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CONTENTS

EDITOR'S COMMENT ON AN ARTICLE IN CRUX

The paper "A problem on lattice points", by B. Leeb and C. Stahlke, which appeared in the April 1987 issue [1987: 104-106], contained the result that *among any 19 points in* $\mathbb{Z}^3$ *there are always three with integer centroid.* I have since been informed that this result was already proved by H. Harborth in 1973 (Ein Extremalproblem für Gitterpunkte, *J. Reine Angew. Math.* 262/263, 356-360) and had also appeared as problem 6298 in the *Amer. Math. Monthly* (solution on pp.279-280 of the April 1982 issue). A more general problem, with references, occurs as problem #93 in W. Moser's *Research Problems in Discrete Geometry*.

I thank those readers, especially W. Moser, who brought these facts to my attention, and apologize for not including them in the published paper. Nevertheless, I hope that the paper's intended purpose, to illustrate several combinatorial techniques for solving problems, was served.

* * *

THE OLYMPIAD CORNER: 90

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this column with some further problems forwarded by Bruce Shawyer that were posed but not used for the 1987 I.M.O. I apologize for any errors in translation or resulting from my editing of the problems.

Belgium 1. Twenty-eight random draws are made from the set

$$\{1,2,3,4,5,6,7,8,9,A,B,C,D,J,K,L,U,X,Y,Z\}$$

containing 20 elements. What is the probability that the sequence

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occurs in that order in the chosen sequence?

Bulgaria 1. A perpendicular dropped from the centre of the circumcircle to the bisector of the angle C of the triangle ABC divides it in the ratio λ. Determine the length of the third side of the triangle if $AC = b$ and $BC = a$. [*Editor's note:* I hope that the circumcircle is the correct circle.]

Bulgaria 2. Let $P_1, P_2, \ldots, P_n$ be n points in the plane. What is the smallest number of segments P_iP_j to join and how should they be selected so that whenever 4 points are chosen a triangle is formed by three of them (with edges amongst the selected segments)?

Bulgaria 3. Find all whole number solutions of the equation

$$[n\sqrt{2}] = [2 + m\sqrt{2}].$$

(Of course $[x]$ means the integer part of x.)

Bulgaria 4. Let $P_1, \ldots, P_{2n+3}$ be $2n + 3$ points selected in the plane so that no four lie on a circle (or on a straight line). Let k be the number of circles (and straight lines) containing three of the points and which partition the remaining $2n$ points equally, with n on each side. Show that

$$k > \frac{1}{\pi}\binom{2n+3}{2}$$

where the binomial coefficient $\binom{2n+3}{2} = \frac{(2n+3)(2n+2)}{2}$.

Finland 1. Let A be an infinite set of integers such that every $a \in A$ is the product of at most 1987 prime numbers. Prove that there is an infinite set $B \subset A$ and a number p such that the greatest common divisor of any two numbers in B is p.

France 1. For each whole number $k > 0$ let $a^k_{n_k} \ldots a^k_0$ $(a^k_{n_k} \neq 0)$ denote the decimal representation of $(1987)^k$. (Thus $n_0 = 0$ and $a^0_{n_0} = 1$ since $(1987)^0 = 1$. Also, $n_1 = 3$, $a^1_3 a^1_2 a^1_1 a^1_0 = 1987$, etc.) Form the infinite decimal

$$x = 0.1\ 1987\ \ a^2_{n_2} \ldots a^2_0\ \ \ldots\ \ a^k_{n_k} \ldots a^k_0\ \ldots\ .$$

Show that x is irrational.

Poland 1. Let ABC be a fixed non-equilateral triangle with the vertices listed anticlockwise. Find the locus of the centroids of those equilateral triangles $A'B'C'$ (the vertices listed anticlockwise) for which the points A, B', C', (respectively A', B, C' and A', B', C) are collinear.

U.S.S.R. 1. The positive quantities α, β and γ are such that $\alpha + \beta + \gamma < \pi$. Prove that a triangle can be formed from segments of length $\sin\alpha$, $\sin\beta$, and $\sin\gamma$ such that the area of the triangle does not exceed

$$(\sin 2\alpha + \sin 2\beta + \sin 2\gamma)/8.$$

U.S.S.R. 2. For each natural number $k \geq 2$ the sequence $a_n(k)$ is generated according to the rule

$$a_0 = k, \quad a_n = \tau(a_{n-1}) \ , \quad n = 1,2,3,\ldots,$$

where $\tau(a)$ is the number of positive integral divisors of a. Find all k for which the sequence $a_n(k)$ does not contain squares of whole numbers.

<u>U.S.S.R. 3</u>. Find the largest value of the expression

$$(a + b)^4 + (a + c)^4 + (a + d)^4 + (b + c)^4 + (b + d)^4 + (c + d)^4$$

where a, b, c and d are real numbers satisfying

$$a^2 + b^2 + c^2 + d^2 \leq 1.$$

* * *

We now turn to solutions to problems posed in past issues of the Corner.

<u>14</u>. [1985: 38] *Proposed by Mongolia.*

Show that there exist distinct natural numbers $n_1, n_2, \ldots, n_k$ such that

$$\pi^{-1984} < 25 - \left(\frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k}\right) < \pi^{-1960}.$$

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Let a and b be given real numbers, $0 < a < b$. We choose an integer n such that $n \geq 1/a$ and $n \geq 1/(b - a)$. Since $1/n \leq a$ and the harmonic series diverges, there is a largest non-negative integer x such that

$$\sum_{i=0}^{x} \frac{1}{n + i} \leq a.$$

Using $n \geq 1/(b - a)$, it follows that

$$a < \sum_{i=0}^{x+1} \frac{1}{n + 1} \leq a + \frac{1}{n + x + 1} < a + \frac{1}{n} \leq b.$$

In our problem, we set $a = 25 - \pi^{-1960}$, $b = 25 - \pi^{-1984}$ and choose n appropriately; then, with x as defined above, the $x + 2$ distinct natural numbers $n, n + 1, \ldots, n + x, n + x + 1$ solve the problem.

<u>19</u>. [1985: 38] *Proposed by Sweden.*

Let a and b be integers. Is it possible to find integers p and q such that the integers $p + na$ and $q + nb$ are relatively prime for any integer n?

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

If $a = 0$ we set $p = q = 1$ and then $\gcd(p + na, q + nb) = 1$, for all n. Similarly, we set $p = q = 1$ if $b = 0$. So we may assume $ab \neq 0$.

Let

$$r = \frac{\mathrm{lcm}(a,b)}{a}, \quad s = \frac{\mathrm{lcm}(a,b)}{b}.$$

Then $\gcd(r,s) = 1$ so there exist integers x, y such that $rx - sy = 1$. Set $p = x$, $q = y$. For any integer n, define $d_n = \gcd(p + na, q + nb)$. Then

$$d_n | r(p + na) - s(q + nb),$$

so

$$d_n | (rp - sq) + n(ra - sb).$$

Also

$$d_n | (rx - sy) + n(\mathrm{lcm}(a,b) - \mathrm{lcm}(a,b)).$$

Hence

$$d_n | 1.$$

Thus $\gcd(p + na, g + na) = 1$ for all integers n.

<u>22</u>. [1985: 38] *Proposed by the U.S.A.*

Determine all pairs (a,b) of positive real numbers with $a \neq 1$ such that

$$\log_a b < \log_{a+1}(b + 1).$$

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Define

$$f(x) \equiv \frac{\ln x}{\ln(x + 1)} \quad \text{for} \quad x > 0.$$

Then

$$f'(x) = \frac{\frac{\ln(x + 1)}{x} - \frac{\ln x}{x + 1}}{(\ln(x + 1))^2} = \frac{(x + 1)\ \ln(x + 1) - x\ln x}{x(x + 1)(\ln(x + 1))^2}.$$

Since, for $x > 0$,

$$(x + 1)\ln(x + 1) > x \ln(x + 1) > x \ln x,$$

$f'(x) > 0$ for $x > 0$. Thus f is increasing for $x > 0$. Thus, if $x,y > 0$, $f(x) > f(y)$ iff $x > y$.

The inequality $\log_a b < \log_{a+1}(b + 1)$ is equivalent to

$$\frac{\ln b}{\ln a} < \frac{\ln(b + 1)}{\ln(a + 1)}. \tag{*}$$

If $a > 1$, we multiply (*) by the positive quantity $\dfrac{\ln a}{\ln(b + 1)}$ to obtain the equivalent inequality

$$f(b) < f(a), \quad \text{or} \quad b > a.$$

If $0 < a < 1$, we again multiply (*) by $\dfrac{\ln a}{\ln(b + 1)}$, but now obtain $f(b) > f(a)$, which is equivalent to $b > a$. Therefore, the set of all admissible pairs is

$$\{(a,b)\colon a > b > 0,\ a > 1\} \cup \{(a,b)\colon a < b,\ 0 < a < 1\}.$$

<u>24</u>. [1985: 39] *Proposed by the U.S.S.R.*

The proper divisors of the natural number n are arranged in increasing order, $x_1 < x_2 < \ldots < x_k$. Find all numbers n such that

$$x_5^2 + x_6^2 - 1 = n.$$

<u>5</u>. [1985: 169] *The 1985 Australian Mathematical Olympiad, Paper II, March 13, 1985.*

Find all positive integers n such that

$$n = d_6^2 + d_7^2 - 1,$$

where $1 = d_1 < d_2 < \ldots < d_k = n$ are all positive divisors of the number n.

Solution by John Morvay, Dallas, Texas, U.S.A. and R.E.W.

The two problems have the same solution since for the former $x_1, x_2, \ldots, x_k$ lists only the proper divisors of n. We present the solution in the notation of 24 [1985: 39].

From $x_5^2 + x_6^2 - 1 = n$ and $x_5|n$, $x_6|n$ we conclude that x_5 and x_6 are coprime and $n = tx_5x_6$. Transforming the equation we have

$$(x_5 - 1)(x_5 + 1) = n - x_6^2 = x_6(tx_5 - x_6)$$

from which we observe that either $x_6 = x_5 + 1$ or x_6 has prime divisors p, q (possibly equal) dividing $x_5 - 1$ and $x_5 + 1$, respectively. The argument breaks down into two main cases.

