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Articles are normally commissioned by the Editors; the Editorial Committee also welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. All correspondence about the contents should be sent to:

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Fame After Death

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At daybreak on 30 May 1832, two adversaries met in an outlying district of Paris to settle an affair of honour. The duel was with pistols at 25 paces. One of the men fell, shot in the stomach, and was left lying where he had fallen. Later that morning he was found by a peasant who took him to hospital, but he died the next day. On 2 June 1832 he was buried in the common ditch at the cemetery of Montparnasse. Over two thousand republicans came to the funeral to pay their last respects to one of their most active members, Évariste Galois, a mere boy of twenty. How surprised those mourners would have been had they known that their hero's name was to be immortalised in mathematics by his outstanding contributions to modern algebra! To commemorate the hundred and fiftieth anniversary of the death of this revolutionary and mathematician extraordinary, we tell the moving story of Évariste Galois.

Galois was born on 25 October 1811 at the village of Bourg-la-Reine near Paris, where his father was later to serve as mayor. His intellectual parents showed no particular talent for mathematics, but he did inherit from them a violent hatred of tyranny. His first twelve years were spent happily and studiously at home, where he received an excellent education from his mother.

In October 1823 Galois was admitted to the college Louis-le-Grand as a fourth-form boarder. During his first term at the college, there was a rebellion, with the students refusing to chant in chapel; the ringleaders were summarily expelled. For the first time in his life he saw tyranny in action, and it was to leave an indelible impression upon the young Galois. Galois's progress during his first two years at the college was excellent, and he won several prizes. Then, however, he lost interest in his studies, due mainly to indifferent and uninspired teaching, with the result that he had to retake his third year. It was while repeating this year that Galois came face to face with mathematics for the first time. Having quickly mastered the elementary mathematics taught in class, he soon realised that the dull routine of the school texts was not for him, and he decided on a bold plan of action: to read the memoirs of some of the greatest mathematicians of the age, such as Legendre, Lagrange and Abel; for only here could he find the satisfaction and stimulation his mind craved. Galois read these difficult works intended for professional mathematicians as if they were novels, an incredible achievement for a boy not yet sixteen years of age.

Galois's recognition of his own genius wrought a profound change in his personality, a change noticed by classmates, teachers and family alike. He was consumed by a burning desire to push back the frontiers of mathematical knowledge, allowing nothing to stand in his way. Boyish cheerfulness gave way to sombre contemplation. Even his classwork in mathematics was mediocre, and both staff and students regarded him as eccentric. Some extracts from his school reports for the years 1827–1828 give a glimpse into Galois's character at this time: 'secretive'; 'argumentative'; 'swayed by his passion for mathematics, he has entirely neglected everything else'; 'conceited with an insufferable affectation of originality'; 'has done his classwork only from fear of punishment'; 'delights in teasing his comrades'; 'does nothing but torment his masters'. It is not surprising that Galois was unpopular, but what is surprising is that no one, other than himself, realised that he was a mathematical prodigy and needed to be treated with great forbearance.

By the age of sixteen Galois was striving to fulfil his boyhood ambition of entering the École Polytechnique in Paris. The École Polytechnique, founded during the French Revolution, was a centre of scientific and mathematical excellence equalled nowhere else in the world, and its students were renowned for their fearless fight against injustice. In such an environment, Galois would be contented and free to develop his genius to the full. Rejecting offers of help from his teacher, he prepared for the entrance examination to the Polytechnique by himself. The result was failure and disappointment. A later commentator put forward the following explanation: 'A candidate of superior intelligence is lost with an examiner of inferior intelligence'.

During his final year at Louis-le-Grand, 1828–1829, Galois attended a course in advanced mathematics given by L. P. E. Richard, a talented mathematician and the first person to recognise his undoubted genius. Under this enlightened teacher Galois was an exemplary student, even though he spent much of his time in private study. On 1 March 1829 his first paper, on continued fractions, was published, and in May 1829 he submitted a memoir on his fundamental research into the theory of polynomial equations to the Academy of Sciences. The Academy appointed Cauchy, the foremost French mathematician at that time, to assess Galois's memoir. He rejected it, and the manuscript has not been seen since. Two further blows were soon to fall. On 2 July 1829 Galois's father committed suicide after being persecuted for his liberal views. Then a month later Galois failed the entrance examination to the Polytechnique for a second time, despite Richard's protestations that he should be admitted on his past achievement. One account relates that during the interview Galois lost his temper and threw a board-duster at the examiner, hitting him in the face.

Thoroughly demoralised and disgusted, Galois decided to enrol at the École Normale Supérieure in October 1829. This establishment, whose function was to train schoolteachers, was much less prestigious than the École Polytechnique. In February 1830 he presented some of his great work on polynomial equations to the Academy of Sciences in the hope of winning the Grand Prize in Mathematics, but the secretary, the distinguished mathematican Fourier, died before he could read it, and the manuscript disappeared. The same year, however, did see the publication by Galois of an important article on the theory of numbers and three short notes.

The three-day revolution of July 1830, which resulted in Louis-Philippe replacing Charles X on the throne of France, marked the final turning point in

Évariste Galois 1811–1832

Galois's short life. Filled with enthusiasm for the uprising, Galois tried to lead his fellow-students on to the streets to join those of the École Polytechnique who were fighting on the barricades, but he was outmanoeuvred by the director of École Normale, who kept his charges under lock and key behind the high walls of the college. This was the final straw for Galois. Now utterly frustrated, he decided to become politically active, championing the outlawed republican cause. The reasons for this frustration are easy to understand. He felt that his shabby treatment by teachers, the repeated losses of his papers and his failures to gain admission to the École Polytechnique were not merely accidents, but a result of a society which prized mediocrity and condemned genius. A lifelong lover of freedom, he became ever more depressed at the oppression he saw all around him; he had experienced this for himself at school, at college and in the hounding of his father. Finally, the initial euphoria of the revolution soon gave way to delusion, when he realised that the new regime was hardly less corrupt than the old one. During the summer of 1830 Galois became politically aware; he formed friendships with republican leaders and himself joined a banned society. The Galois who started his second year at the École Normale in November 1830 was in no mood to accept passively the harsh discipline of the school and the indecision of its director, and he made a blistering attack on the director in an article in the *Gazette des Écoles*. For this he was expelled.

Galois made one last-ditch attempt to gain recognition as a mathematician. He hastily reconstructed the paper on the solution of equations, lost by Fourier, and submitted it to the Academy of Sciences. The distinguished mathematician Poisson, having held on to the memoir for nearly six months, gave his verdict on Galois's most important piece of work: 'incomprehensible'. Galois's anger knew no bounds and he declared: 'If a corpse is needed to stir up the people, I will donate my own'.

Now, freed from all commitments, Galois devoted himself completely to the active support of the republican cause. He joined the artillery of the National Guard and took part in the riots and demonstrations then gripping Paris. On 9 May 1831 a banquet was held by several hundred republicans to protest against the royal decree disbanding the artillery of which Galois had been a member. At the end of the riotous proceedings he held his glass and open knife in one hand and proposed the toast 'To Louis-Philippe'. His companions roared their approval at this threat to the king's life. Alexandre Dumas, who was present at the banquet, made a hurried exit so as not to compromise himself. Galois was arrested the next day and sent to prison. He was acquitted on 15 June 1831 by the assize court; but from then on he was a marked man.

Galois's freedom was to be short-lived. He was arrested again on 14 July 1831 as a preventive measure on the eve of a republican demonstration. After three months' languishing in prison he was found guilty of wearing a uniform of the disbanded artillery, and for this he received a six months' sentence. While in prison he returned to his mathematical investigations. In the cholera epidemic of March 1832 he was moved to a nursing home, where he became a prisoner on parole. This was the time of his first and only love affair; it was short and sad. A few days after the break-up Galois was challenged to a duel over the woman he had loved. Some authors contend that he was tricked into this affair of honour by political enemies wishing to dispose of him.

Galois sensed his life was about to end. On the night of 29 May 1832, the day before the duel, he collected together his manuscripts and dashed off a letter to his friend Auguste Chevalier in which he outlined his principal discoveries. This pathetic letter bears witness to the stress under which it was written: the handwriting is merely scribble, some arguments are so concise that they are very difficult to follow, again and again he writes in the margin: 'I have no time, I have no time.' Galois's mathematical testament contains ideas of the utmost originality, ideas which have taken mathematicians years to appreciate and develop. His greatest achievement was to give a complete solution to one of the most famous unsolved problems of his day: when can the roots of a polynomial equation be constructed from its coefficients by repeated addition, subtraction, multiplication and root extraction? The fertility of his mind, most clearly discerned in the mathematical machinery he invented to solve his problems, reached its zenith in his greatest discovery: the theory of groups. It is ironic that, though his ideas have inspired thousands of research papers and books, his collected works fill only a slender volume of 61 pages.

Throughout the whole of his life, Galois's faith in his innate mathematical abilities remained constant. Even with the ranks of incompetent schoolteachers and some of France's greatest mathematicians apparently ranged against him, his confidence in himself never wavered. This was fortunate for the future of mathematics, because he was right and they were wrong. When he died he was almost unknown as a mathematician, his work unappreciated by all of his contemporaries. But Galois lived, and died, in the hope that one day he would be accorded the recognition he deserved. This hope is expressed in the last prophetic words he wrote on the night before the duel:

> 'Ask Jacobi or Gauss publicly to give their opinion not as to the truth, but as to the importance of these theorems. Later there will be, I hope, some people who will find it to their advantage to decipher all this mess.'

Editor's Note. Dr Webster hopes to write a sequel to this biographical note on Galois, which will appear in a later issue of *Mathematical Spectrum*. The sequel will be a historical survey of the development of the theory of equations, a subject to which Galois made fundamental contributions.

Long Live Kuratowski's Theorem!

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Kazimierz Kuratowski, a Polish mathematician who died recently, has left a legacy to mathematics—his now-classic theorem of graph theory.

Readers are probably familiar with the old chestnut:

'Three houses have each to be connected by pipes or cables to each of the gasworks, electricity power-station and waterworks. Can this be done without any pipe or cable crossing?'

Of course the intended problem is to draw the connecting lines on a piece of paper so that no two cross (or go under the foundations of buildings!) The fact that it is impossible to draw this is an easy consequence of the relationship between the number of vertices (points), edges (lines) and faces (regions) in any such array drawn without lines crossing.

