SKOLIAD No. 83

Robert Bilinski

Please send your solutions to the problems in this edition by *1 May, 2005*. A copy of **MATHEMATICAL MAYHEM Vol. 8** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk (∗) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

We have gradually shortened the deadline for submitting solutions to Skoliad problems. We now have a three month turn-around time, which will be maintained in subsequent issues.

Our items this issue come from a contest organized for the secondary school boards in the Laval region of Quebec by College Montmorency.

Je voudrais remercier André Labelle pour nous avoir fourni gracieusement une copie de ce concours.

Concours Montmorency 2002–2003
Sec V, novembre 2002

1. (∗) Montrez que le produit de deux nombres entiers impairs quelconques est toujours impair. (Attention : des exemples ne suffisent pas.)

**2**. (∗) L'aire d'un cercle inscrit dans un triangle équilatéral est 1 cm². Quelle est l'aire du cercle circonscrit au même triangle équilatéral ?
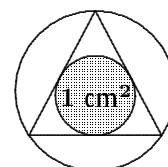
3. (∗) En descendant le courant d'une rivière, un bateau a une vitesse de 30 km/h, tandis qu'en le remontant, sa vitesse n'est que de 22 km/h. Pour franchir la distance séparant les villes Bellerues et Beauparc sur la rive, il met quatre heures de moins qu'en sens inverse. Quelle est la distance entre ces villes ?

4. (∗) En algèbre, il est interdit de simplifier de la sorte : $\dfrac{2\cancel{x}}{\cancel{x} + y} = \dfrac{2}{y}$. En supposant que x et y sont des entiers positifs, montrez que l'égalité est vraie seulement si $x = y = 2$.

5. (∗) On coupe un triangle isocèle AED par BC, une droite parallèle à sa base AD. À quelle hauteur au-dessus de cette base devons-nous couper le triangle pour que l'aire du trapèze $ABCD$ soit la moitié de l'aire du triangle AED sachant que la hauteur de ce dernier est de 4 cm ?
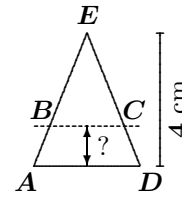
6. (∗) Décomposer en 3 facteurs l'expression : $x^3 - y^3 + x^2 - y^2 + x^2y - xy^2$.

7. (∗) Deux planètes A et B tournent sur des orbites circulaires autour d'un soleil central. La planète A est aujourd'hui exactement entre le soleil et la planète B. Sachant que A et B prennent respectivement 3 ans et 8 ans pour faire une révolution complète autour de leur soleil, évaluer le temps nécessaire pour qu'à nouveau, et pour la première fois, la planète A se retrouve exactement entre le soleil et la planète B.
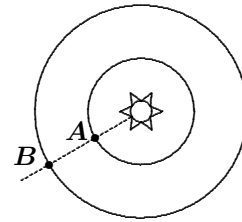
8. (∗) Trois robinets A, B et C sont placés au-dessus d'un bassin. Le tableau donne le temps de remplissage du bassin vide, lorsque seulement deux des trois robinets sont ouverts simultanément. Combien de temps faudra-t-il pour remplir le bassin vide lorsque les trois robinets sont ouverts simultanément ? (Note : On suppose que le débit d'un robinet ouvert est constant.)

A	B	C	Temps
OUVERT	OUVERT	FERMÉ	3 minutes
OUVERT	FERMÉ	OUVERT	4 minutes
FERMÉ	OUVERT	OUVERT	6 minutes

Montmorency Contest 2002–2003
Grade 11, November 2002

1. (∗) Show that the product of any two odd integers is always odd. (Warning: examples are not enough.)

2. (∗) The area of the circle inscribed in an equilateral triangle is 1 cm². What is the area of the triangle's circumscribed circle?
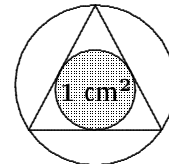
3. (∗) While descending a river, a boat goes 30 km/h; going up-river, the speed is only 22 km/h. To go between the cities of Bellerue and Beauparc on the river takes 4 hours less one way than the other. What is the distance between the cities?

4. (∗) In algebra, the following simplification is not allowed: $\dfrac{2\not{x}}{\not{x}+y} = \dfrac{2}{y}$. Supposing that x and y are positive integers, show that the given equation is true only if $x = y = 2$.

5. (∗) An isosceles triangle AED is cut by a line BC parallel to the base AD. At what height above the base do we have to cut the triangle so that the area of trapezoid $ABCD$ is half that of triangle AED, if the height of the triangle is 4 cm?
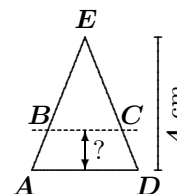
6. (∗) Decompose $x^3 - y^3 + x^2 - y^2 + x^2y - xy^2$ into 3 factors.

7. (∗) Two planets A and B go around a central sun in perfectly circular orbits. Today, planet A is exactly between the sun and planet B. Knowing that A and B take 3 years and 8 years, respectively, to make complete revolutions around their sun, find how long it will take until the next time planet A is exactly between the sun and planet B.
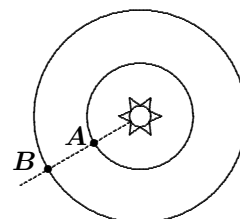
8. (∗) Three faucets A, B, and C are placed above a pan. The table gives the time needed for the pan to fill up when only two of the three faucets are turned on simultaneously. How much time would be needed for the pan to fill up if all three faucets were on at the same time? (Note: Assume that the flow out of each faucet is constant when the faucet is on.)

A	B	C	Time
ON	ON	OFF	3 minutes
ON	OFF	ON	4 minutes
OFF	ON	ON	6 minutes

Next we give solutions to the first annual Hypatia contest for grade 11 students run by the Canadian Mathematical Competition [2004 : 193–196].

2003 Concours Hypatia / 2003 Hypatia Contest
Grade 11 / Sec V, April 16, 2003

1. (a) Quentin has a number of square tiles, each measuring 1 cm by 1 cm. He tries to put these small square tiles together to form a larger square of side length n cm, but finds that he has 92 tiles left over. If he had increased the side length of the larger square to $(n + 2)$ cm, he would have been 100 tiles short of completing the larger square. How many tiles does Quentin have?

(b) Quentin's friend Rufus arrives with a big pile of identical blocks, each in the shape of a cube. Quentin takes some of the blocks and Rufus takes the rest. Quentin uses his blocks to try to make a large cube with 8 blocks along each edge, but finds that he is 24 blocks short. Rufus, on the other hand, manages to exactly make a large cube using all of his blocks. If they use all of their blocks together, they are able to make a complete cube which has a side length that is 2 blocks longer than Rufus' cube. How many blocks are there in total?

Extension to #1: As in Question #1 (a), Quentin tries to make a large square out of square tiles and has 92 tiles left over. In an attempt to make a second square, he increases the side length of this first square by an unknown number of tiles and finds that he is 100 tiles short of completing the square. How many different numbers of tiles is it possible for Quentin to have?

Solution by Alex Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

(a) The area of the square with side n is n^2. This means the number of tiles Quentin has is $n^2 + 92$. The area of the square with side $n + 2$ is $(n + 2)^2$. This means the number of tiles Quentin has is $(n + 2)^2 - 100$. Since the total is the same in both cases, we have:

$$\begin{aligned}
(n+2)^2 - 100 &= n^2 + 92, \\
n^2 + 4n + 4 - 100 &= n^2 + 92, \\
n &= 47.
\end{aligned}$$

The number of tiles Quentin has is $(47)^2 + 92 = 2209 + 92 = 2301$.

(b) Let a be the number of blocks that Quentin has, and let b be the number of blocks Rufus has. Since Quentin is 24 blocks short of making a cube with side 8, the number of blocks Quentin has is $a = 8^3 - 24 = 488$. Rufus makes a complete cube using all of his blocks. If Rufus' cube has side length n, then $b = n^3$. We also know that the blocks of Quentin and Rufus put together will make a cube with side 2 units bigger than the side of the cube that Rufus built. Thus, $a + b = (n + 2)^3$. Therefore,

$$\begin{aligned}
(n+2)^3 &= n^3 + 488, \\
n^3 + 3(n^2)(2) + 3(n)(2^2) + 2^3 &= n^3 + 488, \\
6n^2 + 12n - 480 &= 0, \\
n^2 + 2n - 80 &= 0, \\
(n-8)(n+10) &= 0.
\end{aligned}$$

We cannot have n negative; thus $n = 8$. The total number of blocks is $(n + 2)^3 = 1000$.

Extension: Let a be the total number of tiles. Let n be the number of tiles on each side of the smaller square that Quentin tries to make. Let k be the number by which he increases the cube side to make the second square.

In building his first square, he has 92 tiles left over, implying that $a = n^2 + 92$. In building the second square, with side $(n + k)$, he is 100 tiles short, implying that $a = (n + k)^2 - 100$. Then

$$
\begin{aligned}
n^2 + 92 &= (n + k)^2 - 100\,, \\
n^2 + 92 &= n^2 + 2kn + k^2 - 100\,, \\
k^2 + 2kn - 192 &= 0\,.
\end{aligned}
$$

By the quadratic formula,

$$
k = \frac{-2n \pm \sqrt{(2n)^2 - 4(1)(-192)}}{2} = -n \pm \sqrt{n^2 + 192}\,.
$$

In order to obtain an integer value for k, we need $n^2 + 192$ to be a perfect square. Thus,

$$
\begin{aligned}
n^2 + 192 &= m^2\,, \\
m^2 - n^2 &= 192\,, \\
(m - n)(m + n) &= 2^6 3\,.
\end{aligned}
$$

The divisors of 192 are 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 64, 96, 192. We check the possible pairs of factors:

$m - n$	$m + n$	m	n	a
1	192			
2	96	49	47	2301
3	64			
4	48	26	22	576
6	32	19	13	261
8	24	16	8	156
12	16	14	2	96

We get 5 possible values for a, the number of tiles that Quentin has, namely 2301, 576, 261, 156, 96.

2. Xavier and Yolanda are playing a game starting with some coins arranged in piles. Xavier always goes first, and the two players take turns removing one or more coins from any one pile. The player who takes the last coin wins.

(a) If there are two piles of coins with 3 coins in each pile, show that Yolanda can guarantee that she always wins the game.

(b) If the game starts with piles of 1, 2, and 3 coins [*Ed:* three piles altogether], explain how Yolanda can guarantee that she always wins the game.

Extension to #2: If the game starts with piles of 2, 4, and 5 coins, which player wins if both players always make their best possible move? Explain the winning strategy.

Official solution, adapted by the editor.

(a) Yolanda will win the game if she can guarantee that, at some point when it is her turn to choose, she is selecting coins from just one pile. She will then win the game by removing all of the coins from that pile.

She can guarantee that this will happen by duplicating Xavier's move, but in the other pile. Thus, if Xavier takes 1, 2, or 3 coins, then Yolanda will take the same number of coins from the other pile. Then Xavier will always empty one pile first, and Yolanda will win.

Hence, Yolanda can guarantee that she wins by using a *copying* strategy.

(b) In part (a), we saw that Yolanda could always win the game if she could guarantee that Xavier was choosing when there were two piles with an equal number of coins in each pile.

Starting with piles of 1, 2, and 3 coins, Yolanda can always win, because, after her first turn, she can always give two equal piles (and an empty third pile) back to Xavier. We see this by examining the possibilities for Xavier's first move and Yolanda's first move, as shown in the upper table.

Xavier	Yolanda
$(0, 2, 3)$	$(0, 2, 2)$
$(1, 1, 3)$	$(1, 1, 0)$
$(1, 0, 3)$	$(1, 0, 1)$
$(1, 2, 2)$	$(0, 2, 2)$
$(1, 2, 1)$	$(1, 0, 1)$
$(1, 2, 0)$	$(1, 1, 0)$

Therefore, Yolanda's strategy is to create two equal piles (and a third empty pile) on her first turn, and then win using her strategy from (a).

Extension: We have seen that Yolanda can always win if Xavier chooses first from three piles with 1, 2, and 3 coins (part (b)), or if Xavier chooses first from two piles with an equal number of coins (part (a)).

On his first move, Xavier does not want to create two equal piles (for example, $(2, 4, 4)$ or $(2, 2, 5)$), because Yolanda would then remove the unequal pile and Xavier would be left with two equal piles. Similarly, Xavier does not want to create a situation where Yolanda can reduce immediately to $(1, 2, 3)$, because Yolanda would then win using the strategy from (b).

| Xavier	Yolanda
$(2, 4, 4)$	$(0, 4, 4)$
$(2, 4, 3)$	$(2, 1, 3)$
$(2, 4, 2)$	$(2, 0, 2)$
$(2, 4, 1)$	$(2, 3, 1)$
$(2, 4, 0)$	$(2, 2, 0)$
$(2, 3, 5)$	$(2, 3, 1)$
$(2, 2, 5)$	$(2, 2, 0)$
$(2, 1, 5)$	$(2, 1, 3)$
$(2, 0, 5)$	$(2, 0, 2)$
$(1, 4, 5)$	??
$(0, 4, 5)$	$(0, 4, 4)$

The table above shows the possible first moves for Xavier and the first moves that Yolanda can then make to guarantee a win for herself.

What if Xavier moves to $(1, 4, 5)$? There are then 10 possible moves for Yolanda. If she moves to one of $(0, 4, 5)$, $(1, 1, 5)$, $(1, 0, 5)$, $(1, 4, 4)$, $(1, 4, 1)$, or $(1, 4, 0)$, then Xavier can win by reducing to two equal piles. If she moves to one of $(1, 3, 5)$, $(1, 2, 5)$, $(1, 4, 3)$, or $(1, 4, 2)$, then Xavier can win by reducing to some ordering of $(1, 2, 3)$.

Therefore, Xavier can win by reducing first to $(1, 4, 5)$, then to either two equal piles or some ordering of $(1, 2, 3)$. From then on, he can follow Yolanda's strategy from parts (a) or (b).

Also solved by Alex Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

3. In the diagram, the sphere has a diameter of 10 cm. Also, the right circular cone has a height of 10 cm, and its base has a diameter of 10 cm. The sphere and cone sit on a horizontal surface. If a horizontal plane cuts both the sphere and the cone, the cross-sections will both be circles, as shown. Find the height of the horizontal plane that gives circular cross-sections of the sphere and cone of equal area.
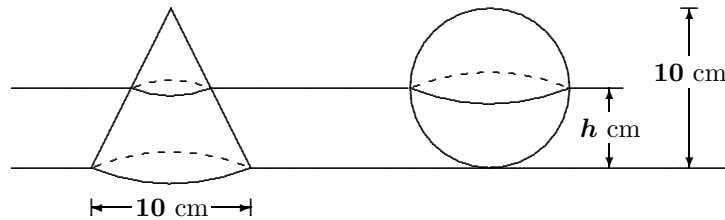
Extension to #3: A sphere of diameter d and a right circular cone with a base of diameter d stand on a horizontal surface. In this case, the height of the cone is equal to the radius of the sphere. Show that, for any horizontal plane that cuts both the cone and the sphere, the sum of the areas of the circular cross-sections is always the same.
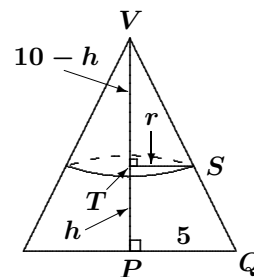
Official solution, adapted by the editor.

In order for the cross-sectional circles to have the same area, they must have the same radius r.

Case 1: Cone.

From V, the vertex of the cone, we draw the principal axis of the cone as shown. By symmetry, triangle VPQ is similar to triangle VTS. Thus,

$$\frac{10}{5} = \frac{10-h}{r},$$
$$2r = 10 - h,$$
$$r = \tfrac{1}{2}(10 - h).$$

Case 2: Sphere.

Let M be the centre of the cross-sectional circle. From O, the centre of the sphere, we draw a line through M. This line is perpendicular to the circle. We have $OP = 5$ (the radius of the circle), $DO = 5$, $DM = h$, and $MP = r$.
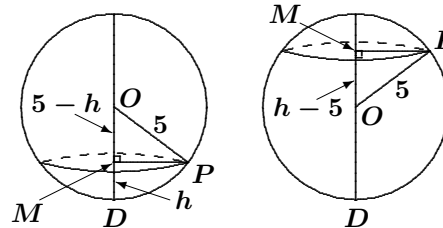
If $h \geq 5$, then $OM = h - 5$; if $h < 5$, then $OM = 5 - h$. In both cases, by Pythagoras,

$$r = \sqrt{OP^2 - OM^2} = \sqrt{5^2 - (5-h)^2} = \sqrt{10h - h^2}.$$

Using the results from cases 1 and 2 above, we have

$$\begin{aligned}
\tfrac{1}{2}(10 - h) &= \sqrt{10h - h^2}\,, \\
(10 - h)^2 &= 4(10h - h^2)\,, \\
5h^2 - 60h + 100 &= 0\,, \\
h^2 - 12h + 20 &= 0\,, \\
(h - 10)(h - 2) &= 0\,.
\end{aligned}$$

Therefore, $h = 10$ or $h = 2$. Thus, the height is 10 cm or 2 cm. (Note that $h = 10$ gives a horizontal plane that just passes through the vertex of the cone and is tangent to the top of the sphere.)

Extension: To avoid fractions, we start by letting $d = 2R$, implying that the cone has radius R and height R. Suppose that a horizontal plane at height h cuts the cone and the sphere in circles of radii r_1 and r_2, respectively. (Note that $h \leq R$ since the cone has height R.)

As in the initial problem, we have, in the cone, $\dfrac{R}{R} = \dfrac{R - h}{r_1}$, or $r_1 = R - h$, and in the sphere,

$$\begin{aligned}
r_2 &= \sqrt{OP^2 - OM^2} \\
&= \sqrt{R^2 - (R - h)^2} \\
&= \sqrt{2Rh - h^2}\,.
\end{aligned}$$
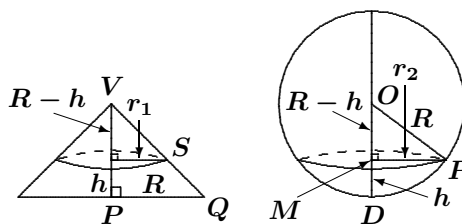
Therefore, the sum of the areas of the two circular cross-sections is

$$\begin{aligned}
\pi r_1^2 + \pi r_2^2 &= \pi(R - h)^2 + \pi\left(\sqrt{2Rh - h^2}\right)^2 \\
&= \pi(R^2 - 2hR + h^2) + \pi(2hR - h^2) = \pi R^2\,,
\end{aligned}$$

which does not depend on h.

Also solved by Alex Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

4. Square $ABCD$ has vertices $A(1, 4)$, $B(5, 4)$, $C(5, 8)$, and $D(1, 8)$. From a point P outside the square, a vertex of the square is said to be visible if it can be connected to P by a straight line that does not pass through the square. Thus, from any point P outside the square, either two or three of the vertices of the square are visible. The visible area of P is the area of the one triangle or the sum of the areas of the two triangles formed by joining P to the two or three visible vertices of the square.

(a) Show that the visible area of $P(2, -6)$ is 20 square units.

(b) Show that the visible area of $Q(11, 0)$ is also 20 square units.

(c) The set of points P for which the visible area equals 20 square units is called the 20/20 set, and is a polygon. Determine the perimeter of the 20/20 set.

Extension to #4: From any point P outside a unit cube, 4, 6, or 7 vertices are visible in the same sense as in the case of the square. Connecting point P to each of these vertices gives 1, 2, or 3 square-based pyramids, which make up the visible volume of P. The 20/20 set is the set of all points P for which the visible volume is 20, and is a polyhedron. What is the surface area of this 20/20 set?

Official solution.

