1$, $G(6) = 2$ (as shown in figure 3).

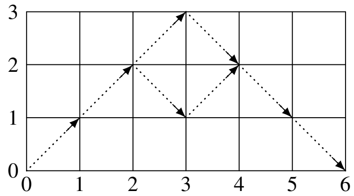


Figure 3

Let us obtain a recurrence relation for $G(2n)$. Let $G^*(2n)$ be the number of routes from $(0,0)$ to $(2n, 0)$ when you are allowed to touch the line $y = 0$, but cannot go below it. The number of routes from $(1,1)$ to $(2n-1, 1)$ when you can touch the line $y = 1$ (but cannot go below it) is the same as the number of routes from $(0,0)$ to $(2n, 0)$ without touching $y = 0$. Hence $G^*(2n-2) = G(2n)$. Now consider a route from $(0,0)$ to $(2n, 0)$ that does not touch $y = 0$ and hits the line $y = 1$ for the *last* time (before the point $(2n-1, 1)$ at the point $(2r+1, 1)$. We see that r can run from 0 to $n-2$. Then:

$$\begin{aligned}
G(2n) &= \sum_{r=0}^{n-2} \{\text{number of routes from } (1,1) \text{ to } (2r+1,1) \\
&\qquad\quad \text{where one may hit the line } y=1\} \\
&\quad \times \{\text{number of routes from } (2r+1,1) \text{ to } (2n-1,1) \\
&\qquad\quad \text{where one may not hit the line } y=1\} \\
&= \sum_{r=0}^{n-2} G^*(2r) \times G(2n-2r-2) \\
&= \sum_{r=0}^{n-2} G(2(r+1)) \times G(2(n-(r+1))) \\
&= G(2) \times G(2(n-1)) + G(2\times 2) \times G(2(n-2)) \\
&\qquad + \dots + G(2(n-1)) \times G(2).
\end{aligned}$$

Denoting $G(2r)$ as G_r, we have $G_1 = 1$, $G_2 = 1$, $G_3 = 2$ and the recurrence relation

$$\begin{aligned}
G_n &= G_1 \times G_{n-1} + G_2 \times G_{n-2} + \dots \\
&\quad + G_r \times G_{n-r} + \dots + G_{n-1} \times G_1.
\end{aligned}$$

So, by looking at the initial values and the recurrence relations, we can combine the three problems and have that

$$G_n = C_n = E_{n+1}.$$

We now solve the recurrence relation for $\{C_n\}$ by using its generating series defined by

$$C(x) = C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n + \dots.$$

Consider

$$\begin{aligned}
[C(x)]^2 &= (C_1 x + C_2 x^2 + \dots + C_n x^n + \dots) \\
&\quad \times (C_1 x + C_2 x^2 + \dots + C_n x^n + \dots) \\
&= (C_1 \times C_1) x^2 + (C_1 \times C_2 + C_2 \times C_1) x^3 \\
&\quad + (C_1 \times C_3 + C_2 \times C_2 + C_3 \times C_1) x^4 + \dots \\
&\quad + (C_1 \times C_n + C_2 \times C_{n-1} + \dots + C_n \times C_1) x^{n+1} \\
&\quad + \dots \\
&= C_2 x^2 + C_3 x^3 + C_4 x^4 + \dots + C_{n+1} x^{n+1} + \dots \\
&= C(x) - x,
\end{aligned}$$

or

$$[C(x)]^2 - C(x) + x = 0.$$

Treating $C(x)$ as a variable we can solve by using the quadratic formula. Hence

$$\begin{aligned}
C(x) &= \tfrac{1}{2} \pm \tfrac{1}{2}(1-4x)^{1/2} \\
&= \tfrac{1}{2} \pm \tfrac{1}{2}\left(1 + \binom{\frac{1}{2}}{1}(-4x) + \binom{\frac{1}{2}}{2}(-4x)^2 \right. \\
&\qquad \left. + \dots + \binom{\frac{1}{2}}{n}(-4x)^n + \dots\right). \qquad (1)
\end{aligned}$$

In the bracket the coefficient of x is $\frac{1}{2}(-4) = -2$ and, for $n > 1$, the coefficient of x^n is

$$\begin{aligned}
&\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\dots(\frac{1}{2}-n+1)}{n!}(-4)^n \\
&\qquad = -\frac{1 \times 3 \times \dots \times (2n-3)}{n!} 2^n.
\end{aligned}$$

Since Catalan numbers are positive, it follows that in (1) the minus sign must be taken. Thus

$$\begin{aligned}
C(x) &= x + x^2 + 2x^3 + \dots \\
&\qquad + \frac{1 \times 3 \times \dots \times (2n-3)}{n!} 2^{n-1} x^n + \dots,
\end{aligned}$$

and so

$$\begin{aligned}
C(1) &= 1, \\
C(n) &= \frac{1 \times 3 \times \dots \times (2n-3)}{n!} 2^{n-1} \qquad (n > 1).
\end{aligned}$$

It is also easily checked that

$$C(n) = \frac{(2n-2)!}{n[(n-1)!]^2} = \frac{1}{n}\binom{2n-2}{n-1},$$

with both formulae also holding for $n = 1$ if, as usual, $0!$ and $\binom{0}{0}$ are interpreted as 1.

There are several other proofs of the formula for C_n. We exhibit one more which arises from the second problem that we showed led to Catalan numbers. Let R_n be the number of different paired products with a binary operation $*$ for a set of n different objects, where all possible orders of the objects are admitted. For example, $R_2 = 2$ (from $a * b$ and $b * a$). Thus R_n counts paired products where all orders are considered, whereas C_n counts paired products when the order is predetermined. As the number of ways of ordering n objects is $n!$,

$$C_n = \frac{R_n}{n!}.$$

We shall obtain a formula for R_n and so immediately arrive at that for C_n.

Let the n objects involved be $f_1, f_2, \ldots, f_n$. When these objects are 'multiplied' in pairs there will be $n-1$ 'multiplications' in a product. Each 'multiplication' will be between two expressions of the form $f * g$. Introducing a new factor f_{n+1} into the product at that particular 'multiplication' sign can be done in four ways:

$$(f_{n+1} * f) * g \text{ or } (f * f_{n+1}) * g$$

or

$$f * (f_{n+1} * g) \text{ or } f * (g * f_{n+1}).$$

Also, for each product P, say, there are two ways of introducing f_{n+1} not at an existing multiplication sign, namely

$$f_{n+1} * P \text{ or } P * f_{n+1}.$$

Hence

$$\begin{aligned} R_{n+1} &= (4(n-1)+2)R_n \\ &= (4n-2)R_n \end{aligned}$$

so, for $n > 1$,

$$R_n = 2 \times 6 \times 10 \times 14 \times \cdots \times (4n-6) \times R_1,$$

giving us that

$$C_n = \frac{1 \times 3 \times 5 \times \cdots \times (2n-3)}{n!} 2^{n-1}$$

for $n > 1$. □

***Isaac Vun**, as a 19-year-old student at Atlantic College, presented this work as part of his Extended Essay in the International Baccalaureate. Isaac came on a scholarship to Atlantic College as a representative of Hong Kong and is now studying at Trinity College, Cambridge. He is not claiming this as new material, only that it was new to him as he investigated it for himself. **Dr Paul Belcher** was his supervisor. He is Head of Mathematics at Atlantic College and recently won his age group of the World Quadrathlon Championships (5 k swim, 20 k kayak, 100 k cycle, 21 k run).*

Braintwister

3. We have the powers

I have just put a particular number in my calculator's memory. I have calculated the sum of that number with its own reciprocal and got a single-digit whole-number answer.

I then squared the particular number in the memory and added that square to its own reciprocal. This time I again got a single-digit whole-number answer, but higher than the previous one.

I then cubed the particular number and added that cube to its own reciprocal. Without using a calculator you should be able to work out. . .

What were my three answers?

(The solution will be published next time.)

