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CONTENTS

THE OLYMPIAD CORNER: 74

M.S. KLAMKIN

All communications about this column should be sent to M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada, T6G 2G1.

This month's problem set consists of the 4th Annual American Invitational Mathematics Examination (AIME). I am grateful to Professor Walter Mientka for these problems. For a description of the AIME, see [1983: 170]. The answers to these problems will be given next month. Any questions or comments about this AIME should be addressed to Professor George Berzsenyi, Department of Mathematics, Lamar University, Beaumont, Texas 77710.

4th Annual American Invitational Mathematics Examination
March 18, 1986 -- Time: 3 hours

1. What is the sum of the solutions of the equation $\sqrt[4]{x} = \dfrac{12}{7 - \sqrt[4]{x}}$?

2. Evaluate the product
$(\sqrt{5} + \sqrt{6} + \sqrt{7})(\sqrt{5} + \sqrt{6} - \sqrt{7})(\sqrt{5} - \sqrt{6} + \sqrt{7})(-\sqrt{5} + \sqrt{6} + \sqrt{7})$.

3. If $\tan x + \tan y = 25$ and $\cot x + \cot y = 30$, what is $\tan(x + y)$?

4. Determine $3x_4 + 2x_5$ if x_1, x_2, x_3, x_4 and x_5 satisfy the system of equations given below:
$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 + x_5 &= 6 \\ x_1 + 2x_2 + x_3 + x_4 + x_5 &= 12 \\ x_1 + x_2 + 2x_3 + x_4 + x_5 &= 24 \\ x_1 + x_2 + x_3 + 2x_4 + x_5 &= 48 \\ x_1 + x_2 + x_3 + x_4 + 2x_5 &= 96. \end{aligned}$$

5. What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

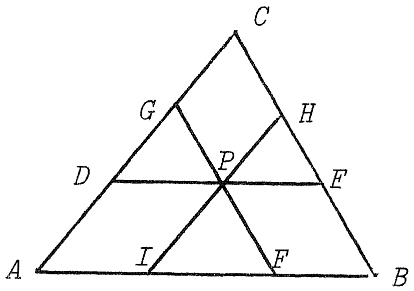
6. The pages of a book are numbered 1 through n. When the page numbers of the book were added, one of the page numbers was mistakenly added twice, resulting in the incorrect sum of 1986. What was the number of the page that was added twice?

7. The increasing sequence 1, 3, 4, 9, 10, 12, 13, ... consists of all those positive integers which are powers of 3 or sums of distinct

powers of 3. Find the 100^{th} term of this sequence (where 1 is the 1^{st} term, 3 is the 2^{nd} term, and so on).

8. Let S be the sum of the base 10 logarithms of all of the proper divisors of 1,000,000. (By a proper divisor of a natural number we mean a positive integral divisor other than 1 and the number itself.) What is the integer nearest to S?

9. In ΔABC shown below, $AB = 425$, $BC = 450$ and $CA = 510$. Moreover, P is an interior point chosen so that the segments DE, FG and HI are each of length d, contain P, and are parallel to the sides AB, BC and CA, respectively. Find d.



10. In a parlor game the "magician" asks one of the participants to think of a three-digit number (abc), where a, b and c represent digits in base 10 in the order indicated. Then the magician asks this person to form the numbers (acb), (bac), (bca), (cab) and (cba), to add these five numbers, and to reveal their sum, N. If told the value of N, the magician can identify the original number, (abc). Play the role of the magician and determine (abc) if $N = 3194$.

11. The polynomial $1 - x + x^2 - x^3 + \ldots + x^{16} - x^{17}$ may be written in the form $a_0 + a_1y + a_2y^2 + a_3y^3 + \ldots + a_{16}y^{16} + a_{17}y^{17}$, where $y = x + 1$ and the a_i's are constants. Find the value of a_2.

12. Let the sum of a set of numbers be the sum of its elements. Let S be a set of positive integers, none greater than 15. Suppose no two disjoint subsets of S have the same sum. What is the largest sum a set S with these properties can have?

13. In a sequence of coin tosses one can keep a record of the number of instances when a tail is immediately followed by a head, a head is immediately followed by a head, etc. We denote these by TH, HH, etc. For example, in the sequence $HHTTHHHHTHHTTTT$ of 15 coin tosses we observe that there are five HH, three HT, two TH and four TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH and five TT subsequences?

14. The shortest distances between an interior diagonal of a rectangular parallelepiped (box), P, and the edges it does not meet are $2\sqrt{5}$, $30/\sqrt{13}$ and $15/\sqrt{10}$. Determine the volume of P.

15. Let ΔABC be a right triangle in the xy-plane with the right angle at C. Given that the length of the hypotenuse AB is 60, and that the medians through A and B lie along the lines $y = x + 3$ and $y = 2x + 4$, respectively, find the area of ΔABC.

*

I now give solutions to some previous problems.

VI ALL-RUSSIAN MATHEMATICAL OLYMPIAD 1979-80 [1982: 72]

VIII Grade

1. A group of tourists decided to sit in a set of buses in such a way that each bus would contain the same number of tourists. At first they tried to sit 22 on each bus, but it turned out that one tourist was left over. But then one bus left empty, and the tourists were able to divide themselves equally among the remaining buses. If each bus holds fewer than 33 people, how many buses and how many tourists were there (originally)?

Solution.

Let n and b denote the number of tourists and buses, respectively. Then $22b = n - 1$ and $k(b - 1) = n$ where $k < 33$. Eliminating n,

$$b = \frac{k+1}{k-22} = 1 + \frac{23}{k-22}\ .$$

Thus (since b is an integer), $k = 23$, $b = 24$ and $n = 529$.

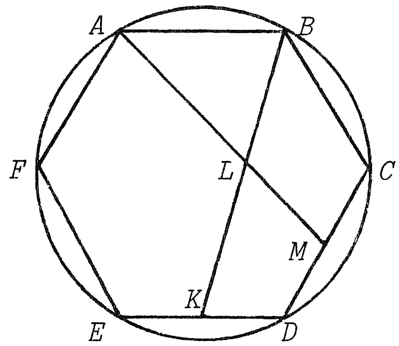
2. Along a segment AB, $2n$ points are chosen which are symmetric in pairs with respect to the midpoint of the segment. Any n of these points are colored blue, and the rest are colored red. Prove that the sum of the distances from the red points to A is equal to the sum of the distances from the blue points to B.

Solution.

We coordinatize the points on a line. Let $A = -K$, $B = K$. Also, let the red points be $R_1, R_2, \ldots, R_n$ and the blue points be $B_1, B_2, \ldots, B_n$. Then the sum of the distances from the red points to A is $\Sigma(R_i + K)$, and the sum of the distances from the blue points to B is $\Sigma(K - B_i)$. These two sums are equal since by centrosymmetry $\Sigma(R_i + B_i)$, the sum of the coordinates of all $2n$ points, is zero.

3. In a regular hexagon $ABCDEF$, points M and K are the midpoints of CD and DE, respectively, and L is the intersection of segments AM and BK. Prove that the area of triangle ABL is equal to that of quadrilateral $MDKL$. Also, find the measure of the angle between lines AM and BK.

Solution.



If we rotate AM 60° about the center of the circle it goes into BK by rotational symmetry. Thus $\angle ALB = 60^\circ$. Also, Area $ABCM$ = Area $BCDK$. Thus Area ABL = Area $MDKL$.

4. If $\{x\}$ denotes the fractional part of x (e.g., $\{7/5\} = 2/5$), how many distinct numbers are there in the sequence

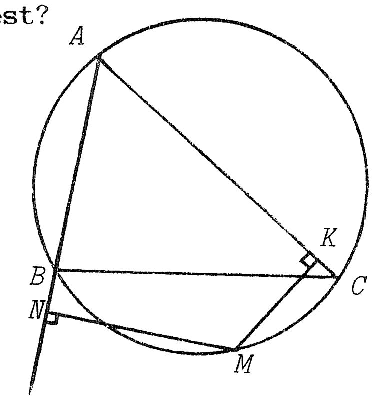
$\{1^2/1980\}$, $\{2^2/1980\}$, $\{3^2/1980\}$, ... , $\{1980^2/1980\}$?

