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IN THIS ISSUE / DANS CE NUMÉRO

Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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THE CONTEST CORNER

No. 46

John McLoughlin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1 mars 2017**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

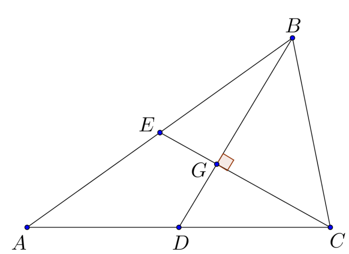
CC226. Dans le tableau qui suit, on a inscrit tous les produits de deux entiers positifs distincts de 1 à 100:

$$\begin{array}{ccccc} 1\cdot 2, & 1\cdot 3 & \dots & 1\cdot 99, & 1\cdot 100 \\ & 2\cdot 3 & \dots & 2\cdot 99, & 2\cdot 100 \\ & & \ddots & \vdots & \vdots \\ & & & & 99\cdot 100 \end{array}$$

Déterminer la somme de tous ces produits.

CC227. Supposons que $\{a_1, a_2, \dots\}$ est une suite géométrique de nombres réels. La somme des n premiers termes est dénotée S_n. Si $S_{10} = 10$ et $S_{30} = 70$, déterminer la valeur de S_{40}.

CC228. Dans le triangle ABC, on sait que $AB = 2\sqrt{13}$ et $AC = \sqrt{73}$, puis que E et D sont les mi points de AB et AC respectivement. De plus, BD est perpendiculaire à CE. Déterminer la longueur de BC.



CC229. Un magasin a en vente des objets aux prix de 10, 25, 50 et 70 sous. Si Sandrine achète 40 objets et dépense sept dollars, quel est le plus grand nombre

possible d'objets à 50 sous dont elle aurait pu faire l'achat ?

CC230. Deux amis se sont mis d'accord pour se rencontrer à la bibliothèque entre 13h00 et 14h00. Ils ont décidé d'attendre 20 minutes l'un pour l'autre. Quelle est la probabilité qu'ils se rencontreront si leurs arrivées sont aléatoires durant l'heure en question et si leurs moments d'arrivée sont indépendants ?

. . . .

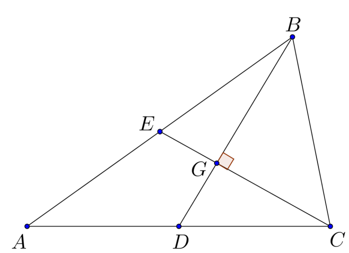
CC226. In the table below we write all the different products of two distinct counting numbers between 1 and 100:

$$\begin{array}{ccccc} 1\cdot 2, & 1\cdot 3 & \ldots & 1\cdot 99, & 1\cdot 100 \\ & 2\cdot 3 & \ldots & 2\cdot 99, & 2\cdot 100 \\ & & \ddots & \vdots & \vdots \\ & & & & 99\cdot 100 \end{array}$$

Find the sum of all of these products.

CC227. Suppose $\{a_1, a_2, \ldots\}$ is a geometric sequence of real numbers. The sum of the first n terms is S_n. If $S_{10} = 10$ and $S_{30} = 70$, determine the value of S_{40}.

CC228. In the triangle ABC, $AB = 2\sqrt{13}, AC = \sqrt{73}$, E and D are the midpoints of AB and AC, respectively. Furthermore, BD is perpendicular to CE. Find the length of BC.



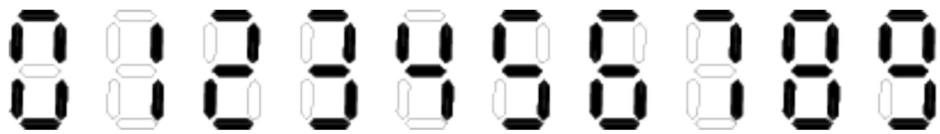
CC229. A store has objects that cost either 10, 25, 50, or 70 cents. If Sharon buys 40 objects and spends seven dollars, what is the largest quantity of the 50 cent items that could have been purchased?

CC230. Two friends agree to meet at the library between 1:00 P.M. and 2:00 P.M. Each agrees to wait 20 minutes for the other. What is the probability that they will meet if their arrivals occur at random during the hour and if the arrival times are independent?

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2015: 41(6), p. 234–235.

CC176. A digital clock shows 4 digits using the following patterns:



Mathew plays the following game: "We subtract the number of lighted segments from the number which is shown. We repeat the operation on the second number and so on ..." For example, since 1234 uses 16 segments, the second number would be $1234 - 16 = 1218$. After performing this operation two times, Mathew gets 2015. What was his starting number?

Originally Problem 14 of the Championnat International des Jeux Mathématiques et Logiques 2014-15.

We received four correct submissions. We present a solution based on the submission by Alyssa Barnett.

We need to perform the inverse operation twice to get the original number.

The largest possible number of segments is 7 per digit; the smallest is 2 per digit. Given a 4 digit number, the maximum number that may be added to 2015 is 28, so the largest the first number (going backwards) can be is $2015 + 28 = 2043$. This tells us that the first two digits of our first number will definitely be 2 0. Since the number of lighted segments for the digits 2 0 is 11, and the last two digits will have at minimum 4 segments and at maximum 14 segments, the full 4-digit number must have 15 to 25 segments. Adding 15 and 25 to 2015, we find that the first number is no less than 2030 and no greater than 2040. Performing the inverse operation on $2030, 2033, \ldots, 2040$ reveals that 2038 yields the desired result: $2038 - 23 = 2015$.

Using the same process, we find that the largest possible second (i.e., original) number is $2038 + 28 = 2066$, which again has 2 and 0 as its first digits and therefore 15 to 25 segments. The original number must be no greater than 2053 and no less than 2063. Performing the inverse operation on $2053, 2054, \ldots, 2063$ reveals that 2057 yields the desired result: $2057 - 19 = 2038$.

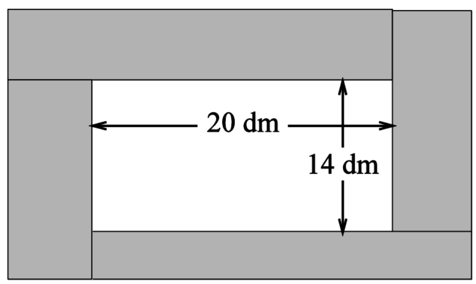
Mathew began with the number 2057.

CC177. An adventurer, born between 1901 and 1955, writes his memoirs when he is between 30 and 60 years of age. He wrote, "On this day celebrating my birthday, an extraordinary fact is made known to me: the weekday is exactly the same as the one I was born on." What was the age of the adventurer when he wrote that sentence?

Originally Problem 13 of the 2015-16 quarter finals of Le Championnat International des Jeux Mathématiques et Logiques.

We received no solutions to this problem.

CC178. A Modern Art painter Rec Tangle has painted the work of art represented here:



The white rectangle in the middle has length 20 dm and width 14 dm. The 4 grey rectangles all have equal areas and their dimensions, in decimeters, are non-zero integers. What is the minimal possible area of each of the grey rectangles? (The drawing is not to scale and a rectangle might be a square.)

Originally Problem 16 of the semi-final of the 2013-14 Championnat International des Jeux Mathématiques et Logiques.

We received two correct solutions and three incorrect solutions. We present the solution by Šefket Arslanagić.

Let t be the height of the top left rectangle, z the width of the top right rectangle, x the width of the bottom left rectangle and y the height of the bottom right rectangle. Then

$$x(14+y) = y(20+z) = z(14+t) = t(20+x) = F,$$

where F is the area of the 4 grey rectangles. We need to find the minimum value of F when x, y, z, and t are positive integers. Suppose the minimum value of F occurs for x_1, y_1, z_1, t_1 with $x_1 \neq z_1$. We may assume that $x_1 > z_1$ and because $x_1(14+y_1) = z_1(14+t_1)$ we must have $t_1 > y_1$. Taking $x = z = z_1, y = t = y_1$ would also give a valid solution with smaller area. This is a contradiction, so the minimum value of F occurs when $x = z$.

This yields

$$x(14+y) = y(20+z) = y(20+x),$$

which gives us $7x = 10y$. In order to minimize, we take $x = 10, y = 7$ which gives area $x(14 + y) = 210$.

CC179. Matthew creates a sequence of numbers starting from the number 7. Every number in his sequence is the sum of the digits of the square of the previous number, plus 1. For example, the second number in his sequence is 14 because $7^2 = 49$ and $4 + 9 + 1 = 14$. What is the 1000th number of Matthew's sequence?

Originally Problem 10 of the 2013-14 quarter final of Le Championnat International des Jeux Mathématiques et Logiques.

We received eight submissions, of which seven were correct. We present a representative solution.

Calculation of the first few terms reveals the sequence

$$7, 14, 17, 20, 5, 8, 11, 5, 8, 11, \ldots,$$

which is easily identified as a 3-cycle from the 5th term onwards. We can represent the n-th term for $n \geq 5$ as

$$a_n = \begin{cases} 5, & \text{if } n = 2 \mod 3, \\ 8, & \text{if } n = 0 \mod 3, \\ 11, & \text{if } n = 1 \mod 3. \end{cases} \qquad n \geq 5$$

Since 1000 is 1 (mod 3), we find $a_{1000} = 11$.

CC180. The pages of a book are numbered $1, 2, 3, \ldots$ A digit that appears in the number of the last page appears 20 times in the set of page numbers of the book. If the book had thirteen pages less, then the same digit would have been used 14 times in total. How many pages does the book have?

Originally Problem 8 of the 2013-14 semi final of Le Championnat International des Jeux Mathématiques et Logiques.

We received 3 correct solutions, and no incorrect solutions. We present the solution of Titu Zvonaru.

The book has 98 pages. It results from the following table:

	# of occurrences of digits									
# of pages	1	2	3	4	5	6	7	8	9	0
110	33	21	21	21	21	21	21	21	21	21
109	31	21	21	21	21	21	21	21	21	20
108	30	21	21	21	21	21	21	21	20	19
107	29	21	21	21	21	21	21	20	20	18
106	28	21	21	21	21	21	20	20	20	17
105	27	21	21	21	21	20	20	20	20	16
104	26	21	21	21	20	20	20	20	20	15
103	25	21	21	20	20	20	20	20	20	14
102	24	21	20	20	20	20	20	20	20	13
101	23	20	20	20	20	20	20	20	20	12
100	21	20	20	20	20	20	20	20	20	11
99	20	20	20	20	20	20	20	20	20	9
98	20	20	20	20	20	20	20	**20**	18	9
97	20	20	20	20	20	20	20	19	17	9
96	20	20	20	20	20	20	19	19	16	9
95	20	20	20	20	20	19	19	19	15	9
94	20	20	20	20	19	19	19	19	14	9
93	20	20	20	19	19	19	19	19	13	9
92	20	20	19	19	19	19	19	19	12	9
91	20	19	19	19	19	19	19	19	11	9
90	19	19	19	19	19	19	19	19	10	9
89	19	19	19	19	19	19	19	19	9	8
88	19	19	19	19	19	19	19	18	8	8
87	19	19	19	19	19	19	19	16	8	8
86	19	19	19	19	19	19	18	15	8	8
85	19	19	19	19	19	18	18	**14**	8	8

THE OLYMPIAD CORNER

No. 344

Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1 mars 2017**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

OC286. On considère quatre joueurs de basketball, A, B, C et D. Au départ, A est en possession du ballon. Il passe le ballon à un autre joueur qui le passe à un autre et ainsi de suite. Combien y a-t-il de façons de faire revenir le ballon à A après exactement **sept** passes ? (Par exemple, A passe le ballon à C qui le passe à B qui le passe à D qui le passe à A qui le passe à B qui le passe à C qui le passe à A.)

OC287. Soit $P(x) = ax^3 + (b-a)x^2 - (c+b)x + c$ et $Q(x) = x^4 + (b-1)x^3 + (a-b)x^2 - (c+a)x + c$ deux polynômes en x, où a, b et c sont des nombres réels non nuls et $b > 0$. De plus, $P(x)$ admet trois zéros réels distincts, x_0, x_1 et x_2, qui sont aussi des zéros de $Q(x)$.

1. Démontrer que $abc > 28$.
2. Sachant que a, b et c sont des entiers non nuls et que $b > 0$, déterminer leurs valeurs possibles.

OC288. Déterminer tous les entiers strictement positifs n de manière que pour tout entier strictement positif a tel que a et n sont premiers entre eux, on ait $2n^2 \mid a^n - 1$.

OC289. Soit a, b, c, d et e des entiers distincts strictement positifs tels que $a^4 + b^4 = c^4 + d^4 = e^5$. Démontrer que $ac + bd$ est un nombre composé.

OC290. Soit ABC un triangle scalène et soit X, Y et Z des points sur les droites respectives BC, AC et AB, de manière que $\measuredangle AXB = \measuredangle BYC = \measuredangle CZA$. Soit P un point d'intersection des cercles circonscrits aux triangles BXZ et CXY. Démontrer que P est situé sur le cercle ayant pour diamètre HG, H étant l'orthocentre du triangle ABC et G étant le centre de gravité de ce triangle.

. . . .

OC286. There are four basketball players A, B, C, D. Initially the ball is with A. The ball is always passed from one person to a different person. In how many ways can the ball come back to A after **seven** moves ? (For example, A passes to C who passes to B who passes to D who passes to A who passes to B who passes to C who passes to A.)

OC287. Let $P(x) = ax^3 + (b-a)x^2 - (c+b)x + c$ and $Q(x) = x^4 + (b-1)x^3 + (a-b)x^2 - (c+a)x + c$ be polynomials of x with a, b, c non-zero real numbers and $b > 0$. Suppose that $P(x)$ has three distinct real roots x_0, x_1, x_2 which are also roots of $Q(x)$.

1. Prove that $abc > 28$,
2. If a, b, c are non-zero integers with $b > 0$, find all their possible values.

OC288.Find all positive integers n such that for any positive integer a relatively prime to n, $2n^2 \mid a^n - 1$.

OC289. Let a, b, c, d, e be distinct positive integers such that

$$a^4 + b^4 = c^4 + d^4 = e^5.$$

Show that $ac + bd$ is a composite number.

