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For Vol. 4 (1978), the support of Algonquin College, the Samuel Beatty Fund, and Carleton University is gratefully acknowledged.

COMPUTER ORIENTED SOLUTIONS

ALAN G. HENNEY and DAGMAR R. HENNEY

A solution of the celebrated Four Color Problem has recently been advanced by Kenneth Appel and Wolfgang Haken, of the University of Illinois, through extensive use of the computer. (See [2] for a discussion of the problem and its solution.) The conjecture had baffled scientists for over 120 years.

One hopes that their approach marks a breakthrough in scientific thought. There are many conjectures which, at the present time, can be proved or disproved only by lengthy computations. Their solution may be hastened by the success of Haken and Appel, which may lead to new methods for humans to interact with computers.

Solutions of the famous "15-14 Puzzle" have appeared recently [1] and the following "Canadian I.Q. Problem" has also been partly analyzed with the assistance of the computer.

A Monte Carlo Approach to Sequential Machine Puzzles or the Canadian I.Q. Problem.

Many popular puzzles that involve a rearrangement of pegs or blocks can be considered models of what are called "sequential machines". Each movement of a piece is an input and each arrangement or state of the pieces is an output. Reference [1] discusses a systematic approach to finding solutions to one type of sequential machine. However, a simpler and often effective approach is the Monte Carlo method (method of statistical trials). Figure 1 shows a popular puzzle which can be solved by the Monte Carlo technique.

Fourteen pegs are arranged as shown on a triangular board with fifteen holes. The objective of the puzzle is to remove one peg at a time by jumping in a straight line. The jumped peg is removed. The process is continued, if possible, until only one peg remains. One popular version uses the starting configuration shown. Fourteen other starting configurations are possible by simply relocating the position of the hole.

```
X X X   X
 X X X X
  X X X
   X X
    X
```

Figure 1

Such puzzles are usually solved manually by trial and error. During the process a person hopes to gain experience indicating the type of spatial arrangement of the pegs that will lead to a successful solution. The game ends when one or more pegs remain on the board in such an arrangement that no more jumps are possible.

After a score of attempts one begins to realize how difficult it is to find a sequence of inputs that will permit the game to continue until thirteen pegs have been removed. Insight gained through experience does not lead one quickly to a solution. Although experience does help one to reduce the number of pegs remaining at the end of the game, one can readily see the value of using an electronic computer to either solve the problem or at least to gather statistical data which can be used in reaching a solution.

The Monte Carlo approach is one way to attack the problem. Monte Carlo is simply a fast automatic method for employing the trial and error process. With an electronic computer the game can be played thousands of times in a few minutes. Although the computer cannot employ insight to eliminate obviously poor moves, statistical information can be easily tabulated and printed to help future programming efforts. Examples of such information will be given later.

The computer program to solve the puzzle of Figure 1 was written in FORTRAN. The technique used is interesting because it can be applied to many types of similar puzzles. It consists of three major parts:

1. input and output routines;
2. a routine that indicates the possible jumps for a peg positioned in each of the fifteen different holes;
3. a method for selecting all possible jumps at each stage of the game and a random number generator for selecting one move randomly from these possible jumps.

The input-output routines enter into the computer such information as the location of the hole prior to the first move, the number of games to be played, and a seed number for the random number generator. An electronic computer generates a set of random numbers relative to a given initial number. Unless the initial seed number is changed the random number generator will produce exactly the same set of random numbers each time it is used. This feature is useful while a computer program is being debugged, but during use each series of games requires a different set of random numbers and hence a different seed number.

The output information can include many parameters. Some obviously important ones are the number of trials, if any, leading to a solution and the sequence of moves resulting in a solution. Also the percentage of games ending with exactly 2, 3,... pegs remaining is interesting. Table 1 is an example of this type of data.

The second part of the computer program is the most tedious to write. To illustrate how it is constructed, identify each position in Figure 1 by a number as shown in Figure 2. For each number identify the possible paths that a peg positioned in that hole could take. For example, arrows indicate that a peg in position 3 can

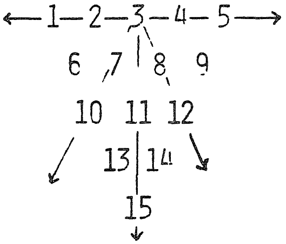

Figure 2

follow only one of 5 different paths. For each possible path (there are 37 in all) instructions are written indicating how the arrangement of the board is changed when that path is followed. It is assumed in each case that the peg-hole arrangement is such that a move can be completed.

The third part of the computer program examines each position in turn and lists the jumps that are possible at this stage of the game. For example, consider Figure 1 at the start of a game with the hole at position 4. The computer program examines each of the 15 positions in turn. Only two of the possible 37 paths can be taken: a peg can jump from position 2 to 4 or from position 11 to 4. Now a random number is generated (a fraction between 0 and 1). If the fraction is greater than ½ the second path is followed, otherwise the first path is chosen. The arrangement is then changed accordingly and the process is repeated, starting with the selection of all possible jumps for the second move. The game ends when no more jumps are possible. If only one peg remains, the game has ended successfully and the sequence of moves is printed. Otherwise the computer program restores the initial arrangement on the peg board and starts another game.

The arrangement shown in Figure 1 was run on a computer three times with the number of games per run increased each time. Table 1 summarizes the results.

Table 1

No. of games	Games ending with indicated no. of pegs remaining (% of total)						
	1	2	3	4	5	6	≥7
20	0	0	20.0	65.0	5.0	5.0	5.0
200	0	2.5	28.5	46.0	10.5	8.5	4.0
2000	0.2	3.4	25.6	44.3	13.7	8.5	4.3

The first computer run limited the number of games to 20 mainly to check the computer program to make certain it was running properly. The number of trials was then increased by a factor of 10. No solutions were found but several games did end with only two pegs remaining. The number of trials was increased again by a factor of 10. The results are shown on the third line of the table. This time four different solutions were obtained. The sequence of jumps for one of these solutions, using the notation of Figure 2, is 2-4, 12-3, 4-2, 10-3, 13-8, 2-4, 5-12, 14-9, 1-10, 4-11, 10-12, 9-14, 15-12.

Table 1 shows that for each computer run a game was most likely to end with 4 pegs remaining. Many games ended with 3 pegs on the board. It was more difficult to play a game leaving only two pegs, and it was very difficult to find a sequence of moves solving the puzzle.

By changing the initial location of the hole one can, using the same computer program, attempt to solve fourteen additional puzzles. The symmetric nature of the peg board allows a reduction in the number of puzzles that must be solved to determine a solution for each of the fourteen remaining puzzles. For example, a puzzle with the initial hole in position 4 is symmetric to that with the initial hole in position 2. Looking at Figure 2 and the solution given above, one can quickly write down a solution to the puzzle with initial hole in position 2. It is 4-2, 10-3, 2-4, 12-3, 14-7, 4-2, 1-10, 13-6, 5-12, 2-11, 12-10, 6-13, 15-10.

Note by observing Figure 2 that each move in the new sequence is symmetrical to that given in the previous sequence. Therefore, to show that each of the fourteen additional puzzles has at least one solution it is only necessary to repeat the computer run for configurations with the initial hole in positions 1, 3, 6, 7, 10, 11, 13, and 15 of Figure 2. Table 2 shows the result of those runs playing 2000 games for each initial configuration.

Table 2

Initial hole position	Games ending with indicated no. of pegs remaining (% of total)						
	1	2	3	4	5	6	≥7
1	0.60	6.25	24.95	39.75	12.00	11.15	5.30
3	0.35	7.80	31.15	34.65	12.55	5.85	7.65
6	0.35	3.65	29.40	43.55	15.55	2.65	4.85
7	0.05	4.25	14.40	46.05	20.75	10.95	3.55
10	0.80	6.90	28.05	43.05	10.90	4.80	5.50
11	0.15	4.10	19.75	43.55	15.65	4.60	12.20
13	0.25	4.10	25.25	46.70	13.15	7.40	3.15
15	0.30	6.30	31.30	36.20	14.25	2.50	9.15

Table 2 shows that each of the remaining fourteen puzzles has at least one solution. The percent of games ending with 1, 2, 3,... pegs left on the board has the same general distribution as that in Table 1. For each configuration less than

1% of the 2000 games played ended in solutions. Solutions using the Monte Carlo technique were found to all fifteen variations of the puzzle.