Solution.

See [1984: 290].

5. From a point M on the circumcircle of a triangle ABC, perpendiculars MN and MK are drawn to lines AB and AC, respectively. For which point M will NK be longest?

Solution.



$ANMK$ is cyclic with circumdiameter AM. Also, $\angle NAK$ is a fixed angle. Consequently, NK will be a maximum when AM is a maximum, i.e. M is antipodal (diametrically opposite) to A.

IX Grade

1. Can the natural numbers from 1 to 30 be arranged in a 5×6 rectangular array in such a way that (a) all columns have the same sum and (b) all rows have the same sum?

Solution.

The sum of all the numbers is $30(1 + 30)/2 = 465$. If the sum of each column is C and the sum of each row is R, then $6C = 5R = 465$. Clearly, this is impossible for (a) the column sums. However, it is possible for (b) the row sums, as indicated in the following array:

1	6	11	20	25	30
2	7	12	19	24	29
3	8	13	18	23	28
4	9	14	17	22	27
5	10	15	16	21	26

2. For which natural numbers n is $2^8 + 2^{11} + 2^n$ a perfect square? [This problem also appeared in the 1981 Hungarian Mathematical Olympiad. See [1981: 267; 1982: 46].]

3. Each vertex of a convex $(2n + 1)$-gon is colored with one of three different colors. No two adjacent vertices are colored the same. Prove that the polygon can be partitioned by nonintersecting diagonals into a set of triangles each of which has its three vertices of different colors. [Diagonals which meet at a vertex are considered nonintersecting.]

Solution by Graham Denham, student, Old Scona Academic High School, Edmonton, Alberta.

We prove a more general result on a convex m-gon where m need not be odd, with the additional condition that each colour appears at least once. For m odd, this condition is redundant.

We use induction on m. For $m = 3$, the result is trivial. Suppose the result holds for $3, 4, \dots, m - 1$. Now consider an m-gon. We have two cases.

Case 1. One of the colours appears only once. Then the other two colours must appear alternately on the other vertices. If we draw all the diagonals from the vertex with the unique colour, a desired triangulation is obtained. Note that the induction hypothesis is not needed in this case.

Case 2. Each colour appears at least twice. We then claim that there exist three consecutive vertices with each a different colour. Otherwise, the colours of the vertices must alternate all the way around the perimeter in just two colours, precluding any appearance of the third colour. Now draw the diagonal which cuts off the triangle defined by those three vertices. The remaining $(m - 1)$-gon still satisfies the induction hypothesis since all three colours are still present by the hypothesis that each colour appears at least twice. Thus by induction the desired triangulation of the m-gon can be accomplished.

4. In expressing the fraction m/n as a decimal, where m and n are natural numbers and $n \leq 100$, a student found, at a certain place after the decimal point, the sequence of digits 167. Show that the student must have made an error.

Solution.

The solution here is essentially the same as that of Andy Liu for problem #6 [1985: 141] in which 501 replaces the sequence 167. As there, it suffices to find the smallest value of n for which $m/n = 0.167\ldots$ for some $m < n$. The six sequences $\{(k+1)/6k\}$, $\{(k+1)/(6k+1)\}$, $\{(k+1)/(6k+2)\}$, $\{(k+1)/(6k+3)\}$, $\{(k+1)/(6k+4)\}$, and $\{(k+1)/(6k+5)\}$ are all decreasing in k. Also, each member of each sequence is $> 1/6 = 0.1666\ldots$ and each sequence approaches $1/6$ in the limit. Since

$$22/130 = 0.169\ldots$$
$$22/131 = 0.167\ldots$$
$$21/125 = 0.168\ldots$$

the desired minimum value of n is 131 and so the student must have made an error.

5. Equilateral triangles ABC and $A'B'C'$ are drawn in a plane (both sets of vertices being labeled clockwise). The midpoints of segments BC and $B'C'$ coincide. Find

(a) the angle between the lines AA' and BB';

(b) the ratio AA'/BB'.

Solution.

Although this problem can be solved by synthetic geometry, we will use complex numbers to demonstrate their utility in certain types of geometry problems. We also do the same for the subsequent problem #5, Grade X.

Consider the Argand diagram with the midpoint M of BC as the origin, MB as the positive real axis, and MA as the imaginary axis. Also, we can assume $BC = 2$. Then the complex number representations for A, B, C are $i\sqrt{3}$, -1, and 1, respectively. If the angle between BC and $B'C'$ is P, then A', B', and C' are $e^{iP}i\sqrt{3}$, $-e^{iP}$, and e^{iP}, respectively. Since $A - A' = i\sqrt{3}(1 - e^{iP})$ and $B - B' = -(1 - e^{iP})$,

(a) the angle between AA' and BB' is 90°,

(b) the ratio $AA'/BB' = \sqrt{3}$.

X Grade

1. For each vertex of a tetrahedron, the point symmetric to that vertex with respect to the centroid of the opposite face is chosen. Find the ratio of the volume of the tetrahedron whose vertices are these new points to that of the original tetrahedron.

Solution.

Our solution is vectorial since this is a particularly good representation for this type of problem. Let A, B, C, D denote vectors from a common origin to the vertices A, B, C, D of the tetrahedron. The centroid of the face opposite to A is given by (B + C + D)/3. The point symmetric to A with respect to the latter centroid is given by

$$A' = (B + C + D)/3 + ((B + C + D)/3 - A)$$

or $A' = 2(S - A)/3 - A$ where $S = A + B + C + D$. The other three points are obtained by symmetric interchange of the letters. Since

$$A' - B' = -5(A - B)/3, \quad \text{etc.},$$

the desired ratio is $(5/3)^3$. Similarly, the ratio of volumes for the analogous problem for an n-dimensional simplex is $((n + 2)/n)^n$.

2. The map of a city has the shape of a convex polygon. Each diagonal of the polygon is a street, and the intersections of the diagonals are intersections of the streets (but the vertices of the polygon are *not* considered to be intersections of streets). Streetcar lines go through the city. Each line goes from one end of a street to the other end, and has stops at each intersection as well as at the endpoints. At each intersection only two streets cross, and a streetcar runs along at least one of them. Show that one can transfer from any intersection to any other, making no more than two transfers. (A transfer may be made whenever two streetcar lines have a common stop.)

Solution by Andy Liu, University of Alberta, Edmonton, Alberta.

Suppose one wants to go from intersection D to intersection W. At least one street through D, say AC, is serviced by a streetcar. Similarly, a streetcar runs along XZ through W. If AC coincides with XZ, no transfer is necessary. If AC intersects XZ, one transfer will suffice. If neither of these cases holds, let B be any vertex between A and C on the opposite side of W and let Y be any vertex between X and Z on the opposite side of D. B and Y exist since A and C cannot be adjacent nor can X and Z. We can assume that AY

and BZ determine a point of intersection, as otherwise we can replace BZ with BX. Now either AY or BZ is serviced by a streetcar. Since each shares a common stop with both AC and XZ, one can get from D to W by making no more than two transfers.

3. Consider the $2k$ numbers

$$2^1 - 1,\ 2^2 - 1,\ \ldots,\ 2^{2k} - 1,$$

where $k \geq 1$. Show that at least one of them is a multiple of $2k + 1$.

Solution by Andy Liu, University of Alberta, Edmonton, Alberta.

Consider $2^1, 2^2, \ldots, 2^{2k}$. None of these are congruent (mod $2k + 1$) to 0 since $(2,2k + 1) = 1$. If no two of them are congruent (mod $2k + 1$) to each other, then for some x between 1 and $2k$ inclusive, $2^x \equiv 1 \pmod{2k + 1}$, and hence $2^x - 1$ will be a multiple of $2k + 1$. Thus suppose that $2^i \equiv 2^j \pmod{2k + 1}$ where $1 \leq i < j \leq 2k$. Then, $2^{j-i} \equiv 1 \pmod{2k + 1}$. The same conclusion now follows as in the first case if we set $x = j - i$.

