Mathematical Spectrum

2005/2006 Volume 38 Number 2

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- Pythagoras — couched in mystery?
- Brahmagupta's problems

A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing.

From the Editor

How Unique is Factorization?

I was interested to receive from Mr S. Hayes, a frequent writer to the editorial office, an item on factorization. He generously offered his thoughts for an Editor's column, so what follows is largely from Mr Hayes, with a few additions of my own. In my younger, keener, days, I even wrote a little book on this subject taken from a course of lectures to second year undergraduates, but sadly it bit the dust! So here goes.

A prime number is a positive integer greater than 1 which cannot be factorized into two smaller numbers. Thus the sequence of primes begins as follows: 2, 3, 5, 7, 11, ... However, 4 and 6 are not prime, since $4 = 2 \times 2$ and $6 = 2 \times 3$. We shall rename this defining property of a prime number by saying that it is *irreducible*. The *fundamental theorem of arithmetic* says that every positive integer greater than 1 can be uniquely factorized into primes (or irreducibles). Thus,

$$60 = 2 \times 2 \times 3 \times 5,$$

and this is the only possible factorization into primes.

Now consider the set E of positive *even* integers. In E, 2 is irreducible, as expected, but so are 6, 10, and 30. In fact, in E, the irreducibles are all the numbers which are twice an odd number. Now, in E,

$$180 = 10 \times 2 \times 2 \times 2 = 6 \times 30,$$

and 180 has two factorizations into irreducibles.

Other similar sets of numbers produce the same bizarre effect. For example, in the set of positive integers which are congruent to $1 (\mathrm{mod}\, 3)$, i.e. in the set $\{1, 4, 7, 10, \ldots\}$,

$$220 = 10 \times 22 = 4 \times 55$$

gives two factorizations into irreducibles. In the set of positive integers which are congruent to $1 (\mathrm{mod}\, 4)$, i.e. $\{1, 5, 9, 13, \ldots\}$,

$$441 = 9 \times 49 = 21 \times 21.$$

Admittedly, these two sets of numbers have the defect that they are not closed under addition, and the set of even integers has the defect that the number 1 does not belong to it, but you cannot have everything. Or can you?

Strange things happen in such number sets. For example, in E we can obtain 10 by adding 2s a certain number of times, namely five, but we can no longer say that $10 = 5 \times 2$. Euclid's algorithm for finding highest common factors no longer works. For example, what is the highest common factor of 60 and 22 in E? According to Euclid's algorithm, we make successive divisions as follows:

$$\begin{aligned} 60 &= 2 \times 22 + 16, \\ 22 &= 16 + 6, \\ 16 &= 2 \times 6 + 4, \\ 6 &= 4 + 2, \\ 4 &= 2 \times 2, \end{aligned}$$

and the highest common factor of 60 and 22 should be 2, the last nonzero remainder. But 2 does not divide 22 in E! The numbers 60 and 22 do not have a highest common factor in E. Indeed, isn't there something even more fundamentally wrong with these and all divisions in E? Can you even say that a number divides itself, since there is no number 1?

But can you have number sets which do not have these defects, yet still do not have unique factorization? Try

$$9 = 3 \times 3 = (2 + \sqrt{-5})(2 - \sqrt{-5})$$

in the set of all complex numbers of the form $a + b\sqrt{-5}$, where a and b are integers. It is no longer obvious that 3, $2 + \sqrt{-5}$, and $2 - \sqrt{-5}$ are irreducible, but they are. For that you will need to dig around a bit. It is a pity that the publishers binned that book!

Mathematical Spectrum **Awards for Volume 37**

Prizes have been awarded to the following student readers for contributions in Volume 37:

Slobodan Radoman

for the article 'I'm Tired: Can We Stop Now?' (with Paul Belcher);

Jonathan Smith

for the article 'Sorting Sequences' and various contributions.

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £50 for letters, solutions to problems and other items.

When is $\cos\theta = \tan\theta$?
When is $\sin\theta = \cot\theta$?

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Bob Bertuello

From Squares to Circles by Courtesy of Einstein

GUIDO LASTERS and DAVID SHARPE

We all know how to add real numbers. But suppose that these real numbers are speeds, with the speed of light denoted by c. Then

$$\tfrac{2}{3}c + \tfrac{3}{4}c = \tfrac{17}{12}c.$$

But, in Einstein's theory of relativity, you cannot have speeds greater than the speed of light. What is to be done? To resolve this difficulty, addition of real numbers is replaced by a new addition, denoted here by $+_c$, and defined by

$$a +_c b = \frac{a+b}{1+ab/c^2},$$

where the addition and multiplication on the right-hand side are the usual addition and multiplication of real numbers. Now

$$\tfrac{2}{3}c +_c \tfrac{3}{4}c = \frac{\frac{17}{12}c}{1+\frac{1}{2}} = \tfrac{17}{18}c,$$

which is still smaller than c. This is true generally. Suppose that $-c < a, b < c$, where we are allowing a and b to be negative to take into account the directional nature of velocity. Then $(c-a)(c-b) > 0$, so that

$$c^2 - (a+b)c + ab > 0,$$

whence

$$(a+b)c < c^2 + ab,$$

so that

$$\frac{a+b}{c^2+ab} < \frac{1}{c},$$

or

$$\frac{a+b}{1+ab/c^2} < c.$$

(Note that $c > a$ and $c > -a$, so that $c > |a| \geq 0$. Also $c > |b| \geq 0$. Hence $c^2 > |ab|$, so that $c^2 > -ab$ and $c^2 + ab > 0$.) Similarly, $(c+a)(c+b) > 0$, so that

$$c^2 + (a+b)c + ab > 0,$$

whence

$$(a+b)c > -(c^2+ab),$$

so that

$$\frac{a+b}{c^2+ab} > -\frac{1}{c},$$

or

$$\frac{a+b}{1+ab/c^2} > -c.$$

Thus, if a and b lie strictly between $-c$ and c, so does $a +_c b$. To put it another way, $+_c$ is a *binary operation on the interval* $(-c, c)$ *of all real numbers strictly between* $-c$ *and* c. There is no reason why c should be the speed of light, it could be any positive real number. It is worth noting that, for fixed a, b,

$$a +_c b \to a + b$$

as $c \to \infty$, so that $+_c$ tends to the usual addition of real numbers as $c \to \infty$.

Suppose that we have three numbers in $(-c, c)$. We change our notation and label them a_1, a_2, a_3. Then

$$\begin{aligned} a_1 +_c (a_2 +_c a_3) &= a_1 +_c \frac{a_2 + a_3}{1 + a_2 a_3/c^2} \\ &= \frac{a_1 + \dfrac{a_2 + a_3}{1 + a_2 a_3/c^2}}{1 + \dfrac{a_1 \dfrac{a_2 + a_3}{1 + a_2 a_3/c^2}}{c^2}} \\ &= \frac{a_1 + a_2 + a_3 + a_1 a_2 a_3/c^2}{1 + (a_1 a_2 + a_1 a_3 + a_2 a_3)/c^2} \\ &= \frac{\dfrac{a_1 + a_2}{1 + a_1 a_2/c^2} + a_3}{1 + \dfrac{\dfrac{a_1 + a_2}{1 + a_1 a_2/c^2} a_3}{c^2}} \\ &= (a_1 +_c a_2) +_c a_3. \end{aligned}$$

This says that this addition is *associative*, i.e. we can add three numbers together and it does not matter which addition we do first. This can be extended to adding any number of terms. Also

$$a +_c b = b +_c a, \quad \text{for all } a, b \in (-c, c),$$

so that this addition is *commutative*. Now,

$$a +_c 0 = a, \quad \text{for all } a \in (-c, c),$$

so that 0 is a *neutral element*, and

$$a +_c (-a) = 0, \quad \text{for all } a \in (-c, c),$$

so that a has a *negative element* $-a$, which is the same negative as with usual addition. These properties can be summarized by saying that $+_c$ makes $(-c, c)$ into an *abelian group*.

Going back for the moment to usual addition, we can obtain all the natural numbers by adding 1 to itself enough times, i.e. $2 = 1 + 1$, $3 = 1 + 1 + 1$, and so on. Let n be any natural number. We shall denote by n_c the result of adding 1 to itself n times in this new addition. Thus we shall need $c > 1$. It turns out that

$$n_c = c\frac{(c+1)^n - (c-1)^n}{(c+1)^n + (c-1)^n}. \tag{1}$$

We can see that this is true if we use induction on n. When $n = 1$, the right-hand side of (1) is equal to 1, and certainly $1_c = 1$. Now, suppose that $n > 1$ and that

$$(n-1)_c = c\frac{(c+1)^{n-1} - (c-1)^{n-1}}{(c+1)^{n-1} + (c-1)^{n-1}}.$$

Then

$$\begin{aligned}
n_c &= (n-1)_c +_c 1 \\
&= c\frac{(c+1)^{n-1} - (c-1)^{n-1}}{(c+1)^{n-1} + (c-1)^{n-1}} +_c 1 \\
&= \frac{c\dfrac{(c+1)^{n-1} - (c-1)^{n-1}}{(c+1)^{n-1} + (c-1)^{n-1}} + 1}{1 + \dfrac{c\dfrac{(c+1)^{n-1} - (c-1)^{n-1}}{(c+1)^{n-1} + (c-1)^{n-1}}}{c^2}} \\
&= c\frac{c((c+1)^{n-1} - (c-1)^{n-1}) + ((c+1)^{n-1} + (c-1)^{n-1})}{c((c+1)^{n-1} + (c-1)^{n-1}) + ((c+1)^{n-1} - (c-1)^{n-1})} \\
&= c\frac{(c+1)^{n-1}(c+1) - (c-1)^{n-1}(c-1)}{(c+1)^{n-1}(c+1) + (c-1)^{n-1}(c-1)} \\
&= c\frac{(c+1)^n - (c-1)^n}{(c+1)^n + (c-1)^n}.
\end{aligned}$$

Thus, assuming that the result is true for $n-1$, we have verified that it is also true for n. Since it is true when $n = 1$, induction tells us that it is true for all n.

We could say that $1_c, 2_c, 3_c, \ldots$ are our new natural numbers! We note that

$$n_c = \frac{(1+1/c)^n - (1-1/c)^n}{(1/c)((1+1/c)^n + (1-1/c)^n)},$$

so that, using l'Hôpital's rule, we have

$$\begin{aligned}
\lim_{c\to\infty} n_c &= \lim_{c'\to 0} \frac{(1+c')^n - (1-c')^n}{c'((1+c')^n + (1-c')^n)} \\
&= \lim_{c'\to 0} \frac{n(1+c')^{n-1} + n(1-c')^{n-1}}{(1+c')^n + (1-c')^n + c'(n(1+c')^{n-1} - n(1-c')^{n-1})} \\
&= \frac{2n}{2} \\
&= n.
\end{aligned}$$

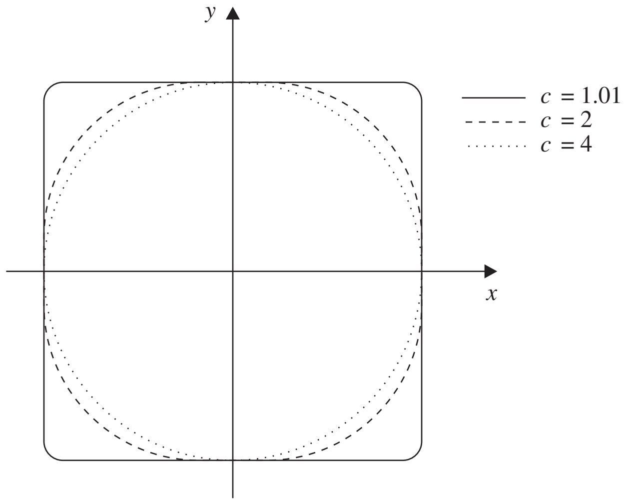

Figure 1

We now come to the point of this article. By analogy with (1), given a function f of a real variable t, for a real number $c > 1$, we can define a new function f_c by

$$f_c(t) = c\frac{(c+1)^{f(t)} - (c-1)^{f(t)}}{(c+1)^{f(t)} + (c-1)^{f(t)}}.$$

As for n_c, $f_c(t) \to f(t)$ as $c \to \infty$. Thus, for example,

$$\cos_c t = c\frac{(c+1)^{\cos t} - (c-1)^{\cos t}}{(c+1)^{\cos t} + (c-1)^{\cos t}},$$

$$\sin_c t = c\frac{(c+1)^{\sin t} - (c-1)^{\sin t}}{(c+1)^{\sin t} + (c-1)^{\sin t}}.$$

We can consider the curve with parametric equations

$$x = \cos_c t, \qquad y = \sin_c t.$$

As $c \to \infty$, this will tend to the curve with parametric equations

$$x = \cos t, \qquad y = \sin t,$$

which is the unit circle. Figure 1 shows the graphs of

$$x = \cos_c t, \qquad y = \sin_c t,$$

for $c = 1.01, 2, 4$. For $c = 1.01$, it is almost a square. With $c = 2$, it is not too far from the unit circle. With our resolution, it is indistinguishable from the unit circle for $c = 4$ and above.

An editor of *Mathematical Spectrum* had fun using DERIVE® to plot the graphs of f_c for functions f other than cosine and sine. You might like to try it!

Guido Lasters *teaches mathematics in a secondary school in Leuven, Belgium, and is a frequent contributor to Mathematical Spectrum.*

David Sharpe *is editor of Mathematical Spectrum. He has officially retired as a lecturer in pure mathematics at the University of Sheffield, but is still requisitioned to give courses there.*

Kaprekar constants

For any four-digit number, Kaprekar (1949) published the following mathematical conjecture:

> Let x be any four-digit number where not all the digits are equal. Order the digits of x from highest to lowest to create a new number $x(\text{large}) = ABCD$ and $x(\text{small}) = abcd$. Subtract $x(\text{large}) - x(\text{small})$ to create a new number y. Repeat the process with y to generate the number z, etc. Kaprekar claimed that the sequence of numbers $x, y, z, \ldots$ always converged to the number 6174. For example, if we try the number 2465, we obtain the sequence
>
> $$4086,\ 8172,\ 7443,\ 3996,\ 6264,\ 4176,\ 6174.$$

Readers may like to prove that the similar process for two-digit numbers always ends in the cycle

$$9,\ 81,\ 63,\ 27,\ 45,\ 9.$$

For three-digit numbers, such a sequence always ends in the single number

$$594.$$

For five-digit numbers, there are three possible end-cycles

1. 85 932, 74 943, 62 964, 71 973, 83 952,
2. 86 922, 75 933, 63 954, 61 974, 82 962,
3. 49 995, 53 955, 59 994.

For six-digit numbers, I have found the following end-cycle:

$$620\,874,\ 851\,742,\ 750\,843,\ 840\,852,\ 860\,832,\ 862\,632,\ 642\,654,\ 420\,876.$$

Shyam Lal College (E)
Delhi University

Vinod Tyagi

From Fermat Numbers to Geometry

MICHAL KŘÍŽEK, FLORIAN LUCA and LAWRENCE SOMER

This is a survey of some of the most beautiful results on Fermat numbers from our book (see reference 1). In 1640, Pierre de Fermat conjectured that all numbers

$$F_m = 2^{2^m} + 1, \quad \text{for } m = 0, 1, 2, \ldots, \tag{1}$$

are prime, which was later found to be incorrect. This statement caused a revolution in number theory and geometry. The first five members of the sequence given by (1) really are prime, i.e.

$$F_0 = 3, \quad F_1 = 5, \quad F_2 = 17, \quad F_3 = 257, \quad F_4 = 65\,537,$$

but in 1742 Leonhard Euler found that $F_5 = 641 \times 6\,700\,417$. The numbers F_m are called *Fermat numbers*. If F_m is prime, we say that it is a *Fermat prime*.

