Crux

Published by the Canadian Mathematical Society.

http://crux.math.ca/

The Back Files

The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

➢ The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.

➢ Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.

➢ Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.

➢ Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.
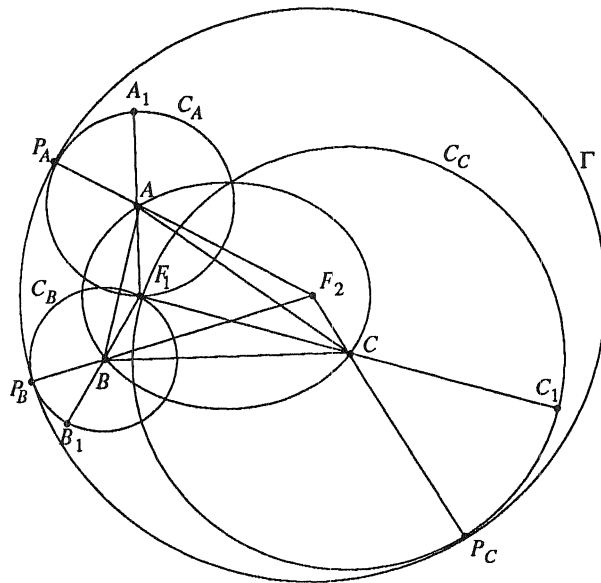
Mathematicorum

A GENERALIZATION OF
EULER'S $R \geq 2r$

Federico Ardila M.

Our purpose here is to extend Euler's classical triangle inequality $R \geq 2r$, replacing the circumcircle of the triangle with an arbitrary ellipse through its vertices. We prove the following:

THEOREM. *Let $\triangle ABC$ be a triangle with inradius r inscribed in an ellipse with constant k (i.e, an ellipse which is the locus of the points such that the sum of its distances to the foci of the ellipse is k). Then $k \geq 4r$.*

This is an extension of Euler's inequality, because the circumcircle of $\triangle ABC$ is an ellipse through its three vertices, both foci coincide with the circumcenter of the triangle, and its constant is $2R$, so from the theorem it will follow that $2R \geq 4r$, or $R \geq 2r$.

Proof. Let F_1 and F_2 be the foci of the ellipse. Let C_A be the circle with center A that passes through F_1. Define analogously C_B and C_C. Extend F_2A to meet C_A beyond A in P_A (it is clear that this point P_A always exists). Define analogously P_B and P_C. Let Γ be the circle with center F_2 and radius k. We have that

$$F_2P_A = F_2A + AP_A = F_2A + AF_1 = k$$

so P_A lies on Γ; and as F_2, A and P_A are collinear, Γ and C_A are tangent in P_A, and analogously for B and C. Extend F_1A to meet C_A again in A_1. Define analogously B_1 and C_1.

Notice that $A_1B_1C_1$ is a triangle, F_1 is a point in its plane, C_A, C_B and C_C are the circles with diameters F_1A_1, F_1B_1 and F_1C_1 and Γ is the circle containing and internally tangent to these three circles. Then using the corrected result of *Crux* 1824 [1994: 55], we obtain that the radius k of Γ is not less than twice the inradius r_1 of $A_1B_1C_1$, i.e., that

$$k \geq 2r_1.$$

But notice that $\triangle A_1B_1C_1$ is a homothety of $\triangle ABC$ with center F_1 and ratio 2, so $r_1 = 2r$, and we obtain

$$k \geq 4r$$

as we wished to prove. $\square$

Applying the condition for equality in *Crux* 1824 mentioned in [1994: 55], we see that equality holds when $\triangle A_1B_1C_1$ is equilateral and F_1 is its center. As $\triangle A_1B_1C_1$ is a homothety of $\triangle ABC$ with center F_1, this occurs when $\triangle ABC$ is equilateral and F_1 is its center. But $AF_1 = BF_1 = CF_1$ and $AF_1 + AF_2 = BF_1 + BF_2 = CF_1 + CF_2$ imply $AF_2 = BF_2 = CF_2$, so F_2 must also be the (circum)center of $\triangle ABC$, so F_1 and F_2 must be coincident, and the ellipse must be a circle. Therefore equality holds when $\triangle ABC$ is equilateral and the ellipse is its circumcircle.

Department of Mathematics (undergraduate student)
Massachusetts Institute of Technology
Cambridge, MA 02139   USA
email: fardila@mit.edu

*   *   *   *   *

THE Nth POWER OF A LION

J. B. Wilker

A friend of mine[1] put together a wonderful computer program to help bring to life one aspect of a complex variables course I taught recently. Believe me, you have never really understood analytic functions as conformal mappings until you have flown a mouse around the z-plane leaving a coloured trail and seen the same-coloured image trail appear simultaneously in the w-plane, like the trail of a mysteriously moving phantom. Mysterious — but not too mysterious! Click to change colours and cross your old trail — sure enough, the phantom crosses his trail too, and at the same angle! Write your

---

[1]M. Horrigan is in charge of the mathematics computer lab at the Scarborough Campus, University of Toronto. In addition to keeping the lab running and encouraging the students with their tasks, she somehow finds time to help the faculty with extraordinary programming requests.
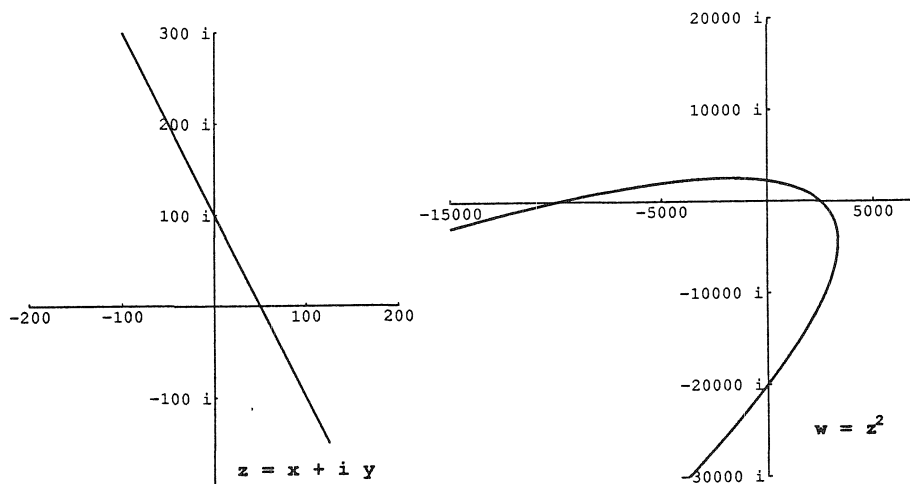
name, and the phantom writes it too, but with artistic distortion. Now you can compute various versions of the square of your signature, or the sine of it — whatever.

Well, you get the idea. It is fun and it is instructive. I want to tell what it taught me, even though I am a little embarrassed not to have realized this thing before I actually saw it on the screen. As soon as I did see it though, I thought of a neat explanation. And the nice thing about the explanation is that it unites aspects of elementary complex variables that do not go together every day, but fit as naturally as hand in glove.

So here we are, looking at the mapping

$$z \to w = z^2.$$

Almost any complex variables text invites its readers to determine the image in the w-plane of the line Re $z = 1$. Some texts vary this a little and ask for the image of Re $z = a$, or Im $z = b$. None that I know asks simply for the image of a line (not through the origin). But this is the first thing you will try with the complexwindows program. And draw that line where you will — the phantom swoops around on an image parabola. (Or is completely off the screen — best to come clean and admit that there is always one more problem of scale.) Back to present business: why the ubiquitous parabola?

The key to the explanation is the transformation $z \to az$, $a \neq 0$. Being the product of a rotation about the origin through an angle arg a and a dilatation centred at the origin with scale factor $|a|$, it is a direct similarity fixing the origin. It maps arbitrary figures to similar figures and, for a suitable choice of a, can map any point different from the origin to any other

such point. In particular it maps lines to lines and, for a suitable choice of a, can map any line not through the origin to the line Re $z = 1$. (The correct choice of a is the reciprocal of the complex number naming the foot of the perpendicular from the origin to the line.)

Now consider the chain of mappings:

$$z \to z' = az$$
$$z' \to w' = (z')^2$$
$$w' \to w = a^{-2}w'.$$

The first is a similarity of the z-plane taking the line in question to the line Re $z = 1$. The second is the square mapping carrying the line Re $z = 1$ to the standard parabola in the w-plane. (I assume we have done the standard exercise.) The third is a similarity of the w-plane taking the standard parabola to another one. The net effect is

$$w = a^{-2}(az)^2 = z^2$$

so squaring must map arbitrary lines not through the origin to parabolas.

THEOREM. *For any integer n the mapping $z \to w = z^n$ carries arbitrary lines not through the origin in the z-plane to mutually similar curves in the w-plane.*

Proof. For any $a \neq 0$ we have

$$w = a^{-n}(az)^n = z^n.$$

Thus the image of any line not through the origin is similar to the image of Re $z = 1$. □

While we are talking about lines, we should remark that the mapping $z \to w = z^n$, $n \neq 0$, carries lines that do pass through the origin in the z-plane either to lines through the origin in the w-plane (n odd) or to doubly covered rays touching the origin (n even).

What about the nth power image of a more general figure such as a signature, or a disk, or (to come to the title) a sketch of a lion? It is too much to hope that arbitrary figures from one of these classes will have similar images. After all, this was not true even for lines. But what can be said is that any two figures that are related by a special similarity $z \to az$ of the z-plane will have nth power images that are similar in the w-plane. This simplified, at least a little bit, such amusing questions as determining the shapes that can arise as squares of circles.

Physical Sciences Division
Scarborough Campus, University of Toronto
Scarborough, Ontario M1C 1A4
Canada

*   *   *   *   *

THE SKOLIAD CORNER
No. 1
R. E. WOODROW

With this number of *Crux* we inaugurate a new section. Over the past years from time to time we have given "Pre-Olympiad" problem sets. It seems a good idea to recognize the interests of those readers who are beginning to try their hand at problem solving or who are looking for materials to use with mathematics clubs and in the classroom. We will be giving contest questions, sometimes in a short answer or multiple choice format, when we will give the answers only the next issue. For other sets we welcome solutions **from pre-university students**. The Skoliad Corner will evolve to meet the response of our readers. I welcome your input, and especially problem sets and solutions for use. To start things off we give the problems of Part I of the Alberta High School Prize Examination for 1994–1995.

ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION
Part I
November 15, 1994

1. You have 5 sticks, of lengths 10, 20, 30, 40 and 50 cm. The number of non-congruent triangles that can be formed by choosing three of the sticks to make the sides is

(a) 3   (b) 5   (c) 8   (d) 9   (e) 10.

