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CONTENTS

THE OLYMPIAD CORNER: 83

R.E. WOODROW

All communications about this column should be sent directly to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this column with two sets of problems. The first is from the 6^{th} *Brazilian Mathematical Olympiad*, September 15, 1984, for which we thank Professor Dr. Angelo Barone Netto, Sao Paulo, Brazil.

The first three problems of the list were proposed to all participants. The last three were chosen by regional committees from a list of 15 problems. Those presented here were chosen for Rio de Janeiro and Sao Paulo.

1. Find all natural numbers n and k such that $(n + 1)^k - 1 = n!$

2. The 289 students enrolled in a course are to be distributed into 17 classes each having 17 students, for each of several units of instruction. No two students may be assigned to the same class for different units. What is the largest number of units for which an assignment is possible following this rule?

3. Let F and F', E and E' respectively be two pairs of opposite faces of a regular dodecahedron. From the centres of these four faces, perpendicular segments of length m are raised outward from the solid. Let A, C, B and D, respectively, be the outer extremes of these perpendicular segments. Show that $ABCD$ is a rectangle and find the ratio of its sides.

4. Let D be a point on the hypotenuse BC of a right triangle ABC. Suppose points E and F, on the legs of the triangle, are such that DE and DF are perpendicular to the corresponding leg. Find the position of D that minimizes the length EF.

5. Let $ABCD$ be a convex quadrilateral, and let E, F, G, H be the centres of the external squares erected upon sides AB, BC, CD, DA respectively. Show that segments EG and FH have the same length and are perpendicular. [*Note*: This appeared as *Crux* 1179(a) [1986: 206].]

6. Figure 1 shows a board used in a game called "One Left". The game starts with one marker on each square of the board except for the central square which has no marker. Let A, B, C (or C, B, A) be three adjacent squares in a horizontal row or a vertical column. If A and B are occupied and C is not, then the marker at A may be jumped to C, and the marker at C removed (see Figure 2). Is it possible to end the game with one marker in the position shown in Figure 3?

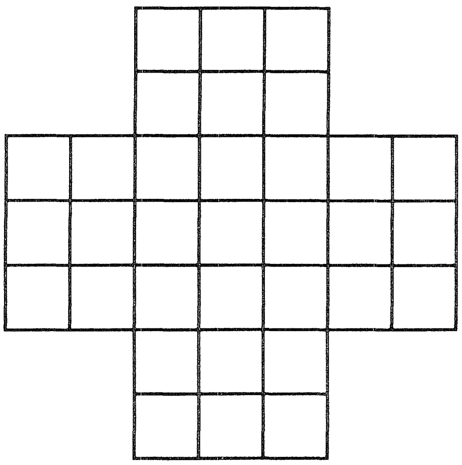


Figure 1

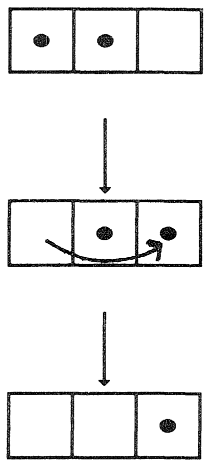


Figure 2

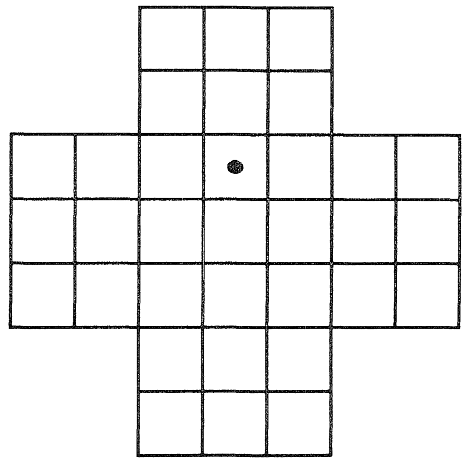


Figure 3

*   *   *

The second set of four problems we pose are from the *Second Balkan Mathematical Olympiad*, May 6, 1985. We thank Ivan Tonov for forwarding them.

1. *Proposed by Bulgaria.*

Let O be the centre of the circle through the points A, B, C, and let D be the midpoint of AB. Let E be the centroid of the triangle ACD. Prove that the line CD is perpendicular to the line OE if and only if $AB = AC$.

2. *Proposed by Romania.*

Let a, b, c, d be real numbers in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ such that

$$\sin a + \sin b + \sin c + \sin d = 1$$

and

$$\cos 2a + \cos 2b + \cos 2c + \cos 2d \geq \frac{10}{3}.$$

Prove that a, b, c, d actually must belong to the interval $\left[0, \frac{\pi}{6}\right]$.

3. *Proposed by Greece.*

Let E denote the set of real numbers, and let S be the set of elements of E of the form $19x + 85y$ where x and y are *positive* integers. Colour the elements of S red and the remaining integer elements of E green. Is there a real number a (not necessarily integral) such that for every two integers b and c if b and c have the same colour then b and c are not symmetrically placed about a (i.e. $|a - b| \neq |a - c|$)?

[*Note*: As we received it this problem is pretty trivial. Any a not of the form $(m + n)/2$ will do. With the restriction $a = (m + n)/2$, m, n integers, what is the answer?]

4. *Proposed by Romania.*

1985 people take part in an international meeting. In any group of three there are at least two individuals who speak the same language. If each person speaks at most five languages, then prove that there are at least 200 people who speak the same language.

* * *

As promised last month, we give the official answers to the *First Round* of the *1986 Dutch Mathematical Olympiad.*

A1. 37.8 A2. A: 26, B: 14, C: 8 A3. 4

A4. 35 A5. 150 A6. 4

B1. 523152 and 523656 B2. 60°

B3. 7 B4. $193^2 + 93^2$ or $213^2 + 23^2$

C1. $n = 11$, $m = 3$ C2. 8 C3. 0

* * *

We next present solutions to some problems posed in the April 1985 Olympiad Corner.

51. [1985: 102] *Proposed by Australia.*

Let P be a regular convex $2n$-gon. Show that there is a $2n$-gon Q with the same vertices as P (but in a different order) such that Q has exactly one pair of parallel sides.

Solution by Aage Bondesen, Royal Danish School of Educational Studies.

The cases $n = 1$ and $n = 2$ are trivial. Assume $n > 2$. Enumerate the vertices of P in order $P_0P_1 \ldots P_{2n-1}$. When is P_0P_i parallel to P_kP_ℓ with $k < \ell$? The only possibilities are that $0 < k < \ell < i$ and $k = i - \ell$, or that $i < k < \ell < 2n$ and $2n - \ell = k - i$.

Now imagine we enumerate the vertices $P_0P_{i_1}P_{i_2} \ldots P_{i_{2n-1}}$ in such a way that the (clockwise) distance $i_{\ell+1} - i_\ell$ on P is either n or $n - 1$ except for one pair whose distance is -1. Also suppose that every vertex, except for the endpoints of the pair at distance -1, has both distances n, $n - 1$ for its two neighbors, and that the neighbors of each end of the side of length one are at distance n. Then it is easy to see that the only parallel sides possible involve as one side the pair at distance -1. This is because two pairs at distance n determine diameters, and were two pairs $P_{i_k}P_{i_{k+1}}$ and $P_{i_\ell}P_{i_{\ell+1}}$ at distance $n - 1$ parallel, we would have $i_k + n \equiv i_\ell \pmod{2n}$ and $i_\ell + n \equiv i_k \pmod{2n}$, thus $P_{i_{k+2}} = P_{i_{\ell+1}}$ and $P_{i_{\ell+2}} = P_{i_{k+1}}$, thus $k + 2 \equiv \ell + 1 \pmod{2n}$ and $\ell + 2 \equiv k + 1 \pmod{2n}$, a contradiction.

For n even the sequence

$$n - 1, n, n - 1, n, \ldots, n, -1, n, n - 1, \ldots, n$$

of distances, with a "-1" in the $n + 1^{\text{st}}$ position, makes the first side parallel to the $n + 1^{\text{st}}$ side since the common distance between endpoints is

$$\frac{2n - (n - 1 + 1)}{2} = \frac{n}{2}.$$

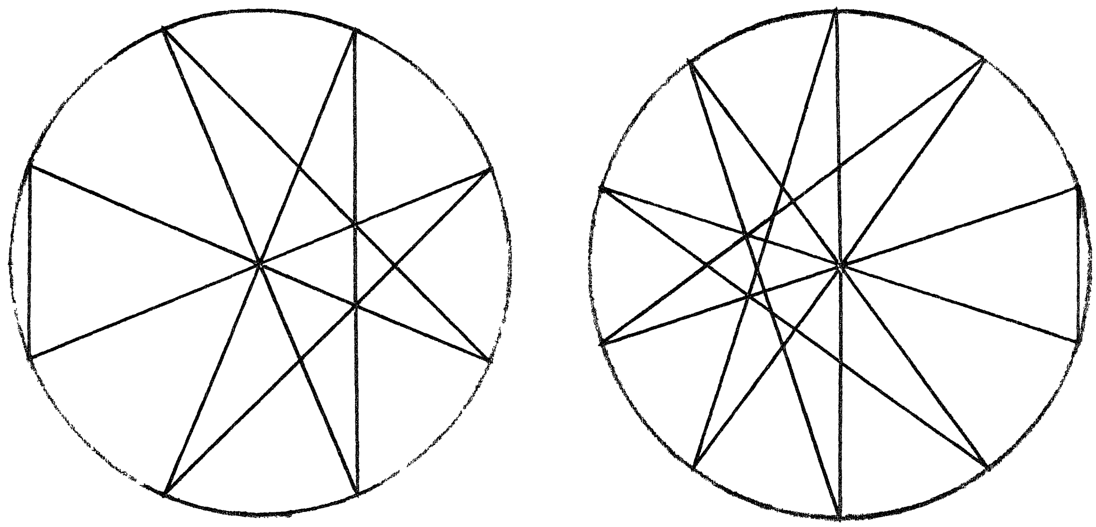
For n odd the sequence

$$n, n - 1, \ldots, n, -1, n, n - 1, \ldots, n, n - 1$$

of distances also produces the desired result, with the first and $n + 1^{\text{st}}$ sides parallel, since the common distance between endpoints is

$$\frac{2n - (n + 1)}{2} = \frac{n - 1}{2}.$$

The situations with $n = 4$ and $n = 5$ are illustrated below.