Although the Monte Carlo technique was used to find solutions to each variation of the puzzle, one must proceed differently to find all possible solutions to any given variation of the puzzle. A systematic approach similar to that discussed in [1] could be used. The computer program would be more difficult to construct than that involving the Monte Carlo method. All combinations of moves that could lead to a solution must be considered. Techniques must be devised to quickly eliminate any combination that cannot result in a solution. Such techniques are necessary to reduce the required computer time.

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1. Alan G. Henney and Dagmar R. Henney, Systematic Solutions of the Famous 15-14 Puzzles, *Pi Mu Epsilon Journal*, Vol. 6, No. 4 (Spring 1976) 197-201.

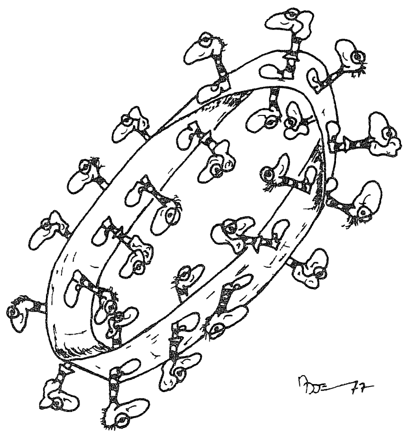
2. Lynn Arthur Steen, Solution of the Four Color Problem, *Mathematics Magazine*, 49 (September 1976) 219-222.

Department of Mathematics, The George Washington University, Washington, D.C. 20052.

* * *

A riddle

What goes and goes and never reaches the door?



Answer: Footies on a Möbius strip.

ANDREJS DUNKELS,
University of Luleå, Sweden.

ALGORITHMS AND POCKET CALCULATORS: SQUARE ROOTS: III AND THE MINIMAX METHOD

CLAYTON W. DODGE

Many functions are accurately approximated by polynomials; generally, the higher the degree of the polynomial, the more accurate is the approximation. As an example, suppose we wish to write a linear approximation to the function f of Figure 1 defined on the interval $[0,1]$. A line through the endpoints of the graph, such as $y = g(x)$ in Figure 2, would be one such approximation. Here $g(0) = f(0)$ and $g(1) = f(1)$, and there is a point $x = c$ in $[0,1]$ such that the error $\varepsilon(x) = f(x) - g(x)$ has maximum magnitude. We search for that line $y = h(x)$ such that $|\varepsilon(x)|$ has as small a maximum value as possible; we wish to minimize the maximum error on the stated interval. Hence the process is called the *minimax method*.

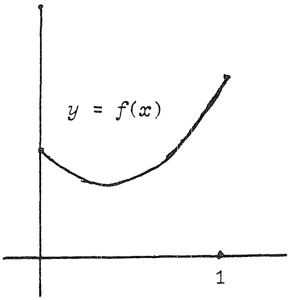


Figure 1

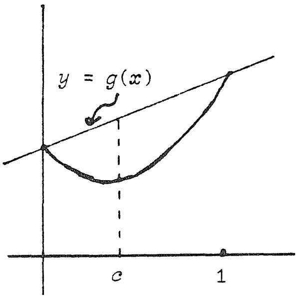


Figure 2

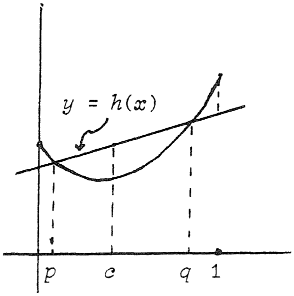


Figure 3

In this case, by lowering the line $y = g(x)$ of Figure 2, we decrease the error $|\varepsilon(c)|$ and increase $|\varepsilon(0)|$ and $|\varepsilon(1)|$. When these three quantities are equal, as shown by the line $y = h(x)$ of Figure 3, we have attained our goal; the error $\varepsilon(x) = f(x) - h(x)$ has the smallest possible maximum value, and this maximum error is attained at least three times: at both endpoints and at one intermediate point. We note that between these points are two values p and q such that the error is zero: $f(p) = h(p)$ and $f(q) = h(q)$.

Rather than working with the pair of simultaneous equations

$$f(p) = h(p) \quad \text{and} \quad f(q) = h(q)$$

and trying to determine what linear polynomial $h(x)$ minimizes the maximum error, we generally encounter less work to let $\varepsilon = \varepsilon(0)$ be that maximum error (or its negative) and solve the trio of equations

$$f(0) = h(0) + \varepsilon, \qquad f(c) = h(c) - \varepsilon, \qquad f(1) = h(1) + \varepsilon.$$

This is the technique of the minimax method. Observe that the error term changes sign from one x-value to the next, as one generally expects would occur.

More generally, an approximating polynomial $h(x)$ of degree n would attain its maximum error $n+2$ times. The big question with the minimax method is how to determine the intermediate c-values at which the maximum error is realized; we need n such values for a degree n approximation. Although algebraic methods may be available, the usual technique, except when n is quite small, is trial. Guess at the locations of $c_1, c_2, \ldots, c_n$, and solve the resulting $n+2$ (linear) equations for the $n+2$ constants, which are ε and the $n+1$ coefficients of the polynomial $h(x)$. Then calculate the actual error at sufficiently many x-values to locate all points of relative maximum error (both positive and negative). Now recalculate the coefficients using these new x-values for the c-values. Usually only two or three repetitions are required to obtain a very good (almost) minimax approximation.

As a specific example, according to Fike [1], in the IBM Fortran library appears a square root algorithm based on Newton's method. To find $\sqrt{x}$ where $x = m \cdot 16^k$ is a base 16 numeral with $1/16 \le m \le 1$, assume a first approximation

$$r_1 = 16^{k/2}p(m) \qquad\qquad \text{if } k \text{ is even}$$

or

$$r_1 = \frac{1}{4} \cdot 16^{(k+1)/2}p(m) \qquad \text{if } k \text{ is odd},$$

where $p(m) = (2+8m)/9$ is the minimax linear approximation to $\sqrt{m}$ on the interval $[1/16, 1]$ whose maximum relative error has been minimized. This algorithm has the advantage that, to obtain double precision square roots, one need only perform the divide-and-average routine (see [1978: 154]) just one extra time.

The reader may enjoy verifying that $p(m) = (2+8m)/9$ is indeed the linear minimax approximation claimed. The maximum relative error

$$ER(m) = \frac{\sqrt{m} - p(m)}{\sqrt{m}}$$

occurs at the endpoints 1/16 and 1 and also at the interior point 1/4.

Pocket calculators do not utilize base 16, but rather *binary-coded decimal* (BCD), in which each decimal digit is separately written in binary notation. Thus, in BCD, we have

$$239 = 0010,0011,1001_{BCD},$$

whereas in base 2,

$$239 = 11101111_2.$$

It is much easier to program a calculator to work in BCD, even though more bits are required, because conversion to decimal for display is very easy and absolutely accurate. Hence, if we are to consider the calculation of square roots on a calculator, we need work only in base ten.

