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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

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Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

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BUT WHAT ABOUT THE y-INTERCEPT OF THE TANGENT LINE?

George F. Feissner and Stephen G. Penrice

Typically a first course in calculus includes a discussion of several elementary results about derivatives which can be interpreted geometrically in terms of slopes of tangent lines. The purpose of this note is to point out some interesting facts concerning the y-intercepts of tangent lines. Many of these results parallel the more familiar ones, and their proofs require only theorems taught in a traditional calculus course. This material is suitable for group projects or students in an honors course.

We will restrict our attention to functions which are differentiable throughout their domains. To simplify our discussion, we introduce the following notation. Let $i_f(x)$ denote the y-intercept of the tangent to f at $(x, f(x))$. Using the fact that $f'(x)$ gives the slope of the curve at this point, one can write the equation of the tangent line in slope-intercept form to obtain $i_f(x) = f(x) - xf'(x)$.

The "traditional" Mean Value Theorem states that if you draw a secant line through two given points on a curve, there is a point between those two points where the tangent to the curve has the same slope as the secant line. As it turns out, there is also a tangent line with the same y-intercept as the secant, provided the x-coordinates of the two given points have the same sign.

Theorem 1. Suppose $0 \notin [a, b]$ and k is the y-intercept of the secant line joining $(a, f(a))$ and $(b, f(b))$. Then there is a number $c \in [a, b]$ such that the tangent to f at c has y-intercept k.

Proof. According to the hypotheses, $(a, f(a))$, $(b, f(b))$, and $(0, k)$ are collinear. Therefore

$$\frac{f(a) - k}{a} = \frac{f(b) - k}{b} .$$

By Rolle's Theorem, there is a number $c \in [a, b]$ such that the derivative of $(f(x) - k)/x$ at c is zero. But then

$$0 = \frac{cf'(c) - f(c) + k}{c^2} = \frac{-i_f(c) + k}{c^2} ,$$

and so $i_f(c) = k$. □

Just as it is of interest to examine where the tangent line has slope equal to zero, it can also be useful to know where the y-intercept of the tangent is zero. It can be seen from the formula for i_f that $f(x)/x$ has a critical point at x if the tangent to f at x has y-intercept zero. This can also be seen geometrically. For example, Figure 1 shows a case where $f(x)/x$ has a local minimum, and Figure 2 shows a case where $f(x)/x$ has a local maximum. Thus we are motivated to discuss some sufficient conditions for the graph of a function to have a tangent line with y-intercept zero.

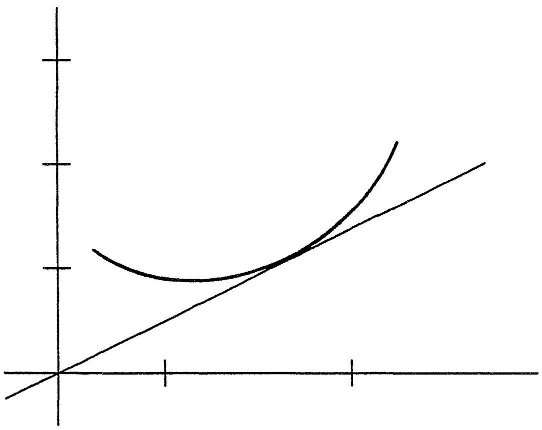

Figure 1

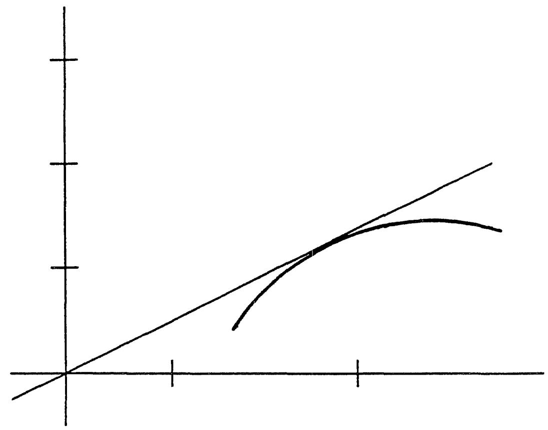

Figure 2

It is well-known that if f is a polynomial of degree n, where n is even and positive, then the graph of f has a horizontal tangent at some point, because f' is a polynomial of odd degree and therefore has a real root. A similar result concerning y-intercepts is true for polynomials of odd degree.

Theorem 2. Let f be a polynomial of odd degree n, where $n \geq 3$. Then there is a tangent line to the graph of f which contains the origin.

Proof. Under this hypothesis, i_f is also a polynomial of degree n; the coefficient on x^n is $(1-n)a$, where a is the coefficient of x^n in f. Thus i_f has a real root. □

It follows immediately from Rolle's theorem that the graph of any periodic function has a horizontal tangent. The corresponding result for y-intercepts holds for non-constant periodic functions.

Theorem 3. Let f be a non-constant periodic function with a continuous first derivative. There is a tangent to the graph of f which contains the origin.

Proof. Let P be the period of f. Choose any x such that $f'(x) < 0$; every interval of the form $[a, a+P]$ contains such a point because, with f non-constant, the interval contains an interior point c at which an extremum occurs, and we may apply the Mean Value Theorem at c and either a or $a+P$ to find x. Since f' is also periodic, if n and m are sufficiently large positive integers,

$$i_f(x-nP) = i_f(x) + nPf'(x) < 0 \quad \text{and} \quad i_f(x+mP) = i_f(x) - mPf'(x) > 0.$$

By the Intermediate Value Theorem, there is some $c \in [x-nP, x+mP]$ such that $i_f(c) = 0$. □

It is easily verified that, like the derivative, i_f is a linear operator on the space of differentiable functions. It is well-known that the eigenvectors of the differentiation operator are the exponential functions ce^{2x}. The eigenvectors of i_f are also easily characterized.

Theorem 4. If λ is a real number, then $\{f(x) = ax^{1-\lambda} \ : \ a \in \mathbb{R}\}$ is the set of functions f such that f is differentiable throughout its domain and $i_f(x) = \lambda f(x)$ for all x in the domain of f.

Proof. It is easily verified that if $f(x) = ax^{1-\lambda}$, then $i_f(x) = \lambda f(x)$. Suppose $i_f(x) = \lambda f(x)$. Using the formula for i_f we obtain the differential equation

$$\frac{f'(x)}{f(x)} = \frac{1-\lambda}{x} .$$

Integrating both sides with respect to x we obtain

$$\ln|f(x)| = C + \ln|x^{1-\lambda}|,$$

where C is an arbitrary constant. Thus, $f(x) = ax^{1-\lambda}$, where a is an arbitrary constant. □

We close with a purely geometric result. Rather than illustrating the parallels between slopes and y-intercepts of tangent lines, it gives an example of how i_f can be applied to a problem that is not obviously a calculus problem.

Theorem 5. Let P be a parabola and let p be a point not in the convex hull of P.

(a) There are exactly two points, p_1 and p_2, on P such that the line containing p and p_i is tangent to P.

(b) Let L be the line containing p which is perpendicular to the axis of symmetry of P. Let $\pi(p_i)$ be the projection of p_i onto L. Then p is the midpoint of the line segment joining $\pi(p_1)$ and $\pi(p_2)$.

Proof. Consider a point p and a parabola P in a plane without coordinate axes. We may impose a set of coordinates so that p is the origin, the y-axis is parallel to P's axis of symmetry, and P is the graph of $y = ax^2 + bx + c$, where a and c are both positive. (Note that if $a > 0$ and $c \leq 0$, then p is in the convex hull of P.) Then L is the x-axis. Note that $i_f(x) = -ax^2 + c$. Thus the only tangents to the graph of f which contain the origin are at $x = (c/a)^{1/2}$ and $x = -(c/a)^{1/2}$. The results follow. □

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THE OLYMPIAD CORNER

No. 157

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

First a correction. When I gave the comment to the solution of 4 [1993: 100; 1994: 154–156] XXV Soviet Math Olympiad, I somehow attributed it to David Vaughan and George Evagelopoulos, when in fact it should be David Vaughan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. My apologies! (The dangers of abbreviations!)

* * *

Next we give the problems of the 1st Mathematical Olympiad of the Republic of China, held at Taipei, May 3–4, 1992. Thanks go to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Newfoundland, for collecting them in Moscow when he was Canadian Team leader to the I.M.O.

1st MATHEMATICAL OLYMPIAD OF THE REPUBLIC OF CHINA (TAIWAN)

First Day — Taipei, May 3, 1992

1. Let A and B be two points on a given circle and M the midpoint of the arc AB. Let ℓ be the tangent line to the circle at A and C the foot of the perpendicular from B to the tangent line ℓ from B. Let the tangent line to the circle at M intersect $\overline{AC}$ and $\overline{BC}$ at A' and B', respectively. Prove that if $\angle BAC < \pi/8$ then $\Delta ABC < 2\Delta A'B'C$.

2. Every positive integer can be expressed as a sum of one or several consecutive positive integers. For each positive integer n, find the total number of the expressions of n as such a sum.

3. If $x_1, x_2, \ldots, x_n$ are n nonnegative numbers, $n \geq 3$, and $x_1 + x_2 + \cdots + x_n = 1$, prove that $x_1^2x_2 + x_2^2x_3 + \cdots + x_n^2x_1 \leq 4/27$.

Second Day — Taipei, May 5, 1992

4. Let r be a positive integer and the sequence $\{a_n\}$ be defined as follows: $a_1 = 1$ and for every positive integer n,

$$a_{n+1} = \frac{n \cdot a_n + 2(n+1)^{2r}}{n+2}\ .$$

Prove that every a_n is an integer and determine the values of n such that a_n is even.

5. If I is the incenter of ΔABC and the line which passes through I and is perpendicular to $\overline{AI}$ intersects $\overline{AB}$ and $\overline{AC}$ at the points P and Q, respectively, prove that the circle which is tangent to $\overline{AB}$ and $\overline{AC}$ at P and Q, respectively, is also tangent to the circumcircle of ΔABC.

6. Determine the maximum value of a positive integer A with the following property: for every permutation of the one thousand positive integers from 1001 to 2000, there always exist 10 consecutive terms whose sum is greater than or equal to A.

* * *

In the June number we gave the questions of the Canadian Mathematical Olympiad [1994: 151]. Next we give the "official solutions". My thanks go to Edward Wang, Chairman of the Canadian Mathematical Olympiad Committee of the Canadian Mathematical Society, for furnishing the problems and solutions.

