

Mathematicorum

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

WHEN DO INTEGERS HAVE A FINITE COMPLETE SEQUENCE OF DIVISORS?

K. R. S. Sastry

The (positive integral) divisors of the integer 12 are

$$d_1 = 1, \quad d_2 = 2, \quad d_3 = 3, \quad d_4 = 4, \quad d_5 = 5, \quad d_6 = 12.$$

It is possible to express every integer k , $1 \leq k \leq 12$, as sums of distinct d_i , $1 \leq i \leq 6$. One way to see this is: $i = d_i$ for $1 \leq i \leq 4$, $5 = d_1 + d_4$, $6 = d_5$, $7 = d_3 + d_4$, $8 = d_2 + d_5$, $9 = d_3 + d_5$, $10 = d_4 + d_5$, $11 = d_1 + d_4 + d_5$, $12 = d_6$. We observe that $d_1 = 1$ and that $d_{m+1} \leq 1 + \sum_{i=1}^m d_i$ for $1 \leq m \leq 5$. The reader can verify that the divisors d_i of the integer 20 also exhibit the above mentioned properties. However, the divisors of the integer 10, namely $d_1 = 1, d_2 = 2, d_3 = 5, d_4 = 10$, do not exhibit the same properties: 4 does not have a representation as a sum of distinct d_i and $d_3 \not\leq 1 + (d_1 + d_2)$.

Brown [1] calls a sequence $\{s_i\}$ of positive integers a *complete* sequence if every positive integer n can be expressed as a sum of distinct terms from the sequence. Assuming $s_1 = 1$ and $\{s_i\}$ non-decreasing, he gives a necessary and sufficient condition for completeness as

$$s_{m+1} \leq 1 + \sum_{i=1}^m s_i, \quad m = 1, 2, 3, \dots . \quad (1)$$

The divisors d_i of $N = 12$ or 20 satisfy the inequalities in (1). So, they form a (finite) complete sequence of divisors for N , i.e. all k , $1 \leq k \leq N$, can be expressed as a sum of distinct divisors of N . $N = 10$ does not have a complete sequence of divisors because its divisors fail to satisfy the inequalities in (1). Hence the natural question is: under what constraints do integers N have a (finite) complete sequence of their divisors? Brown's answer would be to use his inequalities in (1) with the divisors d_i of N in place of s_i , keeping in mind that $\{d_i\}$ is a finite sequence. Our present aim, however, is to propose a constraint on the primes composing N . These constraints are proposed in Theorem 1.

THEOREM 1. Let $N = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ denote the prime decomposition of N . Define

$$\sigma_1 = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \quad \text{and} \quad \sigma_i = \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \sigma_{i-1} \quad \text{for } 1 < i \leq k.$$

Then the divisors of N form a (finite) complete sequence if and only if $p_1 = 2$ and $p_i \leq 1 + \sigma_{i-1}$ for $1 < i \leq k$.

Proof. For $k = 1$, we have $N = p_1^{\alpha_1}$ with divisors $1, p_1, p_1^2, \dots, p_1^{\alpha_1}$. If $p_1 \neq 2$ then we cannot express 2 as a sum of distinct divisors of N . Hence

$p_1 = 2$. It is easy to check that every integer m , $1 \leq m \leq 2^{\alpha_1}$, has a representation as a sum of distinct divisors of N . We note this.

When $k = 1$, the greatest integer obtainable as a sum of distinct divisors of N is

$$\sigma_1 = 2^{\alpha_1+1} - 1. \quad (2)$$

Hence $p_1 = 2$ is both a necessary and a sufficient constraint for $N = 2^{\alpha_1}$ to have a complete sequence of divisors.

When $k = 2$, $N = 2^{\alpha_1} p_2^{\alpha_2}$ has divisors

$$1, 2, 2^2, \dots, 2^{\alpha_1}; \quad p_2, 2p_2, 2^2 p_2, \dots, 2^{\alpha_1} p_2; \quad \dots, 2^{\alpha_1} p_2^{\alpha_2}.$$

If $p_2 > \sigma_1 + 1$ then no integer m such that $\sigma_1 + 1 \leq m < p_2$ is obtainable as a sum of distinct divisors of N because of (2). Hence $p_2 \leq \sigma_1 + 1$. In this case, $p_2, p_2 + 1, \dots, 2p_2 - 1 = p_2 + (p_2 - 1) \leq p_2 + \sigma_1$ are all obtainable as sums of distinct divisors of N . Likewise obtainable are the integers between $2p_2$ and $2^2 p_2$; $2^2 p_2$ and $2^3 p_2$; \dots ; $2^{\alpha_1-1} p_2^{\alpha_2}$ and $2^{\alpha_1} p_2^{\alpha_2}$. This argument also shows that the constraint $p_2 \leq \sigma_1 + 1$ is sufficient for N to have a complete sequence of divisors. We note that the greatest integer obtainable as a sum of distinct divisors of N , when $k = 2$, is σ_2 . The theorem follows as an easy application of induction. \square

A referee called the author's attention to practical numbers defined by Srinivasan [3]. A natural number n is *practical* if and only if for all natural numbers $m \leq n$, m is the sum of distinct proper divisors of n . Note that the set of proper divisors of n excludes n . Our earlier examples, $n = 12$ and $n = 20$, that have complete sequences of divisors are also practical numbers because $12 = 2 + 4 + 6$; $20 = 1 + 4 + 5 + 10$ are representations of n as a sum of distinct proper divisors of n . It is easy to see that 10 is not a practical number. At this point one might be tempted to conclude that the integers N generated by Theorem 1 are all practical numbers. A simple counterexample would be $N = 4$: 4 has a complete sequence of divisors and fails to be practical. More generally, such is the case if $k = 1$ in Theorem 1: $N = 2^{\alpha_1}$ has a complete sequence of divisors and fails to be practical. However, if $k > 1$, then Theorem 1 generates all practical numbers. Theorem 2 answers the question: when are numbers practical?

THEOREM 2. Let S denote the set of natural numbers N given by Theorem 1. Then a natural number n is practical if and only if $n \in S$ and n is not a power of 2.

Proof. The proof of Theorem 2 includes that of Theorem 1 and a bit more. The definition of a practical number makes it clear that the set of practical numbers is a proper subset of S . It is obvious that 2^α , while in S , is not a practical number. Beginning with $k = 2$, the only additional fact to be established is that N is a sum of distinct proper divisors of N . We recall that when $k = 2$, $N = 2^{\alpha_1} p_2^{\alpha_2}$. First we show that $p_2^{\alpha_2}$ has two distinct representations as sums of distinct proper divisors of N . One such representation is $p_2^{\alpha_2}$ itself.

Suppose $\alpha_2 = 1$. Since $p_2 \leq \sigma_1 + 1 = 2^{\alpha_1+1}$, see (2), it follows that (because p_2 is a prime) $p_2 < 2^{\alpha_1+1}$. Hence from Theorem 1, p_2 has a representation as a sum of distinct proper divisors of 2^{α_1+1} . That is,

$$p_2 = 2^{q_1} + 2^{q_2} + \cdots + 2^{q_\ell}, \quad 0 \leq q_i \leq \alpha_1, \quad i = 1, 2, \dots, \ell. \quad (3)$$

If $\alpha_2 > 1$ then the claimed second representation follows from (3),

$$p_2^{\alpha_2} = 2^{q_1} p_2^{\alpha_2-1} + 2^{q_2} p_2^{\alpha_2-1} + \cdots + 2^{q_\ell} p_2^{\alpha_2-1}. \quad (4)$$

Now

$$\begin{aligned} N &= 2^{\alpha_1} p_2^{\alpha_2} = 2^{\alpha_1-1} p_2^{\alpha_2} + 2^{\alpha_1-1} p_2^{\alpha_2} \\ &= 2^{\alpha_1-1} p_2^{\alpha_2} + (1 + 2 + 2^2 + \cdots + 2^{\alpha_1-2}) p_2^{\alpha_2} + p_2^{\alpha_2}, \quad \alpha_1 > 2. \end{aligned} \quad (5)$$

In (5), the first two expressions are all sums of distinct proper divisors of N . The third one, $p_2^{\alpha_2}$, on using (4) is also a sum of distinct proper divisors of N that are not used in the earlier two expressions — these contain terms in $p_2^{\alpha_2-1}$.

We must now show that if $\alpha_1 = 1$ or 2 then N is still practical.

If $\alpha_1 = 1$ then $N = 2p_2^{\alpha_2}$ and $\sigma_1 = 3$. Since the prime $p_2 \leq \sigma_1 + 1 = 4$, p_2 must be 3. So $N = 2(3^{\alpha_2})$. But

$$N = 2(3^{\alpha_2}) = 3^{\alpha_2} + 3^{\alpha_2-1} + 2(3^{\alpha_2-1})$$

is practical.

If $\alpha_1 = 2$ then $N = 2^2 p_2^{\alpha_2} = 4p_2^{\alpha_2}$ and $\sigma_1 = 7$. Since the prime $p_2 \leq \sigma_1 + 1 = 8$, p_2 equals 3, 5, or 7.

(i) $p_2 = 3$ implies that $N = 4(3^{\alpha_2})$. Then

$$N = 4(3^{\alpha_2}) + 2(3^{\alpha_2-1}) + 4(3^{\alpha_2-1})$$

is practical.

