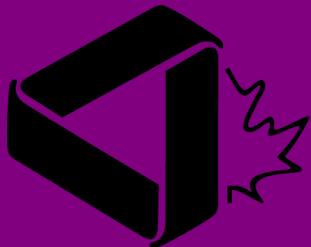


Mathematicorum

Crux

Published by the Canadian Mathematical Society.



<http://crux.math.ca/>

The Back Files

The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

REMARKS ON SOME SYSTEMS OF EQUATIONS

Waldemar Pompe

Consider the following equation:

$$\sqrt[p]{x_1^p + x_2^p + \cdots + x_n^p} = \sqrt[q]{x_1^q + x_2^q + \cdots + x_n^q}, \quad (1)$$

where p , q and n are positive integers ($p \neq q$, $n \geq 2$), and the n -tuple (x_1, x_2, \dots, x_n) consists of real numbers.

If p is even and q is odd, it's easy to show that the only solutions to the equation (1) are of the form $(0, 0, \dots, 0, a, 0, \dots, 0)$, where a is an arbitrary nonnegative real number. (For $p = 2$ and $q = 3$ it was a *Crux* problem, number 1905 [1994: 17], proposed by me. Christopher J. Bradley's short proof [1994: 291] can be easily modified to obtain the generalized version.)

If both the numbers p , q are even, then one also easily obtains that all real solutions to (1) are $(0, 0, \dots, 0, a, 0, \dots, 0)$, where this time a doesn't need to be nonnegative.

So now let p and q be odd. In this case the matter of finding all real solutions to (1) seems to be not that simple. Below the published solution to *Crux* 1905 [1994: 291], Walther Janous asks whether all real solutions of (1) are given by (under possible renumbering of the x_i 's)

$$x_1 = -x_2, \quad x_3 = -x_4, \quad x_5 = -x_6, \quad \text{etc.}, \quad (2)$$

with x_n arbitrary when n is odd. The answer turns out to be negative. It suffices to take $p = 1$, $q = 3$ and $n = 4$. Then $(-5, -1, 7, 8)$ is an example of a solution, which is not of the form (2). (Moreover, this solution has the property that $x_i \neq -x_j$ for all i, j . We will call solutions with this property *nontrivial*.) Knowing this example, one can produce nontrivial solutions to (1) for the larger values of p and q ; for instance $(-\sqrt[3]{5}, -\sqrt[3]{1}, \sqrt[3]{7}, \sqrt[3]{8})$ is a nontrivial solution to (1) with $q = 3p$ and $n = 4$.

Now add some extra equations to (1), that is, consider the following system of equations:

$$\begin{aligned} x_1 + \cdots + x_n &= \sqrt[3]{x_1^3 + \cdots + x_n^3} = \sqrt[3]{x_1^5 + \cdots + x_n^5} \\ &= \cdots = \sqrt[p]{x_1^p + \cdots + x_n^p}, \end{aligned} \quad (3)$$

and assume that p is the biggest odd integer less than n . Is there a nontrivial solution to (3)? Surprisingly, the answer is still "yes"!

THEOREM 1

If p is the biggest odd integer less than n , then there exists a solution to the system (3), such that $x_i \neq -x_j$ for all i, j .

Proof:

Before we construct the example of such a solution, we'll show the following lemma.

Lemma:

Set

$$\sigma_k = \sum_{i=1}^n x_i^k \quad \text{and} \quad \tau_j = \sum_{i_1 < \dots < i_j} x_{i_1} \dots x_{i_j}.$$

Then $\sigma_k = 0$ for all odd k with $1 \leq k < n$ if and only if $\tau_j = 0$ for all odd j with $1 \leq j < n$.

Proof of the Lemma:

It is not too hard to show that

$$\sigma_k - \tau_1 \sigma_{k-1} + \tau_2 \sigma_{k-2} - \dots + (-1)^{k-1} \tau_{k-1} \sigma_1 + (-1)^k k \tau_k = 0 \quad (4)$$

for $1 \leq k < n$ [1]. Therefore using (4) and induction on k (for the "if" part), or on j (for the "only if" part), we obtain the proof of the Lemma. \square

Assume first that n is even; so let $n = 2m$. That means $p = n - 1$. Denote by $-x_{n+1}$ the common value of the roots in (3). Now (3) can be rewritten as the following system of equations:

$$x_1 + \dots + x_{n+1} = 0, \quad x_1^3 + \dots + x_{n+1}^3 = 0, \quad \dots, \quad x_1^{n-1} + \dots + x_{n+1}^{n-1} = 0. \quad (5)$$

Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be distinct positive real numbers. Consider the polynomial

$$\begin{aligned} f(x) &= x(x - \alpha_1)(x + \alpha_1) \dots (x - \alpha_m)(x + \alpha_m) \\ &= x(x^2 - \alpha_1^2) \dots (x^2 - \alpha_m^2) \\ &= x^{n+1} + a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \dots + a_1x, \end{aligned}$$

where $a_k = 0$ for all even k . f has $n + 1$ distinct real roots, so we can choose a real number $c \neq 0$, close enough to 0, such that $f(x) + c$ has still $n + 1$ real roots, say $\beta_1, \beta_2, \dots, \beta_{n+1}$. (The easiest way to see this is to imagine the graph of f intersecting the x -axis in $n + 1$ distinct points and move it up by a small amount.) Thus

$$f(x) + c = x^{n+1} + a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \dots + a_1x + c,$$

which gives $\sum \beta_{i_1} \dots \beta_{i_j} = 0$ for all odd j with $1 \leq j < n + 1$. This is by Viet  's formulae [1], which is a generalization of the theorem that the sum of the roots of the equation $ax^2 + bx + c = 0$ is $-b/a$ and their product is c/a . Hence, according to the Lemma, $\sum \beta_i^k = 0$ for all odd k with $1 \leq k < n + 1$. Therefore $\beta_1, \beta_2, \dots, \beta_{n+1}$ satisfy (5), which means that $\beta_1, \beta_2, \dots, \beta_n$ satisfy (3).

Now we show that this solution is nontrivial, i.e. $\beta_i \neq -\beta_j$ whenever $1 \leq i \leq j \leq n + 1$. Suppose that $\beta_i = -\beta_j$ for some i, j . Then

$$\begin{aligned} f(x) + c &= x^{n+1} + a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \dots + a_3x^3 + a_1x + c \\ &= (x^2 - \beta_i^2)(x^{n-1} + b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + \dots + b_1x + b_0). \end{aligned}$$

Comparing the coefficients of the both polynomials, we obtain in particular that

$$b_{n-2} = 0, \quad b_{n-4} - \beta_i^2 b_{n-2} = 0, \quad b_{n-6} - \beta_i^2 b_{n-4} = 0, \quad \dots, \quad b_0 - \beta_i^2 b_2 = 0,$$

which gives $b_0 = 0$ and consequently $c = 0$, a contradiction. Therefore $\beta_i \neq -\beta_j$ for all $1 \leq i \leq j \leq n+1$.

If n is odd, use the same method to the stronger system of equations

$$\begin{aligned} x_1 + \dots + x_n &= \sqrt[3]{x_1^3 + \dots + x_n^3} = \sqrt[5]{x_1^5 + \dots + x_n^5} \\ &= \dots = \sqrt[p]{x_1^p + \dots + x_n^p} = 0, \end{aligned}$$

which is essentially the same as (5). Theorem 1 is thus proved.

COROLLARY

For any odd positive integers p, q , there exists a solution to (1) with $x_i \neq -x_j$ for all i, j .

Proof:

Take $n > \max(p, q)$ and use Theorem 1. \square

Since the numbers α_i and c in the proof of Theorem 1 were chosen almost freely, we can provide nontrivial solutions having some additional properties (for example exactly two of the x_i 's being equal) by making some particular choices of the α_i 's and c . Also note that making the α_i 's fixed and c variable, we have at least continuum nontrivial and non-proportional n -tuples of solutions to (3).

Let's get deeper into the problem and add another condition to (3). Assume that $p \geq n$. Our question remains the same: Is there a nontrivial solution to (3) in this case? This time the answer is "no".

THEOREM 2

If p is a positive odd integer greater than or equal to n , then all solutions to (3) are either of the form (2) or such that (after possible permutation of the x_i 's)

$$x_1 = 0, \quad x_2 = -x_3, \quad x_4 = -x_5, \quad \text{etc.}, \tag{6}$$

with x_n arbitrary when n is even.

Proof:

Without loss of generality we may assume that p is the least odd integer not less than n . If again $-x_{n+1}$ denotes the common value of the roots in (3), we easily find that (3) is equivalent to (5).

Assume first that n is odd; then $p = n$. Consider the polynomial

$$\begin{aligned} f(x) &= (x - x_1)(x - x_2) \dots (x - x_{n+1}) \\ &= x^{n+1} + a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \dots + a_2x^2 + a_0, \end{aligned}$$

where the last equality follows from the Lemma. Therefore f is an even polynomial, so after possible permutation of the x_i 's, we get

$$x_1 = -x_2, \quad x_3 = -x_4, \quad \dots, \quad x_n = -x_{n+1}.$$

The number $-x_{n+1}$ is chosen to be arbitrary, so x_n is arbitrary as well.

