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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems. The copyright of all published material is vested in the Applied Probability Trust.

Articles are normally commissioned by the Editors; the Editorial Committee also welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. All correspondence about the contents should be sent to:

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Editorial: *Mathematical Spectrum* Awards

The editors of *Mathematical Spectrum* have always placed particular value on the active participation of their younger readers. To demonstrate their appreciation of such involvement, they have decided to institute two annual prizes for contributors who are still at school or are undergraduates in colleges or universities. One prize, to the value of £20, will be awarded for an article published in *Mathematical Spectrum*; another of £10 will be for a letter or the solution of a problem. The winners of these prizes for any one volume of the magazine will be announced in the January issue of the next volume. In any one year the editors may feel obliged to withhold either or both of the prizes, but they hope that, in fact, their difficulties will derive only from an embarrassment of riches. Contributors to the current volume are the first to be eligible for the awards: the decision of the editors will be announced in January 1978.

The 18th International Mathematical Olympiad

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The team from Great Britain to the olympiad held in Lienz, Austria in July 1976 included four who had competed previously, so it was expected to do well. In the event, we achieved our highest place ever, second to the Soviet Union.

The competition, for teams of up to eight pre-university students, consists of three questions to be tackled in a four-hour session on one day followed by three more questions in a similar session on the following day. Correct answers are not sufficient; close reasoning is required and points are lost for any lapses in the rigour of a proof. No charity is shown, with the result that only one student, a Frenchman, scored full marks, one, a Russian girl, obtained 39 points and 36 obtained a single-figure total out of a maximum of 40 points.

The results showing total scores for each team of eight competitors are tabulated below:

	Question							Prizes		
	1	2	3	4	5	6	Total	1st	2nd	3rd
USSR	36	46	46	46	31	45	250	4	3	1
Great Britain	23	35	52	43	11	50	214	2	4	1
USA	25	18	32	44	26	43	188	1	4	1
Bulgaria	22	18	37	36	6	55	174		2	6
Austria	26	16	34	28	8	55	167	1	2	5
France	21	31	27	22	22	42	165	1	3	1
Hungary	27	15	19	38	8	53	160		3	4
East Germany	26	18	28	27	12	31	142		2	3
Poland	25	15	10	42	0	46	138			6
Sweden	19	0	38	27	0	36	120		1	3
Rumania	18	10	13	28	1	48	118		1	3
Czechoslovakia	22	11	13	30	7	33	116		1	3
Yugoslavia	26	10	22	34	4	20	116		1	3
Vietnam	21	9	15	29	10	28	112		1	3
Holland	15	6	9	27	5	16	78			1
Finland	9	3	11	10	0	19	52			1
Greece	2	8	14	12	1	13	50			
Maximum	40	56	64	48	56	56	320			

In addition, Cuba entered three competitors and West Germany made a trial appearance for the first time, with two students.

It is interesting to note how each country fared on each question, the variations reflecting the different mathematics syllabuses in operation. Calculus is not in the school syllabus in many of the Eastern European countries, so no questions requiring calculus are set in the IMO. However, a knowledge of calculus was used to advantage by many competitors in Question 2, and this partly explains the relatively good showing of the British team on this difficult question. On the other hand, one was more likely to find the best solution to Question 4 if no calculus was known. Similarly, some experience of linear algebra proved a real disadvantage in Question 5 which can only be solved successfully by combinatorial methods. A pragmatic approach works well with Question 3, equations and formal argument being used only after considerable numerical investigation; it was here that the British were most effective.

Like other international contests, the IMO aims to foster goodwill and interchange of ideas and information between team members and officials. This year's olympiad certainly fulfilled these objectives, due in no small part to the beauty of the surroundings in the Tyrol, the imaginative organisation of our Austrian hosts and their exceptional generosity.

All countries face the problem of providing stimulus and suitable tuition for the mathematically gifted, often within a more or less comprehensive system of education and usually without the specialisation we are accustomed to in the sixth form. Some (e.g. the Soviet Union and Yugoslavia) have special mathematical schools. Regional and national competitions go some way to identify and encourage talent, and the chance to represent one's country on an enjoyable trip abroad is a further carrot. Tuition by correspondence is used in some countries and most hold training sessions for their IMO teams. (The USA team had three weeks preparation together immediately prior to the olympiad.) Here we have in the past provided no special training and the team has met for the first time at the start of the journey to the olympiad. This year for the first time some problems have been sent at intervals by post to those requesting them, solutions being provided with the next problems. Any British school or sixth-former wishing to take advantage of this service should write to Mr R. C. Lyness, Singleton Lodge, Blackpool FY6 8LT, enclosing a stamped self-addressed envelope.

Appendix 1

The questions in the 1976 olympiad were as follows.

1. (5 points) In a plane convex quadrilateral of area 32 cm^2 the sum of the lengths of two opposite sides and one diagonal is equal to 16 cm.
Determine all possible lengths of the other diagonal.

2. (7 points) Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for $j = 2, 3, \ldots$. Show that, for any positive integer n, the roots of the equation $P_n(x) = x$ are all real and distinct.

3. (8 points) A rectangular box can be filled completely with unit cubes. If one places as many cubes as possible, each with volume 2, in the box so that their edges are parallel to the edges of the box, one can fill exactly 40 per cent of the box. Determine the interior dimensions of all such boxes. ($(2)^{\frac{1}{3}} = 1{\cdot}2599\ldots$.)

4. (6 points) Determine, with proof, the largest number which is the product of positive integers whose sum is 1976.

5. (7 points) Consider the system of p equations in q unknowns, where $q = 2p$,

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1q}x_q &= 0, \\ a_{21}x_1 + \cdots + a_{2q}x_q &= 0, \\ &\cdots \\ a_{p1}x_1 + \cdots + a_{pq}x_q &= 0, \end{aligned}$$

with every coefficient a_{ij} a member of the set $\{-1, 0, +1\}$. Prove that there exists a solution $(x_1, \ldots, x_q)$ of the system such that

(a) all x_j ($j = 1, \ldots, q$) are integers;
(b) there is at least one value of j for which $x_j \neq 0$;
(c) $|x_j| \leqslant q$ ($j = 1, \ldots, q$).

6. (7 points) A sequence $\{u_n\}$ is defined by $u_0 = 2$, $u_1 = 5/2$, $u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$ for $n = 1, 2, \ldots$.
Prove that, for positive integers n,

$$[u_n] = 2^p, \quad \text{where } p = \tfrac{1}{3}(2^n - (-1)^n),$$

and $[x]$ denotes the greatest integer $\leqslant x$.

Appendix 2

Special prizes may be given by the international jury for solutions of particular merit. One only was awarded this year, and this went to J. R. Rickard (City of London School) for generalising the result of Question 5 in the following way.

If '$q = 2p$' is replaced by '$q = kp, k > 1$', and the restriction that $a_{ij} \in \{-1, 0, 1\}$ is replaced by 'a_{ij} is an integer with $|a_{ij}| \leqslant n$, $n \geqslant 1$', then there exists a solution satisfying (a), (b) and

(c′) $|x_j| \leqslant 2r$, where r is the least integer greater than or equal to $\frac{1}{2}(nq)^{1/(k-1)}$.

His proof of the original problem, suitably modified for the generalised problem, went as follows.

Consider the ways of assigning $-r, -r+1, \ldots, -1, 0, 1, \ldots, r$ to each of $x_1, \ldots, x_q$. There are $(2r+1)^q$ such ways.

For each of these, $L_1, \ldots, L_p$, the values of the left-hand sides of the equations are such that $|L_i| \leqslant nqr$. So there are at most $(2nqr + 1)^p$ different sets of values for $L_1, \ldots, L_p$. Now

$$2r + 1 > (nq)^{1/(k-1)},$$

i.e.

$$(2r+1)^{k-1} > nq,$$

and therefore

$$(2r+1)^k > nq(2r+1) \geqslant 2nqr + 1,$$

so that

$$(2r+1)^q > (2nqr+1)^p.$$

From the pigeon-hole principle it follows that there exist two distinct sets of values for $x_1, \ldots, x_q$ which give the same set of values for $L_1, \ldots, L_p$. Call these $y_1, \ldots, y_q$ and $z_1, \ldots, z_q$. Then $X_i = y_i - z_i$ is easily seen to give a set of integer values which satisfy the original equations. Moreover,

$$|X_i| \leqslant |y_i| + |z_i| \leqslant 2r, \quad \text{for } 1 \leqslant i \leqslant q.$$

Two Puzzles

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Both of these puzzles can be made quite easily by even the least practically minded mathematician. For the first one, a small flat box will be useful whose base has dimensions 5×11 (appropriate units). Now draw on thick card the eleven non-convex shapes each of which can be built out of five squares of side-length 1 unit by placing the squares edge to edge. Mirror images are discounted.† Cut them out and see if you can fit them exactly into the box. It is interesting to note that it is, in fact, possible to make a rectangle of size $5 \times x$ from x of the pieces for each x in the range $3 \leqslant x \leqslant 11$. No corresponding result for non-convex 'tetrominoes' (made of four squares instead of five), of which there are just three, is valid. There is a great deal of information about pentominoes and, more generally, about 'polyominoes' in a book on the subject by their inventor Solomon Golomb (see reference 1). Our results are not, I think, mentioned explicitly by Dr Golomb, probably because we have restricted ourselves to the non-convex pentominoes.

