Crux Mathematicorum

Editorial Board

IN THIS ISSUE / DANS CE NUMÉRO

NOTICE TO CRUX READERS

Due to the significant financial limitations of the Canadian Mathematical Society (CMS), effective January 2019, the CMS is not able to continue administration of Crux Mathematicorum (CRUX) as a subscription-based publication. The CMS has undertaken significant efforts to secure support to make CRUX freely available as an online only publication. To date, such support has not been obtained but the CMS continues to seek funding for this initiative to continue the excellent work of the CRUX Editorial Board members. The Society hopes funding can be secured or a partnership can be established to continue to make this unique resource available beyond 2018. If this is not possible, it is most likely future issues of CRUX will not be available.

Denise Charron, Managing Editor

THE CONTEST CORNER
No. 68
John McLoughlin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er mars 2019**.*

La rédaction souhaite remercier Valérie Lapointe, Carignan, QC, d'avoir traduit les problèmes.

CC336. Soit la *matrice de Pascal* $n \times n$ définie de la façon suivante : $a_{1j} = a_{i1} = 1$, et $a_{ij} = a_{i-1,j} + a_{i,j-i}$ pour tout $i, j > 1$. Par exemple, la matrice de Pascal 3×3 est donnée par

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} .$$

Montrer que toute matrice de Pascal est inversible.

CC337. Soit $P(x)$ et $Q(x)$ des polynômes à coefficients réels. Trouver les conditions nécessaires et suffisantes sur N pour garantir que si le polynôme $P(Q(x))$ est de degré N, il existe un nombre réel x tel que $P(x) = Q(x)$.

CC338. Trouver (avec preuve) toutes les solutions entières (x, y) à $x^2 - xy + 2017y = 0$.

CC339. Un *dodécaèdre rhombique* (granatoèdre) a douze faces rhombiques congruentes ; chaque sommet a soit quatre petits angles ou trois grands angles qui s'y rencontrent. Si la mesure du côté est de 1, trouver le volume sous la forme $\frac{p+\sqrt{q}}{r}$, où p, q, et r sont des nombres naturels et r n'a ni de facteur commun avec p ni q.
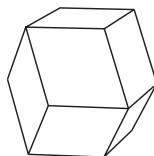
CC340. Soit S l'ensemble des nombres naturels qui divisent 2018^{2018}. De combien de façons peut-on sélectionner trois nombres $\{x, y, z\}$ (pas nécessairement distincts, mais dont l'ordre est sans importance) de S tels que $y = \sqrt{xz}$?

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

CC336. Define the $n \times n$ *Pascal matrix* as follows : $a_{1j} = a_{i1} = 1$, while $a_{ij} = a_{i-1,j} + a_{i,j-i}$ for $i, j > 1$. So, for instance, the 3×3 Pascal matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

Show that every Pascal matrix is invertible.

CC337. Suppose $P(x)$ and $Q(x)$ are polynomials with real coefficients. Find necessary and sufficient conditions on N to guarantee that if the polynomial $P(Q(x))$ has degree N, there exists real x with $P(x) = Q(x)$.

CC338. Find (with proof) all integer solutions (x, y) to $x^2 - xy + 2017y = 0$.

CC339. A *rhombic dodecahedron* has twelve congruent rhombic faces ; each vertex has either four small angles or three large angles meeting there. If the edge length is 1, find the volume in the form $\frac{p + \sqrt{q}}{r}$, where p, q, and r are natural numbers and r has no factor in common with p or q.
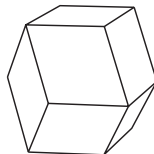
CC340. Let S be the set of natural numbers dividing 2018^{2018}. In how many ways can one select three numbers $\{x, y, z\}$ (not necessarily distinct, but order being irrelevant) from S so that $y = \sqrt{xz}$?

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2017 : 43(8), p. 328–329.

───────────◇◇◇◇◇◇───────────

CC286. The function $f(n) = an + b$, where a and b are integers, is such that for every integer n, $f(3n+1)$, $f(3n)+1$ and $3f(n)+1$ are three consecutive integers in some order. Determine all such $f(n)$.

Problem 1 of the 2008 Alberta High School Mathematics Competition, Part II.

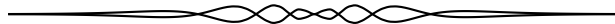*We received five correct solutions. We present the solution of Ivko Dimitrić.*

Since
$$f(3n + 1) = 3an + (a + b),$$
$$f(3n) + 1 = 3an + (b + 1),$$
$$3f(n) + 1 = 3an + (3b + 1)$$

are consecutive integers in some order, then, upon subtracting the common term $3an$ from each, it follows that $a+b$, $b+1$ and $3b+1$ are consecutive integers, whereas b cannot be 0 since these numbers are different. Then $|(3b+1)-(b+1)| = |2b| \geq 2$. We see that $3b + 1$ and $b + 1$ cannot be consecutive integers, so $a + b$ has to be between them and the following are the possibilities :

$$(1) \ \ b + 1 < a + b < 3b + 1 \ \text{ if } b > 0 \ \text{ or}$$
$$(2) \ \ 3b + 1 < a + b < b + 1 \ \text{ if } b < 0.$$

In either case these are three consecutive integers.

In case (1), we have $a + b = b + 2 = 3b$, giving $b = 1$, $a = 2$, so that the function is $f(n) = 2n+1$. In case (2), we have $a+b = 3b+2 = b$, with solution $a = 0$, $b = -1$, so the function is $f(n) = 0 \cdot n + (-1) = -1$. These two are the only such functions.

CC287. In a contest, no student solved all problems. Each problem was solved by exactly three students and each pair of problems was solved by exactly one student. What is the maximum number of problems in this contest ?

Problem 2 of the 2008 Alberta High School Mathematics Competition, Part II.

We received only one solution and it was incorrect. We present a solution created by this question's editor, Allen O'Hara.

Suppose there are n problems on the contest, q_1, q_2, …, q_n.

Each question is solved by 3 different students. Consider the first problem on the test, question q_1. We'll say it was solved by students s_1, s_2, s_3. What other questions might they have solved ? Since every pair of questions has been answered

by exactly one student, we see that the set q_2, q_3, $\cdots$, q_n is partitioned into 3 subsets, A_1, A_2, and A_3. At most one can be empty as then we'd have one student answer all the questions which is prohibited. We may assume that $\|A_1\| \geq \|A_2\| \geq \|A_3\|$, with possibly $\|A_3\| = 0$.

Suppose $\|A_1\| > 2$. Then without loss of generality we can say $q_2, q_3, q_4 \in A_1$ and $q_5 \in A_2$. But then the pairs (q_2, q_5), (q_3, q_5), (q_4, q_5) must be solved by some students. This gives rise to three more distinct students, as none of the q_2, q_3, q_4 can be solved by the same student, since student s_1 has already solved those pairwise groupings. This means that we have at least four students who have solved q_5, a contradiction.

So we have an upper bound, $2 \geq \|A_1\| \geq \|A_2\| \geq \|A_3\|$. Since these sets partition $\{q_2, q_3, \cdots, q_n\}$ this gives $6 \geq n - 1$ and so n is at most 7. Attempting to create a solution with $n = 7$ will give rise to the following possible situation, and so the final answer of 7 problems in the contest is realized.

Student s_1 solves questions q_1, q_2, and q_3.
Student s_2 solves questions q_1, q_4, and q_5.
Student s_3 solves questions q_1, q_6, and q_7.
Student s_4 solves questions q_2, q_4, and q_6.
Student s_5 solves questions q_2, q_5, and q_7.
Student s_6 solves questions q_3, q_4, and q_7.
Student s_7 solves questions q_3, q_5, and q_6.

Editor's comment. It is interesting to note the connection between this solution and Steiner triples.

CC288. The lengths of the sides of triangle ABC are consecutive positive integers. D is the midpoint of BC and AD is perpendicular to the bisector of angle C. Determine the product of the lengths of the three sides.

Problem 15 of the 2011 Alberta High School Mathematics Competition, Part I.

We received six correct solutions to this problem. We present the joint solution of Miguel Amengual Covas and Titu Zvonaru, modified by the editor.
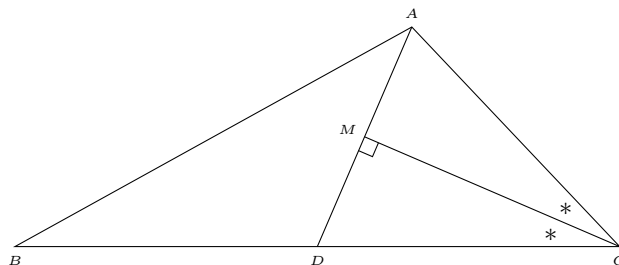
Let the internal bisector of angle C meets AD at M. Since $CM \perp AD$, triangles AMC and DMC are similar.

$\triangle AMC$ and $\triangle DMC$ are congruent since both have a common side $CA = CD$. Thus $CA = \frac{1}{2}BC$. As $CA < BC$, we examine the following three cases :

(i) If $CA = n$ and $BC = n + 1$ then $n = \frac{1}{2}(n + 1)$, which implies that $n = 1$. In this case, $\triangle ACB$ has side lengths 1, 2, 3. However, this case violates the triangle inequality as $AB < AC + BC \Rightarrow 3 < 1 + 2$ does not hold.

(ii) If $CA = n$ and $BC = n + 2$, then $n = \frac{1}{2}(n + 2)$, which implies that $n = 2$. In this case, $\triangle ACB$ has side lengths 2, 4, 3.

(iii) If $CA = n + 1$ and $BC = n + 2$, then $n + 1 = \frac{1}{2}(n + 2)$, which implies that $n = 0$.

Thus the solution to this problem is $4 \cdot 3 \cdot 2 = 24$.

CC289.
A positive integer is said to be *special* if it can be written as the sum of the square of an integer and a prime number. For example, 101 is special because $101 = 64 + 37$. Here 64 is the square of 8 and 37 is a prime number.

a) Show that there are infinitely many positive integers which are special.

b) Show that there are infinitely many positive integers which are not special.

Problem 3 of the 2012 Alberta High School Mathematics Competition, Part II.

We received six correct solutions. We present the solution of David Manes.

For each positive integer n, let p_n be the nth prime. Then

$$p_n + 1 = 1^2 + p_n = 1 + p_n.$$

Hence, $p_n + 1$ is special. Since there are infinitely many primes, the result in part a) follows.

For part b), we will show that for each positive integer n, the integer $(3n + 5)^2$ cannot be written as the sum of a square and a prime. Assume the contrary that

$$(3n + 5)^2 = m^2 + p$$

for some positive integer m and prime p. Then

$$p = (3n + 5)^2 - m^2 = (3n + 5 - m)(3n + 5 + m).$$

If $3n + 5 - m > 1$, then p is not a prime since it is written as the product of two integers greater than 1. Therefore, $3n + 5 - m = 1$ in which case $m = 3n + 4$ and

$$p = 3n + 5 + m = 6n + 9 = 3(2n + 3)$$

and again p is composite since $2n + 3 \geq 5$. Hence, the result in part b) follows.

CC290. Randy plots a point A. Then he starts drawing some rays starting at A, so that all the angles he gets are integral multiples of $10°$. What is the largest number of rays he can draw so that all the angles at A between the rays are unequal, including all angles between non-adjacent rays?

Problem 3 of the 2013 Alberta High School Mathematics Competition, Part II.

We received four correct solutions. We present the solution by Ivko Dimitrić.

If seven or more rays are drawn then there are at least $\binom{7}{2} = \frac{7 \cdot 6}{2} = 21$ different pairs of rays, determining 21 convex angles (angles between $10°$ and $180°$) between them. Their angle measures cannot be all different, since they are integral multiples of $10°$ and there are only 18 such multiples between $10°$ and $180°$ inclusive.

On the other hand, 6 rays can be drawn forming $\binom{6}{2} = 15$ different angles whose measures are integral multiples of $10°$.

