Crux Mathematicorum

Editorial Board

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Crux Mathematicorum

Crux Mathematicorum
with Mathematical Mayhem

EDITORIAL

As you might suspect, the mechanism of entering a university depends on the country where said university is located. In Belarus (where I am originally from), just as in many other countries of the former Soviet Union, the higher education is free – if you're good enough, that is. Your "goodness" is determined by the entrance exams and the competition is fierce. By grade 9, you already know which faculty you will be applying to and you start preparing for the entrance exams: you take evening classes, do endless exercises similar to problems appearing on previous years' exams, you might even switch high schools to the one that has a specialized class for the subject of your choice. Hard sciences are always popular. In my year, Faculty of Mechanics and Mathematics of Belarusian State University had seventeen times the number of students writing the exams than spots available.

You can apply to only one faculty within one top-tier public university. If you don't get in, you can apply to less prestigious universities, and to private universities where you pay for your education (they strategically hold their entrance exams later in the summer). Or you can try again next year unless you are male, in which case your mandatory military service will delay your second attempt by 2 years. You get the idea – the stakes are high.

Each written exam includes a variety of problems: a bit of algebra, planar geometry, solid geometry, word problems. For a sample in English, check out Contest Corner problems in ***Crux*** 40(5) and their solutions in 41(5). If you score high enough on the written exam, you proceed to the oral stage, where you draw 2 or 3 random problems, are given a short time to solve them and then get to discuss your solutions with the admissions committee. Yes, very intimidating. Just to give you some idea of what the problems might be like, here are two from 2009 oral exams to the Faculty of Mechanics and Mathematics of Moscow State University (`http://new.math.msu.su/admission`):

1. Solve the equation

$$\cos 4x + \sin\left(2x - \frac{a\pi}{64}\right) = \sin 3x,$$

   where a is the smallest two-digit natural number which, when written to the right of 20092009, produces a ten-digit number divisible by 36.

2. Find all values of c such that the set of solutions (x, y) of the system

$$\begin{cases} \frac{x^2+y^2-16x+10y+65}{x^2+y^2-14x+12y+79} \leq 0, \\ (x-c)(y+c) = 0 \end{cases}$$

   forms a line segment.

I can't help but wonder, should I give such an examination to my students here in Canada next time I teach a first-year course? Will it be a sobering experience for them as they realize what it takes to get a higher education in another country?

<div align="right">Kseniya Garaschuk</div>

THE CONTEST CORNER
No. 36
Robert Bilinski

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1 août 2016** ; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

CC176. Un réveil numérique affiche 4 chiffres à la manière des chiffres suivants :
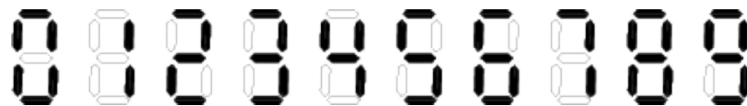
Mathieu joue au jeu suivant : « On soustrait le nombre de segments allumés du nombre affiché. On répète l'opération sur le deuxième nombre et ainsi de suite. . . ». Par exemple, puisque 1234 utilise 16 segments, le deuxième nombre est $1234 - 16 = 1218$. Après avoir joué deux fois, Mathieu obtient 2015 comme troisième nombre. Quel était son nombre de départ ?

CC177. Un aventurier qui a vu le jour entre 1901 et 1955 rédige ses mémoires quand il a entre 30 ans et 60 ans d'âge. Nous y lisons, "En ce jours d'anniversaire, je remarque quelque chose d'extraordinaire : Le jour de la semaine est exactement le même que le jour de ma naissance." Quel est l'âge de l'aventurier quand il écrit cette phrase ?

CC178. Le peintre d'Art Moderne Rec Tangle peint le tableau représenté ici :
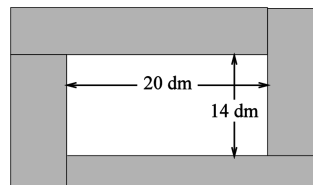
Le rectangle blanc dans le milieu a une longueur de 20 dm et une largeur de 14 dm. Les 4 rectangles gris ont la même aire et leurs dimensions, en décimètres, sont des entiers. Au minimum, quelle est l'aire d'un petit rectangle gris ? (La figure n'est pas á l'échelle. On rappelle qu'un rectangle peut être carré.)

CC179. Matthieu crée une suite de nombres à partir du nombre 7. Un nombre donné de la suite est la somme des chiffres du carré du nombre précédent additionné de 1. Par exemple, le deuxième nombre de sa suite est 14 parce que $7^2 = 49$ et $4 + 9 + 1 = 14$. Quel est le millième nombre dans la suite de Matthieu ?

CC180. Les pages d'un livre sont numérotées $1, 2, 3, \ldots$ Un chiffre qui paraît dans le numéro de la dernière page paraît au total 20 fois dans l'ensemble des numéros des pages du livre. Si le livre comptait treize pages de moins, alors le même chiffre ne serait utilisé que 14 fois au total. Combien le livre compte-t-il de pages ?

·.

CC176. A digital clock shows 4 digits using the following patterns :
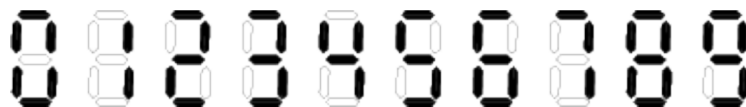
Mathew plays the following game : "We subtract the number of lighted segments from the number which is shown. We repeat the operation on the second number and so on ..." For example, since 1234 uses 16 segments, the second number would be $1234 - 16 = 1218$. After performing this operation two times, Mathew gets 2015. What was his starting number ?

CC177. An adventurer, born between 1901 and 1955, writes his memoirs when he is between 30 and 60 years of age. He wrote, "On this day celebrating my birthday, an extraordinary fact is made known to me : the weekday is exactly the same as the one I was born on." What was the age of the adventurer when he wrote that sentence ?

CC178. A Modern Art painter Rec Tangle has painted the work of art represented here :
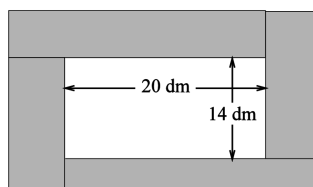
The white rectangle in the middle has length 20 dm and width 14 dm. The 4 grey rectangles all have equal areas and their dimensions, in decimeters, are non-zero integers. What is the minimal possible area of each of the grey rectangles ? (The drawing is not to scale and a rectangle might be a square.)

CC179. Matthew creates a sequence of numbers starting from the number 7. Every number in his sequence is the sum of the digits of the square of the previous number, plus 1. For example, the second number in his sequence is 14 because $7^2 = 49$ and $4 + 9 + 1 = 14$. What is the 1000th number of Matthew's sequence?

CC180. The pages of a book are numbered $1, 2, 3, \ldots$ A digit that appears in the number of the last page appears 20 times in the set of page numbers of the book. If the book had thirteen pages less, then the same digit would have been used 14 times in total. How many pages does the book have?

CONTEST CORNER
SOLUTIONS

CC126. Anna and Ben decided to visit Archipelago with 2009 islands. Some islands are connected by boats which run both ways. Anna and Ben are playing the following game during the trip : Anna chooses the first island on which they arrive by plane ; then Ben chooses the next island to visit. Thereafter, they take turns choosing a new island. When they arrive at an island connected only to islands they have already visited, whoever's turn it is to choose the next island loses. Prove that Anna can always win, no matter how Ben plays and how islands are connected.

Originally from 2009 Tournament of Towns, Spring, A-level, Juniors.

We received no solutions. We present a solution based on the official solution.

We construct a graph, with the vertices representing the islands and the edges representing connecting routes. The graph may have one or more connected components. Since the total number of vertices is odd, there must be a connected component with an odd number of vertices. Anna chooses from this component the largest set of independent edges, that is, edges no two of which have a common endpoint. She will mark these edges.
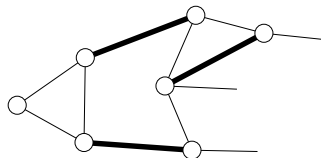
Since the number of vertices is odd, there is at least one vertex which is not incident with a marked edge. Anna will start the tour there.

Suppose Ben has a move. It must take the tour to a vertex incident with a marked edge. Otherwise, Anna could have marked one more edge. Anna simply continues the tour by following that marked edge. If Ben continues to go to vertices incident with marked edges, Anna will always have a ready response. We show that Ben cannot manage to get to a vertex not incident with a marked edge. Suppose otherwise.
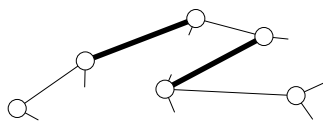
Consider the tour so far. Both the starting and the finishing vertices are not incident with marked edges. In between, the edges are alternately marked and unmarked. If Anna interchanges the marked and unmarked edges on this tour, she could have obtained a larger independent set of edges, a contradiction.

CC127. In the Great Hall of Camelot there is the Round Table with n seats. Merlin summons n knights of Camelot for a conference. Every day, he assigns seats to the knights. From the second day on, any two knights who become neighbours may switch their seats unless they were neighbours on the first day. If the knights manage to sit in the same cyclic order as on one of the previous days, the next day the conference ends. What is the maximal number of days of the conference Merlin can guarantee?

Originally from 2010 Tournament of Towns, Fall, A-level, Juniors.

We received no solutions. We present a solution based on the official solution.

We may assume that the knights are seated from 1 to n in cyclic clockwise order on day 1. Then seat exchanges are not permitted between knights with consecutive numbers (1 and n are considered consecutive). We will assign to each cyclic order a value called the winding number, which will be shown to be invariant under permitted seat exchanges. We obtain the winding number as follows. Merlin has n hats that he gives to the n knights. He starts by giving the first hat to knight 1, then walks clockwise around the table until he reaches knight 2, to which he gives the second hat. He continues in clockwise order giving out the remaining hats to the knights in ascending order, after which he walks on until he reaches knight 1 again. The number of times he has gone around the table is called the winding number of the cyclic order. For instance, in the following example the winding number is 4; in the first round Merlin hands out hats to knights 1 and 2, in the second round to knights 3 and 4, in the third round to knight 5, and in the fourth round to knights 6 and 7.
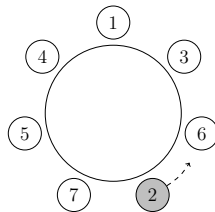
Next we show that the winding number is invariant under seat exchanges. If knight 1 is not involved in the seat exchange, then the hats handed out in each round remain the same. If knight 1 changes seats with knight h, then knight h gets his hat one round earlier or later, but all the other knights get their hats in the same round as before. In both cases the winding number remains constant. Thus to guarantee that the conference lasts at least n days, Merlin just has to seat the knights in $n-1$ arrangements with distinct winding number on the first $n-1$ days. As one solution, Merlin could seat the knights on the k-th day in the cyclic order $1, 2, \ldots, n-k, n, n-1, \ldots, n-k+1$. It is easily verified that this order has winding number k.

To show that Merlin cannot make the conference last any longer, we prove that any two cyclic orders with the same winding number can be transformed into each other via permitted seat exchanges. Since the winding number cannot be any larger than $n-1$, the knights can then force the end of the conference on the n-th day at the latest.

To be precise, we show that any cyclic order with winding number k can be transformed into the order $1, 2, \ldots, n-k, n, n-1, \ldots, n-k+1$ (as a sequence of seat exchanges is reversible, we can then transform any cyclic order into any other cyclic order with the same winding number). Given such a cyclic order, we start making seat exchanges that move knight 2 counter-clockwise towards knight 1 as far as possible. If he encounters knight 3, we move both of them counter-clockwise and so on. By this we obtain a cyclic order starting with $1, h, h-1, \ldots, 2, \ldots$.

