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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

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Mathematical Spectrum Awards for Volume 18

The editors have awarded three prizes to students this year. These go to Joseph McLean for his letters 'On sums of consecutive integers' (page 57) and 'On sums of unlike powers' (pages 88–89), and to Guy Willard and Jeremy Rosten for solutions to problems.

Readers are reminded that awards of up to £30 are available for articles and of up to £15 for letters, solutions to problems, or other items. To qualify, contributors must be students at school, college or university.

Shuttlecock Trajectories in Badminton

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The basic strokes in badminton can be classified into the following categories (cf. reference 1): the *smash*, a powerful overhand stroke barely clearing the net and travelling in a nearly straight line; the *clear*, which lofts the shuttle high into the backcourt; the *drop shot*, a gentle stroke grazing the net and falling close to it; and the *drive*, a variant of the drop shot which is driven with more power from midcourt or backcourt. Whereas the smash is an overhand stroke and the drop shot and drive are underhand strokes, the clear can be either overhand or underhand. Sometimes an overhand drop shot is used when the opponent is out of position in the far backcourt, but such a stroke is inadvisable, since it is susceptible to a 'kill' by the opponent. In this article, we study the trajectories of the four strokes which are used most frequently in high-level competition, viz., the smash, overhand clear, drop shot and underhand clear.

The problem is that of a projectile motion with air resistance varying as the square of the velocity (reference 2). Since the feathers of the shuttle prevent it from rotating, the aerodynamic lift force is zero. When there is no wind, we can find the velocity v of the shuttle as a function of the angle θ which the direction of the velocity makes with the horizontal. The horizontal distance x, vertical distance y and time t at a point on the shuttle's path can be found by simple integrations over θ (cf. reference 3). Since air resistance is anti-parallel to the velocity, we use intrinsic coordinates (reference 3) to obtain the equations of motion at a point along the tangent and normal to the path as

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg\sin\theta - cv^2, \tag{1}$$

and

$$mv\frac{\mathrm{d}\theta}{\mathrm{d}t} = -mg\cos\theta. \tag{2}$$

Here m is the mass of the shuttle, g the acceleration due to gravity and c the quadratic drag coefficient. Eliminating t between equations (1) and (2), we get

$$\frac{\mathrm{d}v}{\mathrm{d}\theta}\cos\theta - v\sin\theta = \frac{cv^3}{mg},$$

which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(v\cos\theta) = \frac{cv^3}{mg}.$$

Integrating both sides, we get

$$\frac{1}{(v\cos\theta)^2} = -\frac{c}{mg}[\ln(\sec\theta+\tan\theta)+\sec\theta\tan\theta]+C.$$

The constant of integration is obtained from the initial conditions $v = v_0$ and $\theta = \theta_0$:

$$C = \frac{1}{v_0^2\cos^2\theta_0} + \frac{c}{mg}[\ln(\sec\theta_0+\tan\theta_0)+\sec\theta_0\tan\theta_0].$$

Hence, the velocity of the projectile as a function of θ only is given by

$$v = \left\{\frac{1}{v_0^2}\frac{\cos^2\theta}{\cos^2\theta_0} - \frac{c\cos^2\theta}{mg}\left[\ln\frac{\sec\theta+\tan\theta}{\sec\theta_0+\tan\theta_0} + \frac{\sin\theta}{\cos^2\theta} - \frac{\sin\theta_0}{\cos^2\theta_0}\right]\right\}^{-\frac{1}{2}}.$$

Since, from equation (2),

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = -v\frac{\sec\theta}{g},$$

we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\theta} = v\cos\theta\frac{\mathrm{d}t}{\mathrm{d}\theta} = -\frac{v^2}{g},$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\theta} = v\sin\theta\frac{\mathrm{d}t}{\mathrm{d}\theta} = -\frac{v^2\tan\theta}{g}.$$

Hence, we get

$$x = -\frac{1}{g}\int_{\theta_0}^{\theta} v^2\, \mathrm{d}\theta, \tag{3}$$

$$y = -\frac{1}{g}\int_{\theta_0}^{\theta} v^2 \tan\theta\, \mathrm{d}\theta, \tag{4}$$

and

$$t = -\frac{1}{g}\int_{\theta_0}^{\theta} v \sec\theta\, \mathrm{d}\theta. \tag{5}$$

Equations (3), (4) and (5) can be solved easily by any standard integration scheme. For a small step size in θ, the trapezoidal scheme provides a simple and convenient method for finding x, y and t, from which the trajectory of the shuttlecock can be determined.

The average mass of a standard shuttle is 5.12 g (cf. reference 1). The terminal velocity of the shuttle has been measured to be $v_{\mathrm{T}} = 6.8\,\mathrm{m/s}$ (reference 2). This means that the drag coefficient, $c = mg/v_{\mathrm{T}}^2 = 0.001\,085\,\mathrm{kg/m}$. Typical trajectories of the four common strokes are calculated fom the above equations and displayed in figure 1. The times of flight are also indicated on the trajectories. The overhand strokes are launched from a height of 2.8 m and 5 m from the net in the backcourt, whereas the underhand strokes are driven from the height of 0.3 m, 2 m from the net in the frontcourt. These launching levels are representative of a player of average height. The launch speeds and angles are as indicated in the caption.

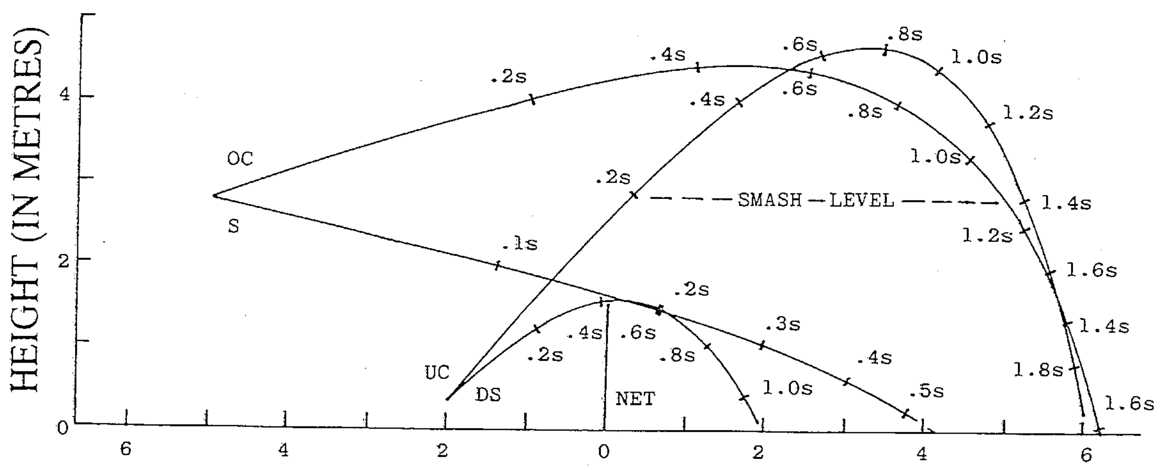

Figure 1. Shuttlecock trajectories in badminton. OC = overhand clear launched at 40 m/s, 20°; S = smash, 60 m/s, −12°; UC = underhand clear, 30 m/s, 50°; DS = drop shot, 9 m/s, 45°.

Calculations indicate that the trajectories of the badminton shuttle are strongly influenced by air resistance, and the speed of the shuttle rapidly diminishes in fractions of a second. The drop shot, underhand clear, and overhand clear constitute the defensive strokes. Their times of flight to the stroke level of the opponent are longer than a second, and therefore give the opponent ample time to react. The smash, on the other hand, is by far the fastest stroke in the game and is most frequently the winning stroke. In figure 1, the smash is driven with an initial speed of 60 m/s (about 134 m.p.h.), which is achievable by a player of moderate strength. It takes about half a second to reach the underhand-stroke level of the opponent, which is just outside the reaction time of an average player. (Note that this is comparable to the flight time of a fast cricket ball bowled at 90 m.p.h.) It is clear that a smash driven from closer range becomes a kill. This is why the clear should be lofted well into the backcourt of the opponent to ensure that the return smash does not become a kill. Of course, the player should also try to prevent the clear from sailing outside the court. In the figure, both the overhand and underhand clears travel over the smash level beyond about 5 m. For the same reason, namely to prevent a kill by the opponent, the drop shot should barely clear the net.

The results of our calculations are quite compatible with the observations of the badminton player and spectator. However, the badminton player inevitably learns his strokes from experience and not from mathematics.

References

1. J. A. Cuddon, *The International Dictionary of Sports and Games* (Schocken Books, New York, 1979), 75–79.

2. M. Peastrel, R. Lynch and A. Armenti, Jr., Terminal velocity of a shuttlecock in vertical fall, *Amer. J. Phys.*, **48** (1980), 511–513.

3. D. A. Quadling and A. R. D. Ramsay, *Elementary Mechanics*, Vol. II (Bell, London, 1966), 41, 518–522.

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Here is our annual puzzle. Express the numbers 1 to 100 in terms of the digits of the year in order, using only the operations $+$, $-$, $\times$, $\div$, $\sqrt{}$, !, brackets and concatenation (i.e. forming 19 from 1 and 9, for example). The editorial office has been hard at work, and has failed with nine of the numbers. Readers who manage more than this are invited to send in their solutions.