Case I. $x_6 = ab$ with $1 < a,b < x$ and $\gcd(a,b) = 1$;

Case II. $x_6 = p^\alpha$ for some prime p.

We show first that Case I is not possible. Then in Case II we shall find two solutions. In both, $x_6 = x_5 + 1$, but we have no easy direct justification of this fact. In each case we consider several subcases.

Case I. Let $x_6 = ab$ with $1 < a,b < x_6$ and $\gcd(a,b) = 1$. Note that $a,b < x_5$.

Subcase I.1. x_5 is not a prime power. Write $x_5 = cd$ with $1 < c,d$ and $\gcd(c,d) = 1$. Now a, b, c, d are pairwise mutually coprime and $\{x_1,x_2,x_3,x_4\} = \{a,b,c,d\}$. Now we cannot have $c \geq a,b$ *and* $d \geq a,b$ (for otherwise $x_5 \geq x_6$). Without loss, let $c < a$. Then $cb < x_6$ is a divisor of n which doesn't divide x_5. However, it is not among a, b, c, d, showing this subcase cannot arise.

Subcase I.2. $x_5 = q^\alpha$, q a prime. If both a and b are prime we must have $\alpha = 3$. Then $q < a$ or $q < b$. Without loss $q < a$. Then qb is a divisor of n which must occur among $\{x_1,x_2,x_3,x_4\}$, which is impossible as $\{a,b,q,q^2\}$ exhaust the list.

Thus we may assume that a is not prime, and write $a = a_1a_2$ with $1 < a_1 \leq a_2$. (Note we allow $a_1 = a_2$.) Then $a_1b < x_6$ so, as above, a_1, $a_1a_2 = a$, b, a_1b are distinct divisors of n which make up $\{x_1,x_2,x_3,x_4\}$ in some order. Then $\alpha = 1$ and b, a_1 are prime and $a_1 = a_2$.

Now $x_5 = q$ is prime and $q > a_1b$ so q is odd. Also x_5 divides $(x_6 - 1)(x_6 + 1)$ so $x_6 = kx_5 + 1$ or $x_6 = kx_5 - 1$. Now

$$a_1^2, b, a_1b < x_5 = q < x_6 = a_1^2b = kq \pm 1$$

implies that $kq \leq aq, bq$ whence $k \leq a, b < q$. From this it is easy to see that $k = 1$ and $x_6 = q + 1$ is the only possibility for $kq \pm 1 = x_6$ to divide $q^2 - 1$. First suppose $(kq - 1)\ell = q^2 - 1$. Then $(k\ell - q)q = \ell - 1$ whence $k\ell = q$, $\ell = 1$ and $k = q$, which contradicts $k < q$; or $q|\ell - 1$, which is impossible unless $k = 1$ (else $kq - 1 \geq q$, $\ell > q$ which gives $(kq - 1)\ell > q^2$). However $k = 1$ is ruled out since $x_6 = x_5 - 1$ violates the order of x_5 and x_6. Next suppose $(kq + 1)\ell = q^2 - 1$. This gives $(k\ell - q)q = -(\ell + 1)$ which yields $k\ell = q$ and $\ell = -1$, $k = q$, an impossibility, or $q|\ell + 1$ whence $\ell = q - 1$, $b = 1$.

Now $x_6 = q + 1$, q odd means that $2|x_6$. But x_5 odd and x_6 even entails from $(x_5^2 - 1) + x_6^2 = n$ that $4 < x_5$, so $a_1 = 2$ and $q = 4b - 1$. Now, however,

$$(4b - 1)^2 + (4b)^2 - 1 = n$$

gives $8|n$. Since b is an odd prime we have $8 \leq 4b$, contradicting that $\{x_1,x_2,x_3,x_4,x_5,x_6\}$ is a_1, a_1^2, b, a_1b, $a_1^2b - 1$, a_1^2b in some order.

This finally shows that Case I cannot arise.

We now turn to

Case II. $x_6 = p^\alpha$, p a prime.

Now $p^\alpha|(x_5 - 1)(x_5 + 1)$ implies $p^\alpha = x_5 + 1$ (since $p^\alpha \geq x_5 + 1$) unless $p = 2$.

Subcase II.1. $p = 2$.

We now consider the possible values of α. Clearly $\alpha \leq 5$.

Subcase II.1(i). $\alpha = 5$. Then $\{x_1,x_2,x_3,x_4\} = 2, 4, 8, 16$ and x_5 is a prime, $16 < x_5 < 32$. Now x_5 divides $x_6^2 - 1 = 31\cdot33$. This gives $x_5 = 31$. Indeed $1984 = 31\cdot64$ is a solution.

Subcase II.1(ii). $\alpha \leq 4$. Then x_5 is not prime. The only composite numbers less than 16 not divisible by 2 are 9 and 15. This implies $\alpha = 4$. However $x_5 = 9$ gives 7 divisors 2, 3, 4, 6, 8, 9, 16 and $x_5 = 15$ gives 9 divisors, 2, 3, 4, 6, 8, 10, 12, 15, 16.

The remaining possibility is

Subcase II.2. $x_5 + 1 = x_6 = p^\alpha$, where p is an odd prime. Then

$$n = p^{2\alpha} - 2p^\alpha + p^2 = 2p^\alpha(p^\alpha - 1).$$

Thus 4 divides n. Therefore $\alpha \leq 3$.

Subcase II.2(i). $\alpha = 1$. Then $x_6 = p$ is prime. Write $x_5 = p - 1 = 2^{\beta}c$ where c is odd. Now if $c > 1$ we either have $c = c_1c_2$ with $c_1 \le c_2$, leaving no place for $2c_1$ among 2, 4, c_1, c or we find $\beta = 3$ and no place for $2c$ among 2, 4, 8, c, again impossible. Thus $c = 1$ and $\beta = 5$. But $2^5 + 1 = 33$ which is not prime.

Subcase II.2(ii). $\alpha = 2$. Now $p \ge 5$ implies $2p$ and $4p$ are less than p^2, giving 5 divisors 2, 4, p, $2p$, $4p$ less than $x_1 = p^2 - 1$. Thus $p = 3$. Now 144 is seen to be a solution with divisors 2,3,4,6,8,9,... .

Subcase II.2(iii). $\alpha = 3$. Clearly $2p$ is not among 2, 4, p, p^2 and this case is impossible.

In summary, the only two values for n are 144 and 1984.

[*Editor's note:* The above argument is based on a solution submitted by Mr. Morvay, that was flawed by an oversight. I hope that I have not furnished an unduly complicated version.]

<u>31</u>. [1985: 71] *Proposed by Bulgaria.*

Prove that, for every natural number n, the binomial coefficient $\binom{2n}{n}$ divides the least common multiple of the numbers $1,2,3,\ldots,2n$.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Let n be a given natural number. Let P_i, $1 \le i \le k$ be all the primes less than or equal to $2n$, and for each i, let a_i be the largest integer s such that $P_i^s \le 2n$. Then

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \prod_{i=1}^{k} \left[P_i^{\sum_{j=1}^{a_i}([2n/P_i^j]-2[n/P_i^j])} \right]$$

where $[x]$ is the greatest integer in x.

Also

$$\mathrm{lcm}(1,2,\ldots,2n) = \sum_{i=1}^{k} P_i^{a_i} .$$

Thus it suffices to prove

$$\sum_{j=1}^{a} ([2n/P_i^j] - 2[n/P_i^j]) \le a_i .$$

Now let x be an arbitrary non-negative real number; then $x = [x] + \{x\}$ with $0 \le \{x\} < 1$. Hence

$$[2x] - 2[x] = [2[x] + 2\{x\}] - 2[x]$$

is 0 or 1. Therefore

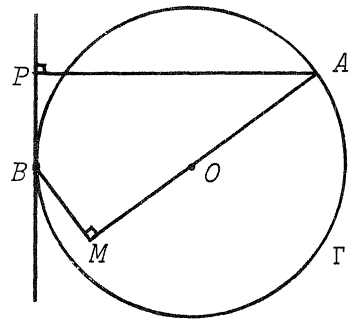
$$\sum_{j=1}^{a_i} ([2n/P_i^j] - 2[n/P_i^j]) \le \sum_{j=1}^{a_i} 1 = a_i ,$$

as required.

<u>33</u>. [1985: 71] *Proposed by Bulgaria.*

Given are a circle Γ and a line ℓ tangent to it at B. From a point A on Γ, a line $AP \perp \ell$ is constructed, with $P \in \ell$. If the point M is symmetric to P with respect to AB, determine the locus of M as A ranges on Γ.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.



Let O be the center of the circle. From isosceles triangle ΔAOB, we have

$$\begin{aligned} \angle BAO &= \angle ABO \\ &= \angle PAB \quad \text{(since } OB \| AP\text{)} \\ &= \angle BAM \quad \text{(by construction of } M\text{).} \end{aligned}$$

Hence A, O, M are collinear, and since $\angle APB = \angle AMB = 90^\circ$, also $\angle OMB = 90^\circ$. The locus of points M with $\angle OMB = 90^\circ$ is the circle with OB as diameter, and so the desired locus lies on this circle.

Conversely, if M lies on this circle, $\angle OMB = 90^\circ$. Let A' be either point of intersection of the line through OM with Γ, and let P' be such that $A'P' \perp \ell$. Then $\angle BMA' = \angle BMO = 90^\circ$ and

$$\begin{aligned} \angle BA'M &= \angle BA'O \quad \text{(by construction of } A'\text{)} \\ &= \angle A'BO \quad \text{(from isosceles } \Delta A'BO\text{)} \\ &= \angle P'A'B \quad (OB \| A'P'). \end{aligned}$$

Also, $A'B = A'B$ so $\Delta P'A'B \equiv \Delta MA'B$ and M is the reflection of P' in $A'B$. Thus the locus is the entire circle on OB as diameter.

<u>57</u>. [1985: 103] *Proposed by Canada.*

Let m and n be nonzero integers. Prove that $4mn - m - n$ can be a square infinitely often, but that this is never a square if either m or n is positive.

Solution by Daniel Ropp, Washington, St. Louis, MO, U.S.A.

