

SKOLIAD No. 81

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Please send your solutions to the problems in this edition by **1 March, 2005**. A copy of **MATHEMATICAL MAYHEM Vol. 7** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk (*) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

We are gradually shortening the deadline for submitting solutions to Skoliad problems. Note in particular that the deadline for the problems in this issue is the same date as for the problems in the previous issue.

Our items this issue come from the 2004 British Columbia Colleges High School Mathematics Competition. My thanks go to Clint Lee of Okanagan University College in Vernon, British Columbia, for forwarding the material to me.

2004 BC Colleges High School Mathematics Contest Senior Final Round – Part B

1. Find the number of different 7-digit numbers that can be made by rearranging the digits in the number 3053345.

2. The number 2004 has only 12 integer factors, including 1 and 2004.

(a) How many distinct factors does 2004^4 have?

(b) If the product of the factors in part (a) is written as 2004^N , find the value of N ?

3. You are given two parallel panes of glass. Each pane will transmit 70%, reflect 20%, and absorb 10% of the light that falls on it. For example, for the portion of a beam of light incident on the pane on the left that follows the path in Figure 1, the fraction transmitted is $0.7 \times 0.7 = 0.49$, but for the portion of the beam following the path shown in Figure 2, the fraction transmitted is $0.7 \times 0.2 \times 0.2 \times 0.7 = 0.0196$. If a light source is placed on one side of the two panes, find the total fraction of light that passes through to the other side.

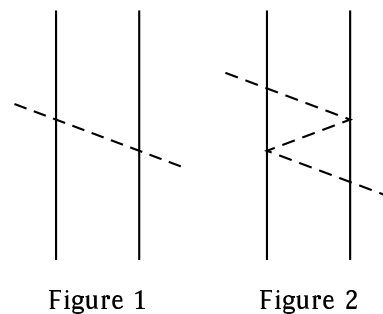


Figure 1

Figure 2

4. Let f be a function whose domain is all real numbers. If

$$f(x) + 2f\left(\frac{x+2001}{x-1}\right) = 4013 - x$$

for all x not equal to 1, find the value of $f(2003)$.

5. An ant is crawling at a rate of 48 centimetres per minute along a strip of rubber which can be infinitely and uniformly stretched. The strip is initially one metre long and one centimetre wide and is stretched an additional one metre at the end of each minute. Assume that when the strip is stretched, the ratio of the distances from each end of the strip remains the same before and after the stretch. If the ant starts at one end of the strip of rubber, find the number of minutes until it reaches the other end.

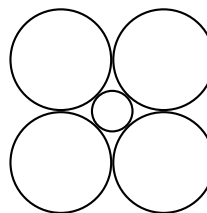
2004 BC Colleges High School Mathematics Contest Junior Final Round – Part B

1. The numbers greater than 1 are arranged in an array, in which the columns are numbered 1 to 5 from left to right, as shown:

(1)	(2)	(3)	(4)	(5)
	2	3	4	5
9	8	7	6	
	10	11	12	13
17	16	15	14	
\vdots	\vdots	\vdots	\vdots	\vdots

- (a) In which column will 2004 fall?
- (b) In which column will 1999 fall?
- (c) In which column(s) could $n^2 + 1$ fall, where n is a positive integer?
2. How many sets of two or more consecutive positive integers have a sum of 105?

3. The centres of four circles of radius 12 form a square. Each circle is tangent to the two circles whose centres are the vertices of the square that are adjacent to the centre of the circle. A smaller circle, with centre at the intersection of the diagonals of the square, is tangent to each of the four larger circles. Find the radius of the smaller circle.



4. The Fibonacci Sequence begins: 1, 1, 2, 3, 5, 8, 13, 21, ... (Each number beyond the second number is the sum of the previous two numbers.) The notation f_n means the n^{th} number; for example, $f_4 = 3$ and $f_7 = 13$.

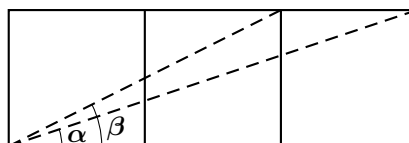
- (a) Which of the following terms in the Fibonacci Sequence are odd? Explain your conclusions.

$$f_{38}, f_{51}, f_{150}, f_{200}, f_{300}$$

- (b) Which of the following terms in the Fibonacci Sequence are divisible by 3? Explain your conclusions.

$$f_{48}, f_{75}, f_{196}, f_{379}, f_{1000}$$

5. The diagram shows three squares. Find the measure of the angle $\alpha + \beta$.



Next we give the solutions to the 2003 Manitoba Mathematics Competition (Concours mathématiques du Manitoba 2003) [2003 : 481–483].

Manitoba Mathematical Contest 2003

Concours Mathématiques du Manitoba 2003

(Senior 4 / 12ième année)

1. (a) Solve the equation $\frac{1}{x} + \frac{1}{x+2} = \frac{1}{x^2+2x}$.

(b) If a and b are non-zero real numbers such that $9a^2 - 12ab + 4b^2 = 0$, find the numerical value of a/b .

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

- (a) Multiplying both sides of the equation by $x^2 + 2x$ yields:

$$\begin{aligned} x + 2 + x &= 1, \\ x &= -\frac{1}{2}. \end{aligned}$$

(b) Dividing both sides by b^2 and factoring yields $\left(3\left(\frac{a}{b}\right) - 2\right)^2 = 0$. Thus, $3\left(\frac{a}{b}\right) - 2 = 0$, which implies that $a/b = 2/3$.

2. (a) Today, Joe's son is $\frac{1}{3}$ of Joe's age. Five years ago he was $\frac{1}{4}$ of Joe's age at that time. How old is Joe's son?

(b) a and b are non-zero real numbers. If the equation $ax^2 + bx + 8 = 0$ has exactly one solution, find the numerical value of b^2/a .

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

(a) Let x represent Joe's son's age now. Then Joe's age now is $3x$. Five years ago, Joe's son's age was $x - 5$ and Joe's age was $3x - 5$. Thus, $4(x - 5) = 3x - 5$, which yields $x = 15$. Hence, Joe's son is 15 years old today.

(b) If a quadratic equation has only one solution, then its discriminant must be zero; that is, $b^2 - 32a = 0$. This means that $b^2/a = 32$.

3. (a) A rectangle is twice as long as it is wide and has a diagonal of length 5. What is its area?

(b) If the perimeter of an isosceles right-angled triangle is 8, what is its area?

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

(a) Let the width and the length of the rectangle be x and $2x$, respectively. Then $\sqrt{x^2 + (2x)^2} = 5$, implying that $x^2 = 5$. Hence, the area of the rectangle is $x(2x) = 2x^2 = 10$.

(b) Let the two equal sides of the triangle have length x . Then the perimeter is equal to $2x + \sqrt{2}x = 8$. This means that $x = \frac{8}{(2 + \sqrt{2})}$; that is, $x = 8 - 4\sqrt{2}$. Thus, the area is equal to $\frac{x^2}{2} = 48 - 32\sqrt{2}$.

4. (a) Find the length of the diameter of a circle whose area is tripled when the length of its radius is increased by 2.

(b) If a and b are real numbers such that $3(2^a) + 2^b = 7\sqrt{2}$ and $5(2^a) - 2^b = 9\sqrt{2}$, find a and b .

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

(a) Let the required length be x . Then $3\left(\frac{x}{2}\right)^2\pi = \left(\frac{x}{2} + 2\right)^2\pi$. Simplifying, we get $x^2 - 4x - 8 = 0$. When we solve this using the quadratic formula, we find that the only positive root is $2 + 2\sqrt{3}$, which is the solution to the problem.

(b) Adding the two equations, we have $8(2^a) = 16\sqrt{2}$, or $2^a = 2\sqrt{2}$. Thus, $a = \frac{3}{2}$. Substituting into the first equation, we get $2^b = \sqrt{2}$, or $b = \frac{1}{2}$.

5. (a) If $\sec \theta + 9 \cos \theta = 6$, what is the numerical value of $\sec \theta$?

(b) The point A lies on the line whose equation is $y = x$. The point B lies on the line whose equation is $y = -x$. The line segment \overline{AB} has length 2. Prove that the mid-point of the segment \overline{AB} lies on a circle of radius 1 with centre at the origin.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

$$\begin{aligned}
 \text{(a)} \quad \sec \theta + 9 \cos \theta &= 6, \\
 \sec \theta + \frac{9}{\sec \theta} &= 6, \\
 \sec^2 \theta - 6 \sec \theta + 9 &= 0, \\
 (\sec \theta - 3)^2 &= 0, \\
 \sec \theta &= 3.
 \end{aligned}$$

(b) Denote the origin by O , and let M be the mid-point of AB . Since $AO \perp BO$, and since, in a right triangle, the length of the median to the hypotenuse is half the length of the hypotenuse, we get $OM = \frac{1}{2}AB = 1$. The result follows.

6. (a) In this problem, O is the origin, A is the point $(3, 1)$ and P is a point in the first quadrant on the graph of $3x - 4y = 0$. If $\angle APO = 45^\circ$, find the area of triangle AOP .

(b) If r , s , and t are real numbers such that $r - 2s + 3t \geq 2$ and $2r + s - 3t \geq 1$, prove that $7r - 4s + 3t \geq 8$.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

(a) The distance from A to OP is $\frac{3(3) - 4(1)}{\sqrt{3^2 + 4^2}} = 1$. We also observe that $OA = \sqrt{3^2 + 1^2} = \sqrt{10}$. Therefore, $\sin AOP = 1/\sqrt{10}$. It follows that $\cos AOP = 3/\sqrt{10}$. Using the Sine Law, we determine that $OP = \frac{OA \sin OAP}{\sin OPA} = 2\sqrt{5} \sin OAP$. But

$$\begin{aligned}
 \sin OAP &= \sin(AOP + OPA) = \sin(AOP + 45^\circ) \\
 &= \frac{\sin AOP + \cos AOP}{\sqrt{2}} = \frac{2}{\sqrt{5}}.
 \end{aligned}$$

Hence, $OP = 4$, and the area of AOP is equal to $\frac{1}{2}OA \cdot OP \sin AOP = 2$.

(b) $7r - 4s + 3t = 3(r - 2s + 3t) + 2(2r + s - 3t) \geq 3(2) + 2(1) = 8$.

7. A right-angled triangle has area 5. The altitude perpendicular to the hypotenuse has length 2. Find the lengths of the three sides of the triangle.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Let the three sides have lengths a , b , and c , where $a \geq b$, and c is the length of the hypotenuse. Using the area formula, we have $\frac{1}{2} \times 2 \times c = 5$; that is, $c = 5$. Using the Pythagorean Theorem, we have $a^2 + b^2 = 25$, and by the area formula, $ab = 10$. Thus,

$$\begin{aligned}
 (a + b)^2 &= a^2 + b^2 + 2ab = 45 & \implies & a + b = 3\sqrt{5}; \\
 (a - b)^2 &= a^2 + b^2 - 2ab = 5 & \implies & a - b = \sqrt{5}.
 \end{aligned}$$

Solving the system, we get $a = 2\sqrt{5}$ and $b = \sqrt{5}$. Therefore, the lengths of the sides are $\sqrt{5}$, $2\sqrt{5}$, and 5.

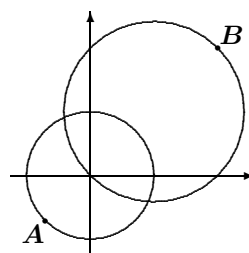
8. A and B are points on the graph of the equation

$$(x^2 + y^2 - 1)\{(x - 1)^2 + (y - 1)^2 - 2\} = 0.$$

What is the largest possible value for the length of the line segment \overline{AB} ? Prove that your answer is correct.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

The graph is the union of two circles, namely $x^2 + y^2 = 1$ and $(x - 1)^2 + (y - 1)^2 = 2$. Letting $A = (-\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$ and $B = (2, 2)$, we note that A is on the first circle, that B is on the second circle, and that $AB = 2\sqrt{2} + 1$. We will show that this is the greatest possible length for a line segment with end-points on the graph.



Suppose first that the two points are on the same circle. Then their distance cannot exceed the diameter, which is either 2 or $2\sqrt{2}$, both of which are less than $2\sqrt{2} + 1$. Now let P be a point on the circle $x^2 + y^2 = 1$, and let Q be a point on the circle $(x - 1)^2 + (y - 1)^2 = 2$. Let C_1 and C_2 be the centres of these circles, respectively. Then $C_1 = (0, 0)$ and $C_2 = (1, 1)$. By the Triangle Inequality, we have $PQ \leq PC_1 + C_1C_2 + C_2Q = 2\sqrt{2} + 1$, and we are done.

9. In this problem O is the origin; P is a point on the graph of $y = x^2$; the coordinates of P are non-zero integers. Prove that the length of the line segment \overline{OP} cannot be an integer.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Let $P(p, p^2)$ be a point on the graph of $y = x^2$, where $p \in \mathbb{Z}$ and $p \neq 0$. Assume for the purpose of contradiction that the length of line segment \overline{OP} is an integer. By the Pythagorean Theorem, $OP = \sqrt{p^2 + p^4} = |p|\sqrt{p^2 + 1}$. But $\sqrt{p^2 + 1}$ is rational if and only if $p^2 + 1$ is a perfect square, which occurs only when $p = 0$ (since no two positive squares are consecutive integers). This is a contradiction. Hence, $OP \notin \mathbb{Z}$.

10. (a) Solve the equation $x^4 - 6x^2 + 9 = (x + 1)^2$.

(b) Solve the equation $x^4 - 7x^2 = 4x - 5$.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

(a) From the given equation, we have

$$0 = (x^2 - 3)^2 - (x + 1)^2 = (x^2 + x - 2)(x^2 - x - 4).$$

The first quadratic expression has roots 1 and -2 , and the second one has roots $(1 + \sqrt{17})/2$ and $(1 - \sqrt{17})/2$. These are the four solutions to the original equation.

(b) From the given equation, we have

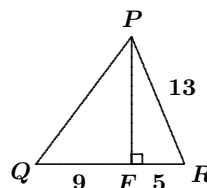
$$0 = x^4 - 7x^2 - 4x + 5 = (x^2 - 3)^2 - (x + 2)^2 = (x^2 + x - 1)(x^2 - x - 5).$$

The first quadratic expression has roots $\frac{-1 + \sqrt{5}}{2}$ and $\frac{-1 - \sqrt{5}}{2}$, and the second has roots $\frac{1 + \sqrt{21}}{2}$ and $\frac{1 - \sqrt{21}}{2}$. These are the four solutions to the original equation.

Next we give the solutions to the 2002 Canadian Open Mathematics Challenge [2004 : 1–5].

Canadian Open Mathematics Challenge, 2002 Part A

1. In triangle PQR , F is the point on QR such that PF is perpendicular to QR . If $PR = 13$, $RF = 5$, and $FQ = 9$, what is the perimeter of $\triangle PQR$?



Solution by Alper Cay, student, Uzman Private School, Kayseri, Turkey.

Denote PF by x and PQ by y . Then

$$x^2 + 5^2 = 13^2 \quad \text{and} \quad x^2 + 9^2 = y^2.$$

The first equation implies that $x = 12$, and the second equation implies that $y = 15$. Therefore, the perimeter of $\triangle PQR$ is $14 + 13 + 15 = 42$.

Also solved by Karthik Natarajan, grade 6 student, Edgewater Park Private School, Thunder Bay, ON; and Alex Wice, grade 10 student, Leaside H.S., Toronto, ON.

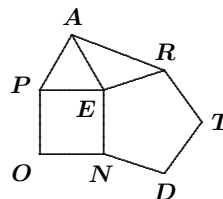
2. If $x + y = 4$ and $xy = -12$, what is the value of $x^2 + 5xy + y^2$?

Solution by Karthik Natarajan, grade 6 student, Edgewater Park Private School, Thunder Bay, ON.

