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ITÉRATION DE σ CONTRÔLÉE PAR LA FONCTION D'EULER

E. Bedocchi

INTRODUCTION Si σ désigne la fonction somme des diviseurs, un problème classique et assez difficile à traiter (voir par exemple [2]) est l'étude des périodes des suites définies par

$$s_0(n) = n \qquad \text{et} \qquad s_i(n) = \sigma(s_{i-1}(n)) - s_{i-1}(n) \quad \text{si } i \geq 1, \tag{1}$$

qui en anglais s'appellent *aliquot sequences.*

Dans cette note on fait une variation sur ce thème, semblable à celle de [1], en étudiant le développement des suites définies par

$$s_0(n) = n \qquad \text{et} \qquad s_i(n) = \sigma(\phi(s_{i-1}(n))) \quad \text{si } i \geq 1, \tag{2}$$

où ϕ est la fonction d'Euler.

Un calcul rapide montre que, pour tout $m \in \mathbb{N}$, il résulte $\sigma(m)/m \leq m/\phi(m)$: il est donc raisonnable de supposer que presque toutes les suites définies suivant (2) sont périodiques.

DONNÉES EXPÉRIMENTALES: Le travail fait à l'ordinateur était comme prévu et montre en outre l'existence d'un nombre très petit de périodes. En voici le compte rendu:

1 - Toute suite $s_i(n)$ avec $1 \leq n \leq 70000$ est périodique.

2 - On a remarqué les périodes suivantes (ou une de leurs rotations):

Périodes de longueur 1:

$\overline{1}$ $\overline{3}$ $\overline{15}$ $\overline{28}$ $\overline{255}$ $\overline{744}$ $\overline{2418}$ $\overline{20440}$ $\overline{65535}$

Périodes de longueur 2:

$\overline{7, 12}$ $\overline{124, 168}$ $\overline{195, 252}$ $\overline{1240, 1512}$ $\overline{5080, 6552}$

$\overline{30855, 36792}$ $\overline{638898, 833280}$ $\overline{5946666, 8124480}$

Périodes de longueur 3:

$\overline{31, 72, 60}$ $\overline{3844, 5376, 4092}$ $\overline{46228, 60984, 58968}$

$\overline{215138, 345600, 319410}$ $\overline{990264, 1038996, 1100736}$

Périodes de longueur 4:

$\overline{1651, 4800, 3066, 2520}$ $\overline{123783, 247380, 185130, 147168}$

Périodes de longueur 5: Aucune

Périodes de longueur 6:

$\overline{6045, 9906, 9920, 12264, 10200, 6138}$

$\overline{67963, 183456, 163520, 163800, 122640, 81880}$

Comme on le voit, c'est une situation pleine de questions. Par exemple, est-ce que chaque suite $s_i(n)$ est périodique? Existe-t-il des périodes de toute longueur? Il s'agit de questions très délicates qui ne trouveront pas une réponse dans ces pages: ici nous mettrons en évidence seulement des faits beaucoup plus modestes.

QUELQUES RÉSULTATS: Les affirmations suivantes sont les fruits, négatifs, de la recherche de familles de suites périodiques.

Proposition 1. Si $n = 2^t p$ avec $t \geq 1$, $p > 2$ est premier et la suite $s_i(n)$ est immédiatement périodique avec une période de longueur 1 , alors $n = 28$.

Proposition 2. Si $n = 3^t p$ avec $t \geq 1$, $p > 3$ est premier et la suite $s_i(n)$ est immédiatement périodique avec une période de longueur 1 , alors $n = 15$.

Les démonstrations sont laissées au lecteur: elles suivent la ligne de celle du théorème d'Euler sur les nombres parfaits.

REMARQUE CURIEUSE: Si $h \in \mathbb{N}$ et pour tout $j \in \mathbb{N}$, $j < h$, $F_j = 2^{2^j}+1$ est premier, alors la suite $s_i(F_h - 2)$ est immédiatement périodique avec une période de longueur 1.

On a en effet

$$\phi(F_h - 2) = \phi(\prod_{j=0}^{h-1} F_j) = \prod_{j=0}^{h-1} 2^{2^j} = 2^{2^h-1}$$

et donc

$$s_1(F_h - 2) = \sigma(\phi(F_h - 2)) = 2^{2^h} - 1 = F_h - 2 = s_0(F_h - 2).$$

C'est dommage que l'hypothèse de Fermat sur les nombres F_j est fausse.

Je conclus en rappelant que tout le travail expérimental a été fait sur un ordinateur 286 IBM-compatible et que les programmes ont été écrits dans le langage UBASIC. À tel propos je remercie vivement le Prof. Y. Kida, auteur de ce langage, qui m'a permis son utilisation.

Références:

[1] E. Bedocchi, Itération contrôlée de la fonction σ, *Tsukuba J. Math.*, 5(1981), pp. 285-289.

[2] R. K. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag New York Inc., 1981.

[3] Y. Kida, UBASIC Version 8.2d, Rikkyo Univ./Math., Ikebukuro Tokyo, 1992.

Dipartimento di Matematica
Università di Bologna
Piazza di P.ta S. Donato 5
40127 BOLOGNA (Italia)

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THE SKOLIAD CORNER

No. 5

R. E. WOODROW

This month we give the problems of the 4th U.K. Schools Mathematical Challenge which was written February 7, 1991. The contest was supported by the National Westminster Bank with the support of the University of Birmingham. Students were allowed one hour, and calculators were not permitted. The contest was aimed at students in the ninth year or below. My thanks go to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Nfld., who collected the contest and forwarded it to me when he was Canadian I.M.O. Team leader in Moscow.

4th U.K. SCHOOLS MATHEMATICAL CHALLENGE

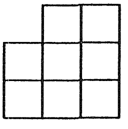
February 7, 1991

Time: 1 hour

1. Four of the options are equal. Which is the odd one out?

(a) $1 \div 9 + 9 \div 1$ (b) $1 \times 9 \div (9 \times 1)$ (c) $1 - 9 + 9 \times 1$ (d) $1 + 9 \div 9 - 1$ (e) $1 \times (9 - 9) + 1$.

2. How many squares are there altogether in this diagram?



(a) 8 (b) 9 (c) 10 (d) 11 (e) 12.

3. I set my video recorder to record the late film from 11:15 p.m. to 1:05 a.m. How many minutes of the new 4 hour tape remained unused?

(a) 50 (b) 110 (c) 130 (d) 150 (e) 190.

4. Four of the following are equal. Which is the odd one out?

(a) $\frac{1}{3} + \frac{5}{7}$ (b) 0.6 (c) $\frac{15}{25}$ (d) 60% (e) $\frac{1}{2} + \frac{1}{10}$.

5. Here are the dates of three mathematicians: Sophie Germain (French) 1776-1831; Sonja Kowalevsky (Russian) 1850-1891; Emmy Noether (German) 1882-1935. Arrange them in order with the shortest lived first (for example, GKN would mean Germain was the youngest when she died and Noether the eldest).

(a) GNK (b) NGK (c) KGN (d) NKG (e) KNG.

6. Looking in at the greengrocer's window it said 'POTATOEƧ'. When I went inside and looked from the other side, what did I see?

(a) ᑫOTATOƎS (b) ƧƎOTATOᑫ (c) POTATOES

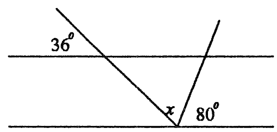
(d) SƎOTATOᑫ (e) SƎOꞱⱯꞱOd.

7. Rachel arranges the fingers of her right hand so that her thumb points upwards, her first finger points north and her second finger points west: we write this for short as "TU, $1N$, $2W$". She then keeps her fingers fixed like this, but can twist her arm and her wrist if she likes. Which of the following arrangements can she *not* achieve? (D=down, S=south, E=east.)

(a) $TD, 1N, 2E$ (b) $TN, 1D, 2W$ (c) $TS, 1E, 2U$
(d) $TE, 1U, 2S$ (e) $TW, 1S, 2D$.

8. Four of the points with these coordinates lie on a single straight line. Which is the odd one out?

(a) $(6, 11)$ (b) $(5, 5)$ (c) $(4, 7)$ (d) $(8, 15)$ (e) $(2, 3)$.



9. How big is the angle x?

(a) 30° (b) 36° (c) 44° (d) 45° (e) 64°.

10. Nine bus stops are equally spaced along a bus route. The distance from the first to the third is 600m. How far is it from the first stop to the last?

(a) 600m (b) 1600m (c) 1800m (d) 2400m (e) 2700m.

11. In 1990 a new size 5p coin was minted. The old size weighed 5.65g while the new size weighs 3.25g. How much lighter will your pocket be if it contains £5 worth of the new size coins instead of the old size?

(a) 2.4000g (b) 24g (c) 12g (d) 240g (e) no difference.

12. A *rod* (sometimes called a *pole*, or a *perch*) was an old unit of length. To measure a rod you were supposed to stand outside a church on a Sunday morning, stop the first sixteen men to come out, and line them up with their left feet in one long line touching toe to heel: the distance from the front to the back of this line was called a *rod*. The *exact* length you got depended on who went to church that day, but it was always more or less the same length. How long was a rod to the nearest metre?

(a) 4m (b) 5m (c) 6m (d) 7m (e) 8m.

13. How many of these statements are true?

(i) $12 \div \frac{1}{2} = 6$ (ii) $3\% = 0.3$ (iii) $\frac{1}{7} < \frac{1}{9}$ (iv) $0.2 \times 0.4 = 0.8$

(a) none (b) one (c) two (d) three (e) four.

14. Sound travels at about 330m/s; light travels so fast that it arrives almost instantaneously. If you time the gap between a flash of lightning and its clap of thunder as 6 seconds, roughly how far away is the storm?

(a) 55m (b) 330m (c) 1km (d) 2km (e) 6km.

15. What is the angle between the two hands of a clock at 2:30?

(a) 100° (b) 105° (c) 110° (d) 120° (e) 135°.

16. Every Maundy Thursday the reigning monarch distributes "Maundy money" to equal numbers of men and women. Last year at Newcastle 64 men and 64 women each received 64p, one penny for each year of the Queen's life. Written as a power of 2, what was the total amount distributed in pence?

(a) 2^6 (b) 2^7 (c) 2^{12} (d) 2^{13} (e) 2^{42}.

17. Snow White wanted to know the mean height of the Seven Dwarfs. So one day she measured them all as they left for work and calculated their mean height (correct to one decimal place) as 112.3cm. Doc complained that she had missed him out and had measured Dopey twice without him noticing. If Doc is 3cm taller than Dopey, what is the mean height of the Seven Dwarfs?

