Crux Mathematicorum

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Editorial Board

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EDITORIAL

"All our knowledge has its origins in our perceptions."
Leonardo da Vinci

If you asked me what was my least favourite class as an undergrad, I would tell you Multivariate Calculus, hands down. All I remember doing in that course is computing double and triple integrals, switching the order of integration, computing numerous partial derivatives and memorizing everything about quadric surfaces. The class was technical, there seemed no apparent reason for anything to be solved the way we were asked to solve it and the methods used were nothing short of arbitrary. (Of course, it didn't help that the class was held at 6-8pm twice a week.) That was my view of the course an as undergrad. Although I still did well in the course and actually enjoyed the next course in Vector Calculus, I knew this branch of mathematics was not for me. So imagine my mixed feelings when I was asked to teach Multivariate Calculus last year.

I said yes. It was time to face my fears, so to speak. I expected to like this course better than I did as an undergrad (low bar for comparison), but I didn't expect to like it quite as much. So I wondered, why was my perception of the material so different this time around? Was it because I was not taking this course for credit and could actually spend time enjoying the math as opposed to being stressed out about it? Was it because I now had friends whose research involved multivariate calculus tools and so it seemed more personally relevant? Was it because I was more mathematically mature and could see more connections within the material itself?

I have experienced similarly diverse feelings with various other things that I have encountered as a young student and later as a more developed mathematician; this includes contest and olympiad problems. While I did participate in these events and did well in them, I can't actually say I ever truly enjoyed them. And I most definitely enjoy them now!

Whatever the reason for my changed appreciation for Multivariate Calculus or math competition problems, it has taught me to give a second chance to things I have made up my mind about. So I urge you to keep an open mind about various types of problems or even whole areas of mathematics that you at some point dismissed as not interesting to you. If you are a fan of inequalities, give geometry problems a chance. If you normally stick to the Problems section of ***Crux***, take a look at our Contest Corner. You might be surprised by the math you discover.

(For a piece that inspired this Editorial, see "Changing the way we think about mathematical ability" by Caroline Junkins, CMS Notes, September 2016.)

Kseniya Garaschuk

THE CONTEST CORNER

No. 43

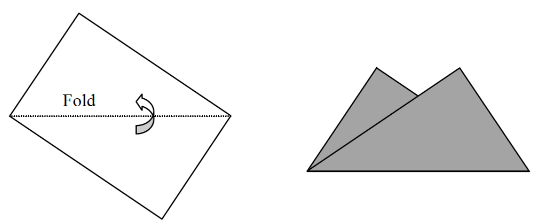
John McLoughlin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

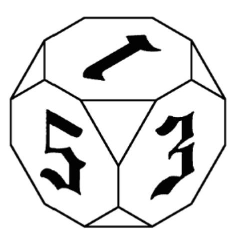
To facilitate their consideration, solutions should be received by the editor by **January 1, 2017**, *although late solutions will also be considered until a solution is published.*

The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

CC211. A rectangular sheet of paper whose dimensions are 12×18 is folded along a diagonal, which creates the M-shaped region drawn at the right. Find the area of the shaded region.



CC212. A cube that is one inch wide has had its eight corners shaved off. The cube's vertices have been replaced by eight congruent equilateral triangles, and the square faces have been replaced by six congruent octagons. If the combined area of the eight triangles equals the area of one of the octagons, what is that area? (Each octagonal face has two different edge lengths that occur in alternating order.)

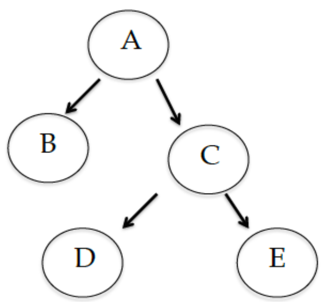


CC213. A pyramid is built from solid unit cubes that are stacked in square layers. The top layer has $1 \times 1 = 1$ cube, the second $3 \times 3 = 9$ cubes and the layer below that has $5 \times 5 = 25$ cubes, and so on, with each layer having two more cubes

on a side than the layer above it. The pyramid has a total of 12 layers. Find the exposed surface area of this solid pyramid, including the bottom.

CC214. The points $(2, 5)$ and $(6, 5)$ are two of the vertices of a regular hexagon of side length two on a coordinate plane. There is a line L that goes through the point $(0, 0)$ and cuts the hexagon into two pieces of equal area. What is the slope of line L?

CC215. Each circle in this tree diagram is to be assigned a value, chosen from a set S, in such a way that along every pathway down the tree the assigned values never increase. That is, $A \geq B, A \geq C, C \geq D, C \geq E$ and $A, B, C, D, E \in S$. (It is permissible for a value in S to appear more than once.) How many ways can the tree be so numbered using only values chosen from the set $S = \{1, \ldots, 6\}$?



(Optional extension: Generalize to a case with $S = \{1, 2, 3, , \ldots, n\}$ by finding an explicit algebraic expression for the number of ways the tree can be numbered.)

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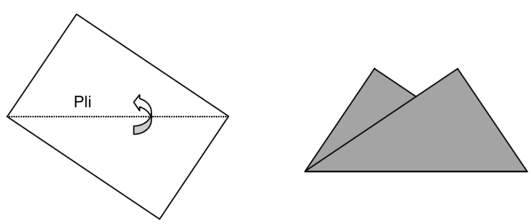
CC211. Une feuille de papier rectangulaire de taille 12×18 est pliée le long de la diagonale, formant ainsi une région en forme de M, telle qu'illustrée. Déterminer la surface de la région ombragée.



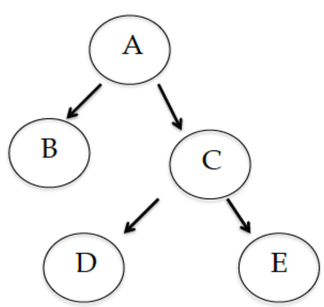
CC212. On a retranché les huit coins d'un cube dont les côtés mesurent chacun un pouce. Les sommets ont ainsi été remplacés par huit triangles équilatéraux congrus, et les faces carrées ont été remplacées par six octogones congrus. Si la

surface totale des huit triangles égale la surface d'un des octogones, quelle est cette surface ? (Chaque face octogonale comporte deux longueurs différentes de côté, en alternance.)

CC213. Une pyramide est construite à partir de cubes de taille unitaire, empilés en tranches carrées. La tranche supérieure comporte $1 \times 1 = 1$ cube, la seconde en a $3 \times 3 = 9$, celle en bas de ça en a $5 \times 5 = 25$, et ainsi de suite, chaque tranche en ayant deux de plus sur chaque côté par rapport à la tranche supérieure. La pyramide a 12 tranches au total. Déterminer la surface externe de cette pyramide, incluant le fond.

CC214. Dans le plan, les points $(2, 5)$ et $(6, 5)$ sont deux sommets d'un hexagone régulier de côté deux. Une certaine ligne L, passant par le point $(0, 0)$, coupe l'hexagone en deux parties de même surface. Quelle est la pente de la ligne L?

CC215. À chaque cercle dans l'arbre indiqué ci-bas on assigne une valeur, choisie dans un ensemble S, de façon à ce que dans chaque chemin vers le bas dans l'arbre les valeurs assignées n'augmentent jamais. C'est-à-dire $A \geq B, A \geq C, C \geq D, C \geq E$ où $A, B, C, D, E \in S$ (Il est permis qu'une valeur dans S apparaisse plus qu'une fois.) De combien de manières peut-on assigner des valeurs à l'arbre si $S = \{1, \ldots, 6\}$?



(Au choix: Généraliser au cas où $S = \{1, 2, 3, , \ldots, n\}$ à l'aide d'une expression algébrique explicite pour le nombre d'assignations.)

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(3), p. 96–97.

CC161. A number n written in base b reads 211, but it becomes 110 when written in base $b+2$. Find n and b in base 10.

A reformulation of #4 of the Santa Clara University High School Mathematics 2001 Contest.

We received ten solutions, of which eight were complete and correct. All eight solutions were nearly identical so we present a composite solution here.

We have that $n = 211$ in base b. This requires $b > 2$ and means

$$n = (211)_b = 2 \cdot b^2 + 1 \cdot b + 1,$$

while $n = 110$ in base $b+2$ gives us

$$n = (110)_{b+2} = 1 \cdot (b+2)^2 + 1 \cdot (b+2) + 0.$$

Equating these two expressions gives

$$\begin{aligned} 2b^2 + b + 1 &= b^2 + 4 + 4b + b + 2 \\ b^2 - 4b - 5 &= 0 \\ b &= -1, 5 \end{aligned}$$

We discard the negative solution both because of the restriction on b and the fact that a base cannot be negative. Using $b = 5$ we can calculate

$$n = (211)_5 = (110)_7 = 56.$$

Therefore $n = 56$ and $b = 5$ in base 10.

CC162. What is the probability that 99 divides a randomly chosen 4-digit palindrome?

A reformulation of #3 from the team section of the 2010 Raytheon MATHCOUNTS State Competition.

We received eight submissions of which four were correct and complete. We present the solution by Titu Zvonaru.

A 4-digit palindrome is a number of the form $\overline{abba}$, with $a = 1, 2, \ldots, 9$ and $b = 0, 1, \ldots, 9$, hence there are 90 numbers which are 4-digit palindrome. Since $\overline{abba} = 1001a + 100b = 11(91a + 10b)$, we deduce that all 4-digit palindromes are divisible by 11. The number $\overline{abba}$ is divisible by 9 if and only if $a+b$ is divisible by 9. If $a+b=9$ we have the possibilities $1881, 2772, 3663, 4554, 5445, 6336, 7227, 8118, 9009$; if $a+b=18$, then there is only the number 9999. The searched probability is $10/90 = 1/9$.

Editor's Comments. Some solvers counted also the case when $a = b = 0$ in the solution, but there is a flaw. Indeed, they have counted a total of 90 palindrome numbers (9 possibilities for the nonzero digit a and 10 possibilities for the digit b), but then they counted the case when $a = b = 0$, a contradiction. The solution could have been consistent if they also counted the degenerate case when $a = 0$ in the total number of palindromes, giving $10 \cdot 10 = 100$ palindrome numbers. In this case we also have $a = 0, b = 0$ and $a = 0, b = 9$, giving the probability $12/100 = 3/25$. The only consistent (but not correct in the strict sense) solutions are the ones given by Kathleen E. Lewis and Hannes Geupel.

CC163. If x is randomly chosen in $[-100, 100]$, what is the probability that $g[f(x)]$ is negative given that $f(x) = x^2 + 3x - 7$ and $g(x) = x^2 - 2x - 99$?

A reformulation of #8 of the 2014 University of North Colorado Math Contest.

We received eight submissions of which seven were correct and complete. We present the solution by Titu Zvonaru.

Since $g(x) = (x+9)(x-11)$, we have $g(x) < 0 \iff x \in (-9, 11)$. It follows that

$$\begin{aligned} g(f(x)) < 0 &\iff -9 < f(x) < 11 \\ &\iff -9 < x^2 + 3x - 7 < 11 \\ &\iff (x+6)(x-3) < 0 \text{ and } (x+1)(x+2) > 0. \end{aligned}$$

Thus, $g(f(x)) < 0 \iff x \in (-6, -2) \cup (-1, 3)$, and the probability is given by the total length of the combined intervals divided by the total length of the domain of x, which is

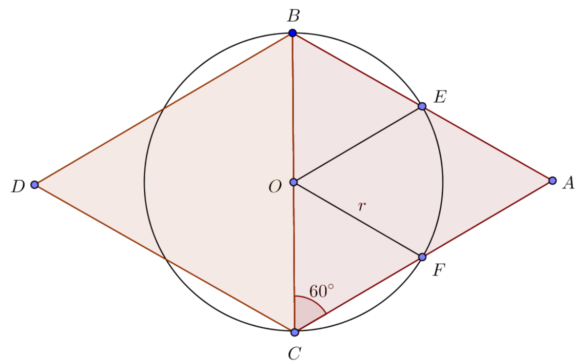
$$\frac{(-2+6)+(3+1)}{100+100} = \frac{1}{25}.$$

CC164. Build two equilateral triangles on the diameter of a circle with radius 5. What is the total area of the circle outside the equilateral triangles? (See the diagram below.)

Proposed by the editor.

We received eleven submissions of which ten were correct. We present the solution by Fernando Ballesta Yagüe, slightly modified by the editor.

Denote the center of the circle by O, the two equilateral triangles by ABC and DBC (with BC being the diameter of the circle), and the intersections of AB and AC with the circle by E and F respectively. Use r for the radius of the circle, and recall that $r = 5$.



As $\triangle ABC$ is equilateral, $\angle BCA = 60°$. Further, $OC = OF = r$, so it follows that $\triangle OCF$ is also equilateral, and $\angle FOC = 60°$.

