5-th Mediterranean Mathematical Competition 2002

1. Determine all positive integers x, y such that $y \mid x^2 + 1$ and $x^2 \mid y^3 + 1$.

2. Suppose x, y, a are real numbers such that $x + y = x^3 + y^3 = x^5 + y^5 = a$. Find all possible values of a.

3. In an acute-angled triangle ABC, M and N are points on the sides AC and BC respectively, and K the midpoint of MN. The circumcircles of triangles ACN and BCM meet again at a point D. Prove that the line CD contains the circumcenter O of $\triangle ABC$ if and only if K is on the perpendicular bisector of AB.

4. If a, b, c are nonnegative real numbers with $a^2 + b^2 + c^2 = 1$, prove that

$$\frac{a}{b^2 + 1} + \frac{b}{c^2 + 1} + \frac{c}{a^2 + 1} \geq \frac{3}{4} \left(a\sqrt{a} + b\sqrt{b} + c\sqrt{c} \right)^2.$$