Crux Mathematicorum

VOLUME 44, NO. 4 April / Avril 2018

IN THIS ISSUE / DANS CE NUMÉRO

133 The Contest Corner: No. 64 *John McLoughlin*

  133 Problems: CC316–CC320

  137 Solutions: CC266–CC270

140 The Honsberger Corner

145 The Olympiad Corner: No. 362

  145 Problems: OC376–OC380

  147 Solutions: OC316–OC320

151 The Method of Indirect Descent (Part II) *Adib Hasan and Thanic Nur Samin*

157 Problem Solving 101: No. 4 *Shawn Godin*

160 Problems: 4331–4340

164 Solutions: 4231–4240

178 Solvers and proposers index

Crux Mathematicorum

Founding Editors / Rédacteurs-fondateurs: Léopold Sauvé & Frederick G.B. Maskell

Former Editors / Anciens Rédacteurs: G.W. Sands, R.E. Woodrow, Bruce L.R. Shawyer, Shawn Godin

Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin

THE CONTEST CORNER

No. 64

John McLoughlin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er septembre 2018**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

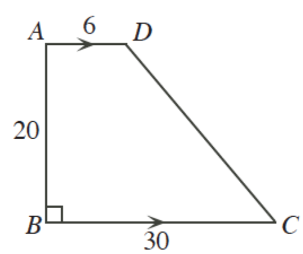
CC294. (Correction.)

a) Démontrer que $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$, A n'étant pas un multiple impair de $\frac{\pi}{2}$.

b) Sachant que $\sin 2A = \dfrac{4}{5}$, déterminer $\tan A$.

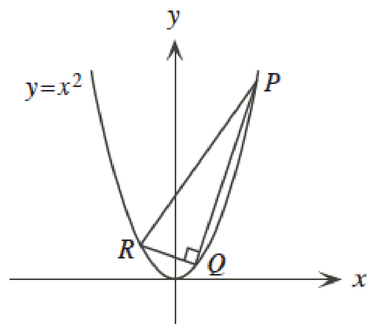
CC316. On dit que trois entiers strictement positifs x, y et z forment un *triplet Trenti* (x, y, z) si $3x = 5y = 2z$. Démontrer que dans tout triplet Trenti (x, y, z), le produit xyz doit être divisible par 900.

CC317. Dans la figure, $ABCD$ est un trapèze, AD est parallèle à BC et BC est perpendiculaire à AB. De plus, $AD = 6$, $AB = 20$ et $BC = 30$.



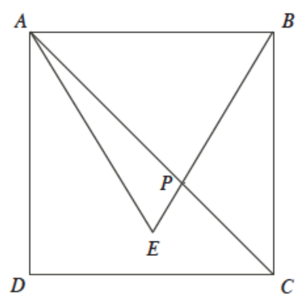
1. Déterminer l'aire du trapèze $ABCD$.
2. Il existe un point K, sur AB, de manière que l'aire du triangle KBC soit égale à l'aire du quadrilatère $KADC$. Déterminer la longueur du segment BK.
3. Il existe un point M, sur DC, de manière que l'aire du triangle MBC soit égale à l'aire du quadrilatère $MBAD$. Déterminer la longueur du segment MC.

CC318. Dans la figure, le triangle rectangle PQR est inscrit dans la parabole d'équation $y = x^2$.



Les points P, Q et R ont pour coordonnées respectives $(p, p^2), (q, q^2)$ et (r, r^2). Démontrer que $2q + p + r = 0$ lorsque p, q et r sont des entiers.

CC319. Dans la figure, le carré $ABCD$ a des côtés de longueur 4 et le triangle ABE est équilatéral. Les segments BE et AC se coupent en P. Déterminer l'aire exacte du triangle APE.



CC320. Une suite formée de m fois la lettre P et de n fois la lettre Q $(m > n)$ est appelée *non prédictive* s'il y a un endroit dans la suite où le nombre de lettres Q comptées à partir de la gauche est supérieur ou égal au nombre de lettres P comptées à partir de la gauche. Par exemple, si $m = 5$ et $n = 2$, les suites $PPQQPPP$ et $QPPPQPP$ sont non prédictives.

1. Soit $m = 7$ et $n = 2$. Déterminer le nombre de suites non prédictives qui commencent par la lettre P.

2. Soit $n = 2$. Démontrer que pour toute valeur de m supérieure à 2, le nombre de suites non prédictives qui commencent par la lettre P est égal au nombre de suites non prédictives qui commencent par la lettre Q.

3. Déterminer combien il existe de suites non prédictives lorsque $m = 10$ et $n = 3$.

. . . .

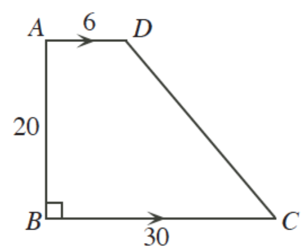
CC294. (Correction.)

a) Prove that $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$, where A is not an odd multiple of $\frac{\pi}{2}$.

b) If $\sin 2A = 4/5$, find $\tan A$.

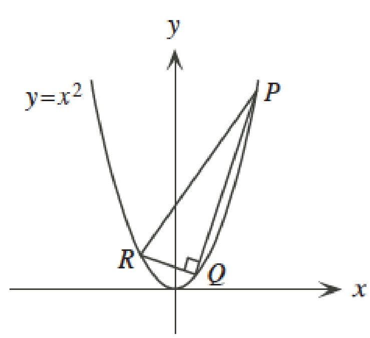
CC316. Positive integers (x, y, z) form a *Trenti-triple* if $3x = 5y = 2z$. Show that for every Trenti-triple (x, y, z) the product xyz must be divisible by 900.

CC317. In the diagram, $ABCD$ is a trapezoid with AD parallel to BC and BC perpendicular to AB. Also, $AD = 6$, $AB = 20$, and $BC = 30$.



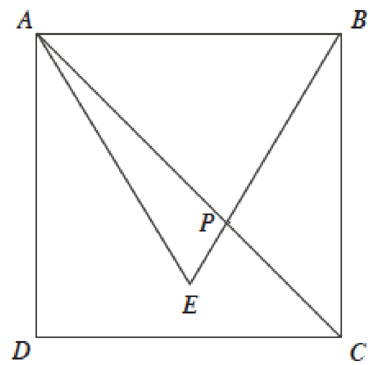
1. Determine the area of trapezoid $ABCD$.
2. There is a point K on AB such that the area of triangle KBC equals the area of quadrilateral $KADC$. Determine the length of BK.
3. There is a point M on DC such that the area of triangle MBC equals the area of quadrilateral $MBAD$. Determine the length of MC.

CC318. Right-angled triangle PQR is inscribed in the parabola with equation $y = x^2$, as shown.



Points P, Q and R have coordinates $(p, p^2), (q, q^2)$ and (r, r^2), respectively. If p, q and r are integers, show that $2q + p + r = 0$.

CC319. In the diagram, square $ABCD$ has sides of length 4, and triangle ABE is equilateral. Line segments BE and AC intersect at P. Determine the exact area of triangle APE.



CC320. A sequence of m P's and n Q's with $m > n$ is called *non-predictive* if there is some point in the sequence where the number of Q's counted from the left is greater than or equal to the number of P's counted from the left. For example, if $m = 5$ and $n = 2$, the sequences $PPQQPPP$ and $QPPPQPP$ are non-predictive.

1. If $m = 7$ and $n = 2$, determine the number of non-predictive sequences that begin with P.
2. Suppose that $n = 2$. Show that for every $m > 2$, the number of non-predictive sequences that begin with P is equal to the number of non-predictive sequences that begin with Q.
3. Determine the number of non-predictive sequences with $m = 10$ and $n = 3$.

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2017: 43(4), p. 124–125.

CC266. Given a 5×5 array of lattice points, draw a set of circles that collectively pass through each of the lattice points exactly once.

Problem from Ross Honsberger's notes (labelled as "Eddie's problem").

We received three correct solutions. We present the solution of Stephen Chow.

Pick any point O in the plane which is not among our given points. Let R be the set of distinct distances between O and the set of points. Draw the set of circles centered at O and having radii from R. It is clear these circles will pass through each of our given points. Since the circles are all concentric, no pair will intersect, and thus no point we are given will appear on two different circles.

Editor's Remark. There are many solutions to the problem as stated. In fact, this was most likely intended as the first step towards a more meaningful and complicated problem. We propose the following two-part extension.

1. Given a 5×5 array of lattice points, show that you can draw a set of 5 circles that collectively pass through each of the lattice points exactly once.
2. Show that this cannot be done with four circles, even if two circles are allowed to pass through the same point.

CC267. Each of ten people around a circle chooses a number and tells it to the neighbour on each side. Thus each person gives out one number and receives two numbers. The players then announce the average of the two numbers they received. Remarkably, the announced numbers, in order around the circle, were $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. What was the number chosen by the person who announced number 6?

Problem from Ross Honsberger's private collection.

We received six correct solutions. We present the solution of Digby Smith.

For $k = 1, 2, \ldots, 10$, let x_k be the number chosen by the person who announced k. The following ten equations hold,

$$\begin{array}{lllll} x_{10} + x_2 = 2, & x_1 + x_3 = 4, & x_2 + x_4 = 6, & x_3 + x_5 = 8, & x_4 + x_6 = 10, \\ x_5 + x_7 = 12, & x_6 + x_8 = 14, & x_7 + x_9 = 16, & x_8 + x_{10} = 18, & x_9 + x_1 = 20. \end{array}$$

It follows that

$$2x_6 = (x_4+x_6)+(x_6+x_8)+(x_{10}+x_2)-(x_2+x_4)-(x_8+x_{10}) = 2, \quad \text{and so} \quad x_6 = 1.$$

CC268. Find the smallest term in the sequence

$$\sqrt{\frac{7}{6}}+\sqrt{\frac{96}{7}},\ \sqrt{\frac{8}{6}}+\sqrt{\frac{96}{8}},\ldots,\sqrt{\frac{n}{6}}+\sqrt{\frac{96}{n}},\ldots,\sqrt{\frac{95}{6}}+\sqrt{\frac{96}{95}}.$$

Problem from Ross Honsberger's private collection.

We received 16 correct solutions. We present the solution of Henry Ricardo.

For any positive integer n, the arithmetic mean - geometric mean inequality yields,

$$\sqrt{\frac{n}{6}}+\sqrt{\frac{96}{n}} \geq 2\sqrt{\sqrt{\frac{n}{6}}\sqrt{\frac{96}{n}}} = 2\sqrt{\sqrt{16}} = 4$$

with equality if and only if $\sqrt{\frac{n}{6}} = \sqrt{\frac{96}{n}}$, or rather $n = 24$. This gives the smallest term in the sequence, $\sqrt{\frac{24}{6}}+\sqrt{\frac{96}{24}} = 4$.

CC269. One is given integers $1, 2, 3, \ldots, n$ and is required to write an ordered sequence with the following properties: no two adjacent elements are the same and within the ordered sequence the subsequence $\ldots a \ldots b \ldots a \ldots b \ldots$ may not appear. For example, if $n = 5$, the sequence 123524 obeys the properties, but 1235243 does not, since the subsequence 2323 occurs.

a) Prove that any sequence obeying these rules must contain an element x which appears only once in the sequence.

b) Write down such a sequence of length $2n - 1$ in which all elements but one occur more than once.

Originally from the Descartes Mathematics Competition, 1976.

We received 3 correct submissions. We present the solution by Steven Chow.

a) It must be assumed that $n \geq 2$ and that the length of the sequence is not 1.

Towards a contradiction, assume that there is a sequence obeying the rules that does not contain an element x which appears only once in the sequence.

Let a be an element such that there is the subsequence $\ldots a \ldots a \ldots$, and there is no subsequence $\ldots x \ldots x \ldots$ between the two a elements. (Otherwise, let $a = x$.) Let b be an element between the two a elements. This exists since no two adjacent elements are the same.

From the assumption that no element x appears only once and the definition of a, b is either to the left of the 'first' a or to the right of the 'second' a, i.e., there is the subsequence $\ldots b \ldots a \ldots b \ldots a$ or $\ldots a \ldots b \ldots a \ldots b$. This is a contradiction.

Therefore any sequence obeying the rules must contain an element x which appears only once in the sequence.

b) The sequence formed by putting j for all integers j from 2 to n, then putting 1, then putting j for all integers j from n to 2, is one such sequence. That is,

$$2, 3, 4, \dots, n, 1, n, \dots, 4, 3, 2.$$

CC270. It is possible to draw straight lines through the point $(3, -2)$ so that for each line the sum of the intercepts is equal to three times its slope. Find the sum of the slopes of all such lines.

Originally problem 28, Junior Mathematics Contest 1975.

We received 9 correct submissions and one incorrect submission. We present the solution by Ricard Peiró i Estruch.

The equation of the line that passes through the point $(3, -2)$ and has slope m is

$$y + 2 = m(x - 3).$$

If $y = 0$, then $x = \frac{2}{m} + 3$. So the point on the x-axis is $(\frac{2}{m} + 3, 0)$. The x-intercept is $\frac{2}{m} + 3$.

If $x = 0$, then $y = -3m - 2$. So the point on the y-axis is $(0, -3m - 2)$. The y-intercept is $-3m - 2$.