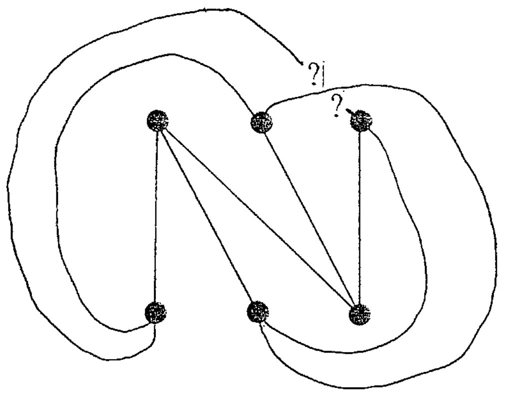


Figure 1

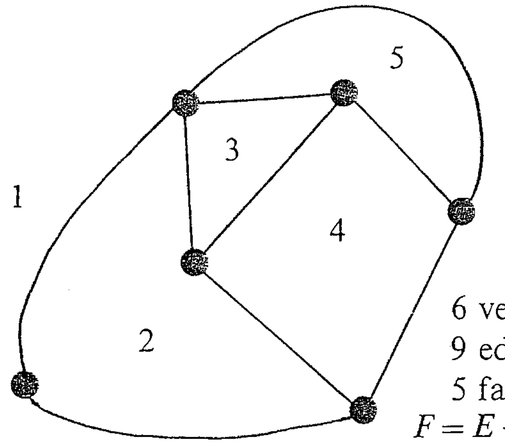


Figure 2

The relationship $F = E - V + 2$ is closely related to Euler's result on the number of faces of a polyhedron. Notice also in Figure 2 that the numbers of edges bounding faces 1, 2, 3, 4 and 5 are 4, 4, 3, 4 and 3 respectively. This gives a total of 18, which is twice the number of edges (which was to be expected, as you can easily see for yourselves). Now if our services and houses array *can* be drawn in the plane, then the number of faces will be

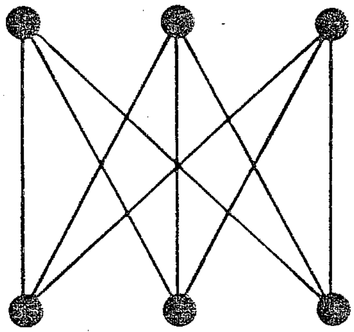
$$F = E - V + 2 = 9 - 6 + 2 = 5.$$

Each of the five faces will require at least four edges in its boundary. But then the total of at least 20 edges used in boundaries is *more* than twice the number of edges, which is not possible.

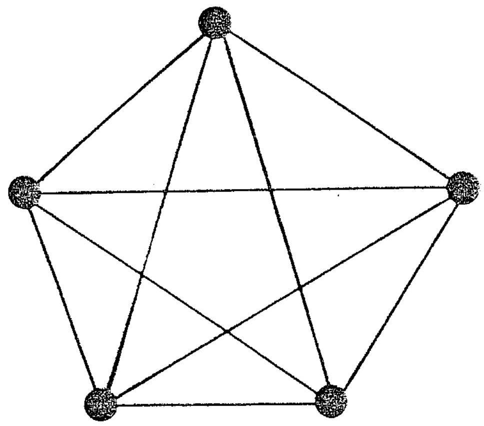
The interested reader can show in a similar way that it is impossible to draw five points on a piece of paper each joined to each of the others by a line, no two of which cross. This area of mathematics (touched on in past *Spectrum* articles†) concerns *planar graphs*, and our configurations (or graphs) are planar if they can be drawn without two of their edges crossing. So the graphs in Figure 3 are not planar (i.e. we cannot re-route their edges and avoid crossings). Nor, therefore, are the graphs in Figure 4. In other words, any graph which contains $K_{3,3}$ or K_5 within it is not planar. The remarkable fact (and much more difficult to prove) is that, if a graph is not planar, then it must have $K_{3,3}$ or K_5 within it, as in the above examples. This converse is the result of the Polish engineer and mathematician, Kazimierz Kuratowski. His difficult proof, using methods of algebra and topology, was published in 1930, and this now-classic result bears his name.

Kuratowski himself died recently, but his result will live for ever as a landmark in graph theory. It enabled other mathematicians to find different characterisations of planar graphs in this quite difficult branch of mathematics where algebra and topology (or 'rubber sheet geometry') meet. Further, its elegant style that a property holds if and only if we exclude two awkward types has led mathematicians to ask

† See, for example, Dr Bryant's article on 'Plangers' in Volume 5, Number 1 of *Mathematical Spectrum*.



$K_{3,3}$



K_5

Figure 3

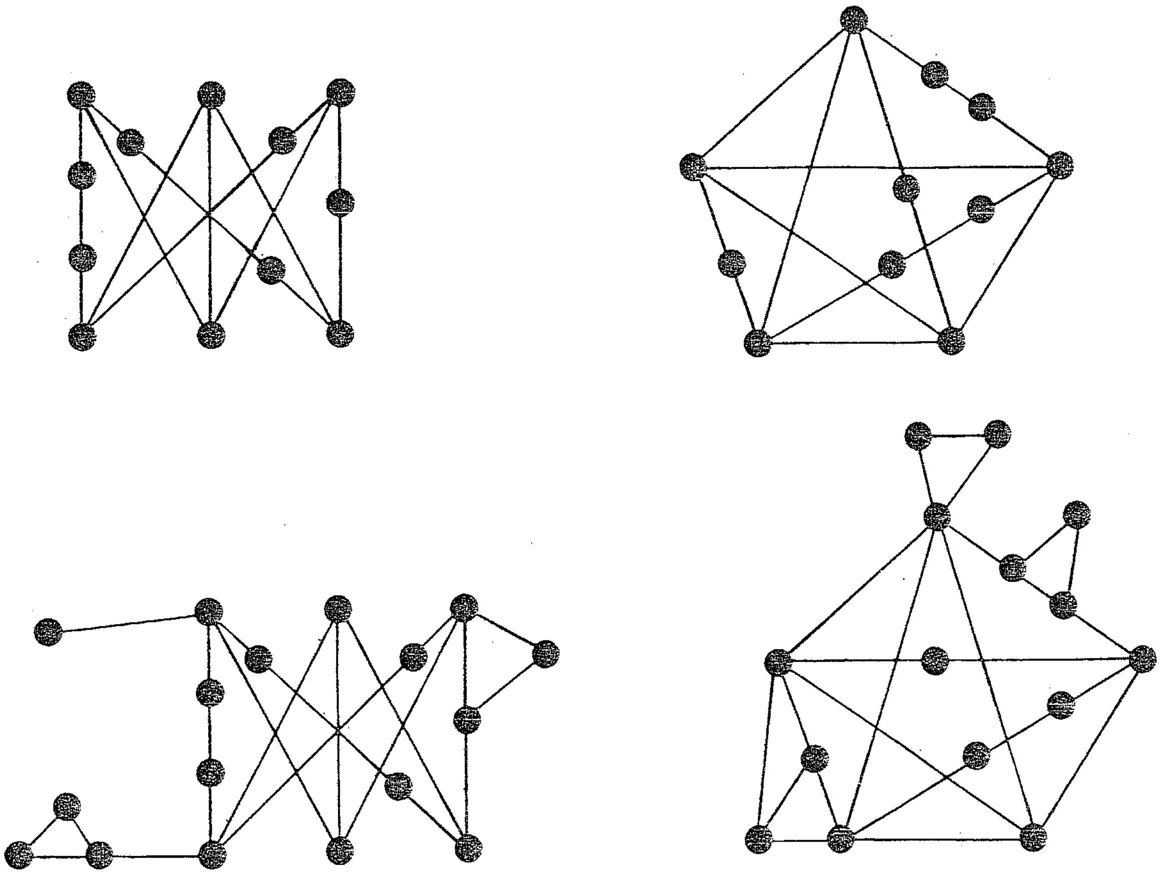


Figure 4

whether other properties of graphs can be similarly characterised. There have been some principal successes in using this technique to characterise certain graphs which arise from vector spaces. The logicians have got their hands on this topic too, and there are several results of the type 'it is impossible to characterise such-and-such graphs by the exclusion of a finite number of special ones'.

One thing is assured: Kuratowski's theorem and its relatives will live for ever!

The Secretary Problem

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The author graduated too many years ago from Cambridge. A postgraduate course in statistics led to jobs with the Medical Research Council, the University of Reading and University College London, before he became Professor at Leicester. His main interests lie in the statistical analysis of data, especially in medical applications. He lists the secretary problem (but not the secretaries themselves) among his too-few hobbies.

1. Introduction

Suppose you are an employer and that you have to choose somebody from amongst n applicants for the job of being your secretary. (In the absence of unisex pronouns, I hope we may take them to be all female without causing undue offence.) You will interview them one by one and after each interview you must decide whether to offer the job to that woman (she is certain to accept!) or to reject her and go on to the next interview. There can be no going back: once rejected, a woman will never accept your subsequent offer. At any stage—after you have interviewed r women, say—you can arrange, or rank, those you have seen in order of desirability for the job, with rank 1 meaning best and rank r worst. You must not change this ranking at any later stage: further interviews result in women simply being 'slotted into' the previous ranking. If you reject all of the first $n-1$ women then you must accept the last one however good or bad she turns out to be. You naturally want to end up having accepted the best applicant. In that case, let's say you *win*; otherwise you *lose* even if you accept a 'near miss' of second or third best. The problem is to find your best strategy, the one that will give you maximum chance of winning (clearly no strategy can hope to guarantee a win). This introduction of chance reminds me to mention that you may assume that the applicants will appear in a random order, so that all of the $n!$ possible orders are equally likely. With this condition, the secretary problem is completely specified.

Although I have stated the problem in its most popular form, it appears in many disguises. It has in its time been called the marriage problem or dowry problem (selecting the most desirable wife), and the pub problem (the most desirable resting place on a long journey). The game of 'googol' introduced by Martin Gardner in *Scientific American* in 1960 (reference 3) is also equivalent. To play this I write down any numbers I like on n slips of paper and present them to you in random order. You have to accept the slip which will turn out to bear the largest number.