The expression $x^2 + 5xy + y^2$ can be written as $(x + y)^2 + 3xy$. Knowing that $x + y = 4$ and $xy = -12$, we can now solve the problem. Since $(x + y)^2 = 16$ and $3xy = -36$, if we add the two, we get our final answer of -20 .

Also solved by Alper Cay, student, Uzman Private School, Kayseri, Turkey; and Alex Wice, grade 10 student, Leaside H.S., Toronto, ON.

3. A regular pentagon is a five-sided figure which has all of its angles equal and all of its side lengths equal. In the diagram, *TREND* is a regular pentagon, *PEA* is an equilateral triangle, and *OPEN* is a square. Determine the size of $\angle EAR$.



Solution by Alper Cay, student, Uzman Private School, Kayseri, Turkey.

We can calculate the measure of the interior angle of a regular polygon by computing $\frac{(n-2)180^\circ}{n}$, where n denotes the number of sides of the polygon. Thus, for a regular pentagon, each interior angle has measure 108° . Let $\theta = \angle AER$ and $\alpha = \angle EAR$. Then $EA = ER$, since all the figures are regular. Hence, $\angle ERA = \alpha$. Therefore, $\theta + 90^\circ + 60^\circ + 108^\circ = 360^\circ$, implying that $\theta = 102^\circ$. In $\triangle AER$ we have $\theta + 2\alpha = 180^\circ$, which gives us $\angle EAR = \alpha = 39^\circ$.

Also solved by Karthik Natarajan, grade 6 student, Edgewater Park Private School, Thunder Bay, ON; and Alex Wice, grade 10 student, Leaside H.S., Toronto, ON.

4. In a sequence of numbers, the sum of the first n terms is equal to $5n^2 + 6n$. What is the sum of the 3rd, 4th, and 5th terms in the original sequence?

Solution by Karthik Natarajan, grade 6 student, Edgewater Park Private School, Thunder Bay, ON.

Since we would like to find out the sum of the 3rd, 4th, and 5th terms in the sequence, it is easiest to first substitute $n = 5$ into $5n^2 + 6n$, which gives the sum of the first 5 terms as $5 \times 5^2 + 6 \times 5 = 155$. To find the sum of the 3rd, 4th, and 5th terms alone, we can eliminate the sum of the 1st and 2nd terms. To find this sum, we substitute $n = 2$ into $5n^2 + 6n$, obtaining 32. Subtracting 32 from 155, we get the answer 123.

Also solved by Alper Cay, student, Uzman Private School, Kayseri, Turkey; and Alex Wice, grade 10 student, Leaside H.S., Toronto, ON.

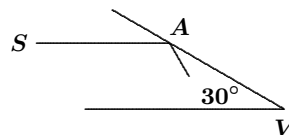
5. If m and n are non-negative integers with $m < n$, we define $m \nabla n$ to be the sum of the integers from m to n , including m and n . For example, $5 \nabla 8 = 5 + 6 + 7 + 8 = 26$. For every positive integer a , the numerical value of $\frac{(2a-1) \nabla (2a+1)}{(a-1) \nabla (a+1)}$ is the same. Determine this value.

Solution by Alper Cay, student, Uzman Private School, Kayseri, Turkey.

$$\frac{(2a-1) \nabla (2a+1)}{(a-1) \nabla (a+1)} = \frac{(2a-1) + 2a + (2a+1)}{(a-1) + a + (a+1)} = \frac{6a}{3a} = 2.$$

Also solved by Karthik Natarajan, grade 6 student, Edgewater Park Private School, Thunder Bay, ON; and Alex Wice, grade 10 student, Leaside H.S., Toronto, ON.

6. Two mirrors meet at an angle of 30° at the point V . A beam of light, from a source S , travels parallel to one mirror and strikes the other mirror at point A , as shown. After a number of reflections, the beam comes back to S . If SA and AV are both 1 metre in length, determine the total distance travelled by the beam.



Solution by Alex Wice, grade 10 student, Leaside H.S., Toronto, ON.

Suppose the beam from S reflected by the first mirror at A meets the second mirror at X and the first mirror again at Y . Then $XV \parallel SA$. Now $\angle XAV = 30^\circ$. Thus, $\angle AXV = 120^\circ$ (by looking at $\triangle AXV$). The supplement of $\angle AXV$ (formed by AX and the second mirror) is 60° , implying that $\angle YXV = 60^\circ$. Now $\angle XYV$ is a right angle, which means that the beam then reverses its path. Hence, the total distance travelled by the beam is $2(SA + AX + XY)$.

Next, we observe that AY is $1/2$ metre (because Y bisects AV , which is 1 metre), and that $\triangle AXV$ is a 30° – 60° – 90° triangle. By known ratios of the sides of such a triangle, $AX = 1/\sqrt{3}$ and $XY = 1/(2\sqrt{3})$. Thus, the total distance travelled by the beam is $2 + \sqrt{3}$.

7. N is a five-digit positive integer. A six-digit integer P is constructed by placing a 1 at the right-hand end of N . A second six-digit integer Q is constructed by placing a 1 at the left-hand end of N . If P is three times Q , determine the value of N .

Solution by Karthik Natarajan, grade 6 student, Edgewater Park Private School, Thunder Bay, ON.

Let N be any 5-digit number. Then we get $10N + 1 = P$ and $N + 100000 = Q$. Since $P = 3Q$, we have $10N + 1 = 3N + 300000$ which becomes $7N = 299999$. Therefore, $N = 42857$.

Also solved by Alex Wice, grade 10 student, Leaside H.S., Toronto, ON.

8. Suppose that M is an integer with the property that if x is randomly chosen from the set $\{1, 2, 3, \dots, 999, 1000\}$, the probability that x is a divisor of M is $\frac{1}{100}$. If $M \leq 1000$, determine the maximum possible value of M .

Solution by Alex Wice, grade 10 student, Leaside H.S., Toronto, ON.

Clearly, M has 10 positive divisors. It is well known that when we write out the prime factorization of a number, if we add 1 to the exponent on each prime factor and multiply the resulting numbers, we obtain the number of divisors. Thus, since the only factorizations of 10 are $10 \cdot 1$ and $5 \cdot 2$, we must have either $M = p^9$ or $M = p^4q$, where p and q are prime numbers. Now we see that $2^9 = 512$ is a candidate for M , while 3^9 is too large. Hence, the largest possible value for M is 512 in the case that M has the form p^9 .

Now let us consider the case that M has the form p^4q . For $p \geq 7$, the numbers we obtain are too large. For $p = 5$, the smallest possible number we can obtain occurs when $q = 2$, which means that $M = 1250$, which is also too large. For $p = 3$, we see that the largest possible value for M occurs when $q = 11$, namely $M = 3^4 \cdot 11 = 891$. For $p = 2$, we similarly see that the largest possible for M occurs when $q = 61$, namely $M = 2^4 \cdot 61 = 976$.

Therefore, across all cases, the largest possible value for M is 976.

Part B

1. Square $ABCD$ has vertices $A(0, 0)$, $B(0, 8)$, $C(8, 8)$, and $D(8, 0)$. The points $P(0, 5)$ and $Q(0, 3)$ are on side AB , and the point $F(8, 1)$ is on side CD .

- What is the equation of the line through Q parallel to the line through P and F ?
- If the line from part (a) intersects AD at the point G , what is the equation of the line through F and G ?
- The centre of the square is the point $H(4, 4)$. Determine the equation of the line through H perpendicular to FG .
- A circle is drawn with centre H that is tangent to the four sides of the square. Does this circle intersect the line through F and G ? Justify your answer. (A sketch is not sufficient justification.)

Solution by Alex Wice, grade 10 student, Leaside H.S., Toronto, ON.

(a) The line PF has slope $\frac{5-1}{0-8} = -\frac{1}{2}$. If the line $y = -\frac{1}{2}x + b$ passes through $(0, 3)$, we must have $b = 3$. Therefore, the equation is $y = -\frac{1}{2}x + 3$.

(b) The line through A and D clearly has equation $y = 0$. Comparing this with the equation from part (a), we get $-\frac{1}{2}x + 3 = 0$, which yields $x = 6$. Thus, the point G is $(6, 0)$. Now the slope of FG is $\frac{1-0}{8-6} = \frac{1}{2}$. If the line $y = \frac{1}{2}x + b$ passes through $(6, 0)$, we see that $b = -3$. Therefore, the equation we seek is $y = \frac{1}{2}x - 3$.

(c) We saw in (b) above that the slope of FG is $\frac{1}{2}$. The slope of a line perpendicular to FG is thus -2 . Now a line perpendicular to FG has an equation of the form $y = -2x + b$. If it also passes through the point $H(4, 4)$, we see that $b = 12$. Hence, the equation is $y = -2x + 12$.

(d) The shortest distance from H to the line FG is found along the line whose equation was determined in part (c) above (that is, the perpendicular distance). Clearly, the radius of the circle is 4. Now the equation of the line through F and G is $y = \frac{1}{2}x - 3$, and the equation from part (c) is $y = -2x + 12$. The point of intersection of these lines is $(6, 0)$, which is G . By the distance formula, the distance from F to G is $\sqrt{(6-8)^2 + (0-1)^2} = \sqrt{5} < 4$. Thus, the circle does not intersect FG .

2. (a) Let A and B be digits (that is, A and B are integers between 0 and 9 inclusive). If the product of the three-digit integers $2A5$ and $13B$ is divisible by 36, determine with justification the four possible ordered pairs (A, B) .

(b) An integer n is said to be a multiple of 7 if $n = 7k$ for some integer k .

(i) If a and b are integers and $10a + b = 7m$ for some integer m , prove that $a - 2b$ is a multiple of 7.

(ii) If c and d are integers and $5c + 4d$ is a multiple of 7, prove that $4c - d$ is also a multiple of 7.

Solution by Alex Wice, grade 10 student, Leaside H.S., Toronto, ON.

(a) First note that $36 = 9 \times 4$. The number $2A5$ has no factor 2, since its last digit is 5. Therefore, the number $13B$ must be divisible by 4. Realizing that $13B = 130 + B$, we see that $B = 2$ or $B = 6$. If B is 2, then $2A5$ need only be divisible by 3 (since 132 is divisible by 3, but not 9). This occurs when $A \in \{2, 5, 8\}$. If $B = 6$, then $2A5$ must be divisible by 9, which implies that $A = 2$. Thus, $(A, B) \in \{(2, 2), (5, 2), (8, 2), (2, 6)\}$.

(b) The following calculations prove (i) and (ii), respectively:

$$\begin{aligned} a - 2b &\equiv a + 5b \equiv 15a + 5b \equiv 5(10a + b) \equiv 0 \pmod{7}; \\ 4c - d &\equiv 4c + 6d \equiv 25c + 20d \equiv 5(5c + 4d) \equiv 0 \pmod{7}. \end{aligned}$$

3. There are some marbles in a bowl. Alphonse, Beryl, and Colleen each take turns removing one or two marbles from the bowl, with Alphonse going first, then Beryl, then Colleen, then Alphonse again, and so on. The player who takes the last marble from the bowl is the loser, and the other two players are the winners.

(a) If the game starts with 5 marbles in the bowl, can Beryl and Colleen work together and force Alphonse to lose?

(b) The game is played again, this time starting with N marbles in the bowl. For what values of N can Beryl and Colleen work together and force Alphonse to lose?

Solution by Alex Wice, grade 10 student, Leaside H.S., Toronto, ON.

Let A , B , C represent the players Alphonse, Beryl, Colleen.

(a) Together B and C can take 2, 3, or 4 marbles. No matter whether A takes 1 or 2 on his turn, B and C working together can take enough to leave only one marble for A (because they can take 3 or 2, respectively). Thus, they can force A to lose.

(b) Define a number N to be *losing* if A can be forced to lose when N marbles remain at the beginning of A 's turn.

Clearly, 1 is losing. But 2 and 3 are not losing, because B and C together must take at least 2 marbles. Also, 4 is not losing because, if A takes 2, B and C together must take at least 2, which means that one of them takes the last one. In part (a) we saw that 5 is losing.

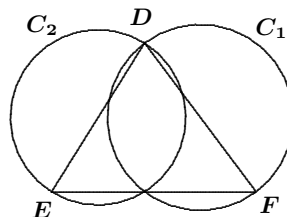
We claim that if X is losing, then $X + 4$ and $X + 5$ are also losing. If A starts with $X + 4$ marbles, then B and C may take either 2 or 3 marbles, depending on whether A takes 2 or 1. If A starts with $X + 5$ marbles, then B and C may take either 3 or 4, depending on whether A takes 2 or 1. Thus, A is forced to start with X marbles on the next round.

Using this idea, we see that 6 is losing (use $X = 1$), as are 9, 10, and 11 (use $X = 5$ and $X = 6$). Furthermore, all numbers greater than or equal to 13 are losing.

We now finish by checking the cases $N = 7, 8$, and 12. For $N = 12$, if A takes 1, then B and C can take 2 to get 9 (losing), and if A takes 2, then B and C can take 4 to force 6 (losing). Therefore, 12 is losing. For $N = 8$, A should take 2, and for $N = 7$, A should take 1 to avoid losing positions.

Therefore, B and C can work together and force A to lose for any N except $N = 2, 3, 4, 7$, and 8.

4. Triangle DEF is acute. Circle C_1 is drawn with DF as its diameter, and circle C_2 is drawn with DE as its diameter. Points Y and Z are on DF and DE , respectively, so that EY and FZ are altitudes of $\triangle DEF$. EY intersects C_1 at P , and FZ intersects C_2 at Q . EY extended intersects C_1 at R , and FZ extended intersects C_2 at S . Prove that P, Q, R , and S are concyclic points.



Solution by Alex Wice, grade 10 student, Leaside H.S., Toronto, ON, modified by the editor.

Let X be the point on EF such that $DX \perp EF$. Then X lies on the circle C_1 , since DF is a diameter of C_1 . Similarly, X lies on C_2 . Thus, not only is DX an altitude of $\triangle DEF$, but X is also the second point of intersection of the circles C_1 and C_2 . Now Y must be an intersection of DF and C_2 , and Z an intersection of DE and C_1 . Let H be the orthocentre of $\triangle DEF$. It is known that the "Power of the Point" Theorem is an "if and only if" result. Thus, if $HP \cdot HR = HQ \cdot HS$, then the points P, Q, R , and S are concyclic. But $HQ \cdot HS = HD \cdot HX$ because D, Q, X, S are all on C_2 and H is the intersection of DX and QS . Similarly, $HP \cdot HR = HD \cdot HX$. Therefore, $HD \cdot HX = HP \cdot HR = HQ \cdot HS$, and the four points are concyclic.

That brings us to the end of another issue. This month's winner of a past Volume of Mayhem is Alex Wice. Congratulations, Alex! Please continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a **Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of *Crux Mathematicorum with Mathematical Mayhem*.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Larry Rice (University of Waterloo), Dan MacKinnon (Ottawa Carleton District School Board), and Ian VanderBurgh (University of Waterloo).

Mayhem Problems

Please send your solutions to the problems in this edition by 1 May 2005. Solutions received after this date will only be considered if there is time before publication of the solutions.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M153. Correction. *Proposed by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

Two similar triangles APB and BQC are erected externally on a triangle ABC . If R is a point such that $PBQR$ is a parallelogram, show that triangles ARC and APB are similar.

M163. *Proposed by the Mayhem Staff.*

Show that it is possible to put positive integer values on the faces of two dice (not necessarily the same on both dice) so that, when the dice are tossed, the outcomes 1, 2, 3, ..., 12 are equally probable.

M164. *Proposed by the Mayhem Staff.*

Consider the following procedure for dividing the three-digit number 375 by 8. Write down the number formed by the first two digits, namely, 37. Multiply this by 2 to get 74. Add to this the units digit of 375 (the original number), obtaining $74 + 5 = 79$. Then divide by 8 to get 9 with a remainder of 7. Add this result (9, remainder 7) to the number 37 (the first two digits of 375) to get your answer: 46, remainder 7. Thus, 375 divided by 8 equals 46 with a remainder of 7.