2. A glass box 7 cm by 12 cm by 18 cm, closed on all six sides, is partly filled with coloured water. When the box is placed on one of its 7 by 12 sides, the water level is 15 cm above the table. When the box is placed on one of its 7 by 18 sides the water level above the table, in centimetres, will be

(a) 7.5   (b) 9   (c) 10   (d) 12.5   (e) none of these.

3. There are four cottages on a straight road. The distance between Lill's and Ted's cottages is 3 kilometres. Both Jack's and Jill's cottages are twice as far from Lill's as from Ted's. In kilometres, the distance between Jack's and Jill's cottages is

(a) 1   (b) 2   (c) 3   (d) 4   (e) 6.

4. Some playing cards from an ordinary deck are arranged in a row. To the right of some King is at least one Queen. To the left of some Queen is at least one other Queen. To the left of some Heart is at least one Spade. To the right of some Spade is at least one other Spade. The minimum number of cards in this row is

(a) 2   (b) 3   (c) 4   (d) 5   (e) 6.

5. Ace, Bea, Cec and Dee have \$16, \$24, \$32 and \$48 respectively. Their father proposed that Ace and Bea share their wealth equally, then Bea and Cec do likewise, and then Cec and Dee. Their mother's plan is the same except that Dee and Cec begin by sharing equally, then Cec and Bea and then Bea and Ace. The number of children who end up with more money under their father's plan than under their mother's is

(a) 0   (b) 1   (c) 2   (d) 3   (e) 4.

6. If n small metal spheres 3 mm in diameter are to be melted down to form one large metal sphere of diameter 10 mm, then the number n satisfies

(a) $n < 5$ (b) $5 \leq n < 10$ (c) $10 \leq n < 15$ (d) $15 \leq n < 20$ (e) $n \geq 20$.

7. The sum of all positive integers which are less than 600 and are not multiples of 3 is

(a) 60,000   (b) 90,000   (c) 120,000   (d) 150,000   (e) 180,000.

8. Five infinite straight lines are to be drawn in the plane. No three lines pass through the same point. The number of non-negative integers which can serve as the total number of points of intersection of the five lines is

(a) less than 7   (b) 7   (c) 8   (d) 9   (e) 11.

9. In a warehouse, a stack of 6 mattresses were piled up. Each mattress was originally 12 cm thick. Each compressed by a third each time an additional mattress was piled on top. The height h of the pile, in centimetres, satisfies

(a) $h < 24$ (b) $24 \leq h < 30$ (c) $30 \leq h < 32$ (d) $32 \leq h < 40$ (e) $h \geq 40$.

10. If $r^2 - r - 10 = 0$, then $(r + 1)(r + 2)(r - 4)$ is
(a) integral   (b) positive and irrational   (c) negative and irrational
         (d) rational but non-integral   (e) non-real.

11. I give Sarah N dollars for getting a good mark in school. Then, since Tim got a better mark, I give him just enough 2 dollar bills so that he gets **more** money than Sarah. Finally, since Ursula got the best mark, I give her just enough 5 dollar bills so that she gets **more** money than Tim. The largest amount of money, in dollars, that Ursula could get is

(a) $N + 2$ (b) $N + 5$ (c) $N + 6$ (d) $N + 7$ (e) none of these.

12. The number of ordered pairs of (positive or negative) integers (x, y) which satisfy

$$8x^3 + y^3 - 12x^2 + 6y^2 + 6x + 12y = 21$$

is

(a) 0   (b) 1   (c) 2   (d) 3   (e) more than 3.

13. Two integers are declared equivalent if both are divisible by the same prime numbers. The number of non-equivalent positive integers less than 25 is

(a) 9   (b) 10   (c) 16   (d) 17   (e) 24.

14. The points (x, y) on the rectangular coordinate plane satisfying $|x| \leq 2$, $|y| \leq 2$ and $||x| - |y|| \leq 1$, where $|a|$ is the absolute value of a, define a region with area

(a) 8   (b) 10   (c) 12   (d) 14   (e) 16.

15. The centre of each of two discs of radius 1 lies on the circumference of the other. The area of the intersection of the discs is

(a) $\frac{\pi}{3} - \frac{\sqrt{3}}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ (d) $\frac{2\pi}{3} - \frac{\sqrt{3}}{3}$ (e) $\frac{2\pi}{3}$.

16. Wilma and Roberta have five coins between them, a dollar, a quarter, a dime, a nickel and a penny. One of them belongs to Wilma and the other four to Roberta. Each tosses her coin or coins. Whoever has more heads wins all the coins. If it is a tie, they toss again. If this game is as fair as possible moneywise, then Wilma must have the

(a) dollar   (b) quarter   (c) dime   (d) nickel   (e) penny.

*   *   *   *   *

THE OLYMPIAD CORNER
No. 161

R. E. WOODROW

All communications about this column should be sent to Professor R. E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

Another year has past and we begin the 1995 volume of *Crux Mathematicorum* with a new and hopefully more appealing format and some new innovations. The Olympiad Corner will remain a regular feature concentrating on "Olympiad level" problem sets and the nice solutions which we can provide largely thanks to our avid readers. Elsewhere we are starting a new regular column which will concentrate on problem materials that may be more directly used in school mathematics classes.

Before launching into the new material let us pause to thank those who contributed to the Corner in 1994. My particular thanks go to Joanne Longworth whose layout skills with LATEX enhance the readability of the copy. It is also time to thank those who have contributed problem sets, solutions, comments, and corrections to the errors which seem intent on slipping by me. Among our contributors we have:

*   *   *

Iran's performance on the International Mathematics Olympiads has recently seen marked improvement. So this number we give the papers of the two stages of the Tenth Iranian Mathematical Olympiad, of 1993. Each stage was comprised of two papers. The papers were three hours in duration at stage one and four at stage 2. My thanks go to Georg Gunther, Wilfred Grenfell College, Corner Brook, Newfoundland, who collected the contest when he was Canadian Team Leader at the I.M.O. in Turkey and forwarded them for the Corner.

10th IRANIAN MATHEMATICAL OLYMPIAD
First Stage Exam
First Paper (Time: 3 hours)

1. Find all integer solutions of

$$\frac{1}{m} + \frac{1}{n} - \frac{1}{mn^2} = \frac{3}{4} \, .$$

2. Let X be a set with n elements. Show that the number of pairs (A, B) such that A, B are subsets of X, A is a subset of B, and $A \neq B$ is equal to:

$$3^n - 2^n.$$

3. Given an equilateral triangle ABC. Let (d) be a straight line outside the triangle. Let O_1 and O_2 be the centres of the circles tangent to $\{(d)$, the line BC, $AB\}$ and $\{(d)$, the line BC, $AC\}$ respectively and outside the triangle. Prove that $O_1B + O_2C = $ constant.

Second Paper (Time: 3 hours)

4. Let a, b, c be rational and one of the roots of $ax^3 + bx + c = 0$ be equal to the product of the other two roots. Prove that this root is rational.

5. Find all primes p such that $(2^{p-1} - 1)/p$ is a square.

6. Let O be the intersection of diagonals of the convex quadrilateral $ABCD$. If P and Q are the centres of circumcircles of AOB and COD, show that

$$PQ \geq \frac{AB + CD}{4}.$$

Second Stage Exam
First Paper (Time: 4 hours)

1. In the right triangle ABC ($A = 90$), let the internal bisectors of B and C intersect each other at I and the opposite sides at D and E respectively. Prove that the area of quadrilateral $BCDE$ is twice the area of the triangle BIC.

2. Given the sequence

$$a_0 = 1, \quad a_1 = 2, \quad a_{n+1} = a_n + \frac{a_{n-1}}{1 + (a_{n-1})^2} \qquad n > 1$$

show that

$$52 < a_{1371} < 65.$$

3. There is a river with cities on both its sides. Some boat lines connect these cities in such a way that each line connects a city of one side to a city on the other side, and each city is joined exactly to k cities on the other side. One can travel between every two cities. Prove that if one of the boat lines is cancelled one can travel between every two cities.

Second Stage Exam
Second Paper (Time: 4 hours)

4. Prove that for each natural number t, 18 divides

$$A = 1^t + 2^t + \cdots + 9^t - (1 + 6^t + 8^t).$$

5. In the triangle ABC we have $A \leq 90$ and $B = 2C$. Let the internal bisector of C intersect the median AM (M is the midpoint of BC) at D. Prove that $\angle MDC \leq 45$. What is the condition for $\angle MDC = 45$?

6. Let X be a nonempty finite set and $f : X \to X$ a function such that for all x in X
$$f^P(x) = x$$
where P is a constant prime. If $Y = \{x \in X : f(x) \neq x\}$, prove that the number of elements of Y is divisible by P.

<div align="center">*   *   *</div>

We now turn to reader's solutions to problems posed in 1993 numbers of the Corner. Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Christopher J. Bradley, Clifton College, Bristol, U. K.; Beatriz Margolis, Paris, France; D. J. Smeenk, Zaltbommel, The Netherlands; and Panos E. Tsaoussoglou, Athens, Greece, sent in solutions to some of the problems of the Canadian Mathematical Olympiad and the United States Mathematical Olympiads which we gave in the June number of the Corner. As we only publish solutions which are quite distinct from the "official" solutions we do not give them all. I do want to discuss a comment and two solutions to one of the problems.