The sum of the intercepts is equal to three times its slope, so:

$$\left(\frac{2}{m} + 3\right) + (-3m - 2) = 3m.$$

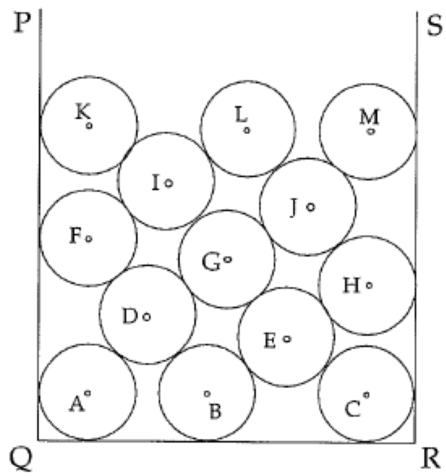
Simplifying, $6m^2 - m - 2 = 0$. Solving the equation, $m = -\frac{1}{2}$ or $m = \frac{2}{3}$.

The sum of the two roots is $\frac{1}{6}$.

THE HONSBERGER CORNER

Les énoncés des problèmes dans cette section ont paru initialement dans 2017: 43(4), p. 130–134. Ce sont quelques-uns des problèmes préférés de Ross Honsberger.

H1. Across the bottom of rectangular wine rack $PQRS$, there is room for more than three bottles (A, B, C) but not enough for a fourth bottle (see the figure). Naturally, bottles A and C are laid against the sides of the rack and a second layer, consisting of just two bottles D and E, holds B in place somewhere between A and C. Now we can lay in a third row of three bottles (F, G, H), with F and H resting against the sides of the rack. Then a fourth layer is held to just two bottles I and J, but a fifth layer can accommodate three bottles (K, L, M).



If the bottles are all the same size, prove that, whatever the spacing of (A, B, C) in the bottom layer, the fifth layer is always perfectly horizontal.

This is problem 44 from "Which Way Did the Bicycle Go? And Other Intriguing Mathematical Mysteries" by Joseph D. E. Konhauser, Dan Velleman and Stan Wagon (Cambridge University Press, 1996).

We received 5 submissions, of which 4 were correct and complete. We present the solution by Steven Chow, modified by the editor.

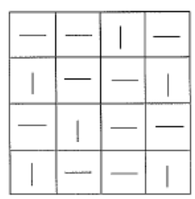
We will write A for the center of the circle representing bottle A, and so on. Since all the bottles are the same size, the distance between the centers of the circles corresponding to any two tangent bottles is twice the common radius, and the point of tangency of any two tangent bottles is the midpoint of the segment connecting the centers.

Note that $\angle BAF = 90°$ (bottles B, A and F are tangent to the rectangular rack). Since A, B, F and G are equidistant from D, the quadrilateral $ABFG$ is cyclic and D is the center of its circumcircle. It follows that $\angle BDF = 2\angle BAF = 180°$, so B, D and F are collinear. Similarly, B, E and H are collinear.

Since all its sides are equal, $BEGD$ is a rhombus, as are $EHJG$, $DGIF$ and $GJLI$. Thus $DG||BE$ and $GJ||EH$; the fact that B, E and H are collinear implies that D, G and J are as well. A similar argument shows F, I and L are also collinear. Since I is on the line segment FL and also the circumcenter of $\triangle FKL$, we have $\angle FKL = 90°$. Bottles K and A are both tangent to the rectangular rack, and so $KL||AB$. Similarly, $ML||CB$. Since the first layer A, B, C is horizontal, this shows that the centers K, L, M are collinear and the fifth layer of bottles is horizontal.

H2. You are given a safe with the lock consisting of a 4×4 arrangement of keys. Each of the 16 keys can be in a horizontal or a vertical position. To open the safe, all the keys must be in the vertical position. When you turn a key, all the keys in the same row and column change positions. You may turn a key more than once.

(a) Given the configuration in the figure below, how do you open the safe?



(b) If you are allowed to turn at most 2002 keys, what is the largest $2n \times 2n$ safe that you can always open?

Problem from Ross Honsberger's private collection.

We received two solutions out of which we present the one by Joel Schlosberg, slightly modified by the editor.

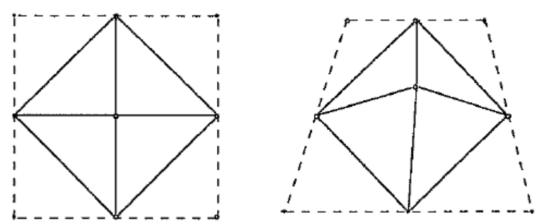
(a) First turn the key in the first row from the top and the second column from the left, then the key in the third row from the top and the third column from the left.

(b) If we choose a key and turn it, as well as each of the $4n-2$ keys that are either in the same row or the same column as that key, then this key changes position $4n-1$ times and thus switches its position. The keys in the same row or column change position $2n$ times and all remaining keys change position 2 times. The only key changed from its original position is the chosen key. By applying this method to each key in the horizontal position, any configuration of the safe can be unlocked.

Note that the order in which keys are turned does not affect the result. Furthermore any two turns of the same key cancel out each other's effect, so each particular key need only be turned at most once. There are $2^{(2n)^2}$ possible configurations for the $(2n)^2$ safe keys (since each key can be in one of two possible positions, horizontal or vertical) and $2^{(2n)^2}$ ways to turn each of the $(2n)^2$ keys at most once. Since for each configuration there exists a way to turn the keys to unlock the safe, each configuration uniquely corresponds to one of the ways to turn the keys.

The configuration needing the most key turns (all $(2n)^2$ keys) is the one with all keys in the horizontal position. In particular, a 46×46 safe can require $46^2 = 2116 > 2002$ key turns to open, while a 44×44 safe can be opened with at most $44^2 = 1936 < 2002$ key turns and is thus the largest $2n \times 2n$ safe that can always be opened with 2002 key turns.

H3. Clearly in the left figure below, the four corners of a square can be folded over to meet at a point without overlapping or gaps; another such quadrilateral is illustrated in the figure on the right.



Determine the necessary and sufficient conditions for such a folding of the corners of a quadrilateral.

Problem from Ross Honsberger's private collection.

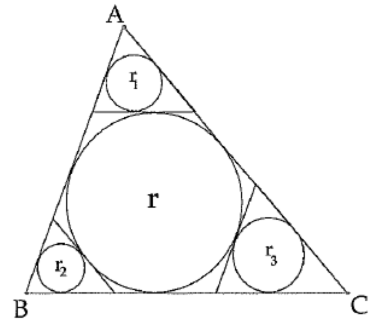
We received four correct submissions. All solvers followed essentially the same approach presented below.

It is necessary and sufficient that the quadrilateral be convex and its diagonals are perpendicular. The convexity is clear, and we assume this property.

Suppose that the folding is possible. Let A, B, C, D be the vertices of the quadrilateral and P, Q, R, S the points where the folds meet the respective sides AB, BC, CD, DA. If the folds meet at O, then $AP = OP = BP$, so that P is the midpoint of AB and triangle AOB is right. Hence $AO \perp BO$. Similarly $BO \perp CO$, $CO \perp DO$ and $DO \perp AO$. Thus AOC and BOD are perpendicular straight lines constituting the diagonals.

On the other hand, suppose that the diagonals AC and BD are perpendicular, meeting at O, and that P, Q, R, S are the midpoints of the sides. Then $PS \| BD$, $PS \perp AC$ and PS is equidistant from A and O. Since O is the reflected image of A about PS, triangles APS and OPS are congruent. Similarly $\Delta BQP \equiv \Delta OQP$, $\Delta CRQ \equiv \Delta ORQ$ and $\Delta DSR \equiv \Delta OSR$, so that triangles APS, BQP, CRQ and DSR fold in to exactly cover rectangle $PQRS$.

H4. Tangents to the incircle of $\triangle ABC$ are drawn parallel to the sides to cut off a little triangle at each vertex (see figure).



Prove that the inradii of the three small circles add up to the inradius of $\triangle ABC$; that is $r_1 + r_2 + r_3 = r$.

From "Problems in plane and solid geometry" by Viktor Prasolov.

We received nine submissions; all of them were based on the observation that the three small triangles are similar to the given triangle ABC, which was likely the "Aha moment" that Ross Honsberger found so attractive about the problem. We present a composite of the similar solutions from C.R. Pranesachar, Joel Scholsberg, Titu Zvonaru, and the Missouri State University Problem Solving Group.

Let h_a be the length of the altitude of ΔABC from A to BC. The length of the corresponding altitude from A to the opposite side of the little triangle at A is $h_a - 2r$, since its side opposite A is on a line parallel to BC and closer to A by a distance of $2r$. Note that because their corresponding sides are parallel, the two triangles are similar; exploiting that similarity, we see that

$$\frac{r_1}{r} = \frac{h_a - 2r}{h_a}.$$

Since the area of $\triangle ABC$ is given by both $\frac{1}{2}ah_a$ and $\frac{1}{2}(a+b+c)r$,

$$\frac{h_a - 2r}{h_a} = \frac{(a+b+c) - 2a}{a+b+c}.$$

Similarly

$$\frac{r_2}{r} = \frac{(a+b+c) - 2b}{a+b+c}, \quad \text{and} \quad \frac{r_3}{r} = \frac{(a+b+c) - 2c}{a+b+c}.$$

Consequently,

$$\frac{r_1 + r_2 + r_3}{r} = \frac{3(a+b+c) - 2a - 2b - 2c}{a+b+c} = 1,$$

as desired.

Editor's comments. Along with his solution, J. Chris Fisher observed that H4 is closely related to Problem 2.6.4 in H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems: San Gaku*, namely,

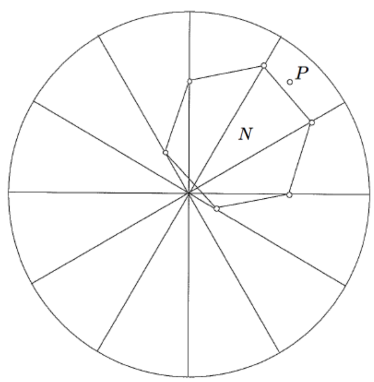
$I(r)$ is the incircle of triangle ABC. The three circles $O_i(r_i)$ $(i = 1, 2, 3)$ are such that $O_1(r_1)$ passes through B and C and touches $I(r)$, surrounding it, and the other two behave in a similar fashion. The circles $O_i'(r_i')$, $(i = 1, 2, 3)$ are such that $O_1'(r_1')$ touches AB and AC and touches $O_1(r_1)$ externally, and the other two circles behave similarly. Show that $r_1' + r_2' + r_3' = r$.

A graphics program suggests that the three small circles are the same in both problems, specifically,

For any triangle ABC let α be the circle through B and C that surrounds the incircle and is tangent to it, while β is a circle tangent to the sides AB and AC. Then β is externally tangent to α if and only if it is also tangent to the line parallel (but not equal) to BC that is tangent to the incircle.

Does anybody see how to prove this conjecture?

H5. A circle is divided into equal arcs by n diameters (see the figure). Prove that the feet of the perpendiculars to these diameters from a point P inside the circle always determine a regular n-gon N.



Problem from Ross Honsberger's private collection.

We received six correct solutions from five individuals. All solvers followed essentially the same approach presented below.

The problem can be solved for any point P in the plane except the centre O of the circle. Let X be the foot of the perpendicular from P to one of the diameters extended. Since $\angle PXO = 90°$, X lies on the circle with diameter PO. Thus, each vertex of N lies on the same circle with diameter OP. Each edge of N subtends at O the common angle $2\pi/n$ between adjacent diameters of the given circle. Hence, the edges of N are equal and N is a regular $n-$gon.

Editor's comments. The Missouri State University Problem Solving Group used complex numbers, beginning with the formula to determine the orthogonal projection $(z+\bar{z}w/\bar{w})/2$ of z onto the complex number w. With z representing P and the given circle centred at the origin, the vertices of N are found to be

$$(z/2) + (\bar{z}/2)(\cos 2k\pi/n + i \sin 2k\pi/n), \quad (0 \leq k \leq n-1).$$

THE OLYMPIAD CORNER

No. 362

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er septembre 2018**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

OC376. Soit m un entier strictement positif et a et b des entiers distincts supérieurs à m^2 et inférieurs à m^2+m. Déterminer tous les entiers d tels que d soit un diviseur de ab et $m^2 < d < m^2+m$.

OC377. Démontrer que l'équation $x - \frac{1}{x} + y - \frac{1}{y} = 4$ n'admet aucune solution dans l'ensemble des nombres rationnels.

OC378. On définit une suite (a_n) par

$$S_1 = 1,\ S_{n+1} = \frac{(2+S_n)^2}{4+S_n}(n = 1,\ 2,\ 3,\ \cdots),$$

S_n étant la somme des n premiers termes de la suite (a_n). Pour tout entier strictement positif n, démontrer que

$$a_n \geq \frac{4}{\sqrt{9n+7}}.$$

OC379. Soit n un entier supérieur ou égal à 3 et $a_1, a_2, \ldots, a_n$ des réels strictement positifs tels que

$$\frac{1}{1+a_1^4} + \frac{1}{1+a_2^4} + \ldots + \frac{1}{1+a_n^4} = 1.$$

Démontrer que

$$a_1 a_2 \ldots a_n \geq (n-1)^{\frac{n}{4}}.$$

OC380. Soit ABC un triangle acutangle dont les hauteurs AD et BE se coupent en H. Soit M le milieu du côté AB. Supposons que les cercles circonscrits aux triangles DEM et ABH se coupent aux points P et Q, P étant du même côté de CH que A. Démontrer que les droites ED, PH et MQ se coupent en un seul point qui est situé sur le cercle circonscrit au triangle ABC.

. . . .

OC376. Let m be a positive integer and a and b be distinct positive integers strictly greater than m^2 and strictly less than m^2+m. Find all integers d such that $m^2 < d < m^2+m$ and d divides ab.