Does this method always work for three-digit numbers? Why, or why not?

M165. *Proposed by Babis Stergiou, Chalkida, Greece.*

If $a, b > 0$, prove that

(a) $\sqrt{ab} \geq \frac{2}{1/a + 1/b}.$

(b) $a^6 + b^6 + 8a^3 + 8b^3 + 2a^3b^3 + 16 \geq 36ab.$

M166. *Proposed by the Mayhem Staff.*

(a) Simplify

$$(3n)^2 + (4n - 1)^2 - (5n - 1)^2, \quad (3n + 2)^2 + (4n)^2 - (5n + 1)^2.$$

(b) Using (a) or otherwise, prove that all positive integers can be represented in the form $a^2 + b^2 - c^2$ where a, b, c are integers and $0 < a < b < c$.

M167. *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Solve the following inequality:

$$(2 + \cos x)(1 + \sin x - \cos x) \geq \cos x(1 + 2 \sin x - \cos x).$$

M168. *Proposed by Neven Jurič, Zagreb, Croatia.*

How many different 3×3 arrays of non-negative integers is it possible to construct so that each of the three horizontal sums and each of the three vertical sums is equal to 7, the first diagonal sum is equal to 10, and the second diagonal sum is equal to 9? (Two arrays which may be transformed into one another by rotations and/or reflections are not considered to be different.)

Here is an example of such an array:

2	3	2
2	4	1
3	0	4

.....

M153. *Correction. Proposé par Yufei Zhao, étudiant, Don Mills Collegiate Institute, Toronto, ON.*

Extérieurement au triangle ABC , on construit deux triangles semblables APB et BQC . Si R est un point tel que $PBQR$ est un parallélogramme, montrer qu'alors les triangles ARC et APB sont semblables.

M163. *Proposé par Équipe de Mayhem.*

Montrer qu'il est possible de mettre des entiers positifs sur les faces de deux pipés (non nécessairement les mêmes pour les deux) de sorte que tous les résultats 1, 2, 3, ..., 12 soient également probables quand on lance les pipés.

M164. *Proposé par Équipe de Mayhem.*

Pour diviser un nombre de trois chiffres par 8, on utilise une procédure dont voici un exemple, partant du nombre 375. On écrit le nombre formé des deux premiers chiffres, à savoir 37, qu'on multiplie par 2 pour obtenir 74. A quoi on ajoute le dernier chiffre de 375 (le nombre initial), donc $74 + 5 = 79$. On divise ceci par 8 pour obtenir 9, reste 7. On ajoute ce résultat à 37 pour obtenir la réponse, 46 reste 7. En résumé, 375 divisé par 8 égale 46 reste 7.

Cette méthode marche-t-elle toujours pour les nombres de trois chiffres ? Pourquoi, ou pourquoi pas ?

M165. *Proposé par Babis Stergiou, Chalkida, Grèce.*

Si a et b sont positifs, montrer que

$$(a) \sqrt{ab} \geq \frac{2}{1/a + 1/b}.$$

$$(b) a^6 + b^6 + 8a^3 + 8b^3 + 2a^3b^3 + 16 \geq 36ab.$$

M166. *Proposé par Équipe de Mayhem.*

(a) Simplifier

$$(3n)^2 + (4n - 1)^2 - (5n - 1)^2, \quad (3n + 2)^2 + (4n)^2 - (5n + 1)^2.$$

(b) À l'aide de (a) ou autrement, montrer que tous les entiers positifs peuvent être représentés sous la forme $a^2 + b^2 - c^2$, où a , b et c sont des entiers et $0 < a < b < c$.

M167. *Proposé par Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Province de Guang Dong, Chine.*

Résoudre l'inéquation suivante :

$$(2 + \cos x)(1 + \sin x - \cos x) \geq \cos x(1 + 2 \sin x - \cos x).$$

M168. *Proposé par Neven Jurić, Zagreb, Croatie.*

Combien de tableaux 3×3 formés d'entiers non négatifs est-il possible de construire de telle sorte que chacune des trois sommes horizontales ainsi que chacune des trois sommes verticales soit égale à 7, que la somme de la première diagonale soit égale à 10 et que la somme de la deuxième diagonale soit égale à 9 ? (Il faut noter que les tableaux formés avec les mêmes nombres mais orientés différemment par rotations et/ou réflexions ne sont pas considérés comme différents.)

Voici un exemple d'un tableau admissible :

2	3	2
2	4	1
3	0	4

Mayhem Solutions

M94. *Proposed by J. Walter Lynch, Athens, GA, USA.*

You have twelve balls which are identical in appearance. Eleven of them are in fact identical, and the other one differs slightly in weight from each of these eleven. Using a balance scale, find the odd ball in only three weighings.

Solution by Bruce Crofoot, University College of the Cariboo, Kamloops, BC.

We will not only find the odd ball in three weighings, but we will also determine whether it is heavier or lighter than the other balls.

We begin by using the scale to compare the weight of any 4 of the balls with the weight of any other 4 (the first weighing).

Case 1: The balls balance.

Then these 8 balls are all standard balls. Label the remaining 4 balls as A , B , C , D . Now use the scale to compare the weight of A and B with the weight of C and a standard ball (the second weighing).

(a) If they balance, then the odd ball is D . By comparing its weight with one of the standard balls (the third weighing), we can decide whether it is heavy or light.

(b) If the balls A and B are heavier than the other balls, then either the odd ball is heavy and is one of A and B , or it is light and is C . We now weigh A against B (the third weighing). If they do not balance, then the heavier ball is the odd ball; if they balance, then C is the odd ball.

(c) If the balls A and B are lighter than the other balls, then we may proceed as in (b), except that the words “heavy” and “light” are interchanged and “heavier” becomes “lighter”.

Case 2: The balls do not balance.

Label the balls on the heavier side as A , B , C , D and those on the lighter side as a , b , c , d . Now use the scale to compare the weight of A , B , and a with the weight of C , D , and b (the second weighing).

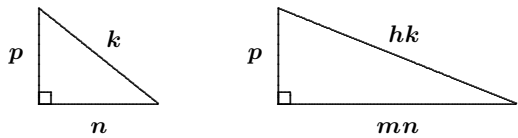
(a) If they balance, then the odd ball is either c or d , and it is light (since it was on the lighter side in the second weighing). We can determine which it is by weighing c against d (the third weighing). The lighter of the two is the odd ball.

(b) If the balls A , B , a are heavier than C , D , b , then either the odd ball is heavy and is one of A and B , or it is light and is b . We now weigh A against B (the third weighing). If they do not balance, then the heavier ball is the odd ball; if they balance, then b is the odd ball;

(c) If the balls C , D , b are heavier than A , B , a , then we may proceed by analogy with (b).

M95. *Proposed by the Mayhem Staff.*

In the figure below, given that h, k, m, n , and p are integers with $h \neq 1$, determine the value of h .



Ed: No solutions have been received. The problem remains open.

M96. *Proposed by the Mayhem Staff.*

Determine the largest possible remainder that is attainable by dividing a three-digit number by the sum of its digits.

Solution by Geneviève Lalonde, Massey, ON.

The largest possible remainder in dividing an integer by an integer d is $d - 1$. Thus, if we do a systematic check starting from the largest sum of digits, we should find our result. From the table below, we see that the largest possible remainder is 24.

Sum of Digits	Possibilities	Remainder
27	999	0
26	998	10
	989	1
	899	15
25	997	22
	979	4
	799	24
	988	13
	898	23
	889	14

M97. *Proposed by the Mayhem Staff.*

A standard 8×8 checkerboard consists of 64 unit squares. A T -shaped tile consists of five unit squares, as shown below. The tile must be placed on the checkerboard to cover exactly five unit squares on the board.

- (a) What is the maximum number of non-overlapping tiles that can be placed on the board in this manner?
- (b) Assuming that overlapping is permitted, what is the minimum number of tiles required to cover the board?



Ed: No solutions have been received. The problem remains open.

M98. *Proposed by the Mayhem Staff.*

We say that N is an *automorphic* number if the value of N^2 ends with the string of digits forming N . For example, 6 is automorphic since 6^2 ends in 6.

- (a) Find all two digit automorphic numbers in base 10.
 (b) Find all two digit automorphic numbers in base 6.

Solution by Robert Bilinski, Outremont, QC, adapted by the editors.

(a) It is a well-known fact that numbers ending in 0, 1, 5, or 6 have the same last digit in their second power.

Case 1: Let $N = 10a$. Then $N^2 = 100a^2$ and N cannot be automorphic.

Case 2: Let $N = 10a + 1$. Then $N^2 = 100a^2 + 20a + 1$. The tens digit of N is given by a and the tens digit of a^2 is given by $2a$. The only digit which is equal to its double is 0, but that would mean that N is a one-digit number.

Case 3: Let $N = 10a + 5$. Then $N^2 = 100a^2 + 100a + 25 = 100(a^2 + a) + 25$. The tens digit of N is a and that of N^2 is 2. Clearly, $N = 25$ is the only two-digit solution.

Case 4: Let $N = 10a + 6$. Then

$$N^2 = 100a^2 + 120a + 36 = 100(a^2 + a) + 10(2a + 3) + 6.$$

We need $2a + 3 \equiv a \pmod{10}$, which gives us $a = 7$. Hence, a second automorphic number is $N = 76$.

Therefore, the only automorphic numbers in base 10 are 25 and 76.

(b) We simply consider each of the 6 possible remainders when N is divided by 6.

Case 1: Let $N = 6b$. Then $N^2 = 36b^2$ always ends in 00 in base 6. These numbers keep their last digit but cannot be automorphic, since the second digit is never b .

Case 2: Let $N = 6b + 1$. Then $N^2 = 36b^2 + 12b + 1$. These numbers keep their last digit, but the second digits are not equal unless $2b = b$; that is, $b = 0$. This would make N a one-digit number.

Case 3: Let $N = 6b + 2$. Then $N^2 = 36b^2 + 24b + 4$. These numbers don't even keep their last digit.

Case 4: Let $N = 6b + 3$. Then $N^2 = 36b^2 + 36b + 9 = 36(b^2 + b) + 6 + 3$ always ends in 13 in base 6. Thus, we get $b = 1$, and we find that an automorphic number is $N = 13$ (base 6).

Case 5: Let $N = 6b + 4$. Then

$$N^2 = 36b^2 + 48b + 16 = 36(b^2 + b) + 6(2b + 2) + 4.$$

These numbers keep their last digit. We need $2b + 2 \equiv b \pmod{6}$, which gives $b = 4$. Hence, a second automorphic number is $N = 44$ (base 6).

Case 6: Let $N = 6b + 5$. Then the last digit of N^2 is 1. The last digit doesn't even repeat.

Therefore, the only numbers written with two digits in base 6 which are automorphic are $13_{(6)} = 9_{(10)}$ and $44_{(6)} = 28_{(10)}$.

M99. *Proposed by the Mayhem Staff.*

Prove that for all positive integers n ,

$$1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + n \cdot \binom{n}{n} = n \cdot 2^{n-1}.$$

I. Similar solutions by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Siwen Sun, grade 12 student, Collège Saint-Louis, Lachine, QC.

Denote by S the left side of the identity given in the statement. Taking into account that $\binom{n}{k} = \binom{n}{n-k}$, for $k = 0, 1, \dots, n$, we have

$$\begin{aligned} S &= 0 \binom{n}{0} + 1 \binom{n}{1} + 2 \binom{n}{2} + \cdots + (n-1) \binom{n}{n-1} + n \binom{n}{n} \\ &= n \binom{n}{0} + (n-1) \binom{n}{1} + (n-2) \binom{n}{2} + \cdots + 1 \binom{n}{n-1} + 0 \binom{n}{n}. \end{aligned}$$

Adding up the preceding expressions, we get $2S = n \sum_{k=0}^n \binom{n}{k} = n2^n$, which gives the desired result.

II. Solution by Robert Bilinski, Outremont, QC.

Since $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$, we have

$$\frac{d}{da}(a+b)^n = n(a+b)^{n-1} = \sum_{k=0}^n \binom{n}{k} k a^{k-1} b^{n-k}.$$

Setting $a = b = 1$ in the last equality, we get the desired result.

M100. *Proposed by the Mayhem Staff.*

Mr. and Mrs. Smith are at a party with three other married couples. Since some of the guests are not acquainted with one another, various handshakes take place. No one shakes hands with his or her spouse, and of course, no one shakes hands with himself or herself. After all of the introductions have been made, Mrs. Smith asks the other seven people how many hands they shook. Surprisingly, they all give different answers. How many hands did Mr. Smith shake?

Solution by Laura Steil, student, Samford University, Birmingham, Alabama, USA.

Since there are eight people at the party, and no person shakes his/her own hand or his/her spouse's hand, the maximum number of handshakes per person was 6. Also, because Mrs. Smith had seven different answers for the number of hands each person shook, the answers had to range from 6 down to 0.

Let A be the person who shook 6 hands, B the person who shook 5 hands, C the person who shook 4 hands, D the person who shook 3 hands, E the person who shook 2 hands, F the person who shook 1 hand, and G the person who shook 0 hands. Since A could not shake hands with G (because G shook 0 hands), then A had to shake hands with B , C , D , E , F , and Mrs. Smith. Also, since G was the only person whose hand A did not shake, then A and G must be a couple.

Since B shook hands with A , we see that B shook 4 other hands. Since B could not have shaken hands with F (because F only shook one hand, and that hand belonged to A), then B must have shaken hands with C , D , E , and Mrs. Smith. Also, B and F must be a couple, because F and G are the only possible partners for B , and G is already paired with A .

Since C shook hands with A and B , we see that C shook 2 other hands. Since C could not have shaken hands with E (because E shook only the hands of A and B), then C shook hands with D and Mrs. Smith. Also, C and E must be a couple, because the only possible partners for C are E , F , and G , and F and G are already paired with B and A , respectively.

Since all the others are paired up, this leaves D and Mrs. Smith as the last couple. Hence, D is Mr. Smith, who therefore shook 3 hands.

One incorrect solution was received.

M101. *Proposed by the Mayhem Staff.*

Find the smallest value of k such that $k!$ ends with 100 zeroes. [Note: $k! = k(k-1)(k-2)\cdots(3)(2)(1)$.]

Solution by Robert Bilinski, Outremont, QC.

Since fives are rarer than twos, we will have 100 zeros at the end of $k!$ when k contains the 100th factor of five among the integers from 1 to k . The first occurrence of the factor 5 comes when $k = 5$; the second occurs when $k = 10$; the third when $k = 15$; the fourth when $k = 20$; whereas $k = 25$ will generate the fifth and sixth. This means that $k = 50$ generates the 11th and 12th factors. And $k = 100$ generates the 23rd and 24th factors. On the other hand, $k = 125$ generates not only the 29th and 30th such factors, but also the 31st factor. Proceeding in this manner, we find that $k = 375$ generates the 93rd factor. Then, $k = 400$ has the 99th factor and $k = 405$ has the 100th factor.

Hence, 405! will be the smallest factorial ending in 100 zeroes.

One incorrect solution was received.

M102. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.

Suppose that $ABCD$ is a parallelogram and that G_A , G_B , G_C , and G_D are the centroids of $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, and $\triangle ABC$, respectively.

Prove that:

1. $G_A G_B G_C G_D$ is a parallelogram;
2. $[G_A G_B G_C G_D] = \frac{1}{9}[ABCD]$, where $[ABCD]$ is the area of $ABCD$.

I. Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

1. Let O be the intersection of diagonals BD and AC . Since CO is the median corresponding to side BD in $\triangle BCD$, we see that the centroid G_A lies on CO and $OG_A = \frac{1}{3}OC$. Similarly, G_B lies on OD and $OG_B = \frac{1}{3}OD$. Then $\triangle COD$ and $\triangle G_A O G_B$ are similar. Hence, $\angle OG_A G_B = \angle OCD$, and therefore,

$$CD \parallel G_A G_B. \quad (1)$$

Similarly, we obtain

$$AB \parallel G_C G_D, \quad (2)$$

$$BC \parallel G_D G_A, \quad (3)$$

$$DA \parallel G_B G_C. \quad (4)$$

Finally, $G_A G_B G_C G_D$ is a parallelogram, because, from (1) and (2), we have $G_A G_B \parallel G_D G_C$, and from (3) and (4), we have $G_A G_D \parallel G_B G_C$.

2. From part 1 above, $\triangle COD$ and $\triangle G_A O G_B$ are similar with ratio $\frac{1}{3}$. Then, $[\triangle G_A O G_B] = \frac{1}{9}[\triangle COD]$, where $[XYZ]$ denotes the area of triangle XYZ . Similarly, $[\triangle G_B O G_C] = \frac{1}{9}[\triangle DOA]$, $[\triangle G_C O G_D] = \frac{1}{9}[\triangle AOB]$, and $[\triangle G_D O G_A] = \frac{1}{9}[\triangle BOC]$. Hence, $[G_A G_B G_C G_D] = \frac{1}{9}[ABCD]$.

II. Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Let $A(a_1, b_1)$, $B(a_2, b_2)$, $C(a_3, b_3)$, and $D(a_4, b_4)$ be the coordinates of the vertices of the given quadrilateral $ABCD$. It is well known that if $ABCD$ is a parallelogram, then (i) $\overrightarrow{AB} \times \overrightarrow{CD} = \vec{0}$ and $\overrightarrow{AD} \times \overrightarrow{BC} = \vec{0}$, and (ii) $[ABCD] = \|\overrightarrow{AB} \times \overrightarrow{AD}\|$, where \times is the vector cross-product, and $[ABCD]$ is the area of the plane figure $ABCD$.

1. The coordinates of G_A , G_B , G_C , and G_D are

$$G_A = \left(\frac{a_2 + a_3 + a_4}{3}, \frac{b_2 + b_3 + b_4}{3} \right),$$

$$G_B = \left(\frac{a_1 + a_3 + a_4}{3}, \frac{b_1 + b_3 + b_4}{3} \right),$$

$$\begin{aligned} G_C &= \left(\frac{a_1 + a_2 + a_4}{3}, \frac{b_1 + b_2 + b_4}{3} \right), \\ G_D &= \left(\frac{a_1 + a_2 + a_3}{3}, \frac{b_1 + b_2 + b_3}{3} \right). \end{aligned}$$

Taking into account (i), it follows that

$$\begin{aligned} \overrightarrow{G_A G_B} \times \overrightarrow{G_C G_D} &= \frac{1}{9} (\overrightarrow{AB} \times \overrightarrow{CD}) = 0 \\ \text{and } \overrightarrow{G_A G_D} \times \overrightarrow{G_B G_C} &= \frac{1}{9} (\overrightarrow{AD} \times \overrightarrow{BC}) = 0. \end{aligned}$$

That is, $G_A G_B G_C G_D$ is a parallelogram.

2. Taking into account (ii), we get

$$[G_A G_B G_C G_D] = \|\overrightarrow{G_A G_B} \times \overrightarrow{G_A G_D}\| = \frac{1}{9} \|\overrightarrow{AB} \times \overrightarrow{AD}\| = \frac{1}{9} [ABCD].$$

M103. *Proposed by the Mayhem Staff.*

Solve for n :

$$100^{1/n} \times 100^{2/n} \times 100^{3/n} \times \cdots \times 100^{2003/n} = 1000.$$

Solution by Gabriel Krimker, grade 9 student, Buenos Aires, Argentina.

Since

$$100^{1/n} \times 100^{2/n} \times \cdots \times 100^{2003/n} = (10^2)^{\frac{1+2+\cdots+2003}{n}} = 10^{\frac{2003 \cdot 2004}{n}},$$

we need only solve $10^{\frac{2003 \cdot 2004}{n}} = 10^3$; whence, $n = \frac{2003 \cdot 2004}{3} = 1338004$.

Also solved by Robert Bilinski, Outremont, QC; and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

M104. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Suppose that $ABCD$ is a parallelogram and that O_A , O_B , O_C , and O_D are the circumcentres of $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, and $\triangle ABC$, respectively.

Prove that:

1. $O_A O_B O_C O_D$ is a parallelogram;
2. parallelograms $ABCD$ and $O_A O_B O_C O_D$ are similar;
3. $AO_B C O_D$ is a parallelogram;
4. $O_A B O_C D$ is a parallelogram;
5. parallelograms $AO_B C O_D$ and $O_A B O_C D$ are similar.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina, adapted by the editors.

If $ABCD$ is a rectangle, then $O_A = O_B = O_C = O_D$. Obviously we must assume that this is not the case.

1. Let M_{CD} , M_{AD} , M_{AB} , and M_{BC} be the perpendicular bisectors of the sides CD , AD , AB , and BC , respectively. Since $AB \parallel CD$, and since $M_{CD} \perp CD$ and $M_{AB} \perp AB$, we must have $M_{CD} \parallel M_{AB}$. Similarly, $M_{BC} \parallel M_{AD}$.

Since O_A and O_B lie on M_{CD} , and O_C and O_D lie on M_{AB} , we get

$$O_A O_B \parallel O_C O_D. \quad (1)$$

Similarly,

$$O_A O_D \parallel O_B O_C, \quad (2)$$

because O_A and O_D lie on M_{BC} , and O_B and O_C lie on M_{AD} . From (1) and (2), we see that $O_A O_B O_C O_D$ is a parallelogram.

2. To prove that parallelograms $ABCD$ and $O_A O_B O_C O_D$ are similar, it suffices to prove that $\triangle ABC$ and $\triangle O_B O_A O_D$ are similar.

We claim it is impossible to have both $O_A O_D \parallel AC$ and $O_A O_B \parallel BD$. If it were true that $O_A O_D \parallel AC$ and $O_A O_B \parallel BD$, then we would have $AC \perp BC$ (since $O_A O_D \perp BC$) and $BD \perp CD$ (since $O_A O_B \perp CD$). This would mean that the adjacent angles BCD and CDA of the parallelogram $ABCD$ are both obtuse, a contradiction. Hence, without loss of generality, we may assume that $O_A O_D$ meets AC .

Let O be the intersection of the diagonals AC and BD , let P be the intersection of $O_A O_D$ with AC , and let Q be the intersection of $O_A O_D$ with the line AD (these must meet, since $O_A O_D \perp BC$ and $BC \parallel AD$). Since O_B and O_D lie on the perpendicular bisector of AC , we have $O_B O_D \perp AC$. On the other hand, $O_A O_D \perp AD$. Then, $\triangle APQ$ and $\triangle O_D P O$ are right triangles and have a common angle $\angle APQ = \angle O_D P O$. Thus, $\triangle APQ$ and $\triangle O_D P O$ are similar. Hence, $\angle PAQ = \angle P O_D O$, and since $\angle PAQ = \angle ACB$, we obtain

$$\angle O_A O_D O_B = \angle ACB \quad (3)$$

Let M_1 be the mid-point of AD , and let M_2 be the mid-point of AB . Note that quadrilateral $AM_1 O_C M_2$ has two right angles, namely $\angle O_C M_1 A$ and $\angle O_C M_2 A$. Therefore, $\angle M_1 A M_2$ and $\angle M_1 O_C M_2$ are supplementary. Since $\angle M_1 O_C M_2 = \angle O_B O_A O_D$, and $\angle BAC$ and $\angle ABC$ are supplementary, we have

$$\angle O_B O_A O_D = \angle ABC \quad (4)$$

From (3) and (4), we see that $\triangle ABC$ and $\triangle O_B O_A O_D$ are similar. Hence, parallelograms $ABCD$ and $O_A O_B O_C O_D$ are similar.

3. Let O be the intersection of the diagonals AC and BD . Note that $OO_B \perp AC$, $OO_D \perp AC$, and O is the mid-point of AC . Thus O , O_B , and

O_D are collinear. Similarly, O , O_A , and O_C are collinear. Hence, O is also the intersection of O_BO_D and O_AO_C .

From part 2, $O_AO_BO_CO_D$ is a parallelogram, and O is the intersection of its diagonals; whence, $OO_B = OO_D$. Since O is the mid-point of both AC and O_BO_D , we see that AO_BCO_D is a parallelogram.

4. Since O is the mid-point of O_AO_C and the mid-point of BD , we observe that BO_ADO_C is a parallelogram.

5. From part 3, we see that AO_BCO_D is a parallelogram and that $O_BA = O_BC$ because O_B is the circumcentre of $\triangle ACD$. Then, AO_BCO_D is a rhombus. Similarly, from part 4, we see that $O_ABO_CO_D$ is a parallelogram and $O_AB = O_AD$ because O_A is the circumcentre of $\triangle BCD$. Then $O_ABO_CO_D$ also is a rhombus. To prove that two rhombi are similar, it suffices to prove that their diagonals are proportional. From part 2, parallelograms $ABCD$ and $O_AO_BO_CO_D$ are similar. Hence,

$$\frac{AC}{BD} = \frac{O_BO_D}{O_AO_C}.$$

The conclusion follows.

Also solved by Robert Bilinski, Outremont, QC.

M105. *Proposed by Andrew Critch, Clarendville High School, Clarendville, NL.*

Suppose that the roots of $P(x) = x^3 - 2kx^2 - 3x^2 + hx - 4$ are distinct, and that $P(k) = P(k+1) = 0$. Determine the value of h .

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Since $P(k) = P(k+1)$, we have

$$\begin{aligned} -k^3 - 4k^2 - 5k + hk + h - 6 &= -k^3 - 3k^2 + hk - 4 \\ h &= k^2 + 5k + 2. \end{aligned}$$

Then $P(x) = x^3 - 2kx^2 - 3x^2 + (k^2 + 5k + 2)x - 4$ and $P(k) = 2(k^2 + k - 2)$. Since k is a root of P , we conclude that $k = 1$ or $k = -2$.

For $k = 1$, we have $P(x) = (x-1)(x-2)^2$; this contradicts the fact that the roots of P are distinct. Thus, $k \neq 1$.

For $k = -2$, we have $P(x) = (x-2)(x+1)(x+2)$. Therefore, $k = -2$ and $h = -4$.

Also solved by Robert Bilinski, Outremont, QC and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Problem of the Month

Ian VanderBurgh, University of Waterloo

Some problems leap off the page and make you want to try to solve them when you first see them. For me, the following was one of those problems. Another thing that I really like about this problem is that it can be used with younger students, since the only tools necessary are logical thinking and addition.

Problem. (2002 Australian Mathematics Competition)

A 4×4 *antimagic* square is an arrangement of the numbers from 1 to 16 (inclusive) in a square, so that the totals of each of the four rows and four columns and two main diagonals are ten consecutive numbers in some order. The diagram below shows an incomplete antimagic square. When it is completed, what number will replace the star?

4	5	7	14
6	13	3	★
11	12	9	
10			

Solving this problem is a great example of just following your nose. I will present one way of doing this—there are certainly other approaches that can be taken. I have tried to explain the reasoning behind each step (which does make the solution a little bit longer than it needs to be).

Solution. The first question to ask is what numbers are missing from the antimagic square. Since the square is supposed to contain the numbers from 1 to 16 (inclusive), we are missing 1, 2, 8, 15, and 16.

The next logical thing to do is to add up any completed rows, columns, and diagonals to get a sense of what the sums should be:

- First row: $4 + 5 + 7 + 14 = 30$
- First column: $4 + 6 + 11 + 10 = 31$
- Diagonal: $10 + 12 + 3 + 14 = 39$

How does this help? We're told that the ten row, column, and diagonal sums are ten consecutive (and thus distinct) positive integers. Since we already have 30, 31, and 39, then these ten sums must be the integers from 30 to 39 (of which there are 10).

The next step that makes sense is to look at the sums of some of the partly completed rows, columns, and diagonals.

- Second row: $6 + 13 + 3 = 22$
- Third row: $11 + 12 + 9 = 32$
- Second column: $5 + 13 + 12 = 30$
- Third column: $7 + 3 + 9 = 19$
- Other diagonal: $4 + 13 + 9 = 26$

What can we conclude from this? Since the sum of the entries in the second row must be between 32 and 38 (30, 31, and 39 are already taken), the star (★) cannot be replaced by 1, 2, or 8; hence, it must be 15 or 16. (We have just narrowed down the number of possibilities for ★ from 5 to 2.) Similarly, the last entry in the third row must be a 1 or 2; the last entry in the second column must be 2 or 8; and the last entry in the third column must be 15 or 16.

However, it is when we reach the diagonal that we can actually reach a definitive conclusion. Since the sum of the the entries on the diagonal is between 32 and 38, the last entry in the diagonal must be 8. Retracing our steps, we see that the last entry in the second column must be 2 and the last entry in the third row must be 1.

Let us regroup and see where we are:

4	5	7	14
6	13	3	★
11	12	9	1
10	2		8

Now our completed rows, columns, and diagonals have sums of 30, 31, 32, 33, 34, and 39 (just add them up!), and we still have to insert 15 and 16. Looking at the third column, the sum of the first three entries is 19; thus, the remaining entry must be 16 (otherwise, the sum would be 34, which we already have elsewhere). Therefore, the completed array must be

4	5	7	14
6	13	3	15
11	12	9	1
10	2	16	8

We can check that all ten row, column, and diagonal sums are distinct. Thus, ★ is replaced by 15.

Pólya's Paragon

Triangular Tidbits (Part 1)

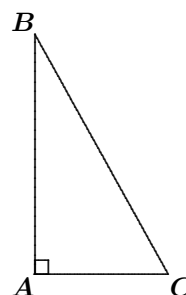
Shawn Godin

The adventurous student who takes the time to look at some high school mathematics textbooks from years gone by will find that the amount of geometry that students are exposed to today is far less than it was 20 to 50 years ago. For the next couple of issues we will look at some properties of triangles that may not be familiar to high school students today.

Just to get us going, we will do a quick review of the definitions of the trigonometric ratios *sine*, *cosine*, and *tangent*. In a right triangle, when referring to one of the acute angles, we will refer to the sides as the *hypotenuse* (across from the right angle), the *opposite* side (across from the angle in question) and the *adjacent* side (the side which is “attached” to the angle and different from the hypotenuse). For example, in the triangle pictured below, the hypotenuse is BC , AB is the side opposite $\angle C$, and AC is the side adjacent to $\angle C$.

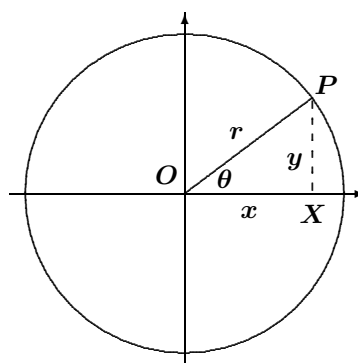
Keeping this terminology in mind, for an acute angle θ , we define:

$$\begin{aligned}\sin \theta &= \frac{\text{opposite}}{\text{hypotenuse}} \\ \cos \theta &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ \tan \theta &= \frac{\text{opposite}}{\text{adjacent}}\end{aligned}$$



We can extend these definitions by considering a circle of radius r centred at the origin O . We will start with a point P on the circle in the first quadrant. Let (x, y) be the coordinates of P . If we look at the right triangle POX , where X is the point on the positive x -axis directly below P , we get:

$$\begin{aligned}\sin \theta &= \frac{y}{r} \\ \cos \theta &= \frac{x}{r} \\ \tan \theta &= \frac{y}{x}\end{aligned}$$



We can use these to extend the definitions so that our ratios are defined for any possible angle. The original definitions can be used to “solve” a right triangle if we have a certain amount of information. Since the trigonometric ratios are well known, their values can be calculated for any angle with an inexpensive calculator (or with a set of trigonometric tables if you happened to find a textbook 20 years old or older in your original search). Unfortunately, we cannot use the original definitions to solve a non-right triangle. Let’s employ some circle geometry to develop the so-called *Law of Sines*.

