Mathematical Spectrum

2002/2003 Volume 35 Number 3

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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year consisting of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing.

From the Editor

Alice in Determinantland

As befits my approaching senility, I spent a fascinating lunchtime reading about what the Mock Turtle learned at school: Reeling and Writhing, the different branches of Arithmetic — Ambition, Distraction, Uglification and Derision — Mystery, ancient and modern, Seaography, Drawling, Stretching and Fainting in Coils, taught by a master called Tortoise 'because he taught us', all the inventions of Lewis Carroll in *Alice in Wonderland*.

Imagine my amazement when, that same afternoon, I was browsing through a volume which came for review called *Linear Algebra Gems* published by The Mathematical Association of America, and came across an article by Wayne Barrett entitled 'Dodgson's identity'. I did not immediately connect the two events until I saw a reference in Barrett's article: C. L. Dodgson, Condensation of Determinants (*Proceedings of the Royal Society of London*, Volume 15 (1866), pp. 150–155) — Rev. Charles Lutwidge Dodgson, of Christ Church College, Oxford, our very own Lewis Carroll.

What is Dodgson's identity? It is best illustrated by an example. Most readers will be able to work out the determinant of a 2×2 matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc\,.$$

Many can manage a 3×3 matrix. Take the example

$$\boldsymbol{M} = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 5 & 6 \end{bmatrix},$$

which Wayne Barrett uses in his article. Dodgson first worked out the four 2×2 'corner determinants' and inserted them into $\boldsymbol{M}$:

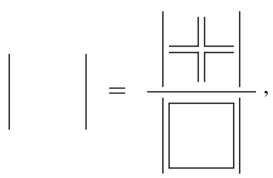
$$\begin{array}{ccccc} 4 & & 2 & & 1 \\ & 8 & & 1 & \\ 2 & & 3 & & 2 \\ & 7 & & 8 & \\ 1 & & 5 & & 6 \end{array}$$

He then dropped the entries round the edge, evaluated the resulting 2×2 determinant and divided by the central entry to give

$$\frac{\begin{vmatrix} 8 & 1 \\ 7 & 8 \end{vmatrix}}{3} = \frac{57}{3} = 19\,.$$

Amazingly, this is the determinant of $\boldsymbol{M}$.

This is Dodgson's identity for the determinant of a 3×3 matrix. It works for larger sizes too. Symbolically we can denote the rule as Barrett does:



where the numerator on the right-hand side is the 2×2 determinant whose four entries are the $(n-1) \times (n-1)$ corner determinants of the original $n \times n$ matrix and the determinant in the denominator is the $(n-2) \times (n-2)$ determinant obtained by 'shaving off' the first and last rows and columns of the original matrix. Of course, the rule only applies if this denominator is not zero, which admittedly is rather a drawback.

Let's try it with a 4×4 matrix. Barrett uses the following matrix:

$$\boldsymbol{M} = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 2 & -1 & 3 & 1 \\ -1 & 2 & 1 & 1 \\ -2 & -5 & 2 & 3 \end{bmatrix}.$$

First insert the 2×2 minors:

$$\begin{array}{ccccccc} 1 & & 3 & & 2 & & -1 \\ & -7 & & 11 & & 5 & \\ 2 & & -1 & & 3 & & 1 \\ & 3 & & -7 & & 2 & \\ -1 & & 2 & & 1 & & 1 \\ & 9 & & 9 & & 1 & \\ -2 & & -5 & & 2 & & 3 \end{array}.$$

Now drop the fringe elements and calculate the 3×3 corner determinants using Dodgson's rule:

$$\begin{array}{ccccc} -7 & & 11 & & 5 \\ & -16 & & 19 & \\ 3 & & -7 & & 2 \\ & 45 & & -25 & \\ 9 & & 9 & & 1 \end{array}.$$

Now

$$\det \boldsymbol{M} = \frac{\begin{vmatrix} -16 & 19 \\ 45 & -25 \end{vmatrix}}{-7} = 65\,.$$

Delightful!

For those who know some elementary matrix theory, the proof of Dodgson's rule is sheer delight. We start with the matrix written in block form:

$$\boldsymbol{M} = \begin{bmatrix} r & \boldsymbol{v} & s \\ \boldsymbol{w} & \boldsymbol{E} & \boldsymbol{x} \\ t & \boldsymbol{y} & u \end{bmatrix}.$$

Here r, s, t, u are the corner entries, $\boldsymbol{v}$ and $\boldsymbol{y}$ and $1 \times (n-2)$ vectors, $\boldsymbol{w}$ and $\boldsymbol{x}$ are $(n-2) \times 1$ vectors, and $\boldsymbol{E}$ is an $(n-2) \times (n-2)$ matrix. Dodgson's rule requires that $\det \boldsymbol{E} \neq 0$, so $\boldsymbol{E}$ has rank $(n-2)$. Thus, its rows are a basis of $\mathbb{R}^{n-2}$, so $\boldsymbol{v}$ and $\boldsymbol{y}$ are linear combinations of the rows of $\boldsymbol{E}$. Hence, we can subtract linear combinations of rows $2, \ldots, n-1$ from rows 1 and n of $\boldsymbol{M}$ to give the matrix

$$\boldsymbol{M}' = \begin{bmatrix} r' & \mathbf{0} & s' \\ \boldsymbol{w} & \boldsymbol{E} & \boldsymbol{x} \\ t' & \mathbf{0} & u' \end{bmatrix}.$$

We can do the same thing with the columns to give the matrix

$$\boldsymbol{M}'' = \begin{bmatrix} r' & \mathbf{0} & s' \\ \mathbf{0} & \boldsymbol{E} & \mathbf{0} \\ t' & \mathbf{0} & u' \end{bmatrix}.$$

Moreover,

$$\det \boldsymbol{M} = \det \boldsymbol{M}' = \det \boldsymbol{M}''$$

and the four $(n-1) \times (n-1)$ corner minors of $\boldsymbol{M}$, $\boldsymbol{M}'$ and $\boldsymbol{M}''$ are the same. But

$$\begin{aligned}\det \boldsymbol{M}'' &= (r'u' - s't') \det \boldsymbol{E} \\ &= \frac{(r' \det \boldsymbol{E})(u' \det \boldsymbol{E}) - (s' \det \boldsymbol{E})(t' \det \boldsymbol{E})}{\det \boldsymbol{E}} \\ &= \frac{\begin{vmatrix} r & \boldsymbol{v} \\ \boldsymbol{w} & \boldsymbol{E} \end{vmatrix}\begin{vmatrix} \boldsymbol{E} & \boldsymbol{x} \\ \boldsymbol{y} & u \end{vmatrix} - \begin{vmatrix} \boldsymbol{v} & s \\ \boldsymbol{E} & \boldsymbol{x} \end{vmatrix}\begin{vmatrix} \boldsymbol{w} & \boldsymbol{E} \\ t & \boldsymbol{y} \end{vmatrix}}{\det \boldsymbol{E}},\end{aligned}$$

and this is $\det \boldsymbol{M}$, which is Dodgson's identity.

According to Wayne Barrett, Dodgson's identity is a special case of two earlier identities by Jacobi and Sylvester, both famous mathematical names. To quote Wayne Barrett: 'It has no doubt been rediscovered independently several times.' Barrett writes how he discovered it in 1978–79. Coincidentally, at the time of writing this, a letter arrived in the *Spectrum* office from a reader in Iran, Hashemi Moosavi, which contains Dodgson's identity for a 3×3 matrix. So it is still being rediscovered!

The volume in which Wayne Barrett's article appears is a treasure trove for teachers and enthusiasts of linear algebra. A rich source of student projects!

Linear Algebra Gems: Assets for Undergraduate Mathematics. Edited by David Carlson, Charles R. Johnson, David C. Lay and A. Duane Porter. Mathematical Association of America, Washington, DC, 2002. Pp. 342. Paperback \$32.95 (ISBN 0-88385-170-9).

Statistical Analyses of Euro Coin Mixing

DIETRICH STOYAN

1. Introduction

Spatial statisticians were offered a unique opportunity on 1 January 2002. In 12 countries of the European Union, the Euro was introduced and coins began to spread across borders, i.e. a spatio-temporal process of coin mixing had started. The methods of spatial statistics seemed to be available for its analysis, although the data basis was likely to be poor. The different Euro coins can be distinguished easily, since their obverses are different in different countries. Banknotes also show a small difference: the letters of the serial numbers are different in the various Euro countries.

The author expected that there would be some form of diffusion of the coins observable in any given country. The foreign coins would spread from the borders to other Euro countries and in the big cities. In the particular case of Germany, the diffusion was likely to start from the borders in the west and south. Furthermore, it was expected that the foreign coins would begin to appear mainly in the big cities, followed by the regions around them, and in towns, and finally in lonely villages. An alternative hypothesis was that such a diffusion would not be observable because of the great degree of mobility of Germans. Thus, the question was 'diffusion versus dispersion'. There was some hope of deciding this question by means of data from Saxony.

As a result, I started the Euromobil project, the aim of which is described in section 2. I was not successful in persuading foreign colleagues to organize similar projects in other Euro countries, but I had yet to learn that colleagues in the Netherlands and Belgium were also studying the coin mixing process. They called their project 'Eurodiffusie'; see reference 1. Their aim was to study the long-term behaviour of the coin mixing or diffusion process for the Netherlands and Belgium. The statistical quantities of main interest were the proportions of foreign coins in the Netherlands and Belgium. This inspired me to prepare Eurodiff, a second German project, which is described in section 6.

Sections 3–5 describe the results of Euromobil. Finally, section 6 reports the results of Eurodiff until the end of June 2002, and discusses some problems of data collecting in both Eurodiff and Eurodiffusie.

2. The Euromobil project

The aim of the project was to collect data about the first contacts of Germans with foreign Euro coins in January and February 2002. In an early phase of the project, there was also the intention of observing banknotes, but it seemed more attractive to consider coins. German volunteers were asked to send by mail, telephone or email information on their contacts with foreign coins, their country of origin and value and the date and location of the contact (German postcode). We accepted that the same person could inform us about several contacts if the location, coin type or value was different. People were asked to report only contacts with change obtained locally, and not with money obtained during holidays in other Euro countries.

Since banks and official agencies did not have much interest in the mixing process (because the coins are considered to be identical so that transport back to the countries of origin is unnecessary), the statistical analysis was based entirely on voluntary observations, depending on the interest and goodwill of individuals. It was clear that only a subgroup of the population would report; namely people who were informed about the project, were active enough to report and willing to pay for postage, telephone or email expenses. My general impression is that younger housewives and pensioners were particularly active. Of course, among the reporters were some who wanted to demonstrate their wide international contacts, as well as hoaxers who reported nonsense.

It was necessary to organize a propaganda campaign for the project. After local newspaper reports in the cities close to Freiberg (Dresden and Chemnitz), a beautiful article was published in the *Frankfurter Sonntagszeitung* of 23 December 2001. This article meant that the Euromobil project was accepted throughout Germany; nearly all regional newspapers in Germany reported on it, at least briefly. Finally, on 8 January 2002, a morning report on the national German TV station ZDF popularized the idea of Euromobil.

All this propaganda resulted in 8026 acceptable reports.

Figure 1 shows the spatial distribution of the reports, which is based on the first two digits of the postcode number; this map indicates that there are large fluctuations in the raw data. In particular, in the neighbourhood of Stuttgart, black and white areas are in direct contact, which is likely to be due to a sampling effect. I phoned the local newpaper in Heilbronn, and learned that they had published nothing more than a very short note about Euromobil.

We therefore decided to smooth the data. Instead of using any ad hoc smoothing method which averages the values in neighbouring postcode regions, we decided to apply modern Bayesian methods. We followed Mollié (see reference 2 and also reference 3), and used the model with ‘spatially structured variation’, namely intrinsic Gaussian autoregression given by her formula (20.7), with parameter $\sigma^2 = 0.01$. This choice was, of course, subjective, simply following our intuitive feeling about a ‘beautiful’ map. Figure 2 shows the final result.

There are two particularly active regions in Germany: the regions around Freiberg and Frankfurt am Main. In the case of Freiberg, we are sure about the influence of the media. The local newspapers reported frequently and so nearly everybody in Freiberg was informed about Euromobil.

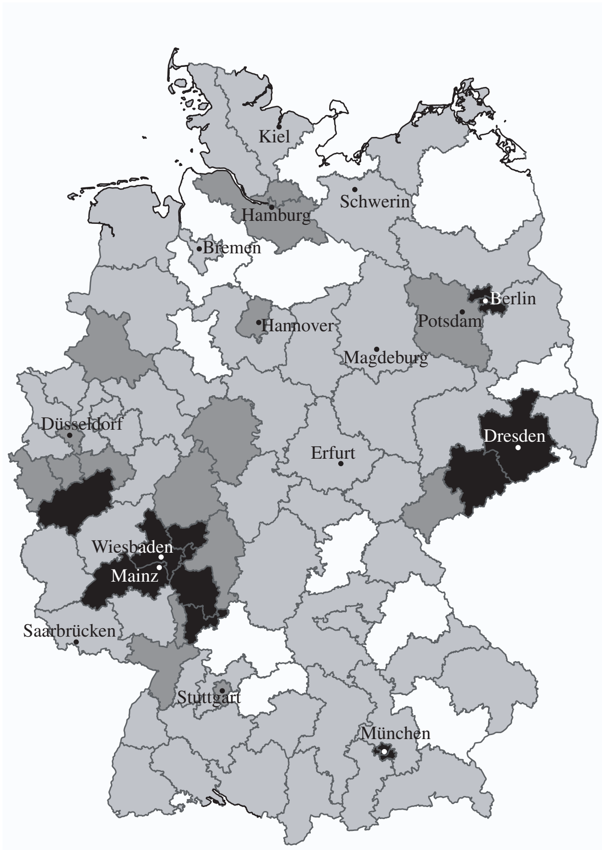


Figure 1. Raw spatial distribution of reports in Germany. Black: more than 18 reports per 100 000 inhabitants; dark grey: 10–18; light grey: 2–10; white: fewer than 2 reports. (The maps were produced by Lutum and Tappert in Bonn.)