VICTOR BRYANT

Domino Words

In a domino word-chain such as

```
BAD
 ADRENALIN
       LINESMAN
            MANGO
               GOOD
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each word after the first starts with the last two or more letters of the previous word. The word chain shown goes from BAD to GOOD with three intermediate words. The fewer intermediate words the better. Devise domino word-chains as follows (the figure in brackets is a target number of intermediate words for you to match or beat).

MAJOR to BLAIR (3)	BLAIR to MAJOR (3)
HEAVEN to HELL (2)	HELL to HEAVEN (2)
BLACK to WHITE (?)	WHITE to BLACK (3)
SANTA to REINDEER (1)	REINDEER to SANTA (4)
ADAMS to PAISLEY (3)	PAISLEY to ADAMS (2)
GOOD to BAD (3)	

J. N. MACNEILL

A Generalized Fibonacci Sequence via Matrices

RUSSELL EULER

A different way of producing properties of the Fibonacci sequence.

The purpose of this article is to use matrices to obtain identities involving the terms of the sequence $\{w_n\}_{n=0}^{\infty}$ defined recursively by $w_0 = 0$, $w_1 = 1$ and

$$w_n = aw_{n-1} + w_{n-2} \tag{1}$$

for $n \geqslant 2$. The first first six terms of the sequence are 0, 1, a, a^2+1, a^3+2a and a^4+3a^2+1. If $a=1$ or $a=2$, then $\{w_n\}$ is the *Fibonacci* or *Pell sequence* respectively. The characteristic equation for (1) is $\lambda^2 - a\lambda - 1 = 0$. The roots of this equation are

$$u = \tfrac{1}{2}\big(a + \sqrt{a^2+4}\big) \quad \text{and} \quad v = \tfrac{1}{2}\big(a - \sqrt{a^2+4}\big).$$

The solution of (1) is

$$w_n = c_1 u^n + c_2 v^n,$$

where c_1 and c_2 are constants (see reference 1, for example). Using the initial conditions, one obtains the system of equations

$$\begin{aligned} w_0 &= c_1 + c_2 = 0, \\ w_1 &= c_1 u + c_2 v = 1. \end{aligned}$$

The solution of this system is

$$c_1 = \frac{1}{\sqrt{a^2+4}}, \quad c_2 = \frac{-1}{\sqrt{a^2+4}}$$

if $a \neq \pm 2\mathrm{i}$. Hence, for $a \neq \pm 2\mathrm{i}$,

$$w_n = \frac{u^n - v^n}{\sqrt{a^2+4}}. \tag{2}$$

The sequence $\{w_n\}_{n=0}^{\infty}$ can be extended to negative values of n as follows. For $n > 0$, define w_{-n} by the analogue of (2), namely

$$\begin{aligned} w_{-n} &= (u^{-n} - v^{-n})/\sqrt{a^2+4}, \\ &= (v^n - u^n)/[\sqrt{a^2+4}(uv)^n]. \end{aligned}$$

Since $uv = -1$,

$$\begin{aligned} w_{-n} &= -(u^n - v^n)/[\sqrt{a^2+4}(-1)^n] \\ &= (-1)^{n-1} w_n. \end{aligned} \tag{3}$$

Some of the terms of the extended sequence $\{w_n\}_{n=-\infty}^{\infty}$ are a^4+3a^2+1, $-a^3-2a$, a^2+1, $-a$, 1, 0, 1, a, a^2+1, a^3+2a, a^4+3a^2+1.

One can show that (1) holds for all integers n as follows. If $n=0$, then (1) is satisfied since both sides of the equation are equal to 0. If $n=1$, then both sides of the equation are equal to 1. If $n<0$, then let $n=-m$, where $m>0$. Then using (3) gives

$$\begin{aligned} w_n - aw_{n-1} - w_{n-2} &= w_{-m} - aw_{-m-1} - w_{-m-2} \\ &= (-1)^{m-1} w_m - a(-1)^m w_{m+1} \\ &\quad - (-1)^{m+1} w_{m+2} \\ &= (-1)^{m-1}[w_m + aw_{m+1} \\ &\quad - (aw_{m+1} + w_m)] \\ &= 0. \end{aligned}$$

The sequence $\{w_n\}_{n=0}^{\infty}$ can also be generated by matrix multiplication as follows.

Theorem 1. If

$$A = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix},$$

then

$$A^n = \begin{bmatrix} w_{n+1} & w_n \\ w_n & w_{n-1} \end{bmatrix}$$

for $n \geqslant 1$.

The theorem is best proved inductively, and the details are left to the reader.

Theorem 1 can be used to obtain identities involving terms of $\{w_n\}$. For example, since $A^{2n} = (A^n)^2$,

$$\begin{aligned} \begin{bmatrix} w_{2n+1} & w_{2n} \\ w_{2n} & w_{2n-1} \end{bmatrix} &= \begin{bmatrix} w_{n+1} & w_n \\ w_n & w_{n-1} \end{bmatrix}^2 \\ &= \begin{bmatrix} w_{n+1}^2 + w_n^2 & w_{n+1}w_n + w_n w_{n-1} \\ w_{n+1}w_n + w_n w_{n-1} & w_n^2 + w_{n-1}^2 \end{bmatrix}. \end{aligned}$$

Equating corresponding entries in the first row gives

$$\begin{aligned} w_{2n+1} &= w_{n+1}^2 + w_n^2 \quad \text{for } n \geqslant 0, \\ w_{2n} &= w_n(w_{n+1} + w_{n-1}) \quad \text{for } n \geqslant 1. \end{aligned}$$

If $a=1$, formulas for Fibonacci numbers analogous to the last two identities are given in reference 2.

More generally, if k is a positive integer, then $A^{kn} = (A^n)^k$ and so

$$\begin{bmatrix} w_{kn+1} & w_{kn} \\ w_{kn} & w_{kn-1} \end{bmatrix} = \begin{bmatrix} w_{n+1} & w_n \\ w_n & w_{n-1} \end{bmatrix}^k.$$

Taking the determinant of the above matrices yields the identity

$$w_{kn+1}w_{kn-1} - w_{kn}^2 = (w_{n+1}w_{n-1} - w_n^2)^k$$

for $n \geqslant 1$. In fact, since $\det A = -1$, both sides are equal to $(-1)^{nk}$. If $a = 1$, the above identity is known as *Cassini's identity*.

As $\det A \neq 0$, A^{-n} may be interpreted as $(A^{-1})^n$ for $n \geqslant 0$, and we have the following theorem, the proof of which is left to the reader.

Theorem 2. If

$$A = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix},$$

then

$$A^{-n} = \begin{bmatrix} w_{-n+1} & w_{-n} \\ w_{-n} & w_{-n-1} \end{bmatrix}$$

for $n \geqslant 0$.

Since this theorem is Theorem 1 with n negative, the identities proved from Theorem 1 also hold for $n < 0$.

It is straightforward to show that $A^2 = I + aA$. Using the binomial theorem we have

$$\begin{aligned} A^{2n} &= (I + aA)^n \\ &= \sum_{k=0}^{n} a^k \binom{n}{k} A^k \end{aligned}$$

for $n \geqslant 0$. If $n = 0$ it is understood that $\binom{0}{0} = 1$. If $a = 0$, let $0^0 = 1$. Now, by Theorem 1,

$$\begin{bmatrix} w_{2n+1} & w_{2n} \\ w_{2n} & w_{2n-1} \end{bmatrix} = \sum_{k=0}^{n} a^k \binom{n}{k} \begin{bmatrix} w_{k+1} & w_k \\ w_k & w_{k-1} \end{bmatrix}.$$

Equating corresponding entries in the first row gives the identities

$$w_{2n+1} = \sum_{k=0}^{n} a^k \binom{n}{k} w_{k+1}$$

and

$$w_{2n} = \sum_{k=0}^{n} a^k \binom{n}{k} w_k$$

for $n \geqslant 0$.

Similarly, $A^{-2} = I - aA^{-1}$ and the above technique gives

$$w_{-2n+1} = \sum_{k=0}^{n} (-a)^k \binom{n}{k} w_{-k+1}$$

and

$$w_{-2n} = \sum_{k=0}^{n} (-a)^k \binom{n}{k} w_{-k}$$

for $n \geqslant 0$.

The above identities can be obtained directly from (2). However, the matrix method is noticeably simpler.

It was noted that (2) is valid for $a \neq \pm 2\mathrm{i}$. If $a = 2\mathrm{i}$, it can be shown that $w_n = n\mathrm{i}^{n-1}$ for $n \geqslant 0$. If $a = -2\mathrm{i}$, then $w_n = n(-\mathrm{i})^{n-1}$ for $n \geqslant 0$. Both of these sequences can be extended for $n < 0$ by defining $w_{-n} = (-1)^{n-1}w_n$. The identity (1) and all other results obtained for $a \neq \pm 2\mathrm{i}$ then also hold for $a = \pm 2\mathrm{i}$.

References

1. R. E. Mickens, *Difference Equations: Theory and Applications* (Van Nostrand Reinhold, New York, 1990) 2nd edn.
2. S. Vajda, *Fibonacci and Lucas Numbers and The Golden Section: Theory and Applications* (Halsted Press, New York, 1990) p. 25. □

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Solution to Braintwister 2

(Moment of madness)

Answer. The calendar is horizontal on Friday 21 February.

Solution. Let the day be a (1–7, 1 = Sunday), let the month be b (1–12) and the number be c (1–31). Then, for example, the 'day' ruler will have $2(a-1)$ cm on the left of the screw and $2(7-a)$ cm on the right. Taking moments, these will contribute $2(a-1)^2$ anticlockwise and $2(7-a)^2$ clockwise. Overall the balancing moments give

$$2(a-1)^2 + 2(b-1)^2 + \tfrac{1}{2}(c-1)^2 = 2(7-a)^2 + 2(12-b)^2 + \tfrac{1}{2}(31-c)^2,$$

which tidies up to $12a + 22b + 15c = 431$. Note that $431 - 22b$ is divisible by 3 and so b is 2, 5, 8 or 11.

Trying $b = 2$ gives $4a + 5c = 129$. But in that month (February 1997) the 1st is on a Saturday (i.e. $c = 1$ and $a = 7$) and throughout the month $c - a$ is -6, 1, 8, 15 or 22. The only value which ensures that $9c = (4a + 5c) + 4(c - a)$ is divisible by 9 is $c - a = 15$, giving $c = 21$ and $a = 6$. So the calendar balances on Friday 21 February.

Trying $b = 5$, 8 and 11 in a similar way leads to no further answers. VICTOR BRYANT

Playing Cards with Buffon

P. GLAISTER

A less painful alternative to Buffon's needle problem.

1. Introduction

The classical needle problem of Buffon is concerned with the probability that a needle will cross a crack when dropped on floorboards. In this article we extend this result to the case when the dropped object is rectangular, e.g. a playing card. We then suggest an obvious further generalisation, and discuss some special cases.

2. The needle problem

We begin with an outline of the classical problem as an introduction to the approach required for our generalisation. Suppose a needle of length l is dropped on a floor of wooden planks of width b, where $l \leqslant b$. The case $l > b$ can be treated similarly (see reference 1). Figure 1 illustrates a plank with such a needle on it. The probability that the needle does not cross a crack is denoted by q, where $0 < q < 1$, and to determine it we consider the distance over which the centre of the needle can land without crossing a crack. Figure 2 shows this distance as a dotted line, and we see that it depends on the *orientation* of the needle.

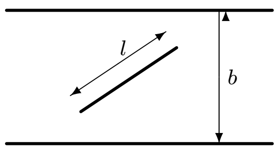

Figure 1

This orientation is measured as the anti-clockwise angle between the horizontal, with left to right being positive, and the needle, one end of which is assigned as the 'point' and the other as the 'eye', with 'eye' to 'point' as the positive direction. Denoting this angle by θ, with $0 \leqslant \theta \leqslant 2\pi$, we see that, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ only, the length of the dotted line in figure 2 is