52. [1985: 102] *Proposed by Belgium.*

If $n > 2$ is an integer and $[x]$ denotes the greatest integer $\leq x$, show that

$$\left[\frac{n(n+1)}{4n-2}\right] = \left[\frac{n+1}{4}\right] .$$

Solutions independently by Beno Arbel, Department of Mathematics, Tel Aviv University; Aage Bondesen, Royal Danish School of Educational Studies; Bob Prielipp, University of Wisconsin, Oshkosh; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Since

$$\frac{n+1}{4} < \frac{n(n+1)}{4n-2}$$

it suffices to show that

$$\frac{n(n+1)}{4n-2} < \left[\frac{n+1}{4}\right] + 1. \qquad (*)$$

Straightforward calculation gives

$$\frac{n(n+1)}{4n-2} = \frac{n+1}{4} + \frac{n+1}{4(2n-1)}$$

and for $n > 2$ we have

$$\frac{n+1}{4(2n-1)} < \frac{1}{4} .$$

Thus

$$\frac{n(n+1)}{4n-2} < \frac{n+1}{4} + \frac{1}{4} .$$

There are two cases:

(i) $n = 4k + r$, $r = 0,1,2$. Then

$$\left[\frac{n+1}{4}\right] + 1 = k + 1$$

while

$$\frac{n+1}{4} + \frac{1}{4} \leq k + 1$$

so (*) follows.

(ii) $n = 4k + 3$. Then

$$\left[\frac{n+1}{4}\right] + 1 = k + 2.$$

But then

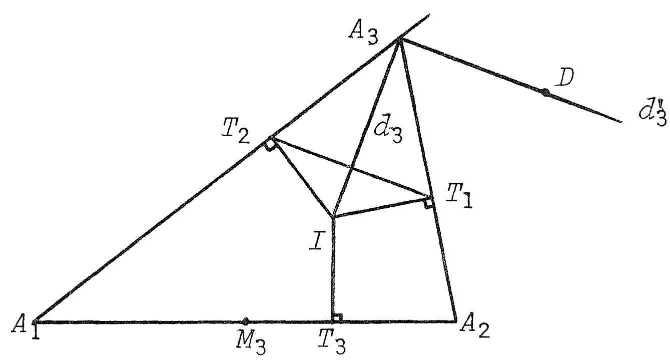
$$\frac{n+1}{4} + \frac{1}{4} = k + 1 + \frac{1}{4}$$

and the inequality (*) holds.

54. [1985: 102] *Proposed by Bulgaria.*

The incircle of triangle $A_1A_2A_3$ touches the sides A_2A_3, A_3A_1, A_1A_2 at the points T_1, T_2, T_3, respectively. If M_1, M_2, M_3 are the midpoints of the sides A_2A_3, A_3A_1, A_1A_2, respectively, prove that the perpendiculars through the points M_1, M_2, M_3 to the lines T_2T_3, T_3T_1, T_1T_2, respectively, are concurrent.

Solution by J.T. Groenman, Arnhem, The Netherlands.



Notice first that $A_3T_2 = A_3T_1$ and that the line d_3 through A_3 and the incentre I bisects $\angle A_3$. Let d_3' bisect the exterior angle at A_3 as indicated, and let D lie on the same side of A_1A_3 as does A_2. Notice that $\angle IA_3T_1 + \angle T_1A_3D = 90^\circ$. Thus $\angle T_1A_3D$ is the complement of $\angle IA_3T_1$. In the isosceles triangle $T_1T_2A_3$ we easily see that $\angle A_3T_1T_2$ is the complement of $\angle IA_3T_1$. Thus T_1T_2 is parallel to d_3'. The perpendicular from M_3 to T_1T_2 is thus perpendicular to d_3' and thus is parallel to d_3. Similarly for $i = 1,2$ we see that the perpendicular from M_i to the corresponding side of $T_1T_2T_3$ is parallel to the angle bisector d_i of $\angle A_i$.

Consider now triangle $M_1M_2M_3$. Now $\Delta M_1M_2M_3$ is similar to $\Delta A_1A_2A_3$ and moreover corresponding sides are parallel. Thus the bisectors e_1, e_2, e_3 of angles M_1, M_2, M_3 are parallel to the bisectors d_1, d_2, d_3 of A_1, A_2, A_3 respectively. These coincide at the centre of $M_1M_2M_3$. But e_1, e_2, and e_3 are just the perpendiculars referred to in the last paragraph, completing the proof.

56. [1985: 102] *Proposed by Canada.*

Given that $a_1, a_2, \ldots, a_{2n}$ are distinct integers such that the equation

$$(x - a_1)(x - a_2) \ \ldots \ (x - a_{2n}) + (-1)^{n-1}(n!)^2 = 0$$

has an integer solution r, show that $r = (a_1 + a_2 + \ldots + a_{2n})/2n$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We first establish two simple lemmas:

Lemma 1. If $d_1 < d_2 < \ldots < d_n$ are distinct positive integers such that $d_1d_2\ldots d_n = n!$ then $d_k = k$ for $k = 1,2,\ldots,n$.

Proof. Since $d_1 \geq 1$ we have $d_k \geq k$ for all k. Then $d_1d_2\ldots d_n \geq n!$ with equality if and only if $d_k = k$ for all k.

Lemma 2. If $b_1, b_2, \ldots, b_{2n}$ are positive integers with $b_1 \leq b_2 \leq \ldots \leq b_{2n}$ where no three consecutive terms are equal and $b_1b_2\ldots b_{2n} = (n!)^2$ then

$$(b_1, b_2, b_3, \ldots, b_{2n}) = (1,1,2,2,\ldots,n,n).$$

Proof. Consider the sequences $b_1, b_3, \ldots, b_{2n-1}$ and $b_2, b_4, \ldots, b_{2n}$. Since $b_1 \geq 1$ we have $b_3 \geq 2$, $b_{2j-1} \geq j$ and $b_1b_3\ldots b_{2n-1} \geq n!$. Similarly $b_{2j} \geq j$ and $b_2b_4\ldots b_{2n} \geq n!$. But $b_1b_2\ldots b_{2n} = (n!)^2$ and hence $b_1b_3\ldots b_{2n-1} = b_2b_4\ldots b_{2n} = n!$ and the conclusion follows from Lemma 1.

Now we prove the main result.

By assumption,

$$(r - a_1)(r - a_2)\dots(r - a_{2n}) = (-1)^n(n!)^2$$

and $r - a_i \neq r - a_j$ for $i \neq j$. Let $b_1, b_2, \dots, b_{2n}$ denote the numbers $|r - a_i|$ arranged in nondecreasing order of magnitude. Then $1 \leq b_1 \leq b_2 \leq \dots \leq b_{2n}$ where no three consecutive terms can be equal. Since $b_1 b_2 \dots b_{2n} = (n!)^2$ we conclude from Lemma 2 that

$$(b_1, b_2, \dots, b_{2n-1}, b_{2n}) = (1,1,2,2,\dots,n,n).$$

Let $a_1', a_2', \dots, a_{2n}'$ denote a rearrangement of $a_1, a_2, \dots, a_{2n}$ such that $|r - a_j'| = b_j$. Then from $b_{2j-1} = b_{2j} = j$ we obtain $|r - a_{2j-1}'| = |r - a_{2j}'| = j$ for $j = 1,2,\dots,n$. Hence for each j one of $r - a_{2j-1}'$ and $r - a_{2j}'$ must be j and the other $-j$. Therefore we get

$$0 = \sum_{j=1}^{n} \{(r - a_{2j-1}') + (r - a_{2j}')\}$$

$$= 2nr - \sum_{j=1}^{n} \left[a_{2j-1}' + a_{2j}'\right]$$

$$= 2nr - \sum_{k=1}^{2n} a_k.$$

From this $r = (a_1 + a_2 + \dots + a_{2n})/2n$ is immediate.