The media in Frankfurt were also active. Two big newspapers wrote about Euromobil and its aims, and several radio stations followed them and published the address of the Freiberg Euro homepage. But in the case of Frankfurt, the role of this city as a traffic node and finance centre also probably had its effect.

However, it would appear, independently of the activities of the media, that the big cities in Germany show a particularly high intensity of reports per number of inhabitants. But we can explain the different intensities of reports in the other German regions as resulting from the degrees of activity of the local media.

It is interesting to compare the temporal distribution of reports of two city regions: Munich and Hamburg, as shown in table 1. Munich exhibits the ‘standard’ behaviour of most postal regions: nearly exponentially decreasing numbers of reports over time. This is plausible: people reported when they considered the foreign coins as a novelty, and ceased to report later, thinking that they would not be the first informants even though they were happy to collect their first foreign coin. In contrast, in Hamburg the maximum report intensity was observed in the second half of January. Here I presume that a certain article in a Hamburg newspaper renewed the interest in reporting. This assumption is supported by the numbers for the Freiberg region, which are similar to those of Hamburg.

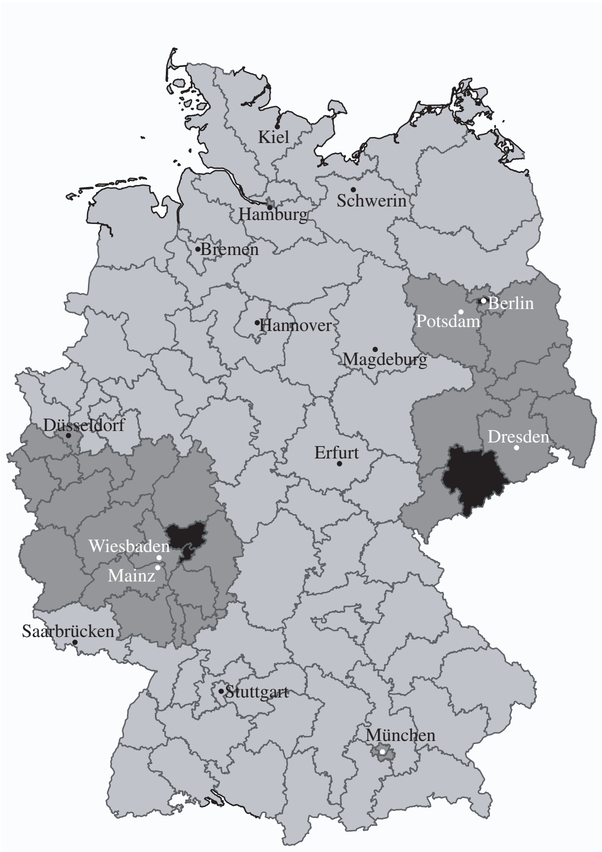


Figure 2. Smoothed spatial distribution of reports in Germany. Black: more than 18 reports per 100 000 inhabitants; dark grey: 10–18; light grey: 2–10.

Table 1. Numbers of reports per 100 000 inhabitants in Hamburg and Munich.

	1–15 Jan.	16–31 Jan.	1–15 Feb.	15–28 Feb.
Hamburg	3.3	4.5	2.3	1.8
Munich	15.6	7.6	4.5	2.9

3. The role of home countries

The numbers of coins produced in the Euro countries vary greatly; small countries produce only a small number of coins, so that these are really rare. (This holds true not only for the coins from Monaco, San Marino and the Vatican, but also for coins from Finland, Greece, Ireland and Luxemburg.) Table 2 shows the percentages for December 2001 according to the European Central Bank.

I did not expect to observe these frequencies in the Euromobil reports. In contrast, the coins from Germany's neighbours, Austria, France, Luxemburg, Belgium and the Netherlands were expected to appear particularly frequently. As table 2 shows, this is the case. Particularly striking is the high percentage from Austria. This may well be explained by the introduction of the Euro in winter time, when many Germans take skiing holidays in the Alps.

Table 2. Percentages of coins produced in the 12 Euro countries and percentages observed in January and February by Euromobil. The values in parentheses are the production percentages without Germany.

Country	Produced %	Observed %
Germany	32.9	
France	15.8 (23.5)	20
Italy	15.4 (22.9)	12
Spain	13.7 (20.4)	12
Netherlands	5.4 (8.0)	16
Belgium	3.8 (5.6)	5
Austria	3.5 (5.2)	26
Greece	2.6 (3.9)	3
Portugal	2.5 (3.7)	2
Ireland	2.1 (3.1)	2
Finland	2.1 (3.1)	1
Luxemburg	0.2 (0.3)	2

A high proportion of Austrian coins was also observed in the Netherlands, this being 2.3% of all coins on 1 March (see reference 1). The explanation is likely to be the same as for Germany. Additionally, many Belgian coins were observed in the Netherlands.

Figures 3 and 4 show the spatial distribution of the percentages of French and Dutch coins in the German postal regions. The spatial relationship between the two countries

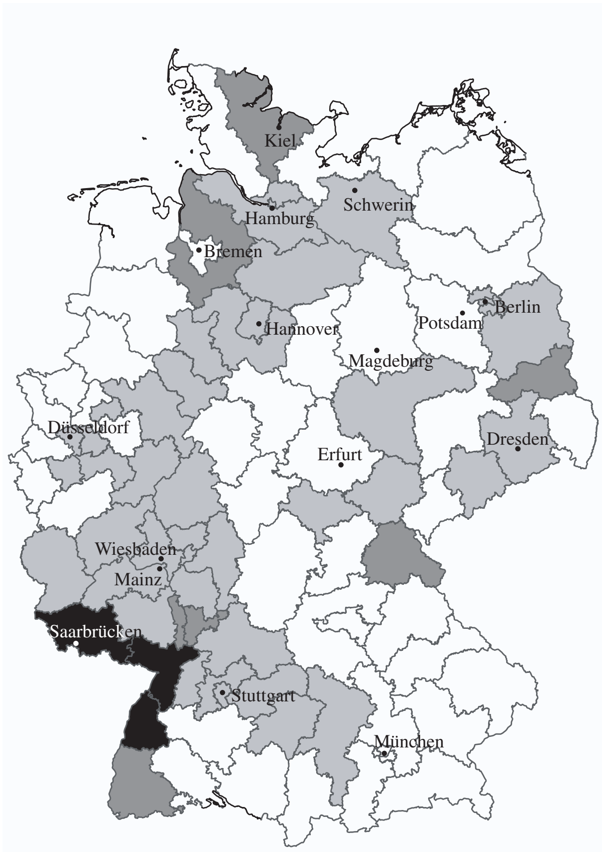


Figure 3. Spatial distribution of French coins in Germany in January and February 2002 in percentages. Black: more than 50%; dark grey: 30–50%; light grey: 15–30%; white: less than 15%.

and the distribution of their coins in Germany seems to be plausible: in regions close to the borders they are particularly frequent, decreasing with increasing distance. Big values in some places far from the borders probably result from small-sample effects. But in the main, the form of data collecting means that these maps are probably not far wrong, since percentages and not absolute values are shown.

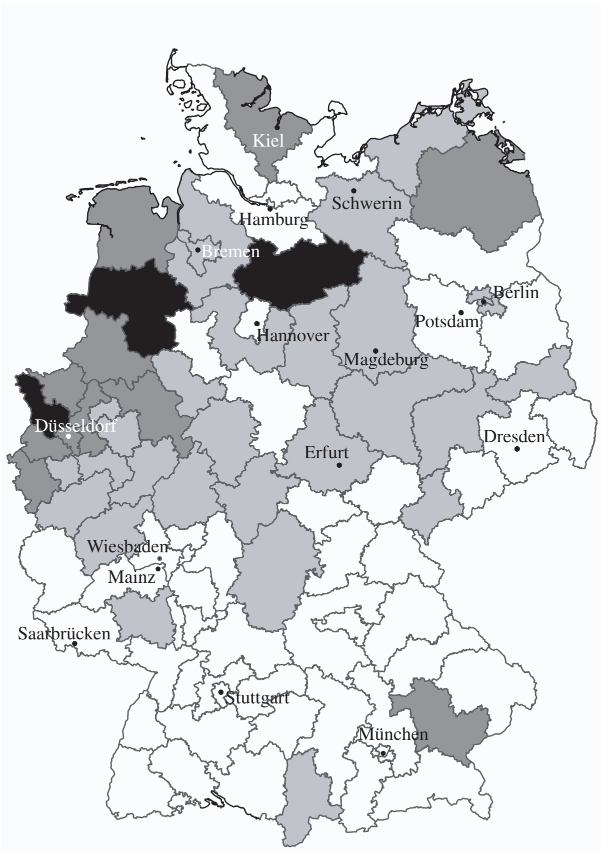


Figure 4. Spatial distribution of Dutch coins in Germany in January and February 2002 in percentages. Black: more than 50%; dark grey: 30–50%; light grey: 15–30%; white: less than 15%.

4. The behaviour of coin values

Some interesting results of Euromobil characterize the use of coins by individuals. Numismatists already knew before the introduction of the Euro that the mobility of small value coins (cent, pfennig) is small, while that of big value coins (pound, mark) is great. In Germany, this could be observed by checking the letters on the coins, indicating their place of production. For example, A denotes Berlin, and today in Berlin the percentage of Euro coins with an A is large. Small value coins usually move between households and shops, while big value coins and banknotes have larger activity regions. If travelling abroad, it seems more reasonable to have some big value coins for use in vending machines, than a purse full of small ones.

Table 3 shows the time-dependent frequencies of three selected coin types. We observed very big differences in their behaviour. Of all coin types, the €2 turned out to be the most frequent. A combination of tables 2 and 3 also works. Indeed the Austrian €2 was the coin which was observed most frequently in Germany (477 out of 8026). Blokland *et al.* (reference 1) report another aspect of the different mobility of coin sorts. On 1 March they observed that 8.3% of the 1¢ coins were of foreign origin, while the values for 10¢ and €2 were 11.7% and 19.6%, respectively.

Table 3. Percentages of produced and observed 1¢, 20¢ and €2 coins in the whole of Germany.

	% observed		
	1¢	20¢	€2
1–10 Jan.	6.0	12.6	20.8
11–20 Jan.	7.2	15.1	19.1
21–31 Jan.	6.9	13.4	19.7
1–10 Feb.	5.1	14.0	17.9
11–20 Feb.	5.4	16.9	18.2
21–28 Feb.	4.1	16.5	20.4
Produced %	17.1	12.0	5.2

There is still another unexpected aspect of the use of €1 and €2 coins. For the 'northern' countries, Austria, Belgium, France, Finland, Ireland, Luxemburg and the Netherlands, the €2 appeared more frequently than the €1, while in the 'southern' countries, Italy, Portugal and Spain, the €1 was more frequent. Table 4 shows the numbers for Italy and France. A χ^2 contingency table test (see reference 4, p. 456) indicates that the differences are significant; the p value is 0.017. It was reported that an Italian radio station had stated that Italians have a tendency to distrust the €2 coins.

I believe that the low quality of the data does not influence the validity of the results given in this section.

Table 4. Observed €1 and €2 from Italy and France.

Country	€1	€2	Total
Italy	200	153	353
France	326	342	668
Total	526	495	1021

5. Is there spatial diffusion?

The results of Euromobil were analysed in several ways in order to find evidence of spatial diffusion of foreign Euro coins in Germany. 'Spatial diffusion' means that the coins move from the borders to the west and south in a north-easterly direction, as well as from the big cities towards the neighbouring countryside. As figures 3 and 4 show, there is indeed diffusion on a 'global scale' in the whole of Germany.

However, on a finer scale, it was not possible to observe such a tendency, as the values for two pairs of regions in Saxony show.

5.1. Freiberg and neighbourhood

The town of Freiberg (48 000 inhabitants) together with some other smaller towns and villages forms the postal region 095, while the neighbouring region 096 consists only of villages, for which Freiberg is the commercial centre. (The number of inhabitants in 096 is much smaller than in 095.) So if there were diffusion, one might expect that the foreign Euro coins would appear in 096 later than in 095. Table 5 shows that this is not the case. Using a χ^2 contingency table test, there is no significant difference between the corresponding numbers.

Table 5. Numbers of reported foreign Euro coins in postal regions 095 and 096.

Region	1–15 Jan.	16–31 Jan.	1–15 Feb.	15–28 Feb.	Total
095	65	114	64	29	272
096	16	31	17	5	69

5.2. Dresden and East Saxony

The city of Dresden (postal regions 010 to 013) is now compared with the region of Bautzen, Görlitz and Zittau (026 to 028) in East Saxony, which is a very large distance from all foreign Euro countries. The numbers of inhabitants are approximately 500 000 and 200 000, respectively, and we know that the media reported on Euromobil in Dresden far more than in East Saxony. Here too, there is no significant difference between the relative numbers of coins as can be seen in table 6. The conclusion is that, at least at the beginning of the coin mixing process (which was observed by Euromobil), there is no diffusion on the local scale. Obviously, travellers were responsible for the first contacts with foreign coins, and today there seems to be little difference in travelling habits of people from villages and cities. Thus, I would characterize the first period of Euro mixing as 'dispersion'.

Table 6. Numbers of reported foreign Euro coins in Dresden and East Saxony.

	1–15 Jan.	16–31 Jan.	1–15 Feb.	15–28 Feb.	Total
Dresden	40	50	34	19	143
East Saxony	4	8	3	1	16

6. Observing the Euro mixing process

It is not easy to get reliable data on the development of the mixing process. No financial institute or bank has observed it statistically, since it is of little interest to them. Once again, private persons were asked to help with my new project Eurodiff. People were asked to inform me (typically once a month) about the numbers of €1 coins from all Euro countries in their purses or cash boxes. Because of the different mobility of the coin types, I decided to consider only €1. The best data have come from people interested in numismatics. On their travels they systematically inspect the cash boxes of shops or restaurants, and buy rolls of coins in banks and analyse them. The results of Eurodiff are published monthly; see http://www.euro.tu-freiberg.de/en/.

