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CONTENTS

THE OLYMPIAD CORNER: 64

M.S. KLAMKIN

I begin with another set of 25 problems which further extends the list of problems submitted by various participating countries (but unused) in past International Mathematics Olympiads. I solicit from all readers elegant solutions to these problems, if possible with extensions or generalizations. Readers submitting solutions should clearly identify the problems by giving their numbers, as well as the year and page number of the issue where they appear.

51. *Proposed by Australia.*

Let P be a regular convex $2n$-gon. Show that there is a $2n$-gon Q with the same vertices as P (but in a different order) such that Q has exactly one pair of parallel sides.

52. *Proposed by Belgium.*

If $n > 2$ is an integer and $[x]$ denotes the greatest integer $\le x$, show that

$$\left[\frac{n(n+1)}{4n-2}\right] = \left[\frac{n+1}{4}\right].$$

53. *Proposed by Belgium.*

If P, Q, and R are polynomials with complex coefficients, not all of them constant polynomials, prove that the identity $P^n+Q^n+R^n \equiv 0$, where n is a natural number, implies that $n < 3$.

[*Note by M.S.K.* As stated, the implication is invalid. Just consider $Q \equiv -P$ and $R \equiv 0$; then $P^3+Q^3+R^3 \equiv 0$. Prove or disprove the implication when the condition "P,Q,R are not all constant polynomials" is changed to "P,Q,R are nonproportional polynomials".]

54. *Proposed by Bulgaria.*

The incircle of triangle $A_1A_2A_3$ touches the sides A_2A_3, A_3A_1, A_1A_2 at the points T_1,T_2,T_3, respectively. If M_1,M_2,M_3 are the midpoints of the sides A_2A_3,A_3A_1, A_1A_2, respectively, prove that the perpendiculars through the points M_1,M_2,M_3 to the lines T_2T_3,T_3T_1,T_1T_2, respectivley, are concurrent.

55. *Proposed by Bulgaria.*

Prove that the volume of a tetrahedron inscribed in a closed right circular cylinder (capped by two disks) of volume 1 does not exceed $2/3\pi$.

56. *Proposed by Canada.*

Given that $a_1,a_2,\ldots,a_{2n}$ are distinct integers such that the equation

$$(x-a_1)(x-a_2)\ldots(x-a_{2n}) + (-1)^{n-1}(n!)^2 = 0$$

has an integer solution r, show that $r = (a_1+a_2+\ldots+a_{2n})/2n$.

57. *Proposed by Canada.*

Let m and n be nonzero integers. Prove that $4mn - m - n$ can be a square infinitely often, but that this is never a square if either m or n is positive.

58. *Proposed by Finland.*

Determine all sequences $\{a_1,a_2,\ldots\}$ such that $a_1 = 1$ and

$$|a_n - a_m| \le \frac{2mn}{m^2+n^2}$$

for all positive integers m and n.

59. *Proposed by Finland.*

A strictly increasing function f defined on $[0,1]$ satisfies $f(0) = 0$, $f(1) = 1$, and

$$\frac{1}{2} \le \frac{f(x+y)-f(x)}{f(x)-f(x-y)} \le 2$$

for all x,y such that $0 \le x\pm y \le 1$. Show that $f(1/3) \le 76/135$.

60. *Proposed by France.*

ABC is an isosceles right triangle with right angle at A. Determine the minimum value of

$$\mathrm{BP} + \mathrm{CP} - \sqrt{3}\mathrm{AP},$$

where P is any point in the plane of the triangle.

61. *Proposed by the Federal Republic of Germany.*

You start with a white balls and b black balls in a container and proceed as follows:

Step 1. You draw one ball at random from the container (each ball being equally likely). If the ball is white, then stop.

Step 2. If the drawn ball is black, then add two black balls to the balls remaining in the container and repeat Step 1.

Let s denote the number of draws until stop. For the cases $a = b = 1$ and $a = b = 2$ only, determine

$$a_n = \Pr(s = n), \qquad b_n = \Pr(s \le n), \qquad \lim_{n\to\infty} b_n,$$

and the expectation $E(s) = \sum_{n\ge1} na_n$.

62. *Proposed by the Federal Republic of Germany.*

A $2\times2\times12$ box is to be filled with twenty-four $1\times1\times2$ bricks. In how many different ways can this be done if the bricks are undistinguishable?

63. *Proposed by Great Britain.*

Prove that the product of five consecutive integers cannot be a perfect square.

64. *Proposed by Great Britain.*

The function $f(n)$ is defined for nonnegative integers n by $f(0) = 0$, $f(1) = 1$, and

$$f(n) = f\left(n - \frac{m(m-1)}{2}\right) - f\left(\frac{m(m+1)}{2} - n\right)$$

for

$$\frac{m(m-1)}{2} < n < \frac{m(m+1)}{2}, \qquad m \geq 2.$$

Determine the smallest integer n for which $f(n) = 5$.

65. *Proposed by Morocco.*

Determine all continuous functions f such that, for all real x and y,

$$f(x+y)f(x-y) = \{f(x)f(y)\}^2.$$

66. *Proposed by the Netherlands.*

Determine positive integers p,q,r such that the diagonal of a block consisting of $p\times q\times r$ unit cubes passes through exactly 1984 of the unit cubes, while its length is a minimum. (The diagonal is said to *pass* through a unit cube if it has more than one point in common with the unit cube.)

67. *Proposed by Poland.*

Let a,b,c be positive numbers such that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} = \frac{\sqrt{3}}{2}.$$

Prove that the system

$$\begin{cases} \sqrt{y-a} + \sqrt{z-a} = 1 \\ \sqrt{z-b} + \sqrt{x-b} = 1 \\ \sqrt{x-c} + \sqrt{y-c} = 1 \end{cases}$$

has exactly one real solution (x,y,z).

68. *Proposed by Poland.*

Let X be an arbitrary nonempty point set in a plane and let $A_1,A_2,\ldots,A_m$ and $B_1,B_2,\ldots,B_n$ be its images under translations (in the plane). If the sets

$A_1, A_2, \ldots, A_m$ are pairwise disjoint and

$$A_1 \cup A_2 \cup \ldots \cup A_m \subset B_1 \cup B_2 \cup \ldots \cup B_n,$$

prove that $m \le n$.

69. *Proposed by Rumania.*

Let $\{a_n\}$ and $\{b_n\}$, $n = 1,2,3,\ldots,$ be two sequences of natural numbers such that, for all $n \ge 1$,

$$a_{n+1} = na_n + 1 \quad \text{and} \quad b_{n+1} = nb_n - 1.$$

Prove that the two sequences can have only a finite number of terms in common.

70. *Proposed by Rumania.*

Let $S_k = x_1^k + x_2^k + \ldots + x_n^k$, where the x_i are real numbers. If

$$S_1 = S_2 = \ldots = S_{n+1},$$

prove that $x_i \in \{0,1\}$ for every $i = 1,2,\ldots,n$.

71. *Proposed by Spain.*

Construct a nonisosceles triangle ABC such that

$$a(\tan B - \tan C) = b(\tan A - \tan C),$$

where a and b are the side lengths opposite angles A and B, respectively.

72. *Proposed by Spain.*

Let P be a convex n-gon with equal interior angles, and let $l_1, l_2, \ldots, l_n$ be the lengths of its consecutive sides. Prove that a necessary and sufficient condition for P to be regular is that

$$\frac{l_1}{l_2} + \frac{l_2}{l_3} + \ldots + \frac{l_n}{l_1} = n.$$

73. *Proposed by Sweden.*

Let $\{a_1, a_2, a_3, \ldots\}$ be an infinite real sequence such that, for all positive integers n and m,

$$a_n \le a_{n+m} \le a_n + a_m.$$

Prove that a_n/n has a limit as $n \to \infty$.

74. *Proposed by the U.S.A.*

A fair coin is tossed repeatedly until there is a run of an odd number of heads followed by a tail. Determine the expected number of tosses.

75. *Proposed by the U.S.A.*

Inside triangle ABC, a circle of radius 1 is externally tangent to the incircle and tangent to sides AB and AC. A circle of radius 4 is externally tangent to the incircle and tangent to sides BA and BC. A circle of radius 9 is externally tangent to the incircle and tangent to sides CA and CB. Determine the inradius of the triangle.

*

I now present solutions and comments for various problems from past columns.

1. [1982: 133] *From the 1982 Australian Mathematical Olympiad.*

If A and B toss $n+1$ and n fair coins, respectively, what is the probability P_n that A gets more heads than B?

Comment by M.S.K.

It is easy to show that $1-P_n$ is the probability that A gets more tails than B. By symmetry $1-P_n = P_n$, so $P_n = 1/2$. For a detailed solution and extensions, see Problem 12-3 [1980: 149, 311].