Hence the area of the circular segment between the chord CF and the circle is equal to the area of a circular sector with central angle $60°$ minus the area of the equilateral $\triangle FOC$, that is

$$\frac{\pi \cdot r^2}{6} - \frac{1}{2} \cdot r^2 \cdot \sin(60°) = \frac{25\pi}{6} - \frac{25\sqrt{3}}{4}.$$

We can reason the same way with $\triangle OBE$, and also with the matching construction on $\triangle DBC$. It follows that the area contained inside the circle but outside the triangles consists of four congruent circular segments, and the total area is

$$4 \cdot \left(\frac{25\pi}{6} - \frac{25\sqrt{3}}{4}\right) = \frac{50\pi}{3} - 25\sqrt{3}.$$

CC165. Georges pays \$50 on each of four gas refills but the prices per litre were \$1.32, \$1.25, \$1.11 and \$1.18 as the price was fluctuating a lot in that time period. What is the average price per litre?

Proposed by the editor.

We received five correct solutions and one incorrect solution. We present the solution of Henry Ricardo.

The quantities of gas purchased were $\frac{\$50}{\$1.32/L}$, $\frac{\$50}{\$1.25/L}$, $\frac{\$50}{\$1.11/L}$, and $\frac{\$50}{\$1.18/L}$.

$$\begin{aligned}
\text{(Average price per litre)} \quad &= \frac{\text{(Total cost of gas)}}{\text{(Total quantity of gas purchased)}} \\
&= \frac{\$200}{\frac{\$50}{\$1.32/L} + \frac{\$50}{\$1.25/L} + \frac{\$50}{\$1.11/L} + \frac{\$50}{\$1.18/L}} \\
&\approx \$1.21/L
\end{aligned}$$

THE OLYMPIAD CORNER

No. 341

Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **January 1, 2017***, although late solutions will also be considered until a solution is published.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC271. A scalene triangle ABC is inscribed within circle ω. The tangent to the circle at point C intersects line AB at point D. Let I be the center of the circle inscribed within $\triangle ABC$. Lines AI and BI intersect the bisector of $\angle CDB$ in points Q and P, respectively. Let M be the midpoint of QP. Prove that MI passes through the middle of arc ACB of circle ω.

OC272. Find all real triples (a, b, c), for which

$$a(b^2 + c) = c(c + ab),$$
$$b(c^2 + a) = a(a + bc),$$
$$c(a^2 + b) = b(b + ca).$$

OC273. Find all functions $f : R \to R$ such that $f(x^{2015} + (f(y))^{2015}) = (f(x))^{2015} + y^{2015}$ holds for all reals x, y.

OC274. Find all triplets (x, y, p) of positive integers such that p is a prime number and $\frac{xy^3}{x+y} = p$.

OC275. Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finishes piling his stones in some manner, he can then perform stone moves, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions $(i, k), (i, l), (j, k), (j, l)$ for some $1 \leq i, j, k, l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i, k) and (j, l) and moving them to (i, l) and (j, k) respectively, or removing one stone from each of (i, l) and (j, k) and moving them to (i, k) and (j, l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.

How many different non-equivalent ways can Steve pile the stones on the grid?

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OC271. Un triangle scalène ABC est inscrit dans un cercle ω. La tangente au cercle au point C coupe la droite AB au point D. Soit I le centre du cercle inscrit dans le triangle ABC. Les droites AI et BI coupent la bissectrice de l'angle CDB aux points respectifs Q et P. Soit M le milieu du segment QP. Démontrer que MI passe au milieu de l'arc ACB du cercle ω.

OC272. Déterminer tous les triplets (a, b, c) de réels tels que

$$a(b^2 + c) = c(c + ab),$$

$$b(c^2 + a) = a(a + bc),$$

$$c(a^2 + b) = b(b + ca).$$

OC273. Déterminer toutes les fonctions $f : R \to R$ qui vérifient

$$f(x^{2015} + (f(y))^{2015}) = (f(x))^{2015} + y^{2015}$$

pour tous réels x, y.

OC274. Déterminer tous les triplets (x, y, p) d'entiers strictement positifs pour lesquels p est un nombre premier et $\frac{xy^3}{x+y} = p$.

OC275. Steve empile m ($m \geq 1$) pierres indifférenciables sur un carrelage $n \times n$. Chaque case du carrelage peut recevoir un nombre arbitraire de pierres. Après avoir terminé d'empiler ses pierres d'une façon quelconque, il peut ensuite accomplir des déplacements de pierres comme suit. On considère quatre cases qui forment les coins d'un rectangle, c.-à-d. les positions $(i, k), (i, l), (j, k), (j, l)$, k, j, k et l étant des entiers tels que $1 \leq i, j, k, l \leq n$, $i < j$ et $k < l$. Un déplacement de pierres consiste à enlever une pierre de chacune des cases (i, k) et (j, l) et les ajouter aux cases respectives (i, l) et (j, k) ou à enlever une pierre de chacune des cases (i, l) et (j, k) et les ajouter aux cases respectives (i, k) et (j, l).

On dit que deux façons d'empiler les pierres sont équivalentes si une façon peut être obtenue à partir de l'autre par une série de déplacements de pierres.

Combien y a-t-il de façons non équivalentes d'empiler les pierres sur le carrelage?

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(1), p. 9–11.

OC211. Find maximum value of

$$|a^2 - bc + 1| + |b^2 - ac + 1| + |c^2 - ba + 1|$$

where a, b, c are real numbers in the interval $[-2, 2]$.

Originally problem 1 of day 2 of 2013 Kazakhstan National Olympiad Grade 11.

Editor's Note. This problem is a duplicate from OC179. The editor used the question from two different Kazakhstan Olympiad contests but did not realize that the question could be repeated across grades. Since the editor received different solutions to this problem as opposed to the first version, the editor will include one here. My apologies.

We received 3 correct submissions. We present the solution by Šefket Arslanagić.

Let $f(a, b, c) = |a^2 - bc + 1| + |b^2 - ac + 1| + |c^2 - ba + 1|$. By symmetry, we may also suppose that $a \geq b \geq c$. Notice immediately that $f(a, b, c) = f(-a, -b, -c)$. By using these two properties, we can assume without loss of generality that at least two values are positive , so $a \geq b \geq 0$.

Further, we can show that $f(a, b, |c|) \leq f(a, b, -|c|)$. This follows since

$$|a^2 - b|c| + 1| \leq a^2 + b|c| + 1 \qquad \text{and} \qquad |b^2 - a|c| + 1| \leq b^2 + a|c| + 1.$$

Hence, since we are looking for the maximum value, we may assume that $a \geq b \geq 0 \geq c$. Next, we show that $f(a, b, c) \leq f(a, b, -2)$. This inequality is equivalent to showing that

$$a^2 - bc + 1 + b^2 - ac + 1 + |c^2 - ab + 1| \leq a^2 + 2b + 1 + b^2 + 2a + 1 + |4 - ab + 1|$$

which is equivalent to

$$|c^2 - ab + 1| \leq 5 - ab + (2 + c)(a + b).$$

If $c^2 - ab + 1 \geq 0$, then the equality is exact by adding the inequalities $c^2 \leq 4$ and $0 \leq (2 + c)(a + b)$. If $c^2 - ab + 1 \leq 0$, then the above becomes via a sequence of if and only if statements

$$\begin{aligned} 2ab &\leq 6 + c^2 + (2 + c)(a + b) \\ 2ab - 2a - 2b + 2 &\leq 8 + c^2 + c(a + b) \\ 2(a - 1)(b - 1) &\leq 8 + c^2 + c(a + b). \end{aligned}$$

This last inequality is true since

$$2(a - 1)(b - 1) \leq 2 \leq 4 + 4 + c^2 + c(a + b).$$

The last inequality holds since

$$4 + c^2 + c(a+b) \geq 4|c| + c(a+b) \geq (a+b)|c| + c(a+b) \geq 0$$

using the fact that $(2-c)^2 \geq 0$ in the first inequality. Therefore, it suffices to find the maximum of the function $f(a,b,-2)$ where $a \geq b \geq 0$. This reduces to finding the maximum of

$$f(a,b,-2) = a^2 + b^2 - ab + 2a + 2b + 7.$$

with $a \in [0,2]$ and $b \in [0,2]$. This maximum must occur when either $a = b$, $b = 0$ or $a = 2$ (by say Calculus). Checking each of these cases reveals that the maximum value is

$$f(2,2,-2) = |4+4+1| + |4+4+1| + |4-4+1| = 19.$$

OC212. Let $ABCDE$ be a pentagon inscribe in a circle (O). Let $BE \cap AD = T$. Suppose the parallel line with CD which passes through T cuts AB, CE at X, Y. If ω is the circumcircle of triangle AXY then prove that ω is tangent to (O).

Originally problem 3 from level X of the 2013 Romanian National Olympiad.

We received 3 correct submissions. We present the solution by Andrea Fanchini.

We have that $\angle AEC = \angle ADC$ because both are inscribed in the same arc of circle. Then since the lines XY and CD are parallel, we have also $\angle ADC = \angle DTY$. Similarly $\angle ECD = \angle TYC = \angle EAD$ and therefore the quadrilateral $ATYE$ is cyclic.

Now if we draw a tangent AW to the circle (O), we have

$$\angle WAB = \angle AEB = \angle AET,$$

but the points A, T, Y and E are concyclic, so we have also

$$\angle AET = \angle AYT = \angle AYX.$$

Therefore AW is also tangent to circle ω completing the proof.

OC213. Suppose $p > 3$ is a prime number and

$$S = \sum_{2 \leq i < j < k \leq p-1} ijk.$$

Prove that $S + 1$ is divisible by p.

Originally problem 4 of the 2013 Indonesian Mathematical Olympiad.

We received 2 correct submissions. We present the solution by Michel Bataille.

From Fermat's Little Theorem, each element of $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$ is a root of the polynomial $x^p - x$ of $\mathbb{Z}_p[x]$. Thus, $x^p - x = x(x-1)(x-2)\cdots(x-(p-1))$ in $\mathbb{Z}_p[x]$. Since $x^p - x = x(x^{p-1}-1)$, it follows that $x^{p-1} - 1 = (x-1)(x-2)\cdots(x-(p-1))$ in $\mathbb{Z}_p[x]$.

This said, let $p(x)$ be the polynomial $(x-2)(x-3)\cdots(x-(p-1))$ of $\mathbb{Z}_p[x]$. Since $(x-1)p(x) = x^{p-1} - 1$, we see that

$$p(x) = x^{p-2} + x^{p-3} + \cdots + x + 1. \tag{1}$$

However, we also have

$$p(x) = x^{p-2} - e_1 x^{p-3} + e_2 x^{p-4} - e_3 x^{p-5} + \cdots - e_{p-2} \tag{2}$$

with, modulo p,

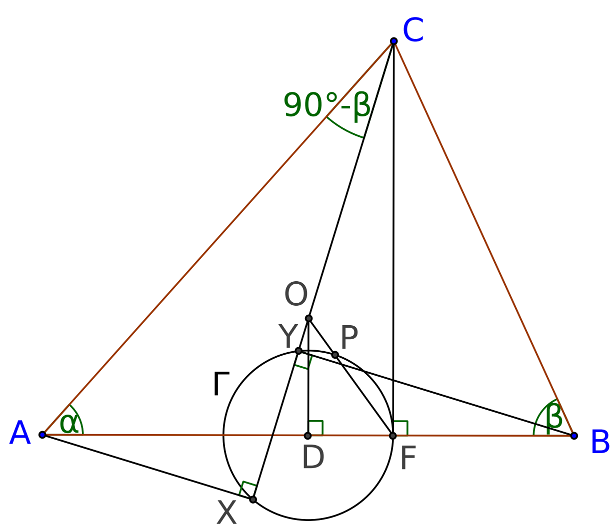
$$e_1 \equiv \sum_{i=2}^{p-1} i, \;\; e_2 \equiv \sum_{2 \le i < j \le p-1} i.j, \;\; e_3 \equiv \sum_{2 \le i < j < k \le p-1} ijk, \ldots$$

Comparing (1) and (2) yields $1 \equiv -e_3 \pmod{p}$. Since $S = e_3$, we conclude that $S + 1 \equiv 0 \pmod{p}$, that is, p divides $S + 1$.

OC214. Let ABC be an acute-angled triangle with $AC \neq BC$, and let O be the circumcentre and F the foot of the altitude through C. Furthermore, let X and Y be the feet of the perpendiculars dropped from A and B respectively to (the extension of) CO. The line FO intersects the circumcircle of FXY a second time at P. Prove that $OP < OF$.

Originally problem 6 of the 2013 South Africa National Olympiad.

We received 2 correct submissions. We present the solution by Oliver Geupel.



Let us denote $\alpha = \angle BAC$ and $\beta = \angle CBA$.

In the isosceles triangle AOC we have $\angle ACO = 90^\circ - \angle COA/2 = 90^\circ - \beta$. As a consequence, in the right triangle AXC we have $CX = AC \sin\beta$. Similarly $CY = BC \sin\alpha$. The altitude from point C in the triangle ABC has length $CF = AC \sin\alpha = BC \sin\beta$. We obtain

$$CF^2 = CX \cdot CY,$$

where the right-hand side is the power of point C with respect to the circumcircle Γ of triangle FXY. Hence the line CF is a tangent to the circle Γ.