The Dutch project Eurodiffusie is similar, but it considers all sorts of coins, not only €1. It publishes daily averages over three days, see http://www.wiskgenoot.nl/eurodiffusie/. Clearly, both projects are vulnerable to wrong information. When one analyses the reports of Eurodiff, one has the impression that many kind persons tend to report too many foreign coins. For example, they report immediately when they obtain foreign Euros or even collect coins in order to make 'good' reports. I must assume that many of the reports are biased towards higher percentages of foreign coins. The figures for Eurodiff were 7.9% foreign coins in April, 4.8% in May and 6.9% in June, based on 4460, 4371 and 3917 €1 coins respectively. One should note that the values for April and May are not comparable: beginning with May, all reports which appeared incorrect were eliminated. (There were, for example, reports saying that a purse contained two €1 coins from each of France, Spain and Portugal, but none from Germany.)

It is difficult to interpret the results for June. As I know well, many small shops collect foreign coins for coin collectors, since in late spring 2002 a coin collection mania prevailed in Germany (and also other Euro countries); also, the foreign coins found by opening coin rolls go into boxes or albums of coin collectors. Thus, the real frequency of foreign coins in circulation may be smaller than the values above. The values for July and August continue to tend to smaller percentages of foreign coins in Germany, so that the influence of coin collectors seems to be obvious.

The data problems become still more obvious from the results of Eurodiffusie. They reported for 9 July that 23% of coins in the Netherlands were foreign and 29% in Belgium. These values are based on 755 and 2291 coins respectively. (Note that the proportion of €1 coins is 9.3% of all coins. Thus, the databases of the Dutch and Belgian values are comparable with that of Eurodiff.) However, for 12 July the percentages were 14% and 25% based on 875 and 803 coins. This shows that there are heavy fluctuations in the reports. If the reports were correct, then they would reflect large heterogeneities in the coin distribution. Assume, for example, that there is a homogeneous mixture of foreign coins in the Netherlands. Then a learned statistician would use the test of the hypothesis H_0: $p_1 = p_2$ as described in reference 4, p. 440. Here $\hat{p}_1 = 0.23$, $\hat{p}_2 = 0.14$, $n_1 = 775$ and $n_2 = 875$, and the application of the test would lead to a clear rejection of the hypothesis of a constant percentage of foreign coins in the Netherlands, since the p value is 2.6×10^{-6}. The reason for the differences is clearly not the time difference of three days. Perhaps there are spatial inhomogeneities, but probably the social origin of the samples and their fluctuating size have the greatest effect.

Acknowledgment

The author thanks Havard Rue for the production of the data for figure 2. An editor helped to make the paper clearer and to improve its language.

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Prickly Pear Meets Its Moth

B. BARNES

A plant–herbivore model for the successful introduction of a non-native species.

There have been many introductions of foreign species of plants and animals into countries across the globe, sometimes with the best of intentions although often with unforeseen and disastrous consequences. However, there have also been success stories, and the following case study examines one spectacular success for the Australian landscape.

The model proposed is a plant–herbivore system, similar to the basic predator–prey model, which describes how a population of cactus plants was destroyed by a moth. It is adapted from references 1–3.

1. Introduction

In 1864 Captain Phillip, on his way to Australia with the First Fleet, stopped off in Brazil to collect the cochineal insect and its host plant, the prickly pear (*Opuntia inermis* and *Opuntia stricta*). He was to introduce them to Australia to ensure a plentiful supply of red dye (from the cochineal insects) for his soldiers' coats. The insects did not fare well in Australia. Several more introductions were tried but all failed. However the cacti, which were also planted as hedges to provide additional stock feed, as well as garden plants, adapted well: far too well. They ran wild. Extensive, dense stands of the cacti spread into the farmland of northern New South Wales and southern Queensland, averaging about 500 plants per acre (or 1250 plants per hectare). They 'walled in' homesteads with growth of impenetrable density, and destroyed the viability of thousands and thousands of square kilometres of farmland.

Attempts were made to eradicate the cacti by spraying them from horseback with arsenic pentoxide and sulphuric acid. Whilst this destroyed those cacti the spray came into contact with, it was but a drop in the ocean. Furthermore, it destroyed the men's clothing, their boots and saddles and finally their horses, which lost their hair and developed sores which would not heal.

In 1925 a moth, *Cactoblastis cactorum*, was introduced from Argentina to combat the cactus growth. Its larvae bore through the plant, their breeding being restricted solely to the cacti. The eggs of the moth are not laid at random, but in clumps of egg sticks on the plants, with each egg stick comprising about 80 eggs. About 1.5 sticks of eggs produce sufficient larvae to destroy a cactus plant.

The imported *Cactoblastis* were bred in Brisbane, Queensland, from where some 3000 million eggs were distributed to farmers. Some plants were hit with large numbers of the moths, receiving many more larvae than were required to destroy them, while others escaped completely. Regardless, within only two years they had virtually wiped out the cacti!

2. The model

A mathematical predator–prey-type model is proposed in reference 1 to describe this plant–herbivore system. Let V be the plant mass (vegetation, in plants per acre) and H the herbivore (moth) population size (in egg sticks per acre). The rate of change of plant mass $V' = \mathrm{d}V/\mathrm{d}t$ depends on the difference between the rate at which the cactus grows and spreads and the rate at which it is eaten by the moths. We assume that its rate of growth is logistic: that is, initially it spreads very rapidly (or grows exponentially), and once established this rate slows to negligible (or the number of plants approaches a maximum sustainable by the environment). A function describing this growth pattern is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = r_1 V\Big(1 - \frac{V}{K}\Big), \tag{1}$$

where parameter r_1 is the intrinsic rate of increase in the cactus plant mass and K is known as the carrying capacity for the population. Note that where $V' = 0$ we have $V = K$ or $V = 0$. Thus, if $V = K$, there is no further change in the cactus density over time. If $V = 0$, then the plants are completely eradicated. The solution to (1) is given in figure 1. Clearly, regardless of the initial number of cactus plants, this figure illustrates that $V \to K$.

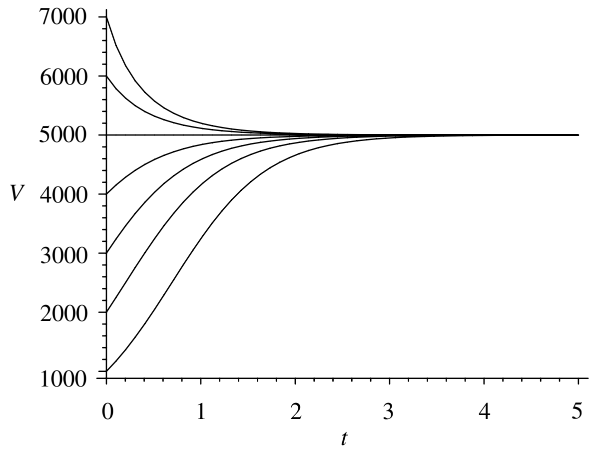

Figure 1. The logistic equation (1), with $r_1 = 2$ and $K = 5000$ plants per acre.

We now establish the rate at which the cacti are destroyed by the moth population H (egg sticks per acre). The plant mass destroyed by each moth depends on the number of plants available, V, and the per capita rate of food intake by the moths which we denote by c_1: thus, $-c_1 V$. So, for the total moth population, the rate of cactus destruction is $-c_1 HV$. There is a slight problem with this however; when the number of cactus plants is very big, this rate gets unrealistically big. To compensate, we introduce a parameter D, associated with the grazing efficiency, and model this rate as $-c_1 HV/(V + D)$. When V is small compared with D, the rate is approximately $-c_1 HV/D$, that is, $-c_1^* HV$, similar to what we had above with $c_1 = c_1^* D$. However, when V is large compared with D, the rate is approximately $-c_1 H$, that is, dependent only on the number of moths and their per capita intake of food. The function $-c_1 V/(V + D)$ is graphed in figure 2, where it is clear that, as the number of cacti V increases, so this rate increases rapidly at first, but then levels out as it approaches a maximum of c_1.

For the cactus plant mass, we now have

$$\frac{\mathrm{d}V}{\mathrm{d}t} = r_1 V\Big(1 - \frac{V}{K}\Big) - c_1 H\Big(\frac{V}{V + D}\Big). \tag{2}$$

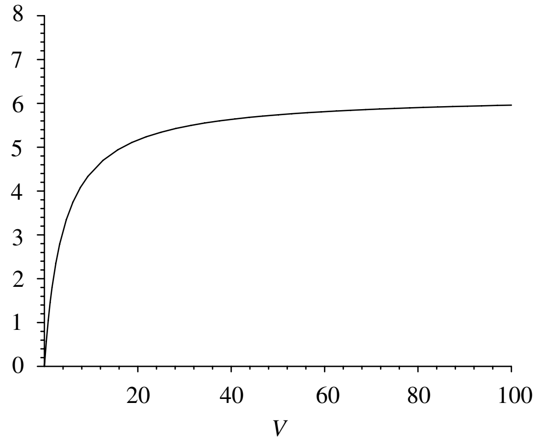

Figure 2. The behaviour of the function $-c_1 V/(V + D)$ when $c_1 = 6.2$ and $D = 4$.

We require a further equation to model the moth population. Just as the maximum sustainable cactus population K is dependent on the environment, so too is the moth population. Thus, we use the logistic equation to model the moths as well. However, the maximum sustainable number of moths (or, carrying capacity K_h) is directly proportional to the available mass of cactus plants V, so that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = r_2 H\Big(1 - \frac{H}{K_h}\Big) = r_2 H\Big(1 - \frac{JH}{V}\Big), \tag{3}$$

where J is the proportionality constant associated with the cactus mass required to sustain a moth at equilibrium (i.e. at $H' = 0$) and r_2 is the per capita rate of increase of the moth population. Equations (2) and (3) now describe the cactus–moth interaction, and comprise our plant–herbivore model.

3. Estimating the parameters

Values for the parameters have been estimated from data collected by Dodd and Monro (see reference 1), and the older unit of measurement, the acre, is retained to make use of the whole numbers from this collected data. (The conversion is 1 hectare = 2.471 acres.)

Since the root stock (1 ton) of the cacti can increase to 250 tons per acre in two years, r_1 is close to 2.7. This estimation follows from assuming that, for small V and $V \ll K$, $V' \approx rV$, so that $V \approx V(0)\mathrm{e}^{rt}$. Thus, since $V(0) = 1$, $r_1 \approx (\ln 250)/2 \approx 2.7$. However, the data used are based on vegetative growth. If sexual reproduction is considered, May (reference 1) suggests this to be an overestimation and that r_1 is 2, at maximum. The carrying capacity K is taken from field data to be 5000 plants per acre.

Laboratory experiments were used to get an estimate for r_2. The laboratory data have 2750 eggs increasing to 100 605 eggs in one year, providing an estimate of 3.6. Field data measurements gave 5000 larvae multiplying to 10 000 000 in two years, suggesting a larger r_2 of 3.8 a year. As above, these estimates follow from $H \approx H(0)\mathrm{e}^{rt}$ giving

$$r_2 \approx \ln \tfrac{100\,605}{2750} \quad \text{and} \quad r_2 \approx \tfrac{1}{2} \ln \tfrac{10\,000\,000}{5000}$$

respectively. Both figures were believed to underestimate the rate and so r_2 was taken as 4. From the work of Monro (see reference 1), J was estimated as 2.23 in units of cactus plants per egg stick.

The parameters c_1 and D were more difficult to estimate. The former should be large, reflecting the damage to the plants by the feeding larvae, while the latter should be small to reflect the efficiency with which the female moths choose plants on which to lay eggs. Following the near-total destruction of the cacti in a period of only two years, the model should predict this crash and then settle to a stable 11 plants per acre, as was observed. For this result $c_1 = 6.2$ and $D = 4$ (reference 2).

4. Results

Certain results are predictable if we consider the values of V and H for which there is no change in plant mass or moth numbers, that is, where the system stabilises with $V' = 0$ and $H' = 0$. Such points are known as equilibrium points, and in our system we are only interested in equilibrium points with positive coordinates since only positive populations are relevant. One point is $(K, 0)$, and the other is given by the intersection of the two lines

$$
\begin{aligned}
H &= \frac{r_1(K - V)(D + V)}{Kc_1} \quad \text{(when } V' = 0\text{)}, \\
H &= \frac{V}{J} \quad \text{(when } H' = 0\text{)}\,.
\end{aligned}
\tag{4}
$$

At $(V, H) = (K, 0)$, the cactus plants have reached their carrying capacity, and the moths have died out; an outcome we would like to avoid. However, substituting the parameter values into (4), and solving to get the second equilibrium point, we get $(V, H) \approx (11, 5)$; that is, a cactus density of 11 plants per acre, and a moth population of 5 egg sticks per acre. A far more attractive outcome.

Since we cannot find a nice analytic solution to the equations (2) and (3), we need to use a software package and computer to solve them for the given parameter values. See figure 3.

Clearly, applying this plant–herbivore model, the dramatic destruction of the cacti after two years is apparent, with the system predicting a stable 11 plants per acre over time, as was observed. Thus, the system settles to the equilibrium point we found above: luckily, the desired outcome.

