Mathematical Spectrum

1993/4
Volume 26
Number 4

A magazine for students and teachers of mathematics in schools, colleges and universities

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

Volume 26 of *Mathematical Spectrum* will consist of four issues, of which this is the fourth. The first was published in September 1993, the second in November 1993 and the third in February 1994.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

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A New Format for *Mathematical Spectrum*

The next issue (Volume 27 Number 1) will have a new look to it. It will have a larger format and a new design, which we think that readers will find attractive. All the regular features will be there, but we hope to include more small items and for this we invite readers to send in their contributions. There will be three issues a year instead of the present four but, because of the larger format, this will not mean any reduction in the amount of material.

We would like to make *Mathematical Spectrum* as widely known as possible. If readers know of any individuals or schools who may be interested to receive a free sample copy, please let the editorial office know. Readers whose subscriptions expire with this volume will receive a renewal order form. We hope that you will all want to renew your subscription. We greatly value the support of our readers, and welcome your comments and suggestions.

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Some Sequences Euclid Would Have Liked

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The author is currently Associate Professor of Mathematics at University College Cork, in Ireland. His research interests include group theory and ring theory. He also works on the history of mathematics, and has written the first full-length biography of George Boole. In a lighter vein, he is also interested in recreational mathematics, puzzles and problem solving, and the humour of mathematics.

Although Euclid showed that the set of prime numbers is infinite, it is well known that there exist gaps 'as large as we please' between successive prime numbers. In other words, given any n, we can produce n consecutive natural numbers all of which are composite. This fact is usually demonstrated by considering the finite sequence of numbers

$$(n+1)!+2, \qquad (n+1)!+3, \qquad \ldots, \qquad (n+1)!+(n+1)$$

and observing that none of these numbers is prime, because $(n+i)!+i$ is divisible by i, for each i $(2 \leqslant i \leqslant n+1)$.

However, even for small n, the numbers in this sequence become very large, so we ask if it is possible to establish the result using smaller numbers. We could of course consider the sequence

$$(n+1)!-(n+1), \qquad (n+1)!-n, \qquad \ldots, \qquad (n+1)!-2,$$

but the saving here is not significant. However, we can achieve our objective with considerably smaller numbers as follows. For each n, define n^* to be the least common multiple of all the elements of the set $\{1, 2, 3, \ldots, n\}$. This sequence begins

$$1, \quad 2, \quad 6, \quad 12, \quad 60, \quad 60, \quad 420, \quad 840, \quad 2520, \quad 2520, \quad 27720, \quad \ldots$$

and does not seem to be very well known—for example, it is not to be found in Sloane's excellent *Handbook of Integer Sequences* (reference 1).

It is clear that n^* increases only when n is a prime power and, for $n = p^r$,

$$n^* = p(n-1)^*.$$

Another way of defining n^* is the following. Let p^{r_p} be the largest power of $p \leqslant n$, i.e.

$$p^{r_p} \leqslant n < p^{r_p+1}.$$

Then

$$n^* = \prod_{p \leqslant n} p^{r_p}.$$

Now consider the sequence

$$(n+1)^*+2, \qquad (n+1)^*+3, \qquad \ldots, \qquad (n+1)^*+(n+1).$$

By the same reasoning as before, this is a sequence of n successive composite numbers.

Note that, *a priori*, we have no reason to believe that $(n+1)^*+1$ is composite. In fact, since $(n+1)^*+1$ is not divisible by any prime less than $n+1$, we might suspect that it is more likely to be prime. Let us examine the first few distinct terms of the sequence $\{T_n\}$, where $T_n = n^*+1$:

$$\begin{aligned}
T_1 &= 2 \text{ (prime)},\\
T_2 &= 3 \text{ (prime)},\\
T_3 &= 7 \text{ (prime)},\\
T_4 &= 13 \text{ (prime)},\\
T_5 &= 61 \text{ (prime)},
\end{aligned}$$

$$
\begin{aligned}
T_7 &= 421 \text{ (prime)},\\
T_8 &= 841 = 29\times 29 \text{ (not just composite, even a square!)},\\
T_9 &= 2\,521 \text{ (prime)},\\
T_{11} &= 27\,721 = 19\times 1459 \text{ (composite)},\\
T_{13} &= 360\,361 = 89\times 4049 \text{ (composite)},\\
T_{16} &= 720\,721 = 71\times 10\,151 \text{ (composite)},\\
T_{17} &= 12\,252\,241 = 1693\times 7237 \text{ (composite)},\\
T_{19} &= 232\,792\,561 \text{ (prime)}.
\end{aligned}
$$

It is natural also to consider the sequence $\{K_n\}$, where $K_n = n^* - 1$, for $n > 2$, and to expect that it will be prime rich. Again we consider only distinct terms:

$$
\begin{aligned}
K_3 &= 5 \text{ (prime)},\\
K_4 &= 11 \text{ (prime)},\\
K_5 &= 59 \text{ (prime)},\\
K_7 &= 419 \text{ (prime)},\\
K_8 &= 839 \text{ (prime)},\\
K_9 &= 2\,519 = 11\times 229 \text{ (composite)},\\
K_{11} &= 27\,719 = 53\times 523 \text{ (composite)},\\
K_{13} &= 360\,359 = 173\times 2083 \text{ (composite)},\\
K_{16} &= 720\,719 = 31\times 67\times 347 \text{ (composite)},\\
K_{17} &= 12\,252\,239 = 29\times 647\times 653 \text{ (composite)},\\
K_{19} &= 232\,792\,559 \text{ (prime)}.
\end{aligned}
$$

Finally, we consider the sequences whose nth terms are given by $n^*+(n+1)$ and $n^*-(n+1)$, where, for obvious reasons, we take n to be even:

$$
\begin{aligned}
2^*+3 &= 5 \text{ (prime)},\\
4^*+5 &= 17 \text{ (prime)},\\
6^*+7 &= 67 \text{ (prime)},\\
8^*+9 &= 849 = 3\times 283 \text{ (composite)},\\
10^*+11 &= 2\,531 \text{ (prime)},\\
12^*+13 &= 27\,733 \text{ (prime)},\\
14^*+15 &= 360\,375 = 3\times 5^3\times 31^2 \text{ (composite)},\\
16^*+17 &= 720\,737 = 29^2\times 857 \text{ (composite)},
\end{aligned}
$$

$$
\begin{aligned}
4^*-5 &= 7 \text{ (prime)},\\
6^*-7 &= 53 \text{ (prime)},\\
8^*-9 &= 831 = 3\times 277 \text{ (composite)},\\
10^*-11 &= 2\,509 = 13\times 193 \text{ (composite)},\\
12^*-13 &= 27\,707 = 103\times 269 \text{ (composite)},\\
14^*-15 &= 360\,345 = 3\times 5\times 24\,023 \text{ (composite)},\\
16^*-17 &= 720\,703 \text{ (prime)}.
\end{aligned}
$$

It is easy to see that each of the sequences $\{(2n)^*+(2n+1)\}$ and $\{(2n)^*-(2n+1)\}$ contains infinitely many composites (can you see why?), but the following questions are likely to be difficult to answer.

1. Does the sequence $\{n^*+1\}$ contain infinitely many primes and infinitely many composites?
2. Does the sequence $\{n^*-1\}$ contain infinitely many primes and infinitely many composites?
3. Does the sequence $\{(2n)^*+(2n+1)\}$ contain infinitely many primes?
4. Does the sequence $\{(2n)^*-(2n+1)\}$ contain infinitely many primes?

We notice that in several cases n^*-1 and n^*+1 are twin primes, so we ask:

5. Are n^*-1 and n^*+1 ever again twin primes for $n > 22$?

Readers are invited to investigate further terms of these sequences, and I should love to be informed of their results.

Reference

1. N. J. Sloane, *The Handbook of Integer Sequences* (Academic Press, London, 1973).

Optimal Crossing of a Desert

WOLFRAM HINDERER, *University of Karlsruhe*

The author wrote this article while in his fifth semester at the University of Karlsruhe, Germany, studying technological mathematics. He likes music (Wagner operas, for example) and programming computer games.

In Volume 25 Number 3, pages 84–86, Dylan Gow considered the problem of a man who wants to cross a desert on a straight track and return to base. He has unlimited fuel at base, but the amount he can carry is limited by the capacity of his Land-Rover's tank. Dylan Gow showed how this was possible. He asked two further questions:

1. What is the most fuel-efficient method with an unlimited number of Land-Rovers?
2. What if the Land-Rover(s) need not return to base?

We first consider one Land-Rover which need not return to base. Whereas Dylan Gow's solution involved a sequence of points equally spaced $\frac{1}{4}d$ apart from the starting point, where d is the distance travelled by the Land-Rover using one tank of fuel, we shall consider a sequence of unequally spaced points starting at Q_0, the *destination* point.

Denote by Q_1 the point distant d on the road from Q_0. If one tank of fuel is available at Q_1, then the Land-Rover can travel from Q_1 to Q_0.

Now let Q_2 be the point on the road distant $\frac{1}{3}d$ back from Q_1. Suppose that two tanks of fuel are available at Q_2. The Land-Rover can fill up from empty at Q_2, travel to Q_1, deposit one third of a tank of fuel, return to Q_2 using the one third of a tank left, arrive at Q_2 empty, fill up there and return to Q_1, still with two thirds of a tank of fuel, where there is one third of a tank already deposited. Thus it can reach Q_1 still having one tank of fuel available. It can therefore travel from Q_2 to Q_0 on two tanks of fuel.

Now consider the point Q_3, which is a distance of $\frac{1}{5}d$ back from Q_2. Figure 1 illustrates how, if three tanks of fuel are available at Q_3, then the Land-Rover can deposit two tanks of fuel at Q_2. The point here is that the Land-Rover covers a total distance d between Q_3 and Q_2, and so uses one tank of fuel, just as it covered the total distance d between Q_2 and Q_1.

