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Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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MATHEMATTIC

No. 3

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

*To facilitate their consideration, solutions should be received by **June 15, 2019**.*



MA11. Let $f(x) = 375x^5 - 131x^4 + 15x^2 - 435x - 2$. Find the remainder when $f(97)$ is divided by 11.

MA12. Ten straight lines are drawn on a two-dimensional plane. Given that three of these lines are parallel to one another, what is the maximum possible number of intersection points formed by the lines?

MA13. How many ways can the letters of the word LETTERKENNY be arranged in a row if the R must stay in the middle position and the letters L,R,K and Y must remain in their current order LRKY? (An example of an arrangement that meets the requirements is ELTTERENKYN.)

MA14. In $\triangle ABC$, the side AB has length 20 and $\angle ABC = 90^\circ$. If the lengths of the other sides must be positive integers, how many such triangles are possible?

MA15. Prove that $43^{43} - 17^{17}$ is divisible by 10. (Do not use Fermat's Little Theorem.)

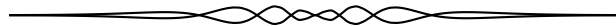
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Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2019**.*

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.



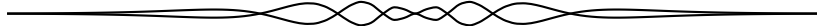
MA11. Soit $f(x) = 375x^5 - 131x^4 + 15x^2 - 435x - 2$. Déterminer le reste lorsqu'on divise $f(97)$ par 11.

MA12. Dix lignes droites sont tracées dans le plan. Sachant que trois de ces lignes sont parallèles les unes aux autres, déterminer le nombre maximum de points d'intersection de ces lignes.

MA13. De combien de manières les chiffres présents dans le nombre 15665235774 peuvent-ils être permutés de façon à ce que le chiffre 2 reste au centre et que les chiffres 1, 2, 3 et 4 restent dans l'ordre initial 1234 ? (Un exemple d'un réarrangement respectant ces contraintes serait 51665257347.)

MA14. Le côté AB du $\triangle ABC$ est de longueur 20; aussi, $\angle ABC = 90^\circ$. Si les longueurs des deux autres côtés doivent être des entiers positifs, déterminer le nombre de triangles possibles.

MA15. Sans utiliser le petit théorème de Fermat, démontrer que $43^{43} - 17^{17}$ est divisible par 10.



CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(5), p. 185–186; and 44(6), p. 234–236.

CC321. Six boxes are numbered 1, 2, 3, 4, 5 and 6. Suppose that there are N balls distributed among these six boxes. Find the least N for which it is guaranteed that for at least one k , box number k contains at least k^2 balls.

Originally Problem 4 from the 2015 Purple Comet! Math Meet.

We received 8 solutions. We present the solution by Ivko Dimitrić, slightly edited.

The least value of N is one more than the largest number M of balls that can be arranged so that for each k , the box number k contains less than k^2 balls. Since the largest value of such number M is sought, the k -th box should contain $k^2 - 1$ balls, thus

$$M = \sum_{k=1}^6 (k^2 - 1) = 0 + 3 + 8 + 15 + 24 + 35 = 85.$$

Therefore, $N = 86$ is the smallest number that will guarantee that box number k contains at least k^2 balls for at least one value of k .

CC322. Suppose that the vertices of a polygon all lie on a rectangular lattice of points where adjacent points on the lattice are at distance 1 apart. Then the area of the polygon can be found using Pick's Formula: $I + \frac{B}{2} - 1$, where I is the number of lattice points inside the polygon, and B is the number of lattice points on the boundary of the polygon. Pat applied Pick's Formula to find the area of a polygon but mistakenly interchanged the values of I and B . As a result, Pat's calculation of the area was too small by 35. Using the correct values for I and B , the ratio $n = \frac{I}{B}$ is an integer. Find the greatest possible value of n . (*Ed.: For more information on Pick's formula, take a look at the article Two Famous Formulas (Part I), **CruX** 43 (2), p. 61–66.*)

Originally Problem 11 from the 2015 Purple Comet! Math Meet.

We received 5 submissions of which 4 were correct and complete. We present the solution by Ivko Dimitrić.

The stated condition gives

$$I + \frac{B}{2} - 1 = B + \frac{I}{2} - 1 + 35,$$

which simplifies to $I - B = 70$. Then $n = \frac{I}{B} = \frac{70}{B} + 1$ is an integer, so $\frac{70}{B}$ must

also be an integer, i. e. B is a positive integer divisor of $70 = 2 \cdot 5 \cdot 7$ and since $B \geq 3$ (a lattice polygon has at least 3 vertices) we have $B \in \{5, 7, 10, 14, 35, 70\}$.

The largest value of n is obtained for the smallest possible value of B . When $B = 5$, then $n = \frac{70}{5} + 1 = 15$, implying $I = nB = 75$, and indeed there is a polygon (quadrilateral) that has exactly $B = 5$ lattice points on the boundary and $I = 75$ points in the interior. Such a polygon can be constructed as follows. Choose an arbitrary lattice point as the origin $(0, 0)$ of a rectangular coordinate system with perpendicular x - and y -axis running along adjacent sides of a unit square cell of the lattice, each containing an infinite sequence of lattice points that are 1 unit apart. The quadrilateral with vertices $(0, -1)$, $(0, 1)$, $(1, 1)$ and $(76, 0)$ contains five points on the boundary (the four vertices and the origin) and 75 points $(k, 0)$, $k = 1, 2, \dots, 75$ along the x -axis in the interior, which are the only points of the lattice inside the polygon. Namely, the side joining vertices $(0, -1)$ and $(76, 0)$ with positive slope makes it impossible for any lattice point below the x -axis to belong to the interior or lie on that side, other than the vertex $(0, -1)$, and the side joining $(1, 1)$ and $(76, 0)$ with negative slope makes it impossible for any lattice point above the x -axis to be in the interior or lie on that side, other than the vertex $(1, 1)$. Thus, all the interior points are those 75 points on the x -axis and the only boundary lattice points are the five mentioned. For such a polygon, the maximum value $n = 15$ is attained.

CC323. Evaluate

$$\frac{\log_{10}(20^2) \cdot \log_{20}(30^2) \cdot \log_{30}(40^2) \cdots \log_{990}(1000^2)}{\log_{10}(11^2) \cdot \log_{11}(12^2) \cdot \log_{12}(13^2) \cdots \log_{99}(100^2)}$$

Originally Problem 14 from the 2015 Purple Comet! Math Meet.

We received 13 submissions of which 11 were correct and complete. We present the solution by Tyler McGilvry-James.

Expand the given expression by the logarithmic power rule:

$$\begin{aligned} & \frac{\log_{10}(20^2) \cdot \log_{20}(30^2) \cdot \log_{30}(40^2) \cdots \log_{990}(1000^2)}{\log_{10}(11^2) \cdot \log_{11}(12^2) \cdot \log_{12}(13^2) \cdots \log_{99}(100^2)} \\ &= \frac{2^{99} \log_{10}(20) \cdot \log_{20}(30) \cdot \log_{30}(40) \cdots \log_{990}(1000)}{2^{90} \log_{10}(11) \cdot \log_{11}(12) \cdot \log_{12}(13) \cdots \log_{99}(100)} \\ &= 2^9 \frac{\log_{10}(20) \cdot \log_{20}(30) \cdot \log_{30}(40) \cdots \log_{990}(1000)}{\log_{10}(11) \cdot \log_{11}(12) \cdot \log_{12}(13) \cdots \log_{99}(100)}. \end{aligned}$$

Consider the two quotients. We use the change of base formula:

$$\begin{aligned} & \log_{10}(20) \cdot \log_{20}(30) \cdot \log_{30}(40) \cdots \log_{990}(1000) \\ &= \frac{\log 20}{\log 10} \frac{\log 30}{\log 20} \cdots \frac{\log 990}{\log 980} \frac{\log 1000}{\log 990} = \frac{\log 1000}{\log 10} = 3 \end{aligned}$$

and

$$\begin{aligned} & \log_{10}(11) \cdot \log_{11}(12) \cdot \log_{12}(13) \cdots \log_{99}(100) \\ &= \frac{\log 11 \log 12}{\log 10 \log 11} \cdots \frac{\log 99 \log 100}{\log 98 \log 99} = \frac{\log 100}{\log 10} = 2. \end{aligned}$$

Thus the original expression can be simplified to $\frac{(3)(2^9)}{2} = 768$. So we see that

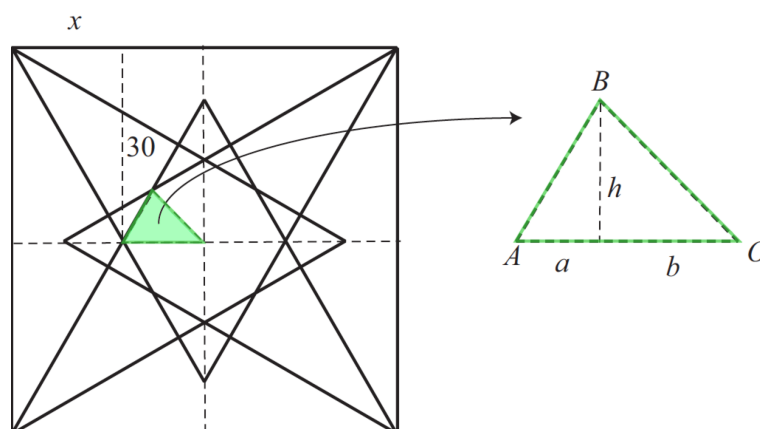
$$\frac{\log_{10}(20^2) \cdot \log_{20}(30^2) \cdot \log_{30}(40^2) \cdots \log_{990}(1000^2)}{\log_{10}(11^2) \cdot \log_{11}(12^2) \cdot \log_{12}(13^2) \cdot \log_{99}(100^2)} = 768.$$

CC324. On the inside of a square with side length 60, construct four congruent isosceles triangles each with base 60 and height 50, and each having one side coinciding with a different side of the square. Find the area of the octagonal region common to the interiors of all four triangles.

Originally Problem 15 from the 2015 Purple Comet! Math Meet.

We received 5 submissions of which four were correct and complete. We present the solution by Ángel Plaza, modified by the editor.

Below we depict the said square and four congruent isosceles triangles while constructing the triangle ABO .



The area of the octagonal region common to the interiors of all triangles is eight times the area of the triangle ABO . By the property of similar triangles $\frac{30}{x} = \frac{50}{30}$, thus $x = 18$. It follows that $|AO| = \frac{60}{2} - x = 12$.

Let us consider $|AO| = a + b$, where a and b are respectively the projections of sides AB and BO over AO . Define C as the point of $h \perp AO$. By symmetry $\angle COB = \frac{\pi}{4}$. This implies that $\angle COB$ is an isosceles triangle, thus $h = b$. By the

property of similar triangles, we have $\frac{h}{a} = \frac{50}{30}$, which implies $a = \frac{3h}{5}$. We observe that

$$12 = a + b = \frac{3h}{5} + h \Rightarrow h = 7.5.$$

Therefore $\text{Area}(ABO) = \frac{12 \cdot 7.5}{2} = 45$. In conclusion, the area of the octagonal region is $8 \cdot 45 = 360$.

CC325. Seven people of seven different ages are attending a meeting. The seven people leave the meeting one at a time in random order. Given that the youngest person leaves the meeting sometime before the oldest person leaves the meeting, the probability that the third, fourth, and fifth people to leave the meeting do so in order of their ages (youngest to oldest) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Originally Problem 26 from the 2015 Purple Comet! Math Meet.

We received 3 submissions of which 1 was correct and complete. We present the solution by Ivko Dimitrić.

Let $\{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$ be the set of persons, appearing in order in which they left the meeting, so p_i was the i th person who left the meeting on a time-line. Let $S = \{p_3, p_4, p_5\}$ be the subset composed of persons who left third, fourth and fifth and let y and o denote the youngest and the oldest person, respectively. Either y or o or both could belong to S and it is also possible that S contains neither y nor o . Let A denote the event that the third, fourth, and fifth persons left the meeting in age-increasing order and let B denote the event that y leaves the meeting before o does. Then we are asked to find the conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

The number of favorable outcomes for B is $\binom{7}{2} \cdot 5! = 21 \cdot 5!$, since (y, o) couple can be chosen in $\binom{7}{2}$ ways among 7 time slots (numbered 1 through 7) for conference leaving and the remaining 5 time slots can be taken arbitrarily by the remaining 5 people in $5!$ ways, for which the age order is irrelevant. To determine the number of favourable outcomes for $A \cap B$ we do the counting depending on whether y and o belong to S .

(1) If $y \in S$ and $o \in S$, then necessarily $p_3 = y$ and $p_5 = o$, since p_3, p_4, p_5 would have to conform to age-ordering where p_4 is older than p_3 but younger than p_5 . With $p_3 = y, p_5 = o$ fixed, the remaining five people can take arbitrarily five remaining time slots in $5!$ ways, since whoever happens to be p_4 will be of age between those of p_3 and p_5 .

(2) If $y \in S$ but $o \notin S$, then $p_3 = y$ per force, since otherwise person p_3 would be older than $y \in \{p_4, p_5\}$, violating age ordering within S . Then $o \in \{p_6, p_7\}$ (2 possibilities) and the positions p_4, p_5 are occupied by an age-ordered pair among

the remaining 5 people in $\binom{5}{2}$ ways, whereas the remaining 3 slots can be taken by the remaining 3 people in $3!$ ways without regard to their ages. This makes for $\binom{5}{2} \cdot 2 \cdot 3! = 5!$ ways.

(3) If $y \notin S$ and $o \in S$, this case is dual to the previous one, where now $p_5 = o$ and there are two possibilities for $y \in \{p_1, p_2\}$. The places for p_3, p_4 can be taken by any age-ordered couple of people in $\binom{5}{2}$ ways and remaining three time-slots are taken by the remaining three people in $3!$ ways. This accounts again for $\binom{5}{2} \cdot 2 \cdot 3! = 5!$ favorable outcomes.

(4) If $y \notin S$ and $o \notin S$, then there are $\binom{4}{2} = 6$ ways of arranging y and o among p_1, p_2, p_6 and p_7 . The places p_3, p_4, p_5 can be taken by any age-ordered triple among the remaining 5 people in $\binom{5}{3}$ ways, whereas the two remaining slots can be then taken arbitrarily in $2!$ ways by the remaining two people. So in this case we have $6 \cdot \binom{5}{3} \cdot 2! = 5!$ ways as well. Thus, the number of favourable outcomes for $A \cap B$ is $4 \cdot 5!$ and the required conditional probability is

$$P(A|B) = \frac{4 \cdot 5!/7!}{21 \cdot 5!/7!} = \frac{4}{21} = \frac{m}{n},$$

so that $m + n = 25$.

CC326. For positive integer n , let a_n be the integer consisting of n digits of 9 followed by the digits 488. For example, $a_3 = 999488$ and $a_7 = 9999999488$. For each given n , determine the largest integer $f(n)$ such that $2^{f(n)}$ divides a_n .

Originally Problem 19 from the Middle School 2013 Purple Comet! Math Meet.

We received eight submissions of which seven were correct and complete. We present the solution of Ivko Dimitric, modified by the editor.

The difference between a_n and the nearest power of ten, 10^{n+3} , is $512 = 2^9$. Algebraically,

$$a_n = 10^{n+3} - 2^9 = 2^{n+3}5^{n+3} - 2^9.$$

We consider the following three cases:

(i) For $1 \leq n < 6$, we have that

$$a_n = 2^{n+3}(5^{n+3} - 2^{6-n}),$$

with the number in the parentheses being odd. Thus, $f(n) = n + 3$.

(ii) If $6 < n$, then

$$a_n = 2^9(2^{n-6}5^{n+3} - 1),$$

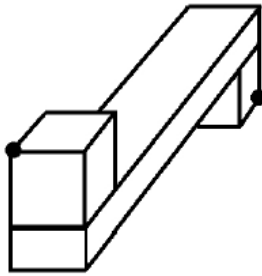
with the number in parentheses being odd. Thus we have that $f(n) = 9$.

(iii) If $n = 6$, then by the binomial expansion theorem

$$\begin{aligned}
 a_n &= 2^9(5^9 - 1) \\
 &= 2^9[(4 + 1)^9 - 1] \\
 &= 2^9 \left[4^9 + 9 \cdot 4^8 + \binom{9}{2} 4^7 + \cdots + \binom{9}{7} 4^2 + 9 \cdot 4 \right] \\
 &= 2^{11} \left[4^8 + 9 \cdot 4^7 + \binom{9}{2} 4^6 + \cdots + \binom{9}{7} 4 + 9 \right].
 \end{aligned}$$

The number inside the brackets is odd. Thus we have that $f(6) = 11$.

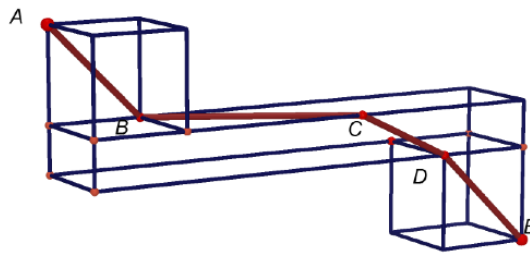
CC327. The diagram below shows a $1 \times 2 \times 10$ duct with $2 \times 2 \times 2$ cubes attached to each end. The resulting object is empty, but the entire surface is solid sheet metal. A spider walks along the inside of the duct between the two marked corners. There are positive integers m and n so that the shortest path the spider could take has length $\sqrt{m} + \sqrt{n}$. Find $m + n$.