Now let n be even; then $p = n + 1$. Consider the same polynomial

$$\begin{aligned} f(x) &= (x - x_1)(x - x_2) \dots (x - x_{n+1}) \\ &= x^{n+1} + a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \dots + a_1x + a_0, \end{aligned}$$

where again the last equality is due to the Lemma. Note that according to (5)

$$f(x_1) + f(x_2) + \dots + f(x_{n+1}) = (n + 1)a_0.$$

On the other hand, since the x_i 's are the roots of f , the sum $f(x_1) + f(x_2) + \dots + f(x_{n+1})$ is equal to 0. Therefore $a_0 = 0$, which means that f is an odd polynomial. The condition $a_0 = 0$ also implies that at least one of the x_i 's equals 0. If $x_{n+1} = 0$, we obtain a solution of the form (2); if $x_i = 0$ for some i with $1 \leq i \leq n$, we get a solution given by (6). This completes the proof of Theorem 2.

Reference:

- [1] Andrzej Mostowski and Marcell Stark, *Introduction to Higher Algebra*, Pergamon Press, 1964, pp. 345–347.

Acknowledgements:

The author offers his cordial thanks to Federico Ardila, student at Massachusetts Institute of Technology, to Krzysztof Chelmiński, Technische Hochschule Darmstadt, Germany, and to the referee, whose suggested corrections made this article much better.

Waldemar Pompe, student
Department of Mathematics, Informatics and Mechanics
University of Warsaw
ul. Banacha 2
02-097 Warszawa, Poland
e-mail: pompe@laser.mimuw.edu.pl

*

*

*

*

*

THE SKOLIAD CORNER

No. 6

R. E. WOODROW

This month we feature a contest from Atlantic Canada. The paper is that of the Fourteenth Annual New Brunswick Junior High School Mathematics Competition which was held on Friday, May 12. It is jointly organized by the Department of Mathematics and Statistics of the University of New Brunswick and by the Faculty of Science and the School of Engineering of l'Université de Moncton. Over 1400 students participate. Students write the test at one of the two campuses in the morning, have a tour of the campus and attend demonstrations before receiving the results in the afternoon at an awards ceremony. Thanks go to Daryl Tingley for sending me a copy of the French and English versions of the paper. Here is an opportunity to try your French and have some fun!

CONCOURS DE MATHÉMATIQUES 12 Mai 1995 — Une heure

Partie A

- 1.** Quelle est la valeur de $\frac{4}{5} + \frac{3}{2} \times \frac{4}{5}$?
 A. $\frac{24}{25}$ B. $\frac{3}{2}$ C. $\frac{8}{5}$ D. $\frac{45}{25}$ E. 2
- 2.** Un magasin a offert un rabais de 25% sur une paire de skis dont le prix initial était de 90 \$. Ce nouveau prix a ensuite été réduit de 10%. Quel était le prix final?
 A. 31,50 \$ B. 55 \$ C. 58,50 \$ D. 60,75 \$ E. 83 \$
- 3.** Quelle est la somme du tiers de 10 et de la demie du tiers de 20?
 A. 5 B. $\frac{20}{3}$ C. 10 D. $\frac{40}{3}$ E. aucune de ces réponses
- 4.** Deux hommes jouent à un jeu de cartes à raison de 10¢ la partie (payé par le perdant.) À la fin, l'un d'eux a gagné 3 parties et l'autre a gagné 30d. Combien de parties ont-ils jouées?
 A. 6 B. 7 C. 8 D. 9 E. information insuffisante
- 5.** Adam a été un enfant pendant le quart de sa vie, un jeune homme pendant le cinquième de sa vie, un adulte pendant le tiers de sa vie et un retraité pendant 13 ans. À quel âge est-il mort?
 A. 42 B. 56 C. 60 D. 120 E. aucune de ces réponses

6. Commençant à 777 et en comptant à l'envers par 7, un élève compte 777, 770, 763, . . . Parmi les nombres suivants, lequel sera compté?

- A. 41 B. 42 C. 43 D. 44 E. 45

7. Cent billes ont été placées dans 3 bols. Le premier et le second bol contiennent un total de 56 billes, le second et le troisième contiennent un total de 70 billes. Combien de billes y-a-t-il dans le troisième bol?

- A. 24 B. 30 C. 36 D. 44 E. information insuffisante

8. Calculez $1 - \frac{1}{1 + \frac{1}{2 - \frac{1}{3}}}$.

- A. $-\frac{1}{2}$ B. $-\frac{3}{5}$ C. $\frac{1}{2}$ D. $\frac{3}{8}$ E. $\frac{5}{8}$

9. 5 pommes et 3 bananes coûtent 2,45 \$. Si les prix des bananes et des pommes étaient inversés, les mêmes fruits coûteraient 3,13 \$. Combien coûteraient 6 pommes et 6 bananes?

- A. 4,20 \$ B. 4,24 \$ C. 4,40 \$ D. 4,80 \$ E. aucune de ces réponses

10. À un banquet où chaque met est servi à tous les invités, chaque plat de riz fournit deux invités, chaque plat de soupe fournit trois invités et chaque plat de viande fournit quatre invités. Combien y-a-t-il d'invités s'il y a 65 plats en tout?

- A. 42 B. 56 C. 60 D. 120 E. aucune de ces réponses

Partie B

11. Nous définissons $a * b$ comme étant le maximum entre $2a$ et $a + b$. Alors, à quoi est égal $(2 * 3) * (3 * 2)$?

- A. 9 B. 10 C. 11 D. 12 E. aucune de ces réponses

12. Dans un groupe d'hommes et de femmes, l'âge moyen est de 31 ans. Si l'âge moyen des hommes est de 35 ans et celui des femmes est de 25 ans, alors quel est le rapport du nombre d'hommes au nombre de femmes?

- A. $\frac{5}{7}$ B. $\frac{7}{5}$ C. $\frac{2}{1}$ D. $\frac{4}{3}$ E. $\frac{3}{2}$

13. Tous les oiseaux volent.

Certains oiseaux sont des moineaux.

Tous les pinsons chantent.

Certains moineaux chantent.

Tous les moineaux et les pinsons sont des oiseaux.

Si toutes les phrases précédentes sont vraies, laquelle des suivantes doit aussi être vraie?

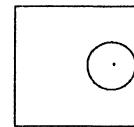
- A. Tous les oiseaux qui volent sont des pinsons.
 B. Tous les moineaux qui volent chantent.
 C. Certains moineaux ne chantent pas.
 D. Les oiseaux qui ne chantent pas ne sont pas des pinsons.
 E. Tous les moineaux ne volent pas.

14. Dans un club, il y a 16 femmes de plus que d'hommes. Si sept fois le nombre de femmes dépasse neuf fois le nombre d'hommes de 32, trouvez le nombre d'hommes.

- A. 4 B. 24 C. 32 D. 42 E. aucune de ces réponses

15. Un cercle de rayon 2 fait un tour complet le long du périmètre à l'intérieur d'un carré de côté 10. Quelle est la distance parcourue par le centre du cercle?

- A. 16 B. 24 C. 32 D. 40 E. aucune de ces réponses

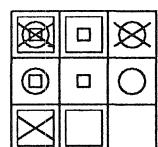


16. Dans une suite de nombre, chaque nombre est obtenu en multipliant le nombre précédent par 2 et en ajoutant le nombre x à ce produit. Si le 6^e nombre est 70 et le 9^e nombre est 609, quelle est la valeur de x ?

- A. 1 B. 3 C. 7 D. 49 E. aucune de ces réponses

17. Parmi les nombres suivants, lequel ne peut pas être exprimé sous la forme $11A + 19B$ avec A et B des entiers positifs?

- A. 30 B. 68 C. 123 D. 211 E. aucune de ces réponses

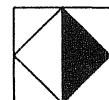


18. Lequel des carrés A , B , C ou D doit venir occuper logiquement la place vide de la figure en bas à droite?

- A. B. C. D. E. information insuffisante

19. On joint les points milieux d'un carré et une partie du carré obtenu est ombragée. Cette partie ombragée représente quelle partie du carré original?

- A. $\frac{1}{8}$ B. $\frac{1}{6}$ C. $\frac{1}{4}$ D. $\frac{1}{3}$ E. $\frac{1}{2}$



20. Une balle qu'on laisse tomber d'une hauteur donnée rebondit à la moitié de cette hauteur. Si la balle est lâchée d'une hauteur de 100 m, quelle distance aura-t-elle parcourue lorsqu'elle touchera le sol pour la 4^e fois?

- A. 137,5 m B. 187,5 m C. 275 m D. 375 m E. information insuffisante

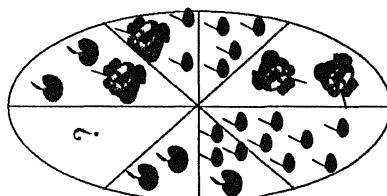
Partie C

21. Si la somme des 100 premiers entiers: $1 + 2 + \dots + 99 + 100 = 5050$, alors quelle est la somme des 50 premiers entiers impairs: $1 + 3 + 5 + \dots + 97 + 99$?

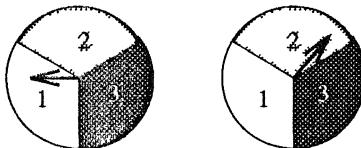
- A. 2500 B. 2524 C. 2525 D. 2550 E. aucune de ces réponses

22. Pommes, cerises et raisins sont disposés sur un plateau de manière à ce que les secteurs opposés contiennent des fruits ayant la même valeur. Pour égaler la valeur de deux grappes de raisins, quels fruits doit-on placer sur le secteur vide?