Consider applying Newton's method to calculator square roots. For $\sqrt{x}$, take $x = m \cdot 10^k$ with k even and $1 \le m < 100$. Then our first approximation would have the form

$$r_1 = 10^{k/2} p(m),$$

where $p(m)$ is an approximation to $\sqrt{m}$ on the interval [1,100]. Suppose we take $p(m)$ linear and minimize the maximum relative error for best operation of Newton's method; say

$$p(m) = am + b.$$

If ε denotes the maximum relative error (or perhaps its negative), then we have

$$\sqrt{1} = 1 = a + b + \varepsilon, \tag{1}$$

$$\sqrt{c} = ac + b - \sqrt{c}\varepsilon, \tag{2}$$

$$\sqrt{100} = 10 = 100a + b + 10\varepsilon \tag{3}$$

to solve simultaneously for a, b, and ε. The general equation for the relative error $ER(m)$ is

$$ER(m) = \frac{(am + b) - \sqrt{m}}{\sqrt{m}} = \frac{am + b}{\sqrt{m}} - 1.$$

In order that the intermediate value $m = c$ be a relative maximum of $|ER(m)|$, we must have the derivative of $ER(m)$ equal to zero. Thus

$$ER'(m) = \frac{am - b}{2m\sqrt{m}},$$

so $ER'(m) = 0$ at $m = b/a$; that is, $c = b/a$. From (1) and (3) we next get

$$10\varepsilon = 10 - 10a - 10b = 10 - 100a - b,$$

so $b = 10a$ and $c = 10a/a = 10$. Now (2) yields

$$\sqrt{10} = 10a + 10a - \sqrt{10}(1 - a - 10a),$$

so

$$a = \frac{2\sqrt{10}}{20 + 11\sqrt{10}} = \frac{2}{2\sqrt{10} + 11} \approx 0.115443,$$

$$b = 10a = \frac{20}{2\sqrt{10} + 11} \approx 1.15443,$$

and

$$\varepsilon = 1 - \frac{22}{2\sqrt{10} + 11} \approx -0.2699.$$

Taking $p(m) = 0.115443(m + 10)$, we arrive at the following values and graph for the relative error (see Figure 4):

m	$p(m)$	$\sqrt{m}$	$ER(m)$
1	1.2699	1.0000	+.270
2	1.3853	1.4142	-.020
5	1.7316	2.2361	-.226
10	2.3089	3.1623	-.270
20	3.4633	4.4721	-.226
50	6.9266	7.0711	-.020
90	11.5443	9.4868	+.217
100	12.6987	10.0000	+.270

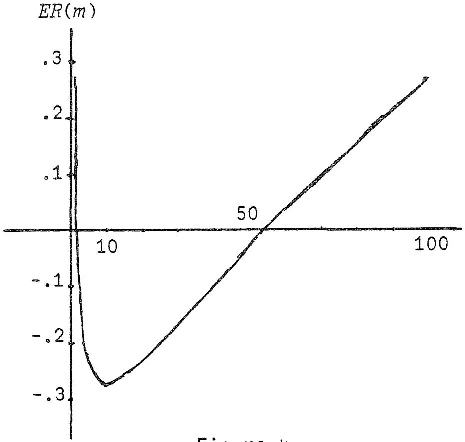


Figure 4

When the error is negative (when $r_1 < \sqrt{x}$), the convergence is not quite as rapid for the divide-and-average method as when $r_1 > \sqrt{x}$, as the algebraic proof showed [1978: 157]. So applying the technique to the extreme case $m = 10$, we get $p(10) = 2.3089$ and

$$r_1 = 2.3089,$$

$$r_2 = 3.319983371,$$

$$r_3 = 3.166023325,$$

$$r_4 = 3.16227987588,$$

$$r_5 = 3.16227766016,$$

the last of which is a satisfactory 10-digit approximation, even for those 10-digit calculators that work internally with 12 or 13 digits.

Curiously, even if we skimp somewhat on the initial accuracy, we can still obtain just as good a final result. If we round our initial estimate to

$$p_1(m) = 0.12(m+10),$$

then $ER(1) = ER(100) = 0.32$ and $ER(10) = -0.24$, increasing the maximum relative error when it is positive but decreasing its absolute value when the relative error is negative. Here again four iterations always suffice for full 10-digit calculator accuracy. For the extreme cases $\sqrt{1}$ and $\sqrt{10}$, we have

$$\begin{array}{lll} r_1 = p_1(1) = 1.32 & \text{and} & r_1 = p_1(10) = 2.4 \\ r_2 = 1.038787879 & & r_2 = 3.283333333 \\ r_3 = 1.000724161 & & r_3 = 3.164509306 \\ r_4 = 1.00000026201 & & r_4 = 3.16227844705 \\ r_5 = 1.00000000000 & & r_5 = 3.16227766016. \end{array}$$

It is reasonable to ask whether a significant improvement would result if a higher degree polynomial first approximation to $\sqrt{m}$ on [1,100] were used. Let us try a quadratic polynomial

$$q(m) = am^2 + bm + c.$$

We may pick four points on [1,100] at which the maximum error is to occur: the two endpoints 1 and 100 and two intermediate values, say 9 and 36. Assuming the maximum relative error is ε, we must have

$$\begin{aligned} 1 &= q(1) - \varepsilon = a + b + c - \varepsilon, \\ 3 &= q(9) + 3\varepsilon = 81a + 9b + c + 3\varepsilon, \\ 6 &= q(36) - 6\varepsilon = 1296a + 36b + c - 6\varepsilon, \\ 10 &= q(10) + 10\varepsilon = 10000a + 100b + c + 10\varepsilon. \end{aligned}$$

With the aid of a calculator and careful algebra, we find that

$$a = -0.0012626, \quad b = 0.2058081, \quad c = 0.9090909,$$

and the maximum relative error ε is

$$\varepsilon = 0.1136.$$

By constructing a table of values for the relative error

$$ER(m) = \frac{am^2 + bm + c - \sqrt{m}}{\sqrt{m}},$$

we locate the extreme values

$$ER(1) = 0.1136, \qquad EP(4.85) = -0.1474,$$

$$ER(49.5) = 0.1375, \quad \text{and} \quad ER(100) = -0.1136.$$

So our original choice of 9 and 36 for intermediate values was incorrect, as expected, and we should use new choices, say 5 and 50.

There seems to be little benefit from further work in this direction, however, since $\varepsilon = 0.1136$ is not sufficiently better than the maximum relative error obtained from the linear approximation; four iterations of Newton's method are still required and the details of the quadratic polynomial would consume much more memory space in a calculator. Hence we drop the search for a higher degree approximation to $\sqrt{m}$.

As an aside comment, if we do find the new approximation $q_1(m)$ using 5 and 50 for intermediate values, we get

$$q_1(m) = -0.0012981m^2 + 0.206932m + 0.9301780$$

with $\varepsilon = 0.1358$. This approximation is quite close to the desired minimax quadratic; extreme error occurs at 1, 4.95, 48.2, and 100, the relative error at 48.2 being 0.1362, sufficiently close to the 0.1358 so that further refinement seems unncessary.

Thus we have found a simple, fast technique for finding square roots by Newton's divide-and-average method starting from a modified minimax linear first approximation. Whether this technique or one similar to it is actually used in a calculator is unknown to me. Such information is a closely-guarded industrial secret, as I have been informed by one well-known calculator company. Surely the method seems reasonable. As we shall see next, however, there is a better method that seems to have gained wide acceptance.

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1. C.T. Fike, *Computer Evaluation of Mathematical Functions*, Prentice-Hall, 1968, pp. 75-76, 92, 111-112, 157, 169, 178, 207.

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* * *

Every theorem in category theory is either a pushover or its dual—a put-on.

From *Category Theory*, by Herrlich and Strecker, Allyn and Bacon, 1973.

A FORMULA FOR POLYTOPES?

DR. A.H. NOËL[1]

For $0 \le i \le n$, let F_i denote the number of i-dimensional faces of an n-simplex. We have the binomial expansion

$$(1+x)^{n+1} = 1 + \sum_{i=0}^{n} F_i x^{i+1},$$

and setting $x = -1$ gives

$$0 = 1 + \sum_{i=0}^{n} (-1)^{i+1} F_i$$

or

$$\sum_{i=0}^{n} (-1)^i F_i = 1. \tag{1}$$

Now let F_i be the number of i-dimensional faces of an n-dimensional hyper-cube. Here we have the binomial expansion

$$(2+x)^n = \sum_{i=0}^{n} F_i x^i$$

and setting $x = -1$ yields again (1).