If $h = n$, we are done. Otherwise, we let knight 1 change seats with knights $h, h-1, \ldots, 3$ in turn, so that knight 2 now sits after knight 1. We repeat the process to move knight 3 to the place after knight 2, and so on, stopping when we arrive at the order $1, 2, \ldots, n-k, n, n-1, \ldots, n-k+1$. The following example illustrates the process.
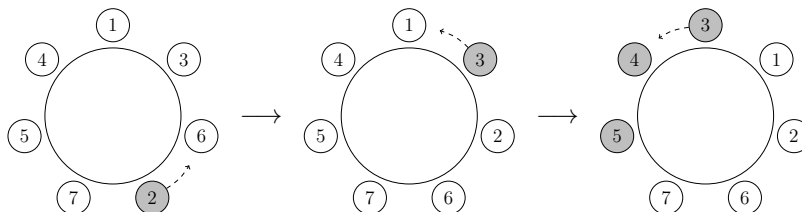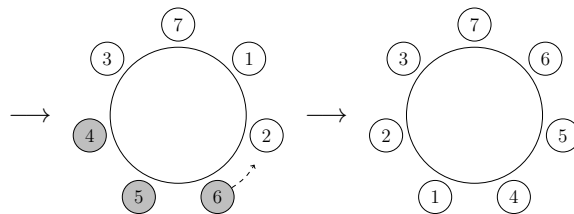
CC128. In one kingdom gold sand and platinum sand are used as a currency. An exchange rate is defined by two positive integers g and p; namely, x grams of gold sand are equivalent to y grams of platinum sand if $x : y = p : g$ (x and y are not necessarily integers). At a day when $g = p = 1001$, the Treasury announced that on each of the following days one of the numbers, either g or p would be decreased by 1 so that after 2000 days $g = p = 1$. However, the exact order in which the numbers decreased is kept unknown. At the day of the announcement a banker had $1\,\mathrm{kg}$ of gold sand and $1\,\mathrm{kg}$ of platinum sand. The banker's goal is to make exchanges so that at the end of this period he would have at least $2\,\mathrm{kg}$ of gold sand and $2\,\mathrm{kg}$ of platinum sand. Can the banker reach his goal for certain as a result of some clever exchanges?

Originally from 2014 Tournament of Towns, Fall, A-level, Seniors.

We received no solutions.

CC129. A closed broken self-intersecting line is drawn in the plane. Each of the links of this line is intersected exactly once and no three links intersect at the same point. Further, there are no self-intersections at the vertices and no two links have a common segment. Can it happen that every point of self-intersection divides both links in half?

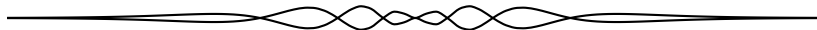*Originally from 2013 Tournament of Towns, Fall, A-level, Seniors.*

We received no solutions.

CC130. Let $P(x)$ be a polynomial such that $P(0) = 1$ and $(P(x))^2 = 1 + x + x^{100}Q(x)$, where $Q(x)$ is also a polynomial. Prove that the coefficient of x^{99} of the polynomial $(P(x) + 1)^{100}$ is zero.

Originally from 2014 Tournament of Towns, Spring, A-level, Seniors.

We received one incomplete submission.

THE OLYMPIAD CORNER
No. 334
Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1 août 2016** ; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.
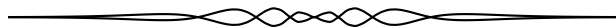
OC236. Pour une fonction $f : \mathbb{R} \to \mathbb{R}$ telle que $f(x)^2 \leq f(y)$ pour tout $x > y$, $x, y \in \mathbb{R}$, démontrer que $f(x) \in [0, 1]$ pour tout $x \in \mathbb{R}$.

OC237. Soit n un entier positif et soit S l'ensemble de tous les entiers dans $\{1, 2, \ldots, n\}$ qui sont relativement premiers avec n. Poser

$$S_1 = S \cap \left(0, \frac{n}{3}\right], \ S_2 = S \cap \left(\frac{n}{3}, \frac{2n}{3}\right], \ S_3 = S \cap \left(\frac{2n}{3}, n\right].$$

Si la cardinalité de S est un multiple de 3, démontrer que S_1, S_2 et S_3 ont la même cardinalité.

OC238. Soit a, b et c des nombres réels positifs tels que $a+b+c = 3$. Démontrer que

$$\frac{a^2}{a + \sqrt[3]{bc}} + \frac{b^2}{b + \sqrt[3]{ca}} + \frac{c^2}{c + \sqrt[3]{ab}} \geq \frac{3}{2}$$

et déterminer quand l'égalité tient.

OC239. Dans une école se trouvent n élèves dont certains sont amis (l'amitié est mutuelle.) Soit a et b les valeurs qui satisfont aux conditions suivantes :

1. Il est possible de séparer les élèves en a équipes telles que deux étudiants de la même équipe sont toujours amis.

2. Il est possible de séparer les élèves en b équipes telles que deux étudiants de la même équipe ne sont jamais amis.

Déterminer la valeur maximale de $N = a + b$ en termes de n.

OC240. Soit ABC un triangle où I est le centre du cercle inscrit ; supposer que ce cercle inscrit est tangent à CA et AB aux points E et F. Dénoter par G et H les réflexions de E et F par rapport à I. Soit Q l'intersection de BC et GH, puis soit M le mipoint de BC. Démontrer que IQ et IM sont perpendiculaires.

. . . .

OC236. Given a function $f \colon \mathbb{R} \to \mathbb{R}$ with $f(x)^2 \leq f(y)$ for all $x, y \in \mathbb{R}$, $x > y$, prove that $f(x) \in [0, 1]$ for all $x \in \mathbb{R}$.

OC237. Let n be a positive integer, and set S be the set of all integers in $\{1, 2, \ldots, n\}$ which are relatively prime to n. Set

$$S_1 = S \cap \left(0, \frac{n}{3}\right], \ S_2 = S \cap \left(\frac{n}{3}, \frac{2n}{3}\right], \ S_3 = S \cap \left(\frac{2n}{3}, n\right].$$

If the cardinality of S is a multiple of 3, prove that S_1, S_2, S_3 have the same cardinality.

OC238. Let a, b, c be positive reals such that $a + b + c = 3$. Prove that

$$\frac{a^2}{a + \sqrt[3]{bc}} + \frac{b^2}{b + \sqrt[3]{ca}} + \frac{c^2}{c + \sqrt[3]{ab}} \geq \frac{3}{2}$$

and determine when equality holds.

OC239. In a school, there are n students and some of them are friends with each other (friendship is mutual). Define a and b to be values which satisfy the following conditions :

1. We can divide students into a teams such that any two students on the same team are friends.

2. We can divide students into b teams such that no two students on the same team are friends.

Find the maximum value of $N = a + b$ in terms of n.

OC240. Let ABC be a triangle with incenter I, and suppose the incircle is tangent to CA and AB at E and F. Denote by G and H the reflections of E and F over I. Let Q be the intersection of BC with GH, and let M be the midpoint of BC. Prove that IQ and IM are perpendicular.

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2014 : 40(4), p. 147–148.
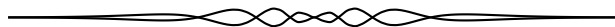
OC176. Solve the following equation

$$y = 2x^2 + 5xy + 3y^2$$

for x and y integers.

Originally problem 2 from the 2013 Greece National Olympiad.

We received nine correct submissions. We present the solution by Matei Coiculescu.

We factor the right side of the equation as

$$y = 2(x + y)^2 + y^2 + xy.$$

Next, we get

$$y(1 - y - x) = 2(x + y)^2.$$

Let $x + y = z$. Then the above equation becomes

$$y(1 - z) = 2z^2$$

or, since z cannot take the value 1,

$$y = \frac{2z^2}{1 - z}.$$

This can be factored as

$$y(1 - z) = 2z^2 - 2 + 2$$

which is equivalent to

$$(1 - z)(y + 2(z + 1)) = 2.$$

From this we conclude,

$$1 - z = \pm 1, \quad 1 - z = \pm 2$$

or $z = 0, 2, -1, 3$. This gives the following solutions (x, y) of the equation : $(0, 0)$, $(10, -8)$, $(-2, 1)$, and $(12, -9)$.

OC177. For any positive integer a, define $M(a)$ to be the number of positive integers b for which $a + b$ divides ab. Find all integer(s) a with $1 \le a \le 2013$ such that $M(a)$ attains the largest possible value in the range of a.

Originally problem 2 from the 2013 Hong Kong National Olympiad.

We received no solutions to this problem.

OC178. Find all nonempty sets S of integers such that $3m - 2n \in S$ for all $m, n \in S$.

Originally problem 2 from day 1 of the 2013 China National Olympiad.

We present the solution by Oliver Geupel. There were no other submissions.

For integers a and $m > 0$ let $[a]_m$ denote the residue class of a modulo m. It is straightforward to check that, for all pairs a, b of integers with $a < b$, the sets $[a]_{b-a}$ and $[a]_{3(b-a)} \cup [b]_{3(b-a)}$ have the required property. Moreover, every singleton $\{a\}$ is a set with the desired property. We are going to prove that this is the complete list of solutions of the problem.

Assume that S is a set with the required property that contains at least two distinct elements. Let $a, b \in S$ such that $a < b$. For any integer k, let's denote

$$c_k = 3(b-a)k + a, \qquad d_k = 3(b-a)k + b.$$

Note both are in S since

$$c_k = 3((b-a)k + a) - 2a \qquad d_k = 3((b-a)k + b) - 2b.$$

We obtain

$$c_{k+1} = 3(b-a)(k+1) + a = 9(b-a)k + 3b - 6(b-a)k - 2a = 3d_k - 2c_k$$

and similarly

$$d_{k-1} = 3c_k - 2d_k, \qquad c_{k-1} = 3d_k - 2c_{k+1}, \qquad d_{k+1} = 3c_k - 2d_{k-1}.$$

By mathematical induction, we deduce that $c_k, d_k \in S$ for every integer k. Hence,

$$[a]_{3(b-a)} \cup [b]_{3(b-a)} \subseteq S.$$

We have proved :

$$\text{If } a, b \in S \text{ then } [a]_{3(b-a)} \cup [b]_{3(b-a)} \subseteq S. \tag{1}$$

Next suppose that

$$[a]_{3(b-a)} \cup [b]_{3(b-a)} \subsetneq S.$$

There is no loss of generality in assuming that $b - a$ is the least positive difference of two elements of the set S. Note that the absolute differences of consecutive elements of the set $[a]_{3(b-a)} \cup [b]_{3(b-a)}$ are $b - a$ and $2(b - a)$ which alternate. Therefore any element $c \in S \setminus \left([a]_{3(b-a)} \cup [b]_{3(b-a)}\right)$ must be exactly in the middle of an interval of length $2(b - a)$ (otherwise we violate the minimality of $b - a$).

Without loss of generality we may put

$$c = b + (b - a) = 2b - a.$$

By the result (1) with a, b substituted by b, c, we obtain

$$[c]_{3(c-b)} = [c]_{3(b-a)} \subset S.$$

Therefore,

$$[a]_{b-a} = [a]_{3(b-a)} \cup [b]_{3(b-a)} \cup [c]_{3(b-a)} \subseteq S.$$

Since $|b - a|$ is minimal, the set S cannot have any more elements. Consequently

$$S = [a]_{b-a}.$$

This completes the proof.

OC179. Find the maximum value of

$$|a^2 - bc + 1| + |b^2 - ac + 1| + |c^2 - ba + 1|.$$

where a, b, c are real numbers in $[-2, 2]$

Originally problem 2 from day 1 of the grade 10 2013 Kazakhstan National Olympiad.

We present the solution by Paolo Perfetti. There were no other submissions.