*

3. [1982: 133] *From the 1982 Australian Mathematical Olympiad.*

Let ABC be a triangle, and let the internal bisector of the angle A meet the circumcircle again at P. Define Q and R similarly. Prove that

$$\mathrm{AP} + \mathrm{BQ} + \mathrm{CR} > \mathrm{BC} + \mathrm{CA} + \mathrm{AB}. \qquad (1)$$

Solution by M.S.K.

Let O,I,R,r be the circumcenter, incenter, circumradius, and inradius, respectively, of the triangle. By the power of a point theorem,

$$\mathrm{AI}\cdot\mathrm{IP} = R^2 - \mathrm{OI}^2 = 2Rr,$$

and so

$$\mathrm{IP} = \frac{2Rr}{\mathrm{AI}} = 2R\sin\frac{\mathrm{A}}{2}.$$

With this and two similar results, inequality (1) is found to be equivalent to

$$\Sigma\left(r\csc\frac{\mathrm{A}}{2} + 2R\sin\frac{\mathrm{A}}{2}\right) > 2R\Sigma\sin\mathrm{A},$$

where the sums are cyclic over A,B,C; or, using

$$r = 4R\sin\frac{\mathrm{A}}{2}\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2} \qquad \text{and} \qquad \sin\frac{\mathrm{A}}{2} = \cos\frac{\mathrm{B+C}}{2}, \text{ etc.},$$

equivalent to

$$\cos\frac{\mathrm{B-C}}{2} + \cos\frac{\mathrm{C-A}}{2} + \cos\frac{\mathrm{A-B}}{2} > \sin\mathrm{A} + \sin\mathrm{B} + \sin\mathrm{C}. \qquad (2)$$

Finally, (2) follows from

$$\cos\frac{B-C}{2} > \cos\frac{B-C}{2}\sin\frac{B+C}{2} = \frac{1}{2}(\sin B + \sin C), \text{ etc. } \square$$

An inequality stronger than (2),

$$\cos\frac{B-C}{2} + \cos\frac{C-A}{2} + \cos\frac{A-B}{2} \geq \frac{2}{\sqrt{3}}(\sin A + \sin B + \sin C),$$

is proved in Problem 613 [1982: 55, 138].

*

4. [1982: 133] *From the 1982 Australian Mathematical Olympiad.*

Find what real numbers d have the following property:

If $f(x)$ is a continuous function for $0 \leq x \leq 1$ and if $f(0) = f(1)$, there exists t such that $0 \leq t < t + d \leq 1$ and $f(t) = f(t+d)$. Equivalently, every continuous graph from (0,0) to (1,0) has a horizontal chord of length d.

Comment by M.S.K.

The graph of every function $f(x)$ continuous for $0 \leq x \leq 1$ such that $f(0) = f(1)$ certainly has a horizontal chord of length $d = 1$; and it follows from the Parallel Chord Theorem of P. Lévy [1] that for such a function there is a t such that $0 \leq t < t+d \leq 1$ for which $f(t) = f(t+d)$, and hence a horizontal chord of length d, only if $d = 1/n$ for some $n = 1,2,3,\ldots$.

Lévy also gave the following example:

$$f(x) = \sin^2\frac{\pi x}{d} - x\sin^2\frac{\pi}{d}, \qquad 0 \leq x \leq 1.$$

Here $f(0) = f(1) = 0$ and, for every t such that $0 \leq t < t+d \leq 1$, we have $f(t) = f(t+d)$, and hence there is a horizontal chord of length d, if $d = 1/n$ for any $n = 1,2,3,\ldots$. This follows from

$$f(t+d) = f(t) - d\sin^2\frac{\pi}{d}.$$

REFERENCE

1. H. Hadwiger and H. Debrunner, *Combinatorial Geometry in the Plane*, Holt, Rinehart & Winston, New York, 1964, p. 23.

*

5. [1982: 134] *From the 1982 Australian Mathematical Olympiad.*

The sequence $p_1, p_2, \ldots$ is defined as follows:

$$p_1 = 2; \; p_n = \text{the largest prime divior of } p_1p_2\ldots p_{n-1} + 1, \quad n \geq 2.$$

Prove that 5 is not a member of this sequence.

Solution by Paul Wagner, Chicago, Illinois.

The definition of the sequence implies that p_n is odd for all $n \geq 2$, and so

$$p_1 p_2 \cdots p_{n-1} \equiv 2 \pmod 4, \qquad n \geq 2. \tag{1}$$

Also, $p_2 = 3$ and $p_n > 3$ for all $n > 2$. Suppose now that $p_r = 5$ for some $r > 2$. Then we must have

$$p_1 p_2 \cdots p_{r-1} + 1 = 5^m$$

for some integer $m \geq 1$. Hence

$$p_1 p_2 \cdots p_{r-1} = 5^m - 1 \equiv 0 \pmod 4,$$

contradicting (1). Therefore 5 is not a member of the sequence.

*

1. [1982: 134] *From the 1982 British Mathematical Olympiad.*

PQRS is a quadrilateral of area A. O is a point inside it. Prove that if

$$2A = OP^2 + OQ^2 + OR^2 + OS^2,$$

then PQRS is a square and O is its centre.

Solution by M.S.K.

We do not assume that the quadrilateral is convex, nor that O is an interior point. From

$$(OP - OQ)^2 + (OQ - OR)^2 + (OR - OS)^2 + (OS - OP)^2 \geq 0, \tag{1}$$

we obtain

$$\begin{aligned} 2A &= OP^2 + OQ^2 + OR^2 + OS^2 \\ &\geq OP\cdot OQ + OQ\cdot OR + OR\cdot OS + OS\cdot OP \\ &\geq 2([OPQ] + [OQR] + [ORS] + [OSP]) \\ &\geq 2A, \end{aligned}$$

where the square brackets denote area, so equality holds throughout here as well as in (1). Hence

$$OP = OQ = OR = OS \quad \text{and} \quad \angle POQ = \angle QOR = \angle ROS = \angle SOP,$$

or, equivalently, PQRS is a square and O is its center. □

Note that one need not assume that O is an interior point of the quadrilateral, since this follows from the above proof.

*

2. [1982: 134] *From the 1982 British Mathematical Olympiad.*

A multiple of 17 when written in the scale of 2 contains exactly three digits 1. Prove that it contains at least six digits 0, and that if it contains exactly seven digits 0, then it is even.

From the official solutions.

In base 2, 17 = 10001. If the given multiple is $17m$ and m contains at most four binary digits, say $m = abcd$ (perhaps with some initial zeros), then we have $17m = abcdabcd$. The number of digits 1 in $17m$ is then even and so cannot equal three. Therefore m must contain at least five binary digits and so $17m$ has at least nine digits. Since exactly three of these are 1's, there must be at least six 0's.

If there are exactly three 1's and seven 0's, and if $17m$ is odd, then

$$17m = 2^9 + 2^x + 1, \qquad \text{where } 1 \le x \le 8.$$

Now for $x = 1,2,\ldots,8$, we find that $2^x \equiv 2,4,8,16,15,13,9,1 \pmod{17}$, respectively, while $2^9+1 \equiv 3$. Hence $2^9 + 2^x + 1 \not\equiv 0 \pmod{17}$, and we have a contradiction. Therefore $17m$ is even.

*

3. [1982: 134] *From the 1982 British Mathematical Olympiad.*

If

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}, \qquad n > 2,$$

prove that

$$n(n+1)^a - n < s_n < n - (n-1)n^b,$$

where a and b are given in terms of n by $an = 1$ and $b(n-1) = -1$.

Solution by A.C.L. Dyson, Leeds Grammar School, England.

By the A.M.-G.M. inequality, for $n \ge 2$,

$$\frac{\frac{n+1}{n} + \frac{n}{n-1} + \ldots + \frac{3}{2} + \frac{2}{1}}{n} > (n+1)^{1/n}$$

$$\Longrightarrow \quad (1+\frac{1}{n}) + (1+\frac{1}{n-1}) + \ldots + (1+\frac{1}{2}) + (1+1) > n(n+1)^{1/n}$$

$$\Longrightarrow \quad n + s_n > n(n+1)^{1/n}.$$

Also, for $n-1 \ge 2$,

$$\frac{\frac{n-1}{n} + \frac{n-2}{n-1} + \ldots + \frac{2}{3} + \frac{1}{2}}{n-1} > (\frac{1}{n})^{1/(n-1)}$$

$$\Longrightarrow \quad (1-\frac{1}{n}) + (1-\frac{1}{n-1}) + \ldots + (1-\frac{1}{3}) + (1-\frac{1}{2}) > (n-1)n^{-1/(n-1)}$$

$$\Longrightarrow \quad n - s_n > (n-1)n^{-1/(n-1)}.$$

Hence, for all $n > 2$,

$$n(n+1)^{a} - n < s_n < n - (n-1)n^{b},$$

where $a = 1/n$ and $b = -1/(n-1)$.