For any integer k, let $m = 2k - 5k$, $n = -1$. Then

$$4mn - m - n = 4(5k^2 - 2k) + (5k^2 - 2k) + 1 = (5k - 1)^2$$

and so $4mn - m - n$ can be square infinitely often.

Suppose now that $mn \neq 0$, $n > 0$ and $4mn - m - n = k^2$, a square. Then

$$k^2 = m(4n - 1) - n \equiv -n \bmod (4n - 1)$$

and

$$(2k)^2 \equiv -4n \equiv -1 \bmod (4n - 1).$$

Now $4n - 1$ is odd and positive and so it cannot have prime factors only of the form $4k + 1$. Thus, there is an integer s and a prime p, with $p = 4s - 1$ and

$p|4n - 1$. Then

$$(2k)^2 \equiv -1 \bmod p, \quad \text{and so} \quad (-1/p) = 1,$$

where (a/p) is the Legendre symbol. But then

$$1 = (-1/p) \equiv (-1)^{(p-1)/2} \bmod p \equiv (-1)^{2s-1} \bmod p \equiv -1 \bmod p.$$

Thus, $2 \equiv 0 \bmod p$. This is a contradiction. Hence n, and similarly m, cannot be positive.

61. [1985: 103] *Proposed by the Federal Republic of Germany.*

You start with a white balls and b black balls in a container and proceed as follows:

Step 1. You draw one ball at random from the container (each ball being equally likely). If the ball is white, then stop.

Step 2. If the drawn ball is black, then add two black balls to the balls remaining in the container and repeat Step 1.

Let s denote the number of draws until stop. For the cases $a = b = 1$ and $a = b = 2$ only, determine

$$a_n = \Pr(s = n)\ , \quad b_n = \Pr(s \le n), \quad \lim_{n\to\infty} b_n,$$

and the expectation $E(s) = \sum_{n\ge1} na_n$.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

The procedure stops after the nth draw iff the nth draw is a white ball while each preceding draw is a black ball. By the instructions of Step 2, for $1 \le k \le n - 1$, there are a white balls and $b + k$ black balls in the container after the kth draw. Hence

$$a_n = \Pr(a = n) = \frac{a}{a + b + n - 1} \prod_{s=1}^{n-1} \frac{b + j - 1}{a + b + j - 1}$$

$$= \frac{a\binom{a + b - 1}{b - 1}}{(a + 1)\binom{a + b + n - 1}{a + 1}}$$

where, as usual, the binomial coefficient

$$\binom{m}{k} = \frac{m!}{k!(m - k)!}\ .$$

(i) For $a = b = 1$,

$$a_n = \frac{1}{n(n + 1)}$$

$$b_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k + 1}\right) = 1 - \frac{1}{n + 1}$$

and

$$\lim_{n\to\infty} b_n = 1.$$

Also $E(s) = \sum_{n=1}^{\infty} na_n = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$

(ii) For $a = b = 2$,

$$a_n = \frac{12}{(n+1)(n+2)(n+3)},$$

$$b_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left[\left[\frac{6}{k+1} - \frac{6}{k+2}\right] - \left[\frac{6}{k+2} - \frac{6}{k+3}\right]\right]$$

$$= \left[3 - \frac{6}{n+2}\right] - \left[2 - \frac{6}{n+3}\right] = 1 - \frac{6}{(n+2)(n+3)},$$

$\lim_{n\to\infty} b_n = 1$, of course, and

$$E(s) = \sum_{n=1}^{\infty} na_n = \sum_{n=1}^{\infty} \left[\left[\frac{6}{n+2} - \frac{6}{n+1}\right] + \left[\frac{18}{n+2} - \frac{18}{n+3}\right]\right]$$

$$= -3 + 6 = 3.$$

72. [1985: 105] *Proposed by Spain.*

Let P be a convex n-gon with equal interior angles, and let $\ell_1, \ell_2, \ldots, \ell_n$ be the lengths of its consecutive sides. Prove that a necessary and sufficient condition for P to be regular is that

$$\frac{\ell_1}{\ell_2} + \frac{\ell_2}{\ell_3} + \ldots + \frac{\ell_n}{\ell_1} = n.$$

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

If P is regular, $\ell_i = \ell_j$, $1 \le i,j \le n$ so

$$\sum_{i=1}^{n} \frac{\ell_i}{\ell_{i+1}} = \sum_{i=1}^{n} 1 = n \quad (\text{where } \ell_{n+1} = \ell_1).$$

Conversely, suppose $\sum_{i=1}^{n} \frac{\ell_i}{\ell_{i+1}} = n$. By the AM-GM inequality

$$\sum_{i=1}^{n} \frac{\ell_i}{\ell_{i+1}} \ge n\left[\prod_{i=1}^{n} \frac{\ell_i}{\ell_{i+1}}\right]^{1/n} = n$$

with equality just in case $\frac{\ell_i}{\ell_{i+1}} = \frac{\ell_j}{\ell_{j+1}}$ for $1 \le i,j \le n$. Since equality does hold, by assumption,

$$\left[\frac{\ell_i}{\ell_{i+1}}\right]^n = \prod_{j=1}^{n} \frac{\ell_j}{\ell_{j+1}} = 1 \quad \text{or} \quad \ell_i = \ell_{i+1}, \quad 1 \le i \le n.$$

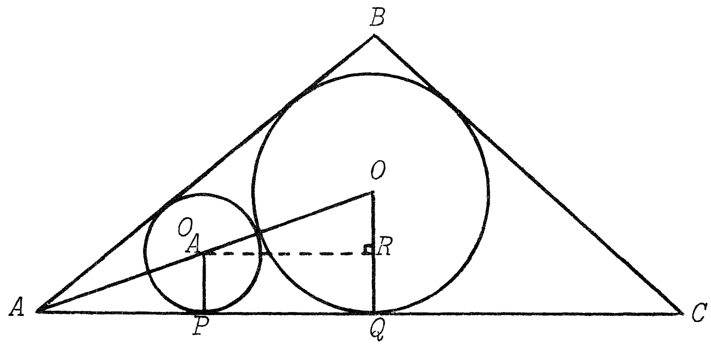
Hence all the ℓ_i are equal, and since the interior angles of P are equal, P is regular.

<u>75</u>. [1985: 105] *Proposed by the U.S.A.*

Inside triangle ABC, a circle of radius 1 is externally tangent to the incircle and tangent to sides AB and AC. A circle of radius 4 is externally tangent to the incircle and tangent to sides BA and BC. A circle of radius 9 is externally tangent to the incircle and tangent to sides CA and CB. Determine the inradius of the triangle.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Let O be the incenter; O_A the center of the circle tangent to AB, AC, and the incircle, P and Q the feet of the altitudes from O_A and O, respectively, to AC, and R the foot of the perpendicular from O_A to OQ.



Since O_A and O are each equidistant from AB and AC, O_A, O, and A are collinear. Let s denote the semiperimeter of ΔABC. Then it is known (and easy to prove) that

(i) $AQ = s - a$

and

(ii) $r^2 s = (s - a)(s - b)(s - c)$

where r is the inradius of ΔABC. Let r_A, r_B and r_C denote the radii of the circles tangent to the incircle and to the two sides of ΔABC passing through A, B, C, respectively. Then

$$O_AP = r_A, \quad OR = OQ - RQ = r - r_A, \quad OO_A = r + r_A,$$

and

$$O_AR = [(OO_A)^2 - (OR)^2]^{1/2} = 2\sqrt{rr_A}\ .$$

By similar triangles ΔAPO_A and ΔAQO,

$$\frac{r_A}{r} = \frac{AP}{AQ} = \frac{AQ - O_AR}{AQ} = 1 - \frac{2\sqrt{rr_A}}{s - a}\ .$$

(Note $OR > 0$ just in case $r > r_A$.) Therefore

$$
\left.\begin{array}{l}
s - a = \dfrac{2\sqrt{rr_A}}{1 - r_A/r}\,. \\[2ex]
\text{Similarly} \\[1ex]
s - b = \dfrac{2\sqrt{rr_B}}{1 - r_B/r}\,, \quad s - c = \dfrac{2\sqrt{rr_C}}{1 - r_C/r}\,.
\end{array}\right\} \qquad (*)
$$

Now (ii) implies

$$(s - a)(s - b)(s - c) = r^2 s = r^2(s - a + s - b + s - c).$$

With (*) this can be written

$$(\sqrt{r_A} + \sqrt{r_B} + \sqrt{r_C})r^2 - (r_B\sqrt{r_A} + r_A\sqrt{r_B} + r_B\sqrt{r_C} + r_C\sqrt{r_B} + r_C\sqrt{r_A} + r_A\sqrt{r_C})r$$
$$+ r_A r_B r_C\left[\frac{1}{\sqrt{r_A}} + \frac{1}{\sqrt{r_B}} + \frac{1}{\sqrt{r_C}}\right] = 0.$$

If $r_A = 1$, $r_B = 4$ and $r_C = 9$ this is equivalent to

$$6r^2 - 72r + 66 = 6(r - 1)(r - 11) = 0.$$

Since $r > r_C = 9$ we have $r = 11$.

* * *

In next month's column we will continue with solutions to problems posed in 1985 that were received in response to the appeal issued last spring. I hope to receive elegant solutions to the Olympiad problems that have appeared this fall.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before July 1, 1988, although solutions received after that date will also be considered until the time when a solution is published.

1291. *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Evaluate

$$\int_0^{\pi/2} \frac{(\cos x)^{\sin x}}{(\cos x)^{\sin x} + (\sin x)^{\cos x}}\, dx.$$

1292*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

It has been shown (see *Crux* 1083 [1987: 96]) that if A, B, C are the angles of a triangle,

$$\frac{2}{\sqrt{3}} \Sigma \sin A \le \sum \cos\left(\frac{B-C}{2}\right) \le \frac{2}{\sqrt{3}} \Sigma \cos \frac{A}{2},$$

where the sums are cyclic. Prove that

$$\sum \cos\left(\frac{B-C}{2}\right) \le \frac{1}{\sqrt{3}}\left[\Sigma \sin A + \Sigma \cos \frac{A}{2}\right],$$

which if true would imply the right hand inequality above.