4. $\mathbb{R}$ being the set of all real numbers, find all functions $F: \mathbb{R} \longrightarrow \mathbb{R}$ which satisfy

$$pF(a) + (1 - p)F(b) \geq F(pa + (1 - p)b)$$

for all a, b, $p \in \mathbb{R}$.

Solution by Andy Liu, University of Alberta, Edmonton, Alberta.

If $(u,F(u))$ and $(v,F(v))$ are any two points on the graph of $y = F(x)$, then by the hypothesis, all the points of the graph lie on or below the line joining the two points. We now show by an indirect proof that all the points of the graph are on a straight line, i.e., F is a linear function. Assume there are three points of the graph which do not lie on a straight line. This gives a contradiction since at least one of the points is above the line joining the other two points.

5. The squares $ABCD$, $A_1B_1C_1D_1$, and $A_2B_2C_2D_2$ are coplanar (and their vertices are labeled counterclockwise). Vertices A and A_1 coincide, and so do vertices C and C_2. Show that $D_1D_2 \perp BM$, where M is the midpoint of B_1B_2, and that $D_1D_2 = 2BM$.

Solution.

As in the previous problem #5, we use a complex number representation. Let A, B, C, D, B_1, and B_2 be given by

$$A = 1,\ B = i,\ C = -1,\ D = -i,\ B_1 = z, \text{ and } B_2 = w.$$

Then

$$A_1 = 1,$$
$$B_1 = z,$$
$$C_1 = B_1 + i(B_1 - 1) = z + i(z - 1),$$
$$D_1 = C_1 + i(C_1 - B_1) = 1 + i(z - 1),$$
$$C_2 = -1,$$
$$D_2 = C_2 + i(C_2 - B_2) = -1 - i(1 + w),$$
$$M = (B_1 + B_2)/2 = (z + w)/2.$$

Then $D_1 - D_2 = 2 + i(z + w)$ and $-2i(B - M) = 2 + i(z + w)$. Thus D_1D_2 is perpendicular to BM and $D_1D_2 = 2BM$.

*

Since I have had some queries concerning problems from the 1985 Dutch Mathematical Olympiad - First Round [1986: 2] (the answers were given last month [1986: 49]), I now give solutions to some of these problems. I am grateful to Andy Liu for them.

<u>B2</u>. On one of the sides of an angle A of 60°, a point P is given such that $AP = 10$. On the other side of the angle a point Q is chosen in such a way that $AP^2 + AQ^2 + PQ^2$ is a minimum. Compute AQ.

Solution.

Let R be a point on AQ extended such that $AQ = QR$. Then using the formula for the length of a median ($4m_a^2 = 2b^2 + 2c^2 - a^2$), we obtain $AQ^2 + PQ^2 = (AP^2 + PR^2)/2$. Since AP is constant, $AP^2 + AQ^2 + PQ^2$ is a minimum if and only if PR is a minimum. This occurs when PR is perpendicular to AR. Since $\angle PAR = 60^\circ$, $AR = 5$ and $AQ = 5/2$.

<u>B3</u>. A rectangular block with edge lengths a, b, c has a volume numerically equal to its total surface area. Furthermore, it is given that a, b, c are integers with $a < b < c$. Determine all possible triples (a,b,c) satisfying these conditions.

Solution.

We have $2(ab + bc + ca) = abc$. If $a \le 2$, the left-hand side exceeds the right-hand side. If $a \ge 6$, it's the other way around, since the left-hand side is in any case at most $6bc$. Hence $a = 3$, 4 or 5.

For $a = 3$,

$$c = 6b/(b - 6) = 6 + \frac{36}{b - 6}.$$

For $b = 7$, 8, 9, and 10, $c = 42$, 24, 18, and 15 respectively. For $b = 11$, c

is non-integral. For $b \geq 12$, $c \leq 12$, contradicting $b < c$.

For $a = 4$,

$$c = 4b/(b - 4) = 4 + \frac{16}{b - 4}\ .$$

For $b = 5$ and 6, $c = 20$ and 12, respectively. For $b = 7$, c is non-integral. For $b \geq 8$, $c \leq 8$, contradicting $b < c$.

When $a = 5$,

$$c = 10b/(3b - 10) = \frac{10}{3} + \frac{100}{3(3b - 10)}\ .$$

For $b = 6$, c is non-integral. For $b \geq 7$, $c < 7$, contradicting $b < c$.

Hence (3,7,42), (3,8,24), (3,9,18), (3,10,15), (4,5,20), and (4,6,12) are the only triples satisfying the hypotheses.

B4. a, b, and c are positive integers of 2, 3, and 5 digits, respectively, all digits being less than 9. The digits of c are distinct. Furthermore, $ab = c$ and adding 1 to each digit does not affect the truth of this equation. Determine a, b, and c.

Solution.

We have $ab = c$ and $(a + 11)(b + 111) = c + 11111$. Eliminating c, we obtain $111a + 11b = 9890$. The only solutions (a,b) satisfying $10 \leq a \leq 88$ and $100 \leq b \leq 888$ are (12,778), (23,667), (34,556), (45,445), (56,334), (67,223) and (78,112). However, only $a = 56$ and $b = 334$ yields a value of c satisfying the hypotheses.

C2. The terms a_n of a sequence of positive integers satisfy

$$a_{n+3} = a_{n+2}(a_{n+1} + a_n),\ n = 1,2,3,\ldots\ .$$

Compute a_7 if it is given that $a_6 = 144$.

Solution.

Let $a_1 = x$, $a_2 = y$, and $a_3 = z$. Then

$$a_4 = z(y + x),$$
$$a_5 = z(y + x)(z + y)$$

and

$$a_6 = 144 = z^2(y + x)(z + y)(y + x + 1).$$

Note that $y + x$ and $y + x + 1$ are two consecutive positive integral divisors of 144, with $y + x \geq 2$. Hence, $y + x = 2$, 3 or 8. The first case leads to a non-integral value of z. The other two cases lead to $x = 2$, $y = 1$, $z = 2$ and $x = 7$, $y = z = 1$, and both give $a_7 = 3456$.

C3. A carpenter saws from a block a polyhedron with 30 vertices and 18 faces. The faces are 5 quadrangles, 6 pentagons, and 7 hexagons. How many interior diagonals does it have? (An interior diagonal connects two vertices not in the same face.)

Solution.

There are $\binom{30}{2} = 435$ segments joining two distinct vertices. Of these, $5\binom{4}{2} + 6\binom{5}{2} + 7\binom{6}{2} = 195$ lie on the faces, with each edge of the polyhedron being counted twice. By Euler's formula, the polyhedron has $30 + 18 - 2 = 46$ edges. Hence the number of space diagonals is $435 - 195 + 46 = 286$.

*

Finally, here is a correction to problem #10 in my November 1985 column [1985: 272]. In the last line, change "n-1?" to "(n-1)!?".

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before November 1, 1986, although solutions received after that date will also be considered until the time when a solution is published.

1116(b) [1986: 27] **(Corrected)** *Proposed by David Grabiner, Claremont High School, Claremont, California.*

Let $g(n)$ be the second smallest positive integer which is not a factor of n. Continue the series $g(n)$, $g(g(n))$, $g(g(g(n)))$,... until you reach 3. What is the maximum length of the series?

1131. *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

Let $A_1A_2A_3$ be a triangle with sides a_1, a_2, a_3 labelled as usual, and let P be a point in or out of the plane of the triangle. It is a known result that if R_1, R_2, R_3 are the distances from P to the respective vertices A_1, A_2, A_3, then a_1R_1, a_2R_2, a_3R_3 satisfy the triangle inequality, i.e.

$$a_1R_1 + a_2R_2 + a_3R_3 \geq 2a_iR_i \ , \ i = 1,2,3. \tag{1}$$

For the a_iR_i to form a non-obtuse triangle, we would have to satisfy

$$a_1{}^2R_1{}^2 + a_2{}^2R_2{}^2 + a_3{}^2R_3{}^2 \geq 2a_i{}^2R_i{}^2$$

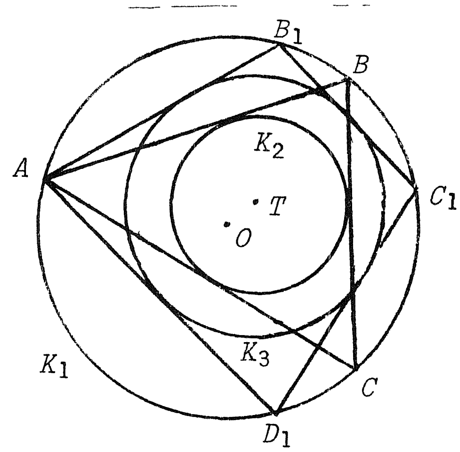
which, however, need not be true. Show that nevertheless

$$a_1{}^2R_1{}^2 + a_2{}^2R_2{}^2 + a_3{}^2R_3{}^2 \geq \sqrt{2}a_i{}^2R_i{}^2$$

which is a stronger inequality than (1).