- A. B. C. D. E. information insuffisante



23. On fait tourner les flèches des deux roulettes ci-dessous. Quelles sont les chances d'obtenir une somme inférieure à 5?



- A. $\frac{1}{3}$ B. $\frac{3}{5}$ C. $\frac{5}{9}$ D. $\frac{2}{3}$ E. information insuffisante

24. Combien de "mots" de trois lettres peuvent être obtenus en utilisant les consonnes *B*, *C*, *D* et les voyelles *A* et *E*? Un "mot" est défini comme étant n'importe quel suite de 3 lettres contenant 2 consonnes et 1 voyelle dans n'importe quel ordre, une lettre pouvant être répétée. Par exemple *ABC* et *CCF* sont des "mots".

- A. 18 B. 27 C. 54 D. 72 E. aucune de ces réponses

25. Une grille rectangulaire est colorée avec les deux couleurs Rouge et Vert de façon à ce que chaque couleur apparaisse deux fois dans chaque rangée et dans chaque colonne. De quelle couleurs doit-on colorer les cases marquées *A* et *B*?

<i>R</i>			<i>V</i>
		<i>V</i>	
	<i>R</i>		<i>B</i>
	<i>A</i>		<i>R</i>

- A. $A = R$ B. $A = R$ C. $A = V$ D. $A = V$ E. information insuffisante

26. Les chiffres de 1 à 4 sont ordonnés de toutes les façons possibles (sans répéter un même chiffre 2 fois) pour faire des nombres de 4 chiffres. Ces nombres sont alors placés en ordre numérique croissant et la liste est divisée en deux moitiés égales. Quel est le dernier nombre de la 1^{re} moitié?

- A. 2314 B. 2134 C. 2431 D. 4123 E. aucune de ces réponses

*

*

*

*

*

Last month we gave the problems of the 4th U.K. Schools Mathematical Challenge. Here are the answers.

- | | | | | |
|-------|-------|-------|-------|-------|
| 1. a | 2. d | 3. c | 4. d | 5. c |
| 6. d | 7. e | 8. b | 9. e | 10. d |
| 11. d | 12. b | 13. a | 14. d | 15. d |
| 16. d | 17. c | 18. d | 19. b | 20. b |
| 21. e | 22. a | 23. c | 24. b | 25. b |

*

*

*

*

*

That completes the space we have this number. Send me your pre-Olympiad material as well as suggestions and comments about the future of the Skoliad Corner.

*

*

*

*

*

THE OLYMPIAD CORNER

No. 166

R. E. WOODROW

All communications about this column should be sent to Professor R. E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin with the 1995 Canadian Mathematical Olympiad which we reproduce with the permission of the Canadian Mathematical Olympiad Committee of the Canadian Mathematical Society. My thanks go to Edward Wang, its chairperson, for sending me the contest, and for agreeing to supply the "official" solutions which we will give in the September number of the Corner.

1995 CANADIAN MATHEMATICAL OLYMPIAD

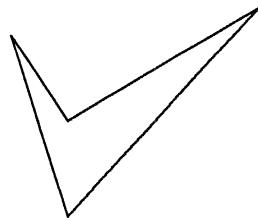
- 1.** Let $f(x) = \frac{9^x}{9^x + 3}$. Evaluate the sum

$$f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + f\left(\frac{3}{1996}\right) + \cdots + f\left(\frac{1995}{1996}\right).$$

- 2.** Let a , b , and c be positive real numbers. Prove that

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}.$$

- 3.** Define a boomerang as a quadrilateral whose opposite sides do not intersect and one of whose internal angles is greater than 180 degrees. (See Figure displayed.) Let C be a convex polygon having s sides. Suppose that the interior region of C is the union of q quadrilaterals, none of whose interiors intersect one another. Also suppose that b of these quadrilaterals are boomerangs. Show that $q \geq b + (s - 2)/2$.



- 4.** Let n be a fixed positive integer. Show that for any nonnegative integer k , the diophantine equation

$$x_1^3 + x_2^3 + \cdots + x_n^3 = y^{3k+2}$$

has infinitely many solutions in positive integers x_i and y .

- 5.** Suppose that u is a real parameter with $0 < u < 1$. Define

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq u \\ 1 - (\sqrt{ux} + \sqrt{(1-u)(1-x)})^2 & \text{if } u \leq x \leq 1 \end{cases}$$

and define the sequence $\{u_n\}$ recursively as follows:

$$u_1 = f(1), \quad \text{and} \quad u_n = f(u_{n-1}) \quad \text{for all } n > 1.$$

Show that there exists a positive integer k for which $u_k = 0$.

* * * *

The next set of problems are from the twentieth-fourth annual United States of America Mathematical Olympiad written April 27, 1995. These problems are copyrighted by the Committee on the American Mathematical Competitions of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C.

Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, N.E., U.S.A. 68588-0322. As always, we welcome your original "nice" solutions and generalizations which differ from the published official solutions. Thanks go to Cecil Rousseau, The University of Memphis, for supplying a TeX file of the contest.

**24th UNITED STATES OF AMERICA
MATHEMATICAL OLYMPIAD**
April 27, 1995
Time Limit: 3.5 hours

1. Let p be an odd prime. The sequence $(a_n)_{n \geq 0}$ is defined as follows: $a_0 = 0$, $a_1 = 1, \dots, a_{p-2} = p - 2$ and, for all $n \geq p - 1$, a_n is the least positive integer that does not form an arithmetic sequence of length p with any of the preceding terms. Prove that, for all n , a_n is the number obtained by writing n in base $p - 1$ and reading the result in base p .

2. A calculator is broken so that the only keys that still work are the sin, cos, tan, \sin^{-1} , \cos^{-1} , and \tan^{-1} buttons. The display initially shows 0. Given any positive rational number q , show that pressing some finite sequence of buttons will yield q . Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.

3. Given a nonisosceles, nonright triangle ABC , let O denote the center of its circumscribed circle, and let A_1 , B_1 , and C_1 be the midpoints of sides BC , CA , and AB , respectively. Point A_2 is located on the ray OA_1 so that ΔOAA_1 is similar to ΔOA_2A . Points B_2 and C_2 on rays OB_1 and OC_1 , respectively, are defined similarly. Prove that lines AA_2 , BB_2 , and CC_2 are concurrent, i.e. these three lines intersect at a point.

4. Suppose q_0, q_1, q_2, \dots is an infinite sequence of integers satisfying the following two conditions:

- (i) $m - n$ divides $q_m - q_n$ for $m > n \geq 0$,
- (ii) there is a polynomial P such that $|q_n| < P(n)$ for all n .

Prove that there is a polynomial Q such that $q_n = Q(n)$ for all n .

5. Suppose that in a certain society, each pair of persons can be classified as either *amicable* or *hostile*. We shall say that each member of an amicable pair is a *friend* of the other, and each member of a hostile pair is a *foe* of the other. Suppose that the society has n persons and q amicable pairs, and that for every set of three persons, at least one pair is hostile. Prove that there is at least one member of the society whose foes include $q(1 - 4q/n^2)$ or fewer amicable pairs.

*

*

*

*

*

For your summer contest pleasure, and to give our readers a chance to submit solutions we give the problems of the second round of the 1992 Dutch Mathematical Olympiad.

1992 DUTCH MATHEMATICAL OLYMPIAD

Second Round

September 18, 1992

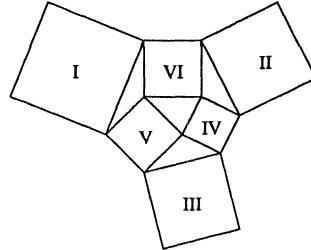
1. Four dice are thrown. What is the chance that the product of the numbers equals 36?

2. In the fraction and its decimal notation (with period of length 4) every letter represents a digit. Different letters denote different digits. The numerator and denominator are mutually prime. Determine the value of the fraction:

$$\frac{ADA}{KOK} = .SNELSNELSNELSNEL\dots$$

[Note. ADA KOK is a famous Dutch swimmer. She won gold in the 1968 Olympic Games in Mexico. SNEL is Dutch for FAST.]

3. The vertices of six squares coincide in such a way that they enclose triangles; see the picture. Prove that the sum of the areas of the three outer squares (I, II and III) equals three times the sum of the areas of the three inner squares (IV, V and VI).



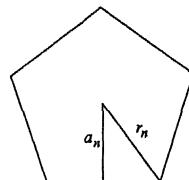
4. For every positive integer n , $n?$ is defined as follows:

$$n? = \begin{cases} 1 & \text{for } n = 1 \\ \frac{n}{(n-1)?} & \text{for } n \geq 2 \end{cases}$$

Prove $\sqrt{1992} < 1992? < \frac{4}{3}\sqrt{1992}$.

5. We consider regular n -gons with a fixed circumference 4. We call the distance from the centre of such a n -gon to a vertex r_n and the distance from the centre to an edge a_n .