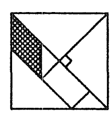
(a) 111.9cm (b) 112.3cm (c) 112.7cm (d) 113.8cm (e) 115.3cm.

18. Each letter stands for a different digit. Which letter has the lowest value?

$$\begin{array}{r} U\ K \\ 4\ + \\ \hline S\ M\ C \\ \hline \end{array}$$

(a) U (b) K (c) S (d) M (e) C.

19. The seven pieces in this 12cm × 12cm square make a Tangram set. What is the area of the shaded parallelogram?

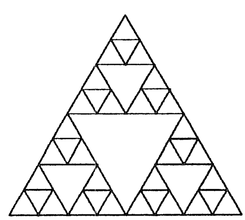


(a) 6cm^2 (b) 12cm^2 (c) 18cm^2 (d) 36cm^2 (e) 144cm^2.

20. A quiz has twenty questions with seven points awarded for each correct answer, two points deducted for each wrong answer and zero for each question omitted. Jack scores 87 points. How many questions did he omit?

(a) 2 (b) 5 (c) 7 (d) 9 (e) 13.

21. The perimeter of a large triangle is 24cm. What is the total length of the black lines used to draw the figure?

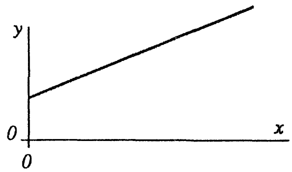


(a) 57cm (b) 66cm (c) 75cm (d) 78cm (e) 81cm.

22. 4 star petrol at s pence per litre costs 3 pence per litre more than unleaded. How many pence does it cost to buy u litres of unleaded petrol?

(a) $u(s-3)$ (b) $s+3$ (c) $s-3$ (d) $us-3$ (e) $(4s+3)u$.

23. Which of the following could this graph *not* represent?



(a) x = time after midnight, y = depth of water in the harbour.
(b) x = time after throwing, y = speed of a stone falling down a well.
(c) x = weight in grams, y = weight in ounces.
(d) x = temperature in °C, y = temperature in °F.
(e) x = age of child, y = height of that child.

24. Fill the empty squares with As, Bs, Cs, Ds, Es so that no line (horizontal, vertical, or inclined at 45°) contains a letter more than once. Which letter goes in the square marked $*$?

	●	A		
	E			B
		C		

(a) A (b) B (c) C (d) D (e) E.

25. If all plinks are plonks and some plunks are plinks, which of the statements X, Y, Z *must* be true?

X: all plinks are plunks
Y: some plonks are plunks
Z: some plinks are not plunks

(a) X only (b) Y only (c) Z only (d) X & Y only (e) Y & Z only.

* * *

In the last number we gave the problems of the 1995 A.I.M.E. As promised we next give the numerical solutions. The problems and their official solutions are copyrighted by the Committee of the American Mathematics Competitions of the Mathematical Association of America, and may not be reproduced without permission. Detailed solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Old Father Hall, University of Nebraska, Lincoln, NE, U.S.A. 68488-0322.

1. 225	**5.** 051	**9.** 616	**13.** 400
2. 025	**6.** 589	**10.** 215	**14.** 378
3. 067	**7.** 027	**11.** 040	**15.** 037
4. 224	**8.** 085	**12.** 005	

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That completes the space we have this month. Send me your pre-Olympiad contests as well as suggestions and comments about the future of the Skoliad Corner.

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THE OLYMPIAD CORNER

No. 165

R. E. WOODROW

All communications about this column should be sent to Professor R. E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The Olympiad Contest we give this month is the Irish Mathematical Olympiad of 1993. My thanks go to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, who collected this contest (and others) when he was the Canadian I.M.O. Team Leader at Istanbul, Turkey.

SIXTH IRISH MATHEMATICAL OLYMPIAD

May 8, 1993 — First Paper

(Time: 3 hours)

1. The real numbers α, β satisfy the equations

$$\alpha^3 - 3\alpha^2 + 5\alpha - 17 = 0, \qquad \beta^3 - 3\beta^2 + 5\beta + 11 = 0.$$

Find $\alpha + \beta$.

2. A natural number n is called **good** if n can be written in a *unique* way simultaneously as the sum $a_1 + a_2 + \cdots + a_k$ and as the product $a_1 a_2 \ldots a_k$ of some $k \geq 2$ natural numbers $a_1, a_2, \ldots, a_k$. (For example 10 is good because $10 = 5 + 2 + 1 + 1 + 1 = 5 \cdot 2 \cdot 1 \cdot 1 \cdot 1$ and these expressions are unique.) Determine, in terms of prime numbers, which natural numbers are good.

3. The line l is tangent to the circle S at the point A; B and C are points on l on opposite sides of A and the other tangents from B, C to S intersect at a point P. If B, C vary along l in such a way that the product $|AB| \cdot |AC|$ is constant, fint the locus of P.

4. Let $a_0, a_1, \ldots a_{n-1}$ be real numbers, where $n \geq 1$, and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be such that $|f(0)| = f(1)$ and each root α of f is real and satisfies $0 < \alpha < 1$. Prove that the product of the roots does not exceed $1/2^n$.

5. Given a complex number $z = x + iy$ (x, y real) we denote by $P(z)$ the corresponding point (x, y) in the plane. Suppose z_1, z_2, z_3, z_4, z_5, α are nonzero complex numbers such that

(i) $P(z_1)$, $P(z_2)$, $P(z_3)$, $P(z_4)$, $P(z_5)$ are the vertices of a convex pentagon $\mathbb{Q}$ containing the origin 0 in its interior and

(ii) $P(\alpha z_1)$, $P(\alpha z_2)$, $P(\alpha z_3)$, $P(\alpha z_4)$, and $P(\alpha z_5)$ are all inside $\mathbb{Q}$. If $\alpha = p + iq$, where p and q are real, prove that $p^2 + q^2 \leq 1$ and that

$$p + q\tan(\pi/5) \leq 1.$$

May 8, 1993 — Second Paper

(Time: 3 hours)

1. Given five points P_1, P_2, P_3, P_4, P_5 in the plane having integer coordinates, prove that there is at least one pair (P_i, P_j) with $i \neq j$ such that the line P_iP_j contains a point Q having integer coordinates and lying strictly between P_i and P_j.

2. Let $a_1, a_2, \ldots a_n,\ b_1, b_2, \ldots b_n$ be $2n$ real numbers, where $a_1, a_2, \ldots, a_n$ are distinct, and suppose that there exists a real number α such that the product

$$(a_i + b_1)(a_i + b_2)\ldots(a_i + b_n)$$

has the value α for all i ($i = 1, 2, \ldots, n$). Prove that there exists a real number β such that the product

$$(a_1 + b_j)(a_2 + b_j)\ldots(a_n + b_j)$$

has the value β for all j ($j = 1, 2, \ldots, n$).

3. For nonnegative integers n, r the binomial coefficient $\binom{n}{r}$ denotes the number of combinations of n objects chosen r at a time, with the convention that $\binom{n}{0} = 1$ and $\binom{n}{r} = 0$ if $n < r$. Prove the identity

$$\sum_{d=1}^{\infty} \binom{n-r+1}{d}\binom{r-1}{d-1} = \binom{n}{r}$$

for all integers n, r with $1 \leq r \leq n$.

4. Let x be a real number with $0 < x < \pi$. Prove that, for all natural numbers n, the sum

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin(2n-1)x}{2n-1}$$

is positive.

5. (a) The rectangle $PQRS$ has $|PQ| = l$ and $|QR| = m$ where l, m are positive integers. It is divided up into lm 1×1 squares by drawing lines parallel to PQ and QR. Prove that the diagonal PR intersects $l + m - d$ of these squares, where d is the greatest common divisor (l, m).

(b) A cuboid (= box) with edges of lengths l, m, n, where l, m, n are postive integers, is divided into lmn $1 \times 1 \times 1$ cubes by planes parallel to its faces. Consider a diagonal joining a vertex of the cuboid to the vertex furthest away from it. How many of the cubes does this diagonal intersect?

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Last number we gave six more Klamkin Quickies. We next give the "quick solutions". Many thanks go to Murray S. Klamkin, University of Alberta for his continued support of the Corner.

SOLUTIONS TO KLAMKIN QUICKIES

7. Determine all integral solutions of the Diophantine equation

$$(x^8 + y^8 + z^8) = 2(x^{16} + y^{16} + z^{16}).$$

Solution. More generally one can find all integral solutions of

$$(x^{2n} + y^{2n} + z^{2n})^2 = 2(x^{4n} + y^{4n} + z^{4n}), \tag{1}$$

where n is a positive integer provided Fermat's equation $x^n + y^n = z^n$ does not have any integer solutions for particular values of $n > 2$ chosen.

Equation (1) can be rewritten as

$$(x^n + y^n + z^n)(y^n + z^n - x^n)(z^n + x^n - y^n)(x^n + y^n - z^n) = 0. \tag{2}$$

The trivial solutions occur for $(x, y, z) = (\pm a, a, 0)$ and permutations thereof.

For $n = 1$, any factor of the left hand side of (2) can be zero.

For $n = 2$, (x, y, z) can be $\pm$ the sides of any integral right triangle $(2mn, m^2 - n^2, m^2 + n^2)$ in any order.

Since Fermat's equation is at least known not to have any non-trivial solutions for all $n > 2$ and < 100 and integral multiples thereof, there are not any non-trivial solutions for at least these cases.

8. Determine all the roots of the quintic equation

$$31x^5 + 165x^4 + 310x^3 + 330x^2 + 155x + 33 = 0.$$

Solution. Since the equation can be rewritten as $(x - 1)^5 = 32(x + 1)^5$,

$$\frac{x-1}{x+1} = 2\omega^r, \quad r = 0, 1, 2, 3, 4$$

where ω is a primitive 5th root of unity. Hence,

$$x = \frac{1 + 2\omega^r}{1 - 2\omega^r}, \quad r = 0, 1, 2, 3, 4.$$

More generally the equation

$$ax^6 + 5bcx^4 + 10ac^2x^3 + 10bc^3x^2 + 5ac^4x + bc^5 = 0$$

is the same as

$$(b - a)(x - c)^5 = (b + a)(x + c)^5.$$

9. If $F(x)$ and $G(x)$ are polynomials with integer coefficients such that $F(k)/G(k)$ is an integer for $k = 1, 2, 3 \ldots$, prove that $G(x)$ divides $F(x)$.