1293. *Proposed by Steve Maurer, Swarthmore College, Swarthmore, Pennsylvania and Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

Solve the following "twin" problems (in both problems, O is the center of the circle and $OA \perp AB$).

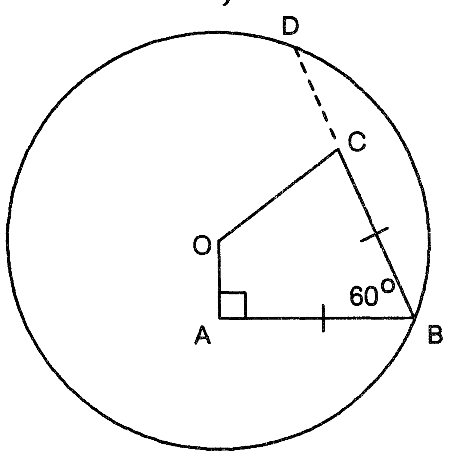


Figure (a)

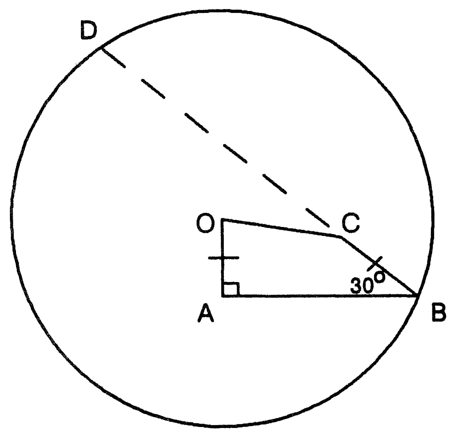


Figure (b)

(a) In Figure (a), $AB = BC$ and $\angle ABC = 60^\circ$. Prove $CD = OA\sqrt{3}$.

(b) In Figure (b), $OA = BC$ and $\angle ABC = 30^\circ$. Prove $CD = AB\sqrt{3}$.

1294. *Proposed by P. Penning, Delft, The Netherlands.*

Find a necessary and sufficient condition on a convex quadrangle $ABCD$ in order that there exist a point P (in the same plane as $ABCD$) such that the areas of the triangles PAB, PBC, PCD, PDA are equal.

1295. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $A_1A_2A_3$ be a triangle with I_1, I_2, I_3 the excenters and B_1, B_2, B_3 the feet of the altitudes. Show that the lines I_1B_1, I_2B_2, I_3B_3 concur at a point collinear with the incenter and circumcenter of the triangle.

1296. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Let r_1, r_2, r_3 be the distances from an interior point of a triangle to its sides a_1, a_2, a_3, respectively, and let R be the circumradius of the triangle. Prove that

$$a_1r_1^n + a_2r_2^n + a_3r_3^n \le (2R)^{n-2}a_1a_2a_3$$

for all $n \ge 1$, and determine when equality holds.

1297. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.* (To the memory of Léo.)

(a) Let $C > 1$ be a real number. The sequence $z_1, z_2, \ldots$ of real numbers satisfies $1 < z_n$ and $z_1 + \ldots + z_n < Cz_{n+1}$ for $n \ge 1$. Prove the existence of a constant $a > 1$ such that $z_n > a^n$, $n \ge 1$.

(b)* Let conversely $z_1 < z_2 < \ldots$ be a strictly increasing sequence of positive real numbers satisfying $z_n \ge a^n$, $n \ge 1$, where $a > 1$ is a constant. Does there necessarily exist a constant C such that $z_1 + \ldots + z_n < Cz_{n+1}$ for all $n \ge 1$?

1298. *Proposed by Len Bos, University of Calgary, Calgary, Alberta.*

Let $A = (a_{ij})$ be an $n \times n$ matrix of positive integers such that $|\det A| = 1$, and suppose that $z_1, z_2, \ldots, z_n$ are complex numbers such that

$$z_1^{a_{i1}} z_2^{a_{i2}} \ldots z_n^{a_{in}} = 1$$

for each $i = 1,2,\ldots,n$. Show that $z_i = 1$ for each i.

1299*. *Proposed by Carl Friedrich Sutter, Viking, Alberta.*

Three real numbers a_1, a_2, a_3 are chosen at random from the interval [0,1] such that $\sum_{i=1}^{3} a_i = 1$. They are then rounded off to the nearest one-digit decimal to form $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$. What is the probability that $\sum_{i=1}^{3} \bar{a}_i = 1$?

1300. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

ABC is a triangle, not right angled, with circumcentre O and orthocentre H. The line OH intersects CA in K and CB in L, and $OK = HL$. Calculate angle C.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

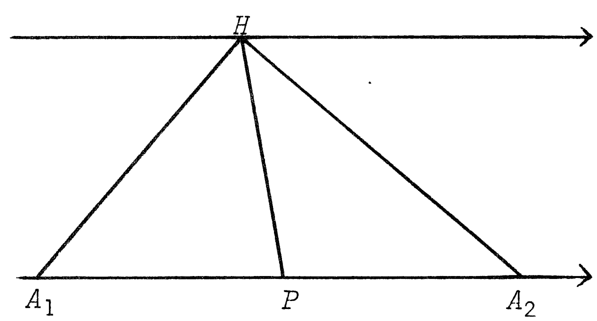
1109. [1986: 13] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

ABC is a triangle with orthocentre H. A rectangular hyperbola with centre H intersects line BC in A_1 and A_2, line CA in B_1 and B_2, and line AB in C_1 and C_2. Prove that the points P, Q, R, the midpoints of A_1A_2, B_1B_2, C_1C_2, respectively, are collinear.

[*Editor's comment.* The most interesting solution submitted to this problem, but also the most mysterious, is that of Jordi Dou. Unable to understand it, I asked my eminent colleague Richard K. Guy for help. This he kindly gave, and what follows is Dou's proof with elaborations (including references) by RKG. Can anyone find a less exotic, but still elegant, proof?]

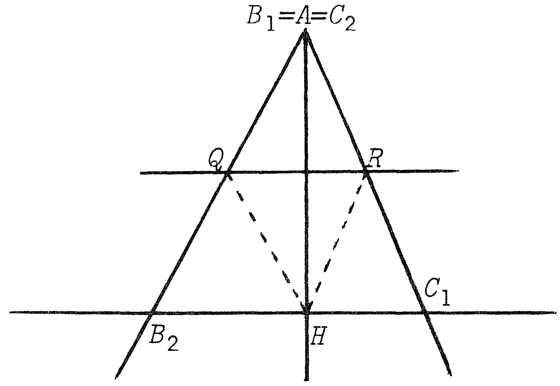
Solution by Jordi Dou, Barcelona, Spain. (Translated and annotated by Richard K. Guy, University of Calgary.)

The locus of the midpoints of a system of parallel chords of a conic is a diameter of the conic. This is true in particular for the pair of asymptotes, so the midpoint of a chord of a hyperbola is also the midpoint of the segment intercepted on the chord by its asymptotes (see Theorem 28(iii), page 60 of [1]). So we may replace the rectangular hyperbola by a pair of perpendicular lines through H, the orthocentre of the triangle ABC.



If such a pair cuts the side BC of the triangle in A_1, A_2, and if P is the midpoint of A_1A_2, then $H\{A_1A_2;P\infty\}$ is a harmonic pencil, and since $\angle A_1HA_2 = \pi/2$, HA_1, HA_2 are the bisectors of the angles formed by HP and the parallel through H to BC (Theorem 35(2), page 68 of [2]). Alternatively, consider the circle on A_1A_2 as diameter. Its centre is P, and since $\angle A_1HA_2 = \pi/2$, H lies on the circle; ΔA_1PH is isosceles and PH makes twice the angle with BC that HA_1 does. In fact, as the pair of perpendicular lines rotates about H, the ray HP, and similarly the rays HQ, HR, rotate through twice the angle. In particular $\angle QHR$ remains constant.

Consider the perpendicular lines when one is through A and the other is parallel to BC. Then P is at infinity in that direction, and the midpoints Q,

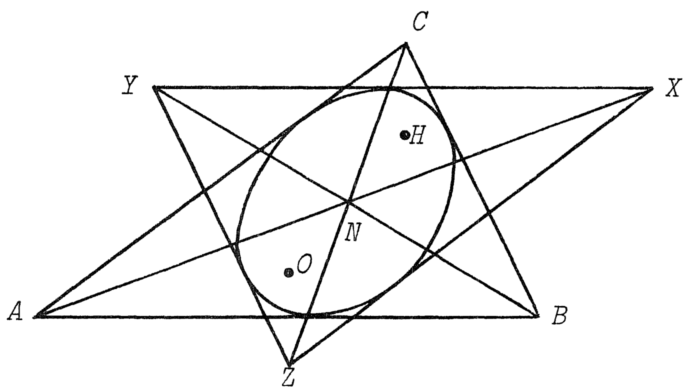


R are on the perpendicular bisector of AH. By reflexion in QR, we see that the constant angle QHR is equal to the angle A of the triangle. Similarly $\angle RHP = B$, $\angle PHQ = C$. Notice that the six angles formed at H by the altitudes of ΔABC are A, B, C, A, B, C. The six angles formed by HP, HQ and HR are the same, but *in opposite cyclic order*.

HQ, HR generate homographic (in fact, congruent) pencils, which intersect CA, AB in homographic ranges. The join, QR, of corresponding points envelops a conic (Theorem 44, Note, page 69 of [3]). Moreover, the constancy of the angle QHR tells us that H is a focus. Similarly RP, PQ envelop the same conic, and PQR is a straight line.

When R is at A, since $\angle RHQ = A$ and $\angle QHP = C$, P will be at C, and AC is a position of the line. Thus the conic touches the sides of the triangle ABC.

We've seen that the perpendicular bisector of AH touches the conic, and similarly so do the perpendicular bisectors of BH and CH. These three lines form a triangle XYZ congruent to ΔABC and obtained from it by rotation through π about N, the common nine-point centre.



By symmetry, N is the centre of the conic and the second focus is O, the circumcentre of ΔABC (and orthocentre of the rotated triangle). The Euler line of ΔABC is the major axis of the conic, which is an ellipse, a hyperbola, or a degenerate pair of points (O and H), according as ΔABC is acute-angled, obtuse-angled, or right-angled. In the last case, H coincides with the right angle, and O is the midpoint of the hypotenuse.