The second puzzle is somewhat similar but is three-dimensional; it is the famous 'Soma cube' invented by the Danish writer Piet Hein. (Soma is the name of a happiness drug in Aldous Huxley's novel *Brave New World.*) A convenient way of making this puzzle is from a box containing twenty-seven equal cubic building blocks. Form all the seven non-convex shapes which can be constructed, each from at most four cubes with appropriate faces glued together. Just one of these consists of three cubes and each of the other six consists of four cubes. The basic puzzle is to fit all of these together to form a $3 \times 3 \times 3$ cube. Most people find this quite difficult at first, but one soon gets the 'feel' of how the pieces fit together and eventually one becomes very skilled at making cubes in many different ways. Martin Gardner has written about the Soma cube in his book *More Mathematical Puzzles and Diversions* (reference 2) and in an article in the *Scientific American* (reference 3). There are very many interesting shapes other than the cube which can be constructed from the seven Soma pieces. Drawings of some of them have been made by Martin Gardner, and readers may like to try constructing these for themselves. Figure 1 shows three more which I have not seen illustrated elsewhere. All of them have a rather pleasing symmetry and I think are quite difficult. There are no holes in these models, so, instead of drawing three-dimensional pictures, I have made plans of them and specified the height (in number of small cubes) in each position.

These puzzles are really extensions of the old Chinese tangram puzzle. Some genuine combinatorial mathematics can be learned from them, and you may be

† These are, in fact, the pieces 2–12 drawn in *Mathematical Spectrum* Vol. 8 No. 2, p. 49. The twelve pieces are called 'pentominoes'.

interested, in particular, to consult the sources mentioned in order to see how 'proofs of impossibility' can be devised. The simplest problem of this kind is to show that a chessboard, from which two opposite corner squares have been cut out, cannot be covered by non-overlapping dominoes each of whose areas is equal to that of two chessboard squares.

1	1	2	1	2	1	1
1	1	2	2	2	1	1
1	1	1	2	1	1	1

1	2	1	1	1	1	1
2	2	2	2	1	1	1
1	2	1	1	1	1	1

2	3	2	2	2	3	2
2	2	1	1	1	2	2

Figure 1

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3. Martin Gardner, *Scientific American*, September 1958.

Odd Couples and Missing Cars

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Many teachers of probability theory will have their own favourite example to demonstrate the importance of computing a conditional probability rather than an unconditional one. In some cases, it is crucial not only to compute a conditional probability, but to condition this probability on the right event. To illustrate these points, we discuss, firstly, a famous legal application of probability theory (reference 1), and secondly, an analogous problem, in which we point out some errors in the first example.

1. The case of the odd couple

In a criminal case in California (reference 2), a couple were found guilty of robbery, partly on the basis of a probability argument, but this decision was reversed on appeal, because the wrong probability had been computed.

Eyewitnesses testified that the robbery had been committed by a couple consisting of a Negro man with a beard and moustache and a Caucasian girl with blonde hair and a ponytail, driving in a partly yellow automobile. The defendants answered this description, but otherwise could not be positively identified as the robbers. Assuming certain probabilities for each of these characteristics, and independence amongst these, the prosecution calculated a probability of one in 12 million that any couple would have these same characteristics.

In reversing the earlier conviction of the couple, the California Supreme Court pointed out that no evidence was given for the accuracy of the probabilities of the individual characteristics, or for the assumption of independence. It also emphasised the 'gravely-misguided' nature of the probability calculation. 'At best, it might yield an estimate as to how infrequently bearded Negroes drive yellow cars in the company of blonde females with ponytails,' but furnished '*absolutely no guidance on the crucial issue: Of the admittedly few such couples, which one, if any, was guilty of committing this robbery?*' (Emphasis theirs.) They calculated that if the probability of a couple having these characteristics was p, then the *conditional* probability that in a population containing n couples there were two or more couples with them, given that there was at least one with them, was

$$(1 - (1 - p)^n - np(1 - p)^{n-1})/(1 - (1 - p)^n). \qquad (1)$$

Setting $p = 1/12{,}000{,}000$ and $n = 12$ million (the robbery was committed in the Los Angeles metropolitan area), this probability equals 0·42. That is, there is a 42% chance that the defendants were not the only couple answering the description of the guilty couple. (The Supreme Court noted the difficulty of defining the population of couples who might have committed the robbery, and of ascertaining its size, even if it could be defined.)

While this example demonstrates clearly the possibly drastic consequences of computing an unconditional probability rather than a conditional one, we claim that:

(a) the probability in equation (1) is not particularly relevant, either; in particular, it is *not* the probability that the accused are innocent;
(b) it is conditional on the wrong event.

To clarify the issues raised, and to justify our claims, we consider an analogous problem, originally set as an exercise for (second-year) university students of probability theory.

2. The case of the missing car

An absent-minded professor at a large university owns a relatively rare car, a model X. He has forgotten where he left it in the car park, and he can never remember the registration number, so he walks at random round the park and stops at the first model X car he sees. What is the probability q that he has found his own car? Assume that there are n ($\simeq 1000$) cars in the car park, and that a fraction p ($\simeq 1/5000$) of all cars are model X ones.

Which (if any) of the following answers is correct? Explain the errors in the incorrect answers.

2.1. Only one of the n cars is the right one, so $q = 1/n$.

2.2. The probability that there is another model X car is p, so $q = 1 - p$.

2.3. The expected number of model X cars in the park is np, which is much less than one, so there is virtually no chance that there could be more than one model X car there, and $q = 1$.

2.4. The car is his if there is only one model X car in the park, and the probability of this is $q = np(1 - p)^{n-1}$.

2.5. Answer 2.4 ignored the known fact that there is at least one model X car in the park. We must take the probability that there is only one conditional on this given event, so

$$q = np(1 - p)^{n-1}/(1 - (1 - p)^n).$$

2.6. If there are k model X cars in the park ($k = 1, 2, \ldots, n$), the professor has a chance of $1/k$ of finding his own. Summing over the possible values of k, we find

$$q = \sum_{k=1}^{n} \frac{1}{k} \binom{n}{k} p^k (1 - p)^{n-k}.$$

2.7. We already know that the professor's car is in the car park, but the identity of the remaining $(n - 1)$ cars is not known. If k of them are model X ones ($k = 0, 1, \ldots, n - 1$), the professor has a chance of $1/(k + 1)$ of finding his own, so that

$$q = \sum_{k=0}^{n-1} \frac{1}{k + 1} \binom{n - 1}{k} p^k (1 - p)^{n-1-k}$$

which reduces to $q = np^{-1}(1 - (1 - p)^n)$.

2.8. Answer 2.6 ignored the known fact that there is at least one model X car in the park. Modifying it to condition the probability on this fact gives

$$q = \sum_{k=1}^{n} \frac{1}{k} \binom{n}{k} p^k (1 - p)^{n-k} \Big/ (1 - (1 - p)^n).$$

Before reading on, readers are invited to assess these 8 answers, or to work out their own solutions, and to decide how the problem is connected with the legal case.

3. Discussion

Answers 2.1 to 2.4 are clearly wrong (why?), and need not be discussed here. Answers 2.5, 2.6 and 2.8 are successive stages in an attempt to solve the problem by calculating a conditional probability. In its most developed form, answer 2.8, he argument assumes that

(i) The probability that there are k model X cars in the park is

$$\binom{n}{k} p^k(1-p)^{n-k}, \quad k = 0, 1, \ldots, n.$$

(ii) If there are k model X cars, the probability that the professor has found his own is $1/k$, $k = 1, 2, \ldots, n$.

(iii) We must take account of the known presence of the professor's car by making q conditional on the event that there is at least one model X car in the park.

This argument is false because it makes q conditional on the wrong event. In fact we know more than that there is at least one model X car in the park: we know that the *professor's* car is in it (indeed, step (ii) of the argument tacitly assumes this). If we treat all n cars in the park as a binomial sample, as implied by step (i) of the argument, then there is no way of calculating the *a priori* probability that the professor's car is in the car park, so we cannot find the right conditional probability.

By applying the binomial model only to the $n - 1$ cars which are not the professor's, answer 2.7 avoids this problem, and so arrives at the best of the 8 solutions.

4. Was the court right?

The professor and the California Supreme Court faced the same problem: a particular couple (car), say A, is known to have certain characteristics; what is the probability that a randomly-chosen couple (car) with these characteristics is A?

The prosecutor's original argument seems to have been similar to our answer 2.2, and was rightly rejected. However, the court's probability (1) is the complement of our answer 2.5, and seems to be open to the criticisms we have made of the general method used in answers 2.5, 2.6 and 2.8. Specifically, the court apparently failed to see the distinction between the events 'there is at least one odd couple' and 'the guilty couple is odd'. As a result, probability (1) is conditional on the wrong event, and based on an inappropriate probabilistic model of the facts.

If we accept that answer 2.7 is the best estimate we can give of the probability that the professor has found his car, we can use the same method to find the probability that the police caught the guilty couple. Taking the previous values of

n and p, we find this probability to be 0·63. In this particular case, there is only a marginal numerical difference between answers 2.5 and 2.7 and so it appears that the Supreme Court made the right decision, even if its own calculations were based on an incorrect probability argument.

5. Final comments

This discussion is intended to bring out a point of principle rather than to give a realistic answer to the very complicated problem faced by the court. For this reason we have not tried to justify the many simplifying assumptions made in all the answers (for instance, the assumption that the population of cars or couples is infinite and the selection random).

