Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

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- Probability in the Human Knot Game
- Always a Cube
- Converting a Fraction into a Decimal

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year and consists of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor

The Best Writing on Mathematics 2011

This is the title of a book which has come our way from Princeton University Press (see reference 1). We pass over the question of whether and how such a claim can be justified. The twenty-six articles included cover a wide spectrum of mathematics. They are probably intended for professional mathematicians, although students will find much of interest.

The British media have recently been lamenting the supposed declining standards of numeracy among the population, and attribute this to the inadequate teaching of Mathematics in schools. Underwood Dudley in the first article asks the question 'What is Mathematics For?', and comes up with a perhaps surprising answer.

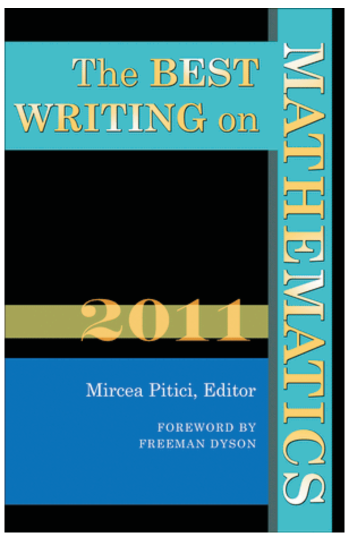
Melvyn B. Nathanson in his article entitled 'One, Two, Many: Individuality and Collectivity in Mathematics' poses the question: Is new Mathematics best produced by individuals working alone or by collaboration with others, even massive collaboration via the internet?

Martin Campbell-Kelly takes a nostalgic look back at Mathematical Tables and gasps with amazement at the labour of love of Henry Briggs who produced the first table of logarithms. How many of us still have a copy of tables of logs, sines, cosines,…?

Reuben Hersh asks whether Jewish people have been and are under- or over-represented in American Mathematics.

David J. Hand writes about modelling the economy and asks: 'Did Over-Reliance on Mathematical Models for Risk Assessment Create the Financial Crisis?' He quotes the statistician George Box: 'All models are wrong, but some are useful'.

Jordan Ellenberg in his article 'Fill in the Blanks: Using Math to Turn Lo-Res Datasets into High-Res Samples' writes on how the mathematical technique of Compressed Sensing can be used to create high-resolution from low-resolution images in, for example, MRI scans. There are gradually improved images of President Obama so produced!



Ivan M. Havel's article 'Seeing Numbers' cites the story of 'prime twins' (not 'twin primes'!) John and Michel with an amazing ability to recognize when six-, eight-, or even up to 20-digit numbers are prime. The neurologist Oliver Sacks was unable to check whether their 20-digit primes are in fact prime!

Ioan James writes on 'Autism and Mathematical Talent' and asks whether the two are at all related.

The final article 'Playing with Matches' considers a problem familiar to all medical graduates scrambling to obtain positions in hospitals and describes algorithms which go a long way to solving this.

There is much besides. All in all, a good read. It is worth looking out for next year's volume!

Reference

1 M. Pitici (ed.), *The Best Writing on Mathematics 2011* (Princeton University Press, 2012).

Sums of powers

$$(n^3+1)^3+(2n^3-1)^3+(n^4-2n)^3=(n^4+n)^3,$$
$$(3n^2)^3+(6n^2-3n+1)^3+(3n(3n^2-2n+1)-1)^3=(3n(3n^2-2n+1))^3,$$
$$(3n^2)^3+(6n^2+3n+1)^3+(3n(3n^2+2n+1))^3=(3n(3n^2+2n+1)+1)^3,$$
$$(2n)^2+(2n+1)^2+(2n+2)^2+(2(3n^2+3n+1))^2=(2(3n^2+3n+1)+1)^2,$$
$$\begin{aligned}(a^2+b^2-2)^2+(c^2+d^2-2)^2+2(ac-bd)^2\\ =(a^2+c^2-2)^2+(b^2+d^2-2)^2+2(ab-cd)^2.\end{aligned}$$

Students' Investigation House,
Shariati Avenue, Sirjan, Iran

Abbas Rouholamini Gugheri

Consecutive numbers divisible by the cubes of consecutive numbers

$$106623=3^3\times 3949,$$
$$106624=4^3\times 1666,$$
$$106625=5^3\times 853.$$

Can you find other similar examples?

Lucknow, India

M. A. Khan

How Many Primes are there Between Consecutive Fibonacci Numbers?

MARTIN GRIFFITHS

In this article we consider a number-theoretic enumeration problem associated with primes and Fibonacci numbers. An asymptotic result is obtained for the number of primes occurring between consecutive Fibonacci numbers. We also discuss the limitations of our result.

1. Introduction

The Fibonacci sequence $\{F_n\}$ is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

for $n \geq 2$, where $F_0 = 0$ and $F_1 = 1$, and starts $0, 1, 1, 2, 3, 5, \ldots$. Let G_n denote the number of primes lying *strictly* between F_n and F_{n+1}. For example, the primes lying between $F_7 = 13$ and $F_8 = 21$ are 17 and 19, so $G_7 = 2$. It is easy to verify that

$$G_4 = 0, \quad G_5 = 1, \quad G_6 = 1, \quad G_7 = 2, \quad G_8 = 3, \quad G_9 = 5,$$

which are the first few terms of the Fibonacci sequence given above. Is this some numerical fluke, or will the correspondence continue? Such questions often form the starting point for mathematical exploration.

It turns out that we do not need to go very far in order to demonstrate, by way of a counterexample, that the conjecture $G_{n+4} = F_n$ for $n \geq 0$ is false; indeed, G_{10} is equal to 7 rather than 8. The question then remains: what *can* we say about G_n? Unfortunately, we are not in a position to be able to provide the reader with an exact answer to the question posed in the title! However, in this article we do make some progress in this regard by obtaining an asymptotic formula for G_n and then considering what this does and does not tell us about the sequence $\{G_n\}$.

2. Some calculations

The aim of this section is to derive an *asymptotic formula* for G_n. By such a formula, we mean one that gives, in the sense described below, ever-better approximations to G_n as n increases without limit. More specifically, the function $f(n)$ is an asymptotic formula for G_n if

$$\lim_{n\to\infty} \frac{G_n}{f(n)} = 1. \tag{1}$$

This may also be expressed as

$$G_n \sim f(n).$$

A key point to note here is that if G_n and $f(n)$ satisfy (1) then it is not necessarily true that the value of $f(n)$ gets closer to that of G_n as n increases. Indeed, for many, if not most, asymptotic formulae this will not be the case. Thus we are unable to say, without being given any further information, much about the absolute error in using $f(n)$ to approximate G_n. What (1) does tell us, however, is that the relative error in using $f(n)$ to approximate G_n tends to zero as n increases without limit.

We shall need two well-known results, each giving us information about some aspect of approximate mathematical behaviour but differing immensely in the amount of effort required to prove them. The first concerns the Fibonacci numbers. In references 1 and 2, for example, we find *Binet's formula*,

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \hat{\phi}^n), \tag{2}$$

where ϕ is the *golden ratio* given by

$$\phi = \frac{1+\sqrt{5}}{2},$$

and

$$\hat{\phi} = \frac{1-\sqrt{5}}{2} = -\frac{1}{\phi}.$$

From (2) it follows that F_n is equal to the nearest integer to $\phi^n/\sqrt{5}$. Note that we may write

$$F_n \sim \frac{\phi^n}{\sqrt{5}}. \tag{3}$$

The second result is in connection with the function $\pi(x)$, which, for $x \in \mathbb{R}$ such that $x > 0$, denotes the number of primes that do not exceed x. For example, $\pi(12) = 5$ since there are five primes that do not exceed 12, namely 2, 3, 5, 7, and 11. The *prime number theorem* may be expressed as

$$\pi(n) \sim \frac{n}{\log n}. \tag{4}$$

By way of our comments above, we know that this may alternatively be written as

$$\lim_{n\to\infty} \frac{\pi(n)\ln n}{n} = 1.$$

The prime number theorem might be regarded as one the crowning achievements of nineteenth-century mathematics. See reference 1 for a simple introduction to this wonderful theorem, reference 3 for a full analytic proof, and reference 4 for an elementary proof.

Some careful analysis, using (3) and (4), shows that

$$\pi(\mathrm{F}_{n+1}) - \pi(\mathrm{F}_n) \to \infty \quad \text{as } n \to \infty.$$

Then, on noting that $\pi(\mathrm{F}_{n+1}) - \pi(\mathrm{F}_n)$ is equal either to $\mathrm{G}_n + 1$ or G_n, depending on whether or not F_{n+1} is prime, we have

$$\begin{aligned}
\mathrm{G}_n &\sim \pi(\mathrm{F}_{n+1}) - \pi(\mathrm{F}_n) \\
&\sim \frac{\phi^{n+1}}{\sqrt{5}\log(\phi^{n+1}/\sqrt{5})} - \frac{\phi^n}{\sqrt{5}\log(\phi^n/\sqrt{5})} \\
&= \frac{\phi^n}{\sqrt{5}}\left(\frac{\phi}{(n+1)\log\phi - \log\sqrt{5}} - \frac{1}{n\log\phi - \log\sqrt{5}}\right) \\
&\sim \frac{\phi^n(\phi-1)}{n\sqrt{5}\log\phi},
\end{aligned}$$

which may in turn be written as

$$\mathrm{G}_n \sim g(n)\mathrm{F}_n, \tag{5}$$

where

$$g(n) = \frac{\phi - 1}{n\log\phi}.$$

3. How useful is our formula?

For the limited range of values of n used in Table 1 at least, it would appear that the estimates for G_n are remarkably good. It is certainly highly plausible, some might even say ‘obvious’, that, for $n \geq 6$, $\{\mathrm{G}_n\}$ is a strictly increasing sequence. However, is (5) sufficient to prove this?

In order to answer this, note first that (5) implies that, for any $\varepsilon > 0$ we choose, there exists some $N \in \mathbb{N}$ such that

$$1 - \varepsilon < \frac{\mathrm{G}_n}{g(n)\mathrm{F}_n} < 1 + \varepsilon \tag{6}$$

Table 1 Comparing G_n with $g(n)\mathrm{F}_n$.

n	G_n	$g(n)\mathrm{F}_n$
10	7	7.06...
11	10	10.39...
12	16	15.41...
13	23	23.01...
14	37	34.58...
15	53	52.22...

for all $n > N$. In the light of this, let us, for the sake of an example, set $\varepsilon = \frac{1}{100}$ and consider some sequence $\{\mathrm{A}_n\}$ satisfying

$$\frac{99g(n)\mathrm{F}_n}{100} < \mathrm{A}_n < \frac{101g(n)\mathrm{F}_n}{100}. \tag{7}$$

Using (2) we have

$$\begin{aligned}\frac{\mathrm{F}_{n+1}}{\mathrm{F}_n} &= \frac{\phi^{n+1} - \hat{\phi}^{n+1}}{\phi^n - \hat{\phi}^n} \\ &\geq \frac{\phi^{n+1} - 1/\phi^{n+1}}{\phi^n + 1/\phi^n} \\ &= \frac{\phi^{2n+2} - 1}{\phi(\phi^{2n} + 1)} \\ &= \frac{\phi^2(\phi^{2n} + 1) - \phi^2 - 1}{\phi(\phi^{2n} + 1)} \\ &= \phi - \frac{\phi^2 + 1}{\phi(\phi^{2n} + 1)}.\end{aligned}$$

It is a straightforward matter to show that

$$\phi - \frac{\phi^2 + 1}{\phi(\phi^{2n} + 1)} \geq \frac{3}{2}$$

when $n \geq 3$, and hence that

$$\frac{\mathrm{F}_{n+1}}{\mathrm{F}_n} > \frac{101(n+1)}{99n}$$

for these values of n. Then, noting that $101 \times 3 \times \mathrm{F}_2 < 99 \times 2 \times \mathrm{F}_3$, we obtain the result

$$101(n+1)\mathrm{F}_n < 99n\mathrm{F}_{n+1}$$

for $n \geq 2$, from which it follows that

$$\frac{101g(n)\mathrm{F}_n}{100} < \frac{99g(n+1)\mathrm{F}_{n+1}}{100} \tag{8}$$

when $n \geq 2$.

We now see that any solution sequence $\{\mathrm{A}_n\}$ must be increasing for $n \geq 2$, although it is clear that A_n cannot be an integer for small values of n. In fact, as may be checked, 15 is the smallest value of $K \in \mathbb{N}$ such that (7) has the potential for admitting integer solutions A_n for all $n \geq K$. It is possible therefore that there exists some sequence $\{\mathrm{A}_n\}$ satisfying (7) such that

$$\mathrm{G}_n = \mathrm{A}_n$$

for all $n \geq 15$. However, this is no more than a possibility; we cannot say for certain that $\{\mathrm{G}_n\}$ is increasing for all $n \geq 15$. This is because although, via (6), there exists some $N \in \mathbb{N}$ such that $\{\mathrm{G}_n\}$ is a solution to (7) for all $n \geq N$, it does not give us any information about the smallest possible value of N for a particular value of ε such as $\frac{1}{100}$. We therefore cannot discount, on the basis of our current results at least, that there exists some large value of n for which

$$G_{n+1} < G_n.$$

Indeed, through (6), (7), and (8), all we are able to say here is that there exists some $N \in \mathbb{N}$ such that $\{G_n\}$ is increasing for all $n \geq N$.

Thus, even upon utilizing some heavyweight mathematical machinery, we have only obtained a partial result, namely that the number of primes between successive Fibonacci numbers is an eventually increasing sequence. This limitation is a consequence of the fact that the prime number theorem only provides information about the behaviour of the primes in the long term, and it demonstrates that even with apparently powerful theorems at our disposal, the primes give up their secrets only with extreme reluctance!

Interested readers might like to consider the potential for generalising the result given here for the Fibonacci numbers by investigating the corresponding situation for the sequence $\{H_n\}$, where

$$H_n = a(1+\varepsilon)^n$$

for positive numbers $a, \varepsilon \in \mathbb{R}$. Furthermore, are we able to say anything about the number of primes appearing between consecutive terms of linear, quadratic, or other types of sequences?

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Martin Griffiths *is a Lecturer in Mathematics Education at the University of Manchester. He has diverse mathematical interests, ranging from combinatorics to mathematical epidemiology. He referees papers for seven different mathematical journals and is also the Reviews Editor of the Mathematical Gazette.*

Self-satisfied food

Lovers of pizza will be relieved to know that the volume of their favourite food is built into its name. Thus a portion of thickness a and radius z will have a volume of pi $zz\,a$.