Solution. By taking k sufficiently large it follows that the degree of F is $\geq$ the degree of G. Then by the remainder theorem,

$$\frac{F(x)}{G(x)} = \frac{Q(x)}{a} + \frac{R(x)}{G(x)}$$

where $Q(x)$ is an integral polynomial, a is an integer, and $R(x)$ is a polynomial whose degree is less than that of $G(x)$. Now $R(x)$ must identically vanish otherwise by taking k sufficiently large, we can make $R(k)/G(k)$ arbitrarily small and this cannot add with $Q(k)/a$ to be an integer.

10. Given that $ABCDEF$ is a skew hexagon such that each pair of opposite sides are equal and parallel. Prove that the midpoints of the six sides are coplanar.

Solution. Since each pair of opposite sides form a parallelogram whose diagonals bisect each other, all three different diagonals are concurrent say at point P. We now let $\mathbf{A}, \mathbf{B}, \mathbf{C}, -\mathbf{A}, -\mathbf{B}, -\mathbf{C}$ be vectors from P to A, B, C, D, E, F, respectively. The successive midpoints (multiplied by 2) are given by

$$\mathbf{A} + \mathbf{B}, \quad \mathbf{B} + \mathbf{C}, \quad \mathbf{C} - \mathbf{A}, \quad -\mathbf{A} - \mathbf{B}, \quad -\mathbf{B} - \mathbf{C}, \quad -\mathbf{C} + \mathbf{A}$$

and which incidentally form another centrosymmetric hexagon. It is enough now to note that $(\mathbf{A} + \mathbf{B}) - (\mathbf{B} + \mathbf{C}) + (\mathbf{C} - \mathbf{A}) = \mathbf{0}$.

11. If a, b, c, d are the lengths of sides of a quadrilateral, show that

$$\frac{\sqrt{a}}{(4 + \sqrt{a})}, \quad \frac{\sqrt{b}}{(4 + \sqrt{b})}, \quad \frac{\sqrt{c}}{(4 + \sqrt{c})}, \quad \frac{\sqrt{d}}{(4 + \sqrt{d})},$$

are possible lengths of sides of another quadrilateral.

Solution. More generally one can show that if $a_1, a_2, \ldots, a_n$ are the lengths of sides of an n-gon, then $F(a_1), F(a_2), \ldots, F(a_n)$ are possible lengths of sides of another n-gon where $F(x)$ is an increasing concave function of x for $x \geq 0$ and $F(0) = 0$.

If a_1 is the largest of the a_i's, then it suffices to show that

$$F(a_2) + F(a_3) + \cdots + F(a_n) \geq F(a_1).$$

By the majorization inequality, we have

$$F(a_2) + F(a_3) + \cdots + F(a_n) \geq F(a_2 + a_3 + \cdots + a_n) + (n-2)F(0).$$

Finally, $F(a_2 + a_3 + \cdots + a_n) \geq F(a_1)$.

Some admissable functions are

$$F(x) = x^\alpha \quad \text{and} \quad \frac{x^\alpha}{k^2 + x^\alpha} \text{ for } 0 < \alpha < 1, \quad \frac{x}{(x + k^2)}, \quad 1 - e^{-k^2 x}, \tanh x.$$

12. Determine the maximum value of the sum of the cosines of the six dihedral angles of a tetrahedron.

Solution. Let **A**, **B**, **C**, **D** be unit outward vectors normal to the faces of a tetrahedron $ABCD$. Then

$$(x\mathbf{A} + y\mathbf{B} + z\mathbf{C} + w\mathbf{D})^2 \geq 0.$$

Expanding out and noting that $\mathbf{A} \cdot \mathbf{B} = -\cos CD$ (here CD denotes the dihedral angle of which the side CD is an edge), etc., we get

$$\begin{aligned} x^2 + y^2 + z^2 + w^2 \geq\ & 2xy\cos CD + 2xz\cos BD + 2xw\cos BC \\ & +2yz\cos AD + 2yw\cos AC + 2zw\cos AB. \end{aligned} \tag{1}$$

Setting $x = y = z = w$, we get that the sum of the cosines of the 6 dihedral angles is ≤ 2. There is equality *iff* $\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = \mathbf{0}$. Since as known

$$F_a\mathbf{A} + F_b\mathbf{B} + F_c\mathbf{C} + F_d\mathbf{D} = \mathbf{0}$$

where F_a denotes the area of the face of the tetrahedron opposite A, etc., it follows that there is equality *iff* the four faces have equal area or that the tetrahedron is isosceles.

Comment. In a similar fashion one can extend inequality (1) to n dimensions and then show that the sum of the cosines of the $n(n+1)/2$ dihedral angles of an n-dimensional simplex is $\leq (n+1)/2$. Here the dihedral angles are the angles between pairs of $(n-1)$-dimensional faces and there is equality *iff* all the $(n-1)$-dimensional faces have the same volume.

* * *

We conclude this issue with solutions from the readers to some of the problems of the 1992 Japan Mathematical Olympiad which we gave in the January 1994 number of the Corner [1994: 7].

1. Let x and y be relatively prime numbers with $xy > 1$, and let n be a positive even number. Prove that $x^n + y^n$ is not divisible by $x + y$.

Solutions by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Wang's solution.

Clearly x and y are either both positive or both negative. We assume first that $x > 0$ and $y > 0$. Let $n = 2k$. Then from $x^n - y^n = (x^2)^k - (y^2)^k$, we see that $x^2 - y^2 \mid x^n - y^n$. Thus $x + y \mid x^n - y^n$. Suppose that $x + y \mid x^n + y^n$ also. Then $x + y \mid 2x^n, 2y^n$ which implies that $x + y \mid \gcd(2x^n, 2y^n) = 2\gcd(x,y)^n = 2$. Thus $x + y = 1$ or 2 neither of which is possible since $xy > 1$. Finally, if $x < 0$ and $y < 0$, then considering $-x$ and $-y$ instead of x and y leads to the conclusion that $-(x+y) \nmid (-x)^n + (-y^n)$ and hence $x + y \nmid x^n + y^n$.

3. Prove the inequality

$$\sum_{k=1}^{n-1} \frac{n}{n-k} \cdot \frac{1}{2k-1} < 4. \qquad (n \geq 2)$$

Corrections by Tim Cross, Wolverley High School, Kidderminster, U.K.; by Edward T.H. Wang and Martin White, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Cross' argument.

Write

$$\begin{aligned} S_n &= \sum_{k=1}^{n-1} \frac{n}{n-k} \cdot \frac{1}{2k-1} = \frac{n}{2n-1} \sum_{k=1}^{n-1} \left(\frac{1}{n-k} + \frac{2}{2k-1} \right) \\ &= \frac{n}{2n-1} \sum_{k=1}^{n-1} \left(\frac{1}{k} + \frac{2}{2k-1} \right) \\ &= \frac{n}{2n-1} \left[\sum_{k=1}^{n-1} \frac{1}{k} + 2 \left(\sum_{k=1}^{2n-2} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{k} \right) \right] = \frac{2n}{2n-1} \sum_{k=1}^{2n-2} \frac{1}{k}. \end{aligned}$$

Since the series $\sum 1/k$ diverges and $2n/(2n-1)$ is a factor greater than 1, it is impossible for S_n not to exceed any finite value for sufficiently large n. In fact the least value of n for which $S_n > 4$ is $n = 15$, when $S_n = 4.06259$ (to five decimal places).

4. Suppose that A is an (m, n)-matrix which satisfies the following conditions:

(1) $m \leq n$;

(2) each element of A is 0 or 1;

(3) if f is an injection from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$, then the $(i, f(i))$-element of A is zero for some i, $1 \leq i \leq m$.

Prove that there exist sets $S \subseteq \{1, \ldots, m\}$ and $T \subseteq \{1, \ldots, n\}$ which satisfy

i) the (i, j)-element is zero for any $i \in S$ and $j \in T$;

ii) $\#(S) + \#(T) > n$.

Comment by Chris Wildhagen, Rotterdam, The Netherlands.

This follows immediately from the well known result from discrete mathematics (Konig-Egerváry) that in a $\{0, 1\}$-matrix the maximum number of independent 1's equals the minimum size of a line cover.

* * *

That completes the May number of the Corner. Don't forget to send me your regional and national Olympiad's for use in the Corner as well as your nice solutions.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **December 1, 1995**, *although solutions received after that date will also be considered until the time when a solution is published.*

2034. [1995: 130] **(Corrected)** *Proposed by Murray S. Klamkin and M. V. Subbarao, University of Alberta.*

(a) Find all sequences $p_1 < p_2 < \cdots < p_n$ of distinct prime numbers such that

$$\left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_2}\right)\cdots\left(1+\frac{1}{p_n}\right)$$

is an integer.

(b) Can

$$\left(1+\frac{1}{a_1^2}\right)\left(1+\frac{1}{a_2^2}\right)\cdots\left(1+\frac{1}{a_n^2}\right)$$

be an integer where $a_1, a_2, \ldots$ are distinct integers greater than 1?

2041. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

P is an interior point of triangle ABC. AP, BP, CP meet BC, CA, AB at D, E, F respectively. Let M and N be points on segments BF and CE respectively so that $BM : MF = EN : NC$. Let MN meet BE and CF at X and Y respectively. Prove that $MX : YN = BD : DC$.

2042. *Proposed by Jisho Kotani, Akita, Japan, and K. R. S. Sastry, Dodballapur, India.*

If A and B are three-digit positive integers, let $A * B$ denote the six-digit integer formed by placing them side by side. Find A and B such that

$$A, \quad B, \quad B-A, \quad A*B \quad \text{and} \quad \frac{A*B}{B}$$

are all integer squares.

2043. *Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

What is the locus of a point interior to a fixed triangle that moves so that the sum of its distances to the sides of the triangle remains constant?

2044. *Proposed by Murray S. Klamkin, University of Alberta.*

Suppose that $n \geq m \geq 1$ and $x \geq y \geq 0$ are such that

$$x^{n+1} + y^{n+1} \leq x^m - y^m.$$

Prove that $x^n + y^n \leq 1$.

2045. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan.*

Show that there are an infinite number of Pythagorean triangles (right-angled triangles with integer sides) whose hypotenuse is an integer of the form $3333\ldots3$.

2046. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Find integers a and b so that

$$x^3 + xy^2 + y^3 + 3x^2 + 2xy + 4y^2 + ax + by + 3$$

factors over the complex numbers.

2047. *Proposed by D. J. Smeenk, Zaltbommel, The Netherlands.*

ABC is a nonequilateral triangle with circumcentre O and incentre I. D is the foot of the altitude from A to BC. Suppose that the circumradius R equals the radius r_a of the excircle to BC. Show that O, I and D are collinear.

2048. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Find the least integer n so that, for every string of length n composed of the letters $a, b, c, d, e, f, g, h, i, j, k$ (repetitions allowed), one can find a nonempty block of (consecutive) letters in which no letter appears an odd number of times.