Therefore the centre of the circle Γ lies on the line AB. Also the centre of the circle Γ lies on the perpendicular bisector p of the segment XY. Since $AC \neq BC$, the line p is not parallel to the line AB so that the lines p and AB are concurrent. The midpoint D of side AB lies on the line p because it is equidistant from the lines AX and BY. Thus, the point D is the centre of the circle Γ.

In the right triangle DFO we have $OD < OF$. The power of point O with respect to the circle Γ is

$$OF \cdot OP = OD^2 - DF^2.$$

Consequently,

$$OP = \frac{OD^2 - DF^2}{OF} < \frac{OF^2}{OF} = OF.$$

OC215. Let $n > 1$ be an integer. The first n primes are $p_1 = 2, p_2 = 3, \ldots, p_n$. Set $A = p_1^{p_1} p_2^{p_2} ... p_n^{p_n}$. Find all positive integers x, such that $\dfrac{A}{x}$ is even, and $\dfrac{A}{x}$ has exactly x divisors.

Originally problem 6 from day 2 of the 2013 South East Mathematical Olympiad.

We present the solution by Konstantine Zelator. There were no other submissions.

First, we claim that the only solution is $x = p_1 p_2 ... p_n$. Note that this is a solution. It suffices to show it is the only one. Since x is a divisor of A, we have that

$$x = \prod_{i=1}^{n} p_i^{e_i} \qquad \text{where } 0 \leq e_i \leq p_i$$

Now, e_1 cannot be 2 since then A/x is odd so $e_1 = 0$ or $e_1 = 1$. Assume towards a contradiction that $e_1 = 0$. Then

$$\frac{A}{x} = 4 \cdot \prod_{i=2}^{n} p_i^{p_i - e_i}$$

and the number of divisors this number has (which by the problem statement is equal to x) is

$$\prod_{i=2}^{n} p_i^{e_i} = x = 3 \prod_{i=2}^{n} (p_i - e_i + 1)$$

Now, if $e_2 = 0$, then the left hand side above is not divisible by 3 but the right hand side is, a contradiction. If $e_2 = 1$, then the left hand side above is divisible by 1 and the right hand side is divisible by at least 3^2 coming from $3(4 - e_1) = 3^2$, another contradiction. If $e_2 = 2$, then the right hand side is even from the $(4 - e_1) = 2$ term but the left hand side is odd, again a contradiction. Lastly, if $e_3 = 3$, then note that $n \geq 3$ must be true by inspection. The remaining terms on the right must all be odd numbers since the left hand side is odd and thus, since each p_i is an odd prime for $i \geq 3$, we have that $(p_i - e_i + 1)$ is odd and hence each e_i is odd. Thus $e_i \geq 1$ for all $i \geq 3$. Hence, since $p_i^{e_i} \geq p_i \geq (p_i - e_i + 1)$, we see that

$$x = \prod_{i=2}^{n} p_i^{e_i} = 3^3 \prod_{i=3}^{n} p_i^{e_i} > 3 \cdot (4-3) \cdot \prod_{i=3}^{n} (p_i - e_i + 1) = x$$

which is a contradiction. Thus $e_1 \neq 0$ and hence $e_1 = 1$. Then as before,

$$\frac{A}{x} = 2 \cdot \prod_{i=2}^{n} p_i^{p_i - e_i}$$

and the number of divisors this number has (which by the problem statement is equal to x) is

$$2 \cdot \prod_{i=2}^{n} p_i^{e_i} = x = 2 \prod_{i=2}^{n} (p_i - e_i + 1).$$

Simplifying gives

$$\prod_{i=2}^{n} p_i^{e_i} = \prod_{i=2}^{n} (p_i - e_i + 1).$$

As before, each of the e_1 terms must be odd otherwise the right hand side is even. Hence $e_i \geq 1$ for all $i \geq 2$. As before, $p_i^{e_i} \geq p_i \geq (p_i - e_i + 1)$ with equality holding if and only if $p_i^{e_i} + e_i = p_i + 1$ and thus, since $e_i \geq 1$, equality holds if and only if $e_i = 1$ for all $i \geq 2$. Thus, $x = p_1 p_2 ... p_n$ completing the proof.

FOCUS ON...

No. 21

Michel Bataille

The Product of Two Reflections in the Plane

Introduction

In this number, we exploit the two following theorems as a tool for geometrical properties or problems. $\mathbf{R}_n$ denotes the reflection in the line n.

(a) If the lines ℓ and m are parallel, then $\mathbf{R}_m \circ \mathbf{R}_\ell = \mathbf{T}_{2\overrightarrow{u}}$, the translation by vector $2\overrightarrow{u}$ where $\overrightarrow{u}$ is the vector orthogonal to ℓ and m such that $\mathbf{T}_{\overrightarrow{u}}(\ell) = m$.

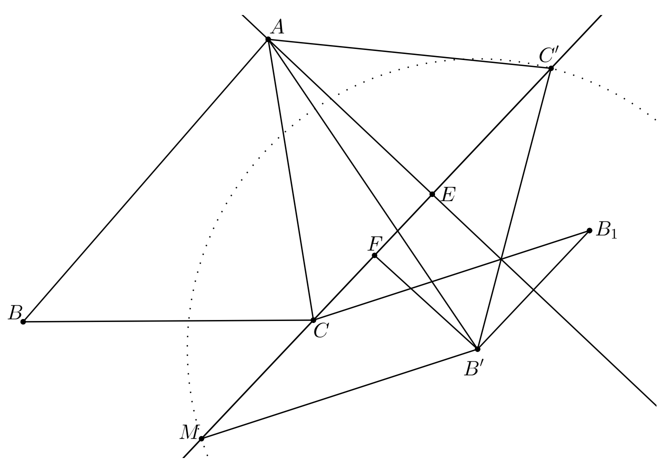
(b) If the lines ℓ and m intersect in O, then $\mathbf{R}_m \circ \mathbf{R}_\ell = \boldsymbol{\rho}_{O,2\theta}$, the rotation with centre O and angle 2θ where θ is the angle such that $\boldsymbol{\rho}_{O,\theta}(\ell) = m$.

Illustrating (a) and (b)

Our first problem offers an opportunity of using both (a) and (b):

> Let ABC be a triangle and $\boldsymbol{\rho}$ be any rotation with centre A. Let $B' = \boldsymbol{\rho}(B), C' = \boldsymbol{\rho}(C)$, and $B_1 = \mathbf{R}_{CA}(B)$. The circle with centre B' and radius $B'C'$ intersects CC' again at M. Show that MCB_1B' is a parallelogram.

Let E be the midpoint of CC'. From (b), $\mathbf{R}_{AE} \circ \mathbf{R}_{CA}$ is a rotation with centre A. In addition, it transforms C into $\mathbf{R}_{AE}(C) = C'$ (since $AC' = AC$, AE is the perpendicular bisector of CC'). Thus, $\mathbf{R}_{AE} \circ \mathbf{R}_{CA} = \boldsymbol{\rho}$ and so $B' = \mathbf{R}_{AE}(B_1)$.



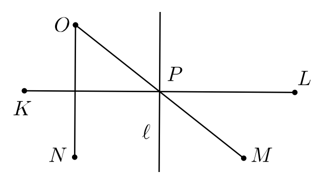
In a similar way, if F is the midpoint of MC', we have $M = \mathbf{R}_{FB'}(C')$. Now, $\mathbf{R}_{AE} \circ \mathbf{R}_{FB'} = \mathbf{T}_{\overrightarrow{B'B_1}}$ (from (a)) and so $\mathbf{T}_{\overrightarrow{B'B_1}}(M) = \mathbf{R}_{AE}(C') = \mathbf{R}_{AE} \circ \boldsymbol{\rho}(C) = \mathbf{R}_{CA}(C) = C$. Thus $\overrightarrow{MC} = \overrightarrow{B'B_1}$ and the result follows.

When ℓ and m are perpendicular

This is the simplest case of (b): the product $\mathbf{R}_m \circ \mathbf{R}_\ell$ is the half-turn about the point O of intersection of ℓ and m. The same is true of $\mathbf{R}_\ell \circ \mathbf{R}_m$. By way of illustration, consider the gist of problem **OC24** [2011 : 275 ; 2012 : 180]:

> Let P be the midpoint of the line segment KL and O be any point not on the line KL. Then, if M is the symmetric of O about P and N the reflection of O in KL, the points K, L, M, N are concyclic.

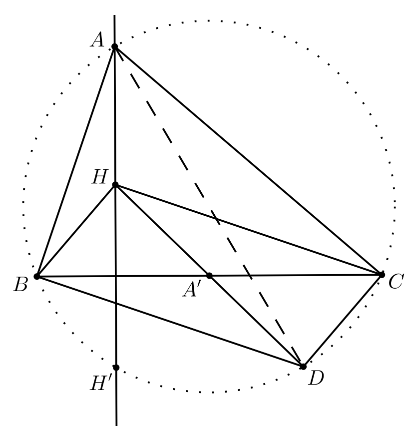
Let $\boldsymbol{\rho}_P$ be the half-turn about P. Then $N = \mathbf{R}_{KL} \circ \boldsymbol{\rho}_P(M)$ that is, $N = \mathbf{R}_\ell(M)$ where ℓ is the perpendicular to KL at P (since $\boldsymbol{\rho}_P = \mathbf{R}_{KL} \circ \mathbf{R}_\ell$). Now, let $\mathcal{C}$ be the circle through K, L, M. As the perpendicular bisector of KL, the line ℓ is a diameter of $\mathcal{C}$, hence an axis of symmetry of $\mathcal{C}$. Since M is on $\mathcal{C}$, $N = \mathbf{R}_\ell(M)$ is on $\mathcal{C}$ as well, and so K, L, M, N are concyclic.



Interestingly, this result readily leads to an unusual proof of the following well-known property of the orthocentre H of a triangle ABC: the reflections of H in the sides of the triangle lie on its circumcircle Γ.

For example, let us show that the reflection H' of H in BC is on Γ.

Let A' be the midpoint of BC. From what we have just proved, it is sufficient to show that the symmetric D of H about A' is on Γ.

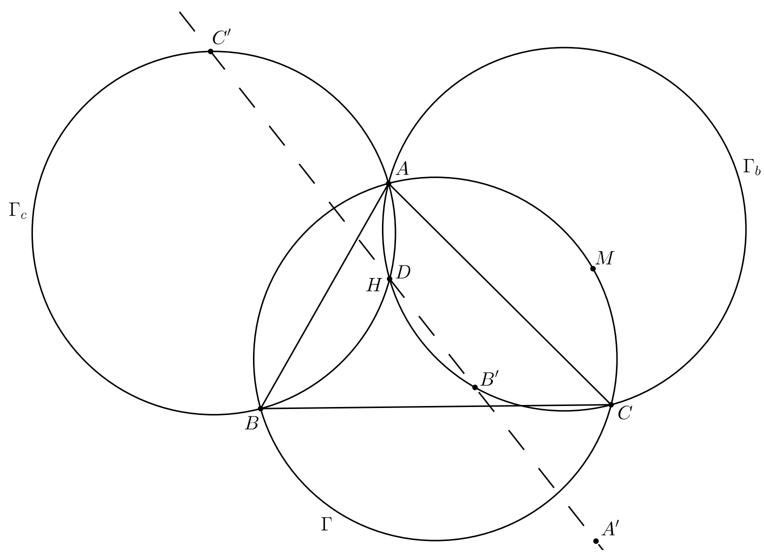


Since BC and HD have the same midpoint, $BHCD$ is a parallelogram. Thus DC is parallel to BH, hence is perpendicular to CA. Similarly, DB is perpendicular to BA and so the circle with diameter AD, which passes through A, B, and C, coincides with Γ. Thus D is on Γ.

Another unusual proof of a well-known property

> Let M be a point of the circumcircle Γ of a triangle ABC and let $A' = \mathbf{R}_{BC}(M)$, $B' = \mathbf{R}_{CA}(M)$, $C' = \mathbf{R}_{AB}(M)$. Then A', B', C' are collinear on a line through the orthocentre H (the Steiner line associated with M).

We discard the obvious case when M is a vertex of the triangle and assume that $M \neq A, B, C$.