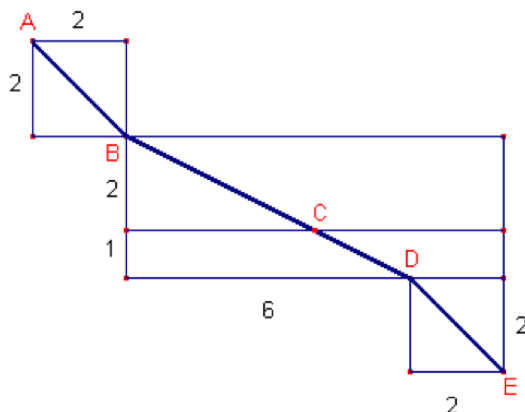
Originally Problem 20 from the 2013 Purple Comet! Math Meet.

We received one submission which was correct and complete. We present the solution of Ricard Peiró.

The shortest path is formed by the polygonal line $ABCDE$, shown in the below figure:



Laid flat, the path is as follows:



The lengths of segments of the path AB , BD , and DE are $2\sqrt{2}$, $3\sqrt{5}$, and $2\sqrt{2}$, respectively. The total length of said path is

$$AB + BD + DE = 4\sqrt{2} + 3\sqrt{5} = \sqrt{32} + \sqrt{45}.$$

Thus $m = 32$, $n = 45$, and $m + n = 77$.

CC328. You have many identical cube-shaped wooden blocks. You have four colours of paint to use, and you paint each face of each block a solid colour so that each block has at least one face painted with each of the four colours. Find the number of distinguishable ways you could paint the blocks. (Two blocks are distinguishable if you cannot rotate one block so that it looks identical to the other block.)

Originally Problem 18 from the 2015 Purple Comet! Math Meet.

We received one solution by C.R. Pranesachar, which we present below in slightly edited form.

We use Pólya's enumeration theorem. For the group of 24 rotations of the cube, the cycle index (for faces) is given by

$$Z(t_1, t_2, t_3, t_4) = \frac{1}{24}(t_1^6 + 6t_1^2t_4 + 3t_1^2t_2^2 + 8t_3^2 + 6t_2^3).$$

Let x, y, z, w be the variables for the number of faces of the cube that are painted with the four colours. We let $\sum x^k$ denote $x^k + y^k + z^k + w^k$ for $1 \leq k \leq 4$. Then the colour-counting polynomial is given by

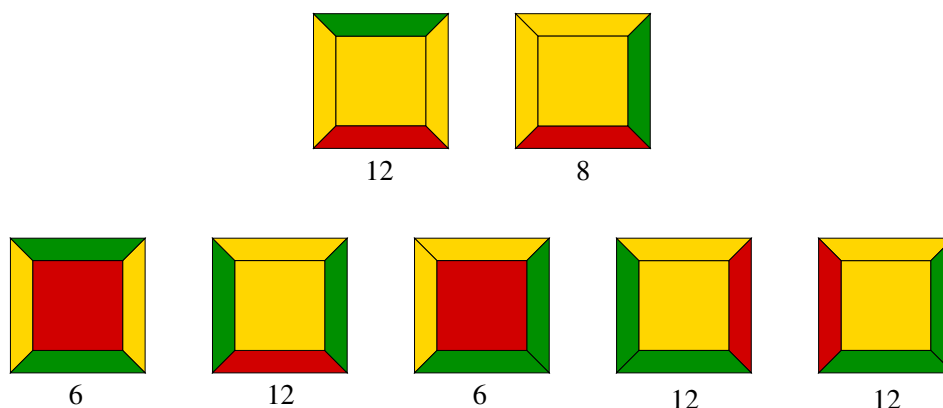
$$\begin{aligned} g(x, y, z, w) &= Z(\sum x, \sum x^2, \sum x^3, \sum x^4) \\ &= \frac{1}{24}(\sum x)^6 + 6(\sum x)^2(\sum x^4) + 3(\sum x)^2(\sum x^2)^2 + 8(\sum x^3)^2 + 6(\sum x^2)^3. \end{aligned}$$

Since all the four colours are to be used, we need the sum of the coefficients of the terms containing all of x, y, z, w in $g(x, y, z, w)$. This occurs only in $(\sum x)^6$ and $3(\sum x)^2(\sum x^2)^2$. Accordingly the sum of the coefficients is equal to

$$\frac{1}{24} \left(\frac{6!}{3!1!1!1!} \cdot 4 + \frac{6!}{2!2!1!1!} \cdot 6 + 3 \cdot 2 \cdot 2 \cdot 6 \right) = \frac{1}{24}(480 + 1080 + 72) = 68,$$

which is the desired answer.

Editor's comment. The possible colourings of the cube can be seen in the figure. The invisible face is coloured with the fourth colour. The number under each configuration indicates how many distinct cubes are obtained by permuting the four colours of each configuration.



CC329. Let a, b, c and d be real numbers such that

$$a^2 + 3b^2 + \frac{c^2 + 3d^2}{2} = a + b + c + d - 1.$$

Find $1000a + 100b + 10c + d$.

Originally Problem 19 from the 2015 Purple Comet! Math Meet.

We received 8 correct solutions. We present the solution by Šefket Arslanagić.

We have

$$\begin{aligned} a^2 + 3b^2 + \frac{c^2 + 3d^2}{2} &= a + b + c + d - 1 \iff \\ a^2 + 3b^2 + \frac{1}{2}c^2 + \frac{3}{2}d^2 - a - b - c - d + 1 &= 0 \iff \\ \left(a - \frac{1}{2}\right)^2 + 3\left(b - \frac{1}{6}\right)^2 + \frac{1}{2}(c-1)^2 + \frac{3}{2}\left(d - \frac{1}{3}\right)^2 &= 0. \end{aligned}$$

From here, we conclude that $a = \frac{1}{2}, b = \frac{1}{6}, c = 1$ and $d = \frac{1}{3}$. It follows that

$$1000a + 100b + 10c + d = 500 + \frac{50}{3} + 10 + \frac{1}{3} = 510 + 17 = 527.$$

CC330. Six children stand in a line outside their classroom. When they enter the classroom, they sit in a circle in random order. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that no two children who stood next to each other in the line end up sitting next to each other in the circle. Find $m + n$.

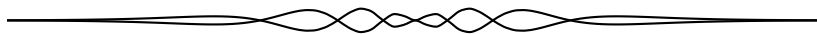
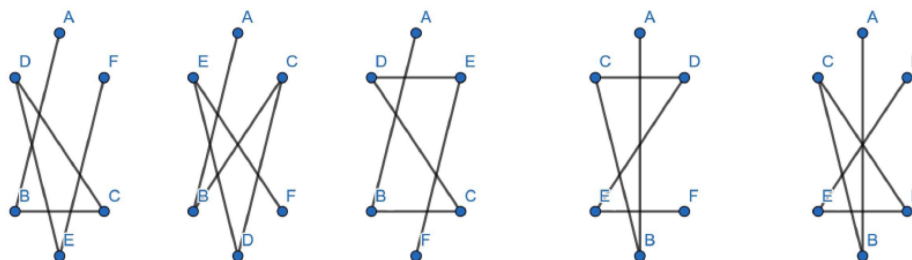
Originally Problem 18 from the Middle School 2013 Purple Comet! Math Meet.

We received 2 correct solutions. Solution by C. R. Pranesachar.

Let the children standing in a line be named A, B, C, D, E, F , in that order, while standing outside the classroom. When they are seated around a circle, in order to satisfy the given non-adjacency condition, we need to take hamiltonian paths along the diagonals of the hexagon and go through all of the vertices, naming them from A to F respectively. We get the 5 diagrams below and their reflections in the vertical line through A . Thus there are $5 \times 2 = 10$ paths only. Since rotation does not change adjacency, we infer that the probability according to the given condition is

$$\frac{10 \times 6}{6!} = \frac{60}{720} = \frac{1}{12}.$$

Thus $m = 1$ and $n = 12$, giving $m + n = 13$.



PROBLEM SOLVING VIGNETTES

No.3

Donald Rideout
Arithmetic of Remainders

Divisibility is a fundamental concept of number theory and is one of the main ideas that sets it apart from other branches of mathematics. The main approach to divisibility questions is through the arithmetic of remainders, or the theory of congruences as it is now commonly known. The concept was first introduced by Carl Friedrich Gauss (1777-1855) in his *Disquisitiones Arithmeticae*; this monumental work, which appeared in 1801 when Gauss was 24 years old, laid the foundations of modern number theory.

We say that a is *congruent to b modulo m* , and we write

$$a \equiv b \pmod{m},$$

if m divides the difference $a - b$; that is, provided $a - b = km$ or $a = b + km$ for some integer k . If $m \nmid (a - b)$, then we say that a is *incongruent to b modulo m* and in this case we write $a \not\equiv b \pmod{m}$.

For example,

$$\begin{aligned} 3 &\equiv 24 \pmod{7} \quad \text{since } 7 \mid (3 - 24), \\ 19 &\equiv -2 \pmod{7} \quad \text{since } 7 \mid (19 + 2), \\ -15 &\equiv -64 \pmod{7} \quad \text{since } 7 \mid (-15 + 64). \end{aligned}$$

The number m is called the *modulus* of the congruence. Congruences with the same modulus behave in many ways like ordinary equations. In particular, if

$$a \equiv b \pmod{m} \text{ and } c \equiv d \pmod{m},$$

then

$$a \pm c \equiv b \pm d \pmod{m} \text{ and } ac \equiv bd \pmod{m}.$$

A warning is in order here. It is not always possible to divide congruences. If $ac \equiv bc \pmod{m}$, it need not be true that $a \equiv b \pmod{m}$. For example, we have $15 \cdot 2 \equiv 20 \cdot 2 \pmod{10}$, but $15 \not\equiv 20 \pmod{10}$. Even more distressing is that we can have $ab \equiv 0 \pmod{m}$ with $a \not\equiv 0 \pmod{m}$ and $b \not\equiv 0 \pmod{m}$. For example, $6 \cdot 4 \equiv 0 \pmod{12}$, while clearly $6 \not\equiv 0 \pmod{12}$ and $4 \not\equiv 0 \pmod{12}$. However, it is permissible to cancel c from the congruence $ac \equiv bc \pmod{m}$ provided that c and m do not have common factors, that is $\gcd(c, m) = 1$.

Let a be an integer. For any positive integer m , by the division algorithm, we have

$$a = mq + r \text{ where } 0 \leq r \leq m - 1,$$

and clearly $a \equiv r \pmod{m}$. The number r is called the *least positive residue* modulo m . Hence, every a is congruent modulo m to one and only one of the integers in the set $\{0, 1, 2, \dots, m - 1\}$, namely the (unique) remainder when divided by m . (Hence the justification of Gauss' phrase *arithmetic of remainders*.) It should be clear now that $a \equiv b \pmod{m}$ if and only if a and b have the same remainders when divided by m . We say that a and b are in the same *equivalence class* modulo m if they have the same remainder. We can think of \equiv as behaving almost exactly like $=$ if we do not make a fuss over the difference between numbers in a particular equivalence class. Hence modulo 10 we see very little difference, so to speak, between 2 and 12 and 202 and -3008 .

We will now see how congruences can be used to solve problems that otherwise might be cumbersome to solve. First note that we can make repeated use of the result that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ imply $a \pm c \equiv b \pm d \pmod{m}$ and $ac \equiv bd \pmod{m}$. For example, if $a \equiv b \pmod{m}$, then $a^n \equiv b^n \pmod{m}$. Hence, for example,

$$\begin{aligned} 10^{17} &\equiv 1^{17} \equiv 1 \pmod{9}, \\ 10^{17} &\equiv (-1)^{17} \equiv -1 \equiv 10 \pmod{11}. \end{aligned}$$

Note that 10^{17} is quite a large number, but we found the remainders quite effortlessly! We quote the limerick by Martin Gardner about the modulus 10:

*There was a young fellow named Ben
Who could only count modulo ten.
He said, "When I go
Past my last little toe,
I shall have to start over again."*

Problem 1 Prove that a number is divisible by 3 if and only if the sum of its digits is divisible by 3, and that an integer is divisible by 9 if and only if the sum of its digits is divisible by 9.

Solution: We prove the rule for divisibility by 9. Let $N = \sum_{k=0}^m a_k 10^k$, where

$0 \leq a_k \leq 9$ and $a_m \neq 0$. Clearly $N \equiv \sum_{k=0}^m a_k \pmod{9}$ since $10^k \equiv 1^k \equiv 1 \pmod{9}$.

Hence $N \equiv 0 \pmod{9}$ if and only if $\sum_{k=0}^m a_k \equiv 0 \pmod{9}$. □

Note that we are instinctively using the following rules for congruences which really need proof: for any modulus m , $a \equiv b$ implies $b \equiv a$, and $a \equiv b$, $b \equiv c$, imply $a \equiv c$.

Problem 2 Given the number 2492, double the units digit and subtract it from the number formed by the other digits. We get $249 - 2 \times 2 = 245$. Repeating this algorithm we get $24 - 2 \times 5 = 14$. Since 14 is clearly divisible by 7, the original number 2492 must be divisible by 7. Prove this rule for checking divisibility by 7.

Solution: Let $N = \sum_{k=0}^m a_k 10^k$ where $0 \leq a_k \leq 9$ and $a_m \neq 0$. Then

$$\begin{aligned} N &= 10 \left(\sum_{k=1}^m a_k 10^{k-1} \right) + a_0 \\ &\equiv 10 \left(\sum_{k=1}^m a_k 10^{k-1} \right) - 20a_0 \pmod{7} \\ &\equiv 10 \left(\sum_{k=1}^m a_k 10^{k-1} - 2a_0 \right) \pmod{7}. \end{aligned}$$

Hence $N \equiv 0 \pmod{7}$ if and only if

$$\sum_{k=1}^m a_k 10^{k-1} - 2a_0 \equiv 0 \pmod{7}$$

since $(10, 7) = 1$. □

Problem 3 Prove that every odd integer other than a multiple of 5 has some multiple that is a string of 1's (called a repunit).

Solution: We will leave the general proof to the reader. We will prove that 7 has a multiple of the form $1111 \dots$. The first 7 + 1 repunit numbers are

$$1, 11, 111, 1111, 11111, 111111, 1111111, 11111111.$$

The residues (remainders) of these numbers modulo 7 are 1, 4, 6, 5, 2, 0, 1, and 4. Since there are eight numbers, we can apply the pigeon-hole principle: the pigeon-holes are labelled with the seven distinct residues modulo 7, and the pigeons are labelled with the 8 repunit residues; that is, we have one more pigeon than pigeon-holes, so two pigeons must share the same hole. So by the pigeon-hole principle, we must have at least two numbers with the same residue (mod 7). In this instance there are two such pairs. The smallest pair is 1 and 1111111. The difference is 1111110, which must be a multiple of 7. Since $7 \nmid 10$, we can divide by 10. Then $111111 = 7 \times 15873$ is a multiple of 7. □

Problem 4 If a and b are odd integers, prove that $a^2 + b^2$ is never a square.

Solution: For any integer c ,

$$c^2 \equiv 0^2, 1^2, 2^2 \text{ or } 3^2 \pmod{4}.$$

That is, $c^2 \equiv 0$ or $1 \pmod{4}$. The odd squares can only be congruent to 1 modulo 4. Hence $a^2 + b^2 \equiv 1 + 1 \equiv 2 \pmod{4}$. But 2 is not a square modulo 4. □

Problem 5 Prove that $a^2 - 11b^2 = 13$ has no integer solutions.

Solution: Modulo 11 we have for any solution a and b that $a^2 \equiv 13 \equiv 2 \pmod{11}$. But the squares modulo 11 are 0, 1, 4, 9, 5, and 3. The number 2 is not in this list! \square

These so-called *Pell* equations have infinitely many solutions in many cases. For example, the equation $a^2 - 1141b^2 = 1$ has infinitely many positive integer solutions, the smallest one being $a = 1,036,782,394,157,223,963,237,125,215$ and $b = 30,693,385,322,765,657,197,397,208$ (26 digits).

Problem 6 Prove that $30|ab(a^4 - b^4)$ for every pair of integers a and b .

Solution: The most efficient way to solve this problem seems to be by using congruences modulo 2, 3, and 5. Consider each number in turn. For example, for the modulus 5, either $5|a$ or $5|b$ or, by checking the numbers $a \equiv 1, 2, 3, \text{ and } 4 \pmod{5}$, we have $a^4 \equiv 1 \pmod{5}$. Similarly for b . Hence $a^4 - b^4 \equiv 1 - 1 \equiv 0 \pmod{5}$. That is, $5|(a^4 - b^4)$. \square

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Don Rideout is a retired math professor from Memorial University of Newfoundland. He can be reached via drideout@mun.ca.



Contact us

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If you have questions, comments or ideas, please feel free to email editors at MathemAttic@cms.math.ca.

OLYMPIAD CORNER

No. 371

The problems in this section appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

*To facilitate their consideration, solutions should be received by **June 15, 2019**.*

Originally the problems were mislabelled. The issue has now been corrected.

OC421. Mim has a deck of 52 cards, stacked in a pile with their backs facing up. Mim separates the small pile consisting of the seven cards on the top of the deck, turns it upside down, and places it at the bottom of the deck. All cards are again in one pile, but not all of them face down; the seven cards at the bottom do, in fact, face up. Mim repeats this move until all cards have their backs facing up again. In total, how many moves did Mim make?