Let us suppose that each modulus is different from zero (so that $|x|$ is twice differentiable) and consider

$$f(a, b, c) = a^2 + b^2 + c^2 - ab - bc - ca + 3.$$

Since $f_{a,a} = f_{b,b} = f_{c,c} = 2$, it follows that f is convex and then the maximum is attained when each variable is $+2$ or -2. Since

$$f(2, 2, 2) = 3, \quad f(2, -2, 2) = 19, \quad f(2, -2, -2) = 19, \quad f(-2, -2, -2) = 3,$$

it follows that the maximum is 19.

Now, let's suppose that

$$a^2 - bc + 1 = 0, \qquad b^2 - ac + 1 > 0, \qquad c^2 - ab + 1 > 0. \tag{1}$$

We need to find the maximum of

$$b^2 + c^2 + 2 - a(b + c)$$

subject to (1). If b and c are free to span $[-2, 2]$, we observe that the maximum is trivially 18; a value attained when $b = c = 2$, $a = -2$. Hence the maximum constrained by (1) is lower because $-a = b = c = 2$ doesn't satisfy (1).

Next let's assume

$$a^2 - bc + 1 = 0, \qquad b^2 - ac + 1 > 0, \qquad c^2 - ab + 1 < 0. \tag{2}$$

We need to find the maximum of

$$b^2 - c^2 + a(b - c) \le b^2 + a(b - c)$$

subject to (2). If b and c are free to span $[-2, 2]$, the maximum is 12; a value attained when $b = c \pm 2$, $a = \mp 2$. Hence the maximum constrained by (2) is lower because it doesn't satisfy (2).

Finally, let's assume

$$a^2 - bc + 1 = 0, \qquad b^2 - ac + 1 < 0, \qquad c^2 - ab + 1 < 0. \tag{3}$$

We need to find the maximum of

$$-b^2 - c^2 + a(b + c) - 2 \leq a(b + c) \leq 6.$$

We don't repeat the above argument.

The other cases when $b^2 - ac + 1 = 0$ and $c^2 - ab + 1 = 0$ are analogous.

The conclusion is that 19 is the searched maximum.

OC180. In an acute triangle ABC, let O be its circumcentre, G be its centroid and H be its orthocentre. Let D be a point on BC with OD perpendicular to BC and E a point on CA with HE perpendicular to CA. Let F be the midpoint of AB. If triangles ODC, HEA and GFB have the same area, find all possible values of the angle $\angle C$.

Originally problem 5 from the 2013 India National Olympiad.

We received four correct submissions. We present the solution by Somasundaram Muralidharan. We include the picture by Andrea Fanchini.
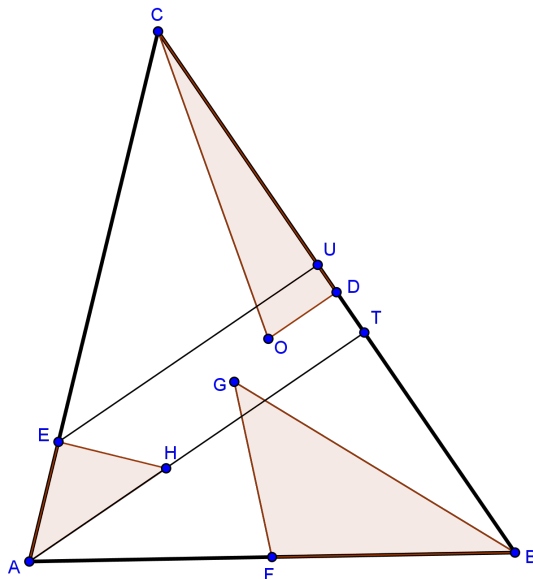
We will show that $\angle C = 45°$ or $\angle C = 60°$.

Let R be the circumradius of the triangle ABC. Since O is the circumcenter and OD is perpendicular to BC, it follows that D is the midpoint of BC, and $\angle COD = \angle A$. Thus, $OD = R\cos A$, $CD = R\sin A$ and area of triangle ODC is $\frac{1}{2}R^2\sin A\cos A$.

Next, we have :

$$\text{area of triangle } GFB = \frac{1}{6} \times \text{area of triangle } ABC = \frac{1}{6} \times \frac{1}{2}bc\sin A.$$

Since the areas of triangles ODC and GFB are equal, we have

$$\frac{1}{2}R^2\sin A\cos A = \frac{1}{6} \times \frac{1}{2}bc\sin A = \frac{1}{12} \times 4R^2\sin B\sin C\sin A.$$

Simplifying, we obtain $3\cos A = 2\sin B\sin C$. Since $A = 180° - B - C$, we get

$$2\sin B\sin C = -3\cos(B+C) = -3(\cos B\cos C - \sin B\sin C)$$

and we have

$$\tan B\tan C = 3.$$

Note that we can divide by $\cos B\cos C$ since the triangle is acute angled, $\angle B \neq 90°$ and $\angle C \neq 90°$.

Since H is the orthocenter and HE is perpendicular to CA, we have $AH^2 + BC^2 = 4R^2$ giving $AH = 2R\cos A$ and $\angle EAH = 90° - \angle C$. Thus, $AE = 2R\cos A\sin C$ and $HE = 2R\cos A\cos C$ and the area of the triangle HEA is given by

$$\frac{1}{2} \times (2R\cos A\sin C)(2R\cos A\cos C) = 2R^2\cos^2 A\sin C\cos C$$

Equating the above area to the area of the triangle COD, we get

$$2R^2\cos^2 A\sin C\cos C = \frac{1}{2}R^2\sin A\cos A$$

and hence

$$4\cos A\sin C\cos C = \sin A.$$

Thus

$$\tan A = 2\sin 2C = \frac{4\tan C}{1 + \tan^2 C}$$

This gives

$$-\tan(B+C) = \frac{4\tan C}{1 + \tan^2 C}$$
$$-\frac{\tan B + \tan C}{1 - \tan B\tan C} = \frac{4\tan C}{1 + \tan^2 C}$$
$$-\frac{\tan B + \tan C}{1 - 3} = \frac{4\tan C}{1 + \tan^2 C}, \qquad \text{since } \tan B\tan C = 3$$

Putting $t = \tan C$ and writing $\tan B = \dfrac{3}{t}$, we have

$$\frac{3}{t} + t = \frac{8t}{1 + t^2}$$

and hence $t^4 - 4t^2 + 3 = 0$. Consequently, $(t^2 - 1)(t^2 - 3) = 0$. Thus, $t = \tan C = 1$ or $\sqrt{3}$ (since the triangle is acute, we discard the negative values). Thus $\angle C = 45°$ or $60°$.

We verify that both values yield equal areas to the triangles ODC, GFB and HEA.

If $\tan C = 1$, then $\tan B = 3$ and $\tan A = 2$. In this case,

$$\text{area of the triangle } ODC = \frac{1}{4} R^2 \sin 2A = \frac{R^2}{5}$$

and

$$\text{area of the triangle } HEA = R^2 \cos^2 A \sin 2C = \frac{R^2}{5}.$$

Also, the area of the triangle GFB equals

$$\frac{1}{3} R^2 \sin A \sin B \sin C = \frac{1}{3} R^2 \times \frac{2}{\sqrt{5}} \times \frac{3}{\sqrt{10}} \times \frac{1}{\sqrt{2}} = \frac{R^2}{5}$$

Thus $\angle C = 45°$ satisfies the given condition.

When $\tan C = \sqrt{3}$, we have $\angle C = 60°$. Since $\tan B \tan C = 3$, it follows that $\tan B = \sqrt{3}$ and $\angle B = 60°$. Thus the triangle is equilateral. In this case, O, H, G coincide and all the triangles ODC, GFB and HEA have equal areas.
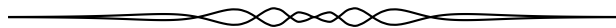
Concurrency and Collinearity

Victoria Krakovna

In this article we consider geometry problems that deal with concurrency and collinearity in the Euclidean plane. To avoid annoying special cases involving parallel lines, we will extend the plane by adjoining a *line at infinity*, a line that by definition consists of exactly one new point on every line of the Euclidean plane. Lines of the Euclidean plane share one of these new points if and only if they are parallel. Using this convention, three or more lines of the Euclidean plane are called *concurrent* if they meet at a single point, which can be either finite (that is, the lines are concurrent in the original Euclidean plane), or infinite (meaning the lines of the Euclidean plane are parallel). Three or more points are said to be *collinear* if they lie on a single straight line. Note that two lines of the extended plane always intersect, while two points always determine a unique line that contains them both (possibly the new line at infinity).

1 Elementary Tools

Here are some tips for concurrency and collinearity questions :

1. You can often restate a concurrency question as a collinearity question, and vice versa. For example, proving that AB, CD and EF are concurrent is equivalent to proving that E, F and $AB \cap CD$ are collinear.

2. A common way of proving concurrency is to consider the pairwise intersections of the lines, and then show that they are the same. A common way of proving collinearity is to show that the three points form an angle of 180°.

3. A particular case of concurrency in the extended plane is parallel lines meeting at their point at infinity. Make sure to be mindful of this case in your solutions of contest problems.

We will be discussing several powerful tools in this lecture : Pappus', Pascal's and Desargues' Theorems. However, you should remember that in questions on concurrency and collinearity, your best friends are good old Ceva and Menelaus.

In what follows, we use signed lengths of segments : that is, the value $\frac{XY}{YZ}$ is positive if $\overrightarrow{XY}$ and $\overrightarrow{YZ}$ are vectors in the same direction and negative otherwise. When Y is the point at infinity of the line XZ, then $\frac{XY}{ZY} = -\frac{XY}{YZ} = 1$.

Theorem 1 (Ceva's Theorem) *In ΔABC, let D, E, F be points different from the vertices on lines the BC, CA, AB, respectively. Then AD, BE, CF are concurrent in the extended plane iff*

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Theorem 2 *(Ceva's Theorem, trigonometric version)* *In $\triangle ABC$, let D, E, F be points different from the vertices on lines the BC, CA, AB, respectively. Then AD, BE, CF are concurrent in the extended plane iff*

$$\frac{\sin \angle BAD}{\sin \angle CAD} \cdot \frac{\sin \angle ACF}{\sin \angle BCF} \cdot \frac{\sin \angle CBE}{\sin \angle ABE} = 1.$$

Theorem 3 *(Menelaus' Theorem)* *In $\triangle ABC$, let D, E, F be points different from the vertices on lines the BC, CA, AB, respectively. Then D, E and F are collinear iff*

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1.$$

Problems

1. (Korea 1997) In an acute triangle ABC with $AB \neq AC$, let V be the intersection of the angle bisector of A with BC, and let D be the foot of the perpendicular from A to BC. If E and F are the intersections of the circumcircle of $\triangle AVD$ with AC and AB, respectively, show that the lines AD, BE, CF are concurrent.

2. (Iran 1998) Let ABC be a triangle and D be the point on the extension of side BC past C such that $CD = AC$. The circumcircle of $\triangle ACD$ intersects the circle with diameter BC again at P. Let BP meet AC at E and CP meet AB at F. Prove that the points D, E, F are collinear.

3. (Turkey 1996) In a parallelogram $ABCD$ with $\angle A < 90^o$, the circle with diameter AC meets the lines CB and CD again at E and F, respectively, and the tangent to this circle at A meets BD at P. Show that P, F, E are collinear.

4. (IMO SL 1995) Let ABC be a triangle. A circle passing through B and C intersects the sides AB and AC again at C' and B', respectively. Prove that BB', CC', and HH' are concurrent, where H and H' are the orthocenters of triangles ABC and $AB'C'$ respectively.

5. (IMO SL 2000) Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that $OD + DH = OE + EH = OF + FH$ and the lines AD, BE, and CF are concurrent.