3. [1993: 159, 196] 1993 *Canadian Mathematical Olympiad.*
In triangle ABC, the medians to the sides AB and AC are perpendicular. Prove that $\cot B + \cot C \geq \frac{2}{3}$.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.
In problem 1265, *Mathematics Magazine*, 61 (1988), 125–126, I showed that the area F of such a triangle does not exceed $3a^2/4$ where $a = \overline{BC}$. Therefore
$$\cot B + \cot C = \frac{c^2 + a^2 - b^2}{4F} + \frac{a^2 + b^2 - c^2}{4F} = \frac{a^2}{2F} \geq \frac{2}{3} .$$

Solution by Christopher J. Bradley, Clifton College, Bristol, U. K..
Take the circumcenter O as origin and write $\overrightarrow{OA} = \mathbf{x}$, $\overrightarrow{OB} = \mathbf{y}$, $\overrightarrow{OC} = \mathbf{z}$. The conditions for the medians to AB and AC to be at right angles is then
$$(\mathbf{z} + \mathbf{x} - 2\mathbf{y}) \cdot (\mathbf{x} + \mathbf{y} - 2\mathbf{z}) = 0.$$
Bearing in mind that $|x|^2 = R^2$ and $\mathbf{y} \cdot \mathbf{z} = R^2 \cos 2\hat{A}$, etc., this provides
$$\sin^2 \hat{B} + \sin^2 \hat{C} = 5 \sin^2 \hat{A}.$$
With this condition and $\hat{A} + \hat{B} + \hat{C} = \pi$, the turning value of $(\cot \hat{B} + \cot \hat{C})$, using Lagrange's multipliers, is when $\hat{B} = \hat{C}$. It then follows that $2 \sin^2 \hat{B} = 5 \sin^2(\pi - 2\hat{B})$ and because $\sin \hat{B} \neq 0, \pi$ one gets $\cos^2 \hat{B} = 1/10$ or $\cot \hat{B} = \cot \hat{C} = 1/3$.
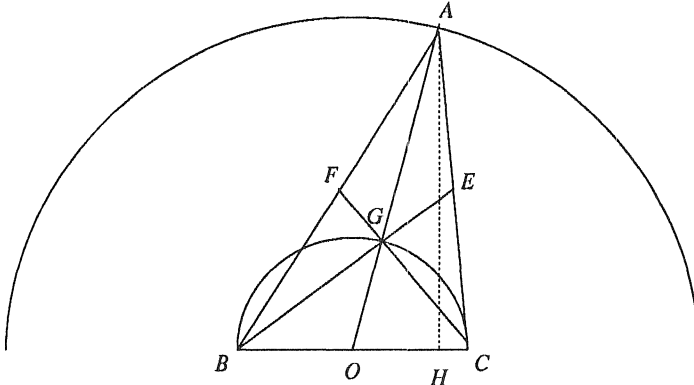
It remains to discover whether $2/3$ is a maximum or minimum and since there is only one turning point it suffices to check some possibilities.
For example $\hat{B} = \pi/2$ gives $1 + \sin^2 \hat{C} = 5 \cos^2 \hat{C}$ or $\cos^2 \hat{C} = 1/3$ and then $\cot \hat{B} + \cot \hat{C} = 0 + 1/\sqrt{2} > 2/3$. Similarly for $\hat{B}$ on the other side of

arc $\cot(1/3)$, say $\hat{B} = \text{arc}\cos(1/\sqrt{3})$, then $\hat{C} = \pi/2$ and $\cot\hat{B} + \cot\hat{C} > 2/3$ again.

Thus $\cot\hat{B} + \cot\hat{C} \geq 2/3$ with equality if and only if $\cot\hat{B} = \cot\hat{C} = 1/3$ (and $\cot\hat{A} = 4/3$).

Solution by D. J. Smeenk, Zaltbommel, The Netherlands.

Assume $BC = 2$ and O is the midpoint of BC. The locus of the centroid G of triangle ABC is the circle centred at O with radius 1. It follows that the locus of A is the concentric circle of radius 3. $AH \perp BC$, H is on BC, so

$$\cot\beta + \cot\gamma = \frac{BH}{AH} + \frac{HC}{AH} = \frac{BC}{AH} \geq \frac{BC}{AO} \geq \frac{2}{3}.$$

Equality occurs when $AB = AC$.

<p style="text-align:center">*   *   *</p>

Next we give solutions to some of the problems of the *Fourth Irish Mathematical Olympiad* [1993: 192–193].

Paper 1

2. Find the polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ satisfying the equation

$$f(x^2) = (f(x))^2$$

for all real numbers x.

Solutions by Seung-Jin Bang, Seoul, Korea; Beatriz Margolis, Paris, France; and by Michael Selby, University of Windsor. We give Margolis' solution.

We show that for each n, either $f(x) = 0$ identically, or $f(x) = x^n$. (In case $n = 0$ this gives the constant function 1).

Assume that f is not identically zero and that the degree of f, $d(f) = n$, i.e. $a_n \neq 0$. Then $f(x) = P_{n-1}(x) + a_n x^n$ and $P_{n-1}(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ is identically zero or has degree k, $0 \leq k < n$. For a contradiction assume that P_{n-1} has degree $k < n$, so $a_k \neq 0$ while $a_{k+1} = a_{k+2} = \cdots = 0$.

From the condition on f,

$$P_{n-1}(x^2) - (P_{n-1}(x))^2 = 2a_n x^n P_{n-1}(x) + a_n^2 x^{2n} - a_n x^{2n}.$$

The left side is a polynomial of degree at most $2k < m + k < 2n$. Hence on the righthand side the coefficients of the two highest powers of x must be zero, i.e. $a_n^2 = a_n$, $2a_n a_k = 0$. This gives $a_n = 1$ and $a_k = 0$ as $a_n \neq 0$, a contradiction. Thus $a_n = 1$, $P_{n-1}(x)$ is identically zero and $f(x) = x^n$.

3. Three operations f, g and h are defined as follows:

$$f(n) = 10n \qquad \text{if } n \text{ is a positive integer}$$
$$g(n) = 10n + 4 \qquad \text{if } n \text{ is a positive integer}$$
$$h(n) = n/2 \qquad \text{if } n \text{ is an even positive integer}$$

Prove that: starting from 4, every natural number can be constructed by performing a finite number of the operations f, g and h in some order. [For example:

$$35 = h(f(h(g(h(h(4)))))). \quad]$$

Comment by Seung-Jin Bang, Seoul, Korea.
This problem was proposed but not used at the 31st I.M.O. in China [1991: 197]. Solutions have appeared in [1993: 8] (solved by S.-J. Bang) and in [1993: 102] (solved by G.A. Kandall).

5. Find *all* polynomials $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ with the following properties:

(i) all the coefficients $a_1, a_2, \ldots, a_n$ belong to the set $\{-1, 1\}$;

(ii) all the roots of the equation $f(x) = 0$ are real.

[Hint: First find the maximum possible value of n].

Solution by Michael Selby, University of Windsor.
Let $r_1, r_2, \ldots, r_n$ represent the roots of $f(x)$. Then

$$\sum_{i=1}^{n} r_i^2 = \left(\sum_{i=1}^{n} r_i \right)^2 - 2 \sum_{i<j} r_i r_j = a_1^2 - 2a_2 = 1 - 2a_2. \qquad (1)$$

By the Arithmetic Mean–Geometric Mean inequality we have

$$\frac{\sum r_i^2}{n} \geq \left(\prod r_i^2 \right)^{1/n} = (a_n^2)^{1/n} = 1$$

with equality if and only if $r_i^2 = 1$ for all i.
From (1) we obtain

$$1 - 2a_2 \geq n.$$

This implies

1. $n \le 3$.
2. If $n = 3$, $r_i = \pm 1$ for all i.
3. If $n = 2, 3$, $a_2 = -1$.

This gives the following polynomials:

$$
\begin{array}{ll}
n = 1 & f(x) = x + 1 \quad \text{or} \quad f(x) = x - 1 \\
n = 2 & f(x) = x^2 + x - 1 \quad \text{or} \quad f(x) = x^2 - x - 1 \\
n = 3 & f(x) = (x - 1)^2(x + 1) = x^3 + x^2 - x - 1 \\
\text{or} & f(x) = (x + 1)^2(x - 1) = x^3 - x^2 - x + 1.
\end{array}
$$

The set of polynomials is

$$\{x - 1, x + 1, x^2 - x - 1, x^2 + x - 1, x^3 + x^2 - x - 1, x^3 - x^2 - x + 1\}.$$

Paper 2

1. The sum of two consecutive squares can be a square ($3^2 + 4^2 = 5^2$).

(a) Prove that the sum of m consecutive squares cannot be a square for the cases $m = 3, 4, 5, 6$.

(b) Find an example of eleven consecutive squares whose sum is a square.

Solutions by Bob Prielipp, University of Wisconsin-Oshkosh; and by Michael Selby, University of Windsor. We use Prielipp's solution although the two are similar.

(a) (1.) Let $S_3 = n^2 + (n+1)^2 + (n+2)^2 = 3n^2 + 6n + 5 = 3(n^2 + 2n + 1) + 2$.

Thus S_3 is of the form $3k + 2$. But a square of an integer is either of the form $3k$ or $3k + 1$. Therefore S_3 cannot be a square.

(2.) Let $S_4 = n^2 + (n + 1)^2 + (n + 2)^2 + (n + 3)^2 = 4n^2 + 12n + 14 = 2(2n^2 + 6n + 7)$.

Because $2n^2 + 6n + 7$ is odd, 2 divides S_4, but 2^2 does not. Therefore S_4 cannot be a square.

(3.) Let $S_5 = n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 + (n+4)^2 = 5n^2 + 20n + 30 = 5(n^2 + 4n + 6)$.

Now $n^2 + 4n + 6 = (n + 2)^2 + 2$ is not a multiple of 5 because $(n + 2)^2$ is of the form $5k$ or $5k + 1$ or $5k - 1$ so $(n + 2)^2 + 2$ is of the form $5k + 2$, $5k + 3$ or $5k + 1$ and 5 divides S_5 but 5^2 does not. So, S_5 is not a perfect square.

(4.) Let $S_6 = n^2 + (n + 1)^2 + \cdots + (n + 5)^2 = 6n^2 + 30n + 55 = 6(n^2 + 5n + 8) + 7$.

Thus S_6 is of the form $12k + 7$ since $n^2 + 5n + 8$ is even. But the square of any number is one of the form $12k$, $12k + 1$, $12k + 4$ or $12k + 9$. So S_6 cannot be a square.

(b) Let $S_{11} = n^2 + (n + 1)^2 + \cdots + (n + 10)^2 = 11n^2 + 110n + 385 = 11(n^2 + 10n + 35)$. In order for S_{11} to be a square we must have that $(n + 5)^2 + 10 = 11t^2$ so it must be the case that

$$(n + 5)^2 - 11t^2 = -10. \tag{$*$}$$

The least positive integer solution (n, t) of $(*)$ is $(n, t) = (18, 7)$. The next smallest positive integer solution of $(*)$ is $(n, t) = (38, 13)$. Direct computation verifies that

  (a) $18^2 + 19^2 + \cdots + 28^2 = 5929 = 77^2$ and
  (b) $38^2 + 39^2 + \cdots + 48^2 = 20449 = 143^2$.

2. Let
$$a_n = \frac{n^2 + 1}{\sqrt{n^4 + 4}} \quad \text{for } n = 1, 2, 3, \ldots$$
and let b_n be the product $a_1 a_2 \ldots a_n$. Prove that
$$\frac{b_n}{\sqrt{2}} = \frac{\sqrt{n^2 + 1}}{\sqrt{n^2 + 2n + 2}}$$

and deduce that
$$\frac{1}{(n+1)^3} < \frac{b_n}{\sqrt{2}} - \frac{n}{n+1} < \frac{1}{n^3}$$

for all positive integers n.