Midsomer Norton, Bath, UK **Bob Bertuello**

Triangular Roots

COLIN FOSTER

This article defines triangular roots of real numbers by analogy with square roots. It shows that the triangular root of a real number can never be purely imaginary: it is either real or complex with a real part of $-\frac{1}{2}$.

We all know about *square roots*. For a nonnegative real number a, we say that b is a square root of a if $a = b^2$. When $a > 0$, it has two square roots and we denote the positive one by $\sqrt{a}$; when $a = 0$, there is just one root: zero. Thus the sequence of perfect squares is $1, 4, 9, 16, \ldots$ and

$$1 = \sqrt{1}, \qquad 2 = \sqrt{4}, \qquad 3 = \sqrt{9}, \qquad 4 = \sqrt{16},$$

and so on. The nth perfect square is the number of dots in an $n \times n$ square array, as shown in figure 1.

Now can we define a *triangular root*? The nth triangular number t_n is defined by

$$t_n = 1 + 2 + 3 + 4 + \cdots + n = \tfrac{1}{2}n(n+1),$$

and is the number of dots in the first n rows of the triangular array shown in figure 2. For a real number a, we say that b is a triangular root of a, and write $b = \mathrm{tr}(a)$, if

$$a = \tfrac{1}{2}b(b+1). \tag{1}$$

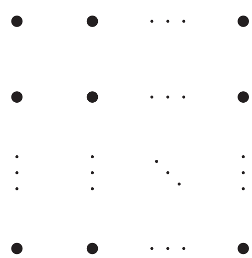

Figure 1

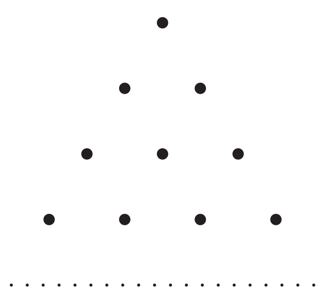

Figure 2

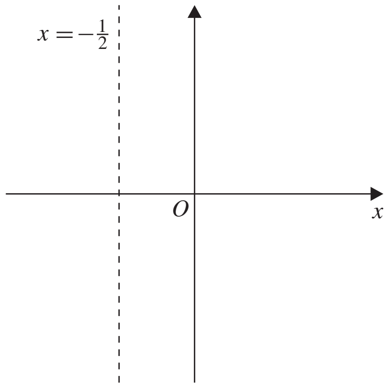

Figure 3

Thus, the sequence of triangular numbers is $1, 3, 6, 10, \ldots$ and

$$1 = \mathrm{tr}(1), \qquad 2 = \mathrm{tr}(3), \qquad 3 = \mathrm{tr}(6), \qquad 4 = \mathrm{tr}(10),$$

and so on.

From (1),

$$b^2 + b - 2a = 0,$$

which is a quadratic equation in b with roots

$$b_1, b_2 = \frac{-1 \pm \sqrt{1+8a}}{2}.$$

As with square roots, we take the positive sign to give

$$\mathrm{tr}(a) = \frac{-1 + \sqrt{1+8a}}{2}.$$

This choice of sign agrees with

$$1 = \mathrm{tr}(1), \qquad 2 = \mathrm{tr}(3), \qquad 3 = \mathrm{tr}(6), \qquad 4 = \mathrm{tr}(10),$$

and so on, and provided that $a \geq -\frac{1}{8}$, it will have a unique real triangular root.

We know that $\sqrt{1+8a}$ is a real number when $1 + 8a \geq 0$ and a purely imaginary number when $1 + 8a < 0$ (e.g. $\sqrt{-4} = 2\mathrm{i}$), and the square roots of real numbers lie on the real and imaginary axes. So if $a \geq -\frac{1}{8}$, the triangular roots lie along the real axis, whereas if $a < -\frac{1}{8}$ then they lie along the line $x = -\frac{1}{2}$ (see figure 3).

Colin Foster *is a Senior Research Fellow in the School of Education at Nottingham University. He has published many books of ideas for mathematics teachers: see www.foster77.co.uk for details.*

Convergence of Pell and Pell–Lucas Series

THOMAS KOSHY

The power series $\sum_{n=0}^{\infty} P_n x^n$ and $\sum_{n=0}^{\infty} Q_n x^n$ converge if and only if $|x| < -\delta$, where P_n and Q_n denote the nth Pell and Pell–Lucas numbers respectively, and $\delta = 1 - \sqrt{2}$. We evaluate each sum for two distinct positive rational numbers x, where $\delta < x < -\delta$.

Introduction

Like Fibonacci and Lucas numbers, Pell and Pell–Lucas numbers are a delightful source for experimentation and exploration for both amateurs and professionals alike. They belong to a much larger family of positive integers and share a multitude of similar properties (see references 1 and 2).

Pell numbers P_n and Pell–Lucas numbers Q_n are often defined recursively:

$$P_0 = 0, \quad P_1 = 1, \qquad\qquad Q_0 = 1 = Q_1,$$
$$P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 2, \qquad\qquad Q_n = 2Q_{n-1} + Q_{n-2}, \quad n \geq 2.$$

The first six Pell numbers are 0, 1, 2, 5, 12, 29; the first six Pell–Lucas numbers are 1, 1, 3, 7, 17, 41.

They can also be defined by the *Binet-like* formulas

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad Q_n = \frac{\gamma^n + \delta^n}{2},$$

where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ are the solutions of the quadratic equation $x^2 = 2x + 1$ (see references 1, 2, and 3).

Both families satisfy a myriad of interesting properties. For example, we have:

(i) $P_n + Q_n = P_{n+1}$,

(ii) $Q_n^2 - 2P_n^2 = (-1)^n$,

(iii) $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$,

(iv) $Q_{n+1}Q_{n-1} - Q_n^2 = 2(-1)^{n-1}$.

These properties can be established using the Binet-like formulas. Properties (iii) and (iv) are the *Cassini-like* formulas. It follows from (ii) that every Q_n is odd. Since $P_{2n} = 2P_nQ_n$ and Q_n is odd, it follows by (i) that $P_n \equiv 0 \pmod 2$ if and only if $n \equiv 0 \pmod 2$.

Using the recurrences, coupled with the initial conditions, it can be shown that Pell and Pell–Lucas numbers can also be generated by the following generating functions:

$$f(x) = \frac{x}{1-2x-x^2} = \sum_{n=0}^{\infty} P_n x^n,$$
$$g(x) = \frac{1-x}{1-2x-x^2} = \sum_{n=0}^{\infty} Q_n x^n.$$

In this article, we will identify the radii of convergence of both power series and then evaluate each infinite sum for two positive rational numbers x. This will yield some surprising Pell dividends.

Convergence of the series $\sum_{n=0}^{\infty} P_n x^n$ and $\sum_{n=0}^{\infty} Q_n x^n$

To minimize the exposition, let $\{S_n\}$ denote an integer sequence satisfying the Pell recurrence: $S_n = 2S_{n-1} + S_{n-2}$. Then $S_n = A\gamma^n + B\delta^n$, where A and B are constants. So we have

$$\sum_{n=0}^{\infty} S_n x^n = A \sum_{n=0}^{\infty} \gamma^n x^n + B \sum_{n=0}^{\infty} \delta^n x^n.$$

The two series on the right-hand side converge if and only if $|x| < 1/|\gamma|$ and $|x| < 1/|\delta|$; that is, if and only if

$$|x| < \min\left(\frac{1}{|\gamma|}, \frac{1}{|\delta|}\right).$$

Since

$$\min\left(\frac{1}{|\gamma|}, \frac{1}{|\delta|}\right) = -\delta,$$

it follows that the series $\sum_{n=0}^{\infty} S_n x^n$ converges if and only if $\delta < x < -\delta$. Consequently, the series $\sum_{n=0}^{\infty} P_n x^n$ and $\sum_{n=0}^{\infty} Q_n x^n$ converge if and only if $\delta < x < -\delta$.

Next we evaluate the sum $\sum_{n=0}^{\infty} P_n x^n$ for two special values of x.

Sum of the series $\sum_{n=0}^{\infty} P_n x^n$

Since $\delta < 0$, we have

$$\begin{aligned}\frac{P_{2k}}{P_{2k+1}} &= \frac{\gamma^{2k} - \delta^{2k}}{\gamma^{2k+1} - \delta^{2k+1}} \\ &< \frac{\gamma^{2k}}{\gamma^{2k+1}} \\ &= \frac{1}{\gamma} \\ &= -\delta.\end{aligned}$$

Thus the power series $\sum_{n=0}^{\infty} P_n x^n$ converges when $x = P_{2k}/P_{2k+1}$.

Using the Cassini-like formula for P_m, we then have

$$\begin{aligned}\sum_{n=0}^{\infty} P_n\left(\frac{P_{2k}}{P_{2k+1}}\right)^n &= \frac{P_{2k}/P_{2k+1}}{1-2P_{2k}/P_{2k+1}-P_{2k}^2/P_{2k+1}^2}\\ &= \frac{P_{2k}P_{2k+1}}{P_{2k+1}(P_{2k+1}-2P_{2k})-P_{2k}^2}\\ &= \frac{P_{2k}P_{2k+1}}{P_{2k+1}P_{2k-1}-P_{2k}^2}\\ &= \frac{P_{2k}P_{2k+1}}{(-1)^{2k}}\\ &= P_{2k}P_{2k+1},\end{aligned}$$

which is an even integer.

For example,

$$\sum_{n=0}^{\infty} P_n\left(\tfrac{2}{5}\right)^n = 2\times 5 \quad\text{and}\quad \sum_{n=0}^{\infty} P_n\left(\tfrac{12}{29}\right)^n = 12\times 29.$$

Notice that

$$\sum_{n=0}^{86} P_n\left(\tfrac{2}{5}\right)^n \approx 9.5 \quad\text{and}\quad \sum_{n=0}^{6445} P_n\left(\tfrac{12}{29}\right)^n \approx 347.5,$$

so the convergence is very slow.

Suppose that we let $x = Q_{2k-1}/Q_{2k}$. Then also $\delta < x < -\delta$. So

$$\sum_{n=0}^{\infty} P_n x^n$$

converges when $x = Q_{2k-1}/Q_{2k}$ also. Consequently, it follows by the Cassini-like formula for Q_m that

$$\sum_{n=0}^{\infty} P_n\left(\frac{Q_{2k-1}}{Q_{2k}}\right)^n = \frac{Q_{2k-1}Q_{2k}}{2},$$

where $k \geq 1$. (Since every Q_n is odd, this sum is *not* an integer.)

For example,

$$\sum_{n=0}^{\infty} P_n\left(\frac{1}{3}\right)^n = \frac{1\times 3}{2} \quad\text{and}\quad \sum_{n=0}^{\infty} P_n\left(\frac{7}{17}\right)^n = \frac{7\times 17}{2}.$$

Notice that

$$\sum_{n=0}^{23} P_n\left(\tfrac{1}{3}\right)^n \approx 1.5 \quad\text{and}\quad \sum_{n=0}^{1195} P_n\left(\tfrac{7}{17}\right)^n \approx 59.5;$$

again the convergence is extremely slow.

Next we evaluate the sum $\sum_{n=0}^{\infty} Q_n x^n$ for the two corresponding values of x.

Sum of the series $\sum_{n=0}^{\infty} Q_n x^n$

Recall that the series $\sum_{n=0}^{\infty} Q_n x^n$ converges if and only if $\delta < x < -\delta$. Suppose that we choose $x = Q_{2k-1}/Q_{2k}$. Since $\delta < x < -\delta$, it follows again by the Cassini-like formula for Q_m that

$$\sum_{n=0}^{\infty} Q_n \left(\frac{Q_{2k-1}}{Q_{2k}}\right)^n = P_{2k-1} Q_{2k},$$

where $k \geq 1$.

Likewise, we also have

$$\sum_{n=0}^{\infty} Q_n \left(\frac{P_{2k}}{P_{2k+1}}\right)^n = Q_{2k} P_{2k+1},$$

where $k \geq 1$.

For example,

$$\sum_{n=0}^{\infty} Q_n \left(\tfrac{7}{17}\right)^n = 5 \times 17 \quad \text{and} \quad \sum_{n=0}^{\infty} Q_n \left(\tfrac{41}{99}\right)^n = 29 \times 99,$$
$$\sum_{n=0}^{\infty} Q_n \left(\tfrac{2}{5}\right)^n = 3 \times 5 \quad \text{and} \quad \sum_{n=0}^{\infty} Q_n \left(\tfrac{12}{29}\right)^n = 17 \times 29.$$

Acknowledgement

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***Thomas Koshy** received his PhD in Algebraic Coding Theory from Boston University. He has authored eight books, including 'Catalan Numbers with Applications' (Oxford University Press, 2009) and 'Triangular Arrays with Applications' (Oxford University Press, 2011). His passions include Fibonacci and Lucas numbers, Catalan numbers, and Pell and Pell–Lucas numbers.*

Variations in *Euclid*[*n*]: The Product of the First *n* Primes Plus One

JAY L. SCHIFFMAN

We undertake an examination encompassing variations on the product of the first n primes plus one utilized in Euclid's classical proof that there are infinitely many primes. In this article, we seek to secure prime outputs from these generalizations and determine if we achieve a rich source for twin prime pairs.

Euclid's classical proof that the number of positive primes is infinite

In Euclid's famous proof that there are infinitely many primes, we assume on the contrary that there are only finitely many primes and achieve a contradiction. Euclid considered the number

$$M = p_1 p_2 p_3 \cdots p_n + 1$$

where $p_1, p_2, \ldots, p_n$ are supposedly all the prime numbers. Observe that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on, and consider the status of this large integer M. None of the primes $p_1, p_2, \ldots, p_n$ divides M, yet M must possess a prime factor, so there must be a prime factor other than $p_1, p_2, \ldots, p_n$. Hence there is no largest prime and the number of primes is infinite.