*

6. [1982: 135] *From the 1982 British Mathematical Olympiad.*

Prove that the number of sequences $a_1a_2\ldots a_n$ with each of their n terms $a_i = 0$ or 1 and containing exactly m occurrences of 01 is

$$\binom{n+1}{2m+1}.$$

Solution by Brian Brunswick, Great Britain.

For each sequence $a_1a_2\ldots a_n$ of the required type, consider the augmented sequence

$$a_0a_1a_2\ldots a_na_{n+1}, \tag{1}$$

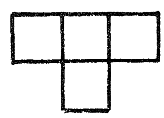
where $a_0 = 1$ and $a_{n+1} = 0$. It is easy to show by induction that each augmented sequence exhibits exactly $2m+1$ occurrences of 01 or 10, consisting of m occurrences of 01 from the original sequence and $m+1$ occurrences of 10 from the augmented sequence. Conversely, every sequence (1) with $a_0 = 1$, $a_{n+1} = 0$, and exactly $2m+1$ occurrences of 01 or 10 gives a sequence $a_1a_2\ldots a_n$ of the desired type when a_0 and a_{n+1} are deleted. The longer sequence is completely specified when the $2m+1$ places for the occurrences 01 or 10 are identified. Since there are $n+1$ possible places a_ia_{i+1}, the desired number is

$$\binom{n+1}{2m+1}.$$

*

4. [1983: 303] *From the 1980 Leningrad High School Olympiad, Third Round.*

Is it possible to arrange the natural numbers from 1 to 64 on an 8×8 checkerboard in such a way that the sum of the numbers in any figure of the form shown on the right is divisible by 5? (The figure can be placed on the board with any orientation.)



Solution by Gali Salvatore, Perkins, Québec.

We show that the arrangement is not possible even if we are allowed to use 64 different natural numbers from 1 to 78.

Suppose the arrangement is possible, and let a,b,c,d,e be the numbers used in any part of the checkerboard having the shape of Figure 1. We then have (congruences throughout are modulo 5)

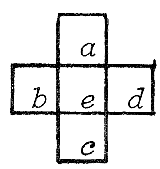


Figure 1

$$a + b + c + e \equiv a + d + c + e \equiv 0 \qquad (1)$$

and

$$b + a + d + e \equiv b + c + d + e \equiv 0. \qquad (2)$$

Now

$$b \equiv d \quad \text{and} \quad a \equiv c \qquad (3)$$

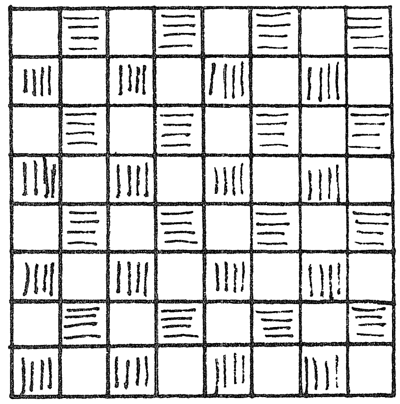


Figure 2

follow from (1) and (2), respectively. We conclude from (3) that the numbers on the 16 squares with horizontal shading in Figure 2 are all congruent to one another modulo 5. None of them is a multiple of 5, for if one of them is then all of them are, and there are only 15 multiples of 5 from 1 to 78. A similar argument shows that no multiple of 5 appears in any of the 16 squares with vertical shading in Figure 2. Therefore no multiple of 5 appears in any of the 32 shaded squares (the black squares, say) of Figure 2. A similar argument shows that no multiple of 5 appears in any of the 32 white squares. Therefore each of the 64 squares of the board is filled by a number that is not a multiple of 5. Since there are only 63 such numbers from 1 to 78, not enough to fill the board, we have the required contradiction.

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P. 383, [1984: 76] *From Középiskolai Matematikai Lapok 67 (1983) 80.*

Does there exist a multiple of 5^{100} which contains no zero in its decimal representation?

Solution by Andy Liu, University of Alberta.

More generally, we prove by induction that, for every $k = 1,2,3,\ldots$, there is a k-digit multiple m_k of 5^k which contains no zero in its decimal representation. For $k = 1$, we can take $m_1 = 5$. We now define

$$m_{k+1} = 10^k t + m_k = 5^k\left(2^k t + \frac{m_k}{5^k}\right),$$

where t is one of 1,2,3,4,5, so chosen that $2^k t + m_k/5^k$ is divisible by 5. Then m_{k+1} is a $(k+1)$-digit multiple of 5^{k+1} which contains no zero in its decimal representation. The first few terms of the sequence are

$$m_1 = 5^1 = 5, \qquad m_5 = 17\cdot 5^5 = 53125,$$

$$m_2 = 5^2 = 25, \qquad m_6 = 29\cdot 5^6 = 453125,$$

$$m_3 = 5^3 = 125, \qquad m_7 = 57\cdot 5^7 = 4453125,$$

$$m_4 = 5\cdot 5^4 = 3125, \qquad m_8 = 37\cdot 5^8 = 14453125.$$

*

P. 384. [1984: 76] *From Középiskolai Matematikai Lapok 67 (1983) 80.*

In a party of 25 members, whenever two members don't know each other, they have common acquaintances among the others. Nobody knows all the others. Prove that if we add the numbers of acquaintances of all members the sum is at least 72. Can the sum be 92? (By *acquaintance* we mean mutual acquaintance.)

Solution by Andy Liu, University of Alberta.

We first point out that no member has exactly one acquaintance, because this sole acquaintance would have to know all the others, and this has been ruled out. To show that the sum, S, of the numbers of acquaintances of all members is at least 72, we assume, on the contrary, that $S \le 70$ (the sum must be even, so it cannot be 71). By the pigeonhole principle, at least one member, say the President, has exactly two acquaintances, say the Secretary and the Treasurer. We refer to the other 22 members as ordinary members. Each of them must know either the Secretary or the Treasurer in order to have access to the President. Hence the sum of the numbers of acquaintances of the Secretary and the Treasurer is at least $2+22 = 24$. The sum of the numbers of acquaintances of the ordinary members is at least $2 \cdot 22 = 44$. Adding the 2 acquaintances of the President, the total count is 70, which must be exact. Hence the Secretary and the Treasurer do not know each other. Moreover, each ordinary member knows exactly one of the Secretary and the Treasurer, and exactly one other ordinary member. Thus they can be classified as Undersecretaries and Undertreasurers. By the pigeonhole principle again, we may assume that there are at least 11 Undersecretaries. In order for the Treasurer to have common acquaintances with each of them, there must be 11 Undertreasurers each knowing a different Undersecretary. However, it is impossible for an Undersecretary and an Undertreasurer who do not know each other to have common acquaintances. This is a contradiction. So $S \ge 72$.

$S = 92$ is possible with a President and a Vice-President who do not know each other, and 23 ordinary members each knowing only the President and the Vice-President.

By a more detailed analysis, Bollobás [1] showed that $S \ge 4n-10$ for a party of n members. When $n = 25$, the *strangest* party (i.e., one with the maximum number of mutual strangers) may consist of a President, a Vice-President, a Secretary, a Treasurer, and 21 ordinary members, where the President knows only the Vice-President and the Secretary, the Vice-President knows only the President and the Treasurer, the Secretary does not know the Treasurer, and each ordinary member knows only the Secretary and the Treasurer. Here $S = 90$, in accordance with the Bollobás result.

REFERENCE

1. Béla Bollobás, *Extremal Graph Theory*, Academic Press, New York, 1978, pp. 173-175.

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1. [1984: 182] *From the 1984 West German Olympiad.*

Given are $2n$ x's in a row. Two players alternately change an x into one of the digits 1, 2, 3, 4, 5, and 6. The second player wins if and only if the resulting $2n$-digit number (in base ten) is divisible by 9. For which values of n is there a winning strategy for the second player?

Solution by Gali Salvatore, Perkins, Québec.

We show that there is a winning strategy for the first player if $n \not\equiv 0 \pmod 9$ and a winning strategy for the second player if $n \equiv 0 \pmod 9$. We recall that a positive integer is divisible by 9 if and only if the sum of its decimal digits is divisible by 9.

Suppose $n \equiv i \pmod 9$, where $0 < i \le 8$. The first player selects his first digit x_0, as explained below, and thereafter he chooses $7-x$ for every x chosen by the second player (except the last). After the second player has chosen his last digit x_1, the sum of the digits is

$$S = x_0 + 7(n-1) + x_1 \equiv x_0 + 7(i-1) + x_1 \pmod 9.$$

The following table gives values of x_0 which will assure the first player of a win, because in each case there is no x_1 for which $S \equiv 0 \pmod 9$.

i	1	2	3	4	5	6	7	8
x_0	1	2	4	6	1	1	3	5

If $n \equiv 0 \pmod 9$, the second player can win by choosing $7-x$ for each x chosen by the first player. At the end of the game, the sum of the digits is

$$S = 7n \equiv 0 \pmod 9.$$

*

2. [1984: 182] *From the 1984 West German Olympiad.*

Determine the sum of the squares of all line segments joining two vertices of a regular n-gon with circumradius 1.