References:

[1] C.V. Durell, *Concise Geometrical Conics*, Macmillan, 1927.

[2] C.V. Durell, *Modern Geometry*, Macmillan, 1920.

[3] C.V. Durell, *Projective Geometry*, Macmillan, 1926.

Also solved by the proposer; and partially by J.T. GROENMAN, *Arnhem, The Netherlands*.

* * *

1166. [1986: 178] *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ontario.* (Dedicated to Léo Sauvé.)

Let A and B be positive integers such that the arithmetic progression $\{An + B: n = 0,1,2,\ldots\}$ contains at least one square. If M^2 $(M > 0)$ is the smallest such square, prove that $M < A + \sqrt{B}$.

I. *Solution by J.T. Groenman, Arnhem, The Netherlands.*

Suppose

$$An + B = m^2 \tag{1}$$

where $n \geq 0$, $m > 0$, and suppose that $m \geq A + \sqrt{B}$. We try to find a smaller solution. If we succeed then for the smallest solution we must have $m < A + \sqrt{B}$.

How do we get this smaller solution? If it is

$$A(n - \lambda) + B = q^2 \tag{2}$$

where $0 < \lambda \leq n$, then from (1) and (2)

$$A\lambda = (m + q)(m - q).$$

We try $A = m - q$, $\lambda = m + q$. Then $\lambda = 2m - A$, and

$$\begin{aligned} A(n - \lambda) + B &= A(n - 2m + A) + B \\ &= An + B - 2mA + A^2 \\ &= m^2 - 2mA + A^2 = (m - A)^2, \end{aligned}$$

so we put $q = m - A$. Since $m \geq A + \sqrt{B}$, $0 < q < A$. Thus we need only show $n - \lambda \geq 0$. But this is equivalent to

$$\begin{gathered} n - 2m + A \geq 0, \\ An - 2mA + A^2 \geq 0, \\ m^2 - B - 2mA + A^2 \geq 0, \\ (m - A)^2 \geq B, \end{gathered}$$

and finally

$$m - A \geq \sqrt{B},$$

which is true.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

From the assumptions follows at once the existence of some integer x such that $x^2 \equiv B \pmod{A}$. As the half-open interval $[\sqrt{B}, \sqrt{B} + A)$ contains A consecutive natural numbers, one of them, say y, has to satisfy $y \equiv x \pmod{A}$. Then $y^2 \equiv B \pmod{A}$ and $\sqrt{B} \leq y < A + \sqrt{B}$. Thus $y^2 = B + An$ for some $n \geq 0$, and, as $M \leq y$, we are done.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; KEE-WAI LAU, Hong Kong; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

* * *

1167. [1986: 179] *Proposed by Jordan B. Tabov, Sofia, Bulgaria.* (Dedicated to Léo Sauvé.)

Determine the greatest real number r such that for every acute triangle ABC of area 1 there exists a point whose pedal triangle with respect to ABC is right-angled and of area r.

Solution by the proposer.

We shall show that the required value of r is $3/2 + \sqrt{3}$.

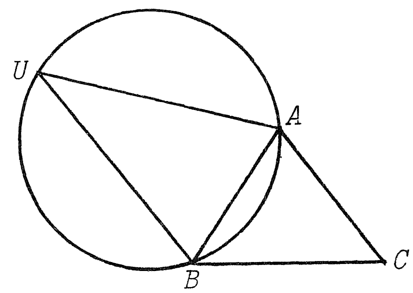
Consider a point U such that

(i) $AU = BU$,

(ii) $\angle AUB = 90^\circ - C$,

and

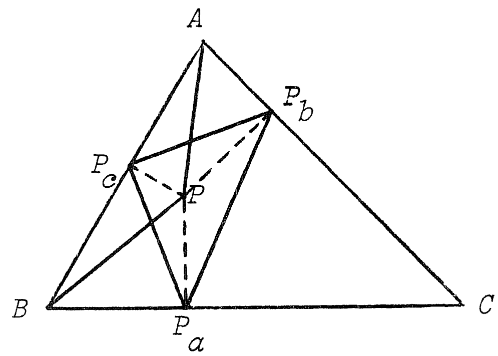
(iii) U and C are on different sides of AB.



We denote the circumcircle of ΔABU by K_c. Then for X on K_c,

$$\angle AXB = \begin{cases} 90^\circ - C & \text{for } X \text{ on the same side of } AB \text{ as } U, \\ 90^\circ + C & \text{otherwise.} \end{cases} \qquad (1)$$

Now let P be a point such that its pedal triangle $P_aP_bP_c$ is right-angled with $\angle P_bP_cP_a = 90^\circ$ (this includes the degenerate cases $P = A$ and $P = B$). We shall show that the locus of P is K_c.



Suppose P is inside ΔABC. Then the quadrilaterals AP_cPP_b and BP_aPP_c are cyclic, therefore $\angle P_bAP = \angle P_bP_cP$, $\angle P_aBP = \angle P_aP_cP$. Hence

$$\begin{aligned} \angle APB &= 180^\circ - (\angle PAB + \angle PBA) \\ &= 180^\circ - (A - \angle P_bAP + B - \angle P_aBP) \\ &= (180^\circ - A - B) + \angle P_bP_cP + \angle P_aP_cP \\ &= C + \angle P_bP_cP_a. \end{aligned}$$

Consequently, by (1), $\angle P_bP_cP_a = 90^\circ$ if and only if

$$\angle APB = C + 90^\circ = 180^\circ - \angle AUB,$$

i.e., if and only if $P \in K_c$. The other case, when P lies outside ΔABC, is quite similar.

Analogously, the locus of the point P such that $\angle P_cP_aP_b = 90^\circ$, respectively $\angle P_cP_bP_a = 90^\circ$, is the circle K_a, respectively K_b, defined similarly. Combining these results, we conclude that $\Delta P_aP_bP_c$ is right-angled if and only if $P \in K_a \cup K_b \cup K_c = K$.

Fix an arbitrary acute triangle ABC of area 1 and consider the function

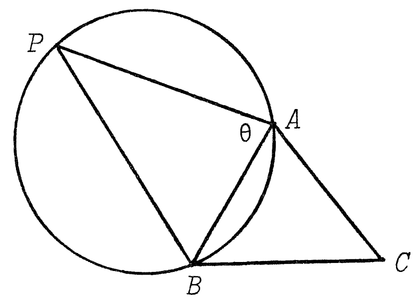
$$F(P) = \text{area of } \Delta P_a P_b P_c$$

defined for P on K. It is continuous, non-negative (with $F(A) = 0$), and, since K is compact and connected, its range is an interval $[0,s]$ where s depends on the triangle ABC. Clearly $r = \min s$, where the minimum is taken over all acute triangles ABC of area 1.

Let $P \in K_c$ and let $\theta = \angle PAB$. Then by (1),

$$\frac{BP}{\sin\theta} = \frac{AB}{\sin(90^\circ - C)} = \frac{AP}{\sin(90^\circ + C - \theta)}$$

if P is outside ΔABC, and



$$\frac{BP}{\sin\theta} = \frac{AB}{\sin(90^\circ + C)} = \frac{AP}{\sin(90^\circ - C - \theta)}$$

if P is inside ΔABC, so that

$$BP = \frac{AB\cdot\sin\theta}{\cos C}$$

and

$$AP = \begin{cases} \dfrac{AB\cdot\cos(C-\theta)}{\cos C} & \text{if } P \text{ is outside } \Delta ABC, \\[2ex] \dfrac{AB\cdot\cos(C+\theta)}{\cos C} & \text{if } P \text{ is inside } \Delta ABC. \end{cases}$$

Remembering that AP_cPP_b is cyclic and $\angle AP_cP = 90^\circ$, we obtain

$$P_cP_b = AP\cdot\sin A = \begin{cases} \dfrac{AB\cdot\sin A\cos(C-\theta)}{\cos C} & \text{if } P \text{ is outside } \Delta ABC, \\[2ex] \dfrac{AB\cdot\sin A\cos(C+\theta)}{\cos C} & \text{if } P \text{ is inside } \Delta ABC, \end{cases}$$

and similarly

$$P_cP_a = BP\cdot\sin B = \frac{\sin B\sin\theta}{\cos C}\,.$$

Hence

$$F(P) = \frac{P_cP_a\cdot P_cP_b}{2} = \frac{AB^2\cdot\sin A\sin B\sin\theta\cos\varphi}{2\cos^2 C}$$

where

$$\varphi = \begin{cases} C-\theta & \text{if } P \text{ is outside } \Delta ABC, \\ C+\theta & \text{if } P \text{ is inside } \Delta ABC. \end{cases}$$

Since the area of ΔABC is 1,

$$1 = \frac{AB\cdot AC\cdot\sin A}{2} = \frac{AB\cdot\sin A}{2}\cdot\frac{AB\cdot\sin B}{\sin C}\,,$$

so

$$F(P) = \frac{\sin C\sin\theta\cos\varphi}{\cos^2 C}$$

$$= \frac{\sin C}{2\cos^2 C}\cdot(\sin(\theta+\varphi)+\sin(\theta-\varphi))$$

$$= \frac{\sin C}{2\cos^2 C}\cdot g(\theta)$$

where

$$g(\theta) = \begin{cases} \sin C + \sin(2\theta - C) & \text{if } P \text{ is outside } \Delta ABC, \\ \sin(2\theta + C) - \sin C & \text{if } P \text{ is inside } \Delta ABC. \end{cases}$$

Thus the maximum value of $g(\theta)$ is $1 + \sin C$, attained when P is outside ΔABC and $2\theta = C + 90^\circ$. Hence the maximum value of $F(P)$ for $P \in K_c$ is

$$\frac{\sin C(1+\sin C)}{2\cos^2 C} = \frac{\sin C}{2(1-\sin C)} = \frac{1}{2}\left[\frac{1}{1-\sin C} - 1\right]. \qquad (2)$$

If we assume $A \le B \le C$, then

$$\max\left\{\frac{1}{1-\sin A}, \frac{1}{1-\sin B}, \frac{1}{1-\sin C}\right\} = \frac{1}{1-\sin C}$$

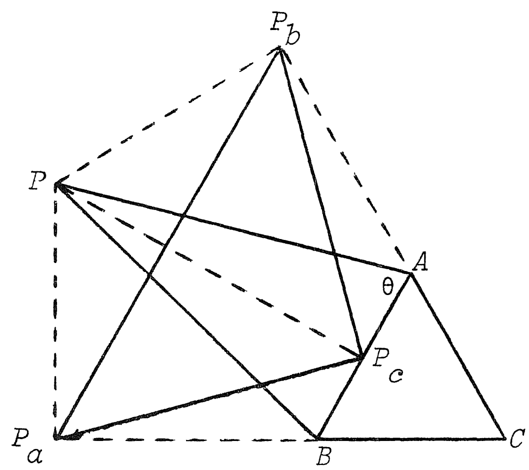
so the maximum value of $F(P)$ over all $P \in K$ is again (2). It remains to note that $60^\circ \le C < 90^\circ$, and therefore by (2)

$$s \ge \frac{\sin 60^\circ(1+\sin 60^\circ)}{2\cos^2 60^\circ} = \frac{3}{2} + \sqrt{3},$$

equality holding for ΔABC equilateral. Therefore

$$r = \min s = 3/2 + \sqrt{3},$$

attained when ΔABC is equilateral and P is outside ΔABC such that $\theta = 75^\circ$.