(ii) $p_2 = 5$ implies that $N = 4(5^{\alpha_2})$. Then

$$N = 4(5^{\alpha_2}) = 5^{\alpha_2} + 2(5^{\alpha_2}) + (5^{\alpha_2-1}) + 4(5^{\alpha_2-1})$$

is practical.

(iii) $p_2 = 7$ implies that $N = 4(7^{\alpha_2})$. Then

$$N = 4(7^{\alpha_2}) = 7^{\alpha_2} + 2(7^{\alpha_2}) + 7^{\alpha_2-1} + 2(7^{\alpha_2-1}) + 4(7^{\alpha_2-1})$$

is practical.

It is now a simple matter to complete the proof of Theorem 2. \square

CONCLUSION: As noted in the proof of Theorem 1, integers greater than n are obtainable as sums of distinct divisors of n . Let $\sigma(N)$ denote the sum of all positive integral divisors of N . Then N is called deficient, perfect, or abundant according as $\sigma(N)$ is less than, equal to, or greater than $2N$ (e.g., see [2,3]). It is easy to see that a deficient number cannot have a complete sequence of divisors.

Problem 1. Prove that an even perfect number has a complete sequence of divisors.

Problem 2. Find an abundant number that does not have a complete sequence of divisors.

Another well known function on natural numbers N is ϕ , the Euler phi-function: $\phi(1) = 1$, $\phi(N)$ equals the number of positive integers less than and prime to N for $N > 1$. Such integers are called totitives of N , see [2].

Problem 3. Find the integers N whose totitives form a complete sequence for N .

Acknowledgement: The author thanks the referees for the suggestions.

References:

- [1] J. L. Brown Jr., Integer representations and complete sequences, *Mathematics Magazine*, Vol. 49 (January 1976), pp. 30-32.
- [2] L. E. Dickson, *History of the Theory of Numbers*, Vol. I, Chelsea Publications (1971), pp. 113, 124.
- [3] R. Honsberger, *Mathematical Gems*, MAA (1973), pp. 112-113.

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BOOKS WANTED

Larry Hoehn (Mathematics and Computer Science, Austin Peay State University, Clarksville, TN 37044, USA) writes: "There is hardly an issue of *Crux Mathematicorum* that does not refer to Roger A. Johnson's *Advanced Euclidean Geometry* at least once. I would love to obtain a copy of this book. Some time ago I wrote to Dover Publications, Inc. hoping that they might still have copies. They don't. If enough others are also looking for this book, could we persuade Dover to reprint it?"

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THE SKOLIAD CORNER

No. 10

R. E. WOODROW

As a problem set this issue we feature the Saskatchewan Senior Mathematics Contest written February 23, 1994. One and a half hours are permitted for the contest, and calculators are allowed. My thanks go to Gareth Griffith, The University of Saskatchewan, and long time organizer of the Saskatchewan Contests for permission to use the contest in the Corner.

SASKATCHEWAN SENIOR MATHEMATICS CONTEST

February 23, 1994

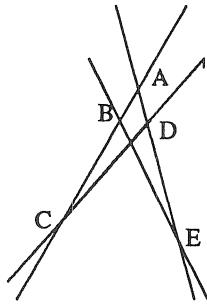
Time: 1.5 hours

- 1.** Solve the equation

$$1 + 68x^{-4} = 21x^{-2}.$$

- 2.** Find the number of divisors of 16128 (including 1 and 16128).

- 3.** In the figure, lines ABC and ADE intersect at A . The points $BCDE$ are chosen such that angles CBE and CDE are equal. Prove that the rectangle whose sides have length AB and AC and the rectangle whose sides have length AD and AE are equal in area.



- 4.** (a) State the domain and range of the functions

$$f(x) = \tan x \quad \text{and} \quad g(x) = \log_a x \quad \text{where } a = \frac{5\pi}{4}.$$

- (b) Determine the smallest value of x for which $\tan x = \log_{5\pi/4} x$.

- 5.** (a) Prove that the system of equations

$$\begin{aligned} x + y &= 1 \\ x^2 + y^2 &= 2 \\ x^3 + y^3 &= 3 \end{aligned}$$

has no solution.

- (b) Determine all values of k such that the system

$$\begin{aligned} x + y &= 1 \\ x^2 + y^2 &= 2 \\ x^3 + y^3 &= k \end{aligned}$$

has at least one solution.

6. (Contributed by Murray Bremner, University of Saskatchewan.)

This problem shows how we may find *all* solutions to the equation $X^2 + Y^2 = Z^2$ where X, Y and Z are positive integers. Such a solution (X, Y, Z) is called a *Pythagorean triple*. If (X, Y, Z) have no common factor (other than 1) we call (X, Y, Z) a *primitive Pythagorean triple*.

Part I

(a) Let a, b be two positive integers with $a > b$. Show that $X = a^2 - b^2$, $Y = 2ab$, $Z = a^2 + b^2$ is a Pythagorean triple.

(b) Now assume that a, b have no common factor and not both are odd. Show that (X, Y, Z) in (a) is a primitive Pythagorean triple. (Hint: Suppose that X, Y, Z have a common factor, p = some prime number. Then p divides $Z + X$ and $Z - X$. Note that $Z + X = 2a^2$ and $Z - X = 2b^2$. So what is p ? But Z must be odd (why?) so p can't be 2).

Part II

(a) Let (X, Y, Z) be any Pythagorean triple. Show that the point $(\frac{X}{Z}, \frac{Y}{Z})$ lies on the unit circle $x^2 + y^2 = 1$.

(b) Let the slope of the line l which joins $(-1, 0)$ to $(\frac{X}{Z}, \frac{Y}{Z})$ be b/a where a, b are positive integers with no common factor and $a > b$. Find the points of intersection of the line l and the unit circle in terms of a, b to show that

$$\frac{X}{Z} = \frac{a^2 - b^2}{a^2 + b^2}, \quad \frac{Y}{Z} = \frac{2ab}{a^2 + b^2}.$$

(c) If (X, Y, Z) is a primitive Pythagorean triple and if a, b are not both odd, show that $X = a^2 - b^2$, $Y = 2ab$, $Z = a^2 + b^2$.

(d) If (X, Y, Z) is primitive Pythagorean triple and if a, b are both odd, $a > b$, we let $r = \frac{1}{2}(a+b)$, $s = \frac{1}{2}(a-b)$.

(i) Prove that r, s are positive integers, that $r > s$, that r, s have no common factor (other than 1) and that r, s are not both odd.

(ii) Let $X' = 2(2rs)$, $Y' = 2(r^2 - s^2)$, $Z' = 2(r^2 + s^2)$. Show that $X = X'/2$, $Y = Y'/2$, $Z = Z'/2$. (Thus $X = 2rs$, $Y = r^2 - s^2$, $Z = r^2 + s^2$).

Remarks. Part I shows that if a, b are positive integers with no common factor and $a > b$ then $X = a^2 - b^2$, $Y = 2ab$, $Z = a^2 + b^2$ is a primitive Pythagorean triple. Part II shows that every primitive Pythagorean triple arises this way (for suitable choice of a, b).

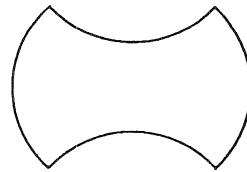
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Last month we gave the problems of the IBM U.K. Junior Mathematical Olympiad 1994. Here are the answers.

- | | | | |
|----------|---------|-----------|-----------|
| A1. 105° | A2. 15 | A3. 13/15 | A4. 17/36 |
| A5. 75° | A6. 32 | A7. 1 | A8. 1332 |
| A9. 180° | A10. 39 | | |

B1. A circle of radius 1 is cut into four equal arcs, which are then arranged to make the shape shown here. What is its area? Explain!

Solution. If we cut the two ends off they will exactly fill the two holes to form a square of side $\sqrt{1^2 + 1^2} = \sqrt{2}$. So the area is $(\sqrt{2})^2 = 2$.



B2. (a) Find three prime numbers such that the sum of all three is also a prime.

(b) Find three positive integers such that the sum of any two is a perfect square. Can you find other sets of three integers with the same property?

Solution. (a) 3, 5, 11. $3 + 5 + 11 = 19$.

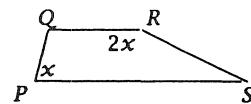
(b) Let's work backward. Start with three squares, k^2, m^2, n^2 . Now call the three numbers x, y, z . We want $x + y = k^2$, $y + z = m^2$ and $z + x = n^2$. This gives

$$x = \frac{k^2 - m^2 + n^2}{2}, \quad y = \frac{k^2 + m^2 - n^2}{2}, \quad \text{and} \quad z = \frac{-k^2 + m^2 + n^2}{2}.$$

For x, y, z to be integers we see that two of k^2, m^2, n^2 must be odd and the remaining one even. So two of k, m, n must be odd and the remaining one even. For x, y, z to be positive integers k^2, m^2, n^2 must be sides of a triangle. For an infinite family of examples let us look for $k, m = k+1, n = k+2$ with k odd. The restriction that x, y, z are positive means $k^2 + (k+1)^2 - (k+2)^2 > 0$, i.e. $(k+1)(k-3) = k^2 - 2k - 3 > 0$ which is true for $k > 3$. So for an example with $k = 5, m = 6, n = 7$ we get

$$x = \frac{5^2 - 6^2 + 7^2}{2} = 19, \quad y = \frac{5^2 + 6^2 - 7^2}{2} = 12, \quad z = \frac{-5^2 + 6^2 + 7^2}{2} = 60.$$

B3. In the trapezium $PQRS$ angle QRS is twice angle QPS , QR has length a and RS has length b . What is the length of PS ? Explain!