One such arrangement is obtained when the rays are drawn so that the angle measures of successive angles formed by pairs of adjacent rays in cyclic order are $30°$, $10°$, $70°$, $50°$, $140°$, $60°$. In this arrangement all the angles between $10°$ and $180°$ except $20°$, $150°$, $180°$ are obtained exactly once by a pair of rays.
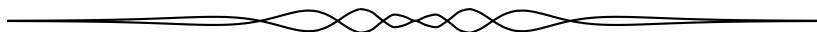
THE OLYMPIAD CORNER
No. 366
Anamaria Savu

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er mars 2019**.*

La rédaction souhaite remercier Valérie Lapointe, Carignan, QC, d'avoir traduit les problèmes.
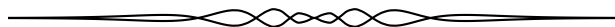
OC396. Montrer qu'il existe une infinité d'entiers positifs m tels que le nombre de facteurs premiers impairs de $m(m+3)$ est un multiple de 3.

OC397. Soit le triangle ABC tel que $\angle A = 45°$ et AM est une médiane. Soit le segment b symétrique au segment AM par rapport à la hauteur BB_1 et le segment c symétrique au segment AM par rapport à la hauteur CC_1. Les segments b et c s'interceptent au point X. Montrer que $AX = BC$.

OC398. Le détective Nero Wolfe enquête sur un crime. Il y a 80 personnes d'intérêt dans cette affaire. Parmi ces personnes, on retrouve le criminel et le témoin du crime (mais on ne sait pas qui est qui). Tous les jours, le détective peut interroger une ou plusieurs des ces 80 personnes. Si parmi les personnes invitées, le témoin est appelé et que le criminel ne l'est pas, le témoin pourra dire qui est le criminel. Est-ce que le détective peut résoudre ce crime en 12 jours ?

OC399. On dit qu'une fonction $f : \mathbb{R} \to \mathbb{R}$ a la propriété $\mathcal{P}$ si pour toute suite de nombres réels $(x_n)_{n \geq 1}$ telle que la suite $(f(x_n))_{n \geq 1}$ converge, alors la suite $(x_n)_{n \geq 1}$ converge. Montrer qu'une fonction surjective avec la propriété $\mathcal{P}$ est continue.

OC400. Soit G un groupe fini ayant la propriété suivante : pour tout automorphisme f de G, il existe un nombre naturel m tel que $f(x) = x^m$ pour tout $x \in G$. Montrer que G est abélien.

. . . .

OC396. Prove that there are infinitely many positive integers m such that the number of odd distinct prime factors of $m(m+3)$ is a multiple of 3.

OC397. In a triangle ABC with $\angle A = 45°$, draw the median AM. The line b is symmetrical to the line AM with respect to the altitude BB_1 and the line c is symmetrical to AM with respect to the altitude CC_1. The lines b and c intersect at the point X. Prove that $AX = BC$.

OC398. Detective Nero Wolfe is investigating a crime. There are 80 people involved in this case, among them one is the criminal and another is a witness of the crime (but it is not known who is who). Every day, the detective can invite one or more of these 80 people for an interview; if among the invited there is the witness, but there is no criminal, then the witness will tell who the criminal is. Can the detective solve the case in 12 days?

OC399. We say that a function $f : \mathbb{R} \to \mathbb{R}$ has the property $\mathcal{P}$ if for any sequence of real numbers $(x_n)_{n\geq 1}$ such that the sequence $(f(x_n))_{n\geq 1}$ converges, then also the sequence $(x_n)_{n\geq 1}$ converges. Prove that a surjective function with property $\mathcal{P}$ is continuous.

OC400. Let G be a finite group having the following property : for any automorphism f of G, there exists a natural number m such that $f(x) = x^m$ for all $x \in G$. Prove that G is abelian.

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2017 : 43(6), p. 246–247.

<div align="center">⎯⎯⎯◇◇◇◇◇◇◇⎯⎯⎯</div>
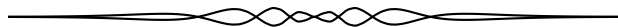
OC336. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $a, b \in \mathbb{N}$, we have

$$(f(a) + b)f(a + f(b)) = (a + f(b))^2.$$

Originally Problem 2 from the 2016 Iranian Mathematical Olympiad (3rd Round).

Michel Bataille notified us that OC336 appeared before in our journal as problem 4040 of 41(4) and solved in 42(4). OC336 received two solutions, both of which were correct, complete and different than previously published solutions. We present the solution by Steven Chow.

We assume that both domain and codomain of the function f are the set of all integers greater than or equal to 1. The equation that defines the function f is

$$(f(x) + y) f(x + f(y)) = (x + f(y))^2. \tag{1}$$

Take $x = a$ and $y = a$ in (1) to obtain $(f(a) + a) f(a + f(a)) = (a + f(a))^2$. Since $a + f(a) \geq 1 + 1$, it follows that for all integers $a \geq 1$

$$f(a + f(a)) = a + f(a). \tag{2}$$

Using mathematical induction, we shall prove that the statement

$$P(x) : \ f(xa) = xf(a) \text{ for all integers } a \geq 1$$

is true for all integers $x \geq 1$. It is obvious that $P(1)$ is true. Assume that for some integer $k \geq 1$, $P(k)$ holds, equivalently $f(ka) = kf(a)$ for all integers $a \geq 1$.

We evaluate (1) at $x = ka$ and $y = a + f(a)$:

$$(f(ka) + a + f(a)) f(ka + f(a + f(a))) = (ka + f(a + f(a)))^2. \tag{3}$$

Because of (2) and $f(ka) = kf(a)$, (3) writes as

$$(a + (k + 1)f(a)) f((k + 1)a + f(a)) = ((k + 1)a + f(a))^2. \tag{4}$$

We evaluate (1) at $x = (k + 1)a$ and $y = a$:

$$(f((k + 1)a) + a) f((k + 1)a + f(a)) = ((k + 1)a + f(a))^2 \tag{5}$$

The right sides of (4) and (5) are the same, implying that the left sides of (4) and (5) are equal; and taking into account that $f((k + 1)a + f(a)) \geq 1$, it follows that $(a + (k + 1)f(a)) = (f((k + 1)a) + a)$, or equivalently $(k + 1)f(a) = f((k + 1)a)$ for any integer $a \geq 1$. Hence $P(k + 1)$ is established; end of induction.

Therefore for all integers $x \geq 1$ and $a \geq 1$, $f(xa) = xf(a)$. Since $f(x) = f(1)x$ and $a + f(a) = f(a + f(a)) = f(1)(a + f(a))$ it follows that $f(1) = 1$. Hence for all integers $x \geq 1$, $f(x) = x$. Indeed, this function satisfies the condition (1).

In conclusion, the only solution of (1) is the identity function, $f(x) = x$ for all integers $x \geq 1$.

OC337. Find all polynomials $P(x)$ with integer coefficients such that

$$P(P(n) + n)$$

is a prime number for infinitely many integers n.

Originally Problem 3 from the 2016 Canadian Mathematical Olympiad.

We received three solutions, all of which were correct and complete. We present the solution by Jose Luis Diaz-Barrero.

Let $P(x)$ be a polynomial with integer coefficients of degree m. That is,

$$P(x) = \sum_{k=0}^{m} a_k x^k$$

with $a_k \in \mathbb{Z}$ for $0 \leq k \leq m$. Next we consider two cases.

First, if $P(x)$ is a constant polynomial equal to p, then $P(P(n) + n) = p$ for any integer n. Hence $P(P(n) + n)$ is a prime number for infinitely many integers n if and only if p is a prime number.

Second, assume that $P(x)$ is not a constant polynomial, hence its degree $\deg(P) \geq 1$. We prove that there exists a polynomial $Q(x)$ such that $P(P(x)+x) = P(x)Q(x)$ and $\deg(Q) = (\deg(P))^2 - \deg(P)$.

$$
\begin{aligned}
P(P(x) + x) - P(x) &= \sum_{k=0}^{m} a_k (P(x) + x)^k - \sum_{k=0}^{m} a_k x^k \\
&= \sum_{k=1}^{m} a_k \left((P(x) + x)^k - x^k \right) \\
&= \sum_{k=1}^{m} a_k P(x) \sum_{l=0}^{k-1} P(x)^l x^{k-1-l} \\
&= P(x) \sum_{k=1}^{m} \sum_{l=0}^{k-1} a_k P(x)^l x^{k-1-l} = P(x)Q(x),
\end{aligned}
$$

with $Q(x) = \sum_{k=1}^{m} \sum_{l=0}^{k-1} a_k P(x)^l x^{k-1-l}$. In addition

$$\deg(Q) = \deg(P(P(x) + x)) - \deg(P) = (\deg(P))^2 - \deg(P).$$

Since $P(P(n) + n) = P(n)Q(n)$ is prime for infinitely many integers n, it follows that either $P(n) = \pm 1$ for infinitely many integers n or $Q(n) = \pm 1$ for infinitely

many integers n. Since $P(x)$ is a polynomial of degree $m \geq 1$, $P(n) = \pm 1$ cannot hold for infinitely many n. So, we have for infinitely many integers n, $Q(n) = \pm 1$, equivalently $\deg(Q) = 0$. Therefore $(\deg(P))^2 - \deg(P) = 0$, $\deg(P) = 1$.

Let $P(x) = a_0 + a_1 x$ for some integers a_0 and a_1. Since

$$P(P(x) + x) = a_0 + a_1(a_0 + a_1 x + x) = P(x)(1 + a_1)$$

it follows that $Q(x) = 1 + a_1 = \pm 1$. The case $1 + a_1 = 1$ leads to $a_1 = 0$ which is impossible since we assumed $\deg(P) \geq 1$. The case $1 + a_1 = -1$ leads to $a_1 = -2$, $P(x) = a_0 - 2x$, and $P(P(x) + x) = a_0 - 2x$. Lastly $P(P(n) + n) = a_0 - 2n$ is prime for infinitely many integers n if and only if a_0 is odd.

Finally, we conclude that constant polynomial $P(x) = p$ with p a prime number or $P(x) = a_0 - 2x$ with a_0 an odd integer are the polynomials $P(x)$ that fulfill the conditions of the statement.

OC338. Let Γ be the excircle of triangle ABC opposite to the vertex A (namely, the circle tangent to BC and to the extensions of the sides AB and AC from the points B and C). Let D be the center of Γ and E, F, respectively, the points in which Γ touches the extensions of AB and AC. Let J be the intersection between the segments BD and EF. Prove that $\angle CJB$ is a right angle.

Originally Problem 3 from the 2016 Italian Mathematical Olympiad.

We received six solutions, all of which were correct and complete. Four solutions were geometrical, one solution used complex numbers, and one solution used barycentric coordinates. We present 2 solutions.

Solution 1, by Mohammed Aassila.

Denote
$$\angle A = \angle BAC, \quad \angle B = \angle CBA, \quad \text{and} \quad \angle C = \angle ACB.$$

Since the angles of a triangle add up to $180°$, we obtain for triangle BDC :

$$
\begin{aligned}
\angle BDC &= 180° - \angle DCB - \angle CBD \\
&= 180° - \frac{180° - \angle C}{2} - \frac{180° - \angle B}{2} \\
&= \frac{180° - \angle A}{2}.
\end{aligned}
$$

Since triangle AEF is isosceles, we obtain :
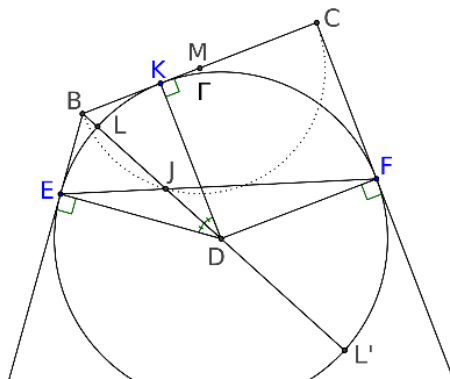
$$\angle JFC = \frac{180° - \angle A}{2}.$$

So the points J, C, F, and D lie on the same circle, hence

$$\angle CJD = 180° - \angle DFC = 90°,$$

and

$$\angle CJB = 180° - \angle CJD = 90°.$$

Solution 2, by Oliver Geupel.

Let K be the point of tangency of the line BC to Γ. Let LL' be the diameter of Γ with the property that the points B, L, and L' are collinear. Let M be the midpoint of BC. Put the picture onto a complex plane such that Γ is the unit circle, and identify each point with the corresponding complex number.