Finally, the above argument shows that if $c_1 < c_2 < \dots < c_{2n}$ are integers such that $|c_1 c_2 \dots c_{2n}| = (n!)^2$ then necessarily

$$(c_1, \dots, c_{2n}) = (-n, -n + 1, \dots, -1, 1, 2, \dots, n)$$

and hence $c_1 c_2 \dots c_{2n} = (-1)^n(n!)^2$. In other words, although the sign $(-1)^{n-1}$ is not necessary for the entire argument to go through, it is nonetheless the correct sign as it is a consequence of the other conditions.

63. [1985: 104] *Proposed by Great Britain.*

Prove that the product of five consecutive integers cannot be a perfect square.

Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen.

The assertion is trivially true if all five integers are negative. If one of them is zero it is false. Thus we should assume that the integers are all positive.

So suppose there are successive *positive* integers a_1, a_2, a_3, a_4, a_5 with $a_1a_2a_3a_4a_5$ a perfect square. If for some $i \in \{1,2,3,4,5\}$ a_i has a prime factor $p \geq 5$ then p divides none of the other a_j's. It follows that the exponent of p in the prime factorization of a_i is even. Consequently, each of a_1, a_2, a_3, a_4, a_5 is of one of the four forms:

(i) a perfect square;

(ii) twice a perfect square;

(iii) three times a perfect square;

(iv) six times a perfect square.

By the pigeon-hole principle two of the a_i's have the same form. Now if two positive integers are of form (ii), (iii) or (iv), their difference is at least 6, which is impossible. If both are of the form (i) their difference is 3 or at least 5. Clearly we can accept only difference 3. But then two of the numbers must be 1 and 4. Then a_1, a_2, a_3, a_4, a_5 are 1, 2, 3, 4, 5 with product 120, which is not a perfect square.

[*Bob Prielipp, University of Wisconsin, Oshkosh also pointed out the error, while John Morvay, Dallas, Texas submitted a different solution.*]

67. [1985: 104] *Proposed by Poland.*

Let a, b, c be positive numbers such that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} = \frac{\sqrt{3}}{2}.$$

Prove that the system

$$\begin{cases} \sqrt{y-a} + \sqrt{z-a} = 1 \\ \sqrt{z-b} + \sqrt{x-b} = 1 \\ \sqrt{x-c} + \sqrt{y-c} = 1 \end{cases}$$

has exactly one real solution (x,y,z).

Comment by George Evagelopoulos, Law student, University of Athens, Greece.

Bjorn M. Poonen, who was a USAMO winner in 1985 and a member of the USA team at the 26th IMO, gave a very nice solution to the problem. His solution was published in *The College Mathematics Journal* (Volume 16, Number 5, November 1985, p.343).

71. [1985: 105] *Proposed by Spain.*

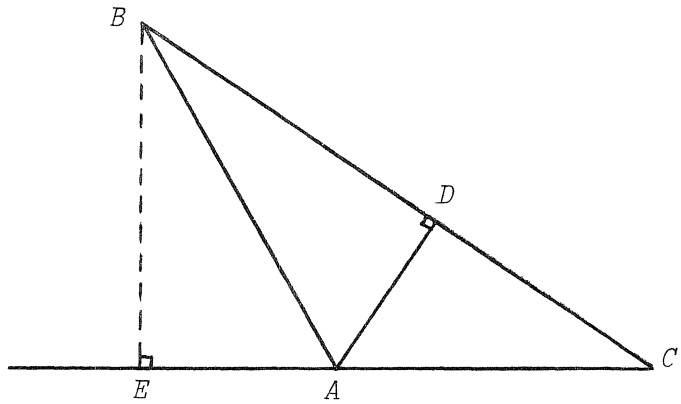
Construct a nonisosceles triangle ABC such that

$$a(\tan B - \tan C) = b(\tan A - \tan C),$$

where a and b are the side lengths opposite angles A and B, respectively.

Solution by J.T. Groenman, Arnhem, The Netherlands.

We look for a solution with an obtuse angle at A and with $a > b$. Refer to the figure, where D denotes the foot of the altitude from A, and E that from B.



Now $a(\tan B - \tan C) = b(\tan A - \tan C)$ is equivalent to

$$a\left(\frac{AD}{DB} - \frac{AD}{DC}\right) = b\left(-\frac{BE}{AE} - \frac{BE}{CE}\right) \tag{1}$$

since A is obtuse. Next $a(AD) = b(BE) = 2$ Area(ABC) so (1) is equivalent to

$$\frac{1}{DB} - \frac{1}{DC} = -\frac{1}{AE} - \frac{1}{CD}\,. \tag{2}$$

Further, $BD = c\cos B$, $DC = b\cos C$, $AE = -c\cos A$, and $CE = a\cos C$, again using the fact that A is obtuse. Hence we have (2) equivalent to

$$\frac{1}{c\cos B} - \frac{1}{b\cos C} = \frac{1}{c\cos A} - \frac{1}{a\cos C}\,. \tag{3}$$

Using

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

we have (3) equivalent to

$$\frac{a}{a^2 + c^2 - b^2} - \frac{a}{a^2 + b^2 - c^2} = \frac{b}{b^2 + c^2 - a^2} - \frac{b}{a^2 + b^2 - c^2}$$

which in turn is equivalent to

$$\frac{ab^2 - ac^2}{a^2 + c^2 - b^2} = \frac{a^2b - bc^2}{b^2 + c^2 - a^2}. \tag{4}$$

Cross multiplying and collecting terms in powers of c gives that (4) is equivalent to

$$(b - a)c^4 - c^2(b^3 - a^3) + ab(b^3 - a^3) + a^2b^2(b - a) = 0. \tag{5}$$

Since we seek a nonisosceles triangle we impose $b \neq a$ to obtain (5) equivalent to

$$c^4 - c^2 (a^2 + ab + b^2) + ab(a + b)^2 = 0. \tag{6}$$

To find *an* example we may take $a = 5b$ to get

$$c^4 - 31c^2b^2 + 180b^4 = 0,$$

giving $c^2 = \left[\frac{31 + \sqrt{241}}{2}\right]b^2$ as one solution. Checking, we see that A is obtuse since

$$b^2 + c^2 - a^2 = \left[1 + \frac{31 + \sqrt{241}}{2} - 25\right]b^2 < 0.$$

As

$$5,\ 1,\ \sqrt{\frac{31 + \sqrt{241}}{2}}$$

are constructible numbers, a nonisosceles example can be constructed.

*　　　　*　　　　*

We have not received solutions to the remaining problems posed in the April 1985 Olympiad Corner. This is also true for the May 1985 issue. Send in your elegant solutions!

*　　　　*　　　　*

We next give some of the solutions to problems posed in the June 1985 number of the Corner. The remaining solutions submitted will be discussed next month.

3. [1985: 169] *The 1984 Dutch Olympiad.*

For $n = 1,2,3,\ldots,$ let

$$a_n = \frac{1\cdot 4\cdot 7\cdot \ldots \cdot(3n - 2)}{2\cdot 5\cdot 8\cdot \ldots \cdot(3n - 1)}.$$

Prove that, for all n,

$$(3n + 1)^{-1/2} \le a_n \le (3n + 1)^{-1/3}.$$

Solution by Beno Arbel, School of Mathematics, Tel Aviv University, Israel.

We shall prove a more general result: for a and b positive real numbers with $a \ge b$,

$$\sqrt{\frac{a}{a + 3nb}} \le \prod_{k=1}^{n} \frac{a + 3(k - 1)b}{a + (3k - 2)b} < \sqrt[3]{\frac{a}{a + 3nb}} \qquad (1)$$

with strict inequalities for $n > 1$, or if $a > b$. The problem follows by setting $a = b = 1$.

We first prove that for each $k = 1,2,\ldots$

$$\frac{a + 3(k - 1)b}{a + (3k - 2)b} < \sqrt[3]{\frac{a + 3(k - 1)b}{a + 3kb}}.$$

This inequality is equivalent to

$$[a + 3(k - 1)b]^2(a + 3kb) < [a + (3k - 2)b]^3.$$

But, from the AM-GM inequality,

$$\begin{aligned}[a + 3(k - 1)b]^2(a + 3kb) &< \left[\frac{2a + 6(k - 1)b + a + 3kb}{3}\right]^3 \\ &= \left[\frac{3a + 9kb - 6b}{3}\right]^3 \\ &= [a + (3k - 2)b]^3\end{aligned}$$

with strict inequality because $a + 3(k - 1)b \ne a + 3kb$. Now

$$\prod_{k=1}^{n} \frac{a + 3(k - 1)b}{a + (3k - 2)b} < \prod_{k=1}^{n} \sqrt[3]{\frac{a + 3(k - 1)b}{a + 3kb}} = \sqrt[3]{\frac{a}{a + 3nb}}$$

the last equality being obtained by cancellation.

For the left inequality of (1) we prove that for each $k = 1,2,3,\ldots$

$$\frac{a + 3(k - 1)b}{a + (3k - 2)b} \ge \sqrt{\frac{a + 3(k - 1)b}{a + 3kb}}$$

which is equivalent to

$$[a + 3(k - 1)b][a + 3kb] \ge [a + (3k - 2)b]^2.$$

Expanding we obtain

$$a^2 + 6akb + 9k^2b^2 - 3ab - 9kb^2 \ge a^2 + 9k^2b^2 + 4b^2 + 6akb - 4ab - 12kb^2$$

which simplifies to give the equivalent inequality

$$b(4 - 3k) \le a.$$

This is true for all k since $b \leq a$. Note that the inequality is strict if $b < a$ or $k \geq 2$. Now

$$\prod_{k=1}^{n} \frac{a + 3(k-1)b}{a + (3k-2)b} \geq \prod_{k=1}^{n} \sqrt{\frac{a + 3(k-1)b}{a + 3kb}} = \sqrt{\frac{a}{a + 3nb}}$$

with strict inequality if $b < a$ or $n \geq 2$. This completes the proof.