Solution by M.S.K.

More generally, let $A_1A_2\ldots A_n$ be a cyclic polygon, with circumradius 1, for which the centroid coincides with the circumcenter O (as happens when the polygon is regular), and let $\vec{a}_i = \vec{OA}_i$, $i = 1,2,\ldots,n$. Then the required sum is

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\vec{a}_i - \vec{a}_j)^2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(2 - 2\vec{a}_i\cdot\vec{a}_j)$$

$$= n^2 - \left(\sum_{i=1}^{n}\vec{a}_i\right)\cdot\left(\sum_{j=1}^{n}\vec{a}_j\right) = n^2,$$

since $\sum_{i=1}^{n} \vec{a}_i/n = \vec{0}$ when O is the centroid.

*

3. [1984: 182] *From the 1984 West German Olympiad.*

Prove that the product of the positive integers a and b is even if and only if there exist positive integers c and d such that

$$a^2 + b^2 + c^2 = d^2. \tag{1}$$

Solution by Daniel Ropp, student, Stillman Valley High School, Illinois.

Suppose ab is even. If one of a,b is even and the other is odd, then a^2+b^2 is odd, and equation (1) is satisfied by the positive integers

$$c = \frac{a^2 + b^2 - 1}{2} \quad \text{and} \quad d = \frac{a^2 + b^2 + 1}{2}.$$

If a and b are both even, then $a^2+b^2 \equiv 0 \pmod 4$, and equation (1) is satisfied by the positive integers

$$c = \frac{a^2 + b^2 - 4}{4} \quad \text{and} \quad d = \frac{a^2 + b^2 + 4}{4}.$$

Conversely, suppose equation (1) has a solution in positive integers. If a and b are both odd, then $a^2+b^2 \equiv 2 \pmod 4$. This implies that $d^2-c^2 \equiv 2 \pmod 4$. However, the square of any integer is congruent to 0 or 1 modulo 4, and so $d^2-c^2 \not\equiv 2 \pmod 4$. This contradiction shows that at least one of a and b, and hence their product ab, is even.

*

2. [1984: 214, 316] *From the 1983 Dutch Olympiad.*

Prove that if n is an odd positive integer, then the last two digits in base ten of $2^{2n}(2^{2n+1} - 1)$ are 28.

II. *Solution by K.S. Murray, Brooklyn, N.Y.*

With $n = 2m+1$, we must show that

$$2^{4m+2}(2^{4m+3} - 1) - 28$$

is divisible by $100 = 2^2 \cdot 5^2$ for $m = 0,1,2,\ldots$, or, equivalently, that

$$2^{8m+3} - 2^{4m} - 7 \equiv 0 \pmod{25}. \tag{1}$$

Now

$$2^4 - 1 \equiv 0 \pmod 5 \implies (2^{4m} - 1)^2 \equiv 2^{8m} - 2^{4m+1} + 1 \equiv 0 \pmod{25}.$$

Thus (1) is equivalent to $2^3(2^{4m+1} - 1) - 2^{4m} - 7 \equiv 0 \pmod{25}$, or to $15(2^{4m} - 1) \equiv 0 \pmod{25}$, and this is clearly true.

*

1. [1984: 214] *From the 1983 Swedish Mathematical Contest.*

The positive integers are summed in groups in the following way:

$$1, \quad 2+3, \quad 4+5+6, \quad 7+8+9+10, \quad \ldots .$$

Find the sum of the nth group.

Comment by Gali Salvatore, Perkins, Québec.

This problem has appeared many times in the past, including in this journal. See Problem 518 [1981: 63-64] where a simple solution is given, and the editor's comment following the solution for references to earlier occurrences of the problem.

*

2. [1984: 214] *From the 1983 Swedish Mathematical Contest.*

Prove that, for all real numbers x and y,

$$\cos x^2 + \cos y^2 - \cos xy < 3.$$

Solution by Aleksandar Zorović, Elizabethtown High School, Eastview, Kentucky.

Since $|\cos\theta| \le 1$, we certainly have $\cos x^2 + \cos y^2 - \cos xy \le 3$, so we need only show that

$$\cos x^2 + \cos y^2 - \cos xy \ne 3 \qquad (1)$$

for all real numbers x and y. Suppose equality holds in (1). Then

$$\cos x^2 = 1, \qquad \cos y^2 = 1, \qquad \cos xy = -1$$

and

$$x = \pm\sqrt{2k\pi}, \qquad y = \pm\sqrt{2l\pi}, \qquad xy = (2m+1)\pi,$$

where k and l are nonnegative integers and m is an integer. It follows that

$$2m + 1 = \pm 2\sqrt{kl},$$

and this is impossible since $2\sqrt{kl}$ is either irrational or an even integer. Hence (1) is established.

*

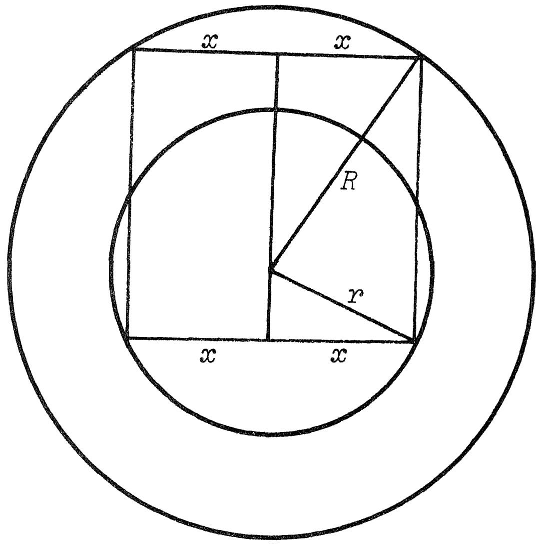
4. [1984: 214] *From the 1983 Swedish Mathematical Contest.*

Two concentric circles have radii r and R. A rectangle has two adjacent vertices on one of the circles. The two other vertices are on the other circle. Determine the length of the sides of the rectangle when its area is maximal.

Solution by K. Seymour, Toronto, Ontario.

It is clear that for maximal area the centre of the circles must not lie outside the rectangle. Referring to the figure, we find that the area function is

$$A(x) = 2x(\sqrt{R^2-x^2} + \sqrt{r^2-x^2}), \qquad 0 \le x \le r.$$



Since $A(x) \geq 0$ for all $x \in [0,r]$ and $A(0) = 0$, it is seen that if the function has a unique critical point in the open interval $(0,r)$, then the corresponding value of the function will be the absolute maximum over the closed interval $[0,r]$, provided this value exceeds $A(r)$. This is indeed what happens, for upon differentiating we find that

$$\frac{dA}{dx} = f(x)(\sqrt{R^2-x^2}\cdot\sqrt{r^2-x^2} - x^2),$$

where $f(x) > 0$ for all $x \in (0,r)$, and the derivative vanishes if and only if

$$x = \frac{Rr}{\sqrt{R^2+r^2}} \in (0,r). \tag{1}$$

Furthermore, for this x the corresponding value of the function is

$$2Rr > 2r\sqrt{R^2-r^2} = A(r).$$

The side lengths of the rectangle of maximal area can now be calculated from (1). They are

$$2x = \frac{2Rr}{\sqrt{R^2+r^2}} \quad \text{and} \quad \frac{2Rr}{2x} = \sqrt{R^2+x^2}.$$

[It would be interesting to have a noncalculus solution for this problem. (M.S.K.)]

*

6. [1984: 214] *From the 1983 Swedish Mathematical Contest.*

Prove that $(x,y) = (1,2)$ is the unique (real) solution of the system

$$\begin{cases} x(x + y)^2 = 9 \\ x(y^3 - x^3) = 7. \end{cases}$$

Comment by M.S.K.

To within a permutation of x and y, this is the same problem as J-10, which is solved in [1981: 109].

*

7. [1984: 215] *From the 1984 British Mathematical Olympiad.*

P, Q, R are arbitrary points on the sides BC, CA, AB, respectively, of triangle ABC. Prove that the triangle whose vertices are the centres of the circles AQR, BRP, CPQ is similar to triangle ABC.

Comment by M.S.K.

This result is the first of several corollaries to the theorem [1]: *All the Miquel triangles of a given point* P *are directly similar, and* P *is the center of similitude or self-homologous point in every case.*

REFERENCE

1. Roger A. Johnson, *Advanced Euclidean Geometry (Modern Geometry)*, Dover, New York, 1960, p. 134.

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3. [1984: 215] *From the 1984 British Mathematical Olympiad.*

(i) Prove that, for all positive integers m,

$$(2-\frac{1}{m})(2-\frac{3}{m})(2-\frac{5}{m})\ldots(2-\frac{2m-1}{m}) \le m!.$$

(ii) Prove that if a,b,c,d,e are positive real numbers, then

$$(\frac{a}{b})^4+(\frac{b}{c})^4+(\frac{c}{d})^4+(\frac{d}{e})^4+(\frac{e}{a})^4 \ge \frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{e}{d}+\frac{a}{e}.$$

Solution by M.S.K.