The primorials, *Euclid*[*n*] and *Euclid*[*n* − 2]

The computer software package MATHEMATICA® refers to the product of the first n primes plus one as $\mathit{Euclid}[n]$. Meanwhile, the product of the first n primes is referred to as a *primorial* (the analog of the factorial for prime numbers) and is denoted by $p_n\# = \prod_{k=1}^{n} p_k$. While the number of primes is infinite, it remains an open problem as to whether there are infinitely many prime outputs generated by $\mathit{Euclid}[n]$. $\mathit{Euclid}[5]$ is prime while $\mathit{Euclid}[6]$ is composite:

$$\mathit{Euclid}[5] = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 1 = 2311,$$

a prime number, whereas

$$\mathit{Euclid}[6] = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031 = 59 \cdot 509,$$

a composite integer. It is known that $\mathit{Euclid}[n]$ yields prime outputs for

$$n = 1, 2, 3, 4, 5, 11, 75, 171, 172, 384, 457, 616,$$
$$1391, 1613, 2122, 2647, 4413, 13494, 31260, 33237.$$

One of the more interesting activities for number theorists is to secure twin prime pairs, i.e. pairs of primes differing by two such as $(3, 5)$, $(5, 7)$, and $(11, 13)$. Consider the product of the first n primes minus one which is $\mathit{Euclid}[n] - 2$. It is thus useful to note the prime outputs

Table 1 The initial twenty outputs generated by $Euclid\,[n]$ and $Euclid\,[n]-2$.

n	$Euclid\,[n]$		$Euclid\,[n]-2$	
1	3	(Prime)	1	
2	7	(Prime)	5	(Prime)
3	31	(Prime)	29	(Prime)
4	211	(Prime)	11·19	
5	2311	(Prime)	2309	(Prime)
6	59·509		30029	(Prime)
7	19·97·277		61·8369	
8	347·27953		53·197·929	
9	317·703763		37·131·46027	
10	331·571·34231		79·81894851	
11	200560490131	(Prime)	228737·876817	
12	181·60611·676421		229·541·1549·38669	
13	61·450451·11072701		304250263527209	(Prime)
14	167·78339888213593		141269·92608862041	
15	953·46727·13808181181		191·53835557·59799107	
16	73·139·173·18564761860301		87337·326257·1143707681	
17	277·3467·105229·19026377261		27600124633·69664915493	
18	223·525956867082542470777		1193·85738903·1146665184811	
19	54730729297·143581524529603		163·2682037·17975352936245519	
20	1063·303049·598841·2892214489673		260681003321·2140320249725509	

for $Euclid\,[n]-2$. It has been established that $Euclid\,[n]-2$ is prime for

$$n = 2, 3, 5, 6, 13, 24, 66, 68, 167, 287, 310, 352, 564, 590, 620, 849, 1552, 1849.$$

In table 1, we list the initial twenty outputs generated by $Euclid\,[n]$ and $Euclid\,[n]-2$ for the composite cases or indicate prime otherwise. All factorizations in this article are obtained with the aid of the aforementioned program MATHEMATICA, Version 8.0.

We observe the three twin prime pairs $(5, 7)$, $(29, 31)$, and $(2309, 2311)$ corresponding to $n = 2$, 3, and 5 respectively. No other twin prime pairs of the form $Euclid\,[n-2]$ and $Euclid\,[n]$ are currently known.

Factorial primes

We next discuss factorial primes which would correspond to prime outputs obtained by considering the product of the first n counting integers plus and minus one. It can be shown that $n!+1$ is prime for

$$n = 1, 2, 3, 11, 27, 37, 41, 73, 77, 116, 154, 320, 340, 399, 472, 872,$$

for $n < 1000$. Similarly $n!-1$ generates prime outputs corresponding to

$$n = 3, 4, 6, 7, 12, 14, 30, 32, 33, 38, 94, 166, 324, 379, 469, 546, 974,$$

for $n < 1000$. We secure only one twin prime pair in this variation; namely 5 and 7 corresponding to $n = 3$. In table 2, we give the first thirty factorizations for both $n!+1$ and $n!-1$.

Table 2 The factorizations of $n!+1$ and $n!-1$, for $1 \le n \le 30$.

n	$n!-1$		$n!+1$	
1	0		2	(Prime)
2	1		3	(Prime)
3	5	(Prime)	7	(Prime)
4	23	(Prime)	5^2	
5	$7 \cdot 17$		11^2	
6	719	(Prime)	$7 \cdot 103$	
7	5039	(Prime)	71^2	
8	$23 \cdot 1753$		$61 \cdot 661$	
9	$11^2 \cdot 2999$		$19 \cdot 71 \cdot 269$	
10	$29 \cdot 125131$		$11 \cdot 329891$	
11	$13 \cdot 17 \cdot 23 \cdot 7853$		39916801	(Prime)
12	479001599	(Prime)	$13^2 \cdot 2834229$	
13	$1733 \cdot 3593203$		$83 \cdot 75024347$	
14	87178291199	(Prime)	$23 \cdot 3790360487$	
15	$17 \cdot 31^2 \cdot 53 \cdot 1510259$		$59 \cdot 479 \cdot 46271341$	
16	$3041 \cdot 6880233439$		$17 \cdot 61 \cdot 137 \cdot 139 \cdot 1059511$	
17	$19 \cdot 73 \cdot 256443711677$		$661 \cdot 537913 \cdot 1000357$	
18	$59 \cdot 226663 \cdot 478749547$		$19 \cdot 23 \cdot 29 \cdot 61 \cdot 67 \cdot 123610951$	
19	$653 \cdot 2383907 \cdot 78143369$		$71 \cdot 1713311273363831$	
20	$124769 \cdot 19499250680671$		$20639383 \cdot 117876683047$	
21	$23 \cdot 89 \cdot 5171 \cdot 4826713612027$		$43 \cdot 439429 \cdot 2703875815783$	
22	$109 \cdot 60656047 \cdot 170006681813$		$23 \cdot 521 \cdot 93799610095769647$	
23	$51871 \cdot 498390560021687969$		$47^2 \cdot 79 \cdot 148139754736864591$	
24	$625793187653 \cdot 991459181683$		$811 \cdot 765041185860961084291$	
25	$149 \cdot 907 \cdot 114776274341482621993$		$401 \cdot 38681321803817920159601$	
26	$20431 \cdot 19739193437746837432529$		$1697 \cdot 237649652991517758152033$	
27	$29 \cdot 3754782569109776607161379 31$		$10888869450418352160768000001$	(Prime)
28	$239 \cdot 156967 \cdot 7798078091 \cdot 1042190196053$		$29 \cdot 10513391193507374500051862069$	
29	$31 \cdot 59 \cdot 311 \cdot 26156201 \cdot 594278556271609021$		$14557 \cdot 218568437 \cdot 2778942057555023489$	
30	$265252859812191058636308479999999$	(Prime)	$31 \cdot 12421 \cdot 82561 \cdot 1080941 \cdot 7719068319927551$	

Fibonorials

Fibonorials or Fibonacci factorials are obtained by taking the product of the first n Fibonacci numbers defined recursively as follows:

$$\mathrm{F}_1 = \mathrm{F}_2 = 1, \qquad \mathrm{F}_n = \mathrm{F}_{n-2} + \mathrm{F}_{n-1}, \quad \text{for } n \ge 3.$$

The first seven Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13. Hence, the first seven Fibonorials are 1, 1, 2, 6, 30, 240, 3120. Consider the product of the first n Fibonacci numbers plus one and the product of the first n Fibonacci numbers minus one. We seek the factorization of the first twenty outputs in each set and determine any possible twin prime pairs that might be generated. It can be shown that

$$\prod_{k=1}^{n} \mathrm{F}_k + 1$$

Table 3 The factorizations for both $\prod_{k=1}^{n} \mathrm{F}_k - 1$ and $\prod_{k=1}^{n} \mathrm{F}_k + 1$, for $1 \leq n \leq 20$.

n	$\prod_{k=1}^{n} \mathrm{F}_k - 1$		$\prod_{k=1}^{n} \mathrm{F}_k + 1$	
1	0		2	(Prime)
2	0		2	(Prime)
3	1		3	(Prime)
4	5	(Prime)	7	(Prime)
5	29	(Prime)	31	(Prime)
6	239	(Prime)	241	(Prime)
7	3119	(Prime)	3121	(Prime)
8	65519	(Prime)	65521	(Prime)
9	$293 \cdot 7603$		$599 \cdot 3719$	
10	$8039 \cdot 15241$		$1373 \cdot 89237$	
11	$7283 \cdot 1497253$		$181 \cdot 60245821$	
12	$37 \cdot 59 \cdot 16349 \cdot 43997$		$631 \cdot 2488505671$	
13	$23 \cdot 43^2 \cdot 8603183137$		$3833 \cdot 12109 \cdot 7882733$	
14	137932073613734399	(Prime)	$7542877 \cdot 18286401013$	
15	8413856490437798399	(Prime)	$28151 \cdot 2988830411153351$	
16	$268969 \cdot 7147787 \cdot 43195491733$		$647 \cdot 736039 \cdot 174384204950897$	
17	$997 \cdot 1216569721 \cdot 109341494997227$		$46906049 \cdot 2827406917310640449$	
18	$67 \cdot 197 \cdot 1439371$ $\cdot 18038312278023659731$		$12071 \cdot 11112487$ $\cdot 2554789735260074513$	
19	$43 \cdot 47609 \cdot 108289$ $\cdot 6463205842260197917093$		$17788657 \cdot 53314033$ $\cdot 1510793641502130721$	
20	33617 $\cdot 2883358827621587061338760979400047$		$3803 \cdot 19379$ $\cdot 1315224707284166971651275222273$	

is prime for $n = 1, 2, 3, 4, 5, 6, 7, 8, 22, 28$, and for no others to $n = 800$. In identical fashion,

$$\prod_{k=1}^{n} \mathrm{F}_k - 1$$

is prime for $n = 4, 5, 6, 7, 8, 14, 15$, and for no others to $n = 1000$. Thus we obtain five pairs of twin primes $(5, 7)$, $(29, 31)$, $(239, 241)$, $(3119, 3121)$, $(65519, 65521)$ corresponding to $n = 4, 5, 6, 7, 8$ respectively. In table 3, we give the initial twenty factorizations for both $\prod_{k=1}^{n} \mathrm{F}_k - 1$ and $\prod_{k=1}^{n} \mathrm{F}_k + 1$. The prime factorizations become extremely large beyond that point.

Variations of Fibonorials to Lucas numbers

Consider the product of the first n Lucas numbers plus or minus one. The Lucas sequence is a Fibonacci-like sequence and satisfies the following recursion relation:

$$\mathrm{L}_1 = 1, \qquad \mathrm{L}_2 = 3, \qquad \mathrm{L}_n = \mathrm{L}_{n-2} + \mathrm{L}_{n-1}, \quad \text{for } n \geq 3.$$

Consider $\prod_{k=1}^{n} \mathrm{L}_k - 1$ and $\prod_{k=1}^{n} \mathrm{L}_k + 1$. It can be seen that $\prod_{k=1}^{n} \mathrm{L}_k + 1$ is prime for $n = 1, 3, 6$ while $\prod_{k=1}^{n} \mathrm{L}_k - 1$ is prime for $n = 2, 3, 4, 6, 12$. Both $\prod_{k=1}^{n} \mathrm{L}_k - 1$ and $\prod_{k=1}^{n} \mathrm{L}_k + 1$ do not generate any new primes to $n = 1000$. There are two known pairs of twin primes $(11, 13)$ and $(16631, 16633)$ corresponding to $n = 3$ and $n = 6$ respectively. We show the factorizations for the first fifteen values of n for both in table 4.

Table 4 The factorizations of $\prod_{k=1}^{n} \mathrm{L}_k - 1$ and $\prod_{k=1}^{n} \mathrm{L}_k + 1$, for $1 \leq n \leq 15$.

n	$\prod_{k=1}^{n} \mathrm{L}_k - 1$		$\prod_{k=1}^{n} \mathrm{L}_k + 1$	
1	0		2	(Prime)
2	2	(Prime)	2^2	
3	11	(Prime)	13	(Prime)
4	83	(Prime)	5·17	
5	13·71		5^2·37	
6	16631	(Prime)	16633	(Prime)
7	61·7907		31·15559	
8	5·17·167·1597		1657·13681	
9	5·5099·67577		89·19358153	
10	41023·5165729		349·607202581	
11	683·150659·409823		13·59·28657·1918607	
12	13579006117811903	(Prime)	5·233·3001·8527·455491	
13	37·61·293·14869·719490007		5·1414932437476000397	
14	151909·495899·79169252521		1997·2986449786660661829	
15	557·2417·6042488139802127243		73·1114358145956612334442553	

Products of figurative numbers plus and minus one

Our concluding excursions take us to respectively consider the products of the first n triangular, square, tetrahedral, and pentagonal numbers plus and minus one determining those outcomes in the range $1 \leq n \leq 1000$ yielding primes. We consider the factorizations for the first ten outcomes for each. The triangular, square, tetrahedral, and pentagonal numbers have the respective closed forms

$$\mathrm{T}_n = \frac{n(n+1)}{2}, \qquad \mathrm{S}_n = n^2, \qquad \mathrm{TH}_n = \frac{n(n+1)(n+2)}{6}, \qquad \mathrm{P}_n = \frac{n(3n-1)}{2}.$$

A MATHEMATICA search revealed that $\prod_{k=1}^{n} \mathrm{T}_k + 1$ is prime for

$$n = 1, 3, 4, 6, 20, 108, 127, 304, 378, 383, 515.$$

On the other hand, $\prod_{k=1}^{n} \mathrm{T}_k - 1$ is prime for

$$n = 2, 3, 4, 5, 35, 367.$$

We find two twin prime pairs corresponding to $n = 3$ and $n = 4$: $(17, 19)$ and $(179, 181)$ respectively. Now, $\prod_{k=1}^{n} \mathrm{S}_k + 1$ is prime for

$$n = 1, 2, 3, 4, 5, 9, 10, 11, 13, 24, 65, 76.$$

It is immediate from factoring the difference of two perfect squares that $\prod_{k=1}^{n} \mathrm{S}_k - 1$ is prime only when $n = 2$. We are led to a single twin prime pair: $(3, 5)$. Our technology generated prime outputs for $\prod_{k=1}^{n} \mathrm{TH}_k + 1$ when $n = 1, 2, 3, 5, 21$. Prime outputs for $\prod_{k=1}^{n} \mathrm{TH}_k - 1$ were secured for $n = 2, 6, 8, 28, 37, 44, 162, 277, 441$. Hence, only one twin prime pair $(3, 5)$ was found. Finally prime outputs occurred for $\prod_{k=1}^{n} \mathrm{P}_k + 1$ when

$$n = 1, 3, 4, 8, 11, 13, 44, 65, 90, 151, 622, 723, 790,$$

Table 5 The factorizations of $\prod_{k=1}^{n} \mathrm{T}_k - 1$, $\prod_{k=1}^{n} \mathrm{T}_k + 1$, $\prod_{k=1}^{n} \mathrm{S}_k - 1$, and $\prod_{k=1}^{n} \mathrm{S}_k + 1$, for $1 \leq n \leq 10$.