Approximating $\sqrt{n}$

SIMON JOHNSON, *Pocklington School, York*

At the time of writing this article, Simon was preparing to take his A levels. He was about to go to Peterhouse College, Cambridge, to read for a degree in mathematics.

On page 84 of Volume 18 Number 3 of *Mathematical Spectrum*, readers were asked to show that, if the rational number a/b (with a and b positive integers) is an approximation to the irrational number $\sqrt{2}$, then $(a+2b)/(a+b)$ is a better approximation. A measure of the accuracy of a/b to $\sqrt{2}$ is given by

$$\epsilon = 2-\frac{a^2}{b^2} = \frac{2b^2-a^2}{b^2}\,.$$

Similarly, the error in the suggested approximation $(a+2b)/(a+b)$ may be given by

$$\epsilon' = 2-\frac{(a+2b)^2}{(a+b)^2} = \frac{a^2-2b^2}{(a+b)^2}\,.$$

Then, for all $a, b > 0$,

$$\epsilon' = -\frac{b^2}{(a+b)^2}\epsilon,$$

so that $|\epsilon'| < |\epsilon|$, and this proves the result.

We may ask the question: if this process is repeated indefinitely, will the limit be $\sqrt{2}$? Suppose we start with any fraction $a_0/b_0 > 0$ and write

$$\frac{a_1}{b_1} = \frac{a_0+2b_0}{a_0+b_0}, \qquad \frac{a_2}{b_2} = \frac{a_1+2b_1}{a_1+b_1}, \qquad \text{etc.,}$$

shall we obtain the limiting value $\sqrt{2}$? We can see that, if there is a limiting value, it must be $\sqrt{2}$. For suppose that $a_r/b_r \to \lambda$ as $r \to \infty$. Then, in

$$\frac{a_{r+1}}{b_{r+1}} = \frac{a_r+2b_r}{a_r+b_r} = \frac{\dfrac{a_r}{b_r}+2}{\dfrac{a_r}{b_r}+1},$$

we can let $r \to \infty$ and obtain

$$\lambda = \frac{\lambda+2}{\lambda+1},$$

which gives $\lambda^2 = 2$, so $\lambda = \sqrt{2}$. But this begs the question in that we do not know that a limit exists.

We shall show that a limit does indeed exist. If we start with a_0/b_0 and form successive iterations, we can use matrix notation and write

$$\begin{bmatrix} a_{r+1} \\ b_{r+1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_r \\ b_r \end{bmatrix}.$$

In fact, whilst we are about it, we may as well deal with rational approximations to $\sqrt{n}$ for any positive integer n and write

$$\begin{bmatrix} a_{r+1} \\ b_{r+1} \end{bmatrix} = A \begin{bmatrix} a_r \\ b_r \end{bmatrix}, \quad \text{where} \quad A = \begin{bmatrix} 1 & n \\ 1 & 1 \end{bmatrix}.$$

Then

$$\begin{bmatrix} a_r \\ b_r \end{bmatrix} = A^r \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}.$$

The best way to work out A^r is to use eigenvalues. Readers who are not familiar with these may like to jump to our formula for A^r below and perhaps verify it by induction on r.

The eigenvalues of A are given by the equation

$$\begin{vmatrix} \lambda - 1 & -n \\ -1 & \lambda - 1 \end{vmatrix} = 0$$

or

$$\lambda^2 - 2\lambda + 1 - n = 0,$$

so that

$$\lambda = 1 \pm \sqrt{1-(1-n)} = 1 \pm \sqrt{n}.$$

An eigenvector for the eigenvalue $1+\sqrt{n}$ is $[\, x \quad y \,]^\top$, given by

$$\begin{aligned} \sqrt{n}x - \quad ny &= 0, \\ -x + \sqrt{n}y &= 0, \end{aligned}$$

so we can choose $[\, \sqrt{n} \quad 1 \,]^\top$. Similarly, $[\, -\sqrt{n} \quad 1 \,]^\top$ is an eigenvector for the eigenvalue $1-\sqrt{n}$. We use these eigenvectors to form the columns of the matrix P, so that

$$P = \begin{bmatrix} \sqrt{n} & -\sqrt{n} \\ 1 & 1 \end{bmatrix}.$$

Thus $P^{-1}AP = D$, where D is the diagonal matrix

$$D = \begin{bmatrix} 1+\sqrt{n} & 0 \\ 0 & 1-\sqrt{n} \end{bmatrix}$$

whose diagonal entries are the eigenvalues of A. Thus $A = PDP^{-1}$, whence

$$\begin{aligned}
A^r &= (PDP^{-1})(PDP^{-1})\dots(PDP^{-1}) \\
&= PD^rP^{-1} \\
&= \frac{1}{2\sqrt{n}}\begin{bmatrix} \sqrt{n} & -\sqrt{n} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (1+\sqrt{n})^r & 0 \\ 0 & (1-\sqrt{n})^r \end{bmatrix}\begin{bmatrix} 1 & \sqrt{n} \\ -1 & \sqrt{n} \end{bmatrix} \\
&= \frac{1}{2\sqrt{n}}\begin{bmatrix} \sqrt{n}(1+\sqrt{n})^r & -\sqrt{n}(1-\sqrt{n})^r \\ (1+\sqrt{n})^r & (1-\sqrt{n})^r \end{bmatrix}\begin{bmatrix} 1 & \sqrt{n} \\ -1 & \sqrt{n} \end{bmatrix} \\
&= \frac{1}{2\sqrt{n}}\begin{bmatrix} \sqrt{n}(1+\sqrt{n})^r+\sqrt{n}(1-\sqrt{n})^r & n(1+\sqrt{n})^r-n(1-\sqrt{n})^r \\ (1+\sqrt{n})^r-(1-\sqrt{n})^r & \sqrt{n}(1+\sqrt{n})^r+\sqrt{n}(1-\sqrt{n})^r \end{bmatrix}.
\end{aligned}$$

Thus, from the equation

$$\begin{bmatrix} a_r \\ b_r \end{bmatrix} = A_r\begin{bmatrix} a_0 \\ b_0 \end{bmatrix},$$

we have

$$\begin{aligned}
\frac{a_r}{b_r} &= \frac{a_0\sqrt{n}[(1+\sqrt{n})^r+(1-\sqrt{n})^r]+b_0n[(1+\sqrt{n})^r-(1-\sqrt{n})^r]}{a_0[(1+\sqrt{n})^r-(1-\sqrt{n})^r]+b_0\sqrt{n}[(1+\sqrt{n})^r+(1-\sqrt{n})^r]} \\
&= \frac{a_0\sqrt{n}\left[1+\left(\frac{1-\sqrt{n}}{1+\sqrt{n}}\right)^r\right]+b_0n\left[1-\left(\frac{1-\sqrt{n}}{1+\sqrt{n}}\right)^r\right]}{a_0\left[1-\left(\frac{1-\sqrt{n}}{1+\sqrt{n}}\right)^r\right]+b_0\sqrt{n}\left[1+\left(\frac{1-\sqrt{n}}{1+\sqrt{n}}\right)^r\right]}.
\end{aligned}$$

We now let $r \to \infty$. Since $|(1-\sqrt{n})/(1+\sqrt{n})| < 1$,

$$\left(\frac{1-\sqrt{n}}{1+\sqrt{n}}\right)^r \to 0 \quad \text{as } r \to \infty.$$

So

$$\frac{a_r}{b_r} \to \frac{a_0\sqrt{n}+b_0n}{a_0+b_0\sqrt{n}} = \sqrt{n}$$

and this establishes our claim that a_r/b_r has a limit as $r \to \infty$, the limit being $\sqrt{n}$.

Thus we have proved that the iteration

$$\frac{a}{b} \to \frac{a+nb}{a+b}$$

will always tend to $\sqrt{n}$ when repeatedly applied with any initial rational number a/b. If we start with a/b and use the iteration

$$\frac{a}{b} \to \frac{ab+nb^2}{a^2+b^2}$$

repeatedly as before, the fraction should tend to $n^{1/3}$, although I have not proved this. Finally, the iteration

$$\frac{a}{b} \to \frac{b^{k-2}(a+nb)}{a^{k-1}+b^{k-1}}$$

should tend to $n^{1/k}$.

Fuzzy Set Theory

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The author received her M.S. and Ph.D. degrees from the University of Tennessee in Knoxville. From 1982 to 1985 she taught at the University of North Carolina—Asheville, and in 1986 had a visiting position at Texas Tech University in Lubbock. Her primary interest is in mathematical modelling of ecosystems, but she enjoys studying many areas of applied mathematics.

Set theory is at the foundation of much modern mathematics. However, set theory is just a subset of a much broader field called fuzzy set theory. Around 1965 Zadeh made the first major contribution to the development of fuzzy set theory. Since then Zadeh and many others have contributed to the advancement of fuzzy set theory (see references 3 and 6). It has made its way into various branches of mathematics including calculus, geometry, algebra and topology. It has also become an increasingly important area for applications. Fuzzy set theory has applications in the areas of decision-making problems in business and management, pattern and language recognition, artificial intelligence and control processes. This increased interest in its applications has been one of the motivating forces behind its advancement.