Until the late 18th Century, Fermat numbers were a mathematical curiosity. Interest in Fermat numbers dramatically increased when the German mathematician Carl Friedrich Gauss (1777–1855) quite unexpectedly found, through investigation of the roots of the equation $z^n = 1$, a theorem that expresses an interesting connection between the Euclidean construction of regular polygons (i.e. construction using only a straight edge and a compass) and the Fermat primes. When Gauss was only nineteen, he wrote a short treatise about the division of a circle into 17 equal parts by geometric means. Here he essentially used the fact that 17 is a Fermat prime. A few years later, Gauss stated a necessary and sufficient condition for the Euclidean construction of regular polygons.

Theorem 1 (Gauss.) *There exists a construction of the regular polygon with n sides by ruler and compass if and only if*

$$n = 2^i F_{m_1} F_{m_2} \cdots F_{m_j},$$

where $n \geq 3$, $i \geq 0$, $j \geq 0$, and $F_{m_1}, F_{m_2}, \ldots, F_{m_j}$ are distinct Fermat primes.

The original proof of Theorem 1, over 50 pages long, is, however, not complete. The necessity was not proved until 1837 by Wantzel. By Theorem 1, the regular n-gons with an odd number of sides can be constructed by ruler and compass for

$$n = 3, 5, 15, 17, 51, 85, 255, 257, \ldots, \tag{2}$$

where $15 = 3 \cdot 5$, $51 = 3 \cdot 17$, $85 = 5 \cdot 17$, $255 = 3 \cdot 5 \cdot 17, \ldots$ are products of Fermat primes.

Investigation of the primality of Fermat numbers thus became an important task. In 1855, the German astronomer Thomas Clausen wrote to Gauss that F_6 is a product of two prime factors. In 1878, the French mathematician François Édouard Anatole Lucas proved the following theorem, which became a powerful tool for finding prime factors of further Fermat numbers.

Theorem 2 (Lucas.) *If a prime p divides F_m for some $m > 1$, then there exists a natural number k such that*

$$p = k2^{m+2} + 1.$$

The usefulness of Theorem 2 can be illustrated by an example, which was treated by A. E. Western in 1903. He wanted to know whether F_{18} is composite. The number of its digits is almost 80 000, since

$$\log_{10}(2^{2^{18}}+1)+1 \approx \log_{10} 2^{2^{18}}+1 = 2^{18}\log_{10} 2+1 \approx 78\,914.$$

Western was searching for a natural number k such that $k2^{20}+1 \mid F_{18}$. The divisibility need only be verified for those k for which $k2^{20}+1$ is a prime. Since the numbers $k2^{20}+1$ are composite for all $k \le 13$ except for $k=7$ and $k=13$, Western easily discovered that divisibility is attained when $k=13$.

How can we verify that $p = 13\cdot 2^{20}+1 = 13\,631\,489$ really divides the gigantic Fermat number F_{18}? This can be easily done via the following chain of congruences:

$$\begin{aligned}
2^{2^5} &= 65\,536^2 &&\equiv 1\,048\,261 \pmod p,\\
2^{2^6} &\equiv 1\,048\,261^2 &&\equiv 3\,164\,342 \pmod p,\\
2^{2^7} &\equiv 3\,164\,342^2 &&\equiv 9\,153\,547 \pmod p,\\
&\ \vdots && \ \vdots\\
2^{2^{17}} &\equiv 1\,598\,622^2 &&\equiv 1\,635\,631 \pmod p,\\
2^{2^{18}} &\equiv 1\,635\,631^2 &&\equiv 13\,631\,488 \pmod p.
\end{aligned}$$

Hence, $2^{2^{18}}+1 \equiv 0 \pmod{13\,631\,489}$.

Due to modern mathematical methods and efficient computers, we know at present that

$$F_m \text{ is composite for } 5 \le m \le 32,$$

even though, for F_{14}, F_{20}, F_{22}, and F_{24}, we do not know of a nontrivial factor. The twenty-fourth Fermat number F_{24}, which has over 5 million decimal digits, was shown to be composite in 1999 (see reference 2). This was the biggest computation ever done to obtain a simple 'yes-or-no' answer. It required 10^{17} computer operations to verify the congruence given by (3), below.

Theorem 3 (Pepin's test.) *For $m \ge 1$, the Fermat number F_m is prime if and only if*

$$3^{(F_m-1)/2} \equiv -1 \pmod{F_m}. \tag{3}$$

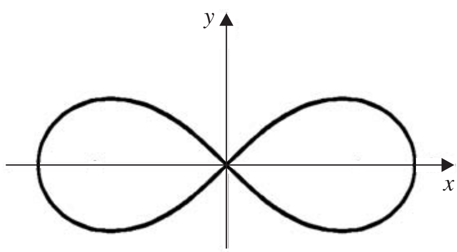

Figure 1 The lemniscate $(x^2+y^2)^2 = \alpha^2(x^2-y^2)$.

$$
\begin{array}{ccccccccccccccccccc}
 & & & & & & & & & 1 & & & & & & & & & \\
 & & & & & & & & 1 & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 0 & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 0 & & 0 & & 0 & & 1 & & & & & \\
 & & & & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & & & & \\
 & & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 1 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 1 & \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \vdots
\end{array}
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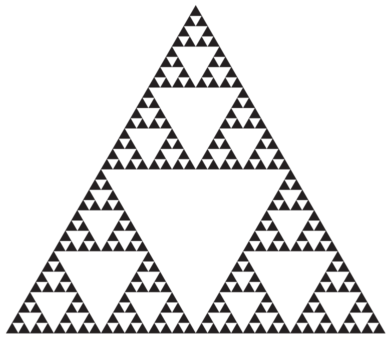

Figure 2 Pascal's triangle modulo 2 and the Sierpiński fractal set.

Pepin, in his original paper of 1877, used the base 5 for $m > 1$ rather than the base 3.

Although many necessary and sufficient conditions for the primality of F_m are known and over 250 factors of the Fermat numbers have been found, no one has been able to discover a general principle that would lead to a definitive answer to the question of whether or not F_4 is the largest Fermat prime. Therefore, up to now, we still do not know whether the current list of constructible regular polygons is complete.

According to reference 3, Gauss also discovered how to divide the lemniscate into five parts of equal length with ruler and compass (i.e. how to construct the division points). This result was later generalized by Niels Henrik Abel. Note that the lemniscate (of Bernoulli) is a curve whose points have a constant product (equal to $\alpha^2/2$) of their distances from two fixed points $(\pm(\alpha/2)\sqrt{2}, 0)$, where $\alpha > 0$ is a real parameter ($\alpha = \sqrt{2}$ in figure 1).

Theorem 4 (Abel.) *The lemniscate can be divided into n equal parts with ruler and compass if $n = 2^i F_{m_1} F_{m_2} \cdots F_{m_j}$, where $i \geq 0$ and $j \geq 0$ are integers and $F_{m_1}, F_{m_2}, \ldots, F_{m_j}$ are distinct Fermat primes.*

For a proof, see reference 3.

We now demonstrate another remarkable connection between number theory and geometry. Write out Pascal's triangle modulo 2 (see figure 2). Then by reading the first 32 rows as the binary expansions of numbers, we get the monotonically increasing sequence

$$1, 3, 5, 15, 17, 51, 85, 255, 257, \ldots .$$

Notice that this is exactly the same sequence as that in (2) (except for the first term), which gives all odd-sided constructible regular polygons in the first 31 terms. Is this not a small miracle? This interesting property of the Fermat numbers was proved by van der Wall (see reference 4). Furthermore, notice that Pascal's triangle modulo 2 has a similar structure to the famous Sierpiński fractal set (see figure 2).

Fermat primes appear not only in connection with constructing regular polygons but also in connection with Heron triangles. Recall that a *Heron triangle* is a triangle such that the lengths of its three sides as well as its area are integers. In the following theorem (see reference 5), we point out an interesting relationship between the Fermat primes and the Heron triangles whose sides are prime powers (see figure 3).

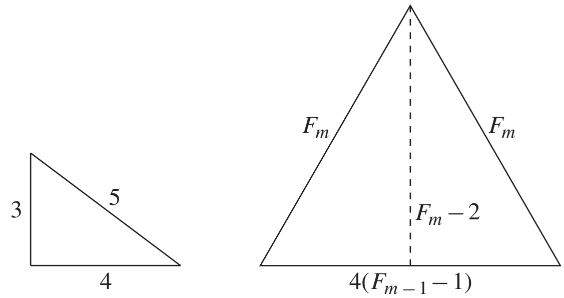

Figure 3 The only possible Heron triangles with sides which are prime powers.

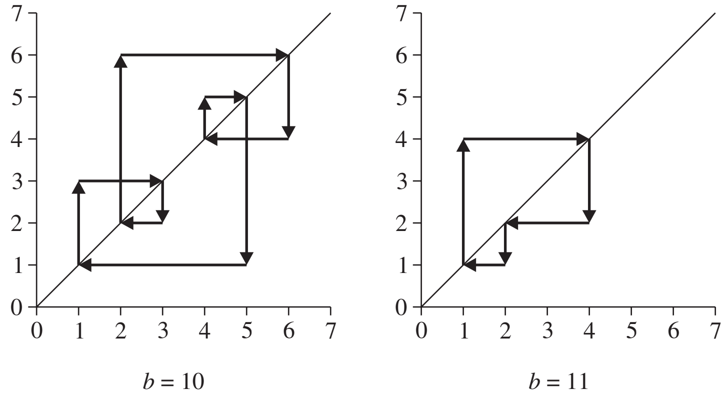

Figure 4 Graphical analysis of $\frac{1}{7}$ for two different bases.

Theorem 5 (Luca.) *Let the lengths of all three sides of a Heron triangle be prime powers. Then the lengths of the sides are either* 3, 4, 5 *or* F_m, F_m, $4(F_{m-1} - 1)$ *for some* $m \geq 1$ *such that* F_m *is prime.*

Further, we introduce another interesting necessary and sufficient condition (see reference 6) for the primality of Fermat numbers, which has a beautiful geometric interpretation. To this end, we first present a graphical procedure that transforms algebraic fractions to images.

Let $b > 1$ and n be positive integers. If r_i is the remainder produced at step i of the base b long division of $1/n$, then the remainder produced at the $(i + 1)$th step obviously satisfies the congruence

$$r_{i+1} \equiv br_i \pmod{n}.$$

Starting with $r_0 = 1$, we get the sequence of remainders $r_0, r_1, r_2, \ldots$ of $1/n$ obtained through long division in base b. We may graphically analyse this fraction. This analysis begins at the point (r_0, r_0), proceeds first vertically, then horizontally, to (r_1, r_1), then moves again vertically, then horizontally, to (r_2, r_2), and continues in this fashion (see figure 4). If the remainder becomes zero at the ith step, then we stop the process. In this way, the sequence of remainders entirely determines the associated graph of the fraction.

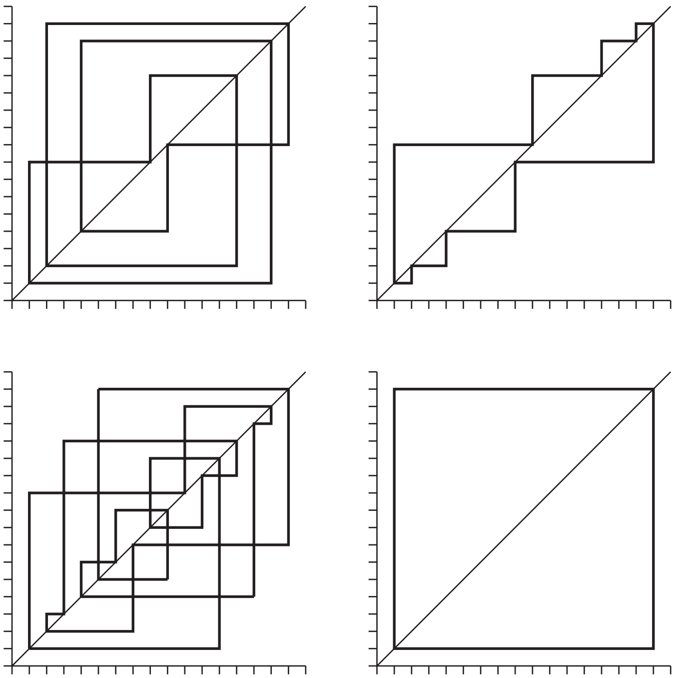

Figure 5 Graphical analysis of $\frac{1}{17}$ for $b = 8, 9, 10, 16$.

Consider, for instance, the fraction $\frac{1}{7}$, which has a base-10 (decimal) expansion of $0.\overline{142\,857}$. The following corresponding sequence of remainders is periodic:

$$\begin{aligned} r_0 &= 1, \\ r_1 &= 3 \equiv 10 \pmod 7, \\ r_2 &= 2 \equiv 30 \pmod 7, \\ r_3 &= 6 \equiv 20 \pmod 7, \\ r_4 &= 4 \equiv 60 \pmod 7, \\ r_5 &= 5 \equiv 40 \pmod 7, \\ r_0 = r_6 &= 1 \equiv 50 \pmod 7, \end{aligned}$$

and so on. In figure 4, we see that the associated graph possesses rotational symmetry with respect to the point $(3.5, 3.5)$ for the base $b = 10$, but the graph is asymmetric for $b = 11$.

An integer $n > 1$ is said to be *perfectly symmetric* if the associated graph of its reciprocal $1/n$ is rotationally symmetric with respect to the point $(n/2, n/2)$ for all bases b such that $b \not\equiv 0, 1 \pmod n$.

Theorem 6 (Jones, Pearce.) *An integer $n > 1$ is perfectly symmetric if and only if $n = 2$ or n is a Fermat prime.*

For the proof, see reference 6. The graphical analysis of the fraction $1/F_m$ for $m = 2$ is illustrated in figure 5.

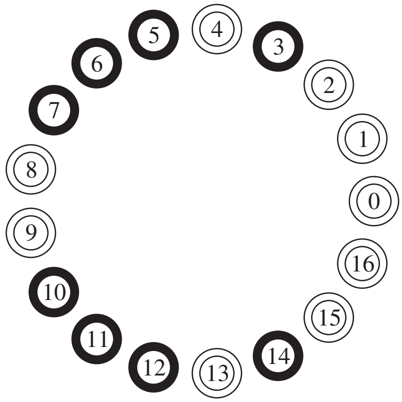

Figure 6 Primitive roots modulo 17 are indicated by a black circle.

Further, let us introduce the *Euler totient function* ϕ. For every $n \in \mathbb{N}$, $\phi(n)$ is defined as the number of all natural numbers not greater than n that are coprime to n, i.e.

$$\phi(n) = \text{card}\{m \in \mathbb{N} \mid 1 \leq m \leq n, \gcd(m, n) = 1\}.$$

Theorem 7 (Euler, Fermat.) *Let* $a, n \in \mathbb{N}$. *Then*

$$a^{\phi(n)} \equiv 1 \pmod{n} \tag{4}$$

if and only if $\gcd(a, n) = 1$.