Remark. One can prove slightly more. For a and b positive real numbers and for all m and n positive integers the following inequality holds:

$$\prod_{k=1}^{n} \frac{a + m(k-1)b}{a + (mk - m + 1)b} < \sqrt[m]{\frac{a}{a + mnb}} .$$

This follows from

$$[a + m(k-1)b]^{m-1}(a + mkb) < [a + (mk - m + 1)b]^{m}$$

for $k = 1,2,\dots,n$, which in turn follows by application of the AM-GM inequality.

[*Comment: This problem was also solved by George Evagelopoulos, Law student, Athens, Greece by a different method.*]

4. [1985: 169] *The 1984 Dutch Olympiad.*

By inserting parentheses in the expression 1: 2: 3, we get two different numerical values, (1: 2): 3 = 1/6 and 1: (2: 3) = 3/2. Now parentheses are inserted in the expression

1: 2: 3: 4: 5: 6: 7: 8.

(a) What are the maximum and minimum numerical values that can be obtained in this way?

(b) How many different numerical values can be obtained in this way?

Solution by Curtis Cooper, Central Missouri State University.

Consider the expression

$$X_1 : X_2 : \dots : X_n .$$

By fully parenthesizing this expression, we can obtain (possibly) different values. A fully parenthesized expression can be transformed into its *post-fix expression* by taking the fully parenthesized expression, removing the left parentheses and colons (:), and replacing the right parentheses by slashes (/). For example, if $n = 5$ the fully parenthesized expression

$$((X_1 : X_2) : ((X_3 : X_4) : X_5))$$

in post-fix form is

$$X_1X_2/X_3X_4/X_5//.$$

We can evaluate a fully parenthesized expression by examining its post-fix expression and using a stack. As we scan the post-fix expression from left to right, we push the elements X_i onto the stack. If a slash is encountered, we take the top two elements of the stack, perform the division of the second stack element by the first, and push the result back on the top of the stack. After the expression is scanned, the item on top of the stack is the value of the fully parenthesized expression. Notice that the final algebraic expression has both a numerator and denominator, since there is always a last slash.

Also note that X_1 is necessarily in the numerator. Also, for $i = 1,2,\ldots,n-1$, X_i and X_{i+1} are both in the numerator or both in the denominator of the fraction if the number of slashes between X_i and X_{i+1} is odd; otherwise X_i and X_{i+1} are in different parts of the fraction. To see this, notice that when X_i is scanned it is put on the stack. An even number of slashes between X_i and X_{i+1} puts X_i in the numerator of the result which is placed on top of the stack; while an odd number puts X_i in the denominator of the result sitting on top of the stack. Now when X_{i+1} is scanned and put on top of the stack, it is in the numerator of the result on top of the stack. This element X_{i+1} will remain in the numerator of a stack element until a slash combines it with the result on the stack containing X_i. If there are an even number of slashes between X_i and X_{i+1}, X_i and X_{i+1} end up in different parts of the fraction; but an odd number of slashes puts X_i and X_{i+1} in the same part of the fraction. Finally, this relationship is preserved throughout the course of the remaining stack operations.

From the above discussion, X_1 is always in the numerator and X_2 is always in the denominator of the result. Also the value of $X_1X_2X_3/X_4/\ldots X_n//$ is

$$\frac{X_1X_3X_4\ldots X_n}{X_2} \tag{1}$$

and the value of $X_1X_2/X_3/X_4/\ldots X_n/$ is

$$\frac{X_1}{X_2X_3\ldots X_n}. \tag{2}$$

Thus for $n = 8$ with $X_1 = 1,\ldots,X_8 = 8$, the maximum value, from (1), is $\frac{8!}{4}$ and the minimum value, from (2), is $\frac{1}{8!}$.

We next construct inductively, for $n \geq 2$, a set S_n of post-fix expressions s so that evaluating s gives a 1-1 correspondence with the collection of algebraic quotients which arise from fully parenthesizing $X_1 : X_2 : \ldots : X_n$. It will be evident that $|S_n| = 2^{n-2}$. With $n = 2$ we set $S_2 = \{X_1X_2/\}$. Note that the only possible quotient is $\frac{X_1}{X_2}$ establishing the claim for $n = 2$. Now suppose S_n has been constructed. Note that every element s of S_n ends with at least one slash. For $s \in S_n$ let s^- denote the string that results when the last slash of s is removed. Form S_{n+1} by collecting expressions of the form $sX_{n+1}/$ and $s^-X_{n+1}//$. Clearly $|S_{n+1}| = 2^{n+1-2}$. For $i = 1,2,\ldots,n-1$ the number of slashes between X_i and X_{i+1} has not changed. Thus different values of s give distinct algebraic expressions. Comparing $sX_{n+1}/$ and $s^-X_{n+1}//$ we note that the parity of the number of slashes between X_n and X_{n+1} is different so that X_n and X_{n+1} are in the same part of the evaluated expression for one case and in different parts for the other. Therefore different post-fix expressions in S_{n+1} give distinct algebraic quotients. To see that all allowable algebraic expressions E arise, notice that each expression is determined by a choice of a subset from $\{X_3,\ldots,X_{n+1}\}$ to act (with X_1) as numerator. This is possible in 2^{n-1} ways so all are accounted for by evaluating post-fix expressions in S_{n+1}.

When we set $X_1 = 1, X_2 = 2,\ldots,X_8 = 8$ and $n = 8$ we must ask when different post-fix expressions give rise to the same rational. This can only occur because of a product of some elements of the numerator being equal to the product of some elements of the denominator. This does not give rise to duplication if one of the numbers involved is 1 or 2 since $X_1 = 1$ must be in the numerator and X_2 in the denominator. Also 5 and 7 obviously cannot be involved. This leaves $3 \cdot 8 = 4 \cdot 6$. There are 2^2 possibilities with 1, 3, 8 in the numerator and 2, 4, 6 in the denominator corresponding to the choice of subset of {5,7} for the numerator. These are also realized by post-fix expressions giving X_4, X_6 in the numerator and X_3, X_8 in the denominator. Thus the correct accounting for (b) is $2^6 - 2^2 = 60$ different numerical values.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before October 1, 1987, although solutions received after that date will also be considered until the time when a solution is published.

<u>1221</u>*. *Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.*

Let u, v, w be nonnegative numbers and let $0 < t \le 2$. If a, b, c are the sides of a triangle and if F is its area, prove that

$$\frac{u}{v+w}(bc)^t + \frac{v}{w+u}(ca)^t + \frac{w}{u+v}(ab)^t \ge \frac{3}{2}\left[\frac{4F}{\sqrt{3}}\right]^t.$$

[See Solution II of *Crux* 1051 [1986: 252].]

<u>1222</u>. *Proposed by George Szekeres, University of New South Wales, Kensington, Australia.*

Evaluate the symmetric $n \times n$ determinant D_n in which $d_{i,i+2} = d_{i+2,i} = -1$ for $i = 1,\ldots,n-2$, $d_{ij} = 1$ otherwise. Also evaluate $\overline{D}_n$ in which $\overline{d}_{ij} = -d_{ij}$ for $i \ne j$, $\overline{d}_{ii} = d_{ii}$. [See the solution to *Crux* 1033 [1987: 90].]

1223. *Proposed by H.S.M. Coxeter, University of Toronto, Toronto, Ontario.*

(a) Show that, if all four faces of a tetrahedron in Euclidean space are right-angled triangles, there must be two vertices at each of which two right angles occur. In other words, the tetrahedron must be an orthoscheme (Coxeter, *Introduction to Geometry*, Wiley, New York, 1969, p.156).

(b) Show that this also holds in hyperbolic space but not in elliptic space.

1224. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

$A_1A_2A_3$ is a triangle with circumcircle Ω. Let $x_1 < X_1$ be the radii of the two circles tangent to A_1A_2, A_1A_3, and arc A_2A_3 of Ω. Let x_2, X_2, x_3, X_3 be defined analogously. Prove that:

(a) $\displaystyle\sum_{i=1}^{3} \frac{x_i}{X_i} = 1$;

(b) $\displaystyle\sum_{i=1}^{3} X_i \ge 3\sum_{i=1}^{3} x_i \ge 12r$,

where r is the inradius of $\Delta A_1A_2A_3$.

1225*. *Proposed by David Singmaster, The Polytechnic of the South Bank, London, England.*

What convex subset S of a unit cube gives the maximum value for V/A, where V is the volume of S and A is its surface area? (For the two-dimensional case, see *Crux* 870 [1986: 180].)

1226. *Proposed by Hidetosi Fukagawa, Yokosuka High School, Aichi, Japan.*

Let *ABCD* be a quadrilateral inscribed in a circle, and let O_1, O_2, O_3, O_4 be the inscribed circles of triangles *BCD*, *CDA*, *DAB*, *ABC* respectively.