1. Evaluate the sum $\sum_{n=1}^{1994}(-1)^n\frac{n^2+n+1}{n!}$.

First Solution. Let S denote the given sum. Then

$$\begin{aligned} S &= \sum_{n=1}^{1994}(-1)^n\left(\frac{n}{(n-1)!}+\frac{n+1}{n!}\right) \\ &= \sum_{n=0}^{1993}(-1)^{n+1}\frac{n+1}{n!}+\sum_{n=1}^{1994}(-1)^n\frac{n+1}{n!} \\ &= -1+\frac{1995}{1994!}. \end{aligned}$$

Second Solution. For positive integers k, define

$$S(k)=\sum_{n=1}^{k}(-1)^n\frac{n^2+n+1}{n!}\ .$$

We prove by induction on k that

$$S(k)=-1+(-1)^k\frac{k+1}{k!}. \qquad (*)$$

The given sum is the case when $k=1994$. For $k=1$, $S(1)=-3=-1-(2/1!)$. Suppose $(*)$ holds for some $k\geq 1$. Then

$$\begin{aligned} S(k+1) &= S(k)+(-1)^{k+1}\frac{(k+1)^2+(k+1)+1}{(k+1)!} \\ &= -1+(-1)^k\frac{k+1}{k!}+(-1)^{k+1}\left(\frac{k+1}{k!}+\frac{k+2}{(k+1)!}\right) \\ &= -1+(-1)^{k+1}\frac{k+2}{(k+1)!} \end{aligned}$$

completing the induction.

2. Show that every positive integral power of $\sqrt{2}-1$ is of the form $\sqrt{m}-\sqrt{m-1}$ for some positive integer m.

$$(\text{i.e. } (\sqrt{2}-1)^2 = 3-2\sqrt{2} = \sqrt{9}-\sqrt{8}.)$$

First Solution. Fix a positive integer n. Let $a = (\sqrt{2}-1)^n$ and $b = (\sqrt{2}+1)^n$. Then clearly $ab = 1$. Let $c = (b+a)/2$ and $d = (b-a)/2$. If n is even, $n = 2k$, then from the Binomial Theorem we get

$$c = \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}(\sqrt{2}^{n-i} + (-1)^i\sqrt{2}^{n-i}) = \sum_{j=0}^{k}\binom{2k}{2j}\sqrt{2}^{2k-2j} = \sum_{j=0}^{k}\binom{2k}{2j}2^{k-j}$$

and

$$\begin{aligned}\frac{d}{\sqrt{2}} &= \frac{1}{\sqrt{2}}\sum_{i=0}^{n}\binom{n}{i}(\sqrt{2}^{n-i} - (-1)^i\sqrt{2}^{n-i})\\ &= \frac{2}{\sqrt{2}}\sum_{j=0}^{k-1}\binom{2k}{2j+1}\sqrt{2}^{2k-2j-1} = \sum_{j=0}^{k-1}\binom{2k}{2j+1}2^{k-j}\end{aligned}$$

showing that c and $d/\sqrt{2}$ are both positive integers. Similarly, when n is odd we see that $c/\sqrt{2}$ and d are both positive integers. In either case, c^2 and d^2 are both integers. Note that $c^2-d^2 = 1/4((b+a)^2-(b-a)^2) = ab = 1$. Hence if we let $m = c^2$, then $m-1 = c^2-1 = d^2$ and $a = c-d = \sqrt{m}-\sqrt{m-1}$.

Second Solution. Let m and n be positive integers. Observe that

$$(\sqrt{2}-1)^n(\sqrt{2}+1)^n = 1 = (\sqrt{m}-\sqrt{m-1})(\sqrt{m}+\sqrt{m-1})$$

and so

$$(\sqrt{2}-1)^n = \sqrt{m}-\sqrt{m-1} \tag{$*$}$$

if and only if $(\sqrt{2}+1)^n = \sqrt{m}+\sqrt{m-1}$. Assuming m and n satisfy $(*)$, then adding the two equivalent equations we get $2\sqrt{m} = (\sqrt{2}-1)^n + (\sqrt{2}+1)^n$ whence

$$m = \frac{1}{4}[(\sqrt{2}-1)^{2n} + 2 + (\sqrt{2}+1)^{2n}]. \tag{$**$}$$

Now we show that the steps above are reversible and that m defined by $(**)$ is a positive integer. From $(**)$ one sees easily that

$$\sqrt{m} = \frac{1}{2}[(\sqrt{2}-1)^n + (\sqrt{2}+1)^n] \quad \text{and} \quad \sqrt{m-1} = \frac{1}{2}[(\sqrt{2}+1)^n - (\sqrt{2}-1)^n],$$

and so $\sqrt{m}-\sqrt{m-1} = (\sqrt{2}-1)^n$ as required. Finally, from the Binomial Theorem,

$$\begin{aligned}(\sqrt{2}-1)^{2n} + (\sqrt{2}+1)^{2n} &= \sum_{k=0}^{2n}\binom{2n}{k}[(-1)^k 2^{(2n-k)/2} + 2^{(2n-k)/2}]\\ &= \sum_{l=0}^{n}\binom{2n}{2l}2^{n-l+1}\end{aligned}$$

which is congruent to 2 modulo 4 since $2^{n-l+1} \equiv 0 \pmod 4$ for all $l = 0, 1, 2, \ldots, n-1$. Therefore, $(\sqrt{2}-1)^{2n} + 2 + (\sqrt{2}+1)^{2n}$ is a multiple of 4, as required.

Third Solution. We show by induction that

$$(\sqrt{2}-1)^n = \begin{cases} a\sqrt{2} - b & \text{where } 2a^2 = b^2 + 1 \quad \text{if } n \text{ is odd} \\ a - b\sqrt{2} & \text{where } a^2 = 2b^2 + 1 \quad \text{if } n \text{ is even.} \end{cases} \qquad (*)$$

Thus, $m = 2a^2$ when n is odd and $m = a^2$ when n is even and the problem is solved.

The induction is as follows:

$$(\sqrt{2}-1)^1 = 1\sqrt{2} - 1 \qquad \text{where } 2(1^2) = 1^2 + 1$$

$$(\sqrt{2}-1)^2 = 3 - 2\sqrt{2} \qquad \text{where } 3^2 = 2(2^2) + 1.$$

Assume $(*)$ holds for some $n \geq 1$, n odd. Then

$$\begin{aligned} (\sqrt{2}-1)^{n+1} &= (a\sqrt{2} - b)(\sqrt{2} - 1) \quad \text{where } 2a^2 = b^2 + 1 \\ &= (2a + b) - (a + b)\sqrt{2} \\ &= A - B\sqrt{2} \quad \text{where } A = 2a + b,\ B = a + b. \end{aligned}$$

Moreover, $A^2 = 2a^2 + 4ab + b^2 + 2a^2 = 2a^2 + 4ab + 2b^2 + 1 = 2B^2 + 1$. Assume $(*)$ holds for some $n \geq 2$, n even. Then

$$\begin{aligned} (\sqrt{2}-1)^{n+1} &= (a - b\sqrt{2})(\sqrt{2} - 1) \quad \text{where } a^2 = 2b^2 + 1 \\ &= (a + b)\sqrt{2} - (a + 2b) \\ &= A\sqrt{2} - B \quad \text{where } A = a + b,\ B = a + 2b \end{aligned}$$

Moreover, $2A^2 = 2a^2 + 4ab + 2b^2 = a^2 + 4ab + 4b^2 + a^2 - 2b^2 = B^2 + 1$.

Fourth Solution. From $(\sqrt{2}-1)^1 = \sqrt{2}-1$, $(\sqrt{2}-1)^2 = 3-2\sqrt{2}$, $(\sqrt{2}-1)^3 = 5\sqrt{2}-7$, $(\sqrt{2}-1)^4 = 17 - 12\sqrt{2}$, etc.

$$\begin{aligned} &(\sqrt{2}-1)^n = s_n\sqrt{2} + t_n \qquad \text{where } s_1 = 1,\ t_1 = -1, \\ &s_{n+1} = (-1)^n(|s_n| + |t_n|), \quad t_{n+1} = (-1)^{n+1}(2|s_n| + |t_n|). \end{aligned} \qquad (*)$$

Note that s_n is positive (negative) if n is odd (even), and t_n is negative (positive) if n is odd (even).

We now show by induction that $(*)$ holds and that each $s_n\sqrt{2} + t_n$ is of the form $\sqrt{m} - \sqrt{m-1}$ for some m.

It is easily verified that $(*)$ is correct for $n = 1$ and 2. Assume $(*)$ holds for some $n \geq 2$. Then

$$(\sqrt{2}-1)^{n+1} = (s_n\sqrt{2} + t_n)(\sqrt{2} - 1) = (t_n - s_n)\sqrt{2} + (2s_n - t_n).$$

If n is odd, then

$$t_n - s_n = -(|t_n| + |s_n|) = s_{n+1}, \qquad 2s_n - t_n = 2|s_n| + |t_n| = t_{n+1}.$$

If n is even, then

$$t_n - s_n = |t_n| + |s_n| = s_{n+1}, \qquad 2s_n - t_n = -2|s_n| - |t_n| = t_{n+1}.$$

We have shown that (*) is correct for all n. Observe now that

$$\begin{aligned}(s_{n+1}\sqrt{2})^2 - t_{n+1}^2 &= 2(s_n^2 - 2s_nt_n + t_n^2) - (4s_n^2 - 4s_nt_n + t_n^2)\\ &= -2s_n^2 + t_n^2 = -((s_n\sqrt{2})^2 - t_n^2).\end{aligned}$$

Since $(s_1\sqrt{2})^2 - t_1^2 = 1$, it follows that $(s_n\sqrt{2})^2 - t_n^2 = (-1)^{n+1}$ for all n.

To complete the proof, it suffices to take $m = (s_n\sqrt{2})^2$, $m - 1 = t_n^2$ when n is odd and $m = t_n^2$, $m - 1 = (s_n\sqrt{2})^2$ when n is even.

3. Twenty-five men sit around a circular table. Every hour there is a vote, and each must respond *yes* or *no*. Each man behaves as follows: on the nth vote, if his response is the same as the response of at least one of the two people he sits between, then he will respond the same way on the $(n+1)$th vote as on the nth vote; but if his response is different from that of both his neighbours on the nth vote, then his response on the $(n+1)$th vote will be different from his response on the nth vote. Prove that, however everybody responded on the first vote, there will be a time after which nobody's response will ever change.