It is well-known that the intersection of tangents from tangency points X and Y to the unit circle is the point $2XY/(X+Y)$. Hence,

$$B = \frac{2EK}{E+K}, \qquad C = \frac{2FK}{F+K}, \qquad M = \frac{B+C}{2} = \frac{K(2EF+EK+FK)}{(E+K)(F+K)}.$$

Then

$$|B-M|^2 = (B-M)\overline{(B-M)} = \frac{K^2(E-F)}{(E+K)(F+K)} \cdot \frac{\frac{1}{K^2}\left(\frac{1}{E}-\frac{1}{F}\right)}{\left(\frac{1}{E}+\frac{1}{K}\right)\left(\frac{1}{F}+\frac{1}{K}\right)}$$
$$= -\frac{K^2(E-F)^2}{(E+K)^2(F+K)^2}.$$

Since $\angle KDL = \angle LDE$, we have $L^2 = EK$. It is a well-known and easily verifiable fact that the intersection of chords $\{X,Y\}$ and $\{Z,U\}$ of the unit circle is the point

$$(XY(Z+U)-ZU(X+Y))/(XY-ZU).$$

We obtain for the intersection J of chords $\{E,F\}$ and $\{L,-L\}$:

$$J = \frac{L^2(E+F)}{EF+L^2} = \frac{EK(E+F)}{EF+EK} = \frac{K(E+F)}{F+K},$$

and

$$|J-M|^2 = (J-M)\overline{(J-M)} = \frac{EK(E-F)}{(E+K)(F+K)} \cdot \frac{\frac{1}{EK}\left(\frac{1}{E}-\frac{1}{F}\right)}{\left(\frac{1}{E}+\frac{1}{K}\right)\left(\frac{1}{F}+\frac{1}{K}\right)}$$
$$= -\frac{K^2(E-F)^2}{(E+K)^2(F+K)^2} = |B-M|^2.$$

Thus, J lies on the circle with center M and radius BM. This circle has CB as diameter, therefore $\angle CJB = 90°$.

Editor's note. Titu Zvonaru notified us that OC338 originally appeared in "Revista Matematica din Timisoara" as problem O.VIII.208 in 2008.

OC339. Let n be any positive integer. Prove that

$$\sum_{i=1}^{n} \frac{1}{(i^2 + i)^{3/4}} > 2 - \frac{2}{\sqrt{n+1}}.$$

Originally Problem 3 from the 2016 Philippines Mathematical Olympiad.

We received 8 solutions, all of which were complete and correct. 7 solutions used AM-GM inequality, of which 4 solutions were completed via mathematical induction. One solution used the monotonicity of a function. We present a solution that was independently proposed by Mohammad Aassila, Steven Chow, and Oliver Geupel.

For all integers $j \geq 1$,

$$\frac{1}{\sqrt{j}} > \frac{1}{\sqrt{j+1}} > 0,$$

so from the AM - GM inequality,

$$\frac{1}{\sqrt{j}} + \frac{1}{\sqrt{j+1}} > \frac{2}{(j^2 + j)^{\frac{1}{4}}}.$$

In the above inequality, multiply both sides by $1/\left(j^2 + j\right)^{\frac{3}{4}}$ to obtain

$$\frac{1}{(j^2 + j)^{\frac{3}{4}}} \left(\frac{2}{\sqrt{j}} + \frac{2}{\sqrt{j+1}} \right) > \frac{4}{j^2 + j}$$

$$= \frac{4}{j} - \frac{4}{j+1}$$

$$= \left(\frac{2}{\sqrt{j}} + \frac{2}{\sqrt{j+1}} \right) \left(\frac{2}{\sqrt{j}} - \frac{2}{\sqrt{j+1}} \right),$$

so

$$\frac{1}{(j^2 + j)^{\frac{3}{4}}} > \frac{2}{\sqrt{j}} - \frac{2}{\sqrt{j+1}}.$$

Therefore

$$\sum_{i=1}^{n} \frac{1}{(i^2 + i)^{\frac{3}{4}}} > \sum_{i=1}^{n} \left(\frac{2}{\sqrt{i}} - \frac{2}{\sqrt{i+1}} \right)$$

$$= 2 - \frac{2}{\sqrt{n+1}}.$$

OC340. Let k be a fixed positive integer. Alberto and Beralto play the following game : Given an initial number N_0 and starting with Alberto, they alternately do the following operation : change the number n for a number m such that $m < n$ and m and n differ, in their base-2 representation, in exactly l consecutive digits for some l such that $1 \leq l \leq k$. If someone can't play, they lose. We say a non-negative integer t is a winning number when the player who receives the number t has a winning strategy, that is, they can choose the next numbers in order to guarantee their own victory, regardless of the options of the other player. Otherwise, we call t a losing number.

Prove that, for every positive integer N, the total of non-negative losing integers less than 2^N is

$$2^{N - \lfloor \frac{\log(\min\{N, k\})}{\log 2} \rfloor}.$$

Originally Problem 3 from the 2016 Brazil National Olympiad Day 1.

We received no solutions for this problem.
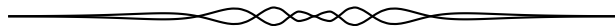
PROBLEM SOLVING 101
No. 6
Margo Kondratieva

This month we have a guest columnist, Margo Kondratieva. In 2017, Margo was the chair of the problems committee for the Canadian Open Mathematics Challenge (COMC) hosted by the CMS each November. You can check out the contest, and past contests on the CMS website at `cms.math.ca/Competitions/COMC`.

. . . .

Let us consider problem C3 from the 2017 COMC.

Problem. Let XYZ be an acute-angled triangle. Let s be the side-length of the square which has two adjacent vertices on side YZ, one vertex on side XY and one vertex on side XZ. Let h be the distance from X to the side YZ and b the distance from Y to Z.
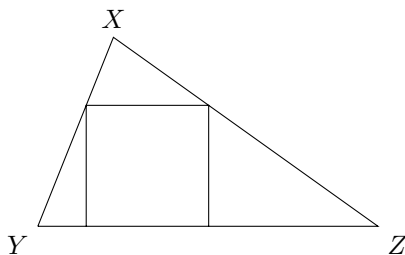
(a) If the vertices have coordinates $X = (2, 4)$, $Y = (0, 0)$ and $Z = (4, 0)$, find b, h and s.
(b) Given that $h = 3$ and $s = 2$, find b.
(c) If the area of the square is 2017, determine the minimum area of triangle XYZ.

Discussion.

As a rule, section C of the contest contains more advanced questions. However, items (a) and (b) are usually comparable to the easier questions from part A or B. Here in order to answer question (a), it could be very useful to draw a grid and place points X, Y, Z in accordance with given coordinates as shown in the Figure 1. Then from the figure it is clear that the distance b between Y and Z is 4, and the distance h from X to the side YZ is 4 as well. Now, we may observe that points $(1, 2)$ and $(3, 2)$ lie on sides XY and XZ respectively and together with points $(1, 0)$ and $(3, 0)$ they define a square that satisfies the conditions of the problem. This square has side $s = 2$.

The lesson learned here is : a precise picture in a geometric problem could lead to a quick visual solution.
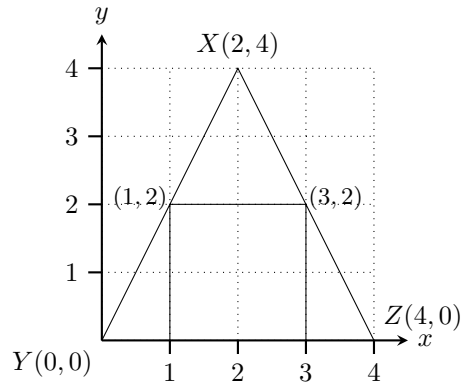
Figure 1 : Visual solution in (a).

For future reference, let us call the vertices of the square P, Q, R, S (see Figure 2). Alternatively to drawing a precise figure in order to find s in (a), we note that PQ is parallel to YZ, and thus triangle XPQ is similar to XYZ. From these similar triangles we have the relation $\dfrac{s}{b} = \dfrac{h-s}{h}$, that is, the ratio of the bases $\dfrac{PQ}{YZ} = \dfrac{s}{b}$ is the same as the ratio of corresponding altitudes $\dfrac{XN}{XM} = \dfrac{h-s}{h}$. Since $h = b = 4$ in (a), we obtain $s = 2$. The above relation also allows us to answer (b) right away : because $h = 3$ and $s = 2$ in (b), we conclude that $b = \dfrac{sh}{h-s} = 6$.

The lesson learned here is : the same formula derived for the general situation could be useful for answering several particular questions.
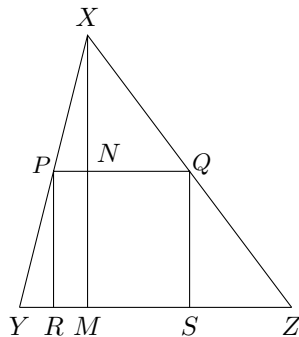
Figure 2 : Similar triangles XPQ and XYZ.

Now we can attempt the more challenging question (c). We should take advantage of what we already learned from either Figure 1 or 2. This might lead us to different approaches : algebraic and geometric. An algebraic approach may start with noting that the area of the triangle is

$$[XYZ] = \frac{bh}{2} = \frac{sh^2}{2(h-s)}.$$

Students who had already studied calculus may want to employ the derivative method in order to find the minimum of this expression. However, the knowledge of calculus is not necessary for solving this problem. For example, one possible approach is to look at the reciprocal of the area and to find the maximum of it instead. The advantage comes from the fact that the reciprocal is a quadratic function in the variable $\frac{1}{h}$, that is

$$\frac{2(h-s)}{h^2 s} = -2\left(\frac{1}{h}\right)^2 + \frac{2}{s}\left(\frac{1}{h}\right),$$

so the maximum is achieved at $\frac{1}{h} = \frac{1}{2} \cdot \frac{1}{s}$ or equivalently, for $h = 2s$. Then $b = 2s$ and the area $[XYZ] = 2s^2 = 4034$.

Another interesting approach is as follows. Let $A = [XYZ]$, the area of the triangle XYZ. We already found above that $A = \frac{bh}{2} = \frac{sh^2}{2(h-s)}$. Rewrite this formula as a quadratic equation in h, namely,

$$sh^2 - 2Ah + 2As = 0.$$

In order for h to be real the discriminant must be non-negative :

$$D = 4A^2 - 8As^2 = 4A(A - 2s^2) \geq 0.$$

This implies $A \leq 0$ or $A \geq 2s^2$. The former inequality is not possible. The latter shows that the minimum value of the area is $A = 2s^2 = 4034$.

The lesson learned here is : it is good to know more advanced techniques but it is even better if you can employ some more elementary smart ideas.

A pure geometric approach could be inspired by Figure 1, where we keep the grid, but remove the labels 1, 2, 3, 4, assuming that after an appropriate rescaling of Figure 1 the area of $PQSR$ is 2017.
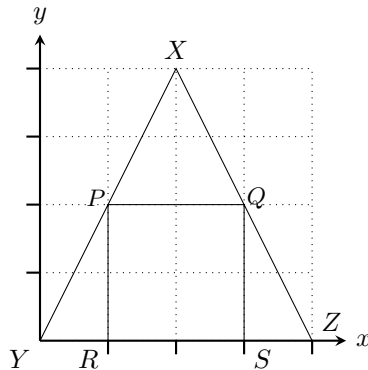

Figure 3 : The area of XYZ is twice the area of $PQSR$.

Now we simply count the squares of the grid : the square $PQSR$ consists of 4 squares of the grid, the area of XPQ is 2 squares of the grid and each of YPR and ZQS has the area of one square of the grid. We conclude that the area of triangle XYZ is exactly twice the area of the square $PQSR$, and thus it is 4034. We will demonstrate now that this is the smallest possible area for XYZ.

First we observe that if we keep the square $PQSR$ in the original place and move the vertex X strictly left (or right) in the position X_1 (Figure 4), the area of the resulting triangle $X_1Y_1Z_1$ is the same as the area of XYZ because their altitudes $XM = X_1M_1$ and the length YZ is the same as Y_1Z_1 (since $YY_1 = XX_1 = ZZ_1$ from congruent triangles $PYY_1 \cong PXX_1$ and $QZZ_1 \cong QXX_1$).