n	$\prod_{k=1}^{n} \mathrm{T}_k - 1$		$\prod_{k=1}^{n} \mathrm{T}_k + 1$	
1	0		2	(Prime)
2	2	(Prime)	2^2	
3	17	(Prime)	19	(Prime)
4	179	(Prime)	181	(Prime)
5	2699	(Prime)	$37 \cdot 73$	
6	$31^2 \cdot 59$		56701	(Prime)
7	$13 \cdot 97 \cdot 1259$		$349 \cdot 4549$	
8	$7559 \cdot 7561$		$4517 \cdot 12653$	
9	$6991 \cdot 367889$		$41 \cdot 43 \cdot 661 \cdot 2207$	
10	$19 \cdot 53 \cdot 2381 \cdot 58997$		$181 \cdot 781520221$	
n	$\prod_{k=1}^{n} \mathrm{S}_k - 1$		$\prod_{k=1}^{n} \mathrm{S}_k + 1$	
1	0		2	(Prime)
2	3	(Prime)	5	(Prime)
3	$5 \cdot 7$		37	(Prime)
4	$5^2 \cdot 23$		577	(Prime)
5	$7 \cdot 11^2 \cdot 17$		14401	(Prime)
6	$7 \cdot 103 \cdot 719$		$13 \cdot 39877$	
7	$71 \cdot 5039$		$101 \cdot 251501$	
8	$23 \cdot 61 \cdot 661 \cdot 1753$		$17 \cdot 95629553$	
9	$11^2 \cdot 19 \cdot 71 \cdot 269 \cdot 2999$		131681894401	(Prime)
10	$11 \cdot 29 \cdot 125131 \cdot 329891$		13168189440001	(Prime)

while $\prod_{k=1}^{n} \mathrm{P}_k - 1$ yielded primes for

$$n = 3, 4, 5, 6, 9, 13, 25, 48, 49, 51, 52, 409, 510.$$

A trio of twin prime pairs are hence found corresponding to $n = 3, 4, 13$ respectively:

$$(59, 61),$$
$$(1319, 1321),$$
$$(23500700015817023999999, 23500700015817024000001).$$

Tables 5 and 6 show the first ten factorizations for these figurative numbers.

Concluding remarks

The use of technological tools as manifested in graphical calculators, computer software, and mathematical websites enables us to ponder and seek new insights. In this article, a variation on the product of the first n primes plus one ensued. A world of possibilities is ripe for neat excursions in experimental mathematics. While we did not secure a large number of twin prime pairs, it was nonetheless fascinating to vary the underpinnings entailed in a popular theorem, expand our search in many cases to the initial 1000 inputs, and view any tangible results. For example, the two prime factors comprising the product of the first eight triangular numbers minus one in table 5 constituted a pair of twin primes given as 7569 and 7571 in

Table 6 The factorizations of $\prod_{k=1}^{n} \mathrm{TH}_k - 1$, $\prod_{k=1}^{n} \mathrm{TH}_k + 1$, $\prod_{k=1}^{n} \mathrm{P}_k - 1$, and $\prod_{k=1}^{n} \mathrm{P}_k + 1$, for $1 \leq n \leq 10$.

n	$\prod_{k=1}^{n} \mathrm{TH}_k - 1$		$\prod_{k=1}^{n} \mathrm{TH}_k + 1$	
1	0		2	(Prime)
2	3	(Prime)	5	(Prime)
3	$3 \cdot 13$		41	(Prime)
4	$17 \cdot 47$		$3^2 \cdot 89$	
5	$3^3 \cdot 17 \cdot 61$		28001	(Prime)
6	1567999	(Prime)	$3 \cdot 29 \cdot 67 \cdot 269$	
7	$251 \cdot 571 \cdot 919$		$47 \cdot 2802383$	
8	15805439999	(Prime)	$13 \cdot 71 \cdot 2957 \cdot 5791$	
9	$19 \cdot 113 \cdot 743 \cdot 1634819$		$278017 \cdot 9380353$	
10	$9551 \cdot 108217 \cdot 555097$		$9833 \cdot 58348161497$	
n	$\prod_{k=1}^{n} \mathrm{P}_k - 1$		$\prod_{k=1}^{n} \mathrm{P}_k + 1$	
1	0		2	(Prime)
2	2^2		$2 \cdot 3$	
3	59	(Prime)	61	(Prime)
4	1319	(Prime)	1321	(Prime)
5	46199	(Prime)	$47 \cdot 983$	
6	2356199	(Prime)	$479 \cdot 4919$	
7	$89 \cdot 1853191$		$179 \cdot 643 \cdot 1433$	
8	$2213 \cdot 6856723$		15173928001	(Prime)
9	1775349575999	(Prime)	$37 \cdot 109 \cdot 5573 \cdot 78989$	
10	$19 \cdot 13548720448421$		$21701 \cdot 11862388301$	

the table, which was totally unexpected. We might conjecture that all members of the family *Euclid* $[n]$ are square-free. A MATHEMATICA search confirmed the truth of this conjecture to $n = 49$. Analogous results hold for the first thirty outputs in the sequence $\prod_{k=1}^{n} \mathrm{F}_k + 1$. Similar behaviour does not hold for $\prod_{k=1}^{n} \mathrm{F}_k - 1$ as well as for the Lucas and factorial analogs. Counter-examples can be readily secured early in each of these sequences (for example, the sixteenth member of $\prod_{k=1}^{n} \mathrm{L}_k - 1$ has 5^2 in its prime factorization) and a number are depicted in tables 3 and 4. The reader is invited to extend the ideas articulated in this article to the products of other figurative numbers plus and minus one as well as additional integer sequences.

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4 http://oeis.org/.

Jay L. Schiffman *has taught mathematics at Rowan University for the past nineteen years. His research interests include number theory, discrete mathematics, and the interface of mathematics and technology as manifested via computer algebra systems and graphical calculators. He also enjoys traveling to conferences to disseminate his research.*

Probability in the Human Knot Game

M. B. RAO, KALYAAN RAO and CHRISTOPHER N. SWANSON

Suppose that n people stand in a circle and randomly join hands inside the circle to form a human link. An interesting problem is to identify what particular link has been formed. One way to attack this problem is to count how many components (knots) are entangled in the human link. In this article, we present the probability distribution for the number of components k formed when n people create a human link. We also present the formulas for the expected number of components formed and the variance of the number of components formed by n people. Finally, we note that the probability distribution can be approximated with a normal distribution if n is large.

Introduction

The Human Knot Game is an icebreaker sometimes played at summer camps, college orientation days, mathematics club meetings, and mathematics classes on knot theory. People stand in a circle, randomly join hands inside the circle and form a 'knot'. The goal is to change the 'knot' into a circle without letting go of each other's hands. Anyone who plays the game numerous times will notice that sometimes the 'knot' cannot be changed into a circle, and sometimes the 'knot' may change into multiple circles.

Formally, *knots* are closed curves in space that do not intersect themselves, and a *link* is just a set of entangled knots, where each knot in a link is called a *component* of the link. Informally, we can think of a knot as a string with its ends glued together and a link with k components as k strings entangled with the ends of each string glued together. Given our description of the Human Knot Game above, some mathematicians may prefer we call it the Human Link Game instead. The probability of forming a particular link in a human link consisting of n people would be quite difficult to calculate. Dreyer (see reference 1) answered this question for $n = 5$ if a person is not allowed to join his/her own hands together. In this article, we answer the following simplified question.

> What is the probability distribution for the number of components k formed when n people create a human link?

Throughout the article, we assume participants randomly join hands, allowing the possibility that a person joins his/her own hands together. In reality, participants do not join hands randomly as described in this article, so the probability distribution we present is probably not reflected in practice.

Spaghetti links and Russian wedding fortunes

Stan Wagon posts Problems of the Week at Macalester College, Saint Paul, MN, USA, that he also emails to people throughout the world. In the midst of performing a search of the mathematical literature to see if someone had solved the problem proposed in this article, we received the following Problem of the Week (see reference 2) submitted by John Guilford.

> **Problem 1007: Play with your food.** You are served a plate containing 100 spaghetti noodles. You randomly grab two ends from the pile and tie them together. Then you repeat this process until there are no ends left. What is the expected number of loops at the end?

If we know the probability distribution for the number of components k when n people form a human link, then the answer to this question is just the mean of this distribution when $n = 100$. John Guilford was unable to locate the reference where he originally saw the problem, but an internet search yielded references 3 and 4, which give similar versions of the spaghetti problem.

John Guilford also brought to our attention another variation of this problem in reference 5, p. 25.

> **78a.** In certain rural areas of Russia fortunes were once told in the following way. A girl would hold six long blades of grass in her hand with the ends protruding above and below; another girl would tie together the six upper ends in pairs and then tie together the six lower ends in pairs. If it turned out that the girl had thus tied the six blades of grass into a ring, this was supposed to indicate that she would get married within a year. What is the probability that a ring will be formed when the blades of grass are tied at random in this fashion? **b.** Solve the same problem for the case of $2n$ blades of grass.

Note that the second girl could initially tie together the upper ends of the $2n$ blades of grass in any way and not affect the probability that a ring will be formed. Once she ties together the upper ends of the grass, we can think of the first girl holding n long blades of grass with the two ends of each long blade of grass protruding below her hands. Treating the long blades of grass as people and the blades' ends as hands, we see this question corresponds to finding the probability that a human link of n people has $k = 1$ component. Similar versions of the $n = 3$ case of this Russian wedding problem (which brings new meaning to 'tying the knot') are in references 6, 7, and 8.

Combinatorial derivation of distribution

Let J_n be the number of ways that n people can join hands if a person's left and right hands are considered distinct. Since there are a total of $2n$ hands to connect, we first pick two hands to connect in $\binom{2n}{2}$ ways. Then we pick two of the remaining hands to connect in $\binom{2n-2}{2}$ ways. We continue to do this until there are only two hands left to connect which can be done in $\binom{2}{2} = 1$ way. Since the order in which these n pairs of hands are joined doesn't matter, and there are $n!$ ways to order these pairs of hands, we see that

$$J_n = \binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2} \Big/ n! = \frac{(2n)!}{2^n n!} = (2n-1)(2n-3)\cdots 3\cdot 1.$$

We can write this last expression for J_n using double factorial notation as $(2n-1)!!$, and this is sequence A001147 in reference 9.

Let $a_{n,k}$ be the number of ways that n people can join hands to form a human link with k components. Note that for $n \geq 1$, $a_{n,0} = 0$ since if we have at least one person, we must

Table 1 Number of ways n people can form a human link with k components.

n	J_n	k = 1	2	3	4	5	6	7	8
1	1	1							
2	3	2	1						
3	15	8	6	1					
4	105	48	44	12	1				
5	945	384	400	140	20	1			
6	10 395	3 840	4 384	1 800	340	30	1		
7	135 135	46 080	56 448	25 984	5 880	700	42	1	
8	2 027 025	645 120	836 352	420 224	108 304	15 680	1 288	56	1

have at least one component. Also note that $a_{n,n} = 1$ since if we have n people, the only way to form a human link with n components is for each person to join his/her own hands. For $1 \leq k \leq n-1$, we claim that $a_{n,k} = a_{n-1,k-1} + 2(n-1)a_{n-1,k}$. To see this, suppose that we have a human link consisting of $n-1$ people at a party. Parker shows up to the party late and wishes to join the human link. He can join his own hands together, thereby creating a human link with one more person and one more component. Otherwise, he can select one of the other $2(n-1)$ hands to grab with his left hand, and then grab with his right hand the hand previously joined to this selected hand. In doing this, Parker will create a human link with one more person and the same number of components since he just joined an existing component. Clearly, this argument is reversible if Parker has to leave the party early and we join the hands that were holding his hands.

Using the recurrence relation and initial conditions in the previous paragraph, we produce table 1. The portion of the table corresponding to values of $a_{n,k}$ is formed by first filling in 1s down the main diagonal. We fill in other entries using the recurrence relation by adding the entry above and to the left of it to $2(n-1)$ times the entry above it (treating missing entries as 0s). For example, $a_{5,3} = 44 + 8 \cdot 12 = 140$. The second column of table 1 gives J_n, the total number of ways n people can join hands, and thus the probability distribution for the number of components is simply found by dividing the $a_{n,k}$ entries by the corresponding value in the J_n column. In particular, the answer to the blades of grass problem is simply the $k = 1$ column divided by the J_n column.

The sequence $a_{n,k}$ is the unsigned sequence A039683 in reference 9 and is given the name *double Pochhammer triangle*. The *Pochhammer symbol* or *rising factorial* is defined as $(x)_n \equiv x(x+1)(x+2)\cdots(x+n-1)$. Similar to the definition of double factorial, we refer to the expression $x(x+2)(x+4)\cdots(x+2n-2)$ as the *double Pochhammer*. The sequence $a_{n,k}$ is the coefficient of x^k in the expansion of $x(x+2)(x+4)\cdots(x+2n-2)$. Using a formula for sequence A039683 in reference 9, we can write $a_{n,k} = 2^{n-k} \cdot S_1(n,k)$, where $S_1(n,k)$ is an unsigned Stirling number of the first kind, or the number of permutations of n elements that contain exactly k cycles with cycles in opposite directions counted as distinct. If we number the n people forming our human link, then each human link with k components has a natural correspondence to a permutation of n elements that contain k cycles. However, we are making a distinction between each person's left and right hands, which essentially means that we care whether each person is facing in towards the center of his/her component or facing out away from the center of his/her component. Since the Stirling numbers count cycles in opposite directions as distinct, we can think of the direction of a particular cycle as corresponding to whether the lowest numbered person in each component is facing in or out of the circle. Thus, we only need to decide which direction the remaining $n-k$ people are facing. Given that there are two directions for each of these people to face, there are 2^{n-k} ways to decide this, explaining the 2^{n-k} factor in the formula.

In particular, the solution to the blades of grass question is $a_{n,1}/J_n$. Using the formula above, $a_{n,1} = 2^{n-1}(n-1)!$. This is sequence A000165 in reference 9, and can be denoted using double factorial numbers as $a_{n,1} = (2n-2)!!$, and thus the answer to the blades of grass question can be written as $(2n-2)!!/(2n-1)!!$. This solution to Problem 78 in reference 5 (see pp. 155–156) also uses Stirling's formula, $n! \approx n^n \mathrm{e}^{-n}\sqrt{2\pi n}$, to show that this probability is approximately $\sqrt{\pi}/2\sqrt{n}$ for large n.