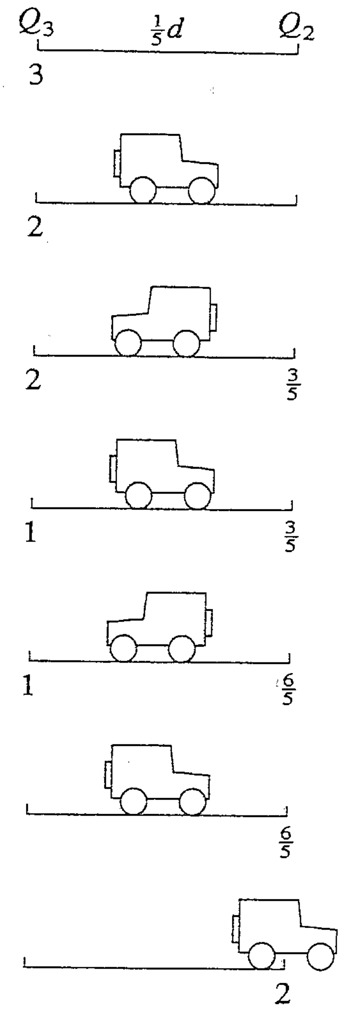

Figure 1

It is now clear how the procedure continues. The point Q_4 is distant $\frac{1}{7}d$ back from Q_3, and four tanks of fuel are needed at Q_4; Q_5 is distant $\frac{1}{9}d$ back from Q_4, and five tanks are needed at Q_5. And so on. Thus, with n tanks of fuel the Land-Rover can cover a range

$$d\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right).$$

If the Land-Rover must return to its starting point, then we replace the points $Q_0, Q_1, Q_2, \ldots$ by $R_0, R_1, R_2, \ldots$, where $R_1R_0 = \frac{1}{2}d$, $R_2R_1 = \frac{1}{4}d$, $R_3R_2 = \frac{1}{6}d$, etc. This is because the Land-Rover now has to traverse the road between R_1 and R_0 twice, between R_2 and R_1 four times, etc. Thus, with n tanks of fuel, it can cover a range

$$r_n = d\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right)$$

and return to its starting point. This compares with the answer

$$s_n = \tfrac{1}{4}(2+\log_2 n)d$$

by Dylan Gow's method, when n is a power of 2. The method given here is more economical than Dylan Gow's method when $n = 2$, and is in fact the optimal solution.

If we have an unlimited number of Land-Rovers and they are not required to return to base, then we can modify the procedure as follows. Let Q_0 be the destination as before, and let Q_1' be the point distant d back from Q_0 on the road. If one tank of fuel and a Land-Rover can reach Q_1', the final journey $Q_1'Q_0$ can be made. Let Q_2' be distant $\tfrac{1}{2}d$ back from Q_1'. If two Land-Rovers and two tanks of fuel can reach Q_2', then they can both proceed full to Q_1', where one Land-Rover can transfer its remaining half-tank of fuel to the other, which can proceed to Q_0. Suppose that three Land-Rovers and three tanks of fuel can reach Q_3', a distance $\tfrac{1}{3}d$ back from Q_2'. They set out full. At Q_2' one Land-Rover divides its remaining two thirds of a tankful of fuel equally between the other two. We now have two Land-Rovers at Q_2', both full, so one of them can reach Q_0. Thus, with n Land-Rovers, we can cover a range of

$$d\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$$

on n tanks of fuel.

If we have an unlimited number of Land-Rovers which must all return to base, a modification of this argument gives the range covered by n tanks of fuel to be

$$d\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right),$$

the same as for a single Land-Rover. If only one Land-Rover must return to base, the range covered by n tanks of fuel by this method is

$$d\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n+1}\right).$$

If all Land-Rovers but one must return to base, the range covered by n tanks of fuel using this method is

$$d\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right),$$

so again there is no advantage in the extra Land-Rovers.

Editor: See the letter from Harold Boas on page 122.

Sums of Powers of Integers

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1. Introduction

In this article we use combinatorial arguments, Bernoulli polynomials and Stirling numbers to find formulas for the sum of the kth powers of the first n integers.

Let n and k be non-negative integers ($n \geqslant 1$) and define

$$s_k(n) = 1^k + 2^k + \cdots + n^k.$$

It is, of course, well known that

$$s_0(n) = n, \qquad s_1(n) = \tfrac{1}{2}n(n+1), \qquad s_2(n) = \tfrac{1}{6}n(n+1)(2n+1).$$

Many writers have considered the problem of evaluating $s_k(n)$ for arbitrary k and n, and many different methods have been used, including the binomial theorem, generating functions, calculus, linear algebra, properties of arithmetic sequences, finite differences, and combinatorial arguments. Many of the results involve a recurrence formula, like

$$\sum_{r=0}^{k} \binom{k+1}{r} s_r(n) = (n+1)^{k+1} - 1, \tag{1}$$

from which we can successively compute $s_0(n)$, $s_1(n)$, $s_2(n)$, etc. Paul (reference 4) gave an interesting combinatorial proof of (1). One of the purposes of the present article is to give a combinatorial proof of a similar recurrence formula, namely,

$$\sum_{r=0}^{k} \binom{k+1}{r} (-1)^{k-r} s_r(n) = n^{k+1}. \tag{2}$$

Formulas (1) and (2) can, of course, easily be proved algebraically. Thus, for (1) we have

$$(n+1)^{k+1} - n^{k+1} = \binom{k+1}{k} n^k + \binom{k+1}{k-1} n^{k-1} + \cdots + \binom{k+1}{0} n^0,$$

$$n^{k+1} - (n-1)^{k+1} = \binom{k+1}{k} (n-1)^k + \binom{k+1}{k-1} (n-1)^{k-1} + \cdots + \binom{k+1}{0} (n-1)^0,$$

$$\cdots \quad = \quad \cdots \quad \cdots$$

$$2^{k+1}-1^{k+1} = \binom{k+1}{k}1^{k}+\binom{k+1}{k-1}1^{k-1}+\cdots+\binom{k+1}{0}1^{0}.$$

Add:

$$(n+1)^{k+1}-1 = \binom{k+1}{k}s_k(n)+\binom{k+1}{k-1}s_{k-1}(n)+\cdots+\binom{k+1}{0}s_0(n).$$

Similarly, for (2):

$$n^{k+1}-(n-1)^{k+1} = \binom{k+1}{k}n^{k}-\binom{k+1}{k-1}n^{k-1}+\cdots+(-1)^{k}\binom{k+1}{0}n^{0},$$

$$(n-1)^{k+1}-(n-2)^{k+1} = \binom{k+1}{k}(n-1)^{k}-\binom{k+1}{k-1}(n-1)^{k-1}+\cdots$$
$$+(-1)^{k}\binom{k+1}{0}(n-1)^{0},$$

$$\cdots \quad = \quad \cdots \quad \cdots$$

$$1^{k+1}-0^{k+1} = \binom{k+1}{k}1^{k}-\binom{k+1}{k-1}1^{k-1}+\cdots+(-1)^{k}\binom{k+1}{0}1^{0}.$$

Add:

$$n^{k+1} = \binom{k+1}{k}s_k(n)-\binom{k+1}{k-1}s_{k-1}(n)+\cdots$$
$$+(-1)^{k}\binom{k+1}{0}s_0(n).$$

We note (see reference 5, chapter 2) that (1) and (2) are essentially 'inverses' of each other.

One of the drawbacks to formulas like (1) and (2) is the fact that we need to know $s_r(n)$ for $r = 0, 1, \ldots, k-1$ in order to find $s_k(n)$. Anderson (reference 2) has found interesting and useful results that allow us to evaluate $s_k(n)$ if we know just $s_{k-1}(n)$. Another purpose of the present article is to show that the results of reference 2 can be proved easily by using elementary properties of Bernoulli polynomials. Finally, we examine a formula which expresses $s_k(n)$ in terms of Stirling numbers of the second kind. This formula, which is apparently not well known, also allows us to find $s_k(n)$ if we know just $s_{k-1}(n)$.

2. A combinatorial proof of (2)

For fixed k and n, define a 'lattice point' to be a $(k+1)$-tuple $(x_1, x_2, \ldots, x_{k+1})$, where each x_i is an integer and $1 \leqslant x_i \leqslant n$. Clearly there are n^{k+1} lattice points, since there are n choices for each of the $k+1$ components $x_1, \ldots, x_{k+1}$.

Let P_i be the property that $x_i \geqslant x_j$ for $j = 1,2,\ldots,k+1$; that is, P_i is the property that a lattice point has its maximum component in the ith position (and perhaps in other positions as well). It is clear that there are no lattice points that have none of the properties $P_1,P_2,\ldots,P_{k+1}$; that is, each point must have a maximum component in at least one position. This accounts for the 0 on the left-hand side of equation (3) below.

Now for $m = 1,2,\ldots,k+1$, we consider the number of lattice points having m of the properties. Let $N(P_{i_1},P_{i_2},\ldots,P_{i_m})$ denote the number of lattice points having properties $(P_{i_1},P_{i_2},\ldots,P_{i_m})$, and let

$$\sum\nolimits_m = \sum N(P_{i_1},P_{i_2},\ldots,P_{i_m}),$$

the sum over all $i_1,i_2,\ldots,i_m$ such that $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant k+1$. Then by the principle of inclusion–exclusion (reference 1, page 67), we have

$$0 = n^{k+1} - \sum\nolimits_1 + \sum\nolimits_2 - \sum\nolimits_3 + \cdots + (-1)^{k+1}\sum\nolimits_{k+1}. \tag{3}$$

Next we show that

$$\sum\nolimits_{k+1-r} = \binom{k+1}{r}(1^r+2^r+\cdots+n^r) = \binom{k+1}{r}s_r(n). \tag{4}$$

There are

$$\binom{k+1}{k+1-r} = \binom{k+1}{r}$$

ways to pick $k+1-r$ positions for the maximum components. If the maximum component is 1, there is one choice for the remaining r positions (they must all be 1); if the maximum component is 2, there are two choices for the each of the remaining r positions (they must be 1 or 2); in general, if the maximum component is j, there are j choices for each of the remaining r positions (the choices are $1,2,\ldots,j$). By adding up all the different cases, we get the right-hand side of (4). Then substituting from (4) into (3), we obtain equation (2). This completes the proof.