Solution. Let RT be parallel to PQ with T on PS . Then $\angle QRT = \angle RTS = \angle QPS = x$. But $\angle QRS = 2x$ so $\angle TRS = x$. Thus $RS = TS = b$. As $PTRQ$ is a parallelogram $PT = a$. Thus $PS = a + b$.

B4. A sequence of fractions obeys the following rule: given any two successive terms a, b of the sequence, the next term is obtained by dividing their product $a \cdot b$ by their sum $a + b$. If the first two terms are $1/2$ and $1/3$, write down the next three terms. What is the tenth term? Explain clearly what is going on, and how you can be sure.

Solution. The first five terms are

$$\frac{1}{2}, \quad \frac{1}{3}, \quad \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2} + \frac{1}{3}} = \frac{1}{3+2} = \frac{1}{5},$$

$$\frac{\frac{1}{3} + \frac{1}{5}}{\frac{1}{3} + \frac{1}{5}} = \frac{1}{5+3} = \frac{1}{8}, \quad \frac{\frac{1}{5} + \frac{1}{8}}{\frac{1}{5} + \frac{1}{8}} = \frac{1}{8+5} = \frac{1}{13}.$$

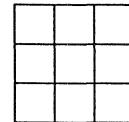
If two successive terms are $1/m$ and $1/n$ the next is

$$\frac{\frac{1}{m} + \frac{1}{n}}{\frac{1}{m} + \frac{1}{n}} = \frac{1}{n+m},$$

so the denominators are 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ... the Fibonacci sequence (1, 1, 2, 3, 5, ...) starting from its third term.

The tenth term of the given sequence is then 1/144.

B5. In this grid, small squares are called *adjacent* if they are next to each other either horizontally or vertically. When you place the digits 1–9 in the nine squares, how many adjacent *pairs* of numbers are there?



You have to arrange the digits 1–9 in the grid so that the total T of all the differences between adjacent pairs is as large as possible. Show how this can be done. Explain clearly why no other arrangement could give a larger total T than yours.

Solution by Bruce Bauslaugh, The University of Calgary; and Derek Kisman, student, Queen Elizabeth High School, Calgary.

Label the squares as shown with a_9 at the centre and a_1, a_2, \dots, a_8 cyclically around the perimeter. We split the differences into two groups: inside differences corresponding to neighbours of a_9 and outside differences between cells on the cycle. For a fixed assignment of 1, 2, ..., 9 to the cells, let T_I be the sum of the inside differences and T_O the sum of the outside differences. So $T = T_I + T_O$.

a_1	a_2	a_3
a_8	a_9	a_4
a_7	a_6	a_5

We next consider maximizing T_O for a fixed assignment X_9 to a_9 . We first claim that for the optimal arrangement the digits X_1, \dots, X_9 assigned to a_1, \dots, a_9 respectively must alternately increase and decrease (cyclically) so that (without loss) $X_1 > X_2 < X_3 > X_4 < X_5 > X_6 < X_7 > X_8 < X_1$. To see this suppose that three cyclically consecutive numbers are in order. If we remove the middle one, say X_i , shrinking the cycle to 7 elements, the sum T_O is unchanged. Now label the 7 remaining elements according as they are larger or smaller than X_i . Since 7 is odd two cyclically consecutive ones must be larger than X_i , or two smaller. Insertion of X_i between them gives an assignment for which T_O is larger.

Now

$$\begin{aligned} T_O &= (X_1 - X_2) + (X_3 - X_2) + (X_3 - X_4) + \cdots + (X_1 - X_8) \\ &= 2((X_1 + X_3 + X_5 + X_7) - (X_2 + X_4 + X_6 + X_8)). \end{aligned}$$

This is clearly maximized by using the 4 largest numbers, different from X_9 , for X_1, X_3, X_5, X_7 and the 4 smallest for X_2, X_4, X_6, X_8 .

Thus for

$$\begin{aligned} X_9 = 9, \quad T_O &\leq 2((8+7+6+5)-(4+3+2+1)) = 32 \\ X_9 = 8, \quad T_O &\leq 2((9+7+6+5)-(4+3+2+1)) = 34 \\ X_9 = 7, \quad T_O &\leq 36 \\ X_9 = 6, \quad T_O &\leq 38 \\ X_9 = 5, \quad T_O &\leq 40. \end{aligned}$$

Now let us regard T_I for these choices of X_9 .

It is easy to check that for

$$\begin{array}{ll} X_9 = 9, \quad T_I \leq 26 & X_9 = 9, \quad T \leq 32 + 26 = 58 \\ X_9 = 8, \quad T_I \leq 22 & X_9 = 8, \quad T \leq 34 + 22 = 56 \\ X_9 = 7, \quad T_I \leq 18 & \text{so that for } X_9 = 7, \quad T \leq 36 + 18 = 54 \\ X_9 = 6, \quad T_I \leq 15 & X_9 = 6, \quad T \leq 38 + 15 = 53 \\ X_9 = 5, \quad T_I \leq 14. & X_9 = 5, \quad T \leq 40 + 14 = 54. \end{array}$$

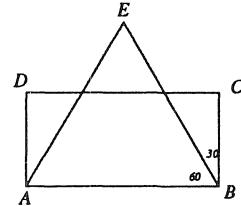
Since replacing X_i by $10 - X_i$ yields the same value for T , these are all we need to check. Thus $T \leq 58$. This is realized by

8	1	5
4	9	2
7	3	6

for example.

B6. In the figure described in problem A5 what fraction of the rectangle is covered by the equilateral triangle ABE ?

Solution. Let EA meet CD at M and EB meet CD at N . Now $\frac{NC}{CB} = \tan 30^\circ = \frac{1}{\sqrt{3}}$ as $\triangle NCB$ is a right triangle with $\angle NBC = 30^\circ$. The area of $\triangle NCB$ is thus $\frac{1}{2}NC \cdot CB = \frac{1}{2\sqrt{3}}(CB)^2$. Similarly the area of $\triangle ADM = \frac{1}{2\sqrt{3}}(CB)^2$. The area covered is therefore



$$AB \cdot BC - 2 \left(\frac{1}{2\sqrt{3}}(CB)^2 \right) = 2(BC)^2 - \frac{1}{\sqrt{3}}(BC)^2 = \left(2 - \frac{1}{\sqrt{3}} \right) (BC)^2.$$

The fraction covered is

$$\frac{\left(2 - \frac{1}{\sqrt{3}} \right) (BC)^2}{2(BC)^2} = 1 - \frac{\sqrt{3}}{6}.$$

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That completes the Skoliad Corner for this month and year. Please send me pre-Olympiad contests and suggestions for the future of this feature of *Crux*.

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THE OLYMPIAD CORNER

No. 170

R. E. WOODROW

All communications about this column should be sent to Professor R. E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We continue this number with the remaining problems proposed to the jury but not used at the 35th I.M.O. in Hong Kong in July 1994. My thanks again go to Andy Liu, The University of Alberta, member of the Problems Selection Committee, and Mathematics Editor of the beautiful booklet of problems and solutions, for forwarding them to me. The booklet does not attribute the country of origin, so that we departed from past practice by not listing that information. Because the booklet has been widely circulated, I would solicit your different, nice solutions.

PROBLEMS PROPOSED BUT NOT USED AT THE 35th I.M.O. IN HONG KONG

More Selected Problems

1. *ABCD* is a quadrilateral with BC parallel to AD . M is the midpoint of CD , P that of MA and Q that of MB . The lines DP and CQ meet at N . Prove that N is not outside triangle ABM .

2. For any positive integer x_0 , three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are defined as follows:

$$(1) \quad y_0 = 4 \text{ and } z_0 = 1;$$

$$(2) \quad \text{if } x_n \text{ is even for } n \geq 0, \quad x_{n+1} = x_n/2, \quad y_{n+1} = 2y_n \text{ and } z_{n+1} = z_n;$$

$$(3) \quad \text{if } x_n \text{ is odd for } n \geq 0, \quad x_{n+1} = x_n - y_n/2 - z_n, \quad y_{n+1} = y_n \text{ and } z_{n+1} = y_n + z_n.$$

The integer x_0 is said to be good if and only if $x_n = 0$ for some $n \geq 1$. Find the number of good integers less than or equal to 1994.

3. In a certain city, age is reckoned in terms of real numbers rather than integers. Every two citizens x and x' either know each other or do not know each other. Moreover, if they do not, then there exists a chain of citizens $x = x_0, x_1, \dots, x_n = x'$ for some integer $n \geq 2$, such that x_{i-1} and x_i know each other. In a census, all male citizens declare their ages, and there is at least one male citizen. Each female citizen only provides the information that her age is the average of the ages of all the citizens she knows. Prove that this is enough to determine uniquely the ages of all the female citizens.

4. There are $n + 1$ fixed positions in a row, labeled 0 to n in increasing order from right to left. Cards numbered 0 to n are shuffled and dealt, one in each position. The object of the game is to have the card i in the i th position for $0 \leq i \leq n$. If this has not been achieved, the following move is executed. Determine the smallest k such that the k -th position is occupied by a card $l > k$. Remove this card, slide all cards from the $(k + 1)$ -st to the l -th position one place to the right, and replace the card l on the l -th position.