(a) The visible area of $P(2, -6)$ is the area of $\triangle ABP$. Now, $\triangle ABP$ has base AB of length 4, and its height is the distance from AB to the point P, which is 10, since AB is parallel to the x–axis. Thus, the area of $\triangle ABP$ is $\frac{1}{2}bh = \frac{1}{2}(4)(10) = 20$ square units; that is, the visible area of P is 20 square units.
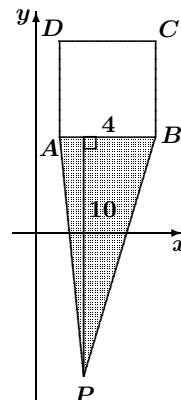
(b) The visible area of $Q(11, 0)$ is the sum of the areas of $\triangle QBC$ and $\triangle QBA$. Now, $\triangle QBC$ has base BC of length 4, and its height is the distance from Q to the line through B and C, which is 6. Thus, the area of $\triangle QBC$ is $\frac{1}{2}bh = \frac{1}{2}(4)(6) = 12$ square units.

Since $\triangle QBA$ has base BA of length 4, and its height is the distance from Q to the line through B and A, which is also 4, the area of $\triangle QBA$ is $\frac{1}{2}bh = \frac{1}{2}(4)(4) = 8$ square units.

The visible area of Q is the sum of these two areas, or 20 square units.
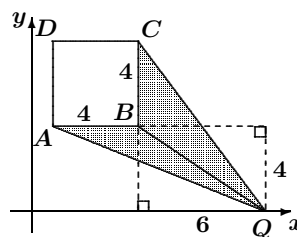
(c) From any point P, there are either 2 or 3 visible vertices of the square. We first consider those points P for which there are 2 visible vertices. Geometrically, these will be points which lie "directly opposite" an edge of the square. That is, they will be the points P that are in one of the following locations:

(i) below the square, with x–coordinate between 1 and 5, inclusive,

(ii) above the square, with x–coordinate between 1 and 5, inclusive,

(iii) left of the square, with y–coordinate between 4 and 8, inclusive, or

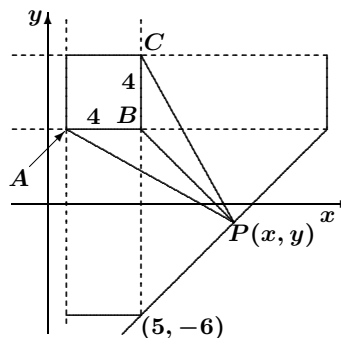(iv) right of the square, with y–coordinate between 4 and 8, inclusive.
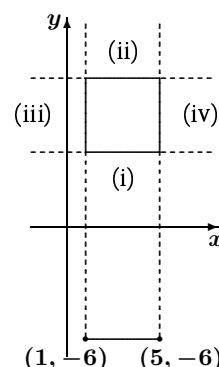
In each of these four cases, the visible area will be a single triangle whose base is a side of the square (of length 4). For the visible area to be 20, the height of this triangle must be 10.

Thus, in case (i), all points P which lie 10 units below the square and have x–coordinates between 1 and 5 will be in the 20/20 set. These are the points P on the line segment joining $(1, -6)$ to $(5, -6)$. (Notice that this includes the point $(2, -6)$ from (a).) This part of the 20/20 set has length 4.

The other three cases will give the same result, by symmetry. Thus, we have four line segments, each of length 4, in the 20/20 set so far.

There are four more regions to consider—regions that do not lie directly opposite an edge of the square. One such region is where $x \geq 5$ and $y \leq 4$. Suppose that a point $P(x, y)$ lies in this region and has a visible area of 20. Then the sum of the areas of $\triangle PBC$ and $\triangle PBA$ is 20.

Now, $\triangle PBC$ has a base of length 4 and a height of $x - 5$, and $\triangle PBA$ has a base of length 4 and a height of $4 - y$. Therefore, for P to be in the 20/20 set, we need

$$\tfrac{1}{2}(4)(x - 5) + \tfrac{1}{2}(4)(4 - y) = 20\,,$$
$$x - 5 + 4 - y = 10\,,$$
$$y = x - 11\,.$$

Thus, in this region, the points in the 20/20 set are the points P on the straight line $y = x - 11$ of slope 1; in other words they are the points on the line segment joining $(5, -6)$ to $(15, 4)$, which has length $10\sqrt{2}$. (Notice that this includes the point $(11, 0)$ from part (b). Also notice that the end-points of this line segment are the end-points of the line segments from regions (i) and (iv) above.)

We can argue by symmetry that the 20/20 set is as shown in the diagram. It is a polygon (an octagon, in fact) with four sides of length 4 and four sides of length $10\sqrt{2}$. Therefore, the perimeter of the 20/20 set is $16 + 40\sqrt{2}$.
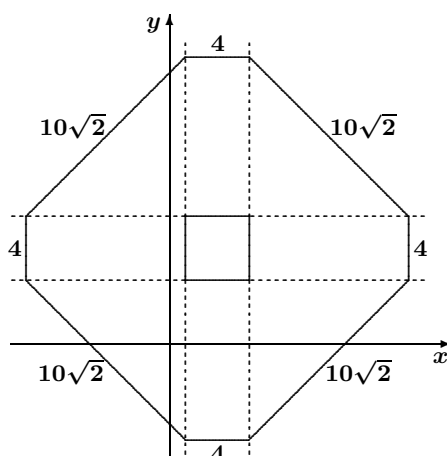
Extension: This is analogous to the 2–dimensional case.

Case 1: 4 visible vertices.

Where will a point P lie so that it has exactly 4 visible vertices? It will lie directly opposite one of the 6 faces of the cube. The visible volume will then be formed by one square-based pyramid. The base of this pyramid is a unit square. Since the volume of a pyramid is one-third the area of the base times the height, then, for a volume of 20, the height of the pyramid must be 60.

Hence, the points P opposite one of the faces which have a visible volume of 20 are all those points on a square (again, of side length 1) which is 60 units from the face. Therefore, the 20/20 set has six square faces, each of area 1.

Case 2: 6 visible vertices.

Where will a point P lie so that it has exactly 6 visible vertices? The 6 visible vertices must be the 6 vertices of two faces which share a common edge, and P will lie in the region which is between the outer edges of these two faces, but not directly above either face. We say that P lies above the edge. Since the cube has 12 edges, there will be 12 such regions.

Consider points P in one of these regions. The visible volume here will be formed by two square-based pyramids, each of which has a base which is a unit square. The visible volume will be one-third the area of the base (which is 1) times the sum of the two heights. Since the visible volume should be 20, the sum of the two heights will be 60.

The set of points above the edge which give a combined height of 60 for the two pyramids will form a rectangle which joins the edges of the two squares over the two adjacent faces. This rectangle will thus have one edge of length 1 and the other edge of length $60\sqrt{2}$ (since the long edge will form the hypotenuse of a right-angled triangle with two legs each of length 60).

(Why does this give us a portion of a plane? If we consider points P which lie in a cross-section which is perpendicular to the two adjacent faces, then we are asking for P to be such that the sum of the heights to two perpendicular line segments is 60, which gives a straight line as in part (b). Sliding this cross-section across the faces gives us a segment of a plane.)

Thus, we have twelve rectangular faces, each of area $60\sqrt{2}$.

Case 3: 7 visible vertices.

Where will a point P have 7 visible vertices? It should lie in a position that has not yet been considered, above one of the 8 vertices of the cube. (This seems sensible since the 20/20 set currently has 8 holes in it!)

This time, the visible volume is made up of three square-based pyramids; hence, we want points P so that the sum of the heights of these three pyramids is 60, as before. This region will again form part of a plane, and it should connect to the three rectangles already formed over the three edges that meet at the vertex in question.

Thus, this segment of the 20/20 set is an equilateral triangle of side length $60\sqrt{2}$. To find the area of this triangle, we join the top vertex to

the mid-point of the base. Each half of the triangle is a 30–60–90 right-angled triangle with a leg of length $30\sqrt{2}$ opposite the 30° angle. Hence, the height of the triangle is $30\sqrt{6}$, implying that the area of the whole equilateral triangle is $1800\sqrt{3}$.

The total surface area of the 20/20 set is $6 + 720\sqrt{2} + 14\,400\sqrt{3}$. A partial sketch of the set is given below.
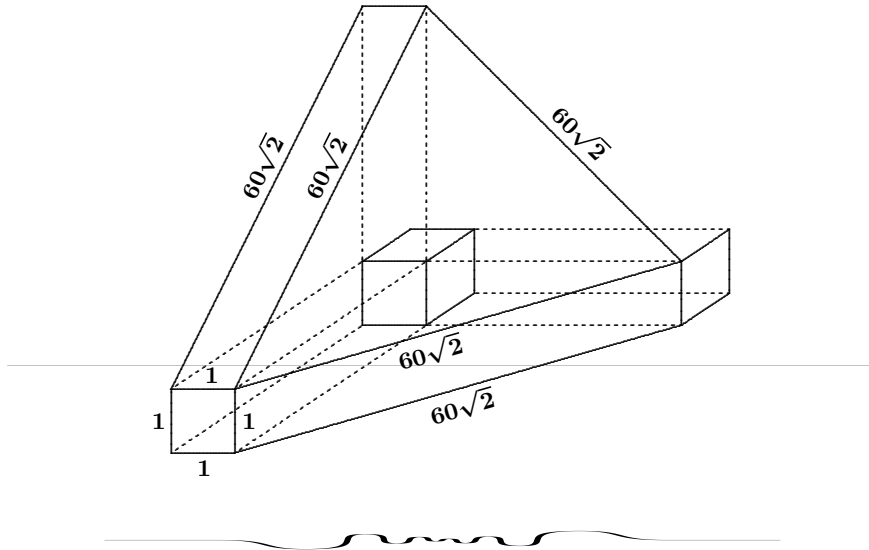
And now, we give the answers to the 2004 British Columbia Colleges High School Mathematics Contest Junior and Senior BC Preliminary Rounds that appeared in Skoliad 79 [2004 : 257–261]

BC Colleges High School Math Contest 2004
Junior Preliminary Round

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
c	a	b	d	b	d	c	b	d	e	d	b	d	b	e

Senior Preliminary Round

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
e	d	b	d	b	b	b	e	c	e	c	a	d	a	d

That brings us to the end of another issue. Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of *Crux Mathematicorum with Mathematical Mayhem*.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Amy Cameron (Carleton University), Dan MacKinnon (Ottawa Carleton District School Board), and Ian VanderBurgh (University of Waterloo).

Mayhem Problems

Please send your solutions to the problems in this edition by **1 July 2005**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M176. *Proposed by the Mayhem Staff.*

How many points (x, y) with positive integral coordinates lie on the curve defined by $x^2 + y^2 + 2xy - 2005x - 2005y - 2006 = 0$?

M177. *Proposed by Babis Stergiou, Chalkida, Greece.*

Let ABC be a triangle with angle A acute. Let BD and CE be two of its altitudes, with D on AC and E on AB. On diameters AB and AC construct circles Γ_1 and Γ_2, respectively. Let M be the point of intersection of Γ_1 and CE, and let N be the point of intersection of Γ_2 and BD, where M and N are interior points of $\triangle ABC$. Prove that $AM = AN$.

M178. *Proposed by the Mayhem Staff.*

Show that, if $10a + b$ is a multiple of 7, then $a - 2b$ must also be a multiple of 7.

**M179**. *Proposed by the Mayhem Staff.*

(a) Find all two-digit numbers that increase by 75% when their digits are reversed.

(b) Show that no three-digit numbers increase by 75% when their digits are reversed.

M180. *Proposed by the Mayhem Staff.*
(a) Show that $121_{(b)}$ is a perfect square in any base $b > 2$.
(b) Determine the smallest value of b for which $232_{(b)}$ is a perfect square.

M181. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A **3–4–5** right triangle has squares drawn outward on each of its sides. Lines are drawn through the outer corners of these squares, as shown. These lines form a triangle ABC. Determine whether $\triangle ABC$ is also a right triangle.

. . · .
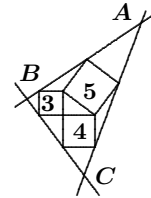
M176. *Proposé par l'Équipe de Mayhem.*

Combien de points (x, y) de coordonnées entières positives se trouvent sur la courbe définie par $x^2 + y^2 + 2xy - 2005x - 2005y - 2006 = 0$?

M177. *Proposé par Babis Stergiou, Chalkida, Grèce.*

Soit ABC un triangle avec un angle aigu en A. Soit BD et CE deux de ses hauteurs, avec D sur AC et E sur AB. On construit deux cercles Γ_1 et Γ_2 avec respectivement AB et AC comme diamètres. Soit M le point d'intersection de Γ_1 avec CE, N celui de Γ_2 avec BD, M et N étant à l'intérieur du triangle ABC. Montrer que $AM = AN$.

M178. *Proposé par l'Équipe de Mayhem.*

Si $10a + b$ est un multiple de **7**, montrer qu'il doit en être de même pour $a - 2b$.

M179. *Proposé par l'Équipe de Mayhem.*
(a) Trouver tous les nombres de deux chiffres qui augmentent de **75%** lorsqu'on permute leurs deux chiffres.
(b) Monter qu'aucun nombre de trois chiffres n'augmente de **75%** lorsqu'on renverse l'ordre de ses chiffres.

M180. *Proposé par l'Équipe de Mayhem.*
(a) Montrer que $121_{(b)}$ est un carré parfait dans toute base $b > 2$.
(b) Déterminer la plus petite valeur de b telle que $232_{(b)}$ soit un carré parfait.

M181. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

On donne un triangle rectangle de côtés **3–4–5**, ainsi que les carrés extérieurs dessinés sur ses côté respectifs. On dessine des droites passant par les coins extérieurs de ces carrés, comme le montre la figure. Ces droites forment un triangle ABC. Déterminer si ce dernier est aussi un triangle rectangle.
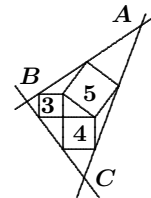
Mayhem Solutions

M119. *Proposed by the Mayhem Staff.*

Andrew, Bernard, and Charles play a game in which the loser has to triple the money of each other player. Three games are played, and the successive losers are Andrew, then Bernard, and finally, Charles. Each player ends with $27. How much money did each person have at the outset?

I. *Solution by Robert Bilinski, Outremont, QC.*

Let a, b, and c be the initial number of dollars that Andrew, Bernard, and Charles held, respectively. The following table represents the evolution of the holdings after each of the three rounds. We take for granted that the total amount of money will always be $a + b + c$.

Start	First Round	Second Round	Third Round
a	$a - 2b - 2c$	$3a - 6b - 6c$	$9a - 18b - 18c$
b	$3b$	$7b - 2c - 2a$	$21b - 6c - 6a$
c	$3c$	$9c$	$25c - 2a - 2b$

Thus, we need to solve the system

$$9a - 18b - 18c = 27,$$
$$21b - 6c - 6a = 27,$$
$$25c - 2a - 2b = 27,$$

which yields the solution $a = 55$, $b = 19$, and $c = 7$.

II. *Solution by Geneviève Lalonde, Massey, ON.*

Let A_n, B_n, and C_n be the respective amounts that Andrew, Bernard, and Charles hold after round n. We are given that $A_3 = B_3 = C_3 = 27$. Working backwards, we get

$$A_2 = \frac{27}{3} = 9$$
$$B_2 = \frac{27}{3} = 9$$
$$C_2 = 27 + 18 + 18 = 63,$$

then,

$$A_1 = \frac{9}{3} = 3$$
$$B_1 = 9 + 6 + 42 = 57$$
$$C_1 = \frac{63}{3} = 21,$$

and finally,

$$A_0 = 3 + 38 + 14 = 55$$
$$B_0 = \frac{57}{3} = 19$$
$$C_0 = \frac{21}{3} = 7.$$

Therefore, Andrew started with $55, Bernard started with $19, and Charles started with $7.

M120. *Proposed by the Mayhem Staff.*

Solve for x, where $0 \leq x < 2\pi$:

$$2^{1+3\cos x} - 10 \times 2^{-1+2\cos x} + 2^{2+\cos x} - 1 = 0\,.$$

Solution by Robert Bilinski, Outremont, QC.

The given equation is equivalent to $2X^3 - 5X^2 + 4X - 1 = 0$, where $X = 2^{\cos x}$. We see that $X = 1$ is a solution; whence, the new equation factors into $(X - 1)(2X^2 - 3X + 1) = 0$. But $X = 1$ is also a root of the second factor. Hence, we have $(X - 1)^2(2X - 1) = 0$, which gives $X = \frac{1}{2}$ as a second solution for X. This means that either $2^{\cos x} = 0$ or $2^{\cos x} = \frac{1}{2}$, which is equivalent to $\cos x = 0$ or $\cos x = -1$. In the range $0 \leq x \leq 2\pi$, the solutions are $x = \frac{\pi}{2}$, $x = \pi$, $x = \frac{3\pi}{2}$.

M121. *Proposed by the Mayhem Staff.*

Use each of the nine digits **1**, **2**, **3**, **4**, **5**, **6**, **7**, **8**, **9** exactly once to form prime numbers whose sum is as small as possible.

Solution by the editors.

The even digits **4**, **6**, and **8** cannot be in the units position of any prime.

Let us first restrict ourselves to primes having one or two digits. Then the leading digits of three of the primes must be the digits **4**, **6** and **8**. Among the two-digit primes there are two primes with leading digit **8** (**83** and **89**), two with **6** (**61** and **67**), and three with **4** (**41**, **43**, and **47**). Since we are looking for the smallest sum, we should try to have the remaining primes be single-digit primes, which means that **9** could not be used. Thus, we must have **89** as one of our two-digit primes. Since **2** and **5** cannot be the units digit of any prime with at least two digits, we should keep them as two of our three single-digit primes. Hence, our set should include the primes **2**, **5**, and **89**.

If **3** is the remaining single-digit prime, then the remaining two-digit primes are **41** and **67**; if **7** is the remaining single-digit prime, then the remaining primes are **43** and **61**. Since the same digits appear in the units column in both sets, the sums must be the same, namely **207**. Clearly, this sum is the smallest possible under the restriction that we only consider primes having one or two digits. The two sets which yield this sum are $\{2, 3, 5, 41, 67, 89\}$ and $\{2, 5, 7, 43, 61, 89\}$.

If we consider allowing primes with more than two digits, then we can surely have only one such, since two would sum to more than **300**, which is too large. If we are to keep the sum below **207**, then the leading digit of this prime must be **1**. Furthermore, the tens digits of three primes must include the digits **4**, **6**, and **8**, but this gives us a sum of more than **280**, which is again too large. Therefore, the smallest sum is **207**, and the above two sets provide the only solutions.

Also solved by Robert Bilinski, Outremont, QC.

Problem of the Month

Ian VanderBurgh, University of Waterloo

Problem (2003/4 British Mathematical Olympiad, Round 1) A set of positive integers is defined to be *wicked* if it contains no three consecutive integers. We count the empty set, which contains no elements at all, as a wicked set. Find the number of wicked subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

The first step in a problem which defines some new terminology is to try to understand the terminology. We write down a few examples of subsets which are wicked and subsets which are not wicked. Wicked subsets include $\{\}$ (the empty set), $\{1, 2, 4, 5, 8\}$, $\{6, 7\}$, and so on. Subsets which are not wicked include $\{2, 3, 4, 5, 7, 8, 9\}$, $\{1, 2, 3\}$, and others.

At this stage, we also want to recall that if a set S has n elements, then the total number of subsets of S, including S itself and the empty set, is 2^n (since each element of S is either in or not in a particular subset).

After a bit of fiddling around, we see that there are quite a few wicked subsets of the set given in the problem. Trying to write them all down is probably not a good idea (unless, of course, we were stuck in that snowstorm that we mentioned in the December issue). Furthermore, there does not seem to be a quick way to characterize these subsets and count them directly.

A good next step is to try finding the number of wicked subsets of a smaller set. Let $S_n = \{1, 2, 3, \ldots, n\}$, and let W_n be the number of wicked subsets of S_n. We want to calculate W_{10}. We first start with something easier.