1132. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

A triangle ABC has circumcircle K_1, with centre O and radius R, and inscribed circle K_2, with centre T and radius r. A third circle K_3 of centre T and radius r_1 has the property that there is a quadrilateral $AB_1C_1D_1$ which is both inscribed in K_1 and circumscribed about K_3. Find r_1 in terms of R and r.



1133. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

The incircle of triangle ABC touches sides BC and AC at points D and E respectively. If $AD = BE$, prove that the triangle is isosceles.

1134. *Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.*

Let n be a positive integer, and consider the set of all n-tuples $(a_1,a_2,\ldots,a_n)$ of integers for which

(i) $0 \leq a_1 \leq a_2 \leq \ldots \leq a_n < n$, and

(ii) $a_1 + a_2 + \ldots + a_n \equiv 0 \pmod{n}$.

Prove that the integers $0,1,2,\ldots,n-1$ each occur the same number of times as coordinates of elements in this set.

1135. *Proposed by Jack Garfunkel, Flushing, N.Y.*

(A variation of an old problem, dedicated to Dr. Leon Bankoff.)

(a) Given equilateral triangles ABC and $A'B'C'$ in the same plane, both labeled counterclockwise, prove that triangle $M_1M_2M_3$ is equilateral, where M_1, M_2, M_3 are the midpoints of AA', BB', CC' respectively.

(b)* Given similar triangles ABC and $A'B'C'$ in the same plane, prove that triangle $M_1M_2M_3$ is similar to triangle ABC, where M_1, M_2, M_3 are as in (a).

1136. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle and D, E, F points on BC, CA, AB respectively. Denote by G_1, G_2, G_3 the centroids of triangles AEF, BDF, CDE respectively. Prove that

$$[G_1G_2G_3] = \frac{2[ABC] + [DEF]}{9}$$

where $[M]$ stands for the area of the figure M.

1137.* *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove or disprove the triangle inequality

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \geq \frac{3\sqrt{3}}{s}$$

where m_a, m_b, m_c are the medians of a triangle and s is its semiperimeter.

1138. *Proposed by L.J. Upton, Mississauga, Ontario.*

You are given four discs A, B, C, D, of identical appearance, but weighing 1, 2, 3, and 4 units not necessarily respectively. Determine the weights of the discs in four weighings on a 2-tray balance (no extra weights supplied).

1139. *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

Let ABC be a triangle and let A', B', C' be the touch points of the nine-point circle with the A-excircle, B-excircle, and C-excircle, respectively. Prove that AA', BB', CC' concur in a point F', and that F' is collinear with the centers of the incircle and nine-point circle.

1140. *Proposed by Jordi Dou, Barcelona, Spain.*

Given triangle ABC, construct a circle which cuts (extended) lines BC, CA, AB in pairs of points A' and A'', B' and B'', C' and C'' respectively such that angles $A'AA''$, $B'BB''$, $C'CC''$ are all right angles.

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S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

999*. [1984: 319] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Let R, r, s be the circumradius, inradius, and semiperimeter, respectively, of an acute-angled triangle. Prove or disprove that

$$s^2 \geq 2R^2 + 8Rr + 3r^2.$$

When does equality occur?

Solution by Leon Bankoff, Los Angeles, California.

From (2.56) of Bottema et al, *Geometric Inequalities*, we read

$$\left[\sum \sin \frac{\alpha'}{2} \right]^2 \leq \sum \cos^2 \frac{\alpha'}{2}$$

where α', β', γ' are the angles of a triangle and the sums are over these three angles. Replacing $\frac{\alpha'}{2}$, $\frac{\beta'}{2}$, $\frac{\gamma'}{2}$ by $90^\circ-\alpha$, $90^\circ-\beta$, $90^\circ-\gamma$ respectively, we obtain

$$(\Sigma \cos\alpha)^2 \leq \Sigma \sin^2\alpha ,$$

where α, β, γ are the angles of an *acute* triangle. Using

$$\Sigma \sin^2\alpha = (\Sigma \sin\alpha)^2 - 2\Sigma \sin\alpha \sin\beta$$

and the known relations

$$\Sigma \sin\alpha = \frac{s}{R}$$

$$\Sigma \sin\alpha \sin\beta = \frac{s^2 + 4Rr + r^2}{4R^2}$$

$$\Sigma \cos\alpha = \frac{R + r}{R}$$

we obtain

$$\left[\frac{R + r}{R}\right]^2 \leq \left[\frac{s}{R}\right]^2 - 2\left[\frac{s^2 + 4Rr + r^2}{4R^2}\right]$$

which is equivalent to the given inequality.

Also solved by O. BOTTEMA, Delft, The Netherlands; D.S. MITRINOVIC and J.E. PECARIC, University of Belgrade, Yugoslavia; and VEDULA N. MURTY, Pennsylvania State University, Middletown, Pennsylvania. There were three partial solutions received.

Most solvers observed that equality holds for the equilateral triangle. HERTA T. FREITAG, Roanoke, Virginia, and J.T. GROENMAN, Arnhem, The Netherlands, proved that the inequality also holds for right triangles, with equality only for the isosceles right triangle.

Mitrinovic and Pecaric point out that the proposed problem has previously appeared as E2388(ii) in the American Mathematical Monthly *79(1972), p.1135 (solution in* AMM *80 (1973), p.1142).*

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<u>1001</u>. [1985: 15] *Proposed by Mark Kantrowitz, student, Maimonides School, Brookline, Massachusetts.*

In the following exact cryptarithmic division, each X can be any of the decimal digits except the single digit represented by S. Restore the digits.

```
        XSXX
   XSX)XXSXXXS
       XSX
       ----
       XXXX
       XXXS
       ----
        XXXX
         XSX
        ----
         XXXS
         XXXS
         ----
```

Solution.

```
        1419
   946)1342374          (S = 4)
       946
       ----
       3963
       3784
       ----
        1797
         946
        ----
         8514
         8514
         ----
```

The solution is unique.

Found by RICHARD I. HESS, Rancho Palos Verdes, California; ALLAN WM. JOHNSON JR., Washington, D.C.; PATRICIA KUSS, Cleveland, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Valencia Community College, Orlando, Florida; KENNETH M. WILKE, Topeka, Kansas; and the proposer. Kuss and Wilke submitted partial arguments for their solutions; all others just sent the solution.