(a) Prove that the game lasts at most $2^n - 1$ moves.

(b) Prove that there exists a unique initial configuration for which the game lasts exactly $2^n - 1$ moves.

5. Let R denote the set of all real numbers and R^+ the subset of all positive ones. Let α and β be given elements in R , not necessarily distinct. Find all functions $f : R^+ \rightarrow R$ such that $f(x)f(y) = y^\alpha f(\frac{x}{2}) + x^\beta f(\frac{y}{2})$ for all x and y in R^+ .

6. N is an arbitrary point on the bisector of $\angle BAC$. P and O are points on the lines AB and AN , respectively, such that $\angle ANP = 90^\circ = \angle APO$. Q is an arbitrary point on NP , and an arbitrary line through Q meets the lines AB and AC at E and F respectively. Prove that $\angle OQE = 90^\circ$ if and only if $QE = QF$.

7. Let x_1 and x_2 be relatively prime positive integers. For $n \geq 2$, define $x_{n+1} = x_n x_{n-1} + 1$.

(a) Prove that for every $i > 1$, there exists $j > i$ such that x_i^t divides x_j^j .

(b) Is it true that x_1 must divide x_j^j for some $j > 1$?

8. On an infinite square grid, two players alternately mark symbols on empty cells. The first player always marks X 's, the second O 's. One symbol is marked per turn. The first player wins if there are 11 consecutive X 's in a row, column or diagonal. Prove that the second player can prevent the first from winning.

9. Prove that for any integer $n \geq 2$, there exists a set of 2^{n-1} points in the plane such that no 3 lie on a line and no $2n$ are the vertices of a convex $2n$ -gon.

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To finish this number of the Corner, we turn to readers' solutions to problems of the 1992 Austrian-Polish Mathematics Competition [1994: 96-98].

1. For natural $n \geq 1$, let $s(n)$ denote the sum of all positive divisors of n . Prove that for every integer $n > 1$ the product $s(n-1)s(n)s(n+1)$ is even.

Solutions by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Wang's solution.

We show equivalently that at least one of $s(n - 1)$, $s(n)$ and $s(n + 1)$ must be even. Let $n = P_1^{t_1} P_2^{t_2} \dots P_k^{t_k}$ be the prime powers decomposition of n where the P_i 's are distinct primes. Then it is well known that

$$s(n) = \prod_{i=1}^k (1 + P_i + P_i^2 + \dots + P_i^{t_i}).$$

If $P_i = 2$, then the factor $p_i = 1 + P_i^2 + \dots + P_i^{t_i}$ is clearly odd. If P_i is odd, then clearly p_i is odd iff t_i is even. Hence $s(n)$ is odd if and only if $n = 2^d q^2$, where $d \geq 0$ and q is odd. Suppose, contrary to what we claim, that $s(n - 1)$, $s(n)$ and $s(n + 1)$ are all odd. Then $n - 1 = 2^a q_1^2$, $n = 2^b q_2^2$ and $n + 1 = 2^c q_3^2$ where $a, b, c \geq 0$ and q_i ($i = 1, 2, 3$) are all odd. We consider two cases:

Case (i): If n is even, then $a = c = 0$. Thus $n - 1 = q_1^2$ and $n + 1 = q_3^2$ which imply $(q_3 - q_1)(q_3 + q_1) = 2$ and thus $q_3 - q_1 = 1$ and $q_3 + q_1 = 2$ which clearly yields no integer solutions, a contradiction.

Case (ii): If n is odd then $b = 0$, $a, c \geq 1$ and $n = q_2^2$. Thus

$$q_2^2 - 2^a q_1^2 = 1. \quad (1)$$

Furthermore $2^c q_3^2 = n + 1 = q_2^2 + 1$. If $c \geq 2$, then $2^c q_3^2 \equiv 0 \pmod{4}$ while $q_2^2 + 1 \equiv 2 \pmod{4}$ since the square of any odd integer is congruent to 1 ($\pmod{4}$). Thus $c = 1$ and from $2q_3^2 = 2^1 q_1^2 + 2$ we get

$$q_3^2 - 2^{a-1} q_1^2 = 1. \quad (2)$$

Note that the equation $x^2 - y^2 = 1$ has no solutions in positive integers since $(x - y)(x + y) = 1$ is clearly impossible. Hence if a is even then (1) is not solvable while if a is odd then (2) is not solvable. Again we have a contradiction.

Remark. The characterization of those n for which $s(n)$ is odd is well known. See, for example, Ex #4 on p. 221 of the book *Elementary Number Theory and its Applications* (3rd ed.) by Kenneth H. Rosen.

3. Prove that for all positive real numbers a, b, c the following inequality holds:

$$2\sqrt{bc + ca + ab} \leq \sqrt{3} \sqrt[3]{(b+c)(c+a)(a+b)}.$$

Solution by Panos E. Tsaousoglou, Athens, Greece.

It is enough to prove that for all positive real numbers a, b, c the following inequality holds:

$$64(bc + ca + ab)^3 \leq 27(a + b)^2(a + c)^2(b + c)^2$$

$$\text{or } 64 \cdot 3(bc + ca + ab)(bc + ca + ab)^2 \leq 81[(a + b)(a + c)(b + c)]^2.$$

It is known that $3(bc + ca + ab) \leq (a + b + c)^2$. Thus, it is enough to prove one of the following equivalent inequalities:

$$8(a + b + c)(bc + ca + ab) \leq 9(a + b)(a + c)(b + c)$$

$$8(a + b)(a + c)(b + c) + 8abc \leq 9(a + b)(a + c)(b + c)$$

$$8abc \leq (a + b)(a + c)(b + c)$$

which is well known to hold.

4. Let k be a natural number and u, v be real numbers. Set

$$P(x) = (x - u^k)(x - uv)(x - v^k) = x^3 + ax^2 + bx + c.$$

(a) For $k = 2$ prove: If a, b, c are rational, then the product uv is rational.

(b) Is that also true for $k = 3$?

Solution by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Expanding $P(x)$ we find that

$$a = -(u^k + uv + v^k), \quad b = (uv)(u^k + u^{k-1}v^{k-1} + v^k), \quad c = -u^{k+1}v^{k+1}.$$

(a) For $k = 2$, we have $a = -(u^2 + uv + v^2)$, $b = uv(u^2 + uv + v^2)$. If $u = v = 0$ then uv is rational. Otherwise

$$a = -\left[\left(u + \frac{v}{2}\right)^2 + \frac{3}{4}v^2\right] \neq 0$$

and thus $uv = -b/a$ which is rational.

(b) For $k = 3$ the implication need not be true; for example, if $u = -\sqrt[3]{2}$ and $v = \sqrt[6]{2}$, then $uv = -\sqrt{2}$ while

$$a = -(u^3 + uv + v^3) = -(-2 - \sqrt{2} + \sqrt{2}) = 2,$$

$$b = uv(u^3 + u^2v^2 + v^3) = -\sqrt{2}(-2 + 2 + \sqrt{2}) = -2$$

and $c = -u^4v^4 = -4$ are all rational.

5. Given a circle k of center M and radius r , let AB be a fixed diameter of k and let K be a fixed point of segment AM . Denote by t the line tangent to k at A . For any chord CD (other than AB) passing through K construct P and Q as the points of intersection of lines BC and BD with t . Prove that the product $AP \cdot AQ$ remains constant as the chord CD varies.

Solution by Toshio Seimiya, Kawasaki, Japan.

Because $\angle BAP = 90^\circ$ and $\angle ACB = 90^\circ$ we get ΔAPB similar to ΔCAB . Thus we have $\frac{AP}{AB} = \frac{CA}{CB}$. Similarly we have $\frac{AQ}{AB} = \frac{DA}{DB}$. Multiplying both sides gives

$$\frac{AP \cdot AQ}{AB^2} = \frac{CA \cdot DA}{CB \cdot DB}. \quad (1)$$

As A, C, B, D are concyclic we get ΔACK similar to ΔDBK and ΔDAK similar to ΔBCK . Thus we have

$$\frac{KA}{BD} = \frac{CK}{BK}, \quad \text{and} \quad \frac{DA}{BC} = \frac{AK}{CK}.$$

Therefore

$$\frac{CA}{BD} \cdot \frac{DA}{BC} = \frac{CK}{BK} \cdot \frac{AK}{CK} = \frac{AK}{BK} \quad (2)$$

From (1) and (2) we get

$$\frac{AP \cdot AQ}{AB^2} = \frac{AK}{BK}.$$

Hence we have $AP \cdot AQ = AB^2 \times \frac{AK}{BK}$, a constant.

6. Let \mathbb{Z} denote the set of all integers. Consider a function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ with the properties:

$$\begin{aligned} f(92 + x) &= f(92 - x) \\ f(19 \cdot 92 + x) &= f(19 \cdot 92 - x) \quad (19 \cdot 92 = 1748) \\ f(1992 + x) &= f(1992 - x) \end{aligned}$$

for all $x \in \mathbb{Z}$. Is it possible that all positive divisors of 92 occur as values of f ?

Solution by Joseph Ling, Department of Mathematics, The University of Calgary.