(a) Show that the centers of these four circles are the vertices of a rectangle.

(b) Show that $r_1 + r_3 = r_2 + r_4$, where r_i is the radius of O_i.

1227. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

Find all angles θ in $[0,2\pi)$ for which

$$\sin\theta + \cos\theta + \tan\theta + \cot\theta + \sec\theta + \csc\theta = 6.4.$$

<u>1228</u>. *Proposed by J. Garfunkel, Flushing, New York and C. Gardner, Austin, Texas.*

If QRS is the equilateral triangle of minimum perimeter that can be inscribed in a triangle ABC, show that the perimeter of QRS is at most half the perimeter of ABC, with equality when ABC is equilateral.

<u>1229</u>. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Characterize all positive integers a and b such that

$$(a,b)^{[a,b]} \le [a,b]^{(a,b)}$$

and determine when equality holds. (As usual, (a,b) and $[a,b]$ denote respectively the g.c.d. and ℓ.c.m. of a and b.)

<u>1230</u>. *Proposed by Jordi Dou, Barcelona, Spain.*

Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let $P = AC \cap BD$ and let s be a line through P cutting AD at E and BC at F. Prove that there exists a conic tangent to AD at E, to BC at F, and twice tangent to Ω.

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S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

<u>1032</u>. [1985: 121] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

The following formulas are given in R.A. Johnson's *Advanced Euclidean Geometry* (Dover, New York, 1960, p.205):

$$IH^2 = 2\rho^2 - 2Rr \quad \text{and} \quad OH^2 = R^2 - 4Rr,$$

where, in Johnson's notation, O, I, H, R, ρ, r are the circumcenter, incenter, orthocenter, circumradius, inradius, and inradius of the orthic triangle, respectively, of a given triangle. Johnson claims (at least tacitly) that these formulas both hold for all triangles. Prove that neither formula holds for obtuse triangles.

I. *Editor's Comment.*

There were two submissions (other than the proposer's) concerning this problem, one agreeing with the proposal and one against. After sitting on this problem for some time, wondering what to do about it, I've decided simply to print the two opposing views, both of which I suspect have some validity, and let the readership decide. Perhaps someone out there would care to contribute the definitive treatise on the subject.

II. *Comment by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

It should be noted that the same error also appears in [1], page 41. It basically stems from an uncareful application of the formula

$$r = 2R \cos A \cos B \cos C$$

which is only valid for acute triangles. In [1], page 27, for obtuse triangles the following formulae are derived. If $A > \pi/2$, then

$$s' = 2R \sin A \cos B \cos C$$

$$F' = -\frac{R^2}{2} \sin 2A \sin 2B \sin 2C$$

where s' and F' denote respectively the semiperimeter and area of the orthic triangle. Therefore,

$$r = \frac{F'}{s'} = -2R \cos A \sin B \sin C$$

for obtuse triangles with $A > \pi/2$.

Reference:

[1] E. Donath, *Die merkwürdigen Punkte und Linien des ebenen Dreiecks*, Berlin, 1969.

III. *Comment by Leon Bankoff, Los Angeles, California.*

The proposer claims that Johnson claims (at least tacitly) that the two formulas involving the orthocenter H hold for all triangles. Johnson does nothing of the sort. He claims explicitly that these formulas were used by Feuerbach in the mainly algebraic proof of his celebrated theorem. Feuerbach claims rightfully that

$$r = 2R \cos A \cos B \cos C$$

(Johnson, 299g, page 191) so that the formulas for IH^2 and OH^2 require a negative value for r when one of the cosines is negative, as would be the case in obtuse triangles.

* * *

1033*. [1985: 121; 1986: 207] *Proposed by W.R. Utz, University of Missouri-Columbia.*

Let D_n be any symmetric determinant of order n in which the elements in the principal diagonal are all 1's and all other elements are either 1's or −1's, and let $\overline{D_n}$ be the determinant obtained from D_n by replacing the non-principal-diagonal elements by their negatives. It is easy to show that $D_2\overline{D_2} = 0$ for all D_2 and $D_3\overline{D_3} = 0$ for all D_3. For which $n > 3$ is it true that $D_n\overline{D_n} = 0$ for all D_n?

Solution by G. Szekeres, University of New South Wales, Kensington, Australia.

As suggested by the Editor [1986: 211], the problem is indeed do-able. The answer is of course that for every $n > 4$ there exists a D_n with $D_n\overline{D_n} \neq 0$. What we need is a sequence of determinants D_n which can be evaluated directly. The most obvious examples are symmetric "circulant" determinants $D_n = (d_{ij})$ in which $d_{i,i+j} = d_{i+j,i} = -1$ for $i = 1,\ldots,n$ and for j in some given subset J_n of $I_n = \{1,\ldots,[n/2]\}$, all other entries of D_n being +1. (The convention used here is that if $i + j > n$ then $d_{i,i+j} = d_{i,i+j-n}$ and $d_{i+j,i} = d_{i+j-n,i}$.) We want to determine J_n so that $D_n\overline{D_n} \neq 0$. Note that $\overline{D_n}$ is also circulant with $\overline{J_n}$ being the complement of J_n in I_n. It is well known (see e.g. Muir, *A treatise on the theory of determinants*, Dover, New York, 1960, p.444) that the value of D_n (for a given J_n) is obtained as follows: Let $w_1 = 1, w_2,\ldots,w_n$ be the nth roots of unity, and let

$$u_i = 1 + \sum_{j=1}^{n-1} \epsilon_j w_i^j, \quad i = 1,\ldots,n$$

where

$$\epsilon_{n-j} = \epsilon_j = -1 \quad \text{if} \quad j \in J_n,$$

$$\epsilon_{n-j} = \epsilon_j = +1 \quad \text{if} \quad j \in \overline{J_n} = I_n - J_n.$$

Then, apart from a ± sign, D_n is equal to $u_1u_2\ldots u_n$, and so $D_n \neq 0$ if and only if each $u_i \neq 0$. In particular we must have $u_1 \neq 0$. But $u_1 = 0$ can only happen if $n = 2m$ is even and

$$2|J_n| = m + 1 \quad \text{if} \quad m \in J_n, \qquad 2|J_n| = m \quad \text{if} \quad m \notin J_n.$$

Similarly for $\overline{D_n} \neq 0$ we must have $\overline{u_1} \neq 0$, where

$$\overline{u_i} = 1 - \sum_{j=1}^{n-1} \epsilon_j w_i^j$$

and ϵ_j is defined as above. $\overline{u_1} = 0$ can only happen if $n = 2m$ is even and

$$2|J_n| = m \quad \text{if} \quad m \in J_n, \qquad 2|J_n| = m - 1 \quad \text{if} \quad m \notin J_n.$$

Thus the condition for $u_1\overline{u_1} \neq 0$ is that either n is odd, or $n = 2m$ and

$$\begin{aligned} &\text{if} \quad m \in J_n \quad \text{then} \quad 2|J_n| \neq m \quad \text{or} \quad m + 1, \\ &\text{if} \quad m \notin J_n \quad \text{then} \quad 2|J_n| \neq m \quad \text{or} \quad m - 1. \end{aligned} \tag{1}$$

Henceforth w will always denote an nth root of unity different from 1, so that

$$1 + w + w^2 + \dots + w^{n-1} = 0.$$

Then $u_i \neq 0$ for $i = 2,\dots,n$ is equivalent to

$$\sum_{j \in J_n} (w^j + w^{-j}) \neq \begin{cases} w^{n/2} & \text{if} \quad n \quad \text{is even and} \quad n/2 \in J_n, \\ 0 & \text{otherwise,} \end{cases}$$

for $w = w_i$, $i = 2,\dots,n$. Similarly $\overline{u_i} \neq 0$ for $i = 2,\dots,n$ is equivalent to

$$1 + \sum_{j \in J_n} (w^j + w^{-j}) \neq \begin{cases} w^{n/2} & \text{if} \quad n \quad \text{is even and} \quad n/2 \in J_n, \\ 0 & \text{otherwise.} \end{cases}$$

So a necessary and sufficient condition for $D_n\overline{D_n} \neq 0$ is

$$\sum_{j \in J_n} (w^j + w^{-j}) \neq \begin{cases} w^{n/2} \quad \text{or} \quad w^{n/2} - 1 & \text{if} \quad n \quad \text{is even and} \quad n/2 \in J_n, \\ 0 \quad \text{or} \quad -1 & \text{otherwise,} \end{cases} \tag{2}$$

for every $w \neq 1$ such that $w^n = 1$, together with condition (1) if n is even.

Consider first the case when $n = p^r$ is a prime power.

Case (i): $p = 3$, $r > 1$. Take $J_n = \{1,2\}$, which is admissible since $2 < n/2$ for $r > 1$. We want to show that

$$w + w^2 + w^{-2} + w^{-1} \neq 0 \text{ or } -1.$$

Supposing

$$w + w^2 + w^{-2} + w^{-1} = 0,$$

then

$$(1 + w)(1 + w^3) = 1 + w + w^3 + w^4 = 0$$

and hence either $w = -1$ or $w^3 = -1$, both of which conflict with $w^{3^r} = 1$. Also, if

$$w + w^2 + w^{-2} + w^{-1} = -1$$

then $w^5 = 1$ which with $w^{3^r} = 1$ gives $w = 1$, contrary to assumption.