Probability arguments have been used in a number of legal cases (reference 3) in an effort to assess the strength of circumstantial evidence. We approve of this in principle, but point out that this use of probability as evidence can be dangerous: it is often more difficult than it appears at first sight to formulate a realistic mathematical model of the facts and to calculate an appropriate probability from this model. Further detailed discussion of these difficulties may be found in references 4, 5 and 6.

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Games of Chance and Probability: A Historical Anecdote

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1. Introduction

High school and college students embarking on their first lessons in probability are led to the probability calculus through problems of cards and dice. They may learn about the famous gambler, the Chevalier de Méré, who posed a problem of dice throws to Blaise Pascal (1623–1662); he in turn consulted Pierre de Fermat (1601–1665) by correspondence about its solution. This, they learn, was how the theory of probability was born. Rarely do they hear the interesting story of Pascal's solutions of the problem of division of stakes given in his letters to Fermat. But it is just in these division of stakes problems that we can catch a glimpse of Pascal's genius as well as his weakness in considering problems of chance. In this paper, I propose to introduce these problems, which teachers may find of interest for classroom use.

2. De Méré and probability

Did the playing of games of chance lead to the development of probability theory? Most writers on this subject seem to think so, and they often credit its beginning to the Chevalier de Méré.

The story is related in a most fascinating manner in an article written by the two distinguished Russian mathematicians, Khinchin and Yaglom, entitled 'The Science of Chance', in the third volume of the Children's Encyclopedia (in Russian). It was reproduced in a recently published book, *Probability Theory, A Historical Sketch* by L. E. Maistrov and translated into English by Samuel Kotz. From this we quote the following:

> A French Knight, Chevalier de Méré, was an ardent dice gambler. He tried to become rich by this game and was constantly thinking of various complicated rules which he hoped would help him reach his goal.
>
> For example, de Méré thought of the following rule: He proposed to throw one die four times in a row and wagered that at least one six would appear; if no six turned up, then the opponent won . . .
>
> However, the other players soon discovered that the game was not fair and they stopped playing with de Méré. It was time to think of some other rules and de Méré devised a new game. He proposed to throw two dice 24 times and bet that two sixes would turn up at least once. But here the Knight made a mistake.
> . . . the chance of losing is greater than one-half. It means that the more the Knight played, the more he was bound to lose. And this is what happened. The more he played, the more he ruined himself and ultimately he ended up impoverished. The most interesting side of this historical anecdote is that, due to these peculiar 'practical inquiries' the theory of calculating random phenomena was initiated.

Was de Méré really the ardent gambler he is represented to be? In contrast to this popular belief, Oystein Ore wrote in his interesting paper, 'Pascal and the Invention of Probability Theory' that 'the distinguished Antoine Gambaud, Chevalier de Méré, Sieur de Baussay, would turn in his grave at such a characterization of his main occupation in life'. He had, in fact, been a prominent figure at the court of Louis XIV where he was an adviser in delicate situations and an arbiter of conflicts, and his works show him to be a philosopher rather than a gambler. Just as Ore disputes the traditional belief that de Méré was a gambler, Maistrov is of the opinion that the emergence of probability had little to do with gambling, and that its development was due to economic factors instead.

3. Pascal's solution of the division of stakes problem

Most elementary textbooks give the impression that the problem posed by de Méré to Pascal, which led to discussions between Pascal and Fermat in 1654, was the problem of obtaining a double six in 24 throws of a pair of dice.

In fact, the bulk of the correspondence between Pascal and Fermat is concerned with the problem of division of stakes; these problems were of such difficulty in the mid-17th century that their solution may be considered a decisive breakthrough in the history of mathematics. From their discussions emerges Fermat's combinatorial method of solving classical probability problems, which students now learn in their first lessons on probability. What is seldom known to students is that the combinatorial method, however simple it may now appear, was possibly not fully understood by Pascal at first, but that Pascal was nevertheless able to arrive at the same conclusions by entirely different methods. These he did not clearly reveal in his letters. At one time when Pascal did not fully appreciate how to apply Fermat's method correctly, he wrote to Fermat as follows:

> the combinatorial method . . . is not general and is not universally sound . . . so that, since you did not know my method when you propounded the problem of points for several players, but only the combinatorial method, I am afraid we shall have different opinions on this matter.

However, in his last letter to Fermat, dated 27 October 1654, Pascal wrote,

> I admire your method for the problem of dividing the stakes all the more so because I understand it perfectly, it is yours entirely and has nothing in common with mine; it reaches the same conclusion very simply.

What are the problems of division of stakes which they discussed and what solutions did Pascal obtain independently of Fermat's combinatorial method? Since their correspondence can be found in F. N. David's *Games, Gods, and Gambling*,we shall merely summarize below for ready reference. Students may wish to find the solution as an exercise.

Problem 1. In eight throws of a die a person is to attempt to throw a six. Suppose he has made three throws without success. What proportion of the stake should he have on condition that he gives up his 4th throw?

Pascal at first thought that the player was entitled to one-sixth of the remainder of the stakes after each throw; hence he should get 125/1296 of the total stakes after three throws. Fermat disagreed, saying that since the first three throws bring nothing to the player, if he agrees not to play the 4th throw, he should receive 1/6 of the total stakes. On this Pascal finally agreed.

Problem 2. Two players, A and B, play a game that requires 3 points to win and each player has a stake of 32 pistoles. How should the sum be divided if they break off at any stage?

Pascal reasoned as follows. After the next play, won or lost, A (the player who has scored a higher point in previous plays) will be assured of the minimum of x pistoles if he loses the next play and a maximum of y pistoles if he wins the next play. His share should therefore be $x + \frac{1}{2}(y - x)$ pistoles while B gets the rest. The following table gives A's share for all three possible cases.

Previous scores A:B	2:1	2:0	1:0
A's share of total stakes	48	56	44
B's loss	16	24	12
B's share	16	8	20

Note that in the above game of 3 points, if the ratio of the scores of A:B is 2:0 when the players decide to discontinue the game, B's share of the total stake is $8/32 = 1/4$ of his own stake. This Pascal calls the value (to the loser) of the last game of three (points). When the ratio of the scores of A:B is 1:0, B's loss to A is $12/32 = 3/8$ of his own stake; this Pascal refers to as 'the proportion for the first game'. He proceeded to make some generalizations for a game between two players requiring n points to win. Thus he wrote to Fermat:

> Now, to make no mystery of it, since you understand it so well and I only wish to see that I have made no mistake, the value (of which I mean only the value of the opponent's money) of the last game of two is double that of the last game of three and four times the last game of four and eight times the last game of five, etc. (see reference 1, page 232).
>
> But the proportion of the first game is not so easy to find: it is as follows, for I do not wish to falsify anything, and here is the method which I have tried out so much, because it pleases me greatly.

For a game of $n + 1$ points, the proportion for the first game, i.e. when the ratio of the scores of A:B is 1:0 and they agree to discontinue the game, then the proportion of B's money that is lost to A, should be, according to Pascal,

$$\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n} \tag{1}$$

For a game of 3 points when $n = 2$ the proportion for the first game has been shown to be 3/8.

Problem 2 and its generalizations can of course be solved by Fermat's combinatorial method. This Pascal admitted and, in fact, he acknowledged that he had used combinatorial methods 'but with a great deal of trouble' to derive the beautiful result in (1). For instance, if the ratio of the scores of A:B is 1:0 in a game that requires three points to win, Pascal realized that Fermat's method consists in first determining 'in how many more games the play will be absolutely decided' (4 for this problem) and then 'in how many ways four games can be arranged between the two players, and one must see how many combinations would make the first win and how many the second and to share out the stakes in this proportion'. The result is 11:5 and hence B's loss to A is 3/8 of his own stakes.

For a game of $n + 1$ points, if the ratio of the scores of A to B is 1:0 when they decide to discontinue the game, it is clear that at most $2n$ more plays would be needed to determine who would be the winner of the game. Thus if each player has stake S and the probability of A's winning the game is m/N, then his share of the total stakes is $2mS/N$ or B's loss to A is $(2m - N)/N$ of his own stakes, where $N = 2^{2n}$ and

$$m = \binom{2n}{n} + \binom{2n}{n+1} + \cdots + \binom{2n}{2n}.$$

By his knowledge of the properties of what is now known as Pascal's triangle, Pascal was able to deduce the simple result in (1) from $(2m - N)/N$.

Problem 3. (A game of two points for three players.) Suppose that the ratios of the scores of three players A, B and C are 1:0:0 so that A will need one more point to win and B and C each need two points when they decide to discontinue the game. Find a fair division of stakes.

Pascal seems to have arrived at the correct solution of this problem by his own method. But when he tried to apply Fermat's combinatorial method he obtained an answer he knew was an unfair division of stakes. He reasoned as follows.

The winner of the game would be decided upon if the players were to play 3 more games. In this case, there are 27 possible combinations of the outcome in the form of (x_1, x_2, x_3) where each $x_i = a$ (if A wins), b (if B wins), c (if C wins), $i = 1, 2, 3$. A combination like (b, a, c) or (a, a, c) makes A the sole winner of the game and there are 13 combinations of this kind. A combination like (a, b, b) makes both A and B winners of the game since this results in both A and B obtaining a total of two points. Pascal thought that, if this happened, A and B should each be given half of the total stakes. There are 3 such combinations. Similarly, there are 3 combinations like (c, a, c) which make both A and C the winners. Hence A should get $(13 + \frac{1}{2} \times 6)/27 = 16/27$ of the stake. Similarly B and C should each get $(4 + \frac{1}{2} \times 3)/27 = 5{\cdot}5/27$ of the stake.