2049*. *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

Let a tetrahedron $ABCD$ with centroid G be inscribed in a sphere of radius R. The lines AG, BG, CG, DG meet the sphere again at A_1, B_1, C_1, D_1 respectively. The edges of the tetrahedron are denoted a, b, c, d, e, f. Prove or disprove that

$$\frac{4}{R} \leq \frac{1}{GA_1} + \frac{1}{GB_1} + \frac{1}{GC_1} + \frac{1}{GD_1} \leq \frac{4\sqrt{6}}{9}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right).$$

Equality holds if $ABCD$ is regular. (This inequality, if true, would be a three-dimensional version of problem 5 of the 1991 Vietnamese Olympiad; see [1994: 41].)

2050. *Proposed by Šefket Arslanagić, Berlin, Germany.*

Find all real numbers x and y satisfying the system of equations

$$2^{x^2+y} + 2^{x+y^2} = 128, \qquad \sqrt{x} + \sqrt{y} = 2\sqrt{2}.$$

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1510*. [1990: 20; 1991: 91; 1993: 50, 204] *Proposed by Jack Garfunkel, Flushing, New York.*

P is any point inside a triangle ABC. Lines PA, PB, PC are drawn and angles PAC, PBA, PCB are denoted by α, β, γ respectively. Prove or disprove that

$$\cot\alpha + \cot\beta + \cot\gamma \geq \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2},$$

with equality when P is the incenter of ΔABC.

IV. *Comment by Federico Ardila, student, MIT, Cambridge, Massachusetts.*

We prove that the constant $1/2$ in Jun-hua Huang's corrected inequality on [1993: 204] is best possible, i.e., that the inequality

$$\cot\alpha + \cot\beta + \cot\gamma \geq t\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right) \qquad (1)$$

fails for some triangles whenever $t > 1/2$.

Using the identities given in Huang's comment [1993: 204], we can rewrite (1) as

$$\frac{s^2 - r^2 - 4Rr}{2sr} + 3\left(\frac{2R^2}{sr}\right)^{1/3} \geq \frac{ts}{r},$$

or, multiplying through by r,

$$\frac{s^2 - r^2 - 4Rr}{2s} + 3\left(\frac{2R^2r^2}{s}\right)^{1/3} \geq ts. \qquad (2)$$

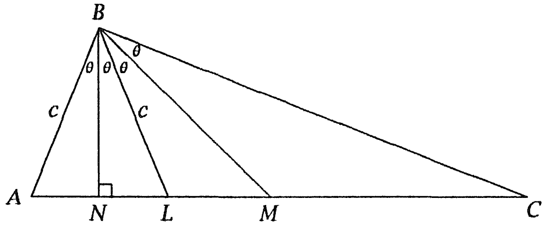
If we put $B = C = 90 - A/2$ and let A go to 0, then $s \to b, R \to b/2$ and $r \to 0$, and therefore the left hand side of (2) goes to $b/2$ while the right hand side goes to tb. Then it follows that (2) fails if $t > 1/2$, as we wished to prove.

* * * * *

1740. [1992: 110; 1993: 94,305] *Proposed by Dan Pedoe, Minneapolis, Minnesota.* (Dedicated in memoriam to Joseph Konhauser.)

In triangle ABC the points N, L, M, in that order on AC, are respectively the foot of the perpendicular from B onto AC, the intersection with AC of the bisector of $\angle ABC$, and the midpoint of AC. The angles ABN, NBL, LBM and MBC are all equal. Determine the angles of ΔABC. [Some comments on the origin of this proposal will be given when a solution is published.]

V. *Solution by K. V. L. N. Rao, New Delhi, India.*



Applying the sine rule to ΔBML,

$$\frac{BM}{\sin(90+\theta)} = \frac{BL}{\sin(90-2\theta)} = \frac{c}{\cos 2\theta},$$

so

$$BM = \frac{c\cos\theta}{\cos 2\theta}.$$

Thus (with $[XYZ]$ denoting the area of ΔXYZ)

$$\begin{aligned}[ABC] &= 2[BMC] = BM \cdot BC\sin\theta \\ &= \frac{c\cos\theta}{\cos 2\theta}\cdot a\sin\theta = \frac{ac\tan 2\theta}{2}. \end{aligned} \qquad (1)$$

But also

$$[ABC] = \frac{1}{2}AB \cdot BC\sin 4\theta = \frac{ac\sin 4\theta}{2}, \qquad (2)$$

so from (1) and (2),

$$\frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta = \sin 4\theta = 2\sin 2\theta\cos 2\theta$$

and thus $\cos^2 2\theta = 1/2$ or $\cos 2\theta = \pm 1/\sqrt{2}$. From this we conclude $2\theta = \pi/4$ (since $4\theta = \angle B < \pi$), i.e. $\theta = \pi/8$. Hence

$$B = 4\cdot\frac{\pi}{8} = \frac{\pi}{2}, \qquad A = \frac{\pi}{2} - \theta = \frac{3\pi}{8}, \qquad C = \frac{\pi}{8}.$$

Francisco Bellot Rosado, I. B. Emilio Ferrari, Valladolid, Spain, has located this problem as the first problem of the 1964 Dutch Olympiad, First Round, as reported in the Romanian journal Gazeta Matematica, *December 1966.*

*   *   *   *   *

1942. [1994: 136] *Proposed by Paul Bracken, University of Waterloo.*
Prove that, for any $a \ge 1$,

$$\left(\sum_{k=0}^{\infty}\frac{1}{(a+k)^2}\right)^2 > 2\sum_{k=0}^{\infty}\frac{1}{(a+k)^3}.$$

Solution by the proposer.

Writing the first few terms of Euler's summation formula (e.g. pp. 524 and 534, Chapter 14 of [1]),

$$f(0)+f(1)+\cdots+f(n)=$$
$$\int_0^n f(x)\,dx+\frac{1}{2}(f(n)+f(0))+\frac{B_2}{2!}(f'(n)-f'(0))+r_1(n) \qquad (1)$$

where

$$|r_1(n)|<|r_1(n)-r_2(n)|=\frac{|B_4|}{4!}|f^{(3)}(n)-f^{(3)}(0)|$$

(and the constants B_2, B_4 are Bernoulli's numbers — see p. 183 of [1]). Let $f(x)=(a+x)^{-2}$ with the first three derivatives given by

$$f'(x)=-2(a+x)^{-3},\quad f''(x)=6(a+x)^{-4},\quad f^{(3)}(x)=-24(a+x)^{-5}.$$

Then, substituting into (1),

$$\sum_{k=0}^{n}\frac{1}{(a+k)^2}=-\frac{1}{a+n}+\frac{1}{a}+\frac{1}{2(a+n)^2}+\frac{1}{2a^2}-\frac{B_2}{(a+n)^3}+\frac{B_2}{a^3}+r_1(n),$$

where

$$|r_1(n)|<|r_1(n)-r_2(n)|=|B_4|\cdot\left|\frac{1}{(a+n)^5}-\frac{1}{a^5}\right|.$$

Taking the limit as $n\to\infty$, and using the values $B_2=1/6$ and $B_4=-1/30$, one has

$$h(a):=\sum_{k=0}^{\infty}\frac{1}{(a+k)^2}=\frac{1}{a}+\frac{1}{2a^2}+\frac{1}{6a^3}+r_1(\infty),$$

where

$$-\frac{1}{30a^5}\le r_1(\infty)\le\frac{1}{30a^5}.$$

Squaring $h(a)$, one obtains the left hand side of the inequality:

$$h(a)^2=\frac{1}{a^2}+\frac{1}{a^3}+\frac{7}{12a^4}+\frac{1}{6a^5}+\frac{1}{36a^6}$$
$$+2\left(\frac{1}{a}+\frac{1}{2a^2}+\frac{1}{6a^3}\right)r_1(\infty)+r_1^2(\infty).$$

To calculate the series on the right hand side of the inequality, set $f(x)=(a+x)^{-3}$ in (1), with derivatives

$$f'(x)=-3(a+x)^{-4},\quad f''(x)=12(a+x)^{-5},\quad f^{(3)}(x)=-60(a+x)^{-6},$$

and again let $n\to\infty$; we obtain

$$\sum_{k=0}^{\infty}\frac{1}{(a+k)^3}=\frac{1}{2a^2}+\frac{1}{2a^3}+\frac{3B_2}{2a^4}+R_1(\infty),$$

so

$$g(a) := 2\sum_{k=0}^{\infty}\frac{1}{(a+k)^3}$$
$$= \frac{1}{a^2}+\frac{1}{a^3}+\frac{3B_2}{a^4}+2R_1(\infty) = \frac{1}{a^2}+\frac{1}{a^3}+\frac{1}{2a^4}+2R_1(\infty)$$

where

$$-\frac{1}{12a^6} = -\frac{|B_4|}{4!}\cdot\frac{60}{a^6} \le R_1(\infty) \le \frac{|B_4|}{4!}\cdot\frac{60}{a^6} = \frac{1}{12a^6}.$$

Now the inequality to be proven is $h(a)^2 - g(a) > 0$. Substituting in the bounds for $r_1(\infty)$ and $R_1(\infty)$ (and $r_1^2(\infty) \ge 0$),

$$h(a)^2 - g(a) > \frac{1}{12a^4}+\frac{1}{6a^5}+\frac{1}{36a^6}-\frac{1}{15a^5}\left(\frac{1}{a}+\frac{1}{2a^2}+\frac{1}{6a^3}\right)-\frac{1}{6a^6}$$
$$= \frac{1}{6a^4}\left[\frac{1}{2}+\frac{1}{a}+\frac{1}{6a^2}-\frac{2}{5a^2}-\frac{1}{5a^3}-\frac{1}{15a^4}-\frac{1}{a^2}\right]$$
$$= \frac{1}{6a^4}\left[\frac{1}{2}+\frac{1}{a}-\frac{1}{5a^2}\left(\frac{37}{6}+\frac{1}{a}+\frac{1}{3a^2}\right)\right].$$

Since $a \ge 1$, one obtains

$$\frac{1}{5}\left(\frac{37}{6}+\frac{1}{a}+\frac{1}{3a^2}\right) \le \frac{1}{5}\left(\frac{37}{6}+1+\frac{1}{3}\right) = \frac{3}{2},$$

hence

$$\frac{1}{2}+\frac{1}{a}-\frac{1}{5a^2}\left(\frac{37}{6}+\frac{1}{a}+\frac{1}{3a^2}\right) \ge \frac{1}{2}+\frac{1}{a}-\frac{3}{2a^2} = \frac{1}{2a^2}(a^2+2a-3) \ge 0$$

for $a \ge 1$, which gives $h(a)^2 - g(a) > 0$.