OC422. A 2017×2017 table is filled with nonzero digits. Among the 4034 numbers whose decimal expansion is formed with the rows and columns of this table, read from left to right and from top to bottom, respectively, all but one are divisible by a prime number p , and the remaining number is not divisible by p . Find all possible values of p .

OC423. There are 100 gnomes with weight $1, 2, \dots, 100$ kg gathered on the left bank of the river. They cannot swim, but they have one boat with capacity 100 kg. Because of the current, it is hard to row back, so each gnome has enough power only for one passage from right side to left as oarsman. Can all gnomes get to the right bank?

OC424. Let n be a nonzero natural number, let $a_1 < a_2 < \dots < a_n$ be real numbers and let b_1, b_2, \dots, b_n be real numbers. Prove that:

- (a) if all the numbers b_i are positive, then there exists a polynomial f with real coefficients and having no real roots such that $f(a_i) = b_i$ for $i = 1, 2, \dots, n$;
- (b) there exists a polynomial f of degree at least 1 having all real roots and such that $f(a_i) = b_i$ for $i = 1, 2, \dots, n$.

OC425. Consider a triangle ABC with $\angle A < \angle C$. Point E is on the internal angle bisector of $\angle B$ such that $\angle EAB = \angle ACB$. Let D be a point on line BC such that $B \in CD$ and $BD = AB$. Prove that the midpoint M of the segment AC is on the line DE .

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 15 juin 2019.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC421. Mim a devant elle un paquet de 52 cartes, dans une pile avec les versos vers le haut. Elle retire la toute petite pile de sept cartes du haut, la renverse en un mouvement et la place au fond du paquet. Les cartes sont ainsi de nouveau en une seule pile, mais les versos ne sont plus tous vers le haut; les sept cartes au fond ont leurs versos vers le bas. Mim répète la même série de déplacements, jusqu'à ce que tous les versos se retrouvent de nouveau vers le haut. Déterminer le nombre de déplacements effectués.

OC422. Un tableau 2017×2017 consiste de chiffres non nuls. Les rangées, lues de gauche à droite, et les colonnes, lues du haut vers le bas, donnent les représentations décimales de 4034 entiers. Pour un certain nombre premier p , tous ces 4034 entiers sont divisibles par p , sauf un seul. Déterminer toute valeur possible de p .

OC423. Se trouvent 100 gnomes de poids $1, 2, \dots, 100$ kg sur la rive gauche d'une rivière. Ils ne peuvent pas nager, mais ils disposent d'un bateau de capacité totale de 100 kg. Aucun gnome peut faire le passage de retour de la rive droite vers la rive gauche plus d'une fois. Est-ce possible pour tous les 100 gnomes de se rendre de la rive gauche à la rive droite ?

OC424. Soit n un nombre naturel non nul et soient $a_1 < a_2 < \dots < a_n$ puis b_1, b_2, \dots, b_n des nombres réels. Démontrer que:

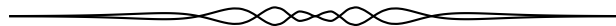
- (a) si tous les nombres b_i sont positifs, alors il existe un polynôme f à coefficients réels et n'ayant aucune racine réelle tel que $f(a_i) = b_i$ pour $i = 1, 2, \dots, n$;
- (b) il existe un polynôme de degré au moins 1 ayant seulement des racines réelles, tel que $f(a_i) = b_i$ pour $i = 1, 2, \dots, n$.

OC425. Soit un triangle ABC tel que $\angle A < \angle C$. Le point E se trouve sur la bissectrice interne de $\angle B$ de façon à ce que $\angle EAB = \angle ACB$. Enfin, soit D un point sur la ligne BC tel que $B \in CD$ et $BD = AB$. Démontrer que le mi point M du segment AC se trouve sur la ligne DE .

OLYMPIAD CORNER

SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(2), p. 52–53, and 44(3), p. 100–101.



OC366. Prove that there exist infinitely many positive integer triples (a, b, c) such that a, b, c are pairwise relatively prime, and $ab + c, bc + a, ca + b$ are pairwise relatively prime.

Originally 2016 China Western Mathematical Olympiad Day 2 Problem 5.

We received 6 submissions. We present 3 solutions.

Solution 1, by Shuborno Das.

Let $a = 2$, $b = 3$ and c be any prime number greater than 3 such that $c \equiv 4 \pmod{5}$. We claim that $ab + c, bc + a, ca + b$ are pairwise co-prime for the above choice. Note that $\gcd(2, 3) = \gcd(2, c) = \gcd(3, c) = 1$. Now,

$$ab + c = c + 6, \quad bc + a = 3c + 2, \quad ca + b = 2c + 3.$$

Therefore,

$$\gcd(c + 6, 3c + 2) = \gcd(3c + 18, 3c + 2) = \gcd(16, 3c + 2) = 1.$$

(Note $\gcd(3, 3c + 2) = 1$).

We have

$$\gcd(c + 6, 2c + 3) = \gcd(2c + 12, 2c + 3) = \gcd(9, 2c + 3) = 1.$$

(Note $\gcd(2, 2c + 3) = 1$).

We also have

$$\gcd(3c + 2, 2c + 3) = \gcd(6c + 4, 2c + 3) = \gcd(6c + 4, 6c + 9) = \gcd(5, 6c + 4).$$

Now, $6c + 4 \equiv c - 1 \pmod{5}$. We have assumed that $c \equiv 4 \pmod{5}$, that is $c - 1 \not\equiv 0 \pmod{5}$, so $\gcd(5, 6c + 4) = 1$. By Dirichlet's theorem, there are infinitely many primes of the form $5k + 4$. This means there are infinitely many triples (a, b, c) satisfying the condition.

Solution 2, by Oliver Geupel.

It is enough to show that, for every positive integer k , the triple

$$(a, b, c) = (10k - 1, 10k, 10k + 1)$$

satisfies the required conditions.

Evidently, a , b , and c are pairwise relatively prime. Let

$$n = 10k, \quad u = ab + c, \quad v = bc + a, \quad w = ca + b.$$

Then,

$$(u, v, w) = (n^2 + 1, n^2 + 2n - 1, n^2 + n - 1).$$

We have

$$(n + 3)u - (n + 1)v = 4,$$

so that every common divisor of u and v is a divisor of 4. Since $u = 100k^2 + 1$ is odd, $\gcd(u, v) = 1$. From the identity

$$(n + 1)v - (n + 2)w = 1,$$

we see that v and w are coprime. Finally, we have

$$(n + 3)u - (n + 2)w = 5,$$

which implies that every common divisor of u and w is a divisor of 5. But u is not divisible by 5. Therefore, u and w are relatively prime, which completes the proof.

Solution 3, by David Manes.

For each positive integer n , let $a_n = 2^n$, $b_n = 3$ and $c_n = 5$. Then

$$(a_n, b_n, c_n) = (2^n, 3, 5)$$

define infinitely many triples such that $\gcd(2^n, 3) = \gcd(2^n, 5) = \gcd(3, 5) = 1$. Therefore, $(2^n, 3, 5)$ are pairwise relatively prime for each n . For a given positive integer n , let

$$x = a_nb_n + c_n = 3 \cdot 2^n + 5, \quad y = b_nc_n + a_n = 3 \cdot 5 + 2^n, \quad z = c_na_n + b_n = 5 \cdot 2^n + 3.$$

Then x, y and z are all odd positive integers. We consider 3 cases.

1) Let $d = \gcd(x, y)$. Then d divides all linear combinations of x and y . In particular, d divides

$$3x - 1 \cdot y = 9 \cdot 2^n - 2^n = 2^{n+3}.$$

Hence, $d = 1$ since the only odd number that divides a power of 2 is 1. Therefore, x and y are relatively prime.

2) Let $d = \gcd(x, z)$. Then d divides the linear combination

$$5x - 3z = 25 - 9 = 2^4.$$

Hence, $d = 1$ so that x and z are relatively prime.

3) Let $d = \gcd(y, z)$. Then d divides

$$5z - y = 24 \cdot 2^n = 3 \cdot 2^{n+3}.$$

The only odd positive divisors of $3 \cdot 2^{n+3}$ are 1 and 3. If $d = 3$, then d divides y and d divides 15 implies d divides $y - 15 = 2^n$, a contradiction. Hence, $d = 1$ so that y and z are relatively prime.

Therefore, x , y and z are pairwise relatively prime for each positive integer n . Summarizing, if for each positive integer n , $(2^n, 3, 5)$ define infinitely many positive integer triples that are pairwise relatively prime, then the triples

$$(3 \cdot 2^n + 5, 5 \cdot 3 + 2^n, 5 \cdot 2^n + 3)$$

are also pairwise relatively prime.

Editor's Comments. These are not the only triples that satisfy the conditions of the problem. Mohammed Aassila found the triples $(a, b, c) = (2, 3, 30k + 5)$, where $k \in \mathbb{N}$, Richard Hess found the triples $(a, b, c) = (2, 3, 30k + 7)$, where $k \in \mathbb{N}$ and the Missouri State University Problem Solving Group found the triples $(a, b, c) = (10k + 5, 10k + 3, 2)$, where $k \in \mathbb{N}$. We leave to the reader the pleasure to check that these triples will do the trick.

OC367. A mathematical contest had 3 problems, each of which was given a score between 0 and 7, inclusive. It is known that, for any two contestants, there exists at most one problem in which they have obtained the same score (for example, there are no two contestants whose ordered scores are 7, 1, 2 and 7, 1, 5, but there might be two contestants whose ordered scores are 7, 1, 2 and 7, 2, 1). Find the maximum number of contestants.

Originally 2016 Italian Mathematical Olympiad Problem 2.

We received 2 submissions. We present the solution by Mohammed Aassila.

Consider the lattice points (x, y, z) such that $x, y, z \in [0, 7]$ and $x, y, z \in \mathbb{Z}$. Colour a lattice point (x, y, z) red if there exists a student who got x , y , and z as their scores for the first, second, and third problems, respectively.

The problem condition is equivalent to the condition that if (x, y, z) is red, then (x, y, k) , (x, k, z) , (k, y, z) are all not red for $k \in [0, 7]$ (with the exception of when the point is equal to (x, y, z)), which correspond to the up-down, left-right, and front-back rows that contain (x, y, z) .

Looking at all lattice points (x, y, k) such that $x, y \in [0, 7]$ and k is a constant, by Pigeonhole principle there exists at most 8 red lattice points, else there exists two red lattice points in the same row, contradiction.

Thus, the maximum possible is $8 \times 8 = 64$ and it remains to show this is attainable. Indeed, just take all points of the form $(x, y, x - y)$ where values are calculated mod 8 and $x, y \in [0, 7]$.

OC368. Let n be a positive integer. Find the number of solutions of

$$x^2 + 2016y^2 = 2017^n$$

as a function of n .

Originally 2016 Korea National Olympiad Day 1 Problem 1.

Problem in not complete as stated. The statement should specify that the solutions sought are positive integer solutions.

We received one correct submission. We present the solution by Shengda Hu.

For $n = 1$, the only integral solutions are $(\pm 1, \pm 1)$.

We see that the following generate integral solutions of the equation for integers $l, m \geq 0$:

$$a_{m,l} + 2b_{m,l}\sqrt{-504} := 2017^m(1 + 2\sqrt{-504})^l \implies a_{m,l}^2 + 2016b_{m,l}^2 = 2017^{l+2m}$$

We prove that for each n the set of integral solutions is

$$\{(\pm a_{m,l}, \pm b_{m,l}), \text{ where } l + 2m = n\}.$$

For $k > 1$, assume that solutions for $n < k$ are given above. Let (u, v) be a solution of the equation for $n = k > 1$, we compute

$$u^2 + 2016v^2 = 2017^k \implies u^2 \equiv v^2 \pmod{2017} \implies u \equiv \pm v \pmod{2017}.$$

Suppose that $u \equiv v \pmod{2017}$ (the other case reduces to this one by changing the sign of v), then

$$\frac{u + 2v\sqrt{-504}}{1 + 2\sqrt{-504}} = \frac{(u + 2v\sqrt{-504})(1 - 2\sqrt{-504})}{2017} = \frac{u + 2016v}{2017} + 2\frac{v - u}{2017}\sqrt{-504},$$

which implies that

$$(a, b) := \left(\frac{u + 2016v}{2017}, \frac{v - u}{2017} \right)$$

is an integral solution to the equation for $n = k - 1$. By the inductive assumption, we see that

$$(a, b) \in \{2017^m(\pm a_l, \pm b_l) : l + 2m = k - 1\}$$

and so

$$a + 2b\sqrt{-504} \in \{\pm 2017^m(1 \pm 2\sqrt{-504})^l : l + 2m = k - 1\}$$

and

$$u + 2v\sqrt{-504} \in \{\pm 2017^m(1 \pm 2\sqrt{-504})^l(1 + 2\sqrt{-504}) : l + 2m = k - 1\}.$$

We compute that

$$\begin{aligned} \pm(1 + 2\sqrt{-504})^l(1 + 2\sqrt{-504}) &= \pm(1 + 2\sqrt{-504})^{l+1} = \pm(a_{0,l+1} + 2b_{0,l+1}\sqrt{-504}), \\ \pm(1 - 2\sqrt{-504})^l(1 + 2\sqrt{-504}) &= \pm 2017(1 - 2\sqrt{-504})^{l-1} \\ &= \pm 2017(a_{0,l-1} - 2b_{0,l-1}\sqrt{-504}). \end{aligned}$$

Thus (u, v) is of the claimed form.

For each (m, l) , among the four solutions $(\pm a_{m,l}, \pm b_{m,l})$, there is only one positive solution. It follows that the number of positive integral solutions to the equation in the problem for $n > 0$ coincides with the number of non-negative integral solutions to the equation below

$$l + 2m = n, \text{ where } l, m \geq 0$$

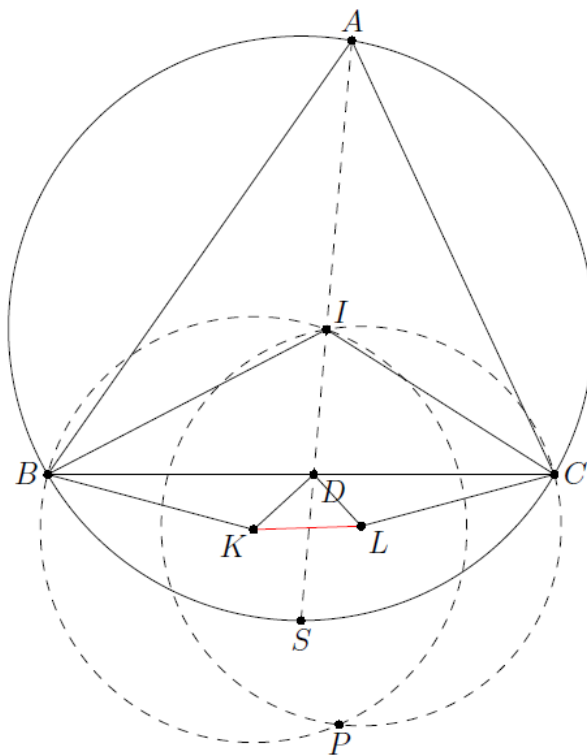
Thus the original equation has $\left\lfloor \frac{n}{2} \right\rfloor + 1$ positive solutions.

We note that the explicit description above also shows that the number of integer solutions to the equation in the problem is $2(n + 1)$.

OC369. Let I be the incenter of $\triangle ABC$. Let D be the point of intersection of AI with BC and let S be the point of intersection of AI with the circumcircle of ABC ($S \neq A$). Let K and L be incenters of $\triangle DSB$ and $\triangle DCS$. Let P be a reflection of I with respect to KL . Prove that $BP \perp CP$.

Originally 2016 Polish Mathematical Olympiad Finals Day 2 Problem 6.

We received 3 submissions. We present the solution by Shuborno Das.



We first prove that $BKDI$ and $CLDI$ are cyclic.

As K and I are incenters of $\triangle DSB$ and $\triangle ABC$, respectively, we have

$$\begin{aligned}\angle BKD + \angle BID &= (180^\circ - \angle DBK - \angle BDK) + \left(\frac{\angle A + \angle B}{2}\right) \\ &= \left(180^\circ - \frac{\angle DBS}{2} - \frac{\angle A}{4} - \frac{\angle B}{2}\right) + \left(\frac{\angle A + \angle B}{2}\right) \\ &= \left(180^\circ - \frac{\angle A}{2} - \frac{\angle B}{2}\right) + \left(\frac{\angle A + \angle B}{2}\right) \\ &= 180^\circ,\end{aligned}$$

where the penultimate equality follows from

$$\angle DBS = \angle CBS = \angle A/2.$$

So, $BKDI$ is cyclic and similarly $CLDI$ is cyclic.

Next, we prove that the circumcenter of $\triangle BPI$ is K and the circumcenter of $\triangle CPI$ is L . We have

$$\angle SBI = \angle SBC + \angle CBI = \frac{\angle A}{2} + \frac{\angle B}{2} = \angle BAI + \angle ABI = \angle SIB,$$

so $SB = SI$. As K is the incenter of $\triangle DSB$, then KS bisects $\angle ISB$, which gives $KI = KB$.

As point P is the reflection of point I over the line KL , then $IK = PK$, so $KB = IK = PK$. Therefore, the circumcenter of $\triangle BPI$ is point K and similarly the circumcenter of $\triangle CPI$ is point L .