2   Power Tools

Theorem 4 *(Pascal's theorem)* *Let A, B, C, D, E, F be points on a circle, in some order. Then if $P = AB \cap DE$, $Q = BC \cap EF$, $R = CD \cap FA$, then P, Q, R are collinear. In other words, if $ABCDEF$ is a cyclic (not necessarily convex) hexagon, then the intersections of the pairs of opposite sides are collinear.*
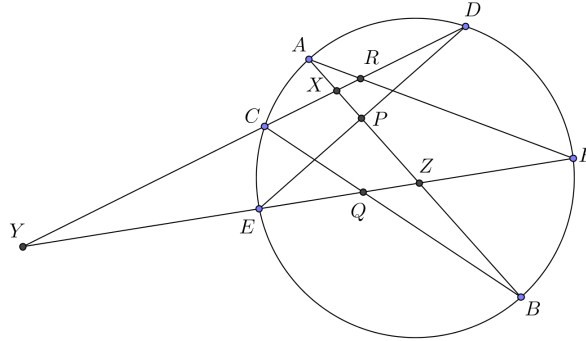
Proof. Let $X = AB \cap CD$, $Y = CD \cap EF$, $Z = EF \cap AB$. We apply Menelaus three times, to lines BC, DE, FA cutting the sides (possibly extended) of $\triangle XYZ$:

$$\frac{XC}{CY} \cdot \frac{YQ}{QZ} \cdot \frac{ZB}{BX} = -1,$$

$$\frac{XD}{DY} \cdot \frac{YE}{EZ} \cdot \frac{ZP}{PX} = -1,$$

$$\frac{XR}{RY} \cdot \frac{YF}{FZ} \cdot \frac{ZA}{AX} = -1.$$

We multiply the three equations, and observe that by Power of a Point

$$AX \cdot BX = CX \cdot DX, \ CY \cdot DY = EY \cdot FY, \ EZ \cdot FZ = AZ \cdot BZ,$$

so after cancellation the product becomes

$$\frac{YQ}{QZ} \cdot \frac{ZP}{PX} \cdot \frac{XR}{RY} = -1,$$

which implies by Menelaus in $\triangle XYZ$ that P, Q, R are collinear. $\square$

Example 1 *On the circumcircle of triangle ABC, let D be the midpoint of the arc AB not containing C, and E be the midpoint of the arc AC not containing B. Let P be any point on the arc BC not containing A, $Q = DP \cap AB$ and $R = EP \cap AC$. Prove that Q, R and the incenter of triangle ABC are collinear.*
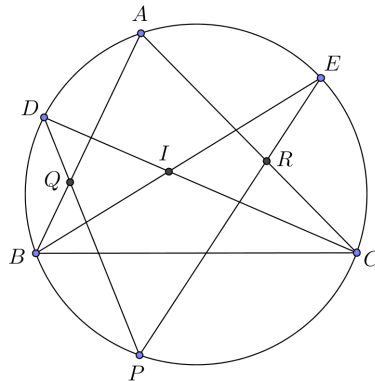
Solution. Since CD and BE are angle bisectors in $\triangle ABC$, they intersect at the incenter I. Now we want to apply Pascal to points A, B, C, D, E, F in such an order that Q, R, I are the intersections of pairs of opposite sides of the resulting hexagon. This is accomplished using the hexagon $ABEPDC$. □

In many problems, you don't have a configuration with six points around a circle. But it is still possible to apply the degenerate case of Pascal's Theorem, where some of the adjacent vertices of the "hexagon" coincide. In the limiting case of vertex A approaching vertex B, the line AB becomes the tangent line at B. We illustrate this with the following simple example.

Example 2 *(Macedonian MO 2001) Let ABC be a scalene triangle. Let a, b, c be tangent lines to its circumcircle at A, B, C, respectively. Prove that points $D = BC \cap a, E = CA \cap b$, and $F = AB \cap c$ exist, and that they are collinear.*
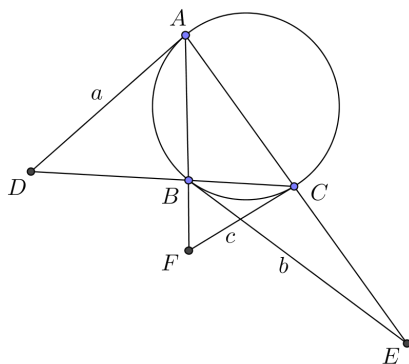
Solution. Apply Pascal to the degenerate hexagon $AABBCC$. Then the sides (in order) are lines a, AB, b, BC, c, CA. The desired conclusion follows. □

Theorem 5 *(Pappus' Theorem) Points A, C, E lie on line l_1, and points B, D, F lie on line l_2. Then $AB \cap DE$, $BC \cap EF$, and $CD \cap FA$ are collinear.*

Exercise 1 *Prove the theorem first for the case where the lines AB, CD, EF form a triangle, and apply Menelaus five times to all five transversals. For the other case, check out the following theorem.*

Theorem 6 *(Desargues' Theorem) Given triangles ABC and $A'B'C'$, let $P = BC \cap B'C'$, $Q = CA \cap C'A'$, $R = AB \cap A'B'$. Then AA', BB', CC' concur in the extended plane iff P, Q, R are collinear. In other words, the lines joining the corresponding vertices are concurrent iff the intersections of pairs of corresponding sides are collinear.*

Proof. ($\Rightarrow$) Suppose AA', BB', CC' concur at a point O. Apply Menelaus to lines $A'B'$, $B'C'$, $C'A'$ cutting triangles ABO, BCO, CAO respectively :

$$\frac{AA'}{A'O} \cdot \frac{OB'}{B'B} \cdot \frac{BR}{RA} = -1,$$

$$\frac{BB'}{B'O} \cdot \frac{OC'}{C'C} \cdot \frac{CP}{PB} = -1,$$

$$\frac{CC'}{C'O} \cdot \frac{OA'}{A'A} \cdot \frac{AQ}{QC} = -1.$$

Multiplying the three equations together, we obtain

$$\frac{AQ}{QC} \cdot \frac{CP}{PB} \cdot \frac{BR}{RA} = -1,$$

and so by Menelaus in triangle ABC, we have that P, Q, R are collinear.
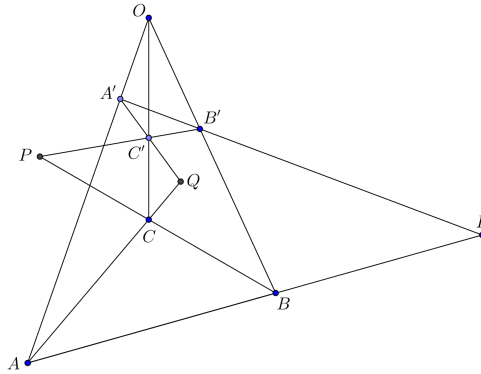
($\Longleftarrow$) Suppose P, Q, R are collinear.
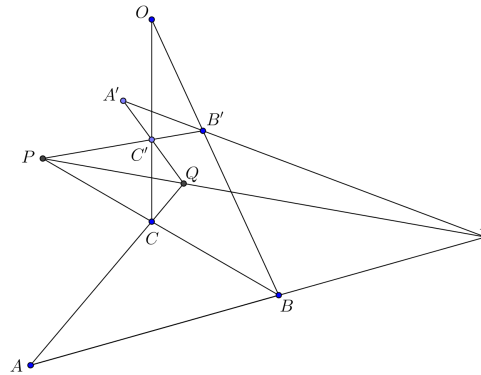
Consider $\triangle QCC'$ and $\triangle RBB'$, and let B correspond to C, B' correspond to C', R correspond to Q. Since the lines through corresponding vertices concur, we apply ($\Rightarrow$) and obtain that $O = BB' \cap CC'$, $A' = QC' \cap RB'$ and $A = QC \cap RB$ are collinear. Therefore, AA', BB', CC' concur at O. $\square$

Exercise 2 *Proofs in three dimensions are usually more difficult than problems in two dimensions, but not always! Suppose that A, B, C, A', B', C' do not lie in one plane. Find an easier proof of Desargues' theorem.*

Example 3 *In a quadrilateral $ABCD$, $AB \cap CD = P$, $AD \cap BC = Q$, $AC \cap BD = R$, $QR \cap AB = K$, $PR \cap BC = L$, $AC \cap PQ = M$. Prove that K, L, M are collinear.*

Solution. Since AQ, BR, CP intersect at D, we apply ($\Rightarrow$) of Desargues' Theorem to triangles ABC and QRP. Then $K = AB \cap QR$, $L = BC \cap RP$, $M = AC \cap QP$ are collinear. □

The great thing about these theorems is that there are no configuration issues whatsoever. Pascal in particular is very versatile – given six points around a circle, the theorem can be applied to them in many ways (depending on the ordering), and some of them are bound to give you useful information.

The following problems use Pascal, Pappus, and Desargues, often repeatedly or in combination. I have attempted to arrange them roughly in order of difficulty.

Problems

6. ([2]) Points A_1 and A_2 lie inside a circle and are symmetric about its center O. Points P_1, P_2, Q_1, Q_2 lie on the circle such that rays A_1P_1 and A_2P_2 are parallel and in the same direction, and rays A_1Q_1 and A_2Q_2 are also parallel and in the same direction. Prove that lines P_1Q_2, P_2Q_1 and A_1A_2 are concurrent in the extended plane.

7. (Australian MO 2001) Let A, B, C, A', B', C' be points on a circle, such that $AA' \perp BC$, $BB' \perp CA$, $CC' \perp AB$. Let D be an arbitrary point on the circle, and let $A'' = DA' \cap BC$, $B'' = DB' \cap CA$ and $C'' = DC' \cap AB$. Prove that A'', B'', C'' and the orthocenter of $\triangle ABC$ are collinear.

8. The extensions of sides AB and CD of quadrilateral $ABCD$ meet at point P, and the extensions of sides BC and AD meet at point Q. Through point P a line is drawn that intersects sides BC and AD at points E and F. Prove that the intersection points of the diagonals of quadrilaterals $ABCD$, $ABEF$ and $CDFE$ lie on a line that passes through point Q.

9. (IMO SL 1991) Let P be a point inside $\triangle ABC$. Let E and F be the feet of the perpendiculars from the point P to the sides AC and AB respectively. Let the feet of the perpendiculars from point A to the lines BP and CP be M and N respectively. Prove that the lines ME, NF, BC are concurrent.

10. Quadrilateral $ABCD$ is circumscribed about a circle. The circle touches the sides AB, BC, CD, DA at points E, F, G, H respectively. Prove that AC, BD, EG and FH are concurrent in the extended plane.

11. (Bulgaria 1997) Let $ABCD$ be a convex quadrilateral such that $\angle DAB = \angle ABC = \angle BCD$. Let H and O denote the orthocenter and circumcenter of the triangle ABC. Prove that D, O, H are collinear.

12. In triangle ABC, let the circumcenter and incenter be O and I, and let P be a point on line OI. Given that A' is the midpoint of the arc BC containing A, let A'' be the intersection of $A'P$ and the circumcircle of ABC. Similarly construct B'' and C''. Prove that AA'', BB'', CC'' are concurrent in the extended plane.

13. In triangle ABC, altitudes AA_1 and BB_1 and angle bisectors AA_2 and BB_2 are drawn. The inscribed circle is tangent to sides BC and AC at points A_3 and B_3, respectively. Prove that lines A_1B_1, A_2B_2 and A_3B_3 are concurrent in the extended plane.

14. (China 2005) A circle meets the three sides BC, CA, AB of a triangle ABC at points D_1, D_2; E_1, E_2; F_1, F_2 respectively. Furthermore, line segments D_1E_1 and D_2F_2 intersect at point L, line segments E_1F_1 and E_2D_2 intersect at point M, line segments F_1D_1 and F_2E_2 intersect at point N. Prove that the lines AL, BM, CN are concurrent in the extended plane.