$$b - \tfrac{1}{2}l\sin\theta - \tfrac{1}{2}l\sin\theta = b - l\sin\theta. \tag{1}$$

The total distance over which the centre of the needle can land is b, the width of the plank.

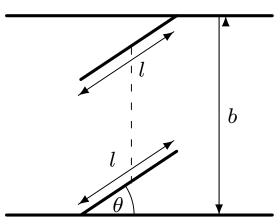

Figure 2

Now, since this distance has a different length for *each* orientation we must calculate q from

$$q = \frac{\text{sum (over } \theta\text{) of distance over which the centre of the needle can land without crossing a crack}}{\text{sum (over } \theta\text{) of possible distance over which the centre of the needle can land}}.$$

We could consider θ to range from 0 to 2π, and determine the distance corresponding to (1) for various ranges of θ. It is easier to use symmetry to sum over θ in the range $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ only, and then multiply the answer by 4. Since this occurs in both the numerator and denominator, the 4 in each will cancel. Thus the sums are determined by integrating both numerator and denominator (with respect to θ) between 0 and $\frac{1}{2}\pi$, so that

$$q = \frac{\int_0^{\pi/2} (b - l\sin\theta)\,\mathrm{d}\theta}{\int_0^{\pi/2} b\,\mathrm{d}\theta} = \frac{\frac{1}{2}\pi b - l}{\frac{1}{2}\pi b} = 1 - \frac{2l}{\pi b}. \tag{2}$$

This is sometimes used as a way of approximating π by determining $p = 1 - q = 2l/\pi b$, the probability of the needle crossing the crack, experimentally.

It is now a straightforward matter to consider the generalisation of this result.

3. The card problem

Let us consider the case where the object dropped is two-dimensional, so that it has *width* as well as length. In practice, it could be a playing card of dimensions $c \times d$. For simplicity, we consider the case where the diagonal of the card is less than the width of the boards, i.e. $\sqrt{(c^2 + d^2)} \leqslant b$. If this is not the case, then the problem becomes much harder.

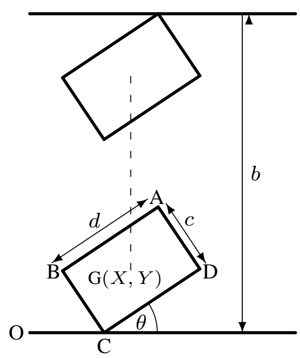

Figure 3

Figure 3 illustrates a plank and the possible distance over which the centre of the card can land without crossing a crack for a particular orientation. The coordinates of the vertices of the labelled card relative to the point O directly below the leftmost point B of the card are

$$\begin{aligned}&\mathrm{A}(d\cos\theta, d\sin\theta + c\cos\theta),\\&\mathrm{B}(0, c\cos\theta),\\&\mathrm{C}(c\sin\theta, 0),\\&\mathrm{D}(c\sin\theta + d\cos\theta, d\sin\theta).\end{aligned}$$

The coordinates of the centre of the card are then obtained by taking the arithmetic mean of the coordinates of its vertices, to give

$$\mathrm{G}(\tfrac{1}{2}(d\cos\theta + c\sin\theta), \tfrac{1}{2}(d\sin\theta + c\cos\theta)) = \mathrm{G}(X, Y).$$

Following the ideas above, we calculate the probability that the card will not cross a crack from

$$q = \frac{\text{sum (over } \theta\text{) of distance over which the centre of the card can land without crossing a crack}}{\text{sum (over } \theta\text{) of possible distance over which the centre of the card can land}}.$$

Referring to figure 3, for a particular orientation given by the angle $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ as before, we see that the distance over which the centre of the card can land without crossing a crack is that shown as the dotted line, whose length is

$$b - 2Y = b - (d\sin\theta + c\cos\theta). \qquad (3)$$

Thus

$$q = \frac{\int_0^{\pi/2} [b - (d\sin\theta + c\cos\theta)]\,\mathrm{d}\theta}{\int_0^{\pi/2} b\,\mathrm{d}\theta} = 1 - \frac{2(c+d)}{\pi b}. \qquad (4)$$

The probability that the card crosses a crack is thus

$$p = 1 - q = \frac{2(c+d)}{\pi b}. \qquad (5)$$

If we have a square card, say $c = d = m$, then

$$p = \frac{4m}{\pi b}.$$

One obvious generalisation is to replace the floorboards with parquet flooring of rectangular blocks. If the blocks are of dimensions $a \times b$, then the probability of crossing a crack is

$$p = \frac{(a+b)(c+d) - \frac{1}{2}(c^2+d^2) - \frac{1}{2}\pi cd}{\frac{1}{2}\pi ab}. \qquad (6)$$

We leave the proof of this as an exercise for the reader. Finally, we note that, by setting $c = l$ and $d = 0$ in (6), the probability that a needle of length l will cross a crack when dropped on rectangular blocks of size $a \times b$ is

$$p = \frac{l(a+b) - \frac{1}{2}l^2}{\frac{1}{2}\pi ab}.$$

Reference

1. J. Gani, The problem of Buffon's needle, *Mathematical Spectrum* **13** (1980) pp. 14–18. □

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38th International Mathematical Olympiad

Teams of school students from 82 countries took part in the 38th IMO held in Mar del Plata, Argentina, 18–31 July, 1997. The top 10 teams and their scores (out of a possible 252 points) were:

China	223
Hungary	219
Iran	217
Russia	202
USA	202
Ukraine	195
Bulgaria	191
Romania	191
Australia	187
Vietnam	183

By adjoining the consecutive integers 183 and 184, we obtain a perfect square:

$$183\,184 = 428^2.$$

Find another consecutive pair with this property.

K. R. S. Sastry
(Dodballapur, India)

For every $n > 1$, find n numbers with greatest common divisor 1 such that no $n-1$ of them have greatest common divisor 1.

Minh Can
(Student, University of Southern California, LA)

Sums of Powers of an Arithmetic Progression

TEMPLE H. FAY and K. R. S. SASTRY

Who enjoys tedious calculations? — Computer algebra systems do!

Introduction

In this article, we look at how to sum the powers of the terms of an arithmetic progression. While this calculation may be accomplished in several ways, we use a generating function approach. The advantage of this approach is twofold. Firstly, the technique is very powerful by being general and is capable of being applied to a wide range of problems. Secondly, the approach permits the use of computer algebra programs, such as DERIVE, *Maple V* and *Mathematica*, to calculate derivatives, factor and simplify expressions. Tedious hand calculations become almost trivial manipulations when employing the software. Readers may like to do computer-based experimentations with our approach to calculate the alternating sums of powers and other variations on this theme.

Given the arithmetic progression

$$a, a+d, a+2d, \dots, a+nd, \dots,$$

where $a \neq 0$, we wish to calculate formulae for the sums of powers of the form

$$a^k + (a+d)^k + \cdots + (a+nd)^k.$$

This is a natural consideration if one views calculation of the sums $1^k + 2^k + \cdots + n^k$ as a special case; calculation of these sums can be traced back to Faulhaber (1631) and J. Bernoulli (1713); see reference 3. (See also references 1, 7 and 8 for recent *Spectrum* articles.)

The situation for $k = 1$ is, of course, well known:

$$\sum_{i=0}^{n}(a+id) = \frac{n+1}{2}(2a+nd).$$

In order to simplify things a bit, we set $a = 1$; this can be done without loss of generality, for we can divide each term by a and obtain a progression with first term 1 and common difference d/a. Then we have the obvious formula

$$\sum_{i=0}^{n}(a+id)^k = a^k \sum_{i=0}^{n}\left(1+i\frac{d}{a}\right)^k.$$

Generating functions

If one associates with each infinite sequence $a_0, a_1, \dots, a_n, \dots$ the formal power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then the sequence $1, 1, \dots, 1, \dots$ is associated with the function

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

The main and crucial observation that makes everything work is that, if we compute the power series expansion for the function $A(x)/(1-x)$, we obtain

$$\frac{A(x)}{1-x} = \sum_{n=0}^{\infty} b_n x^n,$$

where $b_n = a_0 + a_1 + \cdots + a_n$. We will work with the series

$$\frac{1}{1-x^d} = \sum_{n=0}^{\infty} x^{nd}. \tag{1}$$

This is the basic formula from which all others in this article will be derived. We also note that, for each $k \geqslant 1$,

$$\frac{1}{(1-x^d)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^{nd}.$$

The computations

We begin with:

Step 1. Multiply equation (1) by x to obtain

$$\frac{x}{1-x^d} = \sum_{n=0}^{\infty} x^{1+nd}.$$

Step 2. Differentiating this, we have

$$\frac{1+(d-1)x^d}{(1-x^d)^2} = \sum_{n=0}^{\infty}(1+nd)x^{nd}. \tag{2}$$

This differentiation on the left-hand side is easy to do by hand, but as we go on, repeated differentiation becomes very tedious by hand, and this is why we recommend the use of a computer algebra system to perform these operations. Keeping track of the right-hand side will be easy.

Step 3. Now we divide the left-hand side by $1 - x^d$ and multiply the right-hand side by $\sum_{n=0}^{\infty} x^{nd}$, obtaining

$$\frac{1+(d-1)x^d}{(1-x^d)^3} = \sum_{n=0}^{\infty} \alpha_n x^{nd},$$

where $\alpha_n = 1 + (1+d) + \cdots + (1+nd)$.

Step 4. Rewrite the left-hand side as

$$[1+(d-1)x^d]\frac{1}{(1-x^d)^3} = [1+(d-1)x^d] \times \sum_{n=0}^{\infty}\binom{n+2}{2}x^{nd}.$$

Step 5. Equating α_n with the coefficient of x^{nd} in this expression, we have

$$S_1 = \sum_{i=0}^{n}(1+id) = \binom{n+2}{2} + (d-1)\binom{n+1}{2} = \frac{n+1}{2}(2+nd),$$

as was to be expected.

Of course, this is not an elegant way to compute S_1, but this approach does lead to an elegant way to compute the sums of higher powers. Since we are not entering the infinite series into the computer, the user must be aware of what is happening at each step of the algorithm, and thus, throughout the entire derivation, the user is working the problem 'out on paper' while using the computer to perform the difficult or tedious calculations for each step.