- a) Determine a_4, r_4, a_8, r_8 .
- b) Give an appropriate interpretation for a_2 and r_2 .
- c) Prove: $a_{2n} = \frac{1}{2}(a_n + r_n)$ and $r_{2n} = \sqrt{a_{2n}r_n}$



Let $u_0, u_1, u_2, u_3, \dots$ be defined as follows:

$$u_0 = 0, u_1 = 1; u_n = \frac{1}{2}(u_{n-2} + u_{n-1}) \text{ for } n \text{ even and}$$

$$u_n = \sqrt{u_{n-2} \cdot u_{n-1}} \text{ for } n \text{ odd.}$$

d) Determine: $\lim_{n \rightarrow \infty} u_n$.

*

*

*

*

*

For the remainder of this month's Corner we turn to readers' solutions to problems of the Czechoslovak Mathematical Olympiad, Final Round, 1992 [1994: 39].

1. Let $p = (a_1, a_2, \dots, a_{17})$ be any permutation of numbers $1, 2, \dots, 17$. Let k_p denote the greatest index k for which the inequality

$$a_1 + a_2 + \dots + a_k < a_{k+1} + a_{k+2} + \dots + a_{17}$$

holds. Find the greatest and the smallest possible value of k_p and find the sum of all numbers k_p corresponding to all different permutations p .

Solutions by Himadri Choudhury, student, Hunter High School, New York; and by Chris Wildhagen, Rotterdam, The Netherlands. The solutions were very similar, and we give Wildhagen's version.

Clearly k_p has the maximal value of 11 for $p = (1, 2, \dots, 17)$ and the minimal value of 5 for $p = (17, 16, \dots, 1)$.

For any permutation $p = (a_1, \dots, a_{17})$ let $p^* = (a_1^*, a_2^*, \dots, a_{17}^*) = (a_{17}, a_{16}, \dots, a_1)$, the 'reverse permutation'. Thus $a_k^* = a_{18-k}$, $1 \leq k \leq 17$. Note that $1 + 2 + \dots + 17 = 153$. Therefore $a_1 + a_2 + \dots + a_{k_p} \leq 76$, which implies that

$$a_{k_p+1} + \dots + a_{17} \geq 77. \quad (1)$$

Suppose that $a_{k_p+2} + \dots + a_{17} \geq 77$. Then $a_1 + a_2 + \dots + a_{k_p+1} \leq 76$, contradicting the definition of k_p . Thus it follows that

$$a_{k_p+2} + \dots + a_{17} \leq 76. \quad (2)$$

Now (1) and (2) imply that $k_{p^*} = 16 - k_p$, or

$$k_p + k_{p^*} = 16.$$

Since $p \neq p^*$ for all p , it readily follows that $\sum_p k_p = 16 \cdot \frac{1}{2} \cdot 17! = 8 \cdot 17!$

2. Let a, b, c, d, e, f be the lengths of edges of a given tetrahedron and S be its surface area. Prove that

$$S \leq \frac{\sqrt{3}}{6}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2).$$

Solutions by Himadri Choudhury, student, Hunter High School, New York; by Toshio Seimiya, Kawasaki, Japan; and by Panos E. Tsaoussoglou, Athens, Greece. We give Seimiya's solution.

In tetrahedron $ABCD$ we put $AB = a$, $AC = b$, $AD = c$, $BC = d$, $CD = e$ and $BD = f$, and we denote the areas of ΔABC , ΔACD , ΔABD , and ΔBCD by S_1 , S_2 , S_3 and S_4 respectively.

Then the surface area S of the tetrahedron is equal to the sum of S_1 , S_2 , S_3 and S_4 , i.e.

$$S = S_1 + S_2 + S_3 + S_4 \quad (1)$$

Using well known Geometric Inequalities [see item 4.4 of Bottema et al., *Geometric Inequalities*] we get

$$a^2 + b^2 + d^2 \geq 4\sqrt{3}S_1 \quad (2)$$

$$b^2 + c^2 + e^2 \geq 4\sqrt{3}S_2 \quad (3)$$

$$a^2 + c^2 + f^2 \geq 4\sqrt{3}S_3 \quad (4)$$

$$d^2 + e^2 + f^2 \geq 4\sqrt{3}S_4. \quad (5)$$

From (2) + (3) + (4) + (5) we get

$$2(a^2 + b^2 + c^2 + d^2 + e^2 + f^2) \geq 4\sqrt{3}S,$$

by (1).

Hence we have

$$S \leq \frac{\sqrt{3}}{6}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2)$$

as required.

3. Find all natural numbers n which satisfy equalities

$$S(n) = S(2n) = S(3n) = \dots = S(n^2)$$

if $S(x)$ denotes the sum of digits of the number x (in decimal).

Solutions by Himadri Choudhury, student, Hunter High School, New York; and by Bob Prielipp, University of Wisconsin-Oshkosh. We give Prielipp's solution.

The natural numbers n which satisfy our equalities are 1 and $10^m - 1$ for $m = 1, 2, \dots$. To prove this assertion, we begin with some simple, but useful, facts about digital sums. Let $L(x)$ denote the number of large digits (digits greater than or equal to 5) in the number x and let $C(x \oplus y)$ denote the number of carries when x and y are added using the normal algorithm of addition. Then

$$S(2n) = 2S(n) - 9L(n)$$

$$S(m + n) = S(m) + S(n) - 9C(m \oplus n)$$

$$S((10^m - 1) - n) = 9m - S(n).$$

The proofs of these results are fairly straightforward and will be omitted for brevity.

We are now in a position to prove a series of lemmas which will lead to our result.

LEMMA 1. Let $1 \leq n \leq 10^m - 1$ be a positive integer. Then

$$S(n \cdot (10^m - 1)) = 9m.$$

Proof. Let $1 \leq n \leq 10^m - 1$ be a positive integer. Then

$$\begin{aligned} S(n \cdot (10^m - 1)) &= S(n \cdot 10^m - n) \\ &= S(n \cdot 10^m - 10^m + 10^m - 1 + 1 - n) \\ &= S((n-1) \cdot 10^m + ((10^m - 1) - (n-1))) \\ &= S((n-1) \cdot 10^m) + S((10^m - 1) - (n-1)) \\ &= S(n-1) + 9m - S(n-1) \\ &= 9m. \end{aligned}$$

LEMMA 2. Let $S(n) = S(2n)$. Then $9 \mid n$, (9 divides n).

Proof. Let $S(n) = S(2n)$. Then since $S(2n) = 2S(n) - 9L(n)$, $S(n) = 9L(n)$. Hence, $9 \mid S(n)$. Therefore, $9 \mid n$.

LEMMA 3. Let $9 \mid n$ and suppose n has decimal representation $d_m \dots d_1$. Then

$$S(n) \geq 9 \cdot C(n \oplus d_m)$$

with equality if and only if $n = 10^m - 1$.

Proof. Let $9 \mid n$ and suppose n has decimal representation $d_m \dots d_1$. Then since

$$S(n + d_m) = S(n) + d_m - 9C(n \oplus d_m),$$

$$S(n) = S(n + d_m) - d_m + 9C(n \oplus d_m).$$

Now let d_i be the rightmost non-9 digit in n . If d_i does not exist, $n = 10^m - 1$. Thus $S(n + d_m) - d_m = 0$ and we have our result. If d_i exists and $i < m$, then $S(n + d_m) - d_m > 0$. Therefore we have our result. Finally, the case d_i exists and $i = m$ cannot happen, since $9 \mid n$. Thus,

$$S(n) \geq 9 \cdot C(n \oplus d_m)$$

with equality if and only if $n = 10^m - 1$.

LEMMA 4. Let $n > 1$ and

$$S(n) = S(2n) = S(3n) = \dots = S(n^2).$$

Then $n = 10^m - 1$ for some positive integer m .

Proof. Suppose $n > 1$ and

$$S(n) = S(2n) = S(3n) = \dots = S(n^2).$$

It is clear that $n \neq 2, 3, \dots, 9$ and n is not a power of 10. Therefore, let n have decimal representation $d_m \dots d_1$ with $m \geq 2$ and suppose that $n \geq 10^{m-1} + 1$. By Lemma 2, $9 \mid n$. Next, since $S(n) = S((10^{m-1} + 1) \cdot n)$ and $S((10^{m-1} + 1) \cdot n) = S(n) + S(n \cdot 10^{m-1}) - 9C(n \oplus n \cdot 10^{m-1})$, it follows that

$$S(n) = 9 \cdot C(n \oplus d_m).$$

Hence, by Lemma 3, $n = 10^m - 1$.

THEOREM. Let $n > 1$. Then

$$S(n) = S(2n) = S(3n) = \dots = S(n^2)$$

if and only if $n = 10^m - 1$ for some positive integer m .

Proof. The proof follows from Lemmas 1 and 4.

4. Find all solutions of the equation

$$\cos 12x = 5 \sin 3x + 9 \tan^2 x + \cot^2 x.$$

Solutions by Seung-Jin Bang, Seoul, Korea; by Gerd Baron, Technische Universität, Wien, Austria; by Himadri Choudhury, student, Hunter High School, New York; and by D. J. Smeenk, Zaltbommel, The Netherlands. We give Choudhury's solution.