(a) Consider $S_0 = \{\}$. There is only one subset—the empty set itself—and this subset is wicked; hence, $W_0 = 1$.

(b) Consider $S_1 = \{1\}$. There are only two subsets, $\{\}$ and $\{1\}$, both of which are wicked; hence, $W_1 = 2$.

(c) Consider S_2. There are 4 subsets, each wicked; hence, $W_2 = 4$.

(d) Consider S_3. There are 8 subsets, only one of which is not wicked (namely, $\{1, 2, 3\}$); hence $W_3 = 7$.

(e) Consider S_4. There are 16 subsets, only 3 of which are not wicked (namely, $\{1, 2, 3, 4\}$, $\{1, 2, 3\}$, and $\{2, 3, 4\}$); hence, $W_4 = 13$.

We can now proceed in two ways, both of which involve looking at smaller wicked subsets and trying to construct larger ones. This approach seems sensible because we are able to handle the smaller cases.

Solution 1. We start by trying to use the wicked subsets of S_9 to construct the wicked subsets of S_{10}. If A is a wicked subset of S_9, then A is also a subset of S_{10} and is still wicked. Thus, all W_9 wicked subsets of S_9 are also wicked subsets of S_{10}.

Which wicked subsets of S_9 turn into wicked subsets of S_{10} when we put the additional number 10 into them? Those which remain wicked are those

that do not already contain both 8 and 9. Now, how many wicked subsets of S_9 do not contain both 8 and 9? If a wicked subset A contains 8 and 9, then it cannot contain 7, since it is wicked, and the rest of A must be a wicked subset of S_6. Thus, the number of wicked subsets of S_9 containing 8 and 9 equals W_6, the number of wicked subsets of S_6. Therefore, the number of wicked subsets of S_9 which do not contain both 8 and 9 is $W_9 - W_6$.

Now, every wicked subset of S_{10} either does not contain 10 (in which case it is a wicked subset of S_9) or does contain 10 (in which case the set we get by removing 10 is a wicked subset of S_9). Thus,

$$W_{10} \;=\; W_9 + (W_9 - W_6) \;=\; 2W_9 - W_6 \,.$$

In the same way, we see that $W_9 = 2W_8 - W_5$, and $W_8 = 2W_7 - W_4$, and so on. Hence,

$$
\begin{aligned}
W_{10} \;=\;\; & 2W_9 - W_6 \;=\; 2(2W_8 - W_5) - W_6 \;=\; 4W_8 - W_6 - 2W_5 \\
=\;\; & 4(2W_7 - W_4) - W_6 - 2W_5 \;=\; 8W_7 - W_6 - 2W_5 - 4W_4 \\
=\;\; & 8(2W_6 - W_3) - W_6 - 2W_5 - 4W_4 \\
=\;\; & 15W_6 - 2W_5 - 4W_4 - 8W_3 \\
=\;\; & 15(2W_5 - W_2) - 2W_5 - 4W_4 - 8W_3 \\
=\;\; & 28W_5 - 4W_4 - 8W_3 - 15W_2 \\
=\;\; & 28(2W_4 - W_1) - 4W_4 - 8W_3 - 15W_2 \\
=\;\; & 52W_4 - 8W_3 - 15W_2 - 28W_1 \\
=\;\; & 52(13) - 8(7) - 15(4) - 28(2) \;=\; 504 \,.
\end{aligned}
$$

Thus, the number of wicked subsets of S_{10} is 504.

Well, that was certainly better than trying to write out all of the wicked subsets of S_{10}, but it was still a bit painful. There must be a better way.

Solution 2. We look at this in a more general way. Let A be any wicked subset of S_n. How many such sets A do not contain n? If A does not contain n, then A is itself a wicked subset of S_{n-1}. There are W_{n-1} such sets A.

Assume now that A contains n. If A does not contain $n - 1$, then the part of A that is left after removing n is a wicked subset of S_{n-2}. Thus, there are W_{n-2} wicked subsets of S_n which contain n and do not contain $n - 1$. If A does contain $n - 1$, then A cannot contain $n - 2$, since it cannot contain 3 consecutive integers. Hence, the part of A that we get after removing n and $n - 1$ is a wicked subset of S_{n-3}. There are W_{n-3} such sets A.

These are all of the possibilities for A. Thus,

$$W_n \;=\; W_{n-1} + W_{n-2} + W_{n-3} \,.$$

(We can check that $W_3 = W_2 + W_1 + W_0$ and $W_4 = W_3 + W_2 + W_1$.) In other words, each term after the third is the sum of the three previous terms. Now we can write out the sequence starting at W_0 to get 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, ... Thus, $W_{10} = 504$.
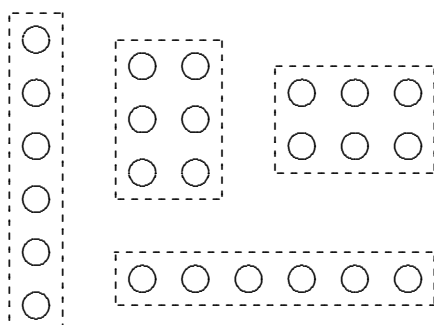
Pólya's Paragon

Fun With Numbers (Part 1)

Shawn Godin

Carl Friedrich Gauss, one of the greatest mathematicians of all time, said "Mathematics is the Queen of the Sciences, and number theory is the Queen of Mathematics". With that thought in mind, we will focus on numbers—the numbers 1, 2, 3, 4, ... that are familiar to most people over the age of three.

Our first stop is in an area that may have given rise to the notion of number in the first place. Suppose we are given a number of counters, such as coins. In how many ways can they be arranged in a rectangular pattern of rows and columns? If we had six counters, for example, we could arrange them in a rectangle in four possible ways, as shown below left. (We consider a 1×6 rectangle to be different from a 6×1 rectangle, and a 2×3 rectangle to be different from a 3×2 rectangle, for reasons to be seen later.) In the table on the right below, we have filled in the number 4 under "Number of Rectangles" corresponding to the case where the number of counters is 6.

Number of Counters	Number of Rectangles
1	?
2	?
3	?
4	?
5	?
6	4
⋮	⋮

Rather than my having all the fun, you should construct your own table like the one above. Try to consider up to 20 counters, at least, and look for patterns in your "Number of Rectangles" column before you read on.

I hope you're not cheating! What did you notice in your table? One thing that jumps out is that 1 is the only number that can be represented by only 1 rectangle. When we classify counting numbers, we put the number 1 in a class of its own. We call it a *unit*.

You should have noticed also that some numbers produce only boring rectangles. The number 3, for example, can only be a column or row of 3. This number and others like it, such as 2, 5, 7, 11, 13, 17, and 19, do not break down. These are the *prime* numbers. The remaining numbers (which are not prime and not 1) can each be represented as rectangles in more than two ways. These are the *composite* numbers.

We have come up with a classification of numbers: every counting number greater than 1 is either prime or composite. This classification is central in number theory, especially, in *multiplicative number theory*, which examines how numbers multiply together to give other numbers.

By now you may have realized that the role of the rectangles in all this discussion is just to help us visualize multiplication. When we form a 3×2 rectangle out of 6 counters, we are *factoring* the number 6 as 3×2, showing that the numbers 3 and 2 divide evenly into 6.

Mathematicians have some language and notation for talking about such things. When one number divides evenly into another number, the first is called a *divisor* of the second. For example, 3 is a divisor of 6. To write this more briefly, we just write $3 \mid 6$. Similarly, we write $5 \nmid 6$ to say that 5 does not divide 6 evenly, or that 5 is not a divisor of 6. (Mathematicians, being notoriously lazy, do not like to write more than they have to!)

Going back to our rectangles, we look at the number of rows that can appear in the rectangles for a particular number, say 6. We see that the possibilities are 1, 2, 3, and 6. These are all the (positive) divisors of 6. Thus, the number of rectangles is just the number of divisors (now you see why we counted both the 2×3 and 3×2 rectangles).

Again, since mathematicians love to make things shorter and easier, they came up with a notation: $d(n)$ is the number of divisors of n. For example, $d(1) = 1$, $d(6) = 4$, $d(19) = 2$, and $d(25) = 3$. The function $d(n)$ is called the *divisor function*.

We have been breaking numbers into pairs of divisors. But suppose we carry this idea further by breaking down the divisors as much as possible. We end up with an underlying principal in number theory:

Fundamental Theorem of Arithmetic. Every positive number greater than 1 can be written as the product of prime numbers in a unique way (up to the order of the primes).

If we agree to arrange the primes from smallest to largest, then there is only one way to write a number as a product of primes. This is the *prime factorization* of the number. For example, the prime factorization of 12 is $12 = 2 \times 2 \times 3 = 2^2 \times 3$.

Notice that prime factorizations would not be unique if we considered 1 to be prime. For example, we could write 6 as 2×3, or as $1 \times 2 \times 3$, or as $1 \times 1 \times \times 2 \times 3$, and so on.

For homework, construct a fairly lengthy table, showing the prime factorization and the number of divisors of each number. Your table should start out as shown. Try to see how the value of $d(n)$ is related to the prime factorization n, and look for relationships among the specific values of $d(n)$.

n	Factorization	$d(n)$
2	2	2
3	3	2
4	2^2	3
⋮	⋮	⋮
12	$2^2 \times 3$	6
⋮	⋮	⋮

Savage Numbers

Ian VanderBurgh

In the final problem of the 2003 Euclid Contest, a positive integer n is defined to be "savage" if the integers $\{1, 2, \ldots, n\}$ can be partitioned into three sets A, B, and C such that

 (i) the sum of the elements in each of A, B, and C is the same,

 (ii) A contains only odd numbers,

 (iii) B contains only even numbers, and

 (iv) C contains every multiple of 3 (and possibly other numbers).

The problem asks solvers to show that 8 is a savage integer, to show that if n is an even savage integer then $\frac{1}{2}(n + 4)$ is an integer, and to find all even savage numbers less than 100.

This was a difficult problem, but it is actually well within our reach to find all savage integers.

We will use the notation $|T|$ to represent the sum of the elements in the set T, and we will define $S = \{1, 2, \ldots, n\}$. There are two summation formulas that we will use:

$$1 + 2 + \cdots + k \;=\; \tfrac{1}{2}k(k + 1)\,, \qquad 1 + 3 + \cdots + (2\ell - 1) \;=\; \ell^2\,.$$

If you have never seen the proofs of these formulas, try them out!

Before we determine which positive integers are savage, let us look at two problems which seem to be only slightly related to the original problem, but which will turn out to be key in determining all savage integers.

Let $A_1 = \{1, 5, 7, 11, 13, 17, \ldots\}$ and $B_1 = \{2, 4, 8, 10, 14, 16, \ldots\}$. That is, A_1 contains all odd positive integers which are not multiples of 3, and B_1 contains all even positive integers which are not multiples of 3.

Problem 1: Show that every even positive integer can be expressed as the sum of one or more elements of B_1.

The solution to this problem and each of the following numbered problems below can be found at the end—this way you will not be tempted to peak!

Now, which positive integers m can be expressed as the sum of one or more elements of A_1? The best way to start is by making a list:

$$1\,,\, 5\,,\, 6 = 1 + 5\,,\, 7\,,\, 8 = 1 + 7\,,\, 11\,,\, 12 = 5 + 7\,,\, 13 = 1 + 5 + 7\,,$$
$$14 = 1 + 13\,,\, 16 = 5 + 11\,,\, 17\,,\, 18 = 1 + 17\,,\, 19\,,\, 20 = 1 + 19\,,$$
$$21 = 1 + 7 + 13\,,\, 22 = 5 + 17\,,\, 23\,,\, 24 = 1 + 23\,,\, \ldots$$

Problem 2: It is clear that we have found all possible sums less than 16. Show now that every positive integer greater than or equal to 16 can be expressed as the sum of one or more elements of A_1.

Having played with these problems, we can see that the sets A_1 and B_1 should be somehow connected to the sets A and B in the original problem. Let us return to the original problem.

We can first use condition (i) from the definition of savage integers to narrow down our hunt for savage integers. Since we are partitioning S into three sets A, B, and C which all have the same sum, then

$$|S| \;=\; |A| + |B| + |C| \;=\; 3|A| \,.$$

Thus, $|S| = \frac{1}{2}n(n+1)$ must be divisible by 3. Then $n(n+1)$ must be divisible by 3, which implies that either n is divisible by 3 or $n + 1$ is divisible by 3.

Problem 3: Show that if n is divisible by 3, then $|C| > \frac{1}{3}|S|$ (that is, the set C is "too big"), which means that n cannot be savage. (Try doing this both algebraically and in a more intuitive way.)

From the result of Problem 3, we see that $n+1$ must be divisible by 3 in order for n to be savage. If n is even, then n must be of the form $n = 6k + 2$ (the general form of an even integer n for which $n + 1$ is divisible by 3); If n is odd, then $n = 6k + 5$ (the general form of an odd integer n for which $n + 1$ is divisible by 3).

We next define three new sets which are subsets of $S = \{1, 2, \ldots, n\}$.

$A_0 \;=\;$ {all odd numbers in S which are not multiples of 3} ,

$B_0 \;=\;$ {all even numbers in S which are not multiples of 3} ,

$C_0 \;=\;$ {all numbers in S which are multiples of 3} .

Based on the conditions (ii), (iii), and (iv), the partitioning of S into three sets A, B, and C can only be accomplished by moving integers out of A_0 and B_0 into C_0, since C must contain every element that is initially in C_0 and A and B cannot contain any of the elements that are initially in B_0 and A_0, respectively.

We now consider the two cases for n to see if it is possible to transform A_0, B_0, and C_0 into A, B, and C.

Case 1: $n = 6k + 2$.

In this case,

$$
\begin{aligned}
S \;&=\; \{1, 2, 3, \ldots, 6k, 6k + 1, 6k + 2\} \,, \\
A_0 \;&=\; \{1, 5, 7, 11, \ldots, 6k - 1, 6k + 1\} \,, \\
B_0 \;&=\; \{2, 4, 8, 10, \ldots, 6k - 2, 6k + 2\} \,, \\
C_0 \;&=\; \{3, 6, 9, \ldots, 6k - 3, 6k\} \,.
\end{aligned}
$$

From this we deduce that

$$
\begin{aligned}
\tfrac{1}{3}|S| \;&=\; \tfrac{1}{3}\big(1 + 2 + 3 + \cdots + (6k + 2)\big) \;=\; \tfrac{1}{3}\big(\tfrac{1}{2}(6k + 2)(6k + 3)\big) \\
&=\; (3k + 1)(2k + 1) \;=\; 6k^2 + 5k + 1 \,,
\end{aligned}
$$

$$\begin{aligned}
|A_0| &= 1 + 5 + 7 + 11 + \cdots + (6k-1) + (6k+1) \\
&= 1 + 3 + 5 + 7 + \cdots + (6k-1) + (6k+1) \\
&\qquad - (3 + 9 + \cdots + (6k-3)) \\
&= (3k+1)^2 - 3(1 + 3 + \cdots + (2k-1)) \\
&= (3k+1)^2 - 3k^2 = 6k^2 + 6k + 1, \\
|B_0| &= 2 + 4 + 8 + 10 + \cdots + (6k-2) + (6k+2) \\
&= 2 + 4 + 6 + 8 + 10 + \cdots + (6k-2) + 6k + (6k+2) \\
&\qquad - (6 + 12 + \cdots + 6k) \\
&= 2(1 + 2 + 3 + \cdots + (3k+1)) - 6(1 + 2 + \cdots + k) \\
&= (3k+1)(3k+2) - 3k(k+1) = 6k^2 + 6k + 2, \\
|C_0| &= 3 + 6 + 9 + \cdots + 6k = 3(1 + 2 + 3 + \cdots + 2k) \\
&= 3\left(\tfrac{1}{2}(2k)(2k+1)\right) = 6k^2 + 3k.
\end{aligned}$$

Thus, in order to create three sets A, B, and C each of whose elements sum to $\frac{1}{3}|S|$, we must move elements of sum k from A_0 into C_0 and elements of sum $k+1$ from B_0 to C_0. We will first examine the case of $n = 6k+5$ in the above way before checking to see if this is possible.

Case 2. $n = 6k + 5$.

In this case,

$$\begin{aligned}
S &= \{1, 2, 3, \ldots, 6k+3, 6k+4, 6k+5\}, \\
A_0 &= \{1, 5, 7, 11, \ldots, 6k+1, 6k+5\}, \\
B_0 &= \{2, 4, 8, 10, \ldots, 6k+2, 6k+4\}, \\
C_0 &= \{3, 6, 9, \ldots, 6k, 6k+3\}.
\end{aligned}$$

Problem 4: Using the same methods as in the previous case, show that

$$\begin{aligned}
\tfrac{1}{3}|S| &= 6k^2 + 11k + 5, & |A_0| &= 6k^2 + 12k + 6, \\
|B_0| &= 6k^2 + 12k + 6, & |C_0| &= 6k^2 + 9k + 3.
\end{aligned}$$

Using the results of Problem 4, if we wish to create three sets A, B, and C each of whose elements sum to $\frac{1}{3}|S|$, we must move elements of sum $k+1$ from A_0 into C_0 and elements of sum $k+1$ from B_0 to C_0.

Where do we go now? In the first case, $n = 6k+2$ will be savage if we can find elements of A_0 which sum to k and elements of B_0 which sum to $k+1$. In the second case, $n = 6k+5$ will be savage if we can find elements of A_0 which sum to $k+1$ and elements of B_0 which sum to $k+1$.

In order to do this we can actually go back to our results from Problems 1 and 2 above. To justify this, let us look at A_0 in the first case. When $n = 6k+2$, the largest element of A_0 is $6k+1$. Since we are trying to determine if there are elements in A_0 which sum to k, we certainly cannot

use the largest element of A_0. On the other hand, we can add even larger elements to A_0 and no effect will be made on whether there are elements that sum to k. In other words, asking whether or not there are elements in $A_0 = \{1,\, 5,\, 7,\, 11,\, \ldots,\, 6k-1,\, 6k+1\}$ which sum to k is the same as asking if there are elements which sum to k in $A_1 = \{1,\, 5,\, 7,\, 11,\, 13,\, 17,\, \ldots\}$. We can make the same argument for extending A_0 to A_1 and B_0 to B_1 in both cases.

In case 1, where $n = 6k+2$, using the results of Problems 1 and 2, we see that there are elements in A_0 which sum to k if and only if k equals 1, 5, 6, 7, 8, 11, 12, 13, 14, or if $k \geq 16$. Also, there are elements in B_0 which sum to $k+1$ if and only if $k+1$ is even, or k is odd. Since we require both conditions to hold in order for n to be savage, it follows that $n = 6k+2$ is savage if and only if k equals 1, 5, 7, 11, 13, or any odd number greater than or equal to 17.

Problem 5: Show that $n = 6k+5$ is savage if and only if k equals 5, 7 or any odd number greater than or equal to 11.

Since in each of the two cases, we have seen that k must be odd and at least 1, we can tidy things up a bit by replacing k with $2\ell + 1$ and stipulating that ℓ must be at least 0. Therefore, the savage integers are

$$n \;=\; 12\ell + 8 \qquad \text{where } \ell \geq 0,\ \ell \neq 1,\ \ell \neq 4,\ \text{and } \ell \neq 7$$
$$\text{and} \qquad n \;=\; 12\ell + 11 \qquad \text{where } \ell \geq 2 \text{ and } \ell \neq 4.$$

Coincidentally, of course, $2003 = 12(166) + 11$, which means that 2003 is a savage integer.

Solutions to the Problems

Problem 1: B_1 contains the number 2, every even number of the form $6j - 2$ with $j \geq 1$, and every even number of the form $6j + 2$ with $j \geq 1$. Hence, the only even positive integers which are not in B_1 already (and thus not obviously the sum of elements from B_1) are those of the form $6j$, which can be obtained by adding $2 + (6j - 2)$. Therefore, every even positive integer is the sum of elements from B_1.