To calculate

$$S_2 = \sum_{i=0}^{n}(1+id)^2,$$

we again follow the steps above, only this time we begin with equation (2).

Step 1. Multiply equation (2) by x to obtain

$$\frac{x+(d-1)x^{d+1}}{(1-x^d)^2} = \sum_{n=0}^{\infty}(1+nd)x^{nd+1}.$$

Step 2. Differentiate to obtain

$$\frac{1+(d^2+2d-2)x^d+(d^2-2d+1)x^{2d}}{(1-x^d)^3} = \sum_{n=0}^{\infty}(1+nd)^2x^{nd}. \quad (3)$$

Step 3. We divide the left-hand side by $(1-x^d)$ and multiply the right-hand side by $\sum_{n=0}^{\infty} x^{nd}$ to obtain

$$\frac{1+(d^2+2d-2)x^d+(d^2-2d+1)x^{2d}}{(1-x^d)^4} = \sum_{n=0}^{\infty}\beta_n x^{nd},$$

where

$$\beta_n = \sum_{i=0}^{n}(1+id)^2.$$

Step 4. We rewrite the left-hand side as

$$[1+(d^2+2d-2)x^d+(d^2-2d+1)x^{2d}]\frac{1}{(1-x^d)^4} = [1+(d^2+2d-2)x^d+(d^2-2d+1)x^{2d}]\sum_{n=0}^{\infty}\binom{n+3}{3}x^{nd}.$$

Step 5. Thus, equating β_n with the coefficient of x^{nd}, we see that

$$\beta_n = \binom{n+3}{3} + (d^2+2d-2)\binom{n+2}{3} + (d^2-2d+1)\binom{n+1}{3}$$

and consequently

$$S_2 = (n+1) + (n+1)nd + \tfrac{1}{6}n(n+1)(2n+1)d^2.$$

If the reader has 'stepped' through this process in these two derivations, using a computer algebra program to perform the differentiations, multiplications and combinatorial calculations, then he or she will have begun to have an appreciation of how the process is repetitive, and of the benefits from using the software.

Naturally, these formulae are a bit more complicated than those for calculating the sums $1^k + 2^k + \cdots + n^k$. As a useful check, one can set $d = 1$ and replace n by $n-1$ to obtain these simpler sums.

The reader now can repeat these steps, beginning with equation (3), to obtain the formulae:

$$\sum_{i=0}^{n}(1+id)^3 = \tfrac{1}{4}\big[n^2(n+1)^2d^3 + 2n(n+1)(2n+1)d^2 + 6n(n+1)d + 4(n+1)\big]$$

$$\sum_{i=0}^{n}(1+id)^4 = \tfrac{n+1}{30}\big[n(6n^3+9n^2+n-1)d^4 + 30n^2(n+1)d^3 + 30n(2n+1)d^2 + 60nd + 30\big]$$

$$\sum_{i=0}^{n}(1+id)^5 = \tfrac{n+1}{12}\big[n^2(n+1)(2n^2+2n-1)d^5 + 2n(6n^3+9n^2+n-1)d^4 + 30n^2(n+1)d^3 + 20n(2n+1)d^2 + 30nd + 12\big].$$

Alternating sums

In exactly the same manner as above, only by employing the formula

$$\frac{1}{1+x^d} = \sum_{n=0}^{\infty}(-1)^n x^{nd},$$

one can generate formulae for the sums

$$S_k^* = 1^k - (1+d)^k + (1+2d)^k - \cdots + (-1)^n(1+nd)^k.$$

The computations here are a bit trickier. For example, to compute

$$S_2^* = \sum_{i=0}^{n} (-1)^i (1+id)^2,$$

we multiply equation (3) by $1/(1+x^d)$, but we rewrite this as $(1+x^d)^2/(1+x^d)^3$ so that, after multiplying, the left-hand side becomes

$$\frac{[1+(d^2+2d-2)x^d + (d^2-2d+1)x^{2d}](1+2x^d+x^{2d})}{(1-x^d)^3(1+x^d)^3}.$$

We then note that

$$\frac{1}{(1-x^d)^3(1+x^d)^3} = \frac{1}{(1-x^{2d})^3},$$

so that we can rewrite the left-hand side as

$$[1+(d^2+2d)x^d + (3d^2+2d-2)x^{2d} + (3d^2-2d)x^{3d} + (d-1)^2x^{4d}] \sum_{n=0}^{\infty} \binom{n+2}{2} x^{2nd}.$$

The right-hand side, after multiplying, is

$$\sum_{n=0}^{\infty} (-1)^n \gamma_n x^{nd}$$

where

$$\gamma_n = \sum_{i=0}^{n} (-1)^i (1+id)^2.$$

Equating the coefficients of x^{2nd}, we have

$$\begin{aligned}\gamma_{2n} &= \binom{n+2}{2} + (3d^2+2d-2)\binom{n+1}{2} + (d-1)^2\binom{n}{2}\\ &= 1 + 2nd + n(2n+1)d^2.\end{aligned}$$

And, equating the coefficients of $x^{(2n-1)d}$, we have

$$\begin{aligned}\gamma_{2n-1} &= (d^2+2d)\binom{n+1}{2} + (3d^2-2d)\binom{n}{2}\\ &= 2nd + n(2n-1)d^2.\end{aligned}$$

It would be interesting to determine formulae for these alternating sums for other values of k. Then, as before, if we put $d = 1$ and replace n by $n-1$, we obtain formulae for the alternating sums

$$1^k - 2^k + 3^k - \cdots + (-1)^{n+1}n^k.$$

Perhaps there are other variations on this theme. Ideas can be found in references 5 and 6.

Some readers may have thought of an alternative approach to calculation of these sums from the formula

$$\sum_{i=0}^{n}(1+id)^k = \sum_{i=0}^{n} 1 + \binom{k}{1} d \sum_{i=0}^{n} i + \binom{k}{2} d^2 \sum_{i=0}^{n} i^2 + \cdots + \binom{k}{k} d^k \sum_{i=0}^{n} i^k.$$

However, our aim has been to demonstrate to the reader a way to incorporate today's available software technology into the learning process. We have guided the reader through these procedures to obtain an understanding of the powerful role of generating functions. And we have attempted to give an appreciation of how the software can be used hand-in-hand with paper and pencil to perform tasks hitherto daunting.

Temple Fay wishes to express his appreciation for the hospitality and support received from the Technikon Pretoria and the University of South Africa while this article was being written.

References

1. O. D. Anderson, Summing powers of integers, *Mathematical Spectrum* **23** (1990/91) pp. 116–121.
2. R. Cook and D. Sharpe, Sums of arithmetic progressions, *Mathematical Spectrum* **27** (1994/5) pp. 30–32.
3. A. W. F. Edwards, Sums of powers of integers: a little history, *Mathematical Gazette* **66** (1982) pp. 22–28.
4. T. H. Fay, A note on the sums $1^k + 2^k + \cdots + n^k$, *Mathematics and Computer Education* **28** (1994) pp. 46–47.
5. T. H. Fay and K. R. S. Sastry, A further note on the sums $1^k + 2^k + \cdots + n^k$, *Mathematics and Computer Education* **29** (1995) pp. 253–261.
6. T. H. Fay and B. L. Piazza, A combinatorial approach to the calculation of the sums $1^k + 2^k + \cdots + n^k$, *Mathematics and Computer Education* **29** (1995) pp. 269–278.
7. F. T. Howard, Sums of powers of integers, *Mathematical Spectrum* **26** (1993/4) pp. 103–109.
8. S. Jafari, Summing the series $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ using Pascal's identity, *Mathematical Spectrum* **26** (1993/4) pp. 50–51.
9. K. R. Sastry, Cubes of natural numbers in arithmetic progression, *Crux Mathematicorum* **18** (1992) pp. 161–164. □

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Regular Polygon Rings

R. A. DUNLAP

The construction of a ring from regular convex polygons in two dimensions is considered. General criteria for combinations of polygons which will form rings are derived and specific examples are given.

Introduction

The construction of a ring from a number of identical regular polygons is an interesting problem which makes use of some fundamental properties of two-dimensional geometry. This problem has been considered by a number of authors for many years. The German artist and scientist Albrecht Dürer (1471–1528) showed an interest in polygon rings and published some erroneous results in his monograph *Unterweysung der Messung* (see references 1 and 2) where he demonstrated that regular pentagons could not be used to construct a closed ring. In fact, such a ring can be constructed, as will be shown in the present article. Dürer's failure to obtain a correct construction was the result of his poorly drawn pentagons. This problem may be considered from a straightforward mathematical approach with some interesting results.

Possible types of rings produced from regular polygons may be divided into two categories:

(1) rings in which the central figure is a regular convex polygon with each of its vertices formed by the intersection of three edges, and

(2) rings in which the central figure is not a regular convex polygon and some of its vertices are formed by the intersection of two edges.

These two categories of constructions will be considered separately below.

Type I rings

In order to understand what figures of this type may be constructed, it is necessary to look at some of the basic properties of regular convex polygons. A regular n-gon (polygon with n sides) has an internal angle at each of its vertices which is given in radians as

$$\theta_n = \pi\left(\frac{n-2}{n}\right). \qquad (1)$$

If a number of n-gons form a closed ring around a central m-gon then each of the vertices of the m-gon will have an interior angle θ_m and will be formed by the intersection of three lines. The sum of the angles between the lines at the vertex will be given by

$$2\pi\left(\frac{n-2}{n}\right) + \pi\left(\frac{m-2}{m}\right) = 2\pi. \qquad (2)$$

The expression given in equation (2) has solutions for n of the form

$$n = \frac{4m}{m-2}. \qquad (3)$$

The class of solutions to this expression is constrained by the fact that both n and m must be integers which are greater than or equal to 3. The possible solutions for n and m are given in table 1. The case $n = 5$ and $m = 10$ is illustrated in figure 1.

Table 1. Allowed values of n and m for type I n-gon rings.

n	m
5	10
6	6
8	4
12	3

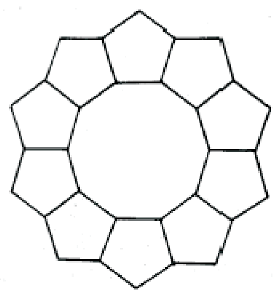

Figure 1. A type I ring of regular pentagons ($n = 5$, $m = 10$).

Type II rings

A typical type II ring is illustrated in figure 2. This figure consists of eight 8-gons (octagons) arranged to form a ring with a central figure which is a non-regular 16-gon (although alternating vertices of the 16-gon may be connected to form a regular 8-gon). In this case the n-gons have an additional vertex between those which intersect with other n-gons. In this case it is not the internal angle of the n-gon which is of most relevance for the construction of the figure but the angle α, as illustrated in figure 3. In this case the condition which must be fulfilled is given by

$$2\alpha + \pi\left(\frac{m-2}{m}\right) = 2\pi \qquad (4)$$

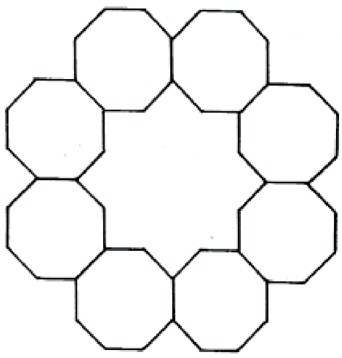

Figure 2. A type II ring with $(m, n) = (8, 8)$ and $k = 1$.

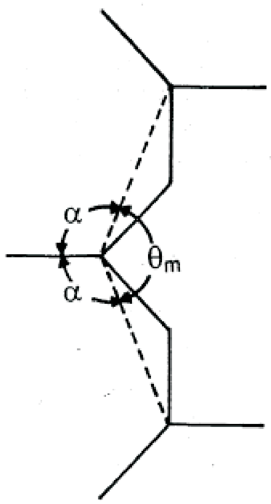