Note that if we replace $k+1$ by k and substract (2) from (1), we have

for k even: $$2\sum_{r=0}^{\frac{1}{2}(k-2)}\binom{k}{2r}s_{2r}(n) = (n+1)^k-n^k-1;$$

for k odd: $$2\sum_{r=1}^{\frac{1}{2}(k-1)}\binom{k}{2r-1}s_{2r-1}(n) = (n+1)^k-n^k-1,$$

which can be compared to similar results in reference 5, page 160.

3. Bernoulli polynomials

Using (1) or (2), we can easily prove by mathematical induction that $s_k(n)$ can be written as a polynomial in n of degree $k+1$, with constant term equal to 0. Thus

$$s_k(n) = \sum_{j=1}^{k+1} a_{k,j}n^j = a_{k,1}n + a_{k,2}n^2 + \cdots + a_{k,k+1}n^{k+1}. \tag{5}$$

Anderson (reference 2) and several other writers have proved, essentially by integrating (5), that

$$a_{k,j} = \frac{k}{j}a_{k-1,j-1} \quad (j = 2, 3, \ldots, k+1). \tag{6}$$

Since $s_k(1) = 1$, we also have

$$a_{k,1} + a_{k,2} + \cdots + a_{k,k+1} = 1. \tag{7}$$

Thus if we know the coefficients of $s_{k-1}(n)$, we can determine the coefficients $a_{k,j}$ of $s_k(n)$ for $j = 2, \ldots, k+1$ from (6) and then compute $a_{k,1}$ from (7).

In the writer's opinion, these results can be most easily proved by using an explicit formula for the coefficients $a_{k,j}$ in terms of Bernoulli numbers. This formula can be derived easily, using only elementary mathematics, in the following way. Define the Bernoulli polynomial $B_k(x)$ by

$$B_k(x) = \sum_{j=0}^{k} \binom{k}{j} B_{k-j}x^j, \tag{8}$$

where the numbers B_j (called the Bernoulli numbers) are defined by $B_0 = 1$ and, for $k > 1$,

$$\sum_{j=0}^{k-1} \binom{k}{j} B_j = 0. \tag{9}$$

Thus $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, etc. Using (8) and (9), it is an easy exercise to prove

$$B_{k+1}(x+1) - B_{k+1}(x) = (k+1)x^k. \tag{10}$$

The details of the proof are given at the end of this section. Now, if we sum both sides of (10) from $x = 0$ to $x = n$, we have the well-known formula: for $k > 0$,

$$\frac{B_{k+1}(n+1) - B_{k+1}}{k+1} = 1^k + 2^k + \cdots + n^k;$$

i.e.,

$$\frac{B_{k+1}(n)-B_{k+1}}{k+1} = 1^k+2^k+\cdots+(n-1)^k. \tag{11}$$

From (8) and (11) we have, for $k > 0$,

$$\begin{aligned} s_k(n) &= n^k + \frac{B_{k+1}(n)-B_{k+1}}{k+1} \\ &= n^k + \frac{1}{k+1}\sum_{j=1}^{k+1}\binom{k+1}{j}B_{k+1-j}n^j, \end{aligned} \tag{12}$$

which gives an explicit formula for the coefficients $a_{k,j}$ defined by (5); that is, for $j > 0$,

$$\begin{aligned} a_{k,j} &= \frac{1}{k+1}\binom{k+1}{j}B_{k+1-j} \quad (\text{if } j \neq k); \\ a_{k,k} &= 1-\frac{1}{2}\left(\frac{k+1}{k+1}\right) = \tfrac{1}{2} \quad (\text{if } k > 0). \end{aligned} \tag{13}$$

It follows easily from (13) that

$$a_{k,j} = \frac{k}{j}a_{k-1,j-1} \quad (j = 2, 3, \ldots, k+1).$$

Note that (13) gives us $a_{k,1} = B_k$ for $k > 1$.

We end this section with a proof of (10).

$$\begin{aligned} B_{k+1}(x+1)-B_{k+1}(x) &= \sum_{j=1}^{k+1}\binom{k+1}{j}B_{k+1-j}\{(x+1)^j-x^j\} \\ &= \sum_{j=1}^{k+1}\binom{k+1}{j}B_{k+1-j}\sum_{r=0}^{j-1}\binom{j}{r}x^r \\ &= \sum_{r=0}^{k}x^r\sum_{j=r+1}^{k+1}\binom{j}{r}\binom{k+1}{j}B_{k+1-j}. \end{aligned} \tag{14}$$

Since

$$\binom{j}{r}\binom{k+1}{j} = \frac{(k+1)!}{r!\,(j-r)!\,(k+1-j)!} = \binom{k+1}{r}\binom{k+1-r}{k+1-j},$$

we can write equation (14) as follows:

$$B_{k+1}(x+1)-B_{k+1}(x) = \sum_{r=0}^{k}\binom{k+1}{r}x^r\sum_{j=r+1}^{k+1}\binom{k+1-r}{k+1-j}B_{k+1-j}. \tag{15}$$

By (9), the inner sum on the extreme right of (15) is 0 unless $r = k$. When $r = k$, the inner sum is 1, and we see that (10) follows.

4. Stirling numbers

In proving that $s_k(n)$ is a polynomial in n of degree $k+1$, Anderson (reference 2) established

$$s_k(n) = c_0 n + c_1 \frac{n(n+1)}{2} + c_2 \frac{n(n+1)(n+2)}{3} + \cdots + c_k \frac{n(n+1)(n+2)\cdots(n+k)}{k+1}, \quad (16)$$

where the c_i's are integers depending on k such that

$$x^k = c_0 + c_1 x + c_2 x(x+1) + \cdots + c_k x(x+1)\cdots(x+k-1). \quad (17)$$

We point out in this section that the c_i's are $\pm$ Stirling numbers of the second kind and that (16) is actually a useful formula in determining $s_k(n)$.

The Stirling number of the second kind, $S(k, j)$, is defined by

$$x^k = \sum_{j=0}^{k} S(k, j) x(x-1)\cdots(x-j+1),$$

so

$$\begin{aligned}(-x)^k &= \sum_{j=0}^{k} S(k, j)(-x)(-x-1)\cdots(-x-j+1) \\ &= \sum_{j=0}^{k} (-1)^j S(k, j) x(x+1)\cdots(x+j-1). \quad (18)\end{aligned}$$

These numbers are very well known; see reference 3, chapter 5, for a good treatment of the many properties and formulas involving $S(k, j)$. A useful recurrence relation is

$$S(k+1, j) = S(k, j-1) + jS(k, j) \quad (j \geqslant 1), \quad (19)$$

with

$$S(k, k) = 1; \qquad S(k, 0) = 0 \quad (k > 0); \qquad S(k, 1) = 1 \quad (k \geqslant 1).$$

Comparing (17) and (18), we see that, for fixed k,

$$c_j = (-1)^{k+j} S(k, j). \quad (20)$$

Thus if we let

$$n^{(j+1)} = n(n+1)\cdots(n+j),$$

we can write, for $k > 0$,

$$s_k(n) = \sum_{j=1}^{k} \frac{(-1)^{k+j} S(k,j)}{j+1} n^{(j+1)}. \tag{21}$$

Furthermore, because of (19) and (21), if we know just $s_{k-1}(n)$ we can write down $s_k(n)$. For example, we have the following:

$$s_1(n) = \tfrac{1}{2}n^{(2)}, \qquad s_2(n) = -\tfrac{1}{2}n^{(2)} + \tfrac{1}{3}n^{(3)},$$

$$s_3(n) = \tfrac{1}{2}n^{(2)} - n^{(3)} + \tfrac{1}{4}n^{(4)}, \qquad s_4(n) = -\tfrac{1}{2}n^{(2)} + \tfrac{7}{3}n^{(3)} - \tfrac{3}{2}n^{(4)} + \tfrac{1}{5}n^{(5)},$$

and so on.

References

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3. L. Comtet, *Advanced Combinatorics* (Reidel, Dordrecht, 1974).
4. J. L. Paul, On the sum of the kth powers of the first n integers, *American Mathematical Monthly* **78** (1971), 271–272.
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Edward James Hannan

29 January 1921–7 January 1994

Ted Hannan, one of the four founding Trustees of the Applied Probability Trust, died suddenly of a heart attack on the evening of 7 January 1994 in Canberra. He was an international authority in time series analysis, and a much valued member of the Advisory Board of *Mathematical Spectrum*. He is survived by his wife Irene and their four children.

To commemorate him, his colleagues hope to establish (through the Australian Academy of Science, of which he was elected Fellow in 1970) a Hannan Medal and Lecture in his honour. Cheques for this purpose made out to 'The Australian Academy of Science' should be sent to Professor J. Gani, Stochastic Analysis Group SMS, Australian National University, Canberra ACT 0200, Australia.

Falling Down a Polygonal Well

TAMARA CURNOW, *Penwith College*

When she wrote this article, Tamara Curnow was a student at Penwith College, where she participated in a multitude of activities. She played the flute in many local and county ensembles, as well as having a love of sport. She often travelled great distances to play hockey or squash in nationwide competitions. Between these pursuits she found time to study for her A-levels and is now reading mathematics at Trinity College, Cambridge.

1. The problem

Take a unit circle, and inscribe an equilateral triangle. In the triangle, inscribe a second circle, then a square, another circle, a regular pentagon and continue in this way adding a circle and the next regular polygon, as shown in figure 1.