Problem 2: The key here is that we can use the numbers 1, 5, and 7 to get the integers 5, 6, 7, and 8, and that every integer of the forms $6j - 1$ and $6j + 1$ is in A_1 for $j \geq 2$. (That is, 11, 13, 17, 19, and so on.) Then, adding our representations for 5, 6, 7, and 8, we get

$$6j + 4 = 5 + (6j - 1), \qquad 6j + 5 = 6 + (6j - 1) = 1 + 5 + (6j - 1),$$
$$6j + 6 = 7 + (6j - 1), \qquad 6j + 7 = 8 + (6j - 1) = 1 + 7 + (6j - 1),$$
$$6j + 8 = 7 + (6j + 1), \qquad 6j + 9 = 8 + (6j + 1) = 1 + 7 + (6j + 1),$$

which give us 6 consecutive integers expressed as the sum of elements of A_1. Using the smallest value of $j = 2$ generates 16, 17, 18, 19, 20, 21; increasing the value of j by 1 gives us the next 6 consecutive integers, and so on, giving us all positive integers.

Problem 3: If n is divisible by 3, then $n = 3k$ for some positive integer k. Thus, $S = \{1, 2, \ldots, 3k\}$ (with $|S| = \frac{1}{2}(3k)(3k+1) = \frac{1}{2}(9k^2 + 3k)$), and C contains at least $\{3, 6, 9, \ldots, 3k\}$, whose sum is

$$3 + 6 + \cdots + 3k = 3(1 + 2 + \cdots + k) = 3\left(\tfrac{1}{2}k(k+1)\right) = \tfrac{1}{2}(3k^2 + 3k),$$

which is greater than $\frac{1}{3}|S| = \frac{1}{2}(3k^2 + k)$. Since $|C|$ must be exactly equal to $\frac{1}{3}|S|$ if n is savage, we conclude that n cannot be divisible by 3. (More intuitively, we can divide the integers from 1 to n into groups of 3 starting with 1. The largest integer in each group (the multiple of 3) must go into the set C, which means that the sum of all of the elements of C must be more than $\frac{1}{3}$ of the total. Therefore, n cannot be savage.)

Problem 4:

$$
\begin{aligned}
\tfrac{1}{3}|S| &= \tfrac{1}{3}(1 + 2 + 3 + \cdots + (6k + 5)) = \tfrac{1}{3}\left(\tfrac{1}{2}(6k+5)(6k+6)\right) \\
&= (6k+5)(k+1) = 6k^2 + 11k + 5\,, \\
|A_0| &= 1 + 5 + 7 + 11 + \cdots + (6k+1) + (6k+5) \\
&= 1 + 3 + 5 + 7 + \cdots + (6k+1) + (6k+3) + (6k+5) \\
&\quad - (3 + 9 + \cdots + (6k+3)) \\
&= (3k+3)^2 - 3(1 + 3 + \cdots + (2k+1)) \\
&= (3k+3)^2 - 3(k+1)^2 = 6k^2 + 12k + 6\,, \\
|B_0| &= 2 + 4 + 8 + 10 + \cdots + (6k+2) + (6k+4) \\
&= 2 + 4 + 6 + 8 + 10 + \cdots + (6k+2) + (6k+4) \\
&\quad - (6 + 12 + \cdots + 6k) \\
&= 2(1 + 2 + 3 + \cdots + (3k+2)) - 6(1 + 2 + \cdots + k) \\
&= (3k+2)(3k+3) - 3k(k+1) = 6k^2 + 12k + 6\,, \\
|C_0| &= 3 + 6 + \cdots + 6k + (6k+3) = 3(1 + 2 + \cdots + (2k+1)) \\
&= 3\left(\tfrac{1}{2}(2k+1)(2k+2)\right) = 6k^2 + 9k + 3\,.
\end{aligned}
$$

Problem 5: In the case where $n = 6k + 5$, there are elements in A_0 which sum to $k + 1$ if and only if $k + 1$ is equal to 1, 5, 6, 7, 8, 11, 12, 13, 14, or if $k \geq 16$. Also, there are elements in B_0 which sum to $k + 1$ if and only if $k + 1$ is even, or k is odd. Therefore, $n = 6k + 5$ is savage if and only if k equals 5, 7, or any odd number greater than or equal to 11.

Ian VanderBurgh
Canadian Mathematics Competition
Centre for Education in Mathematics and Computing
University of Waterloo
Waterloo, Ontario
N2L 3G1
iwtvanderburgh@math.uwaterloo.ca

THE OLYMPIAD CORNER

No. 243

R.E. Woodrow

Another year and another volume of **Crux Mathematicorum** are on the horizon. It is appropriate to thank all those who have contributed to the *Corners* that appeared in 2004 by providing problems, comments, solutions, and generalizations. These include:

Mohammed Aassila   Pavlos Maragoudakis
Miguel Amengual Covas   Vedula N. Murty
Michel Bataille   Alberto Nettel
Robert Bilinski   Luz Palacios
Pierre Bornsztein   Toshio Seimiya
Christopher J. Bradley   Achilleas Sinefakopoulos
Yuming Chen   Chris Small
Bruce Crofoot   Pham van Thuan
José Luis Díaz-Barrero   Edward T.H. Wang
George Evagelopoulos   Kaiming Zhao
Murray S. Klamkin

Thanks again go to Joanne Longworth, who is to be commended for her skill and patience in reading my scribbles and translating them into LaTeX.

With sadness we note that 2004 marked the loss of Murray S. Klamkin, the founder of the *Olympiad Corner* and a staunch supporter throughout. I will particularly miss the Klamkin Quickies (at which I've never been very quick!) and the many elegant solutions and generalizations he would send in regularly.

As a first problem set for the new year, we give the Icelandic Mathematical Contest 2000-2001 (final) of the Association of Science Teachers, The Icelandic Mathematical Society. Thanks go to Chris Small, Canadian Team Leader to the 42nd IMO, for collecting this set for our use.

ICELANDIC MATHEMATICAL CONTEST 2000–2001
Final
March 10, 2001 — Time: 4 hours

1. Let x and y be positive real numbers such that $xy = 1$. Prove that

$$\frac{x}{y} + \frac{y}{x} \geq 2 \,.$$

2. Two circles intersect at points P and Q. A line ℓ that intersects the line segment PQ intersects the two circles at the points A, B, C, and D (in that order along the line ℓ). Prove that $\angle APB = \angle CQD$.

3. Richard is walking up a stair that has 10 steps. With each stride he goes up either one step or two steps. In how many different ways can Richard go up the stairs?

4. In Flora's number-set there are the numbers

$$2^n - 1, \quad 3^{2n} - 1, \quad 4^{3n} - 1, \quad 5^{4n} - 1, \quad 6^{5n} - 1, \quad 7^{6n} - 1, \quad 8^{7n} - 1, \quad 9^{8n} - 1,$$

for each natural number n, and there are no other numbers in the set. How many square numbers does the set contain?

5. Triangle ABC is isosceles with a right angle at B and $AB = BC = x$. Point D on the side AB and point E on the side BC are chosen such that $BD = BE = y$. The line segments AE and CD intersect at the point P. What is the area of the triangle APC, expressed in terms of x and y?

6. How many natural numbers are divisible by 2001 and have exactly 2001 natural divisors?

Next, we give the problems of the Selection Examination of the Greek Mathematical Competitions. Thanks go to Bill Sands, University of Calgary, for obtaining them for our use through participants at IMO 2002.

GREEK MATHEMATICAL COMPETITIONS
Selection Examination for the IMO 2002

1. Find all natural numbers x, y such that y divides $x^2 + 1$ and x^2 divides $y^3 + 1$.

2. Let x, y, a, be real numbers such that

$$x + y = x^3 + y^3 = x^5 + y^5 = a.$$

Determine all the possible values of a.

3. An acute-angled triangle ABC is given. Let M and N be interior points on the sides AC and BC, respectively, and let K be the mid-point of the segment MN. The circumcircles of triangles CAN and BCM meet for the second time at the point D. Prove that the line CD passes through the circumcentre of triangle ABC if and only if the perpendicular bisector of AB passes through K.

4. Prove that the following inequality holds for every triple (a, b, c) of non-negative real numbers with $a^2 + b^2 + c^2 = 1$:

$$\frac{a}{b^2 + 1} + \frac{b}{c^2 + 1} + \frac{c}{a^2 + 1} \geq \frac{3}{4} \left(a\sqrt{a} + b\sqrt{b} + c\sqrt{c} \right)^2.$$

When does equality hold?

To round out this number, we present selected problems from the 16th China Mathematical Olympiad 2001, Hong Kong. Again, thanks go to Chris Small, Canadian Team Leader to the 42nd IMO, for collecting this set.

16th CHINA MATHEMATICAL OLYMPIAD
Selected Problems
First Day, January 13, 2001 – Hong Kong

1. A convex quadrilateral $ABCD$ is inscribed in a circle Γ of radius 1 such that the centre of the circle is inside $ABCD$. The lengths of the longest and the shortest sides of $ABCD$ are a and $\sqrt{4-a^2}$, where $\sqrt{2} < a < 2$.

Lines L_A, L_B, L_C, L_D are tangent to Γ at A, B, C, D, respectively. The pairs of lines L_A and L_B, L_B and L_C, L_C and L_D, L_D and L_A meet at points A', B', C', D', respectively. Find the maximum and minimum values of the ratio of the area of $A'B'C'D'$ to the area of $ABCD$.

2. Let $X = \{1, 2, \ldots, 2001\}$. Find the minimum positive integer m such that, for each m–element subset W of X, there exist $u, v \in W$ (u and v may be the same) with $u + v = 2^k$ for some positive integer k.

3. At each vertex of a regular n-sided polygon, there was a magpie. When scared, all the magpies flew away. After a while they all returned, one to each vertex, but not all to their former positions. Find all positive integers n for which there must be 3 magpies such that the triangle formed by the vertices at which they first stood and the triangle formed by the vertices at which they now stand are both acute triangles, both right triangles, or both obtuse triangles.

Second Day, January 14, 2001 – Hong Kong

4. Let a, b, c, $a+b-c$, $a+c-b$, $b+c-a$, $a+b+c$ be 7 distinct prime numbers such that the sum of two of a, b, c is 800. Let d be the difference between the largest and smallest numbers among the 7 primes. Find the largest possible value of d.

5. A circle with circumference 24 is divided into arcs by 24 equally-spaced points. How many different sets of 8 points may be chosen from among the 24 given points such that the length of the arc between any two of the 8 chosen points is neither 3 nor 8? Show your reasons.

6. Let $a = 2001$, and let A be the set of pairs of positive integers (m, n) such that:

(i) $m < 2a$;   (ii) $2n \mid (2am - m^2 + n^2)$;

(iii) $n^2 - m^2 + 2mn \leq 2a(n - m)$.

Find $\min\limits_{(m,n)\in A} f$ and $\max\limits_{(m,n)\in A} f$, where $f = \dfrac{2am - m^2 - mn}{n}$.

Turning to solutions from our readers, we first present an alternate solution for problem #3 in the Estonian Mathematical Contest 1995–96 [2000 : 6; 2002 : 76–77].

3. Prove that the polynomial $P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$ has no zeroes if n is even and has exactly one zero if n is odd.

Alternate Solution by Skotidas Sotirios, Karditso, Greece.

Note that $P_n(x) \geq 1$ for all $x \geq 0$. Therefore, if P_n has a zero ρ, then $\rho < 0$.

Let $F_n(x) = P_n(x)e^{-x}$. Then F_n has the same zeroes as P_n, and

$$F_n'(x) \; = \; (P_n'(x) - P_n(x))e^{-x} \; = \; -\frac{x^n}{n!}e^{-x} \,. \tag{1}$$

If n is even, then we see from (1) that F_n is decreasing on $\mathbb{R}$. Since $F_n(0) = P_n(0) = 1$, we have $F_n(x) \geq 1$ for all $x \leq 0$. Therefore, F_n has no negative zeroes, and hence, neither does P_n. Thus, P_n has no zeroes at all.

Now assume that n is odd. Then $P_n(x)$, being a polynomial of odd degree, has at least one real zero. From (1), the function F_n is strictly decreasing on the interval $(-\infty, 0]$. Hence, F_n has at most one zero in this interval. It follows that P_n has at most one zero.

Finally, we prove that if P_n has a real zero ρ, then the multiplicity of ρ must be 1. Suppose instead that P_n has a zero ρ of multiplicity $k > 1$. Then $P_n(x) = (x - \rho)^k Q_n(x)$, for some polynomial $Q_n(x)$, and hence,

$$P_n'(x) \; = \; k(x - \rho)^{k-1} Q_n(x) + (x - \rho)^k Q_n'(x) \,.$$

Now $P_n(\rho) = P_n'(\rho) = 0$. Then $0 = P_n(\rho) - P_n'(\rho) = \frac{\rho^n}{n!}$, which implies that $\rho = 0$. But $\rho < 0$, since ρ is a zero of P_n. We have a contradiction.

We turn to the November 2002 number and to solutions to problems of the 35$^{\text{th}}$ Mongolian Mathematical Olympiad 1999, 10$^{\text{th}}$ grade, given [2002 : 416].

1. For any positive integer n show that there exists k such that all the numbers $k \cdot 2^s + 1$, $(1 \leq s \leq n)$ are composite numbers.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution by Bataille.

Given n, the following integer k serves the purpose

$$k \; = \; (2^1 + 1)(2^2 + 1)(2^3 + 1) \cdots (2^n + 1) \; + \; 1 \,.$$

Indeed, for each $s \in \{1, 2, \ldots, n\}$, there is some positive integer a_s such that $k - 1 = (2^s + 1)a_s$. Hence, $k \cdot 2^s + 1 = 2^s(k - 1) + 2^s + 1 = (2^s + 1)(2^s a_s + 1)$ is a composite integer.

2. Given an angle $\angle ABC$ and rays $\ell_1, \ldots, \ell_{n-1}$ dividing the angle into n congruent angles, for a line ℓ denote $\ell \cap (AB) = A_1$, $\ell \cap (BC) = A_{n+1}$ and $\ell \cap \ell_i = A_{i+1}$ for $1 \leq i < n$, show that the quotient

$$\frac{\left(\dfrac{1}{|BA_1|} + \dfrac{1}{|BA_{n+1}|} \right)}{\left(\dfrac{1}{|BA_1|} + \dfrac{1}{|BA_2|} + \cdots + \dfrac{1}{|BA_n|} + \dfrac{1}{|BA_{n+1}|} \right)}$$

is a constant which does not depend on ℓ, and find the value of this constant knowing $\angle ABC = \varphi$.

Solution by Michel Bataille, Rouen, France.

Let $\theta = \angle BA_1A_{n+1}$. For $k = 1, 2, \ldots, n$, the Law of Sines provides

$$\frac{|BA_1|}{|BA_{k+1}|} = \frac{\sin(\angle A_1 A_{k+1} B)}{\sin \theta} = \frac{\sin\left(\theta + \dfrac{k\varphi}{n} \right)}{\sin \theta} .$$

It follows that

$$1 + \frac{|BA_1|}{|BA_{n+1}|} = 1 + \frac{\sin(\theta + \varphi)}{\sin \theta} = \frac{2 \cos\left(\dfrac{\varphi}{2} \right) \sin\left(\theta + \dfrac{\varphi}{2} \right)}{\sin \theta} \qquad (1)$$

and

$$1 + \sum_{k=1}^{n} \frac{|BA_1|}{|BA_{k+1}|} = 1 + \frac{1}{\sin \theta} \sum_{k=1}^{n} \sin\left(\theta + \frac{k\varphi}{n} \right) = \frac{1}{\sin \theta} \sum_{k=0}^{n} \sin\left(\theta + \frac{k\varphi}{n} \right)$$

$$= \frac{1}{\sin \theta} \left(\sin\left(\theta + \frac{\varphi}{2} \right) \frac{\sin\left(\dfrac{(n+1)\varphi}{2n} \right)}{\sin\left(\dfrac{\varphi}{2n} \right)} \right), \qquad (2)$$

where the last line follows by a well-known classical calculation.

Dividing equation (1) by equation (2), we find that the quotient to which the problem refers is equal to

$$\frac{2 \cos\left(\dfrac{\varphi}{2} \right) \sin\left(\dfrac{\varphi}{2n} \right)}{\sin\left(\dfrac{(n+1)\varphi}{2n} \right)} = 1 - \frac{\sin\left(\dfrac{(n-1)\varphi}{2n} \right)}{\sin\left(\dfrac{(n+1)\varphi}{2n} \right)} ,$$

which is independent of θ and, therefore, independent of the line ℓ.

3. Does there exist an infinite sequence $\{a_n\}_{n=1}^{\infty}$ consisting of different natural numbers and such that

(i) $a_n < 1999 \cdot n$ for every n,

(ii) the decimal expression of every a_n does not contain three consecutive 1s?

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We will prove that no such sequence exists.

Let us say that an integer is a *good* integer if it is positive and its decimal representation does not contain three consecutive 1s. For $n \geq 1$, let t_n be the number of good integers which are less than or equal to n.

Suppose there exists a sequence $\{a_n\}_{n=1}^{\infty}$ satisfying the conditions in the problem. Since the terms in the sequence are distinct and each of them is a good integer, we deduce that $t_{1999n} \geq n$ for all positive integers n. Then $\frac{t_{1999n}}{1999n} \geq \frac{1}{1999}$ for all n. But this contradicts the following lemma.

Lemma. $\lim\limits_{n \to \infty} \dfrac{t_n}{n} = 0$.

Proof: For any integer $k \geq 1$, let S_k be the set of good integers having a decimal representation with at most k digits, and let s_k be the number of elements of S_k. For $k \geq 2$, we have $s_k = u_k + v_k + w_k$, where u_k is the number of elements of S_k with rightmost digit not equal to 1, v_k is the number of elements of S_k with the two rightmost digits of the form $x1$, where $x \neq 1$, and w_k is the number of elements of S_k with both rightmost digits equal to 1.

From our definitions, we deduce that

$$u_{k+1} = 9s_k, \quad v_{k+1} = u_k, \quad \text{and} \quad w_{k+1} = v_k.$$

Using these relations, we have $w_{k+3} = v_{k+2} = u_{k+1} = 9s_k$, and then

$$
\begin{aligned}
s_{k+4} &= u_{k+4} + v_{k+4} + w_{k+4} = 9s_{k+3} + u_{k+3} + v_{k+3} \\
&= 10s_{k+3} - w_{k+3} = 10s_{k+3} - 9s_k.
\end{aligned}
$$

Thus, the sequence $\{s_k\}_{k=2}^{\infty}$ satisfies a linear recurrence relation of order 4, and the characteristic equation of this recurrence relation is $P(x) = 0$, where

$$P(x) = x^4 - 10x^3 + 9 = (x-1)(x^3 - 9x^2 - 9x - 9).$$

It is easy to verify that the polynomial $Q(x) = x^3 - 9x^2 - 9x - 9$ has a unique real root r such that $9 < r < 10$, and two conjugate complex roots λ and $\overline{\lambda}$. Since $r\lambda\overline{\lambda} = 9$ and $r > 9$, we must have $|\lambda| < 1$. The solution of the recurrence relation for s_k is

$$s_k = a + b\lambda^k + c\overline{\lambda}^k + dr^k,$$

for some complex numbers a, b, c, d. Since $\lim\limits_{k \to \infty} s_k = \infty$ and $\lim\limits_{k \to \infty} |\lambda|^k = 0$, we deduce that $d \neq 0$. Therefore,

$$s_k = O(r^k). \tag{1}$$

Let n be an integer such that $n \geq 10$, and let k be the number of digits in the decimal representation of n. Thus, $10^{k-1} \leq n < 10^k$. From the definitions of t_n and s_k, we see that $t_n \leq s_k$. Therefore, using (1), we have

$$t_n = O(r^k). \tag{2}$$

Choose any real number $\epsilon > 0$ such that $r + \epsilon < 10$, and suppose that n is sufficiently large so that $\dfrac{\ln(r+\epsilon)}{\ln n} \leq 1 - \dfrac{\ln(r+\epsilon)}{\ln 10}$. Rearranging this inequality, we have

$$\frac{\ln n}{\ln 10} + 1 \leq \frac{\ln n}{\ln(r+\epsilon)} \, .$$

Since $10^{k-1} \leq n$, we also have $k \leq \dfrac{\ln n}{\ln 10} + 1$. Thus, $k \leq \dfrac{\ln n}{\ln(r+\epsilon)}$, and hence,

$$r^k = e^{k \ln r} \leq e^{\frac{\ln n \ln r}{\ln(r+\epsilon)}} = n^\alpha \, ,$$

where $\alpha = \dfrac{\ln r}{\ln(r+\epsilon)}$. Using equation (2), we see that $t_n = O(n^\alpha)$. Since $0 < \alpha < 1$, it follows that $\lim\limits_{n \to \infty} \dfrac{t_n}{n} = 0$. ∎

4. For any prime p and any positive integer n show that $\varphi(p^n - 1)$ is divisible by n, where φ is the Euler function.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Kaiming Zhao and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution by Zhao and Wang.