We will show that the expression on the right has a minimum value of 1, which is obviously the maximum value of $\cos 12x$.

$$9 \tan^2 x + \cot^2 x \geq 2\sqrt{9} = 6, \text{ with equality when } 9 \tan^2 x = \cot^2 x \quad (1)$$

whence $3 \tan x = \cot x$, since both have the same sign. Thus $\tan x = \pm 1/\sqrt{3}$ and $x = 30^\circ + 180^\circ n$, or $x = 150^\circ + 180^\circ m$, where n, m are integers.

$$5 \sin 3x \geq -5 \quad (2)$$

with equality when $\sin 3x = -1$, and $x = 90^\circ + 120^\circ l$, where l is an integer.

Combining (1) and (2) we have

$$5 \sin 3x + 9 \tan^2 x + \cot^2 x \geq 1$$

with equality when $x = 210^\circ + 360^\circ n$ or $x = 330^\circ + 360^\circ n$.

Also note that $1 \geq \cos 12x$ with equality when $x = 30^\circ n$. Since this is consistent with the values of x that minimize the right hand side of the expression we have for our solution that

$$x = 210^\circ + 360^\circ n$$

$$x = 330^\circ + 360^\circ n,$$

for an integer n .

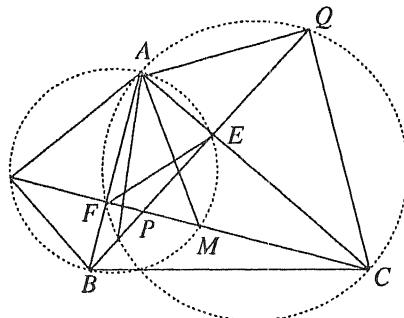
6. In a plane the acute triangle ABC is given. Its altitude through vertex B intersects the circle with diameter AC in points P, Q and the altitude

through point C intersects the circle with diameter AB in points M, N, P, Q . Prove that all the points M, N, P, Q lie on the same circle.

Comment by Himadri Choudhury, student, Hunter High School, New York.

This problem appeared as #5 on the 1990 USAMO. Three nice solutions were given in the MAA solutions pamphlet.

Solutions by Toshio Seimiya, Kawasaki, Japan; and by D. J. Smeenk, Zaltbommel, The Netherlands. We give Smeenk's solution.



It is clear that

$$AM = AN \quad \text{and} \quad AP = AQ, \quad (1)$$

and E and F are the feet of the altitudes to AC and AB . Thus B, C, E, F lie on the circle with diameter BC and

$$AF \cdot AB = AE \cdot AC. \quad (2)$$

Now the angle at N in $\triangle ABN$ is a right angle and we have

$$AN^2 = AF \cdot AB. \quad (3)$$

Similarly from $\triangle ACQ$ we obtain

$$AQ^2 = AE \cdot EC. \quad (4)$$

From (1), (2), (3) and (4) we obtain $AM = AN = AP = AQ$, and

$$AN^2 = AF \cdot AB = bc \cos \alpha = \frac{1}{2}rbc = \frac{1}{2}(-a^2 + b^2 + c^2).$$

Therefore M, N, P and Q lie on a circle centered at A with radius

$$\rho = \sqrt{\frac{-a^2 + b^2 + c^2}{2}}.$$

*

*

*

*

*

That is all the space we have this number. Send me your nice solutions and your Olympiad contests!

*

*

*

*

*

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

Colorado Mathematical Olympiad: The First 10 Years and Further Explorations, by Alexander Soifer. Published by the Center for Excellence in Mathematical Education, Colorado Springs, 1994. ISBN 0-940263-03-3, softcover, 188+ pages, US\$19.95. Reviewed by Andrei Toom, Incarnate Word College, San Antonio, Texas.

According to my knowledge, the first Mathematical Olympiad for high school students was organized in Hungary in the last decade of the 19th century. But the country where Mathematical Olympiads most flourished was Russia. Olympiads, informal classes (called “circles”), popular lectures — all this gave a high school student who lived in Moscow in the 30’s, 50’s, 60’s, 70’s or 80’s rich opportunities for development and provided an excellent start for many future mathematicians (including Alexander and me, for example). The sequel was often frustrating; that is why many of them (us) emigrated to the West and became politically free. But then a sad thing happened: most of us found no way to continue here the same productive combination of research and creative teaching which was so formative for us in Russia and remains a hidden source of strength for so many of us.

Alexander Soifer is a lucky exception. He not only brought Olympiad seeds with him, but also planted and cultivated them, and now the Colorado Mathematical Olympiad is a 10-year reality. Of course, other people also played various vital parts in this story, and Soifer carefully describes their contributions (and settles some personal accounts) in his book. I shall not repeat his interesting Historical Notes; they deserve to be read verbatim.

Still the most important part of the book are the problems of the first ten Olympiads and their solutions.

I believe that the main function of Mathematical Olympiads is to seduce students into thinking. In other words, a good Olympiad problem presents such a special and rewarding experience that some youngsters (later we call them “talented”) become addicted to thinking for the sake of having this exciting experience again and again. From this point of view it is not so important when you solve the problem: during the allocated time or on the next day or even later. You can call this experience insight or you can say that something dawned on you or that some God sent you good advice. Anyway, a good Olympiad problem has some intellectual surprise locked inside as in a puzzle box. For me one of the most rewarding experiences of reading Soifer’s book came from the following problem.

Problem 1.5. (A. Soifer and S. Slobodnik). Forty-one rooks are placed on a 10×10 chessboard. Prove that you can choose five of them that do not attack each other. (We say that two rooks “attack” each other if they are in the same row or column of the chessboard.)

I tried to solve this problem and invented a cumbersome argument, which involved consideration of several cases. The solution provided by the authors is rather long also. But the second solution, found by students, is really beautiful. I shall not rewrite it; read the book or invent it, but remember: it takes only eight lines in the book and you don't need to consider different cases.

Some Olympiad problems can serve as an excellent preparation for the study of professional mathematics. Let us consider two examples.

Problem 5.1. You are given 80 coins. Seventy-nine of these coins are identical in weight, while one is a heavier counterfeit. Using only an equal arm balance, demonstrate a method for identifying the counterfeit coin using only 4 weighings.

An equal arm balance is a device composed of two plates suspended from an arm. By placing a set of coins on each plate, one can determine which set of coins has the greater weight, but can not determine by how much.

This problem belongs to a well-known class which may be called "information theory for children". The maximal number of coins, for which the problem is solvable, is 3^k , where k is the number of weighings. Proof in one direction is based on the fact that every weighing has three possible outcomes. Proof in the other direction is an actual scheme of weighing, which is described in the book (and in many other sources).

There is a useful game in the same vein: I think of an integer number between 1 and 30. You may ask me any questions to which I shall answer only "yes" or "no". (So you should not ask "what is the number ?")! Find out the number using only five questions. (Here, since there are only two possible answers, I should think of a number between 1 and 2^k , where k is the number of questions.) Games, pastimes and puzzles like this are very seminal and I am sure that they contribute a lot to intellectual development of children. Intellectual careers are successful if they start as informal games, leisurely pastimes or even family jokes.

Another example:

Problem 4.3. Each square of a chessboard which is infinite in every direction contains a positive integer. The integer in each square equals the average of the four integers contained in the squares which lie directly above, below, left, and right of it. Show that every square of the board contains the same integer.

The main condition of this problem can be written in the form

$$F_{x,y} = \frac{1}{4}(F_{x-1,y} + F_{x+1,y} + F_{x,y-1} + F_{x,y+1}), \quad (*)$$

where variables $F_{x,y}$ are defined for all $x, y \in \mathbb{Z}$. In Problem 4.3 these variables are positive integers, but we may consider a more general case where

they are any real numbers. This is a discrete analog of the 2-dimensional Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

The two equations have some important common properties, including versions of the maximum principle, whence the idea of how to solve Problem 4.3 comes: just consider the smallest number. I became aware of the discrete Laplace equation (*) as a high school student when some mathematician (whose name I forgot) gave a public lecture which he called “Dirichlet Problem”. And the lecture was quite understandable!

Before a study of differential equations, it is most advisable to study their discrete analogs; but none of the American schools where I have taught differential equations cared about it. The American education reminds me of a speedway where most students drive as fast as they can straight ahead towards graduation. They have no time to enjoy wonderful landscapes. No charming side streets, no mysterious gardens. Only standard road signs in the form of quizzes and tests. Everything that can be omitted, is omitted. Whenever there is a chance to skip a course, students never miss it.¹

Olympiads represent a different, even opposite approach. They concentrate on intellectual difficulties instead of avoiding them. A student who solves a problem like 5.1 or 4.3 is not yet eligible to get a grade in a Computer Science or PDE course, but she deeply understands something that many who have grades, do not. And when she will take these courses, she will understand still more. Who will be a better, more creative scientist or engineer?

Problems given at Olympiads do not need to be really new (although sometimes they are and Dr. Soifer has authored many of them); it is sufficient if they are new for the participants. Using this, Soifer used the rich source of problems provided by many years of the Olympiad activity in Russia. But nobody knows where and when these problems first originated. For example, Soifer attributes the following problem to Tabachnikov and me, because he found it in our book [2]:

Problem 10.2. Four Knights. Four knights are placed on a 3×3 chessboard: two white knights in the upper corners, and two black ones in the lower corners. In one step we are allowed to move any knight in accordance with the chess rules to any empty square. (One knight's move is a result of first taking it two squares in the horizontal or vertical direction, and then moving it one square in the direction perpendicular to the first direction.) Is there a series of steps that ends up with the white knights in diagonally opposite corners, and the black knights in the other pair of opposite corners?