Figure 3. Relevant angles at vertices of type II rings ($n = 8$, $m = 8$ and $k = 1$ shown).

with the additional constraint that

$$\alpha > \tfrac{1}{2}\pi. \tag{5}$$

Here m is the number of n-gons which comprise the ring. Although this is the same as the number of edges of the central figure for the type I rings, this relationship is not in general true. Some basic geometry shows that the angle α is related to the internal angle of the n-gon by

$$\alpha = \pi\left(1 - \frac{3}{n}\right). \tag{6}$$

If we substitute this into equation (4) we obtain

$$n = \frac{6m}{m-2}. \tag{7}$$

It is possible as well to construct a ring with two n-gon vertices between each vertex which intersects with the adjacent n-gon, and this can be readily generalized to consider the case where there are k n-gon vertices between those which intersect with adjacent n-gons. Following the development above it can be shown that n and m are related by the expression

$$n = \frac{2(k+2)m}{m-2}. \tag{8}$$

It is seen, as expected, that equation (7) is a special case of equation (8) with $k = 1$ and that equation (3) is a special case of equation (8) with $k = 0$. There are no constraints on how large n, m or k can be, although they are all constrained to be integers. It is readily seen that there are infinitely many solutions which are consistent with the constraint of equation (8) as follows. For any integer value of $m \geqslant 5$ at least one integer value of n exists ($n = 2m$ for $k = m - 4$). Similarly, for any integer value of $k \geqslant 1$, at least one integer value of n exists ($n = 2m$ for $m = k + 4$). Allowed values of n and m for values of $k \leqslant 6$ are given in table 2. The example in figure 3 is for $k = 1$, $m = n = 8$.

Table 2. Allowed values of n and m for values of $k \leqslant 6$.

$k \rightarrow$	0	1	2	3	4	5	6
m	n	n	n	n	n	n	n
3	12	18	24	30	36	42	48
4	8	12	16	20	24	28	32
5		10			20		
6	6	9	12	15	18	21	24
7				14			
8		8			16		
9						18	
10	5		10		15		20
12				12			
14		7			14		
16						16	
18			9				18
22				11			
26					13		
30						15	
34							17

On the basis of these results given in the table certain empirical relationships between the allowed minimum and maximum values of the parameters n and m for a given k may be expressed as

$$m_{\min} = 3, \tag{9}$$

$$m_{\max} = 4k + 10 = 2n_{\min}, \tag{10}$$

$$n_{\min} = 2k + 5, \tag{11}$$

$$n_{\max} = 6k + 12. \tag{12}$$

The validity of these expressions for all values of k may be shown as follows.

As n, m and k must all be positive integers, the form of the denominator of the right-hand side of equation (8) clearly indicates that $m \geqslant 3$. Equation (8) shows that for $m_{\min} = 3$, n has a value $n = 6k + 12$. It can be seen that the minimum value of m yields the maximum value of n by rewriting equation (8) in the form

$$n = 2(k+2)\left(1 + \frac{2}{m-2}\right). \tag{13}$$

The values of $n_{\min}$ and $m_{\max}$ can be determined by rewriting equation (8) for m as a function of n as

$$m = \frac{2n}{n-2k-4}. \tag{14}$$

Requiring that the denominator in the above expression be positive gives $n_{\min} = 2k+5$ and the corresponding value of $m = 4k+10$. This expression may be rewritten in the form

$$m = 2\left(1 + \frac{2k+4}{n-2k-4}\right), \tag{15}$$

showing that the minimum value of n gives the maximum value of m.

Conclusions

In conclusion, it has been shown that there is an infinite number of rings which can be constructed from regular n-gons but only four of these have a central figure which is also a regular polygon.

References

1. For a translation of Dürer's original manuscript see W. L. Strauss, *The Painter's Manual, 1525* (Abaris Books, New York, 1977).
2. D. W. Crowe, Albrect Dürer and the regular pentagon. In *Five Fold Symmetry* ed. I. Hargittai pp. 465-487 (World Scientific, Singapore, 1992). □

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The Sajdak Conjecture

SIMON R. BLACKBURN

In a recent issue of *Mathematical Spectrum* (reference 1), Filip Sajdak made the following conjecture.

Conjecture 1. Let a and b be co-prime natural numbers between 1 and 9 with a even and b odd. Then every natural number n which is not a multiple of 5 has a multiple m containing only the digits a and b.

This generalises the Roseberry conjecture, which was solved in 1990 (see the references in reference 1). Our aim is to prove Sajdak's conjecture — in fact, we will prove something very slightly more general. We begin with a useful special case.

Lemma 1. Let a and b be natural numbers between 1 and 9 with a even and b odd. Then every number of the form 2^s, where $s \geqslant 1$, has an s-digit multiple t containing only the digits a and b.

The proof of the lemma will actually tell us how to find the number t. We will build t from the bottom up, by picking its least significant digits first. We always start by picking a as the units digit of t. After we have picked k digits of t, we ask: does 2^{k+1} divide the k-digit number we have obtained so far? If it does, we pick a as the next digit. If it does not, we pick b as the next digit. We carry on in this way until we have picked all s digits of t. For example, suppose that $a = 6$ and $b = 9$ and we want to find a multiple t of 16 containing only these digits. We start with 6. Now 6 is not divisible by 4, so we choose 9 as our next digit to produce the number 96. We find that 96 is divisible by 8, so we choose 6 as our next digit to produce 696. Finally, 696 is not divisible by 16, so we choose 9 as our final digit to produce 9696 as our choice for t. Just as we wanted, 9696 is indeed a multiple of 16.

Proof of Lemma 1. We will prove the lemma by induction on s. The lemma is certainly true when $s = 1$ — choosing $t = a$ will do, since a is even.

We now establish the inductive step. Suppose, as an inductive hypothesis, that the lemma is true for $s = k$, where $k \geqslant 1$. Let d be a k-digit number divisible by 2^k which contains only the digits a and b. We know that such an integer exists by our inductive hypothesis. We consider two cases in turn.

Case 1. Suppose that 2^{k+1} divides d. In this case, we set $t = d + a10^k$. Now, t is a $(k+1)$-digit number containing only the digits a and b. Furthermore, t is a multiple of 2^{k+1} as 2^{k+1} divides both d (by assumption) and $a10^k$ (since a is even). So in this case, the inductive hypothesis holds for $s = k+1$.

Case 2. Suppose 2^{k+1} does not divide d, so $d/2^k$ is odd. Since b is odd, $(b10^k)/2^k$ is also odd. We choose $t = d + b10^k$. Now $t/2^k = (d/2^k) + ((b10^k)/2^k)$ is even, because it is the sum of two odd numbers. So $t/2^k$ is divisible by 2, i.e. t is a multiple of 2^{k+1}. Since t is a $(k+1)$-digit number and only contains the digits a and b, the inductive hypothesis for $s = k + 1$ holds in this case also.

We have now proved the hypothesis for $s = k + 1$, assuming the hypothesis for $s = k$. Since the hypothesis holds for $s = 1$, the lemma holds for all $s \geqslant 1$, by induction on s.

We now use the lemma to prove Sajdak's conjecture.

Theorem 1. Let a and b be natural numbers between 0 and 9, with a even and b odd. Then every natural number n which is not a multiple of 5 has a multiple m containing only the digits a and b.

Proof. If $a = 0$, then the theorem follows by Theorem 2 of Filip Sajdak's article, so to prove our theorem it is enough to consider the case when a (and b) are non-zero.

Let n be a natural number which is not divisible by 5. Then we can write n in the form $n = r2^s$ where $s \geqslant 0$ and r is odd. We can find a natural number t which only contains the digits a and b and which is a multiple of 2^s — Lemma 1 shows this when $s \geqslant 1$, and when $s = 0$ taking $t = a$ will do, as a is certainly a multiple of $2^0 = 1$. We are going to construct m by repeating the digits of t several times. Suppose that t has l digits, and consider the number x_c produced by repeating the digits of t a total of c times:

$$x_c = t + 10^l t + 10^{2l} t + \cdots + 10^{(c-1)l} t.$$

We would like to find a value of c such that r divides x_c. We can do this as follows. Consider the remainders $w_0, w_1, \ldots$ when we divide the successive powers of 10^l by r. So

$$\left(10^l\right)^i \equiv w_i \bmod r,$$

where $0 \leqslant w_i < r$ for $i = 0, 1, 2, \ldots$. Eventually we must obtain two remainders which are the same, so there exist integers j and p such that $w_j = w_{j+p}$ where $p > 0$. Then $(10^l)^j \equiv (10^l)^{j+p} \bmod r$. Since 10 and r are coprime, we may divide both sides of this equality by $(10^l)^j$ to obtain $(10^l)^p \equiv 1 \bmod r$. Thus $w_p = w_0$, so $w_{p+i} = w_i$ for $i = 0, 1, 2, \ldots$. We have proved that the sequence $w_0, w_1, \ldots$ repeats after p steps. Now if we take $c = rp$, we find that r divides x_c, because working modulo r:

$$\begin{aligned} x_{rp} &\equiv t\left(1 + 10^l + \cdots + \left(10^l\right)^{rp-1}\right) \\ &\equiv t(w_0 + w_1 + \cdots + w_{rp-1}) \\ &\equiv t(w_0 + \cdots + w_{p-1})r \\ &\equiv 0. \end{aligned}$$

But 2^s also divides x_{rp}, since 2^s divides t. Hence x_{rp} is, in fact, a multiple of $r2^s = n$. Since x_{rp} only contains the digits a and b, setting $m = x_{rp}$ gives us the multiple of n that we require.