Ordinary set theory differs from fuzzy set theory in that ordinary set theory has an all-or-none property—either an element belongs to a set or it does not. For example, suppose that size of an object is to be classified into one of three distinct classes, either small, medium or large. The cut-off between small and medium or medium and large may not be precise. The problem can be handled by considering grades of membership for each object between the values of 0 and 1. An object somewhere between the classes of small and medium may get a value of 0.4 in the class of small objects, a value of 0.7 in the class of medium objects and a value of 0 in the class of large objects. Even if we added a few more categories we would probably never get all the objects to fit into precisely one category (there is always an exception). The exactness of ordinary set theory does not apply to real-world problems, especially when sets are described by adjectives such as best or worst, suitable or unsuitable, etc. In terms of applications, the use of fuzzy set theory has been a means to bridge the gap between mathematical precision and real-world imprecision.

The aim of this article is to give the basic theory of fuzzy sets and to show its usefulness in applications. We begin a brief presentation of fuzzy sets by giving basic definitions and examples of fuzzy sets. A good reference for the basic theory is Kaufmann (see reference 3). First let U represent the universal set, that is the set containing all elements being considered or the reference set.

Definition. A *fuzzy set* in U is a set of pairs consisting of an element $x \in U$ and an assigned value between 0 and 1, inclusive, associated with x. We denote a fuzzy set A of U as follows:

$$A = \{(x, \mu_A(x)) | x \in U\},$$

where the function μ_A assigns a value to each x in U and is referred to as the *membership characteristic function*.

The function μ_A measures the degree of membership in the set A. A value of 1 means the element belongs to the set and a value of 0 means the element does not belong to the set. A value of between 0 and 1 means the element 'partially' belongs to the set. The closer the value is to 1 the more closely it is associated with the set A.

For an ordinary set A, either an element x in U belongs to A or it does not. Thus the membership characteristic function for an ordinary set A is

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

The universal set U and the empty set $\varnothing$ are both ordinary sets. They have membership characteristic functions, $\mu_U(x) = 1$ and $\mu_\varnothing(x) = 0$ for all $x \in U$.

Equality, subset, intersection, union and complement are defined in terms of the function μ_A.

Definitions. The fuzzy set A *equals* the fuzzy set B, $A = B$, if and only if $\mu_A(x) = \mu_B(x)$ for all $x \in U$.

The fuzzy set B is a *subset* of the fuzzy set A, $B \subseteq A$, if and only if $\mu_B(x) \leqslant \mu_A(x)$ for all $x \in U$.

The *intersection* of A and B, denoted $A \cap B$, is the set satisfying $\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$ for all $x \in U$.

The *union* of A and B, denoted $A \cup B$, is the set satisfying $\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$ for all $x \in U$.

The *complement* of A, denoted $\bar{A}$, is the set satisfying $\mu_{\bar{A}}(x) = 1-\mu_A(x)$ for all $x \in U$.

As an example, consider the universal set $U = \{a, b, c\}$ and the fuzzy sets

$$A = \{(a,1), (b,\tfrac{1}{2}), (c,\tfrac{1}{4})\} \quad \text{and} \quad B = \{(a,\tfrac{1}{2}), (b,\tfrac{1}{2}), (c,\tfrac{1}{2})\}.$$

Thus

$$A \cap B = \{(a,\tfrac{1}{2}), (b,\tfrac{1}{2}), (c,\tfrac{1}{4})\}, \qquad A \cup B = \{(a,1), (b,\tfrac{1}{2}), (c,\tfrac{1}{2})\}$$

and

$$\bar{A} = \{(a,0), (b,\tfrac{1}{2}), (c,\tfrac{3}{4})\}.$$

Most of the usual set properties hold, including commutativity, associativity, distributivity, and De Morgan's laws.

Theorem. Let A, B and C represent fuzzy sets in the universal set U. Then

(*a*) $A \cap \varnothing = \varnothing$,

(*b*) $A \cup \varnothing = A$,

(*c*) $A \cap U = A$,

(*d*) $A \cup U = U$,

(*e*) $A \cap B = B \cap A$,

(*f*) $A \cup B = B \cup A$,

(*g*) $(A \cap B) \cap C = A \cap (B \cap C)$,

(*h*) $(A \cup B) \cup C = A \cup (B \cup C)$,

(*i*) $\bar{\bar{A}} = A$,

(*j*) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$,

(*k*) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$,

(*l*) $\overline{A \cup B} = \bar{A} \cap \bar{B}$,

(*m*) $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

We shall prove one of De Morgan's laws (l). The other properties are left for the reader. Following the definitions, we have

$$\begin{aligned}\mu_{\overline{A \cup B}} &= 1-\mu_{A \cup B} = 1-\max\{\mu_A, \mu_B\} \\ &= \min\{1-\mu_A, 1-\mu_B\} = \min\{\mu_{\bar{A}}, \mu_{\bar{B}}\} = \mu_{\bar{A} \cap \bar{B}}.\end{aligned}$$

The use of characteristic functions in proving set properties even for ordinary sets is much simpler than using elements from each set (see reference 4).

Although many of the properties of ordinary sets are true for fuzzy sets, there are properties which do not hold true. For example, $A \cup \bar{A} \neq U$ and $A \cap \bar{A} \neq \varnothing$. You can verify this for yourself by using A and $\bar{A}$ above.

An interesting question is 'how fuzzy is a fuzzy set?' In other words, how different is a fuzzy set from an ordinary set? There are several measures of the 'fuzziness' of a set, each referred to as an index of fuzziness. We present two of these indices here.

Definition. Let A be a fuzzy set. The *ordinary set* S_A *nearest to* A has membership characteristic function

$$\mu_{S_A}(x) = \begin{cases} 0 & \text{if } \mu_A(x) \leqslant 0.5 \\ 1 & \text{if } \mu_A(x) > 0.5 \end{cases} \quad \text{for all } x \in U.$$

If we let $U = \{x_1, x_2, \ldots, x_n\}$ the *index of fuzziness* of the fuzzy set A is

$$I(A) = \frac{2}{n} \sum_{i=1}^{n} |\mu_A(x_i) - \mu_{S_A}(x_i)|.$$

If we let U be the closed interval $[a, b]$ (i.e. all real numbers such that $a \leqslant x \leqslant b$), then the *index of fuzziness* of the fuzzy set A is

$$I(A) = \frac{2}{b-a} \int_a^b |\mu_A(x) - \mu_{S_A}(x)| \, dx.$$

In the previous example, where $U = \{a, b, c\}$, the ordinary set nearest A is $S_A = \{(a, 1), (b, 0), (c, 0)\}$ or, if we use ordinary notation, $S_A = \{a\}$. Thus the index of fuzziness is $I(A) = \frac{1}{2}$. For B the index of fuzziness is $I(B) = 1$. The value of the index for either of the definitions above ranges between 0 and 1, since the minimum value of $|\mu_A(x) - \mu_{S_A}(x)|$ is 0 and the maximum value is $\frac{1}{2}$. The fuzziest set has an index equal to 1 and a non-fuzzy set or ordinary set has an index equal to 0. The set B in the example above is the fuzziest possible set. In fact any set B such that $\mu_B(x) = \frac{1}{2}$ for all $x \in U$ will yield $I(B) = 1$ for either of the indices.

Many applications of fuzzy set theory use only these basic concepts. One application is in the area of ecosystem management. Many concepts and definitions in ecosystem science are imprecise, since ecosystems are large and heterogeneous and even two similar ecosystems can vary markedly in several aspects. Elaborate sampling procedures are required to characterise adequately even one ecosystem—a project which is not feasible. Thus government officials, environmentalists, and members of the community must rely on 'partial' information in formulating policies or making management decisions regarding ecosystems. Fuzzy set theory is a means of quantifying this 'fuzzy' information which can then be analysed by various sectors of society.

A hypothetical problem where fuzzy set theory can be applied is an example presented by Bosserman and Ragade (see reference 2). The Kentucky Department of Natural Resources and Environmental Protection is faced with the task of determining whether parcels of land are suitable for surface mining. The final decision is made by several state agencies and requires consideration of many disparate views. The use of fuzzy set theory can simplify some of the decision-making process.

Let U be the set of all factors considered in deciding whether a parcel of land is unsuitable for surface mining. Eleven factors are listed in the table below; here $U = \{x_1, \ldots, x_{11}\}$.

Table 1
Factors used in deciding the unsuitability of surface mining (see reference 2).

Factors	Description	Range of membership
x_1	Uncommon geological features	0–1
x_2	National natural landmarks	0 or 1
x_3	Fish habitat	0–1
x_4	Wildlife habitat	0–1
x_5	Endangered plants	0 or 1
x_6	Endangered animals	0 or 1
x_7	Wetlands	0–1
x_8	Environmental corridors	0–1
x_9	Natural preserves	0 or 1
x_{10}	Wild rivers	0 or 1
x_{11}	Recreational value	0–1

For each parcel of land A in Kentucky a membership characteristic value $\mu_A(x_i)$ must be assigned for each x_i in U. The officials from various state agencies assign the values—e.g. the Department of Parks and Recreation assigns the value for $\mu_A(x_{11})$ and the Nature Preserves Commission assigns the value for $\mu_A(x_4)$. The values $\mu_A(x_i)$ for $i = 2, 5, 6, 9, 10$ are either 1 or 0, indicating presence or absence, respectively, of the particular feature. Other membership values range between 0 and 1. The larger the membership value, the more unsuitable the parcel of land is for surface mining.