The problem is just one of a class known as 'optimal stopping' problems which is itself a subset of so-called 'sequential decision' problems. These all present you with a sequence of *stages* at each of which you have to make a decision which may well influence the situation you find yourself in at the next stage. Choice of which subjects to study at each stage of your education is a nice example. Indeed, life is really just one big sequential decision problem when you think about it. Such problems are usually very difficult to solve, as it can often happen that what appears to be a good decision today may land you in a mess next week, leaving you wishing you had chosen a decision that looked less attractive at the time. In optimal stopping problems, where your decision is simply whether to stop or to continue to the next stage, much work has been done and many elegant results obtained over the past 20 years. Several practical problems, such as how best to allocate a total budget to a series of competing research projects and how to choose the best of several possible drugs to give to each of a series of patients, have been tackled. The mathematical methods developed to solve such problems have turned out to be useful in other areas, and the whole area is a flourishing one that embraces mathematics, statistics and operations research.

2. Solution of 'standard' problem

Now let's get down to solving the secretary problem. We shall follow the pioneering, and very readable, paper by D. V. Lindley (1961) (reference 5). First, a little notation. After you have interviewed r women, we shall say you are at *stage r*. Your decision whether to accept her or not will clearly only depend on her *rank s* (where she comes in order of desirability amongst those r women). We shall call this her *relative* rank, to distinguish it from her *true* rank t, her position in the league table of all n women. If her relative rank is anything other than 1, she is not the best out of the r you have seen so far, so she can't possibly be best overall and you would be stupid to accept her as you'd then be sure to lose. You only really have a decision to make, then, when $s = 1$. If you decide to accept, you are taking a gamble on whether or not any of the remaining $n - r$ women might turn out even better than this one. Since all orders are equally likely, it is easy to see that the chance of this happening is $(n - r)/n$. Equivalently, your chance of winning if you accept her is r/n. If, on the other hand, you reject her, you must go on to interview the next applicant, so r will increase to $r + 1$. Can we say anything now (at stage r) about what the relative rank s' of the $(r + 1)$th woman will be? Yes. Because of the random order property, it is equally likely to be any one of the values $1, 2, \ldots, r + 1$. We say that your *state* will change from (r, s) to $(r + 1, s')$ since this pair of numbers summarises at each stage the problem you are faced with.

Now comes the crucial idea, first formulated by Richard Bellman (reference 1). It's called the *fundamental principle of dynamic programming* and gives the essential property for your *optimal strategy* (your tactics for maximising your chance of winning):

> An optimal strategy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal strategy with regard to the state resulting from that first decision.

If you find that sentence hard to get your mind round, just think about its not being true. If, for some strategy, the remaining decisions after the first one *didn't* constitute an optimal strategy from that point on, they could be improved upon, and hence so could the overall strategy, so it can't possibly be optimal.

When you are in state (r, s), denote by $V(r, s)$ your chance of winning if you use your optimal strategy from then on. If $s \gg 1$, you will continue. With probability $1/(r+1)$ your next state will be $(r+1, s')$ for $s' = 1, 2, \ldots, r+1$ and your best chance of winning will be $V(r+1, s')$. Since at stage r you don't know what value s' will take, you can only say that your *expected* chance of winning if you continue will be the average of all the possible $V(r+1, s')$'s,

$$\frac{1}{r+1} \sum_{s'=1}^{r+1} V(r+1, s') = v_r \text{ (say)}, \tag{1}$$

since it depends only on r.

Similarly, if $s = 1$ and you decide to continue, your expected chance of winning will again be v_r. If you decide to accept, and hence stop, however, your chance of winning will be r/n. You will obviously choose according to which of these is larger, so at last we can use Bellman's principle to write down the *dynamic programming equations* for our problem. These are

$$V(r, s) = \begin{cases} \max\left[\dfrac{r}{n}, v_r\right] & \text{if } s = 1 \\ v_r & \text{if } s = 2, 3, \ldots, r. \end{cases} \tag{2}$$

This holds for $r = 1, 2, \ldots, n-1$ but $r = n$ needs separate consideration. State (n, s) means you have interviewed all n women, the last was the sth best and you must accept her. You have thus won for sure if $s = 1$ and lost for sure otherwise, so

$$V(n, s) = \begin{cases} 1 & \text{if } s = 1 \\ 0 & \text{if } s = 2, 3, \ldots, n. \end{cases} \tag{3}$$

If we now consider $r = n - 1$ we have from (1)

$$v_{n-1} = \frac{1}{n}$$

and so from (2)

$$V(n-1, 1) = \max\left[\frac{n-1}{n}, \frac{1}{n}\right], \qquad V(n-1, s) = \frac{1}{n} \qquad \text{for } s \geq 2.$$

Since $n \geq 2$, the first term is the larger and the best decision in state $(n-1, 1)$ is to accept and stop. Going on to $r = n - 2$, we get

$$v_{n-2} = \frac{1}{n-1}\left(\frac{n-1}{n} + (n-2)\cdot\frac{1}{n}\right) = \frac{n-2}{n}\left(\frac{1}{n-2} + \frac{1}{n-1}\right)$$

and

$$V(n-2, 1) = \max\left[\frac{n-2}{n}, v_{n-2}\right].$$

If $n \geq 4$ the first term is again greater than the second. Proceeding to $r = n - 3$ we end up comparing

$$\frac{n-3}{n} \text{ with } \frac{n-3}{n}\left(\frac{1}{n-3} + \frac{1}{n-2} + \frac{1}{n-1}\right)$$

and this process of *backward iteration* will clearly carry on until the second term becomes bigger than the first. This will happen when $r = r^*$, say, the integer for which

$$a_r \geq 1 > a_{r+1},$$

where

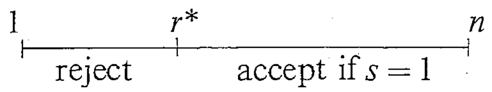
$$a_r = \frac{1}{r} + \frac{1}{r+1} + \cdots + \frac{1}{n-1}.$$

A little more algebra shows that, if we continue the iteration process down below $r = r^*$, the second term remains bigger than the first (and hence your decision in state $(r, 1)$ should be to reject and continue) all the way down to $r = 1$.

Your best overall strategy can now be stated very simply:

> *Reject the first r^* applicants, however good any of them appears to be. After that, accept the first applicant who is better than all those who have gone before.*

In terms of a diagram:



As so often in probability theory, once the solution has been found, it is intuitively obvious, or at least believable. Here the optimal strategy neatly balances out the two kinds of mistake you can make; if you stop too soon the best may be yet to come, while if you continue too long, you may have already rejected the best.

What is the actual chance you will win using your best strategy? It is clearly given by $V(1, 1)$, since you start the whole process in state $(1, 1)$, because the first applicant always has apparent rank 1. It can in principle be obtained by iteratively solving equation (2) from $r = n$ down to $r = 1$. The most interesting case occurs when n is large. Using the approximation

$$\sum_{i=1}^{n} \frac{1}{i} - \log_e n \rightarrow \gamma = 0{\cdot}577\ldots,$$

we see that as, $n \rightarrow \infty$, $r^* \rightarrow \infty$ also, but the ratio $r^*/n \rightarrow e^{-1} = 0{\cdot}387$. It turns out that $V(1, 1) \rightarrow e^{-1}$ as well, and this always strikes me as an amazing result. If you were asked to select the best out of a million or a billion applicants, you might reasonably think this was very much a needle-in-haystack affair and take a poor view of your chance of success. *But you do in fact have slightly more than one chance in three of* *winning*. Try playing a game with any friend you can persuade to give you 5 p if you win while you give him 1 p if you lose. You should make a lot of money.

3. Unknown value of n

Lindley in his original paper remarks that if you use this technique to choose your marriage partner, and if potential partners present themselves to you in a steady stream as you age from 18 to 40 years, you should certainly not accept anybody until you are $18 + e^{-1}(40 - 18) = 26$ years old. Anyone marrying younger than this must be using a sub-optimal strategy! There are two comments on this. Firstly, the optimum is very *insensitive*—if you use a value of r^* very different from the best one ($0{\cdot}25n$ or $0{\cdot}5n$, say), your chance of winning is only a few percent less than the optimal value. Secondly, in real life we do not know in advance how many 'applicants' we are going to be able to 'interview'. Presman and Sonin, two Russian probabilists, solved this problem in 1972. If N denotes the unknown number of applicants, suppose you can specify in advance the probabilities π_n that N takes the value n, for $n = 1, 2, \ldots,$. The problem now contains a further hazard. You may at some stage reject the current applicant and then find there are no more left, in which case you have lost. The general solution to the problem now becomes much more complicated, but there are a few simple special cases. If you initially think that all values of N from 1 up to some limit N_0 are equally likely, for example, the best strategy is of the same form as before, but r^*/N_0 now tends to e^{-2} as $N_0 \to \infty$. This can be used to justify any marriage after age $18 + e^{-2}(40 - 18) = 21$.

4. Alternative problem

You may already be thinking that the whole problem is quite interesting but not very realistic. Is it sensible to use what Lindley calls 'nothing but the best' as your goal? If you ended up with the second or third best applicants, wouldn't you be nearly as pleased as if you had got the very best? Might it not be better to aim at getting a pretty good secretary, that is one with as low a rank (or as high up in the ranking league table) as possible? Lindley's paper was again the first to consider this. We can tackle it by modifying equation (2) as follows:

> Let $V(r, s)$ = the minimum ranking you can achieve starting in state (r, s). In state (r, s), if you decide to continue, the average V value after the next stage is again v_r given by equation (1). If you stop, the probability that the applicant you have accepted will turn out to have *true* rank t is
>
> $$p_t = \frac{\binom{t-1}{s-1}\binom{n-t}{r-s}}{\binom{n}{r}} \qquad (t = s, s+1, \ldots, n+s-r).$$

To see this note that out of all n applicants there are $t - 1$ better than your choice and $n - t$ worse. Since your choice has apparent rank s out of the first r, $s - 1$ out of the $t - 1$ better ones (and $r - s$ out of the $n - t$ worse ones) must also be in the first r.