After his analysis, Pascal wrote to Fermat

> That seems to me to be the way one must solve the problem of points following your combinatorial method, unless you proceed in some other way which I do not know. But if I am not mistaken, this solution is unfair.

This made him suspect that Fermat's method was not sound in general. Fermat wrote to Pascal on 25 September 1654 that the correct solution for this game gives the ratios 17:5:5 instead of Pascal's 16:5·5:5·5, since a combination like (a, b, b) should make A (but not B) winner of the game and stated that his combinatorial method 'is sound and applicable to all cases'. Pascal gave Fermat his reply on 27 October 1654 expressing his admiration for Fermat's method which led to a correct solution very simply. This seems to be their last correspondence on mathematical problems.

On 23 November 1654, a crisis occurred to Pascal while he was in Paris. He took it as a sign from God that he must give up the life he was leading and do no more mathematics, and he retired to Port Royal.

It is interesting to read George Boole's (1815–1864) comments on this matter in his *Laws of Thoughts*:

> The problem which the Chevalier de Méré (a reputed gamester) proposed to the recluse of Port Royal (not yet withdrawn from the interests of science by the more distracting contemplation of the 'greatness and misery of man'), was the first of a long series of problems, destined to call into existence new methods in mathematical analysis, and to render valuable service in the practical concern of life.

4. Postscript

Apparently some of the letters written by Pascal and Fermat to each other have never been found. In a recently published book by the famous Hungarian mathematician Alfred Rényi, *Letters on Probability*, the author stated at the beginning of his book that a French professor of the History of Mathematics, Professor Trouverien, had discovered in 1966 a few more letters written by Pascal to Fermat. He had sent them to Rényi, who translated them into Hungarian; they formed the substance of this book. At the end of the book, however, Rényi confessed that he had fabricated these letters to express his thoughts on probability. The reader should, he writes, be intelligent enough to detect from the name of Professor Trouverien (translated as Findnothing) that no new correspondence between the two great mathematicians in the 17th century had been found.

In one of these fabricated letters by Pascal, Rényi (p. 73) writes the following view on the study of probability:

> It is not enough (though, of course, it is necessary) to have a suitably deep understanding and study of the mathematical theory to be able to immerse onself thoroughly into the mathematics of probability theory and apply it successfully . . . one must first get more closely acquainted with some concrete possibilities of the application of the probability theory, and then strive to acquire a clear and thoroughgoing understanding of the fundamental problems connected with the concept of probability.

With these thoughts, I heartily concur.

Acknowledgment

I wish to thank Dr J. Gani for his suggestions which have greatly improved the presentation of this paper, and to thank Miss Pamela Gonsalves of the University of Guyana, for typing several drafts of the manuscript.

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2. L. E. Maistrov, *Probability Theory, A Historical Sketch* (translated by Samuel Kotz, Academic Press, 1974).
3. Oystein Ore, Pascal and the invention of probability theory, *Amer. Math. Monthly* 67 (1962), 409–419.
4. Alfred Rényi, *Letters on Probability* (translated by Laszlo Vekerdi, Wayne State University Press, Detroit, 1972).

The Race-Track Problem

G. F. A. HOFFMANN de VISME
North Staffordshire Polytechnic

1. The problem

Some time ago the following problem was put to me:

A closed track of integral circumference S has N fences so distributed round it that every length from 1 to S (and no other) exists between pairs of fences. If each length other than S exists between only one pair of fences, obtain a relation between N and S, and find how the fences have to be distributed.

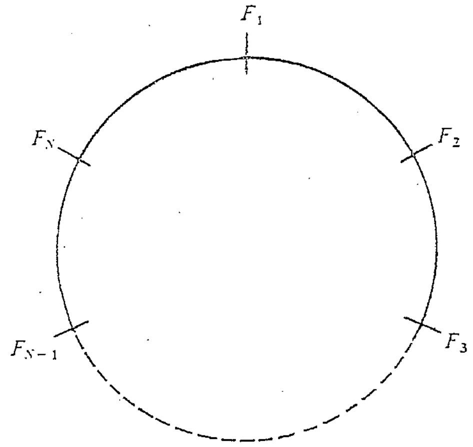

Figure 1. Closed race-track with N fences.

The solution of the first part is simple. Figure 1 shows a closed track with N fences; a segment may start at any fence and finish at any other, or else complete the circuit. A complete circuit counts as one segment of the race track, while two different fences F_i, F_j are the ends of two segments, one corresponding to each of the two directions in which the track may be described. Hence the total number of segments is $1 + N(N-1) = N^2 - N + 1$. These segments have to have lengths $1, 2, \ldots, S$ (without repetition) and so

$$S = N^2 - N + 1.$$

For example, with $N = 4$ the length of the track has to be 13 and the possible fence distributions, as shown in Figure 2, are (1, 2, 6, 4) and (1, 3, 2, 7).

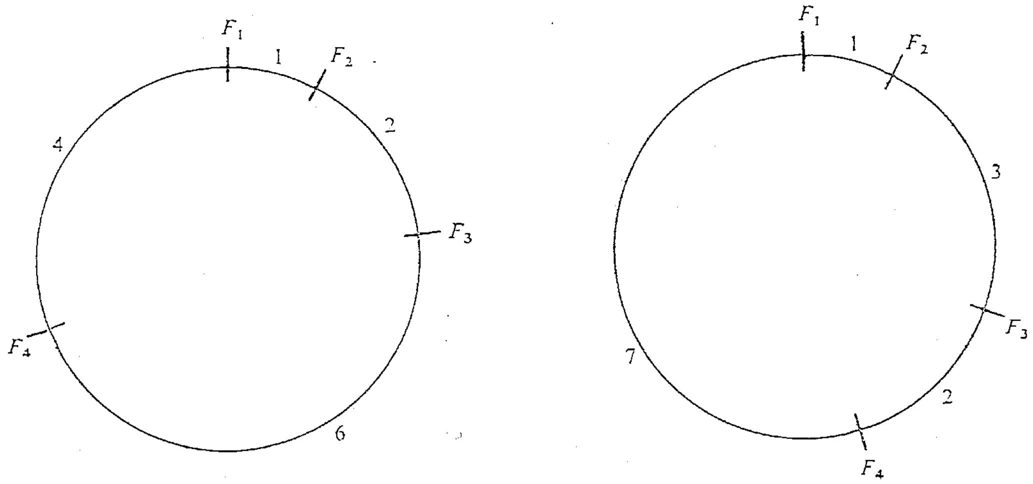

Figure 2. Closed race-tracks with 4 fences.

2. Program for finding solutions

As far as I know there is no analytical solution of the race-track problem, and no theorem exists giving values of N, if any, for which there is no solution. A computer program was therefore developed to find solutions for various values of N. Obviously a program which tests every possible distribution leads to an immense amount of computer work as soon as N becomes at all large. It is therefore necessary to introduce certain refinements.

Let us consider a particular value of N. The fence F_1 can be taken as fixed and then we proceed in a chosen direction, say clockwise, round the track, attempting successively to place $F_2, \ldots, F_N$ in positions that ultimately yield the lengths $1, 2, \ldots, S(= N^2 - N + 1)$. For $m = 1, 2, \ldots, N - 1$, we denote by l_m the length of the segment from F_m to F_{m+1}; and l_N is the length of the segment from F_N to F_1. Since one length between fences has to be 1, we begin by taking $l_1 = 1$. Now suppose that we have obtained a partial solution for $m = p$. By this is meant that we have placed the fences $F_2, \ldots, F_{p+1}$ (i.e. that we have chosen $l_1, \ldots, l_p$) in such a way that no length is produced more than once. We then seek a partial solution for $m = p + 1$. If no partial solution for $p + 1$ exists, then we must go back and seek another partial solution for $m = p$. When we have successfully dealt with the case $m = N$, then we have a solution of the whole problem. On the other hand, if we find ourselves seeking an alternative partial solution for $m = 1$, then we have exhausted all possibilities (since l_1 had been taken to be 1) and there is no solution of the race-track problem.

With the placing of F_2 so that $l_1 = 1$ we have automatically obtained a segment of length $S - 1$ as well as the complete circuit of length S. But for l_2 we need not try out all the remaining lengths $2, 3, \ldots, S - 2 (= N^2 - N - 1)$. For we must have $l_3 \geqslant 2$, then $l_4 \geqslant 3$; and continuing in this way we see that $l_{N-1} \geqslant N - 2$. Therefore the length to be divided between l_2 and l_N is at most

$$(N^2 - N + 1) - (1 + 2 + \cdots + N - 2) = \tfrac{1}{2}(N^2 + N).$$

Now there is no point in taking $l_2 > \frac{1}{4}(N^2 + N)$; for if we do, then $l_N < \frac{1}{4}(N^2 + N)$, so that the anti-clockwise circuit of the race-track (starting with l_1) gives the second segment a length less than $\frac{1}{4}(N^2 + N)$. Thus it is legitimate to assume that $l_2 \leqslant \frac{1}{4}(N^2 + N)$. Since l_2 must be an integer, we therefore have

$$l_2 \leqslant [\tfrac{1}{4}(N^2 + N)];^\dagger$$

and we denote $[\frac{1}{4}(N^2 + N)]$ by $lmax_2$.