Since it was I who put this problem into [2], let me say what motivated me to do it. I was thinking how to turn “non-standard” problems into stan-

¹Needless to say, the situation in Canadian schools isn't much better.—Ed.

dard ones, that is, how to teach students to solve non-standard problems on a regular basis. (Many wise people claimed that this is impossible.) One idea which came to my mind was to train students to translate a problem from one “language” to another, that is, to change the mode of presentation. In Problem 10.2 this means to draw a graph whose vertices are the nine squares of the board and whose edges correspond to possible moves. As soon as you do this, you see the situation. (You “see” it in the two senses: visualize and understand.) Mainly for the same reason Rubinstein included a similar problem in his book [1], pp. 15 and 204. I had found this problem (perhaps in another version) in some old Russian puzzle-book.

The role of problem solving has been debated in American education for many years. Some universities and colleges even offer special courses of “problem solving”. (I wonder what the other courses are for?) Perhaps the greatest achievement of Olympiads for high school students is that they show that teenagers can solve non-standard mathematical problems. Without Olympiads people might think that this is impossible for some “natural” reasons. When I tell my students today that I solved in a public middle school many of the problems they solve in college, they do not react. Perhaps they do not believe me and imagine that Russia is populated mainly by bears and KGB officers. Or, perhaps, they think that Russia is so far away that laws of nature may be different there. If they had taken a Mathematical Olympiad when they were in high school, their opinions might be different.

Some educators believe that every problem students solve must have an immediate practical relevance. Olympiads show clearly how ridiculous this demand is. As a rule, Olympiad problems deal with some imaginary situation, which seems to be very far from practice. It is most practical to organize Olympiads and participate in them, but this practicality is of a higher nature than some students and educators can understand. I have no doubt that most participants of the Colorado Olympiad will become useful scientists, engineers and, I hope, educators, because we need educators who can solve problems, we need them very badly.

Every review must contain some criticism. Although I am quite fond of Alexander's activity, I think that too many of the problems in the Colorado Olympiads are chosen according to his personal taste. An Olympiad must give equal chances to all participants, even those whose tastes may be different from the taste of the organizer. Problems of the Moscow Olympiads were proposed by many people, whose tastes balanced each other. I hope that the same balance will be achieved in Colorado Springs.

About the book: except for a few misprints (e.g., inequalities on pages 150-151), it seems self-explanatory. I hope that it will be translated into many languages, because it is a useful example of a successful transfer of an important cultural phenomenon from one country to another.

References:

- [1] Moshe F. Rubinstein, *Patterns of Problem Solving*, Prentice-Hall, Inc., 1975.
- [2] S. L. Tabachnikov and A. L. Toom, *Didactic Games*, 1987 (in Russian).

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before January 1, 1996, although solutions received after that date will also be considered until the time when a solution is published.

2051. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A convex quadrilateral $ABCD$ is inscribed in a circle Γ with center O . P is an interior point of $ABCD$. Let O_1, O_2, O_3, O_4 be the circumcenters of triangles PAB, PBC, PCD, PDA respectively. Prove that the midpoints of O_1O_3, O_2O_4 and OP are collinear.

2052. *Proposed by K. R. S. Sastry, Dodballapur, India.*

The infinite arithmetic progression $1 + 3 + 5 + 7 + \dots$ of odd positive integers has the property that all of its partial sums

$$1, \quad 1 + 3, \quad 1 + 3 + 5, \quad 1 + 3 + 5 + 7, \quad \dots$$

are perfect squares. Are there any other infinite arithmetic progressions, all terms positive integers with no common factor, having this same property?

2053. *Proposed by Jisho Kotani, Akita, Japan.*

A figure consisting of two equal and externally tangent circles is inscribed in an ellipse. Find the eccentricity of the ellipse of minimum area.

2054. *Proposed by Murray S. Klamkin, University of Alberta.*

Are there any integral solutions of the Diophantine equation

$$(x + y + z)^3 = 9(x^2y + y^2z + z^2x)$$

other than $(x, y, z) = (n, n, n)$?

2055. *Proposed by Herbert Göllicher, Westfälische Wilhelms-Universität, Münster, Germany.*

In triangle ABC let D be the point on the ray from B to C , and E on the ray from C to A , for which $BD = CE = AB$, and let ℓ be the line through D that is parallel to AB . If $M = \ell \cap BE$ and $F = CM \cap AB$, prove that

$$(BA)^3 = AE \cdot BF \cdot CD.$$

2056. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Find a polynomial of degree 5 whose roots are the tenth powers of the roots of the polynomial $x^5 - x - 1$.

2057*. *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

Let P be a point inside an equilateral triangle ABC , and let R_a, R_b, R_c and r_a, r_b, r_c denote the distances of P from the vertices and edges, respectively, of the triangle. Prove or disprove that

$$\left(1 + \frac{r_a}{R_a}\right) \left(1 + \frac{r_b}{R_b}\right) \left(1 + \frac{r_c}{R_c}\right) \geq \frac{27}{8}.$$

Equality holds if P is the centre of the triangle.

2058. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U.K.*

Let a, b, c be integers such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 3.$$

Prove that abc is the cube of an integer.

2059. *Proposed by Šefket Arslanagić, Berlin, Germany.*

Let $A_1A_2\ldots A_n$ be an n -gon with centroid G inscribed in a circle. The lines A_1G, A_2G, \dots, A_nG intersect the circle again at B_1, B_2, \dots, B_n . Prove that

$$\frac{A_1G}{GB_1} + \frac{A_2G}{GB_2} + \cdots + \frac{A_nG}{GB_n} = n.$$

2060. *Proposed by Neven Juric, Zagreb, Croatia.*

Show that for any positive integers m and n , the integer

$$\left[\left(m + \sqrt{m^2 - 1} \right)^n \right]$$

is odd ($[x]$ denotes the greatest integer less than or equal to x).

* * * * *

DID YOU KNOW . . .

— that 6, 66 and 666 are all triangular numbers?

Are there any other triangular numbers consisting of all 6's? Also, 55 is triangular; are there any other triangular numbers whose digits (more than one, of course) are all the same?

— K. R. S. Sastry

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1895. [1993: 295; 1994: 263] *Proposed by Ji Chen and Gang Yu, Ningbo University, China.*

Let P be an interior point of a triangle $A_1A_2A_3$; R_1, R_2, R_3 the distances from P to A_1, A_2, A_3 ; and R the circumradius of $\Delta A_1A_2A_3$. Prove that

$$R_1R_2R_3 \leq \frac{32}{27}R^3,$$

with equality when $A_2 = A_3$ and $PA_2 = 2PA_1$.

II. Comment by the editor.

As mentioned on [1994: 263], this problem is the same as the *Monthly's* earlier problem 10282. The *Monthly* has now published two solutions to this problem, on pages 468–469 of the May 1995 issue, and as these solutions are quite short and attractive, no further solution will be printed here.

The problem was solved by MARCIN E. KUCZMA, Warszawa, Poland; and the proposers. MURRAY S. KLAMKIN, University of Alberta, brought the *Monthly* problem to the editor's attention. WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, found the problem in the survey paper "Addenda to the monograph 'Recent Advances in Geometric Inequalities', I", by D. S. Mitrinović, J. E. Pečarić, V. Volenec, and Chen Ji, in *Journal of Ningbo University*, Vol. 4, No. 2 (December 1991), page 94, item 23.14. In this monograph an even earlier reference is quoted: Chen Ji, Yu Gang, Wang Zhen, Wan Hui-Hua, Problem 55 and solutions, *Bull. Math. (Wuhan)*, 1990, No. 3 (sum No. 224), 17 and 1991, No. 10 (sum No. 243), 42. Readers are reminded to please inform the editor of all relevant references (certainly including earlier appearances) when proposing a problem to *Crux*.

* * * *

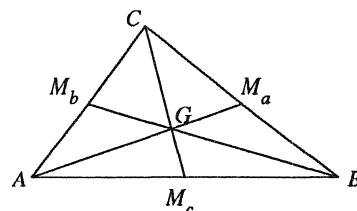
1904. [1994: 16, 289] *Proposed by Kee-Wai Lau, Hong Kong.*

If m_a, m_b, m_c are the medians of a triangle with sides a, b, c , prove that

$$m_a(bc - a^2) + m_b(ca - b^2) + m_c(ab - c^2) \geq 0.$$

II. Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let G be the center of gravity and M_a, M_b, M_c be the midpoints of the sides. We now apply the Möbius-Neuberg inequality (e.g., Mitrinović et al, *Recent Advances in Geometric Inequalities*) to quadrilateral M_cBM_aG and get



$$\overline{BG} \cdot \overline{M_a M_c} \leq \overline{M_c B} \cdot \overline{M_a C} + \overline{BM_a} \cdot \overline{GM_c},$$

i.e.,

$$\frac{2m_b}{3} \cdot \frac{b}{2} \leq \frac{c}{2} \cdot \frac{m_a}{3} + \frac{a}{2} \cdot \frac{m_c}{3},$$

i.e.,

$$2bm_b \leq cm_a + am_c. \quad (1)$$

Thus

$$b^2 m_b \leq \frac{1}{2}(abm_c + bcm_a),$$

and cyclic permutation yields

$$c^2 m_c \leq \frac{1}{2}(bcm_a + cam_b) \quad \text{and} \quad a^2 m_a \leq \frac{1}{2}(cam_b + abm_c).$$

Now adding, we obtain the claimed inequality.