The expected true rank of your choice must therefore be

$$U(r, s) = \sum_{t=s}^{n+s-r} t.p_t = \frac{n+1}{r+1}s.$$

The dynamic programming equations now change from (2) and (3) to

$$V(r, s) = \min\,[U(r, s), v_r] \tag{4}$$

$$V(n, s) = s. \tag{5}$$

Now $U(r, s)$ is an increasing function of s, while v_r does not depend on s, so it is plausible that for each r there is a cut-off value $s^*(r)$ such that you will accept the rth applicant if her apparent rank $s < s^*(r)$ and reject her if $s \geq s^*(r)$. A little algebra quickly shows that, as r increases, $s^*(r+1)$ equals either $s^*(r)$ or $s^*(r)+1$, so your best strategy can be drawn as

1	r_1^*		r_2^*		r_3^*	$\cdots$ n
reject		accept if $s = 1$		accept if $s = 1$ or 2		$\cdots$

This is intuitively appealing. As you get nearer the end, you have to be less and less choosy—a phenomenon often observed in some establishments of higher education in August and September each year.

As before we can ask what is the value of $V(1, 1)$, the best expected rank you can achieve? For finite n the algebra is horrendous, but a limiting value would be useful. The usual way of tackling such problems is to approximate a *difference* equation like (4) by a *differential* equation (merely an equation that involves derivatives) that is easier to solve. Lindley tried this but couldn't make his approximation precise enough. Neither could Chow and others in 1964 (reference 2) (this had to wait until 1975), but they did provide a direct proof, using pages of algebra, showing that

$$V(1, 1) \to \prod_{j=1}^{\infty} \left(\frac{j+2}{j}\right)^{1/(j+1)} = 3{\cdot}8695\ldots \qquad \text{as } n \to \infty.$$

This is again a remarkable result. Asked to pick the best you can out of, say, a billion applicants, you can expect to do rather better than ending up with the fourth-best one.

5. Other variations

There have been over 40 research papers written now on various other forms of the secretary problem. Space only permits a brief list of some of the variations without going into detail about solutions.

Suppose the applicant you decide to choose is not certain to accept your offer but only has probability p of doing so? The probability of winning now tends to $p^{1/(1-p)}$ as a limiting value, but the form of your strategy remains the same.

Suppose you are allowed to go back and offer the job to someone you interviewed a while ago, but that the probability she will accept is only q, hurt pride or getting another job accounting for the remaining $1 - q$. Again, the strategy looks the same, but r^* now satisfies

$$a_{r^*} > 1 - q > a_{r^*+1},$$

and the probability of winning tends to e^{q-1}.

Suppose you are allowed to go back at any time and offer the job to any one of the last m applicants, who is certain to accept. Surprisingly, for fixed m, this is no help and the probability of winning still tends to e^{-1}.

If you combine an unknown value of n with the aim of minimising expected rank, the strategy '*accept if* $s < s^*(r)$, *reject if* $s \geq s^*(r)$' remains the best, but $\{s^*(r)\}$ now no longer needs to be a non-decreasing sequence. When n is equally likely to be anywhere between 1 and N_0, the minimum rank you can achieve no longer remains finite as $N_0 \to \infty$.

The paper by Gilbert and Mosteller (1966) (reference 4) contains a wealth of related problems. They allow you to choose two (or more) applicants so that you win if any of your choices is the best one, or you minimise the average ranks of those you choose. Even more interestingly, they turn the problem into a game for two players. Instead of presenting applicants in random order, I am allowed to choose the order you see them in. My best plot is to write down the integers 1 to n in reverse order and then cycle them around by a random amount, getting, say, 5, 4, 3, 2, 1, n, $n-1$, $n-2, \ldots, 8, 7, 6$. As you see them, the first 5 will have apparent rank 1 but none of the others ever will. Whatever strategy you adopt, your chance of winning will be $1/n$, which is exactly the chance of any strategy that uses sheer guesswork. If you aim to minimise expected rank, the sheer guesswork value is $(n+1)/2$, the average of possible ranks $1, 2, \ldots, n$. I can again hold you down to this level by presenting at each stage either the best or the worst of those remaining to be presented. The proof of that involves things called martingales and their story must be saved for another article.

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Algebra, Machines and Biology

MICHAEL HOLCOMBE, *The Queen's University of Belfast*

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1. Introduction

The whole process of life is concerned with the relationships between organisms and between an organism and its environment. My aim is to describe some mathematical methods for modelling this relationship between an organism and its environment; the approach is deliberately intended to be a general one, so that we can then apply the model to as many different situations as possible. The model is essentially an algebraic one.

We start with the idea of a machine's reacting to the influences of its environment. If we represent the machine by a box and the environmental influences by 'inputs' (see Figure 1), then the machine will 'react' to these inputs in various ways

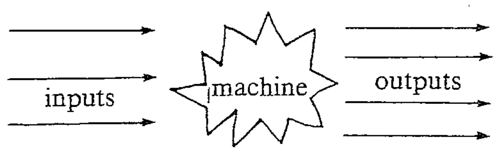


Figure 1

including some which will affect the environment; we call these reactions 'outputs'. The central problem is to relate the inputs and the outputs with the behaviour of the machine. We must first try to describe the machine in some way, and we do this by specifying a *set of internal states* that the machine can be in. At any given time the machine will be in a given internal state, and this will describe the 'internal properties' of the machine. So far I have been a little vague about the idea of machine, internal state, etc., and we should perhaps look at some examples; so we shall now 'build' a simple machine.

	a	*b*	*c*	*d*
0	*a*	*c*	*b*	*a*
1	*b*	*b*	*d*	*b*

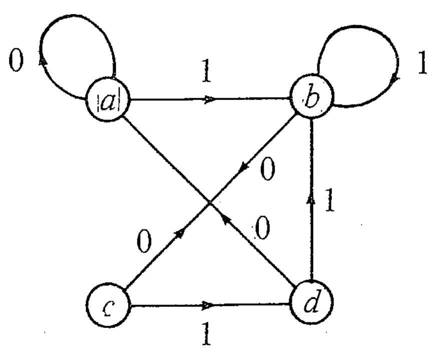


Figure 2

Suppose first that we have a set S consisting of four elements called a, b, c and d. The set S will be called the *set of internal states*. There will be a series of 'inputs' applied to the machine. For the sake of argument, these inputs will be taken from another set A; this time the set is $A = \{0, 1\}$. At any given time the machine will be in one of the four states a, b, c or d. If we apply an input, say 1, to the machine while it is in state a, the machine might then move to another internal state, say b. Another input, say 0, might result in the internal state b changing to c etc. The complete behaviour of the machine can be illustrated either by a table or by a diagram (known as a graph) as in Figure 2.

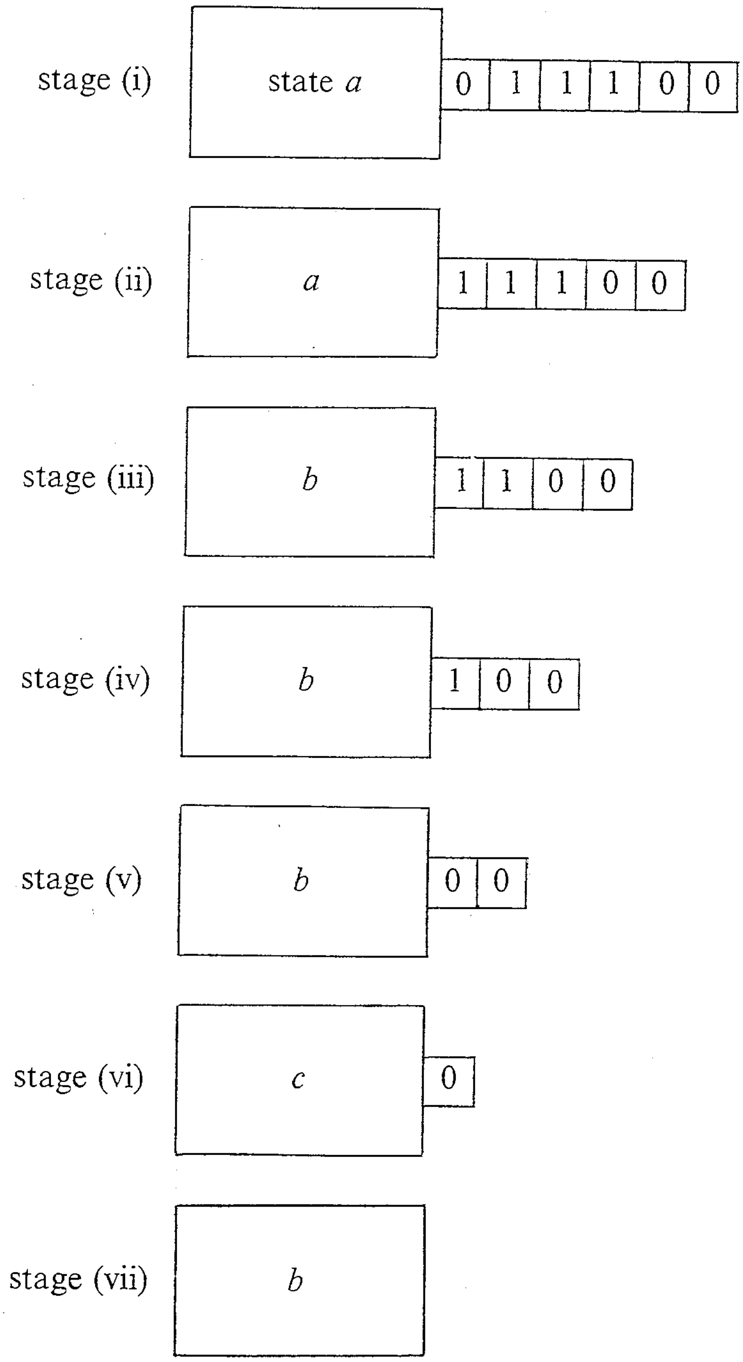


Figure 3

We call such machines *finite-state machines*. They consist of three things:

(i) a finite set S of states,
(ii) a finite set A of inputs,
(iii) a collection of mappings $f_x : S \to S$, one such mapping for each $x \in A$.

In our example there are two mappings $f_0 : S \to S$ and $f_1 : S \to S$; these can be read off from the table or the graph. They are called *next-state functions*. There is no limit to the sizes of the sets S and A, as long as they are *both finite*.

One way to visualize such a machine is to think of it as a box with a tape moving into the box from right to left with symbols from the input set printed on the tape. For example, let us input a tape with 011100 written on it. Suppose that the *initial* internal state of the box is a; what happens as the tape moves in? We illustrate this in Figure 3. The final state is thus b.

Questions. (1) What happens when the same input tape is applied to the box in initial states b, c and d?