Case (ii): $p > 3$, $r > 0$. Take $J_n = \{1\}$. Then $w + w^{-1} = 0$ gives $w^2 = -1$ and so $w^4 = 1$, while $1 + w + w^{-1} = 0$ gives $w^3 = 1$. Both imply (with $w^{p^r} = 1$) that $w = 1$, a contradiction.

Case (iii): $p = 2$, $r > 4$. Take

$$J_n = \{1,2,2^2,\dots,2^{r-1}\}.$$

Then $m = 2^{r-1} \in J_n$, and condition (1) requires that

$$2|J_n| = 2r \neq 2^{r-1}$$

which is so if $r > 4$. Therefore we want to show (2); equivalently, that

$$w + w^2 + w^4 + \dots + w^{2^{r-1}} + w^{-2^{r-2}} + \dots + w^{-1} = 0 \textit{ or } -1 \tag{3}$$

does not hold. Now w must satisfy exactly one of the irreducible equations

$$w + 1 = 0, \quad w^2 + 1 = 0, \quad w^4 + 1 = 0, \quad \dots, \quad w^{2^{r-1}} + 1 = 0.$$

If $w = -1$ then (3) obviously does not hold unless $r = 2$. If $w^2 = -1$, then clearly (3) does not hold unless $r = 2$ or 3. Suppose then that $w^{2^j} + 1 = 0$ for some $1 < j \leq r - 1$. Then if (3) holds,

$$w + w^2 + w^4 + \dots + w^{2^{j-1}} + w^{-2^{j-1}} + \dots + w^{-1}$$

is equal to an integer, giving for w an equation of degree 2^j with integer coefficients and distinct from $w^{2^j} + 1 = 0$, which is impossible.

Case (iv): $n = 16$. Take $J_n = \{1,3,4\}$, and consider

$$S = w + w^3 + w^4 + w^{-4} + w^{-3} + w^{-1}.$$

If $w = -1$ then $S = -2$, if $w^2 = -1$ then

$$S = (w + w^{-3})(1 + w^2) + w^4 + w^{-4} = 2,$$

if $w^4 = -1$ then

$$S = (w^{-1} + w^{-3})(w^4 + 1) + w^4 + w^{-4} = -2,$$

and if $w^8 = -1$ then

$$S = w + w^3 - w^5 - w^7$$

which is certainly not an integer. Hence condition (2) is satisfied in all four cases. Condition (1) is satisfied too since $|J_n| = 3$.

Thus we have a cyclic D_n with $D_n\overline{D_n} \neq 0$ for all prime powers except 2, 3, 4 and 8.

Now suppose we have already found a suitable J_d for some proper divisor d of $n = dq$, $q > 1$. Let $w^n = 1$; then w^q is a dth root of unity, and we may take

$$J_n = qJ_d = \{qx | x \in J_d\}$$

to satisfy condition (2) for every w such that $w^n = 1$. Furthermore $n/2 \in J_n$ if and only if $d/2 \in J_d$, and since in Case (iii) we had $2|J_n| < n/2$ and in Case (iv) we had $2|J_n| < n/2 - 1$, condition (1) will also be satisfied by the above construction. It follows that there exists a cyclic D_n with $D_n\overline{D_n} \neq 0$ for every n which has a prime divisor $p > 3$, or for $n = 2^r3^s$ with $s > 1$, or for $n = 2^r$ or $n = 2^r{\cdot}3$ with $r > 3$. So we have settled all cases except $n = 2$, 3, 4, 6, 8, 12, 24. For 2, 3, 4 and 6 the problem has already been settled by previous solvers, so the only cases left are 8, 12 and 24.

To produce a suitable example for these values, take a D_n with $d_{i,i+2} = d_{i+2,i} = -1$, $i = 1,\ldots,n-2$, all other entries being +1. These D_n and the corresponding $\overline{D_n}$ can be evaluated explicitly and give non-zero values whenever $n \equiv 0$ or 8 (mod 12) [see Problem 1222, this issue]. In the particular cases of 8, 12 and 24 the values are easily calculated to be

$$D_8 = \overline{D_8} = -256,$$
$$D_{12} = -3{\cdot}2^{12}, \quad \overline{D_{12}} = -2^{12},$$
$$D_{24} = -5{\cdot}2^{24}, \quad \overline{D_{24}} = -3{\cdot}2^{24}.$$

Neither of these methods gives an example for $n = 6$.

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1077*. [1985: 249] *Proposed by Jack Garfunkel, Flushing, N.Y.*

For $i = 1,2,3$ let C_i be the center and r_i the radius of the Malfatti circle nearest A_i in triangle $A_1A_2A_3$. Prove that

$$A_1C_1 \cdot A_2C_2 \cdot A_3C_3 \geq \frac{(r_1 + r_2 + r_3)^3 - 3r_1r_2r_3}{3} .$$

When does equality occur?

Editor's comment:

There have been no solutions to this problem submitted as yet.

* * *

1081. [1985: 288] *Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.*

For a given integer $b > 1$, evaluate

$$\int_0^\infty \left\lfloor \log_b \left\lfloor \frac{\lceil x \rceil}{x} \right\rfloor \right\rfloor dx .$$

(The *floor* of x, denoted by $\lfloor x \rfloor$, is the largest integer $\leq x$; and the *ceiling* of x, denoted by $\lceil x \rceil$, is the smallest integer $\geq x$.)

Solution by Zvi Margaliot, Grade 11, A.B. Lucas Secondary School, London, Ontario.

Let's denote

$$C(x) = \left\lfloor \frac{\lceil x \rceil}{x} \right\rfloor ,$$

$$D(x) = \log_b \left\lfloor \frac{\lceil x \rceil}{x} \right\rfloor ,$$

and

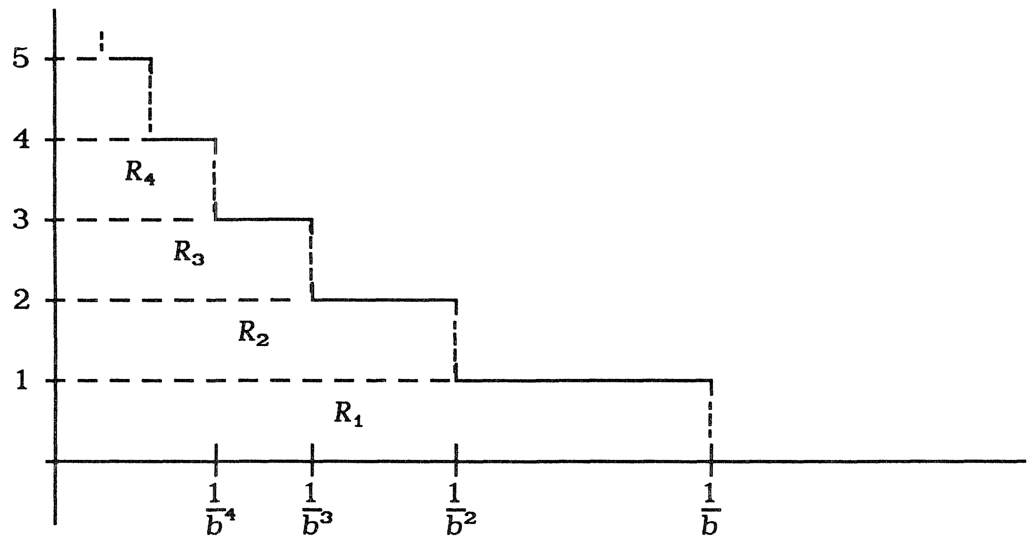
$$E(x) = \left\lfloor \log_b \left\lfloor \frac{\lceil x \rceil}{x} \right\rfloor \right\rfloor .$$

Since $C(x) = 1$ for $x \geq 1$, we observe that $D(x) = 0$ for all $x \geq 1$. Therefore all we have to consider is $0 \leq x < 1$.

For $x = b^{-1}, b^{-2}, \ldots, b^{-n}$, $E(x) = 1,2,\ldots,n$ respectively, since $\lceil b^{-k} \rceil = 1$, $\left\lfloor \frac{1}{b^{-k}} \right\rfloor = b^k$ and $\lfloor \log_b b^k \rfloor = k$. Thus

$$\text{for } b^{-1} < x \leq 1, \quad E(x) = 0,$$
$$\text{for } b^{-2} < x \leq b^{-1}, \quad E(x) = 1,$$
$$\text{for } b^{-3} < x \leq b^{-2}, \quad E(x) = 2, \quad \text{etc},$$

which enables us to sketch $E(x)$:



The area to be calculated will be divided into rectangles $R_1, R_2, \ldots$ as shown. Hence

$$\int_0^\infty E(x)dx = \sum_{i=1}^{\infty} A_i$$

where A_i is the area of R_i. Since $A_i = b^{-i}$ and $b > 1$,

$$\int_0^\infty \left\lfloor \log_b \left\lfloor \frac{[x]}{x} \right\rfloor \right\rfloor = \frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^3} + \ldots = \frac{\frac{1}{b}}{1 - \frac{1}{b}} = \frac{1}{b - 1}\ .$$

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; PAT SURRY, Grade 11, A.B. Lucas Secondary School, London, Ontario; and the proposer.