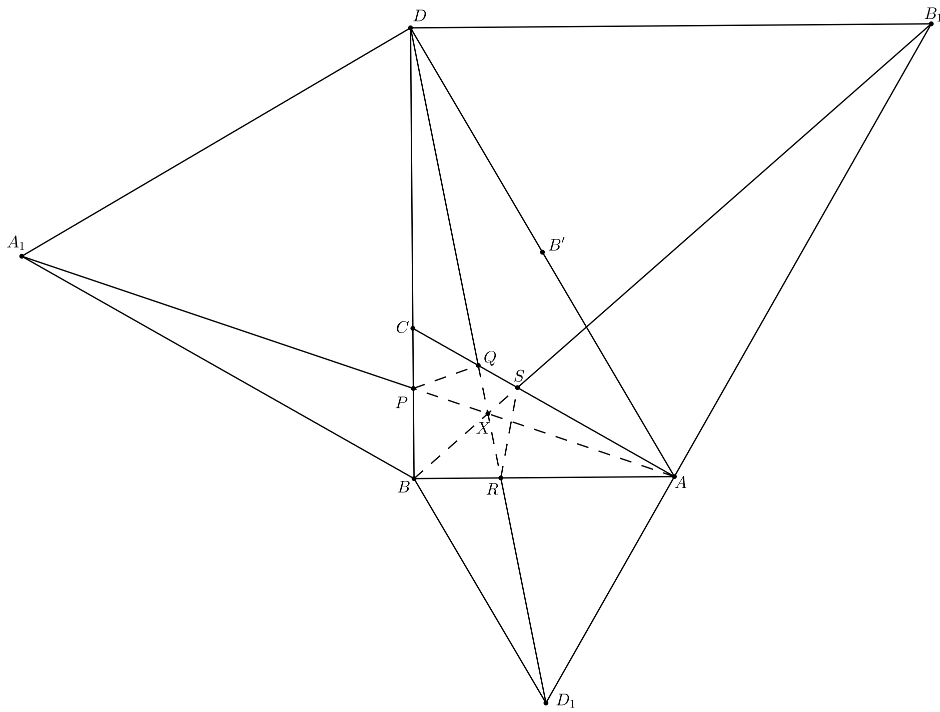
Let Γ_b and Γ_c be the circumcircles of $\Delta AB'C$ and $\Delta AC'B$, respectively. Then $\Gamma_b = \mathbf{R}_{CA}(\Gamma)$ (since Γ is also the circumcircle of ΔAMC) and, similarly, $\Gamma_c = \mathbf{R}_{AB}(\Gamma)$. Let D be the point of intersection other than A of Γ_b and Γ_c. The product $\mathbf{R}_{AB} \circ \mathbf{R}_{CA}$ is a rotation with centre A transforming Γ_b into Γ_c. A rotation being a spiral similarity, it follows from the main result of Focus On... No 12 that $\mathbf{R}_{AB}(C) = \mathbf{R}_{AB} \circ \mathbf{R}_{CA}(C)$ is the point of intersection other than D of the line CD and Γ_c. Thus, CD is perpendicular to AB. In the same way, BD is perpendicular to AC and consequently D coincides with the orthocentre H of ΔABC. To conclude, it suffices to add that since $\mathbf{R}_{AB} \circ \mathbf{R}_{CA}(B') = \mathbf{R}_{AB}(M) = C'$, the points B', C', H are collinear. The same being true of A', B', H and of C', A', H (similarly), we conclude that A', B', C', H are collinear.

Reflections and billiards

A billiards table is a good place for observing reflections. Successive reflections in the cushions are sometimes necessary to score a point! The following problem (a part of *American Mathematical Monthly* problem 10749 posed in 1999) is likely to appeal to the reader who is also a billiards player.

> Let ABC be a triangle with a right angle at B and an angle of $\pi/6$ at

A. Consider a billiard path in the triangle that begins at A, reflects successively off side BC at P, off side AC at Q, off side AB at R, off side AC at S, and then ends at B. Show that AP, QR, and SB are concurrent at a point X and that the angles formed at X measure $\pi/3$.



Without loss of generality, we suppose that the triangle ACB has positive orientation and introduce the following points: $B' = \mathbf{R}_{CA}(B)$, the point D of intersection of the lines BC and AB', and the vertices A_1, B_1, D_1 of equilateral triangles A_1BD, B_1DA, D_1AB constructed outward ΔABD. From the billiard path as described, we have $\mathbf{R}_{BC}(AP) = PQ$, $\mathbf{R}_{CA}(PQ) = QR$, $\mathbf{R}_{AB}(QR) = RS$, and $\mathbf{R}_{CA}(RS) = SB$ so that

$$SB = (\mathbf{R}_{CA} \circ \mathbf{R}_{AB} \circ \mathbf{R}_{CA}) \circ \mathbf{R}_{BC}(AP) = \mathbf{R}_{AB'} \circ \mathbf{R}_{BC}(AP) \qquad (1)$$

(using the easily checked fact that $\mathbf{R}_m \circ \mathbf{R}_\ell \circ \mathbf{R}_m$ is the reflection in the line $\mathbf{R}_m(\ell)$).

Since $\mathbf{R}_{AB'} \circ \mathbf{R}_{BC}$ is the rotation $\boldsymbol{\rho}_{D,\pi/3}$, the point $B_1 = \boldsymbol{\rho}_{D,\pi/3}(A)$ is on the line SB. Likewise, from $AP = \boldsymbol{\rho}_{D,-\pi/3}(SB)$, we obtain that A_1 is on AP. Furthermore, (1) yields $\mathbf{R}_{AB} \circ \mathbf{R}_{CA}(SB) = \mathbf{R}_{CA} \circ \mathbf{R}_{BC}(AP)$, that is, $QR = \boldsymbol{\rho}_{C,2\pi/3}(AP)$ and therefore the points $\boldsymbol{\rho}_{C,2\pi/3}(A) = D$ and $\mathbf{R}_{AB} \circ \mathbf{R}_{CA}(B) = \boldsymbol{\rho}_{A,\pi/3}(B) = D_1$ lie on the line QR. In conclusion, the lines AP, QR, and SB coincide with the lines AA_1, DD_1, and BB_1, respectively. This answers the questions since these lines are concurrent at the Fermat point (say X) of ΔABD, from which each side subtends an angle of $120°$ (a well-known result).

Exercises

Our first exercise is problem **2439** [1999 : 238 ; 2000 : 241]. Three solutions were featured and the reader is asked to find a fourth one! The second exercise is problem **2485** [1999 : 431 ; 2000 : 508], slightly modified. Of course, solutions should use reflections.

1. Suppose that $ABCD$ is a square with side a. Let P and Q be points on sides BC and CD, respectively, such that $\angle PAQ = 45^\circ$. Let E and F be the intersections of PQ with AB and AD, respectively. Prove that $AE + AF \geq 2\sqrt{2}a$. [Hint: first show that $\mathbf{R}_{AP}(B) = \mathbf{R}_{AQ}(D)$.]

2. Let $ABCD$ be a convex quadrilateral with $AB = BC = CD$ and such that AD and BC are not parallel. Let P be the intersection of the diagonals AC and BD. If $AP : BD = DP : AC$, prove that $AB \perp CD$. [Hint: if ℓ, m, n are the perpendicular bisectors of BC, CA, BD, respectively, and O is the circumcentre of ΔBPC, consider $\mathbf{R}_{OC} \circ \mathbf{R}_\ell$ and $\mathbf{R}_n \circ \mathbf{R}_m$.]

The use of coordinate systems before Descartes

Florin Diacu

1 Introduction

It is basic knowledge among mathematicians that René Descartes introduced rectangular coordinate systems in the 17th century, thus creating analytic geometry. This accomplishment was an early example of how a branch of mathematics, algebra, can explore another area, geometry. The division of mathematics into various fields was in its early stages at that time, so this way of looking at Descartes's achievement is a rather contemporary point of view, which reflects our current (though perpetually changing) classification of mathematical subjects.

Descartes published his ideas on this topic in 1637 in the appendix *La géométrie* (see Figure 1) to his book *Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences* (Discourse on the Method of Reasoning Well and Seeking Truth in the Sciences), a philosophical and autobiographical treatise that had a strong influence on the further development of human thought. The concept of *Cartesian coordinate system* was coined to honour Descartes, whose Latanized name was Cartesius. Students of mathematics who have struggled to solve geometry problems using the ancient synthetic methods can get a good feeling about the power of this approach, the more so if they can surmount the computational hurdles that occur sometimes. Apart from this merit, Descartes's contribution opened the way towards laying bridges between branches of mathematics that had developed separately, a feat that is highly regarded among researchers today because it emphasizes the unity of our field of knowledge, in spite of the often overlapping areas in which we divide it.

Some critics, however, contested Descartes's priority for designing this method, since Pierre Fermat had the same idea a few years before Descartes. A quote from E.T. Bell, an American mathematician and colourful historian of mathematics, sheds more light on this issue, [1]: "There is no doubt that he [Fermat] preceded Descartes. But as his work of about 1629 was not communicated to others until 1636, and was published posthumously only in 1679, it could not possibly have influenced Descartes in his own invention, and Fermat never hinted that it had." Since similar simultaneous independent contributions to science and mathematics are common, a phenomenon that has been thoroughly researched, we will not pursue this topic.

What we actually want to emphasize here is a little known fact, namely that hidden rectangular coordinate systems have been used in geometry since antiquity. We will further present two examples of how this idea was applied long before Descartes made it universally known. The first is due to the Greek geometer and astronomer

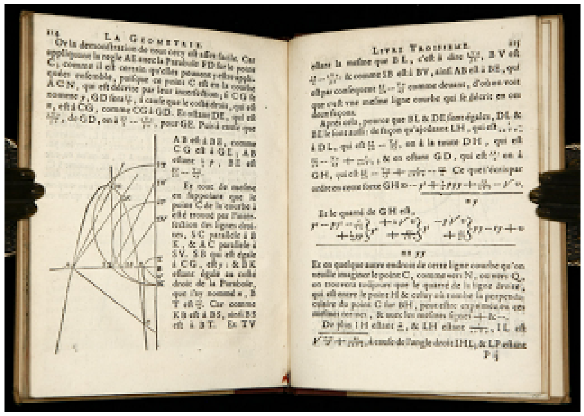


Figure 1: An extract from *La géométrie*, the appendix of *Discourse de la méthode* in which Cartesian coordinates were introduced.

Apollonius of Perga (c. 262–c. 190 BC), whereas the second is drawn from the work of the Persian mathematician Umar ibn Ibrahim al-Khayyami (AD 1048–1131), known among literati as the poet Omar Khayyam.

2 The conics of Apollonius

Apollonius was born in Perga, Pamphylia, today Murtina in Antalya, Turkey, around 262 BC. He studied in Alexandria under disciples of Euclid, where he learned geometry and astronomy, subjects he later taught there. We know very little about his life. The information we can trust appears in the various prefaces of his fundamental treatise, Κωνικά (Conics), whose original was lost (as it happened with most ancient manuscripts), but which reached us mostly through Arabic translations.

It is intuitively easy to understand why the ellipse, parabola, and hyperbola are called conics: if we intersect a cone of two sheets with a plane at suitable angles relative to the symmetry axis, we obtain these curves (along with circles, points, and two intersecting straight lines). Apollonius was the first to coin their names, though other Greek mathematicians studied them earlier. In his treatise, he approached them without using algebra, as we define this branch of mathematics today. So how did he do it? Unlike polygons or circles, conics are difficult to understand without analytic geometry, which describes them through simple quadratic equations. A close look at the work of Apollonius, however, reveals that he used camouflaged coordinate systems, and in the absence of algebraic tools he employed a geometric language. So, with his smart and systematic approach in the study of conics, he anticipated the work of Descartes by almost two millennia.

As an example, we consider now the case of the parabola (see Figure 2), but let us remark that Apollonius used this method to understand hyperbolas and ellipses too (see, e.g., [2]). Let us take a circular cone (Figure 2, left), which we intersect with a plane parallel with the generatrix AC. The result of this intersection is what we call a parabola, the curve FNH. Apollonius wanted to characterize this curve in order to distinguish it from ellipses and hyperbolas. For this purpose he derived the "symptom," which in our modern language is nothing but the relationship between the abscissa and the ordinate of an arbitrary point on the curve. So let M be such a point. A plane through M orthogonal to the symmetry axis intersects the cone along the circle of diameter DE. Denote by O the point at the base of the perpendicular from M to DE. Then ON and OM are also perpendicular to each other. Apollonius wanted to find the relationship between $ON =: x$ and $OM =: y$. Indeed, had he placed a coordinate system with the origin at N in the plane of the parabola (Figure 2, right), M would have had coordinates (x, y), the standard notation we use today to write the equation of the parabola.