Solution. First observe that if two neighbours have the same response on the nth vote, then they both will respond the same way on the $(n+1)$th vote. Moreover, neither will ever change his response after the nth vote.

Let A_n be the set of men who agree with at least one of their neighbours on the nth vote. The previous paragraph says that $A_n \subseteq A_{n+1}$ for every $n \geq 1$. Moreover, we will be done if we can show that A_n contains all 25 men for some n.

Since there are an odd number of men at the table, it is not possible that every man disagrees with both of his neighbours on the first vote. Therefore A_1 contains at least two men. And since $A_n \subset A_{n+1}$ for every n, there exists a $T < 25$ such that $A_T = A_{T+1}$. Suppose that A_T does not contain all 25 men; we shall use this to derive a contradiction. Since A_T is not empty, there must exist two neighbours, whom we shall call x and y, such that $x \in A_T$ and $y \notin A_T$. Since $x \in A_T$, he will respond the same way on the Tth and $(T+1)$th votes. But $y \notin A_T$, so y's response on the Tth vote differs from x's response. In fact, we know that y disagrees with both of his neighbours on the Tth vote, and so he will change his response on the $(T+1)$th vote. Therefore, on the $(T+1)$th vote, y responds the same way as does x. This implies that $y \in A_{T+1}$. But $y \notin A_T$, which contradicts the fact that $A_T = A_{T+1}$. Therefore we conclude that A_T contains all 25 men, and we are done.

4. Let AB be a diameter of a circle Ω and P be any point *not* on the line through A and B. Suppose the line through P and A cuts Ω again in U, and the line through P and B cuts Ω again in V. (Note that in case of tangency U may coincide with A or V may coincide with B. Also, if P is on Ω then $P = U = V$.) Suppose that $|PU| = s|PA|$ and $|PV| = t|PB|$ for some nonnegative real numbers s and t. Determine the cosine of the angle APB in terms of s and t.

Solution. There are three cases to be considered:

<u>Case 1:</u> If P is outside Ω (see Figures I, II and III) then since $\angle AUB = \angle AVB = \pi/2$, we have

$$\cos(\angle APB) = \frac{PU}{PB} = \frac{PV}{PA} = \sqrt{\frac{PU}{PA} \cdot \frac{PV}{PB}} = \sqrt{st}.$$

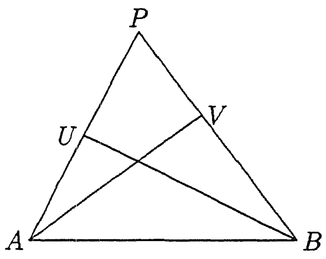

Figure I

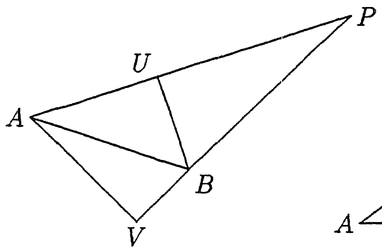

Figure II

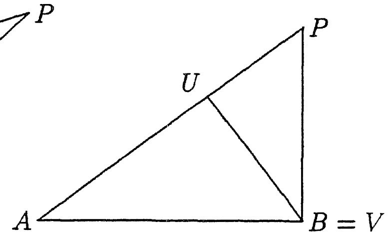

Figure III

<u>Case 2:</u> If P is on Ω (see Figure IV), then

$$P = U = V \Rightarrow PU = PV = 0 \Rightarrow s = t = 0.$$

Since $\angle APB = \pi/2$, $\cos(\angle APB) = 0 = \sqrt{st}$ holds again.

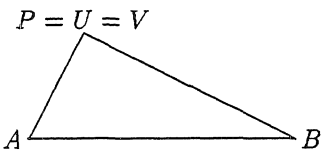

Figure IV

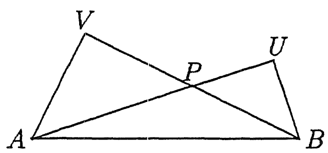

Figure V

<u>Case 3:</u> If P is inside Ω (see Figure V), then

$$\cos(\angle APB) = \cos(\pi - \angle APV) = -\cos(\angle APV) = -\frac{PV}{PA},$$

and

$$\cos(\angle APB) = \cos(\pi - \angle BPU) = -\cos(\angle BPU) = -\frac{PU}{PB}.$$

Therefore,

$$\cos(\angle APB) = -\sqrt{\frac{PU}{PA} \cdot \frac{PV}{PB}} = -\sqrt{st}.$$

5. Let ABC be an acute angled triangle. Let AD be the altitude on BC, and let H be any interior point on AD. Lines BH and CH, when extended, intersect AC and AB at E and F, respectively. Prove that $\angle EDH = \angle FDH$.

First Solution. From A draw a line l parallel to BC. Extend DF and DE to meet l at P and Q, respectively (see Figure I). Then from similar triangles, we have

$$\frac{AP}{BD} = \frac{AF}{FB} \quad \text{and} \quad \frac{AQ}{CD} = \frac{AE}{EC}$$

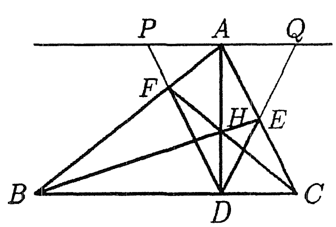

Figure I

or

$$AP = \frac{AF}{FB} \cdot BD \quad \text{and} \quad AQ = \frac{AE}{EC} \cdot CD. \tag{1}$$

By Ceva's Theorem, $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$ and thus

$$\frac{AF}{FB} \cdot BD = \frac{AE}{EC} \cdot CD. \tag{2}$$

From (1) and (2), we get $AP = AQ$ and hence $\Delta ADP \simeq \Delta ADQ$ from which $\angle EDH = \angle FDH$ follows.

Second Solution. Use cartesian coordinates, with D at $(0,0)$, $A = (0,a)$, $B = (-b,0)$, $C = (c,0)$. Let $H = (0,h)$, $E = (u,v)$ and $F = (-r,s)$ where a, b, c, h, u, v, r, s are all positive (see Figure II). It clearly suffices to show that $v/u = s/r$. Since EC and AC have the same slope, we have $v/(u-c) = a/(-c)$. Similarly, since EB and HB have the same slope, $v/(u+b) = k/b$. Thus

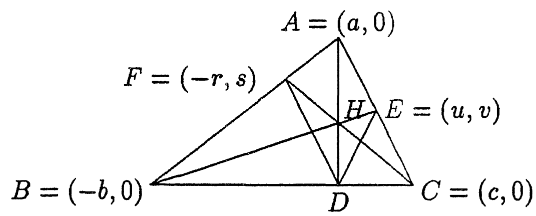

Figure II

$$\frac{v}{a} = \frac{u-c}{-c} = \frac{-u}{c} + 1 \tag{1}$$

and

$$\frac{v}{h} = \frac{u+b}{b} = \frac{u}{b} + 1 \tag{2}$$

From (2) minus (1) we get

$$v\left(\frac{1}{h} - \frac{1}{a}\right) = u\left(\frac{1}{b} + \frac{1}{c}\right)$$

and thus

$$\frac{v}{u} = \frac{\frac{1}{b} + \frac{1}{c}}{\frac{1}{h} - \frac{1}{a}} = \frac{ah(b+c)}{bc(a-h)}\,.$$

With u, v, b, and c replaced by $-r$, s, $-c$ and $-b$ respectively, we have, by a similar argument that

$$\frac{s}{-r} = \frac{ah(-c-b)}{bc(a-h)} \quad \text{or} \quad \frac{s}{r} = \frac{ah(b+c)}{bc(a-h)}\,.$$

Therefore $v/u = s/r$ as desired.

* * *

To finish this number we turn to a problem with some history, Problem 6 of the Final Round of the 1987 Austrian Math. Olympiad [1988: 35]. The solver came upon the problem quite independently when looking for extensions to his article on Factorizing Polynomial pairs, *Mathematical Spectrum*, Vol. 24, No. 3 (1991–1992). He noticed $x^2+x-2 = (x-1)(x+2)$ and set out to characterize the "Self Descriptive Polynomials" which he submitted to *Crux*. Bill noticed the connection to Problem 6 (see his comment to follow) and the problem has become a solution in the Corner!

6. [1988: 35] *18th Austrian Math. Olympiad, Final Round.*
Determine all polynomials

$$P_n(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n,$$

where the a_i are integers and where $P_n(x)$ has as its n zeros precisely the numbers $a_1, \ldots, a_n$ (counted in their respective multiplicities).

Solution by K.R.S. Sastry, Karnataka, Bangalore District, India.
The answer is

(i) $P_n(x) = x^n$; $x^n + 0x^{n-1} + \cdots + 0x + 0 = (x-0)^n$ trivially; $n = 1, 2, 3, \ldots$ nontrivially

(ii) $P_2(x) = x^2 + x - 2 = (x-1)(x+2)$

(iii) $P_3(x) = x^3 + x^2 - x - 1 = (x-1)(x+1)^2$

(iv) $P_n(x) = x^{n-2}p_2(x)$; $n \geq 2$ and

(v) $P_n(x) = x^{n-3}p_3(x)$; $n \geq 3$.

$P_2(x)$ and $P_3(x)$ are determined similarly so we find $P_3(x)$. Note that (iv) and (v) above follow from the easily establishable

Claim. $P_n(x)$ is a solution iff $x^k P_n(x)$ is.

Remark. Because of the claim, it is sufficient to determine solutions $P_n(x)$ that are *not* of the form x^n or $x^k P_{n-k}(x)$. In other words we need to find $P_n(x)$ with $a_n \neq 0$.

Now $P_3(x) = x^3 + a_1 x^2 + a_2 x + a_3 = (x-a_1)(x-a_2)(x-a_3)$. By equating the coefficients of like powers of x we obtain

$$a_1 + a_2 + a_3 = -a_1$$

$$a_1a_2 + a_1a_3 + a_2a_3 = a_2$$

$$a_1a_2a_3 = -a_3, \quad a_3 \neq 0.$$

The above solution has the unique solution

$$a_1 = 1, \quad a_2 = a_3 = -1$$

and hence (iii) follows. We now show the following assertion:

Let $n > 3$ and $a_n \neq 0$. Then

$$P_n(x) = x^n + a_1 x^{n-1} + \cdots + a_n \neq (x-a_1)(x-a_2)\ldots(x-a_n).$$

Proof. Suppose that the equality holds. Then by equating the constant terms, viz. $a_n = (-1)^n a_1 a_2 \ldots a_n$ we see that $a_1 a_2 \ldots a_{n-1} = (-1)^n$.