Since $(p^n - 1, p) = 1$, we have, by Euler's Theorem,

$$p^{\phi(p^n - 1)} \equiv 1 \pmod{p^n - 1} \, ;$$

that is, $p^n - 1 \mid p^{\phi(p^n - 1)} - 1$. The desired result now follows by applying the the following lemma.

Lemma Let a, m, and n be natural numbers. Then $a^n - 1 \mid a^m - 1$ if and only if $n \mid m$.

Proof: The sufficiency is obvious. To prove necessity, we write $m = qn + r$, where q, $r \in \mathbb{N} \cup \{0\}$ such that $0 \leq r < n$. Then

$$a^m - 1 = a^{qn+r} - 1 = a^r((a^n)^q - 1) + a^r - 1 \, .$$

If $a^n - 1 \mid a^m - 1$, then, noting that $a^n - 1 \mid (a^n)^q - 1$, we deduce that $a^n - 1 \mid a^r - 1$. This implies that $r = 0$ and hence $n \mid m$. ∎

Bornsztein gave a proof showing that the conclusion, in fact, holds for all natural numbers $p > 1$. Since $(p^n - 1, p) = 1$ is true even if p is not a prime, it is clear that the proof presented above covers this general case as well.

5. Given a set X with $|X| = n$ and subsets $A_i \subset X$, $(1 \leq i \leq n)$, such that $|A_i| = 3$ and $|A_i \cap A_j| \leq 1$, $(i \neq j)$. Prove that there exists a subset $A \subseteq X$ which does not contain any A_i and $|A| \geq 2\sqrt{n}$.

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

It seems that this problem is the same as one of the problems in the Team Selection Contest for the Italian Team in 1999, for which a solution was given in [2004 : 494]), but the problem has been incorrectly stated here.

6. A point M lies on the side AC of a triangle ABC. The circle γ with the diameter BM intersects the lines AB, BC, at points P, Q, respectively. Find the locus of intersection points of the tangents of the circle γ at the points P, Q, when the point M varies.

Solution by Toshio Seimiya, Kawasaki, Japan.

In the following proof, we assume that $\triangle ABC$ is acute. In other cases, the proof works with minor changes.
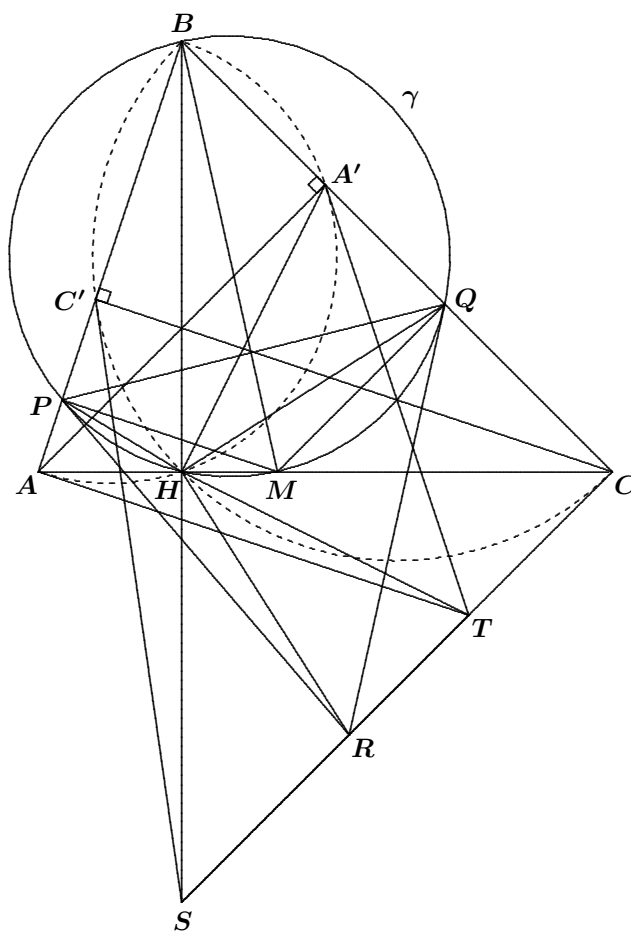
Let H be the second intersection of γ with AC. Then $\angle BHM = 90°$; hence, $BH \perp AC$. Let R be the intersection of the tangents to γ at P and Q. Let A' and C' be the feet of the perpendiculars from A and C to BC and AB, respectively. Let T be the point such that TA and TA' are tangent to the circumcircle of $\triangle BAA'$, and let S be the point such that SC and SC' are tangent to the circumcircle of $\triangle BCC'$.

Since A, B, A', and H are concyclic, we have

$$\angle HA'A \;=\; \angle HBA \;=\; \angle HBP \;=\; \angle HQP\,,$$

and

$$\angle HAA' \;=\; \angle HBA' \;=\; \angle HBQ \;=\; \angle HPQ\,.$$

Hence, $\triangle HAA' \sim \triangle HPQ$. Also, $\triangle TAA' \sim \triangle RPQ$, because

$$\angle TAA' \;=\; \angle TA'A \;=\; \angle ABA' \;=\; \angle PBQ \;=\; \angle RPQ \;=\; \angle RQP\,.$$

Consequently, $\triangle HA'T \sim \triangle HQR$. It follows that $\triangle HA'Q \sim \triangle HTR$. Thus, $\angle HQA' = \angle HRT$.

In the same way, since $\angle HCC' = \angle HQP$ and $\angle HC'C = \angle HPQ$, we have $\triangle HC'C \sim \triangle HPQ$. Also, $\triangle SCC' \sim \triangle RPQ$, and hence, $\triangle HCS \sim \triangle HQR$. Then $\triangle HQC \sim \triangle HRS$. Thus, $\angle HQC = \angle HRS$.

Therefore, $\angle HRT + \angle HRS = \angle HQA' + \angle HQC = 180°$, which shows that the point R is on the segment TS.

Since $\triangle HTR \sim \triangle HA'Q$ and $\triangle HRS \sim \triangle HQC$, we get

$$TR : A'Q \;=\; HR : HQ \;=\; RS : QC\,.$$

Then $TR : RS = A'Q : QC$. Since $AA' \perp BC$ and $MQ \perp BC$, we have $AA' \parallel MQ$, and therefore, $A'Q : QC = AM : MC$. We conclude that $TR : RS = AM : MC$.

When the point M varies on the side AC from A to C, the point R varies on the segment TS from T to S, satisfying $TR : RS = AM : MC$. Therefore, the locus of R is the segment TS.

Next, we pass along a comment about problem #6 of the 35th Mongolian Mathematical Olympiad secondary Level [2002 : 416–417.]

6. For a map f of the plane into itself, it is known that it sends any two points A, B whose distance apart is 1 into 2 points at the same distance apart. Prove that for any natural n, $|A-B| = n$ implies $|f(A) - f(B)| = n$.

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

This result is a special case of the following theorem of Beckman and Quarles [1]: Let f be a function from E_n to E_n, where E_n is a Euclidean space with (finite) dimension $n \geq 2$. If there exists one real number $L > 0$ such that $d\big(f(A), f(B)\big) = L$ whenever A, $B \in E_n$ and $d(A, B) = L$, then f is an isometry of E_n (that is, f preserves distance).

We note that this result does not hold for the real line.

Reference

[1] F.S. Beckman, D.A. Quarles, Jr., On Isometries of Euclidean Spaces, *Proc. Amer. Math. Soc.,* vol 4 (1953), pp. 810–815.

Next we have two solutions to problems of the Team Selection Examination for the 40th IMO (1999, Belgrade) $[2002 : 417\text{--}418]$.

1. Let n be a positive integer and $P(x)$ a polynomial of degree $2n$ such that $P(0) = 1$ and $P(k) = 2^{k-1}$ for $k = 1, 2, \ldots, 2n$. Prove that $2P(2n+1) - P(2n+2) = 1$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's version.

For $k = 0, 1, \ldots, 2n$, let

$$L_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{2n} (x - j).$$

The polynomial $\sum\limits_{k=0}^{2n} \dfrac{P(k)}{L_k(k)} L_k(x)$ has degree at most $2n$ and takes the same values as $P(x)$ at $0, 1, \ldots, 2n$. Thus,

$$P(x) = \sum_{k=0}^{2n} \frac{P(k)}{L_k(k)} L_k(x). \tag{1}$$

Let $P(x) = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_0$. We use (1) to calculate the coefficient a_{2n}:

$$
\begin{aligned}
a_{2n} &= \sum_{k=0}^{2n} \frac{P(k)}{L_k(k)} = \sum_{k=0}^{2n} \frac{P(k)}{(-1)^k k!(2n-k)!} \\
&= \frac{1}{(2n)!} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} P(k) \\
&= \frac{1}{(2n)!} \left(\binom{2n}{0}1 - \binom{2n}{1}1 + \binom{2n}{2}2^1 + \cdots + \binom{2n}{2n}2^{2n-1} \right) \\
&= \frac{1}{2} \frac{1}{(2n)!} \left(1 + \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 2^k \right) \\
&= \frac{1}{2} \frac{1}{(2n)!} \left(1 + (1-2)^{2n} \right) = \frac{1}{(2n)!}.
\end{aligned}
$$

Now, consider $Q(x) = 2P(x) - P(x+1)$. The polynomial $Q(x)$ has the same degree and the same leading coefficient as $P(x)$. From the hypotheses, we must have $Q(0) = 1$ and $Q(k) = 0$ for $k = 1, 2, \ldots, 2n-1$. It follows that

$$Q(x) = a_{2n}(x-1)(x-2)\cdots(x-(2n-1))(x-r)$$

for some real number r. Since $Q(0) = 1$ and $a_{2n} = \dfrac{1}{(2n)!}$, we find that

$$1 = \frac{(-1)^{2n} r}{(2n)!}(2n-1)! = \frac{r}{2n}; \text{ hence, } r = 2n.$$

Thus, $Q(x) = \dfrac{1}{(2n)!}(x-1)\cdots(x-2n)$. Then

$$2P(2n+1) - P(2n+2) \;=\; Q(2n+1) \;=\; \frac{(2n)!}{(2n)!} \;=\; 1\,.$$

2. Let ABC be a triangle such that $\angle A = 90°$ and $\angle B < \angle C$. The tangent at A to its circumcircle k meets the line BC at D. Let E be the reflection of A across BC, X the foot of the perpendicular from A to BE, and Y the mid-point of AX. Let the line BY meet k again in Z. Prove that the line BD is tangent to the circumcircle of triangle ADZ.

Solution by Toshio Seimiya, Kawasaki, Japan.
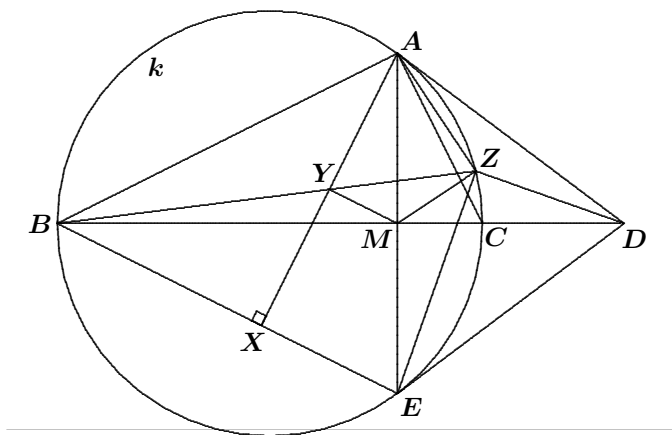
Since BC is a diameter of k, we see that E is a point on k. Let M be the intersection of AE with BC. Then M is the mid-point of AE, and $\angle AMD = 90°$. Since Y and M are mid-points of AX and AE, respectively, we have $YM \parallel XE$; that is, $YM \parallel BE$.

Thus, $\angle ZYM = \angle ZBE = \angle ZAE = \angle ZAM$, which implies that A, Y, M, and Z are concyclic. Hence, $\angle YAM = \angle YZM$. Therefore,

$$\angle BAX \;=\; \angle BAE - \angle YAM \;=\; \angle BZE - \angle YZM \;=\; \angle MZE\,.$$

Also, since $\angle ABX = \angle ABE = \angle EAD = \angle MAD$, we have

$$\angle BAX \;=\; 90° - \angle ABX \;=\; 90° - \angle MAD \;=\; \angle ADM \;=\; \angle EDM\,.$$

Consequently, $\angle MZE = \angle EDM$. Then M, E, D, and Z are concyclic. Thus,

$$\angle ZDM \;=\; \angle ZEM \;=\; \angle ZEA \;=\; \angle ZAD\,.$$

Therefore, BD is tangent to the circumcircle of $\triangle ADZ$.

Now we turn to solutions from our readers to problems given in the December 2002 number of the *Corner*. First are solutions to the XV Gara Nazionale di Matematica 1999 [2002 : 481–482].

1. Given a rectangular sheet with sides a and b, with $a > b$, fold it along a diagonal. Determine the area of the overlapped triangle (the shaded triangle in the picture).
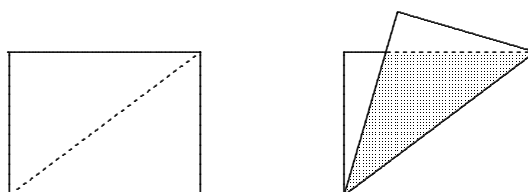

Solved by Marcus Emmanuel Barnes, student, York University; Pierre Bornsztein, Maisons-Laffitte, France; and Bob Serkey, Leonia, NJ, USA. We give the solution by Serkey.

Note that $\triangle ABE \cong \triangle CDE$ (AAS). Thus, $BE = DE$ and $AE = CE$. Letting $x = BE = DE$, we get $a - x = AE = CE$. By the Theorem of Pythagoras, we have $(a - x)^2 + b^2 = x^2$, or $a^2 + b^2 = 2ax$. Then $x = \dfrac{a^2 + b^2}{2a}$.

Using $BE = x$ as a base in $\triangle BDE$, the corresponding height is $CD = b$. Therefore, the area of $\triangle BDE$ is

$$\frac{1}{2}xb \; = \; \frac{1}{2}\left(\frac{a^2 + b^2}{2a}\right)b \; = \; \frac{b(a^2 + b^2)}{4a} \, .$$
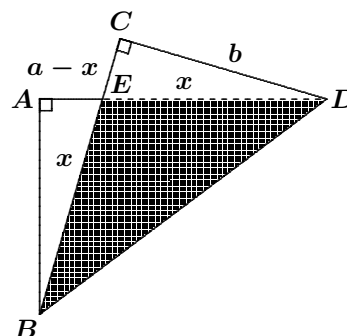
2. A natural number is said to be *balanced* if the number of its decimal digits equals the number of its distinct prime factors (for instance 15 is balanced, whereas 49 is not balanced). Prove that there are only finitely many balanced numbers.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let p_k be the kth prime. Then $p_1 p_2 \cdots p_k > k!$ for all k. For sufficiently large k, we have $k! > 10^k$. Thus, there exists an integer $k_0 > 0$ such that, for each $k \geq k_0$, we have $p_1 p_2 \cdots p_k > 10^k$. For such k, if n is a natural number with exactly k decimal digits, then $p_1 p_2 \cdots p_k > n$, which implies that n cannot have k distinct prime factors and therefore cannot be balanced. Thus, if n is balanced, then n has less than k_0 digits. The conclusion follows.

4. Albert and Barbara play the following game. On a table there are 1999 sticks. Each player in turn must remove from the table some sticks, provided that he removes at least one stick and at most one half of the sticks remaining on the table at the moment of his move. The player who leaves just one stick on the table loses the game. Barbara moves first.

Determine for which of the players there exists a winning strategy, and describe this strategy.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

When there are $k \geq 2$ sticks remaining on the table, we will say that k is a *losing position* if the player who has to play is sure to lose the game, and we will say that k is a *winning position* if the player who has to play can create a losing position for the opponent.

Suppose that k is a losing position. Then $k + 1$, $k + 2$, ..., $2k$ are winning positions, since it is possible to leave k sticks on the table for the opponent. Therefore, $2k + 1$ is a losing position since, according to the rules, the player has to leave at least $k + 1$ and most $2k$ sticks on the table, which gives a winning position to the opponent.

Since $k = 2$ is a losing position, we deduce that the losing positions are the integers U_n, where $U_0 = 2$ and $U_{n+1} = 2U_n + 1$. By induction, it can be shown that $U_n = 3 \times 2^n - 1$ for $n \geq 0$. Note that 1999 does not have the form $3 \times 2^n - 1$, since $1999 + 1$ is not divisible 3. It follows that 1999 is a winning position. Thus, Barbara, who moves first, has a winning strategy.

The strategy is described above. It suffices for Barbara to leave on the table, on consecutive turns, a number of sticks equal to

$$
\begin{aligned}
U_9 &= 3 \times 2^9 - 1 = 1535\,, \\
U_8 &= 3 \times 2^8 - 1 = 767\,, \\
U_7 &= 3 \times 2^7 - 1 = 383\,, \\
U_6 &= 3 \times 2^6 - 1 = 191\,, \\
U_5 &= 3 \times 2^5 - 1 = 95\,, \\
U_4 &= 3 \times 2^4 - 1 = 47\,, \\
U_3 &= 3 \times 2^3 - 1 = 23\,, \\
U_2 &= 3 \times 2^2 - 1 = 11\,, \\
U_1 &= 3 \times 2 - 1 = 5\,, \\
U_0 &= 2\,.
\end{aligned}
$$

6. (a) Determine all pairs (x, k) of positive integers which satisfy the equation

$$3^k - 1 = x^3\,.$$

(b) Prove that if n is an integer greater than 1 and different from 3 there are no pairs (x, k) of positive integers satisfying the equation

$$3^k - 1 = x^n\,.$$

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's write-up.

(a) Clearly, $(x, k) = (2, 2)$ is a solution. We will show that there are no other solutions.

Let x and k be positive integers such that $3^k - 1 = x^3$. Then x is even, since $3^k - 1$ is even, and thus, $x \geq 2$. Furthermore, we have

$$3^k = x^3 + 1 = (x+1)(x^2 - x + 1).$$

It follows that $x^2 - x + 1 = 3^s$ for some integer $s \geq 1$. (We cannot have $s = 0$ because $x \geq 2$.) However, $x^2 - x + 1$ is not congruent to 0 modulo 9, as we can check by considering $x \in \{0, 1, \ldots, 8\}$. Therefore, $s = 1$. Then $x^2 - x + 1 = 3$, giving $x = 2$, from which $k = 2$ follows.