After the membership values have been assigned, decision criteria must be generated to determine unsuitability of a tract of land for surface mining. Here is where fuzzy set theory is applied. One decision criterion may be that if $\mu_A(x_i) = 1$ for some $i = 1, \ldots, 11$, then A is unsuitable for surface mining. However if $\mu_A(x_i) < 1$ for all $i = 1, \ldots, 11$, then all values $\mu_A(x_i)$ must be considered, possibly a weighted sum

$$\sum_{i=1}^{11} W_u \mu_A(x_i),$$

where the value of W_i indicates the relative importance of the factor x_i. Comparisons between parcels of land can be made, for example, if $B \subseteq A$ then B is more suitable than A for surface mining, or if two adjacent parcels are being considered then one would analyse $A \cup B$. Other decision criteria can also be generated.

Other interesting applications of fuzzy sets include the following. In the area of pattern and language recognition, different sounds can be identified via the fuzzy properties of their patterns (see reference 5). The fuzziness of

X-ray images can reveal certain physiological features. Commercially, fuzzy set theory can be applied in control processes for large industrial plants. The use of fuzzy logic on computers is useful in the area of artificial intelligence. Additional areas for applying fuzzy set theory can be found in the references (in particular see reference 1).

The above presentation has been a brief introduction to the theory of fuzzy sets and some of its applications. The intention of this presentation is to excite additional interest in this new, fascinating, and ever-expanding subject in mathematics.

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Odd Squares

The square of an odd number, say x^2, can be written as the difference of two consecutive squares, e.g. $1^2 = 1^2 - 0^2$, $3^2 = 5^2 - 4^2$, $5^2 = 13^2 - 12^2$, $7^2 = 25^2 - 24^2$, etc. To find these, first write x^2 as a sum of two consecutive integers, say $x^2 = a + b$ with $a > b$, e.g. $1^2 = 1 + 0$, $3^2 = 5 + 4$, $5^2 = 13 + 12$, $7^2 = 25 + 24$, etc. Then $x^2 = a^2 - b^2$. Can you prove this generally?

L. B. Dutta
Maguradanga,
Keshabpur,
Jessore,
Bangladesh.

All Known Perfect Numbers are Triangular

Arthur Pounder of Manchester has sent us this observation. A perfect number is a positive integer which is the sum of its positive divisors, excluding the number itself. For example,

$$6 = 1+2+3, \qquad 28 = 1+2+4+7+14,$$

so 6 and 28 are perfect numbers. In fact, they are the first two perfect numbers; we write $P_1 = 6$ and $P_2 = 28$. The next four are $P_3 = 496$, $P_4 = 8128$, $P_5 = 33\,550\,336$ and $P_6 = 8589\,869\,056$, so they increase pretty rapidly. You will notice that these six are all even. All the perfect numbers found so far are even, and no one knows whether an odd perfect number exists.

Perfect numbers go back to the ancient Greeks. In fact, Euclid proved a formula which describes all even perfect numbers; they are those of the form

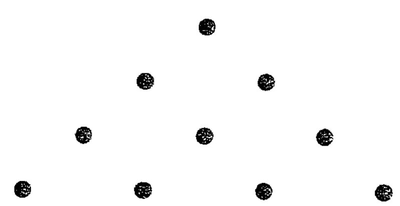
$$2^{p-1}(2^p-1),$$

where 2^p-1 is prime (which means that p must also be prime). For example, $p = 2$ gives $P_1 = 6$, $p = 3$ gives $P_2 = 28$, $p = 5$ gives $P_3 = 496$ and $p = 7$ gives $P_4 = 8128$. But be careful! The number $2^{11}-1$ is not prime (it is 23×89): so $p = 11$ will not give a perfect number. Herein lies the difficulty in finding even perfect numbers. You first have to find a prime number of the form 2^p-1, a so-called *Mersenne prime*. The latest information known to us is that 29 perfect numbers are known, the largest coming from $p = 132\,049$. This was discovered in 1984 by Slowinski; it took a week to calculate on a Cray supercomputer, so we advise readers not to try to work it out on their calculators.

So much for perfect numbers! What about triangular numbers? These are the numbers in the sequence

$$1, \quad 1+2, \quad 1+2+3, \quad 1+2+3+4, \quad \ldots,$$

so-called from the triangular arrangement



It is easy to obtain a formula for the nth triangular number; for

$$\begin{aligned} 2(1+2+\ldots+n) &= (1+2+3+\ldots+n)+[n+(n-1)+(n-2)+\ldots+1] \\ &= (n+1)n, \end{aligned}$$

so the nth triangular number is

$$\tfrac{1}{2}n(n+1).$$

Right, now look at Euclid's formula and see if you can see that *all known perfect numbers are triangular.*

Character Expectation

D. J. COLWELL, J. R. GILLETT AND B. C. JONES, *North Staffordshire Polytechnic*

The authors are all lecturers in mathematics at the North Staffordshire Polytechnic. They are involved with teaching mathematics and statistics to students on engineering and science degree courses.

The ideas explored in this paper were prompted by an aspect of the popular role-playing game Dungeons and Dragons.

1. The standard game

At the start of this game it is necessary for each player to generate a character. For the purposes of the game a character is assumed to have six basic attributes—Intelligence, Charisma, Dexterity, Wisdom, Constitution and Strength. The measure of each of these attributes possessed by a player is determined randomly by rolling dice, the standard method being to roll three unbiased, six-sided dice and then sum the resulting scores. Using this standard method of character generation, what measure can a player expect to attain for each of his or her character attributes?

This question may be readily answered using elementary probability theory. Suppose X is a random variable which takes the face values 1, 2, 3, 4, 5 and 6 of a die, each with probability $\frac{1}{6}$. Then the expected score from one throw of a die is

$$\begin{aligned} E(X) &= \sum_{x=1}^{6} xP(X=x) \\ &= (1\times\tfrac{1}{6})+(2\times\tfrac{1}{6})+(3\times\tfrac{1}{6})+(4\times\tfrac{1}{6})+(5\times\tfrac{1}{6})+(6\times\tfrac{1}{6}) = 3.5. \end{aligned}$$

Hence, since three dice are used to determine a character attribute, the expected measure of each attribute is

$$3E(X) = 10.5.$$

TABLE 1

Sum	3	4	5	6	7	8	9	10
Probability	0.005	0.014	0.028	0.046	0.069	0.097	0.116	0.125
Sum	11	12	13	14	15	16	17	18
Probability	0.125	0.116	0.097	0.069	0.046	0.028	0.014	0.005

The probability distribution of the sum of values obtained on tossing a die three times is given in table 1. This table shows that there is an 81.4 percent chance of obtaining a total score between 7 and 14 and a 48.2 percent chance of obtaining a score between 9 and 12. Hence the total scores tend to cluster around the expected score.

Many people feel that clustering of the attribute measures around 10 and 11 does not give a good basis for a game of Dungeons and Dragons. More excitement can be generated in the game when the characters of the players excel in one or more attributes. On the other hand, a character should not excel in too many attributes, as such 'supercharacters' can also create a boring game.

2. A modified game

A method which is often used to try to meet these requirements is to roll four dice and then sum the scores when the lowest (or lowest equal) score is discarded. Intuitively, this modification of the standard method seems likely to increase the expected score for an attribute. However, the calculation of this expected score is not quite as straightforward as the corresponding calculation for the standard method of character generation.

Suppose the scores of the four dice are X_1, X_2, X_3 and X_4 and that, when ranked, these scores are $X_{(1)}$, $X_{(2)}$, $X_{(3)}$ and $X_{(4)}$, where $X_{(1)} \leqslant X_{(2)} \leqslant X_{(3)} \leqslant X_{(4)}$. Then it is useful to obtain the distribution function of

$$\min(X_1, X_2, X_3, X_4) = X_{(1)}.$$

Clearly

$$P(\min(X_1, X_2, X_3, X_4) = X_{(1)} > x) = [1 - F(x)]^4,$$

where $F(x_i)$ is the cumulative distribution function of X_i $(i = 1, 2, 3, 4)$. Hence, since there are 6^4 possible outcomes when four dice are tossed,

$$P(X_{(1)} > 0) = 1, \qquad P(X_{(1)} > 1) = (\tfrac{5}{6})^4,$$

$$P(X_{(1)} > 2) = (\tfrac{4}{6})^4, \qquad P(X_{(1)} > 3) = (\tfrac{3}{6})^4,$$

$$P(X_{(1)} > 4) = (\tfrac{2}{6})^4, \qquad P(X_{(1)} > 5) = (\tfrac{1}{6})^4,$$

$$P(X_{(1)} > 6) = 0.$$

Thus the probability distribution function for $X_{(1)}$ is

$$P(X_{(1)} = r) = P(X_{(1)} > r-1) - P(X_{(1)} > r)$$
$$= \frac{(7-r)^4-(6-r)^4}{6^4} \quad (r = 1, 2, \ldots, 6),$$

and it may now be deduced that

$$E(X_{(1)}) = \sum_{r=1}^{6} rP(X_{(1)} = r) = \frac{1}{6^4}(6^4+5^4+4^4+3^4+2^4+1^4).$$

Further, using the suggested modification to the standard method, the expected measure of a character attribute is

$$E(X_{(2)}+X_{(3)}+X_{(4)}) = E(X_{(1)}+X_{(2)}+X_{(3)}+X_{(4)}) - E(X_{(1)})$$
$$= 4E(X) - E(X_{(1)})$$
$$= 4\times 3.5 - \frac{1}{6^4}(6^4+\ldots+1^4)$$
$$= 12.2.$$

This expectation is approximately 16.2 percent higher than that given by the standard method. The probability distribution for $X_{(2)}+X_{(3)}+X_{(4)}$ is given in table 2. It may be deduced from this table that there is an 83.6 percent chance of obtaining a character attribute measure between 9 and 16 and 49.9 chance of a measure between 11 and 14.