(2) Construct your own machine by specifying the sets S, A and the table (or graph) that defines the next state functions. Work out a few sequences of events as you input some symbols.

The next stage is to introduce some form of output to the machine. We do this by first specifying a set, B, of output symbols. Then, as the machine changes its state, we arrange for it to produce a symbol from this set as an output. The symbol chosen will depend not only on the state at the time but also on the input received. Returning to our graph diagram, we can now describe the output by writing an appropriate output symbol on each arrow; we distinguish these symbols from the input symbols by writing them in square brackets.

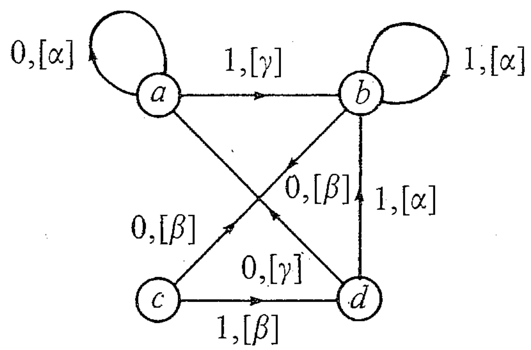


Figure 4

Example. Let $B = \{\alpha, \beta, \gamma\}$ be the finite set of output symbols. Then the machine shown in Figure 4 gives an output of α if it receives an input of 0 while it is in state a, etc. Going back to our box with a tape running into it, we can image that the machine is printing the output symbols on another tape. Let us input the tape with 011100 again into the machine in state a (Figure 5). Verify that the complete output tape is $\alpha\,\gamma\,\alpha\,\alpha\,\beta\,\beta$. The length of the output tape is always the same as the length of the input tape. The output is determined by a collection of mappings $g_x : S \to B$, one such

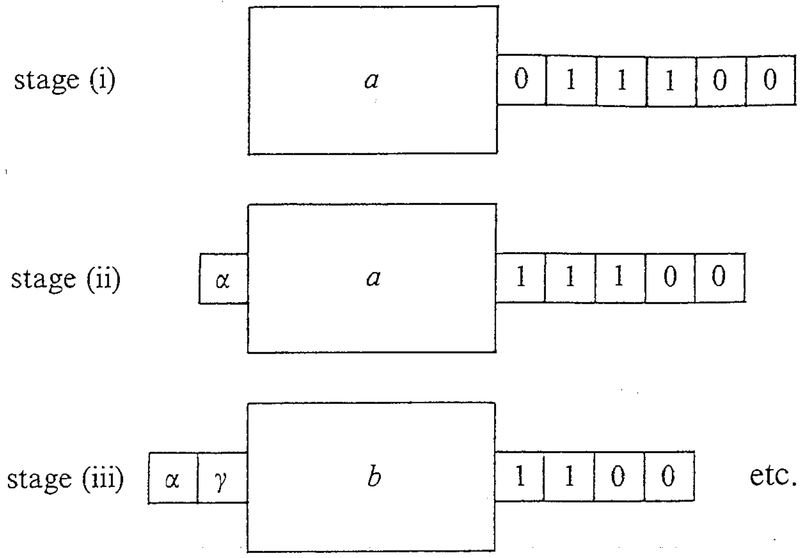


Figure 5

mapping for each $x \in A$, and we can describe these using a table instead of a graph. The example above is described in Table 1. The machines are called *Mealy machines* (after G. Mealy, an American mathematician who first studied this type of machine in 1955). We now study some 'real life' examples of Mealy machines.

TABLE 1

	a	b	c	d
f_0	a	c	b	a
f_1	b	b	d	b
g_0	α	β	β	γ
g_1	γ	α	β	α

2. Electrical circuits. The NAND$_2$ component

This component, of importance in the design of computers, can be regarded as a Mealy machine. (Its name is a standard term in the underlying logical theory of such components.) We have two transistors T_1, T_2 (see Figure 6). The input lines I_1, I_2 each either carry *a current* (denoted by 1) or *no current* (denoted by 0). The output line K, again, either carries a current (1) or no current (0). The input set consists of the following *total inputs* $\{(0,0), (0,1), (1,0), (1,1)\}$, where $(0,0)$ means 0 at I_1, 0 at I_2, $(0,1)$ means 0 at I_1, 1 at I_2, etc. The output set is $B = \{0, 1\}$. The internal states are given by the states of the transistors, i.e. off or on. We have four internal states:

(off, off) meaning T_1 off and T_2 off
(off, on) meaning T_1 off and T_2 on
(on, off) meaning T_1 on and T_2 off
(on, on) meaning T_1 on and T_2 on.

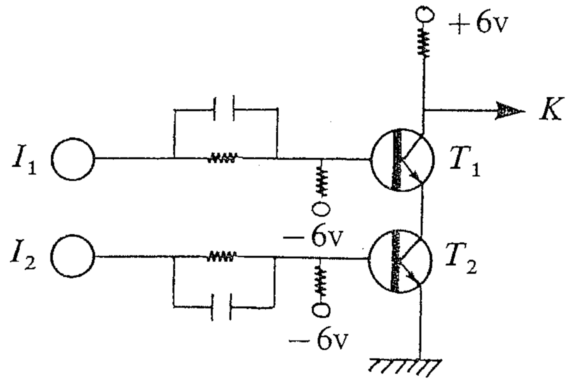


Figure 6

The table for this machine is given in Table 2.

TABLE 2

	(off, off)	(off, on)	(on, off)	(on, on)
$f(0,0)$	(off, off)	(off, off)	(off, off)	(off, off)
$f(0,1)$	(off, on)	(off, on)	(off, on)	(off, on)
$f(1,0)$	(on, off)	(on, off)	(on, off)	(on, off)
$f(1,1)$	(on, on)	(on, on)	(on, on)	(on, on)
$g(0,0)$	1	1	1	1
$g(0,1)$	1	1	1	1
$g(1,0)$	1	1	1	1
$g(1,1)$	0	0	0	0

More complex components clearly give more complex machines, but there is nothing, in theory, to prevent us from regarding computers as Mealy machines.

3. A model of the brain

The brain consists of a collection of specialized nerve cells, called *neurons*, connected together in particular ways. The neurons have certain basic features, a cell *nucleus*, *dendrites* and an *axon* (see Figure 7). Small electrical impulses are passed

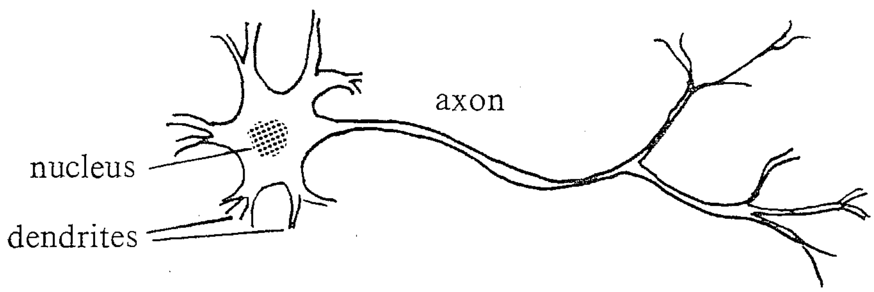


Figure 7

through the system by complex biochemical reactions; they arrive at the dendrites of a neuron, and during a period of time the nucleus of the neuron 'adds up' the impulses arriving at all the dendrites. Some of the impulses excite the neuron, others inhibit it. There is a certain threshold value and, if the total of the exciting impulses minus the total of the inhibiting impulses exceeds the threshold, the neuron produces an electrical impulse which is sent down the axon. The ends of the axon are in connection with dendrites of other neurons. In this way impulses are propagated through the nervous system, which is a vastly complicated interconnection of neurons. A simple model of a neuron could be as shown in Figure 8, with 3 dendrites and a threshold value of 3. As before, we shall regard a 'charge' coming along a dendrite as 1, and 'no charge' will be denoted by 0. The figures $+1$, $+2$, -1 along the dendrites indicate the relative importance of the dendrite, so that an input is multiplied by $+2$ if it arrives along the middle dendrite, -1 if it comes along the lower dendrite, etc. The negative values indicate *inhibitory* dendrites.

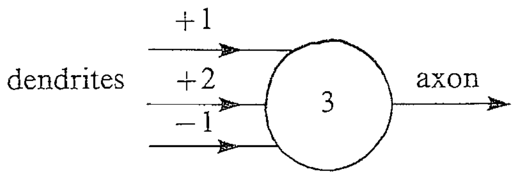


Figure 8

Suppose that there are inputs of 0, 1, 1 respectively along the three dendrites. The combined effect of the inputs at the nucleus will be $0.(+1) + 1.(+2) + 1.(-1) = 1$. This is less than 3 so the axon does *not* fire; however, if the inputs are 1, 1, 0 respectively, then the total is $1.(+1) + 1.(+2) + 0.(-1) = 3$ and the axon does fire. We may regard this neuron as a finite state machine, with two states 'on' and 'off'. By 'on' we mean that the neuron has just fired and we take 'off' to mean that it has not fired. (Recall that we are operating in a discrete time scale so that, if the neuron receives inputs at time $t = 0$, the state of the neuron depends on the inputs received at the point of time immediately preceding $t = 0$.) The combined inputs to the neuron are described by triples; thus $(0, 1, 1)$ and $(1, 1, 0)$ are inputs to the machine. There are different inputs. The table can be calculated, and notice that the initial state of this machine does not affect the final state after the application of an input to the system.

If we connect up a number of neurons to form a simple circuit, such as that shown in Figure 9, we can still consider the complete system as a finite-state machine. Here the state of the machine will be a triple of the form (on, off, on) etc., where 'on' is the state of neuron N_1, neuron N_2 is 'off' and N_3 is 'on' etc. So there are 2^3 states and 2^3 inputs. The machine table can now be drawn up. We shall work out just a couple of entries.