Now, we have

$$\angle BPC = \angle BPI + \angle CPI = \frac{\angle BKI + \angle CLI}{2} = \frac{\angle BDI + \angle CDI}{2} = \frac{\angle BDC}{2} = 90^\circ,$$

which gives the desired conclusion.

OC370. Integers n and k are given, with $n \geq k \geq 2$. You play the following game against an evil wizard. The wizard has $2n$ cards; for each $i = 1, \dots, n$, there are two cards labeled i . Initially, the wizard places all cards face down in a row, in unknown order. You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again.

We say this game is winnable if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds. For which values of n and k is the game winnable?

Originally 2016 USAMO Day 2 Problem 6.

We received 2 submissions. We present the solution by Oliver Geupel.

We assert that the game is winnable if and only if $k < n$.

First, suppose $k < n$. We give a way to win in not more than $n + 3$ moves.

Denote the positions of cards by the integers $1, \dots, 2n$. In the first move, we point to the cards at positions $1, \dots, k$. If the game is not over, we point to the cards at positions $2, \dots, k + 1$ in the second move. By inspecting the card at position $k + 1$ and the distinction of the labels shown in both moves, we can determine the label of the card at position 1. We proceed, if necessary, pointing to cards positions $3, \dots, k + 2$ in the third move and so on. We point to cards at positions $n + 2, \dots, n + k + 1$ in the $(n + 2)$ nd move which gives us enough information to determine the label of the card at position $n + 1$.

At that time, we have learned the labels of $n + 1$ cards. By the Pigeonhole principle, we know two matching cards, and we can point to them and $k - 2$ redundant cards in the $(n + 3)$ rd move, which wins the game.

It remains to show that there is no winning strategy when $k = n$.

Let P_1 be the set of positions of cards we point to in the first move. Assume that there is no match in the first move. Then, after the first move, we have the situation that we know a collection P_1 of n positions with no matching pair of cards, but we do not know the specific position of any label in P_1 . The latter is because the wizard has arbitrarily permuted the n chosen cards.

In the second move, it will not help to point to the set P_1 or to its complement $\overline{P_1}$, which has also no matching pair. So we will point to a combination P_2 of some positions in P_1 and some positions in $\overline{P_1}$. It may happen that the positions chosen from P_1 give exactly the labels that are missing at the positions chosen from $\overline{P_1}$. Then, after the second move, we have the situation that we know a collection P_2 of n positions with no matching pair of cards, but we do not know the specific position of any label in P_2 .

Proceeding this way, it may happen that, after every move, we know a set of n positions with no matching pair of cards, but we do not know the specific position of any label in this set. This prevents us from winning certainly in the next move. So there is no fixed number m of moves, in which we can certainly win the game.

OC371. Let a, b and $c \in \mathbb{R}^+$ such that $abc = 1$. Prove that

$$\frac{a+b}{(a+b+1)^2} + \frac{b+c}{(b+c+1)^2} + \frac{c+a}{(c+a+1)^2} \geq \frac{2}{a+b+c}.$$

Originally 2016 Iranian Mathematical Olympiad 3rd Round Algebra Problem 2.

We received 5 correct submissions. We present the solution submitted independently by Šefket Arslanagić and Shuborno Das.

The inequality in the statement will be obtained via AM-GM inequality and Cauchy-Schwarz inequality.

First, we establish that for any positive real numbers x , y , and z , we have

$$\frac{xyz}{x^3 + y^3 + xyz} + \frac{xyz}{y^3 + z^3 + xyz} + \frac{xyz}{z^3 + x^3 + xyz} \leq 1 \quad (1)$$

Indeed AM-GM inequality implies that

$$\frac{x^3 + x^3 + y^3}{3} \geq x^2y, \quad \frac{x^3 + y^3 + y^3}{3} \geq xy^2$$

and so $x^3 + y^3 \geq x^2y + xy^2$. Similarly

$$y^3 + z^3 \geq y^2z + yz^2 \quad \text{and} \quad z^3 + x^3 \geq z^2x + zx^2.$$

These consequences of AM-GM inequality imply and prove (1)

$$\begin{aligned} & \frac{xyz}{x^3 + y^3 + xyz} + \frac{xyz}{y^3 + z^3 + xyz} + \frac{xyz}{z^3 + x^3 + xyz} \\ & \leq \frac{xyz}{x^2y + xy^2 + xyz} + \frac{xyz}{y^2z + yz^2 + xyz} + \frac{xyz}{z^2x + zx^2 + xyz} \\ & = \frac{xyz}{xy(x + y + z)} + \frac{xyz}{yz(y + z + x)} + \frac{xyz}{zx(z + x + y)} \\ & = \frac{z + x + y}{x + y + z} \\ & = 1. \end{aligned}$$

Next we take $a = x^3$, $b = y^3$, and $c = z^3$ in (1). Taking into consideration that $abc = 1$, we obtain

$$\begin{aligned} & \frac{1}{a + b + 1} + \frac{1}{b + c + 1} + \frac{1}{c + a + 1} \leq 1, \\ 2 = 3 - 1 & \leq 1 - \frac{1}{a + b + 1} + 1 - \frac{1}{b + c + 1} + 1 - \frac{1}{c + a + 1} \quad \text{and,} \\ 4 & \leq \left(\frac{a + b}{a + b + 1} + \frac{b + c}{b + c + 1} + \frac{c + a}{c + a + 1} \right)^2 \end{aligned} \quad (2)$$

Lastly, Cauchy-Schwarz inequality provides an upper bound for the right side of inequality (2), and allows us to write

$$4 \leq [(a + b) + (b + c) + (c + a)] \left[\frac{a + b}{(a + b + 1)^2} + \frac{b + c}{(b + c + 1)^2} + \frac{c + a}{(c + a + 1)^2} \right] \quad (3)$$

After dividing both sides of (3) by $2(a + b + c)$, we obtain the statement inequality. Equality holds if and only if $a = b = c = 1$.

OC372. In the circumcircle of a triangle ABC , let A_1 be the point diametrically opposite to the vertex A . Let A' be the intersection point of A_1' and BC . The perpendicular to the line AA_1 from A' meets the sides AB and AC at M and N , respectively. Prove that the points A, M, A_1 and N lie on a circle whose center lies on the altitude from A of the triangle ABC .

Originally 2016 Spain Mathematical Olympiad Day 1 Problem 3.

We received 4 correct submissions. We present the solution by Ivko Dimitrić.

We solve the question in several cases.

Case 1. If $\angle B = \angle C$, then $\triangle ABC$ is isosceles, $M = B$, and $N = C$. In this case, it is obvious that the statement holds.

Case 2. Assume $\angle B < \angle C$. Let D be the foot of the altitude from A .

Case 2a. Assume $\angle B < \angle C < 90^\circ$. If vertices A, B , and C are arranged in counterclockwise order, then rays AB, AA_1, AD , and AC are also in counterclockwise order. Since AA_1 is a diameter of the circumcircle, we have

$$\angle MBA_1 = \angle ABA_1 = 90^\circ$$

and

$$\angle NCA_1 = 180^\circ - \angle ACA_1 = 90^\circ.$$

In addition, as $AA' \perp MN$, $\angle MA'A_1 = 90^\circ$ and $\angle NA'A_1 = 90^\circ$. Therefore the quadrilaterals MBA_1A' and $NCA'A_1$ are cyclic, inscribed in circles whose centers are midpoints of MA_1 and NA_1 , respectively.

Furthermore, $\angle AMN = \angle BA_1A = \angle C$ as an exterior angle of the cyclic quadrilateral MBA_1A' , and $\angle AA_1N = \angle C$, as $\angle C$ is an exterior angle of cyclic quadrilateral $NCA'A_1$. As $\angle AMN = \angle AA_1N$ it follows that AMA_1N is a cyclic quadrilateral.

Let P be the center of the circumcircle of AMA_1N . We have

$$\angle NAP = \frac{180^\circ - \angle APN}{2} = \frac{180^\circ - 2\angle AMN}{2} = 90^\circ - \angle C.$$

On the other hand, since $AD \perp BC$, $\angle CAD = 90^\circ - \angle C$. Hence $\angle NAP = \angle CAD$ and P belongs to AD . Its location is found as the intersection point of the perpendicular bisector of the diameter AA_1 with altitude AD .

Case 2b. Assume $\angle B < \angle C = 90^\circ$. Then $A' = A_1 = M = B$ and the quadrilateral AMA_1N degenerates into a right triangle, $\triangle ABN$, with hypotenuse, AN , along AC . However AC is the altitude from A of $\triangle ABC$. Hence the circumcenter of quadrilateral AMA_1N , now $\triangle ABN$, belongs to that altitude from A of $\triangle ABC$.

Case 2c. Assume $\angle B < 90^\circ < \angle C$. The proof for this case follows closely the proof of *Case 2a*. If vertices A, B , and C are arranged in counterclockwise order,

then rays AA_1 , AB , AC , and AD are also in counterclockwise order and P and B are on different sides of AC . Moreover, $\angle CAD = \angle C - 90^\circ$ and

$$\angle APN = 2(180^\circ - \angle AMN) = 2\angle BMA' = 2\angle BA_1A = 2(180^\circ - \angle C),$$

leading to

$$\angle NAP = \frac{180^\circ - \angle APN}{2} = \angle C - 90^\circ = \angle CAD,$$

so, again, P belongs to the altitude AD .

OC373. Let a and b be positive integers. Denote by $f(a, b)$ the number of sequences $s_1, s_2, \dots, s_a \in \mathbb{Z}$ such that $|s_1| + |s_2| + \dots + |s_a| \leq b$. Show that $f(a, b) = f(b, a)$.

Originally 2016 Polish Mathematical Olympiad Finals Day 1 Problem 3.

We received 2 correct submissions. We present both solutions modified for editorial presentation.

Solution 1, by Mohammed Aassila.

Let $f(a, b, 0) = 1$. Let k be an integer such that $1 \leq k \leq \min(a, b)$. Let $f(a, b, k)$ be the number of a -tuples, (s_1, s_2, \dots, s_a) , of integers such that $|s_1| + |s_2| + \dots + |s_a| \leq b$ and exactly k of s_1, \dots, s_a are nonzero. Such a tuple is uniquely defined by the k positions of the nonzero s_i 's, the signs, $+$ or $-$, of the nonzero s_i 's, and an ordered k -tuple of positive integers with sum b or less. There are $\binom{a}{k}$ ways to choose the unordered subset of k positions from the fixed set of positions $1, \dots, a$. There are two ways to choose the sign of a specific nonzero s_i , and there are 2^k ways to choose the signs of the nonzero s_i 's. There are $\binom{b}{k}$ ways to choose the ordered k -tuple of positive integers that sum up to b or less. This counting result follows from the next observation. Choosing the ordered k -tuple of nonzero positive integers u_1, \dots, u_k with $u_1 + \dots + u_k \leq b$ is equivalent to choosing k numbers $u_1, u_1 + u_2, \dots, u_1 + \dots + u_k$, disregarding the order, from b numbers $1, \dots, b$.

By the multiplicative principle of counting

$$f(a, b, k) = 2^k \binom{a}{k} \binom{b}{k},$$

for any $a \geq 1$, $b \geq 1$, and $k \geq 1$. Moreover, the above formula holds for $k = 0$, as $f(a, b, 0) = 1$.

Subsequently,

$$f(a, b) = \sum_{k=0}^{\min(a, b)} f(a, b, k) = \sum_{k=0}^{\min(a, b)} 2^k \binom{a}{k} \binom{b}{k}.$$

As the above is symmetric in a and b , it follows that $f(b, a) = f(a, b)$ for any $a \geq 1$, $b \geq 1$.

Solution 2, by Missouri State University Problem Solving Group.

First, $f(a, b) = f(b, a)$ is true if either $a = 1$ or $b = 1$. This is because the number of 1-tuples, (s_1) , with $|s_1| \leq b$ is $2b + 1$, hence $f(1, b) = 2b + 1$. Also, the number of a -tuples such that $|s_1| + \cdots + |s_a| \leq 1$ is $2a + 1$, hence $f(a, 1) = 2a + 1$.

For $a \geq 2$ and $b \geq 2$, $f(a, b) = f(b, a)$ is established as a consequence of the more general result

$$f(a, b) = f(a, b-1) + f(a-1, b) + f(a-1, b-1). \quad (1)$$

Relation (1) follows from writing the set T of a -tuples (s_1, \dots, s_a) of integers such that $|s_1| + \cdots + |s_a| \leq b$ as the union of three mutually exclusive sets. Namely,

$$\begin{aligned} T_1 &= \{(s_1, \dots, s_a) \mid |s_1| + \cdots + |s_a| \leq b-1\}, \\ T_2 &= \{(s_1, \dots, s_a) \mid |s_1| + \cdots + |s_a| = b, s_a > 0\}, \\ T_3 &= \{(s_1, \dots, s_a) \mid |s_1| + \cdots + |s_a| = b, s_a \leq 0\}. \end{aligned}$$

The cardinal of set T_1 is $f(a, b-1)$. The set T_2 is in one-to-one correspondence with

$$\{(s_1, \dots, s_{a-1}) \mid |s_1| + \cdots + |s_{a-1}| \leq b\}$$

and has cardinal $f(a-1, b)$. Lastly, T_3 is in one-to-one correspondence with

$$\{(s_1, \dots, s_{a-1}) \mid |s_1| + \cdots + |s_{a-1}| \leq b-1\}$$

and has cardinal $f(a-1, b-1)$. Relation (1) follows.

Editor's Comments. Solution 2 was submitted as a proof by induction. However, the editor felt that the induction step can be avoided.

OC374. Let p be an odd prime and let a_1, a_2, \dots, a_p be integers. Prove that the following two conditions are equivalent:

- 1) There exists a polynomial $P(x)$ of degree less than or equal to $\frac{p-1}{2}$ such that $P(i) \equiv a_i \pmod{p}$ for all $1 \leq i \leq p$.
- 2) For any natural number $d \leq \frac{p-1}{2}$,

$$\sum_{i=1}^p (a_{i+d} - a_i)^2 \equiv 0 \pmod{p},$$

where indices are taken modulo p .

Originally 2016 China National Olympiad Day 1 Problem 3.

We received one correct submission. We present the solution by Mohammed Aasila with editorial changes.

First, we establish the following two results.

(A) Let p be a prime number, then for natural numbers $k \geq 0$

$$S = \sum_{i=1}^p i^k \equiv \begin{cases} -1 \pmod{p} & \text{if } p-1 \mid k \\ 0 \pmod{p} & \text{if } p-1 \nmid k. \end{cases}$$

Proof of A. If $p-1 \mid k$, the result follows from Fermat's little theorem. For any $1 \leq i \leq p-1$, i is not divisible by p , hence we have $i^{p-1} \equiv 1 \pmod{p}$, and $i^k \equiv 1 \pmod{p}$. Therefore

$$\sum_{i=1}^p i^k \equiv p-1 + p^k \pmod{p} \equiv -1 \pmod{p}.$$

If $p-1 \nmid k$, consider g a primitive root of p . Any odd prime number has a primitive root. A primitive root is an integer number, g , that is coprime with p , such that the powers g, g^2, \dots, g^{p-1} are congruent to $1, 2, \dots, p-1$ modulo p , not necessarily in the specified order. In addition, by Fermat's little theorem we know that $g^{p-1} \equiv 1 \pmod{p}$. Therefore,

$$g^k S = g^k \sum_{i=1}^p i^k \equiv g^k \sum_{i=1}^{p-1} (g^i)^k \pmod{p} \equiv \sum_{i=2}^p (g^i)^k \pmod{p} \equiv S \pmod{p}.$$

Since $p-1 \nmid k$, we have $g^k \not\equiv 1 \pmod{p}$. This together with $g^k S \equiv S \pmod{p}$ implies that $S \equiv 0 \pmod{p}$, which establishes the second part of (A).

(B) Let $Q(x, d) = (P(x+d) - P(x))^2$. Let k be the degree of P as a polynomial in x . Then $Q(x, d) = \sum_m C_m(d)x^m$ can be viewed as a polynomial in x with coefficients $C_m(d)$ that are polynomials in d . The following can easily be established:

(B1) The degree of Q in x is less than or equal to $2(k-1)$.

(B2) The degree of Q in d is less than or equal to $2k$.

(B3) The degree of $C_m(d)$ is less than or equal to $2k-m$.

We now will prove the main results.

(1) \Rightarrow (2)

Fix d . Note that the sum in question is $\sum_{i=1}^p Q(i)$, where $Q(x)$ is the polynomial defined at (B). The degree of Q in x is less than or equal to

$$2(\deg(P) - 1) = 2((p-1)/2 - 1) = p-3.$$

Therefore $\sum_{i=1}^p Q(i)$ is a linear combination of $\sum_{i=1}^p i^k$, where $0 \leq k \leq p-3$. By result (A),

$$\sum_{i=1}^p Q(i) \equiv 0 \pmod{p}.$$

(2) \Rightarrow (1)

Lagrange polynomials and modular multiplicative inverse under p can be used to construct a polynomial P of degree at most $p-1$, such that $P(i) \equiv a_i \pmod{p}$ for any $1 \leq i \leq p$. The polynomial P can be thought of as interpolating the data set of points $\{(i, a_i), 1 \leq i \leq p\} \pmod{p}$.