Next, Apollonius started his geometric reasoning. He noticed that in the circle of diameter DE he could write that

$$OM^2 = OD \cdot OE. \tag{1}$$

Then he constructed the segment NK perpendicular to ON in the plane of the parabola (Figure 2, right), taking its length such that

$$\frac{NK}{NA} = \frac{BC}{AC} \cdot \frac{BC}{AB}, \tag{2}$$

a brilliant choice he probably reached after long reflections on this problem. By similarity of triangles, he noticed on one hand that

$$\frac{BC}{AC} = \frac{OD}{ON}, \tag{3}$$

and on the other hand, after manipulating some proportions, that

$$\frac{BC}{AB} = \frac{OE}{NA}. \tag{4}$$

Substituting the results from (3) and (4) into (2), he obtained that

$$\frac{NK}{NA} = \frac{OE \cdot OD}{NA \cdot ON}.$$

Multiplying NK/NA by ON/ON, i.e. by 1, he got

$$\frac{NK}{NA} = \frac{NK \cdot ON}{NA \cdot ON}.$$

Comparing the numerators of the above two relationships, he concluded that

$$OE \cdot OD = NK \cdot ON,$$

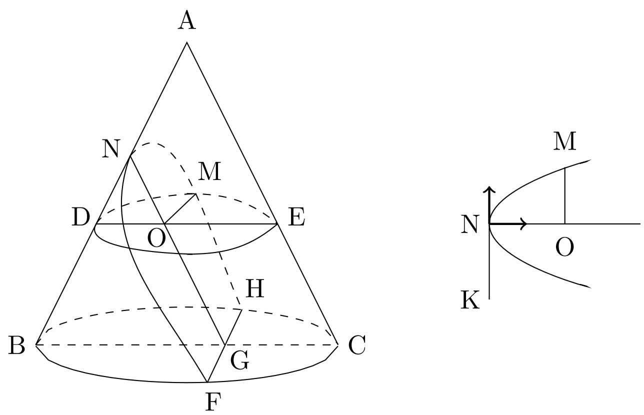


Figure 2: Left: the intersection of the cone with a plane parallel with the generatrix AC yields a parabola. Right: the parabola FNH viewed in its own plane.

which in light of (1) can be written as

$$OM^2 = NK \cdot ON.$$

If we denote the length of NK by p, which is the same as the length of the latus rectum (defined below), then the above relationship becomes

$$y^2 = px,$$

easy to recognize as the equation of a parabola, albeit written in nonstandard form.

So in spite of having no algebra tools at his disposal, Apollonius actually obtained the equation of the parabola. He even had a value for the length of the latus rectum, defined as the chord through the focus of the parabola parallel with the directrix. All in all, he had a very good understanding of conics, which he achieved through the same kind of analysis Descartes's did, but using a very basic language.

3 The cubic equation of al-Khayyami

The mathematician, philosopher, and poet al-Khayyami was born in AD 1048 in Nishapur, Persia, today a city of the same name in northeastern Iran. During his long life he contributed to mathematics, mechanics, astronomy, geography, mineralogy, and philosophy, and left an important body of poetry, which is still widely read. Five centuries before Cardano's formula for the cubic equation was discovered, al-Khayyami approached the problem with innovative methods. Like Apollonius, he used hidden Cartesian coordinates to express his solutions.

At that time all coefficients of an equation were considered positive. (Although negative numbers had been used in China no later than the AD 100, see [2],

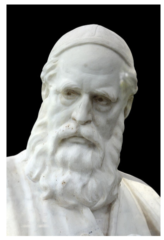


Figure 3: A bust of al-Khayyami in Nishapur, Iran.

p. 198, they became universally accepted only after the 16th century.) So al-Khayyami first listed all possible cubic equations: one binomial, $x^3 = d$; six trinomial, $x^3 + bx^2 = d$, $x^3 + d = bx^2$, $x^3 = bx^2 + d$, $x^3 + cx = d$, $x^3 + d = cx$, $x^3 = cx + d$; and seven tetranomial, $x^3 + bx^2 + cx = d$, $x^3 + bx^2 + d = cx$, $x^3 + cx + d = bx^2$, $x^3 = bx^2 + cx + d$, $x^3 + bx^2 = cx + d$, $x^3 + cx = bx^2 + d$, $x^3 + d = bx^2 + cx$. Then he provided for each of them a solution obtained with the help of a conic section, as we will show in the following example. Let us consider the case of

$$x^3 + cx = d. \tag{5}$$

He first took a segment $AB = \sqrt{c}$ and drew $EB = d/c$ perpendicular to AB (see Figure 4). Then he considered the parabola

$$y = \frac{1}{\sqrt{c}}x^2$$

through D and B, and constructed the semicircle EDB given by

$$\left(x - \frac{d}{2c}\right)^2 + y^2 = \frac{d^2}{4c^2}. \tag{6}$$

The semicircle and the parabola intersect at D. Taking $DF = y_0$ perpendicular to EB, he obtained the segment $FB = x_0$ and claimed that its length, x_0, is a solution of equation (5). Notice the hidden upside-down frame centred at B that occurs here, which allowed us to write the above equations in modern language.

Following [2], let us now prove that x_0 is indeed a solution of equation (5). Since D is on the semicircle, we have

$$x_0\left(\frac{d}{c} - x_0\right) = y_0^2,$$

an equation equivalent to (6) that can be also expressed as

$$\frac{x_0}{y_0} = \frac{y_0}{\frac{d}{c} - x_0}. \tag{7}$$

But D is also on the parabola, so we have

$$x_0^2 = \sqrt{c}y_0,$$

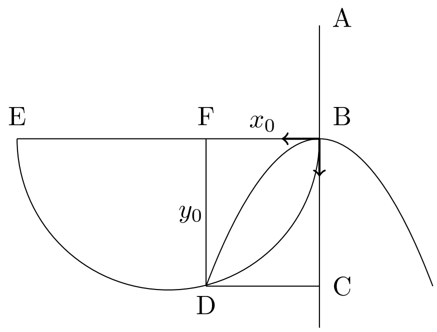


Figure 4: A solution al-Khayyami gave for the equation he described as "a cube and sides are equal to a number," i.e. $x^3 + ax = b$.

which is equivalent to

$$\frac{\sqrt{c}}{x_0} = \frac{x_0}{y_0}.$$

Squaring this equality and using (7), we obtain

$$\frac{c}{x_0^2} = \frac{x_0^2}{y_0^2} = \frac{y_0^2}{\left(\frac{d}{c} - x_0^2\right)^2} = \frac{y_0}{\frac{d}{c} - x_0} \cdot \frac{x_0}{y_0} = \frac{x_0}{\frac{d}{c} - x_0}.$$

But comparing the first and last expressions in the above sequence, we see that x_0 verifies equation (5), a remark that completes the argument.

Of course, al-Khayyami did not write his proof as presented here. He used only words and figures, as it can be seen in one of his manuscripts (Figure 5). Even equation (5) appeared without symbols. He described it as: "a cube and sides are equal to a number." Nevertheless, he did algebra, a branch of mathematics that had recently formed. As in the case of Apollonius, coordinate systems were essential for obtaining these results.

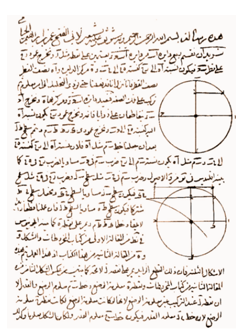


Figure 5: A page of al-Khayyami's manuscript "Cubic equation and intersection of conic sections," kept at the University of Tehran, Iran.

4 Conclusions

The modern language of mathematics is fairly recent. Even Isaac Newton's *Principia*, originally published in 1687, used geometry to express derivatives and integrals. The advantage of introducing specialized notation is tremendous for progress in research, but it also makes understanding difficult for the uninitiated. Since plain language and figures was all that mathematicians used in antiquity and the Middle Ages, the development of the field was very slow. It started booming only after the 18th century, when modern symbolism allowed a better understanding of mathematical objects and of the relationships that govern them.

Nevertheless, the ancients already had some of the ideas that later crystallized and formed new branches of mathematics. Apart from the coordinate system mentioned above, a well-known example is that of the integral, which lies at the foundation of calculus. Some two millennia before the birth of mathematical analysis, Archimedes approximated the length of a circle with regular polygons of many sides, a procedure that leads to an integral in the limit. In a way, these examples resemble the wheel, which was invented in prehistoric times. We added ball bearings and tires, improved and extended their use, but wheels are still based on the same idea. So we should not be too surprised that the ancient Greeks employed the rectangular coordinate system about 2,200 years ago, long before Fermat and Descartes understood its value.

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[2] V.J. Katz, *A History of Mathematics*, 3rd. ed., Addison-Wesley, New York, 2009.

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PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **January 1, 2017**, *although late solutions will also be considered until a solution is published.*

The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

4121. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let s be a fixed real number such that $s \geq 1$. Let a, b, c and d be non-negative numbers that satisfy $a+b+c+d = 4s$ and $ab+bc+cd+da+ac+bd = 6$. Express the minimum value of the product $abcd$ in terms of s.

4122. *Proposed by Daniel Sitaru.*

Prove that for $n \in \mathbb{N}$, the following holds

$$\Big(\frac{e^n-1}{n}\Big)^{2n+1} \leq \frac{(e-1)(e^2-1)(e^3-1)\cdot \ldots \cdot (e^{2n}-1)}{(2n)!}.$$

4123. *Proposed by Michel Bataille.*

In 3-dimensional Euclidean space, a line ℓ is perpendicular to the plane of the acute triangle $A'B'C'$ at its orthocentre K. Let A, B, C be the midpoints of $B'C', C'A'$ and $A'B'$, respectively. Show that $BC > KA$ and if D on ℓ satisfies $KD = \sqrt{BC^2 - KA^2}$, that the tetrahedron $ABCD$ is isosceles. (A tetrahedron is called isosceles if its opposite edges are congruent.)

4124. *Proposed by George Apostolopoulos.*

Let A_1, B_1 and C_1 be points on the sides BC, CA and AB of a triangle ABC such that

$$\frac{A_1B}{A_1C} = \frac{B_1C}{B_1A} = \frac{C_1A}{C_1B} = k.$$

Prove that

$$\left(\frac{AA_1}{BC}\right)^2 + \left(\frac{BB_1}{CA}\right)^2 + \left(\frac{CC_1}{AB}\right)^2 \geq \left(\frac{3k}{k^2+1}\right)^2 \left(\frac{2r}{R}\right)^4,$$

where R and r are the circumradius and the inradius of ABC, respectively.

4125. *Proposed by Stephen Su and Cheng-Shyong Lee.*

Start with a triangle $A_1A_2A_3$ in the Euclidean plane and three nonzero real numbers ℓ_1, ℓ_2, ℓ_3. Define M_k and C_k to be points on the line $A_{k+1}A_{k+2}$ such that

$$\frac{A_{k+1}M_k}{M_kA_{k+2}} = \ell_k \quad \text{and} \quad C_kM_{k+1}||A_kA_{k+1}, \quad k = 1, 2, 3$$

(with subscripts reduced modulo 3 and distances taken to be signed, so that M_k is between A_{k+1} and A_{k+2} precisely when ℓ_k is positive). Denote by R_k the point where C_kM_{k+1} intersects $C_{k+1}M_{k+2}$, $k = 1, 2, 3$. Show that

$$\frac{[R_1R_2R_3]}{[A_1A_2A_3]} = \left(\frac{2 + \ell_1 + \ell_2 + \ell_3 - \ell_1\ell_2\ell_3}{(1 + \ell_1)(1 + \ell_2)(1 + \ell_3)}\right)^2,$$

where square brackets denote area.

4126. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute-angled triangle. Prove that

$$\sum_{\text{cyc}} \frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\sqrt{1 - \tan\frac{A}{2}\tan\frac{B}{2}}} \geq \sqrt{\frac{3}{2}}.$$

4127. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

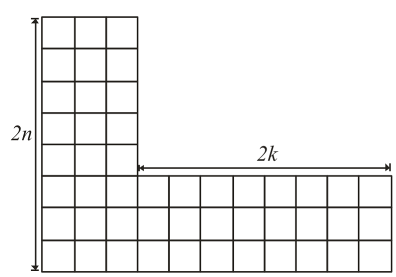
Calculate

$$\lim_{n\to\infty} \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\left(\frac{x}{n}\right) dx,$$

where $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a continuous function.

4128. *Proposed by Valcho Milchev and Tsvetelina Karamfilova.*

Let A_n be the number of domino tilings of a rectangular $3 \times 2n$ grid. Let $L(2n, 2k)$ be the number of domino tilings of the grid composed of two rectangular grids of dimensions $3 \times 2n$ and $3 \times 2k$ with $n \geq 2$ and $k \geq 1$ (depicted below):



Prove that $L(2n, 2n) = A_{2n}$.

4129. *Proposed by Lorean Saceanu.*

Let ABC be an acute-angle triangle and let $\gamma = 3(2-\sqrt{3})$. Prove that

$$\sec A + \sec B + \sec C \geq \gamma + \tan A + \tan B + \tan C.$$

4130. *Proposed by Leonard Giugiuc.*

Let a, b and c be nonnegative real numbers such that $a+b+c = ab+bc+ac > 0$. Prove that

$$\sqrt[n]{a} + \sqrt[n]{b} + \sqrt[n]{c} \geq 2\sqrt[n]{2}$$

for any integer $n \geq 3$ and determine the case for equality to hold.

. . . .

4121. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Soit s une nombre réel tel que $s \geq 1$. Soient a, b, c et d des nombres non négatifs tels que $a+b+c+d = 4s$ et $ab+bc+cd+da+ac+bd = 6$. Déterminer la valeur minimale du produit $abcd$ en terme de s.