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1002. [1985: 15] *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

Let m and n be given natural numbers, where $m \le n$. Evaluate the sum

$$\sum_{j=1}^{m} \frac{\binom{m}{j}}{\binom{n}{j}} \cdot j \quad .$$

I. *Solution by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

Since

$$\frac{\binom{m}{j}}{\binom{n}{j}} = \frac{\binom{n-j}{n-m}}{\binom{n}{m}}$$

and

$$\sum_{t=0}^{b} \binom{a+t}{a} = \binom{a+b+1}{a+1} ,$$

the desired sum is

$$\begin{aligned}
\sum_{j=1}^{m} \frac{\binom{m}{j}}{\binom{n}{j}} \cdot j &= \frac{1}{\binom{n}{m}} \sum_{j=0}^{m} \binom{n-j}{n-m} \cdot j \\
&= \frac{1}{\binom{n}{m}} \sum_{i=1}^{m} \sum_{j=i}^{m} \binom{n-j}{n-m} \\
&= \frac{1}{\binom{n}{m}} \sum_{i=1}^{m} \binom{n-i+1}{n-m+1} \\
&= \frac{1}{\binom{n}{m}} \binom{n+1}{n-m+2} \\
&= \frac{\binom{n+1}{m-1}}{\binom{n}{m}} \\
&= \frac{m(n+1)}{(n-m+1)(n-m+2)} .
\end{aligned}$$

II. *Solution by Frank P. Battles and Laura L. Kelleher, Massachusetts Maritime Academy, Buzzards Bay, MA.*

We consider the more general sum

$$S_\alpha = \sum_{j=1}^{m} \frac{\begin{bmatrix} n \\ j \end{bmatrix}}{\begin{bmatrix} m \\ j \end{bmatrix}} j^\alpha$$

where $m \leq n$ and α is a non-negative integer. The above can be rewritten by some elementary manipulations as

$$S_\alpha = \frac{1}{\begin{bmatrix} n \\ m \end{bmatrix}} \sum_{j=1}^{m} \begin{bmatrix} n - j \\ n - m \end{bmatrix} j^\alpha .$$

By repeated use of the familiar reduction formula

$$\begin{bmatrix} N \\ r \end{bmatrix} = \begin{bmatrix} N - 1 \\ r \end{bmatrix} + \begin{bmatrix} N - 1 \\ r - 1 \end{bmatrix}$$

we obtain the identity

$$\begin{bmatrix} n + \alpha \\ n + \alpha - m + 1 \end{bmatrix} = \sum_{j=1}^{m} \begin{bmatrix} n + \alpha - j \\ n + \alpha - m \end{bmatrix}$$

which we write in the form

$$\frac{1}{\begin{bmatrix} n \\ m \end{bmatrix}} \sum_{j=1}^{m} \begin{bmatrix} n + \alpha - j \\ n + \alpha - m \end{bmatrix} = \frac{1}{\begin{bmatrix} n \\ m \end{bmatrix}} \begin{bmatrix} n + \alpha \\ n + \alpha - m + 1 \end{bmatrix} . \qquad (*)$$

When $\alpha = 0$ it follows directly from (*) that

$$S_0 = \frac{m}{n + 1 - m} .$$

We note that this is problem 258 from the *Two-Year College Mathematics Journal*, September 1983, p.353.

When $\alpha = 1$ (the given problem) it follows from (*) that

$$\frac{1}{\begin{bmatrix} n \\ m \end{bmatrix}} \sum_{j=1}^{m} \begin{bmatrix} n + 1 - j \\ n + 1 - m \end{bmatrix} = \frac{m(n + 1)}{(n + 1 - m)(n + 2 - m)} .$$

Since

$$\begin{bmatrix} n + 1 - j \\ n + 1 - m \end{bmatrix} = \frac{n + 1 - j}{n + 1 - m} \begin{bmatrix} n - j \\ n - m \end{bmatrix} ,$$

the above can be rewritten as

$$(n + 1)S_0 - S_1 = \frac{(n + 1)m}{n + 2 - m} .$$

Substituting for S_0 and solving for S_1 gives

$$S_1 = \frac{m(n + 1)}{(n + 1 - m)(n + 2 - m)} .$$

When $\alpha = 2$ it follows from (*) that

$$\frac{1}{\begin{bmatrix} n \\ m \end{bmatrix}} \sum_{j=1}^{m} \begin{bmatrix} n+2-j \\ n+2-m \end{bmatrix} = \frac{(n+2)(n+1)m}{(n+1-m)(n+2-m)(n+3-m)} .$$

Since

$$\begin{bmatrix} n+2-j \\ n+2-m \end{bmatrix} = \frac{(n+2-j)(n+1-j)}{(n+1-m)(n+2-m)} \begin{bmatrix} n-j \\ n-m \end{bmatrix} ,$$

the above can be rewritten as

$$(n+2)(n+1)S_0 - (2n+3)S_1 + S_2 = \frac{(n+2)(n+1)m}{n+3-m} .$$

Substituting for S_0 and S_1 and solving for S_2 gives

$$S_2 = \frac{m(n+1)(m+n+1)}{(n+1-m)(n+2-m)(n+3-m)} .$$

The above procedure can clearly be continued indefinitely. However, although the denominator of S_α remains simple, the numerator becomes quite unwieldy.

III. *Solution by the proposer.*

Consider a box containing $a = n - m + 1$ white balls and $b = m$ black balls. Balls are drawn from this box at random one by one (without replacement). Let X denote the number of black balls drawn before the first white ball is drawn. Then it is known that

$$E(X) = \frac{b}{a+1} = \frac{m}{n-m+2} . \tag{1}$$

(If all balls were drawn out, this is the average interval between consecutive white balls, counting imaginary initial and terminal white balls. See problem 4, page 178 of J.V. Uspensky's *Introduction to Mathematical Probability*, first edition.)

Since X can take values $0,1,2,\ldots,b$ with respective probabilities

$$\frac{a}{a+b},\ \frac{b}{a+b}\cdot\frac{a}{a+b-1},\ \frac{b}{a+b}\cdot\frac{b-1}{a+b-1}\cdot\frac{a}{a+b-2},\ \ldots$$
$$\ldots,\ \frac{b}{a+b}\cdot\frac{b-1}{a+b-1}\cdot\ \ldots\ \cdot\frac{1}{a+1}\cdot\frac{a}{a},$$

we get

$$E(X) = \frac{a}{a+b}\left[\sum_{i=1}^{b} \frac{\begin{bmatrix} b \\ i \end{bmatrix}}{\begin{bmatrix} a+b-1 \\ i \end{bmatrix}} \cdot i \right]$$

$$= \frac{n-m+1}{n+1}\left[\sum_{i=1}^{m} \frac{\begin{bmatrix} m \\ i \end{bmatrix}}{\begin{bmatrix} n \\ i \end{bmatrix}} \cdot i \right] . \tag{2}$$

(1) and (2) establish that

$$S = \frac{m(n+1)}{(n-m+1)(n-m+2)} .$$

Also solved by CURTIS COOPER, Central Missouri State University, Warrensburg, Missouri; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARK KANTROWITZ, student, Maimonides School, Brookline, Massachusetts; BRUCE WATSON, Memorial University of Newfoundland, St. John's, Newfoundland; and KENNETH M. WILKE, Topeka, Kansas. Janous points out that the proposed problem is item 17, page 629, of A.P. Prudnikow, Ju.A. Brychkow, and O.I. Marichew, Integrals and Series (Elementary Functions), *Nauka, Moscow, 1981 (in Russian).*

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1003*. [1985: 15] *Proposed by M.S. Klamkin, University of Alberta.*

Without using tables or a calculator, show that

$$\ln 2 > \left(\frac{2}{5}\right)^{2/5} .$$

Solution by Jordan B. Tabov, Sofia, Bulgaria.

From the familiar series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \qquad (|x| < 1)$$

and

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \qquad (|x| < 1)$$

we obtain

$$\ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$$

$$= 2x\left[1 + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \ldots\right] .$$

Putting $x = \frac{1}{3}$, we obtain

$$\ln 2 = \frac{2}{3}\left[\sum_{n=0}^{\infty} \frac{1}{(2n+1)3^{2n}}\right] .$$

Thus we have

$$\ln 2 > \frac{2}{3} + \frac{2}{3\cdot 3^3} + \frac{2}{5\cdot 3^5} + \frac{2}{7\cdot 3^7} + \frac{2}{9\cdot 3^9}$$

$$= \frac{2}{3} + \frac{2}{81} + \frac{2}{1215} + \frac{2}{15309} + \frac{2}{177147}$$

$$> 0.6666666 + 0.024691 + 0.001646 + 0.0001306 + 0.000011$$

$$= 0.6931452$$

$$= a .$$

Then

$$a^2 > 0.48045$$

$$a^4 > 0.230832$$

and so

$$a^5 > (0.230832)(0.693145) = 0.16000004664.$$

Hence $(\ln 2)^5 > a^5 > 0.16 = \left[\frac{2}{5}\right]^2$, which is equivalent to the required inequality.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; VEDULA N. MURTY and ELI DAVID, Pennsylvania State University, Middletown, Pennsylvania; BASIL C. RENNIE, James Cook University of North Queensland, Australia; and BOB SALER, Agincourt, Ontario.