Now (1) is known to hold for all simply-connected polytopes for n = 0, 1, 2, 3. If $n = 3$, for example, (1) reduces to (note that $F_3 = 1$)

$$F_0 - F_1 + F_2 = 2, \tag{2}$$

a well-known formula due to an obscure French mathematician named Relue who never received proper credit for it. If it holds as well for $n > 3$, then (1) would provide a nice generalization of (2). But what in the (n-dimensional) world would $(3+x)^{n-1}$, $(4+x)^{n-2}$,... represent?

* * *

Our Own Mother Goose

Jack be nimble,
   Jack be quick,
If you are going to check
   The addition at McDonald's.

CLAYTON W. DODGE

---

[1]This note was found among the unpublished papers of the late Dr. Noël by A. Liu, of University of Alberta, a distant relative of the good doctor.

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before ***January 1, 1979****, although solutions received after that date will also be considered until the time when a solution is published.*

371. *Proposed by Charles W. Trigg, San Diego, California.*

In the following skeleton multiplication, each letter uniquely replaces a digit in the decimal system. Reconstruct the multiplication.

```
                      E A S Y
        M A T H E M A T I C S
        ---------------------
                    M * * * S
                  A * * * C
                H * * * I
              S * * * T
            Y * * * A
          I * * * M
        T * * * E
      C * * * H
    S * * * T
  Y * * * A
I * * * M
-----------------------------
I * * * * * * * * * * * * * S
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372. *Proposed jointly by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.; and Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

A triangle ABC has area 1. Point P is on side a, α units from B; point Q is on b, β units from C; and point R is on c, γ units from A. Prove that, if α/a, β/b, and γ/c are the zeros of a cubic polynomial f whose leading coefficient is unity, then the area of ΔPQR is given by $f(1) - f(0)$.

373. *Proposed by Leroy F. Meyers, The Ohio State University.*

Suppose that the human population of the Earth is increasing exponentially at a constant relative rate k, that the average volume of a person stays at V_0, and that the present population is N_0. If people are assumed packed solidly into a sphere, how long will it be until the radius of that sphere is increasing at the speed of light, c, and what will the radius of the sphere be then?

The following approximate data may be used: $N_0 = 4 \times 10^9$, $k = 1\%/\text{yr}$, 1 yr = 365.25 da, 1 da = 24•60•60 sec; and, in English units, $V_0 = 4$ ft^3 and $c = 186300$ mi/sec, whereas in metric units $V_0 = 0.1$ m^3 and $c = 3 \times 10^8$ m/sec.

(I heard of this problem several years ago. It must be a well-known bit of mathematical folklore.)

374*. *Proposed by Sidney Penner, Bronx Community College, New York.*

Prove or disprove the following

THEOREM. Let R be the set of real numbers and let the function f: $R \to R$ be such that $f''(x)$ exists, is continuous and is positive for every x in R. Let P_1 *and* P_2 *be two distinct points on the graph of f, let L_1 be the line tangent to f at* P_1 *and define L_2 analogously. Let* Q *be the intersection of L_1 and L_2 and let* S *be the intersection of the graph of f with the vertical line through* Q. *Finally, let R_1 be the region bounded by segment* P_1Q, *segment* SQ *and arc* P_1S, *and define R_2 analogously. If, for each choice of* P_1 *and* P_2, *the areas of R_1 and R_2 are equal, then the graph of f is a parabola with vertical axis.*

375. *Proposed by M.S. Klamkin, University of Alberta.*

A convex n-gon P of cardboard is such that if lines are drawn parallel to all the sides at distances x from them so as to form within P another polygon P', then P' is similar to P. Now let the corresponding consecutive vertices of P and P' be $A_1, A_2, \ldots, A_n$ and $A'_1, A'_2, \ldots, A'_n$, respectively. From A'_2, perpendiculars A'_2B_1, A'_2B_2 are drawn to A_1A_2, A_2A_3, respectively, and the quadrilateral $A'_2B_1A_2B_2$ is cut away. Then quadrilaterals formed in a similar way are cut away from all the other corners. The remainder is folded along $A'_1A'_2, A'_2A'_3, \ldots, A'_nA'_1$ so as to form an open polygonal box of base $A'_1A'_2\ldots A'_n$ and of height x. Determine the maximum volume of the box and the corresponding value of x.

376. *Proposed by V.G. Hobbes, Westmount, Québec.*

Isosceles triangles can be divided into two types: those with equal sides longer than the base and those with equal sides shorter than the base. Of all possible isosceles triangles what proportion are long-legged?

377. *Proposed by Michael W. Ecker, Pennsylvania State University.*

For $n = 1, 2, 3, \ldots$, let $f(n)$ be the number of zeros in the decimal representation of n, and let

$$F(p) = \sum_{n=1}^{\infty} \frac{f(n)}{n^p}.$$

Find the domain of F, that is, the real values of p for which the series $F(p)$ converges.

(This problem was suggested by Problem E 2675, *American Mathematical Monthly*, 84 (October 1977) 652.)

378. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

(a) Find four positive decimal integers in arithmetic progression, each having the property that if any digit is changed to any other digit, the resulting number is always composite.

(b)* Can the four integers be consecutive?

(This problem was suggested by Problem 1029* in *Mathematics Magazine*, 51 (January 1978) 69.)

379. *Proposed by Peter Arends, Algonquin College, Ottawa.*

Construct a triangle ABC, given angle A and the lengths of side a and t_a (the internal bisector of angle A).

380. *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

Let P be a point on the graph of $y = f(x)$, where f is a third-degree polynomial, let the tangent at P intersect the curve again at Q, and let A be the area of the region bounded by the curve and the segment PQ. Let B be the area of the region defined in the same way by starting with Q instead of P. What is the relationship between A and B?