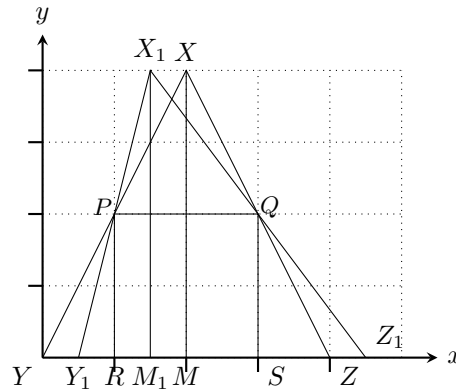Figure 4 : the area of $X_1Y_1Z_1$ is equal to the area of XYZ, which is twice the area of $PQSR$.
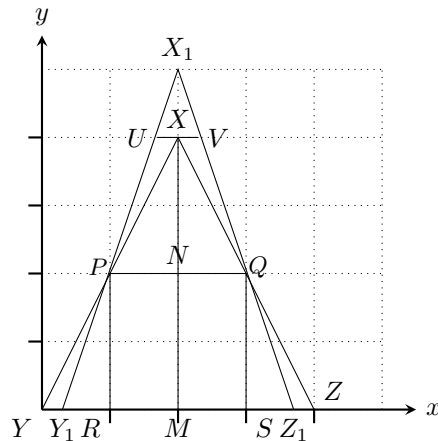

Figure 5 : the area of XYZ is less than the area of $X_1Y_1Z_1$.

Second, if we keep the square $PQSR$ in the original place and move the vertex X strictly up (or down) to the position of X_1, then vertices Y and Z will move to the positions of Y_1 and Z_1 respectively (Figure 5). However now the area of the

triangle $X_1Y_1Z_1$ will be bigger than the area of XYZ by the value of the area of UVX_1, because the following triangles are pairwise congruent : $QVX \cong QZ_1Z$ and $PUX \cong PY_1Y$.

Finally, if X moves in any direction, this could be viewed as a composition of two movements : strictly left or right, which does not change the area of XYZ, and then strictly up or down, which increases the area, thus the minimum area of XYZ is 4034.

The lesson we learned here is : a precise picture could provide an quick idea for answering some questions, but then we need to justify our hypothesis in full generality.

In conclusion, many problems, especially geometrical ones often have several different approaches to find the answer. We have demonstrated some of them and you are welcome to explore your own ! In addition, you may challenge yourself with related questions, for example : how to construct a square $PQSR$, given a particular triangle YXZ ?

Fun with clocks

The 12 dots on the circumference are equally spaced and the only point used inside the circle is its centre. What fraction of each circle is coloured ?
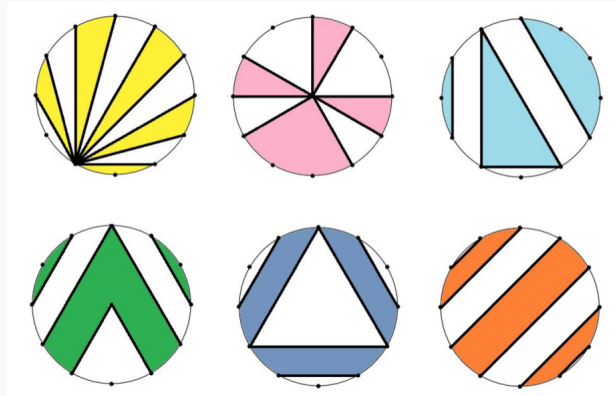
Puzzle by Catriona Shearer (https://twitter.com/Cshearer41).

Inequalities on a Right Trapezoid
Y. N. Aliyev

It often happens that a search for an extremum in a simple synthetic or analytic geometry problem leads to the discovery of some new and interesting inequalities (see e.g. [1, 2, 3]). In the current paper, we present some inequalities which were found while studying an analytic geometry problem. This analytic geometry problem appears at the end of the current paper as Problem 5; other problems using the same inequality are also given.

Proposition 1 *Let $ABCD$ be a trapezoid with right angles at the vertices A and B. Let M be an arbitrary point of side AB. Let k be a line parallel to the lateral side AB which intersects segments MC and MD at points E and F, respectively. If MZ is the altitude and H is the orthocenter of triangle MEF, then*

$$|HZ| \leq MZ \cdot \frac{|AB|^2}{4 \cdot |BC| \cdot |AD|}.$$

Proof. We take coordinates of points as $A(b,0)$, $B(a,0)$, $C(a,c)$, $D(b,d)$, and $M(x_0,0)$.
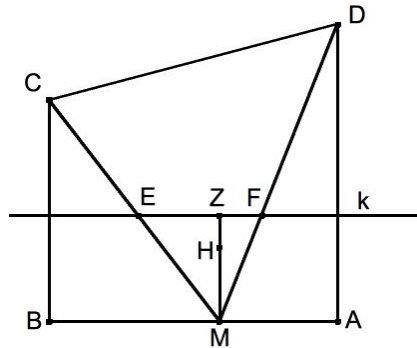
Let the equation of the line k be $y = n$, where $n \leq \min(c,d)$. First, we find the equation of line EH, which is perpendicular to MD :

$$\frac{y-n}{x-x_1} = \frac{x_0 - b}{d},$$

where

$$x_1 = [(c-n)x_0 + na]\,/c$$

is the abscissa of E. Since the abscissa of point H is x_0, to find coordinate y of H we must put this $x = x_0$ value in the equation of the line EH. We obtain

$$cd(y-n) = n(x_0 - a)(x_0 - b).$$

This quadratic function attains its least value when $x_0 = \frac{a+b}{2}$. Therefore,

$$y_{\min} = n - \frac{n(a-b)^2}{4cd}.$$

Hence,

$$|HZ| \le \frac{n(a-b)^2}{4cd} \cdot \frac{|AB|^2}{4 \cdot |BC| \cdot |AD|}.$$

The equality holds when M is the midpoint of AB. $\qquad\square$

Problem 1 *Let $ABCD$ be a trapezoid with right angles at the vertices A and B. Let k_1 and k_2 be lines through the intersection point O of diagonals AC and BD, which are respectively perpendicular and parallel to lateral side AB. Let M be a point on the side AB. Suppose that MC and MD intersect the line k_2 at points E and F, respectively. Then the perpendiculars to MC and MD at the points E and F intersect at point Y. Show that this is on the line k_1.*

Problem 2 *(As above) Denote the intersection of k_1 with AB as X. Show that*

$$|XY| \le \frac{|AB|^2 + 4 \cdot |BC| \cdot |AD|}{4\,(|BC| + |AD|)}.$$

Problem 3 *Let $ABCD$ be a trapezoid with right angles at the vertices A and B. Let M be an arbitrary point of the side AB. Let the altitude CT to the side AD intersect MD at F. If MZ is the altitude and H is the orthocenter of triangle MCF, prove that*

$$4 \cdot |AD| \cdot |HZ| \le |AB|^2.$$
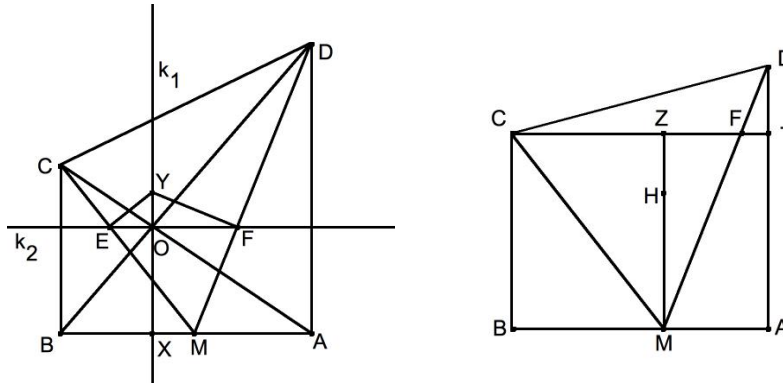
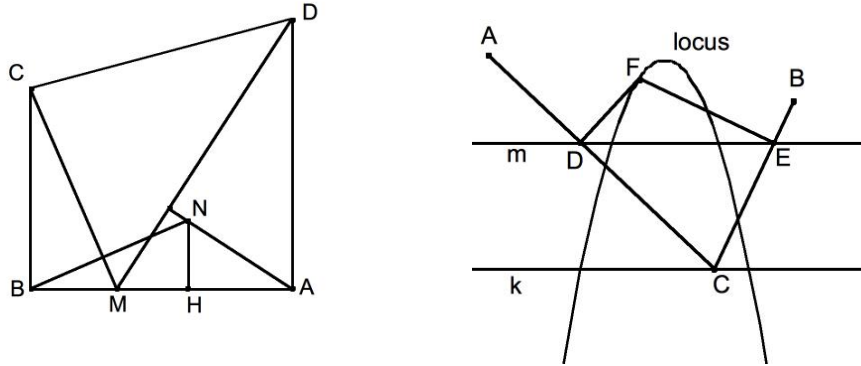Figures for Problem 2 (left) and Problem 3 (right).

Problem 4 *Let $ABCD$ be a trapezoid with right angles at the vertices A and B. Let M be an arbitrary point of side AB. Let the perpendiculars from points A and B to lines MD and MC, respectively, intersect at point N. Prove that if HN is the altitude of the triangle ABN, then*

$$\sqrt{|NH|}\left(\sqrt{|AD|} + \sqrt{|BC|}\right) \le |AB|.$$

Problem 5 *Let k and m be two parallel lines. Let A and B be two fixed points not on k or m. Let C be a generic point on k and let lines CA and CB intersect the line m at D and E, respectively. Find the locus of all points of intersection F of perpendiculars to CA and CB at the points D and E, respectively, as C moves on k. For which locations of lines k, m and points A, B does this locus degenerate to a ray?*

Figures for Problem 4 (left) and Problem 5 (right).

Acknowledgements

References

[1] Y. N. Aliyev, *Hexagons and Inequalities*, Crux Mathematocorum with Mathematical Mayhem 32 (7), 2006, pp. 456–461.

[2] Y. N. Aliyev, *Inequalities involving reciprocals of triangle areas*, Crux Mathematocorum with Mathematical Mayhem 36 (8), 2006, pp. 535–539.

[3] Y. N. Aliyev, *New inequalities on triangle areas*, Journal of Qafqaz University 25 (2009), pp. 129–135.

⋰⋅

The editor would like to thank Max Notarangelo, student at Quest University, for typesetting this article.
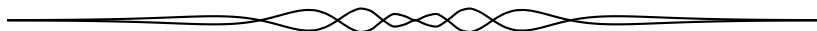
PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er mars 2019**.*

La rédaction souhaite remercier Valérie Lapointe, Carignan, QC, d'avoir traduit les problèmes.

4371. *Proposé par Oai Thanh Dao et Leonard Giugiuc.*

Soit deux segments perpendiculaires passant par l'orthocentre H d'un triangle ABC. Supposons que ces segments interceptent les côtés AB et AC à C_1, B_1 et C_2, B_2, respectivement. Soit M_i, le point milieu de $B_i C_i$ $(i = 1, 2)$ et M le point milieu de BC. Montrer que M, M_1 et M_2 sont collinéaires.

4372. *Proposé par J. Chris Fisher.*

Soit un quadrilatère $ABCD$ dont les angles B et D sont droits, le cercle (D, DA) (de centre D et de rayon DA) qui intercepte le segment AC au point P et le cercle (B, BA) au point Q. Montrer que PQ est perpendiculaire à AB.
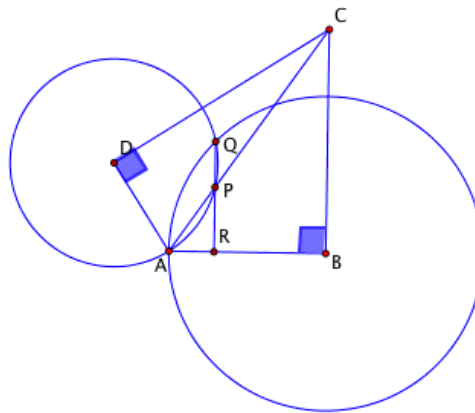
4373. *Proposé par Michel Bataille.*

Soit p un nombre impair premier. Soit q et r le quotient et le reste de la division entière du nombre positif n par p et soit $S_n = \sum_{k=r}^{n} (-1)^{k-r} \binom{k}{r}$. Montrer que $S_n \equiv 0$ (mod p) si et seulement si q est impair et $2^r \equiv 1$ (mod p).