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1004. [1985: 15] *Proposed by O. Bottema, Delft, The Netherlands.*

There exists a right triangle with perimeter p and area F if and only if the positive numbers p and F satisfy what condition?

I. *Solution by Lawrence Somer, George Washington University, Washington, D.C.*

The required condition is that $p \geq (2 + 2\sqrt{2})\sqrt{F}$. It is known and easily provable that among all right triangles of a given area, the one with minimum perimeter is the isosceles right triangle. Let a be one of the legs of the isosceles right triangle with area F. Then $a^2 = 2F$, and so the perimeter is $2a + \sqrt{2a^2} = (2 + \sqrt{2})a = (2 + 2\sqrt{2})\sqrt{F}$. Thus any right triangle of area F and perimeter p must satisfy $p \geq (2 + 2\sqrt{2})\sqrt{F}$.

Conversely, for a given area F, a right triangle of area F can be found with perimeter larger than any given number N: simply choose N for one leg of the triangle and $2F/N$ for the other. By continuity, there exists a right triangle of area F and perimeter equal to any real number greater than or equal to the perimeter of the isosceles right triangle of area F. Thus the condition $p \geq (2 + 2\sqrt{2})\sqrt{F}$ is also sufficient to ensure a right triangle of area F and perimeter p exists.

II. *Solution by Mark Kantrowitz, student, Maimonides School, Brookline, Massachusetts.*

Given a right triangle with sides a, b, c, we have

$$p = a + b + c = a + b + \sqrt{a^2 + b^2}$$

and

$$F = ab/2.$$

Thus from the A.M.-G.M. inequality,

$$\frac{p}{2} = \frac{a + b}{2} + \sqrt{\frac{1}{2}\left[\frac{a^2}{2} + \frac{b^2}{2}\right]}$$
$$\geq \sqrt{ab} + \sqrt{\frac{ab}{2}}$$
$$= (\sqrt{2} + 1)\sqrt{F}$$

with equality if and only if $a = b$. Thus

$$p^2 \geq F(2 + 2\sqrt{2})^2 = F(12 + 8\sqrt{2}).$$

III. *Solution and generalization by Hayo Ahlburg, Benidorm, Alicante, Spain.*

Consider a triangle with sides a, b, c, perimeter p and area F, which means

$$F = \frac{ab \sin C}{2}$$

and

$$p = a + b + c = a + b + \sqrt{a^2 + b^2 - 2ab \cos C}\ .$$

Then

$$p^2 = \left[a + b + \sqrt{a^2 + b^2 - 2ab \cos C}\ \right]^2$$
$$= (a + b)^2 + a^2 + b^2 - 2ab \cos C + 2(a + b)\sqrt{a^2 + b^2 - 2ab \cos C}$$
$$= 2(a + b)^2 - 2ab(1 + \cos C) + 2(a + b)(p - a - b)$$
$$= 2(a + b)p - \frac{4F}{\sin C}(1 + \cos C)$$

and so

$$a + b = \frac{p^2 \sin C + 4F(1 + \cos C)}{2p \sin C}\ .$$

Thus

$$a + \frac{2F}{a \sin C} = \frac{p^2 \sin C + 4F(1 + \cos C)}{2p \sin C}$$

which simplifies to

$$(2p \sin C)a^2 - (p^2 \sin C + 4F(1 + \cos C))a + 4Fp = 0. \qquad (1)$$

Since a is real, this means that

$$[p^2 \sin C + 4F(1 + \cos C)]^2 \geq 16Fp(2p \sin C)$$

or

$$p^4\sin^2C + 8p^2F\sin C(1 + \cos C) + 16F^2(1 + \cos C)^2 \geq 32p^2F\sin C$$

or

$$[p^2\sin^2C - 4F(1 + \cos C)]^2 \geq 16p^2F\sin C(1 - \cos C). \tag{2}$$

Since

$$\begin{aligned} c &= p - (a + b) \\ &= p - \frac{p^2\sin C + 4F(1 + \cos C)}{2p\sin C} \\ &= \frac{p^2\sin C - 4F(1 + \cos C)}{2p\sin C}, \end{aligned} \tag{3}$$

we have

$$p^2\sin C - 4F(1 + \cos C) > 0,$$

and so (2) is equivalent to

$$p^2\sin C - 4F(1 + \cos C) \geq 4p\sqrt{F}\sqrt{\sin C(1 - \cos C)}$$

or

$$p^2\sin C - 4\sqrt{F}\sqrt{\sin C(1 - \cos C)}\,p - 4F(1 + \cos C) \geq 0. \tag{4}$$

Since $p > 0$, (4) is equivalent to

$$\begin{aligned} p &\geq \frac{4\sqrt{F}\sqrt{\sin C(1 - \cos C)} + \sqrt{16F\sin C(1 - \cos C) + 16F\sin C(1 + \cos C)}}{2\sin C} \\ &= \frac{2\sqrt{F}\sqrt{\sin C}}{\sin C}[\sqrt{1 - \cos C} + \sqrt{2}] \\ &= 2\sqrt{\frac{2F}{\sin C}}\left[\sqrt{\frac{1 - \cos C}{2}} + 1\right] \end{aligned}$$

or

$$p \geq \frac{2\sqrt{2}\,(\sin C/2 + 1)}{\sqrt{\sin C}}\sqrt{F}. \tag{5}$$

Equality holds if and only if (1) is a perfect square. Since b satisfies the same equation as a does, namely (1), this means that equality holds in (5) if and only if $a = b$.

Condition (5) is not only necessary but also sufficient. Any positive values p, F, and C ($< 180^\circ$) satisfying (5) yield, according to equations (1) and (3), positive values a, b, c such that

$$a + b + c = p$$
$$\frac{ab\sin C}{2} = F$$

and

$$c = \sqrt{a^2 + b^2 - 2ab\cos C} < a + b,$$

and which are therefore sides of a triangle with area F, perimeter p, and angle C.

To solve the stated problem, put $C = 90°$; then (5) becomes

$$p \geq (2 + 2\sqrt{2})\sqrt{F}$$

which is the required relation.

In the notation of [1], (5) says that for any triangle,

$$s \geq \frac{\sqrt{2}(\sin \gamma/2 + 1)\ \sqrt{F}}{\sqrt{\sin \gamma}}\ .$$

For fixed F, the right side is minimized when $\gamma = 60°$; hence for any triangle,

$$s \geq 3^{3/4}\ \sqrt{F}\ .$$

This is inequality 4.2 of [1].

Reference:

[1] Bottema et al, *Geometric Inequalities*.

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, MA; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; VEDULA N. MURTY, Pennsylvania State University, Middletown, Pennsylvania; MICHAEL M. PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer. Four other solvers submitted solutions which involved the sides of the triangle as well as p *and* F, *but I have interpreted the problem to mean that the condition must be only in terms of* p *and* F. *There were also three incorrect solutions submitted.*

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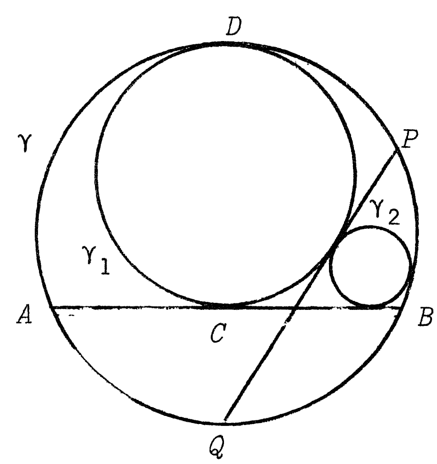
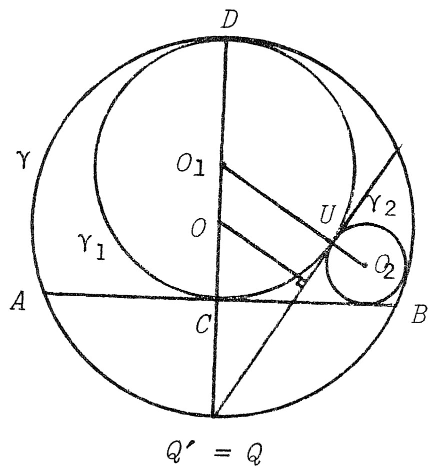
<u>1005</u>. [1985: 16] *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokai-City, Aichi, Japan.*

A chord AB divides a circle γ into two segments. A circle γ_1, of radius r_1, is inscribed in one of the segments, tangent to AB at its midpoint C and to the arc at D, as shown in the figure. A circle γ_2, of radius r_2, is then inscribed in the mixtilinear triangle CBD. The common interior tangent to γ_1 and γ_2 meets circle γ in P and Q. Find the length of PQ in terms of r_1 and r_2.