OC290. Let $\triangle ABC$ be a scalene triangle and X, Y and Z be points on the lines BC, AC and AB, respectively, such that $\measuredangle AXB = \measuredangle BYC = \measuredangle CZA$. The circumcircles of BXZ and CXY intersect at P. Prove that P is on the circle with diameter HG, where H is the orthocenter and G is the barycenter of $\triangle ABC$.

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2015 : 41(4), p. 149–150.

OC226. In a triangle ABC, let D be the point on the segment BC such that $AB+BD = AC+CD$. Suppose that the points B, C and the centroids of triangles ABD and ACD lie on a circle. Prove that $AB = AC$.

Originally problem 1 of the 2014 India National Olympiad.

We received six correct submissions. We present the solution by Titu Zvonaru (similar to Šefket Arslanagić).

As usual, let a, b, c be the sides of ΔABC and let $2s = a + b + c$. Let T be the midpoint of AD and let G_1 and G_2 be the centroids of triangles ABD and ACD respectively. Since $BD - CD = b - c$ and $BD + CD = a$, we see that $BD = s - c$ and $CD = s - b$. Now, BG_qG_2C is cyclic if and only if $TG_1 \cdot TB = TG_2 \cdot TC$. Since medians are divided by the centroid in a $2 : 1$ ratio, we see this holds if and only if $\frac{TB^2}{3} = \frac{TC^2}{3}$. This is true if and only if $4TB^2 = 4TC^2$ which, by the formula for a median's length, holds if and only if

$$2BD^2 + 2BA^2 - AD^2 = 2CD^2 + 2CA^2 - AD^2.$$

Substituting the values from before, this holds if and only if $(s - c)^2 + c^2 = (s - b)^2 + b^2$ which is equivalent to $(b - c)(b + c - a) = 0$. Since $a \neq b + c$ by the triangle inequality, we see that $b = c$ and hence $AB = AC$.

OC227. In a bag there are 1007 black and 1007 white balls, which are randomly numbered 1 to 2014. In every step we draw one ball and put it on the table ; also if we want to, we may choose two different colored balls from the table and put them in a different bag. If we do that we earn points equal to the absolute value of their differences. How many points can we guarantee to earn after 2014 steps ?

Originally problem 1 from day 1 of the 2014 Turkey Mathematical Olympiad.

No submitted solutions.

OC228. Let k be a nonzero natural number and m an odd natural number . Prove that there exist a natural number n such that the number $m^n + n^m$ has at least k distinct prime factors.

Originally problem 4 from day 1 of the 2014 Romanian Team Selection Test.

No submitted solutions.

OC229. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $x, y \in \mathbb{R}^+$,

$$f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) = f(y)$$

Originally problem 4 from day 2 of the 2014 Iran Team Selection Test.

We received two correct submissions. We present the solution by Oliver Geupel.

The function

$$f(x) = \frac{1}{x}$$

is a solution because, for $x, y \in \mathbb{R}^+$,

$$f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) = \frac{1}{(x+1)y} + \frac{x}{(x+1)y} = \frac{1}{y} = f(y).$$

We show that there is no other solution.

Suppose f is any solution of the problem.

To begin with, we prove that

$$f(x) \leq \frac{1}{x} \qquad (1)$$

for every $x > 0$. Suppose that, contrary to our claim, $f(a) > \dfrac{1}{a}$ for some $a > 0$. Putting

$$x = \frac{1}{af(a) - 1}, \qquad y = a,$$

we obtain

$$\frac{x+1}{xf(y)} = y,$$

that is,

$$f\left(\frac{y}{f(x+1)}\right) = f(y) - f\left(\frac{x+1}{xf(y)}\right) = 0,$$

a contradiction. This proves (1) for $x > 0$.

Next we show that, for every $x \geq 1$,

$$f(x) = \frac{1}{x}. \qquad (2)$$

By (1), we have for all $x, y \in \mathbb{R}^+$,

$$f(y) = f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) \leq \frac{f(x+1)}{y} + \frac{x}{x+1} \cdot f(y),$$

so that $yf(y) \leq (x+1)f(x+1)$. It follows that $xf(x)$ is identically constant for $x > 1$, say $xf(x) = c \leq 1$. Moreover, for all $x > 1$, we have

$$\frac{x}{f(x+1)} = \frac{x(x+1)}{c} > 1, \qquad \frac{x+1}{xf(x)} = \frac{x+1}{c} > 1,$$

so that

$$\frac{c^2}{x} = \frac{c^2}{(x+1)x} + \frac{c^2 x}{(x+1)x} = f\left(\frac{x}{f(x+1)}\right) + f\left(\frac{x+1}{xf(x)}\right) = f(x) = \frac{c}{x},$$

which implies $c = 1$. We obtain (2) for every $x > 1$.

By (1), we have $\dfrac{2}{f(1)} \geq 2$, whence

$$f(1) = f\left(\frac{1}{f(2)}\right) + f\left(\frac{2}{f(1)}\right) = \frac{1}{2} + \frac{1}{2}f(1),$$

that is, $f(1) = 1$. This proves that (2) holds for every $x \geq 1$.

Finally, let $P(n)$ denote the assertion that equation (2) is true for every $x \geq 2^{-n}$. It is enough to prove $P(n)$ for all nonnegative integers n. We do so by mathematical induction. We have already established the base case $n = 0$. For the induction step suppose $P(n)$. Let

$$\frac{1}{2^{n+1}} \leq y < \frac{1}{2^n}, \qquad x = \frac{2^n y}{1 - 2^n y}.$$

Then,

$$f(x+1) = \frac{1}{x+1} = 1 - 2^n y$$

and

$$\frac{y}{f(x+1)} = \frac{y}{1 - 2^n y} \geq \frac{1}{2^n}.$$

By (1),

$$\frac{x+1}{xf(y)} \geq \frac{x+1}{x} \cdot y = \frac{1}{2^n}.$$

By induction we deduce

$$f(y) = f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) = \frac{1}{(x+1)y} + \frac{x}{x+1} \cdot f(y)$$

and conclude

$$f(y) = \frac{1}{y},$$

which proves $P(n+1)$ and the proof is complete.

OC230. Find, with justification, all positive real numbers a, b, c satisfying the system of equations :

$$a\sqrt{b} = a + c, b\sqrt{c} = b + a, c\sqrt{a} = c + b.$$

Originally problem 2 of the 2014 Singapore Senior Math Olympiad.

We received seven correct submissions. We present the solution by Albert Stadler.

We claim that $a = b = c = 4$ is the only solution in positive numbers a, b, c. It is easy to verify that this is a solution so we now show this is the only one. Let $a = u^2$, $b = v^2$ and $c = w^2$ in order to eliminate square roots. With these new variables, we need to show that $u = v = w = 2$ is the only positive solution of

$$u^2 v = u^2 + w^2 \qquad v^2 w = v^2 + u^2 \qquad w^2 u = w^2 + v^2$$

Using the AM-GM inequality, we see that

$$u^2 v = u^2 + w^2 \geq 2uw$$

implying that $uv \geq 2w$. Similarly, $vw \geq 2u$ and $wu \geq 2v$. Thus, multiplying the first two inequalities gives

$$(uv)(vw) \geq 4wu$$

implying that $v^2 \geq 4$ and hence $v \geq 2$ (recall we only want positive solutions). Similarly, $u \geq 2$ and $w \geq 2$. Then, the triple of equalities above implies that

$$v - 1 = \left(\frac{w}{u}\right)^2 \qquad w - 1 = \left(\frac{u}{v}\right)^2 \qquad u - 1 = \left(\frac{v}{w}\right)^2$$

and hence, multiplying these together yields $(u-1)(v-1)(w-1) = 1$. However, $u \geq 2$, $v \geq 2$ and $w \geq 2$. Hence, $u = v = w = 2$.

A Surprising Result in Cake-Sharing

Robert Barrington Leigh, YunHao Fu, ZhiChao Li and David Rhee

Amy and Peter are sharing a cake. Amy will cut it into two pieces. Peter then cuts one of the pieces into two. This is followed by a second cut by Amy and a second cut by Peter, so that there will be five pieces, of sizes $0 \le a_1 \le a_2 \le a_3 \le a_4 \le a_5$, with $a_1 + a_2 + a_3 + a_4 + a_5 = 1$. Peter will get the three pieces of sizes $a_1,\ a_3$ and a_5, while Amy will get the remaining two pieces.

What is the maximum amount of the cake Peter can get ?

The answer is surprising !

We first digress and consider the companion problem where Amy will get the three pieces of sizes $a_1,\ a_3$ and a_5, while Peter will get the remaining two pieces. What is the maximum amount of the cake Amy can get ?

First, we prove that Peter can always get $\frac{2}{5}$ of the cake. Suppose Amy cuts the cake into two pieces of sizes x and $1 - x$, where $0 \le x \le \frac{1}{2}$. There are three cases.

Case 1. $\frac{2}{5} \le x \le \frac{1}{2}$.

Peter will cut $1 - x$ into x and $1 - 2x$. Now the three pieces are of sizes $1 - 2x < x = x$. If Amy does not cut either x, neither will Peter. Peter will then be sure of getting x plus a second piece, and $x \ge \frac{2}{5}$. If Amy cuts one of the x's, Peter will cut the other x in the same proportions. Peter will get two pieces which add up to $x \ge \frac{2}{5}$.

Case 2. $\frac{1}{5} \le x < \frac{2}{5}$.

Peter will cut x into $x-\frac{1}{5}$ and $\frac{1}{5}$. Now the three pieces are of sizes $x-\frac{1}{5} < \frac{1}{5} < 1-x$. If Amy does not cut $1 - x$, Peter will cut it in halves. The second smallest piece cannot be less than $\frac{1}{2}(x-\frac{1}{5})$, so Peter will get at least $\frac{1-x}{2} + \frac{1}{2}(x-\frac{1}{5}) = \frac{2}{5}$. Suppose Amy cuts $1 - x$ into y and $1 - x - y$, where $0 \le y \le \frac{1-x}{2}$. Then Peter will cut $1 - x - y$ into $\frac{2}{5} - y$ and $\frac{3}{5} - x$. Now $y + (\frac{2}{5} - y) = \frac{2}{5} = (x - \frac{1}{5}) + (\frac{3}{5} - x)$. Thus Peter will get two pieces which add up to $\frac{2}{5}$.

Case 3. $0 \le x < \frac{1}{5}$.

Peter will cut $1 - x$ into $\frac{1}{5}$ and $\frac{4}{5} - x$. The situation is the same as in Case 2.

We now prove that Amy can always get $\frac{3}{5}$ of the cake. She can start by cutting the cake into two pieces of sizes $\frac{2}{5}$ and $\frac{3}{5}$. There are two cases.

Case 1. Peter cuts $\frac{2}{5}$ into x and $\frac{2}{5} - x$, where $0 \le x \le \frac{1}{5}$.

Amy will cut $\frac{3}{5}$ into x and $\frac{3}{5} - x$. Now the four pieces are of sizes $x = x \le \frac{2}{5} - x < \frac{3}{5} - x$. No matter what Peter does, the size of the second largest piece is at most $\frac{2}{5} - x$ and the size of the fourth largest piece is at most x. Hence Peter gets at most $(\frac{2}{5} - x) + x = \frac{2}{5}$.

Case 2. Peter cuts $\frac{3}{5}$ into x and $\frac{3}{5}-x$, where $0 \le x \le \frac{3}{10}$.

If $0 \le x \le \frac{1}{5}$, Amy will cut $\frac{2}{5}$ into x and $\frac{2}{5}-x$, and the situation is exactly the same as in Case 1. Hence we may assume that $\frac{1}{5} < x \le \frac{3}{10}$. Amy will cut $\frac{3}{5}-x$ into $\frac{1}{5}$ and $\frac{2}{5}-x$. Now the four pieces are of sizes $\frac{2}{5}-x < \frac{1}{5} < x < \frac{2}{5}$. There are four subcases.

- **Subcase 2(a).** Peter cuts $\frac{2}{5}$ into y and $\frac{2}{5}-y$, where $0 \le y \le \frac{1}{5}$.

  Since $y + (\frac{2}{5}-y) = \frac{2}{5} = x + (\frac{2}{5}-x)$, Peter will get two pieces which add up to $\frac{2}{5}$.

- **Subcase 2(b).** Peter cuts x.

  If $\frac{1}{5}$ remains the third largest piece, Amy will get at least $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$. If it becomes the second largest piece, Peter gets at most $\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$.

- **Subcase 2(c).** Peter cuts $\frac{1}{5}$ into y and $\frac{1}{5}-y$, where $0 \le y \le \frac{1}{10}$.

  Since $\frac{2}{5}-x \ge y$, the second smallest piece is at most $\frac{2}{5}-x$. Hence Peter gets at most $(\frac{2}{5}-x)+x=\frac{2}{5}$.

- **Subcase 2(d).** Peter cuts $\frac{2}{5}-x$.

  Amy will get at least $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$.

We now return to the original problem with the surprising answer. We first show that Peter can always get $\frac{35}{53}$ of the cake. Suppose Amy cuts the cake into two pieces of sizes a and b, where $a \ge b \ge 0$ and $a+b=1$. There are four cases.

Case A. $\frac{34}{53} \le a \le 1$ so that $0 \le b \le \frac{19}{53}$.

Peter will cut b into $\frac{b}{2}$ and $\frac{b}{2}$. If Amy does not cut a, Peter will just cut off $\frac{1}{53}$ from another piece, and will get three pieces with total size at least $a+\frac{1}{53} \ge \frac{35}{53}$. Suppose Amy cuts a into $c \le d$. Peter will cut d into $\frac{d}{2}$ and $\frac{d}{2}$, and will get three pieces with total size at least $c+\frac{d}{2}+\frac{b}{2} \ge \frac{a}{4}+\frac{1}{2} \ge \frac{35}{53}$.

Case B. $\frac{33}{53} \le a \le \frac{34}{53}$ so that $\frac{19}{53} \le b \le \frac{20}{53}$.