(i) Our proof is inductive. The result is true for $m = 1$. Suppose it is true for some $m = k \ge 1$. Then

$$(2k-1)(2k-3)\ldots3\cdot1 \le k^k\cdot k!,$$

and the result will be valid for $m = k+1$ if

$$(2k+1)k^k\cdot k! \le (k+1)^{k+1}(k+1)!,$$

or, equivalently, if

$$2k+1 \le (1+\frac{1}{k})^k(k+1)^2.$$

To establish this, we first note that $(1+1/k)^k \ge 2$ for all $k \ge 1$ by the binomial theorem, and then

$$2k+1 < 2(k+1)^2 \le (1+\frac{1}{k})^k(k+1)^2.$$

It follows from our proof that the proposed inequality is always strict except when $m = 1$, when there is equality.

(ii) The following result appears in [1]: for all real nonzero a,b,c,

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \ge \max\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\}.$$

We generalize this as follows. Let n be a positive integer and let x_i, $i = 0,1,\ldots,n$, be $n+1$ positive real numbers (with $x_{n+1} = x_0$). Then

$$S \equiv \left(\frac{x_0}{x_1}\right)^n + \left(\frac{x_1}{x_2}\right)^n + \ldots + \left(\frac{x_n}{x_0}\right)^n \geq \max\left\{\frac{x_0}{x_1}+\frac{x_1}{x_2}+\ldots+\frac{x_n}{x_0},\ \frac{x_1}{x_0}+\frac{x_2}{x_1}+\ldots+\frac{x_0}{x_n}\right\}. \qquad (1)$$

The proposed inequality will then follow when $n = 4$. It is clear that equality holds in (1) for $n = 1$. We now assume that $n > 1$.

We first note that, for any $m > 1$ (all sums run from $i = 0$ to n),

$$\frac{\sum\left(\frac{x_i}{x_{i+1}}\right)^m}{n+1} \geq \left(\frac{\sum\frac{x_i}{x_{i+1}}}{n+1}\right)^m \geq \frac{\sum\frac{x_i}{x_{i+1}}}{n+1}.$$

The first inequality follows from the power mean inequality, and the second from the fact that $\Sigma(x_i/x_{i+1})/(n+1) \geq 1$ by the A.M.-G.M. inequality. In particular, when $m = n$ we get

$$\sum\left(\frac{x_i}{x_{i+1}}\right)^n \geq \sum\frac{x_i}{x_{i+1}}. \qquad (2)$$

We now rewrite S in the form

$$S = \frac{1}{n}\left\{\left(S-\left(\frac{x_0}{x_1}\right)^n\right) + \left(S-\left(\frac{x_1}{x_2}\right)^n\right) + \ldots + \left(S-\left(\frac{x_n}{x_0}\right)^n\right)\right\}.$$

By the A.M.-G.M. inequality, for $i = 0,1,\ldots,n$,

$$\frac{S - \left(\frac{x_i}{x_{i+1}}\right)^n}{n} \geq \frac{x_{i+1}}{x_i},$$

and hence

$$S \geq \sum\frac{x_{i+1}}{x_i}. \qquad (3)$$

Finally, (1) follows from (2) and (3).

REFERENCE

1. M.S. Klamkin, "Asymmetric Triangle Inequalities", *Publikacije Elektrotehničkog Fakulteta Univerziteta U Beogradu (Serija: Matematika I Fizika)*, No. 357 - No. 380 (1971) 42.

*

4. [1984: 215] *From the 1984 British Mathematical Olympiad.*

Let N be a positive integer. Determine, with proof, the number of solutions of the equation

$$x^2 - [x^2] = (x - [x])^2$$

lying in the interval $1 \le x \le N$. (The square brackets denote the greatest integer function.)

Comment by M.S.K.

This problem appeared earlier in the 1982 Swedish Olympiad. For a solution, see [1984: 46].

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5. [1984: 215] *From the 1984 British Mathematical Olympiad.*

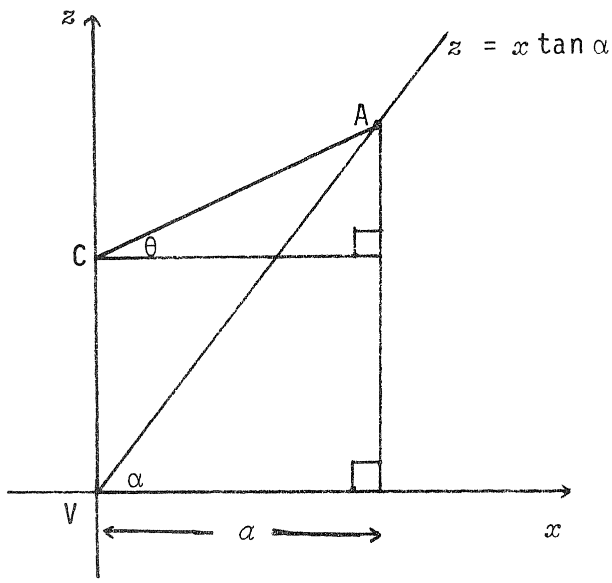
A plane cuts a right circular cone with vertex V in an ellipse E and meets the axis of the cone at C; A is an extremity of the major axis of E. Prove that the area of the curved surface of the slant cone with V as vertex and E as base is

$$\frac{\text{VA}}{\text{AC}}\cdot(\text{area of } E).$$

Solution by M.S.K.

Let the equation of the cone be

$$z^2 = (x^2 + y^2)\tan^2\alpha.$$



If E' is the orthogonal projection of E onto the xy-plane, then it is known that

$$S = \frac{[E']}{\cos\alpha},$$

where the square brackets denote area. Let the plane of E be parallel to the y-axis and consider the traces of the configuration on the xz-plane, as shown in the figure. If θ is the angle that segment AC makes with the x-axis, then $[E'] = [E]\cos\theta$ and

$$S = [E]\cdot\frac{\cos\theta}{\cos\alpha} = [E]\cdot\frac{a/\text{AC}}{a/\text{VA}} = \frac{\text{VA}}{\text{AC}}\cdot[E].$$

*

6. [1984: 215] *From the 1984 British Mathematical Olympiad.*

Let a and m be positive integers. Prove that, if there exists an integer x such that $a^2x - a$ is divisible by m, then there exists an integer y such that both $a^2y - a$ and $ay^2 - y$ are divisible by m.

Solution by Stewart Metchette, Culver City, California.

Clearly, if $m|a$ then y can be any multuple of m. So we assume that $(a,m) = 1$. Then $ax \equiv 1 \pmod{m}$. Since

$$ay^2 - a = a(ay - 1) \quad \text{and} \quad ay^2 - y = y(ay - 1),$$

$y = x + km$ will satisfy the desired conditions for any integer k.

*

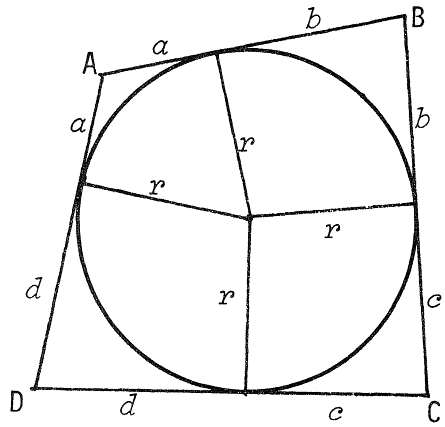
7. [1984: 215] *From the 1984 British Mathematical Olympiad.*

ABCD is a quadrilateral which has an inscribed circle. With the side AB is associated

$$u_{AB} = p_1 \sin(\angle DAB) + p_2 \sin(\angle ABC),$$

where p_1 and p_2 are the perpendiculars from A and B, respectively, to the opposite side CD. Define u_{BC}, u_{CD}, u_{DA} likewise, using in each case perpendiculars to the opposite side. Show that

$$u_{AB} = u_{BC} = u_{CD} = u_{DA}.$$



Solution by K.S. Murray, Brooklyn, N.Y.

Referring to the figure, we have

$$p_1 = (a+d)\sin D \quad \text{and} \quad p_2 = (b+c)\sin C.$$

Hence

$$u_{AB} = (a+d)\sin D \sin A + (b+c)\sin C \sin B.$$

Also, $a/r = \cot(A/2)$, etc., and so

$$\frac{1}{r}\cdot u_{AB} = (\cot\frac{A}{2} + \cot\frac{D}{2})\sin D \sin A + (\cot\frac{B}{2} + \cot\frac{C}{2})\sin C \sin B.$$

This is easily transformed into

$$\begin{aligned}\frac{1}{r}\cdot u_{AB} &= \sin D(1+\cos A) + \sin A(1+\cos D) + \sin C(1+\cos B) + \sin B(1+\cos C)\\ &= \sin A + \sin B + \sin C + \sin D + \sin(A+D) + \sin(B+C)\\ &= \sin A + \sin B + \sin C + \sin D,\end{aligned}$$

since $(A+D) + (B+C) = 360^\circ$. As this result is symmetric in A,B,C,D, we are done.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

MATHEMATICAL SWIFTIES

"$|z_1 + z_2| \le |z_1| + |z_2|$," Tom maintained absolutely.