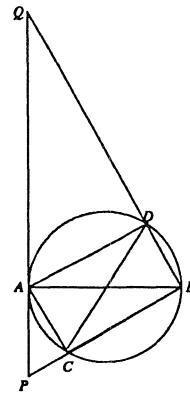
The answer is NO.

Now $f : \mathbb{Z} \rightarrow \mathbb{Z}$ satisfies

- (1) $f(92 + x) = f(92 - x)$
- (2) $f(1748 + x) = f(1748 - x)$, and
- (3) $f(1992 + x) = f(1992 - x)$.

Then, we have

$$\begin{aligned} f(488 + x) &= f(244 + 244 + x) = f(1992 - 1748 + 244 + x) \\ &= f(1992 + 1748 - 244 - x), \text{ by (3)} \\ &= f(1748 + 1992 - 244 - x) \\ &= f(1748 - 1992 + 244 + x), \text{ by (2)} \\ &= f(x) \dots \end{aligned} \quad (4)$$



Then, we have

$$\begin{aligned}
 f(40+x) &= f(1992 - 4 \cdot 448 + x) \\
 &= f(1992 + x), \text{ by repeated application of (4),} \\
 &= f(1992 - x), \text{ by (2)} \\
 &= f(1992 - 4 \cdot 488 - x), \text{ by repeated application of (4)} \\
 &= f(40 - x) \dots
 \end{aligned} \tag{5}$$

So,

$$\begin{aligned}
 f(104+x) &= f(52 + 52 + x) = f(92 - 40 + 52 + x) \\
 &= f(92 + 40 - 52 - x), \text{ by (1)} \\
 &= f(40 + 92 - 52 - x) \\
 &= f(40 - 92 + 52 + x), \text{ by (5)} \\
 &= f(x) \dots
 \end{aligned} \tag{6}$$

Now $8 = 3 \cdot 488 - 14 \cdot 104$. Therefore

$$\begin{aligned}
 f(8+x) &= f(3 \cdot 488 - 14 \cdot 104 + x) \\
 &= f(-14 \cdot 104 + x), \text{ by repeated application of (4)} \\
 &= f(x), \text{ by repeated application of (6).}
 \end{aligned}$$

This shows that f is periodic, and all the possible values of f are in the list $f(0), f(1), f(2), \dots, f(7)$. Finally

$$\begin{aligned}
 f(4+x) &= f(92 - 8 \cdot 11 + x) = f(92 + x), \text{ by periodicity} \\
 &= f(92 - x) \text{ by (1)} \\
 &= f(92 - 8 \cdot 11 + x) = f(4 - x).
 \end{aligned}$$

In particular $f(7) = f(1)$, $f(6) = f(2)$, $f(5) = f(3)$. Hence all the possible values of f are $f(0), f(1), f(2), f(3)$ and $f(4)$. In particular, f assumes no more than 5 function values. However, 92 has 6 positive divisors, namely 1, 7, 4, 23, 46 and 92.

7. We are considering triangles ABC in space.

(a) What conditions must be fulfilled by the angles α, β, γ of triangle ABC in order that there exists a point P in space such that $\angle APB, \angle BPC, \angle CPA$ are right angles?

(b) Let d be the maximum distance among PA, PB, PC and let h be the longest altitude of triangle ABC . Show that $(\sqrt{6}/3)h \leq d \leq h$.

Solution by Toshio Seimiya, Kawasaki, Japan.

(a) By Pythagoras' Theorem we have

$$AB^2 = PA^2 + PB^2, \quad AC^2 = PA^2 + PC^2$$

$$\text{and } BC^2 = PB^2 + PC^2.$$

Hence $AB^2 + AC^2 - BC^2 = 2PA^2 > 0$, thus

$$\angle BAC < 90^\circ, \quad \text{i.e. } \alpha < 90^\circ.$$

Similarly, we get $\beta < 90^\circ$ and $\gamma < 90^\circ$. Conversely, if α , β , and γ are all acute angles, we may prove that there exists a point P such that $\angle APB$, $\angle ABC$, $\angle CPA$ are right angles.

(b) We may without loss of generality assume that $PA \geq PB \geq PC$, so $PA = d$. Because $AB^2 = AP^2 + BP^2 \geq AP^2 + CP^2 = AC^2$ and $AC^2 = AP^2 + CP^2 \geq BP^2 + CP^2 = BC^2$, we have $AB \geq AC \geq BC$.

Let H be the foot of the perpendicular from A to BC , then AH is the longest altitude of $\triangle ABC$, so $AH = h$. As $AP \perp BP$ and $AP \perp CP$, AP is perpendicular to the plane of BPC . Thus $AP \perp BC$ and $AP \perp PH$ so that $AP < AH$, i.e. $d < h$. (1)

Because $AP \perp BP$ and $AH \perp BC$ we get BC perpendicular to the plane of APH . Thus we have $BC \perp PH$.

Let M be the midpoint of BC , then $PH \leq PM$. As $\angle BPC = 90^\circ$ we have $PM = BM = MC = \frac{1}{2}BC$. Hence $2PH \leq 2PM = BC$, so that

$$4PH^2 \leq BC^2 = PB^2 + PC^2 \leq 2PA^2. \quad (2)$$

As $\angle APH = 90^\circ$ we get

$$PH^2 = AH^2 - AP^2 = h^2 - d^2. \quad (3)$$

From (2) and (3) we have

$$4(h^2 - d^2) \leq 2d^2, \quad \text{i.e. } 2h^2 \leq 3d^2,$$

from which we have

$$\frac{\sqrt{6}}{3}h \leq d. \quad (4)$$

From (1) and (4) we obtain $\frac{\sqrt{6}}{3}h \leq d < h$, as required.

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That exhausts our solutions from readers to problems from the April 1994 number of the Corner as well as our space this month. Send me your nice solutions and your Olympiad sets.

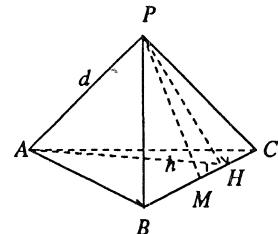
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BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

The Lighter Side of Mathematics, edited by Richard K. Guy and Robert E. Woodrow. Published by The Mathematical Association of America, Washington, DC, 1994. ISBN 0-88385-516-X, 376+ pages, softcover, US \$38.50. *Reviewed by Murray S. Klamkin, University of Alberta.*

This book is the proceedings of the Eugene Strens Memorial Conference on Recreational Mathematics and its History held at the University of Calgary in August 1986 to celebrate the founding of the Strens Collection which is now the most complete library of recreational mathematics in the world.

I had been invited to attend this conference but unfortunately had a previous commitment. To make up for missing this conference, the next best thing was reviewing this proceedings book which on doing so made me realize what I had missed, e.g., some very interesting talks plus getting together with the leading practitioners of recreational mathematics, some of whom were long time colleagues.

One does not normally include a list of contents in a book review, but by doing so, it will give the reader a good indication of the wealth of recreational material here. So if you have any interest in recreational mathematics, this is a book for you. And even if you do not have such an interest, reading this book may give you one.

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Postscript: There is a typographical error in Andy Liu's article on page 206. The number of tetromino es is five and not four. This was corrected in the reprint of this article as Appendix C in the new edition of Golomb's "Polyominoes" [reviewed on [1995: 16-18] — Ed.].

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PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before July 1, 1996, although solutions received after that date will also be considered until the time when a solution is published.

2091. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Four points A, B, C, D are on a line in this order. We put $AB = a$, $BC = b$, $CD = c$. Equilateral triangles ABP , BCQ and CDR are constructed on the same side of the line. Suppose that $\angle PQR = 120^\circ$. Find the relation between a, b and c .

2092. *Proposed by K. R. S. Sastry, Dodballapur, India.*

I take a three-digit base-ten integer (in which the first digit is nonzero) and consider it as a number in a different base. If I convert this new number into base ten, I find that it is exactly twice the original number. In what base does this happen?

2093*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let A, B, C be the angles (in radians) of a triangle. Prove or disprove that

$$(\sin A + \sin B + \sin C) \left(\frac{1}{\pi - A} + \frac{1}{\pi - B} + \frac{1}{\pi - C} \right) \leq \frac{27\sqrt{3}}{4\pi} .$$

2094. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton.*

The problemist Victor Thébault has noted (*American Mathematical Monthly* Vol. 66 (1959), p. 65) an interesting Pythagorean triangle in which the two perpendicular sides are integers having the same digits in reverse order, viz., 88209 and 90288, with hypotenuse 126225.

(a) Can such a Pythagorean triangle be primitive?

(b) Find an example of a primitive Pythagorean triangle in which the hypotenuse and one other side are integers having the same digits in reverse order.

2095. *Proposed by Murray S. Klamkin, University of Alberta.*

Prove that

$$\alpha^x(y - z) + \alpha^y(z - x) + \alpha^z(x - y) \geq 0$$

where $\alpha > 0$ and $x > y > z$.

2096. *Proposed by D. J. Smeenk, Zaltbommel, The Netherlands.*

Triangle $A_1A_2A_3$ has circumcircle Γ . The tangents at A_1, A_2, A_3 to Γ intersect (the productions of) A_2A_3, A_3A_1, A_1A_2 respectively in B_1, B_2, B_3 . The second tangent to Γ through B_1, B_2, B_3 touches Γ at C_1, C_2, C_3 respectively. Show that A_1C_1, A_2C_2, A_3C_3 are concurrent.