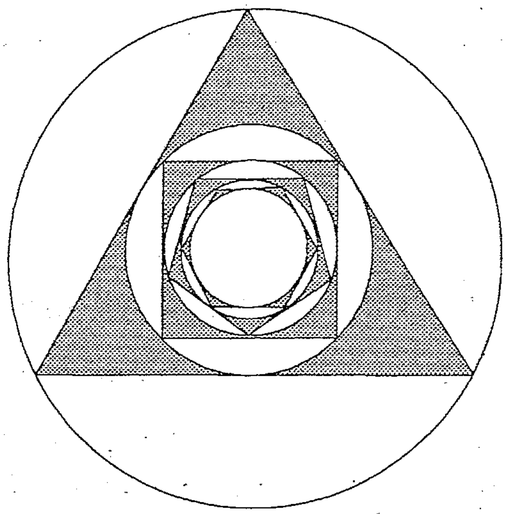


Figure 1

The limiting shape will obviously be either a circle or a point; which will it be, and if it is a circle, what will be its radius?

By simple geometric reasoning, it can be seen that the radius of circle $n-1$, where $n > 2$, is

$$\cos\frac{\pi}{3}\cos\frac{\pi}{4}\cos\frac{\pi}{5}\cos\frac{\pi}{6}\cdots\cos\frac{\pi}{n}. \tag{1}$$

To find the radius of the limiting circle we take the limit of this product as n tends to infinity. Let us denote this infinite product by P. We could try to evaluate P by simple multiplication, observing the value to which the radius of the nth circle appears to converge as $n \to \infty$. However the rate of convergence is very slow as the results in table 1 show.

Table 1

n	Radius	n	Radius
2	1	8	0.206 190 3868
3	0.5	9	0.193 755 5849
4	0.353 553 3906	10	0.184 272 5116
5	0.286 030 7014	11	0.176 808 1801
6	0.247 709 8537	12	0.170 783 5875
7	0.223 178 8664	13	0.165 820 9268

A calculation performed by means of a programmable calculator indicates that the limiting radius is approximately 0.11496, but the error is likely to be considerable owing to the large number of terms required. To achieve a more accurate result, it would appear that we must abandon the pure 'number crunching' method and first obtain a much more rapidly converging process.

2. The power series for $\ln\cos x$

By (1), the logarithm of the $(n-1)$th radius is

$$\ln\cos\frac{\pi}{3}+\ln\cos\frac{\pi}{4}+\cdots+\ln\cos\frac{\pi}{n}. \tag{1}$$

We therefore investigate the function $\ln\cos x$ which is defined for $|x|<\frac{1}{2}\pi$ and which is the sum of a rapidly converging power series. To obtain this series we first aim to express $\cos x$ as an infinite product.

For all x,

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots,$$

and forming the partial sum consisting of the first $n+1$ terms we have

$$\begin{aligned} S_n(x) &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!} \\ &= (1-\alpha_1 x^2)(1-\alpha_2 x^2)\cdots(1-\alpha_n x^2), \end{aligned}$$

since $S_n(x)$ is a polynomial of degree n in x^2. Moreover, the roots of $S_n(x)$ are given by

$$x^2=\frac{1}{\alpha_i}\quad (i=1,2,\ldots,n).$$

Now, as $n\to\infty$, $S_n(x)\to\cos x$ and one would expect the roots of $S_n(x)$ to approach the roots of $\cos x$, namely

$$\pm\tfrac{1}{2}\pi,\qquad \pm\tfrac{3}{2}\pi,\qquad \pm\tfrac{5}{2}\pi,\qquad \ldots,$$

so that the α_i values approach

$$\frac{4}{\pi^2}, \qquad \frac{4}{9\pi^2}, \qquad \frac{4}{25\pi^2}, \qquad \ldots .$$

This now suggests that

$$\cos x = \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{9\pi^2}\right)\left(1 - \frac{4x^2}{25\pi^2}\right)\cdots,$$

i.e. that

$$\cos x = \prod_{r=1}^{\infty}\left(1 - \frac{4x^2}{(2r-1)^2\pi^2}\right). \tag{2}$$

Of course the way we have arrived at the striking formula (2) was purely formal. Nevertheless (2) is true, but a rigorous proof requires quite advanced analysis and even a sketch would take us too far from the objective of this article. However, it would be perverse not to state also the corresponding product for the sine, namely

$$\sin x = x\prod_{r=1}^{\infty}\left(1 - \frac{x^2}{r^2\pi^2}\right),$$

although it is not needed in the present context. This formula, rather than (2), is actually the one that is usually proved in text books, with (2) left as an exercise to be approached in a similar way. See for instance reference 1, p. 209.

If $4x^2/\pi^2 < 1$, i.e. $|x| < \frac{1}{2}\pi$, then each factor in (2) is positive, so that its logarithm exists. Hence

$$\ln \prod_{r=1}^{\infty}\left(1 - \frac{4x^2}{(2r-1)^2\pi^2}\right) = \sum_{r=1}^{\infty} \ln\left(1 - \frac{4x^2}{(2r-1)^2\pi^2}\right)$$

and, since the logarithm is continuous, taking the limit of each side as $n \to \infty$, we obtain

$$\ln\cos x = \sum_{r=1}^{\infty} \ln\left(1 - \frac{4x^2}{(2r-1)^2\pi^2}\right). \tag{3}$$

Now recall that, for $|y| < 1$,

$$\ln(1-y) = -y - \tfrac{1}{2}y^2 - \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 - \cdots .$$

Hence (3) can be written

$$
\begin{aligned}
\ln\cos x = \; & -\frac{4}{\pi^2}x^2 - \frac{4^2}{2\pi^4}x^4 - \frac{4^3}{3\pi^6}x^6 - \cdots \\
& -\frac{4}{3^2\pi^2}x^2 - \frac{4^2}{2\times 3^4\pi^4}x^4 - \frac{4^3}{3\times 3^6\pi^6}x^6 - \cdots \\
& -\frac{4}{5^2\pi^2}x^2 - \frac{4^2}{2\times 5^4\pi^4}x^4 - \frac{4^3}{3\times 5^6\pi^6}x^6 - \cdots \\
& - \;\cdots\; - \;\cdots\; - \;\cdots\; - \cdots .
\end{aligned} \tag{4}
$$

But, if the terms a_{ij} of a double series

$$
\begin{aligned}
& a_{11}+a_{12}+a_{13}+\cdots \\
& +a_{21}+a_{22}+a_{23}+\cdots \\
& +a_{31}+a_{32}+a_{33}+\cdots \\
& +\cdots+\cdots+\cdots+\cdots
\end{aligned}
$$

have constant sign, then the series can be summed by rows or by columns, i.e.

$$
\begin{aligned}
& (a_{11}+a_{12}+a_{13}+\cdots)+(a_{21}+a_{22}+a_{23}+\cdots)+(a_{31}+a_{32}+a_{33}+\cdots)+\cdots \\
& = (a_{11}+a_{21}+a_{31}+\cdots)+(a_{12}+a_{22}+a_{32}+\cdots)+(a_{13}+a_{23}+a_{33}+\cdots)+\cdots
\end{aligned} \tag{5}
$$

when one side in (5) is known to exist. Now, in (4), the sum by rows exists and is $\ln\cos x$, by (3). Hence, summing (4) by columns, we have

$$
\begin{aligned}
\ln\cos x = & -\frac{4x^2}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)-\frac{4^2x^4}{2\pi^4}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right) \\
& -\frac{4^3x^6}{3\pi^6}\left(1+\frac{1}{3^6}+\frac{1}{5^6}+\cdots\right)-\cdots .
\end{aligned}
$$

It is easy to express the bracketed sums in terms of the well-known zeta function

$$\zeta(k) = 1+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\cdots$$

which exists when $k > 1$, since then the series on the right converges; for

$$
\begin{aligned}
1+\frac{1}{3^k}+\frac{1}{5^k}+\cdots &= \left(1+\frac{1}{2^k}+\frac{1}{3^k}+\cdots\right)-\left(\frac{1}{2^k}+\frac{1}{4^k}+\frac{1}{6^k}+\cdots\right) \\
&= \left(1+\frac{1}{2^k}+\frac{1}{3^k}+\cdots\right)-\frac{1}{2^k}\left(1+\frac{1}{2^k}+\frac{1}{3^k}+\cdots\right)
\end{aligned}
$$

$$= \zeta(k) - \frac{1}{2^k}\zeta(k)$$

$$= \frac{2^k - 1}{2^k}\zeta(k).$$

Thus

$$\ln\cos x = -\frac{3}{\pi^2}\zeta(2)x^2 - \frac{15}{2\pi^4}\zeta(4)x^4 - \frac{63}{3\pi^6}\zeta(6)x^6 - \cdots$$

$$- \frac{2^{2n}-1}{n\pi^{2n}}\zeta(2n)x^{2n} - \cdots$$

$$= -\sum_{n=1}^{\infty} \frac{2^{2n}-1}{n\pi^{2n}}\zeta(2n)x^{2n}. \tag{6}$$

3. Evaluation of $\zeta(2n)$

By deriving the power series expansion of $\ln\cos x$ in a different way and equating the two sets of coefficients we shall obtain $\zeta(2n)$ as a rational multiple of π^{2n}.

This time we start with the power series expansion of $\tan x$ (the derivative of $-\ln\cos x$). For $|y| < 1$,

$$\frac{1}{1-y} = 1 + y + y^2 + \cdots$$

and so, for sufficiently small $|x|$,

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left[1 + \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right) + \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)^2 + \cdots\right]$$

$$= x + \left(-\frac{1}{3!} + \frac{1}{2!}\right)x^3 + \left(\frac{1}{5!} - \frac{1}{3!\,2!} - \frac{1}{4!} + \frac{1}{2!^2}\right)x^5 + \cdots$$

$$= x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots. \tag{7}$$

Coefficients of higher powers of x can, of course, be calculated in the same way, though with ever-increasing trouble. However, what is remarkable is that there is a formula for the general coefficient. In fact, for $|x| < \frac{1}{2}\pi$,

$$\tan x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} b_{2n} 2^{2n}(2^{2n}-1)}{(2n)!} x^{2n-1}, \tag{8}$$

where the *Bernoulli numbers* b_m $(m = 1,2,\ldots)$ are defined by

$$b_0 = 1, \qquad \sum_{i=0}^{m} \binom{m+1}{i} b_i = 0 \quad (m = 1,2,\ldots).$$

Thus for $m > 0$, b_m is defined in terms of $b_0,\ldots,b_{m-1}$ and, although the first few numbers are easily obtained, later ones require nasty calculations. It may be shown that

$$b_3 = b_5 = b_7 = \ldots = 0$$

and that the non-zero Bernoulli numbers are alternately positive and negative. Table 2 exhibits the otherwise rather erratic behaviour of the sequence of Bernoulli numbers.