"$\text{Sin}^2 x$ + 1000 has interesting properties," Tom repeated periodically.

"The decimal expansion of 1/7 is rather long," Tom stated indefinitely.

M.S. KLAMKIN

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before November 1, 1985, although solutions received after that date will also be considered until the time when a solution is published.

1031. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Independently solve the alphametics

$$\begin{array}{r} \text{MAN} \\ \underline{\text{WINS}} \\ \text{MAID} \end{array} \qquad \text{and} \qquad \begin{array}{r} \text{MAN} \\ \underline{\text{WEDS}}, \\ \text{WIFE} \end{array}$$

but FIND A MAN AND A WOMAN common to both alphametics.

1032. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

The following formulas are given in R.A. Johnson's *Advanced Euclidean Geometry* (Dover, New York, 1960, p. 205):

$$\mathrm{IH}^2 = 2\rho^2 - 2Rr \qquad \text{and} \qquad \mathrm{OH}^2 = R^2 - 4Rr,$$

where, in Johnson's notation, O,I,H,R,ρ,r are the circumcenter, incenter, orthocenter, circumradius, inradius, and inradius of the orthic triangle, respectively, of a given triangle. Johnson claims (at least tacitly) that these formulas both hold for all triangles. Prove that neither formula holds for obtuse triangles.

1033*. *Proposed by W.R. Utz, University of Missouri-Columbia.*

Let D_n be any symmetric determinant of order n in which the elements in the principal diagonal are all 1's and all other elements are either 1's or -1's, and let $\bar{D}_n$ be the determinant obtained from D_n by replacing the non-principal-diagonal elements by their negatives. It is easy to show that $D_2\bar{D}_2 = 0$ for all D_2 and $D_3\bar{D}_3 = 0$ for all D_3. For which $n > 3$ is it true that $D_n\bar{D}_n = 0$ for all D_n?

1034. *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

Find all positive integers n such that $n^4 + 1$ has a divisor of the form $dn - 1$, where d is a positive integer.

1035*. *From a Trinity College, Cambridge, examination paper dated December 6, 1901.*

If the equations

$$axy + bx + cy + d = 0,$$
$$ayz + by + cz + d = 0,$$
$$azw + bz + cw + d = 0,$$
$$awx + bw + cx + d = 0,$$

are satisfied by values of x, y, z, w which are all different, show that

$$b^2 + c^2 = 2ad.$$

1036. *Proposed by Gali Salvatore, Perkins, Québec.*

Find sets of positive numbers $\{a,b,c,d,e,f\}$ such that, simultaneously,

$$\frac{abc}{def} < 1, \quad \frac{a+b+c}{d+e+f} < 1, \quad \frac{a}{d} + \frac{b}{e} + \frac{c}{f} > 3, \quad \frac{d}{a} + \frac{e}{b} + \frac{f}{c} > 3,$$

or prove that there are none.

1037. *Proposed by (the late) H. Kestelman, University College, London, England.*

If A and B are Hermitian matrices of the same order and A is positive definite, show that AB is similar to a Hermitian matrix and that the latter is positive definite if B is so. Show that if A is assumed only to be positive semi-definite then it may happen that the matrix AB is not similar to any diagonal matrix.

1038. *Proposed by Jordi Dou, Barcelona, Spain.*

Given are two concentric circles and two lines through their centre. Construct a tangent to the inner circle such that one of its points of intersection with the outer circle is the midpoint of the segment of the tangent cut off by the two given lines.

1039. *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

Given are three collinear points O, P, H (in that order) such that OH < 3OP. Construct a triangle ABC with circumcentre O and orthocentre H and such that AP is the internal bisector of angle A. How many such triangles are possible?

1040. *Proposed by Clark Kimberling, University of Evansville, Indiana.*

ABC being a given triangle, describe the locus of all points P such that $\angle ACP = \angle ABP$.

* * *

MATHEMATICAL CLERIHEW

by

ALAN WAYNE, Holiday, Florida

Hermann Amandus Schwarz
(As Philip J. Davis reports)
Indulged in reflections
In complex directions.

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

897. [1983: 313; 1985: 63] *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

If $\lambda > \mu$ and $a \ge b \ge c > 0$, prove that

$$b^{2\lambda}c^{2\mu} + c^{2\lambda}a^{2\mu} + a^{2\lambda}b^{2\mu} \ge (bc)^{\lambda+\mu} + (ca)^{\lambda+\mu} + (ab)^{\lambda+\mu}, \tag{1}$$

with equality just when $a = b = c$.

II. *Extracted from a comment by M.S. Klamkin, University of Alberta.*

The published solution [1985: 63] is incomplete. In a couple of places the solver tacitly assumes that $\lambda+\mu \ge 0$, and this is not guaranteed by the proposal. Still to be proved is that the proposed inequality also holds when $\lambda+\mu < 0$.

In an editor's comment following the solution [1985: 64], it is stated that "$\lambda = 3/2$ and $\mu = 1/2$ give

$$b^2c(b-c) + c^2a(c-a) + a^2b(a-b) \ge 0, \tag{2}$$

an inequality given at the 1983 International Mathematical Olympiad [1983: 207; 1984: 73]." It should be noted that the I.M.O. problem asked for a proof of (2) subject to the condition that a,b,c are the sides of a triangle. Now (2) certainly holds if $a \ge b \ge c > 0$, whether or not a,b,c are the sides of a triangle. But it does not hold, for example, if $a = 1$, $b = 10$, $c = 12$. Since we can have $a < b < c$ in a triangle, it follows that the I.M.O. problem is not a special case of (1).

III. *Completion of proof extracted from the proposer's solution.*

Still to be proved is that (1) holds when

$$\lambda > \mu, \quad \lambda + \mu < 0, \quad a \ge b \ge c > 0. \tag{3}$$

Suppose λ,μ,a,b,c satisfy (3) and let

$$\lambda_1 = -\mu, \quad \mu_1 = -\lambda, \quad a_1 = c^{-1}, \quad b_1 = b^{-1}, \quad c_1 = a^{-1}.$$

Then

$$\lambda_1 > \mu_1, \quad \lambda_1 + \mu_1 > 0, \quad a_1 \ge b_1 \ge c_1 > 0. \tag{4}$$

It follows from (4) and the earlier solution that (1) holds for $\lambda_1,\mu_1,a_1,b_1,c_1$; and since

$$b_1^{2\lambda_1}c_1^{2\mu_1} = a^{2\lambda}b^{2\mu}, \text{ etc.} \qquad \text{and} \qquad (b_1c_1)^{\lambda_1+\mu_1} = (ab)^{\lambda+\mu}, \text{ etc.},$$

it follows that (1) holds when λ,μ,a,b,c satisfy (3).

Klamkin also filled the gap in the earlier solution.

* * *

904. [1984: 19 ; 1985: 90] *Correction.* The display in the penultimate line of the published solution [1985: 91] should, of course, read

$$l \cup m \cup n,$$

not $l \cap m \cap n$, as some typographical gremlins would have it. The statement will appear corrected in any subsequent reprint of the issue concerned and in the 1985 bound volume.

* * *

912. [1984: 53] *Proposed by Shmuel Avital, Technion - Israel Institute of Technology, Haifa, Israel.*

It is not difficult to verify that the nonlinear recursive relation

$$a_{n-1}a_{n+1} = a_n + 1, \qquad n = 1,2,3,\ldots,$$

generates a *periodic* sequence of period length 5 for any given a_0 and a_1. Show how to construct nonlinear recursive relations which will generate periodic sequences of any given period length k.

I. *Comment by M.S. Klamkin, University of Alberta.*

According to Sawyer [1], the proposer's recursive relation was first discovered by R.C. Lyness. In [2], Lyness himself explains how he discovered the relation and shows how to construct nonlinear recursive relations which generate periodic sequences of any given period length k when the first k-3 arbitrary nonzero starting values are given.

II. *Comment by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

For $n = 1,2,3,\ldots$, the following nonlinear recursive relations generate periodic sequences of the stated period length k. In each case, $a_0a_1 \neq 0$ and the value of r is restricted only by the requirement that $a_n \neq 0$ for $n \le k-1$.

$$k = 2, \qquad a_na_{n-1} = r;$$

$$k = 3, \qquad a_n(a_{n-1} + r) = -r^2;$$

$$k = 4, \qquad a_n(a_{n-1} + 2r) = -2r^2;$$

$$k = 5, \qquad a_{n+1}a_{n-1} = r(a_n + r);$$

$$k = 6, \qquad a_n(a_{n-1} + 3r) = -3r^2;$$

$$k = 9, \qquad a_{n+1} + a_{n-1} = |a_n|.$$

The last example is due to Morton Brown [3].