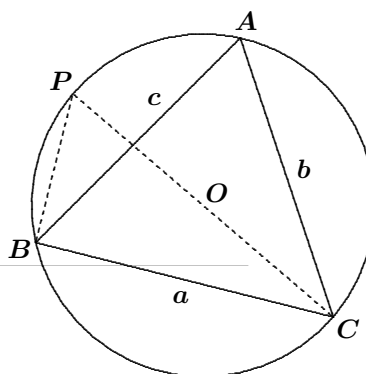
We need two facts.

1. If α and β are two angles with their vertices on the circumference of a circle and subtending the same arc (on the circumference of the circle), then $\alpha = \beta$.
2. If α and β are two angles subtending the same arc and if the vertex of α is on the circumference of a circle and the vertex of β is at the centre of the circle, then $\beta = 2\alpha$. A consequence of this is that the angle subtended at the circumference of a circle by a diameter is 90° .

Now consider any acute triangle ABC . Construct its circumcircle, and let the radius of this circumcircle be R . Denote by a , b , and c the sides opposite angles A , B , and C , respectively. Construct a point P on the circle so that PC is a diameter. Then $\angle A = \angle BPC$ and $\angle PBC = 90^\circ$. Thus,

$$\sin A = \sin BPC = \frac{BC}{PC} = \frac{a}{2R}.$$

That is, $\frac{a}{\sin A} = 2R$. Similar arguments give $\frac{b}{\sin B} = 2R$ and $\frac{c}{\sin C} = 2R$.



We have shown that the following result holds for any *acute* triangle.

Law of Sines. For any acute triangle ABC with circumradius R ,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

For homework, show that the Law of Sines holds for any triangle (not necessarily acute).

Next issue we will continue looking at properties of triangles.

THE OLYMPIAD CORNER

No. 241

R.E. Woodrow

As a first set of problems we give the problems of the 37th Mongolian Mathematical Olympiad, Final Round. Thanks go to Chris Small, Canadian Team Leader to the 42nd IMO, for collecting them.

37th MONGOLIAN MATHEMATICAL OLYMPIAD Final Round 10th Class

1. A sequence $x_1, x_2, \dots, x_{2001}$ of positive real numbers satisfies

$$(i) \quad 3x_{n+1}^2 = 7x_n \cdot x_{n+1} - 3x_{n+1} - 2x_n^2 + x_n,$$

$$(ii) \quad x_{37} = x_{2001}.$$

Find the maximum possible value of x_1 .

2. Prove that, if ABC is an acute-angled triangle, then

$$\frac{a^2 + b^2}{a + b} \cdot \frac{b^2 + c^2}{b + c} \cdot \frac{c^2 + a^2}{c + a} \geq 16 \cdot R^2 \cdot r \cdot \frac{m_a}{a} \cdot \frac{m_b}{b} \cdot \frac{m_c}{c}.$$

3. Suppose that a and b are relatively prime positive integers, a is even, and $a > b$. Prove that there exist infinitely many pairs (m, n) of positive, relatively prime integers such that

$$m \mid a^{n-1} - b^{n-1} \quad \text{and} \quad n \mid a^{m+1} - b^{m+1}.$$

4. Given n points all lying on a line ($n > 3$), find the number of ways of colouring the points with 2 colours, red and blue, such that in each chosen sequence of consecutive points, the difference between the numbers of red and blue points does not exceed 2.

5. The mid-points of the sides of a hexagon with parallel opposite sides are denoted A, B, C, D, E, F in clockwise order. Prove that the points $AB \cap ED$, $BC \cap EF$, and $AC \cap FD$ are collinear.

6. In a 10×10 board, we mark some cells such that each cell has an even number of neighbouring marked cells. Find the maximum possible number of marked cells.

37th MONGOLIAN MATHEMATICAL OLYMPIAD

Final Round

Teachers

1. Given a positive integer n , prove that there exists a polynomial $p(x) \in \mathbb{R}[x]$ of degree n having n distinct real roots and satisfying

$$p(x) \cdot p(4 - x) = p(x(4 - x)).$$

2. Let b_1, \dots, b_n be positive real numbers. Set $a_1 = \frac{b_1}{b_1 + b_2 + \dots + b_n}$ and $a_k = \frac{b_1 + \dots + b_k}{b_1 + \dots + b_{k-1}}$ for $k > 1$. Prove the inequality

$$a_1 + \dots + a_n \leq \frac{1}{a_1} + \dots + \frac{1}{a_n}.$$

3. Let $k \geq 0$ be an integer. Suppose that there exist positive integers n and d , and an odd integer $m > 1$, such that: (i) $d \mid m^{2^k} - 1$, and (ii) $m \mid n^d + 1$. Find all values of m^{2^k-1}/d .

4. On a $2n \times 2n$ board, we mark some cells so that each cell has an even number of neighbouring marked cells. Find the number of all such markings.

5. Chords AC and BD in a circle w intersect at a point E . Another circle is internally tangent to w at a point F and is tangent to the segments DE and EC . Prove that the bisector of angle AFB passes through the incentre of triangle AEB .

6. In a tennis competition involving n players, each player played exactly one match against each other player, scoring 1 for a win, and 0 for a loss. At the end of the competition, the total scores of the players were $r_1 \leq r_2 \leq \dots \leq r_n$. If the winner of a particular match did not end up with a greater total score than the loser of the match, we call the match *wrong*. Let $I = \{i \mid r_i \geq i\}$. Prove that the number of wrong matches is at least $\sum_{i \in I} (r_i - i + 1)$, and that this least value is attainable.

As the second set, we give the problems of the 8th Macedonian Mathematical Olympiad of 2000. Thanks to Chris Small, Canadian team Leader to the 42nd IMO, for collecting them.

8th MACEDONIAN MATHEMATICAL OLYMPIAD

1. Prove that, if $m \cdot s = 2000^{2001}$ where $m, s \in \mathbb{Z}$, then the equation $mx^2 - sy^2 = 3$ has no solution in \mathbb{Z} .

2. Does there exist a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $n \geq 2$,

$$f(f(n-1)) = f(n+1) - f(n)?$$

3. Let ABC be a triangle with no two sides equal, and let k be the circum-circle of $\triangle ABC$. Let t_A, t_B, t_C be tangents to the circle k at the points A, B, C , respectively. Prove that $AB \parallel t_C, AC \parallel t_B, BC \parallel t_A$, and the points $AB \cap t_C, AC \cap t_B, BC \cap t_A$ are collinear.

4. Let M be a finite set and let $\Omega \subseteq \mathcal{P}(M)$ such that:

- (i) If $|A \cap B| \geq 2$ for $A, B \in \Omega$, then $A = B$;
- (ii) There are $A, B, C \in \Omega$ such that $A \neq B \neq C \neq A$ and $|A \cap B \cap C| = 1$;
- (iii) For every $A \in \Omega$ and for every $a \in M \setminus A$, there is a unique $B \in \Omega$ such that $a \in B$ and $A \cap B = \emptyset$.

Prove that there are numbers p and s such that:

- (a) For every $a \in M$ the number of sets which include the point a is p ;
- (b) $|A| = s$ for every $A \in \Omega$;
- (c) $s + 1 \geq p$.

Next we give the Latvian Mathematical Olympiad 2000/2001 Final Grade, 3rd Round. Thanks again go to Chris Small, Canadian Team Leader to the 42nd IMO, for collecting the problems.

LATVIAN MATHEMATICAL OLYMPIAD 2000/2001

Final Grade

3rd Round

1. There are n straight lines in space. The angle between any pair of these lines is the same. Find the maximal value of n .

2. Is the following inequality true for $0 < x_1, x_2, \dots, x_n \leq \frac{1}{2}$ if $n = 2$? If $n = 3$? If $n = 6$?

$$\frac{x_1}{1-x_1} \cdot \frac{x_2}{1-x_2} \cdots \frac{x_n}{1-x_n} \leq \frac{x_1^n + x_2^n + \cdots + x_n^n}{(1-x_1)^n + (1-x_2)^n + \cdots + (1-x_n)^n}$$

3. Is it possible to colour all grid points in the plane white and red so that no rectangle with vertices on grid points of one colour and sides parallel to the grid lines has area from the set $\{1, 2, 4, 8, \dots, 2^n, \dots\}$?

- 4.** Find as small a positive constant α as you can so that the following holds: every triangle can be folded along a line segment such that the area covered by the folded piece does not exceed $\alpha \cdot L$, where L is the area of the original triangle.
- 5.** Prove that for each n there exists a finite graph without triangles such that in each colouring of the vertices with n colours there is an edge with equally coloured endpoints. (A known theorem.)

Next an apology. When sorting materials I discovered that I misfiled a solution of Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON to problems #1, #5, and #8 of the 1998 Ukrainian Mathematical Olympiad [2001 : 294–295], for which we gave solutions [2003 : 441–448].

Now we complete the set of solutions that we have received for the St. Petersburg Contests 1965–1984 given in [2002 : 289–291].

- 24.** An infinite sequence of light bulbs and an infinite sequence of switches are both numbered by the positive integers. Each switch has a finite number of positions. Whether a bulb is on or off depends only on the positions of a finite number of switches. In any setting of the switches, at least one bulb is on. Prove that there exists a finite set of bulbs such that for any setting of the switches, at least one of them is on.

Solution by Pierre Bornsztein, Maisons-Laffitte, France, adapted by the editor.

Denote the bulbs by B_1, B_2, \dots and the switches by S_1, S_2, \dots . Aiming for a contradiction, we suppose that there is no finite set of bulbs with the property that at least one bulb in the set is turned on for each setting of all the switches. We will define a sequence of settings for the switches S_1, S_2, \dots which corresponds to a state in which all the bulbs are off. Thus, we will contradict an hypothesis in the problem.

Since the state of each bulb is determined by finitely many switches, we can fix an increasing sequence of integers m_1, m_2, \dots with the property that, for any given $k \in \mathbb{N}$, the state of the bulbs B_1, \dots, B_k is completely determined by the switches S_1, \dots, S_{m_k} . Among the (finite number of) possible settings for the switches S_1, \dots, S_{m_k} , there must be at least one corresponding to a state in which the bulbs B_1, \dots, B_k are all turned off. (Otherwise, at least one bulb among B_1, \dots, B_k would be on for each setting of the switches S_1, \dots, S_{m_k} , and therefore, at least one of these bulbs would be on for each setting of all the switches, contradicting our initial hypothesis above.)

If $k < \ell$, we say that a setting of S_1, \dots, S_{m_ℓ} extends a setting of S_1, \dots, S_{m_k} if the switches that are involved in both settings (namely, the switches S_1, \dots, S_{m_k}) are set the same in both settings.

We proceed iteratively. First, there must be a setting of S_1, \dots, S_{m_1} which turns off B_1 and which can be extended, for infinitely many values of k , to a setting of S_1, \dots, S_{m_k} which turns off B_1, \dots, B_k . (Otherwise, for sufficiently large k , there would be no setting of S_1, \dots, S_{m_k} which turns off B_1, \dots, B_k .) Choose one such setting of S_1, \dots, S_{m_1} and call it P_1 .

Now, suppose that for some $k \in \mathbb{N}$, we have chosen a setting P_k for S_1, \dots, S_{m_k} which turns off B_1, \dots, B_k and can be extended, for infinitely many $\ell > k$, to a setting of S_1, \dots, S_{m_ℓ} which turns off B_1, \dots, B_ℓ . We then choose a setting P_{k+1} of $S_1, \dots, S_{m_{k+1}}$ which extends our setting P_k , turns off the next bulb, B_{k+1} , and can be extended, for infinitely many $\ell > k+1$, to a setting S_1, \dots, S_{m_ℓ} which turns off B_1, \dots, B_ℓ .

In this iteration, notice that the setting of any given switch S_n is eventually fixed (by step n at the latest, since $n \leq m_n$). Moreover, the bulb B_n is turned off at step n and remains off thereafter. Thus, we obtain a setting of all the switches for which all the bulbs are off—a contradiction.

25. The sum of two continuous periodic functions is a non-constant continuous periodic function. Prove that the periods of these two functions are integral multiples of the period of their sum.

Solution by Pierre Bornshtein, Maisons-Laffitte, France.

The result is false.

Let $f(x) = \cos x - \cos 2x$ and $g(x) = \cos 2x$. Then f and g are continuous periodic functions, with minimal periods $T_f = 2\pi$ and $T_g = \pi$, respectively. Let $h = f + g$. Then $h(x) = \cos x$, which is a non-constant continuous periodic function, with minimal period $T_h = 2\pi$.

Since $T_g \neq kT_h$ for some integer k , the conclusion follows.

26. Four squares on a 25×25 chessboard are called a quartet if their centres form a rectangle with sides parallel to the sides of the board. What is the maximum number of quartets which do not have any common squares?

Solution by Pierre Bornshtein, Maisons-Laffitte, France.

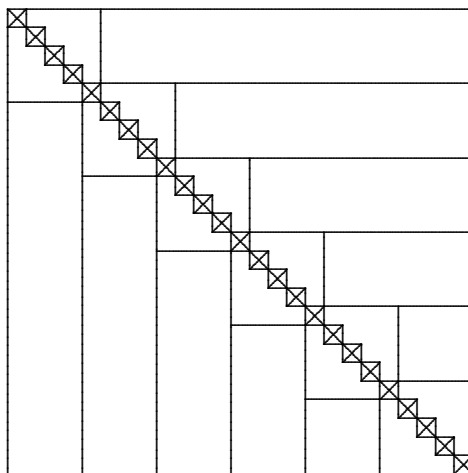
The maximum is 150.

First, note that any quartet uses an even number (0 or 2) of squares from each horizontal line of the board. If the quartets are required to have no common square, then an even number of squares must be used from any horizontal line. Therefore, from any such line, we cannot select more than 24 squares. Consequently, the number of quartets is not more than $\frac{24 \times 25}{4} = 150$.

On the other hand, it is possible to have 150 quartets as follows. For any even integer $p \geq 2$, it is easy to cover a $4 \times p$ rectangle with p pairwise disjoint quartets. Also, it is possible to select 5 pairwise disjoint quartets

from a 5×5 square, as shown on the left below. Then, we just have to cover the main diagonal of the board with six 5×5 squares, two consecutive ones sharing a unit square, as shown on the right below.

	1	2	1	2
3		2	3	2
4	1		1	4
4	5	5		4
3	5	5	3	



28. An irreducible fraction $\frac{x}{y}$ is called a good approximation of a number c if $\left| c - \frac{x}{y} \right| < \frac{1}{y^{100}}$. Prove that in any interval, there is a number with infinitely many good approximations.

Solution by Pierre Bornsztein, Maisons-Laffitte, France, modified by the editors.

Let I be an interval with length $L > 0$. There exists $n \in \mathbb{N}^*$ such that

$$\frac{1}{10^n} < \frac{L}{10}. \quad (1)$$

Then there exists $p \in \mathbb{Z}$ such that

$$\frac{p}{10^n}, \frac{p+1}{10^n} \in I. \quad (2)$$

In the following, we assume that n and p satisfy (1) and (2).

Let $\xi = \sum_{i=n+1}^{\infty} \frac{1}{10^{a^i}}$, for some fixed positive integer $a \geq 2$. It is easy to verify by induction that $a^{n+i} \geq a^n + i$, for every $i \geq 1$. Then

$$0 < \xi = \sum_{i=0}^{\infty} \frac{1}{10^{a^{n+1+i}}} \leq \sum_{i=0}^{\infty} \frac{1}{10^{a^{n+1}+i}} = \frac{1}{10^{a^{n+1}}} \times \frac{10}{9}.$$

From now on, we suppose that n is large enough so that

$$\frac{1}{10^{a^{n+1}}} \times \frac{10}{9} < \frac{1}{10^n}.$$

Letting $c = \xi + \frac{p}{10^n}$, we have $c \in I$.