* * *

PUBLISHING NOTE

We would like to remind our readers of the very fine series of booklets (≈ 90 pages each) entitled 1001 *Problems in High School Mathematics* published from time to time by the Canadian Mathematical Society. They are edited by the well-known Canadian problemists E. Barbeau, M. Klamkin, and W. Moser. Each booklet contains 150 problems and the solutions to 100 of them. Book 1, Book 2, and Book 3 are now available, the last of which was published quite recently and contains several problems from this journal. For more information see [1977: 90] or write to

Canadian Mathematical Society,
3421 Drummond St., Suite 15,
Montréal, Québec, Canada H3G 1X7.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

305. [1978: 11, 180] Late solution: CARL W. NORRIS, Memorial University of Newfoundland.

315. [1978: 35] *Proposed by Orlando Ramos, Havana, Cuba.*

Prove that, if two points are conjugate with respect to a circle, the sum of their powers is equal to the square of the distance between them.

Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.

Let γ be a circle whose equation we assume without loss of generality to be

$$x^2+y^2-r^2=0,$$

and let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ be two points distinct from the origin. Since the polar of a point $P(x',y')\neq(0,0)$ with respect to γ is the line

$$xx'+yy'-r^2=0,$$

it follows that P_1 and P_2 are conjugate points with respect to γ if and only if

$$x_1x_2+y_1y_2-r^2=0. \qquad (1)$$

Now the powers of P_1 and P_2 with respect to γ are respectively

$$p_1=x_1^2+y_1^2-r^2 \quad \text{and} \quad p_2=x_2^2+y_2^2-r^2;$$

hence the following statements (2)-(5) are all equivalent, the last being a consequence of (1):

$$p_1+p_2=|P_1P_2|^2, \qquad (2)$$

$$(x_1^2+y_1^2-r^2)+(x_2^2+y_2^2-r^2)=(x_1-x_2)^2+(y_1-y_2)^2, \qquad (3)$$

$$x_1x_2+y_1y_2=r^2, \qquad (4)$$

P_1 and P_2 are conjugate points with respect to γ. (5)

The equivalence of (2) and (5) proves the stated theorem as well as its converse.

An analogous result clearly holds for a sphere.

Also solved by LEON BANKOFF, Los Angeles, California; ROLAND H. EDDY, Memorial University of Newfoundland; G.C. GIRI, Research Scholar, Indian Institute of Technology, Kharagpur, India; JIM MATTICE, Parkside H.S., Dundas, Ontario; DAN PEDOE, University of Minnesota; SAHIB RAM MANDAN, Indian Institute of Technology, Kharagpur, India (two solutions); HARRY D. RUDERMAN, Hunter College, New York; and the proposer.

Editor's comment.

Pedoe mentioned that this problem is very, very old, and Ruderman located it in Johnson [1].

One solution was less than satisfactory because the solver confused the notions of *conjugate* and *inverse* points; so it may be useful to recall that *every* point on the polar of a point P, not only its inverse point, is conjugate to P.

A synthetic proof is not hard to find, and several of the solutions submitted were of this type. But, as frequently happens with synthetic proofs, in order to have a complete proof one must tiresomely consider several cases: whether, for example, any of the points P_1 and P_2 lie on the circle, or whether one or both of them lie outside the circle. The analytic solution we have featured is valid in every case, and for this we are grateful to Monsieur Descartes.

REFERENCE

1. Roger A. Johnson, *Modern Geometry*, Houghton Mifflin Co., Cambridge, Mass., 1929, p. 105, Corollary *d*.

* * *

316. [1978: 36] *Proposé par Hippolyte Charles, Waterloo, Québec.*

Démontrer l'implication

$$\frac{a-x}{x-b} = \frac{a-d}{b-c}\cdot\frac{c-y}{y-d} \implies \frac{a-y}{y-b} = \frac{a-d}{b-c}\cdot\frac{c-x}{x-d}.$$

I. *Solution by the late R. Robinson Rowe, Sacramento, California.*

We assume that both equations are meaningful, that is, that no denominator vanishes in either one.

The first equation is then equivalent to

$$f(x,y) = 0, \tag{1}$$

where

$$\begin{aligned} f(x,y) &\equiv (a-x)(b-c)(y-d) - (x-b)(a-d)(c-y) \\ &= (a-b+c-d)xy + (bd-ac)(x+y) + bc(a-d) - ad(b-c), \end{aligned}$$

and the second is equivalent to

$$f(y,x) = 0. \tag{2}$$

Since f is symmetric in x and y, it follows immediately that (1) $\iff$ (2).

II. *Solution by Herta T. Freitag, Roanoke, Virginia.*

If we assume that no denominator vanishes in either equation, then the first equation is equivalent to

$$(b-c)(a-x)(y-d) = (a-d)(x-b)(c-y),$$

and subtracting this from the identity

$$(b-c)(d-a)(x-y) \equiv (a-d)(c-b)(x-y)$$

yields

$$(b-c)(x-d)(y-a) = (a-d)(c-x)(b-y),$$

which is equivalent to the second equation.

III. *Solution by David C. Kay, University of Oklahoma.*

Note that the given equation is equivalent to

$$\frac{x-a}{x-b}\cdot\frac{c-b}{c-a} = \frac{y-c}{y-d}\cdot\frac{a-d}{a-c}$$

and also to

$$\frac{x-a}{x-b}\cdot\frac{d-b}{d-a} = \frac{y-c}{y-d}\cdot\frac{b-d}{b-c}.$$

Letting a, b, c, d, x, and y be the coordinates of the points A, B, C, D, X, and Y on some line, each of the above equations yields an equation in cross ratios:

$$(XC,AB) = (YA,CD) = \lambda$$

$$(XD,AB) = (YB,CD) = \mu.$$

Since the same permutations of points will yield the same function of λ (and of μ), we obtain

$$(XB,CA) = (YD,AC) = p_1(\lambda) \qquad (1)$$

$$(XB,AD) = (YD,CB) = p_2(\mu). \qquad (2)$$

Multiplying in (1) and (2),

$$(XB,CA)(XB,AD) = (YD,AC)(YD,CB).$$

Therefore

$$\frac{x-c}{x-a}\cdot\frac{b-a}{b-c}\cdot\frac{x-a}{x-d}\cdot\frac{b-d}{b-a} = \frac{y-a}{y-c}\cdot\frac{d-c}{d-a}\cdot\frac{y-c}{y-b}\cdot\frac{d-b}{d-c},$$

$$\frac{x-c}{x-d}\cdot\frac{b-d}{b-c} = \frac{y-a}{y-b}\cdot\frac{d-b}{d-a},$$

$$\frac{x-c}{x-d}\cdot\frac{a-d}{b-c} = \frac{y-a}{y-b},$$

which is equivalent to the desired equation.