Reference:

[1] Konrad Knopp, *Theory and Application of Infinite Series.*

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California, in a similar way but with some added complications. One comment and one incorrect solution were received.

Expert colleague Len Bos notes that the given inequality is equivalent to

$$(\Psi(a))^2 > -\Psi'(a),$$

where

$$\sum_{k=0}^{\infty}\frac{1}{(x+k)^2} = \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is the digamma function ($\Gamma(x)$ *being the gamma function).*

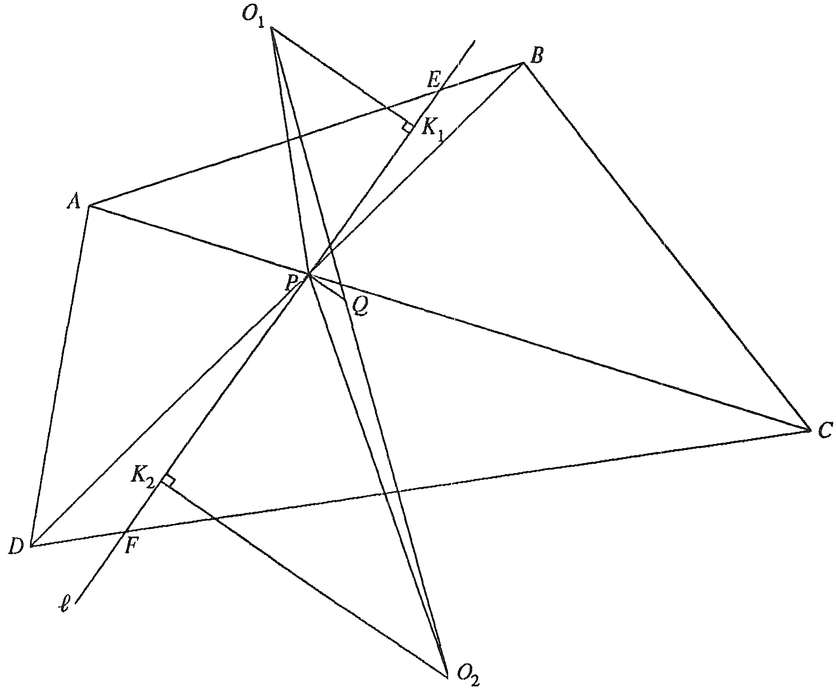
Does anyone have an easier proof, or a reference, for this result?

*   *   *   *   *

1951. [1994: 163] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a cyclic quadrilateral, and P is the intersection of diagonals AC and BD. A line ℓ through P meets AB and CD at E and F respectively. Let O_1 and O_2 be the circumcenters of ΔPAB and ΔPCD, and let Q be the point on O_1O_2 such that $PQ \perp \ell$. Prove that $EP : PF = O_1Q : QO_2$.

Solution by Ashish Kr. Singh, Kanpur, India.



Construct $O_1K_1 \perp \ell$ and $O_2K_2 \perp \ell$, with K_1 and K_2 on ℓ. Then

$$\frac{O_1Q}{QO_2} = \frac{K_1P}{PK_2},$$

so we have to prove $EP/PF = K_1P/PK_2$. Now

$$\begin{aligned}\angle O_1PK_1 &= |\angle O_1PB - \angle K_1PB| = |90^\circ - \frac{1}{2}\angle PO_1B - \angle K_1PB| \\ &= |90^\circ - \angle PAB - \angle K_1PB|,\end{aligned}$$

so

$$\sin\angle PO_1K_1 = \sin(\angle PAB + \angle K_1PB) = \sin(\angle CDP + \angle DPF) = \sin\angle PFC$$

and similarly

$$\sin\angle PO_2K_2 = \sin\angle PEB.$$

Thus [since triangles PAB and PDC are similar and O_1P and O_2P are their respective circumradii]

$$\frac{PK_1}{PK_2} = \frac{O_1P\sin\angle PO_1K_1}{O_2P\sin\angle PO_2K_2} = \frac{O_1P\sin\angle PFC}{O_2P\sin\angle PEB}$$
$$= \frac{PB}{PC}\cdot\frac{\sin\angle PFC/\sin\angle PCD}{\sin\angle PEB/\sin\angle PBA} = \frac{PB}{PC}\cdot\frac{PC/PF}{PB/EP} = \frac{EP}{PF}\,.$$

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; WALDEMAR POMPE, student, University of Warsaw, Poland; and the proposer.

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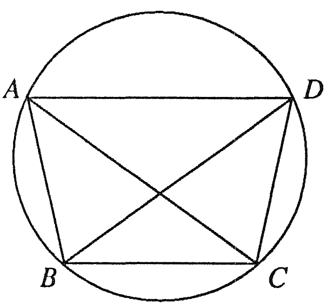
1952. [1994: 163] *Proposed by K. R. S. Sastry, Addis Ababa, Ethiopia.*

The convex cyclic quadrilateral $ABCD$ is such that each of its diagonals bisects one angle and trisects the opposite angle. Determine the angles of $ABCD$.

Solution by Federico Ardila, student, MIT, Cambridge, Massachusetts.

Assume diagonal AC bisects $\angle A$ and trisects $\angle C$, and diagonal BD bisects $\angle D$ and trisects $\angle B$. Then



$$\angle ADB = \angle BDC \quad\Longrightarrow\quad \text{arc}\,AB = \text{arc}\,BC$$

and

$$\angle DAC = \angle BAC \quad\Longrightarrow\quad \text{arc}\,BC = \text{arc}\,CD,$$

so

$$\text{arc}\,AB = \text{arc}\,BC = \text{arc}\,CD = x, \quad \text{say.}$$

Also,

$$\angle ABD = 2\angle DBC \quad\text{or}\quad \angle ABD = \frac{1}{2}\angle DBC,$$

so

$$\text{arc}\,AD = 2x \quad\text{or}\quad \frac{x}{2}\,.$$

If arc $AD = 2x$,

$$360^\circ = x + x + x + 2x = 5x \quad\Longrightarrow\quad x = 72^\circ,$$

so

$$A = D = 72^\circ, \qquad B = C = 108^\circ. \tag{1}$$

If arc $AD = x/2$,

$$360^\circ = x + x + x + \frac{x}{2} = \frac{7x}{2} \quad \Longrightarrow \quad x = \frac{720^\circ}{7},$$

so

$$A = D = \frac{720^\circ}{7}, \qquad B = C = \frac{540^\circ}{7}. \tag{2}$$

Also solved (both solutions) by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; TIM CROSS, Wolverley High School, Kidderminster, U. K.; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; L. J. UPTON, Mississauga, Ontario; and the proposer. One or the other of the solutions was found by ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I. B. Leopoldo Cano, Valladolid, Spain; CAN ANH MINH, student, University of California at Berkeley; DAG JONSSON, Uppsala, Sweden; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; J. A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; and CRISTÓBAL SÁNCHEZ-RUBIO, I. B. Penyagolosa, Castellón, Spain. Two other readers found both solutions plus (unfortunately) a third which wasn't valid.

Dou and Seimiya note that the cyclic quadrilaterals satisfying (1) *and* (2) *are vertices of a regular pentagon and regular heptagon respectively.*

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1953. [1994: 164] *Proposed by Murray S. Klamkin, University of Alberta.*

Determine a necessary and sufficient condition on real constants $r_1, r_2, \ldots, r_n$ such that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq (r_1x_1 + r_2x_2 + \cdots + r_nx_n)^2$$

holds for all real $x_1, x_2, \ldots, x_n$.

Solution by Yijun Yao, student, High School Attached to Fudan University, Shanghai, China.

We show that the condition is

$$\sum_{i=1}^{n} r_i^2 \leq 1. \tag{1}$$

In the given inequality we choose $x_i = r_i$ for $i = 1, 2, \ldots, n$ and we obtain

$$\sum_{i=1}^{n} r_i^2 \geq \left(\sum_{i=1}^{n} r_i^2\right)^2 ,$$

which implies (1). So the condition is necessary.

On the other hand, by Cauchy's inequality we have

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} r_i^2\right) \geq \left(\sum_{i=1}^{n} x_i r_i\right)^2 ,$$

or, assuming (1),

$$\sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} x_i r_i\right)^2 .$$

So the condition is sufficient.

Also solved (often the same way) by FEDERICO ARDILA, student, MIT, Cambridge, Massachusetts; SEUNG-JIN BANG, Seoul, Korea; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; CAN ANH MINH, student, University of California at Berkeley; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; N. T. TIN, Hong Kong; PANOS E. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One incorrect solution was received.

The problem was suggested by problem 1 of the 1988 Chinese Mathematical Olympiad.

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1954. [1994: 164] *Proposed by Vedula N. Murty, Maharanipeta, India.*

Let ABC be a triangle with $\angle A < \pi/2$ and $\angle B \leq \angle C$. The tangents to the circumcircle of ABC at B and C meet at D. Put $\theta = \angle OAD$, where O is the circumcentre. Prove that

$$2\tan\theta = \cot B - \cot C.$$

Solution by Dag Jonsson, Uppsala, Sweden.

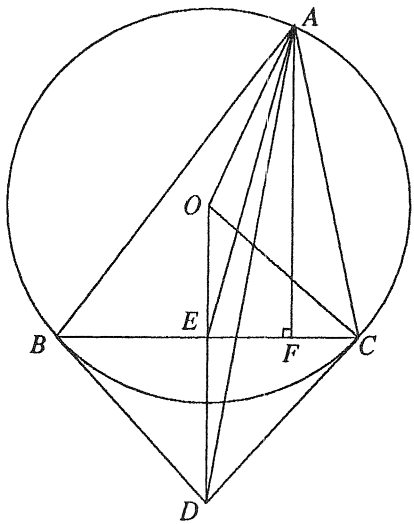
We prove the case where $\angle C \leq \pi/2$. The proof for $\angle C > \pi/2$ is similar.