If $\phi(n)$ is the smallest exponent for which congruence (4) holds, then a is called a *primitive root modulo* n (see figure 6). Let $n > 1$ and a be integers such that $\gcd(a, n) = 1$. If the quadratic congruence

$$x^2 \equiv a \pmod{n}$$

has no solution x, then a is called a *quadratic nonresidue modulo* n.

For an integer $n > 1$ define

$$M(n) = \{a \in \{1, \ldots, n-1\} \mid a \text{ is a primitive root } \pmod{n}\}$$

and

$$K(n) = \{a \in \{1, \ldots, n-1\} \mid \gcd(a, n) = 1 \text{ and } a \text{ is a quadratic nonresidue } \pmod{n}\}.$$

The following necessary and sufficient condition is proved in reference 7.

Theorem 8 (Křížek, Somer.) *The integer* $n \geq 3$ *is a Fermat prime if and only if* n *is odd and* $M(n) = K(n)$.

Assign now to each $n \in \mathbb{N}$ a digraph whose set of vertices is $H = \{0, 1, \ldots, n-1\}$ and for which there is a directed edge from $x \in H$ to $y \in H$ if

$$x^2 \equiv y \pmod{n}.$$

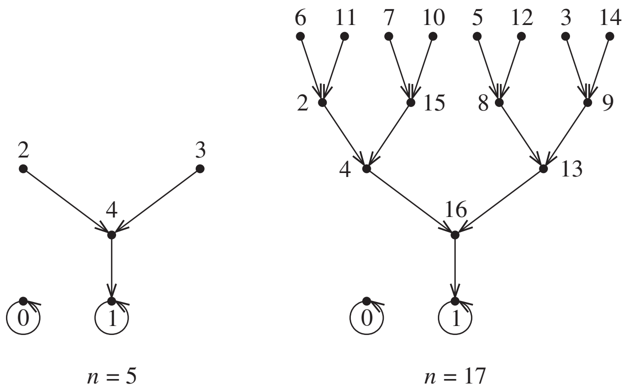

Figure 7 Digraphs corresponding to Fermat primes.

Theorem 9 (Szalay.) *The integer $n \geq 3$ is a Fermat prime if and only if the associated digraph has exactly two components and zero is an isolated fixed point.*

For the proof, see reference 8. This is illustrated in figure 7 for the Fermat primes 5 and 17. All primitive roots modulo a Fermat prime 'sit on the top' of this digraph. According to (3), the number 3 is a primitive root modulo a Fermat prime F_m, $m \geq 1$, so 3 will always sit on the top of the digraph.

Fermat numbers have several useful applications in number theory, e.g. in proving that there exist infinitely many primes and pseudoprimes, and in establishing the existence of Sierpiński numbers (see reference 1). However, there are also more practical applications of F_m. In particular, Fermat numbers are employed in number-theoretic transforms, in binary arithmetic modulo F_m (which leads to fast multiplication of large numbers), in pseudorandom number generators, in hashing schemes, in the chiral Potts model, and in an analysis of bifurcations of the logistic equation which lead to chaos (see reference 1).

Acknowledgements

The authors thank Alena Šolcová for valuable suggestions. This paper was supported by the common Czech–US cooperative research project no. 1P05ME749 of the Ministry of Education of the Czech Republic and Grant no. A 1019201 of the Grant Agency of the Academy of Sciences of the Czech Republic.

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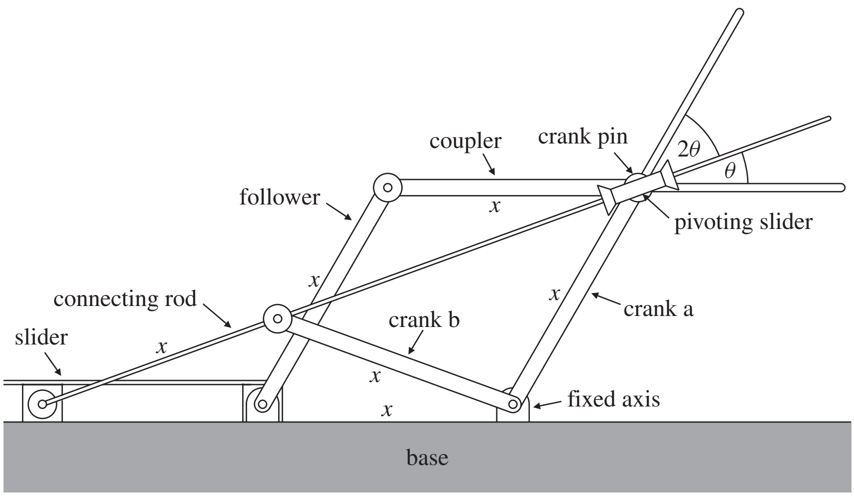
Another device for trisecting an angle
(see also Volume 25, Number 4)



Trisector at angle 3θ position, where x denotes equal lengths.

26 Shull Drive,
Newark, DE 19711, USA

Lyndon O. Barton

Pythagoras — Couched in Mystery?

P. GLAISTER

There are many tales of the life and work of the Pythagoreans. Could this be just one more?

An old problem in calculus is to determine the largest (rectangular) couch that can negotiate a corner in a corridor. Figure 1 shows a corridor of width a which turns a corner to a corridor of width $b \geq a$, and a couch of width $w < a$ and length $l > b$ on its journey around the corner. The angle that the couch makes with the wall is denoted by θ.

If we suppose that the couch is 'telescopic', i.e. of fixed length l and of varying width, then in a typical position the width $w = w(\theta)$ is a function of the angle θ, as shown in figure 1. The couch is positioned so that it touches the corner and both walls, which determines $w(\theta)$. Clearly, as the couch turns the corner, the value of $w(\theta)$ decreases as θ increases, attaining a minimum value, and then increases again. This is shown when the couch moves from the position in figure 1 to that in figure 2. (Note that $w(\theta) \to a$ as $\theta \to 0$ and $w(\theta) \to b$ as $\theta \to \frac{1}{2}\pi$.) Therefore, for a couch of fixed length, the maximum width allowed so that it can completely turn the corner is the minimum value of $w(\theta)$ as θ varies, i.e. $w(\bar{\theta}) = \min_{\theta \in [0,\pi/2]} w(\theta)$, and $\bar{\theta}$ is the angle of inclination of the couch for which w, shown in figures 1 and 2, is smallest. An expression for $w(\theta)$ can be determined by considering the configuration shown in figure 3.

Since

$$\begin{aligned} l\cos\theta &= \mathrm{AC} \\ &= \mathrm{AB} + \mathrm{BC} \\ &= \frac{\mathrm{BE}}{\tan\theta} + b \qquad \text{from } \triangle\,\mathrm{ABE} \\ &= \frac{\mathrm{BD} - \mathrm{ED}}{\tan\theta} + b \\ &= \frac{a - w/\cos\theta}{\tan\theta} + b \qquad \text{from } \triangle\,\mathrm{DFE}, \end{aligned}$$

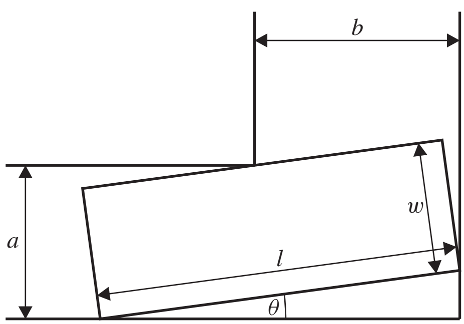


Figure 1

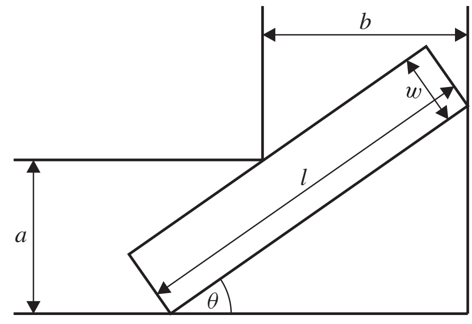


Figure 2

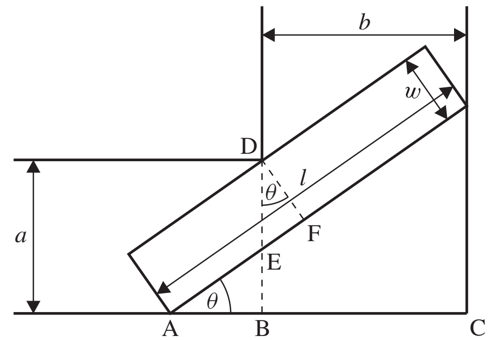


Figure 3

we obtain

$$w = w(\theta) = a\cos\theta + b\sin\theta - l\sin\theta\cos\theta. \tag{1}$$

For a given length, the maximum possible width is therefore found by determining the value $\bar{\theta}$ of θ for which $w(\theta)$ is a minimum. Differentiating (1), setting $w'(\bar{\theta}) = 0$, and simplifying, gives

$$b\cos\bar{\theta} - a\sin\bar{\theta} = l(\cos^2\bar{\theta} - \sin^2\bar{\theta}). \tag{2}$$

In general, for given corridor widths a and b and length of couch l, (2) can be solved for $\bar{\theta}$, possibly graphically, and then the maximum possible width $w(\bar{\theta})$ can be determined from (1). For example, if $a = 3$, $b = 4$, and $l = \sqrt{50}$, as shown in figures 1 and 2, then $\bar{\theta} \approx 40.6°$ and $w(\bar{\theta}) \approx 1.39$.

Now, let us see what the Pythagoreans might have made of this problem, particularly if they had known about calculus.

Pythagoras, particularly, liked to use dimensions which were examples of his triples, for example corridors of width $a = 3$ and $b = 4$, in local units, and always meeting at right-angles with no two narrow corridors meeting one another. He wanted to know how wide his couches, of length $l = 5$ units, could be made so that they could still be moved around the corridors of his dwelling. Without calculus, the Pythagoreans used only trial and error to determine the maximum width allowed, but were disappointed to find that the solution wasn't a whole number. They would have been pleased, however, to learn that a Pythagorean triangle does appear in this problem, which we will reveal shortly.

What they really wanted to know, however, was what dimensions they could make their corridors and length of couches so that they formed a Pythagorean triple, and for which the corresponding maximum allowable width of the couch was integral. We shall also show how this can be achieved.

To begin with, we have from (1) that

$$w(\bar\theta) = 3\cos\bar\theta + 4\sin\bar\theta - 5\sin\bar\theta\cos\bar\theta, \tag{3}$$

where $\bar\theta$ satisfies $w'(\bar\theta) = 0$, i.e.

$$4\cos\bar\theta - 3\sin\bar\theta = 5(\cos^2\bar\theta - \sin^2\bar\theta), \tag{4}$$

from (3) or (2). One way to solve (4) is to use $\cos^2\bar\theta - \sin^2\bar\theta \equiv \cos 2\bar\theta$ and then note that the left-hand side can be written as $R\cos(\bar\theta + \phi)$, where $R\cos\phi = 4$, $R\sin\phi = 3$, i.e. $R = 5$ and $\phi = \tan^{-1}\frac{3}{4}$, so that (4) becomes

$$5\cos\left(\bar\theta + \tan^{-1}\tfrac{3}{4}\right) = 5\cos 2\bar\theta. \tag{5}$$

Clearly the solution $\bar\theta \in [0, \frac{1}{2}\pi]$ of (5) satisfies $\bar\theta + \tan^{-1}\frac{3}{4} = 2\bar\theta$, i.e. $\bar\theta = \tan^{-1}\frac{3}{4}$. (Having found this, an alternative is to rewrite (4) as $\cos\bar\theta(4 - 5\cos\bar\theta) = \sin\bar\theta(3 - 5\sin\bar\theta)$, which is satisfied if $4 - 5\cos\bar\theta = 0 = 3 - 5\sin\bar\theta$, i.e. $\bar\theta = \tan^{-1}\frac{3}{4}$.)

So, for corridors of width $a = 3$ and $b = 4$, and a couch of length $l = 5$, the angle of orientation, $\bar\theta$, when the couch just negotiates the corner is the smaller angle in the $(3, 4, 5)$ Pythagorean triangle. The corresponding maximum allowable width is given by (3) as

$$w(\bar\theta) = 3 \times \tfrac{4}{5} + 4 \times \tfrac{3}{5} - 5 \times \tfrac{3}{5} \times \tfrac{4}{5} = \tfrac{12}{5},$$

which is not an integer, but is rational. Thus, Pythagoras could have scaled-up (or invented new units) so that with $a = 5 \times 3 = 15$, $b = 5 \times 4 = 20$, and $l = 5 \times 5 = 25$, the maximum allowable width is $w = 12$.

So what about the general case? Given a Pythagorean triple (a, b, c), if we suppose that the corridors are of width a and b, with $a \le b$, and the couch is of length $l = c = \sqrt{a^2 + b^2}$, an integer, then the corresponding equations to (3) and (4) are

$$w(\bar\theta) = a\cos\bar\theta + b\sin\bar\theta - c\sin\bar\theta\cos\bar\theta \tag{6}$$

and

$$b\cos\bar\theta - a\sin\bar\theta = c(\cos^2\bar\theta - \sin^2\bar\theta) \tag{7}$$

respectively. Now (7) can be written as

$$\sqrt{a^2 + b^2}\cos\left(\bar\theta + \tan^{-1}\frac{a}{b}\right) = c\cos 2\bar\theta \tag{8}$$

and, since $c = \sqrt{a^2 + b^2}$, the acute angle solution of (8), and hence (7), is $\bar\theta = \tan^{-1}(a/b)$. Alternatively, (7) can be written as $\cos\bar\theta(b - c\cos\bar\theta) = \sin\bar\theta(a - c\sin\bar\theta)$, which is satisfied if $b - c\cos\bar\theta = 0 = a - c\sin\bar\theta$, i.e. $\bar\theta = \tan^{-1}(a/b)$. Either way, the angle of orientation, $\bar\theta$,

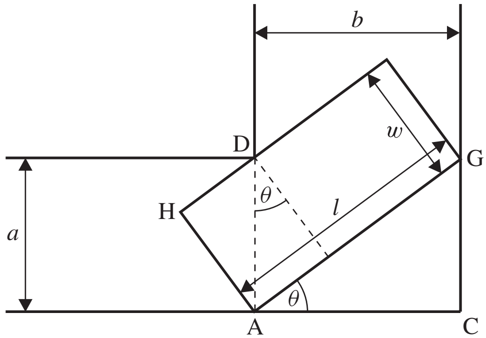


Figure 4

when the couch just negotiates the corner is the smaller angle in the (a, b, c) Pythagorean triangle and, from (6), the corresponding maximum allowable width is

$$w(\bar{\theta}) = a \times \frac{b}{c} + b \times \frac{a}{c} - c \times \frac{a}{c} \times \frac{b}{c} = \frac{ab}{c} = \frac{ab}{\sqrt{a^2+b^2}},$$

which is not necessarily an integer, but is still rational.