Also solved by LEON BANKOFF, Los Angeles, California; W.J. BLUNDON, Memorial University of Newfoundland; RICHARD BURNS and KRISTIN DIETSCHE, East Longmeadow H.S., East Longmeadow, Massachusetts (jointly); CLAYTON W. DODGE, University of Maine at Orono; G.C. GIRI, Research Scholar, Indian Institute of Technology, Kharagpur, India; ALLAN Wm. JOHNSON JR., Washington, D.C.; F.G.B. MASKELL, Collège Algonquin, Ottawa; HERMAN NYON, Paramaribo, Surinam; HARRY D. RUDERMAN, Hunter College, New York; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and KENNETH S. WILLIAMS, Carleton University, Ottawa.

Editor's comment.

Solution III, while not the simplest, is the most illuminating in that it shows one way in which implications such as the proposed one can be generated.

* * *

317. [1978: 36] *Proposed by James Gary Propp, Great Neck, N.Y.*

In triangle ABC, let D and E be the trisection points of side BC with D between B and E, let F be the midpoint of side AC, and let G be the midpoint of side AB. Let H be the intersection of segments EG and DF. Find the ratio EH : HG by means of mass points (see [1976: 55]) or otherwise.

I. *Solution by Richard Burns and Kristin Dietsche, East Longmeadow H.S., East Longmeadow, Massachusetts (jointly).*

Since GF||BC (see Figure 1), triangles FGH and DEH are similar; hence

$$\frac{EH}{HG} = \frac{DE}{GF} = \frac{\frac{1}{3}BC}{\frac{1}{2}BC} = \frac{2}{3}.$$

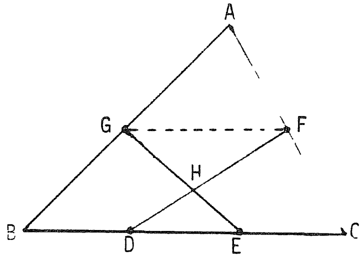

Figure 1

II. *Solution by Harry D. Ruderman, Hunter College, New York.*

More generally, suppose

$$BD : DE : EC = u : 1 : v,$$

$$AG : GB = 1 : r,$$

$$AF : FC = 1 : s,$$

(see Figure 2, where the numbers 1, r, s, u, v indicate *ratios*, not lengths). It will be convenient to assign weights rs, $s+x$, $y+r$ to A, B, C, respectively. These weights can be replaced either by

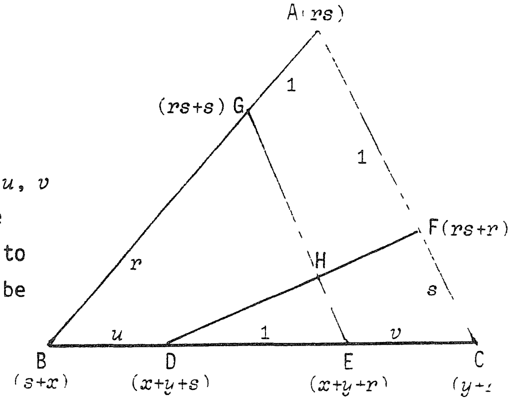

Figure 2

$rs+s$ at G and $x+y+r$ at E or by $rs+r$ at F and $x+y+s$ at D; and with either of these weight assignments the center of mass is at H. Now the following relations hold:

$$x(u+1) = v(y+r), \qquad u(s+x) = y(v+1),$$

and subtracting these equations yields $x+y = rv+su$. Hence

$$\frac{\mathrm{EH}}{\mathrm{HG}} = \frac{rs+s}{x+y+r} = \frac{s(r+1)}{r(v+1)+su}$$

and

$$\frac{\mathrm{DH}}{\mathrm{HF}} = \frac{rs+r}{x+y+s} = \frac{r(s+1)}{s(u+1)+rv}.$$

In particular, for the problem at hand we have $r = s = u = v = 1$, and $\mathrm{EH}:\mathrm{HG} = 2:3$.

Also solved by LEON BANKOFF, Los Angeles, California; STEVEN R. CONRAD, Benjamin N. Cardozo H.S., Bayside, N.Y.; SAHIB RAM MANDAN, Indian Institute of Technology, Kharagpur, India; W.A. McWORTER JR., The Ohio State University; HERMAN NYON, Paramaribo, Surinam; FREDERICK ROTHSTEIN, New Jersey Department of Transportation, Trenton, N.J.; the late R. ROBINSON ROWE, Sacramento, California; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; HARRY SITOMER, Huntington, N.Y.; CHARLES W. TRIGG, San Diego, California; and the proposer.

Editor's comment.

The problem as proposed could be solved easily by traditional methods, as shown in solution I. But the generalization of solution II could hardly be solved more efficiently than by the method of mass points, as shown above.

For those who need a refresher course in the method of mass points, see [1976: 55].

* * *

318. [1978: 36] *Proposed by C.A. Davis in James Cook Mathematical Notes No. 12 (October, 1977), p. 6.*

Given any triangle ABC, thinking of it as in the complex plane, two points L and N may be defined as the stationary values of a cubic that vanishes at the vertices A, B, and C. Prove that L and N are the foci of the ellipse that touches the sides of the triangle at their midpoints, which is the inscribed ellipse of maximal area.

I. *Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India (revised by the editor).*

We will make use of the following lemmas, which describe known properties of the ellipse:

LEMMA 1. *Let* AT_1 *and* AT_2 *be tangents to an ellipse with foci* F_1 *and* F_2 *(see*

Figure 1). *Then*

(a) $\angle F_1AT_1 = \angle F_2AT_2$;

(b) $\angle AF_1T_1 = \angle AF_1F_2$ *and* $\angle AF_2T_2 = \angle AF_2F_1$.

This is a theorem of Poncelet. For a proof, see [2].

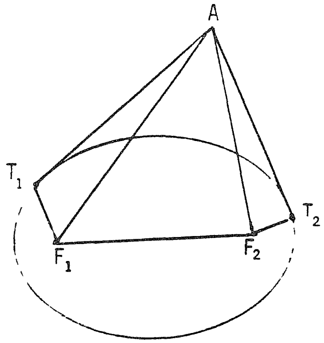

Figure 1

LEMMA 2. *If* F_1 *and* F_2 *are the foci of an ellipse inscribed in* ΔABC, *then* F_1 *and* F_2 *are isogonal conjugate points with respect to* ΔABC. *Conversely, if* F_1 *and* F_2 *are isogonal conjugate points with respect to* ΔABC, *there is a unique inscribed ellipse with foci* F_1 *and* F_2. (See Figure 2.)

Outline of proof. The first part follows from Lemma 1 *(a)* and the fact that the isogonal conjugates of AF_1, BF_1, CF_1 are concurrent in F_2 (see [1]). For the converse, given F_1 and F_2 we can construct T_1, T_2, T_3 by Lemma 1*(b)*; and the unique ellipse through T_1 with foci F_1 and F_2 will be inscribed in ΔABC, the points of tangency being T_1, T_2, T_3.

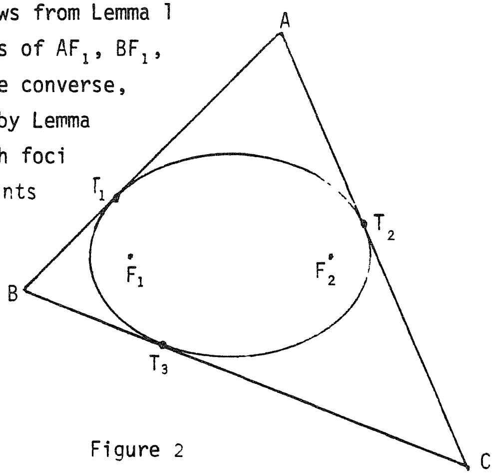

Figure 2

LEMMA 3. *For a given triangle, there is a unique inscribed ellipse whose centre lies at the centroid of the triangle. It is the inscribed Steiner ellipse, which touches the sides of the triangle at their midpoints and is the inscribed ellipse of maximal area.*

This is a well-known result. For a proof, see [3].

Consider now the triangle of the proposal. Let the affixes of A, B, C, L, N be a, b, c, z_1, z_2. The cubic that vanishes at the vertices is then defined by

$$f(z) = (z-a)(z-b)(z-c),$$

and we have

$$\begin{aligned} f'(z) &= (z-b)(z-c) + (z-c)(z-a) + (z-a)(z-b) \\ &= 3z^2 - 2(a+b+c)z + bc + ca + ab \qquad (1) \\ &= 3(z-z_1)(z-z_2). \end{aligned}$$

Now

$$f'(a) = (a-b)(a-c) = 3(a-z_1)(a-z_2);$$

hence

$$\frac{AB}{AL}\cdot\frac{AC}{AN} = \frac{a-b}{a-z_1}\cdot\frac{a-c}{a-z_2} = 3.$$

Thus the sum of the amplitudes of AB/AL and AC/AN is zero, so AL and AN are isogonal conjugates. Similarly BL and BN, and CL and CN, are isogonal conjugates, and so L and N are isogonal conjugate points with respect to ΔABC. It now follows from Lemma 2 that L and N are the foci of a uniquely determined inscribed ellipse. From (1), the centre of this ellipse has affix