4374. *Proposé par Šefket Arslanagić.*

Pour un entier positif donné n, résoudre l'équation

$$|1 - |2 - |3 - \cdots - |\ n - x\ \underbrace{|\cdots\cdots|}_{n \text{ barres verticales}} = 1.$$

4375. *Proposé par George Stoica.*

Soit les deux suites $\{x_n\}$ et $\{y_n\}$ telles que $\{x_n\} \cup \{y_n\} = \mathbb{N}$. Soit $l > 1$ donné. Montrer que $\liminf x_n/n \geq l$ si et seulement si $\limsup y_n/n \leq l/(l-1)$.

4376. *Proposé par Marius Drăgan et Neculai Stanciu.*

Soit A et B deux matrices de $M_n(\mathbb{C})$ telles que $AB = -BA$. Montrer que

$$\det(A^4 + A^2B^2 + 2A^2 + I_n) \geq 0.$$

4377. *Proposé par Tidor Vlad Pricopie et Leonard Giugiuc.*

Soit $x \geq y \geq z > 0$ tels que $x + y + z + xy + xz + yz = 1 + xyz$. Trouver $\min x$.

4378. *Proposé par Tarit Goswami.*

Trouver toutes les valeurs k telles que la limite suivante existe

$$\lim_{n \to \infty} \{k \cdot F_{n+1} - \sum_{i=0}^{n} \tau^i\},$$

où F_n est le $n^{\text{ème}}$ nombre de Fibonacci et τ est le nombre d'or.

4379. *Proposé par Kadir Altintas et Leonard Giugiuc.*

Soit le triangle ABC partageant ses sommets avec trois sommets d'un heptagone régulier. Supposons que B coincide avec le sommet 1, C avec le sommet 2 et A avec le sommet 4. Soit I, le centre du cercle inscrit et G le baricentre de ABC, respectivement. Supposons que BI intercepte AC au point D et que CI intercepte AB au point E. Montrer que les points D, G et E sont collinéaires.

4380. *Proposé par George Apostolopoulos.*

Soit a, b et c les côtés du triangle ABC de rayon de cercle inscrit r et de rayon de cercle circonscrit R. Montrer que

$$a^2 \tan \frac{A}{2} + b^2 \tan \frac{B}{2} + c^2 \tan \frac{C}{2} \leq \frac{3\sqrt{3}R^3(R-r)}{2r^2}.$$

. . . .

4371. *Proposed by Oai Thanh Dao and Leonard Giugiuc.*

Let two perpendicular lines pass through the orthocenter H of a triangle ABC. Suppose they meet the sides AB and AC in C_1, B_1 and C_2, B_2, respectively. Define M_i as the midpoint of B_iC_i $(i = 1, 2)$ and M as the midpoint of BC. Prove that M, M_1 and M_2 are collinear.

4372. *Proposed by J. Chris Fisher.*

Given a quadrangle $ABCD$ with right angles at B and D, let the circle (D, DA) (with center D and radius DA) intersect the line AC at P and the circle (B, BA) at Q. Prove that PQ is perpendicular to AB.
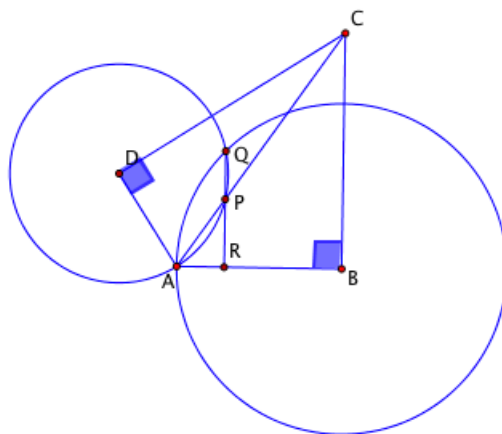
4373. *Proposed by Michel Bataille.*

Let p be an odd prime. Let q and r be the quotient and the remainder in the division of the positive integer n by p and let $S_n = \sum_{k=r}^{n} (-1)^{k-r} \binom{k}{r}$. Show that $S_n \equiv 0 \pmod{p}$ if and only if q is odd and $2^r \equiv 1 \pmod{p}$.

4374. *Proposed by Šefket Arslanagić.*

For a fixed positive integer n, solve the equation

$$|1 - |2 - |3 - \cdots - |\, n - x \underbrace{|\cdots\cdots|}_{n \text{ vertical bars}} = 1.$$

4375. *Proposed by George Stoica.*

Consider two sequences $\{x_n\}$ and $\{y_n\}$ such that $\{x_n\} \cup \{y_n\} = \mathbb{N}$. Let $l > 1$ be given. Prove that $\liminf x_n/n \geq l$ if and only if $\limsup y_n/n \leq l/(l-1)$.

4376. *Proposed by Marius Drăgan and Neculai Stanciu.*

Let A and B be two matrices in $M_n(\mathbb{C})$ such that $AB = -BA$. Prove that

$$\det(A^4 + A^2B^2 + 2A^2 + I_n) \geq 0.$$

4377. *Proposed by Tidor Vlad Pricopie and Leonard Giugiuc.*

Let $x \geq y \geq z > 0$ such that $x + y + z + xy + xz + yz = 1 + xyz$. Find $\min x$.

4378. *Proposed by Tarit Goswami.*

Find all k such that the following limit exists

$$\lim_{n \to \infty} \{k \cdot F_{n+1} - \sum_{i=0}^{n} \tau^i\},$$

where F_n is the nth Fibonacci number and τ is the golden ratio.

4379. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Let triangle ABC share its vertices with three vertices of a regular heptagon; in particular, let B coincide with vertex 1, C with vertex 2, and A with vertex 4. Let I be the incenter and let G be the centroid of ABC, respectively. Suppose BI intersects AC in D and CI intersects AB in E. Show that the points D, G and E are collinear.

4380. *Proposed by George Apostolopoulos.*

Let a, b and c be the side lengths of a triangle ABC with inradius r and circumradius R. Prove that

$$a^2 \tan \frac{A}{2} + b^2 \tan \frac{B}{2} + c^2 \tan \frac{C}{2} \leq \frac{3\sqrt{3}R^3(R - r)}{2r^2}.$$
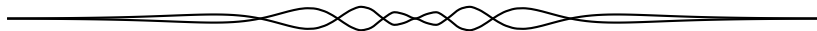
SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2017 : 43(8), pp. 356–360.

4271. *Proposed by Hung Nguyen Viet, supplemented by the Editorial Board.*

(a) Let a, b, c be nonzero real numbers such that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 1.$$

Prove that

$$\sqrt{\frac{(b+c)^2}{a^4} + \frac{(c+a)^2}{b^4} + \frac{(a+b)^2}{c^4}}$$

is a rational function of a, b, c.

(b) (*Suggested by the Editorial Board*). Prove or disprove that the equation

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 1$$

has no rational solution.

We received 9 submissions, all of which provide correct proofs to part (a). For part (b), we received one complete solution, but consider it to be outside the scope of **Crux**. *For a reference on this problem, see A. Bremner, A. Macleod,* An unusual cubic representation problem, *Annales Mathematicae et Informaticae, 43 (2014), pp. 29–41.*

Solution to part (a), by Leonard Giugiuc.

Note first that

$$(a+b+c)\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) = a+b+c \implies$$
$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} + a+b+c = a+b+c \implies$$
$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} = 0. \tag{1}$$

Let $x = \dfrac{a^2}{b+c}$, $y = \dfrac{b^2}{c+a}$, and $z = \dfrac{c^2}{a+b}$. Then $x + y + z = 0$ by (1), and so

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)^2 - 2\left(\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}\right)$$

$$= \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)^2 - 2\frac{x+y+z}{xyz}$$

$$= \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)^2.$$

Hence, $\sqrt{\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{z^2}} = \left|\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right|$. That is,

$$\sqrt{\frac{(b+c)^2}{a^4} + \frac{(c+a)^2}{b^4} + \frac{(a+b)^2}{c^4}} = \left|\frac{b+c}{a^2} + \frac{c+a}{b^2} + \frac{a+b}{c^2}\right|$$

and we are done.

4272. *Proposed by Vaćlav Konečný.*

Let A be a fixed point on the unit circle and let r be a real number such that $0 < r < \frac{1}{2}$. Let $\alpha = \angle BAC$ and $\alpha' = \angle B'AC'$, where ABC and $AB'C'$ are two isosceles triangles with apex A and inradius r that are inscribed in the unit circle. Find $\sin(\alpha/2) + \sin(\alpha'/2)$.

We received nine submissions, all correct. We present two of them.

Solution 1, by Andrew David Ionascu.
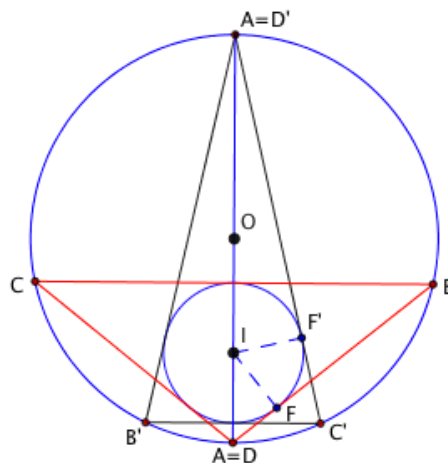
We can imagine that the point A is free to move about the unit circle whose center is O, while a circle of radius $r < \frac{1}{2}$ is fixed with its center I chosen to satisfy Euler's theorem, namely $OI^2 = R^2 - 2Rr = 1 - 2r$ (because we want the

circumradius R of the eventual triangles to equal 1 and I to be their common incenter). Because we want our triangles to be isosceles, the bisector AI of the apex angle at A will necessarily be the perpendicular bisector of the base, and it therefore must pass through O. If we denote the intersections of IO with the unit circle by D (nearest to I as in the figure) and D', A must be taken to coincide with D or D'. Let F denote the foot of the perpendicular from I to AC (and F the foot of the perpendicular from I to AC).

Case 1 : $A \equiv D$. We have $\frac{\alpha}{2} = \angle IAF$ and

$$\sin \frac{\alpha}{2} = \frac{IF}{AI} = \frac{r}{1 - OI} = \frac{r}{1 - \sqrt{1 - 2r}} = \frac{1}{2} + \frac{\sqrt{1 - 2r}}{2}. \tag{1}$$

Case 2 : $A \equiv D'$. We have $\frac{\alpha'}{2} = \angle IAF'$ and

$$\sin \frac{\alpha'}{2} = \frac{IF'}{AI} = \frac{r}{1 + OI} = \frac{r}{1 + \sqrt{1 - 2r}} = \frac{1}{2} - \frac{\sqrt{1 - 2r}}{2}. \tag{2}$$

Adding together (1) and (2), we obtain

$$\sin \frac{\alpha}{2} + \sin \frac{\alpha'}{2} = 1.$$

Solution 2, by Ivko Dimitrić.

We use the familiar formula

$$\cos A + \cos B + \cos C = \frac{r}{R} + 1, \tag{3}$$

expressing the sum of cosines of interior angles of a triangle in terms of the ratio of inradius to circumradius. In our case, A is the angle at the apex and B and C are angles at the base of an isosceles triangle, so that $B = C = 90° - \frac{A}{2}$, whence $\cos B = \cos C = \sin(A/2)$. Moreover, the double-angle formula for cosine gives $\cos A = \cos\left(2 \cdot \frac{A}{2}\right) = 1 - 2\sin^2(A/2)$. Substituting these into (3) we have

$$1 - 2\sin^2(A/2) + 2\sin(A/2) = \frac{r}{R} + 1,$$

which simplifies (after letting $R = 1$ without loss of generality) to

$$\sin^2(A/2) - \sin(A/2) + \frac{r}{2} = 0.$$

This is a quadratic equation in $\sin(A/2)$, which has two real solutions since its discriminant $1 - 2r$ is positive under the assumption $r < 1/2$. Therefore, the two solutions are $\sin(\alpha/2)$ and $\sin(\alpha'/2)$, and by Vieta's formula for the sum of the two roots we get $\sin(\alpha/2) + \sin(\alpha'/2) = 1$.