Just like the lemma above, the theorem gives us a recipe for finding a multiple m. For example, suppose that we want to find a multiple of $n = 112$ $(= 16 \times 7)$ containing only the digits 6 and 9. Since 16 is the highest power of 2 dividing n, we have already calculated that we can use the value $t = 9696$. Calculating the sequence of remainders w_i of the powers of 10^4 modulo 7, we find that the sequence repeats after three steps (the sequence is $1, 4, 2, 1, 4, 2, \ldots$). So the theorem tells us that the 84-digit number consisting of the digits 9696 repeated 3×7 times is a multiple of n. This is far from the smallest number that works: 969696 will do! So, as ever in mathematics, we are left with more questions. When does the construction of our theorem give the smallest possible multiple m of the form we require? Are there better ways of constructing m so as to end up with a smaller number of digits? Finally, what happens when all numbers are expressed in a base other than 10?

Reference

1. F. Sajdak, The Roseberry conjecture, *Mathematical Spectrum* **28** (1995/6) p. 33. □

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The following code is used in a battle. To encode a message, replace letters by numbers (blank by 0, a by 1, b by 2 etc), then arrange the numbers in a two-rowed matrix, then premultiply the matrix by

$$\begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}.$$

The message

$$\begin{pmatrix} 12 & 5 & -18 & 13 & -5 & 4 & 3 & 20 & -1 & -20 \\ -23 & -6 & 58 & -25 & 24 & -5 & -1 & -40 & 3 & 60 \end{pmatrix}$$

is received. Decode it.

Kamlesh Gaya
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Mathematics in the Classroom

Computers in the classroom

There are many problems concerning the use of information technology in mathematics: lack of facilities, the cost of network licences, the fear of what to do if the system crashes during a lesson, the fear of not being able to help a pupil with the software they are using, the cost of maintenance, of the modernisation of equipment, ... Even in a school like ours, where the equipment is fast and reliable, with good access for classes and with a technician on hand, there is the added problem of what software to buy. Then the expense of outlay has to be justified by a belief that learning is being enhanced by the use of the software. We, and our colleagues, firmly believe that computers can enhance learning.

The computer should be viewed as another way to assist learning. It should be added to the list of good practice: investigations, group learning, chalk and talk, learning through experiment and research.

That leaves the choice of software and its compatibility to a schools system and range of pupils. One way to overcome this problem is to create your own software. Figure 1 shows the introduction screen to a program on directed numbers.

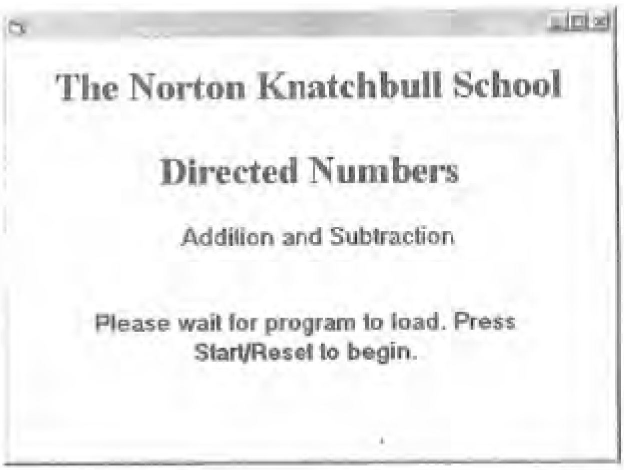

Figure 1

This may seem an extreme solution, but we have found that programs specifically designed to do a task are very useful. They can also be easily adapted to suit the different levels of the pupils in the school. A maximum of thirty minutes a week, for an individual pupil, is a sufficient amount of time to improve learning in this way.

Our two tools for writing programs are Visual Basic and the spreadsheet Excel 5. Most school systems come with a powerful spreadsheet package like Excel, and a PC copy of Visual Basic has proved to be a valuable purchase. These two pieces of software have their advantages and disadvantages over each other, which will not be discussed in depth here. However, the big advantage of Visual Basic is that you can attach an executable file, so that anyone can use the created software on an IBM compatible system using Windows. A disadvantage of both Excel5 and Visual Basic is learning how to use them. The manuals and textbooks seem to tell you only how to analyse the profit and loss of your business in Excel 5 or how to create a database in Visual Basic!

Anyway, one of the authors of this article was surprised how quickly he was able, with the assistance of the other author, to teach himself something new, for the first time since 1976, and that with a background of zero programming knowledge and an inability to use or understand DOS.

As teachers we are constantly being reminded how far mathematically our pupils are behind the rest of the world. There is the call to be traditional. If classes chant multiplication tables the pupils will learn them. Yes, but will they learn anything about numbers?

A program can put some fun into the learning of tables and is also written to improve concentration and reflexes, by including a choice of the time allowed (see figure 2) for a question, before the next randomly generated question comes on screen. Such activity provides good 'exercise' for the brain.

Figure 2

Pupils see it as a challenge to get all the questions correct. But most important is the enjoyment that youngsters get from its use.

Addition, subtraction and division of whole numbers are a variation of the same program. Directed numbers, with randomly changing signs as well as digits, provides a further challenge. Estimations, percentages,... are all written to practise mathematics in a fun way, not to teach mathematics as some of the CDs on the market attempt to do. The programs are written to support teachers, not to replace them. Help is available on screen if the pupil requires it. Figure 3 shows the method used by the authors to find a side using trigonometry, making use of the formulae provided by the examination board used by the department.

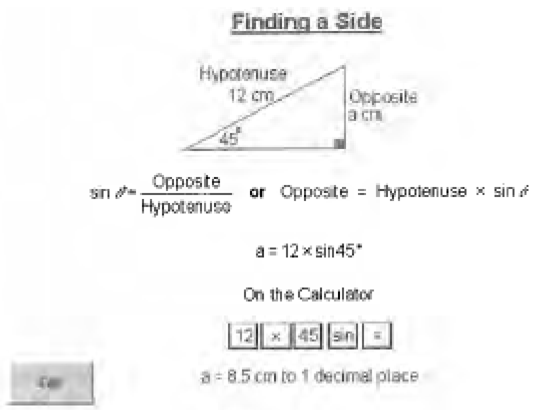

Figure 3

Other programs provide practice at Pythagoras' theorem and mensuration, where a calculator can be provided on screen, as shown in Figure 4.

Figure 4

Visual Basic has proved more successful when calculations are involved, whereas the spreadsheet is better for the study of functions (linear, quadratic, real generic and complex transformations), the simulation of particle movement met in advanced level mechanics, and for investigations. More details can be found in the articles listed in the references.

Smile is a successful and fun program and there are good graphing and transformation packages. However we have discovered little which allows pupils to practise the curriculum without purchasing an all-encompassing CD. Writing your own software using a Help facility to outline the methods used by your own department can fill the gap between programs like Smile and KS3/4 mathematics on CD.

The authors would be happy to send a sample disc, with setup, to any readers who would like to evaluate the software.

References

D. Benjamin, A spreadsheet estimate of π, *Spreadsheet User* **1** number 1 Sheffield Hallam University, May 1994.

D. Benjamin, Infinite series, *Spreadsheet User* **2** number 2 Sheffield Hallam University, November 1995.

D. Benjamin, Shrinking circles, *Mathematical Spectrum* **28** (1995/6) p. 42.

D. Benjamin and J. Dodd, Creating mathematical programs with a spreadsheet, *Spreadsheet User* **4** number 1 Sheffield Hallam University, May 1997. □

David Benjamin and Justin Dodd

Computer Column

Recursion

In an ancient temple in Benares, so the story goes, there are 3 diamond needles. At the beginning of time, God placed 64 discs of gold on one of the needles. The discs were all of different sizes: the largest disc was at the bottom; the second largest was next, and so on. The smallest disc was on top. The Benares monks must transfer the entire tower from one needle to another. There are only three rules. (i) A monk must not move more than one disc at a time. (ii) A monk may not rest a disc anywhere apart from on one of the three needles. (iii) A disc may never be placed on top of a smaller disc. When their task is complete, the world will end.

The Towers of Hanoi is a game based on this story. You may have seen it: you have a small collection of 'doughnuts' and 3 posts onto which you can place them. The idea is to move the tower from one post to another. (The retail version is made of plastic, rather than diamond and gold, but the rules of the game are identical to the monks' task.) Figure 1 shows the start of the game when there are 4 discs.

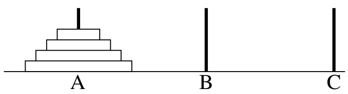

Figure 1

Try playing the game. What is the minimum number of moves, m, for a given number of discs, n? Clearly, if $n = 1$, $m = 1$; and if $n = 2$, $m = 3$. With more work you can convince yourself that if $n = 3$, $m = 7$ and if $n = 4$, $m = 15$. From this you might guess that $m = 2^n - 1$.

Now write a program to solve the game for, say, 10 discs.

Unless you approach the problem inductively, your program is likely to be rather long and involved. The key to an elegant solution is to use *recursion*.

Sooner or later you must move the bottom disc from A to C; at this point, the other 9 discs are stacked in decreasing size order on B. Once the tenth disc is on C it need never be moved again. So the problem of moving 10 discs from A to C reduces to the problem of moving 9 discs from A to B, followed by 1 disc from A to C, followed by 9 discs from B to C. But we can apply this logic to the pile of 9 discs: move a stack of 8 discs from B to A, move the ninth disc to C, then move the stack of 8 discs from A to C. The same logic applies to a pile of 8 discs, and 7 discs, and so on.

In the general case you need to define a function (`Solve`, say) which has four arguments: `N`, the number of discs, and `A`, `B`, `C`, the three pegs. The heart of the function will look something like:

```
Solve(N, A, B, C)
     if N is 0 exit
     Solve(N-1, A, C, B)
     Move from A to C
     Solve(N-1, B, A, C)
```

where `Move` means move the top disc. The function is recursive: it calls itself over and over again. A terminating condition ensures that we do not get into an infinite loop. As you can see, recursion can sometimes save you hundreds of lines of code! **Stephen Webb**

Letters to the Editor

Dear Editor,

Olympic statistics
(Volume 29 Number 3 pages 63–64)

I read with interest the article by Carol Nixon (reference 1) in which her class attempted to predict the winning time of the Men's 100 m at the 1996 Olympics, using linear regression on the times of past winners. It is indeed surprising that Donovan Bailey should have run the fastest 'legal' 100 m of all time, 9.84 s, and yet have failed to achieve, by some considerable margin, the times predicted by the regression models. Since there will presumably be near-unanimity that the fault lies with the models and not with the athlete, the author is right to end her article with a request for an improved model.

The first point to note is that there appears to be a misprint in the source data. (Given that the source was the *Guardian* newspaper, this is perhaps less surprising!) Reference 2 gives the winning time at the 1896 Olympics as 12.0 s, not 12.2 as reported. Repeating the analysis with the corrected figure gives an improved predicted time of 9.68 s (vs 9.64 s) with an r^2 of 0.74 for the linear model, and 9.72 s (vs 9.69 s) for the exponential and log–linear models, with r^2 values of 0.75 and 0.76 respectively.

Secondly, the given data set excludes, apparently quite arbitrarily, the winning times in the Olympics of 1900–1920, 1928, 1932, 1952, 1956, 1964 and 1988 (see references 2 and 3 for details). When these are included, we have 22 data points instead of 10, and we see improved predictions from all three models. The linear model now predicts 9.80 s, with an r^2 of 0.72, and the other two models are very close indeed to the actual performance, with predictions of 9.82 s and r^2 values of 0.74 and 0.75.

However, this is not the end of the story. Although it is far from obvious from casual observation of the graphs, there is a step change in the data. The change is linked to the switch from recording the times to one decimal place (hand timing) to two decimal places (electronic timing). This change occurred in 1960 in the data set as given, and actually from 1952. Hand timing is less accurate than electronic timing, and is considered by the International Amateur Athletics Federation as underestimating the 'true' time by 0.24 s. (See reference 3.) Unfortunately, the effect of reflecting this in the data set is to worsen the models' predictions, although the r^2 values improve. The linear model now predicts 9.74 s, with an r^2 of 0.83, and the other two models both predict 9.77 s, with an r^2 of 0.84.

Finally, despite the above analyses eventually yielding reasonable, albeit still over-optimistic, predictions of the winning time, I have reservations about the value of using regression models in this fashion. The data are not observations over time of a static population, by contrast with, say, the number of students at university in the UK or the annual number of deaths in Greater London from road traffic accidents. The winning time at an Olympics is achieved by the best athlete 'available' from amongst the population of athletes participating. This population differs at each Olympics and might, as the author correctly points out, be affected by extraneous events such as political protests as in 1980, or the aftermath of war as in 1948.

A similar predictive exercise was carried out in reference 4. The author of this book, writing in 1987, attempted to predict the world record in the year 2000 for each athletics event on the Olympic programme. He did this by examining the historic progression of world records from the start of the century. Admittedly, his predictions are not generated using regression techniques, but simply by fitting a curve of best fit to the data subjectively. For some events, this curve is linear; in other events the rate of improvement of the world record has slowed and the curve of best fit reflects this. Nevertheless, the same objection arises: world records occur when particularly talented individuals emerge from the pool of competing athletes; when this might happen, for any given athletics event, is no more predictable than the date of the next San Francisco earthquake. As a consequence, some of the predictions in reference 4 for the year 2000 have already been exceeded; others will — almost certainly — not be acheived until well into the next century.

References

1. C. Nixon, Mathematics in the Classroom, *Mathematical Spectrum* **29** (1996/7) pp.63–64.
2. K. Mays, *Athletics* (Telegraph Publications, London, 1984).
3. P. Matthews (ed.), *Athletics — the International Track and Field Annual, 1997* (SportsBooks, Surbiton, 1997).
4. L. Blackman, *Athletics World Records in the 20th Century* (Book Guild, Lewes, 1988).

Yours sincerely,

MIKE WENBLE
(11 Rue des Jardins,
67590 Schweighouse-sur-Moder
France)

Table 1. Model summary. The actual result: 9.84 s.

	Linear model		Exponential model		Log–linear model	
	Prediction (s)	r^2	Prediction (s)	r^2	Prediction (s)	r^2
Original model	9.64	0.72	9.69	0.74	9.69	0.74
1896 corrected	9.68	0.74	9.72	0.75	9.72	0.76
All years included	9.80	0.72	9.82	0.73	9.82	0.74
Adjustment for hand timing	9.74	0.83	9.77	0.84	9.77	0.84

Dear Editor,

The volume of a carton

Many products are packaged in cartons made from a circular tube by squashing one end to a straight edge (figure 1). I was unable to find a published formula for the volume of such a carton, so I decided to work it out for myself.

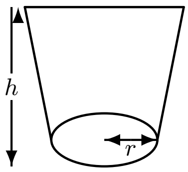

Figure 1

Let r be the radius of the base and denote the height by h. Then the perimeter of any cross section has length $2\pi r$ and the top has length πr. Consider the cross section at a height y above the base. My initial assumption was that this would be an ellipse, but the resulting calculation proved intractable. My second assumption was that the cross section is a pair of parallel lines with semi-circular ends (figure 2).

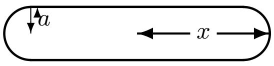

Figure 2

With x, a as shown, the perimeter of this cross section is

$$2(2x-2a)+2\pi a,$$

and this is $2\pi r$, which gives

$$a=\frac{\pi r-2x}{\pi-2}. \tag{1}$$

Note that, as $x\to r$, $a\to r$ which fits the fact that the base is circular; also, as $x\to\frac{1}{2}\pi r$, $a\to 0$ which is consistent with the fact that the top is a straight edge. The area of this cross section is

$$4a(x-a)+\pi a^2. \tag{2}$$

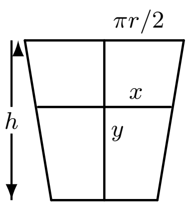

Figure 3

We make a further assumption that the two sides from the points at the end of the top are straight lines (see figure 3). If we denote by y the height of the cross section above the base, then similar triangles give

$$\frac{x-r}{y}=\frac{\frac{1}{2}\pi r-r}{h},$$

which gives

$$x=\frac{\pi-2}{2h}ry+r. \tag{3}$$

If we now substitute for x in (1), we obtain

$$a=r\left(1-\frac{y}{h}\right). \tag{4}$$

We now substitute (3) and (4) in (2) to give the cross sectional area

$$\frac{\pi r^2}{h^2}(h^2-y^2).$$

Hence the volume of the carton is

$$\begin{aligned}\int_0^h \frac{\pi r^2}{h^2}(h^2-y^2)\,dy &= \frac{\pi r^2}{h^2}\left[h^2y-\tfrac{1}{3}y^3\right]_0^h\\ &= \tfrac{2}{3}\pi r^2 h.\end{aligned}$$

It would be interesting to know how nearly this formula agrees with reality.

Yours sincerely,

STELLA DUDZIC
(2 Church Street,
Skipton,
North Yorkshire BD23 2AR)

Dear Editor,

Optimal angle of projection from a height

You might be interested to know that the optimal angle problem discussed by Ali Vahdati (*Mathematical Spectrum* Volume 29 Number 3) was also the subject of an article by Paul Woodruff, Peter Sammut and myself in the *Mathematical Gazette* of November 1996 (and originally in *Simplex*, the in-house magazine of the mathematics department at St Paul's School).

Our approach is somewhat different from Mr Vahdati's, however, in that we eschew the 'classical' calculus approach in favour of three other methods, each of which is elegant in its own way. One of these uses diagonalisation of matrices, the second a vector triangle and some simple geometry and the third some geometrical properties of the parabola, including the fact that tangents at either end of a focal chord meet orthogonally on the directrix. Also we mention the beautiful result that the 'best' direction of projection at shoulder level must be perpendicular to the direction at which the shot hits the ground. This is also true in the inclined plane case.

Yours sincerely,

GERRY LEVERSHA
(St Paul's School,
Lonsdale Road,
London SW13 9JT)

Dear Editor,

Letter from Kamlesh Gaya
(Volume 29 Number 2 page 41)

Gaya raised the problem of finding all natural numbers less than 10^{10} all of whose digits are prime and the sum of whose digits is also prime. Let $\mathbb{N}_P$ be the set of all such numbers.

We note that, if $N \in \mathbb{N}_P$, then all the numbers obtained from the permutations of the digits of N are also in $\mathbb{N}_P$. Hence, in order to find the n-digit numbers in $\mathbb{N}_P$, it is sufficient to consider only the set of n-digit numbers such that any other n-digit number in $\mathbb{N}_P$ can be obtained from one number of this set by the permutation of the digits of this number. We shall call this set a *generating set* of the n-digit numbers in $\mathbb{N}_P$. For example, for the 2-digit numbers in $\mathbb{N}_P$, $\{23, 25\}$ is a generating set, while for the 3-digit numbers in $\mathbb{N}_P$ a generating set is

$$\{223, 227, 335, 337, 553, 557, 773, 775\}.$$

In determining a generating set of the n-digit numbers in $\mathbb{N}_P$, the following observations are useful. Let $N \in \mathbb{N}_P$. Then:

(1) when $n \geqslant 2$, all the digits of N cannot be equal,

(2) $n - 2m$ (where $m \in \mathbb{N}$ with $2m < n$) of the digits cannot be 2.

Let N have n digits $(n > 1)$ and suppose that there are n_1 2's, n_2 3's, n_3 5's and n_4 7's. In order to find a generating set of the n-digit numbers in $\mathbb{N}_P$, we have to find the integers n_1, n_2, n_3 and n_4 such that

$$n_1 + n_2 + n_3 + n_4 = n; \quad 0 \leqslant n_1, n_2, n_3, n_4 \leqslant n - 1, \quad (1)$$
$$2n_1 + 3n_2 + 5n_3 + 7n_4 \text{ is prime,} \quad (2)$$

and corresponding to each such set of numbers there are

$$\frac{n!}{n_1!n_2!n_3!n_4!}$$

positive integers obtained by the permutations of its n digits. Hence the total number of n-digit numbers in $\mathbb{N}_P$ is given by

$$\sum \frac{n!}{n_1!n_2!n_3!n_4!},$$

where the sum is over n_1, n_2, n_3 and n_4 satisfying the conditions (1) and (2).

In the table, we give the number of generators as well as the number of positive integers in the various intervals. From the table, we see that the total number of integers in $\mathbb{N}_P$ and less than 10^{10} is 227 773.

	Number of generators	Number of integers
$1 < N < 10$	4	4
$10 < N < 10^2$	2	4
$10^2 < N < 10^3$	8	24
$10^3 < N < 10^4$	9	80
$10^4 < N < 10^5$	18	290
$10^5 < N < 10^6$	17	984
$10^6 < N < 10^7$	34	3 927
$10^7 < N < 10^8$	35	15 880
$10^8 < N < 10^9$	59	66 870
$10^9 < N < 10^{10}$	52	139 710

Yours sincerely,
A. A. K. Majumdar
(Department of Mathematics,
Jahangirnagar University,
Savar, Dhaka 1342,
Bangladesh)

Dear Editor,

Correction

In a recent issue of *Mathematical Spectrum* (Volume 29 Number 2 page 41) I defined the sequence

$$T_n = p_1 p_2 \ldots p_n + 1,$$

where p_n is the nth prime of the sequence of primes (p_n). I wrote there that $T_7 = 510\,511$ is the first composite number of the sequence (T_n). But actually $T_6 = 30\,031 = 59 \times 509$ is the first composite number of the sequence (T_n). We also note that

$$T_7 = 19 \times 97 \times 277, \qquad T_8 = 9899\,691 = 374 \times 27\,953.$$

Yours sincerely,
A. A. K. Majumdar

Dear Editor,

A Diophantine equation involving factorial numbers

This letter concerns the equation

$$x_1! + \cdots + x_n! = y!$$

where $n \geqslant 2$ is a fixed integer, for which a solution is sought in positive integers $x_1, \ldots x_n, y$, with $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$.

We note first that there can only be finitely many solutions because, for a solution, $y > x_n$ so $y \geqslant x_n + 1$. Hence

$$(x_n + 1)! \leqslant y! = x_1! + \cdots + x_n! \leqslant nx_n!$$

so $x_n + 1 \leqslant n$. Hence $x_1 \leqslant \cdots \leqslant x_n \leqslant n - 1$ and $y! \leqslant nx_n! \leqslant n!$, so $y \leqslant n$. In fact, if $y = n$ then, because $x_1 \leqslant \cdots \leqslant x_n \leqslant n - 1$ and $n \times (n-1)! = n!$, there is only one solution, namely $x_1 = \ldots x_n = n - 1$.