For $k \geq n + 1$, let $\xi_k = \sum_{i=n+1}^k \frac{1}{10^{a^i}}$ and $c_k = \frac{p}{10^n} + \xi_k$. Then, as an irreducible fraction, we have $c_k = \frac{x_k}{y_k}$, with $y_k \leq \max(10^n, 10^{a^k}) = 10^{a^k}$. It follows that

$$\frac{1}{10^{100a^k}} \leq \frac{1}{y_k^{100}}.$$

Moreover, $0 < c - c_k = \xi - \xi_k = \sum_{i=k+1}^{\infty} \frac{1}{10^{a^i}} \leq \frac{10}{9} \times \frac{1}{10^{a^{k+1}}}$. Then, c_k is a good approximation of c if we choose $k \geq n + 1$ such that

$$\frac{10}{9} \times \frac{1}{10^{a^{k+1}}} < \frac{1}{10^{100a^k}}.$$

It suffices to have $a^{k+1} - 1 > 100a^k$, which is clearly satisfied by an infinite number of integers k , as long as $a > 100$. Thus, c has infinitely many good approximations.

29. From each of k points on a plane, a few rays are drawn. No two rays intersect. Prove that one can choose $k - 1$ of the segments connecting these points such that they are disjoint from one another and from any of the rays, except possibly at those k points.

Solution by Pierre Bornsstein, Maisons-Laffitte, France.

We interpret the hypothesis that no two rays intersect as implying that no ray goes through any of the k points, other than the one from which it emanates.

For $k \geq 2$, let \mathcal{P}_k be the claim. "For any configuration of k points in the plane satisfying the assumptions of the problem, it is possible to draw some segments connecting these points so as to form a connected planar simple graph, whose vertices are the k given points, and whose edges are not intersected by any ray (except possibly at the vertices)". In the sequel, such a graph will be called a *good* graph.

The desired conclusion will follow immediately from the proof that \mathcal{P}_k holds and from the well-known result that a connected graph with k vertices has at least $k - 1$ edges.

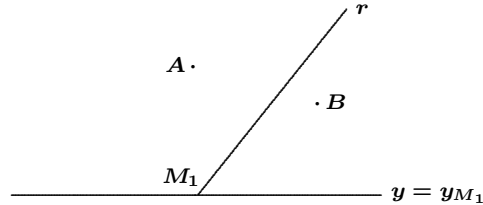
Now, we prove \mathcal{P}_k by induction on k . For $k = 1$, there is nothing to prove. For $k = 2$, let M_1 and M_2 be the given points. Since the segment M_1M_2 is not cut by a ray, the claim \mathcal{P}_2 clearly holds.

Let $k \geq 3$ be fixed. Suppose that \mathcal{P}_q holds for all $q \in \{2, \dots, k - 1\}$. Let us give a configuration with k points satisfying the assumptions of the problem.

Since there is a finite number of points, they determine by pairs only a finite number of directions. Thus, we may choose an orthonormal system of coordinates such that the "vertical" coordinate of the points are pairwise distinct. With no loss of generality, we may suppose that the points are M_1, M_2, \dots, M_k with $y_{M_1} = \min_{i \geq 1} y_{M_i}$ and $y_{M_2} = \min_{i \geq 2} y_{M_i}$. Note that M_1 is the unique point belonging to the half plane $y \leq y_{M_1}$.

In the following, we will say that the ray r (or the segment s) *separates* the points A and B if r (or s) intersects the interior of the line segment AB .

First case. A ray r from M_1 separates two of the points, say A and B .



By the induction hypothesis, we may construct a good graph \mathcal{G}_1 whose vertices are the points on the left side of r including M_1 , and a good graph \mathcal{G}_2 whose vertices are the points on the right side of r including M_1 . Since two rays never intersect, none of the edges of \mathcal{G}_1 (respectively, \mathcal{G}_2) can be separated by a ray issued from a vertex of \mathcal{G}_2 (respectively, \mathcal{G}_1), nor by an edge of \mathcal{G}_2 (respectively, \mathcal{G}_1).

Thus, the graph $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ is a good graph (it is connected via M_1) for the whole set of points.

Second case. No ray from M_1 separates two of the points.

We may use the induction hypothesis to construct a good graph \mathcal{G} whose vertices are M_2, \dots, M_k .

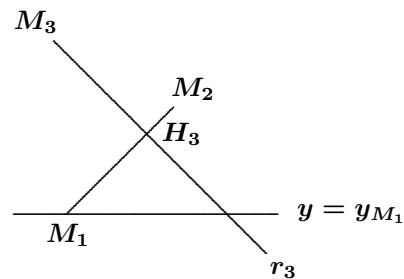
In the following, we will say that the point M *separates* the points A and B if M is an interior point of the line segment AB or if a ray from M separates A and B , or if an edge with end-point M separates A and B .

By adding the vertex M_1 , we separate no pair of points. Now, it suffices to prove that M_1 may be joined by an edge to at least one of the other points, so as to form a good graph.

From the minimality of M_1 and M_2 , no M_i belongs to the interior of the line segment M_1M_2 , and no edge of \mathcal{G} separates M_1 and M_2 .

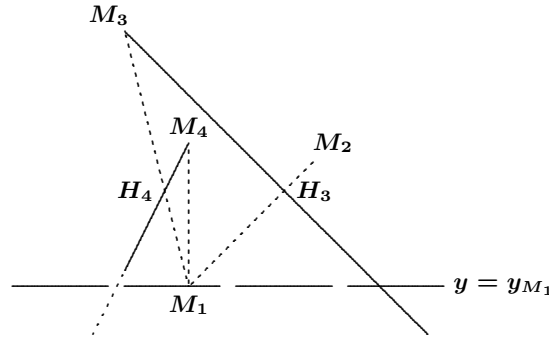
(1) If M_1 and M_2 can be joined, we are done.

(2) If M_1 and M_2 cannot be joined, then there is a ray which separates M_1 and M_2 . Since the number of rays is finite, there is a finite number of intersection points between the interior of the line segment M_1M_2 and the set of rays. With no loss of generality, we may suppose that the ray r_3 issued from M_3 separates M_1 and M_2 , with M_1H_3 minimal, where H_3 is the point in common between r_3 and the interior of line segment M_1M_2 . Note that M_2 is exterior to $\triangle M_1M_3H_3$. Let n_3 be the number of the M_i 's which are not exterior to $\triangle M_1M_3H_3$.



- (3) If M_1 and M_3 can be joined, we are done.
- (4) If not, note that, from the minimality of H_3 and M_2 , the interior of the line segment M_1H_3 cannot be cut by any edge or ray. Moreover, since the segment M_3H_3 belongs r_3 , we see that M_3H_3 cannot be cut by any edge or any other ray. It follows that any ray or edge which separates M_1 and M_3 has one of its end-points in the interior of $\triangle M_1M_3H_3$ or in the interior of the segment M_1M_3 .
- (5) If no ray or edge separates M_1 and M_3 , then they are separated only by points belonging from the interior of M_1M_3 . Let P be the point which separates M_1 and M_3 with M_1P minimal (along M_1M_3). From above, M_1 and P are not separated. We may join M_1 and P by an edge, and we are done.
- (6) If M_1 and M_3 are separated by a ray or an edge, then with no loss of generality, we may suppose that they are separated by an edge or a ray issued from M_4 , which intersects the interior of M_1M_3 at H_4 , with M_1H_4 minimal.

Note that $\triangle M_1M_4H_4$ is included in $\triangle M_1M_3H_3$, and (from the minimality of H_3) the interior of the segment M_1H_3 cannot be cut by any edge or ray. Let n_4 be the number of the M_i 's which are not exterior to $\triangle M_1M_4H_4$.

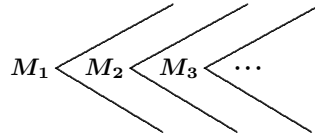


Thus, we are in the same situation as in (5) except that, since M_3 is exterior to $\triangle M_1M_4H_4$, we have $n_4 < n_3$. (*)

Repeating the reasoning above, the process will eventually stop (because of (*)) at some step similar to (3) or (5). Thus, M_1 can be joined by an edge with some M_p , to form a good graph from \mathcal{G} .

This ends the induction step, and proves that \mathcal{P}_k holds.

Remark. The value $k - 1$ is minimal in the sense that it can be achieved, as in the configuration at right.



32. There are 9 points in a 2×2 square. Prove that the distance between some 2 of these points is not greater than 1.

Comment by Pierre Bornshtein, Maisons-Laffitte, France.

This result is due to J. Schaer: *The Densest Packing of 9 Circles in a Square*, Canadian Math. Bulletin, Vol 8, No 3, April 1965, p. 273–277.

34. A straight line passes through the centre of a regular $2n$ -gon. Prove that the sum of distances to this line from the vertices on one side of this line equals the sum of the distances from the remaining vertices.

Solved by Michel Bataille, Rouen, France; and Pierre Bornshtein, Maisons-Laffitte, France. We give Bataille's solution.

Let $A_0, A_1, \dots, A_{2n-1}$ be the points with respective complex affixes $1, z, \dots, z^{2n-1}$, where $z = \exp\left(\frac{2\pi i}{2n}\right) = \exp\left(\frac{\pi i}{n}\right)$. Since any regular $2n$ -gon is similar to the $2n$ -gon $A_0A_1 \dots A_{2n-1}$, it is sufficient to prove the result for $A_0A_1 \dots A_{2n-1}$.

We partition the set of vertices into n pairs: $\{A_0, A_n\}, \{A_1, A_{n+1}\}, \dots, \{A_{n-1}, A_{2n-1}\}$. Since $-z^k = z^n z^k = z^{n+k}$, the two points in each pair are symmetrical about the centre O of $A_0A_1 \dots A_{2n-1}$. Consider one of these pairs $\{A_k, A_{n+k}\}$, and let L be the given line through O . Because the symmetry through O exchanges the two half-planes determined by L , if A_k is on one side of L , then A_{n+k} is on the other side (and if A_k is on L , then A_{n+k} is also on L). Now, since the symmetry through O preserves distances, and since L passes through O , we have $d(A_k, L) = d(A_{n+k}, L)$. The result follows immediately.

39. Each of n lines on a plane is cut by the others into 2 rays and $n - 2$ equal segments. Prove that $n = 3$.

Solution by Pierre Bornshtein, Maisons-Laffitte, France, adapted by the editors.

First, we note that the problem makes sense only for $n \geq 2$. The case $n = 2$ is possible but trivial, since any two intersecting lines cut one another into 2 rays and 0 line segments. This case is apparently disallowed. For $n = 3$, it is easy to come up with 3 lines that satisfy the given conditions.

Now suppose, by way of contradiction, that $n \geq 4$. Note that no three of the given lines have a common point and no two are parallel. In the following, each of the lines which are considered is one of the n given lines.

Let ℓ be one of the lines, and let B, C, D be three consecutive intersection points, in that order, on ℓ . Then $BC = CD$, and none of the lines passes through the interior of the segments BC or CD . Let ℓ_B be a line which intersects ℓ at B , and let ℓ_C and ℓ_D be defined in the same way. Let A be the common point of ℓ_B and ℓ_C , and let E be the common point of ℓ_D and ℓ_C . We claim that one of the lines meets the interior of $\triangle ABC$ or $\triangle ECD$.

Case 1. C between A and E . (Figure 1).

Suppose that our claim is false. Then A, C, E are three consecutive intersection points, in that order, on ℓ_C . Thus, $AC = CE$.

Using Thales' Theorem, since $BC = CD$, we deduce that $\ell_B = AB$ is parallel to $ED = \ell_D$, a contradiction. Thus, our claim is true in this case.

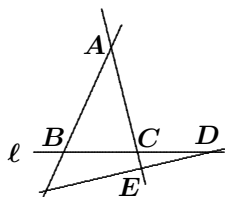


Figure 1

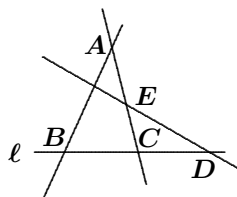


Figure 2

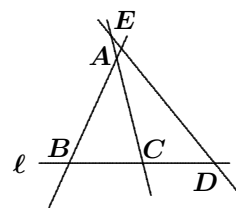


Figure 3

Case 2. E between A and C . (Figure 2)

Then ℓ_D meets the interior of $\triangle ABC$.

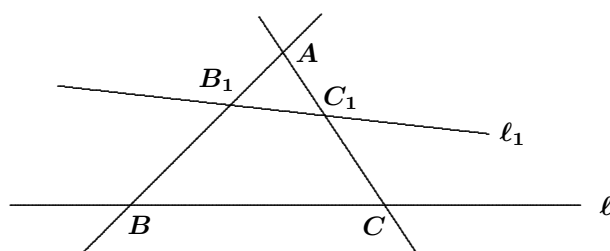
Case 3. A between C and E . (Figure 3)

Then ℓ_B meets the interior of $\triangle ECD$.

Thus, in every case, one of the lines meets the interior of $\triangle ABC$ or $\triangle ECD$. With no loss of generality, we may suppose that one of the lines meets the interior of $\triangle ABC$.

Any line which meets the interior of $\triangle ABC$ is neither ℓ_B nor ℓ_C nor ℓ , and must intersect the interior of each of the segments AB and AC , from above. Then the number of intersection points in the interior of segment AB is equal to the number of intersection points in the interior of segment AC . Let $k \geq 1$ be this common value.

Let B_1 be the intersection point in the interior of AB such that AB_1 is minimal. Let ℓ_1 be the line which meets AB at B_1 , and let C_1 be the intersection of ℓ_1 and the interior of AC . Note that $\ell_1 \neq \ell$.



Case (a). None of the lines meets the interior of $\triangle AB_1C_1$.

Then C_1 is the intersection point in the interior of AC such that AC_1 is minimal. Thus, $\overrightarrow{AB_1} = \frac{1}{k}\overrightarrow{AB}$ and $\overrightarrow{AC_1} = \frac{1}{k}\overrightarrow{AC}$. It follows that the line $\ell_1 = B_1C_1$ is parallel to $BC = \ell$, a contradiction.

Case (b). One of the lines meets the interior of $\triangle AB_1C_1$.

We note that none of the lines intersects the interior of AB_1 , and that the area of $\triangle AB_1C_1$ is strictly less than the area of $\triangle ABC$. We may use the same reasoning as above, with $\triangle AB_1C_1$ in place of $\triangle ABC$. Since the number of triangles XYZ (where X, Y, Z are three intersection points) is finite, this process eventually stops, at which time we obtain the same contradiction as in (a).

In every case, we obtain a contradiction. It follows that $n \leq 3$, and we are done.

40. A strictly increasing sequence $\{a_n\}$ of positive integers is such that $a_2 = 2$ and $a_{mn} = a_m a_n$ if m and n are relatively prime. Prove that $a_n = n$ for all n .

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

This problem is similar to problem #2 of the 24th William Lowell Putnam Competition (1963). A solution may be found in A.M. Gleason, R.E. Greenwood, L.M. Kelly, *The William Lowell Putnam Competition, Problems and Solutions 1938–1964*, MAA, p. 570–571.

43. H is a given point inside a circle. Prove that a fixed circle passes through the mid-points of the sides of any triangle inscribed in the circle and having H as its orthocentre.

Solution by Michel Bataille, Rouen, France.

Let Γ be the given circle and O be its centre. Let $\triangle ABC$ be inscribed in Γ and have H as its orthocentre. Since the circumcentre O and the orthocentre H of $\triangle ABC$ are fixed points, the same is true of its centroid G (because of Euler's relation $\overrightarrow{OG} = \frac{1}{3}\overrightarrow{OH}$, valid in all triangles). The triangle whose vertices are the mid-points of the sides AB, BC, CA is the homothetic of $\triangle ABC$ under the homothety h with centre G and scale factor $-\frac{1}{2}$. Thus, the circumcircle of this triangle is the image of Γ under h and, as such, is the fixed circle with centre O' defined by $\overrightarrow{GO'} = -\frac{1}{2}\overrightarrow{GO}$ and radius $\frac{1}{2}R$, where R is the radius of Γ .