5. Comments

Although the main features of the cactus–moth interactions are predicted with this model, it is with caution that we should accept it as the correct one, or the only useful one. It should be noted that our model does not incorporate many aspects

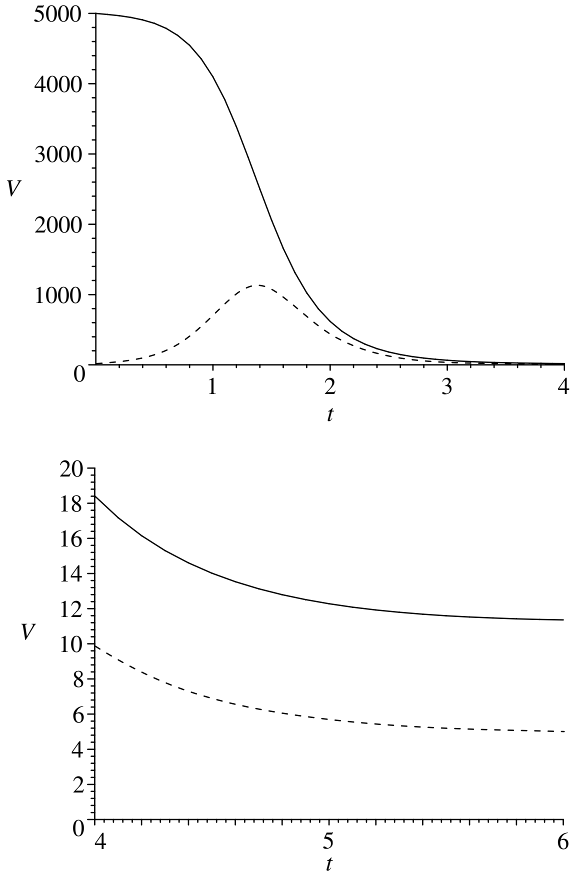

Figure 3. Cactus plants per acre V (solid line) and the moth population H in egg sticks per acre (dashed line) are drawn on the same system of axes for the parameter values given in the text. The crash of the prickly pear cactus in just two years is evident from the first graph. The second graph illustrates the model predictions over time, with $V \to 11$ plants per acre and $H \to 5$ egg sticks per acre.

of the system under study, such as details of the landscape, or specific behavioural traits of the moth.

Today, the cacti can still be found in certain small regions of southern Queensland, but they are well under control and no longer pose a threat to the environment. As a token of their gratitude to the moths, affected communities constructed memorial halls, still in current use, in honour of *Cactoblastis*.

Acknowledgement

This project was funded by the Centre for Mathematics and Its Applications, The Australian National University, Canberra.

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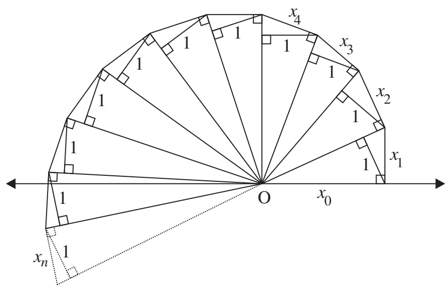
A polygon spiral

Prove that

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 \\ = x_0^2 \times x_1^2 \times x_2^2 \times x_3^2 \times \cdots \times x_n^2.$$

SEYAMACK JAFARI
Bandar Imam, Khozestan, Iran.



The Coupon Collector's Problem and Geological Time

J. GANI

1. A sketch of the coupon collector's problem

An interesting problem in elementary probability theory is the 'coupon collector's problem', mentioned in both Parzen's and Feller's books (references 1 and 2). The problem can be described very simply as follows: M different coupons are to be found in a set of boxes of a particular product, such as cereal packets, and each coupon is assumed to have a probability $1/M$ of being drawn, independently of all others. What is the probability that, after purchasing n boxes, the collector has obtained $i \le M$ different coupons?

Clearly, if one had already collected i different coupons after n purchases, then the next purchase would yield no new coupon with probability $p_{ii} = i/M$, and a different coupon with probability $p_{i,i+1} = (M-i)/M = 1-i/M$. If one sets out these probabilities in a matrix array, with $X(n)$ indicating the random number of different coupons after n purchases, and $X(n+1)$ the number after $n+1$ purchases, then one obtains the matrix $\boldsymbol{P}$ of transition probabilities from $X(n)$ to $X(n+1)$ as

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ i \\ \vdots \\ M-1 \\ M \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \cdots & i & i+1 & \cdots & M \end{array} \\ \begin{bmatrix} 0 & 1 & & & & & \\ & \frac{1}{M} & 1-\frac{1}{M} & & & & \\ & & \ddots & \ddots & & & \\ & & & \frac{i}{M} & 1-\frac{i}{M} & & \\ & & & & \ddots & \ddots & \\ & & & & & 1-\frac{1}{M} & \frac{1}{M} \\ & & & & & & 1 \end{bmatrix} \end{array} \tag{1}$$

with the rows indexed by values of $X(n)$ and the columns by values of $X(n+1)$ and with the sums of probabilities in any row always equalling 1. Note that one goes from 0 to 1 coupon with probability 1, after the purchase of the first box, and remains in the absorbing state M with M different coupons also with probability 1, after purchasing $n \ge M$ boxes, once all M coupons have already been collected.

The random number $X(n)$ of different coupons after n purchases forms what is called a Markov chain, where the value of $X(n+1)$ is clearly dependent on that of $X(n)$. If one starts with 0 coupons at the zeroth trial as we do, then it is easy to see that the probability of collecting $X(n) = 0, 1, 2, \ldots, i, \ldots, M$ different coupons at the nth trial (purchase) will be

$$(1, 0, \ldots, 0)\boldsymbol{P}^n = \left(p_{00}^{(n)}, p_{01}^{(n)}, \ldots, p_{0i}^{(n)}, \ldots, p_{0M}^{(n)}\right),$$

where the first row vector records the initial probability 1 of starting with $X(0) = 0$, and the probabilities $p_{0i}^{(n)}$, $i = 0, \ldots, M$, are effectively the first row of the matrix $\boldsymbol{P}^n$, the nth power of the matrix $\boldsymbol{P}$ in (1). Parzen (reference 1), and more recently McReady and Schwertman (reference 3) have shown that these probabilities can be expressed explicitly as

$$p_{0i}^{(n)} = \frac{M!}{i!\,(M-i)!} M^{-n} \sum_{k=0}^{i} (-1)^k (i-k)^n \frac{i!}{k!\,(i-k)!}. \tag{2}$$

The detailed proof is omitted here, but can be found in references 1 and 3.

Another important feature of the problem is the expected number of purchases n required to obtain i different coupons. If we assume that we already have $X(n-j) = i-1$ different coupons after $n-j$ purchases, where $j < n$, and we get another different coupon after a further j purchases, so that $X(n) = i$, we are basically describing a geometric probability distribution

$$\mathrm{P}\big(X(n) = i \mid X(n-j) = i-1\big) = q^{j-1} p,$$

where $q = (i-1)/M$ and $p = [M-(i-1)]/M$. Now we know that the expected additional number of purchases j required is

$$\sum_{j=1}^{\infty} jq^{j-1}p = p[1+2q+3q^2+\cdots] = \frac{p}{(1-q)^2} = \frac{1}{p} = \frac{M}{M-(i-1)},$$

so that the total expected number of purchases required to obtain i different coupons, starting from 0 coupons, will be the sum

$$\sum_{k=1}^{i} \frac{M}{M-(k-1)}. \tag{3}$$

Let us illustrate the principles involved by examining a simple example in which $M = 3$.

In this case the transition probability matrix (1) takes the form

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so that if we buy 4 boxes, we find that

$$\boldsymbol{P}^4 = \begin{bmatrix} 0 & \frac{3}{81} & \frac{42}{81} & \frac{36}{81} \\ 0 & \frac{1}{81} & \frac{30}{81} & \frac{50}{81} \\ 0 & 0 & \frac{16}{81} & \frac{65}{81} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hence, the probabilities of obtaining 1, 2 or 3 different coupons after 4 purchases are, respectively, $\frac{3}{81}$, $\frac{42}{81}$ and $\frac{36}{81}$. We can verify from (2) that

$$p_{02}^{(4)} = \frac{3!}{2!\,1!}3^{-4}\sum_{k=0}^{2}(-1)^k(2-k)^4\frac{2!}{k!\,(2-k)!}$$

is indeed $\frac{42}{81}$. As for the expected number of purchases required to obtain two different coupons, we see from (3) that it will be

$$\sum_{k=1}^{2} \frac{3}{3-(k-1)} = 1 + \tfrac{3}{2} = \tfrac{5}{2}.$$

2. Lyell's model of geological change

Sir Charles Lyell (1797–1875) was an eminent British geologist, who wrote a pioneering three volume work, *Principles of Geology*, between 1830 and 1833 (reference 4). In it, he attempted to determine the dates of geological deposits of the Tertiary period, the first period of the Cenozoic era, characterized by the appearance of modern flora, apes and other large mammals. He based his dates on the fossil snail shells in an existing collection, and on those found in excavated deposits. Lyell suggested that if, for example, 1000 species were available in an existing collection, and two deposits A and B were excavated in which the numbers of extinct shell species were, respectively, 600 and 300, then deposit A was twice as old as deposit B. In other words, it took twice as long for 600 species to become extinct and replaced by new species as it did for 300. This is referred to as a linear model for geological time determination. Charles Darwin (1809–1882), an avid reader of Lyell's book, was greatly influenced by it in his development of the theory of evolution.

A recent paper by McReady and Schwertman (reference 3) discusses in some detail the mathematical models used by Lyell to date the age of fossil shells found in his excavations. These linear models relied on the ratio

$$\frac{\text{number of extinct species of shells in collection}}{\text{total number of shell species in collection}}$$

to estimate the ratio

$$\frac{\text{lapse of time since the period of the fossils}}{\text{time since the beginning of the Tertiary period}}.$$

We shall attempt to summarize McReady and Schwertman's simpler results; to do so, we have recast Lyell's modelling in a modern formulation to make it more readily accessible to readers.

Lyell's simplified model assumes that, at any given time, the number of different existing species is M, a number which remains constant. At time $n = 0$, the process begins with M different existing species. At the transition $n = 1$, one of these species is replaced by a new species, but at subsequent times $n = 2, 3, \ldots$, the replacement may be either a new species or remain one of the existing ones. Remarkably, the transition probability matrix of the associated Markov chain for the random number $X(n)$ of new species, where n is now the number of transitions, turns out to be precisely the same as that for the number of new coupons given by (1). Thus, the probability that the number of new species $X(n)$ is i is given by (2).

To estimate the expected number T_i of transitions elapsed from any geological epoch until the present appearance of i new species, we may use the result (3):

$$\mathrm{E}(T_i) = \sum_{k=1}^{i} \frac{M}{M-(k-1)}.$$

Following Gould's analysis (reference 5), which assumed a collection of $M = 1000$ shells, of which 966 had become extinct since the Eocene period, the second oldest epoch of the Tertiary period when the rise of mammals occurred, McReady and Schwertman estimated the expected value of T_{966} for the Eocene period (in numbers of transitions) to be

$$\mathrm{E}(T_{966}) = \sum_{k=1}^{966} \frac{1000}{1000-(k-1)} = 3367.26,$$

or roughly 3367 transitions before Lyell's time. If one assumes that a transition occurs roughly every 10 000 years, this would mean that the Eocene period was dated 33 672 600 years before 1830. The current estimate of the date of the Eocene period is approximately 36 000 000 years ago, so that Lyell's approach seems well founded.

I knew very little of Lyell's methods, and found it fascinating to read about his modelling, as well as Stephen Jay Gould's comments on it, in McReady and Schwertman's paper. Those who wish to acquire a deeper knowledge of the origins of scientific geology, and the influence of Lyell's book on Darwin's thinking, will find their paper an invaluable introduction to these topics.

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Integration Simplified

P. GLAISTER

What do the functions $x+1$, $3x^2+x$ and $3x^2-1$ have in common?

The function $f(x) = \mathrm{e}^x$ is the only function (up to a multiplicative constant) for which $f'(x) = f(x)$ for all x, i.e. it does not change under differentiation. Likewise under integration, $\int f(x)\,\mathrm{d}x = f(x) + C$. There are clearly other functions $f(x)$ for which $f'(a) = f(a)$ for specific values of a, i.e. the value of the derivative can be obtained without differentiation. For example, with $f(x) = \sin x$, $f'(a) = f(a)$ when $\tan a = 1$, i.e. $a = (4n+1)\pi/4$, $n \in \mathbb{Z}$, and many more can be generated which satisfy this property. The corresponding result for integrals is less straightforward but much more interesting to study.

Here we seek functions $f(x)$ and limits a and b for which

$$\int_a^b f(x)\,\mathrm{d}x = f(b) - f(a)\,, \tag{1}$$

i.e. the value of the integral can be obtained without integrating. This certainly simplifies the process of integration! (Clearly (1) is satisfied for $f(x) = \mathrm{e}^x$.) A good starting point is to consider the interval $[a, b] = [-1, 1]$, and polynomials for $f(x)$. Taking a linear function $f(x) = \alpha x + \beta$, with α, β constants, (1) becomes

$$\frac{\alpha x^2}{2} + \beta x\Big|_{-1}^{1} = \alpha x + \beta\Big|_{-1}^{1}\,,$$

i.e. $2\beta = 2\alpha$. Thus $f(x) = \alpha(x+1)$, which gives

$$\int_{-1}^{1} (x+1)\,\mathrm{d}x = x + 1\Big|_{-1}^{1} = 2\,.$$

For a quadratic polynomial $f(x) = \alpha x^2 + \beta x + \gamma$, (1) becomes

$$\frac{\alpha x^3}{3} + \frac{\beta x^2}{2} + \gamma x\Big|_{-1}^{1} = \alpha x^2 + \beta x + \gamma\Big|_{-1}^{1}\,,$$

i.e. $\frac{2}{3}\alpha + 2\gamma = 2\beta$. Therefore $f(x) = (3\beta - 3\gamma)x^2 + \beta x + \gamma = \beta(3x^2 + x) - \gamma(3x^2 - 1)$, which gives

$$\int_{-1}^{1} (3x^2 + x)\,\mathrm{d}x = 3x^2 + x\Big|_{-1}^{1} = 2$$

and

$$\int_{-1}^{1} (3x^2 - 1)\,\mathrm{d}x = 3x^2 - 1\Big|_{-1}^{1} = 0\,,$$

and any linear combination of these.