Expected number of components

The spaghetti problem is typically solved using a recurrence relation for the expected number of loops based on whether or not the first knot tied creates a loop (see reference 3). This argument is very similar to our justification of the recurrence relation for $a_{n,k}$. We present a new way to find a formula for the expected number of components based on the probability distribution. Let X_n be the number of components formed by n people. Using our notation, $\mathrm{P}(X_n = k) = a_{n,k}/J_n$. Let μ_n be the expected number of components formed by n people. Then,

$$
\begin{aligned}
\mu_n &= \sum_{k=1}^{n} \frac{k a_{n,k}}{J_n} \\
&= \sum_{k=1}^{n} \frac{k(a_{n-1,k-1} + 2(n-1)a_{n-1,k})}{(2n-1)J_{n-1}} \\
&= \frac{1}{2n-1} \sum_{k=1}^{n} \frac{(k-1)a_{n-1,k-1} + a_{n-1,k-1} + 2(n-1)k a_{n-1,k}}{J_{n-1}} \\
&= \frac{1}{2n-1}\left(\sum_{j=1}^{n-1}\left(\frac{j a_{n-1,j}}{J_{n-1}} + \frac{a_{n-1,j}}{J_{n-1}} \right) + \sum_{k=1}^{n-1} \frac{2(n-1)k a_{n-1,k}}{J_{n-1}} \right) \\
&= \frac{1}{2n-1}(\mu_{n-1} + 1 + 2(n-1)\mu_{n-1}) \\
&= \mu_{n-1} + \frac{1}{2n-1}.
\end{aligned}
$$

Hence, we see that

$$
\mu_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1},
$$

the sum of the reciprocals of the first n odd integers. Thus, not surprisingly, the expected number of components approaches infinity as the number of people forming the link approaches infinity. What seems more surprising is how slow this convergence is. In particular, $\mu_n < 4$ for $n \leq 418$ and $\mu_n < 5$ for $n \leq 3091$. Therefore, we need a plate of 3092 spaghetti noodles to expect to form more than five spaghetti loops when we tie the ends of the spaghetti noodles together. The Euler–Mascheroni constant is

$$\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \ln(n)\right) \sim 0.577\,215\,664\,9,$$

and thus

$$\sum_{m=1}^{n}\frac{1}{m} \sim \ln(n) + \gamma.$$

Note that we can rewrite our formula for the expected number of components as

$$\mu_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n} - \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).$$

Thus, an approximation for the expected number of components is given by

$$\mu_n \sim \ln(2n) + \gamma - \frac{1}{2}(\ln(n) + \gamma) = \frac{\ln(n)}{2} + \ln(2) + \frac{\gamma}{2}.$$

This approximation is accurate to within 0.0001 for $n \geq 15$.

Variance of number of components

The expectation of a function of a discrete random variable $u(X)$ is defined to be

$$\mathrm{E}[u(X)] = \sum_{x} u(x)\,\mathrm{P}(X = x),$$

with the summation being over all possible values of the random variable X. The variance σ^2 of a random variable X with mean μ is defined to be $\sigma^2 = \mathrm{E}[(X-\mu)^2]$ and is often found using the formula $\sigma^2 = \mathrm{E}[X^2] - \mu$. Let σ_n^2 be the variance of the number of components formed by n people. Then

$$\begin{aligned}
\sigma_n^2 &= \mathrm{E}[X_n^2] - \mu_n^2 \\
&= \sum_{k=1}^{n}\frac{k^2 a_{n,k}}{J_n} - \mu_n^2 \\
&= \sum_{k=1}^{n}\frac{k^2(a_{n-1,k-1} + 2(n-1)a_{n-1,k})}{(2n-1)J_{n-1}} - \mu_n^2 \\
&= \frac{1}{2n-1}\left(\sum_{k=1}^{n}\frac{(k-1)^2 a_{n-1,k-1} + (2k-1)a_{n-1,k-1} + 2(n-1)k^2 a_{n-1,k}}{J_{n-1}}\right) - \mu_n^2,
\end{aligned}$$

rewriting k^2 as $(k-1)^2+(2k-1)$. Now, changing the index of summation to $j=k-1$, we get

$$\begin{aligned}
\sigma_n^2 &= \frac{1}{2n-1}\left(\sum_{j=1}^{n-1}\left(\frac{j^2 a_{n-1,j}}{J_{n-1}}+\frac{(2j+1)a_{n-1,j}}{J_{n-1}}\right)+\sum_{k=1}^{n-1}\frac{2(n-1)k^2 a_{n-1,k}}{J_{n-1}}\right)-\mu_n^2\\
&= \frac{1}{2n-1}(\mathrm{E}[X_{n-1}^2]+2\mu_{n-1}+1+2(n-1)\,\mathrm{E}[X_{n-1}^2])-\left(\mu_{n-1}+\frac{1}{2n-1}\right)^2\\
&= \mathrm{E}[X_{n-1}^2]+\frac{1}{2n-1}-\mu_{n-1}^2-\frac{1}{(2n-1)^2}\\
&= \sigma_{n-1}^2+\frac{2(n-1)}{(2n-1)^2}.
\end{aligned}$$

Hence, we see that

$$\sigma_n^2=\frac{2}{3^2}+\frac{4}{5^2}+\frac{6}{7^2}+\cdots+\frac{2(n-1)}{(2n-1)^2},$$

which may be rewritten as

$$\sigma_n^2=1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{2n-1}-\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots+\frac{1}{(2n-1)^2}\right).$$

Thus, the variance of the number of components approaches infinity as the number of people forming the link approaches infinity. Euler showed that

$$\sum_{m=1}^{\infty}\frac{1}{m^2}=\frac{\pi^2}{6}.$$

Using this result, a similar derivation as the last section gives an approximation for the variance of the number of components as

$$\sigma_n^2\sim\frac{\ln(n)}{2}+\ln(2)+\frac{\gamma}{2}-\frac{\pi^2}{8},$$

and this approximation is accurate to within 0.01 for $n\geq 26$.

Alternative derivation of distribution

Recall that X_n is the number of components formed by n people. Consider person n's right hand. The probability that his/her right hand grabs his/her left hand is $1/(2n-1)$, in which case person n forms one component. Otherwise, person n grabs someone else's hand and we can consider this conglomeration of two people to be one 'superperson'. In either case, we still need to decide how to match the $n-1$ 'people' that remain. Thus, we see that $X_n=X_{n-1}+1$ with probability $1/(2n-1)$, and $X_n=X_{n-1}$ with probability $(2n-2)/(2n-1)$. A Bernoulli experiment is a random experiment with two possible outcomes, success or failure, with p denoting the probability of a success. Given a Bernoulli experiment, a Bernoulli random variable Y is equal to 1 when the result of the experiment is a success and 0 when the result is a failure. Considering the connection of person n's right hand as a Bernoulli experiment

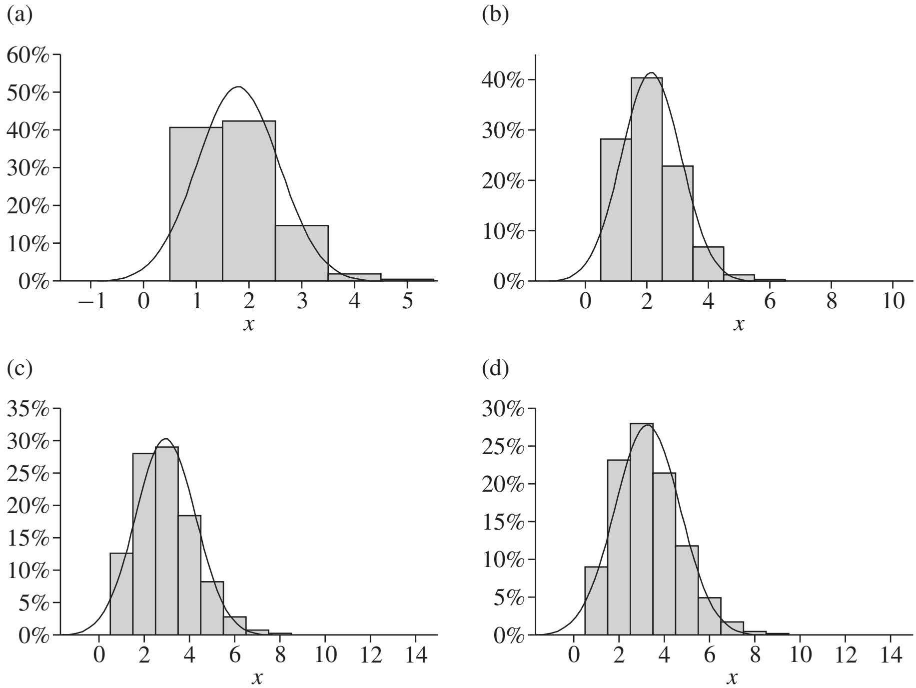

Figure 1 Probability distributions of X_n with normal distribution approximations for (a) $n = 5$, (b) $n = 10$, (c) $n = 50$, and (d) $n = 100$.

with a success being his/her right hand grabs his/her left hand, then denote the corresponding Bernoulli random variable as Y_n with probability of success $p = 1/(2n-1)$. Observe that $X_n = X_{n-1} + Y_n$ with X_{n-1} and Y_n independent random variables. Thus,

$$\begin{aligned} \mathrm{P}(X_n = k) &= \mathrm{P}(X_{n-1} = k-1 \text{ and } Y_n = 1) + \mathrm{P}(X_{n-1} = k \text{ and } Y_n = 0) \\ &= \frac{1}{2n-1}\,\mathrm{P}(X_{n-1} = k-1) + \frac{2n-2}{2n-1}\,\mathrm{P}(X_{n-1} = k). \end{aligned}$$

If we multiply this equation by J_n and simplify, we note that we get the precise recurrence relation we had for $a_{n,k}$ earlier in this article.

One advantage of this derivation of the distribution of the number of components is that we can repeat the argument above to get $X_n = Y_1 + Y_2 + \cdots + Y_n$ with $Y_1, Y_2, \ldots, Y_n$ independent Bernoulli random variables. Thus, we can easily write a computer program to determine the exact distribution of X_n for any n. Furthermore, since the Bernoulli random variable Y_k has mean $1/(2k-1)$ and variance $1/(2k-1) - 1/(2k-1)^2$, we may use the independence of the Y_ks to derive the formulas for the mean and variance of X_n by simply summing the means and variances of the Bernoulli random variables. If an infinite set of random variables $W_1, W_2, W_3, \ldots$ satisfies the Lindeberg condition (see reference 10, p. 298), then their sum is approximately normally distributed if n is large. We can show that the Lindeberg condition holds for the random variables $Y_1, Y_2, \ldots, Y_n$, and thus X_n is approximately normally distributed if n is large. Figure 1 displays the exact probability distributions of X_n for $n \in \{5, 10, 50, 100\}$, with the corresponding normal distributions superimposed.

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M. B. Rao *is a professor of Statistical Genetics and Biomedical Engineering at the University of Cincinnati. He has varied interests ranging from matrix algebra, measure theory, abstract probability, mathematical statistics, and statistics applied to mathematical puzzles. He is the author of two books.*

Kalyaan Rao *is currently a senior at Ohio State University doing a double major in Industrial Engineering and Biomathematics with a minor in entrepreneurship. He hopes to pursue graduate studies.*

Christopher N. Swanson *is an Associate Professor of Mathematics and the Director of the Academic Honors Program at Ashland University in Ashland, Ohio. His research interests are combinatorics and probability. In 2006, Chris received the national Henry L. Alder Award from the Mathematical Association of America in recognition for distinguished teaching by a beginning mathematics faculty member.*

At 12 minutes past 8pm on 20th December this year, the time will be

20:12 on 20/12 of 2012.

At this auspicious time I will be observing one minute's silence before this moment will be lost for ever!

Midsomer Norton, Bath, UK **Bob Bertuello**

Always a Cube

PRITHWIJIT DE

This article starts with a very simple problem on mensuration involving a rectangular box and explores the cases that emerge from it by altering the initial conditions. In all the cases the answer is a cube, and hence the title.

Let us consider the following problem. Suppose that l, b, h are the length, breadth, and height of a rectangular box.

Problem 1 If l, b, h are in arithmetic progression (AP), and if the squares of the lengths of the face-diagonals are also in AP, then what are the ratios $l : b : h$?

Solution 1 To answer this question, assume that $l \geq b \geq h$. As l, b, h are in AP we know that

$$l + h = 2b. \tag{1}$$

The squares of the lengths of the face-diagonals are $l^2 + b^2$, $l^2 + h^2$, and $b^2 + h^2$. Now $l \geq b \geq h$ implies $l^2 + b^2 \geq l^2 + h^2 \geq b^2 + h^2$, and as these are in AP

$$(l^2 + b^2) + (b^2 + h^2) = 2(l^2 + h^2), \qquad \text{i.e. } l^2 + h^2 = 2b^2. \tag{2}$$

Note that l^2, b^2, h^2 are in AP. Eliminating b from (1) and (2) we get $2(l^2 + h^2) = (l + h)^2$, i.e. $(l - h)^2 = 0$, whence $l = h$. Therefore $l = b = h$ and hence $l : b : h = 1 : 1 : 1$. So the rectangular box has to be a cube.

This was a simple problem. Let us tweak it a bit. Instead of assuming that the squares of the lengths of the face-diagonals are in AP, let us assume that their lengths are in AP. Can we still conclude that $l : b : h = 1 : 1 : 1$?

Problem 2 If l, b, h are in AP and $\sqrt{l^2 + b^2}$, $\sqrt{l^2 + h^2}$, $\sqrt{b^2 + h^2}$ are in AP, what are the ratios $l : b : h$?

Solution 2 The new set of equations to contend with is

$$l + h = 2b, \tag{3}$$

$$\sqrt{l^2 + b^2} + \sqrt{b^2 + h^2} = 2\sqrt{l^2 + h^2}. \tag{4}$$

Eliminating b from (3) and (4) leaves us with

$$(5x^2 + 2x + 1)(x^2 + 2x + 5) = (5x^2 - 2x + 5)^2, \tag{5}$$

where $x = l/h$. It may appear that we are in trouble because we are staring at a fourth-degree equation and in general it is not easy to solve such equations. But in this case it is not too difficult to solve (5). Let

$$f(x) = (5x^2 + 2x + 1)(x^2 + 2x + 5) - (5x^2 - 2x + 5)^2.$$

The set of real zeros of $f(x)$ on $[1, \infty)$ is the same as the set of real zeros of $g(x) = f(x)/x^2$ on the same interval. But

$$g(x) = \left(5x + \frac{1}{x} + 2\right)\left(x + \frac{5}{x} + 2\right) - \left(5\left(x + \frac{1}{x}\right) - 2\right)^2 = 16 + 32\left(x + \frac{1}{x}\right) - 20\left(x + \frac{1}{x}\right)^2.$$

The quadratic in $x + 1/x$ on the right-hand side factors as $4(2 - x - 1/x)(2 + 5x + 5/x)$. Thus $g(x) = 0$ if and only if $x + 1/x = 2$, i.e. if and only if $x = 1$, i.e. $l = h$. Therefore once again we end up with $l = b = h$, whence $l : b : h = 1 : 1 : 1$ and the box cannot be anything but a cube.