Now let h be the largest integer such that $h! < n$. Then any solution has $y > h$ because

$$y! = x_1! + \cdots + x_n! \geqslant n > h!$$

Also, there is no solution with $y = n - k$, where $k = 1, \ldots, \frac{1}{2}h! - 1$. For suppose there were. Then $x_1 \leqslant \cdots \leqslant x_n < y = n - k$, so $x_1 \leqslant \cdots \leqslant x_n \leqslant n - k - 1$. If at most $n - k - 1$ of the x_i are $n - k - 1$, then

$$\begin{aligned} x_1! + \cdots + x_n! &\leqslant (k+1)(n-k-2)! \\ &\qquad + (n-k-1)(n-k-1)! \\ &= (k+1)(n-k-2)! \\ &\qquad + (n-k)! - (n-k-1)! \\ &= (n-k)! \\ &\qquad + (n-k-2)!(k+1-n+k+1) \\ &< (n-k)! = y! \end{aligned}$$

because $2(k+1) - n \leqslant n! - n < 0$, which is a contradiction. On the other hand, if at least $n - k$ of the x_i are $n - k - 1$, then

$$\begin{aligned} x_1! + \cdots + x_n! &\geqslant 1! + \cdots + 1! + (n-k) \times (n-k-1)! \\ &= k + (n-k)! \\ &= (n-k)! = y! \end{aligned}$$

again a contradiction.

Consider the case $n = 4$, so that $y \leqslant 4$. When $y = 4$, there is the unique solution

$$(x_1, x_2, x_3, x_4, y) = (3, 3, 3, 3, 4).$$

When $y < 4$, choose h to be the largest integer with $h! < 4$, i.e. $h = 2$. Then $y > h$, so that $y = 3$ is the only possibility. The only solution is

$$(x_1, x_2, x_3, x_4, y) = (1, 1, 2, 2, 3).$$

Hence, there are two solutions when $n = 4$.

It is possible to have more than one solution for a given y. For example, with $n = 9$,

$$6 \times 1! + 3 \times 3! = 2 \times 1! + 5 \times 2! + 2 \times 3! = 4!$$

Yours sincerely,

KENICHIRO KASHIHARA
(Kamitsuruma 4-13-15,
Sagamihara, Kanagawa 228,
Japan)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

30.1 What is the probability that the six numbers in the UK national lottery on a given payout day do not include two consecutive numbers? (The winning numbers are an unordered random choice of six distinct numbers from 1 to 49.)

(Submitted by David Screen, Richard Huish College, Taunton)

30.2 If

$$\begin{aligned} x_1 + x_2 + \cdots + x_{10} &= 1, \\ x_1^2 + x_2^2 + \cdots + x_{10}^2 &= 2, \\ &\vdots \\ x_1^{10} + x_2^{10} + \cdots + x_{10}^{10} &= 10, \end{aligned}$$

what is $x_1^{11} + x_2^{11} + \cdots + x_{10}^{11}$?

(Submitted by David Yates, Preston)

30.3 Find a natural number N such that $\sqrt{N}$ is of the form $M.1997\ldots$.

(Submitted by Toby Gee, The John of Gaunt School, Trowbridge)

30.4 Let L_n denote the nth Lucas number, defined by $L_{n+2} = L_{n+1} + L_n$ with $L_0 = 2$, $L_1 = 1$, and let F_n denote the nth Fibonacci number, defined by $F_{n+2} = F_{n+1} + F_n$ with $F_1 = F_2 = 1$. Prove that, for all n, L_n is not divisible by any Fibonacci number F_a with $a \geqslant 5$.

(Submitted by Mansur Boase, St Paul's School, London)

Solutions to Problems in Volume 29 Number 2

29.5 Prove that a triangle can be triangulated into n similar triangles for every $n \geqslant 6$.

Solution by Toby Gee

We show that a triangle can be decomposed into n triangles, all similar to the original triangle. Firstly note that, if such a decomposition is possible for n, then it is possible for $n+3$; simply join the midpoints of the sides of one of the smaller triangles as in figure 1. Figures 2, 3, 4 show decompositions for $n = 6, 7, 8$, completing the proof.

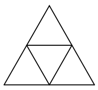

Figure 1

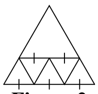

Figure 2

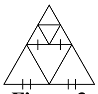

Figure 3

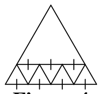

Figure 4

Also solved by Andrew Lobb (St Olave's Grammar School, Orpington).

29.6 Find all prime numbers p for which $2p-1$ and $2p+1$ are both prime.

Solution by Andrew Lobb

If $n \equiv 0 \pmod 3$ and n is prime then $n = 3$. Therefore, if $p \equiv 0 \pmod 3$ then $p = 3$, which is a solution. If $p \equiv 1 \pmod 3$ then $2p+1 \equiv 0 \pmod 3$ so $2p+1 = 3$ so $p = 1$, which is not prime. If $p \equiv 2 \pmod 3$, then $2p-1 \equiv 0 \pmod 3$ so $2p-1 = 3$ so $p = 2$, which is a solution. Thus $p = 2, 3$ are the solutions.

Also solved by Can Minh (University of California, Berkeley), Zhao Yueyang (Millfield School, Somerset), Mike Day (Det Nødvendige Seminarium, Ulfborg, Denmark), Toby Gee, Noah Rosenberg (Rice University, Houston).

29.7 A goat is tethered by a rope to a point on the circumference of a circular field of radius r. The goat can reach exactly half the area of the field. How long is the rope?

Solution by Andrew Lobb

Denote the centre of the field by O and the point to which the rope is attached by P. Denote by R the length of the rope and denote by A and B the points at which the circle centre P radius R cuts the field's circumference. (See figure 5.) Denote by θ the angle APB. Then $\angle\text{AOB} = 2\pi - 2\theta$, so $\angle\text{AOP} = \angle\text{BOP} = \pi - \theta$. Now

$$\cos\theta = -\cos(\pi - \theta) = -\left(\frac{r^2 + r^2 - R^2}{2r^2}\right) = \frac{R^2 - 2r^2}{2r^2},$$

so $2r^2(\cos\theta + 1) = R^2$. The intersection of the field and the circle centre P radius R has area

$$\tfrac{1}{2}\pi r^2 = \tfrac{1}{2}R^2\theta + \tfrac{1}{2}r^2(2\pi - 2\theta) - 2 \times \tfrac{1}{2}r^2 \sin(\pi - \theta).$$

Hence

$$\tfrac{1}{2}\pi = \theta(\cos\theta + 1) + \pi - \theta - \sin\theta,$$

so

$$\theta\cos\theta - \sin\theta + \tfrac{1}{2}\pi = 0.$$

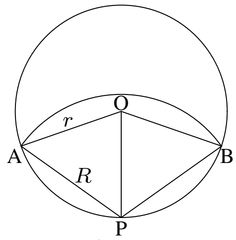

Figure 5

By some iterative process, this gives $\theta \approx 1.906$, so $R \approx 1.16r$.

29.8 Using compasses only, find the centre of a given circle.

The solution was supplied by Eddie Kent, who proposed the problem

Let P be a point on the circumference of the given circle, and draw a circle radius r (say) to cut the given circle at Q and R (figure 6). Draw circles with radius r and centres Q and R to intersect at S (and P). Draw a circle with radius PS (figure 7). Choose a point P′ on the circumference of this circle and draw the circle centre P′ radius r, to cut the circle at Q′ and R′. Draw circles with radius r, centres Q′ and R′, to intersect at S′ (and P′). Then P′S′ is the radius of the given circle and the centre can be located as the intersection of two arcs whose centres are two points of the given circle and of radius P′S′.

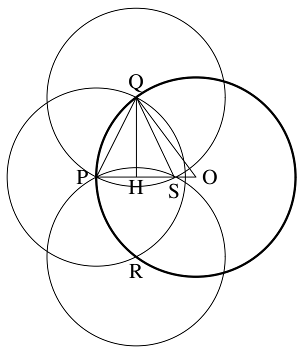

Figure 6

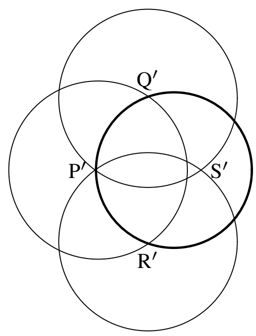

Figure 7

To justify this, denote the centre by O and write $\text{PO} = a$. Denote by H the foot of the perpendicular from Q to PS. Then PH = HS and, by Pythagoras' Theorem,

$$\begin{aligned}\text{QH}^2 = \text{PQ}^2 - \text{PH}^2 &= \text{OQ}^2 - \text{OH}^2 \\ &= \text{OQ}^2 - (\text{OP} - \text{PH})^2 \\ &= \text{OQ}^2 - \text{OP}^2 + 2\text{OP}\cdot\text{PH} - \text{PH}^2.\end{aligned}$$

Hence

$$r^2 = \text{PQ}^2 = 2\text{OP}\cdot\text{PH} = 2a \cdot \text{PH}$$

and

$$\text{PS} = 2\text{PH} = \frac{r^2}{a}.$$

The same calculation for figure 7 gives that

$$\text{P}'\text{S}' = \frac{r^2}{r^2/a} = a.$$

Reviews

The Loom of God: Mathematical Tapestries at the Edge of Time. By CLIFFORD A. PICKOVER. Plenum, New York, 1997. Pp. 292. Hardback (ISBN 0-306-45411-4).

Pickover says that he coined the word 'theomatics' in 1995 to denote the blending of mathematics and religion. This book's main emphasis is claimed to be theomatics.

It is a good-looking book with a very glossy cover, and inside it is full of black and white illustrations both of a mathematical nature and selections from world art, photographs and computer-generated graphics. There is a science fiction story connecting the chapters, with characters from the future discussing the very diverse topics we encounter, a smorgasbord of short chapters and short divisions within chapters meant to be a plenitude of thought-provoking stimuli for the imagination. The basic philosophy of the book, in fact, is said to be that creative thinking is learned by experimenting. Herein we are set going with many patternings in numbers from many different mathematical fields to explore. Topics include: triangular numbers; doomsday; pentagonal numbers; amicable numbers; perfect numbers; Turks and Christians; computer generated poetry and music; Stonehenge; fractals; golden ratio; Kabala.

Aphorisms from eminent mathematicians, scientists, philosophers, theologians and other writers dot the pages and there are five appendices including one compiling responses from newsgroups on the Internet to a posting of 'Goedel's Mathematical Proof of the Existence of God', and another giving Basic programs to experiment with.

Dr Pickover seems himself eminently qualified to be the author of such a book, being a prodigiously working, experienced writer and a winner of many awards. However, I think the book should be given a miss. The packaging is good but the content is of low quality. It appears to me as if the material has been compiled from scraps, fragments and doodles and I do not think it will usefully engage the reader in any coherent and solid mathematical or conceptual ideas and development. It is essentially a book of trivia.

DAVID YATES

Mathematical Analysis and Proof. By DAVID STIRLING. Horwood, Chichester, 1997. Paperback $18.50 (ISBN 1-898563-36-5).

This book aims to introduce students to the idea of 'proof', and in particular to present well-motivated proofs of the elementary results of real analysis.

The early chapters show why proof is necessary and illustrate some of the common techniques, such as induction and contradiction. Naturally enough these chapters tend to be algebraic in nature and they give the reader plenty of practice in the manipulation of mathematical formulae.

Then the work proceeds to the more traditional material of a first analysis course, but with the proofs (and the reasons for them) explained very clearly. All this is interspersed with plenty of down-to-earth calculus examples. However the book also includes some quite technical results on power series and on integration and differentiation which in many institutions would be regarded as second-year material.

For the most part this is a readable introduction to proof which gives a good overview of the approach to analysis; it would therefore make a suitable background reading book for a keen first- or second-year undergraduate. However it is hard to see how it would fit in as a main text book for any one course, partly because of the amount of material in it and partly because it cuts across the boundaries of the way in which algebra, analysis and proof are normally introduced.

University of Sheffield VICTOR BRYANT

Penrose Tiles to Trapdoor Ciphers By MARTIN GARDNER. MAA, Washington, 1997. Pp. 312. Paperback $27.95 (ISBN 0-88385-521-6).

This is the thirteenth collection of Gardner's columns on recreational mathematics from *Scientific American*. As with all of these collections, quality is guaranteed; in this case 21 chapters and 316 pages provide quantity too. This is one of his best books, containing as it does perhaps the widest variety of material.

We have sections on Penrose tiles, which can only tile the plane non-periodically; the two chapters contain a wealth of information not readily available elsewhere. There is information on the Oulipo, a French society of authors with a mathematical bent; and a pair of chapters on trapdoor ciphers which really gives an impression of 'being there'. An impressive introduction to Ramsey theory is followed by sections on logic problems and burr puzzles; elsewhere there are details of hyperbolas and a history of negative numbers. Throughout there are challenging exercises, as well as two chapters devoted to problems.

This is a new edition constituting the first update in eight years. The main text itself does not appear to have been changed, but there are several new references in the bibliography as well as a six-page postscript. This postscript summarises recent developments in the areas of the book and is a welcome addition to it; on its own, however, it does not really justify buying this edition if an older copy is owned.

To conclude, this is perhaps Gardner's best book and is thus essential; but do not buy it for the update alone.

Student, The John of Gaunt School TOBY GEE

Other books received

College Algebra. By WARREN L. RUUD AND TERRY L. SHELL. Worth, New York, 1997. Pp. xviii+669. Hardback $27.95 (ISBN 1-57259-244-3).

A textbook containing a basic course of college algebra designed especially for the US market.

Statistics: Concepts and Controversies. By DAVID S. MOORE. Freeman, New York, 1997. Pp. xvii+526. Paperback $17.95 (ISBN 0-7167-2863-X).

The fourth edition of a book that is designed to introduce statistical ideas to non-mathematicians.

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