4122. *Proposé par Daniel Sitaru.*

Démontrer que pour $n \in \mathbb{N}$, l'inégalité suivante tient

$$\left(\frac{e^n-1}{n}\right)^{2n+1} \leq \frac{(e-1)(e^2-1)(e^3-1)\cdot \ldots \cdot (e^{2n}-1)}{(2n)!}.$$

4123. *Proposé par Michel Bataille.*

Dans l'espace euclidien 3-dimensionnel, une ligne ℓ est perpendiculaire au plan du triangle aigu $A'B'C'$, à son orthocentre K. Soient A, B et C les mi points de $B'C', C'A'$ et $A'B'$, respectivement. Démontrer que $BC > KA$ et que si D sur ℓ satisfait $KD = \sqrt{BC^2 - KA^2}$, alors le tétraèdre $ABCD$ est isocèle. (Un tétraèdre est dit isocèle si ses côtés opposés sont congrus.)

4124. *Proposé par George Apostolopoulos.*

Soient A_1, B_1 et C_1 des points sur les côtés BC, CA et AB du triangle ABC, tels que

$$\frac{A_1B}{A_1C} = \frac{B_1C}{B_1A} = \frac{C_1A}{C_1B} = k.$$

Démontrer que

$$\left(\frac{AA_1}{BC}\right)^2 + \left(\frac{BB_1}{CA}\right)^2 + \left(\frac{CC_1}{AB}\right)^2 \geq \left(\frac{3k}{k^2+1}\right)^2\left(\frac{2r}{R}\right)^4,$$

où R et r sont, respectivement, les rayons des cercles circonscrit et inscrit du triangle ABC.

4125. *Proposé par Stephen Su et Cheng-Shyong Lee.*

Soient un triangle $A_1A_2A_3$ dans le plan euclidien et trois nombres réels non nuls ℓ_1, ℓ_2, ℓ_3. Définissons M_k et C_k comme étant les points sur la ligne $A_{k+1}A_{k+2}$ tels que

$$\frac{A_{k+1}M_k}{M_kA_{k+2}} = \ell_k \quad \text{and} \quad C_kM_{k+1}||A_kA_{k+1}, \quad k = 1, 2, 3$$

(les indices étant réduits modulo 3 et les distances comportant un signe, de façon à ce que M_k se trouve entre A_{k+1} et A_{k+2} précisément lorsque ℓ_k est positif). Dénotons par R_k le point d'intersection de C_kM_{k+1} et $C_{k+1}M_{k+2}$, $k = 1, 2, 3$. Démontrer que

$$\frac{[R_1R_2R_3]}{[A_1A_2A_3]} = \left(\frac{2 + \ell_1 + \ell_2 + \ell_3 - \ell_1\ell_2\ell_3}{(1 + \ell_1)(1 + \ell_2)(1 + \ell_3)}\right)^2,$$

où les crochets dénotent une surface.

4126. *Proposé par Mihaela Berindeanu.*

Soit ABC un triangle aigu. Démontrer que

$$\sum_{\text{cyc}} \frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\sqrt{1 - \tan\frac{A}{2}\tan\frac{B}{2}}} \geq \sqrt{\frac{3}{2}}.$$

4127. *Proposé par D. M. Bătineţu-Giurgiu et Neculai Stanciu.*

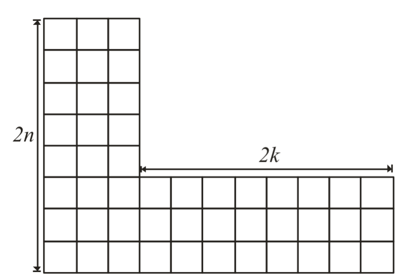
Calculer

$$\lim_{n\to\infty} \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\left(\frac{x}{n}\right) dx,$$

où $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ est une fonction continue.

4128. *Proposé par Valcho Milchev et Tsvetelina Karamfilova.*

Soit A_n le nombre de carrelages par dominos d'une grille $3 \times 2n$. Soit $L(2n, 2k)$ le nombre de carrelages par dominos d'une grille formée de deux grilles rectangulaires de tailles $3 \times 2n$ et $3 \times 2k$, pour $n \geq 2$ et $k \geq 1$, comme ci-bas:



Démontrer que $L(2n, 2n) = A_{2n}$.

4129. *Proposé par Lorean Saceanu.*

Soit ABC un triangle aigu et soit $\gamma = 3(2-\sqrt{3})$. Démontrer que

$$\sec A + \sec B + \sec C \geq \gamma + \tan A + \tan B + \tan C.$$

4130. *Proposé par Leonard Giugiuc.*

Soient a, b et c des nombres réels non négatifs tels que $a+b+c = ab+bc+ac > 0$. Démontrer que

$$\sqrt[n]{a} + \sqrt[n]{b} + \sqrt[n]{c} \geq 2\sqrt[n]{2}$$

pour tout entier $n \geq 3$ et déterminer tout cas où l'égalité tient.

Math Quotes

To the pure geometer the radius of curvature is an incidental characteristic - like the grin of the Cheshire cat. To the physicist it is an indispensable characteristic. It would be going too far to say that to the physicist the cat is merely incidental to the grin. Physics is concerned with interrelatedness such as the interrelatedness of cats and grins. In this case the "cat without a grin" and the "grin without a cat" are equally set aside as purely mathematical phantasies.

Sir Arthur Eddington, "The Expanding Universe." Cambridge University Press, 1988.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(3), p. 119–123.

4021. *Proposed by Arkady Alt.*

Let $(\overline{a}_n)_{n\geq 0}$ be a sequence of Fibonacci vectors defined recursively by $\overline{a}_0 = \overline{a}, \overline{a}_1 = \overline{b}$ and $\overline{a}_{n+1} = \overline{a}_n + \overline{a}_{n-1}$ for all integers $n \geq 1$. Prove that, for all integers $n \geq 1$, the sum of vectors $\overline{a}_0 + \overline{a}_1 + \cdots + \overline{a}_{4n+1}$ equals $k\overline{a}_i$ for some i and constant k.

We received nine correct solutions. We present the solution by David Stone and John Hawkins (joint).

We shall prove that $\bar{a}_0 + \bar{a}_1 + \cdots + \bar{a}_{4n+1} = L_{2n+1}\bar{a}_{2n+2}$, where L_k is the kth Lucas number. We use some easily proven results. Here, F_k is the kth Fibonacci number.

1. $F_0 + F_1 + \cdots + F_m = F_{m+2} - 1$.
2. $F_{4n+2} = L_{2n+1}F_{2n+1}$
3. $F_{4n+3} = L_{2n+1}F_{2n+2} + 1$
4. $\bar{a}_k = F_{k-1}\bar{a}_0 + F_k\bar{a}_1$ for $k \geq 1$.

Therefore,

$$\begin{aligned}
\sum_{k=0}^{m} \bar{a}_k &= \bar{a}_0 + \sum_{k=1}^{m} (F_{k-1}\bar{a}_0 + F_k\bar{a}_1) \\
&= \bar{a}_0 + \left(\sum_{k=1}^{m} F_{k-1}\right)\bar{a}_0 + \left(\sum_{k=1}^{m}\right)\bar{a}_1 \\
&= \bar{a}_0 + (F_{m+1} - 1)\bar{a}_0 + (F_{m+2} - 1)\bar{a}_1 \\
&= F_{m+1}\bar{a}_0 + F_{m+2}\bar{a}_1 - \bar{a}_1 \\
&= \bar{a}_{m+2} - \bar{a}_1
\end{aligned}$$

Hence, with $m = 4n + 1$,

$$\begin{aligned}
\sum_{k=0}^{4n+1} \bar{a}_k = \bar{a}_{4n+3} - \bar{a}_1 &= F_{4n+2}\bar{a}_0 + F_{4n+3}\bar{a}_1 - \bar{a}_1 \\
&= (L_{2n+1}F_{2n+1})\bar{a}_0 + (L_{2n+1}F_{2n+2})\bar{a}_1 \\
&= L_{2n+1}(F_{2n+1}\bar{a}_0 + F_{2n+2}\bar{a}_1) \\
&= L_{2n+1}\bar{a}_{2n+2}.
\end{aligned}$$

Editor's Comments. Various solvers expressed the coefficient of $\bar{a}_{2n+2}$ as L_{2n+1}, $F_{2n} + F_{2n+2}$, and $\frac{F_{4n+2}}{F_{2n+1}}$ and variations of these resulting from different indexing of the Fibonacci sequence. Swylan pointed out that if the word 'constant' is interpreted to mean 'independent of n', then the claim of the problem is false. Perhaps 'scalar' would have been a better word.

4022. *Proposed by Leonard Giugiuc.*

In a triangle ABC, let internal angle bisectors from angles A, B and C intersect the sides BC, CA and AB in points D, E and F and let the incircle of ΔABC touch the sides in M, N, and P, respectively. Show that

$$\frac{PA}{PB} + \frac{MB}{MC} + \frac{NC}{NA} \geq \frac{FA}{FB} + \frac{DB}{DC} + \frac{EC}{EA}.$$

We received eleven submissions, of which seven were correct, two were incorrect, and two were incomplete. We present the solution by Titu Zvonaru.

Define $x = NA = PA, \quad y = PB = MB,$ and $z = MC = NC$; then

$$BC = y + z, \quad CA = z + x, \quad \text{and} \quad AB = x + y.$$

By the angle bisector theorem we have

$$\frac{FA}{FB} = \frac{CA}{BC} = \frac{z+x}{y+z}, \quad \frac{DB}{DC} = \frac{AB}{CA} = \frac{x+y}{z+x}, \quad \text{and} \quad \frac{EC}{EA} = \frac{BC}{AB} = \frac{y+z}{x+y}.$$

We therefore have to prove that for positive real numbers x, y, z,

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{z+x}{y+z} + \frac{x+y}{z+x} + \frac{y+z}{x+y}\ . \tag{1}$$

After clearing denominators what we must prove reduces to

$$x^2y^4 + y^2z^4 + z^2x^4 + x^3y^3 + y^3z^3 + z^3x^3 \ \geq\ x^3yz^2 + x^2y^3z + xy^2z^3 + 3x^2y^2z^2. \tag{2}$$

By the AM-GM inequality we have

$$\begin{array}{rcl} x^2y^4 + y^2z^4 + z^2x^4 & \geq & 3x^2y^2z^2, \\ x^3y^3 + z^3x^3 + z^3x^3 & \geq & 3x^3yz^2, \\ y^3z^3 + x^3y^3 + x^3y^3 & \geq & 3x^2y^3z, \ \text{ and} \\ z^3x^3 + y^3z^3 + y^3z^3 & \geq & 3xy^2z^3\ , \end{array}$$

which together imply that (2) holds. Equality holds if and only if $x = y = z$, which immediately implies that the triangle is equilateral.

Editor's Comments. Most submissions reduced our problem to equation (1), but then algebra caused difficulties with two of the faulty arguments. The solution from Salem Malikić neatly avoided calculations by remarking that (1) is known;

see, for example, the Belarussian IMO Team preparation tests of 1997, where calculations are much simplified by exploiting the cyclic symmetry of the inequalities. Beware, however, that one must not assume noncyclic symmetry (as in one of the incomplete submissions).

4023. *Proposed by Ali Behrouz.*

Find all functions $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}$ with $x > y$, we have

$$f\left(\frac{x}{x-y}\right) + f(xf(y)) = f(xf(x)).$$

We received two correct submissions. We present the solution by Joseph Ling.

It is easy to see that $f(x) = \dfrac{1}{x}$ $\forall x > 0$ satisfies

$$f\left(\frac{x}{x-y}\right) + f(xf(y)) = f(xf(x)) \tag{1}$$

whenever $0 < y < x$. We show that there are no other solutions $f : (0, \infty) \to (0, \infty)$ to (1).

First, we note that f is one-to-one. For if $0 < y < x$ are such that $f(y) = f(x)$, then (1) implies that $f\left(\frac{x}{x-y}\right) = 0$, which is impossible.

Second, we note that $f(x) \le \frac{1}{x}$ for all $x > 0$. For if there exists $x > 0$ such that $f(x) > \frac{1}{x}$, then $y = x - \frac{1}{f(x)}$ satisfies $0 < y < x$. But then $\frac{x}{x-y} = xf(x)$ and (1) will imply that $f(xf(y)) = 0$, which is impossible.

Now, suppose that $0 < y_1 < y_2$. Consider $x = y_2 + \frac{1}{f(y_1)}$. Then $0 < y_1 < y_2 < x$ and (1) implies that

$$f\left(\frac{x}{x-y_2}\right) + f(xf(y_2)) = f(xf(x)) = f\left(\frac{x}{x-y_1}\right) + f(xf(y_1)). \tag{2}$$

By the definition of x, $\frac{x}{x-y_2} = xf(y_1)$. So, (2) is reduced to $f(xf(y_2)) = f\left(\frac{x}{x-y_1}\right)$. Since f is one-to-one, we have $xf(y_2) = \frac{x}{x-y_1}$, and so, $x = y_1 + \frac{1}{f(y_2)}$. Using this and the definition of x, we see that $\frac{1}{f(y_1)} - y_1 = \frac{1}{f(y_2)} - y_2$. Since y_1 and y_2 are arbitrary, $\frac{1}{f(y)} - y$ is independent of y, and so, it must be some constant, say, c. That is,

$$f(y) = \frac{1}{y+c}$$

for all $y > 0$.