OC377. Prove that $x - \frac{1}{x} + y - \frac{1}{y} = 4$ has no solutions over the rationals.

OC378. Define a sequence $\{a_n\}$ by

$$S_1 = 1,\ S_{n+1} = \frac{(2+S_n)^2}{4+S_n}(n = 1,\ 2,\ 3,\ \cdots),$$

where S_n the sum of first n terms of sequence $\{a_n\}$. For any positive integer n, prove that

$$a_n \geq \frac{4}{\sqrt{9n+7}}.$$

OC379. Let $n \geq 3$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$, such that

$$\frac{1}{1+a_1^4} + \frac{1}{1+a_2^4} + \ldots + \frac{1}{1+a_n^4} = 1.$$

Prove that:

$$a_1 a_2 \ldots a_n \geq (n-1)^{\frac{n}{4}}.$$

OC380. Let $\triangle ABC$ be an acute-angled triangle with altitudes AD and BE meeting at H. Let M be the midpoint of segment AB, and suppose that the circumcircles of $\triangle DEM$ and $\triangle ABH$ meet at points P and Q with P on the same side of CH as A. Prove that the lines ED, PH, and MQ all pass through a single point on the circumcircle of $\triangle ABC$.

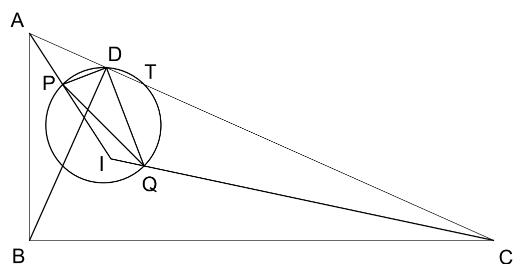
OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2017: 43(2), p. 49–50.

OC316. Let ABC be a right-angled triangle with $\angle B = 90°$. Let BD be the altitude from B. Let P, Q and I be the incenters of triangles ABD, CBD and ABC respectively. Show that the circumcenter of PIQ lies on the hypotenuse AC.

Originally 2015 India National Olympiad, Problem 1.

We received 7 solutions. We present the solution by Mohammed Aassila, slightly modified by the editor.



Denote by Γ the circumcircle of $\triangle PQD$, and by T the other point (besides D) where Γ intersects AC.

Since DP and DQ are angle bisectors, and $BD \perp AC$, we have $\angle PDQ = 90°$ and $\angle QDC = 45°$. By construction, $PDTQ$ is a cyclic quadrilateral, whence $\angle PTQ = \angle PDQ = 90°$ and $\angle QPT = \angle QDT = \angle QDC = 45°$. Hence $\triangle PTQ$ is an isosceles right-angled triangle. Draw the circle Σ that goes through P and Q and has center T. Note that the major arc PQ has length $270°$.

The points A, P, I are collinear (they are all on the bisector of $\angle A$), and so are C, Q, I. Hence $\angle PIQ = \angle AIC = 135°$, which is half of the major arc PQ, implying I is on Σ. Therefore the circle Σ is the circumcircle of $\triangle PIQ$, and the center of Σ (namely, T) is on the hypotenuse AC.

Editor's comments. As pointed out in some of the other solutions, the center T of the circle Σ is also the point of tangency of the incircle of $\triangle ABC$ to the hypotenuse AC.

OC317. In a recent volleyball tournament, 110 teams participated. Every team has played every other team exactly once (there are no ties in volleyball). It turns out that in any set of 55 teams, there is one which has lost to no more than 4 of the remaining 54 teams. Prove that in the entire tournament, there is a team that has lost to no more than 4 of the remaining 109 teams.

Originally 2015 All Russian Olympiad Grade 11, Problem 3.

We received two solutions, out of which we present the one by Oliver Geupel, slightly modified by the editor.

Let $P(n)$ denote the assertion that in every subset of n teams there is one that has lost to no more than 4 of the remaining $n-1$ teams. We prove $P(n)$ for $55 \leq n \leq 110$ by induction.

The base case $n = 55$ is ensured by the hypothesis.

Let us assume that $P(n)$ holds for some n with $55 \leq n \leq 109$ and consider a set $\mathcal{S}$ of $n+1$ teams. We have to show that there is a team in $\mathcal{S}$ that has lost to no more than 4 of the remaining n teams in $\mathcal{S}$.

Let $T_1, T_2, \ldots, T_m$ be all teams in $\mathcal{S}$ that have lost to no more than 5 teams in $\mathcal{S}$. If there is a T_i that has lost to no more than 4 teams in $\mathcal{S}$ we are done, thus we may assume that each T_i has lost to exactly 5 teams in $\mathcal{S}$.

Any team $T \in \mathcal{S}$ has won against some T_i, as otherwise there could be no team in the n-set $\mathcal{S}\backslash\{T\}$ that lost to 4 or less teams, contradicting the induction hypothesis. Since each team T_i lost to exactly 5 teams in $\mathcal{S}$, we thus have $m \geq n+1$. As $n \geq 56$ we obtain $m \geq 12$. But some T_i must have lost to at least $\lceil m/2 \rceil \geq 6$ of the remaining T_j, a contradiction. As a consequence the assumption that every T_i has lost to exactly 5 of the remaining teams in $\mathcal{S}$ was false which completes the induction step.

OC318. Let n be a positive integer and let k be an integer between 1 and n inclusive. Given an $n \times n$ white board, we do the following process.

> We draw k rectangles with integer side lengths and sides parallel to the sides of the $n \times n$ board, and such that each rectangle covers the top-right corner of the $n \times n$ board. Then, the k rectangles are painted black. This process leaves a white figure in the board.

How many different white figures can be formed with k rectangles that cannot be formed with less than k rectangles?

Originally 2015 Mexico National Olympiad Day 2, Problem 2.

We received two solutions. We present the one by Oliver Geupel, expanded by the editor.

We show that there are $\binom{n}{k}^2$ such figures. Consider Cartesian coordinates where the origin is the lower left corner of the white board, the axes are parallel to the sides of the board and the top right corner has coordinates (n,n). Let us consider k black rectangles with lower left corners $(x_1, y_1), \ldots, (x_k, y_k)$. Without loss of generality, $x_1 \leq \cdots \leq x_k$. We claim that the figure cannot be formed with less than k rectangles if and only if

$$x_1 < \cdots < x_k \text{ and } y_1 > \cdots > y_k. \tag{1}$$

If (1) holds then the figure cannot be formed with less than k rectangles as each of the k points (x_i, y_i) has to be covered by a different rectangle. Suppose (1) does not hold. Then there are rectangles i, j with $x_i \leq x_j$ and $y_i \leq y_j$. Then rectangle j is contained in rectangle i and the same figure can be formed by using all rectangles except rectangle j.

As $0 \leq x_i, y_i \leq n-1$, there are $\binom{n}{k}$ choices for $x_1, \ldots, x_k$ and $\binom{n}{k}$ choices for $y_1, \ldots, y_k$. Since the x_i and y_j can be chosen independently of each other, the result follows.

OC319. Let $p > 30$ be a prime number. Prove that one of the following numbers

$$p+1, 2p+1, 3p+1, \ldots, (p-3)p+1$$

is the sum of two integer squares x^2+y^2 for integers x and y.

Originally from the 2015 Iranian Mathematical Olympiad.

We received 2 solutions. We present the solution by Steven Chow.

We prove the stronger result:

> Let $p \geq 7$ be a prime number. Let
>
> $$S = \left\{ jp+1 : 1 \leq j \leq \frac{p-5}{2} \text{ is an integer} \right\}.$$
>
> Prove that one of the numbers in S is the sum of 2 square numbers, i.e., is equal to x^2+y^2 for some integers x and y.

The Legendre symbol and some well known facts are used.

Since $p \geq 7$ is prime, p is congruent to either 1, 3, 5, or 7 modulo 8.

If $p \equiv 1 \pmod 8$ or $p \equiv 3 \pmod 8$, then $p \geq 11$ and either $(-1/p) = 1$ and $(2/p) = 1$, or $(-1/p) = -1$ and $(2/p) = -1$, so $(-8/p) = (-1/p)\,(2/p) = 1$. So there exists an integer $0 \leq y \leq \frac{p-1}{2}$ such that $3^2+y^2 \equiv 1 \pmod p$, and

$$p \geq 11 \implies \left(\frac{p-5}{2}\right)p+1 \geq 3^2 + \left(\frac{p-1}{2}\right)^2 \geq 3^2+y^2,$$

so $3^2+y^2 \in S$.

If $p = 13$, then $2^2+6^2 = 3p+1 \in S$.

If $p \equiv 5 \pmod 8$ and $p \neq 13$, then $p \geq 29$ and $(2/p) = -1$, so $(-24/p) = (2/p)\,(-3/p) = -\,(-3/p)$, so there exists an integer $0 \leq y \leq \frac{p-1}{2}$ such that either 2^2+y^2 or 5^2+y^2 is congruent to 1 (mod p), and

$$p \geq 29 \implies \left(\frac{p-5}{2}\right)p+1 > 5^2 + \left(\frac{p-1}{2}\right)^2 \geq 5^2+y^2, 2^2+y^2,$$

so either $2^2+y^2 \in S$ or $5^2+y^2 \in S$.

Assume now that $p \equiv 7 \pmod 8$.

Since $p \equiv 7 \pmod 8$ is prime, $(-1/p) = -1$ and $(2/p) = 1$.

Let q be the least prime such that $(q/p) = -1$, so $q \geq 3$ and $p \nmid q$.

If $q > p$ or q does not exist, then from prime-power factorization, for all integers $2 \leq j \leq p-1$, $(j/p) = 1$, which is a contradiction (since any square number is congruent modulo p to an element of $\left\{j^2 : 0 \leq j \leq \frac{p-1}{2} \text{ is an integer}\right\}$ which has cardinality $\frac{p+1}{2} < p-2$ since $p \geq 7$).

Therefore $3 \leq q < p$ and

$$\begin{aligned}
\left(\left(-(q-1)^2+1\right)/p\right) &= -\left(\left((q-1)^2-1\right)/p\right) \\
&= -\left((q-2)/p\right)(q/p) \\
&= \left((q-2)/p\right) \\
&= 1
\end{aligned}$$

from the prime-power factorization of $q-2$ and the definition of q (if $q = 3$, then $q-2 = 1$).

Therefore there exists integers $1 \leq x, y \leq \frac{p-1}{2}$ such that $x^2 \equiv (q-1)^2 \pmod p$ and $x^2 + y^2 \equiv 1 \pmod p$. Since $p \geq 7$ is prime,

$$\left(\frac{p-1}{2}\right)^2 + \left(\frac{p-1}{2}\right)^2 \equiv \frac{1}{2} \pmod p$$

is not congruent to $1 \pmod p$, and

$$\left(\frac{p-3}{2}\right)^2 + \left(\frac{p-1}{2}\right)^2 \equiv \frac{5}{2} \pmod p$$

is not congruent to $1 \pmod p$, so

$$p \geq 7 \implies \left(\frac{p-5}{2}\right)p + 1 \geq \left(\frac{p-3}{2}\right)^2 \cdot 2 \geq x^2 + y^2,$$

so $x^2 + y^2 \in S$.

Therefore one of the numbers in S is the sum of 2 square numbers. $\square$

OC320. Let $n \geq 2$ be a given integer. Initially, we write n sets on the blackboard and do a sequence of *moves* as follows:

> choose two sets A and B on the blackboard such that neither of them is a subset of the other, and replace A and B by $A \cap B$ and $A \cup B$.

Find the maximum number of moves in a sequence for all possible initial sets.

Originally 2015 China Girls Mathematics Olympiad Day 2, Problem 8.

No solutions received.

The Method of Indirect Descent (Part II)

Adib Hasan and Thanic Nur Samin

1 Introduction

In the previous article, we introduced a generalization of the method of Infinite Descent [4, 2], which we call indirect descent. To prove a proposition through indirect descent, first we construct a"score function" $s : \mathcal{T} \to \mathbb{N}$ from the set of hypothetical counterexamples into the natural numbers. Next, we construct a transformation $\chi : \mathcal{T} \to \mathcal{T}$, such that $s(\chi(c)) < s(c)$ for all counterexamples $c \in \mathcal{T}$. That is to say, given any counterexample we have a way to generate another one with a lower score. Our work is then done, since even a single counterexample would result in an infinite descending chain of scores $s(c) > s(\chi(c)) > s(\chi^2(c)) > \cdots$, but no such sequence can exist within the natural numbers. We thus conclude that $\mathcal{T}$ is empty, and the proposition true.

In practice, we determine the transformation first, and then define the score function to fit the transformation. That is because the former is often harder to find.

The first article showed the application of indirect descent in Functional Equations and Functional Inequalities. This article will demonstrate its use in Combinatorics and Number Theory.

2 Combinatorial Problems

Example 1 (APMO 2017/1) *We call a* 5*-tuple of integers arrangeable if its elements can be labeled* a, b, c, d, e *in some order so that* $a - b + c - d + e = 29$. *Determine all* 2017*-tuples of integers* $n_1, n_2, \ldots, n_{2017}$ *such that if we place them in a circle in clockwise order, then any* 5*-tuple of numbers in consecutive positions on the circle is arrangeable.*

Solution. (By Thanic.) Replace each number x on the circle with $x - 29$, as it converts the condition $a-b+c-d+e = 29$ into more symmetric $a-b+c-d+e = 0$. Additionally, note that all the numbers on the circle become even after the conversion.