Solution by Michael Selby, University of Windsor.
Observe that
$$\begin{aligned}
n^4 + 4 &= n^4 + 4n^2 + 4 - 4n^2 \\
&= (n^2 + 2)^2 - 4n^2 \\
&= (n^2 + 2n + 2)(n^2 - 2n + 2)
\end{aligned}$$

Also $(k - 1)^2 + 2(k - 1) + 2 = k^2 + 1$ and $(k + 1)^2 - 2(k + 1) + 2 = k^2 + 1$. These observations imply that in the product $a_1 a_2 \ldots a_{k-1} a_k a_{k+1} \ldots a_n$, the numerator of a_k, $1 + k^2$, is cancelled by $\sqrt{(k - 1)^2 + 2(k - 1) + 2}$ in the denominator of a_{k-1} and $\sqrt{(k + 1)^2 + 2(k + 1) + 2}$ in the denominator of a_{k+1}.
Therefore
$$b_n = \frac{\sqrt{2}\sqrt{1 + n^2}}{\sqrt{2 + 2n + n^2}},$$

since all the numerator terms cancel except the numerator of the first and last terms.
To obtain the inequalities, we consider
$$\frac{b_n}{\sqrt{2}} = \frac{\sqrt{1 + n^2}}{\sqrt{n^2 + 2n + 2}}$$

and
$$\begin{aligned}
\frac{b_n}{\sqrt{2}} - \frac{n}{n+1} &= \frac{\sqrt{n^2 + 1}}{\sqrt{n^2 + 2n + 2}} - \frac{n}{n+1} \\
&= \frac{(n + 1)\sqrt{n^2 + 1} - n\sqrt{n^2 + 2n + 2}}{(n + 1)\sqrt{n^2 + 2n + 2}}
\end{aligned} \tag{1}$$

Let r represent the expression in (1). Then r can be written as:

$$r = \frac{2n+1}{(n+1)\sqrt{n^2+2n-}+2((n+1)\sqrt{n^2+1}+n\sqrt{n^2+2n+2})}$$
$$= \frac{2n+1}{n(n+1)(n^2+2n+2)+(n+1)^2\sqrt{n^2+2n+2}\sqrt{n^2+1}} \ .$$

Now

$$r < \frac{2(n+1)}{(n+1)[n^3+(n+1)n^2]} < \frac{2(n+1)}{(n+1)2n^3} = \frac{1}{n^3} \ .$$

To obtain the other inequality:

   This is equivalent to

$$(2n+1)(n+1)^3 > (n+1)[n(n^2+2n+2)+(n+1)\sqrt{n^2+1}\sqrt{n^2+2n+2}]$$

or

$$(2n+1)(n+1)^2 > n(n^2+2n+2)+(n+1)\sqrt{n^2+1}\sqrt{n^2+2n+2}.$$

This is equivalent to

$$n^3+3n^2+2n+1 > (n+1)\sqrt{n^2+1}\sqrt{n^2+2n+2}$$

or

$$\frac{n^2}{n+1}+n^2+n+1 > \sqrt{n^2+1}\sqrt{n^2+2n+2} \ .$$

   If $n = 1$, $7/2 > \sqrt{10}$ is true.
   If $n \geq 2$, $n^2/(n+1) > 1$, then

$$n^2+n+1+\frac{n^2}{n+1} > n^2+n+2.$$

If we can show that $n^2+n+2 > \sqrt{n^2+1}\sqrt{n^2+2n+2}$ for $n \geq 2$, we will be done. $(n^2+n+2) > \sqrt{n^2+1}\sqrt{n^2+2n+2}$ iff $(n^2+n+2)^2 > (n^2+1)(n^2+2n+2)$ iff

$$2(n^2+n+1) > 0, \quad n \geq 2.$$

This is clearly true, hence the inequalities are true.

$$* \qquad * \qquad *$$

   That completes the solutions on file for problems from the September number and fills the space we have available this issue. Send me your nice solutions and olympiad problem sets.

$$* \qquad * \qquad * \qquad * \qquad *$$

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

Polyominoes (Second Edition), by Solomon Golomb. Published by Princeton University Press, 1994. ISBN 0-691-08573-0, 181+ pages, hardcover, US$24.95. *Reviewed by Andy Liu.*
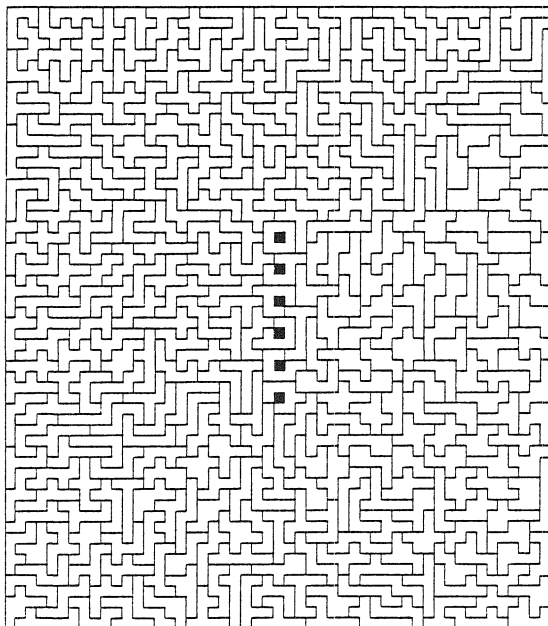
At last, the long awaited reprinting of this classic, originally published by Charles Scribner's Sons in 1965, makes its most welcomed appearance. The definitive treatise on this fascinating subject, Solomon Golomb's book was much sought after. The reviewer obtained his copy of the old edition only after the untimely death of a colleague. Now, one can just pay with money, and it is well worth it!

The author deliberately keeps the changes from the original edition to a minimum. The new features include a preface to the revised edition, two new chapters, two new appendices and a comprehensive bibliography.
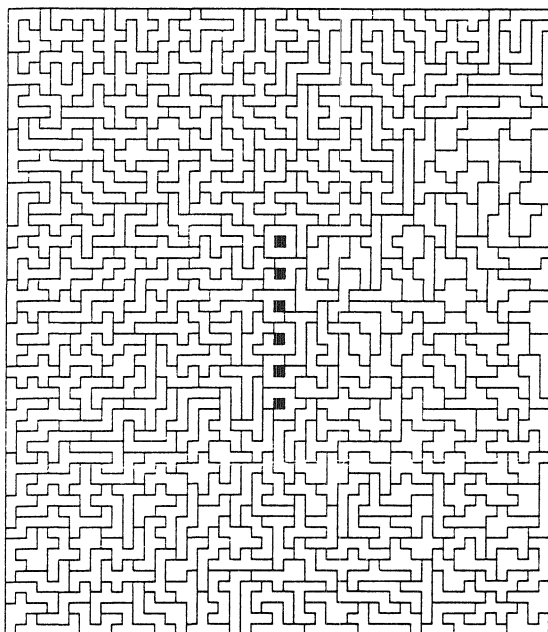
The new Chapter 8 is on tiling rectangles with polyominoes. David Klarner defined the *order* of a polyomino as the minimum number of congruent copies of it that can be assembled to form a rectangle. Some of the new and exciting results are the discovery of sporadic examples of polyominoes of orders 50, 76, 92, 96 and possibly 312 (in the last case, it is not yet proved that no lower number will work), and the systematic construction of polyominoes whose orders are arbitrary multiples of 4. The related concept of odd-order is also considered.

The first problem in the book is the well-known result that if the opposite corners of a chessboard are removed, the remaining part cannot be covered by 31 dominoes. This is true as long as the two squares removed have the same colour. The new Chapter 9 begins with Ralph Gomory's proof of the converse proposition. Next come some constructions from "monster sets", the 369 octominoes and the 1285 enneominoes. Kate Jones of Kadon Enterprises pointed out that although Figure 175 resembles the top diagram of Figure 178, they are not identical. These are reprinted below; readers are challenged to spot the difference! Some generalizations of a theorem of De Bruijn on brick-packing are given for a torus before the chapter concludes with some results on polyhexes.

The new Appendix C, reprinted from [6], contains updated information on the twelve "readers' research" problems in Appendix B. There is a misspelling of the name of Loren Looger of NASA, who is one of the solvers of the notorious "Fifteen Problem" (construction of 4 copies of a polyomino of area 15 using a full set of pentominoes). The new Appendix D gives new bounds for the Klarner Konstant associated with the counting of polyominoes of a given area. The comprehensive bibliography has already fallen behind current literature, which attests to the continual phenomenal growth of the subject Solomon Golomb founded. For instance, [2] contains a solution to

The 369 octominoes in a 51 × 58 rectangle.

Spot the difference!

Problem 81 of Appendix B which is somewhat different from that given in [6].

In the old edition, there was a pocket at the back containing a set of the 12 pentominoes in cheap plastic. The reviewer had always found their presence cumbersome. Perhaps thirty years ago, sets were not readily available. There was in fact a short note on materials if one was interested in constructing sets from scratch. Both the pocket and the note have been eliminated. Nowadays, the most popular sets are the high-quality laser-cut wooden or acrylic ones by Kadon Enterprises (1227 Lorene Drive #16, Pasadena, MD 21122, USA).

As much as the reviewer would like to see the original version preserved in its entirety, there are a few errors which should have been corrected and simpler solutions mentioned. On pages 28 and 29, it is still claimed that 1/3 of the infinite chessboard has to be blocked off to keep out the V-pentomino. It has been proved in [1] and [4] that the correct value is 4/13. Curiously, both references are included in the comprehensive bibliography.

Pages 39 to 42 contain an eight-step outline of a proof that a certain figure cannot be constructed using a set of pentominoes. A much simpler solution due to Mogens Larsen [5] proves an even stronger result. However, only the preliminary draft of his paper is referenced.

The reviewer feels that this book should be on every shelf and in every library. As much as it contains, one can always ask for more. Let us hope that we do not have to wait another thirty years for the next edition. In the meantime, interested readers can also consult [3], [7] and the instruction booklets of Kadon Enterprises' Quintillions, Super Quintillions, Sextillions and especially Poly-5.

References:

[1] F. W. Barnes and J. B. Shearer, Barring rectangles from the plane, *Journal of Combinatorial Theory A* **23** (1982) 9–29.

[2] A. Cibulus, A remarkable figure which cannot be formed by pentominoes, *Mathematics and Informatics Quarterly* **3** (1993) 137–139.

[3] K. Jones, The psycho-aesthetics of combinatorial sets, in *Mathematics for Gifted Students*, ed. A. Liu, Alberta Teachers' Association (1989) 12–23.

[4] M. S. Klamkin and A. Liu, Polyominoes on the infinite checkerboard, *Journal of Combinatorial Theory A* **28** (1980) 7–16.

[5] M. E. Larsen, There are no holes inside a diamond, in *Mathematics for Gifted Students*, ed. A. Liu, Alberta Teachers' Association (1989) 30–36.