Since the a_i are integers it must be that $a_i = \pm 1$, $i = 1, 2, \ldots, n-1$. To be specific suppose that p of $a_1, a_2, \ldots, a_{n-1}$ are $+1$ and the remaining $n-p-1$ are -1. Hence

$$P_n(x) = (x-1)^p (x+1)^{n-p-1}(x-a_n). \tag{1}$$

We use the Binomial theorem and expand the right side of (1).

$$\begin{aligned}
P_n(x) &= \left[x^p - px^{p-1} + \left(\frac{p^2-p}{2}\right)x^{p-2} - \cdots\right]\left[x^{n-p-1} + (n-p-1)x^{n-p-2}\right. \\
&\qquad \left. + \frac{n^2-(2p+3)n+p^2+3p+2}{2}x^{n-p-3} + \cdots\right](x-a_n) \\
&= x^n + (n-2p-1-a_n)x^{n-1} + \frac{1}{2}[n^2-(4p+3)n+4p^2+4p \\
&\qquad\qquad +2-2(n-2p-1)a_n]x^{n-2} + \cdots .
\end{aligned}$$

Recall that $a_1, a_2, \ldots,$ the coefficients of x^{n-1}, x^{n-2} are ± 1. Hence

$$n - 2p - 1 - a_n = \pm 1$$

and

$$\frac{1}{2}[n^2 - (4p+3)n + 4p^2 + 4p + 2 - 2(n-2p-1)a_n] = \pm 1.$$

Now four cases arise:

Case (i). $a_1 = 1 = a_2$. From $a_1 = 1$ we get $a_n = n - 2p - 2$ which we substitute into $a_2 = 1$. After simplification we have the quadratic

$$n^2 - (4p+3)n + 4p^2 + 8p + 4 = 0. \tag{2}$$

Since the discriminant of (2) is $-8p - 7 < 0$, no solution arises.

Case (ii). $a_1 = 1$, $a_2 = -1$. This leads similarly to the quadratic

$$n^2 - (4p+3)n + 4p^2 + 8p = 0$$

with the discriminant $-8p+9$ a nonnegative square (a necessary condition for n to be an integer), for $p = 0$ or 1.

(I) $p = 0 \Rightarrow n = 3$ contradicting the hypothesis that $n > 3$

(II) $p = 1 \Rightarrow n = 3$ or $n = 4$. We consider only $n = 4$. But then $a_n = n - 2p - 2 = 0$ contradicting the assumption that $a_n \neq 0$.

Hence no solution arises in this case too. Similarly we find that in
Case (iii). $a_1 = -1$, $a_2 = 1$; and that in
Case (iv). $a_1 = -1 = a_2$
there is no solution.

Comment by Bill Sands. I was interested when I noticed the problem in allowing *real* coefficients for "self-descriptive" polynomials. In that case there is a solution of degree 4, namely

$$x^4 + x^3 - \left(1 + \frac{1}{b}\right) x^2 + \left(\frac{1}{b} - b\right) x + (b - 1)$$

where $b \approx 1.3247179$ is the real root of $x^3 - x - 1 = 0$. I believe this to be the only solution with nonzero constant term. Moreover, suppose we consider the related problem of finding monic polynomials whose roots are exactly the negatives of the coefficients. This seems unnatural but yields equations without the minus signs. Anyway there is also a degree 4 solution for this problem, namely $x^4 - x^3 - bx^2 + \frac{1}{b}x + (b - \frac{1}{b})$ where b is the same as above! I don't have an explanation for this nice relationship, nor do I have results for degrees greater than 4.

* * *

That completes the September Corner. Send me your problem sets and nice solutions.

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PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **April 1, 1995,** *although solutions received after that date will also be considered until the time when a solution is published.*

1961. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an isosceles triangle with $AB = AC$. We denote the circumcircle of ΔABC by Γ. Let D be the point such that DA and DC are tangent to Γ at A and C respectively. Prove that $\angle DBC \leq 30°$.

1962. *Proposed by Murray S. Klamkin, University of Alberta.*

If A, B, C, D are non-negative angles with sum π, prove that

(i) $\cos^2 A + \cos^2 B + \cos^2 C + \cos^2 D \geq 2 \sin A \sin C + 2 \sin B \sin D$;

(ii) $1 \geq \sin A \sin C + \sin B \sin D$.

1963. *Proposed by K. R. S. Sastry, Dodballapur, India.*

In triangle ABC, one pair of trisectors of the angles B and C meet at the orthocenter. Show that the other pair of trisectors of these angles meet at the circumcenter.

1964. *Proposed by Harvey Abbott and Andy Liu, University of Alberta.*

Find a combinatorial proof that

$$\sum_{i=0}^{n}(-1)^i\binom{n+1}{i}\binom{m(n-i)}{n}=\binom{m+n-1}{n}$$

for all positive integers m and n.

1965*. *Proposed by Ji Chen, Ningbo University, China.*

Let P be a point in the interior of the triangle ABC, and let the lines AP, BP, CP intersect the opposite sides at D, E, F respectively.

(a) Prove or disprove that

$$PD\cdot PE\cdot PF\le\frac{R^3}{8}\ ,$$

where R is the circumradius of ΔABC. Equality holds when ABC is equilateral and P is its centre.

(b) Prove or disprove that

$$PE\cdot PF+PF\cdot PD+PD\cdot PE\le\frac{1}{4}\max\{a^2,b^2,c^2\},$$

where a, b, c are the sides of the triangle. Equality holds when ABC is equilateral and P is its centre, and also when P is the midpoint of the longest side of ABC.

1966. *Proposed by Tim Cross, Wolverley High School, Kidderminster, U.K.*

(a) Find all positive integers $p\le q\le r$ satisfying the equation

$$p+q+r+pq+qr+rp=pqr+1.$$

(b) For each such solution (p,q,r), evaluate

$$\tan^{-1}(1/p)+\tan^{-1}(1/q)+\tan^{-1}(1/r).$$

1967. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U.K.*

ABC is a triangle and P is a point in its plane. The lines through P parallel to the medians of the triangle meet the opposite sides in points U,V,W. Describe the set of points P for which U,V,W are collinear.

1968. *Proposed by Richard Blecksmith and John Selfridge, Northern Illinois University, DeKalb.*

Show how to pack 16 squares of sides $1,2,\ldots,16$ into a square of side 39 without overlapping.

1969. *Proposed by Jerzy Bednarczuk, Warszawa, Poland.*

We have two parallelepipeds whose twelve faces are all congruent rhombi. Must these parallelepipeds be congruent?

1970. *Proposed by Susan Gyd, Nose Hill, Alberta.*

Find an isosceles triangle and a rectangle, both with integer sides, which have the same area and the same perimeter.

*   *   *   *   *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1440.* [1989: 111; 1990: 209] *Proposed by Jack Garfunkel, Flushing, New York.*

Prove or disprove that if A, B, C are the angles of a triangle,

$$\frac{\sin A}{\sqrt{\sin A+\sin B}}+\frac{\sin B}{\sqrt{\sin B+\sin C}}+\frac{\sin C}{\sqrt{\sin C+\sin A}}\le\frac{3}{2}\cdot\sqrt[4]{3}.$$

II. *Comment by Ji Chen and Zhen Wang, Ningbo University, China.*

In his published solution [1990: 209], M. E. Kuczma used the Cauchy inequality to restate the given inequality as

$$2\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right)\le\frac{9\sqrt{3}\,R}{a+b+c}\,, \tag{1}$$

where a, b, c, and R are the sides and circumradius of the triangle. He then proved (1). Here we give a simpler proof of (1).

First we note that

$$2(a^2b+b^2c+c^2a)\le a^3+b^3+c^3+3abc;$$

for letting $a=\max\{a,b,c\}$, we have

$$\sum a^3+3abc-2(a^2b+b^2c+c^2a)=(a-b)(a-c)(a+c-b)+(b-c)^2(b+c-a)\ge 0.$$

Now

$$\begin{aligned}2\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right)&=\frac{2(a^2b+b^2c+c^2a)+2\sum a^2(b+c)+6abc}{(b+c)(c+a)(a+b)}\\&\le\frac{a^3+b^3+c^3+3abc+2\sum a^2(b+c)+6abc}{(b+c)(c+a)(a+b)}\\&=\frac{(a^3+b^3+c^3)+2(b+c)(c+a)(a+b)+5abc}{(b+c)(c+a)(a+b)}\\&=\frac{2s(s^2-3r^2-6Rr)+4s(s^2+r^2+2Rr)+20sRr}{2s(s^2+r^2+2Rr)}\\&=\frac{3s^2-r^2+8Rr}{s^2+r^2+2Rr}\,,\end{aligned} \tag{2}$$

where s is the semiperimeter and r the inradius of the triangle. [The known relations

$$\sum a^3 = 2s(s^2 - 3r^2 - 6Rr), \quad \prod(b+c) = 2s(s^2 + r^2 + 2Rr), \quad abc = 4sRr$$

can be found for example on pp. 52–53 of [2]. —*Ed.*] Now by item 5.4 of [1],

$$s \le 2R + (3\sqrt{3} - 4)r \le \frac{7R + 4r}{2\sqrt{3}}$$

[the second inequality follows from $2r \le R$], and so

$$\frac{9\sqrt{3}\,R}{a+b+c} = \frac{9\sqrt{3}\,R}{2s} \ge \frac{27R}{7R+4r} .$$

Thus by (2), (1) will follow from

$$\frac{3s^2 - r^2 + 8Rr}{s^2 + r^2 + 2Rr} \le \frac{27R}{7R + 4r} ;$$

but

$$\begin{aligned} 27R(s^2 + r^2 + 2Rr) - (7R + 4r)(3s^2 - r^2 + 8Rr) &= (6R - 12r)s^2 - 2R^2r + 2Rr^2 + 4r^3 \\ &= 2(R - 2r)[3s^2 - 2r(R + r)] \ge 0, \end{aligned}$$

where by item 5.8 of [1] we have

$$s^2 \ge r(16R - 5r) > \frac{2}{3}r(R + r).$$

References:

[1] O. Bottema et al, *Geometric Inequalities.*

[2] D.S. Mitrinović, J.E. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities.*

* * * * *

1754*. [1992: 175; 1993: 151] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let n and k be positive integers such that $2 \le k < n$, and let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers satisfying $\sum_{i=1}^n x_i = 1$. Prove or disprove that

$$\sum x_1 x_2 \ldots x_k \le \max\left\{\frac{1}{k^k}, \frac{1}{n^{k-1}}\right\},$$

where the sum is cyclic over $x_1, x_2, \ldots, x_n$. [The case $k = 2$ is known — see inequality (1) in the solution of *Crux* 1662 [1992: 188].]