There were no other solutions received for this problem, and only one response. It seems to be a calculation of that acute triangle of area 1 for which the pedal triangle of the orthocentre is right-angled and has minimal area. Not the question, but, hey, the editor has an open mind[1]. *(The answer given, for anyone who wishes to check it, was a minimum area of* $(\sqrt{2}-1)/2$.)

* * *

1168. [1986: 179] *Proposed by Herta T. Freitag, Roanoke, Virginia.* (Dedicated to Léo Sauvé.)

Let $S = \sum_{i=1}^{k} F_{(2i-1)n}$ where n is odd and F_m denotes a Fibonacci number. Determine a Lucas number L_a such that $L_a S$ is a Fibonacci number.

[1]The empty set being open.

Solution by Kenneth M. Wilke, Topeka, Kansas.

We shall show that L_n satisfies the conditions of the problem and that $L_nS = F_{2kn}$ where S, k, and n are as defined in the problem.

Let m be an odd integer, i a positive integer, and L_j and F_j denote the jth Lucas and Fibonacci numbers respectively. Then we claim that

$$F_{2(i-1)m} + L_mF_{(2i-1)m} = F_{2im}. \tag{1}$$

Letting

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2},$$

it is well known that

$$F_j = \frac{\alpha^j - \beta^j}{\sqrt{5}}, \quad L_j = \alpha^j + \beta^j.$$

Thus

$$\begin{aligned} F_{2(i-1)m} + L_mF_{(2i-1)m} &= \frac{\alpha^{2m(i-1)} - \beta^{2m(i-1)}}{\sqrt{5}} + \frac{(\alpha^m + \beta^m)(\alpha^{(2i-1)m} - \beta^{(2i-1)m})}{\sqrt{5}} \\ &= \frac{\alpha^{2im} - \beta^{2im}}{\sqrt{5}} + \frac{(\alpha^{2m(i-1)} - \beta^{2m(i-1)})(\alpha^m\beta^m + 1)}{\sqrt{5}} \\ &= \frac{\alpha^{2im} - \beta^{2im}}{\sqrt{5}} = F_{2im} \end{aligned}$$

because m is odd and $\alpha\beta = -1$.

Now the formula

$$L_nS = L_n \sum_{i=1}^{k} F_{(2i-1)n} = F_{2kn}$$

can be proved by induction on k. For $k = 1$ we have

$$L_nS = L_nF_n = F_{2n}$$

directly or from (1). For the induction step, (1) provides the necessary argument to show that once we assume the desired result holds for $k - 1$, it must hold for k also.

Also solved by BOB PRIELIPP, University of Wisconsin, Oshkosh, Wisconsin; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

* * *

1169. [1986: 179] *Proposed by Andy Liu, University of Alberta, Edmonton, Alberta; and Steve Newman, University of Michigan, Ann Arbor, Michigan.* [To Léo Sauvé who, like J.R.R. Tolkien, created a fantastic world.]

(i) *The Fellowship of the Ring.* Fellows of a society wear rings formed of 8 beads, with two of each of 4 colours, such that no two adjacent beads are of the same colour. No two members wear indistinguishable rings. What is the maximum number of fellows of this society?

(ii) *The Two Towers*. On two of three pegs are two towers, each of 8 discs of increasing size from top to bottom. The towers are identical except that their bottom discs are of different colours. The task is to disrupt and reform the towers so that the two largest discs trade places. This is to be accomplished by moving one disc at a time from peg to peg, never placing a disc on top of a smaller one. Each peg is long enough to accommodate all 16 discs. What is the minimum number of moves required?

(iii) *The Return of the King*. The King is wandering around his kingdom, which is an ordinary 8 by 8 chessboard. When he is at the north-east corner, he receives an urgent summons to return to his summer palace at the south-west corner. He travels from cell to cell but only due south, west, or south-west. Along how many different paths can the return be accomplished?

Solution by the proposers.

(i) We consider more generally rings of two beads of each of n colours, but for now we allow beads of the same colour to be adjacent, and assume that rotations and reflections of a pattern are considered distinct.

The number of distinct rings is then $(2n)!/2^n$. There are $(2n)!$ permutations of the beads, but we must divide by 2^n to account for the fact that beads of the same colour are indistinguishable.

Let A denote the set of these $(2n)!/2^n$ rings. For $1 \le i \le n$, let A_i denote the subset of rings where the two beads of the ith colour are adjacent (with possibly bead pairs of other colours adjacent as well). The number of rings with no adjacent beads of the same colour is then

$$f(n) = |A| - |A_1 \cup A_2 \cup \ldots \cup A_n|.$$

We claim that for all choices of k colours $i(1), i(2), \ldots, i(k)$,

$$|A_{i(1)} \cap A_{i(2)} \cap \ldots \cap A_{i(k)}| = \frac{(2n-k-1)!(2n)}{2^{n-k}}.$$

Place the first of two adjacent beads of colour $i(1)$ in any of the $2n$ places. For each of the colours $i(2), \ldots, i(k)$, merge the two beads of that colour into a single one, since they have to be adjacent. The $k - 1$ merged beads and the $2(n - k)$ other beads can now be placed in $(2n - k - 1)!$ ways. Finally, we divide by 2^{n-k} to account for the indistinguishable beads.

Since there are $\binom{n}{k}$ subsets of k colours, we have

$$f(n) = \frac{n}{2^{n-1}} \sum_{k=0}^{n} (-1)^k (2n-k-1)! \binom{n}{k} 2^k$$

by inclusion-exclusion. Direct computation yields $f(4) = 744$.

Note that each rotationally distinct pattern has been counted $2n$ times in $f(n)$, except for the $(n - 1)!$ patterns in which the two beads of each colour are diametrically opposite to each other. Each of these $(n - 1)!$ patterns is counted n times. Therefore, the number of rotationally distinct patterns is

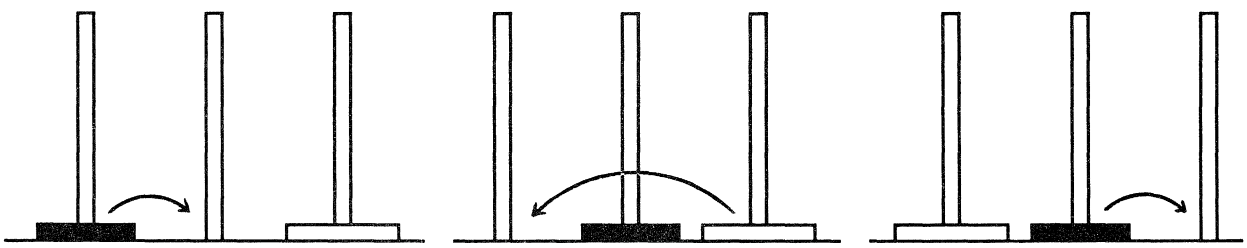
$$g(n) = \frac{f(n) + (n - 1)!n}{2n} = \frac{f(n) + n!}{2n} .$$

Now each rotationally and reflectionally distinct pattern has been counted twice in $g(n)$, except for the $n!/2$ rotationally distinct patterns which remain unchanged after reflection. Each of these $n!/2$ patterns is counted once. Therefore the number of distinguishable rings is given by

$$\frac{g(n) + n!/2}{2} = \frac{f(n) + (n + 1)!}{4n} .$$

In particular, with $n = 4$, the society has at most $(744 + 120)/16 = 54$ members.

(ii) We consider more generally towers of height n. Let $f(n)$ denote the minimum number of moves required. In the following diagram, we record the task for $n = 1$, so that $f(1) = 3$:



(a) (b) (c)

For higher values of n, the bottom (largest) discs still eventually have to be moved in this way. Moreover, during these moves, all smaller discs must lie on top of the stationary large disc. This divides the task into four stages.

We suppose that initially the two towers occupy the left and right pegs in the above diagrams. To go from the original configuration to one in which the large disc on the left peg can be moved as in (a), we have to merge the two towers (minus the bottom discs) into a single tower above the large disc on the right peg. To go from the final move of a large disc, as in (c), to the final configuration, we simply reverse this process. To allow the intermediate moves (b) or (c) of a large disc, in each case we have first to transfer a doubled tower from one peg to another.

Let $g(n)$ denote the minimum number of moves required to merge two towers of n discs as described above, and we use $g_1(n)$ if the merged tower stands on the peg not occupied by either tower before the merger. Let $h(n)$ denote the minimum number of moves required to transfer a doubled tower of $2n$ discs from one peg to another. Note that $g(1) = 1$ and $g_1(1) = h(1) = 2$.