[6] A. Liu, Golomb's Twelve Pentomino Problems, in *The Lighter Side of Mathematics*, ed. R. Guy and R. Woodrow, Mathematical Association of America (1994) 206–211.

[7] G. Martin, *Polyominoes*, Mathematical Association of America (1991).

         ✻   ✻   ✻   ✻   ✻

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **August 1, 1995,** *although solutions received after that date will also be considered until the time when a solution is published.*

2001. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Three similar triangles DBC, ECA, FAB are drawn outwardly on the sides of triangle ABC, such that $\angle DBC = \angle ECA = \angle FAB$ and $\angle DCB = \angle EAC = \angle FBA$. Let $P = BE \cap CF$, $Q = CF \cap AD$, $R = AD \cap BE$. Prove that
$$\frac{QR}{AD} = \frac{RP}{BE} = \frac{PQ}{CF} .$$

2002. *Proposed by K. R. S. Sastry, Dodballapur, India.*
Find a positive integer N such that both N and the sum of the digits of N are divisible by both 7 and 13.

2003. *Proposed by Richard K. Guy, University of Calgary.*
Given any five points in the plane with no three in line and no four on a circle, show that there are (at least) four sets of three points such that the circles through them have just one of the remaining points inside and one outside.

2004. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*
Given are real numbers $a_1, a_2, \ldots, a_n$ with $\sum_{i=1}^{n} a_i = 0$. Determine
$$\sum_{i=1}^{n} \frac{1}{a_i(a_i + a_{i+1})(a_i + a_{i+1} + a_{i+2}) \ldots (a_i + a_{i+1} + \cdots + a_{i+n-2})}$$

where $a_{n+1} = a_1$, $a_{n+2} = a_2$, etc., assuming that the denominators are nonzero.

2005. *Proposed by Murray S. Klamkin, University of Alberta.*

(a) Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be vectors from the circumcenter of a triangle ABC to the respective vertices. Prove that

$$\frac{(\mathbf{B}+\mathbf{C})|\mathbf{B}-\mathbf{C}|}{|\mathbf{B}+\mathbf{C}|} + \frac{(\mathbf{C}+\mathbf{A})|\mathbf{C}-\mathbf{A}|}{|\mathbf{C}+\mathbf{A}|} + \frac{(\mathbf{A}+\mathbf{B})|\mathbf{A}-\mathbf{B}|}{|\mathbf{A}+\mathbf{B}|} = \mathbf{0}. \tag{1}$$

(b)* Suppose that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are vectors from a point P to the respective vertices of a triangle ABC such that (1) holds. Must P be the circumcenter of ABC?

2006. *Proposed by John Duncan, University of Arkansas, Fayetteville; Dan Velleman, Amherst College, Amherst, Massachusetts; and Stan Wagon, Macalester College, St. Paul, Minnesota.*

Suppose we are given $n \geq 3$ disks, of radii $a_1 \geq a_2 \geq \cdots \geq a_n$. We wish to place them in some order around an interior disk so that each given disk touches the interior disk and its two immediate neighbors. If the given disks are of widely different sizes (such as $100, 100, 100, 100, 1$), we allow a disk to overlap other given disks that are not immediate neighbors. In what order should the given disks be arranged so as to maximize the radius of the interior disk? [*Editor's note.* Readers may assume that for any ordering of the given disks the configuration of the problem exists and that the radius of the interior disk is unique, though, as the proposers point out, this requires a proof (which they supply).]

2007. *Proposed by Pieter Moree, Macquarie University, Sydney, Australia.*

Find two primes p and q such that, for all sufficiently large positive real numbers r, the interval $[r, 16r/13]$ contains an integer of the form

$$2^n, \quad 2^n p, \quad 2^n q, \quad \text{or} \quad 2^n pq$$

for some nonnegative integer n.

2008. *Proposed by Jun-hua Huang, The Middle School Attached To Hunan Normal University, Changsha, China.*

Let I be the incenter of triangle ABC, and suppose there is a circle with center I which is tangent to each of the excircles of $\triangle ABC$. Prove that ABC is equilateral.

2009. *Proposed by Bill Sands, University of Calgary.*

Sarah got a good grade at school, so I gave her N two–dollar bills. Then, since Tim got a better grade, I gave him just enough five–dollar bills so that he got *more* money than Sarah. Finally, since Ursula got the best grade, I gave her just enough ten–dollar bills so that she got *more* money than Tim. What is the maximum amount of money that Ursula could have received? (This is a variation of problem 11 on the 1994 Alberta High School Mathematics Contest, First Part; see The Skoliad Corner, this issue.)

2010. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

In triangle ABC with $\angle C = 2\angle A$, line CD is the internal angle bisector (with D on AB). Let S be the center of the circle tangent to line CA (produced beyond A) and externally to the circumcircles of triangles ACD and BCD. Prove that $CS \perp AB$.

<div align="center">

*   *   *   *   *

SOLUTIONS

</div>

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1907. [1994: 17] *Proposed by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*

Find the largest constant k such that

$$\frac{kabc}{a+b+c} \le (a+b)^2 + (a+b+4c)^2$$

for all $a, b, c > 0$.

Solution by N. T. Tin, Hong Kong.

By the A.M.-G.M. inequality,

$$\begin{aligned}
(a+b)^2 + (a+b+4c)^2 &= (a+b)^2 + (a+2c+b+2c)^2 \\
&\ge (2\sqrt{ab})^2 + (2\sqrt{2ac} + 2\sqrt{2bc})^2 \\
&= 4ab + 8ac + 8bc + 16c\sqrt{ab}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{(a+b)^2 + (a+b+4c)^2}{abc} \cdot (a+b+c) &\ge \frac{4ab + 8ac + 8bc + 16c\sqrt{ab}}{abc} \cdot (a+b+c) \\
&= \left(\frac{4}{c} + \frac{8}{b} + \frac{8}{a} + \frac{16}{\sqrt{ab}} \right)(a+b+c) \\
&= 8 \left(\frac{1}{2c} + \frac{1}{b} + \frac{1}{a} + \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{ab}} \right) \left(\frac{a}{2} + \frac{a}{2} + \frac{b}{2} + \frac{b}{2} + c \right) \\
&\ge 8 \left(5\sqrt[5]{\frac{1}{2a^2b^2c}} \right) \left(5\sqrt[5]{\frac{a^2b^2c}{2^4}} \right) = 100,
\end{aligned}$$

again by the A.M.-G.M. inequality. Hence the largest constant k is **100**. For $k = 100$, equality holds if and only if $a = b = 2c > 0$.

Also solved by GERD BARON, Technische Universität Wien, Austria (four solutions!); CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; HIMADRI CHOUDHURY, student, Hunter High School, New York; TIM CROSS, Wolverley High School, Kidderminster, U. K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes,

California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; DAVID E. MANES, State University of New York, Oneonta; BEATRIZ MARGOLIS, Paris, France; WALDEMAR POMPE, student, University of Warsaw, Poland; P. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

*   *   *   *   *

1910. [1994: 17] *Proposed by Jisho Kotani, Akita, Japan.*

The octahedron $ABCDEF$ is inscribed in a sphere so that the three diagonals AF, BD, CE meet at a point, and the centroids of the six (triangular) faces of the octahedron are also inscribed in a sphere. Show that

(i) the orthocenters of the six faces are inscribed in a sphere;

(ii) $(AB \cdot DF + AD \cdot BF)(AC \cdot EF + AE \cdot CF)(BC \cdot DE + CD \cdot BE) = 36V^2$,

where V is the volume of $ABCDEF$.

Solution by Waldemar Pompe, student, University of Warsaw, Poland.

There is something wrong with the problem as stated.

[*Editor's comment by Chris Fisher.* Replace each "six" by an "eight" to get the proposer's original intention. The error was introduced by the *Crux* editor in charge of numbers up to five. He not only believes that an octahedron has six faces, but he thinks that *sexennial* refers to a couple that mates once a year. The other editors specialize in numbers greater than ten, so this problem seems to have fallen between the cracks. Back to you, Waldek.]

Interpreting the problem to read, "the centroids of *six of the (triangular) faces of the octahedron*," we find a simple counterexample to part (ii): Take any sphere with center O and inscribe in its equator the rectangle $BCDE$ (which is not a square) with $\angle BOC$ between the diagonals equal to α. Let the line through O perpendicular to the plane $BCDE$ intersect the sphere in points A and F. Look at the centroids of the faces $FED, FEB, FBC, AED, AEB, ABC$. They form two adjacent rectangular faces of a parallelepiped, so there is a sphere containing them; also, AF, BD and CE have a common point O. By Ptolemy's theorem, part (ii) can be rewritten as $AF \cdot BD \cdot CE = 6V$. Now as α changes, so does the volume of $ABCDEF$, while the product on the left-hand side of the above equality remains constant.

Our problem will become correct if we demand that *seven* of the centroids lie on a sphere.

Proof. Let M be the midpoint of BC, and G_1, G_2 be the centroids of the faces ABC, BCF respectively. Since

$$\frac{MG_1}{G_1A} = \frac{1}{2} = \frac{MG_2}{G_2F},$$

the lines G_1G_2 and AF are parallel and $G_1G_2 = AF/3$. Arguing similarly for any other pair of faces sharing an edge, we see that the eight centroids of the octahedron form a parallelepiped (since each edge is parallel to one of the diagonals of the octahedron, and all edges parallel to the same diagonal are of the same length). Since seven of the eight possible vertices of the parallelepiped lie on a sphere, say S, the parallelepiped must be inscribed in this sphere (six vertices are not enough, as the above counterexample shows), and so it must have rectangles as faces (since the inscribed parallelogram must be a rectangle). It follows that AF, BD, CE are mutually orthogonal.

(i) We show that the orthocenters of all eight faces of the octahedron lie on S. Let O be the common point of the diagonals AF, BD, CE. $BCDE$ is an orthodiagonal quadrilateral inscribed in a circle, so $MO \perp DE$. (This is essentially the theorem of Brahmagupta; its easy proof was provided in the solution to *Crux* 1836 [1994: 84], which is sort of a 2-dimensional version of the present problem.) Set $K = MO \cap ED$. Since $AO \perp OK$ and $MO \perp DE$, AK is an altitude of triangle ADE. Thus ED is perpendicular to the plane $AKFM$, which implies that the planes $AKFM$ and ADE are orthogonal. Analogously, we may consider the orthodiagonal quadrilateral $ADFB$ and show that EL is an altitude of the triangle ADE, and the plane $NELC$ is orthogonal to the face ADE (where N is the midpoint of BF and $L = NO \cap AD$). Thus the points G_2, O and the orthocenter H of the triangle ADE are collinear (they belong to the intersection of the planes $AKFM$ and $NELC$) and OH is perpendicular to the plane ADE (since both the planes $AKFM$ and $NELC$ are orthogonal to the plane ADE).