Peter will cut b into $\frac{18}{53}$ and $b-\frac{18}{53}$. If Amy does not cut a, Peter will just cut off $\frac{2}{53}$ from another piece, and will get three pieces with total size at least $a+\frac{2}{53} \ge \frac{35}{53}$. Suppose Amy cuts a into $c \ge d \ge 0$. We consider four subcases.

- **Subcase B1.** $\frac{18}{53} \ge c \ge \frac{17}{53} \ge d \ge b-\frac{18}{53}$.

  Peter will cut c into d and $c-d$, and will get three pieces with total size at least
  $$\frac{18}{53}+d+\min\left\{c-d, b-\frac{18}{53}\right\}.$$
  In the former instance, it is at least $\frac{18}{53}+c \ge \frac{35}{53}$. In the latter instance, it is at least $d+b=1-c \ge \frac{35}{53}$.

- **Subcase B2.** $\frac{17}{53} \geq c \geq d \geq b - \frac{18}{53}$.

  Peter will cut $b - \frac{18}{53}$ into $\frac{b}{2} - \frac{9}{53}$ and $\frac{b}{2} - \frac{9}{53}$, and will get three pieces with total size at least $\frac{18}{35} + d + \frac{b}{2} - \frac{9}{35} = \frac{62}{53} - \frac{b}{2} - c \geq \frac{35}{53}$.

- **Subcase B3.** $c \geq \frac{18}{53} \geq d \geq b - \frac{18}{53}$.

  Peter will cut $\frac{18}{53}$ into d and $\frac{18}{53} - d$, and will get three pieces with total size at least

  $$c + d + \min\left\{\frac{18}{35} - d, b - \frac{18}{35}\right\}.$$

  In the former instance, it is at least $c + \frac{18}{53} \geq \frac{36}{53}$. In the latter instance, it is at least $a + b - \frac{18}{53} = \frac{35}{53}$.

- **Subcase B4.** $c \geq \frac{18}{53} \geq b - \frac{18}{53} \geq d$.

  Peter will cut $\frac{18}{53}$ into $\frac{9}{53}$ and $\frac{9}{53}$. Since $d \leq b - \frac{18}{53} \leq \frac{2}{53}$, $c \geq \frac{31}{53}$. Hence he will get three pieces with total size at least $c + \frac{9}{53} \geq \frac{40}{53}$.

Case C. $\frac{27}{53} \leq a \leq \frac{33}{53}$ so that $\frac{20}{53} \leq b \leq \frac{26}{53}$.

Peter will cut a into $\frac{27}{53}$ and $a - \frac{27}{53}$. If Amy does not cut $\frac{27}{53}$, Peter will just cut off $\frac{8}{53}$ from another piece. If it is the second largest, then Amy gets two pieces with total size at most $\frac{16}{53}$. Otherwise, Peter will get three pieces with total size at least $\frac{27}{53} + \frac{8}{53} = \frac{35}{53}$. Suppose Amy cuts $\frac{27}{53}$ into $c \geq d$. There are four subcases.

- **Subcase C1.** $\frac{27}{106} \leq c \leq \frac{15}{53}$ so that $\frac{12}{53} \leq d \leq \frac{27}{106}$.

  Peter will cut $a - \frac{27}{53} = \frac{26}{53} - b$ into $\frac{13}{53} - \frac{b}{2}$ and $\frac{13}{53} - \frac{b}{2}$. He will get three pieces with total size $b + d + (\frac{13}{53} - \frac{b}{2}) \geq \frac{35}{53}$.

- **Subcase C2.** $\frac{15}{53} \leq c \leq \frac{18}{53}$ so that $\frac{9}{53} \leq d \leq \frac{12}{53}$.

  Peter will cut c into d and $c - d$. He will get three pieces with total size $b + d + \min\{c - d, a - \frac{27}{53}\}$. In the former instance, it is at least $b + c \geq \frac{35}{53}$. In the latter instance, it is at least $d + 1 - \frac{27}{53} \geq \frac{35}{53}$.

- **Subcase C3.** $\frac{18}{53} \leq c \leq \frac{24}{53}$ so that $\frac{3}{53} \leq d \leq \frac{9}{53}$.

  Peter will cut c into $\frac{c}{2}$ and $\frac{c}{2}$. Amy gets two pieces with total size at most $\frac{c}{2} + \max\{d, a - \frac{27}{53}\}$. In the former instance, it is at most $\frac{27}{53} - c \leq \frac{18}{35}$. In the latter instance, it is at most $\frac{12}{53} + \frac{6}{53} = \frac{18}{53}$.

- **Subcase C4.** $\frac{24}{53} \leq c \leq \frac{27}{53}$ so that $0 \leq d \leq \frac{3}{53}$.

  Peter will cut b into $\frac{b}{2}$ and $\frac{b}{2}$. He will get three pieces with total size at least $c + \frac{n}{2} + \min\{d, a - \frac{27}{53}\}$. In the former instance, it is at least $\frac{27}{53} + \frac{b}{2} \geq \frac{37}{53}$. In the latter instance, it is at least $c + 1 - \frac{27}{53} - \frac{b}{2} \geq \frac{37}{53}$.

Case D. $\frac{1}{2} \leq a \leq \frac{27}{53}$ so that $\frac{26}{53} \leq b \leq \frac{1}{2}$.

Peter will pass. Whichever piece Amy now cuts, Peter will cut off from the larger of the two new pieces a piece equal to $\frac{1}{3}$ of the piece Amy cuts. This piece will be the third largest, and Peter will get three pieces with total size at least $b + \frac{a}{3} = 1 - \frac{2a}{3} \geq \frac{35}{53}$.

The following two preliminary results will be useful.

Lemma 1. Suppose before Peter's final cut, the four pieces have sizes $w,\ x,\ y$ and z in non-ascending order. If $x \leq 2y$, then Amy can get two pieces with total size at least x.

Proof. If Peter cuts either of the smallest two pieces, the second largest piece will have size x. There is nothing further to prove.

Hence we assume that Peter must cut one of the largest two pieces, into two pieces both with size smaller than x. Because $x \leq 2y$, at least one of the new pieces has size less than y. If the original piece with size y is still the third largest, then the largest has size at most w and the smallest has size at most z. Hence Peter gets three pieces with total size at most $w+y+z$, so that Amy will get two pieces with total size at least $1 - w - y - z = x$. On the other hand, if the piece with size y is now the second largest, then the size of each of the two new pieces lies between y and $x - y$. Thus, the second smallest piece has size at least $x - y$, and Amy's two pieces will have total size at least $y + (x - y) = x$. □

Lemma 2. Suppose before Peter's final cut, the four pieces have sizes $w,\ x,\ y$ and z in non-ascending order. If $x \geq 2z$, then Amy can get two pieces with total size at least $\min\{y + z, x + \frac{z}{2}\}$.

Proof. If Peter cuts either of the smallest two pieces, the second smallest piece will have size at least $\frac{z}{2}$ while the second largest piece will have size x. Hence, Amy will get two pieces with total size at least $x + \frac{z}{2}$.

If Peter cuts either of the largest two pieces into two pieces both with size smaller than x, not both can have size smaller than z since $x \geq 2z$. Hence, the second smallest piece has size at least z while the second largest piece has size at least y. Hence, Amy will get two pieces with total size at least $y + z$. □

We now prove that Amy can always get $\frac{18}{53}$ of the cake. She will cut 1 into $\frac{20}{53}$ and $\frac{33}{53}$. There are two cases.

Case A. Peter cuts $\frac{20}{33}$ into $a \geq b \geq 0$.

There are three subcases.

- **Subcase A1.** $\frac{18}{53} \leq a \leq \frac{20}{53}$ so that $0 \leq b \leq \frac{2}{53}$.

  Amy will cut $\frac{33}{53}$ into $\frac{18}{53}$ and $\frac{15}{53}$. In Lemma 1, $w = a,\ x = \frac{18}{53},\ y = \frac{15}{53}$ and $z = b$, with $x \leq 2y$. Hence she will get two pieces with total size at least $\frac{18}{53}$.

- **Subcase A2.** $\frac{17}{53} \leq a \leq \frac{18}{53}$ so that $\frac{2}{53} \leq b \leq \frac{3}{53}$.

  Amy will cut $\frac{33}{53}$ into $\frac{17}{53}$ and $\frac{16}{53}$. In Lemma 2, $w = a,\ x = \frac{17}{53},\ y = \frac{16}{53}$ and $z = b$, with $x \geq 2z$. Now $y + z = \frac{16}{53} + b \geq \frac{18}{53}$ while $x + \frac{z}{2} = \frac{17}{53} + \frac{b}{2} \geq \frac{18}{53}$. Either way, she will get two pieces with total size $\frac{18}{53}$.

- **Subcase A3.** $\frac{10}{53} \le a \le \frac{17}{53}$ so that $\frac{3}{53} \le b \le \frac{10}{53}$.

  Amy will still cut $\frac{33}{53}$ into $\frac{17}{53}$ and $\frac{16}{53}$. The total size of the smallest four pieces is $1 - \frac{17}{53} = \frac{36}{53}$. She will be guaranteed to get at least half of that, which is $\frac{18}{53}$.

Case B. Peter cuts $\frac{33}{53}$ into $a \ge b \ge 0$.

There are seven subcases.

- **Subcase B1.** $\frac{27}{53} \le a \le \frac{33}{53}$, so that $0 \le b \le \frac{6}{53}$.

  Amy will cut a into $\frac{18}{53}$ and $a - \frac{18}{53}$. In Lemma 1, $w = \frac{20}{53}$, $x = \frac{18}{53}$, $y = a - \frac{18}{53}$ and $z = b$, with $x \le 2y$. Hence she will get two pieces with total size at least $\frac{18}{53}$.

- **Subcase B2.** $\frac{51}{106} \le a \le \frac{27}{53}$, so that $\frac{6}{53} \le b \le \frac{15}{106}$.

  Amy will cut a into $\frac{15}{53}$ and $a - \frac{15}{53}$. In Lemma 2, $w = \frac{20}{53}, x = \frac{15}{53}$, $y = a - \frac{15}{53}$ and $z = b$, with $x \ge 2z$. Now $y + z = a + b - \frac{15}{53} = \frac{33}{53} - \frac{15}{53} = \frac{18}{53}$ while $x + \frac{z}{2} = \frac{15}{53} + \frac{b}{2} \ge \frac{18}{53}$. Either way, she will get two pieces with total size at least $\frac{18}{53}$.

- **Subcase B3.** $\frac{25}{53} \le a \le \frac{51}{106}$, so that $\frac{15}{106} \le b \le \frac{8}{53}$.

  Amy will cut a into $a - b - \frac{3}{53}$ and $b + \frac{3}{53}$. If Peter cuts either $b + \frac{3}{53}$ or b, the second smallest piece is at least $\frac{b}{2}$, and Amy will get two pieces with total size at least $a - b - \frac{3}{53} + \frac{b}{2} = a + b - \frac{3}{53} - \frac{3b}{2} \ge \frac{18}{53}$. Suppose Peter cuts either $\frac{20}{53}$ or $a - b - \frac{3}{53}$. If both new pieces are less than b, then Amy will get two pieces with total size at least $b + \frac{3}{53} + \frac{1}{2}(a - b - \frac{3}{53}) = \frac{18}{53}$. If at least one of the new pieces is greater than b, then the second largest piece is at least $b + \frac{3}{53}$ so that she will get two pieces with total size at least $b + \frac{3}{53} + b \ge \frac{18}{53}$.

- **Subcase B4.** $\frac{23}{53} \le a \le \frac{25}{53}$, so that $\frac{8}{53} \le b \le \frac{10}{53}$.

  Amy will pass. If Peter then cuts b, Amy will get at least $\frac{20}{53}$. Hence Peter must cut a or $\frac{20}{53}$. After Peter's final cut, if b is still the third largest, then he gets three pieces with total size at most $a + b + 0 = \frac{33}{53}$, so that Amy will get two pieces with total size at least $\frac{18}{53}$. If b becomes the second smallest, then the second largest is at least $\frac{10}{53}$, and she will get two pieces with total size at least $\frac{10}{53} + b \ge \frac{18}{53}$.

- **Subcase B5.** $\frac{20}{53} \le a \le \frac{23}{53}$, so that $\frac{10}{53} \le b \le \frac{13}{53}$.

  Amy will also pass. In Lemma 1, $w = a$, $x = \frac{20}{53}$, $y = b$ and $z = 0$, with $x \le 2y$. Hence she will get two pieces with total size at least $\frac{20}{53}$.

- **Subcase B6.** $\frac{18}{53} \le a \le \frac{20}{53}$, so that $\frac{13}{53} \le b \le \frac{15}{53}$.

  Amy will still pass. In Lemma 1, $w = \frac{20}{53}$, $x = a$, $y = b$ amd $z = 0$, with $x \le 2y$. Hence she will get two pieces with total size at least $\frac{18}{53}$.

- **Subcase B7.** $\frac{33}{106} \leq a \leq \frac{18}{53}$, so that $\frac{15}{53} \leq b \leq \frac{33}{106}$.

  Amy will cut $\frac{20}{53}$ into $\frac{14}{53}$ and $\frac{6}{53}$. In Lemma 2, $w = a$, $x = b, y = \frac{14}{53}$ and $z = \frac{6}{53}$ with $x \geq 2z$. Now $y + z = \frac{20}{53}$ while $x + \frac{z}{2} = b + \frac{3}{53} \geq \frac{18}{53}$. Either way, she will get two pieces with total size at least $\frac{18}{53}$.