By scaling-up again, with the Pythagorean triple (α, β, γ), $\alpha \le \beta$, and corridors of width $a = \alpha\gamma$ and $b = \beta\gamma$, both integers, and a couch of length $l = \sqrt{a^2+b^2} = \sqrt{\alpha^2\gamma^2 + \beta^2\gamma^2} = \gamma\sqrt{\alpha^2+\beta^2} = \gamma^2$, also an integer (with (a, b, c) a Pythagorean triple too), then the angle $\bar{\theta}$ when the couch just negotiates the corner is the smaller angle in the (a, b, c) and (α, β, γ) Pythagorean triangles with $\tan\bar{\theta} = a/b = \alpha/\beta$, and the maximum allowable width is

$$w(\bar{\theta}) = \frac{ab}{l} = \frac{\alpha\gamma \times \beta\gamma}{\gamma^2} = \alpha\beta,$$

which is an integer.

Finally, having determined that the maximum allowable width for a couch of fixed length $l = \sqrt{a^2+b^2}$ is $w = ab/l$, with the angle for which the couch just negotiates the corner as $\tan^{-1}(a/b)$, we see from figure 4 precisely why this solution is found.

In this position, the lower corner of the couch, A, is in line with the corner, D, with

$$l = \mathrm{AG} = \sqrt{\mathrm{AC}^2 + \mathrm{CG}^2} = \sqrt{a^2+b^2},$$

$$\tan\bar{\theta} = \frac{\mathrm{CG}}{\mathrm{AC}} = \frac{\mathrm{AD}}{\mathrm{DG}} = \frac{a}{b},$$

$$w = \mathrm{AH} = \mathrm{AD}\cos\bar{\theta} = a\frac{b}{\sqrt{a^2+b^2}} = \frac{ab}{\sqrt{a^2+b^2}} = \frac{ab}{c}.$$

I wonder if Pythagoras secretly knew this?

Paul Glaister *lectures in mathematics at Reading University, with interests in computational fluid dynamics, numerical analysis, perturbation methods, as well as mathematics and science education. Having just watched his daughter spend countless hours on her first piece of GCSE maths coursework, he realises just how lucky he is to be of a different generation!*

Brahmagupta's Problems, Pythagorean Solutions and Heron Triangles

K. R. S. SASTRY

Dickson, in the preface to his monumental work *History of the Theory of Numbers* (reference 1), began as follows:

> Diophantine analysis was named after the Greek Diophantus, of the 3rd century, who proposed many indeterminate problems in his arithmetic.

In ancient India too, there was interest in the solution of indeterminate equations. Aryabhata (circa 475–500) considered the summation of arithmetic progressions together with its inversion and the solution of the linear indeterminate equation. Brahmagupta (circa 628) went further and gave the solution not only of the linear indeterminate equation but also of the second degree equations

$$x^2 + y^2 = z^2 \quad \text{and} \quad x^2 - dy^2 = \pm 1,$$

where d is not a perfect square (see references 1 and 2). Our present aim is to state two problems given in reference 2 and find an infinity of solutions to the first problem using Pythagorean triples. An extension of the second problem unexpectedly determines the complete set of Heron triangles.

A recollection of known results

It is well known that all primitive solutions, i.e. the natural number triples having greatest common divisor (gcd) 1, of the Pythagorean equation $x^2 + y^2 = z^2$ are given by $(x, y, z) = (m^2 - n^2, 2mn, m^2 + n^2)$, where m and n are relatively prime natural numbers, $m > n$, and exactly one of m and n is odd. From this, we can deduce *all* Pythagorean triples as follows:

$$x = \lambda(m^2 - n^2) \quad y = \lambda(2mn) \quad z = \lambda(m^2 + n^2) \qquad \lambda = 1, 2, 3, \dots. \tag{1}$$

However, sometimes it is advantageous to let both m and n be odd in (1), Specifically, the transformation $m \to m + n$ and $n \to m - n$ applied to (1) interchanges the values x and y and leaves z invariant when the gcd 2 is removed. In other words, the base x and the height y of the Pythagorean triangle are interchanged. The present discussion needs this advantage so, whenever we use (1) in what follows, m and n are just relatively prime natural numbers.

Two problems from Brahmagupta's Kutakhādyaka (Algebra)

In reference 2 (p. 159), Smith reproduced the following problems to illustrate Brahmagupta's application of algebra in the solution of word problems.

Problem 1 On the top of a certain hill live two ascetics. One of them, being a wizard, travels through the air. Springing from the summit of the mountain he ascends to a certain elevation and proceeds by an oblique descent diagonally to a neighbouring town. The other, walking down the hill, goes by land to the same town. Their journeys are equal. I desire to know the distance of the town from the hill, and how high the wizard rose.

Problem 2 A bamboo 18 cubits high was broken by the wind. Its tip touched the ground 6 cubits from the root. Tell the lengths of the segments of the bamboo.

Evidently, problem 1 has an infinite number of *Pythagorean* solutions. Problem 2 has a single solution because of the numerical data. Furthermore, even in general terms, it is simple to solve. Hence, we will also consider a further extension of problem 2, which leads to a new determination of Heron triangles.

Solution of problem 1

Figure 1 shows the hill by the vertical line segment AB, the town of destination C, and the highest point of ascent D. Smith quoted a commentator who assumed the second ascetic's journey along BA and solved a numerical example. In this solution, we follow the commentator. However, the Editor suggested a more realistic interpretation of the second ascetic's journey along a diagonal path down the hill. We give this as an exercise at the end of this section for the reader to solve. We are looking for Pythagorean solutions, i.e. natural number solutions. Hence, $\triangle$ABC and $\triangle$ADC must both be Pythagorean, i.e. they have sides of integral length, to be determined, subject to the constraint

$$\mathrm{BD} + \mathrm{DC} = \mathrm{BA} + \mathrm{AC}. \tag{2}$$

Invoking (1) we have, for relatively prime pairs (m, n) and (u, v), $m > n$, $u > v$,

$$\begin{aligned} \triangle\mathrm{ABC}: &\quad \mathrm{AB} = \lambda(m^2 - n^2), \quad \mathrm{AC} = \lambda(2mn), \quad \mathrm{BC} = \lambda(m^2 + n^2), \\ \triangle\mathrm{ADC}: &\quad \mathrm{AD} = \mu(u^2 - v^2), \quad \mathrm{AC} = \mu(2uv), \quad \mathrm{DC} = \mu(u^2 + v^2). \end{aligned}$$

From $\mathrm{AC} = \lambda(2mn) = \mu(2uv)$, we may take

$$\lambda = uv, \qquad \mu = mn. \tag{3}$$

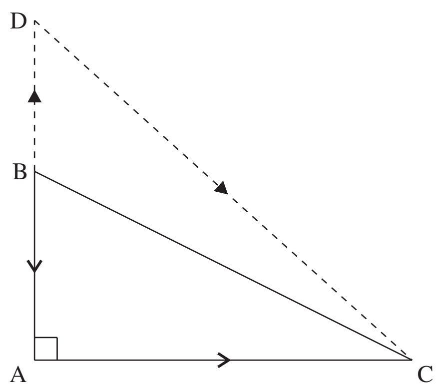


Figure 1 Two equal journeys: BDC and BAC.

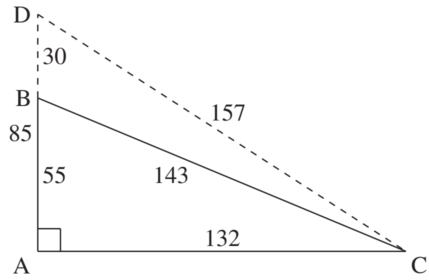


Figure 2 The numerical example $m = 3$ and $n = 2$ gives $30 + 157 = 55 + 132$.

By (2), we obtain $(\mathrm{AD} - \mathrm{AB}) + \mathrm{DC} = \mathrm{AB} + \mathrm{AC}$, which gives

$$\mu(u^2) = \lambda(m^2 + mn - n^2),$$
$$u(mn) = v(m^2 + mn - n^2) \quad \text{(from (3))}.$$

We may now choose $u = m^2 + mn - n^2$, and $v = mn$ which hold for all choices of m and n. These produce Pythagorean triangles ABC and ADC with sides proportional to

$$\begin{aligned} \mathrm{AB} &= (m^2 - n^2)(m^2 + mn - n^2), \\ \mathrm{AC} &= 2mn(m^2 + mn - n^2), \\ \mathrm{BC} &= (m^2 + n^2)(m^2 + mn - n^2), \\ \mathrm{DC} &= m^4 + 2m^3n - 2mn^3 + n^4, \\ \mathrm{AD} &= (m^2 - n^2)(m^2 + 2mn - n^2). \end{aligned} \tag{4}$$

From (4), it is a simple matter to give an infinite number of solutions to problem 1 as follows:

$$\mathrm{AC} = 2mn(m^2 + mn - n^2), \qquad \mathrm{BD} = mn(m^2 - n^2).$$

Incidentally, we note that $\triangle\mathrm{DBC}$ has sides

$$\mathrm{BD} = mn(m^2 - n^2), \qquad \mathrm{BC} = (m^2 + n^2)(m^2 + mn - n^2), \qquad \mathrm{DC} = m^4 + 2m^3n - 2mn^3 + n^4$$

and area given by

$$\tfrac{1}{2}(\mathrm{BD})(\mathrm{AC}) = m^2n^2(m^2 - n^2)(m^2 + mn - n^2),$$

which are all natural numbers, i.e. $\triangle\mathrm{DBC}$ is a *Heron triangle*. However, because of the constraint (2), these do not give *all* Heron triangles. For a numerical illustration, we may put $m = 3$ and $n = 2$ in (4), see figure 2. Furthermore, the Heron triangle DBC generated is (30, 143, 157; 1980).

As the Editor pointed out, in the commentator's solution, it looks as though the second ascetic burrows into the hill! Hence, let us assume that the second ascetic walks down the hill along a straight path (shown in figure 3 by the diagonal BE). As mentioned earlier, we leave it to the reader to determine the Pythagorean triangles ABE and ADC with the property

$$\mathrm{BD} + \mathrm{DC} = \mathrm{BE} + \mathrm{EC}$$

and, hence, to give an infinite number of solutions to problem 1.

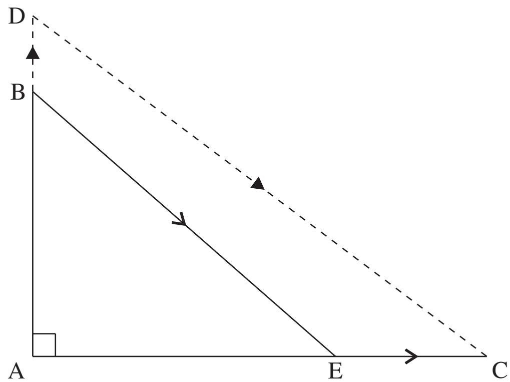


Figure 3 Realistic interpretation of problem 1: $BD + DC = BE + EC$.

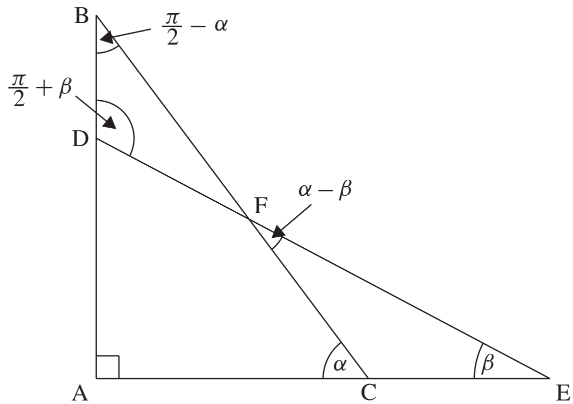


Figure 4 A bamboo tree broken in two ways.

An extension of problem 2

In the following extension of problem 2 we have a (mathematical) bamboo broken in more than one way. Figure 4 shows the bamboo tree broken in two ways. Firstly, the tree ABC is broken at B with the tip touching the ground at C and, secondly, the tree ADE is broken at D with the tip touching the ground at E. To find the lengths of the various segments of the broken bamboo, we need to determine the Pythagorean triangles ABC and ADE. Again, invoking (1), we have

$$\begin{aligned} &\mathrm{AB} = \lambda(p^2 - q^2), \quad \mathrm{AC} = \lambda(2pq), \quad \mathrm{BC} = \lambda(p^2 + q^2), \qquad p > q, \\ &\mathrm{AD} = \mu(r^2 - s^2), \quad \mathrm{AE} = \mu(2rs), \quad \mathrm{DE} = \mu(r^2 + s^2), \qquad r > s. \end{aligned}$$

(i) The original length of the bamboo is the same. This gives $\mathrm{AB} + \mathrm{BC} = \mathrm{AD} + \mathrm{DE}$ or $\lambda(p^2) = \mu(r^2)$. Hence, we may choose $\lambda = r^2$ and $\mu = p^2$ which hold for all

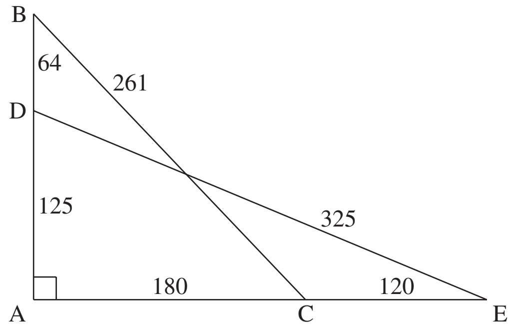


Figure 5 The numerical example $p = 5$, $q = 2$, $r = 3$, and $s = 2$ gives $\mathrm{AB}+\mathrm{BC} = \mathrm{AD}+\mathrm{DE} = 450$.

numerical values of p, q, r and s. This yields $\triangle\mathrm{ABC}$ and $\triangle\mathrm{ADE}$ with

$$\begin{aligned} \mathrm{AB} &= r^2(p^2 - q^2), \quad \mathrm{AC} = 2pqr^2, \quad \mathrm{BC} = r^2(p^2 + q^2), \\ \mathrm{AD} &= p^2(r^2 - s^2), \quad \mathrm{AE} = 2p^2rs, \quad \mathrm{DE} = p^2(r^2 + s^2), \end{aligned} \tag{5}$$

$$\mathrm{AB} + \mathrm{BC} = \mathrm{AD} + \mathrm{DE} = 2p^2r^2.$$

(ii) The assumption that wind broke the bamboo as shown in figure 4 implies that $\mathrm{AD} < \mathrm{AB}$. This yields the following constraint on p, q, r and s:

$$ps > qr. \tag{6}$$

For a numerical illustration, we may put $p = 5$, $q = 2$, $r = 3$, and $s = 2$ in (5). Moreover, $ps = 10 > qr = 6$ satisfies (6). Then the height of the original bamboo is 450. The lengths of the various broken parts are displayed in figure 5. The reader may like to try to solve the following problem.

> A bamboo 100 units tall is broken by the wind. In how many ways can the wind break it in the manner of the extension of problem 2 so that a Pythagorean triangle is formed?