Partial solution by Waldemar Pompe, student, University of Warsaw, Poland.

We first show that the inequality holds if $n \equiv 0, 1$ or $-1 \bmod k$ and n is "not too small".

Case (i): suppose that $n = kl$ for some integer $l \geq 2$. Then

$$\frac{1}{n^{k-1}} = \frac{1}{k^{k-1}l^{k-1}} \leq \frac{1}{k^{k-1}2^{k-1}} \leq \frac{1}{k^k}.$$

Applying the A.M.–G.M. inequality we obtain

$$\begin{aligned}\sum x_1x_2\dots x_k &\leq (x_1 + x_{k+1} + \dots + x_{(l-1)k+1})(x_2 + x_{k+2} + \dots + x_{(l-1)k+2})\dots \\ &\qquad\qquad (x_k + x_{2k} + \dots + x_{lk}) \\ &\leq \left(\frac{x_1 + x_2 + \dots + x_n}{k}\right)^k = \frac{1}{k^k}\end{aligned}$$

with equality, for example, when $x_1 = \ldots = x_k = 1/k$ and $x_j = 0$ for $j > k$.

Case (ii): suppose that $n = lk + 1$ for some integer $l \geq 2$. Then of course $1/n^{k-1} < 1/(k^{k-1}l^{k-1}) \leq 1/k^k$. Since the sum is cyclic, we may assume that $x_1 \leq x_r$ for all $r \in \{1, 2, \dots, n\}$. Again by the A.M.–G.M. inequality we get

$$\begin{aligned}\sum x_1x_2\dots x_k &= (x_1\dots x_k + \dots + x_{n-k+1}\dots x_n) + \sum_{r=3}^{k+1} x_{(l-1)k+r}\dots x_{lk+1}x_1x_2\dots x_{r-2} \\ &\leq (x_1\dots x_k + \dots + x_{n-k+1}\dots x_n) + \sum_{r=3}^{k+1} x_{(l-1)k+r}\dots x_{lk+1}x_{r-1}x_2\dots x_{r-2} \\ &\leq (x_1 + x_{k+1} + \dots + x_{lk+1})(x_2 + x_{k+2} + \dots + x_{(l-1)k+2})\dots \\ &\qquad\qquad (x_k + x_{2k} + \dots + x_{lk}) \\ &\leq \left(\frac{x_1 + x_2 + \dots + x_n}{k}\right)^k = \frac{1}{k^k}\end{aligned}$$

with equality as above.

Case (iii): suppose that $n = lk - 1$. Let either ($k = 3$ and $l \geq 3$) or ($k \geq 4$ and $l \geq 2$). Then still $1/n^{k-1} \leq 1/k^k$. Without loss of generality, we may assume that $x_r \leq x_k$ for all $r \in \{1, 2, \dots, n\}$. In both cases the A.M.–G.M. inequality gives:

$$\begin{aligned}\sum x_1x_2\dots x_k &= \left(\sum_{i=1}^{n-k+1} x_i\dots x_{i+k-1}\right) + x_{(l-1)k+1}x_{(l-1)k+2}\dots x_{lk-1}x_1 \\ &\qquad\qquad + \sum_{r=2}^{k-1} x_{(l-1)k+r}\dots x_{lk-1}x_1\dots x_{r-1}x_r \\ &\leq \left(\sum_{i=1}^{n-k+1} x_i\dots x_{i+k-1}\right) + x_kx_{(l-1)k+2}\dots x_{lk-1}x_1 \\ &\qquad\qquad + \sum_{r=2}^{k-1} x_{(l-1)k+r}\dots x_{lk-1}x_1\dots x_{r-1}x_k\end{aligned}$$

$$\leq (x_1 + x_{k+1} + \cdots + x_{(l-1)k+1}) \ldots (x_{k-1} + x_{2k-1} + \cdots + x_n)$$
$$(x_k + x_{2k} + \cdots + x_{(l-1)k})$$
$$\leq \left(\frac{x_1 + x_2 + \cdots + x_n}{k}\right)^k = \frac{1}{k^k}$$

with equality as in the above two cases.

Now let $k = 3$. If $n \geq 6$, then from (i)–(iii) it follows that $\sum_{i=1}^{n} x_i x_{i+1} x_{i+2} \leq 1/27$ (as usual $x_{n+1} = x_1$, $x_{n+2} = x_2$), so the inequality is true. If $n = 4$ then according to Maclaurin's inequality (e.g., [1]) we get

$$\sqrt[3]{\frac{x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2}{4}} \leq \frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{1}{4},$$

which leads to $x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2 \leq 1/16$, with equality if $x_1 = x_2 = x_3 = x_4 = 1/4$, so the inequality again holds.

Therefore if $k = 3$, only the case $n = 5$ remains unsolved.

Next let $k = 4$. If $n \geq 7$, then from (i)–(iii) it follows that it is enough to prove the inequality when $n \equiv 2 \bmod 4$. So let $n = 4l + 2$, $l \geq 2$. We may assume that for all $r \in \{1, 2, \ldots, n\}$, $x_1 \leq x_r$. Hence in particular, $x_1 \leq x_3$ and $x_1 \leq x_{n-1}$. If $x_2 \leq x_n$, then for all $i \in \{1, 2, \ldots, n\}$, set $a_i = x_{n+3-i}$ (the subscripts are taken modulo n). Otherwise, i.e. when $x_n < x_2$, set $a_i = x_{i-1}$. In both cases $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} x_i = 1$ and

$$\sum_{i=1}^{n} x_i x_{i+1} x_{i+2} x_{i+3} = \sum_{i=1}^{n} a_i a_{i+1} a_{i+2} a_{i+3}$$

with the assumption that $a_1 \leq a_3$ and $a_2 \leq a_4$. Therefore

$$\begin{aligned}
\sum_{i=1}^{n} a_i a_{i+1} a_{i+2} a_{i+3} &= \sum_{i=1}^{n-3} a_i a_{i+1} a_{i+2} a_{i+3} + (a_{n-2}a_{n-1}a_n a_1 + a_{n-1}a_n a_1 a_2 + a_n a_1 a_2 a_3) \\
&\leq \sum_{i=1}^{n-3} a_i a_{i+1} a_{i+2} a_{i+3} + (a_{n-2}a_{n-1}a_n a_3 + a_{n-1}a_n a_3 a_4 + a_n a_1 a_4 a_3) \\
&\leq (a_1 + a_5 + \cdots + a_{n-1})(a_2 + a_6 + \cdots + a_n)(a_3 + a_7 + \cdots + a_{n-3}) \\
&\qquad (a_4 + a_8 + \cdots + a_{n-2}) \\
&\leq \left(\frac{a_1 + a_2 + \cdots + a_n}{4}\right)^4 = \frac{1}{4^4} \quad \left(= \frac{1}{k^k}\right)
\end{aligned}$$

with equality if $x_1 = x_2 = x_3 = x_4 = 1/4$ and $x_j = 0$ for $j \geq 5$, for example.

If $n = 5$, then once again Maclaurin's inequality gives

$$\sqrt[4]{\frac{x_1x_2x_3x_4 + x_2x_3x_4x_5 + \cdots + x_5x_1x_2x_3}{5}} \leq \frac{x_1 + x_2 + \cdots + x_5}{5} = \frac{1}{5}$$

and we obtain $\sum_{i=1}^{5} x_i x_{i+1} x_{i+2} x_{i+3} \le 1/5^3$, with equality when $x_1 = x_2 = \ldots = x_5 = 1/5$.

Therefore if $k = 4$, only the case $n = 6$ remains unsolved.

Note: (i) and (ii) give another proof for $k = 2$ and $n \ge 4$.

Reference:

[1] D. S. Mitrinović, *Analytic Inequalities*, Berlin–Heidelberg–New York, 1970 (Chapter 2.15.1, Theorem 4).

* * * * *

1871. [1993: 234] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let P be a variable point on the arc BC (not containing A) of the circumcircle of ΔABC, and let I_1, I_2 be the incenters of triangles PAB and PAC, respectively. Prove that:

(a) the circumcircle of ΔPI_1I_2 passes through a fixed point;

(b) the circle with diameter I_1I_2 passes through a fixed point;

(c) the midpoint of I_1I_2 lies on a fixed circle.

I. *Solution by Waldemar Pompe, student, University of Warsaw, Poland.*

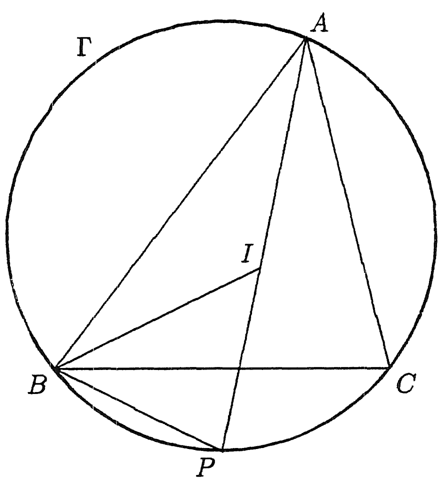


Figure 1

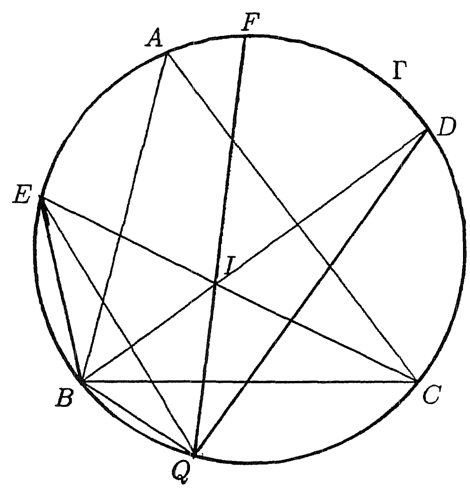


Figure 2

Lemma 1. ABC is a triangle with circumcircle Γ. I is the incenter of ΔABC, P is the midpoint of the arc BC. Then $PI = PB$ $(= PC)$.