15. (IMO SL 1997) Let $A_1A_2A_3$ be a non-isosceles triangle with incenter I. Let ω_i, $i = 1, 2, 3$, be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (the addition of indices being mod 3). Let B_i, $i = 1, 2, 3$, be the second point of intersection of ω_{i+1} and ω_{i+2}. Prove that the circumcentres of the triangles A_1B_1I, A_2B_2I, A_3B_3I are collinear.

3 References and Further reading

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2. Viktor Prasolov, *Problems in Plane and Solid Geometry*. The English translation is available at `ttp://students.imsa.edu/~tliu/Mat`/planegeo.pdf.

3. H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*. The Mathematical Assoc. of America (1967).

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This article (slightly adapted) was originally used as lecture notes and the accompanying handout by the author at the Canadian Mathematical Society Summer IMO Training Camp at Wilfrid Laurier University, Waterloo in summer 2010.
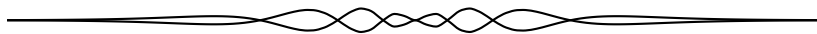
Searching for an Invariant

Y. Ionin and L. Kurlyandchik

This article considers problems of one particular type. Each of the problems deals with some set of numbers or signs together with some pre-specified operations. A thorough reader should discover not only the formal resemblance of the described problems, but also a common approach to their solution.

Problem 1. Ten plus signs and fifteen minus signs are written on the board. You can remove any two signs and replace them with a plus sign if they were the same, and a minus sign if they were different. What sign will remain on the board after 24 such operations?

Solution. Let's substitute every plus sign with a number 1 and every minus with a number -1. The allowed operation then becomes as follows : you remove any two numbers and replace them with their product. Thus, the product of all the numbers on the board stays unchanged. Since this product at the beginning was equal to -1, the number -1 will remain in the end ; that is, a minus sign will remain.

We can solve this problem differently. Let's substitute all plus signs with zeros and minus signs with ones. At the same time, note that the sum of the two removed numbers has the same parity as the number replacing them. Since the total sum of all the numbers was odd in the beginning (namely, equal to 15) , the last number remaining on the board will be odd as well. Hence 1, or minus sign, will remain on the board.

Finally, a third solution to the same problem could be found due to the fact that after each operation the amount of minuses either remains the same or decreases by two. Since the number of minuses was odd in the beginning, a minus sign will also remain in the end. □

Let's analyze the three solutions above. The first one was based on the product staying unchanged ; the second one – on the parity of the sum staying unchanged ; and the third solution – on the parity of the number of minuses staying unchanged. In mathematics, instead of using the word "unchanged", we use the term "invariant". You can say that in every solution we succeeded in finding an invariant : the product of all the numbers, the parity of their sum and the parity of the number of minuses. Solutions to the following problems and exercises are based on the successful invariant choice as well.

Exercise 1. There are several plus signs and minus signs written on the board. You can remove any two signs and replace them with a plus sign if they were the same, and a minus sign if they were different. Prove that the last sign remaining on the board is independent of the order of operations you performed.

Problem 2. A 4×4 table contains plus and minus signs as shown in Figure 1 (left). You can change the signs to the opposite ones simultaneously in all the cells in the same row, column or any line parallel to either of the diagonals (in particular,

FIGURE 1: Tables for Problem 2.

you can always change the sign in any corner cell). Using these operations, is it possible to derive a table which would not contain any minuses at all?

Solution. Let's substitute plus and minus signs with numbers 1 and -1. As an invariant, you can use the product of the numbers located in the shaded cells in Figure 1 (right) : this product always preserves its initial value of -1. This means that there will always be a -1 present in the shaded cells. Therefore, it is impossible to derive a table without minuses. □

Exercise 2. Solve Problem 2 for tables in Figure 2.

FIGURE 2: Tables for Exercise 2.

Problem 3. Several zeros, ones, and twos are written on the board. You can remove any two unequal numbers and replace them with a number distinct from them (that is, replace 0 and 1 with 2, 0 and 2 with 1, 1 and 2 with 0). Prove that after several such operations result in a single number remaining on the board, this number is independent of the order of performed operations.

Solution. Let x_0, x_1 and x_2 denote the initial number of zeros, ones and twos on the board, respectively. By performing the allowed operation once, we change each of these values by 1 and, therefore, we change the parity of all three values as well. When only one number remains on the board, two of x_0, x_1 and x_2 become equal to zero and the third is equal to one. It means that from the very beginning two of those values had the same parity while the third had the different one. Thus, independently of the order of performed operations, at the end only one of x_0, x_1, x_2 can equal 1 and it is the one whose parity differs from the other two. □

This solution illustrates that if x_0, x_1 and x_2 have the same parity to begin with, then the described operations would not result in a single number remaining on the board. Prove that if among x_0, x_1 and x_2 we have both even and odd numbers and, moreover, if at least two of them differ from zero, then there exists a certain order of operations resulting in a single number remaining on the board.

Let's modify the statement of Problem 3 : we now require that the same unequal numbers are erased twice and replaced with a single number distinct from them. Let us assume that again several such operations result in a single number remaining on the board. Based on the initial amount of zeros, ones and twos, is it possible to foresee what number will remain ? Reasoning using parity is not effective here because each operation results in exactly one of the values of x_0, x_1, x_2 changing its parity. Therefore, numbers that start off having the same parity can now end up having different parity. However, you might notice that the numbers x_0, x_1 and x_2 are equivalent modulo 3. The rest of the solution follows that of Problem 3.

Problem 4. Each cell of an 8×8 matrix contains an integer. You can choose any square submatrix of size 3×3 or 4×4 and increase all numbers inside that submatrix by 1. Using these operations, is it always possible to convert the starting matrix to one consisting of numbers all of which are divisible by three ?
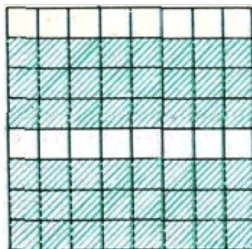
FIGURE 3: Matrix for Problem 4.

Solution. No, not always. Let's find the sum of numbers contained in the shaded cells of Figure 3. Note that any 4×4 submatrix contains 12 shaded cells and any 3×3 submatrix contains 6 or 9 shaded cells. Therefore, after performing the allowed operation, the sum of the numbers in the shaded cells remains unchanged modulo 3. Therefore, if the sum of the numbers was not divisible by 3 from the start, then the shaded cells will always contain numbers not divisible by 3.

Exercise 3. Given the operations of Problem 4, is it possible to convert any starting matrix to one consisting of only odd numbers ?

Problem 5. The numbers $1, 2, 3, \ldots, n$ are arranged in a certain order. You can swap any two adjacent numbers. Prove that after doing any odd number of such operations, you are sure to obtain an arrangement of $1, 2, 3, \ldots, n$ different from the initial one.

Solution. Let $a_1, a_2, a_3, \ldots, a_n$ be a random permutation of the numbers $1, 2, 3, \ldots, n$. We will say that the numbers a_i and a_j create an inversion in this permutation if $i < j$ while $a_i > a_j$; that is, the larger of these numbers precedes the smaller one. By swapping two adjacent numbers, we either increase or decrease the number of inversions by 1. By performing an odd number of such operations, we will change the parity of the number of inversions and hence the arrangement.

Exercise 4. Prove that the conclusion of Problem 5 will remain true if we are allowed to a swap of any two numbers. Hint : prove that you can swap any two numbers by performing the operation described in Problem 5 an odd number of times.

When two numbers from the set of $1, 2, 3, \ldots, n$ are swapped and the rest of the numbers stay where they were, this kind of transition from one permutation to another is called a *transposition*. Then Exercise 4 can be stated as follows : making an odd number of transpositions changes the permutation.

Problem 6. 25 cars start simultaneously in the same direction from different points of a circular motor-racing circuit. According to the race regulations, cars can overtake each other one at a time. The cars finish the race simultaneously at the same points where they started the race. Prove that there were an even number of overtakes during the race.

Solution. Let's paint one car yellow and label other cars with $1, 2, 3, \ldots, 24$ according to their starting position behind the yellow car. Set up a screen in the middle of the circuit ; after each overtake, the screen will display the car numbers in the same order as they follow the yellow car. Then any overtake not involving a yellow car will result in two adjacent numbers swapping positions on the screen.

Let's see what will happen if some car overtakes the yellow one. Before the yellow car was overtaken, the numbers on the screen formed some permutation $a_1, a_2, a_3, \ldots, a_{24}$; after the yellow car was overtaken, the numbers form the permutation $a_2, a_3, a_4, \ldots, a_{24}, a_1$. The same permutation could be obtained after performing 23 consecutive transpositions :

$$a_1, a_2, a_3, \ldots, a_{24} \to a_2, a_1, a_3, \ldots, a_{24} \to a_2, a_3, a_1, \ldots a_{24} \to a_2, a_3, a_4, \ldots, a_{24}, a_1.$$

If a yellow car overtook another car, than a permutation $a_1, a_2, a_3, \ldots, a_{24}$ will become $a_{24}, a_1, a_2, \ldots, a_{23}$. This transition can also be obtained by 23 transpositions.

Thus any overtake is reduced to an odd amount of transpositions. If the total amount of overtakes had been odd, the total amount of permutations would have been odd as well. We can now use the result of Exercise 4 to finish the proof. □

Exercise 5. Five zeros and four ones are placed around a circle in random order. Next, 1 is inserted into the space between two equal numbers and 0 is inserted between two different numbers. Then all the initial numbers are removed. Is it possible to obtain a set of nine zeros by undertaking such an operation several times ?

Exercise 6. Peter tore a piece of paper into ten pieces; some of these pieces he tore into ten pieces again and so on. Could Peter end up with 1975 pieces of paper? 2016 pieces?

Exercise 7. Numbers $1, 2, 3, \ldots, 1975$ are written on the board. You can remove any two numbers and replace them with their sum modulo 13. After several such operations, only one number is left. What number can it be?

Exercise 8. Each number from 1 to 1000000 is replaced with the sum of its digits. The results undergo the same operation until all numbers become single-digit numbers. What digits are there more of: ones or twos?

Exercise 9. Take a three digit number and subtract from it the sum of its digits. Repeat the process 100 times. What number will you get at the end?

Exercise 10. Split a circle into 10 sectors (in a pizza-like fashion) and place a token into each one. In one move, you can move any two tokens into adjacent sectors but so that the tokens move in opposite directions. Can you gather all tokens in the same sector?

Exercise 11.

a) Given a regular 12-gon $A_1 A_2 \ldots A_{12}$, place a minus sign at A_{12} and plus signs at all other vertices. You can pick any three vertices forming a non-right-angled isosceles triangle and swap the signs on its vertices to the opposite ones. Using such operations, can you end up with a minus sign at A_1 and plus signs on all other vertices?

b) Will the result of part a) change if you allow the operation to be performed on any isosceles triangle, including a right-angled one?

Exercise 12. A 4×4 table contains plus and minus signs in every cell. You can change the signs to the opposite ones simultaneously in all the cells in the same row or the same column. Given a starting table, the minimum number of minuses which can result from such an operation is called a *characteristic* of this table. Which values can the characteristic take on?

Exercise 13. 30 tokens, 10 white and 20 black ones, are placed around a circle. You can swap any two tokens separated by three other tokens. Two token arrangements are called *equivalent* if you can get one arrangement from the other one using this operation. How many non-equivalent arrangements are there?