2097. *Proposed by Federico Ardila, student, Massachusetts Institute of Technology, Cambridge.*

Let n be a positive integer and p a prime number. Prove that

$$p^n(p^n - 1)(p^{n-1} - 1)\dots(p^2 - 1)(p - 1)$$

is divisible by $n!$.

2098. *Proposed by John Magill, Brighton, England.*

At the conclusion of our first inter-species soccer tournament, in which each team played each of the others once, the scoresheets, prepared by the Zecropians and the Valudians, were, respectively,

	Won	Drawn	Lost	Goals against	Goals for	Points
Zecropia	c	b	b	cc	ffh	d
Earth Station	b	b	c	fbe	ff	b
Valudia	f	b	f	ah	db	c

	Won	Drawn	Lost	Goals against	Goals for	Points
Valudia	p	x	p	pxq	ppr	r
Zecropia	r	x	x	mm	mrx	q
Earth Station	x	x	r	rqp	pq	x

Each scoresheet is equivalent to the other in that both give the correct values. Each, however, is in the fixed-base positional number systems of those who prepared the scoresheets, each base being less than 10 and greater than 1. Both Valudians and Zecropians use the same operations of addition, subtraction, division and multiplication, and rules of manipulation, as are used by Earth. I have substituted letters for the symbols originally used. Each letter represents a digit, the same digit wherever it appears. Two points were awarded for a win and one for a draw.

As the answer to this puzzle, state the total number of goals scored by each team, and the total number of goals scored against each team — in the base 10 number system.

2099. *Proposed by Proof, Warszawa, Poland.*

The tetrahedron T is contained inside the tetrahedron W . Must the sum of the lengths of the edges of T be less than the sum of the lengths of the edges of W ?

2100. *Proposed by Iliya Bluskov, student, Simon Fraser University, Burnaby, B. C.*

Find 364 five-element subsets A_1, A_2, \dots, A_{364} of a 17-element set such that $|A_i \cap A_j| \leq 3$ for all $1 \leq i < j \leq 364$.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1353. [1988: 174; 1989: 237] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

(a) Find a linear recurrence with constant coefficients whose range is the set of all integers.

(b)* Is there a linear recurrence with constant coefficients whose range is the set of all Gaussian integers (complex numbers $a + bi$ where a and b are integers)?

III. Comment by the editor.

A recent article, "Some problems concerning recurrence sequences" by G. Myerson and A. J. van der Poorten, on pages 698-705 of the October 1995 *American Mathematical Monthly*, answers part (b) in the negative (as expected), and also a similar question (raised by the editor on [1989: 238]) in the negative.

* * * * *

2001. [1995: 19] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Three similar triangles DBC , ECA , FAB are drawn outwardly on the sides of triangle ABC , such that $\angle DBC = \angle ECA = \angle FAB$ and $\angle DCB = \angle EAC = \angle FBA$. Let $P = BE \cap CF$, $Q = CF \cap AD$, $R = AD \cap BE$. Prove that

$$\frac{QR}{AD} = \frac{RP}{BE} = \frac{PQ}{CF}.$$

Solution by Cyrus C. Hsia, student, Woburn Collegiate, Scarborough, Ontario.

If AD , BE and CF are concurrent, the result is obviously true, so assume $P \neq Q \neq R$.

Construct the point G such that AG is parallel and equal to EB , forming the parallelogram $AEBG$. Let the ratio of the corresponding sides of triangles BCD , CAE and ABF be $a : b : c$; i.e., the sides of BCD are ax, ay, az , the sides of CAE are bx, by, bz , and the sides of ABF are cx, cy, cz , as shown.

First we show that triangles BFG and ABC are similar. Since $AEBG$ is a parallelogram, $GB = AE = by$; $BF/BG = c/b = AB/AC$; and $\angle GBF = \angle CAB$ (since $\angle GBA = \angle EAB$ and $\angle EAC = \angle FBA$). Thus the triangles are similar and $GF = ay = DC$.

Further,

$$\begin{aligned}\angle GFC + \angle DCF &= (\angle GFB + \angle BFC) + (\angle DCB + \angle BCF) \\ &= \angle GFB + (\angle BFC + \angle BCF) + \angle DCB \\ &= \angle ABC + (180^\circ - \angle FBC) + \angle FBA \\ &= \angle ABC + 180^\circ - (\angle FBA + \angle ABC) + \angle FBA = 180^\circ.\end{aligned}$$

This means that $GF \parallel DC$ and since $GF = DC$ we have a parallelogram $FGDC$. Thus $GD = FC$ and $GD \parallel FC$. Finally, this means that triangles PQR and GDA are similar; i.e., $QR/DA = RP/AG = PQ/GD$. But $AG = BE$ and $GD = FC$ so $QR/AD = RP/BE = PQ/CF$ as required.

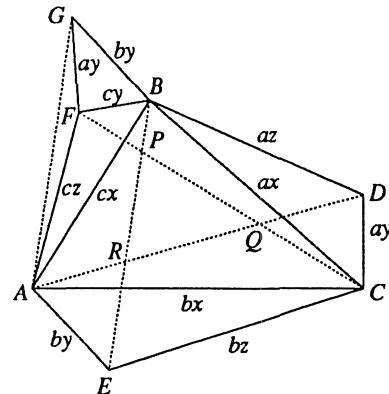
Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I. B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; JORDI DOU, Barcelona, Spain; PETER DUKES, student, University of Victoria, B.C.; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; ASHISH KR. SINGH, Kanpur, India; D. J. SMEENK, Zaltbommel, The Netherlands; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; and the proposer.

Bellot Rosado and López Chamorro point out that this is a generalization of the theorem of Napoleon:

The centroids of equilateral triangles erected (externally) on the sides of any triangle again form an equilateral triangle.

A similar problem, posed by J. Neuberg, was solved in Mathesis, 1928, p. 314:

On the sides of the triangle ABC , outwardly, the directly similar triangles BCA' , CAB' , ABC' are constructed. The segments AA' ,



BB' , CC' are equipolents to the sides of a triangle $A''B''C''$. In which cases are the triangles $A'B'C'$ and $A''B''C''$ similar?

They also point out that there are a number of related problems with similar configurations using various types of triangles, squares, etc.

* * * *

2003. [1995: 19] Proposed by Richard K. Guy, University of Calgary.

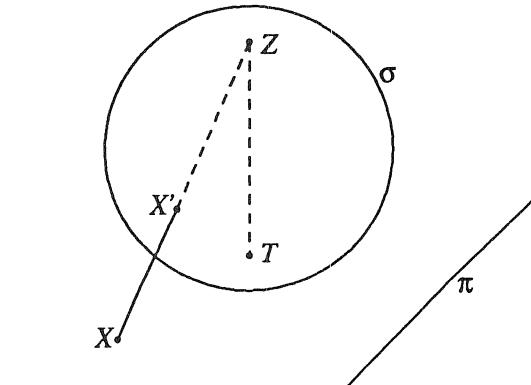
Given any five points in the plane with no three in line and no four on a circle, show that there are (at least) four sets of three points such that the circles through them have just one of the remaining points inside and one outside.

I. *Solution by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas.*

Take any two points A and B and consider the remaining three points, C, D and E . If the remaining points are all on one side of line AB , assume without loss of generality that circle ABD has a larger radius than circle ABC and a smaller radius than circle ABE . Then it contains C and does not contain E . If two of the remaining points are on one side and one, say E , is on the other side, then without loss of generality let the radius of circle ABD be larger than the radius of circle ABC . Consider circle ABD . If it does not contain E , then we have a circle through A, B and D containing C and not E . If it does contain E , then circle ABC contains all the points on E 's side of line AB that circle ABD contained, including E , so we have a circle through A, B and C containing E and not D .

There are $\binom{5}{2} = 10$ choices for A and B , so we get at least ten circles, each counted exactly three times. Hence there are at least four circles.

II. *Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*



Let σ be a sphere touching the plane π of the given points at T [which need not be one of the given points], and let Z be the other endpoint of

the diameter of σ through the point of tangency T . With Z as centre of an inverse stereographic projection, we project π onto $\sigma - \{Z\}$. This yields points A', B', C', D', E' as images of the given points A, B, C, D, E . It is known that this projection maps circles on π into circles on $\sigma - \{Z\}$ and vice versa, and it is evident that a circle separates points P and Q in π if and only if its image separates P' and Q' on σ . Hence we are done if we can show that there are at least four sets of three points out of A', B', C', D', E' such that the circles through them separate the remaining two points on the surface of σ . [Note that, for example, the circle through A', B', C' on σ cannot contain Z because this would mean that A, B, C were collinear in π . — Ed.]

Now we consider the convex hull of $A'B'C'D'E'$. Since no four of the given points A, B, C, D, E are concyclic, no four of the points A', B', C', D', E' are concyclic. Hence the convex hull of $A'B'C'D'E'$ is a polyhedron whose faces are n triangles. Since these triangles have together $3n$ edges, each belonging to two faces, $A'B'C'D'E'$ has $3n/2$ edges. According to Euler's formula we get

$$v - e + f = 2 \implies 5 - \frac{3n}{2} + n = 2 \implies n = 6.$$

Every plane containing one of the 6 faces of $A'B'C'D'E'$ cuts σ along a circle with the remaining two vertices of the polyhedron lying on the same side of this plane. There remain $\binom{5}{3} - 6 = 10 - 6 = 4$ more planes determined by three of the points A', B', C', D', E' which contain no face of the polyhedron. Hence each of these planes cuts the polyhedron into two pieces, with the remaining two points lying on different sides of the plane. Thus we get exactly four circles through three of the five points A', B', C', D', E' on the surface of σ such that in each case the remaining two points are separated by the circle, and the same result holds in the plane π . [Perz also sent in a second solution.—Ed.]