Let G_3 be the centroid of ADE. Since $\angle G_2HG_3 = 90°$ and G_2G_3 is the diameter of the sphere S, H must belong to S.

In the same way we prove that the remaining seven orthocenters lie on S, so part (i) is solved.

(ii) Since AF, BD, CE are mutually orthogonal, $V = (AF \cdot BD \cdot CE)/6$. Applying Ptolemy's theorem to the quadrilaterals $BCDE$, $ACFE$, and $ADFB$, we obtain

$$(AB \cdot DF + AD \cdot BF)(AC \cdot EF + AE \cdot CF)(BC \cdot DE + CD \cdot BE)$$
$$= (AF \cdot BD \cdot CE)^2 = 36V^2,$$

and we are done.

Also solved by P. PENNING, Delft, The Netherlands; and the proposer.
Pompe adds that the problem is nonvoid in the sense that the given octahedron does not have to be regular: any choice of mutually orthogonal chords through an interior point of a sphere provides an example. In fact, he would have preferred that the problem be turned around to read:
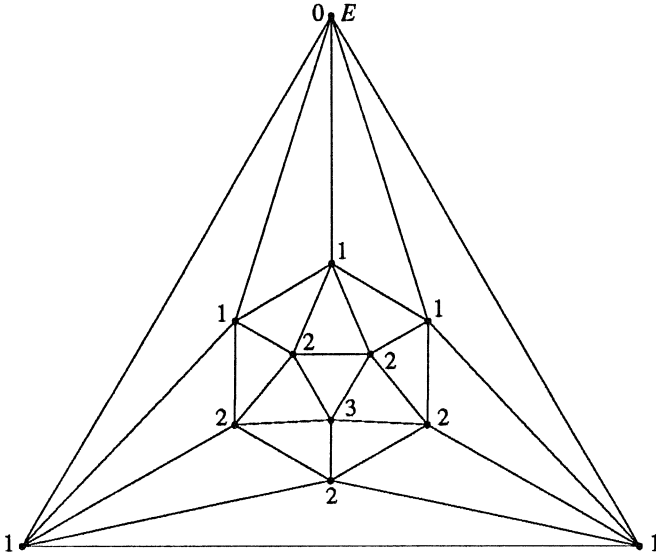
AF, BD and CE are concurrent and mutually orthogonal diagonals of an octahedron ABCDEF inscribed in a sphere. Show that the centroids and the orthocenters of the faces of ABCDEF lie on a common sphere.

*   *   *   *   *

1911. [1994: 47] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Let B and E be two opposite vertices of the regular icosahedron. Consider the random walk over the edge-skeleton of the icosahedron, beginning at B. Each time he arrives at a vertex, the walker continues the walk along any one of the five edges emanating from that vertex, with equal probability of each choice. The walk ends when the walker reaches vertex E. Find the expected length of the walk (in number of edges).

Solution by David G. Poole, Trent University, Peterborough, Ontario.

Group the twelve vertices of the icosahedron into four classes according as their distance from the target vertex E is 0, 1, 2, or 3. By symmetry, the expected length of a random walk to the target from each vertex in a given class is the same. Let a, b, and c respectively denote the expected length of a walk starting at a vertex in class 1, 2 and 3 and ending at the target. The problem as posed asks for c.

We easily obtain the equations

$$a = \frac{1}{5}(2a + 2b) + 1,$$

$$b = \frac{1}{5}(2a + 2b + c) + 1,$$

$$c = b + 1,$$

whose solution is found to be $a = 11$, $b = 14$, $c = 15$.

Comments: Problems related to random walks on graphs in general and polyhedra in particular have been much studied, usually using techniques

from the theory of finite Markov chains (cf. [1, Chapter 7] and [2, 3, 4, 5]). Our solution above treats the given problem as a special case of a quotient Markov chain (see [4] for the case of a dodecahedron). The transition matrix for this is given by

$$P = (p_{ij}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/5 & 2/5 & 2/5 & 0 \\ 0 & 2/5 & 2/5 & 1/5 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where p_{ij} denotes the transition probability from a vertex in class i to a vertex in class j, $0 \le i, j \le 3$. Denoting by Q the matrix obtained by deleting row 0 and column 0 from P, by I the 3×3 identity matrix, and by ξ the vector $(1, 1, 1)^T$, the general theory tells us that the vector of expected path lengths is given by

$$(a, b, c)^T = (I - Q)^{-1}\xi.$$

As a final note, we see that the recurrence time for vertex E (*i.e.* the expected length of a random walk starting and ending at E) is given by $a + 1 = 12$; this is as expected, since it is well-known that the recurrence time for a vertex in a regular graph is just the total number of vertices [2, Corollary 1.2].

References:

[1] M. Gardner, *Mathematical Circus*, Alfred A. Knopf, New York, 1979.

[2] F. Göbel and A. A. Jagers, Random walks on graphs, *Stochastic Processes Appl.* 2 (1974), 311–336.

[3] J. G. Kemeny and J. L. Snell, *Finite Markov chains*, Van Nostrand, New York, 1969.

[4] G. Letac and L. Takács, Random walks on a dodecahedron, *J. Appl. Prob.* 17 (1980), 373–384.

[5] A. R. D. van Slijpe, Random walks on regular polyhedra and other distance-regular graphs, *Statistica Neerlandica* 38 (1984), 273–292.

Also solved by GERD BARON, Technische Universität Wien, Austria; CHRISTOPHER BRADLEY, Clifton College, Bristol, U. K.; JORDI DOU, Barcelona, Spain; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

Hess also solved the problem for the octahedron, cube and dodecahedron, obtaining answers of 6, 10 and 35 respectively.

*   *   *   *   *

1912. [1994: 47] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with $AB \neq AC$. Similar triangles ABD and ACE are drawn outwardly on the sides AB and AC of $\triangle ABC$, so that $\angle ABD = \angle ACE$ and $\angle BAD = \angle CAE$. CD and BE meet AB and AC at P and Q respectively. Prove that $AP = AQ$ if and only if

$$[ABD] \cdot [ACE] = [ABC]^2,$$

where $[XYZ]$ denotes the area of triangle XYZ. (This problem is an extension of *Crux* 1537 [1991: 182].)

Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.

Reflecting B, Q and E around the internal bisector of $\angle BAC$, we get collinear points B^*, Q^* and E^* on the lines AC, AB and AD respectively. Now, since A will lie between P and B and between Q and C if and only if $\angle BAE = \angle DAC > 180°$, $AP = AQ$ if and only if $P = Q^*$; i.e.

$$AP = AQ \quad \Longleftrightarrow \quad B^*, P \text{ and } E^* \text{ are collinear.} \tag{1}$$

From $AB \neq AC$ it follows that $B^* \neq C$ and $E^* \neq D$. Let $AB = c$, $AC = b$, and

$$AD : AB = AE : AC = x : 1,$$

so that $AE^* = AE = xb$ and $AD = xc$. Then from (1), and according to Menelaos' Theorem applied to $\triangle ACD$, $AP = AQ$ if and only if

$$1 = \frac{AE^* \cdot DP \cdot CB^*}{DE^* \cdot CP \cdot AB^*} = \frac{xb \cdot DP \cdot (c - b)}{(xc - xb) \cdot PC \cdot c} = \frac{b \cdot DP}{c \cdot PC},$$

that is,

$$[ACE] : [ABD] = b^2 : c^2 = (PC)^2 : (DP)^2 = [ABC]^2 : [ABD]^2,$$

that is,

$$[ABD] \cdot [ACE] = [ABC]^2.$$

Also solved by FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I. B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; MARCIN E. KUCZMA, Warszawa, Poland; WALDEMAR POMPE, student, University of Warsaw, Poland; D. J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

*   *   *   *   *

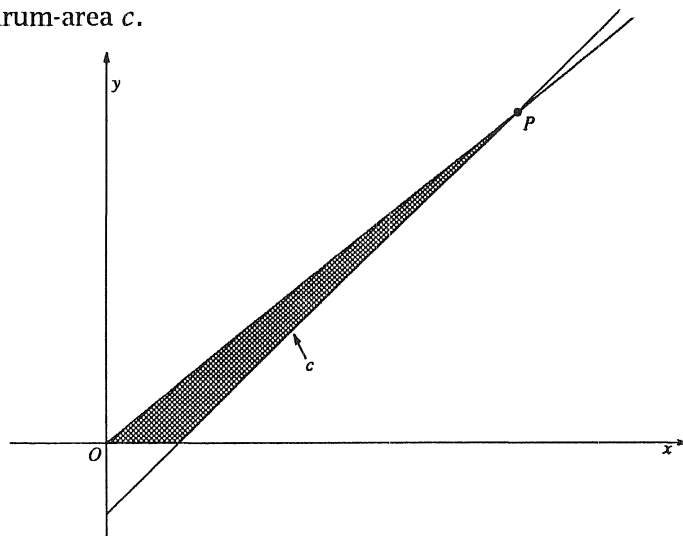**1913.** [1994: 48] *Proposed by N. Kildonan, Winnipeg, Manitoba.*

I was at a restaurant for lunch the other day. The bill came, and I wanted to give the waiter a whole number of dollars, with the difference between what I give him and the bill being the tip. I always like to tip between 10 and 15 percent of the bill. But if I gave him a certain number of dollars, the tip would have been **less** than 10% of the bill, and if instead I gave him one dollar more, the tip would have been **more** than 15% of the bill. What was the largest possible amount of the bill? [*Editor's note to non-North American readers:* your answer should be in dollars and cents, where there are (reasonably enough) 100 cents in a dollar.]

Solution by Robert Geretschläger, Bundesrealgymnasium, Graz, Austria.

Let x be the amount of the bill and y the amount our esteemed colleague would be willing to pay in an integer number of dollars, but which would lamentably result in a cheapskate tip of less than 10%. Paying $\$(y+1)$ would, on the other hand, result in the undue financial stress of doling out a tip of more than 15%. This means that

$$y < \frac{110}{100} \cdot x \quad \text{and} \quad y + 1 > \frac{115}{100} \cdot x.$$

Each of these inequalities defines an open half-plane in cartesian coordinates. The part of the intersection of these half-planes with positive coordinates is the conundrum-area c.