Editor's comment. Note that when $r = \frac{1}{2}$, $I = O$ (in the first solution), which implies that $\triangle ABC$ is equilateral and, consequently, $\sin(\alpha/2) = \sin(\alpha'/2)$ is a repeated root of the quadratic equation of the second solution.

4273. *Proposed by Ion Nedelcu and Leonard Giugiuc.*

Let n be an integer greater than or equal to 3. Prove that for any real numbers $a_i \geq 1, i = 1, 2, \ldots, n$ we have

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \leq n^2 + \sum_{1 \leq i < j \leq n} |a_i - a_j|.$$

We received six correct submissions. We present two different solutions.

Solution 1, by Nghia Doan.

Without loss of generality, we assume that $a_1 \geq a_2 \geq \cdots \geq a_n \geq 1$. Since

$$\sum_{1 \leq i < j \leq n} 1 = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2},$$

we have

$$n^2 + \sum_{1 \leq i < j \leq n} |a_i - a_j| - \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)$$

$$= n^2 - n + \sum_{1 \leq i < j \leq n} \left(a_i - a_j - \left(\frac{a_i}{a_j} + \frac{a_j}{a_i}\right)\right)$$

$$= \sum_{1 \leq i < j \leq n} \left(a_i - a_j - \left(\frac{a_i}{a_j} + \frac{a_j}{a_i} - 2\right)\right)$$

$$= \sum_{1 \leq i < j \leq n} \left(a_i - a_j - \frac{(a_i - a_j)^2}{a_i a_j}\right)$$

$$= \sum_{1 \leq i < j \leq n} (a_i - a_j)\left(1 - \frac{a_i - a_j}{a_i a_j}\right)$$

$$= \sum_{1 \leq i < j \leq n} (a_i - a_j)\left(\frac{a_i a_j - a_i + a_j}{a_i a_j}\right) \qquad (1)$$

For $1 \leq i < j \leq n$, we have $a_i - a_j \geq 0$, and

$$a_i a_j - a_i + a_j = a_i(a_j - 1) + a_j \geq a_j \geq 1,$$

so

$$(a_i - a_j)\left(\frac{a_i a_j - a_i + a_j}{a_i a_j}\right) \geq 0.$$

Hence, from (1) we obtain

$$n^2 + \sum_{1 \leq i < j \leq n} |a_i - a_j| - \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \geq 0$$

and the result follows.

Solution 2, by Oliver Geupel.

The well known Lagrange's identity states that for real numbers x_i and y_i, we have

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) = \left(\sum_{i=1}^{n} x_i y_i\right)^2 + \sum_{1 \le i < j \le n}(x_i y_j - x_j y_i)^2.$$

Setting $x_i = \sqrt{a_i}$ and $y_i = 1/\sqrt{a_i}$ for all i we then have

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) = n^2 + \sum_{1 \le i < j \le n}\left(\sqrt{\frac{a_i}{a_j}} - \sqrt{\frac{a_j}{a_i}}\right)^2 \quad (2)$$

For any fixed pair of indices i and j, let $a = \max\{a_i, a_j\}$ and $b = \min\{a_i, a_j\}$. By hypothesis, $b \ge 1$. Hence,

$$\left(\sqrt{\frac{a_i}{a_j}} - \sqrt{\frac{a_j}{a_i}}\right)^2 = \left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 \le \left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + \frac{(a-b)\,(a(b-1)+b)}{ab}$$

$$= \frac{(a-b)\,((a-b)+a(b-1)+b)}{ab} = \frac{(a-b)ab}{ab}$$

$$= |a_i - a_j|. \quad (3)$$

From (2) and (3), the required inequality follows.

4274. *Proposed by Michel Bataille.*

Let P be a point interior to a triangle ABC and L, M, N be points interior to the line segments PA, PB, PC, respectively. Define α, β, γ by $\overrightarrow{PL} = \alpha\overrightarrow{LA}$, $\overrightarrow{PM} = \beta\overrightarrow{MB}$, $\overrightarrow{PN} = \gamma\overrightarrow{NC}$. Assuming that at least one of α, β, γ is different from 1, construct with straightedge alone the center of mass of A, B, C with respective masses α, β, γ.

We received three correct submissions and feature the solution by Ivko Dimitrić.

There is no need to require that any of the masses be different from 1. The center of mass of point-masses $(A, \alpha), (B, \beta), (C, \gamma)$ — that is, of the vertices of triangle ABC loaded with the specified masses — is the common point of intersection of three lines, each joining the center of mass of any two of these point-masses with the third one. The center of mass (i.e. the balancing point) of points $(A, \alpha), (B, \beta)$ is the point C_1 of AB for which $\alpha|AC_1| = \beta|BC_1|$.

Suspend a unit mass at P and consider the center of mass of $(P, 1), (A, \alpha), (B, \beta)$. The point masses $(P, 1)$ and (A, α) imply a balancing point that divides the segment PA in the ratio $\alpha : 1$ (from P), and hence, coincides with the point L, since we have been given $\overrightarrow{PL} : \overrightarrow{LA} = \alpha : 1$. Then the center of mass of $(P, 1), (A, \alpha), (B, \beta)$ belongs to the segment BL. In the same manner, it also belongs to the segment AM, which means that it is the intersecting point C_2 of $\overline{AM}$ and $\overline{BL}$. Then the segment from the balancing point C_1 of $(A, \alpha), (B, \beta)$ to P must pass through the

same point, which means that the point C_1 can be determined (by straightedge alone) as the point where the line PC_2 meets AB.
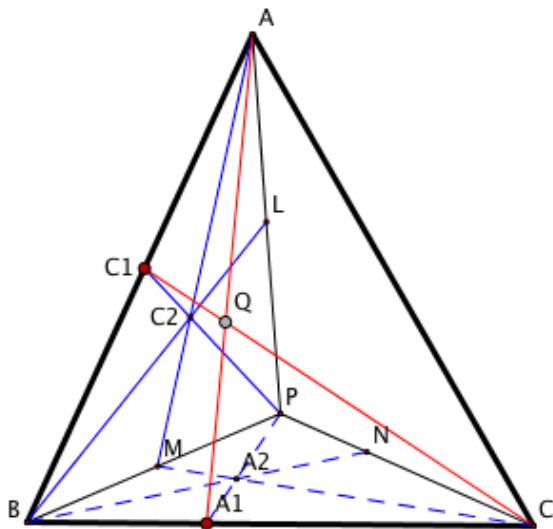
In the same manner one determines the balancing point A_1 of (B, β) and (C, γ) as the intersection of BC with the line PA_2 where $A_2 = \overline{BN} \cap \overline{CM}$ and the balancing point B_1 of (A, α) and (C, γ) as the intersection of AC with the line PB_2 where $B_2 = \overline{AN} \cap \overline{CL}$. Then the center of mass (denoted by Q in the accompanying figure) of all three point-masses $(A, \alpha), (B, \beta), (C, \gamma)$ is the point of intersection of any two (and, therefore, of all three) of the lines AA_1, BB_1 and CC_1, which can be clearly constructed by using straightedge alone.

4275. *Proposed by Leonard Giugiuc.*

Let a, b and c be nonnegative real numbers such that $a + b + c = 3$. Find the best possible k for which the following inequality holds :

$$\left(\frac{ab + bc + ca}{3}\right)^k (ab + bc + ca - abc) \le 2.$$

We received five correct solutions and will feature the solution by Paolo Perfetti.

We suppose $k > 0$. Indeed if $k = 0$ the inequality is

$$ab + bc + ca - abc \le 2$$

Now let us take $c = 0$, $a = b = \dfrac{3}{2}$ and then it does not hold.

Let k be negative. The inequality is

$$\frac{3^{|k|}}{(ab + bc + ca)^{|k|}}(ab + bc + ca - abc) \le 2$$

Now by taking $c = 0$, $a = 2$, $b = 1$ we get

$$\frac{3^{|k|}}{2^{|k|-1}} \leq 2 \iff 3^{|k|} \leq 2^{|k|}$$

which is clearly false.

Now consider $k > 0$. The inequality is linear decreasing in abc and then it suffices to show that it holds when abc assumes its minimum value. The standard theory (see S. Chow, H. Halim and V. Rong, *The pqr Method : Part I*, **Crux**, 43 (5), 2017, pp.210–215) states that the minimum value of abc occurs if at least one among $\{a, b, c\}$ is zero or two variables out of a, b, c, are equal.

Case 1 : $c = 0$. The inequality becomes

$$\left(\frac{ab}{3}\right)^k (ab) \leq 2, \underset{b=3-a}{\iff} \frac{a^{k+1}}{3^k}(3 - a)^{k+1} \leq 2$$

The AGM yields

$$\frac{a^{k+1}}{3^k}(3 - a)^{k+1} \leq \frac{1}{3^k}\left[\frac{(a + 3 - a)^2}{4}\right]^{k+1}$$

so we come to the sufficient condition

$$\frac{1}{3^k}\left[\frac{(a + 3 - a)^2}{4}\right]^{k+1} \leq 2$$

if and only if

$$k \geq \frac{2 \ln 3 - 3 \ln 2}{2 \ln 2 - \ln 3} \sim 0.409$$

and the equality case occurs when $a = 3/2$.

Case 2 : $a = b$. Then $c = 3 - 2a$. Moreover since the inequality is symmetric, we can set $a = b \leq c$ and this implies $0 \leq a \leq 1$. The inequality becomes

$$\left(\frac{a^2 + 2a(3 - 2a)}{3}\right)^k (a^2 + 2a(3 - 2a) - a^2(3 - 2a)) \leq 2$$

or, equivalently,

$$a(a^2 - 3a + 3)a^k(2 - a)^k \leq 1.$$

The derivative of the function $a(a^2 - 3a + 3)$ is $3(a - 1)^2$, so it increases and then $a(a^2 - 3a + 3) \leq 1$ for $0 \leq a \leq 1$. By means of the AGM we come to

$$a(a^2 - 3a + 3)a^k(2 - a)^k \leq 1 \cdot \left(\left(\frac{a + 2 - a}{2}\right)^2\right)^k = 1$$

and this completes the proof.

4276. *Proposed by Daniel Sitaru.*

Let P be a point on the interior of a triangle ABC and let $PA = x, PB = y$ and $PC = z$. Prove that

$$27(ax + by - cz)(by + cz - ax)(cz + ax - by) \le (ax + by + cz)^3.$$

We received 8 correct solutions. We present the solution by Digby Smith.

Let $p = ax$, $q = by$, and $r = cz$. Substituting, expanding, then applying Schur's inequality before applying the AM-GM inequality gives

$$
\begin{aligned}
&(ax + by - cz)(by + cz - ax)(cz + ax - by) \\
&= (p + q - r)(q + r - p)(r + p - q) \\
&= pq(p + q) + qr(q + r) + rp(r + p) - p^3 - q^3 - r^3 - 2pqr \\
&\le pqr \\
&\le \left(\frac{p + q + r}{3}\right)^3,
\end{aligned}
$$

making

$$27(ax + by - cz)(by + cz - ax)(cz + ax - by) \le (ax + by + cz)^3,$$

with equality if and only if $ax = by = cz$.

4277. *Proposed by Tran Quang Hung and Nguyen Le Phuoc.*

Let ABC be a triangle whose incircle (I) touches BC, CA and AB at D, E and F, respectively. Suppose M and N lie on EF such that BM and CN are perpendicular to BC. Finally, suppose DM and DN intersect (I) again at P and Q, respectively, and that BQ cuts CP at R. Prove that DR bisects MN.
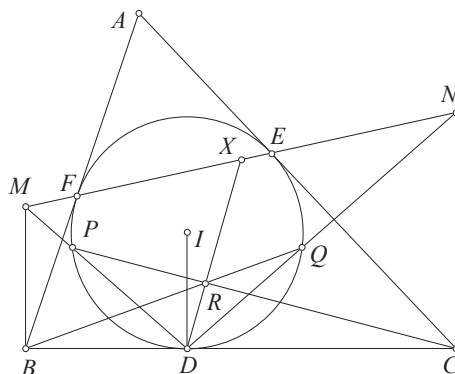
All six submissions that we received were correct; we feature the solution by AN-Anduud Problem Solving Group.