Step 1. Suppose $\mathcal{T} \subseteq \mathbb{Z}^{2017}$ denotes the set of all ordered 2017-tuples of integers that satisfy the problem statement. Notice that

$$a - b + c - d + e = 0 \implies \frac{a}{2} - \frac{b}{2} + \frac{c}{2} - \frac{d}{2} + \frac{e}{2} = 0$$

Since all the numbers on the circle are even, we conclude that

$$(x_1, x_2, \ldots x_{2017}) \in \mathcal{T} \implies \left(\frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_{2017}}{2}\right) \in \mathcal{T}$$

Hence, there is a transformation $\chi : (x_1, x_2, \ldots x_{2017}) \to \left(\frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_{2017}}{2}\right)$ for each element of $\mathcal{T}$.

Step 2. Define a score function $s : \mathcal{T} \to \mathbb{Z}$ as

$$s(x_1, x_2, \ldots, x_{2017}) = \sum_{i=1}^{2017} |x_i|.$$

Note that s is bounded below since $\sum_{i=1}^{2017} |x_i| \geq 0$.

Step 3. Suppose $T \in \mathcal{T}$ and $T = (x_1, x_2, \ldots, x_{2017})$. If there is at least one k so that $x_k \neq 0$, then

$$s(T) = \sum_{i=1}^{2017} |x_i| > \sum_{i=1}^{2017} \left|\frac{x_i}{2}\right| = s(\chi(T))$$

So, applying χ recursively on T will indefinitely decrease s and breach its lower bound. Therefore, all the numbers on the circle has to be 0 after the first replacement. Hence, the only solution is $(29, 29, \ldots 29)$.

□

Example 2 (IMO Shortlist 1994/C3) *Peter has 3 accounts in a bank, each with an integral number of dollars. He is only allowed to transfer money from one account to another so that the amount of money in the latter is doubled.*

a) Prove that Peter can always transfer all his money into two accounts.

b) Can he always transfer all his money into one account?

Solution.(By Pranav A. Sriram [3] (The original solution was adapted to fit the format of this article.)) The second part of the question is trivial - if the total number of dollars is odd, it is clearly not always possible to get all the money into one account. Now, we solve the first part.

Step 1. Suppose $\mathcal{T}$ is the set of all triples (A, B, C) such that $A \leq B \leq C$ are the number of dollars in the accounts at a particular point of time. We want to find a triple in $\mathcal{T}$ containing 0. So, we look for a transformation that reduces the number of dollars in a particular account. We try Euclidean algorithm, since it reduces an ordered pair of natural numbers. This leads us to the following solution.

Assume $B = qA + r$ with $r < A$. Enumerate the account with money A, B, C as 1, 2, 3, respectively. If $A = 0$, we are done. So, assume $A > 0$. $(\overline{m_1 m_2 \ldots m_k})_2$ is the binary representation of q. Transfer money k times to account 1 from accounts 2 or 3. The i^{th} transfer will be from account 2 if $m_i = 1$ and from account 3 otherwise. The number of dollars in the first account starts with A and keeps doubling in each step. Thus we end up transferring

$$A(m_0 + 2m_1 + \ldots + 2^k m_k) = Aq$$

dollars from account 2, and $(2^k - 1 - q)A$ dollars from account 3. So we are left with $B - Aq = r$ dollars in account 2, which now becomes the account with smallest money. Hence,

$$\chi : (A, B, C) \to \begin{cases} \left(r,\ 2^k A - A,\ C + (1+q)A - 2^k A\right), \text{ or,} \\ \left(r,\ C + (1+q)A - 2^k A,\ 2^k A - A\right) \end{cases}$$

Step 2. But in both cases $\min(\chi(T)) \leq r$. So we define $s(T) = \min(A, B, C)$ for each $T \in \mathcal{T}$, so s is indeed bounded below.

Step 3. χ reduces s from A to $r < A$ or even less. Therefore, applying χ on a fixed $T \in \mathcal{T}$ repeatedly, we can reduce s to 0. At that point, all the money will get transferred to two accounts.

□

Remark. In combinatorics, such score functions are called monovariants since they only change in one direction.

3 Number Theoretic Problems

Example 3 (Bulgaria NMO 2005/6 [1]) *Let $a, b, c \in \mathbb{N}$ such that ab divides $c(c^2 - c + 1)$ and $c^2 + 1$ divides $a + b$. Prove that the sets $\{a, b\}$ and $\{c, c^2 - c + 1\}$ must coincide.*

Solution. WLOG assume $a \geq b$. Suppose $c(c^2 - c + 1) = rab$. We want to prove $r = 1$. Assume the contrary, and proceed like this: At first, show that

$$rb^2 + 1 \equiv 0 \pmod{c^2 + 1} \tag{1}$$

$$c \geq 2b \tag{2}$$

$$a > c^2 - c + 1 \tag{3}$$

Set $d = c - b$. Thus $d \geq b$.

After a little investigation, it becomes apparent that $\frac{d}{b}$ will be very large. This makes us wonder how large it can possibly be.

Step 1. Assume

$$\mathcal{T} = \left\{ n \mid \frac{d}{b} \geq n, n \in \mathbb{N} \right\}$$

For any $n \in \mathcal{T}$,

$$c = b + d \geq (n+1)b \implies c^2 + 1 \geq (n+1)^2 b^2 + 1.$$

From 1,

$$rb^2 + 1 \geq c^2 + 1 \geq (n+1)^2 b^2 + 1 \implies r \geq (n+1)^2$$

Hence

$$\frac{c(c^2-c+1)}{ab} = \frac{c^2-c+1}{a} + \frac{d(c^2-c+1)}{ab} = r \geq (n+1)^2 \tag{4}$$

From 3, we know $\frac{c^2-c+1}{a} < 1$. Therefore,

$$\frac{d}{b} > \frac{d(c^2-c+1)}{ab} = r - \frac{c^2-c+1}{a} > (n+1)^2 - 1 = n^2 + 2n$$

implying $n^2 + 2n \in \mathcal{T}$. Therefore, there is a transformation

$$\chi : n \to n^2 + 2n \quad \forall n \in \mathcal{T}$$

Step 2. Set $s(n) = \frac{1}{n}$ $\forall n \in \mathcal{T}$. For a fixed triple (a, b, c), s has a lower bound.

Step 3. Applying χ recursively on any $n \in \mathcal{T}$, it is possible to decrease s below any positive real number. This raises a contradiction. Hence the $r \geq 2$ case is impossible; implying $r = 1$.

Now suppose $a + b > c^2 + 1$. Then

$$2a \geq a + b \geq 2(c^2+1) \implies \frac{c^2-c+1}{a} < 1$$

1 implies $c \leq b$. If $c = b$, we have $a = c^2 - c + 1$, proving the problem statement. Else if $c < b$, then

$$1 = \frac{c(c^2-c+1)}{ab} < \frac{b(c^2-c+1)}{ab} < 1$$

This is again a contradiction. So $a+b = c^2+1$. Now we have $a+b = c+(c^2-c+1)$ and $ab = c(c^2-c+1)$. So we must have $\{a, b\} = \{c, c^2-c+1\}$ □

4 Selected Problems

1. (China NMO 2013/2) Find all nonempty sets S of integers such that $3m - 2n \in S$ for all (not necessarily distinct) $m, n \in S$.

2. (USAMO 2015/6) Consider $0 < \lambda < 1$, and let A be a multiset of positive integers. Let $A_n = \{a \in A : a \leq n\}$. Assume that for every $n \in \mathbb{N}$, the set A_n contains at most $n\lambda$ numbers. Show that there are infinitely many $n \in \mathbb{N}$ for which the sum of the elements in A_n is at most $\frac{n(n+1)}{2}\lambda$. (A multiset is a set-like collection of elements in which order is ignored, but repetition of elements is allowed and multiplicity of elements is significant. For example, multisets $\{1, 2, 3\}$ and $\{2, 1, 3\}$ are equivalent, but $\{1, 1, 2, 3\}$ and $\{1, 2, 3\}$ differ.)

3. (USA Team Selection Test 2002/6) Find all ordered pairs of positive integers (m, n) such that $mn - 1$ divides $m^2 + n^2$.

4. (IMO 2007/5) Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

5. (Thanic) Suppose you are given a bee hive in the shape of a regular hexagon with side length of n hexagons. Some configuration S of the hexagon shaped cells are infected. Then, the infection spreads as follows: a cell becomes infected if and only if at least 3 of its neighbors are infected (two cells are neighbours if they share an edge.) If the entire board eventually becomes infected, prove that $|S| \geq 2n - 1$ (that is, at least $2n - 1$ of the cells were infected initially).

6. (IMO Shortlist 2013/C3) A crazy physicist discovered a new kind of particle which he called an *imon*, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be *entangled*, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

   i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

   ii) At any moment, he may double the whole family of imons in his lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

   Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled.

7. (IMO 2010/5) Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following operations are allowed:

   i) Choose a non-empty box B_j, $1 \leq j \leq 5$, remove one coin from B_j and add two coins to B_{j+1};

   ii) Choose a non-empty box B_k, $1 \leq k \leq 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2}.

   Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly

$$2010^{2010^{2010}}$$

   coins.

References

[1] Bulgaria National Mathematical Olympiad, 2005. https://goo.gl/rCyxEV

[2] Y. Ge, The Method of Vieta-Jumping. http://www.yimin-ge.com/doc/VietaJumping.pdf

[3] P. Sriram, *Olympiad Combinatorics, 2014*. Chapter 1 Example 7.

[4] L. Tat-Wing, The Method of Infinite Descent. *Mathematical Excalibur*, 10(4), 2005. Available at: https://www.math.ust.hk/excalibur/v10_n4.pdf

Adib Hasan
Massachusetts Institute of Technology
notadib@mit.edu

Thanic Nur Samin
Notre Dame College, Bangladesh
thanicsamin@gmail.com

PROBLEM SOLVING 101

No. 4

Shawn Godin

In 1986, I was enrolled in the course C & O 380 : Problem Solving at the University of Waterloo. The course was taught by professor Ross Honsberger and I have already shared the three assignments from this course in earlier installments of this column [2017: 43(4), p. 151 - 153], [2017: 43(8), p. 344 - 346] and [2017: 43(10), p. 441 - 443]. The course was unlike any other math course I had ever taken. On the first class, professor Honsberger presented us with 100 problems that formed the basis of the course. Each class we would discuss 5 of the problems and learn any relevant techniques or theorems related to the problems. The expectation was that students would try to solve the problems before they were discussed in class. This month, we will look at the first 5 problems from the course.

#1. Find the smallest natural number composed only of 1's and 0's which is divisible by 225.

#2. In a certain classroom, there are 5 rows of 5 seats per row arranged in a square. Each student is to change his seat by going either to the seat in front of him, the one behind him, the one to his right, or the one to his left. (Of course, not all possibilities are open to all students). Determine whether this can be done beginning with a full class of students.

#3. By factoring, show that $2222^{5555} + 5555^{2222}$ is divisible by 7.

#4. At any party, show that some pair of guests have exactly the same number of acquaintances among those present.

#5. Two million points in the plane are chosen. Show that, no matter how closely together these points may be chosen, it is always possible to draw a straight line through their midst such that it misses all the points and has a million of them on each side of it.

I wrote about problem #5 a few years ago [2012: 38(5), p. 186-187], so this time we'll look at a couple of important ideas from discrete mathematics through the solutions to problems #2 and #4. I'll leave problems #1 and #3 for you to enjoy.

We'll start with problem #4. For this problem, we will need the following:

Pigeonhole principle (a.k.a. Dirichlet box principle):
If you have n pigeonholes and $m > n$ pigeons, then there must be at least one pigeonhole that contains at least two pigeons.

We can convince ourselves this is true, by example: 5 pigeonholes and 6 pigeons. If we try to keep the pigeons each to their own compartment we do OK until we reach the 5^{th} pigeon. At this point, there will be only one space left, after which all the spaces are full. So no matter what, our last pigeon will have a roommate.

Look at our problem, we will let n represent the number of people at the party. The number of possible acquaintances a person could have could be $0, 1, 2, \ldots, n-1$ (since you are not your own acquaintance). Unfortunately, we have n pigeonholes for our n pigeons, so we cannot (at this point) use the principle.

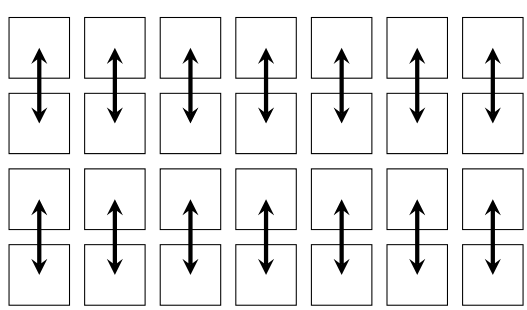
The key, for this problem (just like in our demonstration of the principle at work), is to look at an extreme case. Suppose somebody at the party has 0 acquaintances, what does this mean? Since being an acquaintance is a double edged sword (if you are my acquaintance, then I am yours), this means that *nobody* could have $n-1$ acquaintances. Similarly, if somebody has $n-1$ acquaintances, then nobody has no acquaintances.

As such, we can remove one of our pigeonholes 0 and $n-1$ and label the other "0 or $n-1$". So we do have $n-1$ pigeonholes for our n pigeons and thus at least two people will have the same number of acquaintances. It is interesting to note, that if our "0 or $n-1$" pigeonhole has multiple people in it, you know that they all have 0 acquaintances or they all have $n-1$ acquaintances, but we cannot tell which (unless we ask one of them).

The pigeonhole principle is a simple idea that can be very powerful. You can extend the idea to come up with a condition that there must be a pigeonhole that contains at least 3, or any other number of pigeons. I leave that for you as an exercise.