TABLE 2

Value of $X_{(2)}+X_{(3)}+X_{(4)}$	3	4	5	6	7	8
Probability	0.001	0.003	0.008	0.016	0.029	0.048
Value of $X_{(2)}+X_{(3)}+X_{(4)}$	9	10	11	12	13	
Probability	0.070	0.094	0.114	0.129	0.133	
Value of $X_{(2)}+X_3+X_{(4)}$	14	15	16	17	18	
Probability	0.123	0.101	0.072	0.042	0.016	

3. The game with n dice

The results just obtained confirm that the modification to the standard method for generating character attributes has the desired effect of increasing the chances of a player excelling in several attributes. However, if the character attribute expectation in this modified method is still not considered to be high enough, then the modified process could be investigated for the case when n ($n > 4$) dice are tossed, but only the three highest scores are retained. A decision could then be made as to which value of n gave rise to the most satisfactory character-attribute expectation.

The analysis of the modified method for n dice, which is given in the appendix, allows a table of expectations, table 3, to be constructed. As might have been predicted, the results in this table show that the character attribute expectations climb steadily to 18 as n increases.

TABLE 3

Number of dice used (n)	3	4	5	6	7	8
Character-attribute expectation $E(X_{(n-2)}+X_{(n-1)}+X_{(n)})$	10.5	12.2	13.4	14.3	14.9	15.4
Number of dice used (n)	9	10	11	12	13	
Character-attribute expectation $E(X_{(n-2)}+X_{(n-1)}+X_{(n)})$	15.8	16.1	16.3	16.6	16.7	

Appendix

When n six-sided dice are tossed suppose the resulting scores $X_1,\ldots,X_n$ become, when ranked, $X_{(1)},\ldots,X_{(n)}$, where $X_{(1)} \leqslant X_{(2)} \leqslant \ldots \leqslant X_{(n)}$. The probability distribution for each $X_{(r)}$ $(r = 1,2,\ldots,n)$ may then be obtained in the following way.

$P(X_{(r)} = 1)$

For $1 \leqslant k \leqslant n$,

$$P(1 = X_{(1)} = \ldots = X_{(k)} < X_{(k+1)} \leqslant \ldots \leqslant X_{(n)}) = \frac{1}{6^n}\binom{n}{n-k}5^{n-k}.$$

Hence

$$P(X_{(r)} = 1) = \sum_{k=r}^{n} P(1 = X_{(1)} = \ldots = X_{(k)} < X_{(k+1)} \leqslant \ldots \leqslant X_{(n)})$$
$$= \frac{1}{6^n}\sum_{k=r}^{n}\binom{n}{n-k}5^{n-k}.$$

$P(X_{(r)} = x)$ $(x = 2,3,4,5)$

For each x $(x = 2,3,4,5)$ and $0 \leqslant j < r \leqslant k \leqslant n$,

$$P(X_{(1)} \leqslant \ldots < x = X_{(j+1)} = \ldots = X_{(k)} < \ldots \leqslant X_{(n)})$$
$$= \frac{1}{6^n}\binom{n}{j}(x-1)^j\binom{n-j}{n-k}(6-x)^{n-k}.$$

Hence

$$P(X_{(r)} = x) = \sum_{j=0}^{r-1}\sum_{k=r}^{n} P(X_{(1)} \leqslant \ldots < x = X_{(j+1)} = \ldots = X_{(k)} < \ldots \leqslant X_{(n)})$$
$$= \sum_{j=0}^{r-1}\sum_{k=r}^{n}\frac{1}{6^n}\binom{n}{j}(x-1)^j\binom{n-j}{n-k}(6-x)^{n-k}$$

$$= \frac{1}{6^n} \sum_{j=0}^{n-1} \binom{n}{j} (x-1)^j \sum_{k=0}^{n-r} \binom{n-j}{n-k} (6-x)^{n-k}.$$

$P(X_{(r)} = 6)$

For $0 \leqslant j \leqslant r-1$,

$$P(X_{(1)} \leqslant \ldots \leqslant X_{(j)} < 6 = X_{(j+1)} = \ldots = X_{(n)}) = \frac{1}{6^n}\binom{n}{j}5^j.$$

Hence

$$P(X_{(r)} = 6) = \sum_{j=0}^{r-1} P(X_{(1)} \leqslant \ldots \leqslant X_{(j)} < 6 = X_{(j+1)} = \ldots = X_{(n)}) = \frac{1}{6^n} \sum_{j=0}^{r-1} \binom{n}{j}5^j.$$

Having obtained the probability distribution for each $X_{(r)}$,

$$E(X_{(r)}) = \sum_{x=1}^{6} xP(X_{(r)} = x) \quad (r = 1, \ldots, n)$$

may be readily deduced. Further the character-attribute expectation, namely

$$E(X_{(n-2)} + X_{(n-1)} + X_{(n)}) = E(X_{(n-2)}) + E(X_{(n-1)}) + E(X_{(n)}),$$

may be evaluated. The results for $n = 3, 4, \ldots, 13$ are recorded in table 3.

Approximating e^{π}

P. J. O'Grady, of Warwick School, has sent us the approximation

$$e^{\pi} \simeq \tfrac{10691}{462},$$

so that

$$\pi \simeq \ln \tfrac{10691}{462},$$

a considerably more accurate approximation to π than $3\tfrac{16}{113}$. This gives the 'even more useless result' (to quote Mr. O'Grady)

$$-1 \simeq (\tfrac{10691}{462})^{i},$$

where $i = \sqrt{-1}$. He adds that the fractional part of e^{π} is approximately

$$\frac{5 \times 13}{2 \times 3 \times 7 \times 11},$$

'a rather pleasing combination of the first six primes'. Perhaps other readers have come across similarly 'useless results'—please send them to us if you have.

Lines and Points

DERMOT ROAF, *Exeter College, Oxford*

The author was an undergraduate at Christ College, Oxford and a graduate student in Cambridge. He is Mathematics Fellow at his college, with a particular interest in theoretical physics. His hobbies include bell-ringing (see *Mathematical Spectrum*, Volume 7, pages 60–66) and politics: he is currently Alliance Spokesman on Education on the 'balanced' Oxfordshire County Council.

1. The million-straight-lines problem

Some years ago I read (in a review of a book of Russian puzzles which I cannot now trace) the following puzzle:

> 1. A million straight lines lie in a plane. No two are parallel and through every intersection there are at least three lines. How many points of intersection are there?

The answer is (almost) obvious. Think about it before reading further.

I am sometimes asked to give talks on puzzles to school sixth-forms and I have used this puzzle for several years. Recently I looked for some more puzzles, so read through all those set in *Mathematical Spectrum* and came across this one:

> 2. Show that, given a finite number of points in the plane which do not all lie on the same straight line, there exists a straight line passing through exactly two of the points. [Problem 6.5 in Volume 6, Number 2].

I told my next audience that these were the same problem, but the audience disagreed. The geometry of 'duals' is not known today.

2. Duals

Let us consider the geometry of lines and points in the plane. Think about theorems of collinearity and concurrence which do not use the concept of 'distance' (so do not use ratios).

Q. What is a line?	**A.** The join of two points.
Q. What is a point?	**A.** The intersection of two lines.

The only difficulty is with parallel lines, so we either invent 'points at infinity' (lying on the 'line at infinity') or exclude parallel lines from consideration (as in Problem 1 above). Now any geometrical theorem about lines and points only (no distances) can be rewritten as a theorem about points and lines. Thus Problems 1 and 2 become:

1 (dual). A million points lie in a plane. On every straight line joining these points there are at least three of the points. How many lines are there?

2 (dual). Show that, given a finite number of (non-parallel) lines in the plane which are not all concurrent, there exists a point with exactly two lines through it.