III. *Solution by Leroy F. Meyers, The Ohio State University.*

The proposer's sequence of period length 5 (see also Crux 191 [1977: 77]) has repetend

$$\left(a_0,\ a_1,\ \frac{a_1+1}{a_0},\ \frac{a_0+a_1+1}{a_0a_1},\ \frac{a_0+1}{a_1}\right),$$

provided that none of the terms is 0. The sequence generated by the slightly simpler recurrence relation

$$a_{n-1}a_{n+1} = a_n, \qquad n = 1,2,3,\ldots,$$

has period length 6 and repetend

$$\left(a_0,\ a_1,\ \frac{a_1}{a_0},\ \frac{1}{a_0},\ \frac{1}{a_1},\ \frac{a_0}{a_1}\right),$$

provided again that none of the terms is 0. However, the period may be shorter for certain initial values. In particular, the proposer's sequence has period length 1 if $a_0 = a_1 = (1\pm\sqrt{5})/2$. The simpler sequence has period length 1 if $a_0 = a_1 = 1$; it has period length 2 if

$$a_0 = \frac{-1 \pm i\sqrt{3}}{2} \qquad \text{and} \qquad a_1 = a_0^2 = \frac{-1 \mp i\sqrt{3}}{2};$$

it has period length 3 if $a_0^2 = a_1^2 = 1$ but $a_0a_1 + 1 \neq a_0 + a_1$.

A homogeneous linear recurrence relation

$$c_n = \sum_{j=1}^{r} \alpha_j c_{n-j}, \qquad n \geq r,$$

with initial values $c_0, c_1, \ldots, c_{r-1}$ and constant coefficients $\alpha_1, \alpha_2, \ldots, \alpha_r$, can be converted into a nonlinear relation

$$a_n = \prod_{j=1}^{r} a_{n-j}^{\alpha_j}, \qquad n \geq r,$$

by setting $a_n = b^{c_n}$ for some suitable constant b. If the coefficients α_j of the linear recurrence relation are integers, then no restrictions need be placed on the initial values (other than that they be nonzero). Hence we may first deal with linear recurrence relations, which have a long history.

In terms of vectors and matrices, the linear recurrence relation above can be expressed as

$$v_n = Av_{n-1}, \qquad n \geq r,$$

where (with T for transpose)

$$v_n = (c_n \quad c_{n-1} \quad \ldots \quad c_{n-r+1})^{\mathsf{T}}, \qquad n \geq r-1,$$

and

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{r-1} & \alpha_r \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

The vector of initial values is v_{r-1}. (Note that A is a companion matrix.) Since $v_{n+k} = A^k v_n$ for $k \geq 0$ and $n \geq r-1$, the requirement that the positive integer k be a period length of the sequence is equivalent to the requirement that A^k be the $r \times r$ identity matrix I_r. Now for any companion matrix A, a consequence of the Hamilton-Cayley theorem is that

$$A^r - \sum_{j=1}^{r} \alpha_j A^{r-j} = 0.$$

Hence for each k it is sufficient to choose r and the coefficients $\alpha_1, \alpha_2, \dots, \alpha_r$ so that the polynomial p defined by

$$p(t) = t^r - \sum_{j=1}^{r} \alpha_j t^{r-j}$$

shall divide $t^k - 1$. The trivial choice $r = k$ and $p(t) = t^k - 1$ yields the linear recurrence relation

$$c_n = c_{n-k}, \qquad n \geq k,$$

and exponentiation ($a_n = b^{c_n}$) yields the similar recurrence relation

$$a_n = a_{n-k}, \qquad n \geq k.$$

Since

$$t^k - 1 = (t - 1)(t^{k-1} + \dots + t + 1),$$

another linear recurrence relation of period length k is obtained by letting $r = k-1$ and $p(t) = t^{k-1} + \dots + t + 1$. This yields the relation

$$c_n = -c_{n-1} - c_{n-2} - \dots - c_{n-k+2} - c_{n-k+1}, \quad n \geq k-1,$$

and exponentiation yields the relation

$$a_n = \frac{1}{a_{n-1} a_{n-2} \dots a_{n-k+2} a_{n-k+1}}, \quad n \geq k-1,$$

with repetend

$$(a_0, a_1, \dots, a_{k-2}, \frac{1}{a_0 a_1 \dots a_{k-2}}).$$

Other factors of $t^k - 1$ may be used to generate recurrence relations.

A general linear relation

$$c'_n = \sum_{j=1}^{r} \alpha_j c'_{n-j} + \beta, \quad n \ge r,$$

can also be converted into a nonlinear relation by exponentiation ($a_n = b^{c'_n}$), but it can be obtained from a homogeneous linear relation by setting $c'_n = c_n + \gamma$, where γ is chosen so that

$$\gamma(1 - \sum_{j=1}^{r} \alpha_j) = \beta .$$

The simpler recurrence relation

$$a_{n-1}a_{n+1} = a_n, \quad n = 1,2,3,\ldots,$$

with period length 6, comes from the factor $t^2 - t + 1$ of $t^6 - 1$, and can be generalized to

$$a_{n-1}a_{n+1} = \lambda a_n, \quad n = 1,2,3,\ldots,$$

by setting $\beta = \gamma$ and $\lambda = b^\beta$. This new recurrence relation has repetend

$$(a_0,\ a_1,\ \frac{\lambda a_1}{a_0},\ \frac{\lambda^2}{a_0},\ \frac{\lambda^2}{a_1},\ \frac{\lambda a_0}{a_1}).$$

It is likely that there are many more periodic recurrence relations with simple formulas.

Also solved by FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio. A comment was received from WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.

REFERENCES

1. W.W. Sawyer, "Lyness' periodic sequence", *Mathematical Gazette*, 45 (1961) 207.

2. R.C. Lyness, "Cycles", *Mathematical Gazette*, 45 (1961) 207-209.

3. James F. Slifker, solution to Problem 6439 (proposed by Morton Brown), *American Mathematical Monthly*, 92 (1985) 218.

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913. [1984: 53] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Let

$$f_n(x) = x^n + 2x^{n-1} + 3x^{n-2} + 4x^{n-3} + \ldots + nx + (n+1).$$

Prove or disprove that the discriminant of $f_n(x)$ is

$$(-1)^{n(n-1)/2}\cdot 2(n+2)^{n-1}(n+1)^{n-2}.$$

Solution by John McKay, Concordia University, Montréal, Québec.

Let r_i, $i = 1,2,\ldots,n$, be the roots of f_n and $s_k = \sum_i r_i^k$, $k = 0,1,2,\ldots$. Also, let $V = (v_{ij})$ be the square matrix of order n defined by $v_{ij} = r_j^{i-1}$. Then the required discriminant is

$$\operatorname{disc}(f_n) = \det(VV^{\mathsf{T}}) = \det(W),$$

where $W = (w_{ij})$ with $w_{ij} = s_{i+j-2}$ for $i,j = 1,2,\ldots,n$.

For the function

$$x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n,$$

Newton's identities give $s_0 = n$,

$$s_k + a_1s_{k-1} + \ldots + a_{k-1}s_1 + ka_k = 0, \quad 1 \le k \le n,$$

and

$$s_k + a_1s_{k-1} + a_2s_{k-2} + \ldots + a_ns_{k-n} = 0, \quad k > n.$$

For our function f_n, where $a_i = i+1$, Newton's identities give

$$s_0 = n; \quad s_i = -2, \quad i = 1,2,\ldots,n; \quad s_{n+1} = 2\sum_{t=1}^{n+1} t = n^2 + 3n.$$

The matrix W therefore has the following appearance (the missing elements will not be needed):

$$W = \begin{pmatrix} n & -2 & -2 & \ldots & -2 & -2 \\ -2 & -2 & -2 & \ldots & -2 & -2 \\ -2 & -2 & -2 & \ldots & -2 & s_{n+1} \\ -2 & -2 & -2 & \ldots & s_{n+1} & \\ \vdots & \vdots & \vdots & & & \\ -2 & -2 & s_{n+1} & & & \end{pmatrix}.$$

Subtracting the second row from all the others gives

$$\begin{aligned} \operatorname{disc}(f_n) &= \det(W) = (n+2)\cdot(-2)\cdot(s_{n+1}+2)^{n-2}\cdot(-1)^{\binom{n-2}{2}} \\ &= (-1)^{1+(n-2)(n-3)/2}\cdot 2(n+2)^{n-1}(n+1)^{n-2} \\ &= (-1)^{n(n-1)/2}\cdot 2(n+2)^{n-1}(n+1)^{n-2}, \end{aligned}$$

since

$$1 + \frac{(n-2)(n-3)}{2} = \frac{n(n-1)}{2} - 2(n-2) \equiv \frac{n(n-1)}{2} \pmod 2.$$

Also solved by GEORGE SZEKERES, University of New South Wales, Australia.