Next, suppose that $m \geqslant 3$ and that we have already found a partial solution $(l_1, \ldots, l_{m-1})$. We denote by λ_m the minimum segment length that is still needed, i.e. the least member of the set $\{1, 2, \ldots, S\}$ which is not produced by two of the fences $F_1, \ldots, F_m$. In any final solution of the problem we must have $l_{m+1} \geqslant \lambda_m, \ldots, l_N \geqslant \lambda_m$. Hence

$$l_m \leqslant S - (l_1 + \cdots + l_{m-1}) - (N - m)\lambda_m;$$

and we call the right-hand side $lmax_m$.

A description of the program used is supplied by the flowchart in Figure 3 or by the algorithm below.

$l_1 \leftarrow 1$; $m \leftarrow 2$; $l_2 \leftarrow 2$; $lmax_2 \leftarrow [\frac{1}{4}(N^2 + N)]$;
repeat
  if $m = N$ **then** $\{$*print solution*; $m \leftarrow m - 1$; $l_m \leftarrow l_m + 1\}$ **else**
  if *we can fit segment m* **then**
  $\{$**if** $(m \leftarrow m + 1) \neq N$ **then**
    $\{l_m \leftarrow$ *minimum length needed*;
      if $m \neq 2$ **then** $lmax_m \leftarrow$ *maximum possible*
    $\}$
  $\}$ **else**
  if $(m \leftarrow m - 1) \neq 1$ **then** $l_m \leftarrow l_m + 1$
until $m = 1$.

† If x is any real number, $[x]$ denotes the integral part of x, i.e. the greatest integer $\leqslant x$.

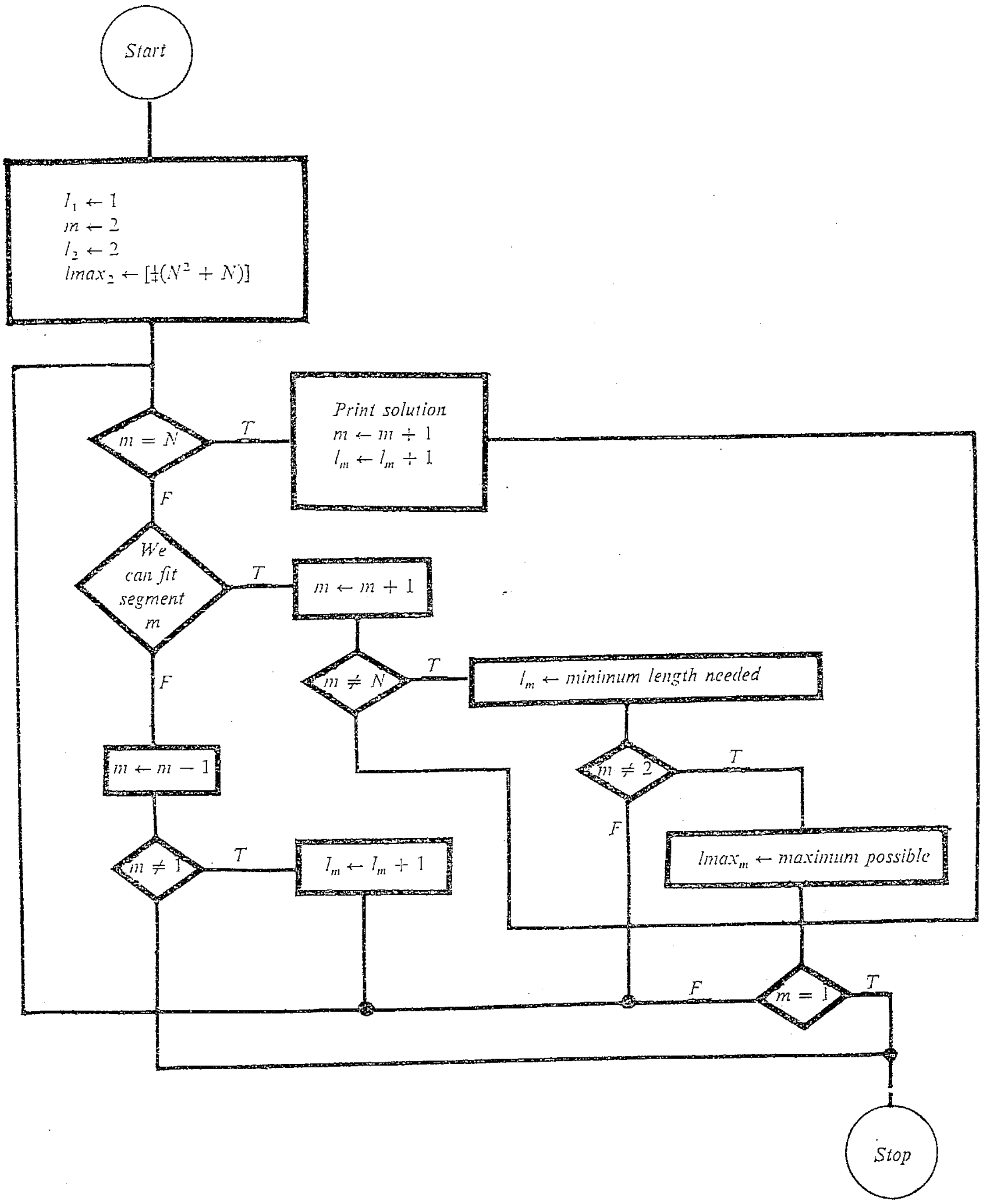

Figure 3. Flowchart of program for producing fence positions.

Here 'we can fit segment m' implies checking that $l_m \leqslant lmax_m$ and that the length l_m currently being considered for the mth segment provides a partial solution compatible with the lengths $l_1, l_2, \ldots, l_{m-1}$ previously given to the first $m - 1$ segments. When the first $m - 1$ segments have been fixed, then, for $1 \leqslant i \leqslant S$, we put

$$\mu_m(i) = \begin{cases} 1 & \text{if the length } i \text{ is obtained from two of } F_1, \ldots, F_m, \\ 0 & \text{if the length } i \text{ is not obtainable from } F_1, \ldots, F_m. \end{cases}$$

Then 'minimum length needed' can be determined from a simple scan of the (continually updated) values of μ_m.

3. The race-track problem and Hadamard difference sets

A Hadamard difference set $D(v, k, \lambda)$ is a set of integers $d_1, d_2, \ldots, d_K$ having the property that all integers $1, 2, \ldots, v - 1$ may be expressed as the difference modulo v between pairs of numbers in the set in exactly λ ways. Thus $(0, 3, 4, 5)$ is a difference set $D(7, 4, 2)$ since

$$\begin{aligned} 1 &\equiv 4 - 3 \equiv 5 - 4 \pmod 7, \\ 2 &\equiv 5 - 3 \equiv 0 - 5 \pmod 7, \\ 3 &\equiv 3 - 0 \equiv 0 - 4 \pmod 7, \\ 4 &\equiv 4 - 0 \equiv 0 - 3 \pmod 7, \\ 5 &\equiv 5 - 0 \equiv 3 - 5 \pmod 7, \\ 6 &\equiv 3 - 4 \equiv 4 - 5 \pmod 7. \end{aligned}$$

The number of differences associated with a difference set $D(v, K, \lambda)$ is $\lambda(v - 1)$. On the other hand, the number of ordered pairs drawn from K numbers is $K(K - 1)$. Hence, if the difference set $D(v, K, \lambda)$ exists, we must have

$$\lambda(v - 1) = K(K - 1);$$

but this condition is not sufficient for the existence of $D(v, K, \lambda)$.

A colleague, Mr W. Roberts, has pointed out to me that solving the race-track problem amounts to finding Hadamard difference sets $D(S, N, 1)$ (with $S - 1 = N(N - 1)$). Consider, for instance, the solution $(1, 2, 10, 19, 4, 7, 9, 5)$ of the race-track problem with $N = 8$ and $S = 57$. If the numbers are successively added and the total sum 57 is replaced by 0, we obtain $(0, 1, 3, 13, 32, 36, 43, 52)$. It is easily checked that this is a difference set $D(57, 8, 1)$; in fact we have

$$\begin{array}{llll} 1 \equiv 1 - 0 & \pmod{57}, & \ldots\ldots\ldots\ldots & \\ 2 \equiv 3 - 1 & \pmod{57}, & 53 \equiv 32 - 36 & \pmod{57}, \\ 3 \equiv 3 - 0 & \pmod{57}, & 54 \equiv 0 - 3 & \pmod{57}, \\ 4 \equiv 36 - 32 & \pmod{57}, & 55 \equiv 1 - 3 & \pmod{57}, \\ \ldots\ldots\ldots\ldots & & 56 \equiv 0 - 1 & \pmod{57}. \end{array}$$

The differences $d_i - d_j \pmod v$ of the terms of a difference set run through all the integers $1, 2, \ldots, v - 1$. Hence multiplication modulo v of each term of a difference set by the same integer m produces a difference set with the same para meters, provided that m is prime to v. It is clear that a similar effect is produced when the same integer is added to all members of a difference set or when one member is replaced by another integer equivalent to it modulo v. For example the solutions of the race-track problem for $N = 6$, $S = 31$ are

$$\begin{array}{ll}
(1, 2, 5, 4, 6, 13), & (1, 13, 6, 4, 5, 2), \\
(1, 2, 7, 4, 12, 5), & (1, 5, 12, 4, 7, 2), \\
(1, 3, 2, 7, 8, 10), & (1, 10, 8, 7, 2, 3), \\
(1, 3, 6, 2, 5, 14), & (1, 14, 5, 2, 6, 3), \\
(1, 7, 3, 2, 4, 14), & (1, 14, 4, 2, 3, 7).
\end{array}$$