Proof. (Figure 1) Clearly, A, I, P are collinear and

$$\angle PBI = \angle PBC + \angle CBI = \angle PAC + \angle IBA = \angle IAB + \angle IBA = \angle BIP,$$

which gives $PI = PB$. $\square$

Lemma 2. ABC is a triangle with circumcircle Γ, I is its incenter. D, E are the midpoints of the arcs CA, AB, respectively and F is the midpoint of the arc CAB. Q is the second intersection of Γ with the line FI. Then triangles EBQ and DIQ are similar.

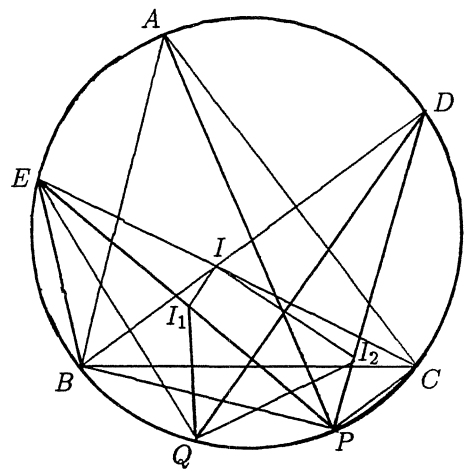
Proof. (Figure 2) Clearly, $\angle BEQ = \angle BDQ = \angle IDQ$. Let $\alpha = \operatorname{arc} AE = \operatorname{arc} EB$, $\beta = \operatorname{arc} CD = \operatorname{arc} DA$. Then

$$\operatorname{arc} EB = (\alpha + \beta) - \beta = \operatorname{arc} CF - \operatorname{arc} CD = \operatorname{arc} DF,$$

which gives $\angle EQB = \angle FQD = \angle IQD$. $\square$

Now we solve the problem.

(a) Assume the same notation as in Lemma 2. We show that I_1, I_2, P, Q are concyclic; then Q is the required fixed point, so we'll be done. We have



$$\angle I_2DQ = \angle PDQ = \angle PEQ = \angle I_1EQ.$$

Moreover (by Lemmas 1 and 2)

$$\frac{EI_1}{EQ} = \frac{EB}{EQ} = \frac{DI}{DQ} = \frac{DC}{DQ} = \frac{DI_2}{DQ}.$$

Therefore triangles EI_1Q and DI_2Q are similar, which gives $\angle I_1QE = \angle DQI_2$. Thus

$$\angle I_1QI_2 = \angle EQD = \angle EPD = \angle I_1PI_2,$$

which shows that I_1, I_2, P, Q are concyclic.

(b) It is enough to prove that $\angle I_1II_2 = 90°$, so that I is the required fixed point. Since $\operatorname{arc} BP + \operatorname{arc} PC = \operatorname{arc} BC$,

$$\angle I_1EI + \angle IDI_2 = \angle PEC + \angle BDP = \angle BAC.$$

Therefore (since $EI = EI_1$ and $DI = DI_2$ by Lemma 1)

$$\begin{aligned}
\angle I_1II_2 &= 360° - (\angle EPD + \angle II_1P + \angle II_2P) \\
&= 360° - \left(\frac{180° - \angle BAC}{2} + \left(90° + \frac{\angle I_1EI}{2}\right) + \left(90° + \frac{\angle IDI_2}{2}\right)\right) \\
&= 360° - 270° \ = \ 90°.
\end{aligned}$$

(c) Let I_3 be the incenter of ΔPBC. Analogously, like in (b), we can prove that $\angle II_2I_3 = \angle I_1I_3I_2 = \angle II_1I_3 = 90°$. Therefore $II_1I_3I_2$ is a rectangle, which shows that the midpoint of II_3 is the midpoint of I_1I_2. So we only need to show that the midpoint of II_3 lies on a fixed circle.

Because of Lemma 1, the I_3's lie on a fixed circle $\mathcal{C}$ with center F (notation of Lemma 2). Consider a homothety H with center I and scale $1/2$. $H(\mathcal{C})$ is another fixed circle, say $\mathcal{C}'$, which contains the midpoints of the II_3's (because of the scale equal to $1/2$). Therefore the midpoint of I_1I_2 lies on the fixed circle $\mathcal{C}'$.

II. *Solution to part* (c) *by the proposer.*

[The proposer first solved parts (a) and (b). The notation in his nice argument for (c) has been changed to agree with Pompe's solution. — *Ed.*]

Let M be the midpoint of I_1I_2. Then $MI = MI_1$ and $EI = EI_1$, thus ME is the bisector of $\angle I_1MI$. Similarly MD is the bisector of $\angle IMI_2$. Therefore $\angle EMD = 90°$, hence M lies on the circle with diameter ED.

Also solved by JORDI DOU, Barcelona, Spain. Parts (b) *and* (c) *only were solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; and P. PENNING, Delft, The Netherlands.*

*   *   *   *   *

1872*. [1993: 234] *Proposed by Murray S. Klamkin, University of Alberta.*

Are there any integer sided non-equilateral triangles whose angles are in geometric progression?

Comment by Václav Konečný, Ferris State University, Big Rapids, Michigan.

We may consider rational sided triangles whose angles are in geometric progression with common ratio $r > 1$. Let $R = r^2 + r + 1$; then we show *there are no such triangles if R or r/R is rational.* This includes all rational values of $r > 1$ and a countable number of irrational values of r. The problem remains to be proved or disproved when both R and r/R are irrational.

Let α be the smallest angle, so $0 < \alpha < \pi/3$. The condition $\alpha + r\alpha + r^2\alpha = \pi$ yields that the three angles of the triangle are

$$\alpha = \frac{1}{r^2+r+1}\pi, \quad \beta = \frac{r}{r^2+r+1}\pi, \quad \gamma = \frac{r^2}{r^2+r+1}\pi.$$

[Note that also $0 < \beta < \pi/3$.] It follows from the law of cosines that if the sides of a triangle are rational then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are rational. In our case, $1/2 < \cos\beta < \cos\alpha < 1$. It is known ([1], [2]) that $\cos(m\pi/n)$ is irrational for rational m/n strictly between 0 and 1/3 [in fact, the only rational values of $\cos k\pi$ for rational k are $0, \pm 1/2$ and ± 1]. Thus if R or r/R is rational, $\cos\alpha$ must be irrational, so there are no such triangles.

References:

[1] R.B. Killgrove and D.W. Koster, Regular polygons with rational area or perimeter, *Mathematics Magazine* **64**, no. 2 (1991) 109–114.

[2] J.M.H. Olmsted, Rational values of trigonometric functions, *Amer. Math. Monthly* **52** (1945) 507–508.

The proposer also observes that no such triangle is possible if r is rational. The problem remains unsolved in general. Any ideas?

One reader apparently misread the problem and found integer sided triangles whose angles are in arithmetic progression. This in fact was a part of the proposer's original problem and was solved by the proposer, but was not included by the editor because such triangles were the subject of Crux 724 [1983: 92].

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1873. [1993: 234] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two circles externally tangent at point P. Let $A_1 \neq P$ be a variable point on $\mathcal{C}_1$ and let $A_2 \neq P$ be on $\mathcal{C}_2$ so that A_1, P, A_2 are collinear. Point A_3 is in the plane of $\mathcal{C}_1$ and $\mathcal{C}_2$ so that $A_1A_2A_3$ is directly similar to a given fixed triangle. Determine the locus of A_3.

Solution by Waldemar Pompe, student, University of Warsaw, Poland.

Let PQ_1 and PQ_2 be the diameters of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively (on the line of centres through P), and Q_3 be the point such that $Q_1Q_2Q_3$ is directly similar to the given fixed triangle. Let $A_1A_2A_3$ be any triangle satisfying the conditions of the problem. From the similar right triangles Q_1PA_1 and Q_2PA_2 we get

$$\frac{Q_1P}{Q_2P} = \frac{A_1P}{A_2P} .$$

So P divides the segments Q_1Q_2 and A_1A_2 in the same ratio and, since $\Delta Q_1Q_2Q_3$ is similar to $\Delta A_1A_2A_3$, it follows that ΔQ_1PQ_3 is similar to ΔA_1PA_3. This means that $\angle Q_1PQ_3 = \angle A_1PA_3$ and

$$\frac{Q_1P}{A_1P} = \frac{Q_3P}{A_3P} ;$$

furthermore, $\angle Q_1PA_1 = \angle Q_3PA_3$ (by subtracting the common angle A_1PQ_3). Consequently, ΔQ_1PA_1 is similar to ΔQ_3PA_3, so that $\angle PA_3Q_3 = \angle PA_1Q_1 = 90°$. Since PQ_3 is fixed, A_3 must lie on the circle $\mathcal{C}$ with diameter PQ_3. It is easy to see that any point of $\mathcal{C}$ different from P can serve as the vertex A_3. Thus the locus of A_3 is the set $\mathcal{C}\backslash\{P\}$, where $\mathcal{C}$ is the circle with diameter PQ_3.

Editor's comments by Chris Fisher. Most solvers showed that for all positions of A_3 on $\mathcal{C}$, the lines A_1A_3 pass through the second point of intersection of $\mathcal{C}_1$ with $\mathcal{C}$, while the lines A_2A_3 pass through the second point where $\mathcal{C}_2$ meets $\mathcal{C}$. Note that P is the *pivot* (or *Miquel*) point of $\Delta A_1A_2A_3$ with respect to the three circles (as discussed, for example, in R.A. Johnson, *Advanced Euclidean Geometry*, Chapter VII); unlike the usual approach where $\Delta A_1A_2A_3$ is fixed and the circles are variable, in our problem the three circles are fixed and the triangle varies, very much like Johnson's corollary 188(d).

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; JORDI DOU, Barcelona, Spain; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

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1874. [1993: 234] *Proposed by Pedro Melendez, Belo Horizonte, Brazil.*
Find the smallest positive integer n such that $n!$ is divisible by 1993^{1994}.