Denote by X the point where the lines (DR) and (MN) meet. We shall show that X is the midpoint of MN. Using standard notation, namely $AB = c$, $BC = a$, $CA = b$, $s = \frac{a+b+c}{2}$, we know that

$$AE = AF = s - a, \ \ BD = BF = s - b, \ \text{and} \ CD = CE = s - c.$$

Let T be the intersection of the lines (MN) and (BC). Applying Menelaus's theorem to the transversal MN (which cuts $\triangle ABC$ at T, F, and E), we have

$$\frac{BT}{TC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{BT}{TC} \cdot \frac{s-c}{s-a} \cdot \frac{s-a}{s-b} = 1,$$

and so,

$$\frac{BT}{TC} = \frac{s-b}{s-c} = \frac{BD}{CD}.$$

Because $\triangle TBM \sim \triangle TCN$ by the (AA) similarity theorem, we have $\frac{BT}{TC} = \frac{MB}{NC}$, which thus implies

$$\frac{MB}{NC} = \frac{BD}{CD}.$$

It follows by the (SAS) similarity theorem that $\triangle MBD \sim \triangle NCD$, and therefore

$$\angle MDB = \angle NDC = \angle QDC.$$

Thus

$$\angle QPD = \angle QDC = \angle MDB = \angle PQD,$$

which implies

$$PD = DQ \quad \text{and} \quad PQ \parallel BC. \tag{1}$$

Let $PQ \cap DX = K$; because $PQ \parallel BC$ (from (1)) we get

$$\frac{PK}{KQ} = \frac{CD}{BD}. \tag{2}$$

Using square brackets to denote areas, we know that

$$\frac{[MDX]}{[PDK]} = \frac{MD \cdot DX}{PD \cdot DK} \ \text{and} \ \frac{[XDN]}{[KDQ]} = \frac{DN \cdot DX}{DQ \cdot DK}.$$

Because $DQ = PD$ (from (1)) we therefore have

$$\frac{[MDX]}{[XDN]} \cdot \frac{[KDQ]}{[PDK]} = \frac{MD}{DN}. \tag{3}$$

We were given

$$\frac{[MDX]}{[XDN]} = \frac{MX}{XN} \ \text{and} \ \frac{[KDQ]}{[PDK]} = \frac{KQ}{PK},$$

which, together with $\frac{KQ}{PK} = \frac{BD}{CD}$ (from (2)), turns (3) into $\frac{MX}{XN} \cdot \frac{BD}{CD} = \frac{MD}{DN}$; consequently,

$$\frac{MX}{XN} = \frac{MD}{DN} \cdot \frac{CD}{BD}. \tag{4}$$

On the other hand, because $\triangle MBD \sim \triangle NCD$,

$$\frac{MD}{DN} = \frac{BD}{CD}. \tag{5}$$

From (4) and (5), we have $\frac{MX}{XN} = 1$, whence X is the midpoint of MN, as desired.

4278. *Proposed by Lorian Saceanu.*

For any positive real numbers x, y and z, show that

$$\sqrt{\frac{y+z}{x}} + \sqrt{\frac{x+y}{z}} + \sqrt{\frac{z+x}{y}}$$

$$= \sqrt{\frac{(y+z)(z+x)(x+y)}{xyz}} + \frac{2(x+y+z)}{\sqrt{x(y+z)} + \sqrt{y(x+z)} + \sqrt{z(x+y)}}.$$

There were 14 correct solutions submitted. We present three variants here.

Solution 1, by Digby Smith, Paolo Perfetti, and Daniel Dan (independently).

Let $a^2 = (y+z)/x$, $b^2 = (z+x)/y$ and $c^2 = (x+y)/z$. We have to verify the equation

$$a + b + c = abc + \frac{a^2 x + b^2 y + c^2 z}{ax + by + cz}.$$

Multiply the terms by the positive quantity $ax + by + cz$ and take the difference of the two sides; the result is

$$a^2 x + b^2 y + c^2 z + (ax + by + cz)(abc - a - b - c)$$
$$= a^2 x + b^2 y + c^2 z + bca^2 x + cab^2 y + abc^2 z - a^2 x - aby$$
$$\quad - caz - abx - b^2 y - bcz - cax - bcy - c^2 z$$
$$= bc(a^2 x - y - z) + ca(b^2 y - z - x) + ab(c^2 z - x - y)$$
$$= 0.$$

Thus, the equation holds.

Solution 2, by Kee-Wai Lau.

Observe that

$$\sqrt{x(y+z)} + \sqrt{y(z+x)} + \sqrt{z(x+y)}$$
$$= \frac{(\sqrt{x(y+z)} + \sqrt{y(z+x)})^2 - (\sqrt{z(x+y)})^2}{\sqrt{x(y+z)} + \sqrt{y(z+x)} - \sqrt{z(x+y)}}$$
$$= \frac{2(\sqrt{xy(y+z)(z+x)} + xy)(\sqrt{xy(y+z)(z+x)} - xy)}{(\sqrt{x(y+z)} + \sqrt{y(z+x)} - \sqrt{z(x+y)})(\sqrt{xy(y+z)(z+x)} - xy)}$$
$$= \frac{2xyz(x+y+z)}{yz(\sqrt{x(y+z)} + xy\sqrt{z(x+y)} + xz\sqrt{y(z+x)} - \sqrt{xyz(y+z)(z+x)(x+y)}}.$$

Therefore

$$\sqrt{\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}}-\sqrt{\frac{(y+z)(z+x)(x+y)}{xyz}}$$

$$=\frac{yz\sqrt{x(y+z)}+xy\sqrt{z(x+y)}+zx\sqrt{y(z+x)}-\sqrt{xyz(y+z)(z+x)(x+y)}}{xyz}$$

$$=\frac{2(x+y+z)}{\sqrt{x(y+z)}+\sqrt{y(z+x)}+\sqrt{z(x+y)}},$$

as desired.

Solution 3, by Leonard Giugiuc.

Let $u = 1/x$, $v = 1/y$ and $w = 1/z$. We have to show that

$$u\sqrt{v+w}+v\sqrt{w+u}+w\sqrt{u+v}$$

$$=\sqrt{(u+v)(v+w)(w+u)}+\frac{2(uv+vw+wu)}{\sqrt{u+v}+\sqrt{v+w}+\sqrt{w+u}}.$$

There exists an acute triangle with sides $a = \sqrt{v+w}$, $b = \sqrt{w+u}$, $c = \sqrt{u+v}$. Denote by s, R, r, $[ABC]$, respectively, the semi-perimeter, circumradius, inradius and area of the triangle. We have that

$$u = \frac{1}{2}(b^2 + c^2 - a^2) = bc\cos A, \qquad v = ca\cos B, \qquad w = ab\cos C,$$

and

$$[ABC]^2 = (1/16)(a+b+c)(a+b-c)(b+c-a)(c+a-b)$$
$$= (1/16)[2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)]$$
$$= (1/4)(uv + vw + wu).$$

Therefore

$$u\sqrt{v+w}+v\sqrt{w+u}+w\sqrt{u+v}-\sqrt{(u+v)(v+w)(w+u)}$$
$$= abc(\cos A + \cos B + \cos C - 1)$$
$$= 4abc\sin(A/2)\sin(B/2)\sin(C/2)$$
$$= 4\left(\frac{abc}{4R}\right)\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$
$$= 4[ABC]r = \frac{8[ABC]rs}{2s} = \frac{8[ABC]^2}{2s}$$
$$= \frac{2(uv + vw + wu)}{\sqrt{u+v}+\sqrt{v+w}+\sqrt{w+u}}.$$

Editor's comments. The introduction of the variables a, b, c as in Solution 1 was popular. The matrix of coefficients for the system of linear equations

$$a^2 x = y + z, \ b^2 y = z + x, \ c^2 z = x + y$$

has characteristic polynomial

$$t^3 - (a^2 + b^2 + c^2)t^2 + (a^2b^2 + b^2c^2 + c^2a^2 - 3)t + (a^2 + b^2 + c^2 + 2 - a^2b^2c^2).$$

Since there was a nontrivial solution (up to a constant factor) given by

$$x : y : z = (b^2 + 1)(c^2 + 1) : (c^2 + 1)(a^2 + 1) : (a^2 + 1)(b^2 + 1),$$

the variables a, b, c were subject to the constraint $a^2b^2c^2 = 2 + a^2 + b^2 + c^2$ or equivalently $(a^2 + 1)^{-1} + (b^2 + 1)^{-1} + (c^2 + 1)^{-1} = 1$.

Some solvers made a complete conversion of the equation to one involving a, b, c, such as

$$\frac{2}{a + b + c - abc} = \frac{a}{a^2 + 1} + \frac{b}{b^2 + 1} + \frac{c}{c^2 + 1},$$

and arrived home after a straightforward technical grind. Two solvers formulated the equation in terms of the symmetric functions $s = a + b + c$, $q = ab + bc + ca$, $p = abc$ and had to verify

$$s - p = \frac{2((1 - q)^2 + (s - p)^2)}{s + qs - 3p + pq},$$

subject to $p^2 = s^2 - 2q + 2$. Solution 1, which involved only a partial conversion of variables, was the most efficient way of handling the situation.

Solution 2 is typical of that provided by five solvers who did not introduce any new variables. This leaves three solutions with more novel approaches. Solution 3 is one of them. In addition, the proposer considered the triangle ABC with sides $(y+z, z+x, x+y)$, semiperimeter s, circumradius R and inradius r. Then it must be shown that

$$2\sqrt{\frac{R}{r}}\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) = 2\sqrt{\frac{R}{r}} + \frac{2s}{\sqrt{a(s-a)} + \sqrt{b(s-b)} + \sqrt{c(s-c)}}.$$

Finally, Bataille imposed the condition $x+y+z = 1$ and set $(x, y, z) = (\cos^2\alpha, \cos^2\beta, \cos^2\gamma)$. Then it was a matter of verifying that

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma + \frac{4}{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}.$$

4279. *Proposed by Leonard Giugiuc.*

Let ABC be an acute angled triangle. Show that

$$\tan A + \tan B + \tan C \geq 4\left(\frac{1}{\sqrt{3} + \cot A} + \frac{1}{\sqrt{3} + \cot B} + \frac{1}{\sqrt{3} + \cot C}\right).$$

We received 13 submissions, all of which are correct. We present a composite of essentially the same solutions by Daniel Dan, Kevin Sotopalacious, and the proposer.

It is well known that $\tan A + \tan B + \tan C \geq 3\sqrt{3}$, which can be easily proved by applying Jensen's Inequality to the convex function $f(x) = \tan x$ on $[0, \pi/2)$. Hence,

$$\sum_{cyc} \tan A \geq \sum_{cyc} \frac{3\sqrt{3} + \tan A}{4}.$$

By AM-HM inequality, we have

$$\frac{3\sqrt{3} + \tan A}{4} = \frac{\sqrt{3} + \sqrt{3} + \sqrt{3} + \tan A}{4} \geq \frac{4}{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\tan A}} = \frac{4}{\sqrt{3} + \cot A},$$

so

$$\sum_{cyc} \frac{3\sqrt{3} + \tan A}{4} \geq 4 \sum_{cyc} \frac{1}{\sqrt{3} + \cot A}.$$

4280. *Proposed by Mihaela Berindeanu.*

Solve the following equation in integers : $x^3 + y^3 - 6xy = 2^z - 8$.

We received 5 correct solutions and 4 incomplete solutions. We present the solution by Andrew David Ionascu.

The set of solutions $[x, y, z]$ is going to be denoted by $\mathcal{S}$. We use the identity

$$a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac).$$

Using this identity, the equation

$$x^3 + y^3 - 6xy = 2^z - 8 \tag{1}$$

becomes

$$x^3 + y^3 + 2^3 - 6xy = (x + y + 2) \cdot (x^2 + y^2 + 2^2 - xy - 2y - 2x) = 2^z.$$

This implies that

$$x + y + 2 = 2^a$$

and

$$x^2 + y^2 + 2^2 - xy - 2y - 2x = 2^b, \tag{2}$$

where $a + b = z$, with a and b integers.