(b) Let n be an integer greater than 1 for which there exist positive integers x, k such that $3^k - 1 = x^n$. We will prove that $n = 3$.

First, we observe that n cannot be even. For, if $n = 2r$, we would have $(x^r)^2 = 3^k - 1 \equiv -1 \pmod{3}$. This is impossible, since a square is congruent to either 0 or 1 modulo 3.

Now set $n = 2r + 1$. Then

$$3^k = x^n + 1 = x^{2r+1} + 1 = (x+1)a,$$

where $a = 1 - x + x^2 - \cdots + x^{2r}$. Note that $x + 1 > 1$ and also $a > 1$ (because $n > 1$). It follows that $x + 1 = 3^u$ and $a = 3^v$ for some positive integers u and v. Then $x \equiv -1 \pmod{3}$ and

$$3^v = a \equiv 1 + 1 + \cdots + 1 \equiv 2r + 1 \pmod{3}.$$

Thus, $2r + 1 \equiv 0 \pmod{3}$; that is, $r \equiv 1 \pmod{3}$. Hence, $r = 1 + 3w$ for some non-negative integer w. Now $3^k = x^{2r+1} + 1 = (x^{1+2w})^3 + 1$. From part (a), we must have $x^{1+2w} = 2$, which implies that $w = 0$. Therefore, $r = 1$ and $n = 3$.

That completes the *Corner* for this issue. Send me your Olympiad contest materials and your nice solutions and generalizations to problems in the *Corner*.

BOOK REVIEWS

John Grant McLoughlin

Duelling Idiots and other Probability Puzzlers
By Paul J. Nahin, published by Princeton University Press, 2002
ISBN 0-691-10286-4 (paperback), 271 pages, US$19.95.
Reviewed by **Jeff Hooper**, *Acadia University, Wolfville, NS.*

Opening this book and reading Nahin's preface, you can immediately sense his love and enthusiasm for probability. The author's informal and engaging style allows even the most challenging problems in the book to be approached using elementary principles from probability. In fact, the entire book is designed around the general thesis of getting the reader to use intuition and creativity in approaching these problems. Although the book contains only 21 problems, each is wrapped in an engaging discussion, often placed in historical context, then given a complete mathematical explanation, followed by additional challenges. Complete solutions are provided.

The book kicks off with a wonderful warm-up problem, *How to Ask an Embarrassing Question*, which explores a common issue in surveying people regarding topics which may be embarrassing. The data must be gathered over time, in individual appointments. But due to the embarrassment factor, participants may not answer the question truthfully. The problem then is to design a technique for gathering accurate data. I first encountered this problem as a student and remember being captivated by it, because, although of critical practical importance, nevertheless the problem has a deceptively simple solution.

As a better example of the types of problems explored, the title problem explores the situation when two "idiots" duel. The initial problem is set up as the classic Russian Roulette problem. Two "idiots", A and B, wish to duel as follows. They place a single bullet in a six-chamber pistol, and take turns. A spins the chamber and fires at B; B then spins the chamber and fires at A. They then repeat this procedure until one fool shoots the other. What is the probability that A will win? And how many trigger pulls can we expect on average? Although the reader may wish to solve this first, the book then gives a thoughtful and careful analysis of this problem, using a geometric series to compute the probability. The real challenge, however, comes next. Nahin then throws out a new spin on this problem: in the new version, each fool now gets two attempts before the gun changes hands. The reader is challenged to answer the same two questions for this new situation.

The remaining problems explore a diverse selection of applications of probability, including such topics as electrical circuits, the likelihood that the underdog will win the World Series, the use of probabilistic simulation to approximate the constants π and e, the design of radios, surprising problems involving balls and urns, the probability of winning a chess match, geometric probability, and finding a lost ship at sea as quickly as possible.

What makes this book particularly delightful, though, are the interesting twists put on classic problems. For instance, the approach to the approximation of π is essentially the Buffon needle problem. Nahin then adds a twist, pointing the reader in the direction of finding a similar method for approximating e.

A similar and startling twist is provided for the following problem. Given the quadratic equation $x^2 + Bx + C$, where B and C are independent, uniformly distributed random variables, what is the probability that the equation has two real roots? This, again, is a problem which can be found in many textbooks. However, Nahin then considers a similar problem involving the quadratic $Ax^2 + Bx + C$, where now A, B, and C are independent, uniformly distributed random variables. As he points out, an obvious approach to this is to divide through by A to get $x^2 + B/Ax + C/A$, and then use the previous idea. But, as always in this book, there's a catch. Although A, B, and C are independent, the fractions B/A and C/A are not!!

Like any excellent problem book, full and careful solutions are provided for all of the problems. Because of its additional emphasis on simulation, the book concludes with an essay on Random Number Generators and their applications, and includes extensive MatLab code which can be used or adapted by students in order to further explore the problems using simulation.

As mentioned earlier, the problems in this book are longer, and more involved, than most found in 'problems' books, and are generally multi-layered. Working through these will usually require more effort on the part of the reader. In a classroom, for instance, these problems could be used for advanced high school students, especially as enrichment problems, and probably as group projects. Although the book's general approach avoids technicalities, some of these problems may require assistance from the teacher.

102 Combinatorial Problems: From the Training of the USA IMO Team
by Titu Andreescu and Zuming Feng, published by Birkhauser, 2003
ISBN 0-8176-4317-6, paperback, 115 pages, US$24.95.
Reviewed by **Richard Hoshino**, *Dalhousie University, Halifax, NS.*

This excellent book consists of 102 carefully selected problems in combinatorics that have been used in the training of the USA team to the International Mathematical Olympiad (IMO). The authors are Titu Andreescu and Zuming Feng, the two leaders of the American IMO team.

The problems provide an in-depth enrichment in combinatorics by reorganizing and enhancing problem-solving tactics and strategies. The topics include combinatorial arguments and identities, generating functions, graph theory, recursive relations, sums and products, probability, number theory, polynomials, theory of equations, complex numbers in geometry,

algorithmic proofs, combinatorial and advanced geometry, functional equations, and classical inequalities. Most of the problems are taken from old contests such as the AHSME and AIME, and several problems are taken from national Olympiads.

The first half of the book consists of 51 Introductory Problems, which will be elementary to an experienced Olympiad–level problem-solver. For those new to writing contests at this level, the first 51 problems provide an excellent foundation for solving difficult combinatorial problems, and important problem-solving ideas and techniques are clearly introduced.

The real strength of this book lies in the second half, which consists of 51 Advanced Problems taken from various national Olympiads and problems shortlisted for the IMO. Multiple solutions are presented for many of the problems, and the questions specifically highlight the insights and techniques necessary to solve IMO–level problems in combinatorics. The problems are well chosen and span all of the major ideas in this field.

There are two criticisms that I must make. First, the book has an awful typesetting error, where superscripts (as in 5th and 2nd) are printed incorrectly, and come out as a funny LaTeXsymbol resembling the letter P. Needless to say, this caused me an enormous amount of confusion as I was reading the solutions. This mistake is consistent throughout the book and happens at least fifty times. Secondly, I would like to have seen more problems! This is an excellent book, but at 115 pages, it is too short, and I feel that the book would have benefited from more problems, especially at the Advanced level.

Nevertheless, the overall assessment of the book is a very positive one. I highly recommend this text to anyone who is striving for the IMO and to anyone who is involved in the coaching and preparation of IMO–level students.

I close with three problems from the book, and I encourage you to try to solve these problems on your own.

1. Prove that among any 16 positive integers not exceeding 100 there are four different ones, a, b, c, d, such that $a + b = c + d$.

2. There are ten cities in Fatland. Two airlines control all of the flights between the cities. Each pair of cities is connected by exactly one flight (in both directions). Prove that one airline can provide two travelling cycles with each cycle passing through an odd number of cities and with no common cities shared by the two cycles.

3. Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ such that $P(2) = n$.

PROBLEMS

Solutions to problems in this issue should arrive no later than **1 September 2005**. *An asterisk (⋆) after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

3001. *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*

Given a, b, c, d, $e > 0$ such that $a^2 + b^2 + c^2 + d^2 + e^2 \geq 1$, prove that

$$\frac{a^2}{b+c+d} + \frac{b^2}{c+d+e} + \frac{c^2}{d+e+a} + \frac{d^2}{e+a+b} + \frac{e^2}{a+b+c} \geq \frac{\sqrt{5}}{3}.$$

3002. *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*

Let r, $s \in \mathbb{R}$ with $0 < r < s$, and let a, b, $c \in (r, s)$. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{3}{2} + \frac{(r-s)^2}{2r(r+s)},$$

and determine when equality occurs.

3003. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC be a triangle with $AB \neq AC$. Let AD be the altitude from A to BC and let BE and CF be the internal angle bisectors of $\angle B$ and $\angle C$, respectively, with E on AC and F on AB. Let B' and C' be the points of intersection of AD with BE and CF, respectively, and let A' be the point where BE intersects CF.

Construct the point Q on BC on the same side of C as B such that $QC = AC$, and construct the point P on BC on the same side of B as C such that $PB = AB$.

Prove that $\triangle A'B'C'$ is similar to $\triangle AQP$.

3004. *Proposed by Mihály Bencze, Brasov, Romania.*

Let R and r be the circumradius and inradius, respectively, of $\triangle ABC$. Prove that

$$\frac{\left(\sqrt{a} - \sqrt{b}\right)^2 + \left(\sqrt{b} - \sqrt{c}\right)^2 + \left(\sqrt{c} - \sqrt{a}\right)^2}{\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2} \leq \frac{4}{9}\left(\frac{R}{r} - 2\right).$$

3005. *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*

Let R and r be the circumradius and inradius, respectively, of $\triangle ABC$. Let h_a, h_b, h_c be the lengths of the altitudes of $\triangle ABC$ issuing from A, B, C, respectively, and let w_a, w_b, w_c be the lengths of the interior angle bisectors of A, B, C, respectively. Prove that

$$\frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \ge 1 + \frac{4r}{R}.$$

3006. *Proposed by Luis V. Dieulefait, Centre de Recerca Matemática, Ballaterra, Spain.*

An old man willed that, upon his death, his three sons would receive the uth, v^{th}, and wth parts of his herd of camels respectively. He had N camels in the herd when he died, where $N + 1$ is a common multiple of u, v, and w. Since the three sons could not divide N exactly into u, v, or w parts, they approached a distinguished **CRUX** problem solver for help. He rode over on his own camel, which he added to the herd. The herd was then divided up according to the old man's wishes. Our **CRUX** problem solver then took back the one camel that remained, which was, of course, his own.

(a)   Find all solutions (u, v, w, N).

(b)⋆ Solve the same problem if there are four sons.

(c)⋆ Let there be k sons. Find an upper bound $f(k)$ on N for the problem to have a solution.

[*Ed*: This is a generalization of Problem 2226 [1997 : 166; 1998 : 186].]

3007. *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be a triangle, and let $A_1 \in BC$, $B_1 \in CA$, $C_1 \in AB$ such that
$$\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B} = k > 0.$$

1. Prove that the segments AA_1, BB_1, CC_1 are the sides of a triangle.

Let T_k denote this triangle. Let R_k and r_k be the circumradius and inradius of T_k. Prove that:

2. $P(T_k) < P(ABC)$, where $P(T)$ denotes the perimeter of triangle T;

3. $[T_k] = \dfrac{k^2 + k + 1}{(k+1)^2}[ABC]$, where $[T]$ denotes the area of triangle T;

4. $R_k = \dfrac{k\sqrt{k}\,P(ABC)}{(k+1)(k^2+k+1)}$;

5. $r_k = \dfrac{k^2+k+1}{(k+1)^2}\,r$, where r is the inradius of $\triangle ABC$.

3008. *Proposed by Mihály Bencze, Brasov, Romania.*

The convex polygon $A_1 A_2 \cdots A_n$ is inscribed in a circle Γ. Let G be the centroid of this polygon. For $k = 1, 2, \ldots, n$, denote by B_k the second point of intersection of the line $A_k G$ with the circle Γ. Prove that

$$\sum_{k=1}^{n} GB_k = \sum_{k=1}^{n} GA_k.$$

3009. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

With I the incentre of $\triangle ABC$, let the angle bisectors BI and CI meet the opposite sides at B' and C', respectively. Prove that $AB' \cdot AC'$ is greater than, equal to, or less than AI^2 according as $\angle A$ is greater than, equal to, or less than $90°$.

3010. *Proposed by Mihály Bencze and Marian Dinca, Romania.*

Let ABC be a triangle inscribed in a circle Γ. Let A_1, B_1, $C_1 \in \Gamma$ such that

$$\frac{\measuredangle A_1 AB}{\measuredangle CAB} = \frac{\measuredangle B_1 BC}{\measuredangle ABC} = \frac{\measuredangle C_1 CA}{\measuredangle BCA} = \lambda,$$

where $0 < \lambda < 1$. Let the inradius and semiperimeter of $\triangle ABC$ be denoted by r and s, respectively; let the inradius and semiperimeter of $\triangle A_1 B_1 C_1$ be denoted by r_1 and s_1, respectively. Prove that

1. $s_1 \geq s$;

2. $r_1 \geq r$;

3. $[A_1 B_1 C_1] \geq [ABC]$, where $[PQR]$ denotes the area of triangle PQR.

3011. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let ABC be a right-angled triangle with right angle at B. Let D be the foot of the perpendicular from B to AC, and let E be the intersection of the bisector of $\angle BDC$ with BC. Let M and N be the mid-points of BE and DC, respectively, and let F be the intersection of MN with BD. Prove that $AD = 2BF$.

3012. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Triangles DBC, ECA, and FAB are constructed outwardly on $\triangle ABC$ such that $\angle DBC = \angle ECA = \angle FAB$ and $\angle DCB = \angle EAC = \angle FBA$. Prove that

$$AF + FB + BD + DC + CE + EA \geq AD + BE + CF.$$

When does equality hold?

3001. *Proposé par Pham Van Thuan, Hanoi, Viêt Nam.*

Etant donné a, b, c, d et $e > 0$ tels que $a^2 + b^2 + c^2 + d^2 + e^2 \geq 1$, montrer que

$$\frac{a^2}{b+c+d} + \frac{b^2}{c+d+e} + \frac{c^2}{d+e+a} + \frac{d^2}{e+a+b} + \frac{e^2}{a+b+c} \geq \frac{\sqrt{5}}{3} \,.$$

3002. *Proposé par Pham Van Thuan, Hanoi, Viêt Nam.*

Soit r, $s \in \mathbb{R}$ avec $0 < r < s$, et soit a, b, $c \in\,]r, s[$. Montrer que

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{3}{2} + \frac{(r-s)^2}{2r(r+s)} \,,$$

et déterminer quand l'égalité a lieu.

3003. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit ABC un triangle avec $AB \neq AC$. Soit AD la hauteur abaissée de A sur BC, et soit BE et CF les bissectrices intérieures des angles en B et C, respectivement, avec E sur AC et F sur AB. Soit B' et C' les intersections respectives de AD avec BE et CF, et soit A' celle de BE avec CF.

Construire le point Q sur BC du même côté de C que B et tel que $QC = AC$, et construire le point P sur BC du même côté de B que C et tel que $PB = AB$.

Montrer que le triangle $A'B'C'$ est semblable au triangle AQP.

3004. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit R et r les rayons respectifs des cercles circonscrit et inscrit du triangle ABC. Montrer que

$$\frac{\left(\sqrt{a} - \sqrt{b}\right)^2 + \left(\sqrt{b} - \sqrt{c}\right)^2 + \left(\sqrt{c} - \sqrt{a}\right)^2}{\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2} \leq \frac{4}{9}\left(\frac{R}{r} - 2\right) \,.$$

3005. *Proposé par Pham Van Thuan, Hanoi, Viêt Nam.*

Soit r et R les rayons du cercle inscrit et circonscrit d'un triangle ABC. Soit h_a, h_b et h_c les longueurs respectives des hauteurs du triangle ABC issues des sommets A, B et C, soit w_a, w_b et w_c les longueurs respectives des bissectrices des angles en A, B et C. Montrer que

$$\frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \geq 1 + \frac{4r}{R} \,.$$

3006. *Proposé par Luis V. Dieulefait, Centre de Recerca Matemática, Ballaterra, Espagne.*

Par testament, un vieil homme avait décidé de partager son troupeau de chameaux entre ses trois fils de sorte qu'ils en reçoivent respectivement la u-ième, v-ième et w-ième partie. À sa mort, il avait N chameaux, avec $N+1$ un commun multiple de u, v et w. Évidemment, les trois fils ne pouvaient procéder au partage prévu. Ils firent donc appel à une personne connue pour ses résolutions de problèmes de **CRUX**, qui amena son propre chameau, pour procéder au partage prévu et repartit avec le chameau qui restait, le sien.

(a)   Trouver toutes les solutions (u, v, w, N).

(b)⋆ Résoudre le même problème avec quatre fils.

(c)⋆ S'il y avait k fils, trouver une borne supérieure $f(k)$ de N pour que le problème ait une solution.

[*Ed* : Ceci est une généralisation du Problème 2226 [1997 : 166 ; 1998 : 186].]

3007. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit ABC un triangle et soit $A_1 \in BC$, $B_1 \in CA$ et $C_1 \in AB$ tels que

$$\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B} = k > 0.$$

1. Montrer que les segments AA_1, BB_1 et CC_1 sont les côtés d'un triangle.

Désignons par T_k ce triangle, par r_k et R_k le rayon des cercles inscrit et circonscrit de T_k. Montrer que

2. $P(T_k) < P(ABC)$, où $P(T)$ désigne le périmètre du triangle T ;

3. $[T_k] = \dfrac{k^2 + k + 1}{(k + 1)^2}[ABC]$, où $[T]$ désigne l'aire du triangle T ;

4. $R_k = \dfrac{k\sqrt{k}\,P(ABC)}{(k + 1)(k^2 + k + 1)}$;

5. $r_k = \dfrac{k^2 + k + 1}{(k + 1)^2}\,r$, où r désigne le rayon du cercle inscrit de $\triangle ABC$.

3008. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Le polygone convexe $A_1 A_2 \cdots A_n$ est inscrit dans un cercle Γ. Soit G le centre de gravité de ce polygone. Pour $k = 1, 2, \ldots, n$, désigner par B_k le second point d'intersection de la droite $A_k G$ avec le cercle Γ. Montrer que

$$\sum_{k=1}^{n} GB_k = \sum_{k=1}^{n} GA_k.$$

3009. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit I le centre du cercle inscrit du triangle ABC et soit B' et C' les points d'intersection respectifs des bissectrices BI et CI avec les côtés opposés. Montrer que $AB' \cdot AC'$ est plus grand, égal ou plus petit que AI^2 suivant que l'angle A est plus grand, égal ou plus petit que $90°$.

3010. *Proposé par Mihály Bencze et Marian Dinca, Roumanie.*

Soit ABC un triangle inscrit dans un cercle Γ. Soit A_1, B_1 et $C_1 \in \Gamma$ tels que

$$\frac{\measuredangle A_1 AB}{\measuredangle CAB} = \frac{\measuredangle B_1 BC}{\measuredangle ABC} = \frac{\measuredangle C_1 CA}{\measuredangle BCA} = \lambda,$$

où $0 < \lambda < 1$. Soit r le rayon du cercle inscrit et s le demi-périmètre du triangle ABC, r_1 et s_1 les grandeurs analogues du triangle $A_1 B_1 C_1$. Montrer que

1. $s_1 \geq s$;

2. $r_1 \geq r$;

3. $[A_1 B_1 C_1] \geq [ABC]$, où $[PQR]$ désigne l'aire du triangle PQR.

3011. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Soit ABC triangle rectangle avec l'angle droit en B. Soit D le pied de la perpendiculaire abaissée de B sur AC, et soit E l'intersection de la bissectrice de l'angle BDC avec BC. Soit M et N les points milieu de BE et DC, et soit F l'intersection de MN avec BD. Montrer que $AD = 2BF$.