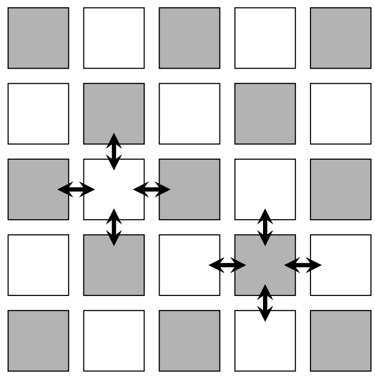
Next, we'll move on to problem #2. I have regularly explored this problem in my classes. It is an interesting problem because, try as you might, you cannot find a solution. Unfortunately, this doesn't prove that there isn't one lurking in the shadows. We need a proof.

If we play around a bit with different configurations, we'll find that if we are using a rectangular array of seats with an *even* number of seats the process can always be done. We will illustrate with a 4 by 7 arrangement.



Since there are an even number of rows, we can pair the rows up so that the pairs of rows switch place, desk by desk. This is one of many different ways that the conditions in the problem can be satisfied.

Even though this process is not possible for an odd number of desks, that still doesn't prove that it cannot be done. The key idea, is that there seems to be a difference between the *parity* of the cases, that is whether there are an even or odd number of desks. Looking at the 5 by 5 case we'll shade every second desk in a checkerboard pattern as shown in the diagram below.



We can see that a person starting in a shaded seat, ends in an unshaded seat and vice versa. Unfortunately, there are 13 shaded seats and only 12 unshaded ones, so one student won't be able to be seated. On the other hand, the students in the 12 unshaded seats have 13 seats to choose from. So the best you can do is move 24 of the 25 students while one student is left without a seat (in the desired place) and one seat is open.

Watch for places you can use the idea of parity or the pigeonhole principle in problems you encounter. There will be "extra" problems in future columns where they will come in handy!

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er septembre 2018**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4331. *Proposé par Daniel Sitaru et Leonard Giugiuc.*

Soit S une sphère de rayon 1. Supposer que la surface de S est colorée à l'aide de 4 couleurs distinctes. Démontrer qu'il existe deux points $X, Y \in S$, de même couleur et tels que $|XY| \in \{\sqrt{3}, \sqrt{3/2}\}$.

4332. *Proposé par S. Muralidharan.*

Soit la famille de tous les cercles de rayon $\frac{1}{2}$ avec centres (i, j) où i et j sont entiers. Démontrer qu'aucune ligne reliant deux de ces centres peut être tangente à un cercle dans cette famille.

4333. *Proposé par Mihai Miculiţa et Titu Zvonaru.*

Soit $ABCD$ un quadrilatère cyclique et soient A_1, B_1 et C_1 les projections orthogonales des points A, B et C sur les lignes BC, CA et AB, respectivement. Dénoter $M = DA \cap BB_1$, $N = DB \cap AA_1$, $P = DC \cap BB_1$, $Q = DB \cap CC_1$, $R = DC \cap AA_1$, puis $S = DA \cap CC_1$. Démontrer que $MN||PQ||RS$.

4334. *Proposé par George Stoica.*

Soient $(a_n)_{n\geq 1}$ et $(x_n)_{n\geq 1}$ des suites de nombres réels telles que $\frac{a_n}{x_n} \searrow 0$. Posons $b_1 = 0$ et

$$b_n = a_1 + \cdots + a_{n-1} - \frac{x_1 + \cdots + x_{n-1}}{x_n} a_n$$

pour $n \geq 2$. Démontrer que la série $\sum_{n=1}^{\infty} a_n$ converge si et seulement si la suite $(b_n)_{n\geq 1}$ converge, puis que, dans un tel cas, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} b_n$.

4335. *Proposé par Leonard Giugiuc.*

Soient a et b deux nombres réels et soit n entier, $n \geq 2$. Démontrer que pour tous nombres réels $x_i, i = 1, \ldots, n$ vérifiant $x_1 + \cdots + x_n = 1$, l'inégalité qui suit est valide:

$$\sqrt[3]{ax_1 + b} + \sqrt[3]{ax_2 + b} + \cdots + \sqrt[3]{ax_n + b} \geq \sqrt[3]{a + b} + (n - 1)\sqrt[3]{b}.$$

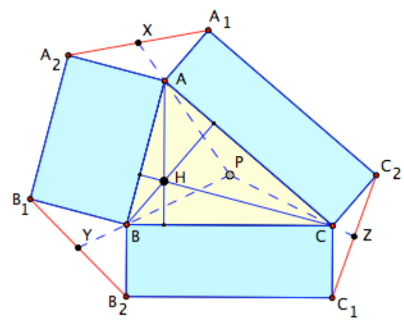
4336. *Proposé par Michel Bataille.*

Pour m et n entiers non négatifs, évaluer l'expression suivante en forme close:

$$\sum_{k=0}^{n}\sum_{j=0}^{m}(j+k+1)\binom{j+k}{j}.$$

4337. *Proposé par Mihaela Berindeanu.*

Soient h_a, h_b, h_c les altitudes émanant des sommets A, B, C du triangle ABC, respectivement. À l'extérieur de ses côtés, érigeons trois rectangles ABB_1A_2, BCC_1B_2, CAA_1C_2, dont les largeurs sont k fois leurs altitudes parallèles, c'est-à-dire,



$$\frac{CC_1}{h_a}=\frac{AA_1}{h_b}=\frac{BB_1}{h_c}=k.$$

Si X, Y, Z sont les mi points des segments A_1A_2, B_1B_2, C_1C_2 respectivement, démontrer que les lignes AX, BY, CZ sont concourantes.

4338. *Proposé par Daniel Sitaru.*

Démontrer que l'inégalité suivante tient, pour tout triangle ABC:

$$2\sum\left|\cos\frac{A-B}{2}\right|\leq 3+\sqrt{3+2\sum\cos(A-B)}.$$

4339. *Proposé par Kadir Altintas et Leonard Giugiuc.*

Soit ABC un triangle acutangle, DEF son triangle orthique, S le point symédian de ABC, puis G le barycentre de DEF. Si D est le pied de l'altitude émanant de A et si K est le point d'intersection de AS et FE, démontrer que D, G et K sont colinéaires.

4340. *Proposé par Digby Smith.*

Soient a, b, c et d des nombres réels positifs tels que

$$a+b+c+d=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

Démontrer que

$$a+b+c+d\geq max\left\{4\sqrt{abcd},\frac{4}{\sqrt{abcd}}\right\}.$$

. . . .

4331. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let S be a unit sphere. Suppose that the surface of S is coloured with 4 distinct colours. Prove that there exist two points $X, Y \in S$ of the same colour with $|XY| \in \{\sqrt{3}, \sqrt{3/2}\}$.

4332. *Proposed by S. Muralidharan.*

Draw the family of circles of radius $\frac{1}{2}$ with centers at (i, j) where i, j are integers. Prove that a line joining centers of any two of these circles cannot be tangent to any circle in the family.

4333. *Proposed by Mihai Miculiţa and Titu Zvonaru.*

Let $ABCD$ be a cyclic quadrilateral and let A_1, B_1 and C_1 be orthogonal projections of the points A, B and C onto the lines BC, CA and AB, respectively. We denote $M = DA \cap BB_1$, $N = DB \cap AA_1$, $P = DC \cap BB_1$, $Q = DB \cap CC_1$, $R = DC \cap AA_1$ and $S = DA \cap CC_1$. Prove that $MN||PQ||RS$.

4334. *Proposed by George Stoica.*

Let $(a_n)_{n\geq 1}$ and $(x_n)_{n\geq 1}$ be sequences of real numbers such that $\frac{a_n}{x_n} \searrow 0$. Put $b_1 = 0$ and

$$b_n = a_1 + \cdots + a_{n-1} - \frac{x_1 + \cdots + x_{n-1}}{x_n} a_n$$

for $n \geq 2$. Prove that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence $(b_n)_{n\geq 1}$ converges and that, in this case, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} b_n$.

4335. *Proposed by Leonard Giugiuc.*

Let a and b be fixed positive real numbers and let $n \geq 2$ be an integer. Prove that for any nonnegative real numbers $x_i, i = 1, \ldots, n$ such that $x_1 + \cdots + x_n = 1$, we have

$$\sqrt[3]{ax_1 + b} + \sqrt[3]{ax_2 + b} + \cdots + \sqrt[3]{ax_n + b} \geq \sqrt[3]{a + b} + (n-1)\sqrt[3]{b}.$$

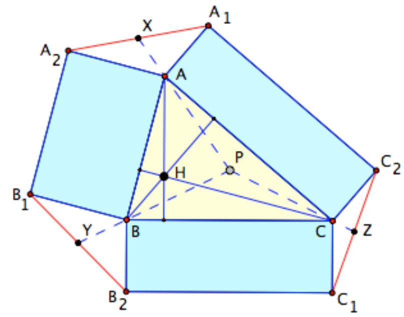
4336. *Proposed by Michel Bataille.*

For non-negative integers m and n, evaluate in closed form

$$\sum_{k=0}^{n} \sum_{j=0}^{m} (j + k + 1) \binom{j+k}{j}.$$

4337. *Proposed by Mihaela Berindeanu.*

Let h_a, h_b, h_c be the altitudes from vertices A, B, C, respectively, of triangle ABC. Erect externally on its sides three rectangles $ABB_1A_2, BCC_1B_2, CAA_1C_2$, whose widths are k times as long as the parallel altitudes; that is,



$$\frac{CC_1}{h_a} = \frac{AA_1}{h_b} = \frac{BB_1}{h_c} = k.$$

If X, Y, Z are the respective midpoints of the segments A_1A_2, B_1B_2, C_1C_2, prove that the lines AX, BY, CZ are concurrent.

4338. *Proposed by Daniel Sitaru.*

Prove that for any triangle ABC, we have

$$2\sum \left| \cos \frac{A-B}{2} \right| \leq 3 + \sqrt{3 + 2\sum \cos(A-B)}.$$

4339. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Suppose ABC be an acute-angled triangle, DEF is an orthic triangle of ABC, S is the symmedian point of ABC, G is the barycenter of DEF. If D is the foot of the altitude from A and K is the point of intersection of AS and FE, prove that D, G and K are collinear.

4340. *Proposed by Digby Smith.*

Let a, b, c and d be positive real numbers such that

$$a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

Show that

$$a + b + c + d \geq max \left\{ 4\sqrt{abcd}, \frac{4}{\sqrt{abcd}} \right\}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2017: 43(4), p. 160–163.

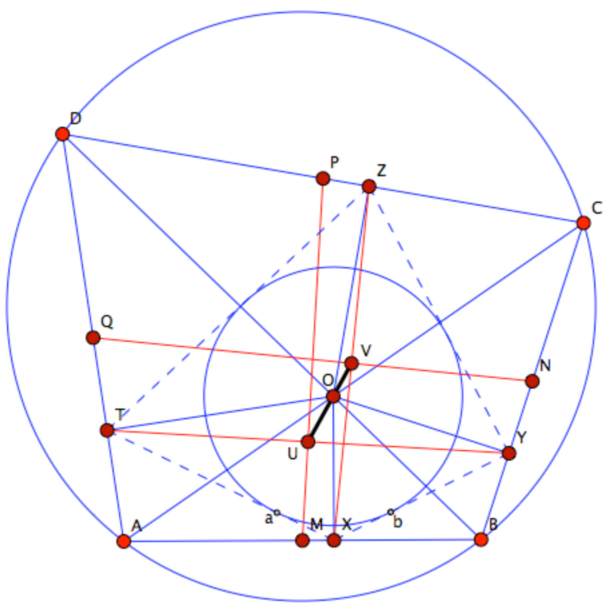
4231. *Proposed by Marius Stănean.*

Let $ABCD$ be a cyclic quadrilateral, $O = AC \cap BD$, M, N, P, Q be the midpoints of AB, BC, CD and DA, respectively, and X, Y, Z, T be the projections of O on AB, BC, CD and DA, respectively. Let $U = MP \cap YT$ and $V = NQ \cap XZ$. Prove that

$$\frac{UO}{VO} = \frac{AB \cdot CD}{BC \cdot DA}.$$

We received two submissions, both of which were correct. We present the solution by Steven Chow.

The following argument is valid for concyclic points A, B, C, and D placed in any order about the circumference. Directed angles modulo π and complex numbers are used.



Since $\measuredangle ATO = \measuredangle OXA = \frac{1}{2}\pi$, the quadruple A, O, T, and X is concyclic. The quadruples $\{B, O, X, Y\}$, $\{C, O, Y, Z\}$, and $\{D, O, Z, T\}$ are likewise concyclic.

Therefore we have

$$\measuredangle OXT = \measuredangle OAT = \measuredangle CAD = \measuredangle CBD = \measuredangle YBO = \measuredangle YXO$$

and the cyclic variations, so the lines XY, YZ, ZT, TX determined by the vertices of the quadrangle $XYZT$ are tangent to a circle with centre O.

It is easy to prove using complex numbers, but there are also synthetic proofs, that MP is the perpendicular bisector of TY, and in particular, U is the midpoint of TY. Refer to some of the slightly generalized solutions to problem 2 of the 2000 USATST on
`https://artofproblemsolving.com/community/c6h326960p1752047.`

Similarly, V is the midpoint of XZ.

Let the unit circle be the circle with centre O that is tangent to the sides of $XYZT$. Let a, b, c, and d be the complex numbers representing its points of tangency on TX, XY, YZ, and ZT, respectively. Let x, y, z, t, u, and v be the complex numbers that represent the points X, Y, Z, T, U, and V, respectively.