Exercise 14. Numbers $1, 2, 3, \ldots, 1975$ are written in increasing order. You can pick any 4 numbers and put them back into the same positions but in the opposite order. Using this operation, can you ever end up with $1975, 1974, \ldots, 3, 2, 1$?

$$\because \because$$
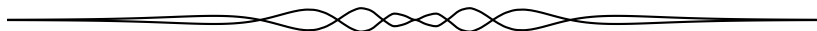
PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1 août 2016** ; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.
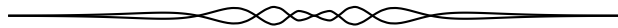
4051. *Proposé par Arkady Alt.*

Soit a, b et c les longueurs des côtés d'un triangle. Démontrer que

$$(a + b + c)\left(a^2b^2 + b^2c^2 + c^2a^2\right) \geq 3abc\left(a^2 + b^2 + c^2\right).$$

4052. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Soit k un réel fixe, $k < 0$, et soit a, b, c et d des réels tels que $a + b + c + d = 0$ et $ab + bc + cd + da + ac + bd = k$. Démontrer que $abcd \geq -k^2/12$ et déterminer les conditions pour qu'il y ait égalité.

4053. *Proposé par Šefket Arslanagić.*

Démontrer que

$$\frac{\cos\alpha\cos\beta}{\cos\gamma} + \frac{\cos\beta\cos\gamma}{\cos\alpha} + \frac{\cos\alpha\cos\gamma}{\cos\beta} \geq \frac{3}{2},$$

α, β et γ étant les angles d'un triangle acutangle.

4054. *Proposé par Mihaela Berindeanu.*

Déterminer un nombre premier p tel que la somme des chiffres du nombre $(p^2 - 4)^2 - 117(p^2 - 4) + 990$ soit minimale.

4055. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Sachant que $x, y > 0, x \neq y$ et $0 < a < b < \frac{1}{2} < c < d < 1$, démontrer que

$$x\left[\left(\frac{y}{x}\right)^a + \left(\frac{y}{x}\right)^d - \left(\frac{y}{x}\right)^b - \left(\frac{y}{x}\right)^c\right] > y\left[\left(\frac{x}{y}\right)^b + \left(\frac{x}{y}\right)^c - \left(\frac{x}{y}\right)^a - \left(\frac{x}{y}\right)^d\right].$$

4056. *Proposé par Idrissi Abdelkrim-Amine.*

Soit n un entier, $n \geq 2$. On considère des réels a_k, $1 \leq k \leq n$, tels que $a_1 \geq 1 \geq a_2 \geq \ldots \geq a_n > 0$ et $a_1 a_2 \ldots a_n = 1$. Démontrer que

$$\sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} \frac{1}{a_k}.$$

4057. *Proposé par Eeshan Banerjee.*

Soit ABC un triangle non obtusangle ayant pour aire Δ. Soit R le rayon de son cercle circonscrit et r le rayon de son cercle inscrit. Démontrer que

$$\Delta < \left(\frac{\frac{1}{r} + 3R + 3}{7} \right)^7.$$

4058. *Proposé par Francisco Javier García Capitán.*

Soit un triangle ABC. Pour tout X sur la droite BC, soit X_b et X_c les centres respectifs des cercles circonscrits aux triangles ABX et AXC et soit P le point d'intersection de BX_c et de CX_b. Démontrer que le lieu géométrique du point P, lorsque X se déplace sur la droite BC, est la conique qui passe par le centre de gravité, l'orthocentre et les sommets B et C du triangle et dont les tangentes à ces sommets sont les symmédianes correspondantes. (À chaque sommet d'un triangle, la symmédiane est l'image de la médiane par une réflexion par rapport à la bissectrice de l'angle.)

4059. *Proposé par Marcel Chiriţa.*

Soit $a, b \in (0, \infty)$, $a \neq b$. Déterminer les fonctions $f : \mathbb{R} \mapsto \mathbb{R} \setminus \{0\}$ telles que

$$f(ax) = e^x f(bx), \quad \forall x \in \mathbb{R}.$$

4060. *Proposé par Michel Bataille.*

Soit

$$f(x, y) = \frac{xy(x + y)}{(1 - x - y)^3}.$$

Déterminer l'image de f lorsque son domaine est le cercle S défini par l'équation $x^2 + y^2 = 1 - 2x - 2y$.

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

4051. *Proposed by Arkady Alt.*

Let a, b and c be the side lengths of a triangle. Prove that

$$(a + b + c)\left(a^2 b^2 + b^2 c^2 + c^2 a^2\right) \geq 3abc\left(a^2 + b^2 + c^2\right).$$

4052. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let $k < 0$ be a fixed real number. Let a, b, c and d be real numbers such that $a + b + c + d = 0$ and $ab + bc + cd + da + ac + bd = k$. Prove that $abcd \geq -k^2/12$ and determine when equality holds.

4053. *Proposed by Šefket Arslanagić.*

Prove that

$$\frac{\cos \alpha \cos \beta}{\cos \gamma} + \frac{\cos \beta \cos \gamma}{\cos \alpha} + \frac{\cos \alpha \cos \gamma}{\cos \beta} \geq \frac{3}{2},$$

where α, β and γ are angles of an acute triangle.

4054. *Proposed by Mihaela Berindeanu.*

Find a prime p such that the number $(p^2 - 4)^2 - 117(p^2 - 4) + 990$ has a minimum digit sum.

4055. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Prove that if $x, y > 0, x \neq y$ and $0 < a < b < \frac{1}{2} < c < d < 1$ then :

$$x\left[\left(\frac{y}{x}\right)^a + \left(\frac{y}{x}\right)^d - \left(\frac{y}{x}\right)^b - \left(\frac{y}{x}\right)^c\right] > y\left[\left(\frac{x}{y}\right)^b + \left(\frac{x}{y}\right)^c - \left(\frac{x}{y}\right)^a - \left(\frac{x}{y}\right)^d\right].$$

4056. *Proposed by Idrissi Abdelkrim-Amine.*

Let n be an integer, $n \geq 2$. Consider real numbers $a_k, 1 \leq k \leq n$ such that $a_1 \geq 1 \geq a_2 \geq \ldots \geq a_n > 0$ and $a_1 a_2 \ldots a_n = 1$. Prove that

$$\sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} \frac{1}{a_k}.$$

4057. *Proposed by Eeshan Banerjee.*

Let ABC be a non-obtuse triangle with circumradius R, inradius r and area Δ. Prove that

$$\Delta < \left(\frac{\frac{1}{r} + 3R + 3}{7}\right)^7.$$

4058. *Proposed by Francisco Javier García Capitán.*

Let ABC be a triangle. For any X on line BC, let X_b and X_c be the circumcenters of the triangles ABX and AXC, and let P be the intersection point of BX_c and CX_b. Prove that the locus of P as X varies along the line BC is the conic through the centroid, orthocenter, and vertices B and C, and whose tangents at

these vertices are the corresponding symmedians. (Recall that a symmedian is the reflection of a median in the bisector of the corresponding angle.)

4059. *Proposed by Marcel Chiriţa.*

Let $a, b \in (0, \infty)$, $a \neq b$. Determine the functions $f : \mathbb{R} \mapsto \mathbb{R} \setminus \{0\}$ such that
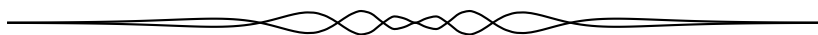
$$f(ax) = e^x f(bx), \quad \forall x \in \mathbb{R}.$$

4060. *Proposed by Michel Bataille.*

Let

$$f(x, y) = \frac{xy(x + y)}{(1 - x - y)^3}.$$

Find the range of $f(x, y)$ when its domain is restricted to the circle S that satisfies the equation $x^2 + y^2 = 1 - 2x - 2y$.

Math Quotes

The discovery in 1846 of the planet Neptune was a dramatic and spectacular achievement of mathematical astronomy. The very existence of this new member of the solar system, and its exact location, were demonstrated with pencil and paper; there was left to observers only the routine task of pointing their telescopes at the spot the mathematicians had marked.

James R. Newman in J. R. Newman (ed.) "The World of Mathematics", New York : Simon and Schuster, 1956.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2014 : 40(6), p. 259–262.

$$\longrightarrow \!\!\!\! \Longleftrightarrow\!\!\!\!\Longleftrightarrow\!\!\!\! \longrightarrow$$

3951. *Proposed by Michel Bataille.*

Let n be a positive integer. Evaluate in closed form

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n+k}{2k}k\cdot 2^{2k}.$$

There were 4 submitted solutions for this problem, all of which were correct. We present the solution by Francisco Perdomo and Ángel Plaza, expanded by the editor.

If S_n is the proposed sum, we will show that

$$S_n = (-1)^{n+1}\binom{2n+2}{3}.$$

The ordinary power series generating function (abbreviated as 'opsgf') of the left-hand side is

$$F(x) = \sum_{n\geq 0}x^n\sum_{k=1}^{n}(-1)^{k-1}\binom{n+k}{2k}k\cdot 2^{2k}$$

$$= \sum_{k}(-1)^{k-1}k\cdot 2^{2k}x^k\sum_{n\geq k}\binom{n+k}{2k}x^{n-k}.$$

We use the following identity, arising from the negative binomial theorem and valid for $k \geq 0$:

$$\sum_{n\geq 0}\binom{n+k}{n}x^n = \frac{1}{(1-x)^{k+1}} \quad .$$

Thus we obtain :

$$F(x) = \sum_{k}(-1)^{k-1}k\cdot 2^{2k}x^k\frac{1}{(1-x)^{2k+1}}$$

$$= \frac{1}{x-1}\sum_{k}k\cdot\left(\frac{-4x}{(x-1)^2}\right)^k.$$

In order to obtain a closed form for the last sum, we introduce a dummy variable y which eventually will be equal to 1. If the opsgf of (a_n), denoted $f(y)$, is given by

$\sum\limits_{n} a_n y^n = f(y)$, then the opsgf of (na_n) is $yDf(y)$, where $Df(y)$ indicates formal differentiation of the series.

Notice that

$$\sum_{k} \left(\frac{-4x}{(x-1)^2}\, y \right)^k = \frac{(x-1)^2}{(x-1)^2 + 4xy}.$$

Let us denote by $g(x, y)$ the series $\sum\limits_{k} k \cdot \left(\frac{-4x}{(x-1)^2} \right)^k y^k$. Then,

$$g(x, y) = \frac{-4x(x-1)^2}{\left((x-1)^2 + 4xy\right)^2},$$

and hence, using that $F(x) = \dfrac{1}{x-1} \cdot g(x, 1)$, it follows that $F(x) = \frac{-4x(x-1)}{(x+1)^4}$.

For the right-hand side of the desired equality, it is easy to prove that $\dbinom{2n+2}{3}$ is the sum of the first n even squares; in particular,

$$\binom{2n+1}{3} = \frac{(2n+2)(2n+1)(2n)}{3!} = \frac{4n(n+1)(2n+1)}{6}$$

$$= 4\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} (2k)^n.$$

Then it has opsgf $h(x) = \dfrac{4x(x+1)}{(x-1)^4}$, by writing the recurrence relation $a_1 = 4$, $a_n = 4n^2 + a_{n-1}$, plugging this relation into the opsgf for a_n, and solving for the opsgf. It follows that the opsgf of $(-1)^{n+1}\dbinom{2n+2}{3}$ will be

$$G(x) = -h(-x) = \frac{-4x(x-1)}{(x+1)^4}.$$

Since $F(x) = G(x)$, the coefficients must be equal, and so the identity is proved.

Editor's Comment. The method of solution above is called the 'Snake Oil Method', due to the late Herbert Wilf in his book *Generatingfunctionology*, a classic book on generating functions. It turns out that its name is relatively appropriate; it is a strong method for calculating summations of this form. In addition, many of the identities used in the proof above can be found in this book.

The proposer made clever use of Chebyshev polynomials of the second kind to give a proof independent of generating functions, and Stadler gave a proof using complex analysis, specifically Cauchy's theorem and residues. Kotronis gave a similar proof using the Snake Oil Method.