It remains to show that $c = 0$. Since $f(x) \le \frac{1}{x}$ for all x, $c \ge 0$. Furthermore, if

$c > 0$, then for any $0 < y < x$, we have

$$\begin{aligned} f\left(\frac{x}{x-y}\right) + f\left(xf\left(y\right)\right) &= \frac{x-y}{x+c\left(x-y\right)} + \frac{y+c}{x+c\left(y+c\right)} \\ &> \frac{x-y}{x+c\left(x+c\right)} + \frac{y+c}{x+c\left(x+c\right)} = \frac{x+c}{x+c\left(x+c\right)} \\ &= f\left(xf\left(x\right)\right), \end{aligned}$$

a contradiction to (1). So, $c = 0$ and our proof is complete.

4024. *Proposed by Leonard Giugiuc.*

Let a, b, c and d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$abc + abd + acd + bcd + 4 \geq a + b + c + d$$

and determine when equality holds.

We received three correct solutions. We present the solution by Titu Zvononu, modified by the editor.

We consider several cases separately.

Case 1. If $a, b, c, d \geq 0$, then by the Cauchy-Schwarz Inequality, we have

$$(a + b + c + d)^2 \leq (1^2 + 1^2 + 1^2 + 1^2)(a^2 + b^2 + c^2 + d^2) = 16.$$

Thus, $a + b + c + d \leq 4$ from which the result follows immediately.

Case 2. If $a, b, c, d \leq 0$, we set $x = -a$, $y = -b$, $z = -c$, and $t = -d$. Then, $x, y, z, t \geq 0$ and we would now like to show that

$$-(xyz + xyt + xzt + yzt) + 4 \geq -(x + y + z + t) \quad \text{or}$$

$$2(x + y + z + t) - (xyz + xyt + xzt + yzt) + 4 \geq x + y + z + t. \tag{1}$$

Since we know $x^2 + y^2 + z^2 + t^2 = 4$, we have by the first case (with a, b, c, d replaced with x, y, z, t, respectively and symbolically), that

$$x + y + z + t \leq 4. \tag{2}$$

Then,

$$\begin{aligned} &4(x + y + z + t) - 2(xyz + xyt + xzt + yzt) \\ &= (x^2 + y^2 + z^2 + t^2)(x + y + z + t) - 2(xyz + xyt + xzt + yzt) \\ &= x(y - z)^2 + y(x - t)^2 + z(x - t)^2 + t(y - z)^2 + x(x^2 + t^2) \\ &\qquad\qquad + y(y^2 + z^2) + z(z^2 + y^2) + t(t^2 + x^2) \\ &\geq 0, \end{aligned}$$

which together with (2) implies (1).

Case 3. If one of a, b, c, d is nonnegative and the other three are nonpositive. Due to the symmetry in the given equation and the one we wish to prove, we may assume that $a \geq 0$ and $b, c, d \leq 0$. Here we let $y = -b$, $z = -c$, and $t = -d$. Then, $a, y, z, t \geq 0$ with $a^2 + y^2 + z^2 + t^2 = 4$ and we wish to prove $ayz + ayt + azt - yzt + 4 \geq a - y - z - t$ or

$$2(y + z + t) + ayz + ayt + azt - yzt + 4 \geq a + y + z + t. \tag{3}$$

Now,

$$\begin{aligned}
&4(y + z + t) + 2(ayz + ayt + azt - yzt) \\
&= (a^2 + y^2 + z^2 + t^2)(y + z + t) + 2(ayz + ayt + azt - yzt) \\
&\geq y(z^2 + t^2) - 2yzt \\
&= y(z - t)^2 \\
&\geq 0,
\end{aligned}$$

so, $2(y + z + t) + ayz + ayt + azt - yzt + 4 \geq 4$, thus, establishing (3), as desired, since $a + y + z + t \leq 4$ (see Case 1).

Case 4. If $a, b \geq 0$ and $c, d \leq 0$, we set $z = -c$ and $t = -d$. Then $a, b, z, t \geq 0$ such that $a^2 + b^2 + z^2 + t^2 = 4$ and we would like to show that

$$-abz - abt + azt + bzt + 4 \geq a + b - z - t \quad \text{or}$$

$$2(z + t) - abz - abt + azt + bzt + 4 \geq a + b + z + t. \tag{4}$$

Now,

$$\begin{aligned}
&4(z + t) - 2(abz + abt - azt - bzt) \\
&= (a^2 + b^2 + z^2 + t^2)(z + t) - 2(abz + abt - azt - bzt) \\
&= z(a - b)^2 + t(a - b)^2 + (z + t)(z^2 + t^2) + 2azt + 2bzt \\
&\geq 0.
\end{aligned}$$

So, $2(z+t) - abz - abt + azt + bzt \geq 4$, thus establishing (4) since $a + b + z + t \leq 4$.

Case 5. If $a, b, c \geq 0$ and $d \leq 0$, we set $t = -d$. Then $a, b, c, t \geq 0$ with $a^2 + b^2 + c^2 + t^2 = 4$ and we would like to show that

$$abc - abt - act - bct + 4 \geq a + b + c - t \quad \text{or}$$

$$2t + abc - abt - act - bct + 4 \geq a + b + c + t. \tag{5}$$

Since $a + b + c + t \leq 4$, to establish (5), it suffices to show that

$$(a^2 + b^2 + c^2 + t^2)t + 2(abc - abt - act - bct) \geq 0. \tag{6}$$

We let L denote the left-hand side of (6) and assume, without loss of generality, that $a \geq b \geq c$. Note that

$$\begin{aligned} L &= t(b-c)^2 + t(a-t)^2 + 2a(t-b)(t-c) \qquad (7)\\ \text{and} \quad L &= t(a-b)^2 + t(c-t)^2 + 2c(t-a)(t-b). \qquad (8) \end{aligned}$$

If $t \leq b$, then from (7) we can see that $L \geq 0$ and if $t \geq b$, then $L \geq 0$ from (8). Hence, we can conclude that (6) is true, as desired.

Examining the five cases, it is readily seen that equality can only hold in Case 5 when $a = b = c = t$; that is, if and only if $(a, b, c, d) = (1, 1, 1, -1)$ and all its permutations.

4025. *Proposed by Dragoljub Miloşević.*

Prove that for positive numbers a, b and c, we have

$$\sqrt[3]{\left(\frac{a}{2b+c}\right)^2} + \sqrt[3]{\left(\frac{b}{2c+a}\right)^2} + \sqrt[3]{\left(\frac{c}{2a+b}\right)^2} \geq \sqrt[3]{3}.$$

We received eleven correct solutions. We present the solution by Salem Madikić.

Let $f(a, b, c)$ denote the left hand side of the given inequality. By the AM-GM Inequality, we have

$$\begin{aligned} \sqrt[3]{\left(\frac{a}{2b+c}\right)^2} = \frac{a}{\sqrt[3]{a(2b+c)^2}} &= \frac{a}{\sqrt[3]{9}\sqrt[3]{a \cdot \frac{2b+c}{3} \cdot \frac{2b+c}{3}}} \\ &\geq \frac{3a}{\sqrt[3]{9}\left(a + \frac{2b+c}{3} + \frac{2b+c}{3}\right)} \\ &= \frac{3\sqrt[3]{3}a}{3a+4b+2c}. \end{aligned}$$

Using similar inequalities involving the other two summands, we then have

$$f(a, b, c) \geq 3\sqrt[3]{3} \sum_{\text{cyc}} \frac{a}{3a+4b+2c}. \qquad (1)$$

Now, by the Cauchy-Schwarz Inequality, we have

$$\left(\sum \left(\sqrt{\frac{a}{3a+4b+2c}}\right)^2\right)\left(\sum \left(\sqrt{a(3a+4b+2c)}\right)^2\right) \geq \sum a^2.$$

So,

$$\sum \frac{a}{3a+4b+2c} \geq \frac{\sum a^2}{\sum a(3a+4b+2c)} = \frac{\sum a^2}{3\sum a^2 + 6\sum ab} = \frac{1}{3}. \qquad (2)$$

Substituting (2) into (1), $f(a, b, c) \geq \sqrt[3]{3}$ follows immediately.

To achieve equality, we must have $3a = 2b+c$, $3b = 2c+a$, and $3c = 2a+b$. Without loss of generality, we may assume that $\max\{a, b, c\} = a$. Then, $3a = 2b+c$ implies $2(a-b) + (a-c) = 0$, so $a = b = c$. Conversely, it is readily checked that if $a = b = c$, then equality holds.

Editor's Comments. Using convexity and Jensen's Inequality, Stadler proved that in general, $\sum \left(\frac{a}{2b+c}\right)^k \geq 3^{1-k}$ for all $k \geq 0$.

4026. *Proposed by Roy Barbara.*

Prove or disprove the following property: if r is any non-zero rational number, then the real number $x = (1+r)^{1/3} + (1-r)^{1/3}$ is irrational.

We received two correct solutions. We present the solution by Joseph DiMuro.

Assume both r and x are rational numbers with $r \neq 0$. Setting $y_1 = (1+r)^{1/3}$ and $y_2 = (1-r)^{1/3}$ we can show that

$$\frac{x^3-2}{3x} = y_1 y_2.$$

That means that y_1 and y_2 are the two solutions for y of $y^2 - xy + \frac{x^3-2}{3x} = 0$. But using the quadratic formula, we also obtain

$$y = \frac{x \pm \sqrt{x^2 - \frac{4x^3-8}{3x}}}{2} = \frac{x}{2} \pm \sqrt{\frac{8-x^3}{12x}}.$$

This shows that if x is rational then y_1 and y_2 are contained in quadratic extensions of $\mathbb{Q}$. On the other hand, if r is rational then $y_1 = (1+r)^{1/3}$ and $y_2 = (1-r)^{1/3}$ are contained in cubic extensions of $\mathbb{Q}$ as well. Both of these can only be true if y_1 and y_2 are rational numbers themselves.

Let $r = \frac{a}{b}$, where a, b are relatively prime non-zero integers. Then $y_1 = (\frac{b+a}{b})^{1/3}$ and $y_2 = (\frac{b-a}{b})^{1/3}$. The fractions $\frac{b+a}{b}$ and $\frac{b-a}{b}$ are in lowest terms, so for them to be perfect cubes, their numerators and denominators must be perfect cubes. Then we have an arithmetic progression $b-a, b, b+a$ of cubes, which is known to be impossible (e.g. see P. Dénes, Über die Diophantische Gleichung $x^l + y^l = cz^l$, *Acta. Math.* **88** (1952) 241-251).

Editor's Comments. The statement that there is no arithmetic progression of three cubes can be proven with elementary number theory and is an interesting exercise.

4027. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a+b+c = 3$. Prove that

$$\frac{ab}{a+ab+b} + \frac{bc}{b+bc+c} + \frac{ac}{a+ac+c} \leq 1.$$

We received 24 submissions of which 22 were correct and complete. We present 5 solutions, each of them insightful in a different way.

Solution 1, by Ali Adnan.

Observe that
$$\sum_{cyc} \frac{ab}{a+ab+b} \le 1 \quad \Longleftrightarrow \quad \sum_{cyc} \frac{9}{\frac{a+b}{ab}+1} \le 9. \tag{1}$$

Now, from Cauchy-Schwarz Inequality,
$$\frac{9}{\frac{a+b}{ab}+1} = \frac{9}{\frac{1}{a}+\frac{1}{b}+1} \le a+b+1,$$

and adding up analogous such inequalities cyclically, (1) follows.

Solution 2, by Ali Adnan.

We note that the inequality is equivalent to
$$\textstyle\sum_{cyc} \frac{a+b}{a+b+ab} \ge 2 \quad \Longleftrightarrow \quad \sum_{cyc} \frac{1}{2+\frac{2ab}{a+b}} \ge 1,$$

which follows easily from the AM-HM and Cauchy-Schwarz Inequalities:
$$\textstyle\sum_{cyc} \frac{1}{2+\frac{2ab}{a+b}} \ge \sum_{cyc} \frac{1}{2+\frac{a+b}{2}} \ge \frac{(1+1+1)^2}{6+a+b+c} = 1,$$

thus completing the proof.

Solution 3, by Henry Ricardo.

We have
$$\begin{aligned} \sum_{cyclic} \frac{ab}{a+ab+b} &= \sum_{cyclic} \frac{1}{\frac{1}{b}+1+\frac{1}{a}} = \frac{1}{3}\sum_{cyclic} \frac{3}{\frac{1}{b}+1+\frac{1}{a}} \\ &\le \frac{1}{3}\sum_{cyclic} \frac{a+b+1}{3} = \frac{1}{3}\left(\frac{2(a+b+c)+3}{3}\right) = 1, \end{aligned}$$

where we have used the harmonic mean-arithmetic mean inequality.

Equality holds if and only if $a=b=c=1$.

Solution 4, by Salem Malikić.