Table 2

b_0	1	b_8	$-\frac{1}{30}$
b_1	$-\frac{1}{2}$	b_{10}	$\frac{5}{66}$
b_2	$\frac{1}{6}$	b_{12}	$-\frac{691}{2730}$
b_4	$-\frac{1}{30}$	b_{14}	$\frac{7}{6}$
b_6	$\frac{1}{42}$	b_{16}	$-\frac{3617}{510}$

The proof of (8) is (like that of (2)) unsuitable for this article; the expansion is obtained on pp. 478–9 of reference 2.

Since a power series may be integrated term by term within its interval of convergence, we have, from (8), for $|x| < \frac{1}{2}\pi$,

$$\begin{aligned}\ln\cos x &= -\int_0^x \tan t \; dt = -\sum_{n=1}^{\infty} \int_0^x \frac{(-1)^{n-1} b_{2n} 2^{2n}(2^{2n}-1)}{(2n)!} t^{2n-1} \, dt \\ &= -\sum_{n=1}^{\infty} \frac{(-1)^{n-1} b_{2n} 2^{2n}(2^{2n}-1)}{(2n)!\, 2n} x^{2n}.\end{aligned} \tag{9}$$

But power series expansions are unique, so the coefficients of x^{2n} in (6) and (9) are identical. Thus, for $n = 1,2,\ldots,$

$$\zeta(2n) = \frac{(-1)^{n-1} b_{2n} 2^{2n-1} \pi^{2n}}{(2n)!}. \tag{10}$$

A short diversion is now of interest. In (10) we have evaluated

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\cdots$$

as a multiple of the Bernoulli number b_{2n}. It is well known that

$$S(n,N) = 1+2^n+3^n+\cdots+N^n$$

is a polynomial in N of degree $n+1$. (For example $S(1,N) = \frac{1}{2}N(N+1)$ and $S(2,N) = \frac{1}{6}N(N+1)(2N+1)$.) What is much less familiar is that the coefficients of the polynomial can be expressed in terms of Bernoulli numbers:

$$S(n,N) = N^n + \frac{1}{n+1}\sum_{i=1}^{n+1}\binom{n+1}{i}b_{n+1-i}N^i.$$

This is proved in the article by Howard in this issue (pages 103–109). See table 3.

4. Evaluation of P

We obtain P by calculating

$$\ln P = \ln\left(\cos\frac{\pi}{3}\cos\frac{\pi}{4}\cos\frac{\pi}{5}\cdots\right) = \sum_{r=3}^{\infty}\ln\cos\frac{\pi}{r}.$$

By (9)

$$\ln P = \sum_{r=3}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n b_{2n}2^{2n}(2^{2n}-1)}{(2n)!\,2n}\left(\frac{\pi}{r}\right)^{2n} \tag{11}$$

and, as we have noted before, we can change the order of summation since a double series of negative terms can be summed by rows or by columns. Hence

$$\begin{aligned}\ln P &= \sum_{n=1}^{\infty}\sum_{r=3}^{\infty}\frac{(-1)^n b_{2n}2^{2n}(2^{2n}-1)}{(2n)!\,2n}\left(\frac{\pi}{r}\right)^{2n}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n b_{2n}2^{2n}(2^{2n}-1)\pi^{2n}}{(2n)!\,2n}\left(\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\cdots\right)\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n b_{2n}2^{2n}(2^{2n}-1)\pi^{2n}}{(2n)!\,2n}\left(\zeta(2n)-1-\frac{1}{2^{2n}}\right),\end{aligned} \tag{12}$$

where $\zeta(2n)$ is given by(10).

For the purposes of calculation it is actually convenient to modify (11) by writing , for a suitable $k > 3$,

$$\ln P = \ln\cos\frac{\pi}{3} + \ln\cos\frac{\pi}{4} + \cdots + \ln\cos\frac{\pi}{k-1} + \sum_{r=k}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n b_{2n} 2^{2n}(2^{2n}-1)}{(2n)!\,2n}\left(\frac{\pi}{r}\right)^{2n},$$

so that (12) is replaced by

$$\ln P = \ln\cos\frac{\pi}{3} + \ln\cos\frac{\pi}{4} + \cdots + \ln\cos\frac{\pi}{k-1} + \sum_{n=1}^{\infty}\frac{(-1)^n b_{2n} 2^{2n}(2^{2n}-1)\pi^{2n}}{(2n)!\,2n}\left(\zeta(2n) - 1 - \frac{1}{2^{2n}} - \cdots - \frac{1}{(k-1)^{2n}}\right). \tag{13}$$

The point of this manipulation is that, when k is reasonably large, the terms

$$\zeta(2n) - 1 - \frac{1}{2^{2n}} - \cdots - \frac{1}{(k-1)^{2n}}$$

are so small that the infinite series $\sum_{n=1}^{\infty}$ in (13), which represents

$$\sum_{r=k}^{\infty} \ln\cos\frac{\pi}{r},$$

is well approximated by a finite sum $\sum_{n=1}^{N}$.

The results we obtained are shown in table 4, from which it can be seen that the limit of the product is certainly very close to 0.114 942 0446. Of course this is only a numerical value, but it is very unlikely that P can be expressed in terms of a finite number of standard mathematical constants.

Table 3

$2n$	Bernoulli number	Zeta function	Coefficient of x^{2n} in expansion (9) of $-\ln\cos x$
2	$\frac{1}{6}$	$\frac{1}{6}\pi^2$	$\frac{1}{2}$
4	$-\frac{1}{30}$	$\frac{1}{90}\pi^4$	$\frac{1}{12}$
6	$\frac{1}{42}$	$\frac{1}{945}\pi^6$	$\frac{1}{45}$
8	$-\frac{1}{30}$	$\frac{1}{9450}\pi^8$	$\frac{17}{2520}$
10	$\frac{5}{66}$	$\frac{1}{93\,555}\pi^{10}$	$\frac{31}{14175}$
12	$-\frac{691}{2730}$	$\frac{691}{63\,851\,288}\pi^{12}$	$\frac{691}{935\,550}$
14	$\frac{7}{6}$	$\frac{2}{18\,243\,225}\pi^{14}$	$\frac{10\,922}{42\,567\,525}$

Then for the shown values of N and k we have the results in table 4.

Table 4

N	k	$\sum_{r=3}^{k-1} \ln\cos\frac{\pi}{r}$	$\sum_{r=k}^{\infty} \ln\cos\frac{\pi}{r}$		P
5	8	−1.499 7817	−0.663 5455	−2.163 3272	0.114 920 464 60
5	12	−1.732 6899	−0.430 6374	−2.163 3272	0.114 942 044 87
5	16	−1.844 3333	−0.318 9940	−2.163 3272	0.114 942 044 84
5	20	−1.909 9499	−0.253 3773	−2.163 3272	0.114 942 044 81
6	20	−1.909 9499	−0.253 3773	−2.163 3272	0.114 942 044 73
7	20	−1.909 9499	−0.253 3773	−2.163 3272	0.114 942 044 66

5. Final Remarks

(a) In the process of finding a good approximation to P we used only a relatively small number of terms of the infinite series in (13). Therefore, it was not absolutely necessary to find the general term in the expansion (9) or the general expression (10) for $\zeta(2n)$. Since these both arose from the power series for $\tan x$, it would have been conceivable to use the elementary method of (7) rather than the difficult to prove result (8). However, the labour necessary to achieve a reasonable degree of accuracy in the calculation of P would have been enormous.

(b) A similar problem to the one we have considered is to find the limiting circle when regular *circumscribing* polygons are alternated with circles, the first of which again has radius 1. It is easy to see that the radius of the limiting circle is $1/P$, which is approximately 8.700 036 6545.

References

1. R. L. Goodstein, *Complex Functions* (McGraw-Hill, London, 1965).
2. M. Spivak, *Calculus*, pp. 478–484 (W. A. Benjamin, Menlo Park, CA, 1967).

What is

$$\left[\frac{1}{\sum_{n=2}^{\infty}\left[\frac{n}{x}\left[\frac{x}{n}\right]\right]}\right],$$

where x is a natural number and [ ] stands for the integral part?