Let OD intersect BC at E. [Then E is the midpoint of BC and $OE \perp BC$.] Since $\triangle OEC \sim \triangle OCD$,

$$\frac{OE}{OA} = \frac{OE}{OC} = \frac{OC}{OD} = \frac{OA}{OD} ,$$

which means that $\triangle AOE \sim \triangle DOA$. Thus

$$\theta = \angle OAD = \angle OEA = \angle EAF.$$



Now if $|AF|$ is h, $|BF|$ is b_1 and $|FC|$ is b_2, then

$$|EC| = \frac{b_1 + b_2}{2}, \quad |EF| = \frac{b_1 - b_2}{2}$$

and thus

$$2\tan\theta = \frac{2|EF|}{|AF|} = \frac{b_1 - b_2}{h} = \cot B - \cot C.$$

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; SEUNG-JIN BANG, Seoul, Korea; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; CYRUS HSIA, student, Woburn Collegiate, Toronto, Ontario; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan (a similar solution); D. J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

For triangles with $\angle A > \pi/2$*, the result holds with* $\tan\theta$ *replaced by* $|\tan\theta|$*, as can be seen from the above proof.*

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1955. [1994: 164] *Proposed by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*

Find all integer solutions of the system of equations

$$x^2 + 9y^2 + 9z^2 + 4u^2 = 1981, \qquad x + y + z + u = 54.$$

Solution by Cyrus Hsia, student, Woburn Collegiate, Scarborough, Ontario.

Consider the first equation minus 72 times the second: we get

$$x^2 - 72x + 9y^2 - 72y + 9z^2 - 72z + 4u^2 - 72u = 1981 - 54(72) = -1907,$$

so

$$(x-36)^2 + (3y-12)^2 + (3z-12)^2 + (2u-18)^2 = 1296 + 144 + 144 + 324 - 1907 = 1.$$

Fortunately, for integer solutions, $x - 36$, $3y - 12$, $3z - 12$, $2u - 18$ are integers as well. Thus one of these is ± 1 and the rest must be 0. Clearly $3y - 12$, $3z - 12$, $2u - 18$ cannot be ± 1 since y, z, u are integers. Thus $x - 36 = \pm 1$ and $3y - 12 = 3z - 12 = 2u - 18 = 0$, and so

$$(x,y,z,u) = (37,4,4,9) \quad \text{or} \quad (35,4,4,9).$$

To satisfy the second given equation, the solution must be

$$(x,y,z,u) = (\mathbf{37},\mathbf{4},\mathbf{4},\mathbf{9}).$$

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; CHARLES ASHBACHER, Cedar Rapids, Iowa; C. J. BRADLEY, Clifton College, Bristol, U. K.; TIM CROSS, Wolverley High School, Kidderminster, U. K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; PETER HURTHIG, Columbia College, Burnaby, B. C.; ESTEBAN INDURAIN, Universidad Pública de Navarra, Pamplona, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; KEE-WAI LAU, Hong Kong; J. A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; CORY PYE, student, Memorial University of Newfoundland, St. John's; N. T. TIN, Hong Kong; PANOS E. TSAOUSSOGLOU, Athens, Greece; EDWARD T. H. WANG, Wilfrid Laurier University, Waterloo, Ontario; YIJUN YAO, student, High School Attached to Fudan University, Shanghai, China; and the proposer.

Tin, Wang and the proposer gave the same solution as Hsia. (Hsia actually made one small mistake which has been corrected free of charge by the editor.) Other methods used include Cauchy's inequality, multivariable calculus, and computers.

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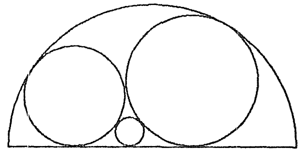
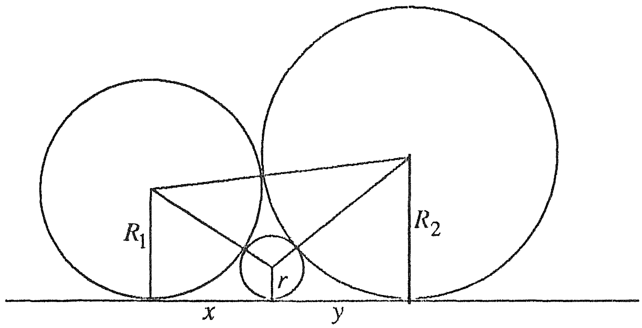
1956. [1994: 164] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

In a semicircle of radius 4 there are three tangent circles as in the figure. Prove that the radius of the smallest circle is at most $\sqrt{2}-1$.



Solution by Federico Ardila, student, MIT, Cambridge, Massachusetts.



Let R_1, R_2, r be the radii of the three circles, as in the figure. Also assume $R_2 \geq R_1$. Clearly then $R_2 \geq R_1 > r$. From the Pythagorean Theorem:

$$x = \sqrt{(R_1+r)^2-(R_1-r)^2} = 2\sqrt{R_1 r}\,,$$

$$y = \sqrt{(R_2+r)^2-(R_2-r)^2} = 2\sqrt{R_2 r}\,,$$

and

$$x+y = \sqrt{(R_2+R_1)^2-(R_2-R_1)^2} = 2\sqrt{R_1R_2}\,.$$

It follows that

$$\sqrt{R_1 r}+\sqrt{R_2 r} = \sqrt{R_1R_2}\,,$$

or

$$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{R_1}}+\frac{1}{\sqrt{R_2}}\,.$$

Now using the result

$$R_1+R_2 \leq 8(\sqrt{2}-1)$$

of the proposer's earlier problem *Crux* 1933 [1995: 96] (scaled by a factor of 4), and the Power Mean Inequality:

$$4r = \left(\frac{r^{-1/2}}{2}\right)^{-2} = \left(\frac{R_1^{-1/2}+R_2^{-1/2}}{2}\right)^{-2} \leq \frac{R_1+R_2}{2} \leq 4(\sqrt{2}-1),$$

so $r \leq \sqrt{2}-1$ as we wished to prove.

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; LEON BANKOFF, Beverly Hills, California; NIELS BEJLEGAARD, Stavanger, Norway; DAVID

DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; PAUL PENNING, Delft, The Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; CRISTÓBAL SÁNCHEZ-RUBIO, I. B. Penyagolosa, Castellón, Spain; TOSHIO SEIMIYA, Kawasaki, Japan; D. J. SMEENK, Zaltbommel, The Netherlands; N. T. TIN, Hong Kong; ALBERT W. WALKER, Toronto, Ontario; and the proposer. There was one incorrect solution sent in.

Most solvers used a similar approach. Bankoff notes an interesting sidelight of this problem can be found in Question 1661, Lady's and Gentleman's Diary, *1841, No. 138, where Mr. Philip Beecroft shows that the locus of the centre of the smaller circle is a parabola whose latus rectum is twice the diameter of the semicircle.*

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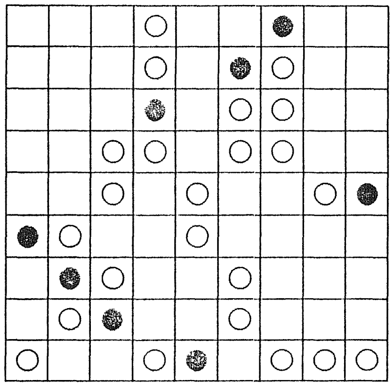
1957. [1994: 164] *Proposed by William Soleau, New York.*

			O			O		
			O		O	O		
			O		O	O		
		O	O		O	O		
		O		O			O	O
O	O			O				
	O	O			O			
	O	O			O			
O			O	O		O	O	O

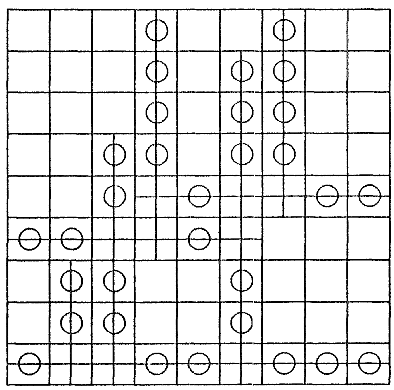
A 9 by 9 board is filled with 81 counters, each being green on one side and yellow on the other. Initially, all have their green sides up, except the 31 marked with circles in the diagram. In one move, we can flip over a block of adjacent counters, vertically or horizontally only, provided that at least one of the counters at the ends of the block is on the edge of the board. Determine a shortest sequence of moves which allows us to flip all counters to their green sides.

Solution by the University of Arizona Problem Solving Lab, Tucson.

We may only flip lines of adjacent counters which are entirely vertical or entirely horizontal, so any two counters which are neither in the same row nor in the same column may only be flipped in distinct moves. Since all counters with their yellow sides initially up must be flipped at least once, if there are n yellow counters on the board no two of which are in the same column or row, the puzzle cannot be solved in less than n moves. We found 8 such counters, as shown in black below:



No more than 8 moves are needed, because the puzzle can be solved as shown below, where a vertical or a horizontal line through a block of squares indicates that all the counters so marked are flipped in one move. Note that all counters that were initially yellow have been flipped once to make them green, and all counters that were originally green either have not been flipped or have been flipped twice, and so will still be green.



Also solved by CURTIS COOPER, Central Missouri State University, Warrensburg; DOUGLAS E. JACKSON, Eastern New Mexico University, Portales; PAVLOS B. KONSTADINIDIS, student, University of Arizona, Tucson; and the proposer. One other reader sent in an 8-move solution without a proof that it is minimal.

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1958. [1994: 164] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Find the tetrahedron of maximum volume given that the sum of the lengths of some four edges is 1.

Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.

There are two possible cases:

Case A: 3 of the 4 edges belong to the same face of the tetrahedron, say to $\triangle ABC$. To maximize the volume of the tetrahedron, the fourth edge (length h) must be perpendicular to $\triangle ABC$, and the area of $\triangle ABC$, which has the perimeter $p = 1 - h$, must be as large as possible. Thus $\triangle ABC$ must be equilateral. Now let x be the length of the sides of $\triangle ABC$. For the volume $v(x)$ of the tetrahedron we get

$$\begin{aligned} v(x) &= \frac{1}{3}\cdot\frac{x^2\sqrt{3}}{4}\cdot(1-3x) = \frac{\sqrt{3}}{12}\cdot\frac{4}{9}\cdot\left(\frac{3x}{2}\right)^2(1-3x) \\ &\le \frac{\sqrt{3}}{27}\cdot\left(\frac{2\cdot 3x/2+(1-3x)}{3}\right)^3 = \frac{\sqrt{3}}{729}\,. \end{aligned}$$

Equality holds if and only if $3x/2 = 1 - 3x$, i.e. if $x = 2/9$, $h = 1/3$.