We can look at this dualisation in coordinate geometry. Consider the equation $lx+my = 1$. If l and m are constant the pairs of values (x, y) which satisfy the equation lie on a straight line. We can describe this line uniquely by the pair $[l, m]$. Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if there exists a line $[l, m]$ such that $lx_1+my_1 = 1$, $lx_2+my_2 = 1$ and $lx_3+my_3 = 1$. But suppose we fix x and y and look at the solutions for l and m. These describe a set of lines $[l, m]$ through the fixed point (x, y). So three lines $[l_1, m_1]$, $[l_2, m_2]$ and $[l_3, m_3]$ are concurrent if there exists a point (x, y) such that $xl_1+ym_1 = 1$, $xl_2+ym_2 = 1$ and $xl_3+ym_3 = 1$.

Now the line $[l, m]$ can be given the point equations

$$x = \frac{l}{l^2+m^2}+\alpha m, \qquad y = \frac{m}{l^2+m^2}-\alpha l,$$

(as α varies). So the equation $l^2+m^2 = 1$ represents a set of lines. Draw some of them. You will find they are the tangents to the circle with point equation $x^2+y^2 = 1$. So we can say that the circle has line equation $l^2+m^2 = 1$.

The condition that three points are collinear is exactly similar to the condition that three lines are concurrent. So theorems involving collinearity dualise to theorems of concurrency. Of course the equation $lx+my = 1$ cannot describe lines through the origin $(0,0)$, but $[0,0]$ does describe the 'line at infinity'; it is easy to see that the parallel lines $[l, m]$ and $[\alpha l, \alpha y]$ are concurrent with $[0,0]$, just as the points (x, y), $(\alpha x, \alpha y)$ and $(0,0)$ are collinear.

There are ways of including the origin and the line at infinity using 'homogeneous co-ordinates in projective geometry', but that, though quite easy, would need more space than *Mathematical Spectrum* can afford.

3. The solutions

I stated earlier that the answer was 'obvious' and comparing problem 1 and problem 2 (dual) the answer to problem 1 is 'exactly one point'. It was 'obvious' because any other answer would be inelegant. Quick sketches show that two, three and four points are all impossible.

M. Ram Murty and V. Kumar Murty of Carleton University, Ottawa solved problem 2 [Volume 7, No. 2 p.68], but their solution does not dualise because it uses distances. I think that problem 1 is easier. Can you solve it? Try before reading further.

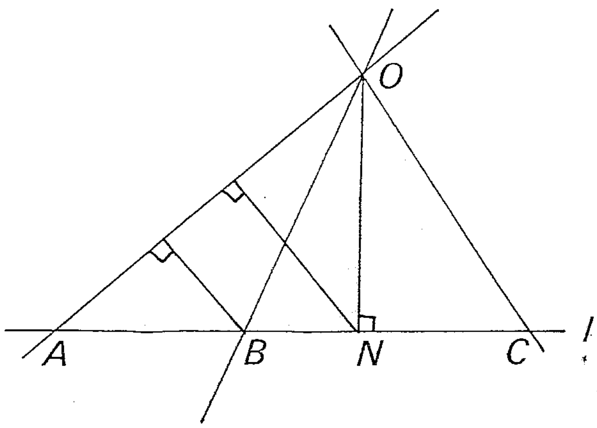
My first solution was:

> Suppose the lines are not all concurrent. Then there must be at least one line l, which does not pass through a particular point of intersection O. As there are at least three lines through O, l must meet them at, say, A, B and C. Consider all such triangles $OABC$. There are a finite number of them, so there is a triangle of smallest area. If this is $OABC$ (with B lying between A and C), consider B. There must be another line through it and this must cut internally either OA or OC. If it cuts OA at D, then the triangle $BODA$ is of the same form as $OABC$, but has a smaller area. Similarly if it cuts OC. So we contradict the hypothesis that $OABC$ has the least area.

I cannot dualise this for problem 2, nor can I dualise the Murty solution to solve problem 1. But the following modification to my proof does solve both problems:

> Consider the perpendicular from O to l at N (see the figure). Consider the triangle $OABC$ which has a perpendicular of the minimum length. Suppose A and B lie on the same side of N (if not, replace A by C in all that follows). Then the perpendicular from B onto OA is shorter than the perpendicular from N onto OA, which is itself less than ON. Thus we contradict the hypothesis. For problem 2 we suppose that there is no line passing through only two of the points, take O to be one of the points and A, B and C to be three of the points lying on a line not passing through O. Then we look for the triangle $OABC$ with smallest perpendicular ON. And again we have the same contradiction.

But this proof is not the dual. Can you find a proof not using lengths which has a dual? Or can you invent an idea for the 'distance' between two lines which intersect to correspond to the idea of the distance between two points which lie on the same line?



Computer Column

MIKE PIFF

Suppose a particle moves according to the rule

$$\ddot{\boldsymbol{r}} = -\beta\dot{\boldsymbol{r}} + \boldsymbol{f}(t),$$

where $\boldsymbol{r} = (x, y)$ denotes its position, $\boldsymbol{f}(t)$ is a random function of time t and β is a damping factor, $0 \leqslant \beta \leqslant 1$. Then we can approximate its motion by sampling at integer times n, letting $\boldsymbol{v}_n$ denote its velocity at time n and take

$$\boldsymbol{v}_{n+1} \triangleq (1-\beta)\boldsymbol{v}_n + \boldsymbol{f}(t).$$

If its position is $\boldsymbol{r}_n$ at time n, we have

$$\boldsymbol{r}_{n+1} \triangleq \boldsymbol{r}_n + \boldsymbol{v}_{n+1},$$

and these equations allow us to simulate its motion. Such a motion is called Brownian motion.

The following BBC program does such a simulation and the Brownian motion is confined to the screen by letting the screen edges act as reflecting barriers. A damping factor of 0 gives virtually a billiard-ball motion, whilst a factor of 0.2 should give quite an interesting effect.

```
 10 INPUT "DAMPING FACTOR? (0..1)",BETA
 20 IF BETA<0 THEN BETA=0
 30 IF BETA>1 THEN BETA=1
 40 BETA=1-BETA
 50 MODE0
 60 VDU19,0,4,0,0,0
 70 VDU19,1,1,0,0,0
 80 VDU 23,1,0;0;0;0;
 90 CLG
100 X=640:Y=512
110 MOVE X,Y
120 VX=0:VY=0
130 VX=0:VY=0
140 REPEAT
150 THETA=2*PI*RND(1)
160 R=RND(1)
170 VX=BETA*VX+R*COS(THETA):VY=BETA*VY+R*SIN(THETA)
180 VX=VX*SGN(1279-X)*SGN(X)
190 VY=VY*SGN(1023-Y)*SGN(Y)
200 X=X+VX:Y=Y+VY
210 DRAW X,Y
220 UNTIL FALSE
230 END
```

Malcolm Smithers, a student of the Open University, has sent us this postcard depicting Pythagoras, posted in Pithagorio on the Greek island of Samos, his birthplace. Perhaps other readers have been to places of similar mathematical interest.

More on 1987

Malcolm Smithers has also written to tell us that 1987 is the 300th prime number. Moreover, it fits neatly into the following sets in which the numeric symbols are used just once:

with zero	$\{2, 5, 1987, 4603\}$,
without zero	$\{2, 5, 1987, 463\}$.

Letters to the Editor

Dear Editor,

Romberg integration and bootstrapping π

As a recent convert to Romberg integration—encouraged by a program that ran FIRST TIME—I want all the Spectrum world to know about it. Simpson's rule is better than the trapezium rule for a fixed odd number of ordinates, but Romberg exploits the latter indefinitely.

If we put

$$I = \int_a^b y \, \mathrm{d}x,$$

then it can be proved, or shown by comparing results for large enough numbers of intervals, that

$$T_n = I + Ah^2 + Bh^4 + Ch^6 + \ldots,$$

where n is the number of intervals in the trapezium rule, $h = (b-a)/n$ and A, B, $C, \ldots$ are constants for particular y, a and b. Set

$$R_n^{(2)} = \tfrac{1}{3}(4T_{2n} - T_n) \qquad \text{to eliminate } A,$$
$$R_n^{(4)} = \tfrac{1}{15}(16R_{2n}^{(2)} - R_n^{(2)}) \quad \text{to eliminate } B, \text{ etc.}$$

Then $R_n^{(4)}$ is better than Simpson's rule, with error $O(h^6)$ *versus* $O(h^4)$. The answers are so good that I stop there.

The recurrence relation

$$\pi_1 = 3, \qquad \pi_{n+1} = \int_0^{\pi_n} x \sin x \, \mathrm{d}x$$

gives steadily better values of π.

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Yours sincerely,

J. L. G. PINHEY
The Perse School,
Cambridge.

Dear Editor,

On sums of two powers

Let $P(n,m,y)$ denote the number of positive integers p for which there are at least m different expressions of the form $p = a_i^n + b_i^n$ for $i = 1,\ldots,m$, where $a_1, b_1, \ldots, a_m, b_m$ are positive integers less than or equal to y whose highest common factor is one. As a demonstration, I give a list of those equations contributing to $P(2,2,13)$:

$$\begin{aligned} 7^2+1^2 &= 5^2+5^2, & 11^2+3^2 &= 9^2+7^2, \\ 8^2+1^2 &= 7^2+4^2, & 12^2+1^2 &= 9^2+8^2, \\ 9^2+2^2 &= 7^2+6^2, & 13^2+1^2 &= 11^2+7^2, \\ 11^2+2^2 &= 10^2+5^2, & 13^2+4^2 &= 11^2+8^2. \end{aligned}$$

Thus $P(2,2,13) = 8$.