Because x is the inverse in the unit circle of the midpoint $\frac{a+b}{2}$ of the segment joining the points corresponding to a and b, we have

$$x = \frac{1}{\overline{\left(\frac{a+b}{2}\right)}} = \frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2ab}{a+b}.$$

Similarly,

$$y = \frac{2bc}{b+c}, \quad z = \frac{2cd}{c+d}, \quad \text{and} \quad t = \frac{2da}{d+a},$$

whence

$$u = \frac{t+y}{2} = \frac{\sum_{\text{cyc}} abc}{(d+a)(b+c)} \quad \text{and} \quad v = \frac{x+z}{2} = \frac{\sum_{\text{cyc}} abc}{(a+b)(c+d)}.$$

Therefore

$$\frac{u}{v} = \frac{(a+b)(c+d)}{(d+a)(b+c)} = \frac{ty}{xz}.$$

Remark. The figure suggests that the points O, U, and V might be collinear. It is now easy to see that this is the case: $\frac{u-0}{v-0} = \frac{(a+b)(c+d)}{(d+a)(b+c)} = \frac{\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{c}+\frac{1}{d}\right)}{\left(\frac{1}{d}+\frac{1}{a}\right)\left(\frac{1}{b}+\frac{1}{c}\right)} = \overline{\left(\frac{u-0}{v-0}\right)}$.

Continuing with our solution, we have $\frac{|u-0|}{|v-0|} = \frac{|t-0||y-0|}{|x-0||z-0|}$, and thus,

$$\frac{UO}{VO} = \frac{TO \cdot YO}{XO \cdot ZO}.$$

Since $\measuredangle BAC = \measuredangle BDC = \measuredangle ODZ = \measuredangle OTZ$ and $\measuredangle ACB = \measuredangle TZO$, $\triangle ABC \sim \triangle TOZ$ and similarly, $\triangle CDA \sim \triangle YOX$, so that

$$\frac{TO \cdot YO}{XO \cdot ZO} = \frac{AB \cdot CD}{BC \cdot DA}.$$

Consequently,

$$\frac{UO}{VO} = \frac{AB \cdot CD}{BC \cdot DA}.$$

Editor's comments. The proposer reported that this problem is a continuation of his earlier problem that appeared as S266 in *Mathematical Reflections*, 2013, issue 3; there he asked for a proof that U, O, V are collinear. Apparently, one must agree to a subscription to be able to view the problem.

4232. *Proposed by Michel Bataille.*

Let n be a positive integer. Prove that

$$\sum_{k=0}^{2n-1} \binom{2n-1+k}{k} \binom{2n-1}{k} \frac{(-1)^k}{2^k} = 0.$$

We received seven solutions to this problem. Presented are two solutions.

Solution 1, by Ángel Plaza.

We will show that the generating function of sequence $(a_m)_{m\geq 0}$ with

$$a_m = \sum_{k=0}^{m} \binom{m+k}{k} \binom{m}{k} \left(\frac{-1}{2}\right)^k$$

only has even terms, from where the result follows.

First of all, notice that $\binom{m+k}{k}\binom{m}{k} = \binom{m+k}{2k}\binom{2k}{k}$, and therefore the generating function for $(a_m)_{m\geq 0}$, say $F(x)$, is

$$\begin{aligned}
F(x) &= \sum_{m\geq 0} x^m \sum_{k=0}^{m} \binom{m+k}{2k} \binom{2k}{k} \left(\frac{-1}{2}\right)^k \\
&= \sum_{k\geq 0} \binom{2k}{k} \left(\frac{-1}{2x}\right)^k \sum_{m\geq 0} \binom{m+k}{2k} x^{m+k} \\
&= \sum_{k\geq 0} \binom{2k}{k} \left(\frac{-1}{2x}\right)^k \frac{x^{2k}}{(1-x)^{2k+1}} \\
&= \frac{1}{1-x} \sum_{k\geq 0} \binom{2k}{k} \left(\frac{-x}{2(1-x)^2}\right)^k \\
&= \frac{1}{1-x} \cdot \frac{1}{\sqrt{1 + \frac{2x}{(1-x)^2}}} \\
&= \frac{1}{\sqrt{1+x^2}}
\end{aligned}$$

where it has been used that (see [1], eq. (2.5.7) page 53)

$$\sum_m \binom{m+k}{2k} x^{m+k} = \frac{x^{2k}}{(1-x)^{2k+1}},$$

and also that (see [1], eq. (2.5.11) page 53)

$$\sum_k \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}.$$

Now, also by [1] (eq. (2.5.11) page 53)

$$\begin{aligned} F(x) = \frac{1}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1-4\left(-\left(\frac{x}{2}\right)^2\right)}} &= \sum_{k\geq 0} \binom{2k}{k} \left(-\left(\frac{x}{2}\right)^2\right)^k \\ &= \sum_{k\geq 0} \binom{2k}{k} \left(\frac{-1}{4}\right)^k x^{2k} \end{aligned}$$

and the problem is done.

Also, as a bonus, we have found that

$$\sum_{k=0}^{2n} \binom{2n+k}{k} \binom{2n}{k} \frac{(-1)^k}{2^k} = \binom{2n}{n} \frac{(-1)^n}{4^n}.$$

Reference: [1] Herbert S. Wilf, *generatingfunctionology*, Academic Press Inc., 2nd ed. 1994.

Solution 2, by Óscar Ciaurri.

We have

$$\begin{aligned} &\sum_{k=0}^{2n-1} \binom{2n-1+k}{k} \binom{2n-1}{k} (-1)^k x^k \\ &\quad = \frac{1}{(2n-1)!} \sum_{k=0}^{2n-1} (2n-1+k)\cdots(k+1) \binom{2n-1}{k} (-1)^k x^k \\ &\quad = \frac{1}{(2n-1)!} \cdot \frac{d^{2n-1}}{dx^{2n-1}} \left(\sum_{k=0}^{2n-1} \binom{2n-1}{k} (-1)^k x^{2n-1+k} \right) \\ &\quad = \frac{1}{(2n-1)!} \cdot \frac{d^{2n-1}}{dx^{2n-1}} \Big(x(1-x) \Big)^{2n-1} \\ &\quad = \frac{1}{(2n-1)!} \cdot \frac{d^{2n-1}}{dx^{2n-1}} \left(\frac{1}{4} - \left(x - \frac{1}{2} \right)^2 \right)^{2n-1} \end{aligned}$$

$$= \frac{1}{(2n-1)!} \cdot \frac{d^{2n-1}}{dx^{2n-1}} \left(\sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{(-1)^k}{4^{2n-1-k}} \left(x - \frac{1}{2}\right)^{2k} \right)$$
$$= \frac{1}{(2n-1)!} \sum_{k=0}^{2n-1} (2k) \cdots (2k-2n+2) \binom{2n-1}{k} \frac{(-1)^k}{4^{2n-1-k}} \left(x - \frac{1}{2}\right)^{2k-2n+1}$$
$$= \sum_{k=0}^{2n-1} \binom{2k}{2n-1} \binom{2n-1}{k} \frac{(-1)^k}{4^{2n-1-k}} \left(x - \frac{1}{2}\right)^{2k-2n+1}.$$

Taking $x = 1/2$ yields the result.

4233. *Proposed by Peter Y. Woo.*

A high-school math teacher required her geometry students to solve this problem without trigonometry: Let ABC be a triangle where $\angle B > 90^\circ$. Denote by M the foot of the altitude from C to AB, and by N the foot of the altitude from B to AC. Then if $AB = 2CM$ and $\angle ABN = \angle CBM$, determine $\angle A$.

We received 17 correct solutions. We present the solution by Roy Barbara.

Set $\alpha = \angle A$. We prove that $\alpha = \frac{\pi}{8} = 22.5^\circ$.

We may take $CM = 1$, so that $AB = 2$. Set $x = BN$, $y = CN$, $t = BM$, and $a = BC$. Similar triangles BMC and CMA yield $\frac{BM}{CM} = \frac{CM}{AM}$; that is, $\frac{t}{1} = \frac{1}{2+t}$; hence

$$t = \sqrt{2} - 1. \tag{1}$$

The right triangle BMC yields $BC^2 = BM^2 + CM^2$; that is, $a^2 = t^2 + 1$, so

$$a^2 = 4 - 2\sqrt{2}. \tag{2}$$

Similar triangles BNA and BMC yield $\frac{BN}{AB} = \frac{BM}{CB}$; that is, $\frac{x}{2} = \frac{t}{a}$; hence,

$$x^2 = \frac{4t^2}{a^2}. \tag{3}$$

The right triangle BCN yields $BC^2 = BN^2 + CN^2$; hence,

$$y^2 = a^2 - x^2. \tag{4}$$

We claim that $x = y$. Indeed, using (4) and (3), we may write

$$x = y \Leftrightarrow x^2 = y^2 \Leftrightarrow x^2 = a^2 - x^2 \Leftrightarrow a^2 = 2x^2 \Leftrightarrow a^2 = \frac{8t^2}{a^2} \Leftrightarrow a^4 = 8t^2 \Leftrightarrow a^2 = 2\sqrt{2}t,$$

which holds by (2) and (1).

Since $x = y$, the right triangle NBC is isosceles, so $\angle NBC = \angle NCB = 45^\circ$. Now, in the right triangle AMC, the angles at A and C are complementary, so that $\alpha + (45^\circ + \alpha) = 90^\circ$, implying that $\alpha = 22.5^\circ$.

4234. *Proposed by Leonard Giugiuc, Daniel Sitaru and Marian Dinca.*

Let x, y and z be real numbers such that $x \geq y \geq z > 0$. Prove that for any $k \geq 0$ we have

$$\frac{4}{x+3y+4k}+\frac{4}{y+3z+4k}+\frac{4}{z+3x+4k} \geq \frac{3}{x+2y+3k}+\frac{3}{y+2z+3k}+\frac{3}{z+2x+3k}.$$

We received four solutions. One of them used MAPLE, which we deem as incomplete. We present the solution by Digby Smith, modified by the editor.

We first prove two auxiliary lemmas:

Lemma 1. If p, q, and r are positive reals such that $p \geq q \geq r$, then

$$p^4q + q^4r + r^4p \geq pq^4 + qr^4 + rp^4.$$

Proof.

$$\begin{aligned}
&p^4q + q^4r + r^4p - (pq^4 + qr^4 + rp^4)\\
&= p^4(q-r) - q^4(p-r) + r^4(p-q)\\
&= p^4(q-r) - q^4\left((p-q)+(q-r)\right) + r^4(p-q)\\
&= (p^4-q^4)(q-r) - (q^4-r^4)(p-q)\\
&= (p-q)(q-r)(p^3+p^2q+pq^2+q^3) - (p-q)(q-r)(q^3+q^2r+qr^2+r^3)\\
&\geq (p-q)(q-r)(p^3+p^2r+pr^2+q^3-q^3-q^2r-qr^2-r^3)\\
&= (p-q)(q-r)\left((p^3-r^3)+(p^2r-p^2r)+(pr^2-qr^2)\right)\\
&= (p-q)(q-r)\left((p^3-r^3)+r(p^2-q^2)+r^2(p-q)\right) \geq 0,
\end{aligned}$$

with equality if and only if $p = q$ or $q = r$ or $p = q = r$.

Lemma 2. If p, q, and r are positive reals, then

$$7(p^3q^2 + q^3r^2 + r^3p^2) + 5(p^2q^3 + q^2r^3 + r^2p^3) \geq 12pqr(pq + qr + rp).$$

Proof. By the AM-GM inequality, we have

$$\begin{aligned}
&7(p^3q^2 + q^3r^2 + r^3p^2) + 5(p^2q^3 + q^2r^3 + r^2p^3)\\
&= (3p^3q^2 + 2q^3r^2 + 2r^3p^2 + 4p^2q^3 + r^2p^3)\\
&\quad + (2p^3q^2 + 3q^3r^2 + 2r^3p^2 + p^2q^3 + 4q^2r^3)\\
&\qquad + (2p^3q^2 + 2q^3r^2 + 3r^2p^2 + q^2r^3 + 4r^2p^3)\\
&\geq 12\sqrt[12]{p^{24}q^{24}r^{12}} + 12\sqrt[12]{p^{12}q^{24}r^{24}} + 12\sqrt[12]{p^{24}q^{12}r^{24}}\\
&= 12(p^2q^2r + pq^2r^2 + p^2qr^2)\\
&= 12pqr(pq + qr + rp),
\end{aligned}$$

with equality if and only if $p = q = r$.