Solution by Dan Sokolowsky, Brooklyn, N.Y.



Let O, O_1, O_2 denote the respective centers of γ, γ_1, γ_2, and r, r_1, r_2 their respective radii. Let γ_2 touch γ_1 at U, and extend DC to meet γ at Q'.

Consider the arbelos ω on diameters DQ' (of γ), DC (of γ_1), and CQ'. The circle γ_2 is then one of ω's "twin circles of Archimedes", and, as is well-known (e.g. R.A. Johnson, *Advanced Euclidean Geometry*, pp.116-117):

(i) the common internal tangent PQ of γ_1 and γ_2 passes through Q', i.e. $Q = Q'$;

(ii) $r_2 = \dfrac{r_1(r - r_1)}{r}$.

Solving (ii) for r, we obtain

$$r = \frac{r_1^2}{r_1 - r_2} . \tag{1}$$

Draw $OW \perp PQ$ and let $x = OW$. Then

$$PQ = 2\sqrt{r^2 - x^2} . \tag{2}$$

Since $O_1U \perp PQ$, $\Delta OWQ \sim \Delta O_1UQ$. Hence

$$\frac{x}{r_1} = \frac{OW}{O_1U} = \frac{OQ}{O_1Q} = \frac{r}{2r - r_1} ,$$

and so

$$x = \frac{rr_1}{2r - r_1} . \tag{3}$$

Substituting (1) in (3) we obtain

$$x = \frac{r_1^2}{r_1 + r_2} . \tag{4}$$

Finally, substituting (1) and (4) in (2) and simplifying, we obtain

$$PQ = \frac{4r_1^2\sqrt{r_1r_2}}{r_1^2 - r_2^2} .$$

Also solved by LEON BANKOFF, Los Angeles, California; HERTA T. FREITAG, Roanoke, Virginia; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

This problem is from a lost 1844 *sangaku* (see the note following the solution to problem 995 [1986: 60]). It is recorded in Professor Fukagawa's book *Collection of Sangaku in Aichi Prefecture* and also in his book *Study of Sangaku*, p.200.

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1006. [1985: 16] *Proposed by Hans Havermann, Weston, Ontario.*

Given a base-ten positive integer of two or more digits, it is possible to spawn two smaller base-ten integers by inserting a space somewhere within the number. We call the left offspring thus created the *farmer* (F) and the value of the right one (ignoring leading zeros, if any) the *ladder* (L). A number is called *modest* if it has an F and an L such that the number divided by L leaves remainder F. (For example, 39 is modest.)

Consider, for $n > 1$, a block of n consecutive positive integers all of which are modest. If the smallest and largest of these are a and b, respectively, and if $a - 1$ and $b + 1$ are not modest, then we say that the block forms a *multiple berth* of size n. A multiple berth of size 2 is called a set of *twins*, and the smallest twins are {411,412}. A multiple berth of size 3 is called a set of *triplets*, and the smallest triplets are {4000026,4000027,4000028}.

(a) Find the smallest quadruplets.

(b)* Find the smallest quintuplets. (There are none less than 25 million.)

Partial solution by Leroy F. Meyers, The Ohio State University, Columbus, Ohio.

The positive integer N is modest if and only if there are positive integers m, F, and L such that

$$N = 10^mF + L, \quad N \equiv F(\text{mod } L), \quad \textit{and } F < L < 10^m.$$

But

$$10^mF + L \equiv F(\text{mod } L) \iff (10^m - 1)F \equiv 0(\text{mod } L) \iff L|(10^m - 1)F.$$

Hence a way to find modest numbers is to choose positive integers m and F (with $F < 10^m - 1$) and then find divisors L of $(10^m - 1)F$ which lie strictly between F and 10^m. Obviously $N < 10^{2m}$. For $m = 1$, the divisors of $10^m - 1$ are 1, 3, and 9, so that the modest numbers obtained in this case are 13, 19, 23, 26, 29, 39, 46, 49, 59, 69, 79 and 89. For $m = 2$, the divisors of $10^m - 1$ are 1, 3, 9, 11, 33 and 99, and there are 350 modest numbers in this case (give or take a few). Twins and higher multiplets occur especially when $10^m - 1$ has many small divisors. The smallest twins occur when $m = 2$: 411, 412. The smallest triplets seem to occur when $m = 6$: 4000026, 4000027, 4000028; the smallest quadruplets seem to occur even earlier, when $m = 6$: 4000011, 4000012, 4000013, 4000014; and quintuplets (possibly the smallest) occur when $m = 18$: $20\cdot 10^{18} + L$, where $35 \leq L \leq 39$. These quintuplets are a result of the favourable factorization

$$10^{18} - 1 = 3^4\cdot 7\cdot 11\cdot 13\cdot 19\cdot 37\cdot 52579\cdot 333667.$$

Note that it is possible for multiplets to be derived from several values of m; for example, the twins 2036, 2037 come from $m = 2$ and $m = 3$, respectively.

The above quadruplets were also found by RICHARD I. HESS, Rancho Palos Verdes, California; and the proposer. The proposer submitted a computer printout showing that the above triplets and quadruplets are minimal. For an earlier problem by the proposer on modest numbers, see #1291, Journal of Recreational Mathematics *17 (1984) p.140.*

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1007. [1985: 16] *Proposed by Leroy F. Meyers, The Ohio State University.*

It is known that every positive rational number can be written as the sum of finitely many reciprocals of distinct positive integers (the Egyptian fraction decomposition). Show that every positive real number can be written as the sum of infinitely many reciprocals of distinct positive integers.

Solution by Richard I. Hess, Rancho Palos Verdes, California.

Consider the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

which is known to diverge. For any positive real number x, start in the harmonic series with the largest term $< x$ and continue to add terms until the next term would cause the running sum to be $\geq x$. The partial sum at this point is less than x by some amount x_1. Now skip to the largest term in the harmonic series which is $< x_1$ and continue as before until the next term would cause the new running sum to be $\geq x$. Define x_2 as x minus the running sum. Continue these steps forever to produce a running sum which converges to x and contains an infinite number of distinct reciprocals.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; EDWIN M. KLEIN, University of Wisconsin, Whitewater, Wisconsin; DAN SOKOLOWSKY, Brooklyn, N.Y.; LAWRENCE SOMER, George Washington University, Washington, D.C.; STAN WAGON, Smith College, Northampton, Massachusetts; and the proposer.

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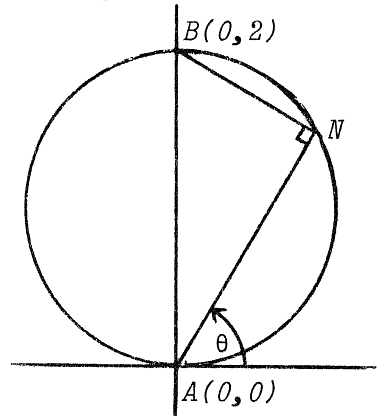
1008. [1985: 16] *Proposed by Jordan B. Tabov, Sofia, Bulgaria.*

A circle γ of diameter AB and two real numbers x and y are given. A variable point N ranges over γ. Find the locus of a point M on the line AN such that $AM = |xAN + yBN|$.



Solution by the proposer.

We let A be the origin and B be the point $(0,2)$. Then we need only find the locus of M for points N in the first quadrant, since by reflecting this locus about the y axis we obtain the entire locus.