Assuming (2), we will prove by contradiction that the degree of P is at most $(p-1)/2 \pmod{p}$.

Assume $\deg(P) \geq (p-1)/2 + 1 \pmod{p}$. Let k be the degree of $P \pmod{p}$, and let R be the coefficient of x^k in P , $R \not\equiv 0 \pmod{p}$. From (B) we know that the degree of Q in x is at most $2(k-1) \leq 2(p-2)$. Note that since (2) is valid for any $1 \leq d \leq (p-1)/2$, in fact, it is valid for $0 \leq d \leq p-1$. Due to (A) and (2) we have that the coefficient of x^{p-1} in $Q(x)$, $C_{p-1}(d) \equiv 0 \pmod{p}$ for any $0 \leq d \leq p-1$. However, $C_{p-1}(d)$ is a polynomial in d of degree at most $2k - (p-1) \leq p-1$. Having more roots modulo p than its degree, it follows that $C_{p-1}(d)$ is the zero polynomial modulo p and that all its coefficients are 0 modulo p .

The coefficient of d^{2k-p+1} , the monomial with largest exponent in $C_{p-1}(d)$, originates from the expansion of $R^2(x+d)^{2k} - 2R^2(x+d)^k x^k$ and is a combination of binomial coefficients

$$R^2 \left(\binom{2k}{p-1} - 2 \binom{k}{2k-p+1} \right).$$

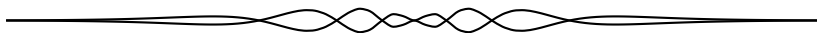
However, notice that p divides $\binom{2k}{p-1}$ and does not divide $\binom{k}{2k-p+1}$, since $p+1 \leq 2k$ and $k \leq p-1$. It follows that $R \equiv 0 \pmod{p}$ which is a contradiction. It must be that $\deg(P) \leq (p-1)/2$.

Editor's Comments. The question and its solution can be better understood when viewed in the finite field, \mathbb{Z}_p , of integers mod p .

OC375. Let $ABCD$ be a non-cyclic convex quadrilateral with no parallel sides. Suppose the lines AB and CD meet in E . Let $M \neq E$ be the intersection of circumcircles of ADE and BCE . Further, suppose that the internal angle bisectors of $ABCD$ form a convex cyclic quadrilateral with circumcenter I while the external angle bisectors of $ABCD$ form a convex cyclic quadrilateral with circumcenter J . Show that I, J, M are collinear.

Originally 2016 Brazil National Olympiad Day 2 Problem 6.

We received no solutions to this problem.



FOCUS ON...

No. 35

Michel Bataille

The Asymptotic Behavior of Integrals

Introduction

Focus On... No 10 [2014 : 21-24] considered the integrals $I_n = \int_0^1 (\phi(x))^n dx$, where the quadratic function $\phi(x) = ax^2 + bx + c$ was supposed to remain positive in $[0, 1]$. The study culminated in the asymptotic behavior of such integrals, that is, the determination of a simple sequence (ω_n) such that $\lim_{n \rightarrow \infty} \frac{I_n}{\omega_n} = 1$, denoted by $I_n \sim \omega_n$. In this number, we keep the same goal but, through various sequences of integrals, present some simple ways to obtain such an asymptotic behavior. We restrict ourselves to elementary problems and methods, referring the reader to [1] for more complicated examples and to [2] for more sophisticated techniques (Laplace's method, for instance).

With monotone sequences

In our first examples, we show how the knowledge of a recursion formula for a monotone sequence of integrals can quickly lead to the sought sequence (ω_n) .

Take the classical integrals

$$I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n dx.$$

Since $(\sin x)^{n+1} \leq (\sin x)^n$ for $x \in [0, \frac{\pi}{2}]$, the sequence (I_n) is nonincreasing. Moreover, an integration by parts yields

$$I_{n+2} = (n+1)(I_n - I_{n+2}) \quad \text{or} \quad (n+2)I_{n+2} = (n+1)I_n.$$

It follows that

$$(n+2)I_{n+2}I_{n+1} = (n+1)I_{n+1}I_n$$

for all nonnegative integers n , showing that the sequence $((n+1)I_{n+1}I_n)$ is constant. Thus

$$(n+1)I_{n+1}I_n = 1 \cdot I_1 \cdot I_0 = \frac{\pi}{2}$$

for all $n \geq 0$.

Now, $I_{n+2} \leq I_{n+1} \leq I_n$ gives

$$\frac{n+1}{n+2}I_n \leq I_{n+1} \leq I_n$$

and using the squeeze principle, we deduce $I_{n+1} \sim I_n$. Then we obtain

$$I_n^2 \sim I_n I_{n+1} = \frac{\pi}{2(n+1)} \sim \frac{\pi}{2n},$$

that is, $I_n \sim \sqrt{\frac{\pi}{2n}}$.

For another example, consider

$$J_n = \int_0^1 \frac{x^n \ln x}{x+1} dx.$$

Since $\lim_{x \rightarrow 0^+} x^n \ln x = 0$ when $n \geq 1$, the integrand is the restriction to $(0, 1]$ of a continuous function on $[0, 1]$ and the integral J_n does exist. For $x \in (0, 1]$ the inequality

$$\frac{x^n \ln x}{x+1} \leq \frac{x^{n+1} \ln x}{x+1}$$

holds (since $x^{n+1} \leq x^n$ and $\ln x \leq 0$), therefore the sequence (J_n) is nondecreasing. In addition, we observe that

$$J_n + J_{n+1} = \int_0^1 x^n \ln x dx = -\frac{1}{(n+1)^2}$$

(readily found by an integration by parts). Now, for all $n \geq 2$, we obtain

$$-\frac{1}{n^2} = J_{n-1} + J_n \leq 2J_n \leq J_n + J_{n+1} = -\frac{1}{(n+1)^2}$$

and so $J_n \sim -\frac{1}{2n^2}$.

Two general results

The following theorem offers two general results of the good-to-be-known category.

Theorem 1 *Let f be a real-valued continuous function on $[0, 1]$. Then*

$$(i) \quad \lim_{n \rightarrow \infty} n \cdot \int_0^1 x^n f(x^n) dx = \int_0^1 f(x) dx,$$

$$(ii) \quad \lim_{n \rightarrow \infty} n \cdot \int_0^1 x^n f(x) dx = f(1).$$

Proof of (i). The change of variables $x = u^{1/n}$ shows that

$$n \cdot \int_0^1 x^n f(x^n) dx = \int_0^1 u^{1/n} f(u) du$$

and, denoting by M the maximum value of the continuous function f on $[0, 1]$, we deduce

$$\begin{aligned} \left| n \cdot \int_0^1 x^n f(x^n) dx - \int_0^1 f(u) du \right| &\leq \int_0^1 |f(u)|(1 - u^{1/n}) du \\ &\leq M \int_0^1 (1 - u^{1/n}) du \\ &= \frac{M}{n+1}. \end{aligned}$$

The result follows.

Proof of (ii). To keep the same level of simplicity, we suppose that f is continuously differentiable on the interval $[0, 1]$ (the reader will find a proof in the general case, together with an example of application, in my solution to problem **4010** [2015 : 28,30 ; 2016 : 43-4]).

Let μ be the maximum of f' on $[0, 1]$. Then, for $x \in [0, 1]$ we have

$$|f(1) - f(x)| = \left| \int_x^1 f'(t) dt \right| \leq \int_x^1 |f'(t)| dt \leq \mu(1 - x).$$

Since

$$\begin{aligned} \left| f(1) - (n+1) \cdot \int_0^1 x^n f(x) dx \right| &= (n+1) \left| \int_0^1 x^n (f(1) - f(x)) dx \right| \\ &\leq (n+1) \int_0^1 x^n |f(1) - f(x)| dx, \end{aligned}$$

we obtain

$$\left| f(1) - (n+1) \cdot \int_0^1 x^n f(x) dx \right| \leq (n+1)\mu \int_0^1 x^n (1-x) dx = \frac{\mu}{n+2}$$

and the conclusion readily follows. \square

An application of (i)

Interestingly, part (i) can be applied twice in a problem of the 2005 Romanian Olympiad proposed in Mathproblems, Vol. 5, Issue 2 (here slightly adapted):

Show that $\lim_{n \rightarrow \infty} n \int_0^1 \frac{x^n}{1+x^{2n}} dx = \frac{\pi}{4}$ and then find $\lim_{n \rightarrow \infty} n \left(\frac{\pi}{4} - n \int_0^1 \frac{x^n}{1+x^{2n}} dx \right)$.

Part (i) directly yields

$$\lim_{n \rightarrow \infty} n \int_0^1 \frac{x^n}{1+x^{2n}} dx = \int_0^1 \frac{dx}{1+x^2} = [\arctan x]_0^1 = \frac{\pi}{4}.$$

Then, integrating by parts, we obtain

$$\begin{aligned} n \int_0^1 \frac{x^n}{1+x^{2n}} dx &= \int_0^1 x \cdot \frac{nx^{n-1}}{1+(x^n)^2} dx \\ &= [x \arctan(x^n)]_0^1 - \int_0^1 \arctan(x^n) dx \\ &= \frac{\pi}{4} - \int_0^1 \arctan(x^n) dx \end{aligned}$$

and the problem now reduces to finding $\lim_{n \rightarrow \infty} n \int_0^1 \arctan(x^n) dx$.

Let f be the continuous function defined on $[0, 1]$ by $f(x) = \frac{\arctan x}{x}$ if $x \neq 0$ and $f(0) = 1$. We observe that $\arctan(x^n) = x^n f(x^n)$ and therefore

$$\lim_{n \rightarrow \infty} n \int_0^1 \arctan(x^n) dx = \lim_{n \rightarrow \infty} n \int_0^1 x^n f(x^n) dx = \int_0^1 f(x) dx = \int_0^1 \frac{\arctan x}{x} dx.$$

The latter integral is equal to the constant of Catalan $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$. Note that

we have proved that $\int_0^1 \frac{x^n}{1+x^{2n}} dx = \frac{\pi}{4n} - \frac{G}{n^2} + o(1/n^2)$.

Some applications of (ii)

As a first example, we examine again the integral $J_n = \int_0^1 \frac{x^n \ln x}{x+1} dx$. The result (ii) gives $\lim_{n \rightarrow \infty} n \cdot J_n = 0$, which does not lead to the sought ω_n . To get round this difficulty, we first integrate by parts:

$$\int_{\varepsilon}^1 \frac{x^n \ln x}{x+1} dx = \int_{\varepsilon}^1 \frac{x \ln x}{1+x} d\left(\frac{x^n}{n}\right) = \left[\frac{x^{n+1} \ln x}{n(1+x)}\right]_{\varepsilon}^1 - \frac{1}{n} \int_{\varepsilon}^1 x^n g(x) dx$$

where $\varepsilon \in (0, 1)$ and $g(x) = \frac{1+x+\ln x}{(1+x)^2}$. Letting $\varepsilon \rightarrow 0^+$, we obtain

$$J_n = -\frac{1}{n} \int_0^1 x^n g(x) dx = -\frac{1}{n^2} \left(n \int_0^1 x^n g(x) dx \right)$$

and, since $\lim_{n \rightarrow \infty} n \int_0^1 x^n g(x) dx = g(1) = \frac{1}{2}$, this confirms the result $J_n \sim -\frac{1}{2n^2}$ found above.

Integration by parts will also play a role in our second example, the integral $K_n = \int_0^1 \frac{x^{2n+2}}{1+x^2} dx$. Incidentally, this integral is equal to $\left| \frac{\pi}{4} - \sum_{k=0}^n \frac{(-1)^k}{2k+1} \right|$, hence

estimates the error when π is approximated *via* the partial sums of Gregory's series. We first prove $K_n \sim \frac{1}{4n}$ and then improve this result by showing that $K_n = \frac{1}{4n} - \frac{1}{4n^2} + o(1/n^2)$. The substitution $x = \sqrt{u}$ leads to

$$K_n = \int_0^1 u^n \frac{\sqrt{u}}{2(1+u)} du$$

and an application of (ii) gives $\lim_{n \rightarrow \infty} nK_n = \frac{1}{4}$, that is, $K_n \sim \frac{1}{4n}$.

Now, let $f(u) = \frac{\sqrt{u}}{2(1+u)}$. First integrating $u^n f(u) = (uf(u)) \cdot u^{n-1}$ by parts on $[\varepsilon, 1]$ and then letting $\varepsilon \rightarrow 0^+$ (as above), an easy calculation leads to

$$K_n = \frac{f(1)}{n} - \frac{1}{n} \int_0^1 u^n \cdot \frac{(3+u)\sqrt{u}}{4(1+u)^2} du = \frac{1}{4n} - \frac{1}{n^2} \left(n \int_0^1 u^n \cdot \frac{(3+u)\sqrt{u}}{4(1+u)^2} du \right)$$

and so $K_n = \frac{1}{4n} - \frac{1}{n^2} \left(\frac{1}{4} + o(1) \right)$, the desired result.

Part (ii) of the theorem can sometimes be used to find the asymptotic behavior of integrals of the form $\int_a^b (h(x))^n dx$. Typically, suppose that h is a positive, continuously differentiable function on $[a, b]$ such that $h'(x) > 0$ for $x \in [a, b]$ and $h(a) = 0, h(b) = 1$. Under these hypotheses, the following holds:

$$\int_a^b (h(x))^n dx \sim \frac{1}{nh'(b)}.$$

Indeed, the function h , being strictly increasing and continuous on $[a, b]$, is a bijection from $[a, b]$ onto $[h(a), h(b)] = [0, 1]$. The change of variables $x = h^{-1}(y)$ yields $dx = \frac{dy}{h'(h^{-1}(y))}$ and so

$$\int_a^b (h(x))^n dx = \int_{h(a)}^{h(b)} y^n \frac{dy}{h'(h^{-1}(y))} = \int_0^1 y^n \frac{dy}{h'(h^{-1}(y))}.$$

Since $\frac{1}{h'(h^{-1}(1))} = \frac{1}{h'(b)}$, the result follows.

Problem 2007 of *Mathematics Magazine* proposed in December 2016 can be solved using the ideas just developed. Here is the statement of this problem and a variant of solution:

Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Evaluate

$$\lim_{n \rightarrow \infty} \left(n \cdot \int_0^1 \left(\frac{2(x - \frac{1}{2})^2}{x^2 - x + \frac{1}{2}} \right)^n f(x) dx \right).$$

Let $\phi(x) = \frac{2(x - \frac{1}{2})^2}{x^2 - x + \frac{1}{2}}$ and $L_n = \int_0^1 (\phi(x))^n f(x) dx$. We prove that

$$\lim_{n \rightarrow \infty} n \cdot L_n = \frac{f(0) + f(1)}{2}.$$

A quick study of $\phi(x) = 2 - \frac{1}{2x^2 - 2x + 1}$ (left to the reader) shows that the restriction ϕ_0 of ϕ to $[0, \frac{1}{2}]$ is a bijection onto $[0, 1]$ with

$$\phi_0^{-1}(u) = \frac{1}{2} \left(1 - \sqrt{\frac{u}{2-u}} \right),$$

and that the restriction ϕ_1 of ϕ to $[\frac{1}{2}, 1]$ is a bijection onto $[0, 1]$ whose inverse is given by

$$\phi_1^{-1}(u) = \frac{1}{2} \left(1 + \sqrt{\frac{u}{2-u}} \right).$$

Now, let

$$U_n = \int_0^{1/2} (\phi_0(x))^n f(x) dx \quad \text{and} \quad V_n = \int_{1/2}^1 (\phi_1(x))^n f(x) dx.$$

The changes of variables $x = \phi_0^{-1}(u)$ in U_n and $x = \phi_1^{-1}(u)$ in V_n and a short calculation give

$$U_n = \frac{1}{2} \int_0^1 u^n g_0(u) du, \quad V_n = \frac{1}{2} \int_0^1 u^n g_1(u) du$$

where $g_k(u) = u^{-1/2}(2-u)^{-3/2}f(\phi_k^{-1}(u))$ ($k = 0, 1$). Thus

$$\lim_{n \rightarrow \infty} n \cdot U_n = \frac{1}{2} g_0(1) = \frac{1}{2} f(0) \quad \text{and} \quad \lim_{n \rightarrow \infty} n \cdot V_n = \frac{1}{2} g_1(1) = \frac{1}{2} f(1)$$

and so

$$\lim_{n \rightarrow \infty} n \cdot L_n = \lim_{n \rightarrow \infty} n(U_n + V_n) = \lim_{n \rightarrow \infty} n \cdot U_n + \lim_{n \rightarrow \infty} n \cdot V_n = \frac{f(0) + f(1)}{2}.$$

As usual we conclude with a bunch of exercises.

Exercises

1. Let $X_n = \int_0^{\frac{\pi}{4}} (\tan x)^n dx$. Compute $X_n + X_{n+2}$ and deduce that $X_n \sim \frac{1}{2n}$.
2. Let $Y_n = \int_0^1 \frac{x^n \ln x}{x^n - 1} dx$. Show that $Y_n \sim \frac{\alpha}{n^2}$ for some positive α .

3. Let $f_1(t) = \frac{1}{1+t}$ and for $k \geq 2$, let $f_k(t) = \frac{d}{dt}(tf_{k-1}(t))$. Prove that for any integer $m \geq 1$,

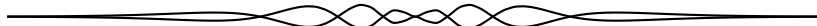
$$\int_0^1 \frac{t^n}{1+t} dt = \frac{f_1(1)}{n} - \frac{f_2(1)}{n^2} + \cdots + (-1)^{m-1} \frac{f_m(1)}{n^m} + o(1/n^m)$$

as $n \rightarrow \infty$.