Case B: No 3 of the 4 edges are sides of a face of the tetrahedron. Let AB, AC, DB and DC be the edges. Necessary conditions for the volume of the tetrahedron to be maximal are that the planes of $\triangle ABC$ and $\triangle BCD$ and the planes of $\triangle ABD$ and $\triangle ACD$ be perpendicular and, for reasons of symmetry, $\triangle ABC$, $\triangle BCD$, $\triangle ABD$ and $\triangle ACD$ be isosceles. Hence it follows that

$$AB = AC = DB = DC = x = \frac{1}{4}\,.$$

Let E and F be the midpoints of BC and AD, respectively. Since $\triangle ADE$ and $\triangle BCF$ are right-angled isosceles triangles with right angles at E and F, respectively, and EF as common altitude, $\triangle ADE$ and $\triangle BCF$ are congruent. We get $AF = EF = BE$ and $BF^2 = 2\cdot EF^2$, so $x^2 = AF^2 + BF^2 = 3\cdot EF^2$ and thus

$$EF = \frac{x\sqrt{3}}{3} = \frac{\sqrt{3}}{12}\,.$$

Since BC and the plane of $\triangle ADE$ are perpendicular [and $BC = 2\cdot BE = 2\cdot EF = \sqrt{3}/6$], it follows that in this case the largest possible volume of the tetrahedron is

$$v = \frac{1}{3}\cdot[ADE]\cdot BC = \frac{1}{3}\cdot EF^2\cdot BC = \frac{1}{3}\cdot\frac{1}{48}\cdot\frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{864}\,.$$

Hence we get the maximal volume $v = \sqrt{3}/729$ in case A; i.e. if one face of the tetrahedron is an equilateral triangle (length of the sides $x = 2/9$) and a fourth edge (length $h = 1/3$) perpendicular to this face.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, *U. K.; JORDI DOU, Barcelona, Spain; P. PENNING, Delft, The Netherlands; YIJUN YAO, student, High School Attached to Fudan University, Shanghai, China; and the proposer. Two other solutions only considered Case A.*

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1959. [1994: 164] *Proposed by John Selfridge, Northern Illinois University, DeKalb.*

Show that there is a (not too large) integer N so that, for every integer $n \geq N$, you can form a square by multiplying together distinct integers between n^2 and $(n+1)^2$. For instance, the product $27 \cdot 28 \cdot 30 \cdot 32 \cdot 35 = 5040^2$ shows that you can do it for $n = 5$. But you can't do it for $n = 6$, so N has to be at least 7.

Solution by the proposer.

We show that $N = 65$ works. Note that

$$n^2 \leq ab < \frac{(2a+1)(b-1)}{2} < a(b+1) < (a+1)(b-1)$$
$$< \frac{(2a+1)(b+1)}{2} < (a+1)b \leq (n+1)^2,$$

where

$$n = n' + 12t, \qquad a = a' + 8t, \qquad b = b' + 18t,$$

t is any integer ≥ 0, and n', a', b' are chosen from the table

n'	65	66	67	68	69	70	71	72	73	74	75	76
a'	43	44	43	45	43	46	49	47	49	47	49	53
b'	99	99	105	103	111	107	103	111	109	117	115	109

Each of the 24 inequalities involving n^2 or $(n+1)^2$ can be quickly checked, and the others follow immediately from $2a + 1 < b < 4a$.

Editor's note. Here are some details. The inequality $2a + 1 < b < 4a$ follows from the inequality $2a' + 1 < b' < 4a'$, which holds for entries in the above table. Similarly, the inequalities $n^2 \leq ab$ and $(a+1)b \leq (n+1)^2$ follow respectively from the pairs of inequalities

$$12n' \leq 9a' + 4b', \qquad n'^2 \leq a'b'$$

and

$$9a' + 4b' \leq 12n' + 3, \qquad (a'+1)b' \leq (n'+1)^2,$$

which also hold for entries from the table. Now noting that the product

$$ab \cdot \frac{(2a+1)(b-1)}{2} \cdot a(b+1) \cdot (a+1)(b-1) \cdot \frac{(2a+1)(b+1)}{2} \cdot (a+1)b$$

(all integers, since b is odd) is a perfect square, we get an answer to the problem for any integer $n \geq 65$, whenever

$$n^2 < ab \qquad \text{and} \qquad (a+1)b < (n+1)^2$$

both hold. And if equality holds in either or both of these cases, the corresponding factor ab or $(a+1)b$ can be dropped from the above product and the resulting smaller product must still be a perfect square, so the result of the problem is verified for all integers ≥ 65. The proposer has also handled the cases $7 \leq n \leq 64$ separately, which shows that $N = 7$ is in fact the smallest value possible.

No other solutions to this problem were received.

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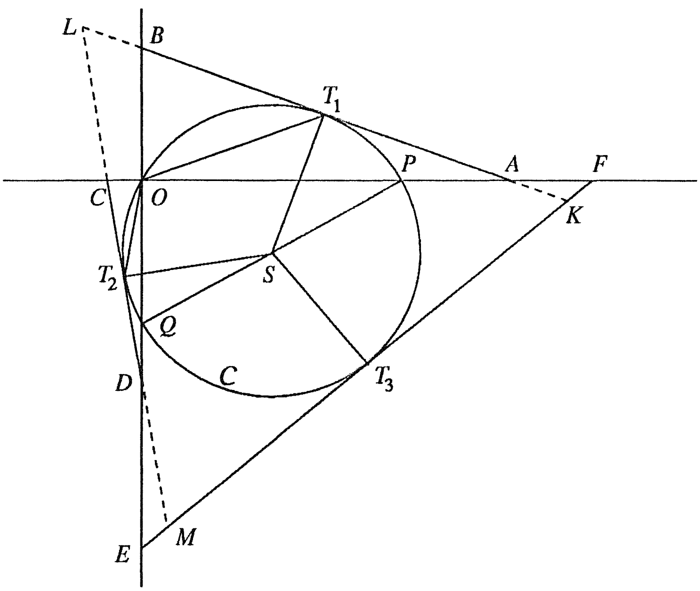
1960. [1994: 165] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Two perpendicular lines and a circle C pass through a common point. Three line segments AB, CD, EF, with endpoints on the two perpendicular lines, are tangent to C at their midpoints. Prove that the length of one segment is equal to the sum of the lengths of the other two.

Solution by Václav Konečný, Ferris State University, Big Rapids, Michigan.



From the figure: as

$$\frac{1}{2}BA = BT_1 = AT_1 = OT_1, \quad CT_2 = DT_2 = OT_2, \quad ET_3 = FT_3 = OT_3,$$

it is sufficient to show that

$$OT_3 = OT_1 + OT_2.$$

Now (since QSP is a diameter)

$$\begin{aligned}\angle T_1ST_2 &= 180^\circ - \angle T_1SP - \angle T_2SQ = 180^\circ - 2\angle T_1OP - 2\angle T_2OQ \\ &= 2[(90^\circ - \angle T_1AO) - \angle T_2DO] = 2(\angle T_1BO - \angle T_2DO) = 2\angle L,\end{aligned}$$

and thus [since $\angle L + \angle T_1ST_2 = 180^\circ$] $\angle L = 60^\circ$. Similarly $\angle K = \angle M = 60^\circ$. Consequently $\Delta T_1T_2T_3$ is also equilateral. By Ptolemy's Theorem on cyclic quadrilaterals we get

$$(OT_3)(T_1T_2) = (OT_1)(T_2T_3) + (OT_2)(T_1T_3) = (OT_1 + OT_2)T_1T_2$$

or $OT_3 = OT_1 + OT_2$.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; JORDI DOU, Barcelona, Spain; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; MARIA ASCENSIÓN LÓPEZ CHAMORRO, I. B. Leopoldo Cano, Valladolid, Spain; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CRISTÓBAL SÁNCHEZ-RUBIO, I. B. Penyagolosa, Castellón, Spain; TOSHIO SEIMIYA, Kawasaki, Japan; D. J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; CHRIS WILDHAGEN, Rotterdam, The Netherlands; YIJUN YAO, student, High School Attached to Fudan University, Shanghai, China; and the proposer. There was one incorrect solution sent in.

The result in the above proof, that $OT_3 = OT_1 + OT_2$ where $\Delta T_1T_2T_3$ is equilateral and O lies on arc T_1T_2 of its circumcircle, is quite familiar and no doubt has appeared in Crux *before.*

Perz notes that the problem is very similar to problem 3 *of the Junior A Level contest in the autumn* 1994 *Tournament of the Towns.*

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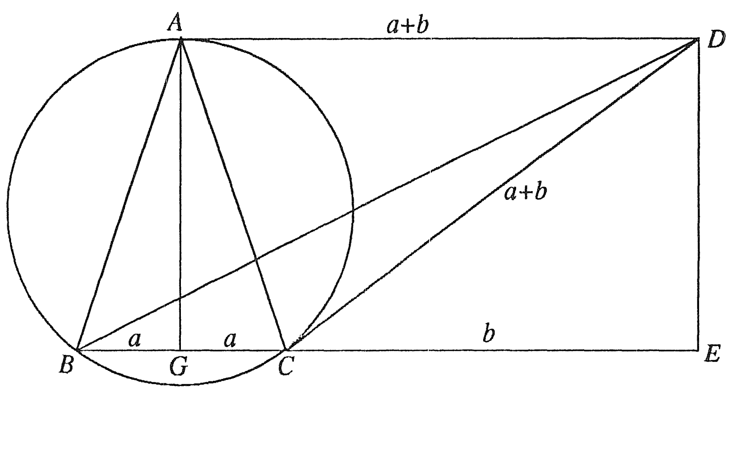
1961. [1994: 193] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an isosceles triangle with $AB = AC$. We denote the circumcircle of ΔABC by Γ. Let D be the point such that DA and DC are tangent to Γ at A and C respectively. Prove that $\angle DBC \leq 30^\circ$.

I. *Solution by Dag Jonsson, Uppsala, Sweden.*



Let G be the midpoint of BC and form the rectangle $AGED$ as in the figure. Let $|BG| = |GC| = a$ and $|CE| = b$. Then $|CD| = |AD| = a + b$ and

$$\begin{aligned}|BD|^2 &= |BE|^2 + (|CD|^2 - |CE|^2) \\ &= (2a+b)^2 + (a+b)^2 - b^2 = 5a^2 + 6ab + b^2.\end{aligned}$$

Now

$$\begin{aligned}\angle DBE \le 30^\circ \quad &\iff \quad |BE| \ge \frac{\sqrt{3}}{2}|BD| \quad \iff \quad 4|BE|^2 \ge 3|BD|^2 \\ &\iff \quad 16a^2 + 16ab + 4b^2 \ge 15a^2 + 18ab + 3b^2 \\ &\iff \quad a^2 + b^2 \ge 2ab,\end{aligned}$$

which of course is true.