Let θ be the angle between AN and the positive x axis, so that $0 \le \theta \le 90^\circ$. Then $AN = 2\sin\theta$ and $BN = 2\cos\theta$.

We may assume that $x \ge 0$ and that x and y are not both 0. Then there is a unique pair (ρ,φ) such that $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $\rho > 0$, and $-90^\circ \le \varphi \le 90^\circ$; (ρ,φ) are just the polar coordinates of the point (x,y).

Now

$$\begin{aligned} AM &= |xAN + yBN| \\ &= |\rho\cos\varphi\cdot 2\sin\theta + \rho\sin\varphi\cdot 2\cos\theta| \\ &= 2\rho|\cos\varphi\sin\theta + \sin\varphi\cos\theta| , \end{aligned}$$

that is, M is one of the two points on the line AN at distance

$$2\rho|\cos\varphi\sin\theta + \sin\varphi\cos\theta|$$

from the origin. We may therefore assume that M has rectangular coordinates

$$(2\rho(\cos\varphi\sin\theta + \sin\varphi\cos\theta)\cos\theta\ ,\ 2\rho(\cos\varphi\sin\theta + \sin\varphi\cos\theta)\sin\theta), \tag{1}$$

as the rest of the locus may be obtained by reflection through the origin.

Let O_1 be the point (y,x), that is,

$$O_1 = (\rho\sin\varphi, \rho\cos\varphi). \tag{2}$$

Then the vector $\overrightarrow{O_1M}$, from (1) and (2), has components

$$\begin{aligned} &(2\rho\cos\varphi\sin\theta\cos\theta + 2\rho\sin\varphi\cos^2\theta - \rho\sin\varphi\ , \\ &\qquad 2\rho\cos\varphi\sin^2\theta + 2\rho\sin\varphi\sin\theta\cos\theta - \rho\cos\varphi) \\ &= (\rho\cos\varphi\sin 2\theta + \rho\sin\varphi(2\cos^2\theta - 1), \\ &\qquad \rho\sin\varphi\sin 2\theta - \rho\cos\varphi(1 - 2\sin^2\theta)) \\ &= (\rho\cos\varphi\sin 2\theta + \rho\sin\varphi\cos 2\theta\ ,\ \rho\sin\varphi\sin 2\theta - \rho\cos\varphi\cos 2\theta) \\ &= (\rho\sin(2\theta + \varphi), -\rho\cos(2\theta + \varphi)), \end{aligned}$$

and so $O_1M = \rho$. Hence M lies on the circle with centre O_1 and radius ρ. Note that from (2), this circle also passes through the origin.

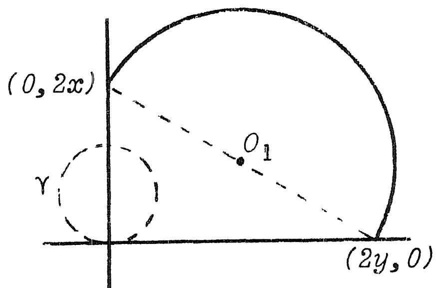
From (1) we get that when $\theta = 0^\circ$,

$$M = (2\rho\sin\varphi, 0) = (2y, 0)$$

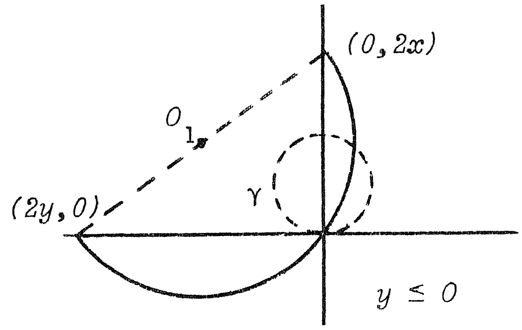
and when $\theta = 90^\circ$,

$$M = (0, 2\rho\cos\varphi) = (0, 2x).$$

It follows that the locus of M is a semicircle, oriented as at the left if $y \geq 0$, and as at the right if $y \leq 0$:

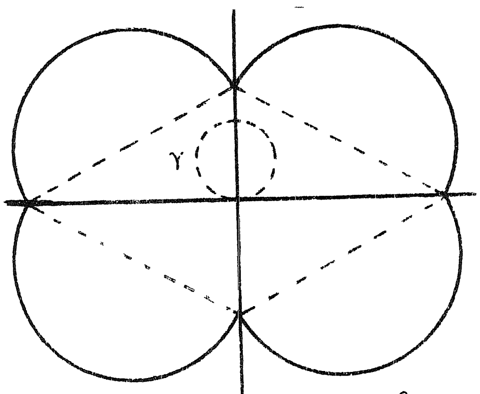


$y \geq 0$

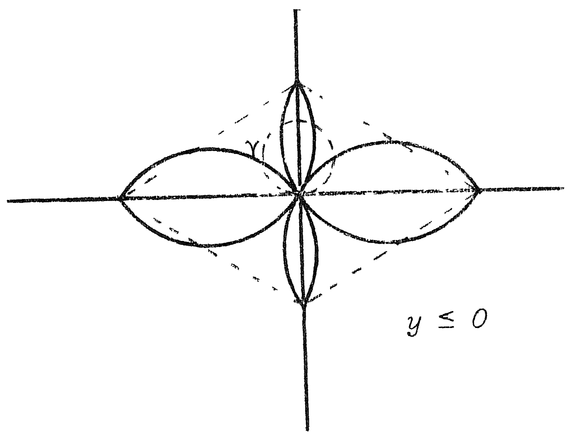


$y \leq 0$

To complete the locus we make it symmetric with respect to the y axis and origin, obtaining

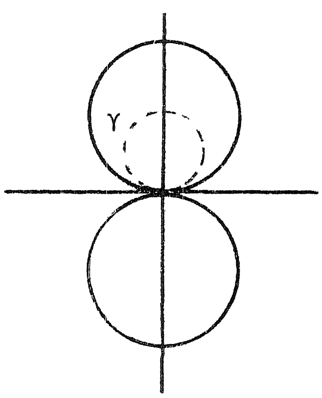


$y \geq 0$

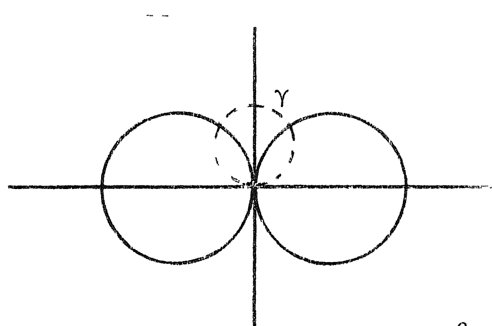


$y \leq 0$

We note two special cases:



$x \neq 0,\ y = 0$



$x = 0,\ y \neq 0$

There was one partial solution submitted.

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1009. [1985: 17] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Prove that every pandiagonal fourth-order magic square can be written in the form

$A + B + C$	$D + B - C$	$D - B + C$	$A - B - C$
$D - B - E$	$A - B + E$	$A + B - E$	$D + B + E$
$A + B - C$	$D + B + C$	$D - B - C$	$A - B + C$
$D - B + E$	$A - B - E$	$A + B + E$	$D + B - E$

.

Solution by Kenneth M. Wilke, Topeka, Kansas.

Maurice Kraitchik, in *Mathematical Recreations* (Second Revised Edition, Dover, 1953) pp.186-187, has shown that all pandiagonal fourth-order magic squares can be written in the form

a	b	c	$S - (a+b+c)$
e	$S - (a+b+e)$	$a - c + e$	$b + c - e$
$\frac{S}{2} - c$	$a+b+c - \frac{S}{2}$	$\frac{S}{2} - a$	$\frac{S}{2} - b$
$\frac{S}{2} - (a-c+e)$	$\frac{S}{2} - (b+c-e)$	$\frac{S}{2} - e$	$a+b+e - \frac{S}{2}$

.

The proposer's form can be obtained by taking

$$a = A + B + C$$
$$b = D + B - C$$
$$c = D - B + C$$
$$e = D - B - E$$

and

$$S = 2A + 2D.$$

For a similar problem, see *Crux* 605 (solution in [1982: 22]).

Also solved by the proposer.

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