Putting the four stages together, we have

$$f(n) = 2g(n - 1) + 2h(n - 1) + 3. \tag{1}$$

Similar analysis yields

$$g(n) = g_1(n - 1) + h(n - 1) + 1, \tag{2}$$

$$g_1(n) = g(n - 1) + 2h(n - 1) + 2, \tag{3}$$

and

$$h(n) = 2h(n - 1) + 2. \tag{4}$$

From (4) we get

$$h(n) = 2^{n+1} - 2. \tag{5}$$

(This is of course just the familiar Tower of Hanoi problem, but with each disc replaced by two discs of the same size, thus doubling the number of moves usually required.)

Eliminating $g_1(n)$ from (2) and (3), and using (5), we obtain

$$\begin{aligned} g(n) &= g(n - 2) + 2h(n - 2) + h(n - 1) + 3 \\ &= g(n - 2) + 2^{n+1} - 3, \end{aligned} \tag{6}$$

and from (2) we also have

$$g(2) = g_1(1) + h(1) + 1 = 2 + 2 + 1 = 5.$$

Inspection of (6) leads us to conjecture that

$$g(n) = A \cdot 2^n + Bn + C$$

for some constants A, B, C, so that

$$A \cdot 2^n + Bn + C = A \cdot 2^{n-2} + B(n - 2) + C + 2^{n+1} - 3.$$

Equating the coefficients, we have $A = 8/3$ and $B = -3/2$. For odd n, we have $g(1) = 1$ so that $C = -17/6$. For even n, we have $g(2) = 5$ so that $C = -8/3$. Putting everything into (1), we obtain

$$f(n) = \begin{cases} \frac{7}{3} \cdot 2^{n+1} - 3n - \frac{10}{3} & n \text{ odd}, \\[2ex] \frac{7}{3} \cdot 2^{n+1} - 3n - \frac{11}{3} & n \text{ even}. \end{cases}$$

In particular, $f(8) = 1167$.

(iii) We consider more generally an n by n board. A path for the King is equivalent to a sequence of S's, W's, and D's, standing respectively for southward, westward, and southwestward moves. Such a sequence must contain i

D's, $n - i - 1$ S's, and $n - i - 1$ W's for some $0 \le i \le n - 1$.

The number of such sequences for a particular i is

$$\binom{2n - i - 2}{i}\binom{2n - 2i - 2}{n - i - 1}$$

(we choose i of the $2n - i - 2$ places for the D's and $n - i - 1$ of the remaining $2n - 2i - 2$ places for the S's, with the rest going to the W's). The total number of such sequences is then

$$\sum_{i=0}^{n-1} \binom{2n - i - 2}{i}\binom{2n - 2i - 2}{n - i - 1} .$$

For $n = 8$, the value is 48639.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California.

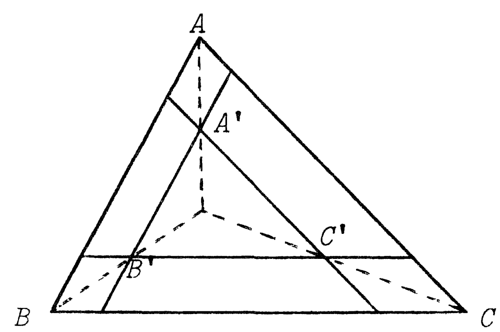
* * *

1170. [1986: 179] *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.* (Dedicated to Léo Sauvé.)

In the plane of triangle ABC, let P and Q be points having trilinears $\alpha_1:\beta_1:\gamma_1$ and $\alpha_2:\beta_2:\gamma_2$, respectively, where at least one of the products $\alpha_1\alpha_2$, $\beta_1\beta_2$, $\gamma_1\gamma_2$ is nonzero. Give a Euclidean construction for the point $P*Q$ having trilinears $\alpha_1\alpha_2:\beta_1\beta_2:\gamma_1\gamma_2$. (A point has trilinears $\alpha:\beta:\gamma$ if its signed distances to sides BC, CA, AB are respectively proportional to the numbers α, β, γ.)

I. *Solution by D.J. Smeenk, Zaltbommel, The Netherlands.*

Since we can construct three line segments of lengths proportional to $\alpha_1:\beta_1:\gamma_1$ (drop perpendiculars from P to the sides of ΔABC), and similarly for $\alpha_2:\beta_2:\gamma_2$, we can by a familiar construction form three line segments of (signed) lengths x, y, z proportional to $\alpha_1\alpha_2$, $\beta_1\beta_2$, $\gamma_1\gamma_2$.



How to find the point $P*Q$? Draw the line $\ell \parallel BC$ at signed distance x from BC. Analogously construct lines $m \parallel AC$ and $n \parallel AB$ at distances y and z from AC and AB respectively. The lines ℓ, m, n form a triangle $A'B'C' \sim \Delta ABC$. The lines AA', BB', and CC' are then concurrent in the point $P*Q$.

II. *Remarks by the proposer.*

The product $*$ is commutative and associative. Thus $(G,*)$ is a group, where G is the set of all points not on the lines BC, CA, AB. The incenter

(1:1:1) serves as the identity of $(G,*)$, and for $P = (\alpha:\beta:\gamma) \in G$, $P^{-1} = (\alpha^{-1}:\beta^{-1}:\gamma^{-1})$. P^{-1} is the isogonal conjugate of P, constructed as follows: reflect line AP about the internal bisector of angle A, and analogously for BP and CP. The reflected lines concur at P^{-1}.

Also solved by the proposer.

* * *

1171*. [1986: 204] *Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.* (Dedicated to Léo Sauvé.)

(i) Determine all real numbers λ so that, whenever a, b, c are the lengths of three segments which can form a triangle, the same is true for

$$(b + c)^{\lambda}\ ,\ (c + a)^{\lambda}\ ,\ (a + b)^{\lambda}\ .$$

(For $\lambda = -1$ we have *Crux* 14 [1975: 28].)

(ii) Determine all pairs of real numbers λ, μ so that, whenever a, b, c are the lengths of three segments which can form a triangle, the same is true for

$$(b + c + \mu a)^{\lambda}\ ,\ (c + a + \mu b)^{\lambda}\ ,\ (a + b + \mu c)^{\lambda}\ .$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We solve the general problem, i.e. part (ii).

First of all we note that always $\mu \geq -1$. Indeed, for $\mu < -1$ we could - in view of the triangle inequality - find a triangle with sides a, b, c such that $a + b + \mu c = 0$, say, so that the numbers

$$(b + c + \mu a)^{\lambda}\ ,\ (c + a + \mu b)^{\lambda}\ ,\ (a + b + \mu c)^{\lambda} \qquad (1)$$

could not be the sides of a triangle.

Next, let $\mu = -1$ and $\lambda \neq 0$. We consider triangles of sides $a = 1$, $b = 2$, $c = 3 - t$, where $t \to 0$. If $\lambda < 0$, then a triangle with sides (1) must satisfy the inequality

$$(a + b - c)^{\lambda} < (b + c - a)^{\lambda} + (c + a - b)^{\lambda},$$

that is,

$$t^{\lambda} < (4 - t)^{\lambda} + (2 - t)^{\lambda},$$

which is impossible as $t \to 0$. For $\lambda > 0$, consider instead

$$(b + c - a)^{\lambda} < (a + b - c)^{\lambda} + (c + a - b)^{\lambda},$$

or

$$(4 - t)^{\lambda} < t^{\lambda} + (2 - t)^{\lambda};$$

as $t \to 0$, this also yields a contradiction.

Moreover, if $\mu = 1$ then all real λ are suitable, as (1) yields an equilateral triangle for all a, b, c.

Thus we assume from now on that $\mu > -1$, $\mu \neq 1$, and put (without loss of generality) $a \leq b \leq c = 1$. We distinguish three cases.

Case I. $\lambda = 0$.

Then all $\mu > -1$ are suitable, since (1) yields equilateral triangles.

Case II. $\lambda > 0$.

We have to study the three possible triangle inequalities

$$(1 + a + \mu b)^{\lambda} < (a + b + \mu)^{\lambda} + (1 + b + \mu a)^{\lambda} \tag{2}$$

$$(a + b + \mu)^{\lambda} < (1 + a + \mu b)^{\lambda} + (1 + b + \mu a)^{\lambda} \tag{3}$$

$$(1 + b + \mu a)^{\lambda} < (a + b + \mu)^{\lambda} + (1 + a + \mu b)^{\lambda}. \tag{4}$$

(2) is always satisfied, as

$$1 + a + \mu b \leq a + b + \mu$$

holds whenever $1 < \mu$ (since $b \leq 1$), and

$$1 + a + \mu b \leq 1 + b + \mu a$$

holds whenever $\mu < 1$ (since $a \leq b$).

For (3) and (4), we put $a + b = s$. Then $1 < s \leq 2$, since $a \leq b \leq c = 1$ and a, b, c are the sides of a triangle. We first consider (3).