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914. [1984: 53] *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

If $a,b,c > 0$, then the equation $x^3 - (a^2+b^2+c^2)x - 2abc = 0$ has a unique positive root x_0. Prove that

$$\frac{2}{3}(a+b+c) \le x_0 < a+b+c.$$

I. *Solution by Andy Liu, University of Alberta.*

Let $f(x) = x^3 - (a^2+b^2+c^2)x - 2abc$. By the continuity of f, the desired result will follow from

$$f(\tfrac{2}{3}(a+b+c)) \le 0 \tag{1}$$

and

$$f(a+b+c) > 0. \tag{2}$$

Now

$$f(\tfrac{2}{3}(a+b+c)) = -\frac{4}{27}X - \frac{2}{27}Y,$$

where

$$X = (b-c)^2(b+c) + (c-a)^2(c+a) + (a-b)^2(a+b)$$

and

$$Y = a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b).$$

Clearly, $X \ge 0$, with equality if and only if $a = b = c$; and $Y \ge 0$ follows by setting $t = 1$ in Schur's inequality

$$a^t(a-b)(a-c) + b^t(b-c)(b-a) + c^t(c-a)(c-b) \ge 0,$$

which holds for any real number t when $a,b,c > 0$, with equality if and only if $a = b = c$. Thus (1) is established, with equality if and only if $a = b = c$.

As for (2), it follows immediately from

$$f(a+b+c) = 2(b+c)(c+a)(a+b).$$

II. *Comment by M.S. Klamkin, University of Alberta.*

For a geometric interpretation of the positive root x_0, see the solution by Howard Eves to Crux 787 [1984: 57].

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer

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918. [1984: 54] *Proposed by John P. Hoyt, Lancaster, Pennsylvania; and Leroy F. Meyers, The Ohio State University.*

Let k be a positive integer and g a polynomial of degree at most k-2. Show that

$$\sum_{n=0}^{\infty} \frac{g(n)}{(n+1)(n+2)\ldots(n+k)}$$

converges to a rational number.

Solution by M.S. Klamkin, University of Alberta.

If $k = 1$, then the series converges to 0, since the zero polynomial is the only one of degree at most -1. If $k \geq 2$, the result is not valid as stated. No doubt the proposers tacitly assumed that $g(n)$ is rational for all integral n. If this is so, then $g(n)$ has the representation

$$g(n) = a_0 + a_1(n+1) + a_2(n+1)(n+2) + \ldots + a_{k-2}(n+1)(n+2)\ldots(n+k-2),$$

where the a_i are all rational (just let $n = -1,-2,\ldots$ successively). The desired result will follow if we show that

$$S = \sum_{n=0}^{\infty} \frac{1}{(n+r)(n+r+1)\ldots(n+k)}$$

is rational for $r = 1,2,\ldots,k-1$. It is well known that S can be written in the form

$$S = \frac{1}{k-r}\sum_{n=0}^{\infty}\left(\frac{1}{(n+r)(n+r+1)\ldots(n+k-1)} - \frac{1}{(n+r+1)(n+r+2)\ldots(n+k)}\right),$$

and this sum telescopes down to

$$S = \frac{1}{k-r}\cdot\frac{1}{r(r+1)\ldots(k-1)} = \frac{1}{k-r}\cdot\frac{(r-1)!}{(k-1)!},$$

which is rational.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands (partial solution); FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; RICHARD PARRIS, Phillips Exeter Academy, New Hampshire; M.A. SELBY, University of Windsor; JORDAN B. TABOV, Sofia, Bulgaria; and the proposers. A comment was received from WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.

Editor's comment.

The proposers noted that this is essentially Problem 106 on page 277 of Konrad Knopp's *Theorie und Anwendung der unendlichen Reihen*, 5th edition (1964).

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919. [1984: 54] *Proposed by Jordi Dou, Barcelona, Spain.*

Show how to construct a point P which is the centroid of triangle A'B'C', where A',B',C' are the orthogonal projections of P upon three given lines a,b,c, respectively.

Solution by J.T. Groenman, Arnhem, The Netherlands; and Jordan I. Tabov, Sofia, Bulgaria (independently).

We assume that the proposer's unstated intention was that the lines a,b,c be coplanar and intersecting in pairs. Let

$$b \cap c = A, \quad c \cap a = B, \quad a \cap b = C,$$

and let G be the centroid of triangle ABC. Then the required point P is the Lemoine point K of the triangle, the isogonal conjugate of G. This is an immediate consequence of the following known result [1]: The symmedian point [Lemoine point] is the median point [centroid] of its own pedal triangle, and it is the only point with that property.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands (second solution); D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

REFERENCE

1. Roger A. Johnson, *Advanced Euclidean Geometry*, Dover, New York, 1960, p. 217 (first theorem and corollary to second theorem).

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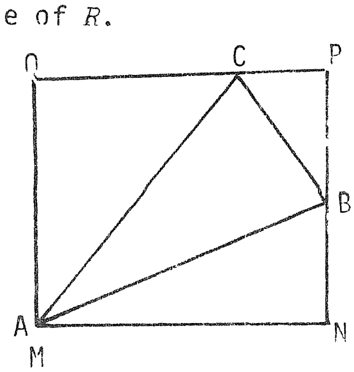
920. [1984: 54] *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

If a triangle of unit base and unit altitude is in the unit square, show that the base of the triangle must be one side of the square.

Solution by the proposer.

The desired result is an immediate consequence of the following

THEOREM. Let $[T]$ be the area of a triangle T inside a rectangle R of area $[R]$. Then $[T] \le [R]/2$, with equality if and only if one side of T coincides with a side of R and the opposite vertex of T is on the opposite side of R.



Proof. Let T be a triangle of maximal area inside R. (The existence of such a triangle is a simple exercise in analysis, for the area is a continuous function of the six coordinates of the vertices on a compact set.) Each side of R must contain at least one vertex of T, for otherwise R could be made smaller, still containing T, and then the whole figure expanded back to the original size of R to give a triangle larger than T in R.

Therefore one vertex of T lies in two adjacent sides, and hence at a vertex, of R, and the configuration is as shown in the figure, with A at M, B on NP, and C on PQ. If either B is at N or C is at Q, the result is clear. If not, then $[T]$ can be increased by moving either B to N or C to Q, contradicting the fact that $[T]$ is maximal.

Also solved by CLAYTON W. DODGE (Maine), JORDI DOU (Spain), J.T. GROENMAN (The Netherlands), RICHARD I. HESS (California), WALTHER JANOUS (Austria), FRIEND H. KIERSTEAD, JR. (Ohio), M.S. KLAMKIN (Alberta), ANDY LIU (Alberta), LEROY F. MEYERS (Ohio), FLORENTIN SMARANDACHE (Maroc), D.J. SMEENK (The Netherlands), DAN SOKOLOWSKY (New York), ESTHER SZEKERES (Australia, two solutions), JORDON B. TABOV (Bulgaria), and KENNETH M. WILKE (Kansas).

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921. [1984: 88] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

In U.S. liquid measure, a gill is 4 fluid ounces (fl oz) and a pint is 4 gills, that is:

$$\begin{array}{r} \text{FLOZ} \\ 4 \\ \hline \text{GILL} \end{array} \quad \text{and} \quad \begin{array}{r} \text{GILL} \\ 4 \\ \hline \text{PINT} \end{array}.$$

Solve these alphametical multiplications independently without reusing the digit 4.

Solution by Charles W. Trigg, San Diego, California.

Clearly, in the first multiplication F = 1 or 2 and 4|LL. Thus LL = 00 or 88, and so LOZ = 025, 075, 872, or 897. If F = 1, then all the LOZ candidates must be rejected, either because of a letter duplication or because an unwanted 4 appears in the product. If F = 2, the only possibilities for LOZ are 075 and 897. In the latter case the product is too large, and the former gives the unique reconstruction $2075 \cdot 4 = 8300$.

In the second multiplication, there are only three values of LL which avoid a 4 or duplicated digits among L, N, and T. Thus:

$$4 \cdot 55 \equiv 20, \quad 4 \cdot 77 \equiv 08, \quad 4 \cdot 88 \equiv 52 \pmod{100}.$$

The first possibility leads to I = 6, G = 1, and P = I; the second leads to I = 9, G = 1, and L = P; and the third and only satisfactory one leads to I = 9, G = 1, and P = 7. The unique reconstruction is $1988 \cdot 4 = 7952$.

Also solved by CLAYTON W. DODGE (Maine), RICHARD I. HESS (California), JACK LESAGE (Ontario), J.A. McCALLUM (Alberta), STEWART METCHETTE (California), GLEN E. MILLS (Florida), STANLEY RABINOWITZ (New Hampshire), KENNETH M. WILKE (Kansas), ANNELIESE ZIMMERMANN (West Germany), and the proposer.

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