Gibbs (and probably others) observed that b *need not be an integer but only a real number greater than 1.*

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1082. [1985: 288] *Proposed by O. Bottema, Delft, The Netherlands.*

The midpoints of the edges A_3A_4, A_4A_1, A_1A_2 of a tetrahedron $A_1A_2A_3A_4$ are B_2, B_3, B_4, respectively; ℓ_1 is the line through A_1 parallel to A_2A_3; and ℓ_2, ℓ_3, ℓ_4 are the lines A_2B_2, A_3B_3, A_4B_4, respectively.

(a) Show that the four lines ℓ_i have two conjugate imaginary transversals t and t'.

(b) If S_i is the intersection of t and ℓ_i, and S_i' that of t' and ℓ_i $(i = 1, 2, 3, 4)$, show that S_i and S_i' are equianharmonic quadruples of points.

Solution by the proposer.

We introduce homogeneous barycentric point coordinates x, y, z, w with respect to the tetrahedron. Then $A_1 = (1,0,0,0)$, etc. The plane at infinity has the equation $x + y + z + w = 0$; hence the point at infinity on A_2A_3 is $(0,1,-1,0)$. Furthermore $B_2 = (0,0,1,1)$, $B_3 = (1,0,0,1)$, $B_4 = (1,1,0,0)$. For the lines ℓ_i we obtain the following (Plücker) line-coordinates p_{ij}:

	p_{12}	p_{13}	p_{14}	p_{34}	p_{42}	p_{23}
ℓ_1	1	−1	0	0	0	0
ℓ_2	0	0	0	0	−1	1
ℓ_3	0	−1	0	1	0	0
ℓ_4	0	0	−1	0	1	0

Two lines p_{ij} and q_{ij} intersect if

$$p_{12}q_{34} + p_{13}q_{42} + p_{14}q_{23} + p_{34}q_{12} + p_{42}q_{13} + p_{23}q_{14} = 0.$$

Hence q_{ij} intersects the four lines ℓ_i if

$$q_{34} - q_{42} = 0, \quad -q_{13} + q_{14} = 0, \quad -q_{42} + q_{12} = 0, \quad -q_{23} + q_{13} = 0.$$

Hence

$$q_{34} = q_{42} = q_{12}\ , \quad q_{13} = q_{14} = q_{23}\ ,$$

or

$$q_{12} = u, \quad q_{13} = v, \quad q_{14} = v, \quad q_{34} = u, \quad q_{42} = u, \quad q_{23} = v.$$

The fundamental equation $\Sigma\, q_{12}q_{34} = 0$ gives us

$$u^2 + uv + v^2 = 0,$$

and therefore

$$u/v = \frac{1}{2}(-1 \pm i\sqrt{3}).$$

These numbers are the imaginary cubic roots of 1. Hence the resulting lines t and t' are conjugate imaginary transversals.

If we introduce the number $\omega = \frac{1}{2}(1 + i\sqrt{3})$, with $\omega^3 = -1$, our roots are $-\omega$ and $-\omega^{-1}$. For one of the transversals, t say, we obtain

$$q_{12} = -\omega, \quad q_{13} = 1, \quad q_{14} = 1, \quad q_{34} = -\omega, \quad q_{42} = -\omega, \quad q_{23} = 1,$$

with $\omega^2 - \omega + 1 = 0$. The intersections of t and ℓ_i are those of t and the planes $w = 0$, $x = 0$, $y = 0$, $z = 0$. Hence

$$S_1 = (1,\omega,-\omega,0),\quad S_2 = (0,-\omega,1,1),\quad S_3 = (\omega,0,1,\omega),\quad S_4 = (1,1,0,\omega).$$

A parameter representation of t is given by

$$x = \lambda,\quad y = \omega(\lambda - 1),\quad z = 1 - \omega\lambda,\quad w = 1,$$

where S_1, S_2, S_3, S_4 correspond to $\lambda = \infty$, $\lambda = 0$, $\lambda = 1$, $\lambda = \omega^{-1}$. Hence the cross ratio (S_1,S_2,S_3,S_4) is equal to $(\infty,0,1,\omega^{-1}) = \omega^{-1}$, which shows that the quadruple of points is equianharmonic; the same holds for the set S_1', S_2', S_3', S_4'.

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1083*. [1985: 288] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Consider the double inequality

$$\frac{2}{\sqrt{3}}\Sigma \sin A \le \Sigma \cos\frac{B-C}{2} \le \frac{2}{\sqrt{3}}\Sigma \cos\frac{A}{2},$$

where the sums are cyclic over the angles A, B, C of a triangle. The left inequality has already been established in this journal (Problem 613 [1982: 55, 67, 138]). Prove or disprove the right inequality.

Solution by Vedula N. Murty, Pennsylvania State University, Middletown, Pennsylvania.

Let

$$\alpha = \frac{\pi - A}{2},\quad \beta = \frac{\pi - B}{2},\quad \gamma = \frac{\pi - C}{2}.$$

Then α, β, γ represent the angles of a triangle. Moreover

$$\Sigma \cos\frac{B-C}{2} = \Sigma\cos(\beta - \gamma),$$

$$\Sigma \cos\frac{A}{2} = \Sigma \sin\alpha.$$

Hence the inequality in question is established if we prove

$$\Sigma\cos(\beta - \gamma) \le \frac{2}{\sqrt{3}}\Sigma\sin\alpha. \tag{1}$$

Using the relations given in *Crux* [1982: 67] we have

$$\Sigma\cos(\beta - \gamma) = \frac{x^2 + y^2 + 2x - 2}{2},$$

$$\Sigma\sin\alpha = y,$$

where $x = r/R$ and $y = s/R$, r, R, and s being the inradius, circumradius, and semiperimeter of a triangle with angles α, β, γ. Thus (1) is established if we show that

$$\frac{x^2 + y^2 + 2x - 2}{2} \le \frac{2y}{\sqrt{3}},$$

or

$$\left(x + \frac{5}{2}\right)\left(x - \frac{1}{2}\right) + \left(y + \frac{\sqrt{3}}{6}\right)\left(y - \frac{3\sqrt{3}}{2}\right) \le 0. \tag{2}$$

It is well known that $0 < x \le \frac{1}{2}$ and $0 < y \le \frac{3\sqrt{3}}{2}$, and hence (2) holds, with equality if and only if $\alpha = \beta = \gamma$. Thus the required inequality holds with equality if and only if $A = B = C$.

Also solved by LEON BANKOFF, Los Angeles, California; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.

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1084. [1985: 288] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Prove that every symmetrical fourth-order magic square can be written as

$F + w + y$	$F - w + z$	$F - w - z$	$F + w - y$
$F - x - y$	$F + x - z$	$F + x + z$	$F - x + y$
$F + x - y$	$F - x - z$	$F - x + z$	$F + x + y$
$F - w + y$	$F + w + z$	$F + w - z$	$F - w - y$

Solution by the proposer.

The algebraic form

$F + A$	$F + B$	$F + C$	$F + D$
$F + J$	$F + K$	$F + L$	$F + M$
$F - M$	$F - L$	$F - K$	$F - J$
$F - D$	$F - C$	$F - B$	$F - A$

(*)

is symmetrical, and becomes a magic square with magic sum $4F$ only if

$$A + D = -B - C \tag{1}$$

$$K + L = -J - M \tag{2}$$

$$A - D = -J + M \tag{3}$$

$$K - L = -B + C. \tag{4}$$

By (1) and (3)

$$A = \frac{1}{2}(-B - C - J + M) \tag{5}$$

$$D = \frac{1}{2}(-B - C + J - M) \tag{6}$$

while by (2) and (4)

$$K = \frac{1}{2}(-B + C - J - M) \tag{7}$$

$$L = \frac{1}{2}(B - C - J - M). \tag{8}$$

Comparing (*) to the square in the problem proposal yields the equations

$$B = -w + z, \quad C = -w - z, \quad J = -x - y, \quad M = -x + y, \tag{9}$$

and so from (5)-(8)

$$A = w + y, \quad D = w - y, \quad K = x - z, \quad L = x + z. \tag{10}$$

Substitution of (9) and (10) into (*) produces the square in the problem proposal. Also, (9) gives

$$w = -\frac{1}{2}(B + C), \quad z = \frac{1}{2}(B - C), \quad x = -\frac{1}{2}(J + M), \quad y = \frac{1}{2}(-J + M)$$

whose substitution into the square in the problem proposal, using (5)-(8), reproduces (*). Hence the square in the problem proposal is universal, that is, it is a form in which any symmetrical fourth-order magic square can be written.

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<u>1085</u>. [1985: 289] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $\sigma_n = A_0A_1\ldots A_n$ be a regular n-simplex in $\mathbb{R}^n$, and let π_i be the hyperplane containing the face $\sigma_{n-1} = A_0A_1\ldots A_{i-1}A_{i+1}\ldots A_n$. If $B_i \in \pi_i$ for $i = 0,1,\ldots,n$, show that

$$\sum_{0\le i<j\le n} |\overrightarrow{B_iB_j}| \ge \frac{n+1}{2}\cdot e,$$

where e is the edge length of σ_n.