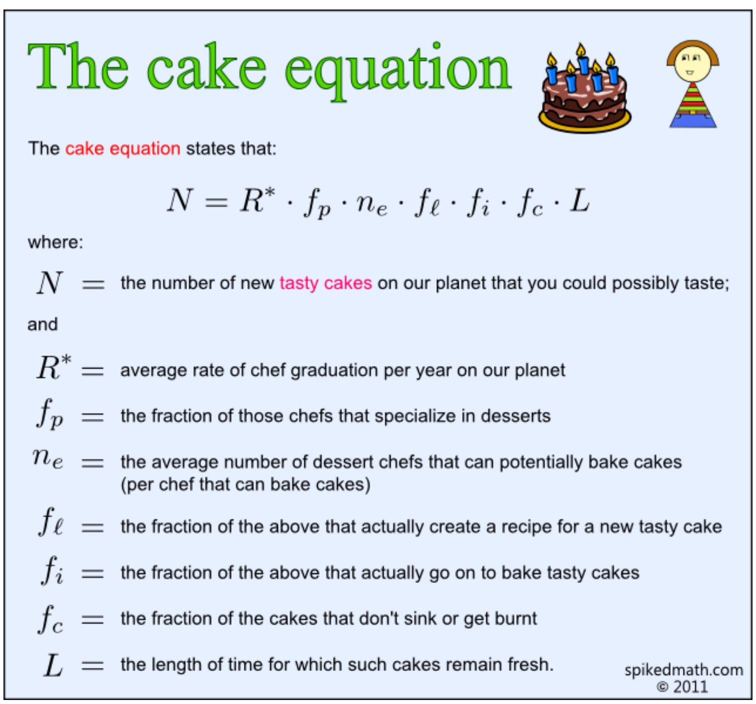
Therefore, Amy can always get $\frac{18}{53}$ of the cake.

. . . .

We dedicate this article to the memory of Robert Barrington Leigh (1986 – 2006), who passed away tragically 10 years ago. Robert was an extremely bright young man who won numerous mathematical and other scientific contests ; he represented Canada at the International Mathematical Olympiad in 2002 and 2003. At the time of his passing, he was enrolled in his final year at the University of Toronto in Specialist Programs both in Mathematics and in Physics and was awarded a posthumous undergraduate degree in June 2007.

This article, found as an unfinished manuscript after Robert's passing, was recently finished by his co-authors.

With apologies to Frank Drake, we present



Useful reference : `https://en.wikipedia.org/wiki/Drake_equation`

Conjugate numbers

N. Vaguten

The reader probably has seen seemingly difficult geometry problems that can be solved nearly instantaneously if one point is replaced by its reflection in some line. Symmetry arguments are very useful in algebra as well.

In this article, we will consider situations where a number of the form $a + b\sqrt{d}$ appears, and the solution involves its *conjugate* $a - b\sqrt{d}$. We will see how this simple step helps to solve problems of varying difficulty in algebra and analysis, from easy estimates to contest and olympiad problems. Many of our examples can serve as a first acquaintance with deep mathematical theories that the reader can then explore further.

Pairs of conjugates appear naturally in solving a quadratic equation in which the discriminant is not a perfect square. For example, the equation $\lambda^2 - \lambda - 1 = 0$ has a pair of conjugate roots $\lambda_1 = \frac{1-\sqrt{5}}{2}$ and $\lambda_2 = \frac{1+\sqrt{5}}{2}$. We will come back to this idea. For now, let us begin by considering the mechanism of "transferring" square roots

... from numerator to denominator and vice versa

Suppose you are solving a problem and you get $\frac{1}{3-\sqrt{7}}$, but the answer at the back of the book is $\frac{3+\sqrt{7}}{2}$. Do not rush into looking for an error : these numbers are equal since $(3+\sqrt{7})(3-\sqrt{7}) = 3^2 - 7 = 2$. Here are some characteristic examples where rationalizing the numerator proves useful.

1. Compute the sum

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}.$$

Indeed, this sum collapses (like a telescope, if you will) if we re-write it as

$$(\sqrt{2}-1) + (\sqrt{3}-\sqrt{2}) + \cdots + (\sqrt{100}-\sqrt{99}) = -1 + 10 = 9.$$

2. Prove that for any natural numbers m and n, we have

$$\left|\frac{m}{n} - \sqrt{2}\right| \geq \frac{1}{\alpha n^2}, \tag{1}$$

where $\alpha = \sqrt{3} + \sqrt{2}$.

Indeed, we have

$$\left|\frac{m - n\sqrt{2}}{n}\right| = \frac{|m^2 - 2n^2|}{(m + n\sqrt{2})n} \geq \frac{1}{(m + n\sqrt{2})n} \tag{2}$$

since the number $|m^2 - 2n^2|$ is a non-zero integer [Ed. The fact that it is impossible to have $m^2 = 2n^2$ is one of the oldest theorem in number theory]. Towards contradiction, suppose (1) does not hold ; that is,

$$m < n\sqrt{2} + \frac{1}{\alpha n}$$

and hence

$$\begin{aligned} n(m + n\sqrt{2}) &< n\left(2n\sqrt{2} + \frac{1}{\alpha n}\right) \\ &= 2n^2\sqrt{2} + \frac{1}{\sqrt{3}+\sqrt{2}} \\ &= 2n^2\sqrt{2} + \sqrt{3} - \sqrt{2} \\ &\le n^2(\sqrt{2}+\sqrt{3}) = \alpha n^2. \end{aligned} \tag{3}$$

However, (1) follows from (2) and (3). Therefore, out assumption is not correct and (1) holds.

Note that (1) shows that the number $\sqrt{2}$ is badly approximated by fractions with small denominators (we will see other approximations of $\sqrt{2}$ later, see example 4). Of course, (1) holds also for all $\alpha > \sqrt{3}+\sqrt{2}$, but the constant α here is not the smallest possible. Analogous inequalities, but with different values for α hold for other square roots.

3. Find the limit of the sequence defined by $a_n = (\sqrt{n^2+1} - n)n$.

We have :

$$a_n = (\sqrt{n^2+1} - n)n = \frac{n}{\sqrt{n^2+1} - n} = \frac{1}{1 + \sqrt{1 + 1/n^2}}.$$

It is now clear that the sequence a_n is decreasing and approaching $1/2$.

4. Consider two sequences $a_n = \sqrt{n+1} + \sqrt{n}$ and $b_n = \sqrt{4n+2}$. Prove that

a) $[a_n] = [b_n]$ (where $[x]$ denotes the integer part of x),

b) $0 < b_n - a_n < \frac{1}{16n\sqrt{n}}$.

Triple irrationality arises in the difference $b_n - a_n$ and we will come back to these types of irrationalities (see problem 8). For now, we will consider $a_n = \sqrt{n+1} + \sqrt{n}$ as a whole. Notice that the number $a_n^2 = 2n+1+2\sqrt{n(n+1)}$ lies between $4n+1$ and $4n+2 = b_n^2$ since $n < \sqrt{n(n+1)} < n+1$. As such, we already proved that $a_n < b_n$, which is the left inequality in part b). Furthermore, the number $b_n^2 = 4n+2$ leaves remainder of 2 when divided by 4 and hence cannot be a perfect square. Therefore, the square root of the integer $[b_n]$ is no more than $4n+1$. Part a) now follows from the inequalities $[b_n] \le \sqrt{4n+1} < a_n < b_n$.

It remains to estimate the difference $b_n - a_n$ from above. Here, we will "transfer" square roots from numerator to denominator twice :

$$\begin{aligned}\sqrt{4n+2}-\sqrt{n}-\sqrt{n+1} &= \frac{2n+1-2\sqrt{n(n+1)}}{\sqrt{4n+2}+\sqrt{n}+\sqrt{n+1}}\\ &= \frac{1}{\sqrt{4n+2}+\sqrt{n}+\sqrt{n+1}}\cdot\frac{1}{2n+1+2\sqrt{n(n+1)}}\\ &\le \frac{1}{(2\sqrt{n}+\sqrt{n}+\sqrt{n})(2n+2n)}\\ &= \frac{1}{16n\sqrt{n}}.\end{aligned}$$

(We got lucky above since $(2n+1)^2 - 4n(n+1) = 1$.)

Replace a plus with a minus

We already mentioned the usefulness of symmetry in geometrical problems. In algebra, one type of symmetry can be achieved by replacing a plus sign with a minus sign. We will use the following fact (a and b are rational, $\sqrt{d}$ is irrational) :

$$(a+b\sqrt{d})^n = p+q\sqrt{d} \quad\Longrightarrow\quad (a-b\sqrt{d})^n = p-q\sqrt{d}. \tag{4}$$

[Ed. This is a special case of the Irrational Conjugate Theorem, which states that if $a+b\sqrt{c}$, (a, b, c rational, $\sqrt{c}$ irrational,) is a root of a polynomial with rational coefficients, then $a-b\sqrt{c}$ is also a root. We can think of this as saying that the field of rational numbers cannot "see" the difference between $a+b\sqrt{c}$ and its conjugate. However, the reader is warned that this is true only for the arithmetic operations $+, -, *, /$, and that conjugates are not interchangeable with regard to other properties (such as inequalities and convergence).]

5. Prove that the equation

$$(x+y\sqrt{5})^4 + (z+t\sqrt{5})^4 = 2+\sqrt{5}$$

does not have rational solutions for x, y, z and t.

Of course, you can find the sum on the left hand side not containing square roots (and equate it to 2) and the coefficient of $\sqrt{5}$ (and equate it to 1). But it is not clear what to do next with the awkward resulting system of equations. Instead, let us use (4) and replace the sign in front of $\sqrt{5}$:

$$(x-y\sqrt{5})^4 + (z-t\sqrt{5})^4 = 2-\sqrt{5}.$$

Now on the left we have a nonnegative number, while on the right we have a negative one.

6. Prove that there are infinitely many pairs (x, y) of natural numbers such that

$$|x^2-2y^2| = 1. \tag{5}$$

In other words, do the hyperbolas $2y^2 - x^2 = 1$ and $2y^2 - x^2 = -1$ go through infinitely many lattice points in the grid ?

Several such pairs are easily found by hand : $(1, 2), (3, 2), (7, 5), (17, 12), \ldots$ But how do we continue such a set ? Can we write down a general formula for these solutions ? To find answers to these questions, let us turn to the number $1 + \sqrt{2}$. The consistent pattern becomes evident in the following table :

n	$(1+\sqrt{2})^n$	x_n	y_n	$x_n^2 - 2y_n^2$	$(1-\sqrt{2})^n$
1	$1+\sqrt{2}$	1	1	$1-2=-1$	$1-\sqrt{2}$
2	$3+2\sqrt{2}$	3	2	$9-8=1$	$3-2\sqrt{2}$
3	$7+5\sqrt{2}$	7	5	$49-50=-1$	$7-5\sqrt{2}$
4	$17+12\sqrt{2}$	17	12	$289-288=1$	$17-12\sqrt{2}$
5	$41+29\sqrt{2}$	41	29	$1681-1682=-1$	$41-29\sqrt{2}$
...	...	...	...	...	...

You can see that the coefficients in $x_n + y_n\sqrt{2} = (1+\sqrt{2})^n$ will give the desired pair. To prove this, take a closer look at the table and use (4) again :

$$x_n - y_n\sqrt{2} = (1-\sqrt{2})^n.$$

Multiplying the last two equalities, we get :

$$x_n^2 - 2y_n^2 = (1+\sqrt{2})^n(1-\sqrt{2})^n = ((1+\sqrt{2})(1-\sqrt{2}))^n = (-1)^n$$

and so the overall expression is equal to 1 or -1. By adding and subtracting the same two equalities, we also get the expressions for x_n and y_n :

$$x_n = ((1+\sqrt{2})^n + (1-\sqrt{2})^n)/2,$$
$$y_n = ((1+\sqrt{2})^n - (1-\sqrt{2})^n)/2\sqrt{2}.$$

In this above problem about integers, can we get away with using only integers and avoiding irrational numbers $1 \pm \sqrt{2}$? Knowing the answer, we can easily express the pair (x_{n+1}, y_{n+1}) through the previous pair (x_n, y_n) : from $x_{n+1} + y_{n+1}\sqrt{2} = (x_n + y_n\sqrt{2})(1-\sqrt{2})$, we get :

$$x_{n+1} = x_n + 2y_n, \quad y_{n+1} = x_n + y_n. \tag{6}$$

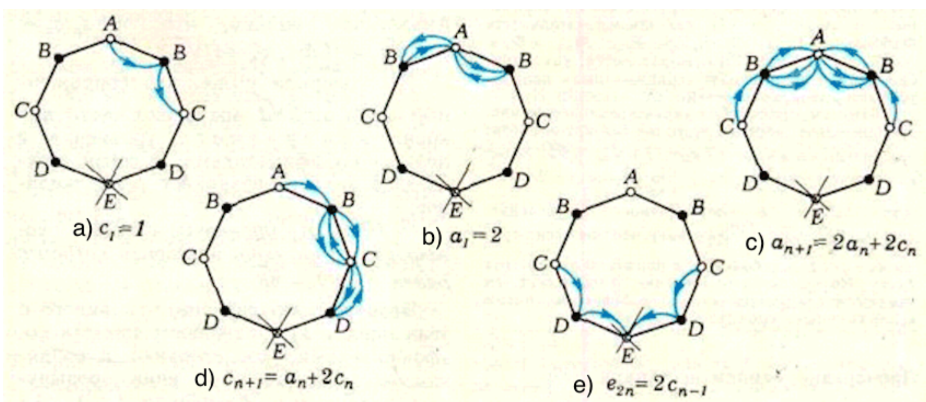
You could have arrived at this recurrence relation from seeing the first couple of solutions and checking that $|x_n^2 - 2y_n^2| = |x_{n+1}^2 - 2y_{n+1}^2|$. By adding the initial condition $x_1 = y_1 = 1$, we can prove by induction that $|x_n^2 - 2y_n^2| = 1$ for all n. Next, by expressing (x_n, y_n) through (x_{n+1}, y_{n+1}) you can prove that this sequence exhausts all solutions to (5) in natural numbers x and y. Similarly, one can solve any *Pell equation* $x^2 - dy^2 = c$, but the initial equation might have several series of solutions.

Recurrence relations like (6) arise not only in number theory, but also in areas like analysis and probability theory. Here is an example of a combinatorial problem like this (originally used in the 1979 International Math Olympiad in London) :

7. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Find the number e_m of distinct paths of exactly m jumps starting at A and ending at E.

Colour the vertices of the octagon in alternating black and white colours. It then becomes clear that $e_{2k-1} = 0$ for all k since the colour of the vertex changes with each jump. Let a_n and c_n denote the number of ways that the frog can get in $2n$ jumps to the vertex A and one of the vertices C, respectively (from the symmetry, it is clear that the frog can get to each vertex C in the same number of ways).