Heron triangles from the twice-broken bamboo

Let us look again at figure 4. The lengths DB and CE are integers. Let F denote the intersection of BC and DE. It is not immediately obvious that BF, DF, FC, and FE all have rational lengths and that the areas of $\triangle\mathrm{DBF}$ and $\triangle\mathrm{CFE}$ are rational too. Indeed they all are rational. In fact, either $\triangle\mathrm{DBF}$ or $\triangle\mathrm{CFE}$ can be used to generate the complete set of Heron triangles. To see this, let $\angle\mathrm{ACB} = \alpha$ and $\angle\mathrm{AED} = \beta$. Then $\angle\mathrm{ABC} = \pi/2 - \alpha$, $\angle\mathrm{BDF} = \pi/2 + \beta$, and $\angle\mathrm{BFD} = \angle\mathrm{CFE} = \alpha - \beta$. Also, from (5) we obtain

$$\sin\alpha = \frac{p^2 - q^2}{p^2 + q^2}, \qquad \cos\alpha = \frac{2pq}{p^2 + q^2}, \qquad \sin\beta = \frac{r^2 - s^2}{r^2 + s^2}, \qquad \cos\beta = \frac{2rs}{r^2 + s^2},$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{2(ps - qr)(pr + qs)}{(p^2 + q^2)(r^2 + s^2)},$$

$$\mathrm{BD} = \mathrm{AB} - \mathrm{AD} = p^2s^2 - q^2r^2.$$

We now consider $\triangle$BDF and apply the sine rule, which gives

$$\frac{\mathrm{BD}}{\sin(\alpha-\beta)}=\frac{\mathrm{DF}}{\cos\alpha}=\frac{\mathrm{BF}}{\cos\beta}.$$

Since BD, $\sin(\alpha-\beta)$, $\cos\alpha$, and $\cos\beta$ are all rational, we see that DF, BF, and the area of $\triangle$DBF (which is equal to $\frac{1}{2}(\mathrm{BD})(\mathrm{BF})(\sin\angle\mathrm{DBF})$) must all be rational. We may now multiply by the least common multiple of the denominators and then divide by the gcd to obtain a new $\triangle$DBF that is similar to the original one. This (new) $\triangle$DBF has integer sides and area, so, it is a Heron triangle. In fact,

$$\mathrm{BD}=(ps-qr)(pr+qs),\qquad \mathrm{DF}=pq(r^2+s^2),\qquad \mathrm{BF}=rs(p^2+q^2),\qquad (7)$$
$$\text{area }\triangle\mathrm{DBF}=pqrs(ps-qr)(pr+qs).$$

The details of deriving (7) are left to the reader. These expressions (7) are precisely the ones obtained by Euler and Schubert, see reference 1. It is known that every Heron triangle is similar to some triangle given by (7). Likewise, we may consider $\triangle$CFE. We then obtain yet another description of Heron triangles as follows:

$$\mathrm{CF}=(p^2+q^2)(r^2-s^2)\qquad \mathrm{FE}=(p^2-q^2)(r^2+s^2)\qquad \mathrm{CE}=2(ps-qr)(pr+qs)$$
$$\text{area }\triangle\mathrm{CFE}=(p^2-q^2)(r^2-s^2)(ps-qr)(pr+qs).$$

Conclusion

Traditionally, Heron triangles have been determined by the *juxtaposition* of two Pythagorean triangles. Strangely, our determination comes as a result of the *intersection* of two Pythagorean triangles. It is remarkable that Heron triangles can be determined in so many different ways! See the references for some diversity in the approach. At the other extreme, equally remarkable is the description of Heron quadrilaterals — the quadrilaterals having integer sides, diagonals and area. The only description that the author knows is due to Kummer, sketched in reference 1. Is it possible to generate Heron quadrilaterals by allowing the wind to break our bamboo in three ways?

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Summing Powers of Roots of an Algebraic Equation

STUART SIMONS

1. Introduction

Consider a general nth degree algebraic equation of the following form:

$$f(x) \equiv x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_{n-1}x + c_n = 0 \tag{1}$$

($c_n \neq 0$), with roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. In this article, we are interested in obtaining an expression for the sum $S_t = \sum_{p=1}^{n} \lambda_p^t$, for a given integer t, in terms of the coefficients $c_1, c_2, \ldots, c_n$. Now, for small values of t, results can be obtained by elementary algebra; thus, $S_1 = -c_1$, $S_2 = (\sum_p \lambda_p)^2 - 2\sum_{p<q} \lambda_p \lambda_q = c_1^2 - 2c_2$. In principle, this method can be extended further making use of the fact that each coefficient c_q is plus or minus the sum of all possible products of q roots, but as t increases beyond 2 the required algebraic manipulations rapidly render this approach impracticable. An alternative recursive technique was developed by Newton (see, for example, reference 1 for details of Newton's formulae) who showed how S_t can be expressed in terms of a linear combination of $S_{t-1}, S_{t-2}, \ldots, S_0$, with coefficients depending on $c_1, c_2, \ldots, c_n$. In principle, repeated application of these recurrence relations should allow S_t to be calculated in terms of the c_q, $1 \leq q \leq n$, but, in practice, the detailed form of the relations prevents us from using them to obtain an explicit formula for S_t in terms of $c_1, c_2, \ldots, c_n$. However, in Section 2, we shall develop a new approach which allows us to derive an expression for S_t as a linear combination of terms, each of which is a specified product of powers of the coefficients c_p. Finally, in Section 3, we provide some applications of our result to particular situations.

2. Derivation of a formula for S_t

We begin by letting $y = 1/x$. It then follows from (1) that y satisfies the equation $1 + c_1 y + c_2 y^2 + \cdots + c_n y^n = 0$, with roots $\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1}$. Thus, for some constant A, $1 + c_1 y + c_2 y^2 + \cdots + c_n y^n = A(\lambda_1^{-1} - y)(\lambda_2^{-1} - y) \cdots (\lambda_n^{-1} - y)$ and, letting $y = 0$, gives $A = \lambda_1 \lambda_2 \cdots \lambda_n$, so that $1 + c_1 y + c_2 y^2 + \cdots + c_n y^n = (1 - \lambda_1 y)(1 - \lambda_2 y) \cdots (1 - \lambda_n y)$. This, in turn, yields

$$\ln(1 + Y) = \ln(1 - \lambda_1 y) + \ln(1 - \lambda_2 y) + \cdots + \ln(1 - \lambda_n y), \tag{2}$$

where

$$Y = c_1 y + c_2 y^2 + \cdots + c_n y^n. \tag{3}$$

For sufficiently small values of $|y|$, we can expand each of the natural logarithm terms on the right-hand side of (2) as a convergent power series in y, based on the standard result

$\ln(1-z) = -\sum_{t=1}^{\infty} z^t/t$. This gives

$$
\begin{aligned}
&\ln(1-\lambda_1 y) + \ln(1-\lambda_2 y) + \cdots + \ln(1-\lambda_n y) \\
&= -\left(\sum_{t=1}^{\infty} \frac{\lambda_1^t y^t}{t} + \sum_{t=1}^{\infty} \frac{\lambda_2^t y^t}{t} + \cdots + \sum_{t=1}^{\infty} \frac{\lambda_n^t y^t}{t}\right) \\
&= -\sum_{t=1}^{\infty} \frac{S_t y^t}{t}.
\end{aligned}
$$

Thus, if we can calculate a_t, the coefficient of y^t in the power series expansion on the left-hand side of (2), we then immediately obtain $S_t = -ta_t$. (We note, in passing, that what we have shown here is equivalent to proving that, for the algebraic equation (1), $g(x) \equiv -\ln(x^n f(1/x))$ is a generating function for $S_t(c_1, c_2, \ldots, c_n)$, in that if $g(x)$ is expressed as a power series in x, then the coefficient of x^t is S_t/t.)

Now, to find a_t, we begin with the series $\ln(1+Y) = \sum_{m=1}^{\infty}(-1)^{m+1}Y^m/m$ and calculate Y^m by applying the multinomial theorem to the right-hand side of (3). This gives

$$
\begin{aligned}
(c_1 y + c_2 y^2 + \cdots + c_n y^n)^m &= \sum_{k_1,\ldots,k_n}^{*} \frac{m!}{k_1!\,k_2!\cdots k_n!}(c_1 y)^{k_1}(c_2 y^2)^{k_2}\cdots(c_n y^n)^{k_n} \\
&= \sum_{k_1,\ldots,k_n}^{*} \frac{m!\,c_1^{k_1} c_2^{k_2}\cdots c_n^{k_n}\, y^{k_1+2k_2+\cdots+nk_n}}{k_1!\,k_2!\cdots k_n!},
\end{aligned}
$$

where $\sum_{k_1,\ldots,k_n}^{*}$ implies a summation over all nonnegative values of $k_1, k_2, \ldots, k_n$ satisfying the constraint $k_1 + k_2 + \cdots + k_n = m$. (Note that here $k_p! = 1$ if $k_p = 0$ and $c_p^{k_p} = 1$ if $c_p = k_p = 0$.) This, in turn, yields

$$
\ln(1+Y) = \sum_{m=1}^{\infty} \sum_{k_1,\ldots,k_n}^{*} \frac{(-1)^{m-1}(m-1)!\,c_1^{k_1} c_2^{k_2}\cdots c_n^{k_n}\, y^{k_1+2k_2+\cdots+nk_n}}{k_1!\,k_2!\cdots k_n!}. \tag{4}
$$

Now, the constraint $k_1+k_2+\cdots+k_n = m$ included in $\sum_{k_1,\ldots,k_n}^{*}$ is effectively cancelled out by the subsequent $\sum_{m=1}^{\infty}$ so that $\sum_{m=1}^{\infty}\sum_{k_1,\ldots,k_n}^{*}$ is equivalent to $\sum_{k_1,\ldots,k_n}$, where this notation implies a summation over *all* nonnegative values of $k_1, k_2, \ldots, k_n$ (without any constraints). Further, this latter summation can be considered as a summation over all nonnegative values of $k_1, k_2, \ldots, k_n$, subject to the constraint

$$
k_1 + 2k_2 + \cdots + nk_n = t, \tag{5}
$$

for a given integer t, followed by a summation over all integer values of t. That is, $\sum_{k_1,\ldots,k_n}$ is equivalent to $\sum_{t=1}^{\infty}\sum_{k_1,\ldots,k_n}'$ where $\sum_{k_1,\ldots,k_n}'$ implies a summation over all nonnegative values of $k_1, k_2, \ldots, k_n$ which satisfy the above constraint (5). Equation (4) can thus be rewritten in the form

$$
\ln(1+Y) = \sum_{t=1}^{\infty}\left(\sum_{k_1,\ldots,k_n}' \frac{(-1)^{m-1}(m-1)!\,c_1^{k_1} c_2^{k_2}\cdots c_n^{k_n}}{k_1!\,k_2!\cdots k_n!}\right) y^t, \tag{6}
$$

where $m = k_1 + k_2 + \cdots + k_n$ and, hence, a_t, the coefficient of y^t in this expansion, is given by the expression in large round brackets in (6). It then immediately follows from our earlier discussion that

$$S_t = t \sum_{k_1,\ldots,k_n}{}' \frac{(-1)^m (m-1)!\, c_1^{k_1} c_2^{k_2} \cdots c_n^{k_n}}{k_1!\, k_2! \cdots k_n!}, \tag{7}$$

which is the basic result we were aiming for. The sum of negative integer powers of the roots may also readily be derived using the above approach by first substituting $X = 1/x$ in (1) to yield $X^n + d_1 X^{n-1} + d_2 X^{n-2} + \cdots + d_n = 0$ where $d_p = c_{n-p}/c_n$, $1 \le p \le n$, $c_0 = 1$. It then follows that S_{-t} is given by (7) with d_p replacing c_p.

3. Applications

We begin by using (7) in order to obtain expressions for S_1, S_2, S_3 in terms of the coefficients $c_1, c_2, \ldots, c_n$. For $t = 1$, (5) can only be satisfied by $k_1 = 1$, $k_p = 0$ ($p \ne 1$), and (7) then gives $S_1 = -c_1$. For $t = 2$, (5) is satisfied by $k_1 = 2$, $k_p = 0$ ($p \ne 1$), or $k_2 = 1$, $k_p = 0$ ($p \ne 2$). Equation (7) then yields $S_2 = 2(\frac{1}{2}c_1^2 - c_2) = c_1^2 - 2c_2$, in accordance with the result given at the beginning of this article. For $t = 3$, there are the following three solutions of (5):

(i) $k_1 = 3,\ k_p = 0\ (p \ne 1)$,

(ii) $k_1 = 1,\ k_2 = 1,\ k_p = 0\ (p \ne 1, 2)$,

(iii) $k_3 = 1,\ k_p = 0\ (p \ne 3)$.

Equation (7) then gives $S_3 = 3(-\frac{1}{3}c_1^3 + c_1 c_2 - c_3) = -c_1^3 + 3c_1c_2 - 3c_3$.

Example 1 We calculate the sum of the fifth powers of the roots of the cubic equation $x^3 - 8x^2 + 19x - 12 = 0$. Here, $n = 3$ and (5) thus becomes $k_1 + 2k_2 + 3k_3 = 5$. This equation has the following five solutions:

(i) $k_3 = 1,\ k_2 = 1,\ k_1 = 0$,

(ii) $k_3 = 1,\ k_2 = 0,\ k_1 = 2$,

(iii) $k_3 = 0,\ k_2 = 2,\ k_1 = 1$,

(iv) $k_3 = 0,\ k_2 = 1,\ k_1 = 3$,

(v) $k_3 = 0,\ k_2 = 0,\ k_1 = 5$.

Since $c_1 = -8$, $c_2 = 19$, $c_3 = -12$, (7) yields

$$\begin{aligned} S_5 = 5\Bigg(& \frac{(-1)^2 1!\,(-8)^0 19^1 (-12)^1}{0!\,1!\,1!} + \frac{(-1)^3 2!\,(-8)^2 19^0 (-12)^1}{2!\,1!\,0!} \\ & - \frac{2!\,(-8)19^2}{1!\,2!} + \frac{3!\,(-8)^3 19^1}{1!\,3!} - \frac{4!\,(-8)^5}{5!}\Bigg) \\ = {} & 1268, \end{aligned}$$

which is readily seen to be correct since the roots of the given equation are $x = 1, 3, 4$.

It is clear that when t is significantly greater than unity, the number of terms in the summation (7) can become very large, and a computer approach may then be called for in evaluating it. However, if a high proportion of the c_n are zero, then the number of nonzero terms in the summation (7) may remain relatively small, since such nonzero terms can only occur if $k_p = 0$ when $c_p = 0$. The position is best illustrated by a specific numerical example.

Example 2 Calculate S_{24} for the equation $x^{10} + x^9 + 2 = 0$. Here, $c_1 = 1$, $c_{10} = 2$, $c_p = 0$, for $2 \le p \le 9$. For nonzero contributions to summation (7), we therefore require that $k_p = 0$ for $2 \le p \le 9$, and thus (5) takes the form $k_1 + 10k_{10} = 24$. It is readily seen that this has only the following three solutions:

(i) $k_1 = 24,\ k_{10} = 0,$

(ii) $k_1 = 14,\ k_{10} = 1,$

(iii) $k_1 = 4,\ k_{10} = 2.$

Hence, we obtain from (7)

$$S_{24} = 24\left(\frac{23!\,1^{24}2^0}{24!\,0!} - \frac{14!\,1^{14}2^1}{14!\,1!} + \frac{5!\,1^4 2^2}{4!\,2!}\right) = 193.$$

Reference

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Before his retirement, ***Stuart Simons*** *was a Reader in Applied Mathematics at the University of London, with interests in transport theory and the mathematical theory of aerosols. His current interests include developing novel approaches to relatively elementary problems.*

Equal differences of squares

$$(3n+4)^2 - n^2 = (3n+5)^2 - (n+3)^2 = (2n^2+6n+5)^2 - (2n^2+6n+3)^2.$$

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Equalising Polygons

ALASTAIR SUMMERS

1. Introduction

Definition 1 Consider the sequence $\{0, 2, 6, 4\}$. Cycle round this sequence in pairs, firstly in a forward direction, and then in reverse. Thus we create the two sequences $\{02, 26, 64, 40\}$ and $\{04, 46, 62, 20\}$. We rewrite the second sequence in the order $\{20, 62, 46, 04\}$ to show more clearly that the elements are identical to those of the first sequence with the digits interchanged. We call these *derived sequences.*

Now sum the cubes of the elements in the derived sequences. We obtain

$$2^3 + 26^3 + 64^3 + 40^3 = 20^3 + 62^3 + 46^3 + 4^3 = 343\,728.$$

We call this phenomenon *equalisation*. We say that here we have a '4-sequence which equalises with index 3'. We call the 4-sequence the *starting-sequence.* In general, the index will be denoted by m and the starting-sequence size by n.