KENICHIRO KASHIHARA
(Kanagawa, Japan)

Computer Column

MIKE PIFF

A universal Turing machine—the listing

Here is the listing described in the last issue of *Mathematical Spectrum.*

```
MODULE Turing; (* (C) Mike Piff, 1992 *)
FROM InOut IMPORT Read, Write, WriteString,
  WriteCard, WriteLongInt, WriteLongCard,
  OpenInput, CloseInput, WriteLn;
FROM Utils IMPORT KeyPressed;
FROM Storage IMPORT ALLOCATE;
CONST
  Left="L"; StayPut="S"; Right="R"; Blank="␣";
  VisBlank=CHR(254); MinState=Blank;
  MaxState="~"; UndefinedState=MinState;
  MinSymbol=Blank; MaxSymbol="~";
  Undefined=MaxSymbol;
TYPE
  Symbols=[MinSymbol..MaxSymbol];
  States=[MinState..MaxState];
  Directions=CHAR;
  SquarePointers=POINTER TO Squares;
  Squares=RECORD
          Symbol:Symbols;
          Next, Last:SquarePointers;
        END;
  Tapes=SquarePointers;
  Transitions=RECORD
            NewSymbol:Symbols;
            Direction:Directions;
            NewState:States;
          END;
  TransitionMatrices=ARRAY States,
    Symbols OF Transitions;
  TuringMachines=RECORD
              Position:Tapes;
              Matrix:TransitionMatrices;
              InitState, State:States;
              TransitionMade:BOOLEAN;
              Square:LONGINT;
              Transition:LONGCARD;
            END;
  Machines=POINTER TO TuringMachines;
VAR
  T:Machines;
PROCEDURE NewSquare(VAR sp:SquarePointers;
  x:Symbols; next, last:SquarePointers);
  BEGIN
    NEW(sp);
    WITH sp↑ DO
      Next:=next; Last:=last; Symbol:=x;
    END;
  END NewSquare;
PROCEDURE ReadState(VAR s:States);
VAR
  x:CHAR;
  BEGIN
    REPEAT
      Read(x);
    UNTIL (x>UndefinedState)&(x<=MaxState);
    s:=States(x);
  END ReadState;
PROCEDURE ReadSymbol(VAR s:Symbols);
VAR
  x:CHAR;
  BEGIN
    REPEAT
      Read(x);
      IF x=VisBlank THEN x:=Blank; END;
    UNTIL (x>=MinSymbol)&(x<=Undefined);
    s:=Symbols(x);
  END ReadSymbol;
PROCEDURE WriteSymbol(s:Symbols);
  BEGIN
    IF s=Blank THEN Write(VisBlank);
    ELSE Write(s); END;
  END WriteSymbol;
PROCEDURE ReadDirection(VAR d:Directions);
VAR
  x:CHAR;
  BEGIN
    REPEAT
      Read(x); x:=CAP(x);
    UNTIL (x=Left)OR(x=Right)OR(x=StayPut);
    d:=Directions(x);
  END ReadDirection;
PROCEDURE MoveLeft(VAR p:SquarePointers);
  BEGIN
    IF p↑.Last=NIL THEN
      NewSquare(p↑.Last,Blank,p,NIL);
    END;
    p:=p↑.Last;
  END MoveLeft;
PROCEDURE MoveRight(VAR p:SquarePointers);
  BEGIN
    IF p↑.Next=NIL THEN
      NewSquare(p↑.Next,Blank,NIL,p);
    END;
    p:=p↑.Next;
  END MoveRight;
PROCEDURE WriteTransition(s:States;x:Symbols;
  t:Transitions);
  BEGIN
    WITH t DO
      WriteString("State␣"); Write(s);
      WriteString("␣reading␣"); WriteSymbol(x);
      IF x#NewSymbol THEN
        WriteString(";␣changes␣it␣to␣");
        WriteSymbol(NewSymbol);
      END;
      IF Direction#StayPut THEN
        WriteString(";␣moves␣");
        IF Direction=Left THEN
          WriteString("left");
        ELSE
          WriteString("right");
        END;
      END;
      IF NewState#s THEN
        WriteString(";␣enters␣state␣");
        Write(NewState);
      END;
    END;
  END WriteTransition;
PROCEDURE ReadTransitions
  (VAR Matrix:TransitionMatrices);
VAR
  s:States;
  x:Symbols;
  i:CARDINAL;
```

```
  BEGIN
    FOR s:=MinState TO MaxState DO
      FOR x:=MinSymbol TO MaxSymbol DO
        WITH Matrix[s,x] DO
          NewSymbol:=Undefined;
          Direction:=StayPut;
          NewState:=UndefinedState;
        END;
      END;
    END;
    i:=0;
    REPEAT
      INC(i);
      ReadState(s);
      IF s≠Undefined THEN
        ReadSymbol(x);
        ReadSymbol(Matrix[s,x].NewSymbol);
        ReadDirection(Matrix[s,x].Direction);
        ReadState(Matrix[s,x].NewState);
        WriteString("T");
        WriteCard(i,1); WriteString(":␣");
        WriteTransition(s,x,Matrix[s,x]);
        WriteLn;
      END;
    UNTIL s=Undefined;
  END ReadTransitions;
PROCEDURE ReadTape(Tape:Tapes);
VAR
  rover:SquarePointers;
  x:Symbols;
  BEGIN
    WriteString("Tape:");
    rover:=Tape;
    REPEAT
      ReadSymbol(x);
      IF x≠Undefined THEN
        rover↑.Symbol:=x;
        MoveRight(rover);
      END;
    UNTIL x=Undefined;
  END ReadTape;
PROCEDURE ReadMachine(VAR T:Machines);
  BEGIN
    WITH T↑ DO
      WriteString
      ("Give␣name␣of␣machine␣file␣(CON␣for␣console)␣");
      OpenInput("DAT");
      WriteString("Initial␣state␣is␣");
      ReadState(InitState); Write(InitState); WriteLn;
      ReadTransitions(Matrix);
      CloseInput;
    END;
  END ReadMachine;
PROCEDURE ClearTape(VAR Tape:Tapes);
VAR
  rover:SquarePointers;
  BEGIN
    rover:=Tape;
    rover↑.Symbol:=Blank;
    WHILE rover↑.Next≠NIL DO
      rover:=rover↑.Next;
      rover↑.Symbol:=Blank;
    END;
    rover:=Tape;
    WHILE rover↑.Last≠NIL DO
      rover:=rover↑.Last;
      rover↑.Symbol:=Blank;
    END;
    Tape:=rover;
  END ClearTape;
PROCEDURE PrepareTape(VAR T:Machines);
  BEGIN
    WITH T↑ DO
      Square:=LONG(0); Transition:=LONG(0);
      ClearTape(Position);
      WriteString
      ("Give␣name␣of␣tape␣file␣(CON␣for␣console)␣");
      OpenInput("DAT");
      ReadTape(Position);
      WriteLn; CloseInput;
      TransitionMade:=TRUE;
    END;
  END PrepareTape;
PROCEDURE Initialise(VAR T:Machines);
  BEGIN
    WriteString
    ("Turing␣Machine␣Simulator,␣(C)␣Mike␣Piff,␣1992");
    WriteLn;
    NEW(T);
    ReadMachine(T);
    NewSquare(T↑.Position,Blank,NIL,NIL);
  END Initialise;
PROCEDURE OutOfRange(T:Machines):BOOLEAN;
  BEGIN
    WITH T↑ DO
      RETURN (State=UndefinedState)
        OR (Position↑.Symbol=Undefined);
    END;
  END OutOfRange;
PROCEDURE MakeTransition(T:Machines);
VAR
  s, ns:States;
  x, nx:Symbols;
  p:SquarePointers;
  d:Directions;
  BEGIN
    WITH T↑ DO
      s:=State;
      x:=Position↑.Symbol;
      p:=Position;
      ns:=Matrix[s,x].NewState;
      nx:=Matrix[s,x].NewSymbol;
      IF (s≠UndefinedState)&(x≠Undefined)
        &(ns≠UndefinedState)&(nx≠Undefined)THEN
        WriteString("Reading␣symbol␣");
        WriteSymbol(x);
        State:=ns;
        Position↑.Symbol:=nx;
        IF nx≠x THEN
          WriteString("␣and␣changing␣it␣to␣");
          WriteSymbol(nx);
        END;
        d:=Matrix[s,x].Direction;
        IF d=Left THEN
          WriteString("␣Moving␣left");
          MoveLeft(Position);
          DEC(Square);
        ELSIF d=Right THEN
          WriteString("␣Moving␣right");
          MoveRight(Position);
          INC(Square);
        END;
        WriteLn;
        TransitionMade:=(s≠State)OR(p≠Position)
          OR(x≠Position↑.Symbol);
      ELSE
        TransitionMade:=FALSE;
      END;
      IF TransitionMade THEN
        INC(Transition); END;
    END;
  END MakeTransition;
PROCEDURE WriteTape(Position:Tapes);
VAR
  rover:SquarePointers;
  x:Symbols;
  BEGIN
    rover:=Position;
    WHILE rover↑.Last≠NIL DO
      rover:=rover↑.Last;
    END;
```

```
      WriteString("...");
      Write(VisBlank); Write(VisBlank);
      Write(VisBlank);
      REPEAT
        IF rover=Position THEN
          Write(CHR(176)); END;
        x:=rover↑.Symbol;
        WriteSymbol(x);
        IF rover=Position THEN
          Write(CHR(176)); END;
        rover:=rover↑.Next;
      UNTIL (rover=NIL);
      Write(VisBlank); Write(VisBlank);
      WriteString("...");
    END WriteTape;
  PROCEDURE DisplayState(T:Machines);
    BEGIN
      WITH T↑ DO
        IF TransitionMade THEN
          WriteString("State is ");
          Write(State);
          WriteString(" and square is ");
          WriteLongInt(Square,1);
          WriteString(" after ");
          WriteLongCard(Transition,1);
          WriteString(" transition");
          IF Transition≠LONG(1) THEN
            Write("s"); END;
          WriteLn;
          WriteTape(Position);
          WriteLn;
        END;
      END;
    END DisplayState;
  PROCEDURE Finished():BOOLEAN;
  VAR Reply:CHAR;
    BEGIN
      WriteLn;
      WriteString("Finished? (Y/N) ");
      REPEAT
        Read(Reply); Reply:=CAP(Reply);
      UNTIL (Reply="Y")OR(Reply="N");
      RETURN Reply="Y";
    END Finished;
  VAR
    Reply:CHAR;
  BEGIN
    Initialise(T);
    REPEAT
      Reply:=Blank;
      PrepareTape(T);
      T↑.State:=T↑.InitState;
      DisplayState(T);
      REPEAT
        MakeTransition(T); DisplayState(T);
        IF KeyPressed() THEN
          Read(Reply);
          Reply:=CAP(Reply);
        END;
      UNTIL OutOfRange(T)
        OR ¬T↑.TransitionMade OR (Reply="Q");
      WriteString("Calculation terminated after ");
      WriteLongCard(T↑.Transition,1);
      WriteString(" transition");
      IF T↑.Transition≠LONG(1) THEN
        Write("s"); END;
      WriteLn;
    UNTIL Finished();
  END Turing.
```

Fermat, again

The following item appeared in a recent *Newsletter* of the London Mathematical Society: see the article 'Fermat's Last Theorem—a theorem at last' in Volume 26 Number 3 of *Mathematical Spectrum.*

> Professor John Coates gave a lecture recently on Fermat's Last Theorem in the series of Isaac Newton Institute seminars at Cambridge. His lecture caused some excitement among those present, as he announced a minor problem with Andrew Wiles' proof. He said that filling this gap will need some work and may take anything from a couple of months to a couple of years to complete. Reports from Princeton give the reassuring news that Andrew Wiles is not too worried about this.