Now we are ready to prove the given inequality. Let $p = x+k$, $q = y+k$, $r = z+k$. Then $p \geq q \geq r > 0$ with $p = q = r$ if and only if $x = y = z$. The given inequality is equivalent, in succession, to

$$\frac{4}{p+3q} + \frac{4}{q+3r} + \frac{4}{r+3p} \geq \frac{3}{p+2q} + \frac{3}{q+2r} + \frac{3}{r+2p},$$
$$\frac{p-q}{(p+3q)(p+2q)} + \frac{q-r}{(q+3r)(q+2r)} + \frac{r-p}{(r+3p)(r+2p)} \geq 0,$$

$$\begin{aligned}(p-q)(q+3r)(q+2r)(r+3p)(r+2p) + (q-r)(p+3q)(p+2q)(r+3p)(r+2p)\\ +(r-p)(p+3q)(p+2q)(q+3r)(q+2r) \geq 0,\end{aligned}$$

or
$$\begin{aligned}&p(q+3r)(q+2r)[(r+3p)(r+2p)-(p+3q)(p+2q)]\\ &\quad + q(r+3p)(r+2p)[(p+3q)(p+2q)-(q+3r)(q+2r)]\\ &\qquad + r(p+3q)(p+2q)[(q+3r)(q+2r)-(r+3p)(r+2p)] \geq 0. \qquad (1)\end{aligned}$$

Let L denote the LHS of (1). Then

$$\begin{aligned}L &= p(q^2+5qr+6r^2)\left((r^2+5rp+6p^2)-(p^2+5pq+6q^2)\right)\\
&\quad + q(r^2+5rp+6p^2)\left((p^2+5pq+6q^2)-(q^2+5qr+6r^2)\right)\\
&\quad + r(p^2+5pq+6q^2)\left((q^2+5qr+6r^2)-(r^2+5rp+6p^2)\right)\\
&= p(q^2+5qr+6r^2)(r^2+5rp+5p^2-5pq-6q^2)\\
&\quad + q(r^2+5rp+6p^2)(p^2+5pq+5q^2-5qr-6r^2)\\
&\quad + r(p^2+5pq+6q^2)(q^2+5qr+5r^2-5rp-6p^2)\\
&= 5pqr(r^2+5rp+5p^2-5pq-6q^2)\\
&\quad + 5pqr(p^2+5pq+5q^2-5qr-6r^2) + 5pqr(q^2+5qr+5r^2-5rp-6p^2)\\
&\quad + p(q^2+6r^2)(r^2+5rp+5p^2-5pq-6q^2)\\
&\quad + q(r^2+6p^2)(p^2+5pq+5q^2-5qr-6r^2)\\
&\quad + r(p^2+6q^2)(q^2+5qr+5r^2-5rp-6p^2)\\
&= pq^2(r^2+5rp+5p^2-5pq-6q^2) + 6r^2p(r^2+5rp+5p^2-5pq-6q^2)\\
&\quad + qr^2(p^2+5pq+5q^2-5qr-6r^2) + 6p^2q(p^2+5pq+5q^2-5qr-6r^2)\\
&\quad + rp^2(q^2+5qr+5r^2-5rp-6p^2) + 6q^2r(q^2+5qr+5r^2-5rp-6p^2)\\
&= 6(p^4q-pq^4+q^4r-qr^4+r^4p-rp^4) + 35(p^3q^2+q^3r^2+r^3p^2)\\
&\quad + 25(p^2q^3+q^2r^3+r^2p^3) - 60(p^2q^2r+pq^2r^2+p^2qr^2)\\
&= 6\left((p^4q+q^4r+r^4p)-(pq^4+qr^4+rp^4)\right)\\
&\quad + 5\left(7(p^3q^2+q^3r^2+r^3p^2)+5(p^2q^3+q^2r^3+r^2p^3)-12pqr(pq+qr+rp)\right)\\
&\geq 0\end{aligned}$$

by Lemma 1 and Lemma 2, completing the proof, with equality if and only if $x = y = z$.

4235. *Proposed by Ruben Dario Auqui and Leonard Giugiuc.*

Let ABC be an isosceles triangle with $BA = BC$ and let I be its incentre. Denote by X, Y and Z the tangency points of the incircle and the sides AB, AC and CB, respectively. A line d passes through I intersecting the segments AX and CZ. Denote by a, b, c, x, y and z the distances from the points A, B, C, X, Y and Z to the line d, respectively. Prove that

$$\frac{a+c}{b} = \frac{x+z}{y}.$$

We received 5 correct solutions. Solution by C. R. Pranesachar.

Let the line d meet AC extended in point O and make an angle θ with OA. Let also $AB = BC = p,\ CA = q,$. Therefore, $CY = CZ = AY = AX = \frac{q}{2}$. Further $BZ = BX = p - \frac{q}{2}$. Note that $\angle A = \angle C$. One has $\angle OSR = C - \theta$. Hence, we successively have

$$OC = c\csc\theta,\quad OA = q + c\csc\theta,\quad a = OA\sin\theta = q\sin\theta + c,\quad OY = \frac{q}{2} + c\csc\theta,$$

$$y = OY\sin\theta = \frac{q}{2}\sin\theta + c,\quad CS = c\csc(C-\theta),\quad SZ = \frac{q}{2} - c\csc(C-\theta),$$

$$z = SZ\sin(C-\theta) = \frac{q}{2}\sin(C-\theta) - c,$$

$$\begin{aligned} b &= SB\sin(C-\theta) = (SZ + BZ)\sin(C-\theta) \\ &= \left[\frac{q}{2} - c\csc(C-\theta) + p - \frac{q}{2}\right]\sin(C-\theta) = p\sin(C-\theta) - c, \end{aligned}$$

$$AT = a\csc(C+\theta),\quad TX = AX - AT = \frac{q}{2} - a\csc(C+\theta),$$

$$x = TX\sin(C+\theta) = \left[\frac{q}{2} - a\csc(C+\theta)\right]\sin(C+\theta) = \frac{q}{2}\sin(C+\theta) - a.$$

If r is the inradius of the triangle, then $r\cot\theta = c\csc\theta + \frac{q}{2}$, from which

$$c = r\cos\theta - \frac{q}{2}\sin\theta.$$

At this stage we can assume that for suitable real numbers u, v, $p = u^2 + v^2$ and $q = 4uv$. [One way to see that this is possible is to solve this system of equations. This involves solving a quadratic equation; that its discriminant is positive follows from the triangle inequality in the form $q < 2p$.] Then the perpendicular distance from B to AC is $u^2 - v^2$. Also

$$\sin C = \frac{u^2 - v^2}{u^2 + v^2},\ \cos C = \frac{2uv}{u^2 + v^2}.$$

With these substitutions, we get

$$r = 2uv \cdot \frac{u-v}{u+v};$$

$$a = \frac{2uv}{u+v} \cdot [(u-v)\cos\theta + (u+v)\sin\theta];$$

$$b = (u^2+v^2)\left(\frac{u-v}{u+v}\right)\cos\theta;$$

$$c = \frac{2uv}{u+v} \cdot [(u-v)\cos\theta - (u+v)\sin\theta];$$

$$x = \frac{2uv}{u^2+v^2} \cdot \frac{u-v}{u+v}(v^2\sin\theta + 2uv\cos\theta - u^2\sin\theta);$$

$$y = 2uv\left(\frac{u-v}{u+v}\right)\cos\theta;$$

$$z = \frac{2uv}{u^2+v^2} \quad \frac{u-v}{u+v}(u^2\sin\theta + 2uv\cos\theta - v^2\sin\theta).$$

With these values of a, b, c, x, y, z, we get

$$\frac{a+c}{b} = \frac{4uv}{u^2+v^2} = \frac{x+z}{y},$$

whence the result follows.

4236. *Proposed by Nguyen Viet Hung.*

Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\sqrt[k]{\frac{(k^2+2)^2}{k^4+4}}\right).$$

We received 15 *solutions. We present* 4 *solutions.*

Solution 1, by Henry Ricardo.

Using Bernoulli's inequality, we have

$$1 \le \sqrt[k]{\frac{(k^2+2)^2}{k^4+4}} = \sqrt[k]{1+\frac{4k^2}{k^4+4}} \le 1 + \frac{4k}{k^4+4} < 1 + \frac{4}{k^3}.$$

Thus

$$1 \le \frac{1}{n}\sum_{k=1}^{n}\sqrt[k]{\frac{(k^2+2)^2}{k^4+4}} < 1 + \frac{4}{n}\sum_{k=1}^{n}\frac{1}{k^3} = 1 + O\left(\frac{1}{n}\right),$$

and the desired limit is 1 as $n \to \infty$.

Solution 2, by Arkady Alt.

First note that

$$\frac{\left(k^2+2\right)^2}{k^4+4} > 1 \implies \sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}} > 1$$

where $k \in \mathbb{N}$. By Bernoulli Inequality,

$$\sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}} = \left(1+\frac{4k^2}{k^4+4}\right)^{1/k} \leq 1+\frac{4k^2}{k^4+4}\cdot\frac{1}{k} = 1+\frac{4k}{k^4+4},$$

where $k \in \mathbb{N}$.

Thus, for any natural k we have the double inequality

$$1 < \sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}} \leq 1+\frac{4k}{k^4+4}$$

and therefore

$$n < \sum_{k=1}^{n}\sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}} \leq n+\sum_{k=1}^{n}\frac{4k}{k^4+4}.$$

Noting that

$$\begin{aligned}
\frac{4k}{k^4+4} &= \frac{4k}{\left(k^2-2k+2\right)\left(k^2+2k+2\right)} \\
&= \frac{1}{k^2-2k+2}-\frac{1}{k^2+2k+2} \\
&= \frac{1}{(k-1)^2+1}-\frac{1}{(k+1)^2+1} \\
&= b_k - b_{k+1},
\end{aligned}$$

where $b_k := \dfrac{1}{(k-1)^2+1}+\dfrac{1}{k^2+1}$, we obtain

$$\sum_{k=1}^{n}\frac{4k}{k^4+4} = \sum_{k=1}^{n}(b_k-b_{k+1}) = b_1-b_{n+1} = \frac{3}{2}-\left(\frac{1}{n^2+1}+\frac{1}{(n+1)^2+1}\right).$$

Since

$$n < \sum_{k=1}^{n}\sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}} \leq n+\frac{3}{2}-\left(\frac{1}{n^2+1}+\frac{1}{(n+1)^2+1}\right),$$

then

$$1 < \frac{1}{n}\sum_{k=1}^{n}\sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}} < 1+\frac{1}{n}\left(\frac{3}{2}-\left(\frac{1}{n^2+1}+\frac{1}{(n+1)^2+1}\right)\right).$$

By the Squeeze Theorem,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}}\right)=1.$$

Solution 3, by Leonard Giugiuc.

Let $x_n=\sum\limits_{k=1}^{n}\sqrt[k]{\dfrac{(k^2+2)^2}{k^4+4}}$ and $y_n=n$ for all $n\geq 1$. By the Stolz-Cesàro Theorem,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}}\right)=\lim_{n\to\infty}\frac{x_n}{y_n}=\lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}}=\lim_{n\to\infty}\sqrt[n]{\frac{\left(n^2+2\right)^2}{n^4+4}}.$$

Since $1<\dfrac{\left(n^2+2\right)^2}{n^4+4}<2$, by the Squeeze Theorem, we get

$$\lim_{n\to\infty}\sqrt[n]{\frac{\left(n^2+2\right)^2}{n^4+4}}=1.$$

Solution 4, by Michel Bataille.

For each positive integer n, let $u_n=\left(\frac{(n^2+2)^2}{n^4+4}\right)^{1/n}$. We show that the required limit, that is,

$$\lim_{n\to\infty}\frac{u_1+u_2+\cdots+u_n}{n}$$

is 1.
Since $\lim\limits_{n\to\infty}\frac{4n^2}{n^4+4}=0$, we have, as $n\to\infty$,

$$\ln(u_n)=\frac{1}{n}\ln\left(1+\frac{4n^2}{n^4+4}\right)\ \sim\ \frac{1}{n}\cdot\frac{4n^2}{n^4+4}\ \sim\ \frac{4}{n^3}.$$

It clearly follows that $\lim\limits_{n\to\infty}\ln(u_n)=0$ and so $\lim\limits_{n\to\infty}u_n=1$. The Cesàro mean of the sequence (u_n) has the same limit, which means that

$$\lim_{n\to\infty}\frac{u_1+u_2+\cdots+u_n}{n}=1,$$

as claimed.

4237. *Proposed by Cristinel Mortici and Leonard Giugiuc.*

For an integer $n \geq 2$, find all $a_1, \ldots, a_n, b_1, \ldots, b_n$ so that

i) $0 \leq a_1 \leq \ldots \leq a_n \leq 1 \leq b_1 \leq \ldots \leq b_n$;

ii) $\sum_{k=1}^{n} (a_k + b_k) = 2n$; and

iii) $\sum_{k=1}^{n} (a_k^2 + b_k^2) = n^2 + 3n$.

There were six correct solutions. We present the solution of Prithwijit De.

For $1 \leq k \leq n$, let $u_k = 1 - a_k$ and $v_k = b_k - 1$. Then $\sum_{k=1}^{n} u_k = \sum_{k=1}^{n} v_k$ and

$$\begin{aligned}\sum_{k=1}^{n}(u_k^2 + v_k^2) &= \sum_{k=1}^{n}[(1-a_k)^2 + (b_k - 1)^2] \\ &= \sum_{k=1}^{n}[(a_k^2 + b_k^2) - 2(a_k + b_k) + 2] = n^2 + n.\end{aligned}$$

Since $0 \leq u_k^2 \leq u_k \leq 1$ and $v_k \geq 0$ for each k, $\sum_{k=1}^{n} u_k^2 \leq \sum_{k=1}^{n} u_k \leq n$, so that

$$\begin{aligned}n^2 = (n^2 + n) - n &\leq \sum_{k=1}^{n}(u_k^2 + v_k^2) - \sum_{k=1}^{n} u_k^2 \\ &= \sum_{k=1}^{n} v_k^2 \leq \left(\sum_{k=1}^{n} v_k\right)^2 = \left(\sum_{k=1}^{n} u_k\right)^2 \leq n^2.\end{aligned}$$

All the inequalities in the foregoing chain must be equalities, each $u_k = 1$ and all but one of the v_k must be 0. Hence $v_1 = \cdots = v_{n-1} = 0$ and $v_n = n$. Therefore the only solution to the equation pair under the constraint is $a_1 = \cdots = a_n = 0$, $b_1 = \cdots = b_{n-1} = 1$, $b_n = n + 1$.