In this array the right-hand solutions are the left-hand ones taken in the opposite direction round the track. The corresponding difference sets are

$$\begin{array}{ll}
a = (0, 1, 3, 8, 12, 18), & \bar{a} = (0, 1, 14, 20, 24, 29), \\
b = (0, 1, 3, 10, 14, 26), & \bar{b} = (0, 1, 6, 18, 22, 29), \\
c = (0, 1, 4, 6, 13, 21), & \bar{c} = (0, 1, 11, 19, 26, 28), \\
d = (0, 1, 4, 10, 12, 17), & \bar{d} = (0, 1, 15, 20, 22, 28), \\
e = (0, 1, 8, 11, 13, 17), & \bar{e} = (0, 1, 15, 19, 21, 24).
\end{array}$$

Multiplying the terms of a by 2 (mod 31) we get (0, 2, 6, 16, 24, 5) or, in ascending order, (0, 2, 5, 6, 16, 24). We can also replace 0 by 31 and 2 by 33 to obtain (in ascending order) (5, 6, 16, 24, 31, 33); and if we subtract 5 from each term we finally have $(0, 1, 11, 19, 26, 28) = \bar{c}$. In a similar way, starting with multiplication by 2 (mod 31) we can turn $\bar{c}$ into d, d into $\bar{b}$, $\bar{b}$ into $\bar{e}$, and $\bar{e}$ back into a.

Certain other multiplying factors lead to sequences of the same type. If, however, the terms of any of the above difference sets are multiplied by 5 (mod 31), the same difference set results. The number 5 is said to be a *multiplier* of the set $D(31, 6, 1)$ and it is noteworthy† that, whenever a difference set $D(v, K, \lambda)$ has been found to have a multiplier, the multiplier has always turned out to be a prime divisor of $K - \lambda$. It is not yet known under what circumstances a difference set possesses a multiplier; nor has it actually been proved that a multiplier is necessarily a prime divisor of $K - \lambda$.

† See G. F. A. Hoffmann de Visme, *Binary Sequences* (E.U.P., 1971), p. 76.

Dear Editor,

Heronian triangles

The problem raised by K. R. S. Sastry (reference 1) has been considered by many mathematicians—see L. E. Dickson (reference 2).

Rational triangles can be easily specified by one of two methods.

(i) Since each altitude must be rational, a scalene (or isosceles) triangle may be obtained by putting together two Pythagorean triangles with one side adjacent to the right angle in common. The other sides adjacent to the right angle can either be added or subtracted.

(ii) If the sides and the area are rational, so must be the sine and cosine of each angle and hence the tangent of the half angle. Choosing for two of these simple fractions in their lowest terms, p/q and r/s, where $qs \neq pr$, we obtain the third half-angle tangent as $(qs - pr)/(ps + qr)$. This gives the ratios of the sides as:

$$pq(r^2 + s^2):rs(p^2 + q^2):(qs - pr)(ps + qr)$$

with semiperimeter $qs(ps + qr)$ and area $pqrs(qs - pr)(ps + qr)$. (See reference 3.)

There seems some suggestion in Dickson that this formula gives all the triangles with integral sides, but, in fact, common factors arise if p or q have a factor in common with r or s and also they may arise through common factors of $r^2 + s^2$ and $p^2 + q^2$ which may also divide into $qs - pr$ and $ps + qr$, e.g. when $p, q, r, s = 1, 2, 3, 4$.

One such triangle has sides 13, 14, 15. None of these numbers can be expressed either as $2(p^2 + q^2)(r^2 + s^2)$ or as $4(p^2 + q^2)(r^2 + s^2)$.

It would seem clear that the formulae given at the end of Sastry's article are not able to produce all Heronian triangles.

References

1. K. R. S. Sastry, Heronian triangles, *Math. Spectrum* 8 (1975/76), 77–80.
2. L. E. Dickson, *History of the Theory of Numbers* (Carnegie Institute, Washington D.C., 1919–23), Volume 2, p. 191.
3. H. Schubert, *Die Ganzzahligkeit in der algebraischen Geometrie* (Leipzig, 1905).

Yours sincerely,

DONOVAN TAGG
(Department of Mathematics, University of Lancaster)

Dear Editor,

A formula for Heronian triangles

Mr Sastry, in his recent article (Vol. 8 No. 3, pp. 77–80), mentioned that he knew of no set of expressions that yields all Heronian triangles (i.e. those whose sides and area are all integers). However it is not difficult to obtain the following pair of results which constitutes a partial solution of the problem.

(i) If x, y, z are positive integers, the triangle with sides

$$(x + y)|xy - z^2|, \quad x(y^2 + z^2), \quad y(z^2 + x^2), \tag{*}$$

is Heronian.

(ii) Every Heronian triangle is similar to one with sides given by (*).

To prove (i) denote the numbers in (*) by a, b, c respectively and put $s = \frac{1}{2}(a + b + c)$. It is easily checked that, when $xy > z^2$, $s = xy(x + y)$, $s - a = (x + y)z^2$, $s - b = x(xy - z^2)$, $s - c = y(xy - z^2)$; and, when $xy < z^2$,

$$s = (x + y)z^2, \quad s - a = xy(x + y), \quad s - b = y(z^2 - xy), \quad s - c = x(z^2 - xy).$$

Hence, in either case, the area $\{s(s - a)(s - b)(s - c)\}^{\frac{1}{2}}$ of the triangle with sides a, b, c is

$$xyz(x + y)|xy - z^2|.$$

Thus, when x, y, z are positive integers, the sides of the triangle and the area all have integral values.

To prove (ii), suppose that the triangle with sides a, b, c and with area z is Heronian. Put $s = \frac{1}{2}(a + b + c)$ and

$$l = s - a, \quad m = s - b, \quad n = s - c,$$

so that

$$z = \{lmn(l + m + n)\}^{\frac{1}{2}}$$

is an integer. Moreover s must be an integer; for if s is half an odd integer, then l, m, n and $l + m + n$ are also half odd integers and z cannot be an integer. It follows that l, m, n are integers.

We have

$$lmn = z^2/s = (z/s)^2(l + m + n).$$

This gives

$$n = \frac{(z/s)^2(l + m)}{lm - (z/s)^2},$$

so that the sides are

$$l + m, \quad \frac{z^2(l + m)}{s^2lm - z^2} + m, \quad \frac{z^2(l + m)}{s^2lm - z^2} + l,$$

which are in the ratio

$$(l + m)(s^2lm - z^2) : l(z^2 + s^2m^2) : m(s^2l^2 + z^2).$$

Multiplying these numbers by s and putting $x = sl$, $y = sm$ we finally see that a, b, c are in the ratio

$$(x + y)(xy - z^2) : x(y^2 + z^2) : y(z^2 + x^2);$$

and these numbers are of the required form (with $xy > z^2$).

I am not altogether happy with these expressions because I feel one ought to be able to find a symmetrical set. The corresponding Pythagorean result is, of course, *ipso facto* unsymmetrical.

Yours sincerely,

A. R. PARGETER
(Blundell's School, Tiverton, Devon)

Dear Editor,

Derived sequences

In his interesting letter (Volume 8, No. 3, pp. 86–88) M. I. Wenble considered various sequences (a_n) and the corresponding derived sequences (c_n) and (d'_n) which were called the α-derived and β-derived sequences, respectively. For simplicity I shall consider only the α-derived sequence (c_n). This was defined as

$$c_n = b_{n+1} - b_n, \quad \text{where} \quad b_n = a_{n+1}/a_n.$$

Hence the α-derived sequence of $(|a_n|)$ is $(A_n) = (|b_{n+1}| - |b_n|)$. In all this one must, of course, assume that $a_n \neq 0$. Using numerical evidence Mr Wenble conjectured that the α-derived sequence of (a_n) has limit l if and only if the α-derived sequence of $(|a_n|)$ has limit $|l|$. The conjecture in this form is false, but it is natural to separate the 'if' from the 'only if'.

Consequently I invite your readers to prove the following results, the first being true for any complex sequence (b_n), not necessarily of the form a_{n+1}/a_n with $a_n \neq 0$.†

(1) If $b_{n+1} - b_n \to l$ as $n \to \infty$, then (i) $b_n/n \to l$ and (ii) $|b_{n+1}| - |b_n| \to |l|$.
(2) There is a sequence (a_n) such that $A_n \to 0$, but (c_n) diverges.

It is, of course, (2) which refutes the 'only if' part of Mr Wenble's conjecture.

In connection with the original example of $a_n = n^n$ it is worth noting that, since $b_{n+1} - b_n \to e$ as $n \to \infty$, the result (1) (i) shows that $b_n/n \to e$. Moreover, if $d_n = c_{n+1} - c_n$, then $nd_n \to 0$ as $n \to \infty$. However, to prove this is rather more difficult. The crucial step is to note that, if f is twice differentiable, then

$$f(n+2) - 2f(n+1) + f(n) = \tfrac{1}{2}\{f''(p_n) + f''(q_n)\}$$

for some p_n in the interval $(n, n+1)$ and some q_n in the interval $(n+1, n+2)$; this follows from the second mean value theorem which is likely to figure in a first-year university analysis course. It is then only necessary to show that $xf''(x) \to 0$ as $x \to \infty$ when $f(x) = x^x/(x-1)^{x-1}$; and here the method of proof is that used by Mr Wenble in the Appendix of his letter.