Multiplying equation (2) by 2, that equation can be rewritten as

$$(x - y)^2 + (x - 2)^2 + (y - 2)^2 = 2^{b+1}. \tag{3}$$

A power of 2 can be written as a sum of 3 squares in a unique way, depending on the exponent. For b even,

$$2^{b+1} = 2^b + 2^b + 0^2,$$

and for b odd,
$$2^{b+1} = 2^{b+1} + 0^2 + 0^2.$$

[*Editor's note* : We prove as follows that the decomposition of 2^k into three squares must have at least one zero term. This is clear for 2^0 and 2^1 ; assuming it for 2^k, suppose that $2^{k+2} = u^2 + v^2 + w^2$, with u, v, w all positive. Then 4 divides the left-hand side, so u, v, w must all be even, since if two of them are odd, the right-hand side would be congruent to 2 mod 4. Writing $u = 2x$, $v = 2y$, $w = 2z$ gives $2^k = x^2 + y^2 + z^2$, with x, y, z all positive, a contradiction. An analogous proof shows that for k even, the decomposition contains a single non-zero term and is thus unique. For k odd, supposing that $2^{k+2} = u^2 + v^2$, with $u > v$ likewise leads to a contradiction, again ensuring uniqueness.]

We have two cases.

Case 1 : (b even). Since b is even, $b = 2w$, and
$$2^{b+1} = (2^w)^2 + (2^w)^2 + 0^2 = (x - y)^2 + (x - 2)^2 + (y - 2)^2,$$

so that one of the numbers $x - y$, $x - 2$, $y - 2$ is 0.

(a) Assume first that $x - y = 0$. Then $x = y$, and $x - 2 = y - 2 = \pm(2^w)$. Thus
$$\begin{aligned} 2^a &= x + y + 2 = 2 \pm 2^w + 2 \pm 2^w + 2 \iff \\ 2^a &= 6 \pm 2(2^w) = 2(3 \pm 2^w) \iff \\ 2^{a-1} &= 3 \pm 2^w \iff \\ 2^{a-1} &\mp 2^w = 3. \end{aligned}$$

This implies that $a - 1 = 0$ or $w = 0$. If $a - 1 = 0$, then $w = 1$, $x = y = 0$, and $z = 3$. We have then the first solution $[0, 0, 3] \in \mathcal{S}$. If $w = 0$, then a must be 2 or 3, giving two more solutions : $[1, 1, 2] \in \mathcal{S}$, $[3, 3, 3] \in \mathcal{S}$.

(b) Assume that $x - 2 = 0$. The equations $y = 2 \pm 2^2$ and $x + y + 2 = 2^a$ become
$$2^a = 2 + 2 + \pm 2^w + 2 \iff 2^a = 6 \pm 2^w 2(3 \pm 2^{w-1})$$

and
$$2^{a-1} = 3 \pm 2^{w-1} \iff 2^{a-1} \mp 2^{w-1} = 3.$$

This implies that $a - 1 = 0$ or $w - 1 = 0$. If $a - 1 = 0$, then $w = 2$, $y = -2$, and $z = 5$. We now have our fourth solution, $[2, -2, 5] \in \mathcal{S}$. If $w - 1 = 0$, then a must be 2 or 3. This gives two more solutions : $[2, 0, 4] \in \mathcal{S}$, $[2, 4, 5] \in \mathcal{S}$.

(c) Now assume that $y - 2 = 0$. This situation is the same as assuming $x - 2 = 0$, except that the roles of x and y are interchanged. Therefore, we have three more solutions : $\{[-2, 2, 5], [0, 2, 4], [4, 2, 5]\} \subset \mathcal{S}$.
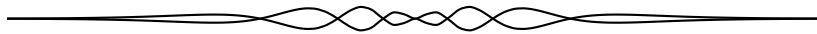
Case 2 : (b odd). Since b is odd, $b = 2w + 1$, so that
$$2^{b+1} = (2^{w+1})^2 + 0^2 + 0^2 = (x - y)^2 + (x - 2)^2 + (y - 2)^2.$$

In this case, there are no solutions since if two of these numbers $x - y$, $x - 2$, $y - 2$ are 0, then the third is also. This means $2^{b+1} = 0$, a contradiction.
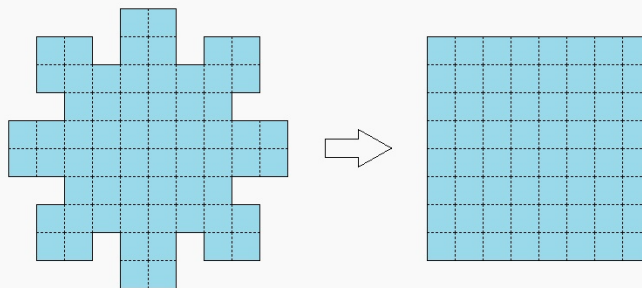
In conclusion, the solution set is

$$\mathcal{S} = \{[0, 0, 3], [1, 1, 2], [3, 3, 3], [2, -2, 5], [2, 0, 4], [2, 4, 5], [-2, 2, 5], [0, 2, 4], [4, 2, 5]\}.$$
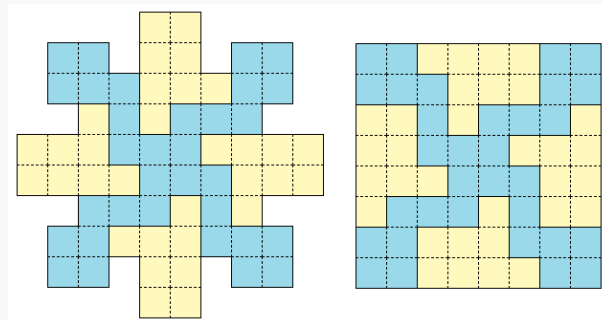
Quadrature of the figure : solution

Problem. Take a grid paper and cut out the figure shown below on the left. Can you cut it into 5 pieces and arrange them to form an 8×8 square ?

Solution.

Puzzle by Nikolai Avilov.

AUTHORS' INDEX

(Bold font indicates featured solution.)

Proposers

Kadir Altintas, Turkey, and Leonard Giugiuc, Romania : 4367, 4379
George Apostolopoulos, Athens, Greece : 4380
Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina : 4374
Michel Bataille, Rouen, France : 4363, 4373
Mihaela Berindeanu, Bucharest, Romania : 4369
Oai Thanh Dao, Vietnam, and Leonard Giugiuc, Romania : 4362, 4371
Ovidiu Furdui and Alina Sîntămărian, Cluj-Napoca, Romania : 4368
Tarit Goswami, West Bengal, India : 4378
Leonard Giugiuc, Romania, and Sladjan Stankovik, Macedonia : 4370
Marius Drăgan and Neculai Stanciu, Romania : 4365, 4376
J. Chris Fisher, Cleveland Heights, Ohio, USA : 4372
Tidor Vlad Pricopie and Leonard Giugiuc, Romania : 4377
Daniel Sitaru, Romania : 4366
George Stoica, Saint John, NB : 4364, 4375
Andrew Wu, St. Albans School, McLean, Virginia, USA : 4361

Solvers - individuals

Mohammed Aassila, Strasbourg, France : 4268, 4277, **OC333**, OC336, OC337, **OC338**, **OC339**
Arkady Alt, San Jose, CA, USA : 4264, 4275, 4278, 4279
Miguel Amengual Covas, Cala Figuera, Mallorca, Spain : CC282, **CC283**, CC285, **CC288**, OC338
Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina : 4264, 4267, 4269, 4271, 4276, CC282, CC284, CC285, OC339
Fernando Ballesta Yagüe, Murcia, Spain : **CC281**, CC282, **CC283**
Roy Barbara, Lebanese University, Fanar, Lebanon : 4264, 4270, 4271, 4278, 4279, 4280
Michel Bataille, Rouen, France : 4263, 4265, **4266**, 4267, 4268, 4269, **4270**, 4274, 4278
Mihaela Berindeanu, Bucharest, Romania : 4268
Paul Bracken, University of Texas, Edinberg, TX, USA : 4262, 4270, OC339
Charles Burnette, Taipei, Taiwan : CC282
Steven Chow, Scarborough, ON : CC281, CC282, **CC283**, CC284, **OC331**, OC333, **OC334**, **OC335**, **OC336**, OC337, OC338, **OC339**
Dan Daniel, National College Gheorghe Titeica, Romania : 4265, 4269, 4272, **4278**, **4279**, CC282, CC284, **CC285**, CC286, CC288, CC289, CC290
Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India : 4262, 4269
Ivko Dimitrić, Pennsylvania State University Fayette, PA, USA : **4262**, **4268**, 4269, **4272**, **4274**, CC281, CC282, CC283, CC284, CC285, **CC286**, CC288, CC289, **CC290**, OC339
José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain : **OC337**
Nghia Doan, Ottawa, ON : 4264, 4268, 4269, **4273**, 4275, 4276, 4278, 4279, 4280
Andrea Fanchini, Cantù, Italy : 4266, 4268, 4272, 4277, CC285, OC338
Oliver Geupel, Brühl, Germany : 4262, 4265, 4269, **4273**, 4278, 4279, **OC338**, **OC339**

Leonard Giugiuc, Drobeta Turnu Severin, Romania : 4262, 4263, 4265, 4267, 4268, 4269, 4270, 4271, 4272, 4276 (2 solutions), **4278**, **4279**

Richard Hess, Palos Verdes, CA, USA : 4270, CC281, CC282, CC283, CC284, CC285, CC288, CC289, CC290

Nguyen Viet Hung, Vietnam : 4269, 4271

Andrew David Ionascu, Columbus, Georgia, USA : **4263**, 4269, **4272**, **4280**

Václav Konečný, Big Rapids, MI, USA : 4268, 4272, 4274

Kee-Wai Lau, Hong Kong, China : 4264, **4267**, 4269, 4270, 4272, 4277, **4278**, 4279

Kathleen Lewis, Brikama, Republic of the Gambia : CC281, CC282, CC283, CC284

David E. Manes, SUNY at Oneonta, NY, USA : CC281, CC284, CC286, **CC289**, 4269

Madhav R. Modak, Pune, India : 4268, 4271, 4277

Julio Cesar Mohnsam and Rikelme Griep, Pelotas, Brazil : CC282

Allen O'Hara, University of Western Ontario, ON : CC287

Kevin Sato Palacios, Huarmey, Peru : **4279**

Ricard Peiró i Estruch, Valècia, Spain : CC282, CC285

Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy : **4264**, 4269, 4270, **4275**, **4278**, 4279, CC285

Angel Plaza, University of Las Palmas de Gran Canaria, Spain : CC282, **4267**, OC339

C.R. Pranesachar, Bangalore, India : 4264, 4265, 4268, 4271, 4278

Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam : 4279

Henry Ricardo, Tappan, NY, USA : CC282

Barbara Roy, Lebanese University, Fanar, Lebanon : 4272

Lorian Saceanu, Harstad, Norway : 4278

Volkhard Schindler, Berlin, Germany : 4271

Ashton Short, San Angelo, TX, USA : CC282

Daniel Sitaru, Drobeta Turnu-Severin, Mehedinti, Romania : 4265, 4276

Digby Smith, Mount Royal University, Calgary, AB : 4262, 4264, 4265, 4267, 4269, 4270, 4273, 4275, **4276** (2 solutions), **4278**, 4279, CC281, CC282, CC283, OC339

Muralidharan Somasundaram, Chennai, India : 4268

Marius Stănean, Zalău, Romania : 4266

Anna Valkova Tomova, Varna, Bulgaria : 4268, 4271

C Watts, San Angelo, Texas, USA : CC282

Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA : OC331, CC282, CC283, CC285, CC286, CC288, CC289

Titu Zvonaru, Cománeşti, Romania : **4268** (two solutions), 4272, 4278, 4279, CC281, **CC284**, CC285, CC286, **CC288**, CC289, OC338

Solvers - collaborations

AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia : 4264, 4265, 4267, 4269, 4270, 4271, 4273, **4277**, 4279, 4280

Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, USA : 4265, 4273, 4276, 4278

Leonard Giugiuc and Drobeta Turna Severin, Romaina : 4275

Ino Nedelcu and Leonard Giugiuc, Romania : 4273

Dorin Marghidanu and Leonard Giugiuc, Romania : 4264

Tran Quang Hung and Nguyen Le Phuoc, Hanoi, Vietnam : 4277

Missouri State University Problem Solving Group, Springfield, MO : 4264, 4270, 4280, CC290