II. *Solution by Waldemar Pompe, student, University of Warsaw, Poland.*

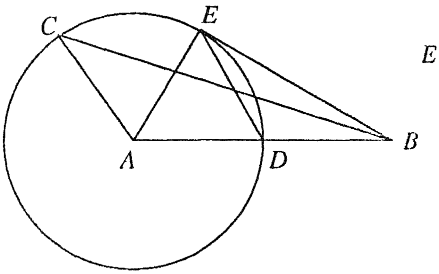


Figure 1

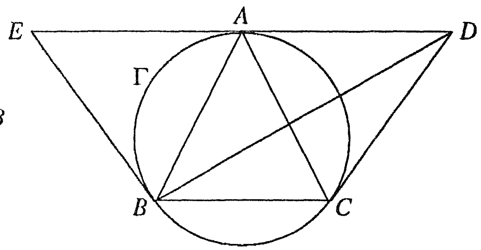


Figure 2

Lemma. Given triangle ABC with $AB = 2AC$. Then $\angle ABC \le 30^\circ$.

Proof. (This can be easily shown using the law of sines, but there is a nice geometric proof which doesn't use trigonometry.) See Figure 1. Let D be the midpoint of AB and let BE be a tangent to the circle with center A and radius $AD = AC$. Then $AE = AD = DE$ (since D is the circumcenter of the right triangle ABE), which shows that ΔADE is equilateral. Thus $\angle ABC \le \angle ABE = 30^\circ$.

Now the solution to the problem.

Let E be the symmetric point to D with respect to the point A (see Figure 2). Since ΔABC is isosceles, the line DE is tangent to Γ at A. Therefore $2BE = DE$. Using the lemma on the triangle BDE we get

$$\angle DBC = \angle BDE \le 30^\circ,$$

as desired.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain (two solutions); FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; ŠEFKET ARSLANAGIĆ, Berlin, Germany; SAM BAETHGE, *Science Academy, Austin, Texas; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I. B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; HIMADRI CHOUDHURY, student, Hunter High School, New York; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; F. O. FARID, University of Calgary; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; JOSEPH LING, University of Calgary; ANDY LIU, University of Alberta; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzi-gymnasium, Graz, Austria; CRISTÓBAL SÁNCHEZ-RUBIO, I. B. Penyagolosa, Castellón, Spain; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, New York; D. J. SMEENK, Zaltbommel, The Netherlands; and the proposer.*

The proposer's solution was similar to Pompe's.

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1962. [1994: 193] *Proposed by Murray S. Klamkin, University of Alberta.*

If A, B, C, D are non-negative angles with sum π, prove that

(i) $\cos^2 A + \cos^2 B + \cos^2 C + \cos^2 D \geq 2\sin A \sin C + 2\sin B \sin D$;

(ii) $1 \geq \sin A \sin C + \sin B \sin D$.

Solution by Himadri Choudhury, student, Hunter High School, New York.

(i) Since $A + B + C + D = \pi$,

$$\cos(A + C) = \cos(\pi - (B + D)) = -\cos(B + D),$$

so

$$\cos A \cos C - \sin A \sin C = -\cos B \cos D + \sin B \sin D,$$

i.e.

$$\cos A \cos C + \cos B \cos D = \sin A \sin C + \sin B \sin D. \qquad (1)$$

Substituting this into the inequality (i) yields the equivalent inequality

$$\cos^2 A + \cos^2 B + \cos^2 C + \cos^2 D \geq 2\cos A \cos C + 2\cos B \cos D,$$

which is equivalent to

$$(\cos A - \cos C)^2 + (\cos B - \cos D)^2 \geq 0,$$

which is obviously true.

(ii) We will show that

$$2 \geq 2\sin A \sin C + 2\sin B \sin D.$$

Substituting (1) into the above we get equivalently

$$\begin{aligned} 2 &\geq \sin A \sin C + \sin B \sin D + \cos A \cos C + \cos B \cos D \\ &= \cos(A - C) + \cos(B - D), \end{aligned}$$

which is true since $|\cos x| \leq 1$ for all real x.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; ŠEFKET ARSLANAGIĆ, Berlin, Germany; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; BEATRIZ MARGOLIS, Paris, France; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; J.-B. ROMERO MÁRQUEZ, Universidad de Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D. J. SMEENK, Zaltbommel, The Netherlands; CHRISTIAN WOLINSKI, Halifax, Nova Scotia; and the proposer.

Janous notes that the left-hand sides of (i) *and* 2×(ii) *don't compare in general, that is,* $\sum \cos^2 A$ *can be* < 2 *or* > 2.

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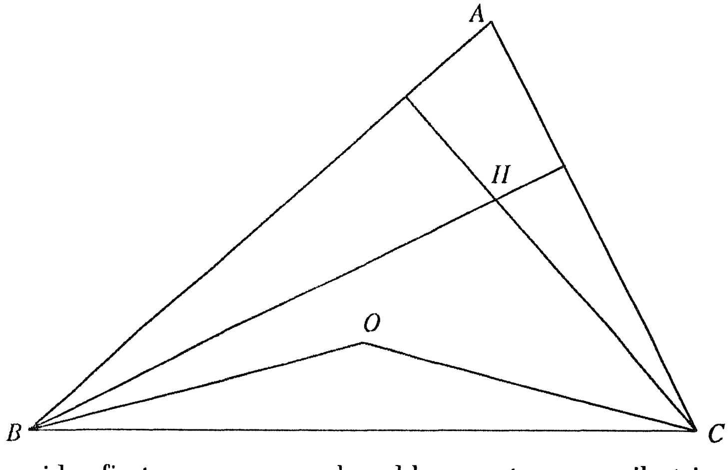
1963. [1994: 194] *Proposed by K. R. S. Sastry, Dodballapur, India.*

In triangle ABC, one pair of trisectors of the angles B and C meet at the orthocenter. Show that the other pair of trisectors of these angles meet at the circumcenter.

Solution by P. Penning, Delft, The Netherlands.



Consider first a more general problem: not necessarily trisectors, but two lines that make equal angles with the sides (these are called *isogonal* lines). [So in the figure, $\angle ABH = \angle OBC$ and $\angle ACH = \angle OCB$.—*Ed.*] It is known and easy to show that if one pair intersects in the orthocentre H, the other pair intersects in the circumcentre O; i.e., H and O are isogonally related. [They are *isogonal conjugates*: e.g., see pages 153 and 163 of Johnson's *Advanced Euclidean Geometry*. — *Ed.*]

Let A, B, C be the angles of the triangle in degrees, and let $\angle HBC = \lambda B$ and $\angle HCB = \mu C$. Since H is the orthocentre,

$$\lambda B = 90^\circ - C \quad \text{and} \quad \mu C = 90^\circ - B.$$

Solve for B and C, and then A:

$$B = \frac{90(1-\mu)}{1-\lambda\mu}, \quad C = \frac{90(1-\lambda)}{1-\lambda\mu}, \quad A = \frac{90(\lambda+\mu-2\lambda\mu)}{1-\lambda\mu}. \tag{1}$$

For the other intersection we find:

$$\angle OBC = (1-\lambda)B = \frac{90(1-\lambda)(1-\mu)}{1-\lambda\mu} = 90^\circ - A,$$

$$\angle OCB = (1-\mu)C = \frac{90(1-\lambda)(1-\mu)}{1-\lambda\mu} = 90^\circ - A,$$

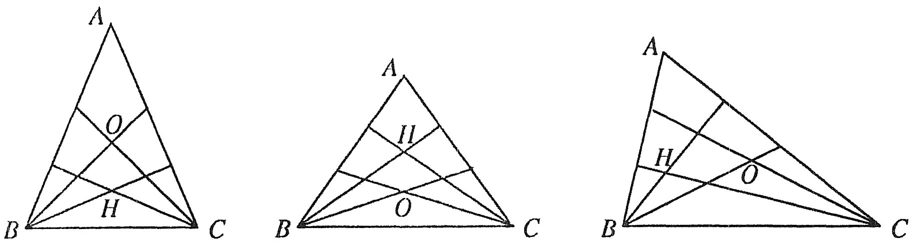
and thus

$$\angle BOC = 180^\circ - \angle OBC - \angle OCB = 2A.$$

So [since $\angle OBC = \angle OCB$ means $OB = OC$] O is the circumcentre, in agreement with the isogonal relationship between H and O.

In the problem, λ and μ are $1/3$ or $2/3$, which from (1) poses conditions on triangle ABC:

if $\lambda = \mu = 1/3$, then $A = 45^\circ$, $B = C = 67.5^\circ$;
if $\lambda = \mu = 2/3$, then $A = 72^\circ$, $B = C = 54^\circ$;
if $\lambda = 2/3$ and $\mu = 1/3$, then $A = (450/7)^\circ$, $B = (540/7)^\circ$, $C = (270/7)^\circ$.



Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; HIMADRI CHOUDHURY, student, Hunter High School, New York; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; GOTTFRIED PERZ, Pestalozzi-gymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; D. J. SMEENK, Zaltbommel, *The Netherlands; PAUL YIU, Florida Atlantic University, Boca Raton; and the proposer. Three incomplete solutions were sent in.*

Most solvers noted that only three triangles satisfy the condition of the problem. Heuver and Seimiya also pointed out that the result follows immediately because H and O are isogonal conjugates.

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1964. [1994: 194] *Proposed by Harvey Abbott and Andy Liu, University of Alberta.*

Find a combinatorial proof that

$$\sum_{i=0}^{n}(-1)^i\binom{n+1}{i}\binom{m(n-i)}{n}=\binom{m+n-1}{n}$$

for all positive integers m and n.

Solution by the proposers.

Suppose there are $mn-n$ shares, to be distributed among $n+1$ shareholders so that nobody has m or more shares. Let S denote the set of all distributions without the constraint, and let A_i denote the set of those in which the ith shareholder has at least m shares, $0 \le i \le n$. It follows from the Principle of Inclusion-Exclusion that

$$\begin{aligned}|\overline{A_0}\cap\overline{A_1}\cap\cdots\cap\overline{A_n}| &= |S|-\sum_{i=0}^{n}|A_i|+\sum_{i<j}|A_i\cap A_j|-\cdots\\ &= \binom{mn-n+(n+1)-1}{(n+1)-1}-\binom{n+1}{1}\binom{mn-m-n+(n+1)-1}{(n+1)-1}+\cdots\\ &= \binom{mn}{n}-\binom{n+1}{1}\binom{m(n-1)}{n}+\cdots\\ &= \sum_{i=0}^{n}(-1)^i\binom{n+1}{i}\binom{m(n-i)}{n}.\end{aligned}$$

On the other hand, since $mn-n=(m-1)(n+1)-(m-1)$, we could give $m-1$ shares to each shareholder at the beginning, and reclaim $m-1$ shares from them arbitrarily. This can be done in

$$\binom{m-1+(n+1)-1}{(n+1)-1}=\binom{m+n-1}{n}$$

ways.

No other solutions to this problem were received.

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