Editor's note. The Möbius–Neuberg inequality is just an extension of Ptolemy's theorem and appeared in *Crux* recently on [1995: 29]. The above solution was received (following an earlier, incorrect solution by Janous) *after* Marcin Kuczma's solution was published on [1994: 289], but *before* Janous would have seen it. The above solution is the same as Kuczma's except that Janous's proof of (1) is much simpler.

Incidentally, Walther also informs the editor that he was married on April 8! Congratulations (from all of us).

* * * * *

1940. [1994: 108; 1995: 107] *Proposed by Ji Chen, Ningbo University, China.*

Show that if $x, y, z > 0$,

$$(xy + yz + zx) \left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2} \right) \geq \frac{9}{4}.$$

II. Red-faced retraction by the editor.

Well, it doesn't happen too often, but the editor was really asleep at the switch this time! Šefket Arslanagić, Berlin, Germany, has pointed out that the published proof by Marcin Kuczma [1995: 107] is fatally flawed. In particular, the application of Chebyshev's inequality is incorrect because the inequality is the wrong way around. Below is what the editor hopes is a correct proof, taken from the ones originally sent in. These may not have been done "as neatly" as Marcin's, as the editor unwisely noted on [1995: 107], but they were certainly done "more correctly"! My apologies.

Marcin also apologizes, and has sent the editor two correct solutions of this problem. Surprisingly, in the two months elapsed since the original

solution appeared, no other readers seem to have noticed the error, probably because they have grown to take Marcin's usual accurate and elegant solutions for granted.

And, by the way, if anyone finds a "nice" solution of this problem, the editor would be interested to see it!

III. Solution by Kee-Wai Lau, Hong Kong.

Without loss of generality suppose that $\min(x, y) \geq z > 0$. Let

$$s = \frac{x+y}{2z} \quad \text{and} \quad t = \frac{xy}{z^2}.$$

It suffices to show that

$$(2s+t)\left(\frac{1}{4s^2} + \frac{4s^2 - 2t + 2 + 4s}{(1+2s+t)^2}\right) \geq \frac{9}{4} \quad (1)$$

whenever $1 \leq t \leq s^2$. [Editor's note. The calculations are:

$$2s+t = \frac{x+y}{z} + \frac{xy}{z^2} = \frac{xy + yz + zx}{z^2}$$

and

$$\begin{aligned} \frac{4s^2 - 2t + 2 + 4s}{(1+2s+t)^2} &= \frac{z^2[(x+y)^2 - 2xy + 2z^2 + 2z(x+y)]}{(z^2 + xy + yz + zx)^2} \\ &= \frac{z^2[(y+z)^2 + (z+x)^2]}{[(y+z)(z+x)]^2} = z^2\left(\frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right), \end{aligned}$$

and thus (1) is just the original inequality; furthermore

$$(x-y)^2 \geq 0 \implies 4xy \leq (x+y)^2 \implies t = \frac{xy}{z^2} \leq \frac{(x+y)^2}{4z^2} = s^2,$$

and $\min(x, y, z) = z$ implies that $t \geq 1$.]

For fixed s and $1 \leq t \leq s^2$ let

$$\begin{aligned} f(t) &= t^3 - (17s^2 - 6s - 2)t^2 + (16s^4 - 36s^3 + 2s^2 + 8s + 1)t \\ &\quad + 32s^5 - 4s^4 - 12s^3 - s^2 + 2s. \end{aligned}$$

It is easy to check that (1) is equivalent to $f(t) \geq 0$. [Editor's note. (1) is equivalent to

$$(2s+t)[(1+2s+t)^2 + 4s^2(4s^2 - 2t + 2 + 4s)] \geq 9s^2(1+2s+t)^2,$$

which simplifies to $f(t) \geq 0$.] Now

$$\begin{aligned} \frac{d^2f}{dt^2} &= 6t - 2(17s^2 - 6s - 2) \\ &\leq 6s^2 - 2(17s^2 - 6s - 2) = -4(7s^2 - 3s - 1) < 0 \end{aligned}$$

for $s \geq 1$ [and thus f is concave down for $s \geq 1$]. Also

$$\begin{aligned} f(1) &= 32s^5 + 12s^4 - 48s^3 - 16s^2 + 16s + 4 \\ &= 4(s-1)(8s^4 + 11s^3 - s^2 - 5s - 1) \geq 0 \end{aligned}$$

and

$$f(s^2) = 2s^5 - 4s^3 + 2s = 2s(s^2 - 1)^2 \geq 0,$$

since $s \geq 1$. Hence for any $s \geq 1$, $f(t) \geq 0$ for all $1 \leq t \leq s^2$, and this completes the solution of the problem.

* * * *

1965*. [1994: 194] *Proposed by Ji Chen, Ningbo University, China.*

Let P be a point in the interior of the triangle ABC , and let the lines AP , BP , CP intersect the opposite sides at D , E , F respectively.

(a) Prove or disprove that

$$PD \cdot PE \cdot PF \leq \frac{R^3}{8},$$

where R is the circumradius of $\triangle ABC$. Equality holds when ABC is equilateral and P is its centre.

(b) Prove or disprove that

$$PE \cdot PF + PF \cdot PD + PD \cdot PE \leq \frac{1}{4} \max\{a^2, b^2, c^2\},$$

where a , b , c are the sides of the triangle. Equality holds when ABC is equilateral and P is its centre, and also when P is the midpoint of the longest side of ABC .

Editor's note.

No solutions to this problem have been received. For part (a), using respectively item 12.39 of Bottema et al, *Geometric Inequalities*, and *Crux* 1895 ([1994: 263] and this issue), we get

$$PD \cdot PE \cdot PF \leq \frac{1}{8} AP \cdot BP \cdot CP \leq \frac{1}{8} \cdot \frac{32}{27} R^3 = \frac{4}{27} R^3;$$

can anyone at least improve this, if not reduce the constant $4/27$ all the way to $1/8$?

* * * *

1966. [1994: 194] *Proposed by Tim Cross, Wolverley High School, Kidderminster, U. K.*

(a) Find all positive integers $p \leq q \leq r$ satisfying the equation

$$p + q + r + pq + qr + rp = pqr + 1.$$

(b) For each such solution (p, q, r) , evaluate

$$\tan^{-1}(1/p) + \tan^{-1}(1/q) + \tan^{-1}(1/r).$$

Solution by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

(a) There are exactly 3 solutions, given by

$$(p, q, r) = (2, 4, 13), \quad (2, 5, 8) \quad \text{and} \quad (3, 3, 7).$$

Note first that

$$(p - 1)(q - 1)(r - 1) = pqr - (pq + qr + rp) + (p + q + r) - 1$$

which becomes, using the given equation,

$$(p - 1)(q - 1)(r - 1) = 2(p + q + r - 1). \quad (1)$$

If $p \geq 4$, then $4 \leq p \leq q \leq r$ implies

$$(p - 1)(q - 1)(r - 1) \geq 9(r - 1) \quad \text{and} \quad 2(p + q + r - 1) \leq 2(3r - 1);$$

and since

$$9(r - 1) - 2(3r - 1) = 3r - 7 > 0,$$

(1) cannot hold in this case. Thus $p < 4$. Since $p = 1$ clearly does not satisfy (1), we have $p = 2$ or 3 . When $p = 2$, (1) becomes $(q - 1)(r - 1) = 2(q + r + 1)$ or $(q - 3)(r - 3) = 10$. Thus $q - 3 = 1, r - 3 = 10$ or $q - 3 = 2, r - 3 = 5$, yielding two solutions: $(2, 4, 13)$ and $(2, 5, 8)$. When $p = 3$, (1) becomes $(q - 1)(r - 1) = q + r + 2$ or $(q - 2)(r - 2) = 5$ which yields the third solution: $(3, 3, 7)$.

(b) The value is $\pi/4$ in all cases. To see this, set $A = \tan^{-1}(1/p)$, $B = \tan^{-1}(1/q)$ and $C = \tan^{-1}(1/r)$. Since

$$0 < \frac{1}{r} \leq \frac{1}{q} \leq \frac{1}{p} < 1,$$

we have $0 < C \leq B \leq A < \pi/4$ and thus $0 < A + B + C < 3\pi/4$. From the well known formula for $\tan(x + y)$ one easily deduces that for all x, y, z with $x + y + z \neq k\pi + \pi/2$ (where k denotes an integer),

$$\tan(x + y + z) = \frac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - (\tan x \tan y + \tan y \tan z + \tan z \tan x)}.$$

Thus

$$\tan(A + B + C) = \frac{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - \frac{1}{pqr}}{1 - \left(\frac{1}{pq} + \frac{1}{qr} + \frac{1}{rp}\right)} = \frac{pq + qr + rp - 1}{pqr - (p + q + r)} = 1.$$

Hence $A + B + C = \pi/4$, that is,

$$\tan^{-1}(1/p) + \tan^{-1}(1/q) + \tan^{-1}(1/r) = \frac{\pi}{4}.$$

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, and MARIA ASCENSION LÓPEZ CHAMORRO, I. B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; HIMADRI CHOUDHURY, student, Hunter High School, New York; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; KEE-WAI LAU, Hong Kong; JOSEPH LING, University of Calgary; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; R. P. SEALY, Mount Allison University, Sackville, New Brunswick; D. J. SMEENK, Zaltbommel, The Netherlands; PANOS E. TSAOUSOGLOU, Athens, Greece; DAVID VELLA, Skidmore College, Saratoga Springs, New York; PAUL YIU, Florida Atlantic University, Boca Raton; and the proposer. There was one incomplete and one incorrect solution. In addition one solver correctly solved part (a), but interpreted the ambiguous \tan^{-1} as cotangent. Most solutions were variations on the above.