More generally, with $f(x) = \sum_{i=0}^{n} a_i x^i$ for odd n,

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \sum_{i=0}^{n} a_i \frac{x^{i+1}}{i+1}\Big|_{-1}^{1} = \sum_{j=0}^{(n-1)/2} \frac{2a_{2j}}{2j+1}$$

and

$$f(x)\Big|_{-1}^{1} = \sum_{i=0}^{n} a_i x^i\Big|_{-1}^{1} = \sum_{j=0}^{(n-1)/2} 2a_{2j+1}\,.$$

Therefore, (1) becomes

$$\sum_{j=0}^{(n-1)/2} \frac{2a_{2j}}{2j+1} = \sum_{j=0}^{(n-1)/2} 2a_{2j+1}\,. \tag{2}$$

Rearranging (2) gives

$$a_n = \sum_{j=0}^{(n-1)/2} \frac{a_{2j}}{2j+1} - \sum_{j=0}^{(n-1)/2-1} a_{2j+1}\,,$$

and thus

$$\begin{aligned} f(x) &= \sum_{i=0}^{n} a_i x^i \\ &= \sum_{j=0}^{(n-1)/2} a_{2j} x^{2j} \\ &\quad + \sum_{j=0}^{(n-1)/2-1} a_{2j+1} x^{2j+1} + a_n x^n \\ &= \sum_{j=0}^{(n-1)/2} a_{2j} x^{2j} + \sum_{j=0}^{(n-1)/2-1} a_{2j+1} x^{2j+1} \\ &\quad + x^n \left(\sum_{j=0}^{(n-1)/2} \frac{a_{2j}}{2j+1} - \sum_{j=0}^{(n-1)/2-1} a_{2j+1} \right) \\ &= \sum_{j=0}^{(n-1)/2} a_{2j} \left(\frac{x^n}{2j+1} + x^{2j} \right) \\ &\quad + \sum_{j=0}^{(n-1)/2-1} a_{2j+1} (x^{2j+1} - x^n)\,. \end{aligned}$$

Therefore

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = f(1) - f(-1) = 2 \tag{3}$$

holds for

$$f(x) = x^n + (2j+1)x^{2j} \quad \text{and} \quad f(x) = x^n - x^{2j+1}$$

for odd n and $j = 0, 1, \ldots, (n-1)/2$, and for any linear combination of these. (Note that for $j = (n-1)/2$, $x^n - x^{2j+1} \equiv 0$ and (3) is automatically satisfied.) The case $n = 1$ has already been considered, and with $n = 3$, $f(x) = x^3 + 1,\ x^3 + 3x^2,\ x^3 - x$.

We leave readers to consider the case when n is even, as well as other intervals and functions.

The author lectures in mathematics at Reading University. His most recent achievements on the mathematical front concern his two children. Despite showing a total lack of interest in talking about mathematics problems with him, his daughter is one of a handful in her year who have been regular attenders at the school puzzle club, and his son was so enthusiastic about a shape puzzle which was set for homework that it now features on the family website (http://www.glaisters.btinternet.co.uk/mark/mindex.htm).

Trisecting an angle

I have designed two devices for dividing an arbitrary angle into three equal sections. The first consists of three strips, AB, OA and OC, equal in length. The point C can slide along a groove in AB and B can slide along a groove in the horizontal strip through O, as shown in figure 1. Prove that $\beta = \frac{1}{3}\alpha$.

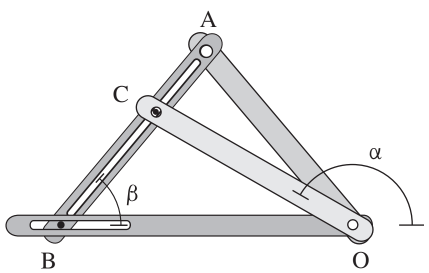

Figure 1.

A similar device is shown in Volume 25, Number 4, p. 105, when the angle α is acute.

The second device consists of strips AB and BC fixed at right angles and strips AC and BD with an axis at B and pins in A, D and C which can slide along grooves as shown in figure 2. The distance AC in equal to 2BD. Prove that $\beta = \frac{1}{3}\alpha$.

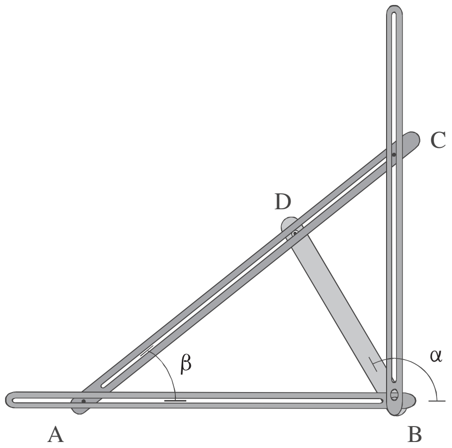

Figure 2.

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University of Kragujevac

Some Fibonacci-like Sequences

RANDALL J. SWIFT

A random look at Fibonacci sequences

Each term of the familiar Fibonacci sequence

$$1, 1, 2, 3, 5, 8, \ldots$$

is obtained as the sum of the two previous terms. This sequence has many beautiful properties and even has a journal, *The Fibonacci Quarterly*, devoted to its study. Many of its fundamental properties are considered in reference 1. In this article, we consider some sequences related to the Fibonacci sequence and pose some interesting questions for further study.

The Fibonacci sequence can be written in the form of a recurrence relation. Specifically, since the nth term $x(n)$ relies on only the two previous terms, we can write

$$x(n) = x(n-1) + x(n-2)\,. \tag{1}$$

To get the recurrence relation going, we need to specify the first two terms. Here $x(0) = 1$ and $x(1) = 1$.

Recurrence relations such as (1) are useful for describing a sequence, but are often difficult to use when we wish to compute a term for a large value of n. For instance, to find the 100th term of the Fibonacci sequence, we would have to compute the previous 99 terms. It is thus desirable to express $x(n)$ in a *closed form*, that is, a form that depends only upon n.

Finding a closed form for the recurrence relation (1) is similar to finding the solution of a homogeneous linear differential equation with constant coefficients. The details of the method can be found in reference 2.

We apply the method to the Fibonacci recurrence by rewriting it as a difference equation

$$x(n) - x(n-1) - x(n-2) = 0\,,$$

and then substituting $x(n) = C\lambda^n$, where C is a constant, to obtain

$$C\lambda^n - C\lambda^{n-1} - C\lambda^{n-2} = 0\,.$$

Factoring out $C\lambda^{n-2}$ gives the *characteristic equation*

$$\lambda^2 - \lambda - 1 = 0\,,$$

a quadratic which we can solve easily. The solution gives the *eigenvalues* for the recurrence relation. The eigenvalues can be viewed as 'growth rates' of the recurrence relation. For the Fibonacci sequence, they are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}\,.$$

Notice that $\lambda_1 \approx 1.618$ and $\lambda_2 \approx -0.618$, so $\lambda_1 > \lambda_2$. We call λ_1 the *dominant* eigenvalue. In general, the root of a characteristic equation with largest modulus will be termed the dominant eigenvalue and denoted λ^*.

The value

$$\lambda_1 = \frac{1+\sqrt{5}}{2}$$

is also known in geometry as the *golden ratio*.

The closed form of $x(n)$ is

$$x(n) = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

where we know that the initial values are $x(0) = 1$ and $x(1) = 1$. Using these we have the following simultaneous equations for the constants C_1 and C_2:

$$C_1 + C_2 = 1 \quad \text{and} \quad C_1\frac{1+\sqrt{5}}{2} + C_2\frac{1-\sqrt{5}}{2} = 1\,.$$

Solving gives $C_1 = (5+\sqrt{5})/10$ and $C_2 = (5-\sqrt{5})/10$. Thus, the closed-form solution to the Fibonacci recurrence relation (1) is

$$x(n) = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{2}$$

One surprising feature of this closed-form solution (2) is that, despite the fact that all of its factors are irrational, it produces integer values for every n.

The long-term behaviour of the Fibonacci sequence can be determined by (2). Considering each of its terms, we see that, since

$$\frac{1+\sqrt{5}}{2} > 1 \quad \text{and} \quad \frac{1-\sqrt{5}}{2} < 1\,,$$

the behaviour of the sequence as n increases resembles that of exponential growth with a base of $(1+\sqrt{5})/2$, which is the value of the dominant eigenvalue. Moreover,

$$\lim_{n\to\infty} \sqrt[n]{x(n)} = \frac{1+\sqrt{5}}{2}\,. \tag{3}$$

In general, the growth of a recurrence relation will be determined by its dominant eigenvalue λ^* with an expression similar to (3); indeed,

$$\lim_{n\to\infty} \sqrt[n]{|x(n)|} = \lambda^*\,. \tag{4}$$

Problem 1. Suppose that we generate a sequence using the Fibonacci recurrence relation

$$x(n) = x(n-1) + x(n-2)$$

with different initial values; for instance, the sequence generated with $x(0) = 2$ and $x(1) = 4$. Do these initial conditions change the value of the dominant eigenvalue λ^*? What if $x(0) = a$ and $x(1) = b$, where a and b are any real numbers?

The ideas we have just considered can be applied to some Fibonacci-like sequences. Consider the 'tri-bonacci' sequence

$$1, 1, 1, 3, 5, 9, 17, \ldots,$$

where each term in the sequence is obtained as a sum of the previous three terms. The recurrence relation for this sequence is

$$x(n) = x(n-1) + x(n-2) + x(n-3) \tag{5}$$

where we specify the first three terms of the sequence as $x(0) = 1$, $x(1) = 1$ and $x(2) = 1$.

The eigenvalues for this recurrence relation are obtained from the cubic polynomial

$$\lambda^3 - \lambda^2 - \lambda - 1 = 0,$$

which has solution

$$\begin{aligned}
\lambda_1 &= \tfrac{1}{3} + \tfrac{1}{3}\big(19 - 3\sqrt{33}\big)^{1/3} + \tfrac{1}{3}\big(19 + 3\sqrt{33}\big)^{1/3},\\
\lambda_2 &= \tfrac{1}{3} - \tfrac{1}{6}\big(1 + \mathrm{i}\sqrt{3}\big)\big(19 - 3\sqrt{33}\big)^{1/3}\\
&\quad - \tfrac{1}{6}\big(1 - \mathrm{i}\sqrt{3}\big)\big(19 + 3\sqrt{33}\big)^{1/3},\\
\lambda_3 &= \tfrac{1}{3} - \tfrac{1}{6}\,\big(1 - \mathrm{i}\sqrt{3}\big)\,\big(19 - 3\sqrt{33}\big)^{1/3}\\
&\quad - \tfrac{1}{6}\big(1 + \mathrm{i}\sqrt{3}\big)\big(19 + 3\sqrt{33}\big)^{1/3}.
\end{aligned}$$

Computing the modulus of each of these eigenvalues, we find that $\lambda_1 \approx 1.83929$, while both λ_2 and λ_3 have modulus less than 1. We also note that they have the same modulus, as they are complex conjugates. Thus, the dominant eigenvalue for the tri-bonacci sequence is $\lambda^* = \lambda_1 \approx 1.83929$.

Problem 2. Obtain a closed form for the tri-bonacci recurrence relation (5) with the initial conditions $x(0) = 1$, $x(1) = 1$ and $x(2) = 1$. Note that you may need to use a computer algebra system or some other appropriate method to solve and simplify the equations.

We now consider the k-bonacci sequence, where each term in the sequence is obtained as a sum of the previous k terms. Specifically,

$$x(n) = \sum_{j=1}^{k} x(n-j)\,, \tag{6}$$

where the first k-terms of the sequence are specified as

$$x(0) = x(1) = \cdots = x(k-1) = 1.$$

The eigenvalues for the k-bonacci sequence (6) can be obtained by solving the polynomial

$$\lambda^k - \sum_{j=0}^{k-1} \lambda^j = 0.$$

For increasing values of k this is a difficult task, and for $k \geq 5$ requires numerical procedures. Our interest lies in the dominant eigenvalue λ^* of the k-bonacci sequence.

Table 1 shows the dominant eigenvalue λ^* for the k-bonacci sequence for k from 2 to 10. Numerically, we see that the dominant eigenvalue λ^* approaches 2 as k increases. It is not too difficult to show, using a geometric series argument, that λ^* does indeed converge to 2. In effect, this says that, for large k, the k-bonacci sequence experiences growth as a doubling of each of its subsequent terms.

Table 1. The dominant eigenvalue of the k-bonacci sequence.

k	λ^*
2	1.61803
3	1.83929
4	1.92756
5	1.96595
6	1.98358
7	1.99196
8	1.99603
9	1.99803
10	1.99902

Problem 3. For the k-bonacci sequence, show that the dominant eigenvalue λ^* converges to 2 as $k \to \infty$. Give a heuristic argument that, for large k, $x(n) \approx 2x(n-1)$ so that this dominant eigenvalue can be interpreted as a successive doubling of the terms of the sequence.

The discussion thus far has centred around the notion of the dominant eigenvalue for Fibonacci-like sequences. We will now give a surprising generalization.

Consider the recurrence relation

$$x(n) = z_1 x(n-1) + z_2 x(n-1)\,, \tag{7}$$

with $x(0) = 1$ and $x(1) = 1$ and where z_1 and z_2 are independent Bernoulli random variables

$$z_i = \begin{cases} 1 & \text{with probability } p\,, \\ -1 & \text{with probability } q\,, \end{cases} \tag{8}$$

with $p + q = 1$. If we take $p = 1$, then (7) becomes the Fibonacci recurrence relation (1). For different values of p, the terms of the sequence behave very differently from the Fibonacci sequence. In fact, since we have introduced a random factor into the recurrence relation, each time we apply the recurrence relation (7) a different sequence is generated.

If we let $p = \frac{1}{2}$, some examples of sequences are 1, 1, 0, 1, -1, -2, -3, 1, ... and 1, 1, 2, 3, -1, -4, 5, -9, ... The surprising result here is that these two different sequences have the same dominant eigenvalue!

Unlike the sequences we have considered above, we cannot obtain a characteristic polynomial expression for the eigenvalues of (7). However, we can obtain the dominant eigenvalue numerically using (4).