Yours sincerely,

I. J. Maddox
(Department of Pure Mathematics,
The Queen's University of Belfast)

† See Problem 9.6 on p. 64.

Dear Editor,

Three-group methods for linear curve fitting

The method of fitting a straight line described in a recent letter by P. J. Bishop (reference 1) is related to a resistant technique of exploratory data analysis, and your readers may be interested in some discussion and historical background.

The basic technique, as given by Bishop, is to put the (x, y) points in order according to their x-coordinates, divide them into two groups of equal size so that those in the lower set have smaller x-coordinates than those in the upper set, and then obtain the line by joining the centroids of the two groups. It turns out that this method was proposed in 1940 by Abraham Wald (reference 2) for situations in which both x and y are subject to error. In 1942 Nair and Shrivastava presented (for the case in which only y is subject to error) their 'method of group averages' (reference 3) which forms three groups of equal size and uses the centroids of the outer groups to determine the line. (It may seem wasteful to discard the information in the middle group, but the result is, in fact, a more accurate estimate of the slope.) They also generalized this technique to the fitting of polynomials of higher degree. As an improvement on Wald's method M. S. Bartlett (reference 4) later independently rediscovered this three-group method for fitting straight lines (again for the case of both x and y subject to error).

Recently J. W. Tukey suggested (reference 5), as an exploratory technique, using three equal-sized groups, calculating the (x, y) summary point (median x, median y) within each group, and basing the fitted slope on the outer summary points and the fitted intercept on all three summary points. The data of Bishop's second example provide a convenient illustration:

Data

low group:	(8, 1380), (9, 1292), (10, 1174);
middle group:	(11, 1113), (12, 1066), (13, 978), (14, 904);
high group:	(15, 820), (16, 752), (17, 661).

Summary points

low:	(9, 1292);
middle:	(12·5, 1022);
high:	(16, 752).

From the low and high summary points we easily find the slope to be $(752 - 1292)/(16 - 9) = -77{\cdot}14$. Removing this fitted slope from the three summary points and averaging the resulting y-coordinates we obtain a fitted intercept of

$$(1986{\cdot}3 + 1986{\cdot}3 + 1986{\cdot}3)/3 = 1986{\cdot}3.$$

Two general properties of this technique are now apparent. First, the use of group medians instead of group means provides protection against straying or wild values (as long as they are relatively few—at most one per group in the 10-point example). This underlies the notion of *resistance*: a technique or fit is resistant if perturbing (even drastically) a small fraction of the data produces only a small change in the result. Second, part of the 'price' of resistance in this technique is that it may be necessary to iterate. That is, the residuals from the initial fitted line replace the original y-coordinates, and the iteration step (more than one is rarely necessary) gives corrections to the fitted slope and intercept. In the example these corrections are $-1{\cdot}31$ and $12{\cdot}01$, respectively.

Even though its calculation is quite simple, the resistant line is not intended as a direct replacement for the usual least-squares line. Instead it provides a valuable alternative and might often be preferred in the early stages of analysing a set of data, when one may not yet know whether the least-squares procedure is justified.

References

1. P. J. Bishop, A simple method of linear curve fitting, *Math. Spectrum* **8** (1975/76), 62.
2. A. Wald, The fitting of straight lines if both variables are subject to error, *Ann. Math. Statist.* **11** (1940), 284–300.
3. K. R. Nair and M. P. Shrivastava, On a simple method of curve fitting, *Sankhyā* A **6** (1942), 121–132.

4. M. S. Bartlett, Fitting a straight line when both variables are subject to error. *Biometrics* 5 (1949), 207–212.
5. J. W. Tukey, *Exploratory Data Analysis* (limited preliminary edition) (Addison-Wesley Publishing Company, Reading, Mass., 1970).

Yours sincerely,

DAVID C. HOAGLIN
(Harvard University, Cambridge, Mass., U.S.A.)

Dear Editor,

Comments on linear curve fitting

The method of fitting a straight line to a set of points by dividing the points into two subsets of equal size and joining the centroids of the two subsets to produce the required line, as advocated recently in your pages by Bishop (reference 1), is identical with that of Wald (reference 2), who also provides a method for computing confidence limits for the estimated slope. Wald's method was intended to be used for obtaining an estimate of the best linear *functional* relation between two variables, i.e. a regression between variables whose true values are unobservable because they are masked by errors.

Improvements to Wald's method have been suggested by Nair and colleagues (reference 3) and Bartlett (reference 4). Both these latter suggestions involve dividing the original set of n points in three groups, the two extreme groups being chosen to be equal and as near $n/3$ in size as possible. Nair and colleagues simply joined the centroids of the two extreme groups to produce the required line, whereas Bartlett used this method to obtain the slope, but positioned the line so that it passed through the centroid of the full set of n points.

Ricker (reference 5), in his comprehensive review of linear regression procedures in which one or both of the variables are subject to errors of measurement and/or inherent variability, shows that none of the suggested methods provides an unbiased estimate of the functional regression, and advocates use of the geometric mean regression (*la relation d'allométrie*) of Teissier (reference 6) for this purpose. Ricker shows by example that the slope obtained by using either the Wald method or the Nair–Bartlett method is very close to the slope of the *predictive* regression of Y on X, i.e. the usual least squares regression when X is assumed to be free of error. The examples given by Bishop also show close agreement between the Wald method and least squares.

It is perhaps ironic that methods proposed to solve the problem of estimating the functional regression, while failing to provide a satisfactory solution for that problem, provide a simple alternative to least squares for estimating the predictive regression.

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1. P. J. Bishop, A simple method of linear curve fitting, *Math. Spectrum* 8 (1975/76), 62.
2. A. Wald, The fitting of straight lines if both variables are subject to error, *Ann. Math. Statist.* **11** (1940), 284–300.
3. K. R. Nair and M. P. Shrivastava, On a simple method of curve fitting, *Sankhyā* **6** (1942), 121–132; K. R. Nair and K. S. Banerjee, A note on fitting of straight lines when both variables are subject to error, *Sankhyā* **6** (1942), 331.
4. M. S. Bartlett, Fitting a straight line when both variables are subject to error, *Biometrics* 5 (1949), 207–212.

5. W. E. Ricker, Linear regressions in fishery research, *J. Fisheries Res. Board Can.* **30** (1973), 409–434.
6. G. Teissier, La relation d'allométrie: sa significance statistique et biologique, *Biometrics* 4 (1948), 14–48.

Yours sincerely,

D. A. Ratkowsky
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Hobart, Tasmania)

Dear Editor,

Motion of a bubble

'In a non-viscous liquid, what is the motion of a small bubble of gas (of constant size) rising from the bottom.'

In Volume 7, Number 3 (p. 101), Major W. H. Carter posed the above problem.

For an isolated small bubble (less than 0·05 cm) the shape will remain spherical and its velocity is

$$U = \frac{1}{9}\frac{\rho g a^2}{\mu}$$

where U = velocity, ρ = density of liquid, g = gravitational acceleration, a = radius of sphere, μ = viscosity of liquid and we have neglected the air density.

For larger bubbles (the ones we can easily observe) the bubbles become elliptical and unstable causing oscillations in its shape and consequently a spiral motion occurs during their vertical ascent.

For a mathematical discussion of this topic, the reader is referred to pp. 367–370 of G. K. Batchelor's book, *An Introduction to Fluid Dynamics* (Cambridge University Press).

Yours sincerely,

J. R. Blake
(C.S.I.R.O. Division of Mathematics and Statistics,
Canberra, Australia)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

Problems

9.4. (Submitted by C. J. Knight, University of Sheffield.) What is the expression in base 7 of the square root of the number whose expression in base 7 is 14,641?

9.5. (Submitted by B. G. Eke, University of Sheffield.) The triangle T_1 lies inside the triangle T_2. Show that the perimeter of T_1 is shorter than that of T_2.

9.6. (Submitted by I. J. Maddox, Queen's University of Belfast.) (i) Let (b_n) be a sequence of complex numbers such that $b_{n+1} - b_n \to l$ as $n \to \infty$. Show that $b_n/n \to l$ and that $|b_{n+1}| - |b_n| \to |l|$ as $n \to \infty$.

(ii) Let (a_n) be a sequence of non-zero real numbers, and put $b_n = a_{n+1}/a_n$ for $n = 1, 2, 3, \ldots$. Put $c_n = b_{n+1} - b_n$, $c'_n = |b_{n+1}| - |b_n|$. Show that it is possible for c'_n to tend to zero as $n \to \infty$ but for the sequence (c_n) to diverge. (Part (ii) disproves a conjecture posed by M. I. Wenble in his letter in Vol. 8 No. 3. See the letter by I. J. Maddox in this issue.)