$$\frac{z_1+z_2}{2} = \frac{a+b+c}{3};$$

hence the centre lies at the centroid of the triangle and, from Lemma 3, the ellipse is the inscribed Steiner ellipse, as required.

II. *Solution by O. Bottema, Delft, The Netherlands.*

This problem is well-known in Dutch mathematical literature. As far as I know it was first proved by van den Berg [5] and other proofs were later given by Scheffer [4] and van Veen [6]. Here is still another proof.

Let $A_1'A_2'A_3'$ be a triangle in the complex plane; B_1', B_2' the zeros of the derivative of the cubic whose zeros are the vertices of the triangle; and E' the inscribed Steiner ellipse, which touches the sides of the triangle at their midpoints. It is well-known that E' is the inscribed ellipse of maximal area and that its center coincides with the centroid of the triangle.

All displacements in the complex plane are transformations of the form

$$z' = \alpha z + \beta, \tag{2}$$

where α and β are complex numbers with $|\alpha| = 1$, and there is always a displacement (2) which transforms E' into an ellipse E with standard equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a \geq b.$$

Under (2), A_1', A_2', A_3', B_1', B_2' are transformed into A_1, A_2, A_3, B_1, B_2, where B_1, B_2 are the zeros of the derivative of the cubic with zeros at A_1, A_2, A_3; and E is the inscribed Steiner ellipse of triangle $A_1A_2A_3$.

The transformation $x' = a^{-1}x$, $y' = b^{-1}y$ transforms E into the unit circle c; it is an affine transformation with the property that tangents remain tangents and midpoints remain midpoints. The triangles with c as their inscribed Steiner ellipse are the equilateral triangles inscribed in the circle C with center 0 and radius 2. Hence the vertices of the triangles with E as their Steiner ellipse are

$$A_k = (2a\cos\phi_k,\ 2b\sin\phi_k), \qquad k = 1,\ 2,\ 3$$

with $\phi_1 = \phi$ (arbitrary), $\phi_2 = \phi + 2\pi/3$, $\phi_3 = \phi - 2\pi/3$. Thus

$$\Sigma\cos\phi_k = \Sigma\sin\phi_k = 0,$$

which confirms that the centroid coincides with the center 0 of E. Furthermore,

$$\Sigma\cos^2\phi_k = \Sigma\sin^2\phi_k = \frac{3}{2} \pm \frac{1}{2}\Sigma\cos 2\phi_k = \frac{3}{2}.$$

Consider now the cubic equation whose roots z_k are the points A_k, so that

$$z_k = 2a\cos\phi_k + 2ib\sin\phi_k, \qquad k = 1,\ 2,\ 3.$$

The equation reads

$$z^3 + d_1z^2 + d_2z + d_3 = 0, \tag{3}$$

with

$$d_1 = -\Sigma z_k = 0$$

and

$$d_2 = z_2z_3 + z_3z_1 + z_1z_2 = -\frac{1}{2}(z_1^2 + z_2^2 + z_3^2) = -2\Sigma(a\cos\phi_k + ib\sin\phi_k)^2.$$

Since

$$2\Sigma\cos\phi_k\sin\phi_k = \Sigma\sin 2\phi_k = 0,$$

we obtain

$$d_2 = -2\left(\frac{3}{2}a^2 - \frac{3}{2}b^2\right),$$

and (3) becomes

$$z^3 - 3(a^2 - b^2)z + d_3 = 0. \tag{4}$$

The roots of the derivative of (4) are $z = \pm\sqrt{a^2 - b^2}$; they coincide therefore with the foci of E.

Editor's comment.

A solution of this problem, which included a proof of Lemma 3 in our solution I, was published in *James Cook Mathematical Notes No.* 14 (January 1978). It was by Basil C. Rennie, the editor of *JCMN*.

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4. M. Scheffer, *Mathematica*, A7 (1938-39) 285-288.

5. F.J. van den Berg, Over het verband tusschen de wortels eener vergelijking en die van haar afgeleide (On the relationship between the roots of an equation and those of its derivative), *N. Arch. v. Wisk.*, IX (1882) 10.

6. S.C. van Veen, *Mathematica*, A8 (1939-40) 14-16.

* * *

319. [1978: 36] *Proposed by Leigh Janes, Rocky Hill, Connecticut.*

If a solution of the following type has not yet occurred in a student paper, it soon will. "Cancelling the exponents" yields

$$\frac{37^3+13^3}{37^3+24^3} = \frac{37+13}{37+24} = \frac{50}{61},$$

which is correct.

Find necessary and sufficient conditions for the positive integer triple (A,B,C) to satisfy

$$\frac{A^3+B^3}{A^3+C^3} = \frac{A+B}{A+C}.$$

I. *Solution by David Karr, student, Bronx High School of Science, Bronx, N.Y.*

For positive A, B, and C (integers or not), we have

$$\begin{aligned}\frac{A^3+B^3}{A^3+C^3} = \frac{A+B}{A+C} &\iff (A+C)(A^3+B^3) = (A+B)(A^3+C^3)\\ &\iff A^2-AB+B^2 = A^2-AC+C^2\\ &\iff (B-C)(A-B-C) = 0;\end{aligned}$$

so the required necessary and sufficient condition is:

$$B = C \quad \text{or} \quad A = B+C.$$

II. *Comment by Allan Wm. Johnson Jr., Washington, D.C.*

There is ample equipment in the high school algebra toolbox to construct more examples of "cancelling the exponents".

(a) To find positive integers A, B, C, D such that

$$\frac{A^3 + B^3}{C^3 + D^3} = \frac{A + B}{C + D},$$

we first clear fractions to obtain the equivalent condition

$$A^2 - AB + B^2 = C^2 - CD + D^2. \tag{1}$$

Next we call upon the following identity [1]: if either

$$r = mp - nq, \quad s = np + (m - n)q \quad \text{or} \quad r = mq - np, \quad s = nq + (m - n)p,$$

then

$$(m^2 - mn + n^2)(p^2 - pq + q^2) = r^2 - rs + s^2.$$

This furnishes the following parametric representation of solutions of (1):

$$A = mp - nq, \quad B = np + (m - n)q, \quad C = mq - np, \quad D = nq + (m - n)p.$$

For example, for $m = 4$, $n = 1$, $p = 2$, $q = 3$, we get

$$\frac{5^3 + 11^3}{10^3 + 9^3} = \frac{5 + 11}{10 + 9}.$$

(b) For

$$\frac{A^2 + B^2}{A^2 + C^2} = \frac{A + B}{A + C},$$

we assume $B \neq C$ to eliminate trivial solutions and clear fractions to obtain the equivalent condition

$$A^2 - (B + C)A - BC = 0, \tag{2}$$

from which

$$A = \frac{1}{2}[B + C \pm \sqrt{(B + C)^2 + 4BC}\,].$$

To make the radicand a perfect square, we seek solutions of

$$s^2 = (B + C)^2 + 4BC = (B + 3C)^2 - 8C^2,$$

an equation of the form $s^2 + 2t^2 = u^2$ to which we apply the following identity [2]:

$$(p^2 - 2q^2)^2 + 2(2pq)^2 = (p^2 + 2q^2)^2.$$

This yields the following two parametric representations for solutions of (2):