Applying $(1, 0, 1)$ to (off, off, off) will yield the state (off, off, off) because the threshold of N_1 is not reached and similarly N_2 will remain off as no charge reaches it since N_1 is off. Finally N_3 remains off. There is no charge at O_1 or at O_2. Applying

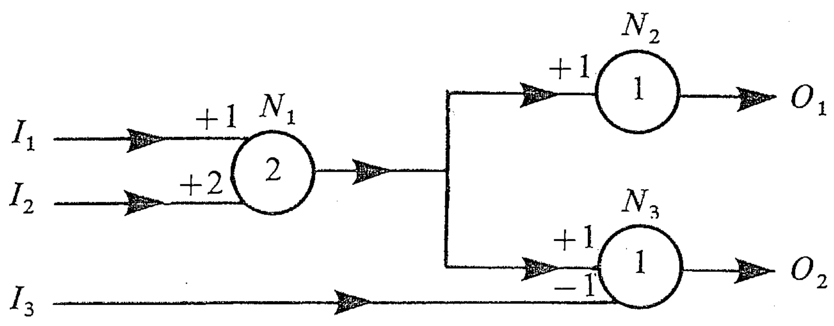


Figure 9

(1, 0, 1) to (on, off, on) we see that again N_1 must then turn off, N_2 receives a charge so is then on and N_3 becomes off since the charge in I_3 neutralizes the charge produced by N_1 being on at the start. We get (off, on, off). There is thus a charge at O_1 but not at O_2.

The pair (O_1, O_2) can be thought of as an output from the system and so we can consider such a neural network as a Mealy machine. Our model of the brain, being a large neural network, is thus another example of a Mealy machine. We must not get carried away, however, with this model; it is based on so many arbitrary assumptions that we must be careful how we apply it in neurophysiology. However, these models have played an important role in the theory of computers.

Mealy machines have arisen in many other biological situations. General biological cells and groups of cells can be considered as Mealy machines; the internal states are vectors representing the concentrations of many chemicals, temperatures, etc. The inputs and outputs are also vectors involving similar quantities. The state tables are clearly vastly complicated, and it is not practicable to do detailed work with these models. They are useful, however, in the study of the way in which cells and groups of cells organize.

One very fruitful area of research is into the way cells can repair themselves. An interesting model known as a *metabolism-repair system* has been developed to look at this problem; the mathematics, however, becomes rather advanced.

One other area we shall now look at is the modelling of biochemical reactions.

4. Metabolic pathways

Consider the Krebs cycle (Figure 10), which is the series of chemical reactions that enable animals to convert carbohydrates into useable energy. (Don't worry about the chemical details!)

For each reaction to take place there must be present molecules of a certain enzyme, denoted by $E_1, E_2, \ldots, E_9$. (These are really just chemical catalysts.) For some stages of the reaction other chemicals called *coenzymes*, C_1, C_2, C_3, are required. These coenzymes combine with certain hydrogen atoms and pass these on in subsidiary reactions to release the energy from the cycle. We shall first of all rename the chemicals—oxalacetic acid, citric acid, . . . etc.—as $S_1, S_2, \ldots, S_9$. Now we delete all those reactions *not* involving coenzymes to obtain the cycle shown in Figure 11. The coenzymes C_1, C_2, C_3 are regarded as inputs to a state machine with states S_1,

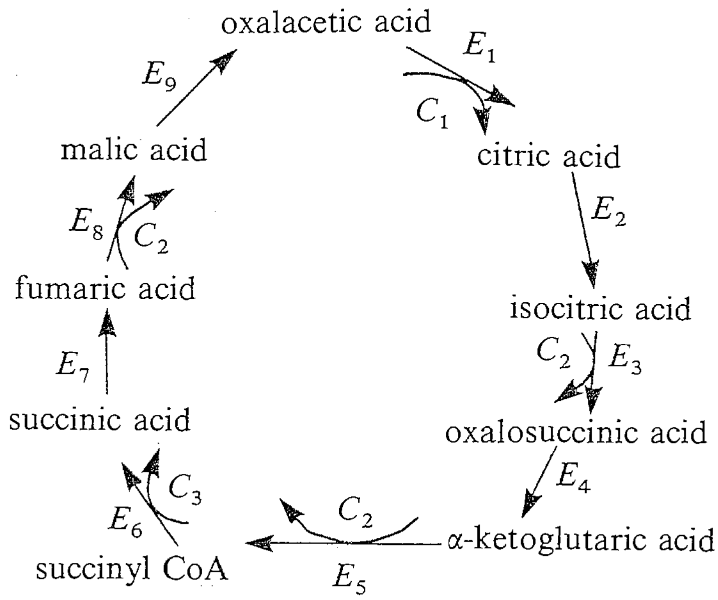


Figure 10

S_3, S_5, S_6, S_8. The state table (Table 3) can now be drawn up if we realize that the coenzymes affect only the reactions indicated in Figure 11, so that, for example, input C_2 will not change the state S_1 etc.

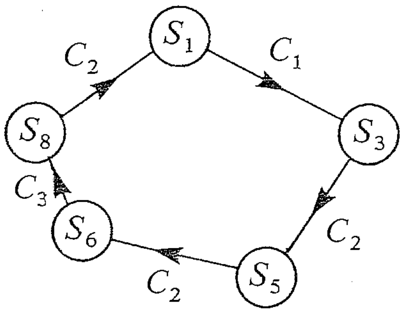


Figure 11

TABLE 3

	S_1	S_3	S_5	S_6	S_8
C_1	S_3	S_3	S_5	S_6	S_8
C_2	S_1	S_5	S_6	S_6	S_1
C_3	S_1	S_3	S_5	S_8	S_8

This procedure can also be carried out with far more complex metabolic pathways. The resulting machine may then be analyzed *algebraically*, and the algebraic consequences are then reinterpreted in the original biochemical context. This is an area where research is currently still at an early stage.

Our final biological application involves a model of the growth of an organism.

5. Lindenmeyer systems

Table 4 is reminiscent of a finite-state machine with states 0 and 1 and inputs also 0 and 1. However, notice that one of the entries in the state table is 11, which is not a state in the accepted sense. What we are going to do is exhibit a model which allows machines to 'grow' in a way that is similar to the way organisms or groups of cells grow, namely by *cell division*.

TABLE 4

inputs		states 0	states 1
	0	0	11
	1	0	0

We imagine first that we have our cell behaving like a state machine with two possible states 0 or 1. Suppose that the cell is in state 1 and receives a 'beneficial' input from the environment, namely the input 0. The cell has divided into two identical cells, both in state 1, side by side (Figure 12). Now we want to apply another

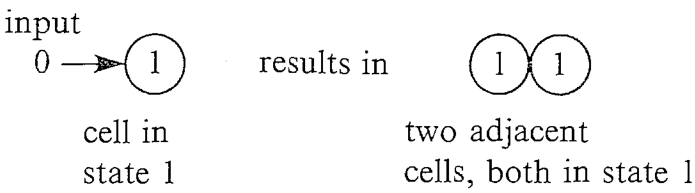


Figure 12

beneficial input 0 to the left-hand cell it will then divide again to form two cells in state 1. What happens to the right-hand cell? We make the basic assumption that cells on the right-hand side of other cells behave like machines that receive as inputs the *state* of their immediate left-hand neighbour. This happens at the same time as the left-hand cell receives its environmental input. The biological justification for this assumption is that the right-hand cell is influenced very strongly by what happens in its left-hand neighbour (see Figure 13).

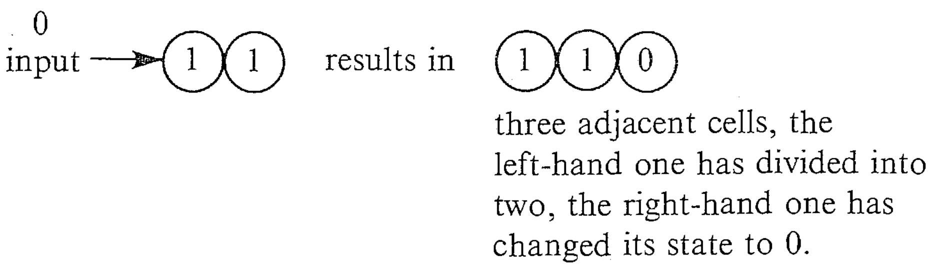


Figure 13

Following through a sequence of beneficial inputs gives us Table 5. The organism is growing. Figure 14 shows a very crude model of the way a root tip of a plant might grow.

TABLE 5

input	state of system
0	1
0	1 1
0	1 1 0
0	1 1 0 0
0	1 1 0 0 0 1 etc.

Obviously this is a very crude model, but with the help of computers very accurate models of the growth of plants, snail shells etc. have been developed. They help us to develop a *framework* for understanding the process of cellular growth. The first person to study this particular type of model was A. Lindenmeyer in 1968.

There are many other applications of machine theory in biology and also to some important problems in psychology, including the theory of learning, question-answering, etc.

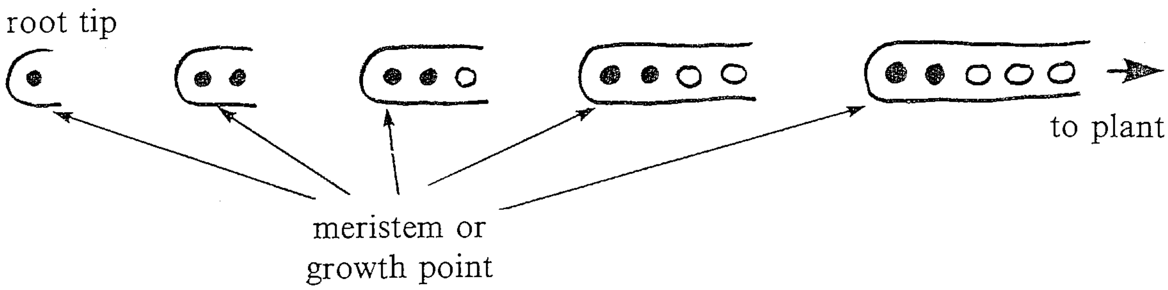


Figure 14. A developing root tip.

We have used most of our space on an examination of the applications of machines, and have only mentioned the algebra of these machines in passing. Unfortunately the algebra is rather difficult, but basically there are two questions of interest. Firstly, given a machine, can we find a 'simpler machine' that will 'do the same thing'? Secondly, can we find a way of joining together simple machines in some way so that they behave like the given machine? Positive answers exist for both questions.

In the first case techniques exist for reducing the number of states to a minimum. The second question leads to methods of constructing machines from mathematical groups and to the construction of certain very simple machines by connecting them together in parallel and series in suitable ways.