Solution by Cyrus Hsia, student, Woburn Collegiate, Scarborough, Ontario.
The smallest positive integer n such that $n!$ is divisible by 1993^{1994} is $n = \mathbf{1993^2}$.
The largest number of factors of a prime p that divides $n!$ is

$$\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \cdots,$$

where $\lfloor\ \rfloor$ denotes the greatest integer function. Note that 1993 is [was? — *Ed.*] prime. We need to show that for any $n < 1993^2$, say $n = 1993^2 - k$ (k a positive integer), the largest number of factors of 1993 that divides $n!$ is less than 1994. We have

$$\left\lfloor \frac{1993^2 - k}{1993} \right\rfloor + \left\lfloor \frac{1993^2 - k}{1993^2} \right\rfloor + \left\lfloor \frac{1993^2 - k}{1993^3} \right\rfloor + \cdots \leq 1992 + 0 + 0 + \cdots < 1994,$$

therefore the required n is $\geq 1993^2$. Now for $n = 1993^2$, the largest number of factors of 1993 that divides $n!$ is

$$\left\lfloor \frac{1993^2}{1993} \right\rfloor + \left\lfloor \frac{1993^2}{1993^2} \right\rfloor + \left\lfloor \frac{1993^2}{1993^3} \right\rfloor + \cdots = 1993 + 1 + 0 + \cdots = 1994.$$

Therefore $(1993^2)!$ is divisible by 1993^{1994}, and $n = 1993^2$ is the smallest.

Also solved by H.L. ABBOTT, University of Alberta; FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; CHARLES ASHBACHER, Cedar Rapids, Iowa; SEUNG-JIN BANG, Seoul, Korea; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; TIM CROSS, Wolverley High School, Kidderminster, U.K.; KEITH EKBLAW, Walla Walla Community College, Walla Walla, Washington; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; DAVID HANKIN, John Dewey H.S., Brooklyn, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; PAVLOS B. KONSTADINIDIS, student, University of Arizona, Tucson; KEE-WAI LAU, Hong Kong; DAVID E. MANES, State University of New York, Oneonta; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; DAVID R. STONE, Georgia Southern University, Statesboro; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Several solvers (including the proposer in his original problem) mention the more general result that p^2 is the smallest n so that $n!$ is divisible by p^{p+1}, for any prime p.

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1875. [1993: 235] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

The iterates $f^2, f^3, \ldots$ of a mapping f of a set A into itself are defined by: $f^2(x) = f(f(x))$, $f^3(x) = f(f^2(x))$, and in general $f^{i+1}(x) = f(f^i(x))$. Assuming that A has n elements ($n \geq 3$), find the number of mappings $f : A \to A$ such that f^{n-2} is a constant while f^{n-3} is not a constant.

Solution by Robert Geretschläger, Bundesrealgymnasium, Graz, Austria.

Assume $A = \{a_1, a_2, \ldots, a_n\}$. We first note that it is not possible for f to map k elements of A onto each other cyclically, i.e.

$$f(a_1) = a_2, \quad f(a_2) = a_3, \quad \ldots, \quad f(a_{k-1}) = a_k, \quad f(a_k) = a_1,$$

as this would mean that each of the elements $a_1, \ldots, a_k$ is represented in the range of f^i for every i, in contradiction to the fact that f^{n-2} is to be a constant. Similarly, there can be no two elements a_1 and a_2 which are mapped onto themselves for the same reason. There is therefore a unique $a_n \in A$ with the property $f(a_n) = a_n$. (If there were none, a cycle would result, by the pigeon-hole principle.) Sufficient iterations of f must eventually map every element of A onto this a_n, since all other elements are mapped onto other elements by f.

Since f^{n-3} is not a constant, an element a_i must exist for which $f^{n-3}(a_i) \neq a_n$. This means that $f^j(a_i) \neq a_n$ for each $1 \leq j \leq n-3$. Each element $f^j(a_i)$ must be different, since there would otherwise exist either a cycle or a second a_i which f maps onto itself, other than a_n. f therefore implies an ordering of $n-1$ elements of A, starting with this a_i and ending in a_n, as $f^{n-2}(a_i) = a_n$. We assume these elements to be $a_2, a_3, \ldots, a_n$. The remaining element a_1 can be mapped onto any one of the elements $a_3, \ldots, a_n$ (not a_2, as this would mean $f^{n-2}(a_1) = a_{n-1} \neq a_n$).

Since a_1 can be any of the n elements of A, the others can be ordered in $(n-1)!$ ways, and a_1 can be mapped onto $n-2$ different elements of A, it would seem that there are $n(n-2)(n-1)!$ mappings. We have, however, counted those mappings twice for which $f(a_1) = f(a_2) = a_3$. Since the elements a_1 and a_2 can be chosen freely, and the other elements put in any order, there are $\binom{n}{2} \cdot (n-2)!$ mappings of this type, and so the total number of mappings f satisfying the given conditions is

$$n(n-2)(n-1)! - \frac{n(n-1)}{2} \cdot (n-2)! = n!\left(n - 2 - \frac{1}{2}\right) = \frac{n!(2n-5)}{2}.$$

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. There was one incorrect solution sent in.

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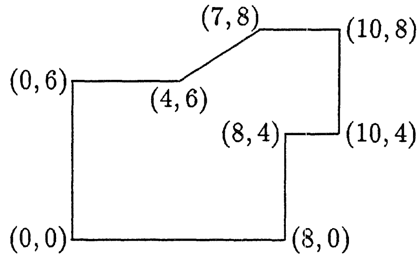
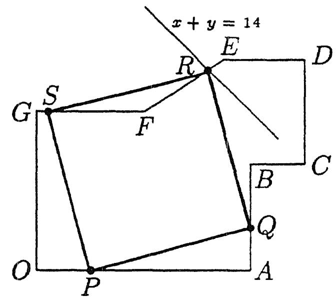
1876. [1993: 235] *Proposed by Bernardo Recamán, United World College of Southern Africa, Mbabane, Swaziland.*

Find four points on the boundary of the following octagon which are the corners of a square.



Solution by Jordi Dou, Barcelona, Spain.



Label the points as in the diagram. Suppose that the vertices P, Q, R, S of the square are on the respective sides OA, AB, EF, FG of the octagon, and consider the simple property: *if J, K, L are three consecutive vertices (counterclockwise) of a square, then $x_J - x_K = y_K - y_L$ where x_N, y_N are the coordinates of N.* We have

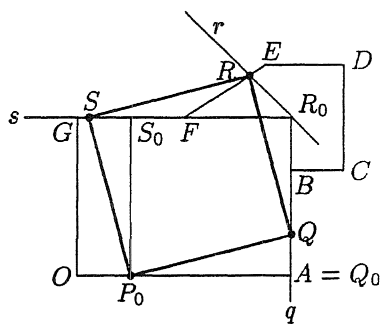
$$8 - x_R = x_Q - x_R = y_R - y_S = y_R - 6,$$

so $x_R + y_R = 14$. Thus R is on the line $x + y = 14$. R is also on EF, which has equation

$$y - 6 = \frac{2}{3}(x - 4).$$

Solving these, we get $x_R = 6.4$, $y_R = 7.6$. From $x_R - x_S = y_S - y_P$ we have $6.4 - x_S = 6 - 0$ or $x_S = 0.4$. Since $\overrightarrow{SP} = \overrightarrow{RQ}$, we find that $y_Q = 1.6$ and $x_P = 2$. Summarizing:

$$P = (2,0), \quad Q = (8,1.6), \quad R = (6.4,7.6), \quad S = (0.4,6).$$



Here is another solution. Put $Q_0 = A$, $R_0 = AB \cap GF$, P_0 on OA such that $AP_0 = AR_0$ (so $x_{P_0} = 2$), and S_0 on R_0G so that $R_0S_0 = R_0Q_0$. Then $P_0Q_0R_0S_0$ is a square. Consider the lines $q = Q_0B$, $s = S_0G$, and r the perpendicular to P_0R_0 through R_0. With centre P_0, rotate the lines P_0Q_0, P_0R_0, P_0S_0 through an angle α; then the intersections Q', R', S' with q, r, s give us a square $P_0Q'R'S'$. If the angle α is $\angle R_0P_0R$ where $R = r \cap EF$, we obtain the square $PQRS$ we are looking for.

Also solved by H.L. ABBOTT, University of Alberta; CHARLES ASHBACHER, Cedar Rapids, Iowa; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; CHARLES

R. DIMINNIE, St. Bonaventure University, St. Bonaventure, N. Y.; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; PAVLOS B. KONSTADINIDIS, student, University of Arizona, Tucson; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; and CHRIS WILDHAGEN, Rotterdam, The Netherlands. One incorrect solution was sent in.

Geretschläger's and Perz's solutions were very similar to Dou's first and second solutions, respectively!

There is an old and fairly well known unsolved problem: does every simple closed curve in the plane contain four points which are the vertices of a square? *The answer is known to be yes for all "nice" curves, certainly including all polygons like the one in this problem. See pages* 51–52 *of Croft, Falconer and Guy's* Unsolved Problems in Geometry *(Springer–Verlag,* 1991*), or pages* 58–65 *and* 137–144 *of Klee and Wagon's* Old and New Unsolved Problems in Plane Geometry and Number Theory *(MAA,* 1991*) for more information.*

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1878*. [1993: 235] *Proposed by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*

Given two triangles ABC and $A'B'C'$, prove or disprove that

$$\frac{\sin A'}{\sin A}+\frac{\sin B'}{\sin B}+\frac{\sin C'}{\sin C}\le 1+\frac{R}{r}\ ,$$

where r, R are the inradius and circumradius of triangle ABC.

Solution by Kee-Wai Lau, Hong Kong.

We prove the inequality.

Let a, b and c be the lengths of triangle ABC. Since $a = 2R\sin A$ etc., we need only show that

$$\frac{\sin A'}{a}+\frac{\sin B'}{b}+\frac{\sin C'}{c}\le\frac{R+r}{2Rr}\ .$$

In [1], M.S. Klamkin proves that for $x, y, z > 0$ we have

$$x\sin A'+y\sin B'+z\sin C'\le\frac{1}{2}(xy+yz+zx)\sqrt{\frac{x+y+z}{xyz}}\ .$$

So by substituting $x = 1/a$, $y = 1/b$, $z = 1/c$ we obtain

$$\frac{\sin A'}{a}+\frac{\sin B'}{b}+\frac{\sin C'}{c}\le\frac{1}{2}\frac{(a+b+c)\sqrt{ab+bc+ca}}{abc}=\frac{\sqrt{s^2+r^2+4Rr}}{4Rr}\ ,$$

where $s = (a+b+c)/2$ is the semiperimeter, and the formulas $ab+bc+ca = s^2+r^2+4Rr$ and $abc = 4sRr$ are known (e.g. [2], page 52). Thus it remains to show that $\sqrt{s^2+r^2+4Rr}\le 2(R+r)$ or

$$s^2\le 4R^2+3r^2+4Rr.$$

However, the last inequality is a well-known inequality of J.C. Gerretsen (e.g. [2], page 50) and this completes the solution.