Consider the following graphics :



We have :

a) To get from A to C in two jumps can only be done in one way, so $c_1 = 1$.

b) You can get from A to A in two jumps in two different ways, so $a_1 = 2$.

c) You can get from C to A in two ways and from A in two ways, so $a_{n+1} = 2a_n + 2c_n$.

d) You can get to C from A in one way and from C in two ways, so $c_{n+1} = a_n + 2c_n$.

e) You can get from C to E in two ways, so $e_{2n} = 2c_{n-1}$.

You can verify that $a_1 = 2$, $c_1 = 1$ and

$$\begin{cases} a_{n+1} = 2a_n + 2c_n, \\ c_{n+1} = a_n + 2c_n. \end{cases} \tag{7}$$

Then we have $e_{2n} = 2c_{n-1}$.

How can we find the formula for a_n and c_n ? Let us re-write the recurrence (7) as

$$a_{n+1} + c_{n+1}\sqrt{2} = (a_n + c_n\sqrt{2})(2 + \sqrt{2}) \tag{8}$$

and, as you might have already guessed,

$$a_{n+1} - c_{n+1}\sqrt{2} = (a_n - c_n\sqrt{2})(2 + \sqrt{2}) \tag{9}$$

From here, by induction using (7), we get

$$a_n + c_n\sqrt{2} = (2 + \sqrt{2})^{n-1}(a_1 + c_1\sqrt{2}) = (2 + \sqrt{2})^n$$

and

$$a_n - c_n\sqrt{2} = (2 - \sqrt{2})^{n-1}(a_1 - c_1\sqrt{2}) = (2 - \sqrt{2})^n.$$

Therefore, $c_n = ((2 + \sqrt{2})^n - (2 - \sqrt{2})^n)/2\sqrt{2}$ and since $e_{2n} = 2c_{n-1}$, we finally get

$$e_{2n} = ((2 + \sqrt{2})^{n-1} - (2 - \sqrt{2})^{n-1})/\sqrt{2}$$

with $e_{2n-1} = 0$. Problem is solved. It only remains unclear how, given the original problem, one could have dreamt up the idea to use formulas containing $\pm\sqrt{2}$. (The original Olympiad problem did have the final answer in its statement.)

Actually, the conjugate numbers would have appeared automatically if we used basic linear algebra to solve (7). First, one could figure out what geometric progressions $a_n = a_0\lambda^n$ and $c_n = c_0\lambda^n$ satisfy the recurrence : the values for which such progressions exist are called *characteristic values* arising from the so-called *characteristic equation.* For (7), the characteristic equation is $\lambda^2 + 2 - 4\lambda = 0$ and its roots are exactly $2 \pm \sqrt{2}$. Knowing these roots, we can get the solutions to the recurrence as linear combination of these roots. It is therefore not surprising that many integer recurrences with characteristic equation having integer coefficients (such as (6), (7) and the Fibonacci sequence) contain conjugate irrational numbers.

Note that the greater characteristic number determines how fast the sequence is growing : for large n in example 7 we have $e_n \approx (2 + \sqrt{2})^n/\sqrt{2}$ or, alternatively, $\lim_{n\to\infty}(e_{n+1}/e_n) = 2+\sqrt{2}$. For problem 6, we have an analogous observation that $\lim_{n\to\infty}(x_n/y_n) = \sqrt{2}$ shows that both summands in $x_n+y_n\sqrt{2}$ are approximately equal for large n. We will see an interesting generalization of this fact in the following problem with many conjugate irrationalities.

Change all the signs

8. Let $(1 + \sqrt{2} + \sqrt{3})^n = q_n + r_n\sqrt{2} + s_n\sqrt{3} + t_n\sqrt{6}$, where q_n, r_n, s_n and t_n are all integers. Find the following limits :

$$\lim_{n\to\infty} r_n/q_n, \quad \lim_{n\to\infty} s_n/q_n, \quad \text{and} \quad \lim_{n\to\infty} t_n/q_n.$$

We can express $(q_{n+1}, r_{n+1}, s_{n+1}, t_{n+1})$ in terms of (q_n, r_n, s_n, t_n) using the fact that

$$q_{n+1} + r_{n+1}\sqrt{2} + s_{n+1}\sqrt{3} + t_{n+1}\sqrt{6} = (1 + \sqrt{2} + \sqrt{3})(q_n + r_n\sqrt{2} + s_n\sqrt{3} + t_n\sqrt{6}).$$

But our previous experience tells us that easier formulas will arise not for q_n, r_n, s_n and t_n, but for some of their combinations. We already know one such combination : namely, that $q_n + r_n\sqrt{2} + s_n\sqrt{3} + t_n\sqrt{6} = (1+\sqrt{2}+\sqrt{3})^n$. It is not hard to imagine what others will be like. Together with $\lambda_1 = 1+\sqrt{2}+\sqrt{3}$, consider other conjugates $\lambda_2 = 1-\sqrt{2}+\sqrt{3}$, $\lambda_3 = 1+\sqrt{2}-\sqrt{3}$ and $\lambda_4 = 1-\sqrt{2}-\sqrt{3}$. Then

$$\begin{aligned} q_n - r_n\sqrt{2} + s_n\sqrt{3} - t_n\sqrt{6} &= \lambda_2^n, \\ q_n + r_n\sqrt{2} - s_n\sqrt{3} - t_n\sqrt{6} &= \lambda_3^n, \\ q_n - r_n\sqrt{2} - s_n\sqrt{3} + t_n\sqrt{6} &= \lambda_4^n. \end{aligned}$$

We can now express q_n, r_n, s_n, t_n through $\lambda_1, \lambda_2, \lambda_3, \lambda_4$:

$$\begin{aligned} q_n &= (\lambda_1^n + \lambda_2^n + \lambda_3^n + \lambda_4^n)/4, \\ r_n &= (\lambda_1^n - \lambda_2^n + \lambda_3^n - \lambda_4^n)/4\sqrt{2}, \\ s_n &= (\lambda_1^n + \lambda_2^n - \lambda_3^n - \lambda_4^n)/4\sqrt{3}, \\ t_n &= (\lambda_1^n - \lambda_2^n - \lambda_3^n + \lambda_4^n)/4\sqrt{6}. \end{aligned}$$

Now note that $\lambda_1 > |\lambda_2|, \lambda_1 > |\lambda_3|, \lambda_1 > |\lambda_4|$. Hence,

$$\lim_{n\to\infty} r_n/q_n = \lim_{n\to\infty} \frac{1-(\lambda_2/\lambda_1)^n + (\lambda_3/\lambda_1)^n - (\lambda_4/\lambda_1)^n}{1+(\lambda_2/\lambda_1)^n + (\lambda_3/\lambda_1)^n + (\lambda_4/\lambda_1)^n} \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Analogously, we find that $\lim_{n\to\infty} s_n/q_n = \frac{1}{\sqrt{3}}$ and $\lim_{n\to\infty} t_n/q_n = \frac{1}{\sqrt{6}}$.

We mentioned earlier that conjugate numbers $a \pm b\sqrt{d}$ often arise as roots of a quadratic equation with integer coefficients. In light of the previous problem, the following one arises naturally :

9. Find a quartic equation with integer coefficients with $1+\sqrt{2}+\sqrt{3}$ as one of its roots.

In light of the previous example (and using the same notation), we can suspect that the roots of this equation will be $\lambda_1, \lambda_2, \lambda_3$ and λ_4. As such, we can write that equation as

$$(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)(x-\lambda_4) = 0,$$

or, equivalently,

$$(x-1-\sqrt{2}-\sqrt{3})(x-1+\sqrt{2}-\sqrt{3})(x-1-\sqrt{2}+\sqrt{3})(x-1+\sqrt{2}+\sqrt{3}) = 0.$$

After some algebraic manipulations, we get :

$$\begin{aligned} ((x-1)^2 - 5 - 2\sqrt{6})((x-1)^2 - 5 + 2\sqrt{6}) &= 0, \\ (x^2-2x-4)^2 - 24 &= 0, \\ x^4 - 4x^3 - 4x^2 - 16x - 8 &= 0. \end{aligned}$$

In fact, this is the lowest degree equation with integer coefficients with a root $\lambda_1 = 1+\sqrt{2}+\sqrt{3}$: prove this fact.

We will end this article with several exercises which continue some of the themes that we have touched upon. Some of them, however, require innovative ideas. [Ed. Note that this method does have its limitations ; in particular, one should be very careful when using it with problems involving inequalities or limits.]

Exercises.

1. Which is larger : $\sqrt{1979}+\sqrt{1980}$ or $\sqrt{1978}+\sqrt{1981}$?

2. Prove that for all positive x we have

$$|\sqrt{x^2+1}-x-1/2x|<\frac{1}{8x^3}.$$

3. Sketch the graph of the function $y=\sqrt{x^2-1}$ and prove that for all $|x|\geq 1$ we have

$$0<|x|-\sqrt{x^2-1}\leq\frac{1}{|x|}.$$

4. In the formula $\sqrt{2}=1+\frac{1}{\sqrt{2}+1}$, replace the $\sqrt{2}$ in the denominator of the right-hand side with the same expression to get

$$\sqrt{2}=1+\cfrac{1}{2+\frac{1}{\sqrt{2}+1}}.$$

In this formula, again replace the lowest $\sqrt{2}$ by $1+\frac{1}{\sqrt{2}+1}$ and so on n times. If we now replace the remaining $\sqrt{2}$ by 1 or 2, we will get two rational numbers p_n and q_n. Prove that $\sqrt{2}$ is bounded by these two numbers and that $\lim_{n\to\infty}p_n=\lim_{n\to\infty}q_n=\sqrt{2}$. (Did we see these numbers in one of the previous problems ?)

5. Prove that the equations $x^2-3y^2=1$ and $x^2-3y^2=2$ have infinitely many solutions in integers.

6. Prove that the function $y=\ln(\sqrt{1+x^2}+x)$ is odd and sketch its graph.

7. a) Prove that for any natural number n, we have

$$2(\sqrt{n+1}-1)<1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}<2\sqrt{n}-1.$$

b) Prove that the sequence

$$U_n=1+\frac{1}{\sqrt[4]{2^3}}+\frac{1}{\sqrt[4]{3^3}}+\cdots+\frac{1}{\sqrt[4]{n^3}}-4\sqrt[4]{n}$$

is decreasing and has a limit.

8. a) Prove that the sequence $\{(2+\sqrt{3})^n\}$, where $\{x\}$ denotes the fractional part of x, converges and find its limit.

b) What are the first 100 digits in the decimal expansion of $(\sqrt{50}+7)^{100}$?

9. Prove that for any natural number d, which is not a perfect square, there exists α such that for any m and n,

$$\left|\frac{m}{n} - \sqrt{d}\right| \geq \frac{1}{\alpha n^2}.$$

10. Prove that for any natural number n, the number $[(35+\sqrt{1157})^n/2^n]$ is divisible by 17 and in general for any natural numbers k and n, the number $[(2k+1+\sqrt{4k^2+1})^n/2^n]$ is divisible by k.

11. Prove that for any $p > 2$, there exists β such that for any n we have

$$\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{2+\beta}}}}}_{n \text{ radicals}} = \beta^{2^n} + \beta^{-2^n}.$$

12. Prove that the sequence $b_m = 1 + 17m^2$ contains infinitely many perfect squares.

13. Find a quadratic equation with integer coefficients with a root $(3+\sqrt{5})/4$.

14. Find a fourth degree equation with roots $\pm\sqrt{p} \pm \sqrt{q}$ and solve it as a biquadratic equation. By comparing the solution with the given roots, derive the following well-known formulas for the double radicals :

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} \pm \sqrt{\frac{a-\sqrt{a^2-b}}{2}},$$

where $a^2 > b > 0$ and $a > 0$.

15. Rationalize the denominators of the following fractions :

$$\text{a) } \frac{1}{1+\sqrt{2}+\sqrt{3}}; \qquad \text{b) } \frac{1}{\sqrt{10}+\sqrt{14}+\sqrt{21}+\sqrt{15}}.$$

16. A frog can jump from any vertex to any other vertex of an equilateral triangle ABC. Find the number a_n of distinct paths of exactly n jumps starting at A and ending at A. Prove that $\lim_{n\to\infty} a_{n+1}/a_n$ exists and find it.

This article appeared in Russian in Kvant, 1980(2), p. 26–32. It has been translated and adapted with permission.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1 mars 2017***.*

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4151. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Soit n un entier tel que $n \geq 2$. Déterminer les nombres réels t tels que

$$(a_1a_2\dots a_n)^t + (b_1b_2\dots b_n)^t + (c_1c_2\dots c_n)^t \leq 1$$

pour tout $a_i, b_i, c_i > 0$ où $a_i + b_i + c_i = 1$, $i = 1, 2, \dots, n$.

4152. *Proposé par Daniel Sitaru.*

Démontrer que si $a, b, c \in (0, \infty)$ alors :

$$\ln(1+a)^{\ln(1+b)^{\ln(1+c)}} \leq \ln^3(1+\sqrt[3]{abc}).$$

4153. *Proposé par Michel Bataille.*

Soit $ABCDEFG$ un heptagone régulier inscrit dans un cercle de rayon r. Démontrer que

$$\frac{1}{AB^3 \cdot BD} - \frac{1}{BD^3 \cdot DG} + \frac{1}{DG^3 \cdot GA} = \frac{1}{r^4}.$$

4154. *Proposé par Leonard Giugiuc.*

Déterminer X, une matrice 3×3 à coefficients entiers, telle que

$$X^4 = 3\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

4155. *Proposé par Mihaela Berindeanu.*

Démontrer que pour tout triangle ABC avec côtés de longueurs a, b, c et demi périmètre p, l'inégalité qui suit est valide :

$$\sqrt{\frac{2(p-a)}{c}} + \sqrt{\frac{2(p-b)}{a}} + \sqrt{\frac{2(p-c)}{b}} \geq \frac{p^2}{a^2+b^2+c^2-p^2}.$$

4156. *Proposé par José Luis Díaz–Barrero.*

Soient $x_1, x_2, \ldots, x_n$ des nombres réels positifs tels que $x_1 x_2 \ldots x_n = 1$. Démontrer que

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\left(\sqrt{x_k}+\sqrt{x_{k+1}}\right)^4}{x_k+x_{k+1}} \geq 8,$$

où les indices sont considérés modulo n.