3952. *Proposed by Bill Sands.*

Three points O, A, B are on a flat field at coordinates $O(0,0)$, $A(-a, 0)$, $B(0, -b)$, where $a > b > 0$. A balloon is hovering some distance above a point X in the field, where $X = (x, -x)$ for some real number x. To an observer in the balloon, the triangle OAB "looks" equilateral. Show that

$$x = \frac{ab(a+b)^2}{2(a^3 - b^3)} \,,$$

and find the height of the balloon.

We received three correct submissions. We present the solution by Oliver Geupel.

We use three-dimensional Cartesian coordinates so that the given points are $O(0,0,0)$, $A(-a, 0, 0)$, and $B(0, -b, 0)$. Let $P(x, -x, h)$ be the position of the balloon (where $h > 0$ denotes its height), and let p denote the distance $|OP|$. Then, $p^2 = |OP|^2 = 2x^2 + h^2$. By the Law of Cosines we have

$$\cos^2 \angle APO \;=\; \frac{\left(|AP|^2 + |OP|^2 - |AO|^2\right)^2}{4\,|AP|^2\,|OP|^2} = \frac{(p^2 + ax)^2}{p^2(p^2 + 2ax + a^2)}, \qquad (1)$$

$$\cos^2 \angle BPO \;=\; \frac{(p^2 - bx)^2}{p^2(p^2 - 2bx + b^2)}, \qquad (2)$$

$$\cos^2 \angle APB \;=\; \frac{(p^2 + (a-b)x)^2}{(p^2 + 2ax + a^2)(p^2 - 2bx + b^2)}. \qquad (3)$$

If two line segments QQ' and RR' "look" as if they had equal lengths, then the angles $\angle QPQ'$ and $\angle RPR'$ would have equal measures to an observer at P. Hence, by hypothesis, the terms in (1), (2), and (3) are equal.

From (1) and (2) we obtain

$$(p^2 + ax)^2(p^2 - 2bx + b^2) - (p^2 - bx)^2(p^2 + 2ax + a^2)$$

$$= (a+b)(p^2 - x^2)(2abx + (b-a)p^2) = 0,$$

whence $2abx + (b - a)p^2 = 0$; that is,

$$p^2 = \frac{2abx}{a - b}.$$

Plugging this into (1) and (3), we get

$$\frac{(a+b)^2 x}{2ab(2x + a - b)} \;=\; \cos^2 \angle APO = \cos^2 \angle APB = \frac{(a^2 + b^2)^2 x^2}{a^2 b^2 (2x + a - b)^2},$$

$$ab(2x + a - b)(a+b)^2 \;=\; 2(a^2 + b^2)^2 x,$$

$$x \;=\; \frac{ab(a+b)^2}{2(a^3 - b^3)}.$$

This completes the proof.

Finally, $h^2 = p^2 - 2x^2 = \frac{a^2 b^2 (a+b)^2 (a^2 + b^2)}{2(a^3 - b^3)^2}$, so that

$$h = \frac{ab(a+b)}{a^3 - b^3} \sqrt{\frac{a^2 + b^2}{2}}$$

is the height of the balloon.

3953. *Proposed by An Zhen-ping.*

Let ABC be a triangle with sides a, b, c and area Δ. Prove that

$$a^2 \cos \frac{\angle B - \angle C}{2} + b^2 \cos \frac{\angle C - \angle A}{2} + c^2 \cos \frac{\angle A - \angle B}{2} \geq 4\sqrt{3}\Delta.$$

We received nine correct solutions. We present the one given by Cao Minh Quang.

Note first that

$$\frac{a}{b+c} = \frac{\sin A}{\sin B + \sin C} = \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}} = \frac{\sin \frac{A}{2} \cos \frac{A}{2}}{\sin \frac{\pi - A}{2} \cos \frac{B-C}{2}} = \frac{\sin \frac{A}{2}}{\cos \frac{B-C}{2}}.$$

Hence $a \cos \frac{B-C}{2} = (b+c) \sin \frac{A}{2}$ and the given inequality is equivalent, in succession, to

$$\sum a(b+c) \sin \frac{A}{2} \geq 4\sqrt{3}\Delta,$$

$$\sum ab \left(\sin \frac{A}{2} + \sin \frac{B}{2} \right) \geq 4\sqrt{3}\Delta,$$

$$\sum \frac{2\Delta}{\sin C} \left(\sin \frac{A}{2} + \sin \frac{B}{2} \right) \geq 4\sqrt{3}\Delta,$$

$$\sum \frac{1}{\sin C} \left(\sin \frac{A}{2} + \sin \frac{B}{2} \right) \geq 2\sqrt{3}. \tag{1}$$

By the AM-GM inequality we have

$$\sum \frac{1}{\sin C} \left(\sin \frac{A}{2} + \sin \frac{B}{2} \right) \geq 3 \sqrt[3]{\frac{\Pi(\sin \frac{A}{2} + \sin \frac{B}{2})}{\Pi \sin A}}$$

$$\geq 3 \sqrt[3]{\frac{\Pi 2 \sqrt{\sin \frac{A}{2} \sin \frac{B}{2}}}{\Pi \sin A}} = 3 \sqrt[3]{\frac{8\Pi \sin \frac{A}{2}}{8\Pi \sin \frac{A}{2} \cos \frac{A}{2}}}$$

$$= 3 \sqrt[3]{\frac{1}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}}. \tag{2}$$

It is well known (Ed. : see item 2.28 on p. 26 of *Geometric Inequalities* by O. Bottema et al.) that

$$\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \leq \frac{3\sqrt{3}}{8}. \tag{3}$$

Substituting (3) into (2) we then have

$$\sum \frac{1}{\sin C}\left(\sin \frac{A}{2} + \sin \frac{B}{2}\right) \geq 3\sqrt[3]{\frac{8}{3\sqrt{3}}} = 3\left(\frac{2}{\sqrt{3}}\right) = 2\sqrt{3},$$

establishing (1) and completing the proof.

3954. *Proposed by Dao Hoang Viet.*

Solve the inequality

$$\sqrt{4x^2 - 8x + 5} + \sqrt{3x^2 + 12x + 16} \geq 6\sqrt{x} - x - 6.$$

We received 13 correct solutions, two incorrect submissions and one incomplete submission. We present the solution by Digby Smith.

The solution set consists of all nonnegative real numbers, as we shall show in the following. Note that we need $x \geq 0$ in order for the right hand side of the inequality to be defined. Moreover, for all nonnegative real numbers x we have

$$\sqrt{4x^2 - 8x + 5} + \sqrt{3x^2 + 12x + 16} = \sqrt{4(x-1)^2 + 1} + \sqrt{3(x+2)^2 + 4}$$
$$\geq 1 + 2 = 3.$$

On the other hand, $6\sqrt{x} - x - 6 = -(\sqrt{x} - 3)^2 + 3 \leq 3$, completing the proof.

3955. *Proposed by Roy Barbara.*

Let p be a prime number. Suppose that $\sqrt{p} = a + b$, where a and b are algebraic real numbers. Prove or disprove that $\sqrt{p}$ must lie in (at least) one of the fields $\mathbb{Q}(a)$, $\mathbb{Q}(b)$.

We received three correct solutions. We present the solution by Mohammed Aassila.

We will prove that $\sqrt{p}$ must lie in (at least) one of the fields $\mathbb{Q}(a)$, $\mathbb{Q}(b)$. If $a \in \mathbb{Q}$, then $\sqrt{p} = a + b \in \mathbb{Q}(b)$. Similarly, if $b \in \mathbb{Q}$, then $\sqrt{p} \in \mathbb{Q}(a)$. We assume, in the rest of the proof, that a and b are irrational numbers. We also assume, without loss of generality, that $\deg(a) \leq \deg(b)$ over $\mathbb{Q}$. Let $P(X) = \sum_{i=0}^{n} \lambda_i X^i$ be the minimal polynomial of a (with $\lambda_n \neq 0$). We have

$$0 = P(a) = P(\sqrt{p} - b) = \sum_{i=0}^{n} \lambda_i \sum_{k=0}^{i} \binom{i}{k} \sqrt{p}^k (-b)^{i-k}.$$

By rearranging the terms with respect to the parity of k, we obtain an expression of the form $Q(b)\sqrt{p} + R(b) = 0$, with $Q, R \in \mathbb{Q}[X]$. Furthermore, $\deg(Q) = n-1$, and the coefficient of b^{n-1} in Q (obtained when $i = n$ and $k = 1$) is $(-1)^{n-1}n\lambda_n \neq 0$. Hence $Q(b) \neq 0$, and

$$\sqrt{p} = -\frac{R(b)}{Q(b)} \in \mathbb{Q}(b).$$

3956. *Proposed by Dragoljub Milošević.*

Let m_a, m_b and m_c be the lengths of medians, w_a, w_b and w_c be the lengths of the angle bisectors, r and R be the inradius and the circumradius. Prove that

a) $\dfrac{m_a}{w_a} + \dfrac{m_b}{w_b} + \dfrac{m_c}{w_c} \leq 1 + \dfrac{R}{r}$ is true for all acute triangles,

b) $\dfrac{13}{4} - \dfrac{r}{2R} \leq \dfrac{m_a}{w_a} + \dfrac{m_b}{w_b} + \dfrac{m_c}{w_c}$ is true for triangles.

We received three correct solutions. We present the solution by the proposer.

In his solution to **Crux** problem **2963** Vol. 31 (7), p. 351 (first appearing in Vol. 30 (7) p. 367 and p. 370), Arkady Alt proved the following double inequalities :

$$\frac{(b+c)^2}{4bc} \leq \frac{m_a}{w_a} \leq \frac{b^2+c^2}{2bc},$$

with equality if and only if $b = c$. Using this and two other inequalities we obtain

$$\frac{1}{4abc} \sum_{cyc} a(b+c)^2 \leq \sum_{cyc} \frac{m_a}{w_a} \leq \frac{1}{2abc} \sum_{cyc} a(b^2+c),$$

$$\text{or} \quad \frac{1}{4abc}\left((a+b+c)(ab+bc+ca)+3abc\right) \leq \sum_{cyc} \frac{m_a}{w_a}$$

$$\leq \frac{1}{2abc}\left((a+b+c)(ab+bc+ca)-3abc\right). \qquad (1)$$

Using $a+b+c = 2s$ (where s denotes the semiperimeter) and the well-known facts that $abc = 4Rrs$, and $ab + bc + ca = s^2 + 4Rr + r^2$, (1) becomes

$$\frac{1}{16Rrs}\left(2s(s^2+4Rr+r^2)+12Rrs\right) \leq \sum_{cyc} \frac{m_a}{w_a}$$

$$\leq \frac{1}{8Rrs}\left(2s(s^2+4Rr+r^2)-12Rrs\right),$$

$$\text{or} \quad \frac{1}{8Rr}\left(s^2+10Rr+r^2\right) \leq \sum_{cyc} \frac{m_a}{w_a} \leq \frac{1}{4Rr}\left(s^2-2Rr+r^2\right). \qquad (2)$$

By items 5.7 and 5.8 on p. 50 of [1], we have

$$16Rr - 5r^2 \leq s^2, \qquad (3)$$

$$\text{and} \quad 2s^2 \leq \frac{R(4R+r)^2}{2R-r}. \qquad (4)$$

Substituting (3) into the left inequality in (2), we get

$$\sum_{cyc} \frac{m_a}{w_a} \geq \frac{1}{8Rr}(26Rr - 4r^2) = \frac{13}{4} - \frac{r}{2R},$$

establishing b).