Having conquered this problem without encountering much of difficulty, let us move on and cover more ground. What if l, b, and h are in geometric progression (GP) and the squares of the face-diagonals are in AP?

Problem 3 If l, b, h are in GP and $l^2 + b^2$, $l^2 + h^2$, $b^2 + h^2$ are in AP, what are the ratios $l : b : h$?

Solution 3 Now (1) changes to

$$b^2 = lh, \tag{6}$$

but (2) remains unaltered. From (2) and (6) it follows immediately that $(l - h)^2 = 0$, i.e. $l = h$ and hence $l = b = h$. Thus once again $l : b : h = 1 : 1 : 1$. The answer remains the same if we assume that $l^2 + b^2$, $l^2 + h^2$, and $b^2 + h^2$ are in GP instead of AP.

Problem 4 If l, b, h are in GP and $l^2 + b^2$, $l^2 + h^2$, $b^2 + h^2$ are in GP, what are the ratios $l : b : h$?

Solution 4 Just observe that (2) changes to

$$(l^2 + b^2)(b^2 + h^2) = (l^2 + h^2)^2. \tag{7}$$

Upon simplification (7) reduces to $l^4 + l^2h^2 + h^4 = b^2(l^2 + h^2 + b^2)$. To further simplify this, observe that $l^4 + l^2h^2 + h^4 = (l^2 + lh + h^2)(l^2 - lh + h^2)$ and recall $b^2 = lh$. Using these two pieces of information in the simplified form above we get

$$l^2 - 2lh + h^2 = 0,$$

i.e. $(l - h)^2 = 0$, and we conclude once again that $l : b : h = 1 : 1 : 1$.

It is easy to see that the same conclusion is reached by assuming that l, b, h are in GP and $\sqrt{l^2 + b^2}$, $\sqrt{l^2 + h^2}$, $\sqrt{b^2 + h^2}$ are in GP because $\sqrt{l^2 + b^2}$, $\sqrt{l^2 + h^2}$, $\sqrt{b^2 + h^2}$ are in GP if and only if $l^2 + b^2$, $l^2 + h^2$, $b^2 + h^2$ are in GP, and we are back to the previous problem.

We now consider a few more problems which may appear more challenging in terms of the algebra involved.

Problem 5 If l, b, h are in AP and $\sqrt{l^2 + b^2}$, $\sqrt{l^2 + h^2}$, $\sqrt{b^2 + h^2}$ are in GP, what are the ratios $l : b : h$?

Solution 5 The relevant equations are

$$l + h = 2b, \tag{8}$$

$$\sqrt{l^2 + b^2}\sqrt{b^2 + h^2} = l^2 + h^2. \tag{9}$$

Eliminating b from (8) and (9) we get

$$16(l^4 + l^2h^2 + h^4) = (l + h)^2(5l^2 + 2lh + 5h^2).$$

Setting $x = l/h$ gives

$$16(x^4 + x^2 + 1) = (x^2 + 2x + 1)(5x^2 + 2x + 5).$$

Expanding, simplifying, and factorising we get

$$(x - 1)^2(11x^2 + 10x + 11) = 0.$$

Thus $x = 1$ or $l = h$. Hence $l = b = h$ and we obtain a cube.

Problem 6 If l, b, h are in harmonic progression (HP) and $l^2 + b^2$, $l^2 + h^2$, $b^2 + h^2$ are in AP, what are the ratios $l : b : h$?

Solution 6 In this case the equations to be dealt with are

$$\frac{1}{l} + \frac{1}{h} = \frac{2}{b}, \tag{10}$$

$$l^2 + h^2 = 2b^2. \tag{11}$$

After eliminating b from (10) and (11) we are left with

$$\left(\frac{l}{h} + 1\right)^2 + \left(\frac{h}{l} + 1\right)^2 = 8.$$

Setting $x = l/h$ and simplifying we obtain

$$\left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) - 8 = \left(x + \frac{1}{x} - 2\right)\left(x + \frac{1}{x} + 4\right) = 0.$$

Thus $x + 1/x - 2 = 0$ implying $x = 1$ or $l = h$. Hence $l = b = h$ and we obtain a cube.

Problem 7 If l, b, h are in HP and $l^2 + b^2$, $l^2 + h^2$, $b^2 + h^2$ are in HP, what are the ratios $l : b : h$?

Solution 7 Here we have

$$\frac{1}{l} + \frac{1}{h} = \frac{2}{b}, \tag{12}$$

$$\frac{1}{l^2 + b^2} + \frac{1}{b^2 + h^2} = \frac{2}{l^2 + h^2}. \tag{13}$$

Writing (13) as

$$\frac{1}{b^2 + h^2} - \frac{1}{l^2 + h^2} = \frac{1}{l^2 + h^2} - \frac{1}{l^2 + b^2}$$

and simplifying, we obtain

$$\frac{l^2 - b^2}{b^2 + h^2} = \frac{b^2 - h^2}{l^2 + b^2},$$

whence

$$l^4 + h^4 = 2b^4. \tag{14}$$

Note that (14) shows that l^4, b^4, h^4 are in AP. From (12) we get $b = 2lh/(l+h)$. Using this in (14) we get after simplification

$$\left(\frac{l}{h}+1\right)^4 + \left(\frac{h}{l}+1\right)^4 = 32. \tag{15}$$

Putting $x = l/h$ in (15) we obtain

$$(1+x)^4(1+x^4) = 32x^4.$$

The polynomial equation obtained after expanding and simplifying is

$$1 + 4x + 6x^2 + 4x^3 - 30x^4 + 4x^5 + 6x^6 + 4x^7 + x^8 = 0. \tag{16}$$

Dividing both sides of (16) by x^4 we get

$$\left(x^4 + \frac{1}{x^4}\right) + 4\left(x^3 + \frac{1}{x^3}\right) + 6\left(x^2 + \frac{1}{x^2}\right) + 4\left(x + \frac{1}{x}\right) - 30 = 0. \tag{17}$$

So, how do we solve it without software? First observe that

$$4\left(x^3 + \frac{1}{x^3}\right) + 4\left(x + \frac{1}{x}\right) = 4\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right). \tag{18}$$

Also

$$\left(x^4 + \frac{1}{x^4}\right) + 6\left(x^2 + \frac{1}{x^2}\right) = \left(x^2 + \frac{1}{x^2}\right)^2 + 6\left(x^2 + \frac{1}{x^2}\right) - 2. \tag{19}$$

Using (18) and (19) in (17) transforms it to

$$\left(x^2 + \frac{1}{x^2}\right)\left(x^2 + \frac{1}{x^2} + 4\left(x + \frac{1}{x}\right) + 6\right) - 32 = 0. \tag{20}$$

We are in familiar territory now. Using

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$$

and setting $t = x + 1/x$ in (20) we get

$$(t^2 - 2)(t+2)^2 - 32 = 0. \tag{21}$$

The left-hand side of (21) factors as $(t-2)(t^3 + 6t^2 + 14t + 20)$. The second factor cannot be zero as $t > 0$. Thus (21) is satisfied if and only if $t = 2$, implying $x = 1$ or $l = h$. Hence $l = b = h$ and we end up once again with a cube.

Problem 8 If l, b, h are in AP and $l^2 + b^2$, $l^2 + h^2$, $b^2 + h^2$ are in HP, what are the ratios $l : b : h$?

Solution 8 Equations (1) and (13) are relevant in this problem. Eliminating b from these two equations we get

$$8(l^4 + h^4) = (l + h)^4.$$

As before we set $x = l/h$ and simplify to obtain

$$7\left(x + \frac{1}{x}\right)^2 - 4\left(x + \frac{1}{x}\right) - 20 = \left(x + \frac{1}{x} - 2\right)\left(7x + \frac{7}{x} + 10\right) = 0.$$

Thus $x + 1/x = 2$ and $x = 1$, implying $l = h$ which leads to $l = b = h$.

Problem 9 If l, b, h are in HP and $l^2 + b^2$, $l^2 + h^2$, $b^2 + h^2$ are in GP, what are the ratios $l : b : h$?

Solution 9 The pertinent equations are

$$\frac{1}{l} + \frac{1}{h} = \frac{2}{b},$$
$$(l^2 + b^2)(b^2 + h^2) = (l^2 + h^2)^2.$$

Eliminating b and setting $x = l/h$ we end up with

$$4x^2(1 + 2x + 6x^2 + 2x^3 + x^4) = (1 + x^2 + x^4)(1 + x)^4. \tag{22}$$

Dividing both sides of (22) by x^4 yields

$$4\left(\left(x^2 + \frac{1}{x^2}\right) + 2\left(x + \frac{1}{x}\right) + 6\right) - \left(x^2 + \frac{1}{x^2} + 1\right)\left(x + \frac{1}{x} + 2\right)^2 = 0. \tag{23}$$

Write $x^2 + 1/x^2 = (x + 1/x)^2 - 2$ and set $t = x + 1/x$ in (23) to get

$$t^4 + 4t^3 - t^2 - 12t - 20 = 0. \tag{24}$$

The polynomial on the left-hand side of (24) factors as

$$(t - 2)(t^3 + 6t^2 + 11t + 10).$$

As $t > 0$, (24) is satisfied if and only if $t = 2$ implying $x = 1$ or $l = h$. Once again we see that $l = b = h$.

What happens if l, b, h are in GP and $l^2 + b^2$, $l^2 + h^2$, $b^2 + h^2$ are in HP? Do we get a cube? We leave it to the reader.

We conclude by making an observation. In most of the problems considered above we have reduced a higher degree polynomial equation in one variable, for instance x, to a lower degree polynomial equation in a new variable $t = x + 1/x$ and extracted the root we sought.

Prithwijit De *is a faculty member at Homi Bhabha Centre for Science Education, Tata Institute of Fundamental Research, Mumbai. He holds a PhD in Statistics from University College Cork, Ireland. He enjoys mathematical problem-solving and recreational mathematics.*

Converting a Fraction into a Decimal by Hand Calculation

KING SHUN LEUNG

It is time-consuming to convert a fraction whose denominator has two or more digits into a decimal by hand calculation. In this article we present two efficient conversion methods due to A. C. Aitken. One method applies to fractions of the form $x = l/(k \times 10^n - 1)$ while the other applies to $x = l/(k \times 10^n + 1)$, where k is a single-digit number. Aitken's methods outdo the traditional method as they involve only successive division by the single-digit number k.

Introduction

To convert a fraction such as $\frac{5}{23}$ into a decimal ($\frac{5}{23} = 0.\dot{2}17\,391\,304\,347\,826\,086\,956\,\dot{5}$) by hand calculation is time-consuming and prone to errors. This is because the traditional long division method is a process that involves successive division by a two-digit number 23. In reference 1, A. C. Aitken did the conversion in two more efficient ways involving easier divisions. We reformulate his methods as described below.

Method 1

First write $\frac{5}{23} = \frac{15}{69}$ to obtain the denominator one less than a multiple of 10. Now write

$$\frac{15}{69} = \frac{15 + \frac{15}{69}}{70}.$$

Put $\frac{15}{69} = 0.abcd\ldots$. Now

$$0.abcd\ldots = \tfrac{1}{70}(15.abcd\ldots) = 0.2\ldots,$$

so $a = 2$. Then

$$0.2bcd\ldots = \tfrac{1}{70}(15^1.2bcd\ldots) = 0.21\ldots,$$

so $b = 1$. Now

$$0.21cd\ldots = \tfrac{1}{70}(15^1.2^5 1cd\ldots) = 0.217\ldots.$$

Continuing in this way, it is seen that, beginning with 15, at each stage, the number is divided by 7 and the remainder is placed in front of the quotient, giving the sequence

15, 12, 51, 27, 63, 09, 21, 03, 30, 24, 33, 54, 57, 18, 42, 06, 60, 48, 66, 39, 45, 36, 15,

after which the sequence repeats. The sequence of quotients (the unit digits in this sequence after the first term) gives the decimal representation of

$$\tfrac{5}{23}: \tfrac{5}{23} = 0.\dot{2}17\,391\,304\,347\,826\,086\,956\,\dot{5}.$$

This method can be used for all fractions which can be expressed in the form

$$x = \frac{l}{k \times 10^n - 1} = \frac{l + x}{k \cdot 10^n}.$$

It will involve successive division by k, so will mean less mental effort, especially when k is a single-digit number.

As a second example,

$$\frac{1}{19} = \frac{1 + \frac{1}{19}}{20}.$$

The method involves successive division by 2, again placing the remainder in front of the quotient at each stage, giving the sequence

$$10,\, 05,\, 12,\, 06,\, 03,\, 11,\, 15,\, 17,\, 18,\, 09,\, 14,\, 07,\, 13,\, 16,\, 08,\, 04,\, 02,\, 01,\, 10,$$

giving $\frac{1}{19} = 0.\dot{0}52\,631\,578\,947\,368\,42\dot{1}$, which is much easier than successive division by 19.