Further reading

Here are a few books that you may find interesting:

M. A. Arbib, *Brains, Machines and Mathematics* (McGraw-Hill, New York, 1964).

M. Minsky, *Computation, Finite and Infinite Machines* (Prentice-Hall, London, 1967).

G. Birkhoff and T. C. Bartee, *Modern Applied Algebra* (Van Nostrand, Princeton, 1970).

Letters to the Editor

Dear Editor,

Mathematics and sport

The article by Burghes (*Mathematical Spectrum* Volume 13 Number 2) on mathematics and sport produced the result that the relationship between the world athletic record time T and distance D was of the form $T = KD^{1\cdot 1}$. It was then suggested that this analysis could be used to determine a runner's best racing distance, or to find an overall running champion (said to be Don Quarrie, for his 19·8 seconds over 200 metres).

This seems naive, both athletically and mathematically. A quick check would show that even if the 200-metre record were only 23 seconds (a time which even I could achieve, a few years ago!) it would still be superior to the 1500-metre and mile records, according to the above formula. The truth is that all that these 'handicapped' times ($T/D^{1\cdot 1}$) show is the lack of fit of the chosen model. Since we know beforehand that it takes roughly twice as long to run a distance $2D$ as it takes to run D, it is not very illuminating to plot T against D, or even $\log T$ against $\log D$. It is surely more enlightening to plot something like average speed (T/D) against $\log D$. This shows, what most athletes would confirm, that there are three regions: (i) the sprints (60 to 400 metres) in which the initial acceleration up to full speed plays a significant role, and in which full speed falls off quadratically with $\log D$; (ii) the long-distance races (2000 metres to marathon), in which average speed decreases linearly with $\log D$; and (iii) the middle-distance events which can be seen as a transitional stage between (i) and (ii). No single formula can be expected to apply to the whole range of distances from 60 metres to marathon, and hence the residuals left after the fitting of any simple model such as $T = KD^{1\cdot 1}$ merely demonstrate the lack of fit of the model itself rather than any relative merits of the records. Only neighbouring distances can be compared, I would suggest, so that for example Sebastian Coe's mile record of 3 minutes 47·33 seconds can be seen to be 'better' than Steve Ovett's 1500-metre record of 3 minutes 31·36 seconds by consideration of the speed–distance plot between 1000 and 2000 metres.

Yours sincerely

M. J. Maher
(University of Sheffield)

Dear Editor,

The derivative of the characteristic polynomial of a matrix

The following result may be of interest to your readers:

$$\frac{d}{dx}\{\ln(\det(xI - A))\} = \operatorname{trace}(xI - A)^{-1},$$

where A is a square matrix, so that $\det(xI - A)$ is the characteristic polynomial of A, and ln denotes the natural logarithm.

An example may help to illustrate this result. Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

so that

$$\det(xI - A) = \begin{vmatrix} x-1 & -2 \\ -2 & x-1 \end{vmatrix} = x^2 - 2x - 3 = (x+1)(x-3).$$

Hence

$$\frac{d}{dx}\{\ln(\det(xI-A))\}=\frac{2x-2}{(x+1)(x-3)}.$$

On the other hand,

$$(xI-A)^{-1}=\frac{\operatorname{adj}(xI-A)}{\det(xI-A)}=\frac{1}{(x+1)(x-3)}\begin{pmatrix}x-1 & 2\\ 2 & x-1\end{pmatrix},$$

so that

$$\operatorname{trace}(xI-A)^{-1}=\frac{2x-2}{(x+1)(x-3)},$$

and we have verified our result in this particular example.

To prove the result generally, let B be a square matrix whose elements are differentiable functions of x. If we denote by b_{ij} the (i,j)th element of B and by B_{ij} the (i,j)th cofactor of B, then it may be proved by differentiating the formula for $\det B$ that

$$\frac{d}{dx}\{\det B\}=\sum_i\sum_j\frac{db_{ij}}{dx}B_{ij}.$$

If now $B=xI-A$, this gives

$$\frac{d}{dx}\{\det B\}=\sum_i B_{ii}=\operatorname{trace}(\operatorname{adj}B)=\operatorname{trace}((\det B)B^{-1})$$
$$=(\det B)\operatorname{trace}(B^{-1}).$$

Hence

$$\frac{1}{\det(xI-A)}\frac{d}{dx}\{\det(xI-A)\}=\operatorname{trace}(xI-A)^{-1},$$

which gives the required result.

Yours sincerely,

J. M. H. Peters
(Department of Mathematics,
Liverpool Polytechnic)

Dear Editor,

Inaccurate clocks

Keith Austin's question 'Why is a clock that never goes better than one that loses 1 minute per day?' ('A survey of mathematical puzzles', *Mathematical Spectrum* Volume 14 Number 2) is incorrectly posed. The word 'better', unqualified in any way, used in connection with clocks can mean only 'of more value as a time-keeper'. A clock that never goes has no value as a time-keeper.

C. L. Dodgson's original question was roughly as follows: If we agree that the better of two clocks is the one that more often shows the correct time and are offered the choice of two clocks, one that loses a minute a day and one that does not run at all, which one should we accept? The answer to this question is, of course, the one that does not run at all.

Yours sincerely,

Eric Holland
(18 Easenby Avenue, Kirkella, North Humberside HU10 7JP)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and home address and also the postal address of your school, college or university.

Problems

14.7. (From the Hungarian Olympiad, 1981) The angles of the convex hexagon $ABCDEF$ are equal. Prove that

$$AB - DE = EF - BC = CD - FA.$$

14.8. A man takes his obstinate dog for a walk. The man walks in a straight line, and the lead is always taut. Initially, the lead is at right angles to the direction in which the man walks. How far has the dog walked when the man has walked a distance d? (This leads on from Problem 14.4 in the last issue.)

14.9. (Submitted by D. J. Roaf, Exeter College, Oxford) After the second ballot for the election of the Deputy Leader of the Labour Party, it was announced that Mr Benn had received from members of the Parliamentary Party 10·241 % of the electoral college vote. A television commentator said that it would be interesting to know how many MPs had actually voted for Mr Benn, but that this would not be known until the next day. Is this correct? The total electoral vote of the Parliamentary Labour Party was 30 %, divided in proportion to those who voted for each candidate, and there were, at that time, 254 Labour MPs, not all of whom voted. How many MPs voted for Mr Benn if we assume that the figure of 10·241 was correct to $\pm 0{\cdot}0005$?

When the actual votes were disclosed later, it was found that the figure of 10·241 had been miscalculated; it should have been 10·240. Can you find the actual figures from this information?

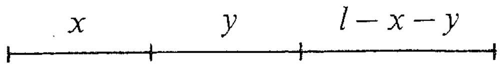
Solutions to Problems in Volume 14, Number 1

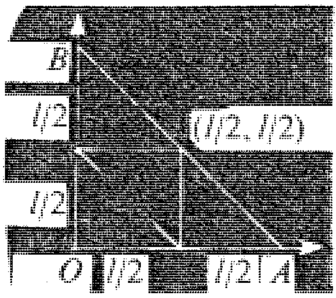
14.1. A straight rod is cut at random into three pieces. What is the probability that one of the pieces is at least half as long as the original rod?

Solution

The simplest approach is perhaps the best in solving this problem. Suppose the rod is of length l and is cut into three pieces of lengths x, y, $l-x-y$. Then $0 \le x+y \le l$, so the possible values of x, y are given by the coordinates of the points in the triangle OAB as shown in the diagram. We shall obtain a piece at least half as long as the rod provided that either $x \geq l/2$ or $y \geq l/2$ or $l - x - y \geq l/2$, i.e. either $x \geq l/2$ or $y \geq l/2$ or $x + y \leq l/2$, i.e. provided that the point (x, y) lies in one of the shaded triangles. Since the four smaller triangles all have the same area, this gives a probability of 3/4.

Also solved by P. C. Macey (Peterhouse, Cambridge), M. G. Sykes (Huddersfield New College) who also discussed the generalisation of this problem to n cuts.





14.2. There are five people in a room. Given any two of them, there is just one other who is a friend of them both. Show that there is just one person who is a friend of all the rest. Having established the result for five people, extend your solution to the case of seven people.

Solution

The problem was originally posed by Victor Bryant to accompany his article 'Bonds of Friendship', and it is his solution that we give here. Firstly, the people cannot all be friends of each other. Assume that A is not a friend of B and that their mutual friend is C. Now B and C have a mutual friend D, who cannot be A. The mutual friend of C and D is B, not A, so A and D cannot be friends. The final person E must be the mutual friend of A and C. It is an easy matter to see that the only bonds of friendship are those shown by the unbroken lines in Figure 1 and there is just one person, C, who is a friend of all the rest. Similar arguments show that the seven-person case can be represented as in Figure 2. Victor Bryant also pointed out that this result generalises to any odd number of people. This general problem was considered by P. C. Macey.

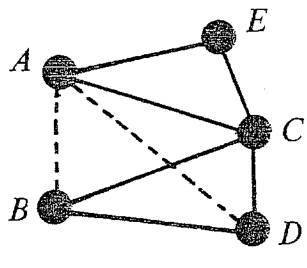


Figure 1

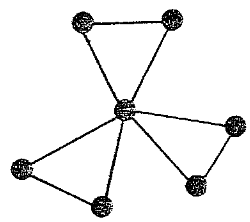


Figure 2

14.3. Given four distinct primes p_1, p_2, p_3, p_4 and four integers q_1, q_2, q_3, q_4, prove that the determinant

$$\begin{vmatrix} p_1\alpha_1 + q_1 & p_2\alpha_2 + q_2 \\ p_3\alpha_3 + q_3 & p_4\alpha_4 + q_4 \end{vmatrix}$$

takes all integral values as the integers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ vary.

Solution by P. C. Macey (Peterhouse, Cambridge)
The given determinant is equal to

$$\alpha_1 p_1(p_4\alpha_4 + q_4) - \alpha_2 p_2(p_3\alpha_3 + q_3) + q_1(p_4\alpha_4 + q_4) - q_2(p_3\alpha_3 + q_3).$$

If m_1, m_2 are coprime integers, then it is well known that there exist integers s, t such that $sm_1 + tm_2 = 1$, from which we see that the equation

$$\alpha_1 m_1 + \alpha_2 m_2 = n$$

is solvable in integers for all integers n. Hence it is sufficient to show that integers α_3, α_4 can be chosen so that $p_1(p_4\alpha_4 + q_4)$ and $p_2(p_3\alpha_3 + q_3)$ are coprime. In particular, this will be so if

$$p_1(p_4\alpha_4 + q_4) - p_2(p_3\alpha_3 + q_3) = 1,$$

i.e. if $\alpha_4 p_1 p_4 - \alpha_3 p_2 p_3 = 1 - p_1 q_4 - p_2 q_3$. Since $p_1 p_4$ and $p_2 p_3$ are coprime, α_3 and α_4 can indeed be chosen to satisfy this condition. (In fact, we do not require that p_1, p_2, p_3, p_4 be prime, only that $p_1 p_4$ and $p_2 p_3$ be coprime.)