Now consider $n = 2$ and $m = 2$ and let $P(2,2,y) = p(y)$. Then

$$\begin{aligned} p(20) &= 22, & p(50) &= 178, & p(80) &= 487, \\ p(25) &= 39, & p(55) &= 226, & p(85) &= 554, \\ p(30) &= 56, & p(60) &= 259, & p(90) &= 624, \\ p(35) &= 83, & p(65) &= 315, & p(95) &= 693, \\ p(40) &= 107, & p(70) &= 361, & p(100) &= 767. \\ p(45) &= 143, & p(75) &= 431, & & \end{aligned}$$

(I should like to point out that all numerical results were obtained using my brother's Spectrum 48K in BASIC, a very slow process.) I realise that there cannot possibly be an exact formula for $p(y)$ (the above list was obtained by counting solutions), but I ask if there is an approximate form giving the correct order of magnitude [as in the case that $\pi(x) \approx x/\log(x)$].

Now let $n = 2$ and $m = 3$ and let $P(2,3,y) = q(y)$. The smallest y such that $q(y) \neq 0$ is $y = 18$ ($18^2+1^2 = 17^2+6^2 = 15^2+10^2$). We have

$$\begin{aligned} q(20) &= 2, & q(50) &= 29, & q(80) &= 92, \\ q(25) &= 3, & q(55) &= 37, & q(85) &= 108, \\ q(30) &= 7, & q(60) &= 45, & q(90) &= 120, \\ q(35) &= 11, & q(65) &= 56, & q(95) &= 139, \\ q(40) &= 15, & q(70) &= 65, & q(100) &= 154. \\ q(45) &= 22, & q(75) &= 76, & & \end{aligned}$$

Again, is there an approximate form?

For $n = 2$, more interesting cases are $m \geqslant 4$. Let $P(2,4,y) = r(y)$. Then

$$r(20) = 0, \quad r(60) = 13, \quad r(90) = 50,$$
$$r(40) = 2, \quad r(70) = 23, \quad r(100) = 66.$$
$$r(50) = 7, \quad r(80) = 33,$$

I list all the equations contributing to $r(50)$:

$$33^2+4^2 = 32^2+\ 9^2 = 31^2+12^2 = 24^2+23^2,$$
$$40^2+5^2 = 37^2+16^2 = 35^2+20^2 = 29^2+28^2,$$
$$43^2+6^2 = 42^2+11^2 = 38^2+21^2 = 34^2+27^2,$$
$$46^2+3^2 = 42^2+19^2 = 45^2+10^2 = 35^2+30^2,$$
$$47^2+1^2 = 43^2+19^2 = 41^2+23^2 = 37^2+29^2,$$
$$49^2+2^2 = 47^2+14^2 = 46^2+17^2 = 38^2+31^2,$$
$$49^2+8^2 = 47^2+16^2 = 44^2+23^2 = 41^2+28^2.$$

Now let $Q(n,m)$ be the smallest value of y such that $P(n,m,y) \neq 0$. We have $Q(2,2) = 7$, $Q(2,3) = 18$ and $Q(2,4) = 33$. Also, $Q(2,5) = 73$, the smallest contributing expressions being

$$73^2+14^2 = 71^2+22^2 = 70^2+25^2 = 62^2+41^2 = 55^2+50^2,$$

and $Q(2,6) = 74$, the smallest contributing expressions here being

$$74^2+7^2 = 73^2+14^2 = 71^2+22^2 = 70^2+25^2 = 62^2+41^2 = 55^2+50^2.$$

At the time of writing, I have taken y up to 220 for $n = 2$. Thus

$$P(2,2,220) = 3936, \quad P(2,3,220) = 925,$$
$$P(2,4,220) = 519, \quad P(2,5,220) = 77,$$
$$P(2,6,220) = 59, \quad P(2,7,220) = P(2,8,220) = 5.$$

Thus $Q(2,7) \geqslant 101$. I hope to proceed much further in this case.

I have also considered $n = 3$ as far as $y = 550$. I have $P(3,2,550) = 268$ and $P(3,3,550) = 0$. The same questions can be asked of this case. Most especially, what is the value of $Q(3,3)$?

Note. For $n = 4$, $Q(4,2) = 158$ since $133^4+134^4 = 158^4+59^4$ is the smallest such equation. I fear my computer is not up to the task of investigating $n \geqslant 4$. In fact I would venture that finding the value of $Q(4,3)$ is a major task for the fastest computers in the world.

Finally may I ask that interested readers check that my numbers for $P(n,m,y)$ are accurate and perhaps let me know of improved results.

Yours sincerely,

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Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

19.4. (Submitted by János Sütö, Kelvin Hall Comprehensive School, Kingston upon Hull)
How many triangles are there whose vertices are vertices of a given $(2n+1)$-sided regular convex polygon and which contain the centre of the circle circumscribing the polygon?

19.5. (Submitted by Adrian Hill, Trinity College, Cambridge)
Evaluate the infinite product

$$\sqrt{\tfrac{1}{2}}\times\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}\times\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}}\times\dots.$$

(Hint: Use cosines.)

19.6. (Submitted by Lászlo Cseh, a student at Babes Bolyai University, Romania)
Find all poitive integers x for which $d(8x) = 4p$, where p is a prime and $d(n)$ denotes the number of positive divisors of the positive integer n.

19.7. (Submitted by Miklòs Bòna, a student at the University of Budapest)
Twenty-five points are given in a right-angled triangle whose smallest angle is 30° and whose hypotenuse has length 1. Prove that three of the points can be chosen so that they lie inside a semicircle, the diameter of the whole circle being no larger than 0.29.

Solutions to Problems in Volume 18, Number 3

18.7. Twenty non-overlapping squares lie inside a square of side 1. Show that there are four of these squares the sum of the lengths of whose sides does not exceed $2/\sqrt{5}$.

Solution by Adrian Hill (Trinity College, Cambridge)

Consider the four smallest squares and denote the lengths of their sides by a_1, a_2, a_3 and a_4 and the sum of their areas by A. Clearly $A \leqslant \frac{4}{20} = \frac{1}{5}$. Now

$$\begin{aligned}
&(a_1-a_2)^2+(a_1-a_3)^2+(a_1-a_4)^2+(a_2-a_3)^2+(a_2-a_4)^2+(a_3-a_4)^2\\
&\quad= 3(a_1^2+a_2^2+a_3^2+a_4^2)-2(a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4)\\
&\quad= 4(a_1^2+a_2^2+a_3^2+a_4^2)-(a_1+a_2+a_3+a_4)^2\\
&\quad= 4A-(a_1+a_2+a_3+a_4)^2,
\end{aligned}$$

so that

$$4A-(a_1+a_2+a_3+a_4)^2 \geqslant 0,$$

$$\Rightarrow \qquad a_1+a_2+a_3+a_4 \leqslant 2\sqrt{A} \leqslant 2/\sqrt{5}.$$

18.8. Obtain a result connecting the areas of the faces of a right-angled tetrahedron.

Solution by Adrian Hill (Trinity College, Cambridge)

Take the origin at the right-angled vertex of the tetrahedron and the other vertices at $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. The area A of the face opposite the right-angled vertex is $\frac{1}{2}|(\boldsymbol{a}-\boldsymbol{c})\wedge(\boldsymbol{a}-\boldsymbol{b})|$, so that $A^2 = \frac{1}{4}(\boldsymbol{b}\wedge\boldsymbol{a}+\boldsymbol{c}\wedge\boldsymbol{b}+\boldsymbol{a}\wedge\boldsymbol{c})^2 = \frac{1}{4}[(\boldsymbol{b}\wedge\boldsymbol{a})^2+(\boldsymbol{c}\wedge\boldsymbol{b})^2+(\boldsymbol{a}\wedge\boldsymbol{c})^2]$ (since the normals to the other three faces are perpendicular to each other). Thus

$$A^2 = B^2+C^2+D^2,$$

where B, C and D are the areas of the other three faces.

Also solved by Philip Wadey (University of Exeter) and Guy Willard (The Haberdashers' Aske's School, Elstree).

18.9. Show that

$$\int_{x_1}^{x_2} \exp(2\pi \mathrm{i}x)\,\mathrm{d}x$$

is zero if and only if x_2-x_1 is an integer.

A rectangle is subdivided into smaller rectangles whose sides are parallel to the sides of the large rectangle. By considering the double integral

$$\iint \exp[2\pi\mathrm{i}(x+y)]\,\mathrm{d}x\,\mathrm{d}y$$

over the large rectangle, or otherwise, show that, if at least one of the sides of each of the smaller rectangles is an integer, then at least one of the sides of the large rectangle is an integer.