References:

[1] M.S. Klamkin, On a triangle inequality, *Crux Mathematicorum* 10 (1984) 139–140.

[2] D.S. Mitrinović, J.E. Pečarić and V. Volenec, *Recent Advances in Geometric Inequalities.*

Also solved by G.P. HENDERSON, Campbellcroft, Ontario; and MURRAY S. KLAMKIN, University of Alberta, whose solution was practically the same as Lau's.

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1879. [1993: 235] *Proposed by Jisho Kotani, Akita, Japan.*

Show that for any integer $n \geq 3$ there is a polynomial $x^n + ax^2 + bx + c$, with $a, b, c \neq 0$, which has three of its roots equal.

Solution by J.A. McCallum, Medicine Hat, Alberta.

If r, say, is the triple root of the polynomial, then the polynomial, its first and also its second derivative will go to zero when r is substituted for x. Thus

$$r^n + ar^2 + br + c = 0,$$

$$nr^{n-1} + 2ar + b = 0,$$

$$n(n-1)r^{n-2} + 2a = 0.$$

Taking these in reverse order we get

$$\begin{aligned}
a &= \frac{-n(n-1)}{2}r^{n-2}, \\
b &= n(n-1)r^{n-1} - nr^{n-1} = n(n-2)r^{n-1}, \\
c &= \left(-1 + \frac{n(n-1)}{2} - n(n-2)\right) r^n = -\frac{(n-1)(n-2)}{2}r^n.
\end{aligned}$$

So, by choosing a value we want for the triple root r, and using these results for a, b and c, we can form a polynomial of the indicated type for any integral value of $n \geq 3$.

Also solved by H.L. ABBOTT, University of Alberta; FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; SEUNG-JIN BANG, Seoul, Korea; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; CHARLES R. DIMINNIE, St. Bonaventure University, St. Bonaventure, N.Y.; F.J. FLANIGAN, San Jose State University, San Jose, California; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; PETER HURTHIG, Columbia College, Burnaby, B.C.; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; DAVID E. MANES, State University of New York, Oneonta; BEATRIZ MARGOLIS, Paris, France; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, *student, University of Warsaw, Poland; DAVID R. STONE, Georgia Southern University, Statesboro; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer. Four other readers sent in incorrect or incomplete solutions.*

Ardila and Flanigan note that for any integers $n > k \geq 2$ there is a polynomial $x^n + a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ with all $a_i \neq 0$ which has $k+1$ of its roots equal. However there are apparently no "nice" expressions for the a_i in general.

Some solvers observe that the required polynomial is unique (as is clear from the above proof) and that the triple root actually occurs just three times.

Readers may like to investigate similar problems, like: find other polynomials with just four nonzero coefficients (perhaps required to be integers), one of which is the constant term, which have a root occurring three times. Or: is there such a polynomial which has two different double roots? If not, what is the minimum number of nonzero coefficients required?

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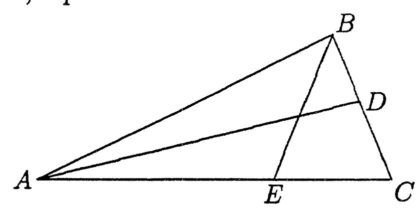
1880. [1993: 235] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

AD and BE are angle bisectors of triangle ABC, with D on BC and E on AC. Suppose that $AD = AB$ and $BE = BC$. Determine the angles of ΔABC.

Solution by Miguel Angel Cabezón Ochoa, Logroño, Spain.

$AD = AB$ implies



$$B = \frac{180 - A/2}{2} = 90 - \frac{A}{4},$$

and $BE = BC$ implies

$$C = \frac{180 - B/2}{2} = 90 - \frac{B}{4} = 90 - \frac{90}{4} + \frac{A}{16} = \frac{135}{2} + \frac{A}{16}.$$

Thus $A + B + C = 180°$ implies

$$180 = A + 90 - \frac{A}{4} + \frac{135}{2} + \frac{A}{16} = \frac{13}{16}A + \frac{315}{2},$$

so

$$A = \frac{360}{13} = \frac{2\pi}{13} \text{ radians}, \qquad B = 90 - \frac{90}{13} = \frac{1080}{13} = \frac{6\pi}{13} \text{ radians},$$

$$C = \frac{135}{2} + \frac{45}{26} = \frac{900}{13} = \frac{5\pi}{13} \text{ radians}.$$

Also solved by H.L. ABBOTT, University of Alberta; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; ŠEFKET ARSLANAGIĆ, Nyborg, Denmark; SAM BAETHGE, Science Academy, Austin, Texas; SEUNG-JIN BANG, Seoul, Korea; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J.

BRADLEY, Clifton College, Bristol, U.K.; TIM CROSS, Wolverley High School, Kidderminster, U.K.; CHARLES R. DIMINNIE, St. Bonaventure University, St. Bonaventure, N.Y.; JORDI DOU, Barcelona, Spain; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; CYRUS HSIA, Woburn Collegiate, Scarborough, Ontario; L. J. HUT, Groningen, The Netherlands; NEVEN JURIĆ, Zagreb, Croatia; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; J. A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; D. J. SMEENK, Zaltbommel, The Netherlands; L. J. UPTON, Mississauga, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer. One incorrect solution was sent in.

Amengual considers also the case that one or both of AD and BE are external angle bisectors, and obtains two more solutions:

$$A = \frac{180}{13}, \quad B = \frac{540}{13}, \quad C = \frac{1620}{13} \quad \textit{and} \quad A = \frac{900}{13}, \quad B = \frac{360}{13}, \quad C = \frac{1080}{13}.$$

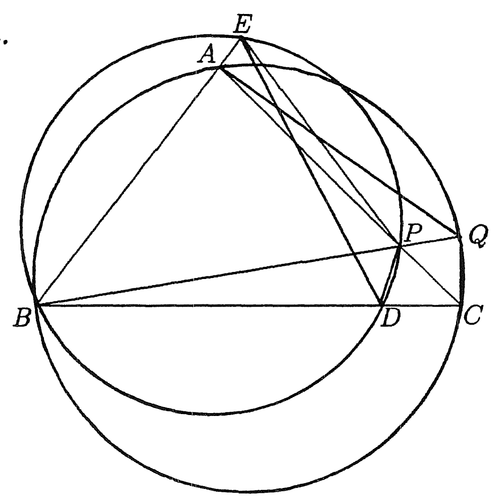
The second of these triangles is congruent to the triangle solving the original problem. Therefore, ΔABC of the problem also satisfies $CF = CA$, where CF is the external angle bisector of angle C and F lies on AB. No reader seems to have noticed this! Similarly, Amengual's first triangle has the property that all three external angle bisectors are equal in length to the sides.

* * * * *

1881. [1993: 264] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with $BC > CA > AB$. D is a point on side BC, and E is a point on BA produced beyond A, so that $BD = BE = CA$. Let P be a point on side AC such that E, B, D, P are concyclic, and let Q be the second intersection of BP with the circumcircle of ΔABC. Prove that $AQ + CQ = BP$.

I. *Solution by Jordi Dou, Barcelona, Spain.*



Let R and R' be the circumradii of BCA and BDE. Since $\angle PBE = \angle QBA$, $\angle PBD = \angle QBC$, and $\angle EBD = \angle ABC$, it follows that

$$\frac{PE}{QA} = \frac{PD}{QC} = \frac{DE}{CA} = \frac{R'}{R}$$

[i.e., equal angles inscribed in two circles will subtend chords proportional in length to the radii of the circles.—*Ed.*]. Then $QA + QC = PB$ is equivalent to

$$\frac{PE}{QA}\cdot QA+\frac{PD}{QC}\cdot QC=\frac{DE}{CA}\cdot PB,$$

or, multiplying by $BD = BE = CA$,

$$PE\cdot BD+PD\cdot BE=PB\cdot DE,$$

and this is luckily the theorem of Ptolemy applied to $BDPE$.

[*Editor's note.* This is the 350th solution Dou has sent in to *Crux*!]

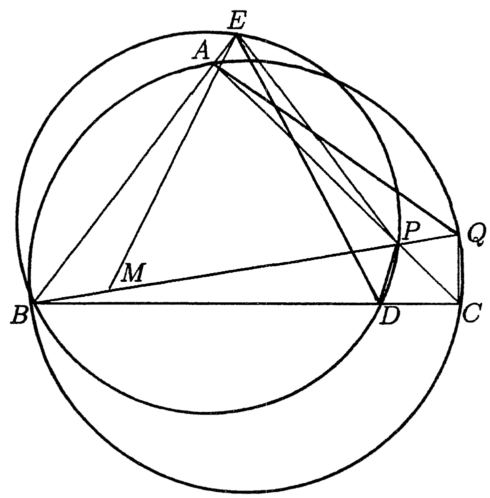
II. *Solution by Cyrus Hsia, student, Woburn Collegiate, Scarborough, Ontario.*

Note that

$$\angle CAQ=\angle CBQ \quad (=\alpha)$$

and

$$\angle ACQ=\angle ABQ \quad (=\beta).$$



Let M be the point on BQ such that $\angle BEM = \alpha$. Since $BE = CA$ we have $\Delta BEM \cong \Delta CAQ$, therefore

$$BM = CQ \quad \text{and} \quad ME = QA. \quad (1)$$

Now consider ΔPME and ΔDBE. Since $\angle DEP = \angle DBP = \alpha$,

$$\angle MEP=\angle MED+\alpha=\angle BED,$$

and also

$$\angle PME=\angle MBE+\angle MEB=\beta+\alpha=\angle DBE.$$

Therefore $\Delta PME \sim \Delta DBE$. Now ΔPME is isosceles since ΔDBE is ($BD = BE$), so $MP = ME$. Finally from (1),

$$CQ+AQ=BM+ME=BM+MP=BP.$$

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; KEE-WAI LAU, Hong Kong; PAUL PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; and the proposer.

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Short articles intended for publication should be sent to Dr. Hanson, contest problem sets and solutions to Olympiad Corner problems should be sent to Dr. Woodrow and other problems and solutions to Dr. Sands.

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