On the other hand, if we substitute (4) into the right inequality in (2), then we get

$$\sum_{cyc} \frac{m_a}{w_a} \le \frac{1}{8Rr}\left(\frac{R(4R+r)^2}{2R-r} - 4Rr + r^2\right) = \frac{16R^3 + 9Rr^2 - 2r^3}{8Rr(2R-r)}. \qquad (5)$$

Note that $\dfrac{16R^3 + 9Rr^2 - 2r^3}{8Rr(2R-r)} \le 1 + \dfrac{R}{r} = \dfrac{R+r}{r}$ is equivalent, in succession, to

$$16R^3 + 9Rr^2 - 2r^3 \le 8R(2R^2 + Rr - r^2),$$
$$9Rr^2 - 2r^3 \le 8R^2r - 8Rr^2,$$
$$8R^2 - 17Rr + 2r^2 \ge 0,$$
$$(8R - r)(R - 2r) \ge 0,$$

which is true since $R \ge 2r$ by Euler's Theorem. Hence, from (5) we obtain

$$\sum_{cyc} \frac{m_a}{w_a} \le 1 + \frac{R}{r},$$

which establishes a). Note that equality holds in both inequalities if and only if $a = b = c$.

Reference : [1] O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.

Editor's Comments. The proposer remarked that the proposed inequalities gave a refinement of the result

$$3 \le \sum_{cyc} \frac{m_a}{w_a} \le \frac{3R}{2r}$$

in *Recent Advances in Geometric Inequalities* by D. S. Mitrinović, J. E. Pečarić, and V. Volenec (Kluwer Academic Publisher, 1989), IX. 11.18, p. 219. The solution given by the proposer actually showed that the inequalities hold for all triangles. This was pointed out by Alt and Arslanagić.

3957. *Proposed by Zhihan Gao, Edward Wang and Dexter Wei.*

There are n^2 lamps arranged in an $n \times n$ array. Each lamp is equipped with a switch, which, when pressed, will change the status of the lamp as well as all the lamps in the same row and the same column from on to off or vice versa. This is called an "operation". Initially, all n^2 lamps are off.

Prove that there is a finite sequence of operations which will turn all the n^2 lamps on and determine the value $f(n)$ of the smallest number of operations necessary for all the n^2 lamps to be on.

We received seven correct solutions for this problem. We present the solution by Joseph DiMuro.

We present a more general solution, for the case where the array of lamps is $m \times n$, for positive integers m and n. Let $f(m, n)$ be the smallest number of operations necessary to turn all lamps from off to on in an $m \times n$ array.

First of all, note that the order in which the switches are pressed makes no difference, and pressing a switch twice does not affect any of the lamps. So any optimal solution will press each switch either once or not at all; we just need to determine which switches to press once.

Assume first that m and n are both even. Let X be any lamp, and let S be the set of lamps that are in the same row or column as X (including X itself). At the start, an even number of lamps in S are on (namely, 0); at the end, an odd number of lamps in S will be on (namely, $m + n - 1$). So at some point, we need to press a switch that changes the parity of the number of lamps in S that are on. But the only such switch is X's switch itself, which toggles all $m + n - 1$ lamps in S. All other switches will toggle an even number of lamps in S: either m, or n, or 2. So we must press X's switch. But X was chosen arbitrarily, so we must press all the switches. (This will turn on all the lamps, because each lamp will be toggled $m + n - 1$ times, an odd number.) So $f(m, n) = mn$.

Next, assume m and n are both odd; without loss of generality, assume $m \leq n$. Pressing all of the m switches in one column will turn on all the lamps; all lamps in that column will be toggled m times, and all other lamps will be toggled once. And that turns out to be the optimal solution. If fewer than m switches are pressed, then there will be at least one row and at least one column where no switches are pressed; a lamp at the intersection of that row and that column will never be turned on. So if m and n are odd, and if $m \leq n$, then $f(m, n) = m$. (More generally, if m and n are odd, then $f(m, n) = \min\{m, n\}$.)

Finally, assume m and n have different parities. Without loss of generality, assume m is odd and n is even. As with the last case, we can turn on all the lamps by pressing the m switches in a single column. And that turns out to be the optimal solution, for the following reasons:

If fewer than m switches are pressed, there will be a row where no switches are pressed. Each lamp in that row will be turned on if and only if an odd number of switches are pressed in that lamp's column. So we need to press an odd number of switches in every column. Now consider any lamp X whose switch we do press. We need to press an odd number of switches in X's column (including X's switch itself), so for X to be on at the end, we need to press an odd number of switches in X's row (including X's switch itself). There are an even number of lamps in each row, so there is a lamp in X's row (call it Y) whose switch we will not press. But then we will have pressed an odd number of switches in Y's row and an odd number of switches in Y's column (and not Y's switch itself), so Y will be off at the end. Thus, no scheme where fewer than m switches are pressed will work.

Final conclusion: if m and n are both even, then $f(m, n) = mn$; otherwise, $f(m, n)$ is whichever of m and n is the smaller odd number. Which means, in the original statement of the problem, $f(n) = n^2$ if n is even, and $f(n) = n$ if n is odd.

3958. *Proposed by Marcel Chiriţă.*

Let $A, B \in M_2(\mathbb{R})$, that is A and B are 2×2 matrices over reals. Prove that if $BA = 0$ and $\det(A^2 + AB + B^2) = 0$, then $\det(A + xB) = 0$ for all positive real numbers x.

We received 13 correct submissions. We present the solution by the Skidmore College Problem Group.

The restriction on x is unnecessary. We will show that

$$\det(A + xB) = 0 \quad \text{for all real } x. \tag{1}$$

First, observe that because $BA = 0$, then $A^2 + AB + B^2 = (A + B)^2$. So the hypothesis $\det(A^2 + AB + B^2) = 0$ implies $\det\left((A + B)^2\right) = 0$, whence

$$\det(A + B) = 0. \tag{2}$$

Next, observe that since $BA = 0$, equation (2) implies that $0 = \det(B(A + B)) = \det(B)^2$, whence

$$\det B = 0. \tag{3}$$

Similarly, we have $0 = \det((A + B)A) = \det(A)^2$, so that

$$\det A = 0. \tag{4}$$

Now, let us look at what the hypotheses say in terms of the entries of A and B. So let

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}.$$

Then the condition $BA = 0$ becomes

$$\begin{bmatrix} a_1 b_1 + a_3 b_2 & a_2 b_1 + a_4 b_2 \\ a_1 b_3 + a_3 b_4 & a_2 b_3 + a_4 b_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5}$$

Also, the determinant of $A + xB$ can be expanded :

$$
\begin{aligned}
\det(A + xB) &= \det\left(\begin{bmatrix} a_1 + xb_1 & a_2 + xb_2 \\ a_3 + xb_3 & a_4 + xb_4 \end{bmatrix}\right) \\
&= a_1 a_4 - a_2 a_3 + x^2 b_1 b_4 - x^2 b_2 b_3 + xa_1 b_4 - xa_2 b_3 - xa_3 b_2 + xa_4 b_1 \\
&= (\det B)\, x^2 + (a_1 b_4 - a_2 b_3 - a_3 b_2 + a_4 b_1)\, x + \det(A) \tag{6}
\end{aligned}
$$

Although this isn't necessary, we can, by using (5), rewrite (6) as

$$\det(A + xB) = (\det B)\, x^2 + tr(A) tr(B) x + \det(A), \tag{7}$$

where $tr(A)$ indicates the trace of the matrix A. Notice that with A and B fixed, (3), (4) and (7) combined yield

$$\det(A + xB) = tr(A) tr(B) x, \tag{8}$$

which is a linear polynomial. But (2) shows that $x = 1$ is a zero of this polynomial, while (4) shows that $x = 0$ is also a zero. It follows, since the linear polynomial has two distinct zeros, that $\det(A + xB)$ is the zero polynomial, which implies (1). Alternatively, just observe that (2) combined with (8) yields that $tr(A)tr(B) = 0$. Either way, the proof of (1) is complete.

Editor's Comments. Note that the real numbers in the statement of the problem can be replaced by an arbitrary field.

3959. *Proposed by Dragoljub Milošević.*

Suppose that in a $\triangle ABC$, we have $\angle A = 40°$ and $\angle B = 60°$. If $\dfrac{BC}{AC} = k$, prove that $3k^3 - 3k + 1 = 0$.

We received 23 correct solutions. We present the solution by Matei Coiculescu.

From the law of sines in the triangle ABC, we have

$$\frac{BC}{\sin 40°} = \frac{AC}{\sin 60°},$$

or, denoting $\sin 40° = x$,

$$\frac{BC}{AC} = \frac{\sin 40°}{\sin 60°} = \frac{x}{\frac{\sqrt{3}}{2}} = k.$$

This gives us $x = \frac{k\sqrt{3}}{2}$. It is known that

$$\sin(3\theta) = 3\sin\theta - 4(\sin\theta)^3.$$

For $\theta = 40°$, we have $\frac{\sqrt{3}}{2} = 3x - 4x^3$. Substituting $x = \frac{k\sqrt{3}}{2}$ into this equation, we have

$$4\left(\frac{k\sqrt{3}}{2}\right)^3 - 3\left(\frac{k\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} = 0,$$

which simplifies to

$$3k^3 - 3k + 1 = 0.$$

3960. *Proposed by George Apostolopoulos.*

Let a, b, c be nonnegative real numbers such that $a + b + c = 4$. Prove that

$$\frac{a^2 b}{3a^2 + b^2 + 4ac} + \frac{b^2 c}{3b^2 + c^2 + 4ab} + \frac{c^2 a}{3c^2 + a^2 + 4bc} \geq \frac{1}{2}.$$

The problem first appeared with a typo : the inequalty in the statement of the problem was the wrong way. Six submissions gave the correct statement and provided a solution. Two others gave a counterexample. One submission was flawed. We

present the solution obtained independently by Cao Minh Quang and Titu Zvo-naru.

Since $a^2 + b^2 \geq 2ab$ and

$$\frac{1}{x+y} \leq \frac{1}{4}\left(\frac{1}{x} + \frac{1}{y}\right)$$

for $x, y > 0$, we have that

$$\frac{a^2b}{3a^2 + b^2 + 4ac} \leq \frac{a^2b}{2a^2 + 2ab + 4ac} = \frac{ab}{2(a+b+2c)} \leq \frac{1}{8}\left(\frac{ab}{a+c} + \frac{ab}{b+c}\right),$$

with equality if and only if $a = b$.

In the event that $c = 0$, only the first term of the left side survives and is less than or equal to $(1/8)(a+b) = 4$ with equality if and only if $(a, b, c) = (2, 2, 0)$. The cases $a = 0$ and $b = 0$ can be handled similarly, equality occurring if and only if $(a, b, c) = (0, 2, 2), (2, 0, 2)$. The left side is undefined if more than one variable vanishes.

If a, b, c are all nonzero, then using the inequalities analogous to the foregoing for the other terms, we find that the left side is less than or equal to

$$\frac{1}{8}\left(\frac{ab}{a+c} + \frac{ab}{b+c} + \frac{bc}{a+b} + \frac{bc}{a+c} + \frac{ac}{b+c} + \frac{ac}{a+b}\right)$$
$$= \frac{1}{8}\left(\frac{bc+ac}{a+b} + \frac{ab+ac}{b+c} + \frac{ab+bc}{a+c}\right)$$
$$= \frac{a+b+c}{8} = \frac{1}{2},$$

with equality if and only if $(a, b, c) = (4/3, 4/3, 4/3)$.

Editor's Comments. Three solvers obtained the upper bound $ab(4+c)^{-1} + bc(4+a)^{-1} + ca(4+b)^{-1}$ for the left side. Two of them exploited the related inequality $xy(1+z)^{-1} + yz(1+x)^{-1} + zx(1+y)^{-1} \leq 1/4$ for $x, y, z > 0$ and $x + y + z = 1$. Michel Bataille noted that this inequality was Problem 3350 appearing in **Crux** Vol. 34 (4) p. 241, 243 and Vol. 35 (4) p. 253.
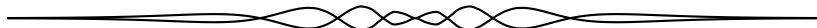
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