The highest corner P of this area is the point where the lines

$$y = \frac{11}{10} \cdot x \quad \text{and} \quad y = \frac{23}{20} \cdot x - 1$$

intersect. This is the point

$$x = 20, \quad y = 22.$$

28

The highest dollar value y in the open conundrum-area c is therefore $21, and since

$$y + 1 > \frac{115}{100} \cdot x,$$

we have

$$x < \frac{100}{115} \cdot (y + 1) = \frac{20}{23} \cdot 22 = \$19.13\frac{1}{23}.$$

The largest possible amount of the bill to launch our friend into such a lamentable state of fiscal confusion is thus **$19.13**. If he pays $21, the tip is $1.87, which works out to slightly more than 9.77%. On the other hand, if he pays $22, the tip is $2.87, which works out to an entirely unconscionable 15.0026%.

How to get out of this morass? I suggest splurging and going for the 15.0026% tip. Make the poor schlub of a waiter happy, for heaven's sake. After all, he or she may be a future colleague working his or her way through school. And even if that doesn't happen to be the case, remember that heavy tippers stand a better chance of getting that burger without the side order of extra gristle (or worse) next time.

Also solved by AL ADAMS, University of Pittsburg, Johnstown, Pennsylvania; BILL CORRELL JR., student, Denison University, Granville, Ohio; TIM CROSS, Wolverley High School, Kidderminster, U. K.; KEITH EKBLAW, Walla Walla, Washington; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; DAVID HANKIN, John Dewey High School, Brooklyn, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; PETER HURTHIG, Columbia College, Burnaby, B.C.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID E. MANES, State University of New York, Oneonta; J. A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; R. P. SEALY, Mount Allison University, Sackville, New Brunswick; P. E. TSAOUSSOGLOU, Athens, Greece; EDWARD T.H. WANG and MARTIN WHITE, Wilfrid Laurier University, Waterloo, Ontario; and the proposer. Three incorrect solutions were received, one (at least) probably due to misunderstanding the problem, for which the editor, and our Central Canadian proposer, offer apologies.
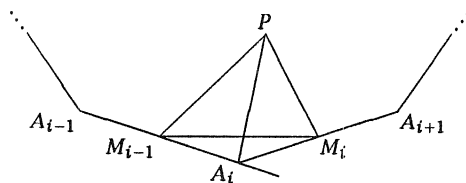
*   *   *   *   *

1914. [1994: 48] *Proposed by K. R. S. Sastry, Addis Ababa, Ethiopia.*

Let $A_1A_2\ldots A_n$ be a regular n-gon, with $M_1, M_2, \ldots, M_n$ the midpoints of the sides. Let P be a point in the plane of the n-gon. Prove that

$$\sum_{i=1}^{n} PM_i \geq \cos(180°/n) \sum_{i=1}^{n} PA_i.$$

Solution by Toshio Seimiya, Kawasaki, Japan.

Let M_i be the midpoint of side $A_i A_{i+1}$ ($i = 1, 2, \ldots, n$), where $A_{n+1} = A_1$ and $M_0 = M_n$. Applying the well-known extension of Ptolemy's theorem (e.g., see p. 213 of [1] or Chapter XIII of [2]) to quadrilateral $PM_{i-1}A_iM_i$, we have

$$PM_{i-1} \cdot A_iM_i + PM_i \cdot A_iM_{i-1} \geq PA_i \cdot M_{i-1}M_i. \tag{1}$$

Because $A_iM_i = A_iM_{i-1}$, and the exterior angle at A_i is equal to $360°/n$, we get

$$\angle A_iM_iM_{i-1} = \angle A_iM_{i-1}M_i = \frac{180°}{n} ,$$

and thus

$$M_{i-1}M_i = 2A_iM_i \cos \frac{180°}{n} .$$

Therefore we have from (1)

$$PM_{i-1} + PM_i \geq \frac{M_{i-1}M_i}{A_iM_i} \cdot PA_i = 2 \cos \frac{180°}{n} \cdot PA_i.$$

Hence we have

$$2 \sum_{i=1}^{n} PM_i = \sum_{i=1}^{n} (PM_{i-1} + PM_i) \geq 2 \cos \frac{180°}{n} \sum_{i=1}^{n} PA_i,$$

and therefore the result follows.

References:

[1] N. Altshiller Court, *College Geometry*.
[2] D.S. Mitrinović, J.E. Pečarić and V. Volenec, *Recent Advances in Geometric Inequalities*.

Also solved (about the same way) by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; and the proposer.

As Klamkin points out, and the above proof shows, equality holds if and only if P is the centre of the polygon, since equality holds in (1) if and only if P lies on the circle containing M_{i-1}, A_i and M_i, and these circles (for $i = 1, \ldots, n$) intersect at the centre. Also, the result holds for points P outside the plane of the polygon as well, since Ptolemy's inequality still applies.

Compare this inequality with item 16.9 of Bottema et al, Geometric Inequalities (see also pp. 421–424 of [2]), which says

$$\cos\left(\frac{180°}{n}\right)\sum_{i=1}^{n} PA_i \ge \sum_{i=1}^{n} w_i,$$

where P is an interior point of the convex (not necessarily regular) n-gon $A_1 \ldots A_n$, and w_i is the length of the angle bisector of P in triangle PA_iA_{i+1}.

*   *   *   *   *

1915. [1994: 48] *Proposed by Richard K. Guy, University of Calgary.*
Is $\binom{n}{r}$ ever relatively prime to $\binom{n}{s}$ for $0 < r < s < n$? (This is not a new problem. Its history will be revealed when a solution is published.)

Solution by Marcin E. Kuczma, Warszawa, Poland.
I happen to know the solution. The answer is "never" and it follows immediately from the identity

$$\binom{n}{r}\binom{n-r}{s} = \binom{n}{s}\binom{n-s}{r}.$$

[Because if $\binom{n}{r}$ and $\binom{n}{s}$ were relatively prime then $\binom{n}{r}$ would have to divide into $\binom{n-s}{r}$, which is impossible since $\binom{n}{r} > \binom{n-s}{r}$.—*Ed.*]

I did not invent this myself; I was told the problem, along with this proof, some time before I saw it in *Crux*. I have also come across the reference to a publication of Erdős and Szekeres ["Some number theoretic problems on binomial coefficients", *Australian Math. Soc. Gazette* 5(1978) 97–99].

Editor's note. The Erdős–Szekeres paper is indeed the source of the problem, which seems to have received some belated and well-deserved publicity the last couple of years! According to the proposer, the above proof is probably the Erdős–Szekeres one too, though neither he nor the editor has seen the paper.

Also solved by H. L. ABBOTT, University of Alberta; DAVID E. MANES, State University of New York, Oneonta; and the proposer. Both Manes and the proposer gave the above proof. The problem appears as B31 *in the proposer's* Unsolved Problems in Number Theory (Second Edition), *Springer-Verlag*, 1994.

*   *   *   *   *

1916. [1994: 48] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*
Let $A_1A_2\ldots A_{n+1}$ be a regular simplex inscribed in a unit sphere in n-dimensional space, and let P be a point on the sphere. Prove that

$$\sum_{i=1}^{n+1} (A_iP)^4 = \frac{4(n+1)^2}{n}.$$

Solution by Murray S. Klamkin, University of Alberta.

More generally we show that if P is a point at a distance λ from the center of the unit sphere, then

$$\sum_{i=1}^{n+1} (A_i P)^4 = (n+1)\left((1+\lambda^2)^2 + \frac{4\lambda^2}{n}\right),$$

which reduces to the given result for $\lambda = 1$.

Let **A** denote a vector from the center of the sphere to point A, etc. Then

$$(A_i P)^2 = |\mathbf{A_i} - \mathbf{P}|^2 = |\mathbf{A_i}|^2 + |\mathbf{P}|^2 - 2\mathbf{A_i} \cdot \mathbf{P} = 1 + \lambda^2 - 2\mathbf{A_i} \cdot \mathbf{P},$$

so

$$(A_i P)^4 = (1+\lambda^2)^2 - 4(1+\lambda^2)\mathbf{A_i} \cdot \mathbf{P} + 4(\mathbf{A_i} \cdot \mathbf{P})^2,$$

and so [using $\sum_{i=1}^{n+1} \mathbf{A_i} = 0$]

$$\sum_{i=1}^{n+1} (A_i P)^4 = (n+1)(1+\lambda^2)^2 - 4(1+\lambda^2)\left(\sum_{i=1}^{n+1} \mathbf{A_i}\right) \cdot \mathbf{P} + 4\sum_{i=1}^{n+1}(\mathbf{A_i} \cdot \mathbf{P})^2$$

$$= (n+1)(1+\lambda^2)^2 + 4\sum_{i=1}^{n+1}(\mathbf{A_i} \cdot \mathbf{P})^2. \tag{1}$$

To evaluate the last sum, we first express **P** in the form $\sum x_i \mathbf{A_i}$, where the x_i are scalars. Then

$$\lambda^2 = \mathbf{P} \cdot \mathbf{P} = \sum_{i=1}^{n+1} x_i^2 \mathbf{A_i}^2 + 2\sum_{i<j} x_i x_j \mathbf{A_i} \cdot \mathbf{A_j} = \sum_{i=1}^{n+1} x_i^2 - \frac{2}{n}\sum_{i<j} x_i x_j$$

$$= \left(\sum_{i=1}^{n+1} x_i\right)^2 - 2\sum_{i<j} x_i x_j - \frac{2}{n}\sum_{i<j} x_i x_j = S_1^2 - \frac{2(n+1)}{n} T_2, \tag{2}$$

where

$$S_1 = \sum_{i=1}^{n+1} x_i \quad \text{and} \quad T_2 = \sum_{i<j} x_i x_j,$$

and we have used

$$\mathbf{A_i} \cdot \mathbf{A_j} = -\frac{1}{n} \quad \text{for} \quad i \ne j. \tag{3}$$

[*Editor's note.* For instance, the following proof is borrowed from Christopher Bradley's solution. Since the simplex is regular, all the $\mathbf{A_i} \cdot \mathbf{A_j}$ are equal, thus

$$0 = \left(\sum_{i=1}^{n+1} \mathbf{A_i}\right)^2 = \sum_{i=1}^{n+1} \mathbf{A_i}^2 + 2\sum_{i<j} \mathbf{A_i} \cdot \mathbf{A_j} = n + 1 + \frac{2(n+1)n}{2}\mathbf{A_i} \cdot \mathbf{A_j},$$

and (3) follows.] Also from (3),

$$\mathbf{A_i} \cdot \mathbf{P} = \mathbf{A_i} \cdot \sum_{j=1}^{n+1} x_j \mathbf{A_j} = x_i - \frac{1}{n}\left(\sum_{j=1}^{n+1} x_j - x_i\right) = \frac{(n+1)x_i - S_1}{n},$$

so

$$n^2(\mathbf{A_i} \cdot \mathbf{P})^2 = (n+1)^2 x_i^2 - 2(n+1)x_i S_1 + S_1^2$$

and thus

$$n^2 \sum_{i=1}^{n+1} (\mathbf{A_i} \cdot \mathbf{P})^2 = (n+1)^2(S_1^2 - 2T_2) - 2(n+1)S_1^2 + (n+1)S_1^2$$

$$= n(n+1)\left(S_1^2 - \frac{2(n+1)}{n}T_2\right).$$

Hence from (2),

$$\sum_{i=1}^{n+1} (\mathbf{A_i} \cdot \mathbf{P})^2 = \frac{(n+1)\lambda^2}{n}$$

and finally, from (1),

$$\sum_{i=1}^{n+1} (A_i P)^4 = (n+1)(1+\lambda^2)^2 + \frac{4(n+1)\lambda^2}{n} = (n+1)\left((1+\lambda^2)^2 + \frac{4\lambda^2}{n}\right).$$