44. A strictly increasing sequence $\{x_n\}$ of positive integers is such that for all $n > 1982$,

$$x_1^3 + x_2^3 + \cdots + x_n^3 = (x_1 + x_2 + \cdots + x_n)^2.$$

Prove that $x_n = n$ for all n .

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let $S_n = \sum_{i=1}^n x_i$, for $n > 0$. Let $n \geq 1983$. We have

$$x_{n+1}^3 + S_n^2 = \sum_{i=1}^{n+1} x_i^3 = (S_n + x_{n+1})^2 = S_n^2 + x_{n+1}^2 + 2x_{n+1}S_n,$$

and hence, $x_{n+1}^2 = x_{n+1} + 2S_n$. We deduce that, for $n \geq 1984$,

$$\begin{aligned} x_{n+1}^2 &= x_{n+1} + 2x_n + 2S_{n-1} = x_{n+1} + 2x_n + (x_n^2 - x_n) \\ &= x_{n+1} + x_n^2 + x_n; \end{aligned}$$

that is, $(x_{n+1} - x_n - 1)(x_{n+1} + x_n) = 0$. Then $x_{n+1} = x_n + 1$ (since the x_i 's are positive). It follows that, for every $n \geq 0$, we have

$$x_{1984+n} = a + n, \quad (1)$$

where $a = x_{1984}$.

Let

$$S = \sum_{i=1}^{1983} x_i^3 - \sum_{i=1}^{a-1} i^3 \quad \text{and} \quad K = \sum_{i=1}^{1983} x_i - \sum_{i=1}^{a-1} i.$$

For $n \geq 1984$, we have

$$\begin{aligned} \sum_{i=1}^n x_i^3 &= \sum_{i=1}^{1983} x_i^3 + \sum_{i=1984}^n x_i^3 = \sum_{i=1}^{1983} x_i^3 - \sum_{i=1}^{a-1} i^3 + \sum_{i=1}^{x_n} i^3 \quad (\text{using (1)}) \\ &= S + \left(\frac{x_n(x_n + 1)}{2} \right)^2. \end{aligned}$$

Similarly

$$\begin{aligned} \left(\sum_{i=1}^n x_i \right)^2 &= \left(\sum_{i=1}^{1983} x_i + \sum_{i=1984}^n x_i \right)^2 = \left(\sum_{i=1}^{1983} x_i - \sum_{i=1}^{a-1} i + \sum_{i=1}^{x_n} i \right)^2 \\ &= \left(K + \frac{x_n(x_n + 1)}{2} \right)^2. \end{aligned}$$

Then $S = K^2 + Kx_n(x_n + 1)$; that is,

$$Kx_n(x_n + 1) = S - K^2. \quad (2)$$

Since $\{x_i\}$ is an increasing sequence of positive integers, we deduce that $x_i \geq i$ for all i . It follows that

$$\lim_{n \rightarrow +\infty} x_n(x_n + 1) = +\infty.$$

Then, from (2), we must have $K = 0$; that is,

$$\sum_{i=1}^{1983} x_i = \sum_{i=1}^{a-1} i.$$

Since $a - 1 = x_{1984} - 1 \geq x_{1983}$, each x_i on the left side appears also on the right side. It follows that there is no term on the right side other than those which appear on the left side. Thus, $x_{1984} - 1 = 1983$, and

$$\{1, 2, \dots, 1983\} = \{x_1, x_2, \dots, x_{1983}\}. \quad (3)$$

Then $a = 1984$, and hence, using (1), we have $x_n = n$ for $n \geq 1984$. Also, using (3), we have $x_n = n$ for $n \leq 1983$. The conclusion follows.

45. Let $P(z)$ and $Q(z)$ be complex polynomials, one of which is not constant. Every root of $P(z)$ is also a root of $Q(z)$ and vice versa. Every root of $P(z) - 1$ is also a root of $Q(z) - 1$ and vice versa. Prove that $P = Q$.

Solution by Michel Bataille, Rouen, France.

Suppose that $P(z)$ has degree $m \geq 1$. Since any root of $P(z)$ is also a root of $Q(z)$, we have $\deg Q(z) \geq 1$ as well. Without loss of generality, we will assume that $m \geq \deg Q(z)$. Let u_1, u_2, \dots, u_k be the distinct complex roots of $P(z)$, with respective multiplicities r_1, r_2, \dots, r_k (so that $r_1 + r_2 + \dots + r_k = m$). Likewise, let v_1, v_2, \dots, v_ℓ be the distinct complex roots of $P(z) - 1$, with respective multiplicities s_1, s_2, \dots, s_ℓ .

For $j = 1, 2, \dots, k$ and $i = 1, 2, \dots, \ell$, we have $P(u_j) = 0$ and $P(v_i) = 1$. Therefore, each of the u_j 's is different from each of the v_i 's. It follows that the derivative $P'(z)$, which is divisible by each $(z - u_j)^{r_j-1}$ and by each $(z - v_i)^{s_i-1}$, is divisible by the product

$$\prod_{j=1}^k (z - u_j)^{r_j-1} \prod_{i=1}^{\ell} (z - v_i)^{s_i-1}.$$

The latter has degree

$$(r_1 - 1) + \dots + (r_k - 1) + (s_1 - 1) + \dots + (s_\ell - 1) = 2m - (k + \ell),$$

and $P'(z)$ has degree $m - 1$. Thus, $m - 1 \geq 2m - (k + \ell)$; that is, $k + \ell \geq m + 1$.

This said, consider $R(z) = P(z) - Q(z) = (P(z) - 1) - (Q(z) - 1)$. The degree of $R(z)$ is at most m and $u_1, \dots, u_k, v_1, \dots, v_\ell$ are $k + \ell$ distinct roots of $R(z)$. Since $k + \ell \geq m + 1$, we must have $R = 0$; that is, $P = Q$.

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

This problem is similar to problem #7 of the 16th William Lowell Putnam Competition (1956). A solution may be found in A.M. Gleason, R.E. Greenwood, L.M. Kelly, *The William Lowell Putnam Competition, Problems and Solutions 1938–1964*, MAA, p. 431–432.

That completes the *Corner* for this issue. Send me your nice solutions to recent problems for use in upcoming issues as well as Olympiad Contests.

BOOK REVIEW

John Grant McLoughlin

Mathematical Puzzles: A Connoisseur's Collection

By Peter Winkler, published by A.K. Peters, Ltd., 2003

ISBN 1-56881-201-9, paperback, 180 pages, US\$18.00.

Reviewed by **Peter Hardy**, University of Maine at Farmington, Farmington, ME.

The preface to Peter Winkler's *Mathematical Puzzles* begins as follows: "These puzzles are not for everyone. To appreciate them, and to solve them, it is necessary—but not sufficient—to be comfortable with mathematics." I would tend to agree; these puzzles are not for your average armchair puzzler. Although I believe that this collection of puzzles may be appreciated by anyone with an interest in problem solving, to actually derive the solutions to them requires an above-average level of mathematical proficiency, let alone comfort.

The book is organized into chapters based upon problem type, from combinatorics, probability, and geometry to algorithms and game theory. At the end of the book the author includes some problems he considers "toughies" as well as some unsolved problems. Each chapter includes one sample problem with solution to get the reader in the proper mindset for the group of problems which follow. Very little mathematical instruction is included, though an occasional hint is dropped, such as the use of the pigeon-hole principle in the combinatorial puzzles. With the exception of the unsolved problems, possible solutions to every problem in the book are given at the end of each chapter. It was interesting to this reader that the author also included the source of each problem, many of which come from international mathematics competitions.

In selecting puzzles for this collection the author writes that a puzzle "should be elementary and easy to state, it should not be easy to solve, and it should boast at least one solution which is elementary and easily convincing". Indeed, there are very few puzzles in the book which can be solved either easily or through some mental sleight-of-hand. A few exceptions are included in the first chapter entitled "Insight". For example, in *The Attic Lamp Switch*, we are given the problem "A downstairs panel contains three on-off switches, one of which controls the lamp in the attic—but which one? Your mission is to do something with the switches, then determine after one trip to the attic which switch is connected to the attic lamp."

The only other relatively easy puzzles to solve (if one happens to have a world map handy) are included in the chapter entitled "Geography(!)", which the author admits "does not belong in the book. Some of the puzzles are mathematical in nature, to be sure, but really they are here because mathematical puzzle solvers seem to enjoy them." Examples include *The Phone Call*: "A phone call is made from an East Coast state to a West Coast

state and it's the same time of day at both ends. How can this be?" and *Department of Odd Names*: "What distinction is held by the point of land called West Quoddy Head, Maine?"

The rest of the puzzles in this collection range from hard to very challenging, and many a solution requires the proof of some conjecture. Although it is conceivable that there are many possible proofs to these problems, most of the "elementary" proofs posed by the author cannot be either determined or understood without a strong mathematical foundation in a wide range of topics from calculus, set theory, and probability to combinatorics, game theory, and college geometry. As can be expected, the level of difficulty encountered in each section of the book is inversely proportional to the reader's comfort level with the topic under consideration. I found the geometrical problems particularly challenging, including a problem called *Line Through Two Points*: "Suppose X is a finite set of points on the plane, not all on one line. Prove that there is a line passing through exactly two points of X ."

Even before the reader reaches the "toughies" and unsolved problems in the last two chapters, he or she is likely to have little hair left to pull out after delving into the realm of game theory and algorithms in the previous four chapters. A few of these puzzles occupied my mind for hours stretching into days, at which point I reluctantly read Peter Winkler's solution, comforting myself for my lack of resolve with the knowledge that I may never have come up with his particular solution on my own.

The determined and sufficiently prepared puzzle solver will derive countless hours of entertainment and enjoyment from Peter Winkler's delightful puzzle collection. Although any lover of mathematical puzzles will find something in this book to tickle his or her fancy, I believe that the difficulty level of the problems contained in the book makes it most suitable for the bright high school or undergraduate mathematics student, or anyone equally well prepared in the art of mathematical proof.

PROBLEMS

Solutions to problems in this issue should arrive no later than **1 May 2005**. An asterisk (*) after a number indicates that a problem was proposed without a solution.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

2976. Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Let $a, b, c \in \mathbb{R}$. Prove that

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq (ab + bc + ca)^3.$$

2977. Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.

Let a_1, a_2, \dots, a_n be positive real numbers, let $r = \sqrt[n]{a_1 a_2 \cdots a_n}$, and let

$$E_n = \frac{1}{a_1(1+a_2)} + \frac{1}{a_2(1+a_3)} + \cdots + \frac{1}{a_n(1+a_1)} - \frac{n}{r(1+r)}.$$

(a) Prove that $E_n \geq 0$ for

(a₁) $n = 3$;

(a₂) $n = 4$ and $r \leq 1$;

(a₃) $n = 5$ and $\frac{1}{2} \leq r \leq 2$;

(a₄) $n = 6$ and $r = 1$.

(b)★ Prove or disprove that $E_n \geq 0$ for

(b₁) $n = 5$ and $r > 0$;

(b₂) $n = 6$ and $r \leq 1$.

2978★. Proposed by Christopher J. Bradley, Bristol, UK.

Let $ABCD$ be a cyclic quadrilateral. The internal bisectors of angles A and B meet at P . Points Q, R, S are similarly defined by a cyclic change of letters. It is easy to show that $PQRS$ is a cyclic quadrilateral. Suppose that the circles $ABCD$ and $PQRS$ have centres O and X , respectively. Let AC meet BD at E . Prove that O, E , and X are collinear. Prove also that $PR \perp QS$.

2979. Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.

$$\text{If } e_n = \left(1 + \frac{1}{n}\right)^n, \text{ find } \lim_{n \rightarrow \infty} \left(\frac{2n(e - e_n)}{e}\right)^n.$$

2980. Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.

Let Γ be a semicircle with centre O on diameter AB . Let C be the mid-point of the semicircular arc \widehat{AB} . Let P be an arbitrary point on the semicircle different from both A and B .

Determine all points Q on the semicircle such that if the lines BP and AQ intersect at a point S , then C is the orthocentre of $\triangle SPQ$.

2981★. Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.

Find all pairs of positive integers a and b such that a divides $b^2 + b + 1$, and b divides $a^2 + a + 1$.

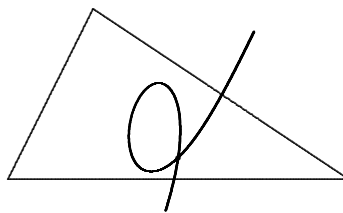
2982★. Proposed by Bruce Sawyer, Memorial University of Newfoundland, St. John's, NL.

In a given triangle ABC , points D, E, F are taken on the sides BC, CA, AB , respectively, such that

$$BD : DC = CE : EA = AF : FB = \frac{1 - \lambda}{\lambda},$$

where λ is a constant. (If $0 < \lambda < 1$, then the points are interior to the sides; if $\lambda < 0$ or $\lambda > 1$, then the points are exterior to the sides; if $\lambda = 0$ or $\lambda = 1$, then the points are coincident with the vertices A, B, C .)

It is easy to see that the centroid of $\triangle DEF$ is a fixed point as λ varies. The curve in the figure is the locus of the circumcentre of $\triangle DEF$ as λ varies. Determine this curve.



2983. Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.

Let $a_1, a_2, \dots, a_n < 1$ be non-negative real numbers satisfying

$$a = \sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}} \geq \frac{\sqrt{3}}{3}.$$

Prove that

$$\frac{a_1}{1 - a_1^2} + \frac{a_2}{1 - a_2^2} + \dots + \frac{a_n}{1 - a_n^2} \geq \frac{na}{1 - a^2}.$$

2984. Proposed by Mihály Bencze, Brasov, Romania.

Prove that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{ij(i+j)} = 2 \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

2985. Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.

Let a and b be real numbers, and let

$$L = \lim_{n \rightarrow \infty} n \left(1 - \frac{a}{n} - \frac{b \ln(n+1)}{n} \right)^n.$$

Prove that

$$L = \begin{cases} \infty & \text{if } b < 1, \\ e^{-a} & \text{if } b = 1, \\ 0 & \text{if } b > 1. \end{cases}$$

2986. Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Given three points A, B, C in the plane such that none of the angles of triangle ABC exceeds 165° , what is the minimum number of circular arcs needed to construct the circumcircle, using only a compass and no straight-edge? (The answer should be fewer than 22.)

2987. Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.

Let Γ be a circle with centre O , and let A, B be two points on the circle. The points A, O , and B determine a minor arc \widehat{AB} with $0 < \theta \leq \pi$, where $\theta = \angle AOB$. Let P be any fixed point in the interior of the minor arc \widehat{AB} , let O_1 be the centre of the circle which is tangent to OA, OP , and the minor arc \widehat{AB} at the point T_1 , and let O_2 be the centre of the circle which is tangent to OB, OP , and the minor arc \widehat{AB} at the point T_2 .

(a) Prove that $\frac{\pi}{2} - \frac{\theta}{4} < \angle O_1 P O_2 < \frac{\pi}{2}$.

(b) Prove that the lines $O_2 T_1, O_1 T_2$, and OP are concurrent.

(c)★ Let S be the intersection of the lines AT_1 and BT_2 ; let K be the intersection of the lines AT_2 and BT_1 ; let H be the intersection of the tangent lines to the arc \widehat{AB} at A and B ; and let T be the intersection of the tangent lines to the arc \widehat{AB} at T_1 and T_2 . Prove that the points S, K, H , and T are collinear.

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2976. *Proposé par Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Soit a, b et $c \in \mathbb{R}$. Montrer que

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq (ab + bc + ca)^3.$$

2977. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit a_1, a_2, \dots, a_n des nombres réels positifs, soit $r = \sqrt[n]{a_1 a_2 \cdots a_n}$, et soit

$$E_n = \frac{1}{a_1(1+a_2)} + \frac{1}{a_2(1+a_3)} + \cdots + \frac{1}{a_n(1+a_1)} - \frac{n}{r(1+r)}.$$

(a) Montrer que $E_n \geq 0$ pour

(a₁) $n = 3$;

(a₂) $n = 4$ et $r \leq 1$;

(a₃) $n = 5$ et $\frac{1}{2} \leq r \leq 2$;

(a₄) $n = 6$ et $r = 1$.