With $p = \frac{1}{2}$, it can be shown using the software package MATHEMATICA® that the dominant eigenvalue for (7) is

$\lambda^* \approx 1.13198$. This value, which appears to be irrational, is obtained by generating a large number of terms of the sequence (7) and then applying the expression (4).

Recently, Viswanath (reference 3) proved that the dominant eigenvalue of the sequence generated by (7) in the case $p = \frac{1}{2}$ is indeed the number $1.13198824\ldots$. The proof uses products of random matrices, the Stern–Brocot tree, and a fine computer calculation. It is by no means a trivial proof and, indeed, was part of Viswanath's PhD dissertation.

Table 2. The dominant eigenvalue of (7) for increasing values of p.

P	λ^*	P	λ^*
0	1.0	0.55	1.13106
0.05	1.02647	0.60	1.13301
0.10	1.05108	0.65	1.14058
0.15	1.07228	0.70	1.15045
0.20	1.09312	0.75	1.16645
0.25	1.10717	0.80	1.19537
0.30	1.12088	0.85	1.24438
0.35	1.12368	0.90	1.31584
0.40	1.13038	0.95	1.42701
0.45	1.13294	1.0	1.61801
0.50	1.13193		

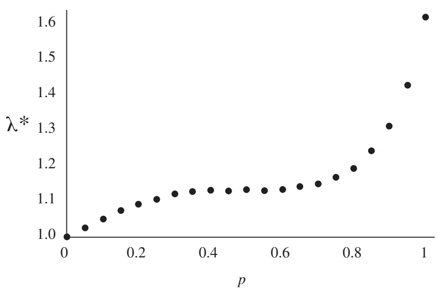

Figure 1. Dominant eigenvalue for different values of p for the sequence generated by (7).

In general, numerical results do not provide a proof. However, they can be used to obtain insight into the behaviour of a problem. Consider the natural question of the value of the dominant eigenvalue for the sequence generated by (7) for values of p different from $\frac{1}{2}$. Table 2 and figure 1 illustrate the dominant eigenvalue of the sequence generated by (7). The eigenvalues are obtained numerically for varying values of p. It is interesting to note the general shape of this graph and that, as p approaches 1, the eigenvalue approaches the dominant eigenvalue of the Fibonacci sequence $(1+\sqrt{5})/2 \approx 1.618$.

These random Fibonacci sequences are indeed rather remarkable, as they possess dominant eigenvalues for each p. An electronic copy of a MATHEMATICA 3.0 file that performs the analysis described in this article, as well as some additional related ideas, is available from the author. We close with some problems for further study.

Problem 4. The same analysis applies to the k-bonacci-like sequence generated by

$$x(n) = \sum_{j=1}^{k} z_j x(n-j)\,, \tag{9}$$

where the first k-terms of the sequence are specified as

$$x(0) = x(1) = \cdots = x(k-1) = 1$$

and where the z_i are independent Bernoulli random variables given by (8). Using a software package, determine the value of the dominant eigenvalues for each of the sequences generated by (9) with $k = 3$ for varying values of the probability p. Create a graph similar to figure 1.

Problem 5. Generalize problem 4 to determine the value of dominant eigenvalues for each of the sequences generated by (9) for any k and for varying values of the probability p. Create graphs similar to figure 1 for increasing k. Surprised by the behaviour?

Problem 6. Along these same lines, one can consider Fibonacci 'sub'-recurrence relations. For instance, the tribonacci recurrence relation

$$x(n) = x(n-1) + x(n-2) + x(n-3)$$

has the following subcases where only two of the three terms in the relation appear:

$$\begin{aligned} x_1(n) &= x_1(n-1) + x_1(n-2)\,,\\ x_2(n) &= x_2(n-1) + x_2(n-3)\,,\\ x_3(n) &= x_3(n-2) + x_3(n-3)\,. \end{aligned}$$

Determine the dominant eigenvalues for each of these subcases. Interpret the relative size of each of these growth rates in terms of each of the sequences. Explore these questions for subcases of the k-bonacci sequence.

Problem 7. Following the analysis of the sequence (7), generalize problem 6 to the case of random subsequences. Explore and have fun!

References

1. R. Dunlap, *The Golden Ratio and Fibonacci Numbers* (World Scientific, River Edge, NJ, 1997).
2. S. Goldberg, *Introduction to Difference Equations*, 2nd edn (Dover, New York, 1986).
3. D. Viswanath, Random Fibonacci sequences and the number 1.13198824..., *Math. Comp.* **69** (2000), pp. 1131–1155.

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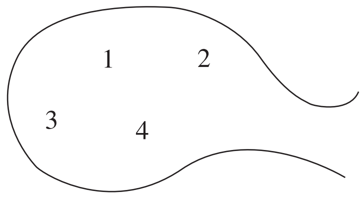
Mathematics in the Classroom

The following worksheet was devised initially for my GCSE Intermediate re-takers and just went through the roof!

Equation solving

Put each of the numbers in the bag into a different circle to make an equation.

$$\bigcirc x + \bigcirc = \bigcirc x + \bigcirc$$



How many different equations can you make? Solve them all, and see what possible values of x arise as solutions.

Try placing your own numbers into the bag. What is the largest possible number of different solutions for x?

Suppose now that the bag contains the distinct positive whole numbers a, b, c, d. Let S be the set

$$\{p : p \text{ is the solution for } x \text{ in one of the possible equations}\}.$$

Show that, if p is in S, then so is $1/p$ and so is $-p$. What is the largest number of positive integers that S can contain? Using your own values for a, b, c and d, give an example of how this maximum number may be attained. Without doing any further calculation, find the other members of S in this case.

Show that the positive numbers 1, 2, 3 and $2k$ will always give three distinct positive integers in S for k a whole number greater than 2.

Show that the positive numbers 1, 2, 4 and $6k+5$ will always give three distinct positive integers in S for k a whole number greater than 0.

Show that the positive numbers 1, 2, 5 and $12k+10$ will always give three distinct positive integers in S for k a whole number greater than or equal to 0.

What about 1, 2, r and s? Can we always find a value for s that will give us three distinct positive integer solutions for any natural number r? Notice that the integer solutions for when 1, 2, 4, 11 are in the bag are the same as for when 1, 2, 3, 8 are there. Is this a coincidence?

Show that, for any distinct natural numbers a, b and for any distinct natural numbers c, d greater than 1,

$$a, \qquad b, \qquad a + c(b-a),$$
$$c^2 d(b-a) - c(b-a)(d+1) + b$$

is a quartet that gives three integer solutions for the equations. Show that, in this case, these integer solutions can be written as

$$x, \qquad y, \qquad \frac{xy-1}{x-y}.$$

Paston College, Norfolk **Jonny Griffiths**

Wrong method, right answer

Once, when I was teaching calculus to intermediate-level students, I asked them to evaluate the limit below. One student replied quickly and his answer was correct, but I was surprised to see his solution:

$$\begin{aligned}
&\lim_{\alpha\to\beta} \frac{\sin^2\alpha - \sin^2\beta}{\alpha^2-\beta^2} \\
&\quad = \lim_{\alpha\to\beta} \frac{(\sin\alpha+\sin\beta)(\sin\alpha-\sin\beta)}{(\alpha+\beta)(\alpha-\beta)} \\
&\quad = \lim_{\alpha\to\beta} \frac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{(\alpha+\beta)(\alpha-\beta)} \\
&\quad = \lim_{\alpha\to\beta} \frac{\sin(\alpha+\beta)}{(\alpha+\beta)} \frac{\sin(\alpha-\beta)}{(\alpha-\beta)} \\
&\quad = \frac{\sin 2\beta}{2\beta}\,.
\end{aligned}$$

Another solution which is not less surprising is:

$$\begin{aligned}
\lim_{\alpha\to\beta} \frac{\sin^2\alpha - \sin^2\beta}{\alpha^2-\beta^2} &= \lim_{\alpha\to\beta} \frac{\sin(\alpha^2-\beta^2)}{(\alpha+\beta)(\alpha-\beta)} \\
&= \lim_{\alpha\to\beta} \frac{\sin\big((\alpha+\beta)(\alpha-\beta)\big)}{(\alpha+\beta)(\alpha-\beta)} \\
&= \lim_{\alpha\to\beta} \frac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{(\alpha+\beta)(\alpha-\beta)} \\
&= \lim_{\alpha\to\beta} \frac{\sin(\alpha+\beta)}{(\alpha+\beta)} \frac{\sin(\alpha-\beta)}{(\alpha-\beta)} \\
&= \frac{\sin 2\beta}{2\beta}
\end{aligned}$$

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Computer Column

Mental computing

For millions of years, humans (and their ancestors) have had at their disposal computers which are far more powerful than any PC: their own brains. As little as twenty years ago, their workings were deeply mysterious; we knew about their physical structure, but we had no idea how they we able to do so much. The bigger picture is still far from clear, but we are at last beginning to see where all that power comes from.

The brain is composed of a large number (about 10^{10}) of basic cells, called neurons. Each neuron can be thought of as a very small computer, with many inputs (called dendrites) and one output (called an axon). The axon branches repeatedly, each branch going on to join up with a dendrite of another neuron. The connection between an axon and a dendrite isn't a physical connection as such: there is a gap between the two, called a synapse. The signal coming down the axon causes a chemical (known as a neurotransmitter) to be released, which then causes the dendrite on the other side of the gap to open up a number of tiny pores. Ions (charged atoms) flow into the dendrite through these, changing its voltage. The voltage in the main body of the cell then changes accordingly and, whenever the voltage reaches a certain threshold, a voltage signal is released down its axon.

This may all seem quite complex — and there's plenty more to tell — but the basic behaviour of neurons is actually very simple, as summarised in figure 1.

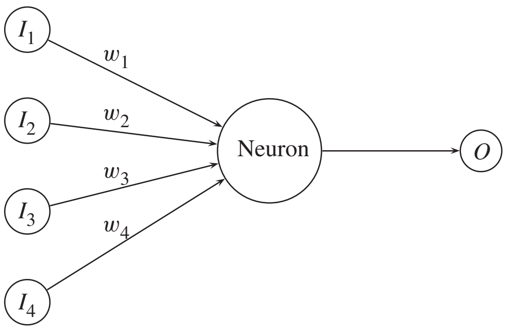

Figure 1. Basic model of a neuron. The output O is 1 if $\sum_i w_i I_i > T$ and is 0 otherwise, where the w_i are the weights and T is the threshold.

A good way to think of a neuron is as a detector: if it 'detects' enough input, then it gives an output (called a spike); otherwise it does nothing. A crucial feature in all this is the existence of the weights, which represent the efficiency of the synapses in transmitting the signal from dendrite to axon. These can be thought of as indicating the importance that should be placed on the different inputs; as we learn, these weights change to represent what we've learned.

Another crucial feature of neurons is that their output can either be 'excitatory' or 'inhibitory' (i.e. positive or negative), with each neuron only providing one type of output. As we will see, it is only through having both types of neuron that the brain can be as powerful as it is.

Viewed from a computational angle, a neuron can be thought of as a general (and adaptable) type of logic gate. Logic gates, which form the basis of all man-made computers, are tiny electrical circuits which, like the neuron, have inputs (which can be on or off, describing a binary 1 or 0) and an output. An important theoretical result about logic gates is that only a few different types are needed in order to perform any possible computation. In fact, it is possible to use just one type of gate: the NAND gate. This has two inputs, and gives an output unless both inputs are on. For any given set of inputs and outputs, it is possible to build a network of NAND gates that, given the inputs, produces the outputs. In other words, any possible function from integers to integers can be represented with NAND gates.

Following this line of thought, we rapidly hit a difficulty in thinking of brains as flexible computers: it doesn't look as if one neuron could possibly represent a NAND gate. As presented, more input means more output, not less! A single neuron with suitable weights *can* represent some types of gate, such as an OR gate (which gives an output if at least one input is on) or an AND gate (which gives an output only if both inputs are on), but 'negative' gates like NAND are a problem.

This is where inhibition comes in: by connecting together inhibitory and excitatory neurons, we can do much more. Figure 2 shows a pair of neurons, one excitatory (e) and one inhibitory (i), which, working together, can solve problems like these.

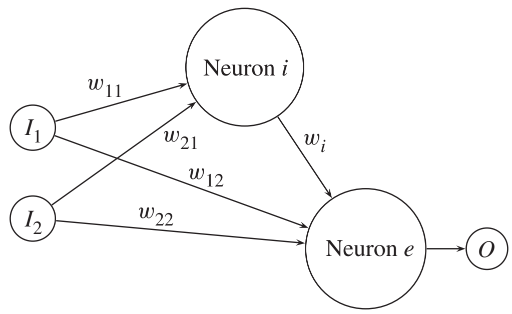

Figure 2. Co-operative pair of neurons.

If, for example, we arrange for neuron i to be active only when both the inputs are (i.e. to be an AND gate) and use the inhibitory output to suppress neuron e, then we can replicate another type of logic gate, XOR (which is on only if one or other input is on, but not both). The inhibitory neuron has given us the last piece of the puzzle: how to get less output from more input. Replicating a NAND gate is even easier than this: rather than connecting the inputs to neuron e, we just have to connect e to a source which is always on, making it active unless it is suppressed by neuron i.

Given how many neurons there are in a human brain, we clearly have very powerful computers in our heads, but how do we learn? There are two main mechanisms that have been proposed for this: Hebbian learning and error-driven learning.

Hebbian learning (first proposed by Donald Hebb in 1949) works to emphasise the correlations between neurons, by strengthening the weights between neurons that are often active at the same time. This is a bit like making the connection between a lightning flash and a thunder clap: because they tend to occur together, we come to feel that there must be a connection between them. Eventually, we start to expect the thunder when we see the lightning. Similarly, if it so happens that a neuron is active more often than not when we see a cat, then Hebbian learning will act to press the neuron into service as a 'cat detector'. (This is not as crazy as it may sound: the brain has many highly specialised neurons, some of which act as 'face detectors' for people we know well.)