In Volume 36, Number 3, p. 69, the problem was given with the four numbers at the vertices of a square as illustrated below.

$$\begin{matrix} 0 & 2 \\ 4 & 6 \end{matrix}$$

Throughout this article, we write D for the sum of the powers going in a clockwise or forward direction minus the sum going in the opposite direction. It follows that, by equalisation, we have $D = 0$.

2. A generalisation

We now generalise the concept of equalisation, keeping the index $m = 3$. So far, we have considered double-digit decimal numbers. For example, if the first two elements are a and b then the first element of the first derived sequence is $10a+b$. What if we replace the coefficients $(10, 1)$ here by (p, q)? It turns out that it makes no difference to the conclusion, so I will do this. Letting $p = 0$ and $q = 0$, or $p = q$ leads to trivial cases in which the same sequence of cubes is repeated.

If the coefficient set is $(k, 1)$, then I will call this base k, so the base in the original problem is 10.

Taking the sequence $\{a, b, c, d\}$, we obtain

$$\begin{aligned} D &= (pa + qb)^3 + (pb + qc)^3 + (pc + qd)^3 + (pd + qa)^3 \\ &\quad - (pb + qa)^3 - (pc + qb)^3 - (pd + qc)^3 - (pa + qd)^3 \\ &= 3(p^2q - pq^2)(a^2b - ab^2 + b^2c - bc^2 + c^2d - cd^2 + d^2a - da^2) \\ &= 3pq(p - q)(a - c)(b - d)(a + c - b - d). \end{aligned}$$

The cases $a = c$, $b = d$, $p = 0$, $q = 0$, and $p = q$ are trivial, so a necessary and sufficient condition for *nontrivial equalisation* is

$$a + c - b - d = 0.$$

If we restrict ourselves to single-digit elements, this yields 50 essentially distinct nontrivial equalising sequences.

To illustrate what can be achieved, we present the following examples.

Example 1 The sequence $\{1, 4, 5, 2\}$ meets the criterion for equalisation since $1+5 = 4+2$. With $p = 5$ and $q = -1$, we obtain $1^3 + 15^3 + 23^3 + 9^3 = 3^3 + 5^3 + 21^3 + 19^3$. Both sums are equal to $16\,272$.

Example 2 The sequence $\{23, 37, 19, 5\}$, with $p = 100$ and $q = 1$, meets the criterion for equalisation since $23 + 19 = 37 + 5$, and so $2337^3 + 3719^3 + 1905^3 + 523^3 = 3723^3 + 1937^3 + 519^3 + 2305^3$. This example illustrates the fact that double digits may be used instead of single digits in the original problem. We can have three digits or more too.

3. A further generalisation

It is possible to generalise the problem further by introducing three or more coefficients. So, for example, we may use p, q, and r with the sequence $\{a, b, c, d\}$ and index 3 when

$$\begin{aligned} D = {} & (pa + qb + rc)^3 + (pb + qc + rd)^3 + (pc + qd + ra)^3 + (pd + qa + rb)^3 \\ & - (pc + qb + ra)^3 - (pd + qc + rb)^3 - (pa + qd + rc)^3 - (pb + qa + rd)^3, \end{aligned}$$

which simplifies to

$$D = 3q(p - r)(p - q + r)(a - c)(b - d)(a + c - b - d).$$

It follows that, for index 3, equalisation is not affected by the introduction of an extra coefficient.

Note that if $p = r$ or $q = p + r$ then $D = 0$. The case $p = r$ is trivial. I shall use the notation (p, q, r) for a coefficient set. I will denote the coefficient-set size by v.

Example 3 The sequence $\{0, 2, 6, 4\}$, with $m = 3$ and coefficient set $(100, 10, 1)$ gives

$$264^3 + 640^3 + 402^3 + 26^3 = 462^3 + 46^3 + 204^3 + 620^3 = 345\,526\,128,$$

and, with coefficient set $(1000, 100, 10, 1)$, we obtain

$$2640^3 + 6402^3 + 4026^3 + 264^3 = 462^3 + 2046^3 + 6204^3 + 4620^3 = 346\,064\,110\,128.$$

So, going back to the original problem, we may take blocks of 3 or 4 consecutive digits instead of the original pairs, and still achieve equalisation.

Also, if we take $q = p+r$, for example $(p, q, r) = (2, 5, 3)$, then any 4-sequence equalises. For example, $\{1, 3, 5, 6\}$ gives $32^3 + 49^3 + 43^3 + 26^3 = 28^3 + 46^3 + 47^3 + 29^3 = 247\,500$. Note also that $32 + 49 + 43 + 26 = 28 + 46 + 47 + 29$, which is true, in general, of derived sets.

4. The cases $m = 1$ and $m = 2$

For indices 1 and 2, equalisation always occurs no matter what coefficient set is used. This can easily be proved.

5. The case $m = 4$

I now consider 4-sequences with index $m = 4$ and coefficient-set size $v = 2$. In this section, where $v = 2$, I shall replace p/q by k. This is equivalent to dividing D by q^3. It will not therefore affect the occurrence of equalisation. The cases $p = 0$ and $q = 0$, $p = \pm q$, and hence also $k = 0$ or $k = \pm 1$ are trivial, so need not be considered.

For brevity, I will use the notation $\sum_c (a^u b - ab^u)$ to mean a cyclic sum so, for example, summing round the starting sequence $\{a, b, c, d\}$ we obtain

$$\sum\nolimits_c (a^u b - ab^u) = (a^u b - ab^u) + (b^u c - bc^u) + (c^u d - cd^u) + (d^u a - da^u).$$

The obvious equivalent definitions will apply to any other such sum.

We first note the general algebraic identity

$$\sum\nolimits_c (a^u b - ab^u) = (a^u - c^u)(b - d) - (a - c)(b^u - d^u),$$

for a 4-sequence $\{a, b, c, d\}$. Hence, for $\{a, b, c, d\}$ with $m = 4$ and base k, we obtain

$$D = \sum\nolimits_c \{(ka + b)^4 - (kb + a)^4\} = 4(k^3 - k) \sum\nolimits_c (a^3 b - ab^3).$$

Hence, $D = 0$ if and only if $\sum_c (a^3 b - ab^3) = 0$, given that $k \neq 0$ or $k \neq \pm 1$. But

$$\begin{aligned} \sum\nolimits_c (a^3 b - ab^3) &= (a^3 - c^3)(b - d) - (a - c)(b^3 - d^3) \\ &= (a - c)(b - d)(a^2 + ac + c^2 - b^2 - bd - d^2). \end{aligned}$$

Hence, the required condition for equalisation is $a^2 + ac + c^2 = b^2 + bd + d^2$. Avoiding trivial solutions we need to find integers which can be expressed in more than one way in the form $a^2 + ac + c^2$.

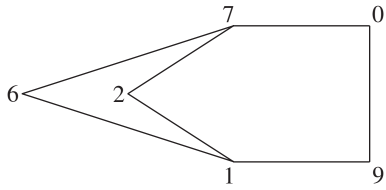
If we restrict ourselves to single-digit elements a, b, c, and d, then there are two solutions given by $49 = 0^2 + 0 \cdot 7 + 7^2 = 3^2 + 3 \cdot 5 + 5^2$ and $91 = 1^2 + 1 \cdot 9 + 9^2 = 5^2 + 5 \cdot 6 + 6^2$. These yield the 4-sequences $\{0, 3, 7, 5\}$ and $\{1, 5, 9, 6\}$ and so, using base 10, we obtain

$$3^4 + 37^4 + 75^4 + 50^4 = 30^4 + 73^4 + 57^4 + 5^4 = 39\,764\,867.$$

If we change the base then we will get other sums. For example, if the base is 2 for $\{0, 3, 7, 5\}$, then we get $3^4 + 13^4 + 19^4 + 10^4 = 5^4 + 17^4 + 17^4 + 6^4$.

6. More generalisations

We can extend the principle of equalisation to longer sequences, so that $n > 4$, and get infinitely more equalising sequences. There are exactly four nontrivial equalising sequences consisting

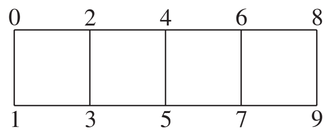


of single digits of size $n = 5$ with index $m = 3$, namely $\{1, 2, 7, 0, 9\}$, $\{1, 6, 7, 0, 9\}$, $\{2, 3, 8, 0, 9\}$, and $\{2, 7, 8, 0, 9\}$. Hence, for example,

$$12^3 + 27^3 + 70^3 + 9^3 + 91^3 = 21^3 + 72^3 + 7^3 + 90^3 + 19^3 = 1\,118\,711.$$

Adjoining equalising sequences (as illustrated below) produces another equalising sequence. For example, adjoining $\{1, 6, 7, 2\}$ to $\{1, 2, 7, 0, 9\}$ produces $\{1, 6, 7, 0, 9\}$, for $m = 3$.

Rather a nice result (as illustrated below) can be obtained from $\{0, 2, 3, 1\} + \{2, 4, 5, 3\} + \{4, 6, 7, 5\} + \{6, 8, 9, 7\} = \{0, 2, 4, 6, 8, 9, 7, 5, 3, 1\}$, for $m = 3$.



It will be seen that the four component sets are equalising, since elements at opposite vertices for each set have the same total, as required in Section 2. So we obtain

$$\begin{aligned} &2^3 + 24^3 + 46^3 + 68^3 + 89^3 + 97^3 + 75^3 + 53^3 + 31^3 + 10^3 \\ &\quad = 20^3 + 42^3 + 64^3 + 86^3 + 98^3 + 79^3 + 57^3 + 35^3 + 13^3 + 1^3 \\ &\quad = 2\,644\,785. \end{aligned}$$

In fact, in general, the $2n$-sequence $\{a_1, a_2, \ldots, a_n, a_n+\alpha, a_{n-1}+\alpha, \ldots, a_1+\alpha\}$ equalises where α is any constant, for $m = 3$.

Note that, since we may start with an equalising 5-sequence to get odd sized sequences, we may construct equalising sequences of any size greater than 3 and index 3, as each new adjoining 4-sequence increases the sequence size by 2. For example, attach $\{2, 3, 8, 0, 9\}$ to $\{3, 1, 6, 8\}$ with $m = 3$ to obtain the 7-sequence $\{2, 3, 1, 6, 8, 0, 9\}$. Using base 10, we then find that

$$\begin{aligned} 23^3 + 31^3 + 16^3 + 68^3 + 80^3 + 9^3 + 92^3 &= 32^3 + 13^3 + 61^3 + 86^3 + 8^3 + 90^3 + 29^3 \\ &= 1\,651\,903. \end{aligned}$$

7. Base-specific equalising sets

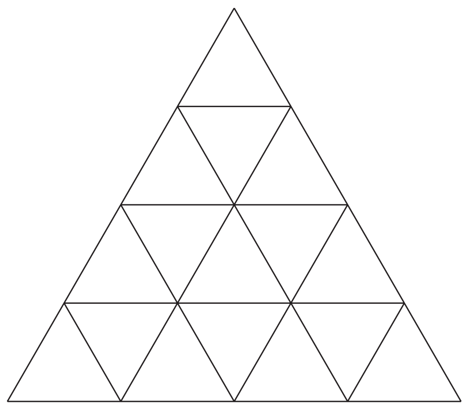
I have also found a number of rules for base-specific equalising sets, for example, for sequence size $n = 3$ and index $m = 4$, if the coefficient set is $\{p, q, r\}$ such that $p + q + r = 0$ then any sequence equalises. For example, with starting-sequence $\{-1, 0, 2\}$ and coefficient set $\{1, -3, 2\}$, we obtain $3^4 + (-8)^4 + 5^4 = 0^4 + (-7)^4 + 7^4 = 4802$. Also, the case $n > 3$, $m = 3$, and coefficient set $\{p, q, r\}$, such that $q = p + r$, always yields an equalising set. This is illustrated at the end of Section 3 and proved there for $n = 4$.

8. A final question

I leave the reader with a question. Can this analysis be used to find integers that may be expressed as the sum of *two* distinct cubes in two different ways?

Alastair Summers *has taught mathematics for the last 30 years at Stamford School in Lincolnshire, and before that near Birmingham and in Nottingham. He is now semi-retired. His interest in problem-solving has increased during this time, partly stimulated by the many interesting articles and letters in the pages of Mathematical Spectrum, and partly by attempting the International Maths Olympiad Problems sent out by the United Kingdom Mathematics Trust.*

How many triangles?



12/15 Kamardanga Rly Quarter,
Kolkata, 700046, India

Bablu Chandra Dey

If

$$f(2x) = 6x + \sin(f(x)) \qquad \text{and} \qquad f(0) = 0,$$

what is $f'(0)$?

10 Shahid Azam Lane,
Makki Abad Avenue, Sirjan, Iran

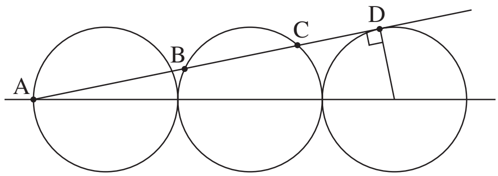
Abbas Roohol Amini

Mathematics in the Classroom

Finding the length of part of a tangent

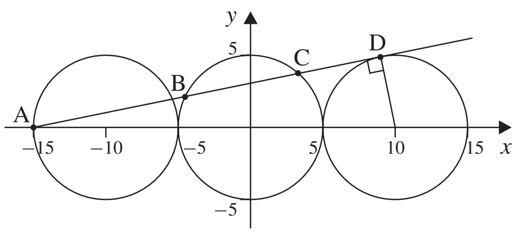
A. Gardiner (see reference 1) offered the following problem without solution, but suggested that the mathematics needed should be within the grasp of an able student.

In the figure, where the radius of each circle is 5 and line AD is a tangent to the right-most circle, what is the length of the line BC?



In what follows, I describe my solution to this problem.

Firstly, redraw the above figure with axes and think graphically.



The middle circle now has the following equation:

$$x^2 + y^2 = 25.$$

To find the equation of the line AD we need the gradient of the line and a set of coordinates of a point on the line. We know that the coordinates of point A are $(-15, 0)$.