4157. *Proposé par Michael Rozenberg, Leonard Giugiuc et Daniel Sitaru.*

Soient a, b et c des nombres réels positifs tels que $a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = k$ pour un certain nombre réel positif $k \geq 3$. Déterminer la valeur minimale de $ab+bc+ca$ en termes de k.

4158. *Proposé par George Apostolopoulos.*

Soient m_a, m_b et m_c les longueurs des médianes d'un triangle ABC où r est le rayon du cercle inscrit. Démontrer que

$$\frac{m_a+m_b+m_c}{\sin^2 A+\sin^2 B+\sin^2 C} \geq 4r.$$

4159. *Proposé par Michel Bataille.*

Démontrer que

$$\cosh x+\cosh y+\cosh(x+y) \leq 1+2\sqrt{\cosh x\cosh y\cosh(x+y)}$$

pour tous nombres réels x, y. Pour quels couples (x, y) l'égalité tient-elle ?

4160. *Proposé par Leonard Giugiuc et Marian Cucoanes.*

Soit ABC un triangle dont le rayon du cercle circonscrit est R, le rayon du cercle inscrit est r et le centre du cercle inscrit est I. Soient D, E et F les centres des cercles circonscrits des triangles IBC, ICA et IAB respectivement. Démontrer que

$$\frac{Aire(DEF)}{Aire(ABC)} = \frac{R}{2r}.$$

. . . .

4151. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let n be an integer with $n \geq 2$. Find the real numbers t such that

$$(a_1a_2\ldots a_n)^t+(b_1b_2\ldots b_n)^t+(c_1c_2\ldots c_n)^t \leq 1$$

for all $a_i, b_i, c_i > 0$ with $a_i+b_i+c_i = 1$, $i = 1, 2, \ldots, n$.

4152. *Proposed by Daniel Sitaru.*

Prove that if $a, b, c \in (0, \infty)$ then :

$$\ln(1+a)^{\ln(1+b)^{\ln(1+c)}} \leq \ln^3(1+\sqrt[3]{abc}).$$

4153. *Proposed by Michel Bataille.*

Let $ABCDEFG$ be a regular heptagon inscribed in a circle with radius r. Prove that

$$\frac{1}{AB^3 \cdot BD} - \frac{1}{BD^3 \cdot DG} + \frac{1}{DG^3 \cdot GA} = \frac{1}{r^4}.$$

4154. *Proposed by Leonard Giugiuc.*

Find a 3×3 matrix X with integer coefficients such that

$$X^4 = 3\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

4155. *Proposed by Mihaela Berindeanu.*

Show that in any triangle ABC with side lengths a, b, c and semi-perimeter p we have :

$$\sqrt{\frac{2(p-a)}{c}} + \sqrt{\frac{2(p-b)}{a}} + \sqrt{\frac{2(p-c)}{b}} \geq \frac{p^2}{a^2+b^2+c^2-p^2}.$$

4156. *Proposed by José Luis Díaz–Barrero.*

Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1 x_2 \ldots x_n = 1$. Prove that

$$\frac{1}{n}\sum_{k=1}^{n} \frac{\left(\sqrt{x_k} + \sqrt{x_{k+1}}\right)^4}{x_k + x_{k+1}} \geq 8,$$

where the subscripts are taken modulo n.

4157. *Proposed by Michael Rozenberg, Leonard Giugiuc and Daniel Sitaru.*

Let a, b and c be positive real numbers such that $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = k$ for some positive real number $k \geq 3$. Find the minimum value of $ab + bc + ca$ in terms of k.

4158. *Proposed by George Apostolopoulos.*

Let m_a, m_b and m_c be the lengths of medians of a triangle ABC with inradius r. Prove that

$$\frac{m_a + m_b + m_c}{\sin^2 A + \sin^2 B + \sin^2 C} \geq 4r.$$

4159. *Proposed by Michel Bataille.*

Prove that

$$\cosh x + \cosh y + \cosh(x+y) \leq 1 + 2\sqrt{\cosh x \cosh y \cosh(x+y)}$$

for any real numbers x, y. For which pairs (x, y) does equality hold ?

4160. *Proposed by Leonard Giugiuc and Marian Cucoanes.*

Let ABC be a triangle with circumradius R, inradius r and incenter I. Let D, E and F be the circumcenters of the triangles IBC, ICA and IAB, respectively. Prove that

$$\frac{Area(DEF)}{Area(ABC)} = \frac{R}{2r}.$$

Math Quotes

Geometry enlightens the intellect and sets one's mind right. All of its proofs are very clear and orderly. It is hardly possible for errors to enter into geometrical reasoning, because it is well arranged and orderly. Thus, the mind that constantly applies itself to geometry is not likely to fall into error. In this convenient way, the person who knows geometry acquires intelligence.

Ibn Khaldun in "The Muqaddimah. An Introduction to History", Princeton University Press.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015 : 41(6), p. 260–263.

4051. *Proposed by Arkady Alt.*

Let a, b and c be the side lengths of a triangle. Prove that

$$(a+b+c)\left(a^2b^2+b^2c^2+c^2a^2\right) \geq 3abc\left(a^2+b^2+c^2\right).$$

We received eleven correct solutions. We present two solutions.

Solution 1, by Michel Bataille.

A key to a solution is contained in solution 2 to ***Crux*** problem **3991** published in 41 (1) (December 2015).

Setting $a = \frac{y+z}{2}$, $b = \frac{z+x}{2}$, $c = \frac{x+y}{2}$ transforms the proposed inequality into

$$x^5+y^5+z^5+x^2y^2z+x^2yz^2+xy^2z^2 \geq x^3y^2+x^2y^3+y^3z^2+y^2x^3+z^3x^2+z^2x^3, \quad (1)$$

where x, y, z are positive real numbers. The general Schur inequality is

$$u^r(u-v)(u-w)+v^r(v-w)(v-u)+w^r(w-u)(w-v) \geq 0$$

for $u, v, w \geq 0$ and r real. We take $r = \frac{1}{2}$ and $u = x^2$, $v = y^2$, $w = z^2$ and obtain

$$x(x^2-y^2)(x^2-z^2)+y(y^2-z^2)(y^2-x^2)+z(z^2-x^2)(z^2-y^2) \geq 0.$$

Expanding and arranging directly leads to (1).

Solution 2, by Titu Zvonaru.

By Consequence 16.3, p. 156 from [1], all symmetric three-variable polynomials of degree less than or equal to five achieve their maximum and minimum values on $\mathbb{R}^*$ at (a, b, c) if and only if $(a-b)(b-c)(c-a) = 0$ or $abc = 0$. It thus suffices to prove the given inequality for $b = c$ and $c = 0$.

If $b = c$, then we have to prove that

$$\begin{aligned}
(a+2b)(2a^2b^2+b^4) &\geq 3ab^2(a^2+2b^2)\\
2a^3b^2+ab^4+4a^2b^3+2b^5 &\geq 3a^3b^2+6ab^4\\
4a^2b^3+2b^5 &\geq a^3b^2+5ab^4\\
b^2(a-b)^2(2b-a) &\geq 0,
\end{aligned}$$

which is true by the triangle inequality.

If $c = 0$, then we have to prove that

$$(a+b)a^2b^2 \geq 0,$$

which is true.

Equality holds if and only if $a = b = c$.

[1] Z. Cvetkovski, *Inequalities - Theorems, Techniques and Selected Problems*, Springer-Verlag, 2012.

4052. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let $k < 0$ be a fixed real number. Let a, b, c and d be real numbers such that $a+b+c+d = 0$ and $ab+bc+cd+da+ac+bd = k$. Prove that $abcd \geq -k^2/12$ and determine when equality holds.

We received six submissions, of which five were correct and complete. We present the solution by Oliver Geupel, slightly modified by the editor.

It is sufficient to consider the case when three of the numbers a, b, c, d have the same sign. Otherwise, $abcd \geq 0 > -k^2/12$, and we're done.

By symmetry, there is no loss of generality in assuming that it is a, b, and c which have the same sign. Then $a+b+c+d = 0$ gives us $d = -(a+b+c)$, so

$$\begin{aligned}
k &= ab+bc+cd+da+ac+bd \\
&= (a+b+c)d + (ab+bc+ca) \\
&= -(a+b+c)^2 + (ab+bc+ca) \\
&= -(a^2+b^2+c^2+ab+bc+ca),
\end{aligned}$$

and the inequality we want to prove can be rewritten as

$$-abc(a+b+c) \geq -(a^2+b^2+c^2+ab+bc+ca)^2/12,$$

or, equivalently,

$$12abc(a+b+c) \leq (a^2+b^2+c^2+ab+bc+ca)^2. \tag{1}$$

Let $x = |a|$, $y = |b|$, $z = |c|$, so that x, y, z are nonnegative real numbers. Since a, b and c have the same sign by assumption,

$$\begin{aligned}
&abc(a+b+c) = xyz(x+y+z), \text{ and} \\
&a^2+b^2+c^2+ab+bc+ca = x^2+y^2+z^2+xy+yz+zx,
\end{aligned}$$

so it is sufficient to prove that (1) holds with a,b and c replaced by x, y and z, respectively.

By the AM-GM Inequality (where the right hand side is treated as a sum of 8 terms, with terms repeated as indicated by the coefficients), we have

$$8x^2yz \le x^4 + 2x^3y + 2x^3z + 3y^2z^2. \tag{2}$$

The equality holds only when $x^4 = x^3y = x^3z = y^2z^2$, that is when either $x = y = z$ or $x = yz = 0$. Summing up inequality (2) and its two cyclic variants, and adding terms to both sides so we can complete the square on the right hand side, we obtain

$$12xyz(x+y+z) \le (x^2+y^2+z^2+xy+yz+zx)^2, \tag{3}$$

where the equality holds if and only if $x = y = z$. This shows that (1) holds, and thus concludes the proof that $abcd \ge -k^2/12$.

It follows from the preceding steps that the equality holds if and only if three of the four numbers a, b, c, and d are equal. A straightforward computation (from $a+b+c+d=0$ and $abcd = -k^2/12$) shows that the common value is $\pm\sqrt{-k/6}$, and that the fourth number has the value $\mp 3\sqrt{-k/6}$.

4053. *Proposed by Šefket Arslanagić.*

Prove that

$$\frac{\cos\alpha\cos\beta}{\cos\gamma} + \frac{\cos\beta\cos\gamma}{\cos\alpha} + \frac{\cos\alpha\cos\gamma}{\cos\beta} \ge \frac{3}{2},$$

where α, β and γ are angles of an acute triangle.

We received 13 correct solutions. We present a composite of essentially the same solution by José Luis Díaz-Barrero, Dionne Bailey, Elsie Campbell, and Charles R. Diminnie (joint), Henry Ricardo, and Lorian Saceanu.

Since $\alpha, \beta, \gamma \in (0, \frac{\pi}{2})$ we have

$$\begin{aligned}
\frac{\cos\alpha\cos\beta}{\cos\gamma} &= \frac{\cos\alpha\cos\beta}{\cos\gamma} \cdot \frac{\tan\alpha + \tan\beta}{\tan\alpha + \tan\beta} \\
&= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\gamma} \cdot \frac{1}{\tan\alpha + \tan\beta} \\
&= \frac{\sin(\pi - \gamma)}{\cos\gamma} \cdot \frac{1}{\tan\alpha + \tan\beta} \\
&= \frac{\tan\gamma}{\tan\alpha + \tan\beta}.
\end{aligned}$$

By the well-known Nesbitt's Inequality which states that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}$$

with equality if and only if $a = b = c$, we then have

$$\sum_{\text{cyc}} \frac{\cos\alpha\cos\beta}{\cos\gamma} = \sum_{\text{cyc}} \frac{\tan\gamma}{\tan\alpha + \tan\beta} \ge \frac{3}{2}$$

with equality if and only if $\alpha = \beta = \gamma$; in other words, if the given triangle is equilateral.

Editor's comment. Both Ricardo and Diminnie et al pointed out that the current problem is the special case when $m = 0$ of problem #5381 in the January 2016 issue of *School, Science and Mathematics* :

If A, B, C are angles of an acute triangle, then

$$\sum \left(\frac{\cos A \cos B}{\cos C} \right)^{m+1} \geq \frac{3}{2^{m+1}}$$

for all nonnegative integers m.

Interestingly, this general problem was proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu, two regular contributors to the ***Crux*** problem section. Ricardo actually gave a proof to the general inequality as well as a proof for the above case. The general proof uses the same argument in the featured solution above together with the power mean inequality and it appeared as solution 3 in the April 2016 issue of that journal (see `www.ssma.org/publications`).

4054. *Proposed by Mihaela Berindeanu.*

Find a prime p such that the number

$$(p^2 - 4)^2 - 117(p^2 - 4) + 990$$

has a minimum digit sum.

We received eight correct and complete solutions, all of which were very similar. We present the solution by Joseph DiMuro.

Let

$$f(p) = (p^2 - 4)^2 - 117(p^2 - 4) + 990.$$

If $p \neq 3$ is a prime, then $p = 3n \pm 1$ for some integer n. Then

$$p^2 - 4 = 9n^2 \pm 6n - 3,$$

a multiple of 3 ; thus, $(p^2 - 4)^2$ is a multiple of 9. But 117 and 990 are also multiples of 9. So $f(p)$ is a multiple of 9, as is its digit sum.

If we had $f(p) = 0$, then by the quadratic formula we would have

$$p^2 - 4 = \frac{117 \pm \sqrt{9729}}{2},$$

which has no integer solutions, so $f(p) \neq 0$. Therefore, the digit sum of $f(p)$ must be at least 9 when p is a prime other than 3.

However, the digit sum of $f(3) = 430$ is 7, so the minimum digit sum is 7, attained for $p = 3$.