Error-driven learning, by contrast, makes the brain learn by trial and error. When we do something right, the weights between the neurons that were active in making the decision are strengthened; otherwise, they are weakened. (The mechanism for doing this — the release of a neurotransmitter called dopamine — is also what makes us feel good when we succeed.) In other words, we remember good strategies and forget bad ones.

These two strategies work well together, in that error-driven learning drives us towards good solutions to problems, while Hebbian learning drives us towards solutions that represent the essential relationships between their parts. As a result, our solutions to problems come to resemble a series of logical steps, making it much easier to generalise from situations we know about to situations we've never experienced before.

This has been a bit of a whistle-stop tour, but I hope it has given you a flavour of how mysterious the brain really is. Its component parts appear so simple, and yet their combination produces behaviour so complex that it has enabled us to travel to the Moon, experience joy and sadness, and even to wonder why; just try doing that with even the fastest PC!

Peter Mattsson

Letters to the Editor

Dear Editor,

A Tale of Two Series — A Dickens of an Integral

In my letter in Volume 34, Number 2 which follows on from A. G. Summers' letter in Volume 33, Number 3, it was asked if there are any differentiable functions f such that

$$\int_0^1 f(x)\,\mathrm{d}x = \sum_{x=1}^{\infty} f(x).$$

I have thought of the following solution to this problem.

Consider the function

$$f(x) = T(x)\cos(2\pi x) + Q(x)\sin^2(2\pi x).$$

Suitable choices for $T(x)$ and $Q(x)$ will solve the problem. Then

$$\sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{\infty} T(x).$$

If, for example, we let $T(x) = 1/(x+1)^2$, then

$$\sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{\infty} \frac{1}{(x+1)^2} = -1 + \frac{1}{6}\pi^2.$$

If we put $Q(x) = \beta \mathrm{e}^x$, then

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 \frac{\cos(2\pi x)}{(x+1)^2}\,\mathrm{d}x + \beta \int_0^1 \mathrm{e}^x \sin^2(2\pi x)\,\mathrm{d}x,$$

and this will be equal to $\sum_{x=1}^{\infty} f(x)$ when

$$\beta = \frac{-1 + \pi^2/6 - \int_0^1 \cos(2\pi x)/(x+1)^2\,\mathrm{d}x}{\int_0^1 \mathrm{e}^x \sin^2(2\pi x)\,\mathrm{d}x}.$$

This gives

$$\begin{aligned}\beta &= \frac{1}{8}\,\frac{-\frac{3}{2} + \frac{1}{6}\pi^2 + 2\,\mathrm{Si}(4\pi)\pi - 2\,\mathrm{Si}(2\pi)\pi}{\pi^2(\mathrm{e}-1)}(1+16\pi^2)\\ &= 0.714449695\end{aligned}$$

to nine decimal places, where

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}t.$$

A different choice for both $T(x)$ and $Q(x)$ may have resulted in a simpler looking (closed) form for β.

An easier alternative is to take any integrable even function g, such as $g(x) = x^2$. Then $g(x - \frac{1}{2})$ is even and $\sin(2\pi x)$ is odd about $x = \frac{1}{2}$, so $g(x-\frac{1}{2})\sin(2\pi x)$ is odd and

$$\int_0^1 g\big(x - \tfrac{1}{2}\big)\sin(2\pi x)\,\mathrm{d}x = 0 = \sum_{x=1}^{\infty} g\big(x-\tfrac{1}{2}\big)\sin(2\pi x).$$

Yours sincerely,

Milton Chowdhury
(16 Caledonian Avenue,
Layton,
Blackpool FY3 8RB.)

Dear Editor,

The Destiny Stone

In the Open University's *Open Eye* magazine, Autumn 2002, Jim Taggart quotes an equation that represents an elliptical bridge with a span of 17 ft 6 in which he built in a Scottish botanical garden. For posterity, Jim engraved a rock (The Destiny Stone) with the following:

$$\frac{x^2}{1102}+\frac{y^2}{602}=1\,,$$
$$-\sqrt{105}<x<\sqrt{105}\,.$$

I wondered if points with positive integer-valued coordinates could be found that fit this curve. I found two.

Yours sincerely,
Bob Bertuello
(12 Pinewood Road,
Midsomer Norton,
Bath BA3 2RG.)

Dear Editor,

The Fibonacci sequence

Inspired by Bob Bertuello's letter in Volume 35, Number 1, and by the Editor's column in the same issue, I have just discovered the following results empirically. Are they true in general? Can any reader provide proofs?

Consider the Fibonacci sequence defined by $u_1=u_2=1$ and $u_n=u_{n-1}+u_{n-2}$ for $n>2$.

(i) If p is a prime, $p\neq 5$, then either $p\,|\,u_{p-1}$ and $p\,|\,u_p-1$ or $p\,|\,u_{p+1}$ and $p\,|\,u_p+1$.

(ii) The former holds if the last digit of p is 1 or 9 and the latter if it is 3 or 7.

(iii) If n is any power of 5 or $12k$ where k is composite, or 1, 2, 3 or 5, then $n\,|\,u_n$. This last rule will work for certain k only. Which ones?

I have tested results (i), (ii) and (iii) as far as $n=307$ using EXCEL®.

(iv) If $n\,|\,u_n$ holds for all powers of 5, then I think this may have a bearing on the extension of Bob Bertuello's result to the last m digits of the Fibonacci sequence numbers. I leave it to the reader to investigate further!

Also, following the Editor's column in Volume 34, Number 3, I found the following generalisation of the Perrin sequence.

(v) The relation $u_n=u_{n-k}+u_{n-k-1}+\cdots+u_{n-m}$ for $n\geq m$, with starting values $u_0=m$, $u_1=u_2=\cdots=u_{k-1}=0$, $u_k=k$, $u_n=u_{n-1}+u_{n-k}+1$ for $k<n<m$ has the same property.

(vi) The sequence for the starting values, if extended indefinitely, will also yield another set of sequences with the same property for each possible value of k. With $k=2$ this reduces to the well-known Lucas sequence less 1. The term for $n=0$ is ignored, i.e. we have $1,3,4,7,11,18,\ldots$ reducing to $0,2,3,6,10,17,\ldots$.

All these results follow using the method of proof outlined in the Editor's column in Volume 35, Number 1, much more simply than by the methods I used, which took me an enormous effort to discover! The proof of result (v) requires a transformation $v_n=u_n+1$ first. Essentially my methods were based on a consideration of the generating functions of the sequences.

(vii) It is also the case that $u_{ap}\equiv u_a \bmod(p)$ holds for all the sequences I have mentioned, which yields $p\,|\,u_p$ as a special case. This is another example of an embedding of a sequence within itself. Compare with Bob Bertuello's example in Volume 35, Number 1.

Since drafting this letter I have made a good deal of progress on result (iii). I think I can provide a proof that, if n is a power of 5, then $n\,|\,u_n$, and give details of multiples of 12 in most cases which fit this rule too. I still have more work to do on this. Some of the work here relates to Bob Bertuello's article too.

Yours sincerely,
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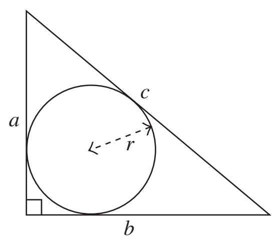
An international dispute



In about 300 BC, Greek mathematicians found a method for working out the radius of a circle inscribed in a right angle triangle, as illustrated:

$$r=\frac{b+a-c}{2}\,.$$

This was proved using Euclidean theorems (congruent triangles, circle theorems, and so on).

Somewhat later, in about AD 250, the Chinese mathematician Liu Hui gave a different method in *The Nine Chapters on the Mathematical Art*:

$$r=\frac{a\times b}{a+b+c}\,.$$

His proof uses the dissection and translation of various parts of triangles until the answer can be seen, a bit like doing a jigsaw puzzle.

Who are right, the Chinese or the Greeks?

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

35.9 *The box problem extended.* Given a square sheet of card, side 1 m, what is the maximum volume of the lid-less box obtained by cutting four equal squares from the corners of the sheet and folding up the four flaps? The box-maker uses the four cut-out squares and cuts boxes for these in the same proportions as the original box. And so on. Which value of x gives the maximum volume of all the boxes now, and by how much does the maximum volume increase?

(Submitted by Jonny Griffiths, Paston College, Norfolk)

35.10 If $x, y, z, t > 1$, prove that

$$\frac{1}{\log_x y^3 z} + \frac{1}{\log_y z^3 t} + \frac{1}{\log_z t^3 x} + \frac{1}{\log_t x^3 y} \geq 1 .$$

When does equality occur?

(Submitted by Mihály Bencze, Săcele-Négyfalu, Romania)

35.11 Let $a_1, \ldots, a_n$ $(n \geq 2)$ be distinct real numbers and let m be a positive integer smaller than n. Prove that

$$\sum_{k=1}^{n} \frac{a_k^m}{\prod_{j \neq k}(a_k - a_j)} = \begin{cases} 1 & \text{if } m = n-1 , \\ 0 & \text{if } m < n-1 . \end{cases}$$

(Submitted by M. A. Khan, Lucknow, India)

35.12 Prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

$$= \frac{1}{a_2 a_3} \begin{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix} \\ \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_2 & d_2 \\ a_3 & d_3 \end{vmatrix} \\ \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix} & \begin{vmatrix} a_3 & c_3 \\ a_4 & c_4 \end{vmatrix} & \begin{vmatrix} a_3 & d_3 \\ a_4 & d_4 \end{vmatrix} \end{vmatrix} ,$$

where $a_2, a_3 \neq 0$, and extend this to the determinant of a 5×5 matrix.

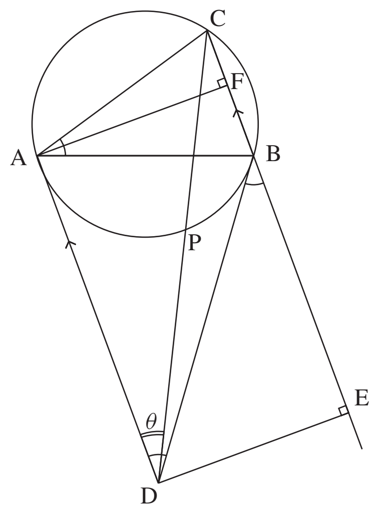
(Submitted by Hashemi Moosavi, Iran)

Solutions to Problems in Volume 35 Number 1

35.1 Triangles ABC (with angles denoted by A, B and C) and ABD are similar but not congruent and do not intersect. If AD is parallel to BC and $\theta = \angle\mathrm{ADC}$, express $\cot\theta$ in terms of $\cot \mathrm{A}$, $\cot \mathrm{B}$ and $\cot \mathrm{C}$.

If CD meets the circumcircle of triangle ABC again at P, which is inside the triangle ABD, deduce further that P has a symmetrical relationship with the triangle ABD.

Solution by J. A. Scott, who proposed the problem



We have

$$\angle\mathrm{BAD} = \mathrm{B} , \quad \angle\mathrm{ADB} = \angle\mathrm{DBE} = \mathrm{A} , \quad \angle\mathrm{ABD} = \mathrm{C}$$

and

$$\angle\mathrm{BCD} = \angle\mathrm{ADC} = \theta .$$

Let E, F be the feet of the perpendiculars from D, A respectively to BC (produced). Then

$$\mathrm{BC} = \mathrm{BF} + \mathrm{CF} = \mathrm{AF}(\cot \mathrm{B} + \cot \mathrm{C}) = \mathrm{DE}(\cot \mathrm{B} + \cot \mathrm{C}) ,$$

and

$$\mathrm{CE} = \mathrm{DE} \cot\theta = \mathrm{BC} + \mathrm{BE} = \mathrm{DE}(\cot \mathrm{B} + \cot \mathrm{C} + \cot \mathrm{A}) ,$$

so

$$\cot\theta = \cot \mathrm{A} + \cot \mathrm{B} + \cot \mathrm{C} .$$

Now, since they are angles in the same segment,

$$\angle\mathrm{PAB} = \angle\mathrm{PCB} = \theta$$

and

$$\angle PBA = \angle PCA = C - \theta\,,$$

so that

$$\angle PBD = \angle ABD - \angle PBA = C - (C - \theta) = \theta\,.$$

Hence

$$\angle PAB = \angle PBD = \angle PDA = \theta\,.$$

(The point P is called a 'Brocard point' of the triangle ABD.)

35.2 Let $a_0, a_1, \ldots, a_{2n}$ be positive real numbers. Prove that there exist at least $(n!)^2$ distinct permutations σ of $\{0, 1, \ldots, 2n\}$ such that

$$a_{\sigma(2n)}x^{2n} + \cdots + a_{\sigma(1)}x + a_{\sigma(0)} > 0 \qquad (1)$$

for every real number x.

Solution by Hassan Shah Ali, who proposed the problem

Without loss of generality, we may suppose that

$$a_{2n} \geq a_{2n-1} \geq \cdots \geq a_1 \geq a_0 > 0\,.$$

Since (1) holds for all permutations σ when $x \geq 0$, we only need consider (1) for $x < 0$ or, equivalently,

$$\sum_{i=0}^{2n} a_{\sigma(i)}(-x)^i$$

when $x > 0$. Consider all permutations σ such that $\sigma(0) = n$, $\sigma(2i) \in \{n+1, \ldots, 2n\}$ and $\sigma(2i-1) \in \{0, \ldots, n-1\}$ for $i = 1, \ldots, n$. There are $(n!)^2$ such permutations. Then $a_{\sigma(2i)} \geq a_{\sigma(0)} \geq a_{\sigma(2i-1)} > 0$, so that, with $x > 0$,

$$\left(\frac{a_{\sigma(2i)}}{a_{\sigma(0)}} - 1\right)x \geq 0 \geq \frac{a_{\sigma(2i-1)}}{a_{\sigma(0)}} - 1\,.$$

Hence, for $x > 0$ and $i = 1, \ldots, n$,

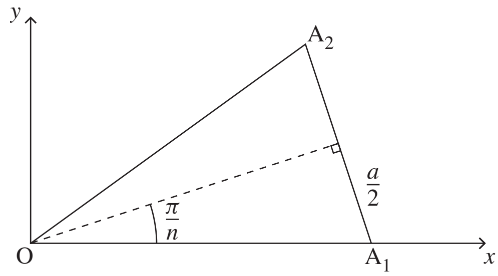
$$\frac{a_{\sigma(2i)}}{a_{\sigma(0)}}\,x^{2i} - \frac{a_{\sigma(2i-1)}}{a_{\sigma(0)}}\,x^{2i-1} \geq x^{2i} - x^{2i-1}\,.$$

Thus, for $x > 0$,

$$\begin{aligned}
&\sum_{i=0}^{2n} a_{\sigma(i)}(-x)^i \\
&\quad = a_{\sigma(0)}\left[\sum_{i=1}^{n}\left(\frac{a_{\sigma(2i)}}{a_{\sigma(0)}}\,x^{2i} - \frac{a_{\sigma(2i-1)}}{a_{\sigma(0)}}\,x^{2i-1}\right) + 1\right] \\
&\quad \geq a_{\sigma(0)}\sum_{i=0}^{2n}(-x)^i = a_{\sigma(0)}\frac{x^{2n+1}+1}{x+1} \\
&\quad > 0\,.
\end{aligned}$$

35.3 Let $A_1A_2\ldots A_n$ be a regular n-sided polygon. Show that $PA_1^2 + PA_2^2 + \cdots + PA_n^2$ is the same for all points P on the inscribed circle of the polygon.