Letters to the Editor

Dear Editor,

The 1994 puzzle

A possible extension to the puzzle presented in Volume 26 Number 3 is to create the numbers 1 to 100 (or 150) using as few √ and ! signs as possible (so that, if the set were being generated by computer, the program would be as efficient as possible, √ and ! requiring more processing than +, −, × and ÷).

My best effort yields 99 √ and 25 !, but I am sure that this can be bettered.

Yours sincerely,
MIKE WENBLE
(8 Carling Road,
Conning Common,
Oxon RG4 9TG)

Dear Editor,

Crossing deserts

The problem about how to cross a desert by establishing fuel caches, discussed by Dylan Gow ('Flyaway', Volume 25 Number 3 pages 84–86) and Wolfram Hinderer ('Optimal crossing of a desert', pages 100–102 of this issue), has been studied before. It seems to have come into currency half a century ago in connection with military logistics.

Solutions of various versions of the problem have appeared in the *American Mathematical Monthly* (N. J. Fine, 'The jeep problem', Volume 54 (1947), pages 24–31; C. G. Phipps, 'The jeep problem: a more general solution', same volume, pages 458–462; David Gale, 'The jeep once more or jeeper by the dozen', Volume 77 (1970), pages 493–501, and a correction in Volume 78 (1971), pages 644–645). On your side of the ocean, see the *Mathematical Gazette* (G. G. Alway, 'Crossing the desert', Volume 41 (1957), page 209).

Yours sincerely,
HAROLD P. BOAS
(Department of Mathematics,
Texas A&M University,
College Station, TX 77843-3368, USA)

Prizes for Student Contributors

The Editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £25 for letters, solutions to problems, and other items.

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues, and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

26.10 (Submitted by Sinefakopoulos Achilleas, University of Athens)
Let $m \geqslant 13$ be an integer and $f\colon \mathbb{N} \to \mathbb{N}$ a mapping such that

$$f(n) = \begin{cases} n-m+13 & (\text{if } n > m^2), \\ f(f(n+m-12)) & (\text{if } n \leqslant m^2). \end{cases}$$

Find all m such that $f(1994) = 1994$.

26.11 (Submitted by Gregory Economides, University of Newcastle upon Tyne Medical School)
Find the sum of the infinite series

$$1 - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{15} + \tfrac{1}{17} - \tfrac{1}{23} + \tfrac{1}{25} - \cdots.$$

26.12 (Submitted by Wolfram Hinderer, University of Karlsruhe)
Prove that, for $n > 2$,

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \tfrac{1}{2}\log_2 n.$$

(See the article by Wolfram Hinderer in this issue.)

Solutions to Problems in Volume 26 Number 2

26.4 Let ABC be an acute-angled triangle and let D and E be the points on BC such that $\angle ADB$ is a right angle and $\angle DAB = \angle EAC$. Prove that

$$\text{area } \triangle EAC > \text{area } \triangle DAB \Leftrightarrow AC > AB.$$

Solution by W. A. Rose (University of Cambridge)
For either diagram,

$$\begin{aligned} \text{area } \triangle EAC > \text{area } \triangle DAB &\Rightarrow \text{area } \triangle ACD > \text{area } \triangle ABE \\ &\Rightarrow \tfrac{1}{2}AC\times AD\sin(\theta\pm\alpha) > \tfrac{1}{2}AB\times AE\sin(\theta\pm\alpha) \\ &\Rightarrow \frac{AC}{AB} > \frac{AE}{AD} = \frac{1}{\cos\alpha} \geqslant 1 \\ &\Rightarrow AC > AB. \end{aligned}$$

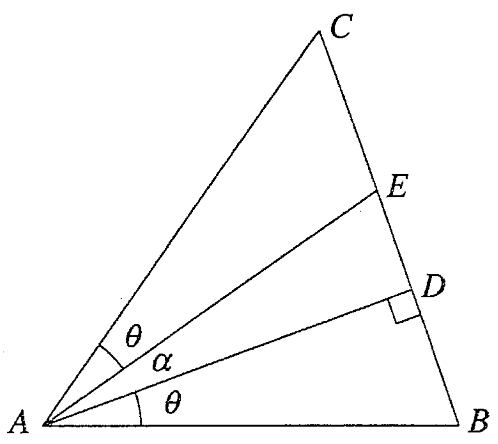


Figure 1

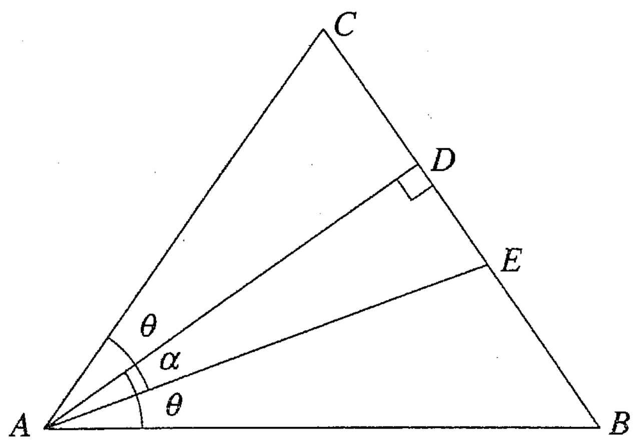


Figure 2

(Since $AC > AB$, figure 2 cannot occur.) Conversely, suppose that $AC > AB$, so that figure 1 occurs. If we put $AD = 1$, then

$$AC = \frac{1}{\cos(\alpha+\theta)}, \qquad AB = \frac{1}{\cos\theta},$$

$$\text{area } \triangle EAC = \tfrac{1}{2} AC \times AE \sin\theta = \frac{\sin\theta}{2\cos(\alpha+\theta)\cos\alpha},$$

$$\text{area } \triangle DAB = \tfrac{1}{2} AD \times DB = \tfrac{1}{2}\tan\theta.$$

Then

$$\begin{aligned} AC > AB &\Rightarrow \frac{1}{\cos(\alpha+\theta)} > \frac{1}{\cos\theta} \\ &\Rightarrow \frac{\sin\theta}{2\cos(\alpha+\theta)} > \tfrac{1}{2}\tan\theta \\ &\Rightarrow \frac{\sin\theta}{2\cos(\alpha+\theta)\cos\alpha} > \tfrac{1}{2}\tan\theta \\ &\Rightarrow \text{area } \triangle EAC > \text{area } \triangle DAB. \end{aligned}$$

Also solved by Gregory Economides, Sammy and Jimmy Yu (middle school students, Vermillion, South Dakota) and Khalid Khan (London School of Economics).

26.5 The *Smarandache function* $\eta : \mathbb{N} \to \mathbb{N}$ is defined by $\eta(n)$ = the smallest positive integer m such that n divides $m!$.
(a) Calculate $\eta(p^{p+1})$, where p is prime.
(b) Find all possible positive integers n such that $\eta(n) = 10$.
(c) Prove that, for every real number k, there is a positive integer n such that

$$\frac{n}{\eta(n)} > k.$$

Does $n/\eta(n) \to \infty$ as $n \to \infty$?

Solution by W. A. Rose

We use the fact that the largest exponent of p to divide $m!$ is

$$\left[\frac{m}{p}\right]+\left[\frac{m}{p^2}\right]+\cdots.$$

(a) The largest exponent of p to divide $p^2!$ is

$$\left[\frac{p^2}{p}\right]+\left[\frac{p^2}{p^2}\right]=p+1.$$

The largest exponent of p to divide $(p^2-1)!$ is

$$\left[\frac{p^2-1}{p}\right]=p-1.$$

Hence $\eta(p^{p+1})=p^2$.

(b) $\eta(n)=10$ means that $n\nmid 9!$ but $n\mid 10!$. The largest exponent of 2 dividing 10! is

$$[\tfrac{10}{2}]+[\tfrac{10}{4}]+[\tfrac{10}{8}]=5+2+1=8,$$

which is not the largest exponent of 2 dividing 9!. The largest exponent of 3 dividing 10! is

$$[\tfrac{10}{3}]+[\tfrac{10}{9}]=4,$$

which is also the largest exponent of 2 dividing 9!. The largest exponent of 5 dividing 10! is $[\frac{10}{5}]=2$, which is not the largest exponent of 5 dividing 9!. The largest exponent of 7 dividing 10! is $[\frac{10}{7}]=1$, which is also the largest exponent of 7 dividing 9!. Hence those natural numbers n for which $\eta(n)=10$ are given by

$$n=2^{\alpha}3^{\beta}5^{\gamma}7^{\delta},$$

where $\alpha\in\{0,1,\dots,8\}$, $\beta\in\{0,1,2,3,4\}$, $\gamma\in\{0,1,2\}$, $\delta\in\{0,1\}$ and either $\alpha=8$ or $\gamma=2$.

(c) From (a), if $n=p^{p+1}$ then

$$\frac{n}{\eta(n)}=\frac{p^{p+1}}{p^2}=p^{p-1},$$

which increases without bound as p increases. However, $\eta(p)=p$ because $p\mid p!$ but $p\nmid(p-1)!$, so $p/\eta(p)=1$. Since there are infinitely many prime numbers, $n/\eta(n)\not\to\infty$ as $n\to\infty$.