4055. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Prove that if $x, y > 0, x \neq y$ and $0 < a < b < \frac{1}{2} < c < d < 1$ then :

$$x\left[\left(\frac{y}{x}\right)^a+\left(\frac{y}{x}\right)^d-\left(\frac{y}{x}\right)^b-\left(\frac{y}{x}\right)^c\right] > y\left[\left(\frac{x}{y}\right)^b+\left(\frac{x}{y}\right)^c-\left(\frac{x}{y}\right)^a-\left(\frac{x}{y}\right)^d\right].$$

We received three correct solutions and feature two of them that are similar.

Solution 1, by Michel Bataille.

With $t = \frac{x}{y}$, the inequality can be re-written as

$$(t^d + t^{1-d}) - (t^c + t^{1-c}) > (t^b + t^{1-b}) - (t^a + t^{1-a}). \tag{1}$$

Let us fix $t > 0, t \neq 1$ and set $f(u) = t^u$ and $g(u) = f(u) + f(1-u)$ so that (1) is just

$$g(d) - g(c) > g(b) - g(a). \tag{2}$$

From the Mean Value Theorem, we have

$$g(b) - g(a) = (b-a)g'(\alpha), \qquad g(d) - g(c) = (d-c)g'(\beta)$$

for some $\alpha \in (a, b)$ and $\beta \in (c, d)$.

Since $g'(u) = (\ln t)(t^u - t^{1-u}) = (\ln t)(f(u) - f(1-u))$, (2) becomes

$$(d-c)(\ln t)(f(\beta) - f(1-\beta)) > (b-a)(\ln t)(f(\alpha) - f(1-\alpha)). \tag{3}$$

Applying the Mean Value Theorem again, we have $f(\beta)-f(1-\beta) = (2\beta-1)(\ln t)t^{\sigma}$ and $f(\alpha) - f(1-\alpha) = (2\alpha-1)(\ln t)t^{\tau}$ with σ between β and $1-\beta$ and τ between α and $1-\alpha$.

Substituting into (3) and because $(\ln t)^2 > 0$, we are reduced to proving

$$(d-c)(2\beta-1)t^{\sigma} > (b-a)(2\alpha-1)t^{\tau}. \tag{4}$$

Now, on the one hand $d-c > 0,\ t^{\sigma} > 0,\ 2\beta - 1 > 0$ (note that $\beta \in (c, d)$, hence $\beta > \frac{1}{2}$) and on the other hand, $b-a > 0,\ t^{\tau} > 0,\ 2\alpha - 1 < 0$ (since $\alpha \in (a, b)$). Thus,

$$(d-c)(2\beta-1)t^{\sigma} > 0 > (b-a)(2\alpha-1)t^{\tau}$$

and (4) follows.

Solution 2, by Daniel Sitaru and Leonard Giugiuc.

Let $f : [0, 1] \to \mathbb{R}, f(\alpha) = \frac{x^{1-\alpha}y^{\alpha}+x^{\alpha}y^{1-\alpha}}{2}$, with $x, y \in (0, \infty), x \neq y$. We have :

$$\lim_{\substack{\alpha\to 0\\ \alpha>0}} f(\alpha) = \frac{x+y}{2}, \quad \lim_{\substack{\alpha\to 1\\ \alpha<1}} f(\alpha) = \frac{x+y}{2}.$$

Since

$$f'(\alpha) = \frac{1}{2}(\ln y - \ln x)(x^{1-\alpha}y^{\alpha} - x^{\alpha}y^{1-\alpha}),$$

then

$$f'(\alpha) = 0 \quad \Rightarrow \quad x^{1-\alpha}y^{\alpha} = x^{\alpha}y^{1-\alpha} \quad \Rightarrow \quad \left(\frac{x}{y}\right)^{1-2\alpha} = 0 \Rightarrow a = \frac{1}{2}.$$

Therefore, $\min f(\alpha) = f\left(\frac{1}{2}\right) = \sqrt{xy}$ and $f(a) > f(b) > f\left(\frac{1}{2}\right)$, $\quad f\left(\frac{1}{2}\right) < f(c) < f(d)$. By adding, we get $f(a) + f(d) > f(b) + f(c)$ or

$$\begin{aligned} \frac{x^{1-a}y^a + x^a y^{1-a}}{2} + \frac{x^{1-d}y^d + x^d y^{1-d}}{2} &> \frac{x^{1-b}y^b + x^b y^{1-b}}{2} \\ &> \frac{x^{1-b}y^b + x^b y^{1-b}}{2} + \frac{x^{1-c}y^c + x^c y^{1-c}}{2}. \end{aligned}$$

This results in

$$x\left[\left(\frac{y}{x}\right)^a + \left(\frac{y}{x}\right)^d - \left(\frac{y}{x}\right)^b - \left(\frac{y}{x}\right)^c\right] > y\left[\left(\frac{x}{y}\right)^b + \left(\frac{x}{y}\right)^c - \left(\frac{x}{y}\right)^a - \left(\frac{x}{y}\right)^d\right].$$

4056. *Proposed by Idrissi Abdelkrim-Amine.*

Let n be an integer, $n \geq 2$. Consider real numbers a_k, $1 \leq k \leq n$ such that $a_1 \geq 1 \geq a_2 \geq \ldots \geq a_n > 0$ and $a_1 a_2 \ldots a_n = 1$. Prove that

$$\sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} \frac{1}{a_k}.$$

We received eight solutions of which six were correct. We present the solution by Roy Barbara.

Note first that if $0 < a, b \leq 1$, then

$$a - \frac{1}{a} + b - \frac{1}{b} \geq ab - \frac{1}{ab}. \tag{1}$$

Indeed, multiplying by ab, (1) is equivalent, in succession, to

$$\begin{array}{rccl} & a^2b - b + ab^2 - a & \geq & (ab)^2 - 1 \\ \text{or} & (a+b)(ab-1) & \geq & (ab+1)(ab-1) \\ \text{or} & (1-ab)(1-a)(1-b) & \geq & 0, \end{array}$$

which is true.

We now prove the given inequality by using induction on $n \geq 2$.

The case when $n = 2$ is trivial. Suppose the inequality holds for some $n \geq 2$, and let a_i, $i = 1, 2, \cdots, n+1$ satisfy $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1}$ and $a_1 a_2 \cdots a_n a_{n+1} = 1$. We need to prove that

$$\left(\sum_{k=1}^{n-1} a_k - \sum_{k=1}^{n-1} \frac{1}{a_k}\right) + \left(a_n - \frac{1}{a_n}\right) + \left(a_{n+1} - \frac{1}{a_{n+1}}\right) \geq 0. \tag{2}$$

Set $b_n = a_n a_{n+1}$.

Then clearly $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_{n-1} \geq b_n$ and $a_1 a_2 \cdots a_{n-1} b_n = 1$.

By the induction hypothesis, we have

$$\left(\sum_{k=1}^{n-1} a_k - \sum_{k=1}^{n-1} \frac{1}{a_k}\right) + \left(b_n - \frac{1}{b_n}\right) \geq 0. \tag{3}$$

Hence, to get (2), it suffices to prove that

$$\left(a_n - \frac{1}{a_n}\right) + \left(a_{n+1} - \frac{1}{a_{n+1}}\right) \geq \left(b_n - \frac{1}{b_n}\right),$$

that is,

$$\left(a_n - \frac{1}{a_n}\right) + \left(a_{n+1} - \frac{1}{a_{n+1}}\right) \geq \left(a_n a_{n+1} - \frac{1}{a_n a_{n+1}}\right),$$

which is true by (1) as $0 < a_n, a_{n+1} \leq 1$.

Editor's comment. The proposer of the current problem remarked that the given inequality was inspired by the following inequality due to Leonard Giugiuc : $(\sum a_k)^2 \geq n \sum \frac{1}{a_k}$, where the a_k's satisfy the same conditions given in the current problem.

4057. *Proposed by Eeshan Banerjee.*

Let ABC be a non-obtuse triangle with circumradius R, inradius r and area Δ. Prove that

$$\Delta < \left(\frac{\frac{1}{r} + 3R + 3}{7}\right)^7.$$

We received four correct solutions. We present a composite of very similar solutions by Michel Bataille and Andrea Fanchini.

With AM-GM, we have that

$$\frac{\frac{1}{r} + R + R + R + 1 + 1 + 1}{7} \geq \sqrt[7]{\frac{R^3}{r}},$$

so it suffices to prove that

$$\Delta < \frac{R^3}{r}.$$

From Euler's inequality $R \geq 2r$ and the inequality $R \geq \frac{2s}{3\sqrt{3}}$, we have

$$\frac{R^3}{r} \geq \frac{4r^2 \cdot 2s}{r \cdot 3\sqrt{3}} = \frac{8\sqrt{3}}{9} \cdot rs = \frac{8\sqrt{3}}{9} \cdot \Delta > \Delta,$$

completing the proof.

4058. *Proposed by Francisco Javier García Capitán.*

Let ABC be a triangle. For any X on line BC, let X_b and X_c be the circumcenters of the triangles ABX and AXC, and let P be the intersection point of BX_c and CX_b. Prove that the locus of P as X varies along the line BC is the conic through the centroid, orthocenter, and vertices B and C, and whose tangents at these vertices are the corresponding symmedians. (Recall that a symmedian is the reflection of a median in the bisector of the corresponding angle.)

We received two submissions, both of which were correct. We feature the solution by Michel Bataille with a few details added from the proposer's solution.

As usual, set $BC = a, CA = b, AB = c$. Should $a = b$, then CX_b will be the perpendicular bisector of AB, which immediately implies that this line is the locus of P; similarly, should $a = c$, then the locus of P would be the perpendicular bisector of AC. Assume therefore that $a \neq b, c$. We shall see that under this further assumption the locus is a hyperbola. For our argument we shall use barycentric coordinates relative to (A, B, C), and the following notation :

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2}.$$

For later use, here are a few readily checked relations satisfied by these numbers :

$$S_B + S_C = a^2, \quad S_C + S_A = b^2, \quad S_A + S_B = c^2,$$

and

$$\begin{aligned} a^2 S_A + S_B S_C &= b^2 S_B + S_C S_A = c^2 S_C + S_A S_B \\ &= S_A S_B + S_B S_C + S_C S_A = \frac{1}{2}(a^2 S_A + b^2 S_B + c^2 S_C). \end{aligned}$$

If $(f : g : h)$ is the point at infinity of a line ℓ, then $(gS_B - hS_C : hS_C - fS_A : fS_A - gS_B)$ is the point at infinity of the perpendiculars to ℓ. With the help of this property, we easily obtain the equations of the perpendicular bisector ℓ_1 of AB and ℓ_2 of AC,

$$\ell_1 : c^2 x - c^2 y + (a^2 - b^2) z = 0, \qquad \ell_2 : b^2 x + (a^2 - c^2) y - b^2 z = 0.$$

If $X = (0 : \beta : \gamma)$ with $\beta + \gamma = 1$, the point at infinity on AX is $(-1 : \beta : \gamma)$, hence the one on the perpendiculars to AX is $(\beta S_B - \gamma S_C : \gamma S_C + S_A : -S_A - \beta S_B)$. It follows that the perpendicular bisector m of AX is

$$x(\beta^2 S_B + \gamma^2 S_C + S_A) + y(\gamma^2 a^2 - c^2) + z(\beta^2 a^2 - b^2) = 0.$$

Note that

$$\begin{aligned}
&\beta^2 S_B + \gamma^2 S_C + S_A \\
&= \beta^2 S_B + (1-\beta)^2 S_C + S_A \\
&= a^2\beta^2 - 2\beta S_C + b^2 = a^2\beta^2 - \beta(a^2+b^2-c^2) + b^2 = \beta c^2 + \gamma b^2 - a^2\beta\gamma.
\end{aligned}$$

Let $x_1 = a^2(S_A + \beta S_B),\ y_1 = a^2\beta S_A + b^2 S_B,\ z_1 = c^2(S_C - a^2\beta)$. It is rather long but easy to check that (x_1, y_1, z_1) satisfies both the equations of ℓ_1 and m. Thus $X_b = (x_1 : y_1 : z_1)$. Similarly, we obtain $X_c = (x_2 : y_2 : z_2)$ with $x_2 = a^2(S_A + \gamma S_C),\ y_2 = b^2(S_B - a^2\gamma),\ z_2 = a^2\gamma S_A + c^2 S_C$.

The equations of $CX_b : xy_1 - yx_1 = 0$ and $BX_c : xz_2 - zx_2 = 0$ then provide $P = (u : v : w) = (x_1x_2 : x_2y_1 : x_1z_2)$ so that

$$v = \frac{a^2\beta S_A + b^2 S_B}{a^2(S_A + \beta S_B)} \cdot u, \quad w = \frac{a^2\gamma S_A + c^2 S_C}{a^2(S_A + \gamma S_C)} \cdot u,$$

from which we obtain

$$a^2\beta = \frac{b^2 S_B u - a^2 S_A v}{vS_B - uS_A}, \qquad a^2\gamma = \frac{c^2 S_C u - a^2 S_A w}{wS_C - uS_A}.$$

Eliminating β, γ (through $\beta + \gamma = 1$) yields a necessary and sufficient condition on u, v, w for P to belong to the desired locus, namely

$$\begin{aligned}
u^2 S_A(a^2 S_A + b^2 S_B + c^2 S_C) - uvc^2(a^2 S_A + S_B S_C) - wub^2(a^2 S_A + S_B S_C)& \\
+vwa^2(S_A S_B + S_B S_C + S_C S_A) = 0;&
\end{aligned}$$

that is, $2u^2 S_A - c^2 uv - b^2 uw + a^2 vw = 0$.

Thus, the locus of P is the conic Γ with equation

$$(b^2 + c^2 - a^2)x^2 - c^2 xy + a^2 yz - b^2 zx = 0.$$

Note that because we have assumed that $a \neq b, c$, the discriminant of the conic (namely, $\frac{a^2}{4}(b^2 - a^2)(c^2 - a^2)$) is nonzero; because the coefficient of y^2 is zero, this nondegenerate conic must be a hyperbola, as claimed. Let $\mathcal{C}(x, y, z)$ be the left-hand side of the equation. We readily find that

$$\mathcal{C}(0,1,0) = \mathcal{C}(0,0,1) = \mathcal{C}(1,1,1) = \mathcal{C}(S_B S_C, S_C S_A, S_A S_B) = 0,$$

hence Γ passes through B, C, G, H (respectively), where $G = (1 : 1 : 1)$ and $H = (S_B S_C : S_C S_A : S_A S_B)$ denote the centroid and the orthocenter of ABC. In addition, the equation of the tangent to Γ at $(x_0 : y_0 : z_0)$ is

$$2xx_0 S_A - \frac{1}{2}c^2(x_0 y + xy_0) - \frac{1}{2}b^2(x_0 z + xz_0) + \frac{1}{2}a^2(y_0 z + yz_0) = 0.$$

In particular, the tangent to Γ at B is $c^2 x - a^2 z = 0$, a line passing through the Lemoine point $K = (a^2 : b^2 : c^2)$ and through B. Therefore this tangent is the symmedian through vertex B. Similarly, the tangent to Γ at C is the symmedian through vertex C.