4238. *Proposed by Mihaela Berindeanu.*

Let ABC be a triangle with M an arbitrary point on BC. If X, Y are centers of inscribed circles in $\triangle ABM$ and $\triangle AMC$ and if there exists $Z \in (AM)$ for which $BXZA$, $CYZA$ are cyclic quadrilaterals, find $m,\ n \in \mathbb{R}$ leading to $\overrightarrow{AM} = m\overrightarrow{AB} + n\overrightarrow{AC}$.

The statement of the problem came with an unfortunate typographical error: the second circle CYZ should have contained A as a fourth point (and not B). Computer graphics suggest that the requested values of m and n for the problem as stated would be unpleasant functions involving the angles of ΔABC. We received two solutions to the intended problem (even though it was not the problem that the authors claimed to be solving). We present the solution by Daniel Dan, corrected by the editor.

We assume that A is a point common to the circles BXZ and CYZ and will first prove that the triangles BMX and ZMX must be congruent (using directed angles so that we require no assumption that Z lies between A and M). In the first circle we have

$$\angle XBA = \angle XZA = \angle XZM.$$

Because X is the incenter of ΔABM, we have

$$\angle XBA = \angle MBX, \quad \angle XMB = \angle ZMX, \quad \text{and} \quad \angle BAX = \angle XAZ.$$

The equality of the last pair of angles implies that they are subtended by congruent chords $BX = ZX$. As a consequence, we have $\Delta XBM \cong \Delta XZM$ (by angle-side-angle).

Under the assumption that the points C, Y, Z, A are concyclic, the same argument with B and X replaced by C and Y, respectively, gives us

$$\Delta YCM \cong \Delta YZM.$$

Consequently $BM = ZM = CM$, so that M must be the midpoint of BC, and we conclude that $m = n = \frac{1}{2}$.

4239. *Proposed by Leonard Giugiuc and Abdilkadir Altintas.*

Let ABC be a triangle with centroid G. Denote by D and E the midpoints of the sides BC and AC, respectively. If the quadrilateral $CDGE$ is cyclic, prove that

$$\cot A = \frac{2AC^2 - AB^2}{4 \cdot Area(ABC)}.$$

We received 13 submissions, all of which were correct and complete. We present the solution by Ricard Peiró i Estruch.

Denote by a, b and c the lengths of the sides of $\triangle ABC$, and by m_a the length of the median AD. From the formula for the length of a median, $m_a^2 = \frac{2b^2+2c^2-a^2}{4}$.

By the cosine law, $\cos A = \frac{b^2+c^2-a^2}{2bc}$; from the sine formula for area, $\sin A = \frac{2 \cdot Area(ABC)}{bc}$. It follows that $\cot A = \frac{b^2+c^2-a^2}{4 \cdot Area(ABC)}$, and so it suffices to prove that for the given triangle $2c^2 = a^2 + b^2$.

Since G is the intersection of the medians of $\triangle ABC$ we have $AG = \frac{2}{3}AD$. Applying the power of a point theorem to point A with respect to the circumscribed circle of quadrilateral $CDGE$ we get $AE \cdot AC = AG \cdot AD \Rightarrow \frac{1}{2}b \cdot b = \frac{2}{3}m_a \cdot m_a$. Substituting the earlier formula for m_a and simplifying gives $2c^2 = a^2 + b^2$, concluding the proof.

4240. *Proposed by Michael Rozenberg and Leonard Giugiuc.*

Let a, b and c be positive real numbers such that $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Prove that $1 + a + b + c \geq 4abc$.

We received 11 submissions, all correct. We present the solution by Titu Zvonaru, modified slightly by the editor.

Let $p = a + b + c$, $q = ab + bc + ca$, and $r = abc$. Then we are to prove that

$$\sqrt{\frac{pr}{q}} + p \geq \frac{4q}{p}, \tag{1}$$

under the assumption that $pr = q$. Since $pq \geq 3\sqrt[3]{r} \cdot 3\sqrt[3]{r^2} = 9r$ by the AM-GM inequality, we have

$$\sqrt{\frac{pr}{q}} + p - \frac{4q}{p} = \frac{\sqrt{pqr}}{q} + p - \frac{4q}{p} \geq \frac{3r}{q} + p - \frac{4q}{p} = \frac{3pr + p^2q - 4q^2}{pq}. \tag{2}$$

Now,

$$\begin{aligned} 3pr + p^2q - 4q^2 &= 3pr + q(p^2 - 4q) \\ &= 3abc(a+b+c) + (ab+bc+ca)((a+b+c)^2 - 4(ab+bc+ca)) \\ &= 3abc(a+b+c) + (ab+bc+ca)(a^2+b^2+c^2-2ab-2bc-2ca) \\ &= a^3b + ab^3 + b^3c + bc^3 + c^3a + ca^3 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2 \\ &= ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2 \geq 0, \end{aligned}$$

which implies $\sqrt{\frac{pr}{q}} + p - \frac{4q}{p} \geq 0$ by (2), and (1) follows. The equality holds if and only if $a = b = c = 1$.

Editor's comments. The solution featured above is the only proof received which is both elementary and straightforward. Arkady Alt, Steven Chow, and Madhav Modak all gave a proof based on the more advanced Schur's Inequality.

AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Mohammad Aassila, Strasbourg, France : 4330
Kadir Altintas, Turkey, and Leonard Giugiuc, Romania : 4323, 4339
Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina : 4340
Michel Bataille, Rouen, France : 4324, 4336
Mihaela Berindeanu, Bucharest, Romania : 4329, 4337
Marius Drăgan, Mircea cel Batran Technical College, Bucharest, Romania : 4322
Leonard Giugiuc, Drobeta Turnu Severin, Romania : 4335
Leonard Giugiuc and Diana Trailescu, Romania : 4321
Tran Quang Hung, Hanoi University of Science, Hanoi, Vietnam: 4326
Van Khea, Cambodia, and Leonard Giugiuc, Romania : 4328
Mihai Miculiţa and Titu Zvonaru, Romania : 4333
Somasundaram Muralidharan, Chennai, India : 4332
Daniel Sitaru, Drobeta Turnu-Severin, Romania : 4327, 4338
Daniel Sitaru and Leonard Giugiuc, Romania : 4331
George Stoica, Saint John, NB : 4334
Alessandro Ventullo, Milan, Italy : 4325

Solvers - individuals

Mohammed Aassila, Strasbourg, France : **OC312**, OC314, OC315, **OC316**, OC317
Arkady Alt, San Jose, CA, USA : 4223, **4225**, **4226**, **4236**, 4240
Miguel Amengual Covas, Cala Figuera, Mallorca, Spain : **4230**, 4239
Jean-Claude Andrieux, Beaune, France : OC311, OC316, **4222**
Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina : CC262, **CC265**, OC312, 4221, 4225
Roy Barbara, Lebanese University, Fanar, Lebanon : **4227**, **4229**, **4233**, 4239, 4240
Michel Bataille, Rouen, France : OC311, OC312, OC314, OC316, **4221**, 4222, 4223, 4224, 4225, **4226**, 4227, **4228**, 4229, 4230, 4232, **4236**, 4239
Brian D. Beasley, Presbyterian College, Clinton, USA : 4224
Mihály Bencze, Bucharest, Romania : **4228**
Mihaela Berindeanu, Bucharest, Romania : 4222, 4238
Paul Bracken, University of Texas, Edinburg, TX, USA : 4224, 4226
Charles Burnette, Academia Sinica, Taipei, Taiwan : CC270
Steven Chow, Albert Campbell Collegiate Institute, Scarborough, ON : **CC266**, CC267, CC268, **CC269**, CC270, OC311, OC312, OC313, OC314, OC316, OC318, **OC319**, **H1**, H2, H3, H4, H5, 4221, 4222, 4223, 4224, 4225, 4230, **4231**, 4233, 4235, 4237, 4239, 4240
Óscar Ciaurri, Universidad de la Rioja, La Rioja, Spain : **4232**, 4236
Qinyu Cui, La Prairie, QC : OC313
Daniel Dan, National College Gheorghe Titeica, Romania : CC262, **CC263**, CC264, CC265, CC268, OC312, OC313, 4223, 4233, 4235, 4237, **4238**, 4239
Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India : CC267,

OC311, OC312, **4221**, 4222, **4224**, 4233, **4237**
Paul Deierman, Southeast Missouri State University, Cape Girardeau, MO, USA : 4230, 4232
William Digout, Moncton, NB : CC268
Ivko Dimitrić, Pennsylvania State University Fayette, PA : CC261, CC262, CC263, CC264, CC265, CC266, CC267, CC268, CC269, CC270, **OC311**, OC312, H1, H3, H4, H5, 4224, 4233
Joseph DiMuro, Biola University, La Mirada, CA, USA : 4233
Nghia Doan, Moncton, NB : 4223, 4233, 4237
Madison Estabrook, Missouri State University, Springfield, Missouri, USA : CC268, 4224
Andrea Fanchini, Cantù, Italy : OC311, OC316, **CC262**, CC265, H4
J. Chris Fisher, Cleveland Heights, Ohio, USA : H4
Oliver Geupel, Brühl, NRW, Germany : OC311, OC312, OC313, OC314, **OC315**, OC316, **OC317**, **OC318**, H1, H4, 4221, 4222, 4223, 4224, 4225, 4236, 4227
Leonard Giugiuc, Drobeta Turnu Severin, Romania : 4221, 4222, 4223, **4226** (2 solutions), 4228, 4229, 4233, **4236**
Richard Hess, Palos Verdes, CA, USA : CC261, CC262, CC263, CC264, CC265, CC266, CC267, CC268, CC270, 4236
Hoang Le Nhat Tung, Hanoi, Vietnam : 4223
David Huckaby, Angelo State University, San Angelo, TX, USA : 4233
Doddy Kastanya, Toronto, ON : CC268
Kee-Wai Lau, Hong Kong, China : 4225, 4230, 4236, 4239, 4240
Kathleen Lewis, Brikama, Republic of the Gambia : CC262, CC265
Martin Lukarevski, University Goce Delcev, Stip, Macedonia : 4225
David E. Manes, SUNY at Oneonta, Oneonta, NY, USA : CC258, CC262, CC263, CC264, CC265, CC268, CC270, OC312
Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India : 4222, 4223, 4224, 4230, 4233, 4239, 4240
Kordali Mohammad, Talayedaran, Ahvaz, Iran : CC268
Somasundaram Muralidharan, Chennai, India : CC265, 4222, **4224**
Theoklitos Parayiou, Lyseum Polemidion, Limassol, Cyprus : 4222, 4223, 4225
Ricard Peirói Estruch, Valècia, Spain : CC262, CC265, CC268, **CC270**, 4233, **4239**
Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy : CC268, OC312, OC313, 4221, 4223, 4224, 4225, 4232, 4234, 4236, 4240
Kevin Soto Palacios, Huarmey, Peru : 4221
Ángel Plaza, University of Las Palmas de Gran Canaria, Spain : **CC262**, CC264, CC268, OC312, **4232**, 4236
C.R. Pranesachar, Indian Institute of Science, Bangalore, India : H1, H3, **H4**, H5, 4221, 4223, 4225, 4230, 4233, **4235**, 4236, 4239
Henry Ricardo, Tappan, NY, USA : **CC268**, 4224, 4232, **4236**
Joel Schlosberg, Bayside, NY : **CC262**, CC265, CC267, CC268, CC269, CC270, **H2**, H3, **H4**, H5, 4224
Daniel Sitaru, Drobeta Turnu-Severin, Mehedinti, Romania : 4226
Digby Smith, Mount Royal University, Calgary, AB : **CC261**, CC262, **CC263**, CC264, CC265, **CC267**, CC268, OC312, OC313, 4221, 4223, 4224, 4225, **4226**, 4230, 4233, **4234**, 4235, 4236, 4237, 4239, 4240
Trey Smith, Angelo State University, San Angelo, TX, USA : **4229**
David Stone, Georgia Southern University, Statesboro, GA, USA : 4233
Sladjan Stankovic, Kumanovo, Macedonia : 4236

Marius Stănean, Zalău, Romania : 4231
Nguyen Hung Viet, Vietnam : 4221, 4236
Peter Y. Woo, Biola University, La Mirada, CA, USA : 4222, 4233, 4240 (2 solutions)
Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA : CC270
Titu Zvonaru, Comăneşti, Romania : CC262, **CC263**, **CC264**, CC265, CC268, CC270, OC316, OC312, **OC313**, **H4**, 4222, **4223**, 4224, 4225, 4230, 4233, 4239, **4240**

Solvers - collaborations

AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia : 4232, 4239
Ruben Dario Auqui, Peru, and Leonard Giugiuc, Romania : 4235
Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, USA : 4221, 4223, 4224
M.Bello, M. Benito, O. Ciaurri, E. Fernandez and L. Roncal, Logroño, Spain : **4226**, 4236
Leonard Giugiuc, Daniel Dan, and Daniel Sitaru : 4225
Leonard Giugiuc, Daniel Sitaru, and Marian Dinca : 4234
Leonard Giugiuc and Dan Marinescu, Romania : 4227
Leonard Giugiuc and Kadir Altintas : 4239
Missouri State University Problem Solving Group : CC262, OC319, **H4**, H5, 4229, 4233
Cristinel Mortici and Leonard Giugiuc, Romania : 4237
Miguel Ochoa Sanchez, Peru and Leonard Giugiuc, Romania : 4230
Michael Rozenberg, Israel, and Leonard Giugiuc, Romania : 4240