$$\begin{aligned} A &= p(p - q) \\ B &= (p - q)(p - 2q) \\ C &= pq \end{aligned} \qquad \text{and} \qquad \begin{aligned} A &= q(2q - p) \\ B &= (2q - p)(q - p) \\ C &= pq \end{aligned}$$

For example, if we use $p = 3$, $q = 1$ in the first representation and $p = 1$, $q = 3$ in the second, we get

$$\frac{6^2+2^2}{6^2+3^2} = \frac{6+2}{6+3} \quad \text{and} \quad \frac{15^2+10^2}{15^2+3^2} = \frac{15+10}{15+3}.$$

(c) To find solutions of

$$\frac{A^2+B^2}{C^2+D^2} = \frac{A+B}{C+D}, \tag{3}$$

we first set $k = (A^2+B^2)/(A+B)$, which is equivalent to

$$A^2 - kA + B(B-k) = 0,$$

giving

$$A = \frac{1}{2}(k + \sqrt{k^2+4kB-4B^2}).$$

Proceeding as in (b) to make $k^2+4kB-4B^2 = (k+2B)^2 - 8B^2$ a square, we find

$$A = p(p-q), \qquad B = pq, \qquad k = p^2 - 2pq + 2q^2.$$

Solving the last equation for p gives $p = q + \sqrt{k-q^2}$, which requires $k = q^2 + r^2$. We note that if k can be expressed as a sum of two squares in two ways, then two different pairs (p,q) result: a pair (p_1,q_1) to solve $(A^2+B^2)/(A+B) = k$ and a pair (p_2,q_2) to solve $(C^2+D^2)/(C+D) = k$. Using

$$k = (m^2+n^2)(s^2+t^2) = (ms+nt)^2 + (mt-ns)^2 = (mt+ns)^2 + (ms-nt)^2,$$

we obtain the following parametric representation of solutions of (3):

$$\begin{aligned} A &= p_1(p_1-q_1) & \qquad & p_1 = m(s+t) - n(s-t) \\ B &= p_1q_1 & \text{where} \qquad & q_1 = ms+nt \\ C &= p_2(p_2-q_2) & \qquad & p_2 = m(s+t) + n(s-t) \\ D &= p_2q_2 & \qquad & q_2 = mt+ns. \end{aligned}$$

As an example, for $m=2$, $n=1$, $s=3$, $t=2$, we obtain

$$\frac{9^2+72^2}{44^2+77^2} = \frac{9+72}{44+77}.$$

The above discussions, of course, give no assurance that all solutions of (a), (b), and (c) have been found.

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REFERENCES

1. R.D. Carmichael, *Diophantine Analysis*, Dover, New York, 1959, p. 25.

2. J.V. Uspensky and M.A. Heaslet, *Elementary Number Theory*, McGraw-Hill, New York, 1939, p. 393.

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320. [1978: 36] *Proposed by Dan Sokolowsky, Antioch College.*

The sides of ΔABC are trisected by the points P_1, P_2, Q_1, Q_2, R_1, R_2, as shown in the figure below. Show that:

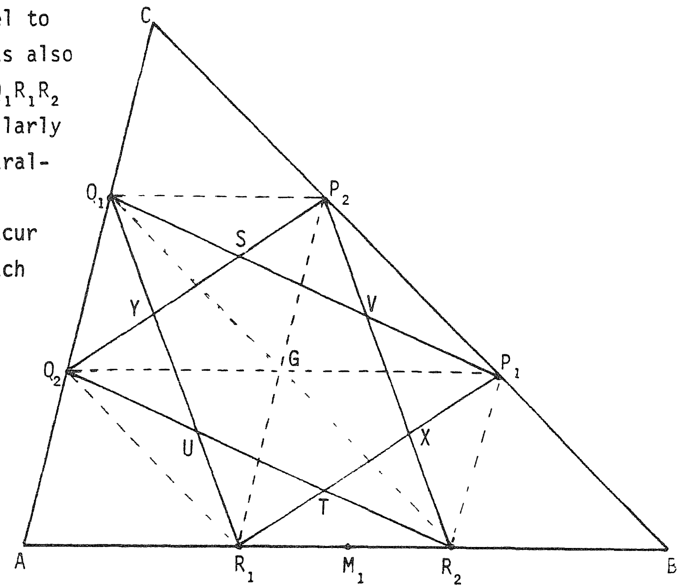
(a) $\Delta P_1Q_1R_1 \cong \Delta P_2Q_2R_2$;

(b) $|P_1Q_1R_1| = \frac{1}{3}|ABC|$, where the bars denote area;

(c) the sides of Δs $P_1Q_1R_1$ and $P_2Q_2R_2$ trisect one another;

(d) If M_1 is the midpoint of AB, then C, S, T, M_1 are collinear.

I. *Solution by Clayton W. Dodge, University of Maine at Orono.*



Because Q_1R_1 is parallel to and 2/3 the length of CM_1, as also is P_2R_2, it follows that $P_2Q_1R_1R_2$ is a parallelogram; and similarly $Q_2R_1P_1P_2$ and $R_2P_1Q_1Q_2$ are parallelograms. Their diagonals Q_1R_2, R_1P_2, and P_1Q_2 all concur at a point G that bisects each diagonal. Also, G is the center of a half-turn that carries $\Delta P_1Q_1R_1$ into $\Delta Q_2R_2P_2$, so these two triangles are congruent (in that order rather than the order given in part (a) of the proposal). Hence part (a).

Because $AR_1 = AB/3$ and $AQ_1 = 2AC/3$, we have $|AQ_1R_1| = \frac{1}{3}\cdot\frac{2}{3}\cdot|ABC|$. From this and two similar results, we get

$$|AQ_1R_1| = |BR_1P_1| = |CP_1Q_1| = \tfrac{2}{9}|ABC|,$$

and part (b) follows by subtraction.

P_1Q_1 and P_2Q_2 are medians of ΔCQ_2P_1, so S is its centroid, and P_1Q_1 and P_2Q_2 trisect each other. Similarly Q_1R_1 and Q_2R_2, and R_1P_1 and R_2P_2, trisect each other. The half-turn above takes S into T, so T trisects R_2Q_2 and R_1P_1, and similarly other trisection points are taken into trisection points. This proves (c).

The similarity that takes ΔCQ_2P_1 into ΔCAB shows that C, S, and M_1 are collinear; and the half-turn then shows that C, T, and M_1 are collinear. Hence (d).

It would not be hard to show, in addition, that G is the common centroid of Δs $P_1Q_1R_1$, $P_2Q_2R_2$, and ABC.

II. *Extracted from a comment by Sahib Ram Mandan, Indian Institute of Technology, Kharagpur, India.*

If the sides of ΔABC are internally n-sected by P_1, P_2, Q_1, Q_2, R_1, R_2 (so that $BP_1 : BC = CP_2 : CB = 1/n$, etc.), the corresponding results are:

(a) $\Delta P_1Q_1R_1$ is *not* congruent to $\Delta Q_2R_2P_2$ unless $n = 3$ or ΔABC is equilateral.

(b) $|P_1Q_1R_1| = |P_2Q_2R_2| = \dfrac{n^2 - 3n + 3}{n^2}\cdot|ABC|$.

(c) The sides of Δs $P_1Q_1R_1$ and $P_2Q_2R_2$ divide each other in the same ratios. Specifically (see figure),

$$P_2S : SQ_2 = Q_2U : UR_2 = R_2X : XP_2 = 1 : (n-1)$$

and

$$P_2V : VR_2 = R_2T : TQ_2 = Q_2Y : YP_2 = (n-2) : (n-1).$$

(d) C, S, T, and M_1 are collinear for all n.

For all n, also, the Δs $P_1Q_1R_1$, $P_2Q_2R_2$, and ABC have a common centroid G.

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Editor's comment.

Trigg mentioned Problem 3707 in [1], which generalizes part (b) of our comment II. In our notation, it states that if P_1, P_2, Q_1, Q_2, R_1, R_2 are equidistant from the midpoints of the sides upon which they lie (the distances not being necessarily the same on the three sides), then $|P_1Q_1R_1| = |P_2Q_2R_2|$. A solution to this problem has not yet been published.

REFERENCE

1. *School Science and Mathematics*, 77 (December 1977) 714, Problem 3707, proposed by Fred A. Miller.

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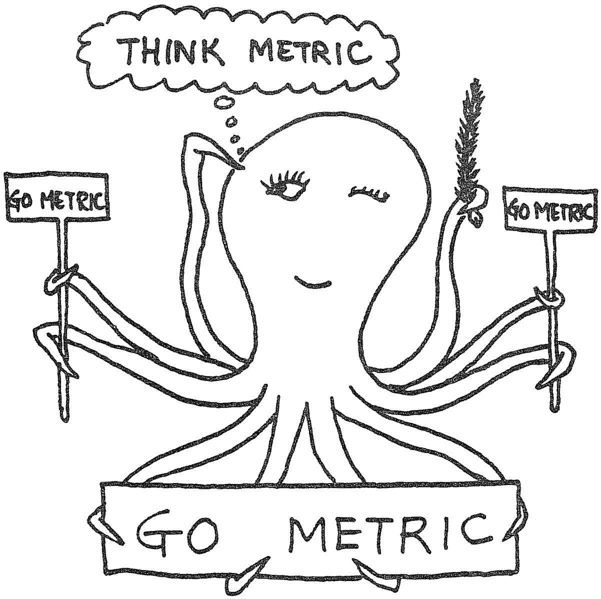

Drawing by
Supaced.

THE DECAPUS

Behold the Decapus, my friends.
This son of Hecto has ten ends,
One of which he used, I think,
To draw himself with his own ink.
A Metric Monster, he is urging
You to become acquainted with
His family, both sires and offspring,
In fact, with his whole kin and kith.

EDITH ORR

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As this is written, the Canadian postal workers are on strike, so the Great Postmaster-in-the-Sky only knows when readers will receive this issue.

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