Comment. This result also appears on page 208 of *Ptolemy's Legacy*, an unpublished manuscript of Stanley Rabinowitz. He gets the proof by imbedding the simplex in a cubic lattice of one higher dimension so that the coordinates of each vertex are all zero but one. He also gives related results including sums of sixth powers for regular polyhedra and polytopes.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; P. PENNING, Delft, The Netherlands; STANLEY RABINOWITZ, Westford, Massachusetts; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

The manuscript of Rabinowitz mentioned above is soon to be published in book form by MathPro Press (so says the author/publisher, anyway). No doubt it will be reviewed in Crux *when the time comes, so watch this space. Incidentally, Rabinowitz's solution to this problem consisted of an earlier draft of this manuscript, "only" 70 pages long, the solution to Crux 1916 appearing on page 66, together with the helpful comment "feel free to edit out one or two intermediate results if you find it a bit long for publication"!*

* * * * *

1917. [1994: 48] *Proposed by Joaquín Gómez Rey, I.B. Luis Buñuel, Alcorcón, Madrid, Spain.*

For any positive integer n, evaluate a_n/b_n where

$$a_n = \sum_{k=1}^{n} \tan^2 \frac{k\pi}{2n+1}, \qquad b_n = \prod_{k=1}^{n} \tan^2 \frac{k\pi}{2n+1}.$$

Solution by David Hankin, John Dewey High School, Brooklyn, New York.
Using De Moivre's theorem

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n,$$

one finds easily that

$$\sin n\theta = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1}\theta \sin^{2k+1}\theta,$$

so

$$\sin(2n+1)\theta = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \cos^{2n-2k}\theta \sin^{2k+1}\theta$$

$$= \tan\theta \cos^{2n+1}\theta \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \tan^{2k}\theta.$$

Thus

$$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \tan^{2k}\theta = 0 \qquad \text{for } \theta = \frac{j\pi}{2n+1}, \quad 1 \le j \le n,$$

so $\tan^2 \frac{j\pi}{2n+1}$, $1 \le j \le n$, are the roots of $\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1}x^k = 0$, and thus also of $\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k}x^{n-k} = 0$. Since a_n and b_n are the sum and product of the roots, respectively, we have

$$a_n = \binom{2n+1}{2} = n(2n+1) \quad \text{and} \quad b_n = \binom{2n+1}{2n} = 2n+1,$$

and so

$$\frac{a_n}{b_n} = n.$$

Remark. The *USSR Olympiad Problem Book* by Shklarsky, Chentzov, and Yaglom contains several similar problems, in particular, # 231a on page 53 in the English translation of the third Russian edition. The solution above is essentially theirs.

Also solved by FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I. B. Leopoldo Cano, Valladolid, Spain;

CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; TIM CROSS, Wolverley High School, Kidderminster, U. K.; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; PETER HURTHIG, Columbia College, Burnaby, B.C.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY KLAMKIN, University of Alberta; ALEX LEE, student, Choate Rosemary Hall, Wallingford, Connecticut; JOSEPH LING, University of Calgary; DAVID E. MANES, State University of New York, Oneonta; CHRIS WILDHAGEN, Rotterdam, The Netherlands; PAUL YIU, Florida Atlantic University, Boca Raton; and the proposer. One incorrect solution, and one correct answer without proof, were also sent in.

About half of the solvers use approaches which are either very similar to the one given above or variations of it. The USSR Olympiad Problem Book is also quoted by Heuver and Manes. Using known results in the book Integrals and Series (Elementary Functions) [in Russian] by A. P. Prudnikov et al, Janous obtains the following similar results: if $r(f, N)$ stands for the fraction a_n/b_n of the problem with tan replaced by the function f and the denominators in a_n and b_n replaced by N, then

$$r(\sin x, 2n + 1) = 4^{n-1}; \qquad r(\cos x, 2n + 1) = (2n - 1)4^{n-1};$$

$$r(\sin x, 2n) = r(\cos x, 2n) = \left(\frac{n-1}{n}\right)2^{2n-3};$$

$$r(\cot x, 2n + 1) = \frac{n(4n^2 - 1)}{3}; \qquad and$$

$$r(\tan x, 2n) = r(\cot x, 2n) = \frac{(n - 1)(2n - 1)}{3}.$$

*   *   *   *   *

1918. [1994: 48] *Proposed by D. J. Smeenk, Zaltbommel, The Netherlands.*

ABC is a triangle with circumcentre O and incentre I, and K, L, M are the midpoints of BC, CA, AB respectively. Let E and F be the feet of the altitudes from B and C respectively.

(a) If $\overline{OK}^2 = \overline{OL}^2 + \overline{OM}^2$, show that E, F and O are collinear, and determine all possible values of $\angle BAC$.

(b) If instead $\overline{OK} = \overline{OL} + \overline{OM}$, show that E, F and I are collinear, and determine all possible values of $\angle BAC$.

Solution by Toshio Seimiya, Kawasaki, Japan.

Because $\overline{OK} > \overline{OL}, \overline{OM}$ in either case, we have $\angle A < 90°$. Letting R be the circumradius of $\triangle ABC$, we get

$$\overline{OK} = R \cos A, \quad \overline{OL} = R \cos B, \quad \overline{OM} = R \cos C. \qquad (1)$$

[This requires the use of directed distances when B or C is obtuse.— *Ed.*]

(a) As $\overline{OK}^2 = \overline{OL}^2 + \overline{OM}^2$, we have

$$\cos^2 A = \cos^2 B + \cos^2 C. \tag{2}$$

The trilinear coordinates of E, F and O are

$$E(\cos C, 0, \cos A), \quad F(\cos B, \cos A, 0), \quad O(\cos A, \cos B, \cos C).$$

By virtue of (2) we get

$$\begin{vmatrix} \cos C & 0 & \cos A \\ \cos B & \cos A & 0 \\ \cos A & \cos B & \cos C \end{vmatrix} = \cos A(\cos^2 C + \cos^2 B - \cos^2 A) = 0.$$

Hence E, F and O are collinear.

From (2) we have

$$\cos^2 A = \frac{1}{2}(1 + \cos 2B + 1 + \cos 2C) = 1 + \cos(B + C)\cos(B - C)$$
$$= 1 - \cos A \cos(B - C) \geq 1 - \cos A,$$

because $\cos A > 0$. Therefore $\cos^2 A + \cos A - 1 \geq 0$, from which we have

$$\cos A \geq \frac{\sqrt{5} - 1}{2}.$$

Hence we get

$$0 < A \leq \cos^{-1}\left(\frac{\sqrt{5} - 1}{2}\right) \approx 51.82°.$$

Under this condition there exists a triangle satisfying $\overline{OK}^2 = \overline{OL}^2 + \overline{OM}^2$.

[*Editor's note.* As other solvers point out, (2) holds for the degenerate triangle $A = 0°$, $B = 45°$, $C = 135°$, and also for the triangle

$$A = \cos^{-1}\left(\frac{\sqrt{5} - 1}{2}\right), \quad B = C,$$

so both bounds for A are obtained (or approached). It's not hard to show that any value for A in between can be attained too.]

(b) From (1), $\overline{OK} = \overline{OL} + \overline{OM}$ implies

$$\cos A = \cos B + \cos C. \tag{3}$$

The trilinear coordinates for I are $I(1, 1, 1)$. By virtue of (3) we get

$$\begin{vmatrix} \cos C & 0 & \cos A \\ \cos B & \cos A & 0 \\ 1 & 1 & 1 \end{vmatrix} = \cos A(\cos C + \cos B - \cos A) = 0,$$

hence E, F and I are collinear.

From (3) we have

$$\cos A = 2 \cos\left(\frac{B+C}{2}\right) \cos\left(\frac{B-C}{2}\right) = 2 \sin\frac{A}{2} \cos\left(\frac{B-C}{2}\right) \le 2 \sin\frac{A}{2} \, ,$$

thus

$$1 - 2 \sin^2\frac{A}{2} \le 2 \sin\frac{A}{2} \, .$$

Therefore

$$\sin\frac{A}{2} \ge \frac{\sqrt{3}-1}{2} \, ,$$

and hence we have

$$A \ge 2 \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) \approx 42.94° .$$

Editor's note. This bound is again attained for the isosceles triangle. Seimiya now just gives 90° as an upper bound for A. All other solvers give $A \le 60°$, though only the proposer supplies a proof visible to the editor. Here is a proof picking up where Seimiya leaves off. From $B - C \le B \le 180° - A$ (assuming $B \ge C$) we get

$$\cos\frac{B-C}{2} \ge \cos\left(90° - \frac{A}{2}\right) = \sin\frac{A}{2} \, .$$

Thus

$$1 - 2 \sin^2\frac{A}{2} = \cos A = 2 \sin\frac{A}{2} \cos\frac{B-C}{2} \ge 2 \sin^2\frac{A}{2} \, ,$$

whence $\sin^2(A/2) \le 1/4$, so $A/2 \le 30°$ or $A \le 60°$ as claimed. Solvers give the degenerate triangle $A = 60°$, $B = 120°$, $C = 0°$ to show this bound is approached.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; P. PENNING, Delft, The Netherlands; and the proposer. One incorrect solution was sent in.

Francisco Bellot Rosado, I. B. Emilio Ferrari, Valladolid, Spain, notes that problem E161 of the American Math. Monthly *(solution on p. 47 of the 1936 volume) asks for a construction of the triangles of part* (a). *He also finds a problem of Victor Thébault, on p. 130 of the 1923 volume of the Belgian journal* Mathesis, *which gives several properties of the triangles of part* (b), *some of which are:*

(i) *the circumradius is equal to the radius of the excircle opposite A;*

(ii) *O is collinear with the feet of the internal bisectors of angles B and C;*

(iii) *the foot A' of the altitude from A lies on the line IO;*

(iv) *the circle with AA' as diameter passes through the nine-point centre of ABC.*

     *   *   *   *   *