4. Let $Z_n = \int_0^1 (ax^2 + bx + c)^n dx$ where a, b are negative numbers and $a + b + c$ is positive. Use this number of Focus On... to obtain $Z_n \sim \frac{c^{n+1}}{-nb}$ already found in No. 10.

References

- [1] O. Furdui, *Limits, Series, and Fractional Integrals*, Springer, 2013, pp. 9-16.
- [2] H. S. Wilf, *Mathematics for the Physical Sciences*, Wiley, 1962, chapter 4.



PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **June 15, 2019**.



4421. *Proposed by Peter Y. Woo. (Correction.)*

Show that the area of the largest $30^\circ - 60^\circ - 90^\circ$ triangle that fits inside the unit square is greater than $1/3$.

4422. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Let ABC be a scalene triangle with incenter I and nine-point center N . Find $\angle A$ given that A, N and I are collinear.

4423. *Proposed by Mihaela Berindeanu.*

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a twice differential function such that

$$f(x) + f''(x) = -x \cdot c^x \cdot f'(x)$$

for all real values of x and an arbitrary constant c . Find $\lim_{x \rightarrow 0} x \cdot f(x)$.

4424. *Proposed by Marius Drăgan and Neculai Stanciu.*

Let $k \in \mathbb{N}$ such that $9k + 9$, $9k + 10$ and $9k + 13$ are not perfect squares. Prove that

$$[\sqrt{k+x} + \sqrt{k+x+1} + \sqrt{k+x+2}] = [\sqrt{9k+7}]$$

for all $x \in [0, 7/9]$, where $[a]$ denotes the integer part of number a .

4425. *Proposed by Nguyen Viet Hung.*

Prove the following identities

$$(a) \tan^3 \theta + \tan^3(\theta - 60^\circ) + \tan^3(\theta + 60^\circ) = 27 \tan^3 3\theta + 24 \tan 3\theta,$$

$$(b) \frac{1}{1 + \tan \theta} + \frac{1}{1 + \tan(\theta - 60^\circ)} + \frac{1}{1 + \tan(\theta + 60^\circ)} = \frac{3 \tan 3\theta}{\tan 3\theta - 1}.$$

4426. *Proposed by Michel Bataille.*

Let distinct points A, B, C on a rectangular hyperbola \mathcal{H} be such that $\angle BAC = 90^\circ$. A point M of \mathcal{H} , other than A, B, C , is called *good* if the triangles MAB and MAC have the same circumradius. Show that either infinitely many M are good or a unique M is good. Characterize the triangle ABC in the former case and find M and the common circumradius in the latter one.

4427. *Proposed by Max A. Alekseyev.*

Prove that the equation

$$u^8 + v^9 + w^{14} + x^{15} + y^{16} = z^8$$

has infinitely many solutions in positive integers with $\gcd(u, v, w, x, y, z) = 1$.

4428. *Proposed by Leonard Giugiuc.*

Let ABC be a triangle and let O be an arbitrary point in the same plane. Let A', B' and C' be the reflections of A, B and C in O . Prove that

$$\frac{AB' \cdot B'C}{AB \cdot BC} + \frac{BC' \cdot C'A}{BC \cdot CA} + \frac{CA' \cdot A'B}{CA \cdot AB} \geq 1.$$

4429. *Proposed by Lorian Saceanu.*

Let a, b, c be positive real numbers. Prove that

$$\sqrt{\frac{a^2 + b^2 + c^2}{2(ab + bc + ca)}} \geq \frac{a + b + c}{\sqrt{a(b+c)} + \sqrt{b(a+c)} + \sqrt{c(a+b)}}.$$

4430. *Proposed by Leonard Giugiuc.*

Let $s \geq \frac{28}{3}$ be a fixed real number. Consider the real numbers a, b, c and d such that

$$a + b + c + d = 4 \quad \text{and} \quad a^2 + b^2 + c^2 + d^2 = s.$$

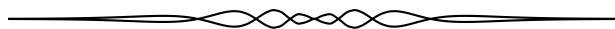
Find the maximum value of the product $abcd$.

.....

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.



4421. *Proposé par Peter Y. Woo. (Correction.)*

Démontrer que pour un triangle $30^\circ - 60^\circ - 90^\circ$ situé dans le carré unitaire, la surface doit être supérieure à $1/3$.

4422. *Proposé par Kadir Altintas et Leonard Giugiuc.*

Soit ABC un triangle scalène dont le centre du cercle inscrit est I et le centre du cercle des neuf points est N . Déterminer $\angle A$, prenant pour acquis que A , N et I sont colinéaires.

4423. *Proposé par Mihaela Berindeanu.*

Soit $f : \mathbb{R} \mapsto \mathbb{R}$ deux fois différentiable, telle que pour toutes valeurs réelles x

$$f(x) + f''(x) = -x \cdot c^x \cdot f'(x),$$

puis une constante arbitraire c . Déterminer $\lim_{x \rightarrow 0} x \cdot f(x)$.

4424. *Proposé par Marius Drăgan et Neculai Stanciu.*

Soit $k \in \mathbb{N}$ tel que $9k + 9$, $9k + 10$ et $9k + 13$ ne sont pas des carrés parfaits. Démontrer que

$$[\sqrt{k+x} + \sqrt{k+x+1} + \sqrt{k+x+2}] = [\sqrt{9k+7}]$$

pour tout $x \in [0, 7/9]$, où $[a]$ dénote la partie entière de a .

4425. *Proposé par Nguyen Viet Hung.*

Démontrer les identités suivantes:

$$(a) \tan^3 \theta + \tan^3(\theta - 60^\circ) + \tan^3(\theta + 60^\circ) = 27 \tan^3 3\theta + 24 \tan 3\theta,$$

$$(b) \frac{1}{1 + \tan \theta} + \frac{1}{1 + \tan(\theta - 60^\circ)} + \frac{1}{1 + \tan(\theta + 60^\circ)} = \frac{3 \tan 3\theta}{\tan 3\theta - 1}.$$

4426. *Proposé par Michel Bataille.*

Les points distincts A, B et C se trouvent sur une hyperbole rectangulaire \mathcal{H} , de façon à ce que $\angle BAC = 90^\circ$. Un point M sur \mathcal{H} , distinct de A, B et C , est dit *bon* si les cercles circonscrits des triangles MAB et MAC ont le même rayon r . Démontrer qu'il y a infiniment de bons M ou qu'il n'y en a qu'un seul. Caractériser le triangle ABC si le premier scénario tient ; identifier M et la valeur du rayon commun r si c'est plutôt le deuxième scénario qui tient.

4427. *Proposé par Max A. Alekseyev.*

Démontrer que l'équation

$$u^8 + v^9 + w^{14} + x^{15} + y^{16} = z^8$$

possède un nombre infini de solutions entières positives u, v, w, x, y, z telles que $\gcd(u, v, w, x, y, z) = 1$.

4428. *Proposé par Leonard Giugiuc.*

Soient ABC un triangle, puis O un point arbitraire dans le même plan. Soient A', B' et C' les réflexions de A, B et C par rapport à O . Démontrer que

$$\frac{AB' \cdot B'C}{AB \cdot BC} + \frac{BC' \cdot C'A}{BC \cdot CA} + \frac{CA' \cdot A'B}{CA \cdot AB} \geq 1.$$

4429. *Proposé par Lorian Saceanu.*

Soient a, b et c des nombres réels positifs. Démontrer que

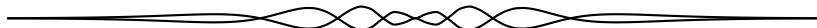
$$\sqrt{\frac{a^2 + b^2 + c^2}{2(ab + bc + ca)}} \geq \frac{a + b + c}{\sqrt{a(b+c)} + \sqrt{b(a+c)} + \sqrt{c(a+b)}}.$$

4430. *Proposé par Leonard Giugiuc.*

Soit $s \in \mathbb{R}$, $s \geq \frac{28}{3}$ et considérer des nombres réels a, b, c et d tels que

$$a + b + c + d = 4 \quad \text{et} \quad a^2 + b^2 + c^2 + d^2 = s.$$

Déterminer la valeur maximale du produit $abcd$.



SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2018: 44(3), p. 115–119 and 44(4), p.160–163.

An asterisk (★) after a number indicates that a problem was proposed without a solution.



4321. Proposed by Leonard Giugiuc and Diana Trailescu.

Find the greatest positive real number k such that

$$(a^2 + b^2 + c^2 + d^2 + e^2)^2 \geq k(a^4 + b^4 + c^4 + d^4 + e^4)$$

for all real numbers a, b, c, d and e satisfying $a + b + c + d + e = 0$.

There were 4 correct solutions and one submission using Maple. We present the solution obtained independently by AN-anduud Problem Solving Group and Digby Smith.

We prove that

$$(a^2 + b^2 + c^2 + d^2 + e^2)^2 \geq \frac{20}{13}(a^4 + b^4 + c^4 + d^4 + e^4)$$

with equality if and only if $\{a, b, c, d, e\} = \{4r, -r, -r, -r, -r\}$ for some real number r .

Suppose that $a^2 = \max\{a^2, b^2, c^2, d^2, e^2\}$, and let $4x^2 = b^2 + c^2 + d^2 + e^2$. Then $x^2 \leq a^2$ and the left side of the inequality is $(a^2 + 4x^2)^2$. By the Cauchy-Schwarz inequality (or the quadratic-arithmetic means inequality),

$$a^2 = (b + c + d + e)^2 \leq 4(b^2 + c^2 + d^2 + e^2) = 16x^2$$

and

$$\begin{aligned} (a^2 - 4x^2)^2 &= [(b + c + d + e)^2 - (b^2 + c^2 + d^2 + e^2)]^2 \\ &= 4(bc + bd + be + cd + ce + de)^2 \\ &\leq 24(b^2c^2 + b^2d^2 + b^2e^2 + c^2d^2 + c^2e^2 + d^2e^2). \end{aligned}$$

Equality occurs if and only if $b = c = d = e = -a/4$.

Therefore

$$\begin{aligned} &12(b^4 + c^4 + d^4 + e^4) \\ &= 12(b^2 + c^2 + d^2 + e^2)^2 - 24(b^2c^2 + b^2d^2 + b^2e^2 + c^2d^2 + c^2e^2 + d^2e^2) \\ &\leq 192x^4 - (a^4 - 8a^2x^2 + 16x^4) \\ &= 176x^4 + 8a^2x^2 - a^4. \end{aligned}$$

Since $x^2 \leq a^2 \leq 16x^2$, we have

$$\begin{aligned} 0 &\geq 16(16x^2 - a^2)(x^2 - a^2) \\ &= 256x^4 - 272x^2a^2 + 16a^4 \\ &= 5(12a^4 + 176x^4 + 8a^2x^2 - a^4) - 39(a^4 + 8a^2x^2 + 16x^4) \\ &\geq 60(a^4 + b^4 + c^4 + d^4 + e^4) - 39(a^2 + 4x^2)^2, \end{aligned}$$

from which

$$39(a^2 + b^2 + c^2 + d^2 + e^2)^2 \geq 60(a^4 + b^4 + c^4 + d^4 + e^4).$$

The desired result follows.

4322. *Proposed by Marius Drăgan.*

Let a, b, c be the side lengths of a triangle, x, y, z be positive numbers and let $u = bz - cy$, $v = ay - bx$, $w = cx - az$. Prove that $uv + vw + wu \leq 0$.

There were 5 correct solutions. We present two of them.

Solution 1, by Michel Bataille and Digby Smith, independently.

Since $au + cv + bw = 0$, we have that

$$\begin{aligned} b(uv + vw + wu) &= buv + bw(u + v) \\ &= buv - (au + cv)(u + v) \\ &= -[au^2 + (a + c - b)uv + cv^2]. \end{aligned}$$

The discriminant of the quadratic form, namely,

$$\begin{aligned} (a + c - b)^2 - 4ac &= (a^2 + b^2 + c^2) - 2(ab + bc + ca) \\ &= -[(a + b - c)(a + c - b) + (b + c - a)(b + a - c) + (c + a - b)(c + b - a)], \end{aligned}$$

is negative because of the inequality involving the sides of a triangle. Since a, b, c are positive, the quadratic form is never positive. Thus $uv + vw + wu \leq 0$.

Solution 2, by the proposer.

The expression $uv + vw + wu$ can be written as $-\frac{1}{2}(x, y, z)M(x, y, z)^T$, where

$$M = \begin{pmatrix} 2bc & c(c - b - a) & b(b - c - a) \\ c(c - b - a) & 2ac & a(a - b - c) \\ b(b - c - a) & a(a - b - c) & 2ab \end{pmatrix}.$$

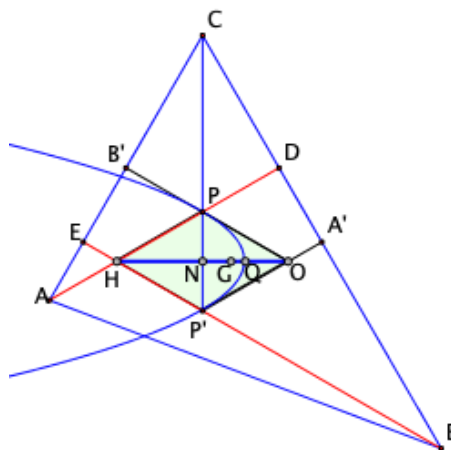
The principal minors are $2bc$,

$$\frac{c^2}{4} [4ab - (c - b - a)^2] = \frac{c^2}{4} [(c + a - b)(c + b - a) + 2c(a + b - c)]$$

and $\det M$. Because of the triangle inequality and because M annihilates the nontrivial vector $(a, b, c)^T$, these minors are nonnegative. Therefore, by Sylvester's Criterion, M is positive semidefinite and the result follows.

4323. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Let ABC be a triangle with $\angle C = 60^\circ$. Let H denote the orthocenter, G the centroid, N the nine-point circle center and O the circumcenter of ABC . Let Q be the midpoint of NO . Prove that the parabola with vertex at Q and focus at G is tangent to the perpendicular bisector of both AC and BC .



We received 4 submissions, all of which were correct, and feature the solution by Ivko Dimitrić, modified by the editor.

Let A' and B' be the midpoints of sides BC and CA whose lengths are a and b , respectively, and assume that $a > b$ as in the picture (since for the points H, N, G, Q , and O to be distinct we cannot have $a = b$). Let D and E be the feet of the altitudes from A and B and let P and P' be the points of intersection of the altitudes AD, BE with the perpendicular bisectors of the sides AC and BC , respectively. Then the quadrilateral $OPHP'$ is a parallelogram.

Since $\angle C = 60^\circ$, from the right triangles ACD and BCE we get $CD = \frac{1}{2}CA = \frac{b}{2}$ and $CE = \frac{1}{2}CB = \frac{a}{2}$. Hence,

$$DA' = CA' - CD = \frac{1}{2}(a - b) \quad \text{and} \quad B'E = CE - CB' = \frac{1}{2}(a - b),$$

and the parallelogram $OPHP'$ is a rhombus (because the distances between its parallel sides are the same). Because the sides of $\angle POP'$ are perpendicular to the sides of $\angle ACB$, we have $\angle POP' = 60^\circ$. Since a diagonal of a rhombus bisects the angles at the pair of vertices that it connects, we conclude that

$$\angle POH = \angle HOP' = 30^\circ.$$

Because the points are arranged on the Euler line OH of any triangle so that the centroid G divides the segment joining the 9-point center N to the circumcenter O in the ratio $1 : 2$, we can introduce coordinates with the origin at the midpoint Q of NO and

$$N = (0, 3), G = (0, 1) \text{ and } O = (0, -3).$$

It follows that the line OP , which makes an angle of 30° with respect to the y -axis, has slope $\tan 60^\circ = \sqrt{3}$ and equation $y = \sqrt{3}x - 3$, while the parabola with focus at G and vertex at Q has equation $y = \frac{1}{4}x^2$.

It is easily confirmed that OP is tangent to the parabola at $P(2\sqrt{3}, 3)$, as desired. By symmetry the other perpendicular bisector, namely OP' , is likewise tangent to the parabola (at $P'(-2\sqrt{3}, 3)$).

Editor's comments. Properties of triangles with a 60° angle are discussed in the article "Recurring *Crux* Configurations 3" [37:7, November 2011, pages 449-453]. One of the results there confirms a theorem that is suggested by the figure displayed above: the line PP' (which is the perpendicular bisector of OH) is the bisector of $\angle ACB$. This is part of Problem 2855 [2003: 316; 2004: 308-309]. Note that the reflection in PP' fixes C and interchanges O with H , and the line CB with the line CA . This observation shows that the key result in the featured solution above, namely

If exactly one of the angles of a triangle is 60° then the perpendicular bisectors of the sides adjacent to that vertex form an angle that is bisected by the Euler line OH .

is equivalent to the analogous theorem with the perpendicular bisectors replaced by the altitudes to those sides. The proof of that theorem was Problem M1046 from the 1987 U.S.S.R journal *Kvant* [appearing in *Crux* 1988: 165; 1990: 103].

4324. *Proposed by Michel Bataille.*

Let f be a continuous, positive function on $[0, 1]$ such that

$$\mathcal{S} = \left\{ \int_0^1 (f(x))^n dx : n \in \mathbb{N} \right\}$$

is bounded above. Find the value of $\sup \mathcal{S}$.