Solution independently by Adrian Hill (Trinity College, Cambridge) and Yik-Man Chiang (University College, London)

$$\begin{aligned}\int_{x_1}^{x_2} \exp(2\pi\mathrm{i}x)\,\mathrm{d}x &= \frac{1}{2\pi\mathrm{i}}[\exp(2\pi\mathrm{i}x)]_{x_1}^{x_2}\\ &= \frac{1}{2\pi\mathrm{i}}\exp(2\pi\mathrm{i}x_1)\{\exp[2\pi\mathrm{i}(x_2-x_1)]-1\},\end{aligned}$$

and this is zero if and only if

$$\exp[2\pi\mathrm{i}(x_2-x_1)] = 1 \quad\Leftrightarrow\quad x_2-x_1 \text{ is an integer.}$$

Assume that at least one of the sides of each of the smaller rectangles is an integer. Then

$$\begin{aligned}\iint\limits_{\substack{\text{large}\\ \text{rectangle}}} \exp[2\pi\mathrm{i}(x+y)]\,\mathrm{d}x\,\mathrm{d}y &= \sum_{\substack{\text{small}\\ \text{rectangles}}} \int \exp(2\pi\mathrm{i}x)\,\mathrm{d}x \int \exp(2\pi\mathrm{i}y)\,\mathrm{d}y\\ &= 0 \quad \text{(by the first part).}\end{aligned}$$

Thus one of

$$\int \exp(2\pi\mathrm{i}x)\,\mathrm{d}x, \qquad \int \exp(2\pi\mathrm{i}y)\,\mathrm{d}y$$

must be zero over the large rectangle, so one of its sides is an integer.

Also solved by Philip Wadey (University of Exeter) and Guy Willard (The Haberdashers' Aske's School, Elstree).

Book Reviews

Algebraic Structures. By C. F. GARDINER. Ellis Horwood, 1986. Pp. 280. £14.50.

This is a sequel to the author's *Modern Algebra* (Ellis Horwood, 1981), which was reviewed in *Mathematical Spectrum* Volume 15, page 96. As such, it is well beyond the stage that most of our readers have reached in their mathematical journey, so a detailed review is not appropriate here. It covers a great deal of ground, with chapters on groups, (containing, for example, the Sylow theorems), rings and fields (including Galois theory), linear groups, and a final chapter on computational group theory which concentrates on the Todd–Coxeter coset-enumeration algorithm developed in 1936 and includes computer programs suitable for use on a microcomputer. This last chapter is a novel feature. The author writes in a clear style familiar to readers from his articles in *Spectrum*, but it has to be said that this is a very hasty journey through some very substantial mathematics. For example, to cover Noetherian rings in five pages may well be a record, but perhaps not a very helpful one. There is a serious lack of examples in the text in the first two chapters, which would have aided understanding. Most students meeting this material for the first time will probably need a more leisurely journey, with time to pause and look around at some very beautiful mathematics.

University of Sheffield — D. W. SHARPE

Maths A–Z. Edited by P. J. F. HORRIL. Longman, 1986. Pp. vi+166. Paperback £3.95.

Dictionaries are primarily reference books to be used when there is a need to know something quickly; and yet at the same time they should be sufficiently thorough and interesting to become absorbing reading. This book is intended to cover the 12–18 age range and so needs to be intelligible to the youngest reader and to give precise and accurate definitions to the A-level student. There are some helpful entries and, in general, the diagrams are clear and informative. However, many of the definitions are clumsily worded and, if the reader did not know the meaning of the term already, he would be little wiser for the reading. Here are some of the entries:

Absolute value is defined as 'modulus 1, 2'. It is not made clear that this is a cross-reference to the Modulus definition, parts 1 and 2 on page 93.

and is defined as 'a logical connective denoted by $\wedge$'. There is no other reference to logical symbols except in the definition of Boolean.

Bound gives simply '1. Lower Bound, 2. Upper Bound.' No cross-reference is given to the definition of Upper Bound on page 150 and the term Lower Bound does not appear anywhere else.

In the foreword it is stated that cross-references to other entries are shown in small capitals, but this is not really sufficient—a reference to the pages involved would be much clearer.

Mapping Why not say that mapping is another word for a function?

Axiom is defined as a postulate—which is quite true but not very informative.

Rational numbers It would be useful to mention terminating and recurring decimals.

The appendix of mathematical notation is useful. The potted biographies of mathematicians would probably be better also in an appendix.

There are several mathematical dictionaries and this book is cheaper than most. However, I see only little use for such a book for students preparing for O-level (and GCSE) and A-level. Such students should build their own notebooks, putting into them definitions which they need to know and diagrams where appropriate (culled from textbooks and occasionally given as notes in class). They can include solutions of standard problems and indeed anything that the student finds interesting. These books, as I know from my experience of teaching, are much treasured and much used in the lead-up to examinations and for final revision. A copy of a suitable mathematical dictionary could be in a class library for occasional reference.

Maths A–Z would have some uses in a library which does not already contain a mathematical dictionary, but it is not sufficiently interesting for browsing.

Formerly of Wycombe Abbey School M. V. BILSBORROW

Mastering Statistics with your Microcomputer. By CONALL BOYLE. Macmillan Master Series, Macmillan Education Ltd, London, 1986. Pp. xiv+155. Hardback £10.00. Paperback £3.50.

The aim of this book is to give the reader all the essential information to explore and master statistics on a microcomputer. Also, the blurb claims, it is ideal for those who are looking for something worthwhile to do on their home computers.

Certainly the book is an extremely readable introduction to elementary statistics, covering ground as far as the correlation coefficient, least squares, and smoothing techniques for time series, before passing the reader on to more advanced reading. Its text is backed up by numerous programs in BASIC which, in themselves, should give the reader first, a feel for statistical programming at an elementary level, and second, a core of short programs for elementary applications. A noteworthy device is the inclusion of various tasks designed for self-learning. The book closes with a review of the various statistical packages available on minicomputers and mainframe computers.

However, there are a number of errors in exposition which might puzzle the observant reader. For instance, on page 68 a geometrical explanation is given for the formula for the standard deviation, but in fact what is being described is the *variance*, since a square root is nowhere mentioned. On page 82, an example is given with eleven students sitting an English and a Maths exam, but then suddenly Maths becomes French, and acquires a twelfth student who has opted out by the bottom of the page! The formula for the intercept in a linear least-squares fit on page 119 has a square missing, though the accompanying diagram is correct. The book also suffers from a first-edition crop of printing errors, all trivial and easily corrected.

In summary, this book is an easy introduction to statistics for the non-mathematical reader, as well as being worthwhile initiation into using a computer to do a job of work rather than just to play games with. I can recommend it with few reservations. For me the highlights were the revelations about a house-price index used by the D.O.E., based on building society figures, which went seriously adrift when they forgot that the banks had started lending on mortgages and had cornered the top of the market; and the (apocryphal?) figures which seem to indicate that half of the statisticians in Britain earn over £40 000 a year, and all statisticians in Germany earn at least £50 000 a year!

University of Sheffield MIKE PIFF

The Shape of Space. By JEFFREY R. WEEKS. Marcel Dekker, Inc., New York and Basel, 1985. Pp. x+324. $59.50.

From our privileged position in three-dimensional space we observe obvious differences between the surface of a plane and the surface of a sphere. We are able to observe them extrinsically. However, a two-dimensional being living on one of these surfaces cannot take this 'bird's-eye view' of his world and must deduce its properties from within. He is forced to take an intrinsic viewpoint. In this way he might discover that on the plane the angle sum of any triangle is 180° while on the sphere the angle sum of any triangle is greater than 180°. From this he might deduce the facts that are so obvious to us with our extrinsic viewpoint—that the plane is flat while the sphere is curved.

This intrinsic view of the plane and the sphere is considered in the books *Flatland*, by E. A. Abbott, and *Sphereland*, by D. Burger, and this is the starting point of the book currently under review. However, *The Shape of Space* quickly moves on to consider the construction and the intrinsic geometry of all of the familiar compact two-dimensional manifolds: the torus, the Klein bottle, the projective plane and all of their connected sums. The value of restricting ourselves to the intrinsic viewpoint now becomes apparent as the author moves on to consider three-dimensional manifolds. Since these are locally three-dimensional we can no longer step outside of them to take a bird's-eye view. Nevertheless, our intrinsic feeling for the geometry of these strange spaces is quickly and effectively developed through lots of examples and puzzles. The author introduces us to the higher-dimensional relatives of the sphere, the torus etc., and to the elliptic and hyperbolic geometries associated with them. Finally he moves on to consider the possible geometry of the universe we actually live in.

The style of the author is personal and encouraging. The text moves at a lively pace, developing the ideas throughout by the active involvement of the reader. It contains no technical mathematics and the exercises might more accurately be called puzzles. Nevertheless, I believe the author does succeed in his aim to show how to visualise higher dimensions, and on the way he gives simple pictorial expositions of the Euler number and the Gauss–Bonnet formula as well as an elementary exposition of the ideas behind Thurston's recent results on the geometries of 3-manifolds.

I enjoyed this book and recommend it as a stimulating and mind-bending experience. My only objections concern the way the publishers have presented the text. The words are so freely spaced on the pages that it could very easily have been produced as a 150-page book. Thus it is a worthwhile read, but very expensive. I object on the grounds of cruelty to trees, and to your pocket.

University of Exeter PETER FIRBY

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