As the gradient of AD is the tangent of the angle that the line AD makes with the positive x-axis, then the gradient is $5/\mathrm{AD}$. Also, $\mathrm{AD}^2 = 25^2 - 5^2 = 600$. Thus, the gradient is $5/\sqrt{600} = 1/2\sqrt{6}$. Hence, the equation of line AD is

$$y = \frac{x + 15}{2\sqrt{6}}.$$

We now solve the equations of the line and the circle to find the coordinates of B and C as follows:

$$x^2 + \left(\frac{x + 15}{2\sqrt{6}}\right)^2 = 25,$$

i.e.

$$25x^2 + 30x + 225 = 600,$$

which simplifies to

$$5x^2 + 6x - 75 = 0.$$

The x-coordinates of B and C, x_{B} and x_{C}, are the roots of this quadratic equation. The y-coordinates of B and C, y_{B} and y_{C}, are the roots of the equation

$$(2\sqrt{6}y - 15)^2 + y^2 = 25,$$

which simplifies to

$$25y^2 - 60\sqrt{6}y + 200 = 0.$$

Now, the length BC is

$$\begin{aligned}\sqrt{(x_{\mathrm{B}} - x_{\mathrm{C}})^2 + (y_{\mathrm{B}} - y_{\mathrm{C}})^2} &= \sqrt{(x_{\mathrm{B}} + x_{\mathrm{C}})^2 - 4x_{\mathrm{B}}x_{\mathrm{C}} + (y_{\mathrm{B}} + y_{\mathrm{C}})^2 - 4y_{\mathrm{B}}y_{\mathrm{C}}} \\ &= \sqrt{\left(\frac{-6}{5}\right)^2 - 4\frac{-75}{5} + \left(\frac{60\sqrt{6}}{25}\right)^2 - 4\frac{200}{25}} \\ &= 8.\end{aligned}$$

If anyone knows a purely geometrical solution to this problem then I should be interested to hear of it.

Reference

1 A Gardiner, *Mathematical Puzzling* (UK Mathematics Foundation, University of Birmingham, 1996).

The Sixth Form College, Solihull **Carsten Ghedia**

Sums of primes

- Is it possible to use six different digits to form two prime numbers totalling 1 000?
- Is it possible to use eight different digits to form two prime numbers totalling 10 000?

12 Pinewood Road, Midsomer Norton **Bob Bertuello**
Bath BA3 2RG, UK

Letters to the Editor

Dear Editor,

Curious Cubes

I refer to Vinod Tyagi's 'Curious Cubes' (see Volume 37, Number 3, p. 111). These are not so curious when viewed in the following way.

1. The sum of the digits gives the residue modulo 9. Hence, for example, $8^3 = 512$ and $5+1+2=8$ can only occur because $8^3 \equiv 8 \mod 9$. In general, $x^3 \equiv x \mod 9$, i.e. $x \equiv 0$ or $\pm 1 \mod 9$, is a necessary condition for x to have a 'curious cube'.

2. Now consider, say, those cubes which have exactly four digits, i.e. for which $1\,000 \leq x^3 < 10\,000$ or $10 \leq x \leq 21$. The only candidates for 'curiosity' in this range are 10, 17, 18, and 19. Of these, only 17^3 and 18^3 are curious.

3. These arguments can be honed further. As another example, consider six-digit cubes. Now $10^5 \leq x < 10^6 \implies 47 \leq x \leq 99$. The maximum possible digit total is $6 \times 9 = 54$. So, if $x > 54$ then its cube cannot be curious. So, only 53^3 and 54^3 are candidates and, not surprisingly, neither is curious.

4. A similar argument shows that curious cubes with more than six digits are impossible.

5. The complete list of nonnegative integers having curious cubes is 0, 1, 8, 17, 18, 26, and 27. I have excluded 9 from Vinod's list because he takes an extra step to reach curiosity and then omits to do the same for 18.

6. In some other number bases, curiosity is sometimes achieved with greater frequency. For example, in base 7 the integers having curious cubes are 0, 1, 2, 4, 8, 9, 11, 12, 15, 16, and 19. This greater frequency is not surprising since a necessary condition for curiosity is $x^3 \equiv x \mod 6$ (analogous to $x^3 \equiv x \mod 9$ in base 10). This happens to be true for all x.

As an amusing postscript, when converted to base 7

$$1^3 = 1, \quad 2^3 = 11, \quad 4^3 = 121, \quad 8^3 = 1\,331, \quad 16^3 = 14\,641, \ldots$$

and

$$1 = 1, \quad 1+1 = 2, \quad 1+2+1 = 4, \quad 1+3+3+1 = 8, \quad 1+4+6+4+1 = 16, \ldots$$

so all these cubes are curious!

Yours sincerely,

Alastair Summers
(57 Conduit Road
Stamford
Lincolnshire PE9 1QL
UK)

Dear Editor,

Kepler's polygonial well (see Volume 26, Number 4, pp. 110–118)

If Leibniz saw curves as infinitely sided polygons and Newton viewed them as dynamic mechanical loci, Kepler's unusual homage to Platonic and Archimedian solids and their shadows is retrospectively the more intriguing. It is worth a long view of Kepler's tormented mind as is his saying 'truth, like light, will out'.

His truths were very precious to future thinkers, given his historical background of the atrocities of the Thirty Years' War. His overcoming constant semi-starvation and religious persecution makes him to me, a hero, seeking simplicity and a just God in the heavens.

Yours sincerely,

John Macnaughton
(9 Argyle Street
Inverness IV2 3BA
UK)

Dear Editor,

An alternative proof for the sum of the first n triangular numbers

The sum of the first n triangular numbers can be expressed as follows:

$$\begin{aligned}\sum_{r=1}^{n} \mathrm{T}_r = {} & \;1 \\ & +1+2 \\ & +1+2+3 \\ & \;\vdots \\ & +1+2+3+\cdots+n.\end{aligned}$$

The sum of the first n squares can be expressed as follows:

$$\begin{aligned}\sum_{r=1}^{n} r^2 = 1+2+3+\cdots+n \\ +2+3+\cdots+n \\ +3+\cdots+n \\ \vdots \\ +n.\end{aligned}$$

When we add $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} \mathrm{T}_r$, we obtain

$$\begin{aligned}\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} \mathrm{T}_r = {} & \;1+2+3+\cdots+n \\ & +1+2+3+\cdots+n \\ & +1+2+3+\cdots+n \\ & \;\vdots \\ & +1+2+3+\cdots+n \quad \text{(going down } n+1 \text{ times),}\end{aligned}$$

which can be written as $(n+1)\sum_{r=1}^{n} r$.

Therefore,

$$(n+1)\sum_{r=1}^{n} r = \sum_{r=1}^{n} \mathrm{T}_r + \sum_{r=1}^{n} r^2,$$

so that

$$\begin{aligned}\sum_{r=1}^{n} \mathrm{T}_r &= (n+1)\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2 \\ &= \tfrac{1}{2}n(n+1)(n+1) - \tfrac{1}{6}n(n+1)(2n+1) \\ &= \tfrac{1}{6}n(n+1)(3(n+1) - (2n+1)) \\ &= \tfrac{1}{6}n(n+1)(n+2).\end{aligned}$$

Yours sincerely,

Joel Murang
(Stonyhurst College
Stonyhurst
Clitheroe
Lancashire BB7 9PZ
UK)

Dear Editor,

Expressing the sum of two cubes as a square

Referring to Muneer Jebreel's letter in Volume 37, Number 3, a family of solutions to the Diophantine equation

$$x^3 + y^3 = k^2$$

is given by

$$x = p(p^3 + q^3), \qquad y = q(p^3 + q^3), \qquad k = (p^3 + q^3)^2,$$

for all integers p and q.

Yours sincerely,

M. A. Khan
(C/o A. A. Khan
Manager Regional Office
Indian Overseas Bank
Ashok Marg
Lucknow
India)

Dear Editor,

Right- and left-prime numbers

A prime number p is said to be a *right-prime number* if the numbers xp are not prime for all digits x from 1 to 9, where xp denotes the 'concatenation' of x and p. Thus, 5 is prime but $15, 25, \ldots, 95$ are not prime, so that 5 is a right-prime number. The only other right-prime number less than 1 000 is 773. Are there others? Are there infinitely many?

A prime number p is said to be a *left-prime number* if py is not prime for $y = 1, 3, 7, 9$. (Clearly, py is not prime for other digits y between 0 and 9.) The only left-prime number less than 1 000 is also 773. Are there others? Are there infinitely many?

A prime number p is said to be a *mid-prime number* if xpy is not prime for all x from 1 to 9 and $y = 1, 3, 7, 9$. There are three mid-prime numbers less than 3 600, namely 773, 1 103, 1 301. Are there others? Are there infinitely many?

Starting with a prime number p, we can look for the length of the longest chain of prime numbers $px_1, px_1x_2, \ldots, px_1x_2\cdots x_n$, where the x_i are 1, 3, 7 or 9. We call n the *left-length of* p, and denote it by $l(p)$. There are two chains from 2 of length 7, namely

$$2,\quad 23,\quad 233,\quad 2\,339,\quad 23\,399,\quad 233\,993,\quad 2\,339\,933,\quad 23\,399\,339$$

and

$$2,\quad 29,\quad 293,\quad 2\,939,\quad 29\,399,\quad 293\,999,\quad 2\,939\,999,\quad 29\,399\,999$$

(there are no longer chains), so that $l(2) = 7$. The longest chain from 43 is

$$43,\quad 439,\quad 4\,391,\quad 43\,913,\quad 439\,133,\quad 4\,391\,339,$$

so that $l(43) = 5$.

Yours sincerely,

Seyamack Jafari
(PO Box 161
Razi Petrochemical Complex
Bandar Imam
Khozestan
Iran)

Dear Editor,

Thought-provoking tesseracts

To complement *Curious cubes* by Vinod Tyagi (Volume 37, Number 3, p. 111), I came up with the following 'thought-provoking tesseracts':

$$1^4 = 1, \qquad 1,$$
$$4^4 = 256, \qquad 2+5+6 = 13, \qquad 1+3 = 4,$$
$$7^4 = 2\,401, \qquad 2+4+0+1 = 7,$$
$$9^4 = 6\,561, \qquad 6+5+6+1 = 18, \qquad 1+8 = 9,$$
$$22^4 = 234\,256, \qquad 2+3+4+2+5+6 = 22.$$

Yours sincerely,

Joel Murang
(Stonyhurst College
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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

38.5 Determine all natural numbers n for which there is a permutation $(a_1, \ldots, a_n)$ of $1, \ldots, n$ such that $a_1 + \cdots + a_k$ is divisible by k for $k = 1, \ldots, n$.

(Submitted by H. A. Shah Ali, Tehran, Iran)

Mr Shah Ali wonders whether there is a permutation of the whole set of natural numbers with this property.

38.6 Solve the equation

$$(\sqrt{6} - \sqrt{5})^x + (\sqrt{3} - \sqrt{2})^x + (\sqrt{3} + \sqrt{2})^x + (\sqrt{6} + \sqrt{5})^x = 32.$$

(Submitted by Mihály Bencze, Brasov, Romania)

38.7 Let x, y, z be positive real numbers and let A, B, C be the angles of a triangle. Prove that

$$x^2 + y^2 + z^2 \geq 2yz \cos A + 2zx \cos B + 2xy \cos C.$$

(Submitted by Abbas Roohol Aminy, Sirjan, Iran)

38.8 A set of n points in the plane or in space is partitioned into two non-empty subsets. If P and Q are the mean points of these subsets, show that all such lines PQ are concurrent.

(Submitted by Guido Lasters, Tienen, Belgium)

Solutions to Problems in Volume 37 Number 3

37.9 Prove that

$$\sqrt{(a+b)(c+d)(e+f)} + \sqrt{ace} \geq \sqrt{adf} + \sqrt{bcf} + \sqrt{bde},$$

for positive real numbers $a, \ldots, f$.

Solution by Mihály Bencze, Brasov, Romania

For real numbers x, y, z, we obtain

$$\begin{aligned}(1+x^2)(1+y^2)(1+z^2) - (yz+zx+xy-1)^2 \\ &= 1 + (x^2+y^2+z^2) + (y^2z^2+z^2x^2+x^2y^2) + x^2y^2z^2 \\ &\quad - (y^2z^2+z^2x^2+x^2y^2) + 2(yz+zx+xy) - 2xyz(x+y+z) - 1 \\ &= (x+y+z-xyz)^2 \\ &\geq 0,\end{aligned}$$

so that

$$\sqrt{(1+x^2)(1+y^2)(1+z^2)} \geq yz+zx+xy-1. \tag{1}$$

If we put $x^2 = b/a$, $y^2 = d/c$, $z^2 = f/e$, and multiply by $\sqrt{ace}$, the result follows.

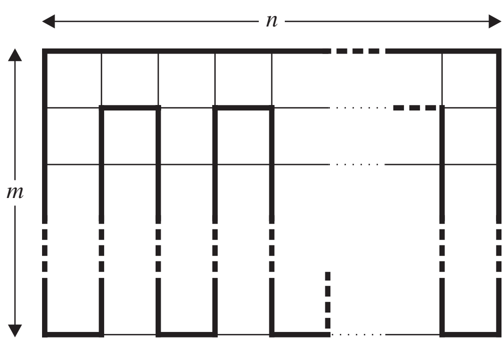
Mihály Bencze gave the following three applications of (1) (which are equivalent to the inequality in the question):

$$1 + \cosh u \cosh v \cosh w \geq \sinh v \sinh w + \sinh w \sinh u + \sinh u \sinh v,$$
$$1 + \sec u \sec v \sec w \geq \tan v \tan w + \tan w \tan u + \tan u \tan v,$$
$$1 + \operatorname{cosec} u \operatorname{cosec} v \operatorname{cosec} w \geq \cot v \cot w + \cot w \cot u + \cot u \cot v.$$

37.10 There are mn houses at the junctions of an $m \times n$ rectangular grid of roads. For which values of m and n is it possible for one of the householders to visit every other house just once and return to his own house?

Solution by James West, Hills Road Sixth Form College, Cambridge

If $m = n = 1$, then it is trivially possible. If $m = 1$ and $n > 1$ or if $m > 1$ and $n = 1$, then it is clearly impossible. If m or n is even, then it is possible as shown in the figure. (In the figure, n is taken to be even.)



Now suppose that $m, n > 1$ are both odd. Denote the number of moves to the left, right, up, and down by l, r, u, and d respectively. If it is possible to visit each house just once and return to the starting point, then

$$l + r + u + d = mn,$$

which is odd. But, to return to the starting point, $l = r$ and $u = d$, so this is impossible.