Editor's comments. The proposer noted the following theorem to be a consequence of his problem :

> If a conic passes through the vertices B and C of a triangle ABC, while the tangents at those points are the corresponding symmedians, then the centroid of the triangle lies on the conic if and only if the orthocenter does also.

4059. *Proposed by Marcel Chiriţa.*

Let $a, b \in (0, \infty)$, $a \neq b$. Determine the functions $f : \mathbb{R} \mapsto \mathbb{R} \setminus \{0\}$ such that

$$f(ax) = e^x f(bx), \quad \forall x \in \mathbb{R}.$$

We received four submissions of which three were correct and complete. We present the solution by Michel Bataille.

We show the following :

> Let $\mathcal{P}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R} \setminus \{0\}$ that are periodic with period $\ln(b/a)$ and let $u : t \mapsto u(t) = \ln(|t|)$ for $t \neq 0$. Then the solutions are the functions $t \mapsto g(t) \cdot e^{\frac{t}{a-b}}$ where the function g is defined by
>
> $$g(t) = p(u(t)) \ (t > 0), \quad g(0) = \alpha, \quad g(t) = q(u(t)) \ (t < 0)$$
>
> for some $p, q \in \mathcal{P}$ and some $\alpha \in \mathbb{R} \setminus \{0\}$.

First, a remark : if we set $g(x) = f(x) \cdot e^{-\frac{x}{a-b}}$, a simple calculation shows that solving the given equation boils down to solving the functional equation $g(ax) = g(bx)$ for functions $g : \mathbb{R} \to \mathbb{R} \setminus \{0\}$, f being then defined by $f(x) = g(x) \cdot e^{\frac{x}{a-b}}$ for $x \in \mathbb{R}$. Since $\frac{x}{a}$ takes all real values when x does, substituting $\frac{x}{a}$ for x even reduces the problem to seeking functions $g : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ such that

$$g(x) = g\left(x \cdot \frac{b}{a}\right) \tag{1}$$

for all $x \in \mathbb{R}$. Let g be a solution. Then, for $x > 0$, we have

$$g(e^{\ln x}) = g(e^{\ln x + \ln(b/a)}),$$

that is, $p(\ln x) = p(\ln x + \ln(b/a))$ if we set $p = g \circ \exp$. Since $\ln x$ takes all real values as x describes $(0, \infty)$, it follows that p is periodic with period $\ln(b/a)$ and does not take the value 0, i.e. $p \in \mathcal{P}$, and that $g(x) = p(\ln x) = p(\ln(|x|)$ for positive x.

Similarly, for $x < 0$, we have

$$g(-e^{\ln(-x)}) = g(-e^{\ln(-x) + \ln(b/a)})$$

that is, $q(\ln(-x)) = q(\ln(-x) + \ln(b/a))$ where $q = g \circ (-\exp)$ is an element of $\mathcal{P}$, so that $g(x) = q(\ln(-x)) = q(\ln(|x|)$.

Conversely, define g by

$$g(t) = p(u(t))\ (t > 0), \quad g(0) = \alpha, \quad g(t) = q(u(t))\ (t < 0)$$

where $p, q \in \mathcal{P}$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Then, g takes only nonzero values and, if $x > 0$, $g\left(x \cdot \left(\frac{b}{a}\right)\right) = p\left(\ln\left(x \cdot \left(\frac{b}{a}\right)\right)\right) = p(\ln x + \ln(b/a)) = p(\ln x) = g(x)$ and if $x < 0$, then $g\left(x \cdot \left(\frac{b}{a}\right)\right) = q\left(\ln\left(-x \cdot \left(\frac{b}{a}\right)\right)\right) = q(\ln(-x) + \ln(b/a)) = q(\ln(-x)) = g(x)$. Thus, the equality $g\left(x \cdot \left(\frac{b}{a}\right)\right) = g(x)$ holds for all real x (it is obvious for $x = 0$) and so g satisfies (1) with $g(x) \neq 0$ for all real x. The proof is complete.

Remark. The solutions which are continuous at 0 are the functions $t \mapsto \alpha e^{\frac{t}{a-b}}$ where α is a nonzero real constant : with the above notations, f is continuous at 0 if and only if g is. So, we consider a solution g of (1), with g continuous at 0. Suppose first that $b < a$. Then,

$$g\left(x \cdot \left(\frac{b}{a}\right)^k\right) = g\left(x \cdot \left(\frac{b}{a}\right)^{k+1}\right)$$

if k is a positive integer; hence, by an immediate induction, we see that $g(x) = g\left(x \cdot \left(\frac{b}{a}\right)^n\right)$ for all positive integers n. Since $0 < \frac{b}{a} < 1$, we have $\lim\limits_{n\to\infty} x \cdot \left(\frac{b}{a}\right)^n = 0$ and so $\lim\limits_{n\to\infty} g\left(x \cdot \left(\frac{b}{a}\right)^n\right) = g(0)$. As a result, $g(x) = g(0)$ for any $x \in \mathbb{R}$. If $b > a$, the treatment is similar using the equation $g(x) = g\left(x \cdot \frac{a}{b}\right)$ which holds for all x as well. Thus g is a constant function. Conversely, any constant function $\mathbb{R} \to \mathbb{R} \setminus \{0\}$ is obviously a solution of (1).

Editor's comments. Roy Barbara proposed and solved the following generalization of the problem. Set $k = a/b$ and $t = bx$ ($t \in \mathbb{R}$). If $\varphi : \mathbb{R} \to \mathbb{R}^*$ is a function with $\varphi(0) = 1$ and $k \in (0, \infty)$, $k \neq 1$, determine all the functions $f : \mathbb{R} \to \mathbb{R}^*$ satisfying

$$f(kt) = \varphi(t) \cdot f(t) \qquad \forall t \in \mathbb{R}.$$

Unfortunately, Marcel Chiriţa passed away on 29 February 2016 and he cannot enjoy the beautiful solution given above. We will miss him and we will miss his precious contribution to the journal.

4060. *Proposed by Michel Bataille.*

Let

$$f(x, y) = \frac{xy(x + y)}{(1 - x - y)^3}.$$

Find the range of $f(x, y)$ when its domain is restricted to the circle S that satisfies the equation $x^2 + y^2 = 1 - 2x - 2y$.

Five correct solutions were submitted from four people. Two others made incorrect submissions. We present two solutions.

Solution 1, following the approach of Kee-Wai Lau.

Suppose that $(x,y) \in S$ and let $t = x+y$. Then

$$(t+2)^2 = 2(x^2+y^2+2x+2y-1)+6-(x-y)^2 = 6-(x-y)^2 \le 6,$$

so that $-2-\sqrt{6} \le t \le -2+\sqrt{6} < 1$, with equality possible only if $x=y$. Since

$$2xy = (x+y)^2 + 2(x+y) - 1 = t^2+2t-1,$$

we have

$$f(x,y) = \frac{t(t^2+2t-1)}{2(1-t)^3}.$$

The condition for S can be written as

$$(x+1)^2+(y+1)^2 = 3,$$

so that S is a circle with centre $(-1,-1)$ that intersects the line $y=x$ at

$$\left(\frac{1}{2}(-2-\sqrt{6}), \frac{1}{2}(-2-\sqrt{6})\right) \quad \text{and} \quad \left(\frac{1}{2}(-2+\sqrt{6}), \frac{1}{2}(-2+\sqrt{6})\right).$$

These points correspond to the limiting values of t. From the geometry, it is easily seen that as (x,y) ranges over S, the variable t assumes all values in the closed interval $[-2-\sqrt{6}, -2+\sqrt{6}]$. [Alternatively, using the theory of the quadratic, one can determine that the system

$$\begin{cases} (x+1)+(y+1) = t+2, \\ (x+1)^2+(y+1)^2 = 3, \end{cases}$$

is solvable for real values of x and y if and only if $(t+2)^2 \le 6$.]

Since $f(x,y) = -\sqrt{6}/18$ when $t = -2-\sqrt{6}$ and $f(x,y) = \sqrt{6}/18$ when $t = -2+\sqrt{6}$, and since f is continuous in t, the range of f includes the closed interval $[-\sqrt{6}/18, \sqrt{6}/18]$.

Observe that

$$\begin{aligned} f(x,y) + \frac{\sqrt{6}}{18} &= \frac{(9-\sqrt{6})t^3 + (18+3\sqrt{6})t^2 - (9+3\sqrt{6})t + \sqrt{6}}{18(1-t)^3} \\ &= \frac{(t+2+\sqrt{6})[(9-\sqrt{6})t^2 + (6-4\sqrt{6})t + (3-\sqrt{6}]}{18(1-t)^3}. \end{aligned}$$

Since the quadratic factor, having zero discriminant, is the square of a linear polynomial, and since $1-t>0$, $f(x,y) \ge -\sqrt{6}/18$ when $t \ge -2-\sqrt{6}$. Likewise

$$\begin{aligned} f(x,y) - \frac{\sqrt{6}}{18} &= \frac{(9+\sqrt{6})t^3 + (18-3\sqrt{6})t^3 + (-9+3\sqrt{6})t - \sqrt{6}}{18(1-t)^3} \\ &= \frac{(t+2-\sqrt{6})[(9+\sqrt{6})t^2 + (6+4\sqrt{6})t + (3+\sqrt{6})]}{18(1-t)^3} \le 0, \end{aligned}$$

so that $f(x,y) \le \sqrt{6}/18$ when $t \le -2+\sqrt{6}$. Thus the range of f is exactly $[-\sqrt{6}/18, \sqrt{6}/18]$.

Solution 2, by the proposer.

Let $u = x^2 + y^2$ and

$$a = \frac{2x}{u+1} \qquad b = \frac{2y}{u+1} \qquad c = \frac{u-1}{u+1}.$$

When $(x,y) \in S$, we have that

$$u+1 = 2(1-x-y) \quad \text{and} \quad u-1 = x^2+y^2-1 = -2(x+y).$$

It can be checked that

$$a+b+c=0, \quad a^2+b^2+c^2=1 \quad \text{and so} \quad ab+bc+ca = -\frac{1}{2}.$$

Also

$$abc = -\frac{xy(x+y)}{(1-x-y)^3} = -f(x,y).$$

Thus, the real numbers a, b, c are the roots of the polynomial

$$t^3 - \frac{1}{2}x + f(x,y).$$

Since the roots are all real, we must have $4 \cdot (1/8) \ge 27(f(x,y))^2$, so that

$$-\frac{1}{\sqrt{54}} \le f(x,y) \le \frac{1}{\sqrt{54}}$$

and $f(x,y)$ belongs to the closed interval $[-\sqrt{6}/18, \sqrt{6}/18]$.

Conversely, let $p \in [-\sqrt{6}/18, \sqrt{6}/18]$. We show that $p = f(x,y)$ for some $(x,y) \in S$. This is true for $p=0$, since $f(0,-1+\sqrt{2}) = 0$. Suppose $p \ne 0$. Consider the polynomial $t^3 - \frac{1}{2}t + p$ and let a, b, c be its roots. Since the discriminant condition $4 \cdot (1/8) \ge 27p^2$ holds, the roots are all real. Since $a+b+c=0$ and $ab+bc+ca = -1/2$, we have that $a^2+b^2+c^2 = 1$. Since $abc \ne 0$, $-1 < c < 1$, so that $c = (v-1)/(v+1)$ for some $v > 0$. Now, let

$$x = \frac{a(1+v)}{2} \qquad \text{and} \qquad y = \frac{b(1+v)}{2}.$$

Then $x^2+y^2 = v = 1-2x-2y$, so that $(x,y) \in S$. Moreover

$$\begin{aligned} f(x,y) = \frac{xy(x+y)}{(1-x-y)^3} &= \frac{ab}{4}(v+1)^2 \cdot \frac{(v+1)(a+b)}{2} \cdot \frac{8}{(v+1)^3} \\ &= ab(a+b) = -abc = p. \end{aligned}$$

Editor's Comments. The proposer submitted a second solution that followed the same strategy as Lau, except that he analyzed the behaviour of the function $t(t^2 + 2t - 1)(1-t)^{-3}$ by calculus. Paul Bracken used Lagrange Multipliers and located the maximum and minimum values of $f(x, y)$ as well as eight other critical points on S given by the equations $(x+1)^2+(y+1)^2 = 3$ and $(x^2+y^2)(x+y) = xy+x+y$.

Paul Deiermann looked at the more general restriction $(x + 1)^2 + (y + 1)^2 = a$ where $0 < a < 9/2$, and parameterized the points of S by

$$(x, y) = (-1 + \sqrt{a}\cos\theta, -1 + \sqrt{a}\sin\theta).$$

He found $f(x, y)$ to be equal to

$$\frac{1}{2} \cdot \frac{[q^2 - 2q + 2 - a][-2 + q]}{(3 - q)^3},$$

where $q = \sqrt{2a}\cos(\theta - \pi/4)$ satisfies $-\sqrt{2a} \le q \le \sqrt{2a}$. He then analyzed this function, identifying its values at the endpoints and the two critical points of the interval $[-\sqrt{2a}, \sqrt{2a}]$. In the case $a = 3$ of the problem, the global maximum is achieved at both an endpoint and a critical point, while the global minimum is achieved at the other endpoint and critical point. He adds that *Mathematica* graphs suggest that $a = 3$ is the only value of a where f achieves both global extrema at an endpoint and a critical point at the same time.

AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Solvers - individuals

Solvers - collaborations

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