* * * *

1967. [1994: 194] *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*

ABC is a triangle and P is a point in its plane. The lines through P parallel to the medians of the triangle meet the opposite sides in points U, V, W . Describe the set of points P for which U, V, W are collinear.

Solution by Toshio Seimiya, Kawasaki, Japan.

The medians concur at the centroid G of $\triangle ABC$; we assume that $PU \parallel AG$, $PV \parallel BG$, $PW \parallel CG$, and that U, V, W are collinear. Let $\Delta A'B'C'$ be an equilateral triangle in the same plane. There exists an affine transformation that sends A, B, C into A', B', C' . Denoting the image points by primes, we have that G' is the centroid of $\Delta A'B'C'$, $P'U' \parallel A'G'$, $P'V' \parallel B'G'$, $P'W' \parallel C'G'$, and that U', V', W' are collinear. Moreover, because $\Delta A'B'C'$ is equilateral we get $A'G' \perp B'C'$, $B'G' \perp C'A'$, and $C'G' \perp A'B'$, so that U', V' and W' are the feet of the perpendiculars from P' to the sides of $\Delta A'B'C'$. Simson's theorem says that these points are collinear if and only if P' lies on the circumcircle of $\Delta A'B'C'$. [Editor's note by Chris Fisher. Some solvers prefer to attribute the theorem to Wallace instead of Simson; the story behind the confusion is told, for example, in Coxeter and Greitzer, *Geometry Revisited*, p. 41.] The inverse transformation takes the circumcircle Γ' of $\Delta A'B'C'$ to the ellipse Γ which passes through A, B, C and has centre G . As P' lies on Γ' if and only if P lies on Γ , this ellipse is the set we are looking for.

Also solved (usually the same way) by JORDI DOU, Barcelona, Spain; J. CHRIS FISHER, University of Regina; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; MARIA ASCENSIÓN LÓPEZ CHAMORRO, I. B. Leopoldo Cano, Valladolid, Spain; P. PENNING, Delft, The Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; D. J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

The ellipse Γ is called the Steiner ellipse of ΔABC . Penning describes it as tangent to each line through a vertex that is parallel to the opposite side; Konečný constructs three further points on Γ as the points of intersection of the lines parallel to the medians that pass through the vertices (i.e., the points on the other end of a diameter through a vertex).

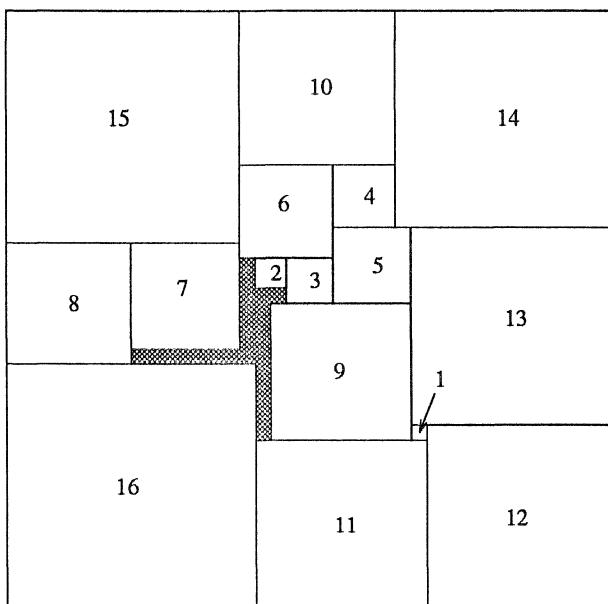
* * * *

1968. [1994: 194] Proposed by Richard Blecksmith and John Selfridge, Northern Illinois University, DeKalb.

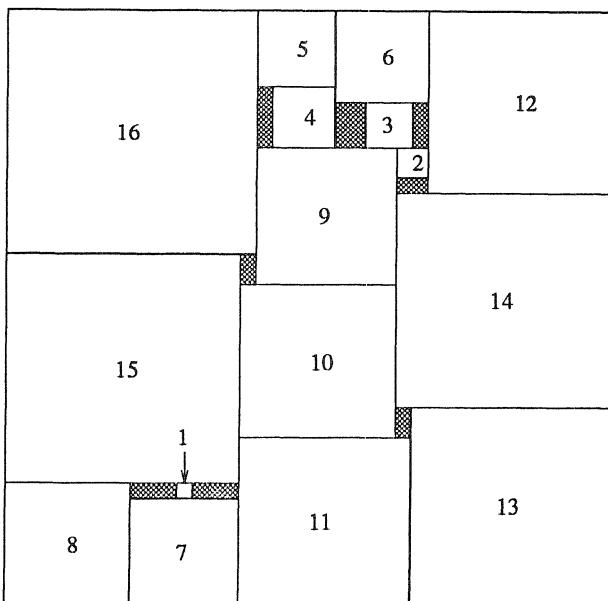
Show how to pack 16 squares of sides 1, 2, ..., 16 into a square of side 39 without overlapping.

Two solutions.

(a)



(b)



Comments by the Editor. For lack of any better criterion, the editor has chosen to print the received solutions with (a) the least and (b) the greatest number of connected unused regions. Solution (a), or one like it, was sent in by Engelhaupt, Jonsson and Konstadinidis, and a couple of other solutions easily turn into this one when some squares are slid around. Based on the solutions received, there seems to be a unique solution having only one connected unused region, up to rotations and reflections, and except for the position of the 2×2 square. The largest number of unused regions in any solution received is five, sent in by four solvers; however, by moving squares around in one of these solutions (Perz's) the editor was able to achieve eight unused regions, as shown in solution (b). Can anyone do better?

Incidentally, suppose you pack 16 squares of arbitrary sizes without overlapping into a square of arbitrary size. What is the largest possible number of unused regions? (Let's say the sides of the small squares should all be parallel to the sides of the large square. Or does it make a difference?)

Solved by CHARLES ASHBACHER, Cedar Rapids, Iowa; TIM CROSS, Wolverley High School, Kidderminster, U. K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; PETER HURTHIG, Columbia College, Burnaby, B. C.; IGNOTUS, Panama; DAG JONSSON, Uppsala, Sweden; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; PAVLOS B. KONSTADINIDIS, student, University of Arizona, Tucson; JOHN L. LEONARD, University of Arizona, Tucson; JOSEPH LING, University of Calgary; GOTTFRIED PERZ, Pestalozigymnasium, Graz, Austria; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, New York; and the proposers.

In item D5, page 113, of Croft, Falconer and Guy, Unsolved Problems in Geometry (Springer-Verlag, 1991), there is a table giving the smallest square into which the squares of integer sides 1 to n will pack, for $n = 1$ to 17. (The first unknown case is $n = 18$.) The 39×39 square is listed as the answer for $n = 16$, but no picture is given. The same table appears in Chapter 11 of Martin Gardner's Mathematical Carnival (reprinted by the MAA in 1989). Also in this chapter is a related problem, which many readers may recall, and which is based on the fact that $1^2 + 2^2 + \dots + 24^2 = 70^2$. The problem is to pack some or all of the 24 squares of sides 1 to 24 into the 70×70 square so as to cover as much area as possible. The answer: all but the 7×7 square can be squeezed in.

* * * * *

LETTER TO THE EDITOR

Dear Dr. Sands,

I enclose a copy of our note with Paul Erdős in the *Australian Math. Soc. Gazette* (1978) to which Kuczma refers in *Crux* 21, p. 30 [re *Crux* 1915 — Ed.]. May I add two amusing incidents relating to the history of the problem. It was at a conference (I think in Canberra) that Paul asked me the problem first, at tea-break; he actually thought it might be non-trivial. During the lecture that followed I did produce a proof (essentially the same as Kuczma) so after the talk I casually walked up to Paul and with straight face said: it looks like an interesting problem, how much would you offer for a solution? Oh, 5 dollars, was his prompt reply, whereupon I put out my hand with a grin. I must say Paul did pay up, the only occasion I ever earned money from him in this way — admittedly not very honestly.

The second incident happened two years ago in Keszthely, Hungary, at one of the numerous conferences to celebrate Paul's 80th birthday. In the intervening 16 years we both forgot about the problem and Paul brought it up again, but this time both of us thought it might be a difficult problem (evidently I lost a few brain-cells in between), and we left it at that. It took a few weeks before Paul realized (accidentally coming through it in his notebooks) that we had in fact written it up in the *Gazette* 16 years earlier, of course together with some other problems and conjectures of Paul about binomial coefficients.²

Regards,

George Szekeres
University of New South Wales
Kensington, NSW 2033, Australia

* * * * *

²Editor's note: one such conjecture is that, for any $1 \leq i < j \leq n/2$, there is always a prime number $\geq i$ which divides into both $\binom{n}{i}$ and $\binom{n}{j}$.