37.11 The $n \times n$ matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are such that $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{A}\boldsymbol{B}$, $\boldsymbol{B}+\boldsymbol{C}=\boldsymbol{B}\boldsymbol{C}$, $\boldsymbol{C}+\boldsymbol{A}=\boldsymbol{C}\boldsymbol{A}$. Prove that

$$(\boldsymbol{A}-\boldsymbol{I}_n)^2+(\boldsymbol{B}-\boldsymbol{I}_n)^2+(\boldsymbol{C}-\boldsymbol{I}_n)^2=3\boldsymbol{I}_n.$$

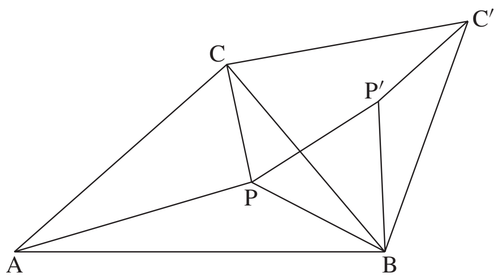
Solution by Hussein Islami-Arshagi and Kayvan Najarian, University of North Carolina, Charlotte

The equations $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{A}\boldsymbol{B}$ and $\boldsymbol{B}+\boldsymbol{C}=\boldsymbol{B}\boldsymbol{C}$ give $\boldsymbol{A}\boldsymbol{C}+\boldsymbol{B}\boldsymbol{C}=\boldsymbol{A}\boldsymbol{B}\boldsymbol{C}$ and $\boldsymbol{A}\boldsymbol{B}+\boldsymbol{A}\boldsymbol{C}=\boldsymbol{A}\boldsymbol{B}\boldsymbol{C}$, so that $\boldsymbol{A}\boldsymbol{B}=\boldsymbol{B}\boldsymbol{C}$. Similarly, $\boldsymbol{B}\boldsymbol{C}=\boldsymbol{C}\boldsymbol{A}$. Hence $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{C}=\boldsymbol{C}+\boldsymbol{A}$, so that $\boldsymbol{A}=\boldsymbol{B}=\boldsymbol{C}$. Hence $2\boldsymbol{A}=\boldsymbol{A}^2$, $2\boldsymbol{B}=\boldsymbol{B}^2$, $2\boldsymbol{C}=\boldsymbol{C}^2$, so that

$$\begin{aligned}
&(\boldsymbol{A}-\boldsymbol{I}_n)^2+(\boldsymbol{B}-\boldsymbol{I}_n)^2+(\boldsymbol{C}-\boldsymbol{I}_n)^2\\
&\quad=\boldsymbol{A}^2-2\boldsymbol{A}+\boldsymbol{I}_n+\boldsymbol{B}^2-2\boldsymbol{B}+\boldsymbol{I}_n+\boldsymbol{C}^2-2\boldsymbol{C}+\boldsymbol{I}_n\\
&\quad=3\boldsymbol{I}_n.
\end{aligned}$$

37.12 A triangle ABC has largest angle less than $\frac{2}{3}\pi$ radians. Prove that the angles subtended by the sides of the triangle at an interior point P are equal when the sum $\mathrm{AP}+\mathrm{BP}+\mathrm{CP}$ is a minimum. If $\mathrm{AP}:\mathrm{BP}:\mathrm{CP}=1:2:4$ and $\mathrm{AP}+\mathrm{BP}+\mathrm{CP}$ is a minimum, find the angles A, B, C.

Solution by J. A. Scott, who proposed the problem



Rotate the triangle BPC to $\mathrm{BP'C'}$ through $\pi/3$ radians away from A. This creates equilateral triangles $\mathrm{BPP'}$ and $\mathrm{BCC'}$, and

$$\mathrm{AP}+\mathrm{BP}+\mathrm{CP}=\mathrm{AP}+\mathrm{PP'}+\mathrm{P'C'}.$$

This will be minimum when $\mathrm{APP'C'}$ is a straight line, i.e. when $\angle\mathrm{APB}=2\pi/3$ and $\angle\mathrm{BP'C'}=2\pi/3$, i.e. when $\angle\mathrm{APB}=\angle\mathrm{BPC}=\angle\mathrm{CPA}=2\pi/3$. Put $\mathrm{AP}=1$, $\mathrm{BP}=2$, $\mathrm{CP}=4$, and let $\mathrm{AP}+\mathrm{BP}+\mathrm{CP}$ be a minimum. Then, from $\Delta\mathrm{BPC}$,

$$\mathrm{BC}^2=2^2+4^2-2\times2\times4\times\left(-\tfrac{1}{2}\right)=28.$$

Similarly, $\mathrm{CA}^2=21$, $\mathrm{AB}^2=7$, so that $\mathrm{BC}=2\sqrt{7}$, $\mathrm{CA}=\sqrt{3}\sqrt{7}$, $\mathrm{AB}=\sqrt{7}$. Thus, $\angle\mathrm{A}=\pi/2$, $\angle\mathrm{B}=\pi/3$, $\angle\mathrm{C}=\pi/6$.

Reviews

Exploring, Investigating and Discovering in Mathematics. By Vasile Berinde. Birkhäuser, Basel, 2004. Paperback, 265 pages, $49.95 (ISBN 3-76437-019-X).

This book arose from the author's desire and need to write journal and competition questions. His central claim is that every problem can be seen as a source of others, and from these new problems can spring others, and so on. Berinde implies that he would like to see the word 'exhaustive' removed from the mathematicians' dictionary: for him, a problem is never finished. In his introduction, he uses the language of mining to explain his approach. Each problem must be explored for its 'deposit of precious metals', and then once the 'mother lode' has been found, many other directions to be explored emerge. For example, he begins one chapter with the following problem: construct, using ruler and compasses, a triangle having the same area as a given convex quadrilateral. From this, a convex pentagon may be considered. A convex polygon? Now in three dimensions: a tetrahedron with the same volume as a convex polyhedron? The final problem in the section becomes this: construct a square having the same area as a given triangle.

Berinde sees this multiplication of problems as a marvellously creative activity, as the title of his book suggests, and he writes with passion over the need to encounter this approach to research earlier in the life of a mathematician.

Much of the material is simple to state, but hard to solve. Berinde does (just about) keep his promise not to progress beyond elementary mathematics, but most of this book will be hard going for school or college students, unless they are Olympiad material. However, university students who are prepared to invest time in this book will reap rewards in technical skill. Indeed, Berinde claims that his book could easily form the basis of a course in mathematical methodology. He calls too for recognition of the creativity that lies in all of us: 'sparks of our creator and reflections of the divine act of creation'. Whatever our calling in life, Berinde says, finding the secret of creation is the path to spiritual self-fulfilment. It is clear that it has been a spiritual endeavour for Berinde to write this book, a task for which we should thank him.

Paston College, Norfolk **Jonny Griffiths**

Historical Modules for the Teaching and Learning of Mathematics. Edited by Victor J. Katz and Karen Dee Michalowicz. MAA, Washington, DC, 2004. CD-ROM, $41.95 (ISBN 0-88385-741-3).

This CD-ROM contains a vast resource of interactive lessons across many key topics and at many different levels that encompass both GCSE, AS-level, and A-level.

Much of the lesson material is provided for you and added interest is gained by the constant referral to pertinent historical mathematical discoveries. The appropriate level of each activity is indicated in relation to American schools so this needs to be mapped onto our levels, but any teacher will have an immediate feel for the level that meets the needs of his or her students. Consequently, this is a small problem compared to the amount of work that has been done for you in the material contained on this CD-ROM. This is a very valuable and useful resource indeed.

Solihull Sixth Form College **Donald Mason**

Musings of the Masters: An Anthology of Mathematical Reflections. Edited by Raymond G. Ayoub. MAA, Washington, DC, 2004. Hardback, 288 pages, $46.50 (ISBN 0-88385-549-6).

This book is a collection of 17 essays by famous mathematicians of the 20th Century. Each essay is preceded by a short biography of the author and a preface by the editor. The book is divided into four parts: Mathematics and the Intellect, Mathematics and Human Understanding, Mathematics and Society, and Miscellaneous. The essays are philosophical in nature and do not contain mathematical notation or equations. For example, there are discussions as to whether mathematics is invented or discovered. The essays are described as 'humanistic' in nature, and the desire is expressed that they are accessible to the general reader who may not have technical knowledge of mathematics. Some of the essays read easier than others, but none of them could be considered to be light reading. As a Mathematician, I felt that I ought to read such articles on mathematical philosophy. However, they were read out of a feeling of duty, rather than of pleasurable choice. At the end of the book, I had a strong desire to work with mathematical symbols that are well defined and lead to definite conclusions, rather than words and their interpretations. This perhaps says more about me than it does about the book.

Atlantic College **Paul Belcher**

First Lessons for Graphics Calculators. By Alan Graham and Barrie Galpin. A+B Books, Corby, 2005. Paperback, 90 pages, £30.00 (ISBN 0-9541020-5-3).

This book is a classroom manual to get teachers and students started using a graphics calculator. There are 20 lessons dealing with such topics as Number, Algebra, Shape, and Data Handling. The worksheets can be photocopied for use by the class, so you only need to buy one copy of the book; hence its price! It is written for the TI-83 and TI-84 families of graphics calculators. (Instructions could be modified for other makes and models.) The book can be bought directly from the publishers at A+B Books, 15 Top Lodge, Fineshade, Corby NN17 3BB, UK, (www.AplusB.co.uk).

The Calculus Gallery. Masterpieces from Newton to Lebesgue. By William Dunham. Princeton University Press, 2005. Hardback, 288 pages, £18.95 (ISBN 0-691-09565-5).

This lovely book is essentially a history of the development of the basic theory of calculus between the 17th and 20th Centuries. Professor Dunham gives portraits of, and some biographical information about, his 13 heroes, and prints facsimiles of the original statements of their key results. He gives their original proofs, indicating their weaknesses, and shows how these were remedied by the development of analysis in the 19th Century and of the Lebesgue integral in the 20th Century. He gives examples of pathological behaviour (of series and functions) to show what motivated this development. Along the way there are many incidental pleasures, such as Liouville's transcendental number and Cantor's original proof of the non-enumerability of the real numbers (quite different from his later 'diagonal' proof).

This beautifully written book can be recommended for university students, to whom it will give a sense of the perspective of the subject, and to teachers of calculus (who will also be reminded of their long-past college studies). It would be valuable in a school library where it will give a bright student a glimpse of a fascinating world where mathematics is no longer just putting numbers into formulae.

Retired Mathematics Teacher **Norman Routledge**

R. L. Moore: Mathematician & Teacher. By John Parker. MAA, Washington, DC, 2004. Hardback, 380 pages, $45.95 (ISBN 0-88385-550-X).

This is a biography of a mathematician who is remembered as much for his teaching, known universally as the Moore method (no textbooks, no lectures, no conferring), and the many mathematicians he educated both directly and indirectly, as he is for his research. Moore himself summed up his method in just eleven words 'that student is taught the best who is told the least'. Having never encountered Moore before, on reading these words I was immediately drawn to him as a man after my own heart. He once described himself as 'Mr Chips with attitude'!

This is a very detailed account of most aspects of Moore's working life, some of which I confess to having found tedious to read. Having said that, the book does contain some interesting tales, not least the ones concerning the political machinations typical of university departments, particularly those with separate 'pure' and 'applied' sections and containing the odd prima donna. The wrangling that went on in The University of Texas as Moore's Department of Pure Mathematics attempted to resist a merger with the Department of Applied Mathematics, and the feud he had with another eminent mathematician in his department (who subsequently 'defected' to the Applied Department), together with their subsequent efforts to avoid having to converse with one another, make for amusing reading.

The remainder of the book, however, is likely only to be of interest to those who have heard of Moore and are interested to know what made him tick, or to the mathematically inclined who enjoy reading how some stars in the field operate.

The University of Reading **P. Glaister**

Mathematics Elsewhere. By Marcia Ascher. Princeton University Press, 2004. Paperback, 207 pages, £10.95 (ISBN 0-691-12022-6).

How can hierarchy and equality be merged using cycles of relations and cycles of changing roles in a little Basque community? The answer challenges our views of hierarchy and equality.

The aim of this book is to introduce mathematical ideas that occur in traditional or small-scale cultures, to examine their differences and similarities, and thus to find out if there are characteristics of mathematics that modern mathematical culture considers to be universal but that are not. Marcia Ascher had written a book before on the same topic, and was encouraged to write this second one by the interest shown in her first work. Here, she presents 13 extended examples and mentions several more.

How can the Balinese handle ten parallel weeks of different lengths, the interaction of which determines what is to be expected from a certain day? How can people from the Marshall islands create maps and navigate on the vast ocean between the islands using only a deep understanding of the behaviour of winds and waves? Divination, time structures, models and maps, systems of relationships, decorative figures on the thresholds in Tamil Nadu, India: the examples are fascinating. However, the analysis of their substantial relationships does not go very deep, not much further than distinguishing the usage of linear and cyclic structures or natural and artificial time periods. But the examples speak for themselves, and their curiosity is more cultural and anthropological than mathematical.

The text is very enjoyable and usually easy to read. At only a few points did I feel that it was too formal (do we really need the explanation of switching cycles and the multiple use of the XOR operation to explain how the sum of some 0s and 1s can be considered to be 1 if it is odd and 0 if it is even?). Before each example, the author explains the necessary mathematical background. Although these topics are not difficult and they do not go much above school level, if they have not been seen before it might be difficult to handle them at once. But the cultural aspects of this book can be enjoyed without a perfect understanding of the mathematics behind it. This book is excellent for everyone with an average mathematical knowledge and some interest in the humanities.

Student, Atlantic College **Balázs Gosztonyi**

A Concise Oxford Dictionary of Mathematics. Edited by Christopher Clapham and James Nicholson. Oxford University Press, 2005. Paperback, 464 pages, £9.99 (ISBN 0-19-860742-3).

The third edition of this extremely useful reference book, with more than 500 new entries over the previous edition, is indispensable for all students of mathematics.

Advancing Maths for AQA: Statistics 2 & 3. By Roger Williamson and Gill Buque. Heinemann, Oxford, 2nd edn., 2004. Paperback, 326 pages, £15.50 (ISBN 0-435-51340-0).

Purpose-written for the AQA syllabus and updated to meet the requirements of the changes effective from September 2004, this is an easy book to follow. Grey boxes on the right-hand side of the page give helpful hints for understanding each topic and for answering questions. The important definitions are boxed, with the word being defined in bold to make it stand out. I personally have found this feature helpful as it makes the definitions easy to find and lets me know straight away what they mean.

Questions within and at the end of topics improve knowledge and understanding, and develop the skills required for success in the AS-level examinations. Answers are provided at the back of the book. Diagrams, sketches and examples aid the exposition and add interest to the book. The absence of colour, which is what I expect from a maths textbook, lets it down. Overall, it is a good book to learn from and will prepare the candidate thoroughly for the ensuing exams.

Student, Solihull Sixth Form College **Andrew Armstrong**

Advancing Maths for AQA: Core Maths 3. By Greg Atwood, Alistair Macpherson, Bronwen Moran, Joe Petran, Geoff Staley and Dave Wilkins. Heinemann, Oxford, 2nd edn., 2004. Paperback, 169 pages, £10.99 (ISBN 0-435-51099-1).

Aimed at students studying the third pure maths module of the Edexcel A-level mathematics syllabus, this book covers eight topics, each with its own colour-coded chapter. This useful feature enables quick access when the book is used for revision. Key points at the end of each chapter summarise the contents and make for easy revision. A plethora of worked examples is present; these are explained well with clear annotations in yellow boxes. Useful pink boxes containing key points and hints are also dotted throughout each chapter.

As well as plenty of questions for the reader to work through, there is also an exam-style paper at the end of the book, helpful in reviewing what has been learned from the topics and when preparing for the examination. The book is finished off nicely with lists of relevant formulae, symbols, and notation.

Overall, I think that this is a good book which is very clear and coherent, and would be useful to students as both a topic book and a revision guide — studying made as easy as it can be!

Student, Solihull Sixth Form College **Phillip Yeomans**

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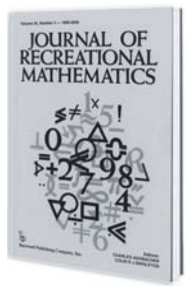

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