We received 6 submissions of which 5 were correct and complete. We present the solution by Ivko Dimitrić.

Since a continuous function on a closed interval attains global maximum, we let $c \in [0, 1]$ be a point where f attains its maximum value $M = f(c)$. Assume that $M > 1$ and let $q = \frac{1+M}{2} > 1$. By continuity, there must exist an interval $[a, b] \subset [0, 1]$ containing c such that $f(x) \geq q$ on $[a, b]$. Indeed, if that is not so, we would have a sequence $\{x_n\}$ approaching c for which $f(x_n) < q$ for every n . But then

$$M = f(c) = \lim_{n \rightarrow \infty} f(x_n) \leq q = \frac{1+M}{2},$$

which is not possible. Consequently, we have

$$\int_0^1 (f(x))^n dx \geq \int_a^b (f(x))^n dx \geq \int_a^b q^n dx = q^n(b-a).$$

Since $q^n(b-a) \rightarrow \infty$ as $n \rightarrow \infty$, the set \mathcal{S} would be unbounded, contrary to the given condition. Hence, our assumption $M > 1$ cannot hold and we have $f(x) \leq M \leq 1$ for all $x \in [0, 1]$. Therefore, $(f(x))^n \leq f(x)$ and

$$\int_0^1 (f(x))^n dx \leq \int_0^1 f(x) dx, \quad \text{hence} \quad \sup_{n \in \mathbb{N}} \mathcal{S} = \int_0^1 f(x) dx$$

when \mathbb{N} is defined as the set of integers that are greater or equal than 1.

Editor's comments. Several comments included in the submissions are worth a mention. First, Kathleen Lewis indicated that if the set of natural numbers is assumed to include 0, then the supremum of \mathcal{S} is 1. Additionally, Roy Barbara pointed out that under the same hypotheses the lower bound of the set \mathcal{S} can be found. If we denote by μ the Lebesgue measure on $[0, 1]$, then $f^{-1}\{1\}$ is a closed, compact set and $\mu(f^{-1}\{1\})$ exists. The infimum of \mathcal{S} is $\mu(f^{-1}\{1\})$.

4325. Proposed by Alessandro Ventullo.

Solve in real numbers the system of equations:

$$\begin{aligned} x^4 - 2y^3 - x^2 + 2y &= -1 + 2\sqrt{5} \\ y^4 - 2x^3 - y^2 + 2x &= -1 - 2\sqrt{5}. \end{aligned}$$

We received 10 correct and complete submissions. We present the solution by the proposer. Similar solutions were submitted by Šefket Arslanagić and the group of Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith.

Adding the two equations, we get

$$\begin{aligned} (x^4 - 2x^3 - x^2 + 2x + 1) + (y^4 - 2y^3 - y^2 + 2y + 1) &= 0 && \Longleftrightarrow \\ (x^2 - x - 1)^2 + (y^2 - y - 1)^2 &= 0 && \Longleftrightarrow \\ x^2 - x - 1 = 0 \quad \text{and} \quad y^2 - y - 1 &= 0. \end{aligned}$$

So, $x, y \in \left\{ \frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2} \right\}$. Let $\alpha = \frac{1 - \sqrt{5}}{2}$ and $\beta = \frac{1 + \sqrt{5}}{2}$. Since

$$\begin{aligned} \alpha^2 &= \alpha + 1, \\ \beta^2 &= \beta + 1, \\ \alpha^3 &= \alpha(\alpha + 1) = 2\alpha + 1, \\ \beta^3 &= \beta(\beta + 1) = 2\beta + 1, \\ \alpha^4 &= \alpha(2\alpha + 1) = 3\alpha + 2, \\ \beta^4 &= \beta(2\beta + 1) = 3\beta + 2, \end{aligned}$$

then

$$\alpha^4 - 2\beta^3 - \alpha^2 + 2\beta = (3\alpha + 2) - 2(2\beta + 1) - (\alpha + 1) + 2\beta = 2\alpha - 2\beta - 1 = -1 - 2\sqrt{5}$$

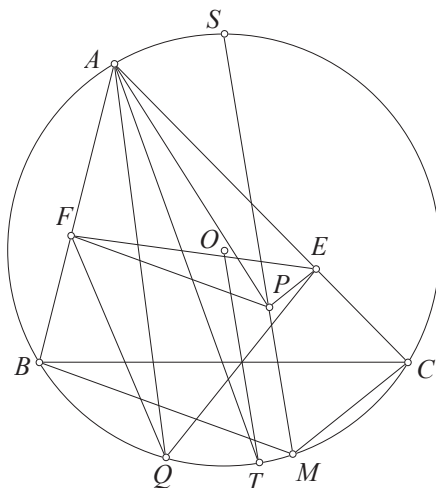
and by symmetry,

$$\beta^4 - 2\alpha^3 - \beta^2 + 2\alpha = 2\beta - 2\alpha - 1 = -1 + 2\sqrt{5}.$$

It follows that $x = \frac{1 + \sqrt{5}}{2}$ and $y = \frac{1 - \sqrt{5}}{2}$.

4326. *Proposed by Tran Quang Hung.*

Let ABC be a triangle inscribed in circle (O) . Suppose S is the midpoint of arc BC containing A , T is a point on arc BC not containing A , M is on (O) such that $SM \parallel OT$, P is a point on SM . Let points E and F lie on CA and AB , respectively, such that $PE \parallel MC$ and $PF \parallel MB$. Finally, let Q be on (O) such that AT is bisector of $\angle PAQ$. Prove that $QE = QF$.



We received 4 submissions, of which 3 were correct and one was incomplete. We feature the proposer's solution modified by the editor, who introduced directed angles. The proposer's original argument depended upon his diagram.

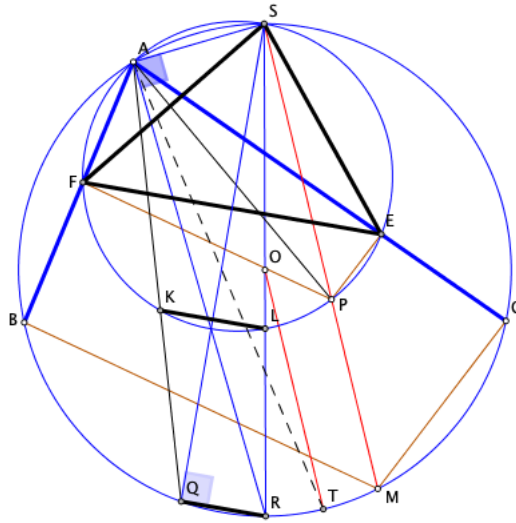
All angles are directed (between 0° and 180°). Observe that $\angle AFP = \angle ABM = \angle ASM = \angle ASP$, whence F lies on circle (ASP) . Similarly, E also lies on (ASP) . Because AS is the external bisector of $\angle FAE (= \angle BAC)$, we have

$$SE = SF. \quad (1)$$

Define K and L to be the points where AQ and SO again intersect (ASP) , while we let R be the other end of the diameter of (O) through S . We see $\angle AKL = \angle ASL = \angle ASR = \angle AQR$. From this

$$QR \parallel KL \quad \text{and (because } SR \text{ is a diameter of } (O)) \quad QR \perp QS.$$

We will prove that $KL \parallel EF$, in which case we would have $QS \perp EF$ so that from (1) we will be able to conclude, finally, that $QE = QF$.



Indeed, we have by angle chasing,

$$\begin{aligned}\angle RAL &= \angle RAT + \angle TAL \\ &= \angle RAT + (\angle TAP + \angle PAL) \\ &= \angle RAT + \angle QAT + \angle PSL \\ &= \angle RAT + \angle QAT + \angle MSR \\ &= \angle RAT + \angle QAT + \angle TOR \\ &= \angle QAT + \angle RAT + 2\angle TAR \\ &= \angle QAT + \angle TAR = \angle QAR.\end{aligned}$$

Therefore AR bisects $\angle QAL = \angle KAL$; but AR also bisects $\angle BAC = \angle FAE$, so that $KL \parallel EF$ as claimed. Now we are done.

4327. *Proposed by Daniel Sitaru.*

Prove the following inequality for all $x > 0$:

$$\arctan(x) \arctan\left(\frac{1}{x}\right) < \frac{\pi}{2(x^2 + 1)}.$$

We received 8 submissions besides the original proposal. All submitted solutions pointed out that the conclusion in the given problem was incorrect. This error was caused by a small typo when the problem was printed. The right hand side of the given inequality should be $\frac{\pi x}{2(x^2+1)}$ instead of $\frac{\pi}{2(x^2+1)}$. The printed inequality

was clearly incorrect since if $x = \sqrt{3}$, then

$$\arctan(x) \cdot \arctan\left(\frac{1}{x}\right) = \left(\frac{\pi}{3}\right) \left(\frac{\pi}{6}\right) = \frac{\pi^2}{18} > \frac{\pi}{8} = \frac{\pi}{2((\sqrt{3})^2 + 1)}.$$

This was given by a few solvers. In addition, several solvers also provided, with proof, the correct inequality. We present the proof by Michel Bataille, enhanced by the editor.

Let

$$f(x) = \frac{\pi x}{2(x^2 + 1)} - \arctan(x) \cdot \arctan\left(\frac{1}{x}\right)$$

for $x > 0$. Since $\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$, we have

$$f(x) = \frac{\pi x}{2(x^2 + 1)} - \frac{\pi}{2} \arctan(x) + (\arctan(x))^2.$$

We need to show that $f(x) > 0$ for all $x > 0$.

Since $f\left(\frac{1}{x}\right) = f(x)$, it suffices to show that $f(x) > 0$ for $x \in (0, 1]$.

By straightforward computations, we have

$$\begin{aligned} f'(x) &= \frac{\pi}{2} \left(\frac{1-x^2}{(x^2+1)^2} - \frac{1}{x^2+1} \right) + \frac{2\arctan(x)}{x^2+1} \\ &= \frac{\pi}{2} \cdot \frac{-2x^2}{(x^2+1)^2} + \frac{2\arctan(x)}{x^2+1} \\ &= \frac{2g(x)}{x^2+1}, \end{aligned}$$

where $g(x) = \arctan(x) - \frac{\pi}{2} \cdot \frac{x^2}{x^2+1}$.

Now,

$$g'(x) = \frac{1}{x^2+1} - \frac{\pi x}{(x^2+1)^2} = \frac{x^2 - \pi x + 1}{(x^2+1)^2},$$

so setting $g'(x) = 0$ we have

$$x = \frac{1}{2} \left(\pi \pm \sqrt{\pi^2 - 4} \right).$$

Since $\frac{1}{2} \left(\pi + \sqrt{\pi^2 - 4} \right) > 1$, we have $x = \frac{1}{2} \left(\pi - \sqrt{\pi^2 - 4} \right)$.

Let $\alpha = \frac{1}{2} \left(\pi - \sqrt{\pi^2 - 4} \right)$. Then $\alpha \in (0, 1)$, $g'(\alpha) = 0$, $g'(x) > 0$ for $x \in (0, \alpha)$, and $g'(x) < 0$ for $x \in (\alpha, 1]$.

Since $g(1) = \arctan(1) - \frac{\pi}{4} = 0 = g(0)$, it follows that $g(x) > 0$ for all $x \in (0, 1)$ so f is increasing on $(0, 1]$. Finally, since $\lim_{x \rightarrow 0^+} f(x) = 0$, we conclude that $f(x) > 0$ for all $x \in (0, 1]$ as claimed.

4328. *Proposed by Van Khea and Leonard Giugiuc.*

A circle I is inscribed in a triangle ABC and the points of tangency on the sides BC, CA and AB are D, E and F , respectively. The rays AD, BE and CF cut the circle I in points X, Y, Z , respectively. Prove that

$$\frac{1}{\frac{AX}{XD} + \frac{1}{4}} + \frac{1}{\frac{BY}{YE} + \frac{1}{4}} + \frac{1}{\frac{CZ}{ZF} + \frac{1}{4}} = 4.$$

We received 6 submissions, all of which were correct, and present the solution by C.R. Pranesachar.

Let the tangents from the vertices of triangle ABC to its incircle be denoted by

$$x = AF = AE, \quad y = BD = BF, \quad \text{and} \quad z = CD = CE.$$

Then by Stewart's theorem,

$$(y+z)AD^2 = y(z+x)^2 + z(x+y)^2 - \frac{yz}{y+z}(y+z)^2.$$

Simplification gives

$$AD^2 = x^2 + \frac{4xyz}{y+z}.$$

By the tangent-secant theorem, we have $AX \cdot AD = AF^2 = x^2$. Hence $AX = \frac{x^2}{AD}$.

Also

$$XD = AD - AX = AD - \frac{x^2}{AD} = \frac{AD^2 - x^2}{AD}.$$

Therefore

$$\frac{AX}{XD} = \frac{x^2}{AD^2 - x^2} = \frac{x^2}{\frac{4xyz}{y+z}} = \frac{x(y+z)}{4yz},$$

with analogous expressions for $\frac{BY}{YE}$ and $\frac{CZ}{ZF}$. Hence

$$\begin{aligned} \frac{1}{\frac{AX}{XD} + \frac{1}{4}} + \frac{1}{\frac{BY}{YE} + \frac{1}{4}} + \frac{1}{\frac{CZ}{ZF} + \frac{1}{4}} &= \frac{1}{\frac{x(y+z)}{4yz} + \frac{1}{4}} + \frac{1}{\frac{y(z+x)}{4zx} + \frac{1}{4}} + \frac{1}{\frac{z(x+y)}{4xy} + \frac{1}{4}} \\ &= \frac{4yz}{yz + zx + xy} + \frac{4zx}{yz + zx + xy} + \frac{4xy}{yz + zx + xy} \\ &= 4. \end{aligned}$$

This completes the proof.

Editor's Comments. Both Pranesachar and Volkhard Schindler reported that a similar problem was proposed by Abdul Hanjnan of Chennai, India:

Prove that

$$\frac{AX}{XD} + \frac{BY}{YE} + \frac{CZ}{ZF} = \frac{R}{r} - \frac{1}{2}.$$

It appeared as Problem 12027 in the *American Mathematical Monthly* **125**:3, page 276. Curiously, both problems appeared in the March 2018 issue of their respective journals and had the same deadline.

4329. *Proposed by Mihaela Berindeanu.*

For $x, y, z \geq 1$, show that

$$\frac{\log_2 xy}{(\log_2 2z)^2} + \frac{\log_2 yz}{(\log_2 2x)^2} + \frac{\log_2 xz}{(\log_2 2y)^2} \geq \frac{\log_2 xyz}{1 + (\log_2 \sqrt[3]{xyz})^2}.$$

We received 9 submissions, all of which were correct. We present the solution by the AN-anduud Problem Solving Group.

Let $a = \log_2 x$, $b = \log_2 y$, and $c = \log_2 z$. Then $a, b, c > 0$ and the given inequality is equivalent to

$$\frac{a+b}{(1+c)^2} + \frac{b+c}{(1+a)^2} + \frac{c+a}{(1+b)^2} \geq \frac{a+b+c}{1 + \left(\frac{a+b+c}{3}\right)^2}$$

or

$$\sum_{cyc} \frac{a+b}{2(a+b+c)} \cdot \frac{1}{(1+c)^2} \geq \frac{a+b+c}{2\left(1 + \left(\frac{a+b+c}{3}\right)^2\right)} \quad (1)$$

Let $f(x) = \frac{1}{(1+x)^2}$, $x \geq 0$. Then $f''(x) = \frac{6}{(1+x)^4} > 0$ so f is convex on $[0, \infty)$. By Jensen's Inequality we have

$$\sum_{cyc} \frac{a+b}{2(a+b+c)} \cdot f(c) \geq f\left(\frac{(a+b)c + (b+c)a + (c+a)b}{2(a+b+c)}\right) = f\left(\frac{ab+bc+ca}{a+b+c}\right)$$

or

$$\sum_{cyc} \frac{a+b}{2(a+b+c)} \cdot \frac{1}{(1+c)^2} \geq \frac{1}{\left(1 + \frac{ab+bc+ca}{a+b+c}\right)^2} \quad (2)$$

By AM-GM Inequality, we have

$$\begin{aligned} 1 + \left(\frac{a+b+c}{3}\right)^2 &\geq 2\left(\frac{a+b+c}{3}\right) \\ &= \frac{2}{3} \cdot \frac{(a+b+c)^2}{a+b+c} \\ &\geq \frac{2}{3} \cdot \frac{3(ab+bc+ca)}{a+b+c} \\ &= \frac{2(ab+bc+ca)}{a+b+c}. \end{aligned} \quad (3)$$

Also,

$$(a + b + c)^2 \geq 3(ab + bc + ca)$$

implies that

$$\frac{a + b + c}{3} \geq \frac{ab + bc + ca}{a + b + c}. \quad (4)$$

From (3) and (4) we have

$$1 + 2\left(\frac{a + b + c}{3}\right)^2 \geq \frac{2(ab + bc + ca)}{a + b + c} + \left(\frac{ab + bc + ca}{a + b + c}\right)^2$$

or

$$2\left(1 + \left(\frac{a + b + c}{3}\right)^2\right) \geq \left(1 + \frac{ab + bc + ca}{a + b + c}\right)^2 \quad (5)$$

From (2) and (5), we then obtain (1), completing the proof.