Book Reviews

What is the Name of this Book? By RAYMOND SMULLYAN. Penguin Books Ltd., 1981. Pp. 255. £1·95 paperback.

This splendid book is now available in paperback form. It is, as one might guess from its title, about logical puzzles, some of which are conundrums, some are jokes and some have a substantial logical and mathematical content for which careful thinking is required. All the problems are given in an entertaining style and setting, and there are comprehensive (and comprehensible) solutions.

Here is a fairly typical problem to whet your appetite. It comes from the section entitled 'From the Files of Inspector Craig'.

Was It a Wise Thing to Say?

On a small island a man was being tried for a crime. Now, the court knew that the defendant was born and bred on the neighbouring island of knights and knaves. (We recall that knights always tell the truth and knaves always lie.) The defendant was allowed to make only one statement in his own defence. He thought for a while and then came out with this statement: 'The person who actually committed this crime is a knave'.

Was this a wise thing for him to have said? Did it help or injure his case? Or did it make no difference?

Buy the book today!

University of Durham HUGH NEILL

Rethinking Mathematical Concepts. By ROGER F. WHEELER. Ellis Horwood Ltd., Chichester, 1981. Pp. 314. £17·50.

This book is written for student teachers and those already teaching mathematics, but its aim differs from that of other books in the area. It does not tell the reader how to teach mathematics but makes him consider the way he would teach it and why. It contains a range of mathematics covering much of the school curriculum. Each chapter is concerned with a particular mathematical topic; the author picks out several points relating to that topic and discusses them further, pointing out where difficulties and confusion may occur. He considers the different methods of teaching them, giving the merits and disadvantages of each, but leaves the reader to decide for himself which method is best. He also discusses the different notations in use, in particular pointing out how bad notation or misuse of notation can lead to a misunderstanding of the principles involved. Many examples are given, a large number of which involve mathematics beyond the school curriculum. The reader is required to understand the relevant topics, and so would need to have studied the subject beyond school level to make full use of the book.

This book is designed to encourage the reader to think and to use his own ability rather than to teach him mathematics. The examples used make many interesting and useful points not only to aid the reader to teach others but also for his own understanding of the subject. For this reason the book would probably be interesting to anyone studying mathematics beyond school level. It is rather expensive but certainly worth the attention of anyone involved in or considering teaching mathematics.

University of Durham SUSAN WOODS

Sets, Functions and Logic. By K. J. DEVLIN. Chapman and Hall Ltd., London, 1981. Pp. ix + 90. £6·50 hardback; £2·95 paperback.

The author claims that this book is to provide the first-year student 'with a solid foundation in the basic logical concepts necessary for most of the subjects encountered in a university mathematics course'.

Chapter 1 introduces mainly the basic ideas of the language of mathematics. The chapter works through the definitions for various basic mathematical statements and symbols with good examples on how to use these in context and in various combinations. Truth tables are introduced, and negation of statements is considered, and finally the chapter is concluded by examples of various proofs by contradiction and induction, bringing together various parts of the chapter in the methods. As a student who studied a traditional mathematics syllabus at school I found this chapter extremely useful as well as easy to understand, since although my own 'definitions' for the basic statements were adequate for school mathematics, they were found wanting when I started university mathematics.

Chapter 2 introduces the idea of sets along with the notations used when working with them (including the use of Venn diagrams). I felt that intersection and union of sets were considered in particularly good detail. The chapter then goes on to discuss functions as defined using the idea of sets, and their various properties, e.g. bijective functions, invertible functions, and composition of functions. I think invertibility was dealt with slightly too briefly; a few examples of working with inverses would have been helpful.

The main topics discussed in Chapter 3 are the concepts of real numbers and their properties. We are introduced to upper bounds, absolute values, intervals and sequences, with various examples of each. My only criticism of this chapter is that I found the example showing the completeness property of the real line rather confusing. This could have been improved greatly if some of the steps in the proof had been explained instead of just stated.

The final chapter deals with the major aspects of complex numbers, discussing the various ways of representing complex numbers and also how to work with them, for instance, to solve equations. Argand diagrams are mentioned along with conjugate complex numbers and de Moivre's Theorem. I found the chapter well explained and easily understood if studied carefully. It is not a very detailed study of complex numbers, but intentionally so, as the author only wished to illustrate the basic ideas and properties of complex numbers.

My only criticism of the book in general is that there are no answers to any of the set exercises. I found that, when working through the exercises, especially on notation questions in Chapter 1, answers to selected questions would have been helpful to check that I was answering the more difficult questions correctly. Apart from this, I found the book both interesting and very useful. It is written in a style which is enjoyable to read, and the basic concepts I found were explained very well. I would recommend the book to any first-year student who is a little uncertain in some of the basic concepts of university mathematics.

Collingwood College, University of Durham I. STOKOE

Statistics: A Foundation Course for Accountancy and Business Studies Students. By STANLEY LETCHFORD. Gee & Co. (Publishers) Limited, London, 1980. Pp. 184. £3·95.

This book describes a series of fairly useful techniques, but does little to explain theoretical concepts. Its use in A-level statistics courses is therefore limited.

The pace of the early chapters is quite leisurely, but from Chapter 8 the student will find the material less easy to grasp because many definitions and technical terms are introduced too rapidly. Even after studying Chapters 8 and 9 thoroughly the student will be equipped to tackle only a limited range of problems. More attention to topics such as the normal distribution and small samples, and help in deriving the various formulae, would have been useful.

The book includes chapters on index numbers, forecasting, regression, and rank correlation. Although a student of moderate mathematical ability would obtain from it a fair idea of the scope of statistics, he would need help in mastering the more technical details.

Prior Pursglove College, Guisborough, Cleveland S. KING

The Mathematics of Games and Gambling. By EDWARD PACKEL. The Mathematical Association of America, 1981. Pp. x + 141. £5·60.

This book is part of the New Mathematical Library, which describes itself as 'a series written by professional mathematicians in order to make some important mathematical ideas interesting and understandable to a large audience of high school students and laymen'. The author's approach to this aim is to take a wide ranging sample of topics, but not to treat them just at a superficial level. He dips into areas such as dice, card games, horse-racing and lotteries, and also considers some more esoteric aspects of game theory such as the 'majority game' of voting within committees. The snippets within each topic are taken at some depth, and pointers are given to other areas. The reader is encouraged to extend many of the ideas through some imaginative exercises, and a useful bibliography refers to more detailed treatments of each subject.

The mathematics in this book is well treated, and is clearly illustrated by a fairly extensive use of tables. A prior knowledge of elementary probability theory would perhaps help, since this topic is approached at some speed.

Within a general overview of the history and current practices of gambling, the author tries to present an impartial view of the morality of gambling. He looks at various ways of 'beating the system', but the figures all seem to show that the gambler has an uphill struggle in his attempts to win over the 'house edge'. Anyone who reads this book with the hopes of finding how to make a fortune by gambling will be sorely disappointed!

Maidenhill School A. F. RICHARDS

Numerical Analysis: the Mathematics of Computing, 2nd edition. By W. A. WATSON, T. PHILIPSON and P. J. OATES. Edward Arnold (Publishers) Ltd., Londo , 1981. Pp. 206. £5·95.

'It is a matter of regret that there are still many advanced level students who are not introduced to numerical analysis.'—the words of P. J. Oates in his preface to this book. And in the spirit of this statement this book provides good, sound reading to those dipping into the subject for the first time.

The contents assure us that we will find all we expect of an advanced-level text—calculating aids and their restrictions, errors, nested multiplication, evaluation and tabulation of functions, sketches of simple functions (Chapter 3), iterative methods, differences, the solution of linear simultaneous equations, roots of polynomial equations, linear interpolation and numerical integration.

First impressions of the book are favourable—a large number of diagrams which, on inspection, amply illustrate the text, a splendid collection of worked examples and a healthy diet of exercises for the student at the end of each chapter . . . with answers provided at the back. Closer scrutiny reveals, in general, a clarity of notation and a text eminently suitable for a student newly embarking on an advanced-level course. In fact the authors have bent over backwards to ensure that any techniques prerequisite to the development of the numerical analysis are explained, from their beginning, in the amazing Chapter 3 which, in the course of 30 pages, introduces polynomials and their graphs, curve sketching, trigonometry—yes, from scratch—exponential, logarithmic and hyperbolic functions, limits, and maxima and minima. It is rather like being asked a question on family history and starting the answer with Adam and Eve. Surely even a talented student would struggle with the concentration of those ideas.

In a text with so many strengths it comes as a surprise to discover the occasional looseness. In Section 2.4.3, for example, I do not like '. . . neglecting terms involving powers of a', and I think that words to the effect 'provided the function is well-behaved' should be added. I should like to see a more rigorous analysis of rates of convergence of the successive approximations to the roots of an equation using Newton's method—the intelligent student may feel cheated here—and it is a pity that the analysis on page 80 is wrong, or, at least, incorrectly expressed. I find it mildly irritating that graphs, with axes labelled x and y, should have those same axes referred to in the text as OX and OY. The reader may be amused at the description of the Gauss–Seidel method of solving simultaneous equations, which puts me in mind of that well-known explanation of the game of cricket—'You have two sides, one out in the field and one in. Each man that's in the side that's in goes out and when he's out he comes in. . . ., etc'. There are some misprints but generally the publishers have done a good job.

In conclusion, this is a book the students will understand. A copy will be on the shelves of *my* school library and I would be happy to distribute it as a textbook (when my Board introduces numerical analysis!) if my department could afford the £5·95 that this modest-sized paperback costs.

Monmouth School

M. V. BRADLEY

Mathematical Circus. By MARTIN GARDNER. Penguin Books Ltd., 1981. Pp. xiii, 272. £1·95.

This book by Martin Gardner consists of material originally published in *Scientific American* and needs no special review. It is splendid value at £1·95.

University of Durham

HUGH NEILL

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