Using the inequality between arithmetic and geometric mean for positive reals x and y we have
$$x+xy+y \ge 3\sqrt[3]{x^2y^2}$$
with equality if and only if $x=xy=y$, that gives $x=y=1$ and implying that
$$\frac{xy}{x+xy+y} \le \frac{\sqrt[3]{xy}}{3}.$$

Using this inequality we have

$$\frac{ab}{a+ab+b}+\frac{bc}{b+bc+c}+\frac{ca}{c+ca+a}\le\frac{\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca}}{3}.$$

Now, using Power-mean inequality, we have

$$\frac{\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca}}{3}\le\sqrt[3]{\frac{ab+bc+ca}{3}}\le\sqrt[3]{\frac{\frac{(a+b+c)^2}{3}}{3}}=1.$$

where in the last step we used the well known inequality

$$3(ab+bc+ca)\le(a+b+c)^2.$$

This completes our proof.

In order to achieve equality we must have $a=b=c=1$. It is easy to verify that this is indeed an equality case.

Solution 5, by Leonard Giugiuc.

The inequality is equivalent to

$$\frac{1}{\frac{1}{a}+\frac{1}{b}+1}+\frac{1}{\frac{1}{b}+\frac{1}{c}+1}+\frac{1}{\frac{1}{c}+\frac{1}{a}+1}\le 1.$$

By AM-HM Inequality,

$$\frac{1}{a}+\frac{1}{b}\ge\frac{4}{a+b},\qquad\frac{1}{b}+\frac{1}{c}\ge\frac{4}{b+c},\qquad\frac{1}{c}+\frac{1}{a}\ge\frac{4}{c+a}.$$

From here,

$$\frac{1}{\frac{1}{a}+\frac{1}{b}+1}+\frac{1}{\frac{1}{b}+\frac{1}{c}+1}+\frac{1}{\frac{1}{c}+\frac{1}{a}+1}\le\frac{1}{\frac{4}{a+b}+1}+\frac{1}{\frac{4}{b+c}+1}+\frac{1}{\frac{4}{c+a}+1}.$$

But

$$\frac{1}{\frac{4}{a+b}+1}+\frac{1}{\frac{4}{b+c}+1}+\frac{1}{\frac{4}{c+a}+1}=\frac{a+b}{a+b+4}+\frac{b+c}{b+c+4}+\frac{c+a}{c+a+4}.$$

Since the function $f(x)=\dfrac{x}{x+4}$ is concave if $x>0$, then by Jensen's Inequality we get

$$f(a+b)+f(b+c)+f(c+a)\le 3f\left(\frac{2(a+b+c)}{3}\right)=3f(2)=1.$$

So,

$$\frac{a+b}{a+b+4}+\frac{b+c}{b+c+4}+\frac{c+a}{c+a+4}\le 1.$$

4028. *Proposed by Michel Bataille.*

In 3-dimensional Euclidean space, a line ℓ meets orthogonally two distinct parallel planes $\mathcal{P}$ and $\mathcal{P}'$ at H and H'. Let r and r' be positive real numbers with $r \leq r'$; let $\mathcal{C}$ be the circle in $\mathcal{P}$ with center H, radius r, and let $\mathcal{C}'$ in $\mathcal{P}'$ be similarly defined. For a fixed point M' on $\mathcal{C}'$, find the maximum distance between the lines ℓ and MM' as M moves about the circle $\mathcal{C}$ (where the distance between two lines is the minimum distance from a point of one line to a point of the other).

We received four correct solutions and will feature two of them that are quite similar except that the first makes use of coordinates.

Solution 1, by Oliver Geupel.

We prove that the required maximum distance is r. We use Cartesian coordinates such that $H' = (0,0,0)$, $M' = (r', 0, 0)$, and $H = (0, 0, h)$ where $h \in \mathbb{R}$. For every point M on $\mathcal{C}$, the distance between ℓ and MM' is not greater than $|MH| = r$ (because that distance is, by definition, the length of the shortest among all segments joining a point of ℓ to a point of MM', which is therefore at most $|MH|$). Moreover, the distance between two non-intersecting lines is measured along a line that is perpendicular to both. Put

$$M = \left(\frac{r}{r'} \cdot r,\ \sqrt{1 - \frac{r^2}{r'^2}} \cdot r,\ h\right),$$

which is on $\mathcal{C}$. We have $\overrightarrow{HM} \cdot \overrightarrow{HH'} = 0$ and

$$\overrightarrow{M'M} \cdot \overrightarrow{HM} = \left(\frac{r^2}{r'} - r',\ \sqrt{1 - \frac{r^2}{r'^2}} \cdot r,\ h\right) \cdot \left(\frac{r^2}{r'},\ \sqrt{1 - \frac{r^2}{r'^2}} \cdot r,\ 0\right) = 0,$$

so that HM is perpendicular to both ℓ and MM'. Therefore the distance between the lines ℓ and MM' is $|HM| = r$, which completes the proof.

Solution 2, by Edmund Swylan.

For every point M on $\mathcal{C}$, let $\mathcal{Q}$ be the plane orthogonal to ℓ that contains a point P of MM' nearest to ℓ. Let $O, \mathcal{D}, \mathcal{D}', N, N'$ be the orthogonal projections of $\ell, \mathcal{C}, \mathcal{C}', M, M'$, respectively, onto $\mathcal{Q}$. The distance between the lines ℓ and MM' projects to $|PO|$. Our problem is thereby reduced to a 2-dimensional problem:

> Given circles $\mathcal{D}$ and $\mathcal{D}'$ in the same plane with common centre O and radii r and r', a fixed point N' on $\mathcal{D}'$, and a point N moving about $\mathcal{D}$, what is the maximum distance from O to NN'?

The answer is r.

For $r < r'$ the maximum is achieved if and only if NN' is tangent to $\mathcal{D}$. For $r = r'$ it is achieved if and only if $N = N'$ and the line NN' degenerates into a point which occurs when MM' is parallel to ℓ.

4029. *Proposed by Paul Bracken.*

Suppose $a > 0$. Find the solutions of the following equation in the interval $(0, \infty)$:

$$\frac{1}{x+1} + \sum_{n=1}^{\infty} \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} = x - a.$$

We received four correct solutions and will feature two different ones.

Solution 1. We present a composite of the very similar solutions by Arkady Alt and the proposer, Paul Bracken. Another similar solution was received from Oliver Geupel.

It is clear that

$$\frac{1}{x} - \frac{1}{x+1} = \frac{1}{x(x+1)}, \quad \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)(x+2)} = \frac{2}{x(x+1)(x+2)},$$

and

$$\frac{n!}{x(x+1)(x+2)\cdots(x+n)} - \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} = \frac{(n+1)!}{x(x+1)(x+2)\cdots(x+n+1)}.$$

It therefore follows by induction that

$$\frac{1}{x} - \frac{1}{x+1} - \sum_{k=1}^{n-1} \frac{k!}{(x+1)(x+2)\dots(x+k+1)} = \frac{n!}{x(x+1)(x+2)\dots(x+n)}.$$

However, for $x > 0$,

$$\lim_{n\to\infty} \frac{n!}{x(x+1)\dots(x+n)} = 0,$$

since

$$\frac{n!}{x(x+1)(x+2)\cdots(x+n)} = \frac{1}{x\,(x+1)\left(\frac{x}{2}+1\right)\cdots\left(\frac{x}{n}+1\right)}$$

and

$$(x+1)\left(\frac{x}{2}+1\right)\cdots\left(\frac{x}{n}+1\right) > 1 + x\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

Hence the left-hand side of the original equation is given by

$$\frac{1}{x+1} + \sum_{n=1}^{\infty} \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} = \frac{1}{x}.$$

Therefore the original equation is equivalent to $x^2 - ax - 1 = 0$. This quadratic equation has the following unique solution in $(0, \infty)$:

$$x_r = \frac{1}{2}(a + \sqrt{a^2+4}).$$

Solution 2, by Albert Stadler.

We note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} \\
&= \sum_{n=1}^{\infty} \frac{\Gamma(x+1)\Gamma(n+1)}{\Gamma(x+1)(x+2)\cdots(x+n+1)} \\
&= \sum_{n=1}^{\infty} \frac{\Gamma(x+1)\Gamma(n+1)}{\Gamma(x+n+1)} \\
&= \sum_{n=1}^{\infty} \beta(x+1,n+1) \\
&= \sum_{n=1}^{\infty} \int_0^1 t^x(1-t)^n\,dt \\
&= \int_0^1 t^{x-1}(1-t)\,dt \\
&= \frac{1}{x} - \frac{1}{x+1}, \quad x>0.
\end{aligned}$$

The original equation is therefore equivalent to $\frac{1}{x} = x - a$ or $x^2 - ax - 1 = 0$. This quadratic equation has exactly one positive root, which is $x_r = \frac{1}{2}(a + \sqrt{a^2+4})$.

4030. *Proposed by Paolo Perfetti.*

a) Prove that $4^{\cos t} + 4^{\sin t} \geq 5$ for $t \in [0, \frac{\pi}{4}]$.

b) Prove that $6^{\cos t} + 6^{\sin t} \geq 7$ for $t \in [0, \frac{\pi}{4}]$.

There were six submitted solutions for this problem, four of which were correct. We present the solution by Michel Bataille.

Lemma. Let $u(t) = \sin t\cos t(\cos t + \sin t)$. Then, u is an increasing function on $[0, \frac{\pi}{4}]$ with $u(0) = 0$ and $u(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$.

Proof. $u(0) = 0, u(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ are immediate. A simple calculation gives the derivative of u:

$$u'(t) = (\cos t - \sin t)((\cos t + \sin t)^2 + \sin t\cos t).$$

For $t \in (0, \frac{\pi}{4}), \cos t > \sin t$, hence $u'(t) > 0$ and so u is increasing on $[0, \frac{\pi}{4}]$. □

a) Let $f(t) = 4^{\cos t} + 4^{\sin t}$. We show that f is increasing on $[0, \frac{\pi}{4}]$ (the required result then follows since $f(0) = 5$). To this end, we prove that $f'(t) > 0$ for all $t \in (0, \frac{\pi}{4})$. We calculate

$$f'(t) = (\ln 4)4^{\cos t}\cos t\left(4^{\sin t - \cos t} - \frac{\sin t}{\cos t}\right)$$

so that it is sufficient to prove that $\phi(t) > 0$ for $t \in (0, \frac{\pi}{4})$ where

$$\phi(t) = (\sin t - \cos t)(\ln 4) - \ln(\sin t) + \ln(\cos t).$$

Now, we easily obtain $\phi'(t) = \frac{(\ln 4)u(t)-1}{\sin t \cos t}$ with, from the lemma,

$$(\ln 4)u(t) - 1 < \frac{\sqrt{2}\ln 4}{2} - 1 < 0.$$

Therefore, $\phi'(t) < 0$ for $t \in (0, \frac{\pi}{4})$ and $\phi(t) > \phi\left(\frac{\pi}{4}\right) = 0$, as desired.

b) Similarly, we introduce $g(t) = 6^{\cos t} + 6^{\sin t}$ whose derivative has the same sign as $\psi(t) = (\sin t - \cos t)(\ln 6) - \ln(\sin t) + \ln(\cos t)$. Here,

$$\psi'(t) = \frac{\ln 6}{\sin t \cos t} \cdot \left(u(t) - \frac{1}{\ln 6}\right),$$

and since $0 < \frac{1}{\ln 6} < \frac{\sqrt{2}}{2}$, $u(t) - \frac{1}{\ln 6}$ (and so $\psi'(t)$) vanishes at a unique t_0 in $(0, \frac{\pi}{4})$. From the lemma, we deduce that $\psi'(t) < 0$ if $0 < t < t_0$ and $\psi'(t) > 0$ if $t_0 < t < \frac{\pi}{4}$. Thus, ψ is decreasing on $(0, t_0]$ and increasing on $[t_0, \frac{\pi}{4})$. Since $\psi(\frac{\pi}{4}) = 0$, we must have $\psi(t_0) < 0$, and since $\lim_{t \to 0^+} \psi(t) = \infty$, we deduce that for some $\alpha \in (0, t_0)$, we have $\psi(t) > 0$ if $t \in (0, \alpha)$, $\psi(\alpha) = 0$ and $\psi(t) < 0$ if $t \in (\alpha, \frac{\pi}{4})$. Thus, $g'(t) > 0$ if $t \in (0, \alpha)$ and $g'(t) < 0$ if $t \in (\alpha, \frac{\pi}{4})$ and so $g(t) \geq (\min(g(0), g(\pi/4)) = 7$ for all $t \in [0, \frac{\pi}{4}]$.

Editor's Comments. It turns out that AM-GM is too weak to prove this inequality when used right at the beginning; the resulting right-hand-side is too small. However, one may use AM-GM in a step of the proof, as A. Stadler did, and have things work out well; the Stadler solution is an impressive use of Taylor series and clever bounds. As well, the 'general' inequality, $a^{\cos(t)} + a^{\sin(t)} \geq a + 1$, is not true over the required interval for every $a > 1$; plotting it for $a = 10$, for example, shows this.

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