Solution by the proposer.

Let O be the origin and let $\vec{X} = \overrightarrow{OX}$ for a point X. Using barycentric coordinates we have

$$\overrightarrow{B_i} = \sum_{k=0}^{n} p_{ik}\,\overrightarrow{A_k}$$

where $p_{ii} = 0$ and $\sum_{k=0}^{n} p_{ik} = 1$. Thus

$$\begin{aligned}
\overrightarrow{A_jA_i}^2 - \overrightarrow{B_iB_j}\cdot\overrightarrow{A_jA_i} &= (\overrightarrow{A_i} - \overrightarrow{A_j} + \overrightarrow{B_i} - \overrightarrow{B_j})(\overrightarrow{A_i} - \overrightarrow{A_j}) \\
&= (\overrightarrow{B_i} - \overrightarrow{A_j} + \overrightarrow{A_i} - \overrightarrow{B_j})(\overrightarrow{A_i} - \overrightarrow{A_j}) \\
&= \left[\sum_{k=0}^{n} p_{ik}\,\overrightarrow{A_k} - \overrightarrow{A_j} + \overrightarrow{A_i} - \sum_{k=0}^{n} p_{jk}\,\overrightarrow{A_k}\right](\overrightarrow{A_i} - \overrightarrow{A_j}) \\
&= \left[\sum_{k=0}^{n} p_{ik}(\overrightarrow{A_k} - \overrightarrow{A_j}) + \sum_{k=0}^{n} p_{jk}(\overrightarrow{A_i} - \overrightarrow{A_k})\right](\overrightarrow{A_i} - \overrightarrow{A_j}) \\
&= \sum_{\substack{k=0\\k\neq j}}^{n} p_{ik}\,\overrightarrow{A_jA_k}\cdot\overrightarrow{A_jA_i} + \sum_{\substack{k=0\\k\neq i}}^{n} p_{jk}\,\overrightarrow{A_iA_k}\cdot\overrightarrow{A_iA_j}. \qquad (1)
\end{aligned}$$

But $|\overrightarrow{A_rA_s}| = e$ and $\angle A_rA_sA_t = 60^\circ$ for distinct r, s, t; thus (1) becomes

$$\begin{aligned}
e^2 - \overrightarrow{B_iB_j}\cdot\overrightarrow{A_jA_i} &= e^2\cos 60^\circ\cdot\sum_{\substack{k=0\\k\neq i,j}}^{n} p_{ik} + e^2\cos 60^\circ\cdot\sum_{\substack{k=0\\k\neq i,j}}^{n} p_{jk} \\
&= \frac{e^2}{2}(1 - p_{ij} + 1 - p_{ji})
\end{aligned}$$

so that

$$\overrightarrow{B_iB_j}\cdot\overrightarrow{A_jA_i} = \frac{e^2}{2}(p_{ij} + p_{ji}).$$

Using

$$\overrightarrow{B_iB_j}\cdot\overrightarrow{A_jA_i} \le |\overrightarrow{B_iB_j}|\cdot|\overrightarrow{A_jA_i}| = e|\overrightarrow{B_iB_j}|$$

we have

$$|\overrightarrow{B_iB_j}| \ge \frac{e}{2}(p_{ij} + p_{ji}),$$

and hence

$$\begin{aligned}
\sum_{0\le i<j\le n} |\overrightarrow{B_iB_j}| &\ge \frac{e}{2}\sum_{0\le i<j\le n}(p_{ij} + p_{ji}) \\
&= \frac{e}{2}(n+1).
\end{aligned}$$

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1086. [1985: 289] *Proposed by M.S. Klamkin, University of Alberta.*

The medians of an n-dimensional simplex $A_0A_1\ldots A_n$ in $\mathbb{R}^n$ intersect at the centroid G and are extended to meet the circumsphere again in the points $B_0,B_1,\ldots,B_n$, respectively.

(a) Prove that

$$A_0G + A_1G + \ldots + A_nG \le B_0G + B_1G + \ldots + B_nG.$$

(b)* Determine all other points P such that

$$A_0P + A_1P + \ldots + A_nP \le B_0P + B_1P + \ldots + B_nP.$$

I. *Solution and generalization of (a) by George Tsintsifas, Thessaloniki, Greece.*

Let P be a point which is inside the simplex and is also in the (closed) ball with diameter OG, where O is the circumcenter of the simplex. Let B_i denote the intersection of A_iP with the circumsphere. We prove that

$$A_0P + A_1P + \ldots + A_nP \le B_0P + B_1P + \ldots + B_nP.$$

In particular, putting $P = G$ we get (a).

From the triangle OPG we get

$$(OG)^2 \ge (OP)^2 + (PG)^2,$$

so

$$(n + 1)^2(OG)^2 \ge (n + 1)^2((OP)^2 + (PG)^2). \tag{1}$$

Leibniz's formula for the polar moment of inertia asserts that for any point X,

$$\sum_{i=1}^{n+1} \lambda_i(A_iX)^2 = (QX)^2 + \sum_{j<i}^{1,n+1} \lambda_i\lambda_j(A_iA_j)^2$$

where Q is a point with barycentric coordinates $\lambda_1,\lambda_2,\ldots,\lambda_{n+1}$. We put Q = G and X = O. Leibniz's formula is transformed into

$$R^2 = (OG)^2 + \sum_{j<i}^{1,n+1} (A_iA_j)^2/(n + 1)^2, \tag{2}$$

where R is the circumradius. Therefore from (1) and (2) we have

$$(n + 1)^2R^2 - \sum_{j<i}^{1,n+1} (A_iA_j)^2 \ge (n + 1)^2((OP)^2 + (PG)^2).$$

Hence

$$(n+1)^2(R^2 - (OP)^2) \geq \sum_{j<i}^{1,n+1} (A_iA_j)^2 + (n+1)^2(PG)^2$$

or

$$(n+1)\mathfrak{D}(P) \geq (n+1)(PG)^2 + \sum_{j<i}^{1,n+1} (A_iA_j)^2/(n+1), \tag{3}$$

where $\mathfrak{D}(P)$ is the power of the point P with respect to the circumsphere.

Putting $X = P$ and $Q = G$ in Leibniz's formula, we find

$$\sum_{i=1}^{n+1} a_i^2 = (n+1)(PG)^2 + \sum_{j<i}^{1,n+1} (A_iA_j)^2/(n+1), \tag{4}$$

where $a_i = A_iP$. From (3) and (4) we get

$$(n+1)\mathfrak{D}(P) \geq \sum_{i=1}^{n+1} a_i^2 ,$$

or

$$(n+1)^2\mathfrak{D}(P) \geq (n+1) \sum_{i=1}^{n+1} a_i^2 .$$

The well known arithmetic mean and root-mean-square inequality of the numbers $a_1, a_2, \ldots, a_{n+1}$ gives

$$\frac{\sum_{i=1}^{n+1} a_i^2}{n+1} \geq \left[\frac{\sum_{i=1}^{n+1} a_i}{n+1} \right]^2 .$$

Hence

$$(n+1)^2\mathfrak{D}(P) \geq \left[\sum_{i=1}^{n+1} a_i \right]^2 .$$

But from the A.M.-G.M. inequality follows

$$\left[\sum_{i=1}^{n+1} a_i \right]\left[\sum_{i=1}^{n+1} 1/a_i \right] \geq (n+1)^2$$

or

$$\left[\sum_{i=1}^{n+1} a_i \right]\left[\sum_{i=1}^{n+1} 1/a_i \right]\mathfrak{D}(P) \geq \left[\sum_{i=1}^{n+1} a_i \right]^2$$

or

$$\left[\sum_{i=1}^{n+1} 1/a_i \right]\mathfrak{D}(P) \geq \sum_{i=1}^{n+1} a_i . \tag{5}$$

Now, letting $b_i = B_iP$, and bearing in mind that $a_ib_i = \mathfrak{D}(P)$ for all i, from (5) we immediately conclude

$$\sum_{i=1}^{n+1} b_i \geq \sum_{i=1}^{n+1} a_i .$$

II. *Comments by the proposer.*

There is equality in part (a) if and only if all the $|A_i|$ are equal, i.e., if and only if the centroid G coincides with the circumcenter. For $n = 2$ this requires the triangle to be equilateral. For $n \geq 3$, the simplex need not be regular, although for $n = 3$ the tetrahedron will have to be isosceles.

It is to be noted that the inequality for the special case $n = 2$ corresponds to Problem E2959, *American Mathematical Monthly* 89 (1982) 498, proposed by Jack Garfunkel.

It is also to be noted that more generally if one is given n points $A_1, A_2, \ldots, A_n$ lying on an m-dimensional sphere and with centroid G, then

$$A_1G + A_2G + \ldots + A_nG \leq B_1G + B_2G + \ldots + B_nG$$

where B_i is the point where A_iG extended intersects the sphere.

Also solved (part (a)) by the proposer.

A solution to part (a), by P. Boente, appears in the Monthly, *May* 1985, *page* 360, *as a generalization of problem E2959 mentioned above.*

Part (b) remains open. Tsintsifas' generalization of (a) presumably is not what was meant in (b).

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