4330★. *Proposed by Mohammed Aassila.*

Let a and b be integers such that $a^2 - 20b + 24 = 0$. Find the complete set of solutions of the following equation over integers:

$$5x^2 + axy + by^2 = 11.$$

There were 8 correct solutions and 1 incorrect submission.

Since $a^2 + 4$ is a multiple of 10, $a = 10c \pm 4$ for some integer c , whereupon $b = 5c^2 \pm 4c + 2$.

Multiplying the equation by 20 and completing the square yields

$$(10x + ay)^2 = 4(55 - 6y^2).$$

Since $55 - 6y^2$ has to be square, $y = \pm 1$ or $y = \pm 3$. Since (x, y) satisfies the equation if and only if $(-x, -y)$ does, we consider the cases $y = -1$ and $y = -3$.

Suppose $y = -1$. Then

$$10x - a = 14 \quad \text{or} \quad 10x - a = -14.$$

In the first case, a must have the form $10c - 4$, whence $x = c + 1$. In the second case, $a = 10c + 4$ and $x = c - 1$.

Suppose $y = -3$. Then

$$10x - 3a = 2 \quad \text{or} \quad 10x - 3a = -2.$$

If $10x - 3a = 2$, then $a = 10c - 4$ and $x = 3c - 1$. If $10x - 3a = -2$, then $a = 10c + 4$ and $x = 3c + 1$.

Thus, for the equation to be solvable, we need $(a, b) = (10c \pm 4, 5c^2 \pm 4c + 2)$. The solutions of

$$5x^2 + (10c + 4)xy + (5c^2 + 4c + 2)y^2 = 11$$

are

$$(x, y) = (c - 1, -1), (-c + 1, 1), (3c + 1, -3), (-3c - 1, 3)$$

and the solutions of

$$5x^2 + (10c - 4)xy + (5c^2 - 4c + 2)y^2 = 11$$

are

$$(x, y) = (c + 1, -1), (-c - 1, 1), (3c - 1, -3), (-3c + 1, 3).$$

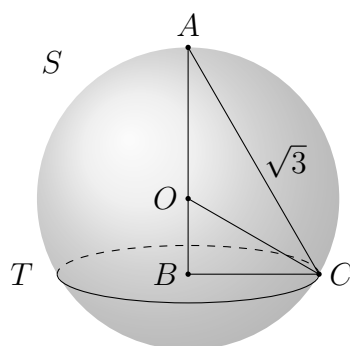
These work.

4331. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let S be a unit sphere. Suppose that the surface of S is coloured with 4 distinct colours. Prove that there exist two points $X, Y \in S$ of the same colour with $|XY| \in \{\sqrt{3}, \sqrt{3}/2\}$.

We received 8 solutions. We present the solution by Srihari Ramanujan, slightly edited.

Consider the unit sphere S , centred at O . Let A be a point on the sphere. The points on S which are at a distance of $\sqrt{3}$ away from A form a circle T centred at B with radius BC (for any point C on T), as shown in the figure below.



Since $\angle OBC$ is a right angle, we have

$$AC^2 - AB^2 = OC^2 - OB^2$$

which yields $OB = 1/2$ (using $OC = 1$ and $AB = 1 + OB$) and

$$BC^2 = OC^2 - OB^2 = 3/4.$$

Therefore the diameter of T is $\sqrt{3}$.

Now consider a square $CDEF$ inscribed in T . The diagonal has length $\sqrt{3}$ and thus the sidelength is $\sqrt{3}/2$. So the distances between the four points C , D , E , and F are either $\sqrt{3}/2$ or $\sqrt{3}$, whereas the distance between A and any of the four points is $\sqrt{3}$. By the pigeonhole principle two of the points A , C , D , E , and F must receive the same colour, yielding two points on S of the same colour of distance $\sqrt{3}$ or $\sqrt{3}/2$.

4332. *Proposed by S. Muralidharan.*

Draw the family of circles of radius $\frac{1}{2}$ with centers at (i, j) where i, j are integers. Prove that a line joining centers of any two of these circles cannot be tangent to any circle in the family.

Of the 13 submissions, 12 were correct; we feature a shortened version of the solution by the Missouri State University Problem Solving Group.

Assume that P , Q , and R are points with integer coordinates such that the line PQ is tangent to the circle centered at R with radius $\frac{1}{2}$. After a translation, we may assume $P = (0, 0)$, $Q = (a, b)$ and $R = (h, k)$. We may also assume $\gcd(a, b) = 1$ since otherwise we may replace Q with

$$\left(\frac{a}{\gcd(a, b)}, \frac{b}{\gcd(a, b)} \right).$$

If we consider the vectors $v = \overrightarrow{PQ}$ and $u = \overrightarrow{PR}$, then the distance from R to the line PQ is

$$\frac{|u \times v|}{|v|} = \frac{1}{2}.$$

Since

$$|u \times v| = |ak - bh| \quad \text{and} \quad |v| = \sqrt{a^2 + b^2},$$

this gives

$$4(ak - bh)^2 = a^2 + b^2.$$

But the only way for $a^2 + b^2$ to be a multiple of 4 is if both a and b are even, which would contradict the assumption that $\gcd(a, b) = 1$.

More generally, if the radius of the circles is a rational number $r = p/q$ with $\gcd(p, q) = 1$, then there is a line through the centers of two circles that is tangent to a third circle if and only if $q = 1$ or all the prime factors of q are of the form $4i + 1$.

Editor's comments. The authors provided a proof of their generalization, but it is simply the featured solution together with an exercise in elementary number theory that relies on two related facts: the congruence

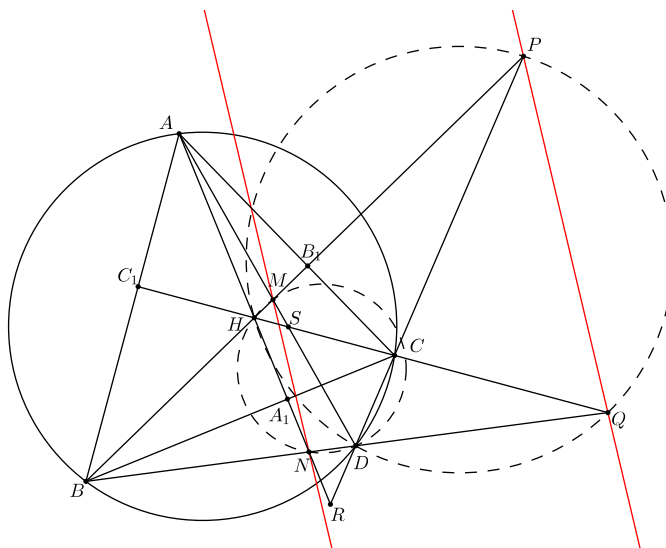
$$x^2 \equiv -1 \pmod{4i + 3}$$

has no solution, and a prime of the form $4i + 1$ can be written as the sum of two squares.

4333. *Proposed by Mihai Miculița and Titu Zvonaru.*

Let $ABCD$ be a cyclic quadrilateral and let A_1, B_1 and C_1 be orthogonal projections of the points A, B and C onto the lines BC, CA and AB , respectively. We denote $M = DA \cap BB_1$, $N = DB \cap AA_1$, $P = DC \cap BB_1$, $Q = DB \cap CC_1$, $R = DC \cap AA_1$ and $S = DA \cap CC_1$. Prove that $MN \parallel PQ \parallel RS$.

We received 4 submissions, all of which were correct, and feature the solution by Michel Bataille.



Let H be the orthocenter of the triangle ABC . Since A, B, C, D are concyclic, we have

$$\angle(DB, DA) = \angle(CB, CA) \neq 0 \pmod{\pi}$$

and so

$$\begin{aligned} \angle(DN, DM) &= \angle(DB, DA) = \angle(CB, CA) = \angle(HA_1, HB_1) = \angle(HN, HM) \\ &\neq 0 \pmod{\pi} \end{aligned}$$

(using $HA_1 \perp CB$ and $HB_1 \perp CA$).

It follows that H, D, M, N lie on a circle. In the same way, H, D, P, Q lie on a circle.

Now, the result $MN \parallel PQ$ is deduced from

$$\begin{aligned} \angle(MN, PQ) &= \angle(MN, ND) + \angle(DQ, QP) \quad (N, D, Q \text{ collinear}) \\ &= \angle(HM, HD) + \angle(HD, HP) \quad (\text{concyclicity}) \\ &= \angle(HM, HP) \\ &= 0 \pmod{\pi} \quad (H, M, P \text{ collinear}). \end{aligned}$$

Similarly, we prove that $RS \parallel PQ$ and the required result follows.

4334. *Proposed by George Stoica.*

Let $(a_n)_{n \geq 1}$ and $(x_n)_{n \geq 1}$ be sequences of real numbers such that $\frac{a_n}{x_n} \searrow 0$. Put $b_1 = 0$ and

$$b_n = a_1 + \cdots + a_{n-1} - \frac{x_1 + \cdots + x_{n-1}}{x_n} a_n$$

for $n \geq 2$. Prove that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence $(b_n)_{n \geq 1}$ converges and that, in this case, $\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} b_n$.

There was one correct solution, one partial and one incorrect solution. We present both the correct and the partial solutions.

Full solution, by Madhav Modak.

We will suppose that all the elements of the sequences $\{a_n\}$ and $\{x_n\}$ are positive.

Let $u_n = a_1 + a_2 + \cdots + a_n$ and $v_n = x_1 + x_2 + \cdots + x_n$ for each positive integer n . Then

$$b_n = a_1 + a_2 + \cdots + a_{n-1} - \frac{x_1 + \cdots + x_{n-1}}{x_n} a_n = u_n - v_n \frac{a_n}{x_n}.$$

Using the fact that when $p/q \leq r/s$ for two positive fractions, then

$$\frac{p}{q} \leq \frac{p+r}{q+s} \leq \frac{r}{s},$$

and applying an induction argument, we have that, for $n > m \geq 1$,

$$\frac{a_n}{x_n} \leq \frac{a_{m+1} + \cdots + a_n}{x_{m+1} + \cdots + x_n} \leq \frac{a_{m+1}}{x_{m+1}}$$

whence

$$(v_n - v_m) \frac{a_n}{x_n} \leq u_n - u_m. \quad (1)$$

Suppose that $\sum a_n$ converges. Let $\epsilon > 0$ be given. Fix p so that, for all $n \geq p$, $0 < u_n - u_p < \epsilon/2$. Now fix $q \geq p$, so that for $n \geq q$, $v_p(a_n/x_n) < \epsilon/2$. Then

$$0 < v_n \frac{a_n}{x_n} = v_p \frac{a_n}{x_n} + (v_n - v_p) \frac{a_n}{x_n} \leq v_p \frac{a_n}{x_n} + (u_n - u_p) < \epsilon.$$

Hence $\lim_{n \rightarrow \infty} v_n(a_n/x_n) = 0$, and therefore $\lim_{n \rightarrow \infty} b_n$ exists and is equal to

$$\lim_{n \rightarrow \infty} u_n = \sum_{n=1}^{\infty} a_n.$$

On the other hand, suppose $\sum a_n$ diverges. Let $M > 0$ be given, and choose p so that $u_p > M + 1$. Select $q \geq p$ so that $0 < v_p(a_q/x_q) < 1$. Then, by (1),

$$b_q = u_q - v_q \frac{a_q}{x_q} > u_p - v_p \frac{a_q}{x_q} > M + 1 - 1 = M.$$

Hence the sequence $\{b_n\}$ is unbounded and therefore diverges.

Partial solution, by the proposer.

We prove that the convergence of $\sum a_n$ implies the convergence of $\{b_n\}$, and also that $\sum a_n = \lim b_n$. This proof is valid when a_n and x_n are allowed to take both positive and negative values in tandem.

Since

$$b_n = a_1 + \cdots + a_n - \frac{x_1 + \cdots + x_n}{x_n} a_n$$

for $n \geq 2$, it suffices to prove that $a_n(x_1 + \cdots + x_n)/x_n$ converges to 0. For $n > m$, define $s_n = a_{m+1} + \cdots + a_n$.

Let $\epsilon > 0$ and choose $m > 0$ such that $|s_n| < \epsilon/4$ for $n > m$. Then

$$\begin{aligned} & \left| \frac{a_n}{x_n} (x_{m+1} + \cdots + x_n) \right| \\ &= \frac{a_n}{x_n} \left| \frac{x_{m+1}}{a_{m+1}} \cdot a_{m+1} + \cdots + \frac{x_n}{a_n} \cdot a_n \right| \\ &= \frac{a_n}{x_n} \left| \frac{x_{m+1}}{a_{m+1}} \cdot s_{m+1} + \frac{x_{m+2}}{a_{m+2}} \cdot (s_{m+2} - s_{m+1}) + \cdots + \frac{x_n}{a_n} \cdot (s_n - s_{n-1}) \right| \\ &= \frac{a_n}{x_n} \left| \left(\frac{x_{m+1}}{a_{m+1}} - \frac{x_{m+2}}{a_{m+2}} \right) \cdot s_{m+1} + \cdots + \left(\frac{x_{n-1}}{a_{n-1}} - \frac{x_n}{a_n} \right) \cdot s_{n-1} + \frac{x_n}{a_n} \cdot s_n \right| \\ &\leq \frac{a_n}{x_n} \left[\left(\frac{x_{m+2}}{a_{m+2}} - \frac{x_{m+1}}{a_{m+1}} \right) |s_{m+1}| + \cdots + \left(\frac{x_n}{a_n} - \frac{x_{n-1}}{a_{n-1}} \right) |s_{n-1}| + \frac{x_n}{a_n} |s_n| \right] \\ &\leq \left(\frac{\epsilon}{4} \right) \frac{a_n}{x_n} \left[\left(\frac{x_{m+2}}{a_{m+2}} - \frac{x_{m+1}}{a_{m+1}} \right) + \cdots + \left(\frac{x_n}{a_n} - \frac{x_{n-1}}{a_{n-1}} \right) + \frac{x_n}{a_n} \right] \\ &\leq \left(\frac{\epsilon}{4} \right) \frac{a_n}{x_n} \left(2 \cdot \frac{x_n}{a_n} - \frac{x_{m+1}}{a_{m+1}} \right) < \frac{\epsilon}{2}. \end{aligned}$$

Since $a_n/x_n \rightarrow 0$, there is an index $p > m$ such that

$$\left| \frac{a_n}{x_n} (x_1 + \cdots + x_m) \right| < \frac{\epsilon}{2}$$

for all $n > p$. Thus

$$\left| \frac{a_n}{x_n} (x_1 + \cdots + x_n) \right| \leq \left| \frac{a_n}{x_n} (x_1 + \cdots + x_m) \right| + \left| \frac{a_n}{x_n} (x_{m+1} + \cdots + x_n) \right| < \epsilon$$

for all $n > p$. The desired result follows.

Comment by the editor. In the problem, there seems to be a tacit assumption that the sequences $\{a_n\}$ and $\{x_n\}$ both be positive. As we have seen, the result holds when $\sum a_n$ is convergent but not necessarily positive (with the x_n changing sign in tandem with a_n). However, it is possible for $\sum a_n$ to diverge while $\{b_n\}$ converges.

Let

$$a_n = (-1)^{n-1} \quad \text{and} \quad x_n = (-1)^{n-1}n$$

for each positive integer n , so that $a_n/x_n = 1/n$. Then, with the notation of the first solution,

$$\begin{aligned} u_n &= 1 \quad \text{and} \quad v_n = \frac{1}{2}(n+1) \quad \text{when } n \text{ is odd,} \\ u_n &= 0 \quad \text{and} \quad v_n = -\frac{1}{2}n \quad \text{when } n \text{ is even.} \end{aligned}$$

Thus $b_n = 1 - \frac{n+1}{2n}$ when n is odd and $b_n = 0 + \frac{n}{2n}$ when n is even. Hence $\{b_n\}$ converges to the limit $\frac{1}{2}$.

However, with the same sequence $\{a_n\}$ and $x_n = (-1)^{n-1}2^{n-1}$, the sequence $\{b_n\}$ will diverge.

4335. *Proposed by Leonard Giugiuc.*

Let a and b be fixed positive real numbers and let $n \geq 2$ be an integer. Prove that for any nonnegative real numbers $x_i, i = 1, \dots, n$ such that $x_1 + \dots + x_n = 1$, we have

$$\sqrt[3]{ax_1 + b} + \sqrt[3]{ax_2 + b} + \dots + \sqrt[3]{ax_n + b} \geq \sqrt[3]{a + b} + (n-1)\sqrt[3]{b}.$$

We received 11 correct solutions. We present the solution by AN-anduud Problem Solving Group.

Consider the function

$$f(x) = \sqrt[3]{ax + b}, \quad x \geq 0.$$

Note that $f(x)$ function is concave on $[0, +\infty)$. Using $x_k = (1 - x_k) \cdot 0 + x_k \cdot 1$, we have

$$f(x_k) \geq (1 - x_k)f(0) + x_k f(1) \iff \sqrt[3]{ax_k + b} \geq (1 - x_k) \cdot \sqrt[3]{b} + x_k \cdot \sqrt[3]{a + b}.$$

Adding the above inequalities and using $x_1 + x_2 + \dots + x_n = 1$, gives our inequality. Equality holds only for $\{x_1, x_2, \dots, x_n\} = \{1, 0, \dots, 0\}$ and permutations.