3012. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

On construit les triangles DBC, ECA et FAB à l'extérieur des côtés du triangle ABC de telle sorte que les angles $DBC = ECA = FAB$ et $DCB = EAC = FBA$. Montrer que

$$AF + FB + BD + DC + CE + EA \ \geq \ AD + BE + CF.$$

Quand y a-t-il égalité ?

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologize for omitting the name of CARL LIBIS, University of Rhode Island, Kingston, RI from the list of solvers of 2862(a).

2653. [2001 : 336; 2002 : 342] *Proposed by Walther Janous, Ursulinen-gymnasium, Innsbruck, Austria.*

For whole numbers $n \geq 0$ and $N \geq 1$, evaluate the (combinatorial) sum

$$S_N(n) := \sum_{k \geq n} \binom{N}{2k}\binom{k}{n} .$$

III. *Solution by Michel Bataille, Rouen, France.*

For $N = 0, 1, 2, \ldots$, let T_N be the Nth Chebyshev polynomial. One way to define T_N is by the relation $T_N(\cos\theta) = \cos(N\theta)$. The degree of T_N is N, the coefficient of x^N in $T_N(x)$ is 2^{N-1}, and $y = T_N(x)$ is a solution of the differential equation $(x^2 - 1)y'' + xy' = N^2 y$. Using this last property of T_N, the following explicit formula (found in [1]) may be derived:

$$T_N(x) = \sum_{n=0}^{\lfloor N/2 \rfloor} (-1)^n \frac{N}{N-n}\binom{N-n}{n} 2^{N-2n-1} x^{N-2n} . \tag{1}$$

Since $T_N(\cos\theta) = \cos(N\theta)$ and

$$\cos(N\theta) + i\sin(N\theta) = (\cos\theta + i\sin\theta)^N ,$$

we have $T_N(x) = \sum_{k=0}^{\lfloor N/2 \rfloor} \binom{N}{2k} x^{N-2k}(x^2 - 1)^k$. Expanding $(x^2 - 1)^k$ and changing the order of summation, we obtain

$$\begin{aligned} T_N(x) &= \sum_{n=0}^{\lfloor N/2 \rfloor} (-1)^n \left(\sum_{k \geq n} \binom{N}{2k}\binom{k}{n} \right) x^{N-2n} \\ &= \sum_{n=0}^{\lfloor N/2 \rfloor} (-1)^n S_N(n) x^{N-2n} . \end{aligned}$$

Comparing with (1) yields $S_N(n) = \dfrac{N \cdot 2^{N-2n-1}}{N-n}\binom{N-n}{n}$.

Reference

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, 1972, p. 775.

2901★. [2004 : 38, 41] *Proposed by Stanley Rabinowitz, Westford, MA, USA.*

Let I be the incentre of $\triangle ABC$. The circles d, e, f inscribed in $\triangle IAC$, $\triangle IBC$, $\triangle ICA$ touch the sides AB, BC, CA at the points D, E, F, respectively. The line IA is one of the two common internal tangents between the circles d and f. Let ℓ be the other common internal tangent. Prove that ℓ passes through the point E.

[This problem was suggested by experiments using *Geometer's Sketchpad*.]

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

This result is known. It is a nice consequence of Steiner's solution of the celebrated Malfatti's Problem. However, Steiner left his solution without a proof. Hart later provided a proof, in which our current result is established. For details and original references, see J.L. Coolidge, *A treatise on the Circle and the Sphere*, Chelsea, 1971, 174–176.

Also solved by MICHEL BATAILLE, Rouen, France; and Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain.

Bataille supplied the reference to Hart's original proof: Hart, Geometrical Investigation of Steiner's solution of Malfatti's Problem, Quarterly Journal of Mathematics, vol. I, 1856.

2902. [2004 : 38, 41] *Proposed by Stanley Rabinowitz, Westford, MA, USA.*

Let P be a point in the interior of $\triangle ABC$. Let D, E, F be the feet of the perpendiculars from P to BC, CA, AB, respectively. If the three quadrilaterals $AEPF$, $BFPD$, $CDPE$ each have an incircle tangent to all four sides, prove that P is the incentre of $\triangle ABC$.

Nearly identical solutions by Michel Bataille, Rouen, France; Peter Y. Woo, Biola University, La Mirada, CA, USA; Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

Since $\angle PEA = \angle PFA = 90°$, the Pythagorean Theorem gives us $PA^2 = PF^2 + FA^2 = PE^2 + EA^2$. Thus,

$$(AF + PE)(AF - PE) = (AE + PF)(AE - PF). \tag{1}$$

Since $AFPE$ has an incircle, we have

$$AF + PE = AE + PF. \tag{2}$$

Hence, (1) implies $AF - PE = AE - PF$. Subtracting this from (2), we get $PE = PF$. Similarly, $PD = PE$. Thus, $PD = PE = PF$, and P is the incentre of $\triangle ABC$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; KIM UYEN TRUONG, California State

University, Fullerton, CA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

The idea behind all the submitted solutions is the observation that if a quadrangle $PQRS$ has (a) right angles at Q and S and (b) an incircle tangent to all four sides, then it is symmetric about PR: $\triangle PQR \cong \triangle PSR$, and the centre of the circle inscribed in $PQRS$ lies on PR.

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**2903**. [2004 : 39, 41] Proposed by Stanley Rabinowitz, Westford, MA, USA.

Three disjoint circles A_1, A_2, and A_3 are given in the plane, none being interior to any other. The common internal tangents to A_j and A_k are α_{jk} and β_{jk}.

If the α_{jk} are concurrent, prove that the β_{jk} are also concurrent.

[This is a known result—but not well-known.]

Solution by Michel Bataille, Rouen, France.

Assume that α_{12}, α_{23}, and α_{31} are concurrent at I. Let J be the point of intersection of β_{12} and β_{23}. Let t denote the tangent to A_1 which passes through J and is different from β_{12}. We must prove that t is also tangent to A_3.
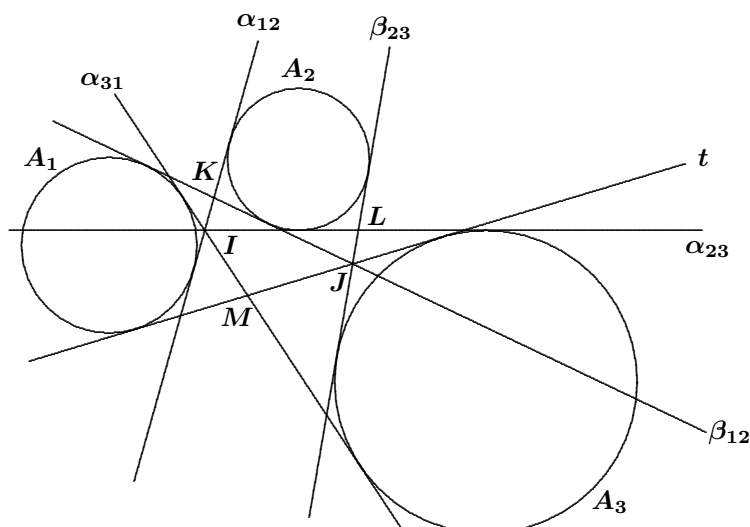
Let K be the intersection of α_{12} and β_{12}, L the intersection of α_{23} and β_{23}, and M the intersection of α_{31} and t. Let d and d' be the lengths of the tangents to A_2 from K and L, respectively. Then $IK + d = IL - d'$ and $JK - d = JL + d'$, from which we have $IK + JK = IL + JL$.

Similarly, by considering the tangents to A_1 from K and M, we obtain $IK + JK = IM + JM$. Thus, $IL + JL = IM + JM$.

Now, assume that the line t is not tangent to A_3. Let t' be the tangent to A_3 which passes through J and is not β_{23}. Let M' be the intersection of t' with α_{31}. As above, $IM' + JM' = IL + JL$, from which it follows that

$IM' - IM = JM - JM'$. As a result, we have $|JM - JM'| = MM'$, which contradicts the Triangle Inequality $|JM - JM'| < MM'$. Therefore, t must be tangent to A_3.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Both Bellot Rosado and Janous refer to works by I.F. Shagyrin [1], who passed away in May of 2004. The proposer also cited a reference to a related problem [2].

[1] I.F. Shagyrin, *Problems in Plane Geometry*, Mir Publ., Moscow, 1988, p. 134 and p. 331.

[2] F. G.-M., Exercices de Géométrie—comprenant l'exposé des méthodes géométriques et 2000 questions résolues, sixième édition, J. De Gigord, Paris, 1920 , Problem 758, p. 326.

2904. [2004 : 39, 42] *Proposed by Mohammed Aassila, Strasbourg, France.*

Suppose that $x_1 > x_2 > \cdots > x_n$ are real numbers. Prove that

$$\sum_{k=1}^{n} x_k^2 - \sum_{1 \le j < k \le n} \ln(x_j - x_k) \ge \frac{n(n-1)}{4}(1 + 2\ln 2) - \frac{1}{2}\sum_{k=1}^{n} k \ln k .$$

Comment by Michel Bataille, Rouen, France.

This interesting but difficult problem is not new: it is solved by Jean-Pierre Hormière in *Revue des mathématiques de l'enseignement supérieur, RMS*, Nº5, Janvier 2000, pp. 615–7.

The starting point was actually a problem posed at the oral examination of l'École Polytechnique in 1997 and solved by Jean-Marie Monier and Jean-Louis Garcin in *RMS (Revue de mathématiques spéciales)* Nº 9/10, Mai–Juin 1998, pp. 1095–7. Jean-Pierre Hormière gave a new solution in *RMS* Nº 2, Octobre 1998, pp. 338–40 and proposed a generalized form of the problem (the same as 2904) that he solved in 2000. He determined the minimum value of the expression on the left side of the inequality via some polynomials related to Hermite polynomials.

Ed.: The proposer submitted a solution which invokes Hermite polynomials. He explained that he submitted this problem because it was, in his opinion, not well known (at least not to **CRUX with MAYHEM** readers), and because the only known solution uses Hermite polynomials. He was hoping that some of our readers could find a different (and, hopefully, more elementary) proof. Since no such solution has been received, we consider this problem to be still open.

In the statement of the problem, the condition "$1 \le j \le k \le n$" was clearly a typo for "$1 \le j < k \le n$" (this has been corrected in the problem statement above). This was pointed out by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, who also submitted comments showing that the given inequality is in its sharpest form if, on the left side of the inequality, the summation $\sum_{k=1}^{n} x_k^2$ is replaced by $\frac{1}{n} \sum_{1 \le j < k \le n} (x_j - x_k)^2$.

2905. [2004 : 39, 42] *Proposed by Titu Zvonaru, Bucharest, Romania.*

The lines joining the vertices to the centroids of the squares constructed externally over the sides of a triangle are concurrent (at Vecten's point). Suppose that Vecten's point is also the symmedian point. Characterize the triangle.

Solution by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.

The trilinear coordinates of both the Vecten point and the symmedian point can be found, for example, in Clark Kimberling's *Encyclopedia of Triangle Centers* (http://faculty.evansville.edu/ck6/encyclopedia/ETC.html).

The Vecten point, $X(485)$, has trilinear coordinates

$$\frac{1}{\sin A + \cos A} : \frac{1}{\sin B + \cos B} : \frac{1}{\sin C + \cos C} \, ,$$

and those of the Symmedian point, $X(6)$, are $\sin A : \sin B : \sin C$. Equating these two points, we obtain

$$\frac{\sin A}{\sin B} = \frac{\sin B + \cos B}{\sin A + \cos A} = \frac{\sqrt{2}\sin(B + 45°)}{\sqrt{2}\sin(A + 45°)} \, ,$$

$$\sin(A + 45°)\sin A = \sin(B + 45°)\sin B \, ,$$

$$\tfrac{1}{2}\left(\cos 45° - \cos(2A + 45°)\right) = \tfrac{1}{2}\left(\cos 45° - \cos(2B + 45°)\right) \, ,$$

$$\cos(2A + 45°) = \cos(2B + 45°) \, . \tag{1}$$

Of course, the cyclic permutations of (1) must hold also. To satisfy (1), it follows that either $A = B$ or $(2A + 45°) + (2B + 45°) = 360°$. This last condition is equivalent to $A + B = 135°$. Consequently, all of the following conditions must hold:

(i) $A = B$ or $A + B = 135°$,   (ii) $B = C$ or $B + C = 135°$,

(iii) $C = A$ or $C + A = 135°$,   (iv) $A + B + C = 180°$.

From (i) to (iv), it is easy to see that there are only two possibilities:

(a) $A = B = C = 60°$, or

(b) $A = B = 45°$, $C = 90°$ (and its cyclic permutations).

We conclude that only equilateral triangles and right-angled isosceles triangles meet the conditions of the problem.

Comment. A good website for a discussion of the Vecten point and a list of references is Alex Bogomolny's web page at

http://www.cut-the-knot.org/ctk/BridesChair.shtml .

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; BRA; JACQUES CHONÉ, Nancy, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There were also two incomplete solutions.

2906. [2004 : 39, 42] *Proposed by Titu Zvonaru, Bucharest, Romania.*

Suppose that $k \in \mathbb{N}$. Find $\min_{n \in \mathbb{N}} \left(\dfrac{2}{n} + \dfrac{n^2}{k} \right)$.

Composite of similar solutions by Jacques Choné, Nancy, France; and Chip Curtis, Missouri Southern State College, Joplin, MO, USA.

Fix $k \in \mathbb{N}$, and let $f_k(x) = \dfrac{2}{x} + \dfrac{x^2}{k}$. Since $f_k'(x) = \dfrac{2(x^3 - k)}{kx^2}$, we see that $f_k(x)$ is decreasing on $\left[1, \sqrt[3]{k}\right]$ and increasing on $\left[\sqrt[3]{k}, \infty\right)$. Thus, on the interval $[1, \infty)$, the function $f_k(x)$ is minimized at $\sqrt[3]{k}$. Therefore, the minimum value of $f_k(x)$ on $\mathbb{N}$ occurs at either $\lfloor \sqrt[3]{k} \rfloor$ or $\lfloor \sqrt[3]{k} \rfloor + 1$.

For any $m \in \mathbb{N}$, we have

$$
\begin{aligned}
f_k(m) - f_k(m+1) &= \frac{2}{m} + \frac{m^2}{k} - \frac{2}{m+1} - \frac{(m+1)^2}{k} \\
&= \frac{2k - m(m+1)(2m+1)}{m(m+1)k},
\end{aligned}
$$

and hence, $f_k(m) \leq f_k(m+1)$ if and only if $k \leq \frac{1}{2}m(m+1)(2m+1)$. Letting $m = \lfloor \sqrt[3]{k} \rfloor$, we have

$$
\min_{n \in \mathbb{N}} \left(\frac{2}{n} + \frac{n^2}{k} \right) = \begin{cases} \dfrac{2}{m} + \dfrac{m^2}{k} & \text{if } k \leq \frac{1}{2}m(m+1)(2m+1), \\ \dfrac{2}{m+1} + \dfrac{(m+1)^2}{k} & \text{if } k > \frac{1}{2}m(m+1)(2m+1). \end{cases}
$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; JOEL SCHLOSBERG, Bayside, NY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Almost all the submitted solutions are similar to the one presented above, although the final answer can be given in several slightly different, but equivalent, forms.

**2907**. [2004 : 39, 42] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Rhombus $ABCD$ has incircle Γ with centre O. Circle Γ touches sides AB and AD at M and N, respectively. Suppose that a tangent to Γ meets the segments AM and AN at E and F, respectively, and that EF intersects BC and CD at P and Q, respectively.

Prove that $[AMON]$ is the geometric mean of $[AEF]$ and $[CPQ]$.

($[Z_1 Z_1 \ldots Z_n]$ denotes the area of the n–gon $Z_1 Z_2 \ldots Z_n$.)

Solution by Michel Bataille, Rouen, France.

Let $\sigma = \frac{1}{2}(AF + FE + EA)$ and $s = \frac{1}{2}(CP + PQ + QC)$. Since Γ is the excircle opposite angle A of $\triangle AEF$, then $AN = AM = \sigma$. Note that Γ is also the incircle of $\triangle CPQ$. Let r and ρ be the inradii of $\triangle CPQ$ and $\triangle AEF$, respectively. Then $[CPQ] = rs$ and $[AEF] = \rho\sigma$.
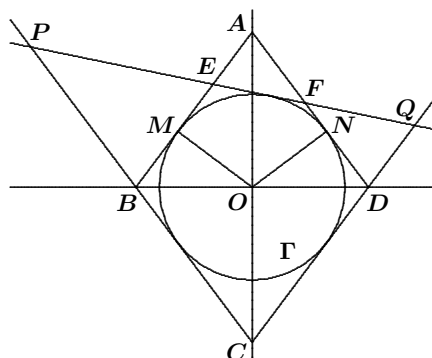
Since $DA \parallel CB$ and $AB \parallel DC$, triangles AEF and CQP are similar. Hence, $\frac{\rho}{r} = \frac{\sigma}{s}$. Then

$$[AEF][CPQ] = (\rho\sigma)(rs) = (\rho s)(r\sigma) = (r\sigma)^2 = [AMON]^2,$$

which completes the proof.

2908. [2004 : 39, 42] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

On the sides of triangle ABC, triangles DBC, ECA, FAB, are constructed externally such that $\angle DBC = \angle DCB = \angle ECA = \angle FBA = \phi$ and $\angle CAE = \angle BAF = \theta$, where $\angle BAC + \theta < 180°$, $\angle ABC + \phi < 180°$ and $\angle ACB + \phi < 180°$.

Prove that $[ABDE] = [CAFD]$. ($[Z_1 Z_1 \ldots Z_n]$ denotes the area of the n–gon $Z_1 Z_2 \ldots Z_n$.)

I. *Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

We first observe that $\triangle BDC$ is isosceles. Therefore, we may set $x = BD = DC$; moreover, $\triangle ABF$ and $\triangle ACE$ are similar, which implies that $BF/CE = c/b$. We also have the following equalities:

$$
\begin{aligned}
[ABDE] &= [ABDCE] - [CED] \\
&= [ABC] + [BDC] + [ACE] - [CED], \\
[CAFD] &= [CAFBD] - [BDF] \\
&= [ABC] + [BDC] + [AFB] - [BDF].
\end{aligned}
$$

Therefore, we will need to prove that

$$[ACE] + [BDF] = [AFB] + [CED]. \tag{1}$$

The relevant areas are

$$[ACE] \ = \ \tfrac{1}{2}b \cdot CE \cdot \sin \phi \,, \qquad [BDF] \ = \ \tfrac{1}{2}x \cdot \tfrac{c}{b} \cdot CE \cdot \sin(2\phi + B) \,,$$

$$[AFB] \ = \ \tfrac{1}{2}c \cdot \tfrac{c}{b} \cdot CE \cdot \sin \phi \,, \qquad [CED] \ = \ \tfrac{1}{2}x \cdot CE \cdot \sin(2\phi + C) \,.$$

Substitution in (1) and simplification reduces the problem to proving that

$$b^2 \sin \phi + xc \sin(2\phi + B) \ = \ c^2 \sin \phi + xb \sin(2\phi + C) \,.$$

Expanding the sines of the angle sums, and observing that

$$xc \cos 2\phi \sin B \ = \ xb \cos 2\phi \sin C$$

(by the Sine Law in $\triangle ABC$), we further reduce the problem to

$$b^2 \sin \phi + xc \sin 2\phi \cos B \ = \ c^2 \sin \phi + xb \sin 2\phi \cos C \,.$$

Since $\sin \phi$ is non-zero (in order for the problem to be non-trivial), and since $2x \cos \phi = a$, the last equation is equivalent to

$$b^2 + ac \cos B \ = \ c^2 + ab \cos C \,,$$

which, by the Cosine Law, reduces to the identity $ab^2 + ac^2 = ac^2 + ab^2$, and we are done.

II. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*
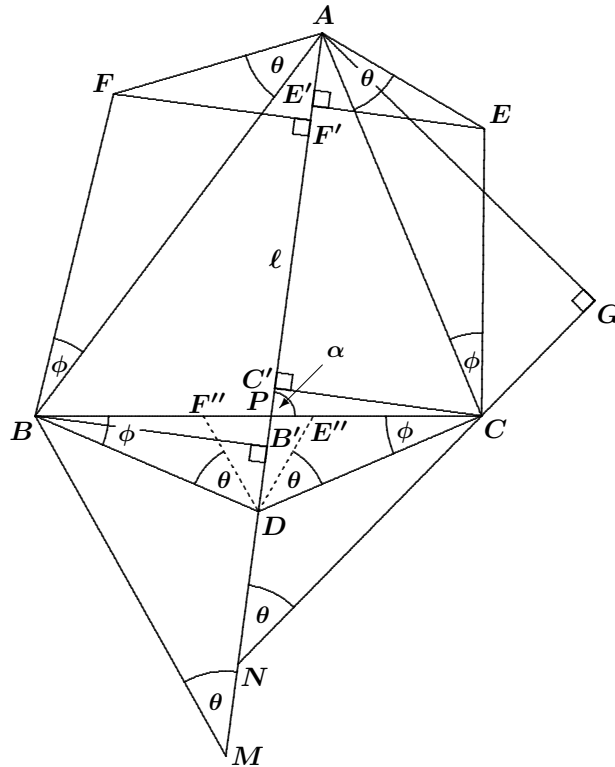
Let AD cut BC at P with $\angle APC = \alpha$. Without loss of generality, we may assume $\alpha < \pi/2$. Let B', C', E', F' be the respective projections of B, C, E, F onto the line AD. We may think of A as a variable point on the fixed line $\ell = PD$, and ϕ, θ, B, and C as being fixed. Therefore, we have

$$[ABDE] \;=\; [ABD] + [ADE] \;=\; \frac{AD \cdot (BB' + EE')}{2} \,,$$

and

$$[AFDC] \;=\; [AFD] + [DCA] \;=\; \frac{AD \cdot (CC' + FF')}{2} \,.$$

Hence, it is sufficient to prove that

$$BB' - CC' \;=\; FF' - EE' \,.$$

Let N be the point of ℓ such that the directed angle from CN to ℓ is θ and let M be the point on ℓ such that the directed angle from ℓ to BM is θ. Let G be the projection of A onto CN. Then $\angle NAG + \angle CAE = \frac{\pi}{2}$. Hence, triangles AGC and $EE'A$ are similar. Thus,

$$EE' \;=\; \frac{AG \cdot AE}{AC} \;=\; \frac{AN \cdot \sin\theta \sin\phi}{\sin(\theta + \phi)} \,,$$

where we have also used the Law of Sines in $\triangle AEC$. Similarly,

$$FF' \;=\; \frac{AM \cdot \sin\theta \sin\phi}{\sin(\theta + \phi)} \,.$$

Hence,

$$FF' - EE' \;=\; \frac{MN \cdot \sin\theta \sin\phi}{\sin(\theta + \phi)} \,,$$

which is independent of the length AN. Without loss of generality, we may therefore move A to D. Then E and F become E'' and F'', respectively, on BC such that triangles BDF'' and CDE'' are congruent. (We shall take the distance $E''P$ to be negative if E'' and F'' lie on the same side of ℓ.) Then

$$
\begin{aligned}
FF' - EE' \;&=\; \sin\alpha(F''P - E''P) \\
&=\; \sin\alpha((BF'' + F''P) - (CE'' + E''P)) \\
&=\; \sin\alpha(BP - CP) \;=\; BB' - CC' \,,
\end{aligned}
$$

which completes the proof.

2909. [2004 : 40, 42] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given a convex quadrilateral $ABCD$, a line parallel to AD meets segments AB, AC, BD, CD, at E, F, G, H, respectively.

Prove that $[EBCF] : [GBCH] = EF : GH$. ($[Z_1 Z_1 \ldots Z_n]$ denotes the area of the n–gon $Z_1 Z_2 \ldots Z_n$.)

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Without loss of generality, we may assume that the distance from C to AD is less than or equal to the distance from B to AD. Let the line through C parallel to AD meet AB and BD at points X and Y, respectively. Then

$$\frac{EF}{XC} = \frac{AE}{AX} = \frac{DH}{DC} = \frac{HG}{CY},$$

which implies that

$$\frac{EF}{GH} = \frac{CX}{CY}.$$
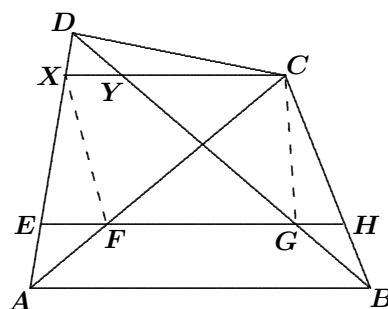
Using triangles of the same height, we obtain

$$\frac{[EFX]}{[GHC]} = \frac{EF}{GH} \quad \text{and} \quad \frac{[XCF]}{[YCG]} = \frac{CX}{CY} = \frac{[XCB]}{[YCB]},$$

where $[\Delta]$ denotes the area of the polygon Δ. Then

$$\frac{[EFX]}{[GHC]} = \frac{[XCF]}{[YCG]} = \frac{[XCB]}{[YCB]} = \frac{EF}{GH}.$$

Thus,

$$\frac{[EBCF]}{[GBCH]} = \frac{[EFX] + [XCF] + [XCB]}{[GHC] + [YCG] + [YCB]} = \frac{EF}{GH},$$

which completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2910. [2004 : 40, 43] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In triangle ABC, let D, E, F, be the mid-points of sides BC, CA, AB, respectively. Let M, N, P be points on the segments FD, FB, DC, respectively, such that $FM : FD = FN : FB = DP : DC$.

Prove that AM, EN, FP are concurrent.

I. *Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

Since every statement in the problem is affine-invariant, we may assume without loss of generality that $\triangle ABC$ is equilateral.

Let $AM \cap EN = X$ and $AM \cap FP = Y$. Then $\triangle AFM \cong \triangle EFN$, and hence, $\angle MAF = \angle NEF$, implying that quadrilateral $AFXE$ is cyclic. Similarly, $\triangle AEM \cong \triangle FEP$ gives $\angle EAM = \angle EFP$, implying that quadrilateral $AFYE$ is cyclic.

It follows immediately that $X = Y$, which means that AM, EN, and FP are concurrent.

II. *Solution by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

Let $\lambda = \dfrac{FM}{FD} = \dfrac{FN}{FB} = \dfrac{DP}{DC}$ (the given ratio). Let $S = AM \cap FE$ and $T = FP \cap AC$. These points of intersection exist, since $0 < \lambda < 1$. Since $\triangle AFE \cong \triangle FBD \cong \triangle EDC \cong \triangle DEF$, we have

$$\frac{FS}{SE} = \frac{FM}{AE} = \frac{FM}{FD} = \lambda. \tag{1}$$
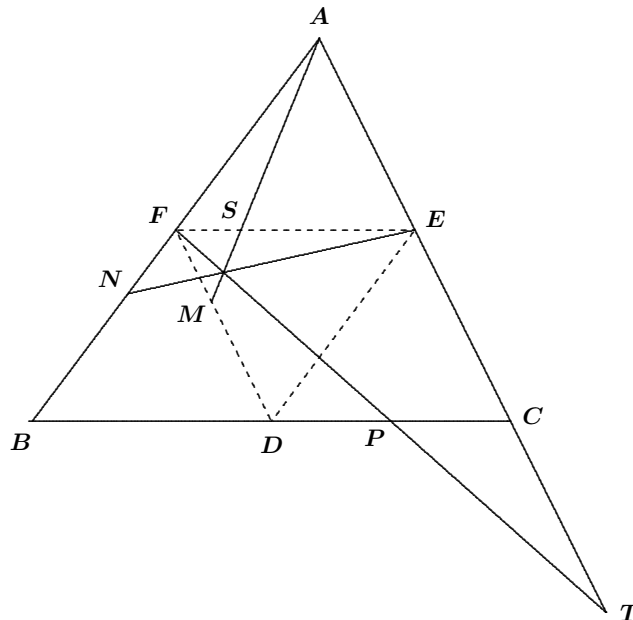
Note that

$$\frac{FD}{CT} \;=\; \frac{DP}{PC} \;=\; \frac{DP}{DC - DP} \;=\; \frac{\dfrac{DP}{DC}}{1 - \dfrac{DP}{DC}} \;=\; \frac{\lambda}{1-\lambda}\,,$$

and hence (taking account of the orientation of the segments),

$$\frac{ET}{TA} \;=\; -\frac{FD + CT}{2\,FD + CT} \;=\; -\frac{\dfrac{FD}{CT} + 1}{2\,\dfrac{FD}{CT} + 1} \;=\; -\frac{\dfrac{1}{1-\lambda}}{\dfrac{1+\lambda}{1-\lambda}} \;=\; -\frac{1}{1+\lambda}\,.$$

Furthermore,

$$\frac{AN}{NF} \;=\; -\frac{AF + FN}{FN} \;=\; -\left(1 + \frac{AF}{FN}\right) \;=\; -\left(1 + \frac{FB}{FN}\right) \;=\; -\frac{1+\lambda}{\lambda}\,.$$

Through the vertices of $\triangle AFE$, we have three cevians—an interior cevian AM and two exterior cevians EN and FT. Using equation (1) along with the last two equations above, we obtain

$$\frac{FS}{SE} \cdot \frac{ET}{TA} \cdot \frac{AN}{NF} \;=\; \lambda \cdot \left(-\frac{1}{1+\lambda}\right) \cdot \left(-\frac{1+\lambda}{\lambda}\right) \;=\; 1\,.$$

By the converse of Ceva's Theorem, the lines AM, EN, and FP are concurrent.

III. *Solution par Jacques Choné, Nancy, France.*

Soit α la valeur commune de rapports définis dans l'énoncé. (Ainsi $\alpha \in [0,1]$.) On trouve que les coordonnées barycentriques (par rapport à (A, B, C)) des points M, N et P sont respectivement

$$\left(\frac{1-\alpha}{2}, \frac{1}{2}, \frac{\alpha}{2}\right),\; \left(\frac{1-\alpha}{2}, \frac{1+\alpha}{2}, 0\right) \text{ et } \left(0, \frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right).$$

Les coordonnées barycentriques d'un point de la droite (A, M) sont donc de la forme $\left((1-x) + x\frac{1-\alpha}{2}, x\frac{1}{2}, x\frac{\alpha}{2}\right)$ et celles d'un point de la droite (E, N) de la forme $\left((1-y)\frac{1}{2} + y\frac{1-\alpha}{2}, y\frac{1+\alpha}{2}, (1-y)\frac{1}{2}\right)$.

On trouve que les coordonnées barycentriques de l'intersection des droites (A, M) et (E, N) sont de la forme précédente avec $x = \dfrac{1+\alpha}{1+\alpha+\alpha^2}$ et $y = \dfrac{1}{1+\alpha+\alpha^2}$. De même les coordonnées barycentriques de l'intersection des droites (A, M) et (F, P) vérifient :

$$\begin{aligned}
&\left((1-x') + x'\frac{1-\alpha}{2}, x'\frac{1}{2}, x'\frac{\alpha}{2}\right) \\
&\quad = \left((1-z)\frac{1}{2}, (1-z)\frac{1}{2} + z\frac{1-\alpha}{2}, z\frac{1+\alpha}{2}\right);
\end{aligned}$$

on trouve ainsi $x' = \dfrac{1 + \alpha}{1 + \alpha + \alpha^2}$ et $z = \dfrac{\alpha}{1 + \alpha + \alpha^2}$.

Puisque $x = x'$ et que x, y et z sont compris entre 0 et 1, les segments AM, EN et FP sont concourants au point de coordonnées barycentriques

$$\left(\frac{1 + \alpha^2}{2(1 + \alpha + \alpha^2)}, \frac{1 + \alpha}{2(1 + \alpha + \alpha^2)}, \frac{\alpha(1 + \alpha)}{2(1 + \alpha + \alpha^2)} \right) .$$

Also solved by AUSTRIAN IMO TEAM 2004; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Bucharest, Romania; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.
Some solvers used vectors, and some used Cartesian methods.
Janous asked what happens if the points D, E, and F are not mid-points but are such that $BD : BC = CE : CA = AF : AB = \mu$, where $\mu \in \mathbb{R}$ and $\mu \notin \{0, \frac{1}{2}, 1\}$. He gave a solution using Cartesian methods. Perhaps our readers can give a proof using one of the featured methods.

2911. [2004 : 40, 43] *Proposed by Mihály Bencze, Brasov, Romania.*

(a) If z, $w \in \mathbb{C}$ and $|z| = 1$, prove that

$$(n - 1) \sum_{k=1}^{n} |w + z^k| \geq \sum_{k=1}^{n-1} (n - k) |1 - z^k| .$$

(b) If $x \in \mathbb{R}$, prove that

$$(n - 1) \sum_{k=1}^{n} |\cos(kx)| \geq \sum_{k=1}^{n-1} (n - k) |\sin(kx)| .$$

Solution by Michel Bataille, Rouen, France.

(a) Let z, $w \in \mathbb{C}$, with $|z| = 1$. For all positive integers j and k,

$$|1 - z^k| = |z^j| \cdot |1 - z^k| = |z^j - z^{j+k}| \leq |z^j + w| + |z^{j+k} + w| ,$$

using the Triangle Inequality. Then, for all positive integers n and k with $k < n$,

$$(n - k)|1 - z^k| = \sum_{j=1}^{n-k} |1 - z^k| \leq \sum_{j=1}^{n-k} \left(|z^j + w| + |z^{j+k} + w| \right)$$

$$= \sum_{j=1}^{n-k} |z^j + w| + \sum_{j=k+1}^{n} |z^j + w| .$$

Summing over k from 1 to $n - 1$, and making the change of index $\ell = n - k$ in the second double sum on the right side, we obtain

$$\sum_{k=1}^{n-1}(n-k)|1-z^k| \;\leq\; \sum_{k=1}^{n-1}\sum_{j=1}^{n-k}|z^j+w| + \sum_{\ell=1}^{n-1}\sum_{j=n-\ell+1}^{n}|z^j+w|$$

$$= \sum_{k=1}^{n-1}\sum_{j=1}^{n}|z^j+w| \;=\; (n-1)\sum_{j=1}^{n}|z^j+w|\,.$$

(b) Letting $z = e^{-2ix}$ and $w = 1$ in (a) yields

$$\sum_{k=1}^{n-1}(n-k)|1-e^{-2ikx}| \;\leq\; (n-1)\sum_{k=1}^{n}|1+e^{-2ikx}|\,.$$

The required result follows, since

$$|1 + e^{-2ikx}| \;=\; \left|e^{-ikx}(e^{ikx}+e^{-ikx})\right| \;=\; |e^{ikx}+e^{-ikx}| \;=\; 2|\cos(kx)|\,,$$

and similarly $|1 - e^{-2ikx}| = 2|\sin(kx)|$.

Also solved by ARKADY ALT, San Jose, CA, USA; JACQUES CHONÉ, Nancy, France; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2912. [2004 : 40, 43] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\lim_{n\to\infty}\frac{1}{\sqrt[n]{n!}}\log_a\left(\sum_{k=1}^{a^n}(1+k)^{a^{-n}}\right) \;=\; e\,,$$

where $a \geq 2$ is a positive integer.

Solution by Michel Bataille, Rouen, France.

Since

$$\lim_{n\to\infty}\ln\left(\sqrt[n]{\frac{1}{n}\cdot\frac{2}{n}\cdots\frac{n}{n}}\right) \;=\; \lim_{n\to\infty}\frac{1}{n}\left(\sum_{k=1}^{n}\ln(k/n)\right)$$

$$= \int_0^1\ln(x)\,dx \;=\; -1\,,$$

we have $\dfrac{1}{\sqrt[n]{n!}} \sim \dfrac{e}{n}$ as $n \to \infty$. Thus, it suffices to prove that

$$\log_a\left(\sum_{k=1}^{a^n}(1+k)^{a^{-n}}\right) \;\sim\; n \quad\text{as}\quad n\to\infty\,. \tag{1}$$

Now,

$$a^n 2^{1/a^n} \leq \sum_{k=1}^{a^n} (1+k)^{a^{-n}} \leq a^n(1+a^n)^{1/a^n},$$

and therefore, using the fact that $\log_a$ is increasing,

$$n + \frac{\log_a(2)}{a^n} \leq \log_a\left(\sum_{k=1}^{a^n}(1+k)^{a^{-n}}\right) \leq n + \frac{n}{a^n} + \frac{1}{a^n}\log_a\left(1+\frac{1}{a^n}\right).$$

The squeeze principle now shows that

$$\lim_{n\to\infty} \frac{1}{n}\log_a\left(\sum_{k=1}^{a^n}(1+k)^{a^{-n}}\right) = 1;$$

that is, (1).

Also solved by MIHÁLY BENCZE, Brasov, Romania; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2913. [2004 : 40, 43] *Proposed by Mihály Bencze, Brasov, Romania.*

If a, b, $c > 1$ and $\alpha > 0$, prove that

$$a^{\sqrt{\alpha\log_a b}+\sqrt{\alpha\log_a c}} + b^{\sqrt{\alpha\log_b a}+\sqrt{\alpha\log_b c}} + c^{\sqrt{\alpha\log_c a}+\sqrt{\alpha\log_c b}}$$
$$\leq \sqrt{abc}\left(a^{\alpha-\frac{1}{2}} + b^{\alpha-\frac{1}{2}} + c^{\alpha-\frac{1}{2}}\right).$$

Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; and Bogdan Ioniţă and Titu Zvonaru, Bucharest, Romania.

Since a, b, $c > 1$ and $\alpha > 0$, the AM–GM Inequality implies that

$$\sqrt{\alpha\log_a b} + \sqrt{\alpha\log_a c} \leq \frac{\alpha+\log_a b}{2} + \frac{\alpha+\log_a c}{2}$$
$$= \alpha + \tfrac{1}{2}\log_a bc = \alpha + \log_a\sqrt{bc}$$
$$= \alpha - \tfrac{1}{2} + \log_a\sqrt{bc} + \log_a\sqrt{a}$$
$$= \alpha - \tfrac{1}{2} + \log_a\sqrt{abc},$$

where equality occurs if and only if $b = c = a^\alpha$. Since $a > 1$,

$$a^{\sqrt{\alpha\log_a b}+\sqrt{\alpha\log_a c}} \leq a^{\alpha-\frac{1}{2}+\log_a\sqrt{abc}} = a^{\alpha-\frac{1}{2}}\sqrt{abc},$$

with equality if and only if $b = c = a^\alpha$. Similarly,

$$b^{\sqrt{\alpha \log_b a} + \sqrt{\alpha \log_b c}} \leq b^{\alpha - \frac{1}{2}}\sqrt{abc},$$

with equality if and only if $a = c = b^\alpha$, and

$$c^{\sqrt{\alpha \log_c a} + \sqrt{\alpha \log_c b}} \leq c^{\alpha - \frac{1}{2}}\sqrt{abc},$$

with equality if and only if $a = b = c^\alpha$. Therefore,

$$a^{\sqrt{\alpha \log_a b} + \sqrt{\alpha \log_a c}} + b^{\sqrt{\alpha \log_b a} + \sqrt{\alpha \log_b c}} + c^{\sqrt{\alpha \log_c a} + \sqrt{\alpha \log_c b}}$$
$$\leq \sqrt{abc}\left(a^{\alpha - \frac{1}{2}} + b^{\alpha - \frac{1}{2}} + c^{\alpha - \frac{1}{2}}\right),$$

with equality if and only if $a = b = c$ and $\alpha = 1$.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.