If $\lambda \geq 1$, then by the general mean-inequality applied to means M_{λ} and M_1, we get

$$(1 + a + \mu b)^{\lambda} + (1 + b + \mu a)^{\lambda} \geq 2[1 + (\mu + 1)s/2]^{\lambda}$$

(with equality if and only if $a = b$ or $\lambda = 1$). For (3) we thus have to solve the inequality

$$2[1 + (\mu + 1)s/2]^{\lambda} > (s + \mu)^{\lambda},$$

that is,

$$2^{1-1/\lambda} < \frac{2 + s + \mu s}{s + \mu} =: f(s), \tag{5}$$

for $1 < s \leq 2$. Because

$$f'(s) = \frac{(s + \mu)(1 + \mu) - (2 + s + \mu s)}{(s + \mu)^2} = \frac{(\mu + 2)(\mu - 1)}{(s + \mu)^2},$$

we infer that f decreases on $(1,2]$ if $-1 < \mu < 1$ and increases on $(1,2]$ if $\mu > 1$. Thus if $-1 < \mu < 1$ then

$$f(s) \geq f(2) = 2 > 2^{1-1/\lambda},$$

so (5) holds for all $\lambda \geq 1$. If $\mu > 1$ then

$$f(s) > f(1) = \frac{3 + \mu}{1 + \mu},$$

so (5) is true if

$$\frac{3 + \mu}{1 + \mu} \geq 2^{1-1/\lambda},$$

which leads to

$$\lambda \le \frac{\log 2}{\log[(2\mu + 2)/(\mu + 3)]}.$$

If $0 < \lambda < 1$, then for (3) we consider a fixed s, $1 < s \le 2$, and the function

$$g(a) = (1 + a + \mu s - \mu a)^\lambda + (1 + s - a + \mu a)^\lambda, \quad s - 1 \le a \le s/2,$$

which is just the right side of (3) with $b = s - a$. Then

$$g'(a) = \lambda(1 - \mu)[(1 + a + \mu s - \mu a)^{\lambda-1} - (1 + s - a + \mu a)^{\lambda-1}]$$

and

$$g''(a) = \lambda(\lambda - 1)(1 - \mu)^2[(1 + a + \mu s - \mu a)^{\lambda-2} + (1 + s - a + \mu a)^{\lambda-2}].$$

Thus $g''(a) < 0$, i.e. g is concave and attains its minimum value at an endpoint. Since

$$\begin{aligned} g(s - 1) &= (s + \mu)^\lambda + (2 + \mu s - \mu)^\lambda \\ &\le 2\left[\frac{2 + \mu s + s}{2}\right]^\lambda \\ &= \left[1 + \frac{s}{2} + \frac{\mu s}{2}\right]^\lambda + \left[1 + \frac{s}{2} + \frac{\mu s}{2}\right]^\lambda = g(s/2) \end{aligned}$$

by the general mean-inequality applied to M_λ and M_1, we have

$$g(a) \ge g(s - 1) > (s + \mu)^\lambda$$

for all a, $s - 1 \le a \le s/2$, which implies (3).

For inequality (4), if $\mu > 1$ we have

$$1 + b + \mu a \le a + b + \mu \quad \text{and} \quad 1 + b + \mu a \le 1 + a + \mu b$$

(since $a \le b \le 1$), either one of which implies (4) at once. Now let $-1 < \mu < 1$ and consider the function

$$h(a) = (1 + s - a + \mu a)^\lambda - (1 + a + \mu s - \mu a)^\lambda$$

for fixed s and $s - 1 \le a \le s/2$. Then

$$h'(a) = \lambda(\mu - 1)[(1 + s - a + \mu a)^{\lambda-1} + (1 + a + \mu s - \mu a)^{\lambda-1}] < 0 \tag{6}$$

so h decreases. Thus for (4) we need only consider

$$h(s - 1) < (a + b + \mu)^\lambda,$$

that is,

$$(2 + \mu s - \mu)^\lambda < (s + \mu)^\lambda + (s + \mu)^\lambda = 2(s + \mu)^\lambda,$$

or

$$2^{1/\lambda} > \frac{\mu s + 2 - \mu}{s + \mu} =: k(s), \quad 1 < s \le 2. \tag{7}$$

Since

$$k'(s) = \frac{(s + \mu)\mu - (\mu s + 2 - \mu)}{(s + \mu)^2} = \frac{(\mu + 2)(\mu - 1)}{(s + \mu)^2} \tag{8}$$

and $-1 < \mu < 1$, k decreases. Thus

$$k(s) < k(1) = \frac{2}{\mu + 1},$$

and (7), hence (4), holds if and only if

$$\frac{2}{\mu + 1} \leq 2^{1/\lambda},$$

implying

$$\lambda \leq \frac{\log 2}{\log[2/(\mu + 1)]} .$$

Case III. $\lambda < 0$.

We put $\lambda = -\upsilon$, where $\upsilon > 0$, and rewrite inequalities (2), (3), (4) accordingly. Then (2) follows much as it did in Case II.

For (3) and (4), we again let $a + b = s$, $1 < s \leq 2$. By the general mean-inequality applied to M_υ and M_{-1}, we get

$$\left[\frac{1}{1 + a + \mu b}\right]^\upsilon + \left[\frac{1}{1 + b + \mu a}\right]^\upsilon \geq 2\left[\frac{2}{2 + s + \mu s}\right]^\upsilon .$$

For (3) we thus have to solve the inequality

$$2\left[\frac{2}{2 + s + \mu s}\right]^\upsilon > \frac{1}{(s + \mu)^\upsilon} ,$$

that is,

$$2^{1+1/\upsilon} > \frac{2 + s + \mu s}{s + \mu} = f(s). \tag{9}$$

It was already determined that f decreases on $(1,2]$ if $-1 < \mu < 1$ and increases on $(1,2]$ if $\mu > 1$. Thus if $\mu > 1$ then

$$f(s) \leq f(2) = 2,$$

so (9) holds with no further limitation for υ. If $-1 < \mu < 1$ then

$$f(s) < f(1) = \frac{3 + \mu}{1 + \mu} ,$$

so (9) is true if

$$\frac{3 + \mu}{1 + \mu} \leq 2^{1+1/\upsilon},$$

yielding

$$\upsilon \leq \frac{\log 2}{\log[(\mu + 3)/(2\mu + 2)]} ,$$

i.e.

$$\lambda \geq \frac{\log 2}{\log[(2\mu + 2)/(\mu + 3)]} .$$

For (4), if $-1 < \mu < 1$ we have

$$1 + b + \mu a \geq a + b + \mu \quad \text{and} \quad 1 + b + \mu a \geq 1 + a + \mu b$$

(since $a \leq b \leq 1$), either one of which implies (4). Now let $\mu > 1$, and consider again the function $h(a)$ defined above for $s - 1 \leq a \leq s/2$ where s is fixed. Then from (6), $h'(a) < 0$ so h is decreasing on $[s - 1, s/2]$. Hence for (4) we need only consider

$$h(s - 1) < (a + b + \mu)^\lambda,$$

i.e.

$$(2 + \mu s - \mu)^\lambda < (s + \mu)^\lambda + (s + \mu)^\lambda = 2(s + \mu)^\lambda$$

or

$$2^{-1/\nu} < \frac{\mu s + 2 - \mu}{s + \mu} = k(s). \tag{10}$$

From (8), since $\mu > 1$, k increases on $(1,2]$. Thus

$$k(s) > k(1) = \frac{2}{\mu + 1},$$

and (10), hence (4), holds if and only if

$$\frac{2}{\mu + 1} \geq 2^{-1/\nu}$$

yielding

$$\nu \leq \frac{\log 2}{\log[(\mu + 1)/2]},$$

i.e.

$$\lambda \geq \frac{\log 2}{\log[2/(\mu + 1)]}.$$

Summarizing the results from above we get the following possibilities for λ and μ:

if $\mu = -1$, $\lambda = 0$

if $-1 < \mu < 1$, $\dfrac{\log 2}{\log[(2\mu + 2)/(\mu + 3)]} \leq \lambda \leq \dfrac{\log 2}{\log[2/(\mu + 1)]}$

if $\mu = 1$, λ is arbitrary

if $\mu > 1$, $\dfrac{\log 2}{\log[2/(\mu + 1)]} \leq \lambda \leq \dfrac{\log 2}{\log[(2\mu + 2)/(\mu + 3)]}$.

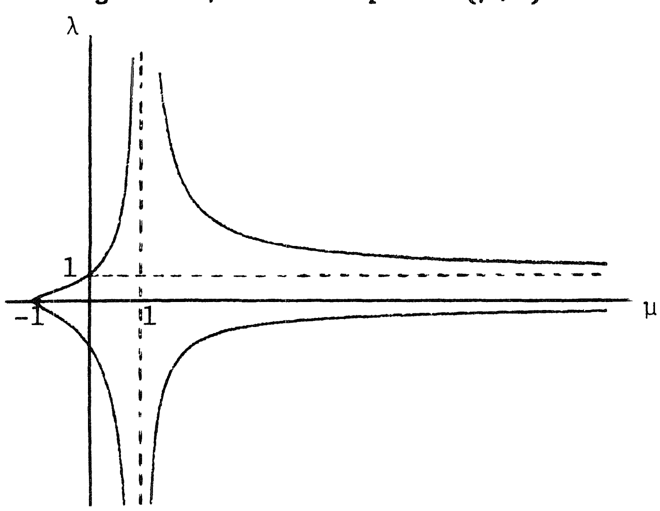
Concerning part (i), we get via $\mu = 0$ that

$$-1.7 \approx \frac{\log 2}{\log(2/3)} \leq \lambda \leq 1.$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; and MURRAY S. KLAMKIN, University of Alberta.

Klamkin's proof was almost complete, with just one case left open.

Hess supplied no proofs for his answer, but did contribute a picture of the resulting set of ordered pairs (μ,λ):



Every point inside the "cross" works. The "boundary" contains degenerate cases.

Janous and Klamkin both gave some further generalizations, none of which the editor had the energy to record.

Frankly, the editor wishes he had asked for part (i), and left it at that!

* * *

<u>1172</u>. [1986: 205] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Show that for any triangle ABC, and for any real $\lambda \geq 1$,

$$\sum (a + b)\sec^{\lambda}(C/2) \geq 4(2/\sqrt{3})^{\lambda}s,$$

where the sum is cyclic over ΔABC and s is the semiperimeter.

Solution by Murray S. Klamkin, University of Alberta.

By the power mean inequality,

$$\frac{\sum (a + b)\sec^{\lambda}(C/2)}{4s} = \frac{\sum (a + b)\sec^{\lambda}(C/2)}{\sum (a + b)}$$

$$\geq \left[\frac{\sum (a + b)\sec(C/2)}{\sum (a + b)}\right]^{\lambda}$$

for any $\lambda \geq 1$. The desired result now follows from the inequality

$$\sum (a + b)\sec(C/2) \geq \frac{2}{\sqrt{3}} \sum (a + b). \tag{1}$$

Letting $a = 2R \sin A$, etc., (1) becomes

$$\sum (\sin A + \sin B)\sec(C/2) \geq \frac{4}{\sqrt{3}} \sum \sin A.$$

Then using

$$\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}, \text{ etc.,}$$

we get

$$\sum \cos \frac{A - B}{2} \geq \frac{2}{\sqrt{3}} \sum \sin A.$$

But this inequality is known (*Crux* 613 [1982: 55, 67, 138]).

Also solved by the proposer.

* * *

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Collection (volumes 1-10) ______________________________

_____ × 75$ = ___________$ ______________________________

Total de votre remise ___________$