Now consider $\frac{1}{38}$. It is no longer possible to obtain a denominator one less than a multiple of 10. The denominator is $4 \times 10 - 2$, and we can write

$$\frac{1}{38} = \frac{1 + \frac{2}{38}}{40}.$$

Put $\frac{1}{38} = 0.0abcd\ldots$. Then

$$0.0abcd\ldots = \frac{1 + 2 \times 0.0abcd\ldots}{40} = \frac{1.0\ldots}{40} = 0.02\ldots,$$

giving $a = 2$. Now

$$0.02b\ldots = \frac{1 + 2 \times 0.02\ldots}{40} = \frac{1.0^24\ldots}{40} = 0.026\ldots,$$

giving $b = 6$. Next

$$0.026c\ldots = \frac{1 + 2 \times 0.026c\ldots}{40} = \frac{1.0^25^12\ldots}{40} = 0.0263\ldots,$$

giving $c = 3$. These three divisions are as follows.

$$4 \,\big|\, \underline{1 \quad 0} \qquad\qquad 4 \,\big|\, \underline{2 \quad 4} \qquad\qquad 4 \,\big|\, \underline{1 \quad 2}$$
$$2 \quad \text{rem } 2 \qquad\qquad 6 \quad \text{rem } 0 \qquad\qquad 3 \quad \text{rem } 0$$

The general rule at each stage is to divide by 4, record the quotient, multiply this dividend by 2, and add to its tens digit the remainder to form the next number in the sequence. This gives the sequence with the quotient at each stage in brackets:

$$10(2),\, 24(6),\, 12(3),\, 06(1),\, 22(5),\, 30(7),\, 34(8),\, 36(9),\, 18(4),\, 28(7),$$
$$14(3),\, 26(6),\, 32(8),\, 16(4),\, 08(2),\, 04(1),\, 02(0),\, 20(5),\, 10,$$

after which the sequence repeats. For example, the fifth term of the sequence is 22, which on division by 4 gives 5 remainder 2. Multiply 5 by 2 to give 10 and add 2 to the tens digit to give 30. So the next term in the sequence is 30. The quotients give the decimal expansion of $\frac{1}{38}$: $\frac{1}{38} = 0.\dot{0}26\,315\,789\,473\,684\,210\,\dot{5}$.

Method 2

Aitken (see reference 1) also found the decimal expansion of $\frac{5}{23}$ in the following way. He wrote

$$\frac{5}{23} = \frac{435}{2001} = \frac{434 + (1 - \frac{435}{2001})}{2000}.$$

Write $\frac{5}{23} = 0.abcd\ldots$. Then $0.abcd\ldots = 0.217\ldots$, so $abc = 217$. Then

$$\begin{aligned} 0.217def\ldots &= 0.217 + \frac{0.999\ldots - 0.217def\ldots}{2000} \\ &= 0.217 + \frac{999-217}{2\times 10^6} + \cdots \\ &= 0.217391\ldots. \end{aligned}$$

Then

$$0.217391ghi\ldots = 0.217391 + \frac{999-391}{2\times 10^9} + \cdots = 0.217391304\ldots.$$

Next

$$0.217391304\ldots = 0.217391304 + \frac{999-304}{2\times 10^{12}} + \cdots = 0.217391304347 + \frac{1}{2\times 10^{12}} + \cdots,$$

whence

$$0.217391304347\ldots = 0.217391304347 + \frac{1999-347}{2\times 10^{15}} + \cdots = 0.217391304347826\ldots,$$

and so on. We can set out the steps more succinctly as follows:

$$\frac{434\ \ 782\ \ 608\ \ 695\ \ 1652\ \ 173\ \ 1913\ \ 1043\ \ 1478\ldots}{217\ \ 391\ \ 304\ \ 347^1\ \ 826\ \ 086^1\ \ 956^1\ \ 521^1\ \ 739\ldots}.$$

This gives $\frac{5}{23} = 0.\dot{2}17\,391\,304\,347\,826\,086\,956\,\dot{5}$. Since we started with the fraction $\frac{5}{23}$, the decimal must repeat by the 22nd decimal places as is seen. The procedure is summarised as follows. Start with 434 and at each stage divide the number by 2 and take the resulting number from 999, or 1999 if the previous number is odd, to give the next number on the top row. The numbers on the bottom row are the quotients of the numbers on the top row when divided by 2, and these give the decimal expansion of $\frac{5}{23}$.

We illustrate the method with one more example,

$$\frac{4}{31} = \frac{3 + (1 - \frac{4}{31})}{30}.$$

We have

$$\frac{3\ \ 8\ \ 27\ \ 0\ \ 9\ \ 6\ \ 7\ \ 17\ \ 24\ \ 1\ \ 19\ \ 13\ \ 15\ \ 4\ \ 18\ \ 3\ldots}{1\ \ 2^2\ \ 9\ \ 0\ \ 3\ \ 2\ \ 2^1\ \ 5^2\ \ 8\ \ 0^1\ \ 6^1\ \ 4^1\ \ 5\ \ 1^1\ \ 6\ \ 1\ldots}.$$

Start with 3 and at each stage divide the number by 3 and take the resulting number from 9, or 19 (29) if the previous number is one (two) more than a multiple of 3, to give the next number on the top row. The quotients of the numbers on the top row when divided by 3 are recorded on the bottom row, which gives $\frac{4}{31} = 0.\dot{1}29\,032\,258\,064\,51\dot{6}$.

This method can be used to find the decimal expansion of a fraction of the form

$$x = \frac{l}{k \times 10^n + 1},$$

where k lies between 1 and 9, using the equation

$$x = \frac{(l-1) + (1-x)}{k \times 10^n}.$$

Reference

1 A. C. Aitken, The art of mental calculation: with demonstrations, *J. Trans. Soc. Eng. London* **44** (1954), pp. 295–309.

King Shun Leung *is an assistant professor at The Hong Kong Institute of Education. His research interests include number theory, fractals, tilings, mathematical games, and the mathematics of paper-folding.*

Letters to the Editor

Dear Editor,

Periodicity of the Fibonacci sequence

In Volume 43, Number 3, pp. 120–124, Jay L. Schiffman showed that the Fibonacci sequence repeats modulo any given positive integer and gave the periodicity for divisibility by all the numbers up to 100. Thus, for example, the periodic length modulo 5 is 20. When I was 17, studying at High School under the guidance of my teacher Mr Yadollah Mohammadi, I proved the following periodicity formula for Fibonacci numbers.

If $m = a^2 - a - 1$ $(a \in \mathbb{N})$ and $d = (m, 2a-1)$, the highest common factor of m and $2a-1$, then

$$\mathrm{F}_{\phi(m)+n} \equiv \mathrm{F}_n \left(\operatorname{mod} \frac{m}{d}\right).$$

Here ϕ is the Euler ϕ-function, so that $\phi(m)$ is the number of integers between 1 and m which are coprime to m. If $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ in prime factors, then

$$\phi(m) = m\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right).$$

To prove this, we first show that

$$a^n \equiv a\mathrm{F}_n + \mathrm{F}_{n-1} \pmod{m}.$$

Also $(1-a)^2-(1-a)-1=a^2-a-1$, so that

$$(1-a)^n \equiv (1-a)\mathrm{F}_n + \mathrm{F}_{n-1} \pmod{m}.$$

Hence

$$a^n + (1-a)^n \equiv \mathrm{F}_n + 2\mathrm{F}_{n-1} \pmod{m}$$

and

$$a^n - (1-a)^n \equiv (2a-1)\mathrm{F}_n \pmod{m}.$$

Since $m=a^2-a-1$, $(a,m)=1$. Also $m=-a(1-a)-1$, so that $(1-a,m)=1$. Hence, by Euler's theorem

$$a^{\phi(m)} \equiv (1-a)^{\phi(m)} \equiv 1 \pmod{m}.$$

Hence

$$\mathrm{F}_{\phi(m)} + 2\mathrm{F}_{\phi(m)-1} \equiv 2 \pmod{m}$$

and

$$(2a-1)\mathrm{F}_{\phi(m)} \equiv 0 \pmod{m}.$$

Hence, if $d=(m,2a-1)$,

$$\mathrm{F}_{\phi(m)} \equiv 0 \left(\operatorname{mod} \frac{m}{d}\right) \quad \text{and} \quad \mathrm{F}_{\phi(m)-1} \equiv 1 \left(\operatorname{mod} \frac{m}{d}\right).$$

Also

$$\mathrm{F}_{\phi(m)+1} = \mathrm{F}_{\phi(m)} + \mathrm{F}_{\phi(m)-1} \equiv 0+1 \equiv 1 \left(\operatorname{mod} \frac{m}{d}\right).$$

But

$$\mathrm{F}_{a+b} = \mathrm{F}_a\mathrm{F}_{b-1} + \mathrm{F}_b\mathrm{F}_{a+1},$$

so that

$$\mathrm{F}_{\phi(m)+n} = \mathrm{F}_{\phi(m)}\mathrm{F}_{n-1} + \mathrm{F}_n\mathrm{F}_{\phi(m)+1} \equiv \mathrm{F}_n \left(\operatorname{mod} \frac{m}{d}\right),$$

as asserted.

Note that

$$4m = 4a^2-4a-4 = (2a-1)^2-5,$$

so that $(m,2a-1) \mid 5$ and $d=1$ or $d=5$.

As an example, let $a=4$. Then $m=11$, $d=1$, and $\phi(11)=10$, so that $\mathrm{F}_{10+n} \equiv \mathrm{F}_n \pmod{11}$.

Yours sincerely,

Abbas Rouholamini Gugheri
(Students' Investigation House
Shariati Avenue
Sirjan
Iran)

Dear Editor,

The area of a double-circle quadrilateral

In a triangle ABC, if T_A, T_B, and T_C are the tangent points on BC, CA, and AB respectively of the inscribed circle, the radius of the circumcircle is R, the radius of the inscribed circle is r, the area of triangle ABC is Δ, and the area of triangle $T_AT_BT_C$ is Δ_T, then

$$\Delta_T = \frac{r}{2R}\Delta. \tag{1}$$

To see this, put $AB = c$, $BC = a$, and $CA = b$. Then

$$\begin{aligned}
\Delta_T &= \tfrac{1}{2}r^2\sin(180^\circ - A) + \tfrac{1}{2}r^2\sin(180^\circ - B) + \tfrac{1}{2}r^2\sin(180^\circ - C)\\
&= \tfrac{1}{2}r^2(\sin A + \sin B + \sin C)\\
&= \frac{1}{2}r^2\left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}\right)\\
&= \frac{r}{2R}\,\frac{1}{2}r(a+b+c)\\
&= \frac{r}{2R}\Delta.
\end{aligned}$$

I can obtain a similar formula to (1) for a *double-circle quadrilateral*, i.e. a quadrilateral with circumscribed and inscribed circles (see reference 1).

Theorem 1 *Let $ABCD$ be a double-circle quadrilateral, T_A, T_B, T_C, and T_D, be tangent points on AB, BC, CD, and DA respectively of the inscribed circle, the radius of the circumcircle be R, the radius of the inscribed circle be r, the area of $ABCD$ be $\triangle$, and the area of $T_AT_BT_CT_D$ be $\triangle_T$. Then*

$$\triangle_T = \frac{r^2 + r\sqrt{r^2+4R^2}}{4R^2}\triangle.$$

Proof Put $AB = a$, $BC = b$, $CD = c$, $DA = d$. Then (see reference 1)

$$A + C = B + D = 180^\circ, \qquad a + c = b + d = s, \qquad \triangle = rs,$$

$$\begin{aligned}
\triangle_T &= \tfrac{1}{2}r^2\sin(180^\circ - A) + \tfrac{1}{2}r^2\sin(180^\circ - B) + \tfrac{1}{2}r^2\sin(180^\circ - C) + \tfrac{1}{2}r^2\sin(180^\circ - D)\\
&= \tfrac{1}{2}r^2\sin A + \tfrac{1}{2}r^2\sin B + \tfrac{1}{2}r^2\sin A + \tfrac{1}{2}r^2\sin B\\
&= r^2\sin A + r^2\sin B.
\end{aligned}$$

Since

$$\begin{aligned}
\triangle &= \tfrac{1}{2}ab\sin B + \tfrac{1}{2}cd\sin D = \tfrac{1}{2}(ab+cd)\sin B,\\
\triangle &= \tfrac{1}{2}ad\sin A + \tfrac{1}{2}bc\sin C = \tfrac{1}{2}(ad+bc)\sin A,
\end{aligned}$$

we have

$$\sin A = \frac{2\triangle}{ad+bc}, \qquad \sin B = \frac{2\triangle}{ab+cd},$$

so

$$\begin{aligned}\triangle_T &= r^2\sin A + r^2\sin B\\ &= r^2\left(\frac{2\triangle}{ad+bc}+\frac{2\triangle}{ab+cd}\right)\\ &= \frac{2\triangle r^2(ab+cd+ad+bc)}{(ad+bc)(ab+cd)}\\ &= \frac{2\triangle r^2 s^2}{(ad+bc)(ab+cd)}\\ &= \frac{2\triangle^3}{(ad+bc)(ab+cd)}\\ &= \frac{\triangle}{2}\sin A\sin B.\end{aligned}$$

By reference 2 we have

$$\sin A\sin B = \frac{r^2 + r\sqrt{r^2+4R^2}}{2R^2}.$$

Hence,

$$\triangle_T = \frac{r^2 + r\sqrt{r^2+4R^2}}{4R^2}\triangle.$$

Yours sincerely,

Zhang Yun
(Sunshine High School
Xi’an Jiaotong University
Xi’an City
Shaanxi Province
China)

References

1 Z. Yun, Some properties of a double circle quadrilateral, *Math. Spectrum* **37** (2004/2005), pp. 57–60.
2 Z. Yun, Euler’s inequality revisited, *Math. Spectrum* **40** (2007/2008), pp. 119–121.

Dear Editor,

An integer replacement problem

Readers may be familiar with the following well known result. Let n be a positive integer greater than 3. Then, in the identity

$$(1+2+3+\cdots+(n-1)+n)^2 = 1^3+2^3+3^3+\cdots+(n-1)^3+n^3, \tag{1}$$

if $n-1$ is replaced by 2 on both sides, (1) remains valid. This motivated us to think of a generalization.

We thought of the following problem. Find a triple of positive integers (n, k, m) such that $k \neq m$, $\max(k, m) < n$, and if k is replaced by m in (1), the result remains valid.

One solution of the problem is given by the triple $(n, n-1, 2)$ for every $n > 3$. We thus need the triple (n, k, m) such that

$$\left(\frac{n(n+1)}{2} - k + m\right)^2 = \left(\frac{n(n+1)}{2}\right)^2 - k^3 + m^3,$$

i.e.

$$n(n+1) = m^2 + k^2 + mk - m + k.$$

For every pair of positive integers x and a, the following four sets of triples of the form (n, k, m) provide a solution to the above problem:

Set I:	$n = (a^2 + 3a + 3)x + a + 1,$	$k = a(a+2)x + a,$	$m = (2a+3)x + 2,$
Set II:	$n = (a^2 + 3a + 3)x - a - 2,$	$k = a(a+2)x - a - 2,$	$m = (2a+3)x,$
Set III:	$n = (3a^2 + 3a + 1)x + 3a + 1,$	$k = a(3a+2)x + 3a,$	$m = (2a+1)x + 2,$
Set IV:	$n = (3a^2 + 3a + 1)x - 3a - 2,$	$k = a(3a+2)x - 3a - 2,$	$m = (2a+1)x.$

We observe that if we put $x = 0$ and $a = n - 1$ in Set I, we get the solution discussed at the start of this letter.