Solution by Zhang Yun, who proposed the problem



Denote by a the length of a side of the polygon. Then, with axes as shown, for $r = 1, \ldots, n$,

$$A_r = \left(\frac{a}{2}\operatorname{cosec}\frac{\pi}{n}\cos\frac{2(r-1)\pi}{n},\ \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}\sin\frac{2(r-1)\pi}{n}\right).$$

With $\angle A_1OP = \theta$,

$$P = \left(\frac{a}{2}\cot\frac{\pi}{n}\cos\theta,\ \frac{a}{2}\cot\frac{\pi}{n}\sin\theta\right).$$

Then

$$\begin{aligned}
PA_r^2 &= \left(\frac{a}{2}\cot\frac{\pi}{n}\cos\theta - \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}\cos\frac{2(r-1)\pi}{n}\right)^2 \\
&\quad + \left(\frac{a}{2}\cot\frac{\pi}{n}\sin\theta - \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}\sin\frac{2(r-1)\pi}{n}\right)^2 \\
&= \frac{a^2}{4}\cot^2\frac{\pi}{n} + \frac{a^2}{4}\operatorname{cosec}^2\frac{\pi}{n} \\
&\quad - \frac{a^2}{2}\cot\frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n}\cos\frac{2(r-1)\pi}{n}\cos\theta \\
&\quad - \frac{a^2}{2}\cot\frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n}\sin\frac{2(r-1)\pi}{n}\sin\theta\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
&PA_1^2 + \cdots + PA_n^2 \\
&\quad = \frac{na^2}{4}\cot^2\frac{\pi}{n} + \frac{na^2}{4}\operatorname{cosec}^2\frac{\pi}{n} \\
&\quad - \frac{a^2}{2}\cot\frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n}\cos\theta \\
&\quad \times\left(1 + \cos\frac{2\pi}{n} + \cdots + \cos\frac{2(n-1)\pi}{n}\right) \\
&\quad - \frac{a^2}{2}\cot\frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n}\sin\theta\left(\sin\frac{2\pi}{n} + \cdots + \sin\frac{2(n-1)\pi}{n}\right).
\end{aligned}$$

Put

$$z = \cos\frac{2\pi}{n} + \mathrm{i}\sin\frac{2\pi}{n}\,.$$

Then

$$\begin{aligned}
&\left(1 + \cos\frac{2\pi}{n} + \cdots + \frac{2(n-1)\pi}{n}\right) \\
&\qquad + \mathrm{i}\left(\sin\frac{2\pi}{n} + \cdots + \sin\frac{2(n-1)\pi}{n}\right) \\
&\qquad = 1 + z + z^2 + \cdots + z^{n-1} \\
&\qquad = \frac{1-z^n}{1-z} \\
&\qquad = 0 \quad \text{because } z^n = 1\,.
\end{aligned}$$

Hence,

$$1+\cos\frac{2\pi}{n}+\cdots+\cos\frac{2(n-1)\pi}{n}$$

and

$$\sin\frac{2\pi}{n}+\cdots+\sin\frac{2(n-1)\pi}{n}$$

are both zero, and $\mathrm{PA}_1^2+\cdots+\mathrm{PA}_n^2$ is independent of θ.

35.4 Let ABCD be a regular tetrahedron. Show that $\mathrm{PA}^2+\mathrm{PB}^2+\mathrm{PC}^2+\mathrm{PD}^2$ is the same for all points P of the inscribed sphere of the tetrahedron. (The centre of the inscribed sphere is in the ratio $1:3$ from a face and the opposite vertex.)

Solution by Zhang Yun, who proposed the problem

Choose axes so that O is the centre of the inscribed sphere of the tetrahedron, D lies on the z-axis and AD lies in the plane $y=0$. Denote the length of the sides of the tetrahedron by a. Then

$$\mathrm{A}=\left(\frac{2}{3}\frac{a\sqrt{3}}{2},0,-\frac{a}{4}\sqrt{\frac{2}{3}}\right),$$
$$\mathrm{B}=\left(-\frac{1}{3}\frac{a\sqrt{3}}{2},\frac{a}{2},-\frac{a}{4}\sqrt{\frac{2}{3}}\right),$$
$$\mathrm{C}=\left(-\frac{1}{3}\frac{a\sqrt{3}}{2},-\frac{a}{2},-\frac{a}{4}\sqrt{\frac{2}{3}}\right),$$
$$\mathrm{D}=\left(0,0,\frac{3a}{4}\sqrt{\frac{2}{3}}\right).$$

We can put

$$\mathrm{P}=\left(\frac{a}{4}\sqrt{\frac{2}{3}}\cos\theta\cos\phi,\frac{a}{4}\sqrt{\frac{2}{3}}\sin\theta\cos\phi,\frac{a}{4}\sqrt{\frac{2}{3}}\sin\phi\right).$$

A calculation now shows that

$$\mathrm{PA}^2+\mathrm{PB}^2+\mathrm{PC}^2+\mathrm{PD}^2=\frac{5}{3}a^2.$$

Reviews

Indra's Pearls: The Vision of Felix Klein. By David Mumford, Caroline Series and David Wright. Cambridge University Press, 2002. Pp. 396. Hardback £29.95 (ISBN 0-521-35253-3).

This is a highly unusual and visually stunning book. It tells the story of the authors' collaboration over the last twenty years in a computer-aided exploration of the properties of what are known as limit sets of Kleinian groups. These fractal-like geometrical objects are generated by a process first described by Felix Klein over a century ago who confessed that 'the imagination seems to fail utterly when we try to form a mental image of the result.' The rapid development of computer graphics has provided a spur to current research in this area. This book supplies both the mathematical tool-kit (symmetry groups of transformations and complex numbers) and the computing pseudo-code to enable the reader to understand and draw these limit sets. It is written in an engrossing, homely style with interesting historical snapshots of the many mathematicians involved. Most of the symmetry groups are generated by two transformations of the form $z\mapsto(az+b)/(cz+d)$ with two associated intricate invariant sets of complex numbers. This produces a 'Swiss cheese'-like tile which, under the action of the group, tiles all of the complex plane except for the limit set, which is itself also invariant. The properties of this tiling are sensitive to the specific values of a,b,c,d in the transformations involved, and it is the striking range of behaviour which forms the main theme of the later chapters.

And what of the title? The 'world within a world within a world …' aspect of these geometrical configurations is suggestive to the authors of the ancient Buddhist myth of Indra's net in which the whole of space is covered by a spider-like web, the pearls of which are reflected in all of the other pearls in the web.

I found this beautiful book an absolutely compelling read. It is one of a very small number of books that successfully scales the Mount Everest of popularisation where the challenge is to guide the reader in the ascent from base-camp to the frontiers of research without skating over too many details.

Tonbridge School, Kent NICK LORD

Mathematical Olympiads: Problems and Solutions from around the World 1999–2000. Edited by Titu Andreescu and Zuming Feng. MAA, Washington, DC, 2002. Pp. 321. Paperback $28.50 (ISBN 0-88385-805-3).

This book is suitable for pupils in the last two years of their schooling. It contains challenging mathematical problems and their solutions for 24 national mathematics competitions and 8 regional contests in 1999. The countries vary from Belarus, France and Russia, to the UK, USA and Vietnam, revealing the standard of international mathematics, although, curiously, Germany is not amongst the countries included. The regional contests vary from Asian Pacific Mathematical Olympiad to the Czech and Slovak match (sounds more like ice-hockey!).

For any enthusiast the problems represent a treasure trove and the authors are to be congratulated on bringing so much together in a small space. Any school library will welcome a copy and mathematics teachers should set one of the contests for their classes at the end of term. In other subjects you cannot easily pit your wits against the intelligentsia of other countries, but in mathematics you can. It could be an eye-opener to the better pupils to realise what the standard is internationally.

For the year 2000, only the problems are given (a total of 213 from 22 national contests and 5 regional contests), but for the solutions, students have to wait until next year's volume!

The index classifies the problems as algebra, combinatorics, combinatorial geometry, combinatorial number theory, combinatorics and sets, graph theory, functional equations, geometry inequalities and number theory.

The solutions should only be looked at after the pupil has tried his or her hardest to solve the problem over an extended period, as the benefit of the solution is often directly proportional to the effort that has gone into solving the problem — a quick look at the clinical solution often leaves the difficulties unappreciated. The solver will often have had to erect quite a lot of 'scaffolding' that can eventually be knocked down — a reading of the given solution will not reveal the scaffolding.

There is a useful glossary of the main mathematical results needed although, for example, while pointing out that $A \geq G$ and $A \geq H$, it does not point out that the order is $A \geq G \geq H$ where A, G and H are the arithmetic, geometric and harmonic means respectively. Also, for the root mean square inequality, it would be nice if it was pointed out that this is saying that the dispersion about the mean of a set of numbers is positive, as then the inequality can be 'seen' to be true at a glance; or that the Cauchy inequality is saying that the cosine of the angle between two lines (vectors) is less than or equal to 1. Indeed, the book could be improved, in my view, if the authors showed how some of the solutions could be 'seen' (as in my experience good mathematicians often have strong visual imaginations), or if a statement was made of what the solutions 'boil down' to. For example, a number of the problems depend on finding a minimum for $f(x, y)$, but no two-dimensional surface plots — where the solution can be seen — are shown. Likewise a problem may boil down to similar triangles or to a problem requiring integer solutions subject to constraints. But these are only minor criticisms.

Edinburgh DAVID FORFAR

USA and International Mathematical Olympiads 2001. Edited by TITU ANDREESCU AND ZUMING FENG. MAA, Washington, DC, 2002. Pp. 120. Paperback $16.50 (ISBN 0-88385-089-6).

This book contains the problems used for the selection of the USA team in 2001, and for the IMO of that year, with solutions and hints for all of the problems. It also contains a guide to other material useful for advanced problem solving.

For each of the six IMO problems, a number of different solutions are given, illustrating the intrinsic beauty of such problems in that there are many different ways to prove the same theorem. For those who have attempted the problems, the multitude of solutions is of great interest, so that one can then go back over one's work and see those places where, if an idea had been pushed a little harder, it would have arrived at a solution. For those who are relatively new to problem solving, this book serves to illustrate the inherent beauty in it.

Although this book is of interest for the many solutions given to each of the problems, I would not recommend it to those who are attempting to improve their problem-solving skills, as it only contains a limited number of problems. Furthermore, it contains a few glaring typographical errors, so that those not familiar with the problems may occasionally find a solution difficult to follow. I would recommend it to those who are interested in the specific problems of IMO 2001, and those interested in problem solving in general. The book is more a record of events than a problem-solving textbook.

Student, Berkhamsted Collegiate School PAUL JEFFERYS

The Inquisitive Problem Solver. By PAUL VADERLIND, RICHARD GUY AND LOREN LARSON. MAA, Washington, DC, 2002. Pp. 344. Paperback $34.95 (ISBN 0-88385-806-1).

This book, aimed at those interested in mathematical problem solving, contains over two hundred engaging problems, and solutions. Included are hints for the harder problems, and in some cases possible generalisations are explored. The solutions given emphasise the necessity of proof and, for the most part, cut to the heart of the question, showing the intrinsic beauty of the mathematics involved.

The level of problems in this book is lower than that of the harder olympiad problems. The main requirements for solving the problems are clear thinking and insight, not mathematical knowledge, so the book is accessible to anyone interested in problem solving. The problems could also be useful stimulus material for teachers interested in stretching their brighter pupils.

I would definitely recommend this to anyone new to problem solving, as it presents the topic in a clear and interesting manner. However, those more experienced may not find it challenging enough.

Student, Berkhamsted Collegiate School PAUL JEFFERYS

Other books received

Inequalities from Complex Analysis. By JOHN P. D'ANGELO. MAA, Washington, DC, 2002. Pp. xvi+264. Hardback $39.95 (0-88385-033-8).

Ergodic Theory of Numbers. By KARMA DAJANI AND COR KRAAIKAMP. MAA, Washington, DC, 2002. Pp. 212. Paperback $39.95 (ISBN 0-88385-034-6).

Discrete Mathematics for New Technology. By R. GARNIER AND J. TAYLOR. Institute of Physics, Bristol, 2002. Pp. 749. Paperback £25.00 (ISBN 0-7503-06521).

Guide to Mathematical Methods. By JOHN GILBERT AND CAMILLA JORDAN. Palgrave Macmillan, Basingstoke, 2002. Pp. 419. Paperback £19.99 (ISBN 0-333-79444-3).

Mathematical Spectrum

2002/2003 Volume 35 Number 3

ISSN 0025-5653

Published by the Applied Probability Trust
Printed by Pear Tree Press Ltd, Stevenage, Herts, UK