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; JORDI DOU, Barcelona, Spain; PETER DUKES, student, University of Victoria, B.C.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; DOUGLAS E. JACKSON, Eastern New Mexico University, Portales; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; and the proposer. One incorrect solution was received.

The problem was suggested by problem 23, pages 48–51 of Ross Honsberger's Mathematical Morsels (MAA, 1978), which starts with $2n + 3$ points in the plane (in general position) and asks for only one circle passing through three of them such that n of the remaining points are inside and n outside the circle. Mathematical Morsels gives two proofs, one along the lines of Solution I and one using inversion; interestingly, the solutions received for the present problem (with the exception of Solution II) were about evenly split between these two methods. The proposer used inversion to obtain at least $\lceil (n+1)(2n+3)/3 \rceil$ such circles, which gives the answer 4 for the special case $n = 1$, which was his proposal.

Dou and the proposer also show that there never are five circles with the property given in the problem, but Perz had the best argument for this.

* * * *

2004. [1995: 19] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Given are real numbers a_1, a_2, \dots, a_n with $\sum_{i=1}^n a_i = 0$. Determine

$$\sum_{i=1}^n \frac{1}{a_i(a_i + a_{i+1})(a_i + a_{i+1} + a_{i+2}) \dots (a_i + a_{i+1} + \dots + a_{i+n-2})}$$

where $a_{n+1} = a_1$, $a_{n+2} = a_2$, etc., assuming that the denominators are nonzero.

[Editor's note: the condition $n \geq 2$ was accidentally left out of the problem statement.]

Solution by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.
Let

$$\begin{aligned} S &= \sum_{i=1}^n \frac{1}{a_i(a_i + a_{i+1}) \dots (a_i + a_{i+1} + \dots + a_{i+n-2})} \\ &= \sum_{i=1}^n \frac{1}{a_{i+1}(a_{i+1} + a_{i+2}) \dots (a_{i+1} + a_{i+2} + \dots + a_{i+n-1})} \end{aligned}$$

be the given sum. Let $b_i = \sum_{r=1}^i a_r$ so $b_n = 0$. Also, let $p = \prod_{r=1}^{n-1} b_r$. Since for all $j \neq i$,

$$\begin{aligned} b_j - b_i &= b_n - b_i + b_j = a_{i+1} + a_{i+2} + \dots + a_n + a_1 + a_2 + \dots + a_j \\ &= a_{i+1} + a_{i+2} + \dots + a_n + a_{n+1} + \dots + a_{n+j}, \end{aligned}$$

we have

$$\begin{aligned} S &= \sum_{i=1}^n \prod_{\substack{1 \leq j \leq n \\ j \neq i}} \frac{1}{b_j - b_i} = \frac{1}{p} + \sum_{i=1}^{n-1} \prod_{\substack{1 \leq j \leq n \\ j \neq i}} \frac{1}{b_j - b_i} \\ &= \frac{1}{p} + \sum_{i=1}^{n-1} \left(-\frac{1}{b_i} \prod_{\substack{1 \leq j \leq n-1 \\ j \neq i}} \frac{1}{b_j - b_i} \right). \end{aligned}$$

Using Lagrange's interpolation, we let

$$F(x) = \sum_{i=1}^{n-1} \left(-\frac{1}{p} \prod_{\substack{1 \leq j \leq n-1 \\ j \neq i}} \frac{b_j - x}{b_j - b_i} \right)$$

so that $F(b_k) = -1/p$ for all $1 \leq k \leq n-1$. Note that $F(x)$ is a polynomial in x of degree at most $n-2$, yet it is a constant at $n-1$ distinct values. Hence $F(x)$ is a constant, $F(x) \equiv -1/p$. Since

$$-\frac{1}{p} \prod_{\substack{1 \leq j \leq n-1 \\ j \neq t}} \frac{b_j}{b_j - b_t} = -\frac{1}{b_t} \prod_{\substack{1 \leq j \leq n-1 \\ j \neq t}} \frac{1}{b_j - b_t},$$

we get $F(0) = S - 1/p$ and thus

$$S = F(0) + \frac{1}{p} = 0.$$

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

Walther Janous sends in the correct answer without proof but recalls that the result (or something equivalent) is known. He also points out the somewhat similar problem 5 of the 1989 Canadian Mathematical Olympiad [1989: 200].

* * * * *

2005. [1995: 20] *Proposed by Murray S. Klamkin, University of Alberta.*

(a) Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be vectors from the circumcenter of a triangle ABC to the respective vertices. Prove that

$$\frac{(\mathbf{B} + \mathbf{C})|\mathbf{B} - \mathbf{C}|}{|\mathbf{B} + \mathbf{C}|} + \frac{(\mathbf{C} + \mathbf{A})|\mathbf{C} - \mathbf{A}|}{|\mathbf{C} + \mathbf{A}|} + \frac{(\mathbf{A} + \mathbf{B})|\mathbf{A} - \mathbf{B}|}{|\mathbf{A} + \mathbf{B}|} = \mathbf{0}. \quad (1)$$

(b)* Suppose that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are vectors from a point P to the respective vertices of a triangle ABC such that (1) holds. Must P be the circumcenter of ABC ?

[Editor's note. In rephrasing the proposer's original problem, the editor accidentally left off the condition that the triangle ABC is acute in part (a). In the end this did not deter most solvers from solving this part anyway, but my apologies to the others.]

I. *Solution to part (a) by Christopher J. Bradley, Clifton College, Bristol, U. K.*

The lengths of

$$\frac{\mathbf{B} + \mathbf{C}}{|\mathbf{B} + \mathbf{C}|} |\mathbf{B} - \mathbf{C}|, \quad \frac{\mathbf{C} + \mathbf{A}}{|\mathbf{C} + \mathbf{A}|} |\mathbf{C} - \mathbf{A}|, \quad \frac{\mathbf{A} + \mathbf{B}}{|\mathbf{A} + \mathbf{B}|} |\mathbf{A} - \mathbf{B}|$$

are

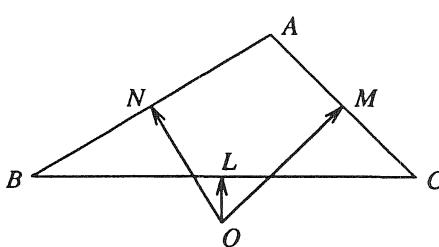
$$|\mathbf{B} - \mathbf{C}| = a, \quad |\mathbf{C} - \mathbf{A}| = b, \quad |\mathbf{A} - \mathbf{B}| = c,$$

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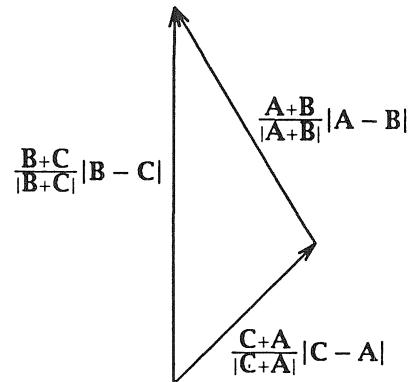
so a triangle with sides equal to these lengths must be congruent to triangle ABC . With circumcentre as origin the directions of these vectors are perpendicular to BC, CA, AB respectively. So the given equation (1) is simply a mapping of the relation

$$(\mathbf{B} - \mathbf{C}) + (\mathbf{C} - \mathbf{A}) + (\mathbf{A} - \mathbf{B}) = \mathbf{0}$$

under rotation by 90° . This is true always provided O is internal to triangle ABC . The figure below illustrates the situation for O an external point, where evidently a sign has to be adjusted appropriately.



$\overrightarrow{OL}, \overrightarrow{OM}, \overrightarrow{ON}$ are in directions
 $\mathbf{B} + \mathbf{C}, \mathbf{C} + \mathbf{A}, \mathbf{A} + \mathbf{B}$

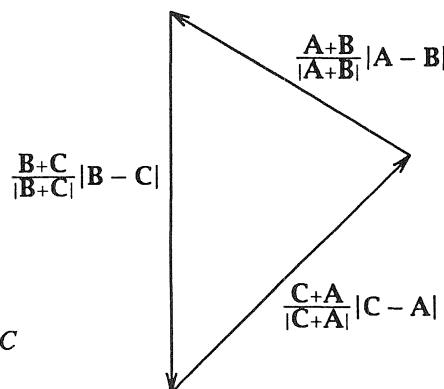
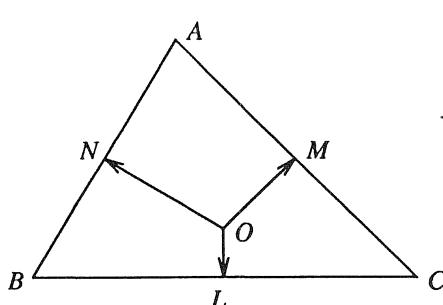


It can be seen that for $\angle A$ obtuse one has

$$\frac{(\mathbf{C} + \mathbf{A})|\mathbf{C} - \mathbf{A}|}{|\mathbf{C} + \mathbf{A}|} + \frac{(\mathbf{A} + \mathbf{B})|\mathbf{A} - \mathbf{B}|}{|\mathbf{A} + \mathbf{B}|} = \frac{(\mathbf{B} + \mathbf{C})|\mathbf{B} - \mathbf{C}|}{|\mathbf{B} + \mathbf{C}|}.$$

(When $\angle A = 90^\circ$, $\mathbf{B} + \mathbf{C}$ vanishes and the relation degenerates and requires further interpretation.)