Yours sincerely,

K. S. Bhanu* **and M. N. Deshpande**
(*Department of Statistics
Institute of Science
Nagpur, 440001
India)

Dear Editor,

Reversing digits and divisibility by 121

It is well known that a natural number is divisible by 11 if and only if the alternating sum of its digits is divisible by 11. This follows from the congruence

$$a_0 + 10a_1 + 10^2a_2 + \cdots + 10^n a_n \equiv a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n \pmod{11}.$$

For example, $627 = 57 \times 11$ and $6 - 2 + 7 = 11$. It follows that, if a number is divisible by 11, then so is the number obtained by reversing its digits, which we call its *reverse*. Thus $726 = 66 \times 11$.

Now consider numbers which are divisible by 121. They will be divisible by 11, so the alternating sum of their digits will be divisible by 11. In all the examples I have tried, if this alternating sum is zero, then the reverse is also divisible by 121; and if the alternating sum is not zero, then the reverse is not divisible by 121. For example,

$$57596 = 121 \times 476, \qquad 5 - 7 + 5 - 9 + 6 = 0,$$
$$69575 = 121 \times 575,$$

whereas $9224314 = 121 \times 76234$, $9 - 2 + 2 - 4 + 3 - 1 + 4 = 11$ and 4134229 is not divisible by 121. Is this always the case and, if so, can any reader provide a proof?

Yours sincerely,

M. A. Khan
(c/o A. A. Khan
Regional Office
Indian Overseas Bank
Ashok Marg
Lucknow
India)

[An article will appear in the next issue which answers Mr Khan's interesting question – Ed.]

Table 1

n	F_n	$4\mathrm{F}_n + 5$
1	1	$9 = 3^2$
2	1	$9 = 3^2$
4	3	$17 = 4^2 + 1$
5	5	$25 = 5^2$
6	8	$37 = 6^2 + 1$
9	34	$141 = 12^2 - 3$
10	55	$225 = 15^2$
11	89	$361 = 19^2$

Dear Editor,

A conjecture about Fibonacci numbers

I have been investigating the sequence $4\mathrm{F}_n + 5$, where F_n is the nth Fibonacci number. Table 1 shows values of this sequence for various values of n. It is intriguing that the result is often a perfect square or is near to a perfect square. My computer gives perfect squares when $n = 108, 132, 146$, and when n is larger than 149, it always gives a perfect square! Be that as it may, I propose the conjecture that

$$\lim_{n\to\infty}(\sqrt{4\mathrm{F}_n + 5} - \lfloor\sqrt{4\mathrm{F}_n + 5}\rfloor) = 0,$$

where $\lfloor x \rfloor$ denotes the integer part of x.

Yours sincerely,

Abbas Rouholamini Gugheri
(Students' Investigation House
Shariati Avenue
Sirjan
Iran)

Dear Editor,

The harmonic mean, a recurrence relation, and a complex oddity

The harmonic mean, H, of n positive real numbers $x_1, x_2, \ldots, x_n$ is defined by

$$\frac{1}{H} = \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \Big/ n.$$

Thus, if we denote the harmonic mean of $1, 2, \ldots, n$ by H_n,

$$\frac{1}{H_n} = \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right) \Big/ n.$$

If we write $h_n = 1/H_n$, this gives us the recurrence relation

$$h_n = \left((n-1)h_{n-1} + \frac{1}{n}\right) \Big/ n, \quad \text{for } n > 1, \text{ with } h_1 = 1.$$

Thus, $h_2 = \frac{3}{4}$, $h_3 = \frac{11}{18}$, $h_4 = \frac{25}{48}$, $h_5 = \frac{137}{300}$, and so on, so that $H_1 = 1$, $H_2 = \frac{4}{3}$, $H_3 = \frac{18}{11}$, $H_4 = \frac{48}{25}$, $H_5 = \frac{300}{137}$, and so on.

The harmonic mean of two positive real numbers a, b is

$$\left(\left(\frac{1}{a} + \frac{1}{b}\right) \Big/ 2\right)^{-1} \quad \text{or} \quad \frac{2ab}{a+b}.$$

This cannot be equal to their sum, for then

$$\frac{2ab}{a+b} = a + b;$$

whence, $a^2 + b^2 = 0$, which is impossible. However, it *is* possible if a, b are allowed to be complex numbers, ($a \neq 0$, $b \neq 0$, $a + b \neq 0$). All that is required for this is that $a = \pm \mathrm{i} b$. For example, the harmonic mean of 1 and i is equal to their sum.

Yours sincerely,
Bob Bertuello
(12 Pinewood Road
Midsomer Norton BA3 2RG
UK)

Dear Editor,

A variant of Fermat's last theorem

It is well known that the equation

$$x^3 + y^3 = z^3$$

has no natural number solutions. But what about the equation

$$x^{2^3} + y^{2^3} = z^{3^3}\ ?$$

Yours sincerely,
Muneer Karama
(Hebron Education Officer
Jerusalem
Box 19149
UNRWA)

Dear Editor,

Summing a finite series of Fibonacci numbers

Problem 43.11 in Volume 43, Number 3 was to sum the finite series

$$\mathrm{F}_1\mathrm{F}_2 + \mathrm{F}_2\mathrm{F}_3 + \mathrm{F}_3\mathrm{F}_4 + \cdots + \mathrm{F}_n\mathrm{F}_{n+1},$$

where F_n denotes the nth Fibonacci number. I ask what is the sum of the finite series

$$S_n = \mathrm{F}_1^2\mathrm{F}_2 + \mathrm{F}_2^2\mathrm{F}_4 + \mathrm{F}_3^2\mathrm{F}_6 + \cdots + \mathrm{F}_n^2\mathrm{F}_{2n}.$$

Using $F_{2n} = F_{n+1}^2 - F_{n-1}^2$ for $n > 1$, we have

$$\begin{aligned} S_n &= F_1^2 F_2 + F_2^2(F_3^2 - F_1^2) + F_3^2(F_4^2 - F_2^2) + \cdots + F_n^2(F_{n+1}^2 - F_{n-1}^2) \\ &= F_n^2 F_{n+1}^2. \end{aligned}$$

Yours sincerely,
Abbas Rouholamini Gugheri
(Students' Investigation House
Shariati Avenue
Sirjan
Iran)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st March will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

45.1 For positive numbers a, b with $a > b > 0$, prove that

$$\frac{a^2 + b^2}{a - b} \geq 2\sqrt{2ab}.$$

(Submitted by Abbas Rouholamini, Sirjan, Iran)

45.2 The x-coordinates of the endpoints of an arc of the circle, centre the origin and radius r, are a and b. Express the x-coordinate of the midpoint of the arc in terms of a, b, and r.

(Submitted by Gregory Akulov, Luther College High School, Regina, and Alex Akulov, University of Waterloo, Canada)

45.3 Given a triangle ABC and a point P in the same plane as the triangle, the centroids of triangles BCP, CAP, ABP are denoted by L, M, N respectively. What is the connection between the areas of the triangles LMN and ABC?

(Submitted by Zhang Yun, Xi An Jiao Tong University, Sunshine High School, China)

45.4 A cubic curve has equation $y = x^3 + ax^2 + bx + c$, where the polynomial has real coefficients and three real roots. Show that

(a) the x-coordinate of the point of inflection is the mean of the roots,

(b) the sum of the slopes of the normals at the roots is zero,

(c) it has two-fold symmetry about its point of inflection.

(Submitted by Michael Wragg, Totton Sixth Form College, Southampton, UK)

Solutions to Problems in Volume 44 Number 2

44.5 Let P_1 be the point $(x_1, 0)$ on the x-axis. The straight line through P_1 perpendicular to the x-axis meets the curve $y = a^x$ (where $a > 0$, $a \neq 1$) at $Q_1 = (x_1, y_1)$. The tangent to the curve at Q_1 meets the x-axis at $P_2(x_2, 0)$. This is repeated for P_2, and so on, to obtain the sequences $\{x_n\}$ and $\{y_n\}$. Show that the sequence $\{x_n\}$ is an arithmetic progression and $\{y_n\}$ is a geometric progression. Carry out a similar procedure using the parabola $y = ax^2$, where $a > 0$.

Solution by Ben McDonnell, Hills Road Sixth Form College, Cambridge, UK

From figure 1,

$$\tan\theta_n = \left(\frac{\mathrm{d}}{\mathrm{d}x}(a^x)\right)_{x=x_n} = a^{x_n}\ln a,$$

so that

$$\frac{y_n}{x_n - x_{n+1}} = \tan\theta_n = a^{x_n}\ln a$$

and

$$x_n - x_{n+1} = \frac{y_n}{a^{x_n}\ln a} = \frac{1}{\ln a}.$$

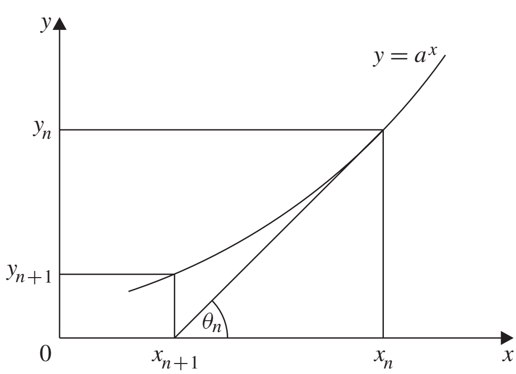

Figure 1

Hence $\{x_n\}$ is an arithmetic progression. Also,

$$\frac{y_n}{y_{n+1}} = \frac{a^{x_n}}{a^{x_{n+1}}} = a^{x_n - x_{n+1}} = a^{1/\ln a},$$

so that $\{y_n\}$ is a geometric progression.

For the parabola $y = ax^2$,

$$\tan\theta_n = \left(\frac{\mathrm{d}}{\mathrm{d}x}(ax^2)\right)_{x=x_n} = 2ax_n,$$

so that

$$x_n - x_{n+1} = \frac{y_n}{2ax_n} = \frac{x_n}{2}$$

and $x_n = 2x_{n+1}$, so that $\{x_n\}$ is a geometric progression; and

$$\frac{y_n}{y_{n+1}} - \frac{ax_n^2}{ax_{n+1}^2} = 4,$$

so that $\{y_n\}$ is also a geometric progression.

44.6 The triangular number 120 is the sum of four consecutive powers of 2 ($120 = 8 + 16 + 32 + 64$). Prove that, for every positive integer n, there is a triangular number which is the sum of n consecutive powers of 2.

Solution by Abbas Rouholamini, Sirjan, Iran

Denote by T_k the kth triangular number. Then

$$T_k = 2^a + 2^{a+1} + \cdots + 2^{a+n-1}$$

if and only if

$$\tfrac{1}{2}k(k+1) = 2^a(2^n - 1),$$

or

$$k(k+1) = 2^{a+1}(2^n - 1).$$

Choose $k = 2^n - 1$ and $a = n - 1$ to satisfy this.

Also solved by David Christopher, The American College, Madurai, India.

44.7 The function f satisfies $|f(x)| \le \frac{1}{2}$ and $|f'(x)| \le 1/b^2$ for all $x \in [a, b]$, where $0 < a < b$. If $|f(a)| \ge 1/\sqrt{a}$, prove that $|f(b)| \ge 1/\sqrt{b}$.

Solution by Spiros Andriopoulos, who proposed the problem

Multiply the inequalities to give $|f(x)f'(x)| \le 1/2b^2$, so that $f(x)f'(x) \ge -1/2b^2$. Hence

$$\int_a^b f(x)f'(x)\,\mathrm{d}x \ge \int_a^b \left(-\frac{1}{2b^2}\right)\mathrm{d}x,$$

so that

$$\left[\frac{1}{2}f(x)^2\right]_a^b \ge -\frac{1}{2b^2}(b-a),$$

whence

$$f(b)^2 - f(a)^2 \geq -\frac{b-a}{b^2}.$$

But $f(a)^2 \geq 1/a$, so that

$$\begin{aligned} f(b)^2 &\geq -\frac{b-a}{b^2} + \frac{1}{a} \\ &= \frac{a^2+b^2-ab}{ab^2} \\ &\geq \frac{ab}{ab^2} \\ &= \frac{1}{b} \end{aligned}$$

since $(a-b)^2 \geq 0$. Hence $|f(b)| \geq 1/\sqrt{b}$.

44.8 For a given positive integer n, find the minimum value of

$$\left(1+\frac{1}{\sin^n \alpha}\right)\left(1+\frac{1}{\cos^n \alpha}\right),$$

for $0 < \alpha < \pi/2$.

Solution by Abbas Rouholamini, who proposed the problem

Since $0 < \alpha < \pi/2$, we can write $1/\sin^n \alpha = a^2$ and $1/\cos^n \alpha = b^2$ for some positive real numbers a, b. Then the expression is

$$\begin{aligned} (1+a^2)(1+b^2) &= a^2b^2 + a^2 + b^2 + 1 \\ &= (ab+1)^2 + (a-b)^2 \\ &\geq (ab+1)^2. \end{aligned}$$

Now

$$ab = \frac{1}{\sin^{n/2}\alpha \cos^{n/2}\alpha} = \frac{2^{n/2}}{\sin^{n/2} 2\alpha} \geq 2^{n/2},$$

so the expression is greater than or equal to $(2^{n/2}+1)^2$. When $\alpha = \pi/4$, the expression is equal to this, so this is the minimum value.

Solution by Spiros Andriopoulos

Put

$$f(x) = \left(1+\frac{1}{\sin^n x}\right)\left(1+\frac{1}{\cos^n x}\right).$$

Then

$$\begin{aligned} f'(x) &= \frac{n\sin x}{\cos^{n+1} x}\left(1+\frac{1}{\sin^n x}\right) - \frac{n\cos x}{\sin^{n+1} x}\left(1+\frac{1}{\cos^n x}\right) \\ &= \frac{n}{\sin^{n+1} x \cos^{n+1} x}(\sin^{n+2} x - \cos^{n+2} x + \sin^2 x - \cos^2 x). \end{aligned}$$

For $0 < x < \pi/4$, $\sin x < \cos x$, so that $f'(x) < 0$. For $\pi/4 < x < \pi/2$, $\sin x > \cos x$, so that $f'(x) > 0$. Hence the minimum value of $f(x)$ occurs when $x = \pi/4$, and so is $(1+2^{n/2})^2$.

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