Also solved by Gregory Economides.

26.6 Evaluate

$$\int_{x=1}^{\infty}\int_{y=1/x}^{2x}2y^2\exp\left\{-\left(x+\frac{1}{x}\right)y\right\}\,dy\,dx.$$

Solution by W. A. Rose

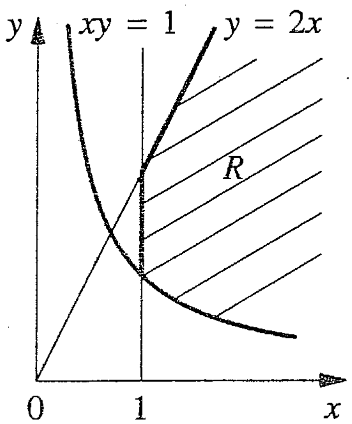
We make the substitution $u = xy$, $v = y/x$. Now

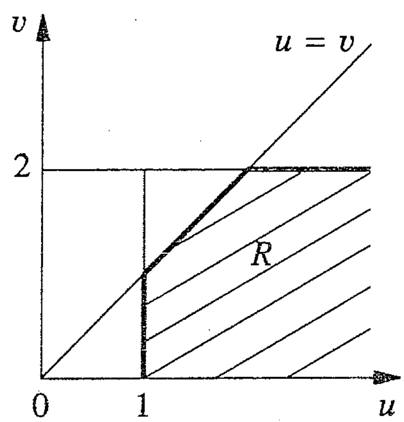
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -\dfrac{y}{x^2} & \dfrac{1}{x} \end{vmatrix} = \frac{2y}{x},$$

so

$$I = \iint_R u e^{-u} e^{-v} \, du \, dv,$$

where R is the region shown in the diagram.





Hence $I = I_1 - I_2$, where

$$\begin{aligned}
I_1 &= \int_1^\infty u e^{-u} \, du \int_0^2 e^{-v} \, dv \\
&= [-(u+1)e^{-u}]_1^\infty [-e^{-v}]_0^2 \\
&= 2e^{-1}(1-e^{-2}), \\
I_2 &= \int_1^2 \int_u^2 u e^{-u} e^{-v} \, dv \, du \\
&= \int_1^2 u e^{-u} [-e^{-v}]_u^2 \, du \\
&= \int_1^2 u e^{-u} [e^{-u} - e^{-2}] \, du \\
&= \int_1^2 u e^{-2u} \, du - e^{-2} \int_1^2 u e^{-u} \, du \\
&= [-(\tfrac{1}{2}u + \tfrac{1}{4})e^{-2u}]_1^2 - e^{-2}[-(u+1)e^{-u}]_1^2 \\
&= \tfrac{3}{4}e^{-2} - \tfrac{5}{4}e^{-4} - e^{-2}(2e^{-1} - 3e^{-2}).
\end{aligned}$$

This gives

$$I = 2e^{-1} - \tfrac{3}{4}e^{-2} - \tfrac{7}{4}e^{-4}.$$

Also solved by Khalid Khan.

Reviews

200% of Nothing: An Eye-Opening Tour through the Twists and Turns of Math Abuse and Innumeracy. By A. K. DEWDNEY. Wiley, Chichester, 1993. Pp. 192. Hardback £12.95 (ISBN 0-471-57776-6).

This book is intended as a more serious, modernized version of Darell Huff's classic *How to Lie with Statistics*. As that book was, this one is also intended for the mathematically aware layman.

Dewdney believes that a major problem facing Americans is that of innumeracy—an unwillingness to understand basic mathematical ideas. The book is intended to show how prevalent this is and, at the end, to show some ways of getting round it.

Unfortunately, it falls into the same trap as books designed to encourage literacy fall into. Those who are numerate may buy the book, but will use the various examples of 'math abuse' collated inside as a source of amusement rather than as an exposé of a major problem. Those who are unselfconsciously innumerate will be put off by the front cover, and it is only those who have a desire to cease to be innumerate who will get an advantage out of the book.

The bulk of the book consists of a series of examples of 'math abuse'—manipulation of figures in a way which is wrong, but not obviously so—collected by a team of 'math detectives'. Although the problem is serious, treating it in this way will inevitably attract pedants. It does, as is shown by the first example quoted—in which a light bulb advertisement claiming '200% savings on energy' is thoroughly ridiculed, despite the fact that even the most innumerate, on seeing such an advertisement, would be sure that something was wrong.

There are some amusing examples, such as the NSA poster which said that it would take a mere 120 years, counting 24 hours a day, to count from 1 to a googol. In fact, in that amount of time, even with a fast computer, one would be hard pressed to exceed 10^{15}!

There are, however, some worrying ones—for example, the advert stating that 'the Tijuana clinic has cured 50 patients of inoperable cancer', without any statement of how many people it had failed to cure, and the statement in a schooling report that 'although test scores were down by 60%, they have since gone up by 70%'—a superficially encouraging statement, but one which actually means that they are still down by 32%.

The silliest example is an advertisement from an electricity company—'96% of streets in the US are underlit, and 88% of crime takes place on underlit streets'. If you work out what this means, you would be well advised to move to an underlit street—the well-lit streets have 3.27 times as much crime!

Having given the reader these examples, Dewdney proceeds to show that lotteries will on average lose you 98.7 pence in the pound, and comprehensively ridicules the various services which tell you that the number 29 has not come up in a lottery for several years, and therefore should be bet upon (because it's due), or avoided (because it's unlucky)!

He then proceeds, in the final chapters, to give a relatively tedious exposition of 'some of the ideas that cause problems'. These turn out to be exponential notation and the idea of factorials, and the style of the chapter seems to have been mischosen. Anyone with a GCSE in mathematics will look with despair at the long-winded way in which the ideas are set out, whereas those who need the help will be discouraged because of the textbook-like style.

The author of this book is addressing a valid threat, and one about which he evidently feels very strongly, but I thing that the job of reducing this threat lies with teachers, rather than with professional mathematicians who regard deceptive claims as humorous, not serious. Get this book if what concerns you is the gullibility of the population and the cynicism of the advertising agent, but for a guide to not being fooled by mathematics, get *How to Lie with Statistics* from the library. Besides, its funnier.

Sixth form Winchester College THOMAS WOMACK

Satan, Cantor and Infinity and other Mind-Boggling Puzzles. By RAYMOND SMULLYAN. Oxford University Press, Oxford, 1993. Pp. viii+270. Paperback £7.99 (ISBN 019-286161-1).

This is a book of paradoxes and puzzles mostly related to logic, infinity and self-reference. The author begins with various puzzles about an island inhabited by truth-tellers and liars. We are guided through this island and the other strange lands by a logician known as the sorcerer. The sorcerer then moves on to meta-puzzles. These are puzzles in which we are not given enough information to solve the puzzle, but knowing that we cannot solve the puzzle gives us enough information to do so!

The next stop is the island of robots. We are introduced to the formal system that governs robot behaviour and have to find the programs (strings of capital letters) necessary for certain robots such as the self-dismantling suicidal robot, or the robot that creates a robot that creates another robot that destroys its grandparent! The point to this is revealed when the author takes us on to a parallel formal system where a surprisingly simple explanation of Gödel's incompleteness theorem is furnished.

The book rounds off with a discussion of infinity which then leads on to a discussion of sets and cardinality and what the author (or at least his *alter ego* the sorcerer) considers to be the grand unsolved problem of mathematics—the (general) continuum hypothesis.

This book can be frustrating at times, but with a little perseverance answering the questions can be very rewarding. Little previous knowledge is required and the book is a great primer for anyone intending to read *Gödel, Escher, Bach*, or indeed anyone interested in logic and meta-mathematics.

Student, London School of Economics KHALID KHAN

Other book received

Oscillations in Nonlinear Systems. By JACK K. HALE. Dover, New York, 1993. Pp. ix+180. Paperback £7.95 (ISBN 0-486-67362-6).

A graduate text which is a republication of a book first published in 1963.

LONDON MATHEMATICAL SOCIETY

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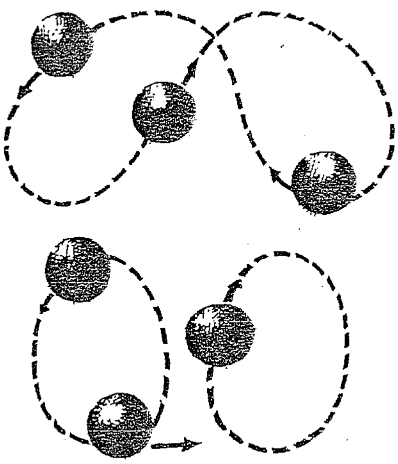

Dr Colin Wright

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"Juggling is an ancient art that has baffled and entertained countless people. In this talk we discover a richness and variety of mathematical structure only recently discovered in this skill."

STRATHCLYDE UNIVERSITY Commences at 2.00 pm, 3.00 pm refreshments, ends 4.30. Lecture Room 1, McCance Building, University of Strathclyde, 16 Richmond Street, Glasgow G1 1XQ. Admission is free. For further information contact Dr C. Constanda (041 552 4400 ext 3714) or Dr A. McBride (041 552 4400 ext 3647) at the Department of Mathematics, University of Strathclyde.

MANCHESTER UNIVERSITY Commences at 7.00 pm, 8.00 pm refreshments, ends at 9.30 pm. Lecture Theatre B, Roscoe Building, University of Manchester. Admission free, with ticket in advance. Apply by Friday 10 June to Dr M. Prest, Department of Mathematics, University of Manchester M13 9PL. A stamped addressed envelope would be appreciated.

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CONTENTS

ISSN 0025–5653

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Editor—Mathematical Spectrum,
Hicks Building,
The University,
Sheffield S3 7RH, UK.

Published by the Applied Probability Trust
Typeset by The Pi-squared Press, Nottingham, UK
Printed by Galliard (Printers) Ltd, Great Yarmouth, UK