Solutions to Problems in Volume 8, Number 3

8.7. If m, n are odd integers, show that $m^2 - n^2$ is divisible by 8.

Solution by Roger Hale (Hymers College, Hull)
Put $m = 2r - 1$, $n = 2s - 1$, where r, s are integers. Then

$$\begin{aligned} m^2 - n^2 &= (m+n)(m-n) \\ &= (2r + 2s - 2)(2r - 2s) \\ &= 4(r + s - 1)(r - s). \end{aligned}$$

If r, s are both even or both odd, then $r - s$ is even and $m^2 - n^2$ is divisible by 8. If one of r, s is odd and the other even, then $r + s - 1$ is even and again $m^2 - n^2$ is divisible by 8. Thus $m^2 - n^2$ is always divisible by 8.

Also solved by Stuart Southall (King Edward VI School, Birmingham), Giles Orton (King Edward VII School, Sheffield), Alan Greenwood (King Edward VII School, Sheffield), Stephen Whiteside (University of Sheffield), M. W. Friend (Hymers College, Hull), John Hampton (The Open University), Lim Tiong (University of Singapore), M. J. Chapman (University of Bristol).

8.8. Is it possible for three consecutive binomial coefficients to be (a) in arithmetic progression, (b) in geometric progression?

Solution by Lim Tiong
(a) This is possible; an example is

$$\binom{7}{1}, \binom{7}{2}, \binom{7}{3}.$$

(b) This is impossible. For suppose that

$$\binom{n}{r-1}, \binom{n}{r}, \binom{n}{r+1}$$

are in geometric progression. Then

$$\binom{n}{r-1}\binom{n}{r+1} = \binom{n}{r}^2,$$

which gives $n = -1$.
Also solved by Stuart Southall, Roger Hale.

8.9. The polynomial f has complex coefficients, and all its roots have positive real parts. Show that all the roots of the derivative of f have positive real parts.

Solution
Put

$$f = (x - z_1)(x - z_2) \ldots (x - z_n),$$

so that $\operatorname{Re}(z_1), \ldots, \operatorname{Re}(z_n) > 0$. Then

$$f' = \sum_{k=1}^{n} (x - z_1) \ldots (x - z_{k-1})(x - z_{k+1}) \ldots (x - z_n).$$

Suppose that f' has a root z such that $\operatorname{Re}(z) \leqslant 0$. Then $z \neq z_1, \ldots, z_n$, and

$$0 = \frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \frac{1}{z - z_k} = \sum_{k=1}^{n} \frac{\bar{z} - \bar{z}_k}{|z - z_k|^2}.$$

Put $z = x + iy$, $z_k = x_k + iy_k$ $(1 \leqslant k \leqslant n)$ in real and imaginary parts. Then

$$\operatorname{Re}\left\{\sum_{k=1}^{n} \frac{\bar{z} - \bar{z}_k}{|z - z_k|^2}\right\} = \sum_{k=1}^{n} \frac{x - x_k}{|z - z_k|^2} < 0,$$

and this is impossible.

Introductory Linear Algebra with Applications. By BERNARD KOLMAN. Collier Macmillan, London, 1976. Pp. 426. £8·00.

This book is aimed at first- or second-year undergraduates in an American university, and is designed for those not specialising in mathematics. The exposition is clear, and proceeds at a leisurely pace with numerous examples. Some proofs are omitted; for example, the proof that the eigenvalues of a symmetric matrix are real.

Matrices and determinants are introduced via sets of simultaneous linear equations; vector spaces and linear transformations are then developed, and the connection made clear. The linear algebra proper is concluded with a discussion of eigenvectors and eigenvalues, stopping short of the Jordan normal form.

Linear programming is introduced from a geometric point of view, and an account is given of the simplex method. Applications of the preceding material are then discussed in some detail, including quadratic forms, graph theory and the theory of games. The final chapter is on numerical linear algebra, emphasizing the role of the computer.

There are many practical (i.e. numerical) exercises, and some theoretical ones. The specialist mathematician would probably find the pace rather pedestrian, and the occasional lack of proof frustrating, but the book seems very good for the type of student for whom it was written.

College of St. Hild and St. Bede, Durham C. KEARTON

A First Course in Abstract Algebra (second edition). By JOHN B. FRALEIGH. Addison-Wesley Publishing Company, Inc., London, 1976. Pp. xviii+455. £10·50.

Those who know the first edition of this book will welcome this second edition. The only substantial change is the inclusion of an early section on equivalence relations and the consequent modification of text and examples.

For those who do not know the book, it is an extremely readable introduction to groups and fields which develops the theory required to handle the insolvability of the quintic. It is a splendid book for first-year students to read and to learn from, for the author appears to be very sympathetic to the needs and the difficulties encountered by students. In addition the pace is gentle and there are many worked examples and well graded exercises as well as hints and answers.

The book is most thoroughly recommended.

University of Durham H. NEILL

Elementary Differential Equations with Linear Algebra. By ROSS L. FINNEY and DONALD R. OSTBERG. Addison-Wesley Publishing Company, Inc., London, 1976. Pp. xii + 477. £10·50.

This book is an extensively revised and modernised version of *Elementary Differential Equations* by Donald L. Kreider, Robert G. Kuller and Donald R. Ostberg, first published eight years ago. It is in many ways an excellent book. The arguments are laid out with great care and attention to detail. Examples taken from many fields are used to illustrate the material throughout the book and each section has a long list of exercises. These are carefully graded, beginning with purely mathematical problems and moving on to those which arise in practical applications. The authors clearly have a deep understanding of the subject and make a commendable attempt to convey their appreciation of its beauties and subtleties.

In general there is a uniformity of background knowledge and the student will learn many new things in addition to the solution of differential equations. It is a pity that more effort was not given to providing an easier connection with the students' first course in calculus. A student who can differentiate and integrate might not really appreciate what is involved when he is told that to solve $(d^2y/dt^2) = -g$ he has to integrate the equation to obtain $(dy/dt) = -gt + c_1$.

A more fundamental problem with books of this kind is that it is not easy to see how they will fit in to the average undergraduate training in mathematics. Most students, for example, will meet a complete treatment of linear algebra independently from courses in applied calculus and differential equations. The treatment of linear algebra in this book will certainly be inadequate for such courses. Nevertheless, as a textbook for courses on elementary differential equations and as a book to be available for mathematics undergraduates, this volume is well worth its price.

University of Durham E. J. SQUIRES

Basic Topology. By DAN E. CHRISTIE. Collier Macmillan, London, 1976. Pp. 256. £9·00.

This book is aimed at students who are not necessarily specialising in mathematics, and adopts the teaching practice of R. L. Moore, whereby the reader is intended to supply proofs for many of the theorems.

After a somewhat lengthy treatment of elementary set theory from an axiomatic viewpoint, interior operators and continuity are discussed, again axiomatically. It is not until page 98 that the usual axioms for a topology, in terms of open sets, appear. There follows a treatment of closed sets, subspaces, connectedness, compactness, product and quotient spaces, separation axioms, and metric spaces. Finally, the Brouwer Fixed-point Theorem is proved for a closed triangle, and metrizability using Urysohn's lemma is discussed.

On the whole, the exposition is clear. The relationships between the axioms for the interior operator, open sets, closed sets, and closure operator are discussed fully. The book is divided into forty-seven sections, each of which ends with a set of exercises.

College of St. Hild and St. Bede, Durham C. KEARTON

Fundamentals of Topology. By BENJAMIN T. SIMS. Collier Macmillan, London, 1976. Pp. x + 179. £8·50.

The main part of the book is an introduction to general topology. The usual material is included: bases, countability, separation axioms, compactness, connectivity, metrizability and completeness. The emphasis seems at times a little surprising: for example, quotient spaces are in a section starred as optional, whereas proximity spaces are not. Otherwise, the treatment of the subject is fairly standard.

The final two chapters introduce homotopy and singular homology theory. It was disappointing here to be referred to other books for the proofs of some results; in particular, for that of the excision theorem. This part of the book could with advantage be expanded, especially in view of the slimness of the volume relative to its price.

The author adopts the teaching style of R. L. Moore, leaving the proofs of some theorems as exercises for the reader. The content and treatment are suited to an undergraduate in his second or third year.

College of St. Hild and St. Bede, Durham C. KEARTON

Notes on Contributors

Colin Goldsmith is a graduate of Cambridge. After some years as an Instructor Officer in the Royal Navy he went to teach at Marlborough College, where he has remained ever since. He became head of the Mathematics Department, and is now a housemaster. Part author and editor of many SMP books, he is also vice-chairman of the National Committee for Mathematical Contests. He was deputy leader of the British Team at the 1976 International Mathematical Olympiad.

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Malcolm Clark is a Lecturer in Mathematics at Monash University in Victoria, Australia. He graduated at the University of Melbourne and has also held posts in the Operations Research Section of Imperial Chemical Industries Ltd, Melbourne, and at the University of Sheffield. His current research interest is applied statistics, with emphasis on applications in archaeology.

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Peter Tan is an Associate Professor of Mathematics at Carleton University, Ottawa. He obtained his first degree in civil engineering at Sun Yat Sen University in China and his Ph.D. in mathematical statistics at the University of Toronto. His main teaching and research interests are in probability, mathematical statistics and the history of mathematics.

G. F. A. Hoffmann de Visme is a graduate in physics of the University of London. His varied career has included not only a university lectureship, but also a period in industry and the post of Education Officer in the Royal Air Force. Since 1971 he has been head of the Department of Electrical and Electronic Engineering at the North Staffordshire Polytechnic. His interest in computing largely dates from 1970, when he attended some undergraduate lectures on the subject.

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