When O is internal the situation is satisfactory, as the diagrams below show.



Now relation (1) holds.

[Bradley then proves part (b). — Ed.]

II. Solution to part (b) by John Vlachakis, Athens, Greece.

[Vlachakis first proves part (a) in the case that ABC is acute. — Ed.]

Let K, L, M be the midpoints and a, b, c the lengths of the sides BC, CA, AB respectively. Then, since $B + C = 2\vec{PK}$, etc., (1) becomes

$$a \frac{\vec{PK}}{|\vec{PK}|} + b \frac{\vec{PL}}{|\vec{PL}|} + c \frac{\vec{PM}}{|\vec{PM}|} = 0$$

or

$$a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3 = 0 \quad (2)$$

where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are unit vectors in the direction of $\vec{PK}, \vec{PL}, \vec{PM}$ respectively. We deduce from (2) that P must be in the interior of the triangle.

Now let

$$a\mathbf{u}_1 = \vec{PQ}, \quad b\mathbf{u}_2 = \vec{PR}, \quad c\mathbf{u}_3 = \vec{PS}$$

so that $\vec{PQ} + \vec{PR} + \vec{PS} = \mathbf{0}$ and $|\vec{PQ}| = a, |\vec{PR}| = b, |\vec{PS}| = c$ (see the figure). Suppose that $\vec{PQ} + \vec{PR} = \vec{PT}$ where T is the fourth vertex of the parallelogram $PRTQ$. This means that $|\vec{PT}| = |\vec{PR} + \vec{PQ}| = |\vec{PS}|$, and \vec{PT} and \vec{PS} are parallel and in opposite directions. [Thus from $|\vec{PT}| = |\vec{PS}| = c, |\vec{PR}| = b$, and $|\vec{RT}| = |\vec{PQ}| = a$, triangles ABC and PTR must be congruent. — Ed.] Thus $\angle RPT = \angle A$ and so

$$\angle SPR = 180^\circ - \angle A.$$

Also $\angle PRT = \angle C$ and so (by the parallelogram $PRTQ$)

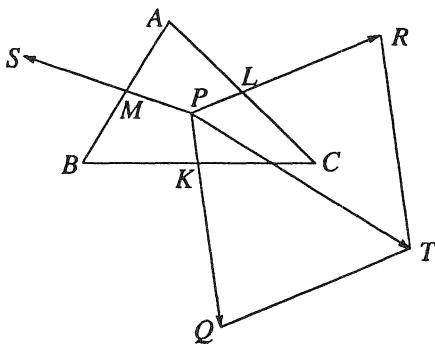
$$\angle RPQ = 180^\circ - \angle C.$$

Finally,

$$\angle SPQ = 180^\circ - \angle B.$$

These imply that P has the following property: the quadrilaterals $AMPL$, $BMPK$ and $CKPL$ are all cyclic. But this is a property that only the circumcentre O possesses. So $P = O$ and [since P is an interior point] ABC must be acute.

Also solved (both parts, (a) for acute triangles) by FEDERICO ARDILA, student, MIT, Cambridge, Massachusetts; G. P. HENDERSON, Campbellcroft, Ontario; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; WALDEMAR POMPE, student, University of Warsaw, Poland; and TOSHIO SEIMIYA, Kawasaki, Japan. Part (a) only solved (for acute triangles) by CLAUDIO ARCONCHER, Jundiaí, Brazil; PETER DUKES, student, University of Victoria, B. C.; P. PENNING, Delft, The Netherlands; ASHISH KR. SINGH,



Kanpur, India; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. Counterexamples to part (a) for nonacute triangles were given by J. K. FLOYD, Newnan, Georgia; RICHARD I. HESS, Rancho Palos Verdes, California; and WALther JANous, Ursulinengymnasium, Innsbruck, Austria.

* * * * *

YEAR-END WRAPUP

Yet another year has gone by, and it is once again time to record some of the comments, late solutions, etc. which the editor has received from readers over the past twelve months or so.

1969 [1995: 238]. Chris Fisher sends in a small clarification of the published solution (by Richard I. Hess): namely, in order to get two nonisomorphic parallelepipeds with isomorphic rhombic faces, the angle θ must be restricted to $60^\circ < \theta < 120^\circ$, $\theta \neq 90^\circ$. Chris also points out that Hess makes use of the symmetry of any squashed or elongated cube to get that the parallelepipeds are nonisomorphic.

1981 [1995: 255]. Waldek Pompe points out that the observation by Konečný (mentioned by the editor on [1995: 256]) that I lies on the circle with centre D and radius DC , has appeared in *Crux* before, and in more generality; for example, see Lemma 1 in Waldek's solution of *Crux* 1871 [1994: 199].

Late solutions were received from Hayo Ahlburg, Benidorm, Alicante, Spain (1992); Federico Ardila, student, Massachusetts Institute of Technology, Cambridge (1986, 1988, 1991, 1992); Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario (1903); and Walther Janous, Ursulinengymnasium, Innsbruck, Austria (1928, 1932, 1935, 1943, 1944, 1945(a), 1948, 1950).

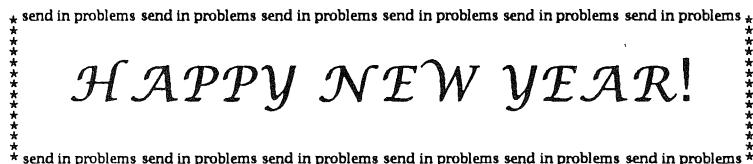
Many thanks to the following people for their assistance to the editor and other members of the Editorial Board during 1995, in giving advice regarding problems, articles, and solutions: *ED BARBEAU, LEN BOS, JOHN CIRIANI, ROLAND EDDY, PETER EHLLERS, DOUG FARENICK, BRUCE GILLIGAN, WALther JANous, STEVE KIRKLAND, MURRAY KLAMKIN, CLAUDE LAFLAMME, MICHAEL LAMOUREUX, CINDY LOTEN, JOANNE MACDONALD, JUDY MACDONALD, PIETER MOREE, STANLEY RABINOWITZ, DIETER RUOFF, JONATHAN SCHAEER, JIM TOMKINS, MICHAEL TSATSOMEROS, CHARLTON WANG, SIMING ZHAN*. The editor would also like to thank *JAN CERNY, MIKE LOGOZAR, and ARUNAS SALKAUSKAS* for their efforts during the last couple of years to get *Crux* produced in a new format and on a better computer. The result has been a more attractive product and fewer headaches for the editor!

As usual, special thanks are due to the members of the *Crux* Editorial Board for their constant contributions to the quality of our journal. In particular, and for the record, the following Board members had the major role in editing and writing up the published solution for the following problems,

appearing in 1995: *CATHY BAKER* — 1939, 1954, 1958, 1980, 1983, 1993 and 2001; *CHRIS FISHER* — 1910, 1919, 1928, 1935, 1967, 1979, 1987 and 1997; *JIM TOTTEN* — 1933, 1948, 1956, 1966, 1978, 1992 and 2000; *ED WANG* — 1917, 1922, 1936, 1944, 1989, 1996 and 2004. In addition, the efforts of *ANDY LIU* and *DENIS HANSON* on behalf of *Crux* are visible issue after issue, in the vitality of the regular *Crux* features Book Reviews and Articles. There is also *RICHARD GUY*'s vast and eclectic knowledge (not to mention library!), which the editor must have tapped almost daily over the past ten years. And two more people, not usually mentioned here: *GRAHAM WRIGHT*, for his substantial (though invisible) contributions to *Crux*; and my co-editor and colleague *ROBERT WOODROW*, obviously for nine years of Olympiad Corners and one year of Skoliad Corners!

Finally, thanks again to *Crux* typist *JOANNE LONGWORTH* for devoting innumerable lunch hours, evenings and weekends putting *Crux* into L^AT_EX.

And now an admission, no longer possible to delay: this is my final issue as editor of *Crux*; after ten years I wish to have time for other things, and *Crux* passes on to other hands, namely the capable ones of Bruce Shawyer, who will be introducing himself to you next issue. If you are on email and want to say hello, you can write him at *cruxeditor@cms.math.ca*! I will not be disappearing from *Crux* entirely, I hope, because (among other things) I intend to propose problems regularly, knowing first-hand how much *Crux* is in need of a regular supply of problems from its readers, and I urge you all to do the same. (And, by the way, how about sending Denis more articles, maybe some unusual ones on teaching, or math contests, for example? It is in your power to make *Crux* into what you want.) There is no denying the amount of time and work required to edit a journal like *Crux*, but it is equally true that the rewards are many and great. For this I thank all of you out there who have kept my mailbox (and more recently my email) brimming with your letters. So for the last time, fellow *Crux* fans,



* send in problems *

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