(b)★ Montrer ou réfuter que $E_n \geq 0$ pour

(b₁) $n = 5$ et $r > 0$;

(b₂) $n = 6$ et $r \leq 1$.

2978★. *Proposé par Christopher J. Bradley, Bristol, GB.*

Soit $ABCD$ un quadrilatère cyclique. Les bissectrices intérieures des angles A et B se coupent en P . On définit les points Q, R et S de manière semblable en procédant à un changement cyclique des lettres. Il est facile de montrer que $PQRS$ est un quadrilatère cyclique. Supposons que les cercles $ABCD$ et $PQRS$ ont leur centre en O et X , respectivement. Soit E le point d'intersection de AC avec BD . Montrer que O, E et X sont colinéaires. Montrer aussi que $PR \perp QS$.

2979. *Proposé par Ovidiu Furdui, étudiant, Western Michigan University, Kalamazoo, MI, USA.*

$$\text{Si } e_n = \left(1 + \frac{1}{n}\right)^n, \text{ trouver } \lim_{n \rightarrow \infty} \left(\frac{2n(e - e_n)}{e}\right)^n.$$

2980. *Proposé par Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Province de Guang Dong, Chine.*

Soit Γ un demi-cercle de centre O et de diamètre AB . Soit C le point milieu de l'arc semi-circulaire \widehat{AB} . Soit P un point arbitraire sur le demi-cercle, mais différent aussi bien de A que de B .

Déterminer tous les points Q sur le demi-cercle de sorte que si les droites BP et AQ se coupent en un point S , alors C est l'orthocentre du triangle SPQ .

2981★. *Proposé par Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Province de Guang Dong, Chine.*

Trouver toutes les paires d'entiers positifs a et b de sorte que a divise $b^2 + b + 1$, et b divise $a^2 + a + 1$.

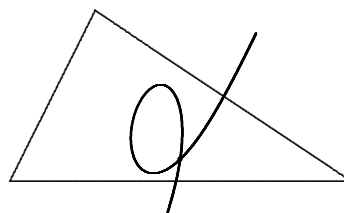
2982★. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Dans un triangle ABC , on choisit des points D , E et F sur les côtés BC , CA et AB , respectivement, de telle sorte que

$$BD : DC = CE : EA = AF : FB = \frac{1 - \lambda}{\lambda},$$

où λ est une constante. (Si $0 < \lambda < 1$, les points se trouvent entre les sommets; si $\lambda < 0$ ou $\lambda > 1$, les points sont sur les prolongements des côtés; si $\lambda = 0$ ou $\lambda = 1$, les points coïncident avec les sommets A , B , C .)

Il est facile de voir que le centre de gravité du triangle DEF est un point fixe lorsque λ varie. La courbe dans la figure est le lieu des centres du cercle circonscrit au triangle DEF lorsque λ varie. Déterminer cette courbe.



2983. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit $a_1, a_2, \dots, a_n < 1$ des nombres réels non-négatifs satisfaisant

$$a = \sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}} \geq \frac{\sqrt{3}}{3}.$$

Montrer que

$$\frac{a_1}{1 - a_1^2} + \frac{a_2}{1 - a_2^2} + \dots + \frac{a_n}{1 - a_n^2} \geq \frac{na}{1 - a^2}.$$

2984. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Montrer que

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{ij(i+j)} = 2 \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

2985. *Proposé par Ovidiu Furdui, étudiant, Western Michigan University, Kalamazoo, MI, USA.*

Soit a et b des nombres réels et soit

$$L = \lim_{n \rightarrow \infty} n \left(1 - \frac{a}{n} - \frac{b \ln(n+1)}{n} \right)^n.$$

Montrer que

$$L = \begin{cases} \infty & \text{si } b < 1, \\ e^{-a} & \text{si } b = 1, \\ 0 & \text{si } b > 1. \end{cases}$$

2986. *Proposé par Peter Y. Woo, Biola University, La Mirada, CA, USA.*

On donne trois points A, B, C dans le plan de sorte qu'aucun des angles du triangle ABC n'excède 165° ; quel est le nombre minimum d'arcs de cercle nécessaires pour construire le cercle circonscrit, en n'utilisant qu'un compas et pas de règle? (La réponse devrait être plus petite que 22.)

2987. *Proposé par Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Province de Guang Dong, Chine.*

Soit Γ un cercle de centre O , et soit A et B deux points sur le cercle. Les points A, O et B déterminent un arc AB d'ouverture AOB mesurée par l'angle θ avec $0 < \theta \leq \pi$. Soit P un point fixé à l'intérieur du petit arc \widehat{AB} , soit O_1 le centre du cercle tangent à la fois à OA , OP , et au petit arc \widehat{AB} au point T_1 , et soit O_2 le centre du cercle tangent à la fois à OB , OP , et au petit arc \widehat{AB} au point T_2 .

(a) Montrer que $\frac{\pi}{2} - \frac{\theta}{4} < \angle O_1 P O_2 < \frac{\pi}{2}$.

(b) Montrer que les droites $O_2 T_1$, $O_1 T_2$ et OP sont concourantes.

(c)★ Soit S l'intersection des droites AT_1 et BT_2 ; soit K l'intersection des droites AT_2 et BT_1 ; soit H l'intersection des tangentes à l'arc \widehat{AB} en A et B ; et soit T l'intersection des tangentes à l'arc \widehat{AB} en T_1 et T_2 . Montrer que les points S, K, H et T sont colinéaires.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2872. [2003 : 400] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given an acute angled triangle ABC with orthocentre H and circumcentre O , suppose that D is a point on the side AC such that $CD = 2AD$, that DO meets BC at E , and that $HO \parallel BC$. Prove that

(a) $DO = OE$, and (b) $DE = CE$.

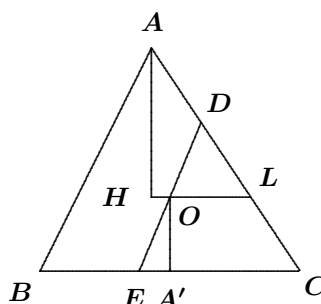
Solution by Michael Parmenter, Memorial University of Newfoundland, St. John's, NL.

Extend HO to meet AC at L . Let A' be the mid-point of BC . Since $AH \perp BC$ and $OA' \perp BC$, we deduce that $AH \parallel OA'$. It is well known that $|AH| = 2|OA'|$ (see [1]). Since $AH \parallel OA'$ and $HO \parallel BC$, we conclude that $|AL| = 2|LC|$. Thus,

$$|AD| = |DL| = |LC|.$$

Since $OL \parallel EC$ and $|DL| = |LC|$, we obtain $|OD| = |OE|$, as required for (a).

Note that $|AO| = |OC|$, because O is the circumcentre. Consequently, $\angle OCA = \angle OAC$. Since we also have $|AL| = |CD|$, it follows that $\triangle AOL$ is congruent to $\triangle COD$. Then $|OL| = |OD|$, and hence $\angle OLD = \angle ODL$. But $\angle OLD = \angle ECD$, since $OL \parallel EC$. Therefore, $\angle EDC = \angle ODL = \angle ECD$. This implies that $|EC| = |ED|$, as required for (b).



Reference

[1] Arthur Baragar, *A Survey of Classical and Modern Geometries: with Computer Activities*, Prentice Hall, 2001, pp. 44–45.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, student, New York University, NY, USA; ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

For this problem, the complexity of the solution seemed to increase in proportion to the extent to which trigonometry or vectors were used.

2873. [2003 : 400] *Proposed by Kee-Wai Lau, Hong Kong, China.*

Find all positive integers n such that the system of equations

$$\begin{aligned}x + y + z &= 3, \\x^2 + y^2 + z^2 &= 3, \\x^n + y^n + z^n &= 3.\end{aligned}$$

has the unique solution $x = y = z = 1$.

Solution by Michel Bataille, Rouen, France.

We show that the given system has the unique solution $x = y = z = 1$ if and only if $n = 3$, $n = 4$, or $n = 5$.

For $n = 1$ or $n = 2$, the system reduces to $x + y + z = 3$, and $x^2 + y^2 + z^2 = 3$. It is readily checked that $(x, y, z) = (2, 1 + \omega, 1 + \omega^2)$ is also a solution when $\omega = e^{\frac{2\pi i}{3}}$.

Suppose now that $n \geq 3$, and let (x, y, z) be a solution to the system. The numbers x, y, z are the roots of the polynomial

$$p(X) = X^3 - \sigma_1 X^2 + \sigma_2 X - \sigma_3,$$

where $\sigma_1 = x + y + z = 3$, $\sigma_2 = xy + yz + zx = \frac{1}{2}(\sigma_1^2 - (x^2 + y^2 + z^2)) = 3$, and $\sigma_3 = xyz$. Thus,

$$p(X) = X^3 - 3X^2 + 3X - \sigma_3.$$

Let $s_n = x^n + y^n + z^n$. Using the fact that $p(x) = x^3 - 3x^2 + 3x - \sigma_3 = 0$, we obtain $x^3 = 3x^2 - 3x + \sigma_3$. By substituting $X = y$ and $X = z$, we have

$$\begin{aligned}s_3 &= (3x^2 - 3x + \sigma_3) + (3y^2 - 3y + \sigma_3) + (3z^2 - 3z + \sigma_3) = 3\sigma_3, \\s_4 &= (3x^3 - 3x^2 + \sigma_3 x) + (3y^3 - 3y^2 + \sigma_3 y) + (3z^3 - 3z^2 + \sigma_3 z) \\&= 12\sigma_3 - 9,\end{aligned}$$

and similarly,

$$s_5 = 30\sigma_3 - 27.$$

When the third equation of the system is $x^n + y^n + z^n = 3$ with $n = 3$, $n = 4$, or $n = 5$, it follows that $\sigma_3 = 1$. Hence, $p(X) = (X - 1)^3$. Thus, the solution is necessarily $x = y = z = 1$.

Now suppose $n \geq 6$ and let $m = \lfloor n/3 \rfloor$. Note that $m \geq 2$. Choose a complex root r of the polynomial

$$q_n(X) = \binom{n}{3m} X^{m-1} + \binom{n}{3m-3} X^{m-2} + \cdots + \binom{n}{6} X + \binom{n}{3},$$

and let ρ be a cubic root of r in \mathbb{C} . Note that $\rho \neq 0$.

We prove that $x = 1 + \rho$, $y = 1 + \rho\omega$, $z = 1 + \rho\omega^2$ is a solution of the system. (Thus, the system has a solution other than $x = y = z = 1$).

Clearly, $x + y + z = 3 + \rho(1 + \omega + \omega^2) = 3$. Similarly, $x^2 + y^2 + z^2 = 3$. Observe that $1 + \omega^k + \omega^{2k} = 3$ if $k \equiv 0 \pmod{3}$ and $1 + \omega^k + \omega^{2k} = 0$ if $k \equiv 1 \pmod{3}$ or $k \equiv 2 \pmod{3}$. In addition to satisfying the first two equations of the system, we have

$$\begin{aligned} x^n + y^n + z^n &= (1 + \rho)^n + (1 + \rho\omega)^n + (1 + \rho\omega^2)^n \\ &= 3 + \sum_{k=1}^n \binom{n}{k} \rho^k (1 + \omega^k + \omega^{2k}) \\ &= 3 + 3 \sum_{j=1}^m \binom{n}{3j} \rho^{3j} = 3 + 3rq_n(r). \end{aligned}$$

Therefore, $x^n + y^n + z^n = 3$ since $q_n(r) = 0$. The result follows.

Also solved by CHRISTOPHER J. BRADLEY, Bristol, UK; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; JOEL SCHLOSBERG, student, New York University, NY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

[Ed: If one assumes that x , y , and z are real, then the solution $x = y = z = 1$ is the unique solution for all n . Solutions of this fact were submitted by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; and KENNETH M. WILKE, Topeka, KS, USA.]

2874. [2003 : 401] Proposed by Vedula N. Murty, Dover, PA, USA.

Let a , b , and c denote the side lengths BC , CA , and AB , respectively, of triangle ABC , and let s , r , and R denote the semi-perimeter, inradius, and circumradius of the triangle, respectively. Let $y = s/R$ and $x = r/R$.

Show that

1. $\sum_{\text{cyclic}} \sin^2 A = 2 \iff y - x = 2 \iff \triangle ABC \text{ is right-angled;}$
2. $\sum_{\text{cyclic}} \sin^2 A > 2 \iff y - x > 2 \iff \triangle ABC \text{ is acute-angled;}$
3. $\sum_{\text{cyclic}} \sin^2 A < 2 \iff y - x < 2 \iff \triangle ABC \text{ is obtuse-angled.}$

I. Comment by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

These are fairly familiar results, and can be found, together with some other equivalent characterizations, in items 11.26–11.28 on page 102 of [1].

[Ed.: The proposed results were obtained by C. Ciamberlini in 1943.]

II. *Comment by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

All twelve implications follow immediately from the known identities:

$$\begin{aligned}\sum_{\text{cyclic}} \sin^2 A &= 2 + 2 \cos A \cos B \cos C \\ &= 2 + \frac{(s - 2R - r)(s + 2R + r)}{2R^2}.\end{aligned}$$

III. *Solution by Titu Zvonaru, Bucharest, Romania (abridged by the editor).*

In [2003 : 82–83], the proposer of this problem established the identity below:

$$y^2 - (x + 2)^2 = 4 \cos A \cos B \cos C.$$

Since it is well known that

$$\sum_{\text{cyclic}} \sin^2 A = 2 + 2 \cos A \cos B \cos C,$$

the proposed implications follow immediately from these two identities.

Reference.

[1] O. Bottema et al., *Geometric Inequalities*, Groningen, 1969.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; JOE HOWARD, Portales, NM, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; PANOS E. TSAOUSSOGLU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2875. [2003 : 401] *Proposed by Michel Bataille, Rouen, France.*

Suppose that the incircle of $\triangle ABC$ is tangent to the sides BC , CA , AB , at D , E , F , respectively.

Prove that $EF^2 + FD^2 + DE^2 \leq \frac{s^2}{3}$, where s is the semiperimeter of $\triangle ABC$.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

We use two inequalities from the book [1]:

$$s^2 \geq 27r^2 \quad (5.11 \text{ in } [1]) \quad (1)$$

$$a^2 + b^2 + c^2 \leq 9r^2 \quad (5.13 \text{ in } [1]), \quad (2)$$

where a , b , and c are the sides of $\triangle ABC$ and r is its inradius. We note that the incircle of $\triangle ABC$ is the circumcircle of $\triangle DEF$. Applying the inequality (2) to $\triangle DEF$, we obtain

$$EF^2 + FD^2 + DE^2 \leq 9r^2. \quad (3)$$

The inequalities (1) and (3) give

$$EF^2 + FD^2 + DE^2 \leq \frac{s^2}{3}.$$

Reference.

[1] O. Bottema et al., *Geometric Inequalities*, Groningen, 1969

Also solved by ARKADY ALT, San Jose, CA, USA (2 solutions); MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CON AMORE PROBLEM GROUP, The Danish University of Education Copenhagen, Denmark; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; VEDULA N. MURTY, Dover, PA, USA; JOEL SCHLOSBERG, student, New York University, NY, USA; ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

Janous showed that the general inequality $EF^\lambda + FD^\lambda + DE^\lambda \leq (s^\lambda/3^{\lambda-1})$ is true for $0 < \lambda \leq 6$ and has conjectured that it is true for all $\lambda > 0$.

2876. [2003 : 464] Proposed by Toshio Seimiya, Kawasaki, Japan.

Given $\triangle ABC$ with incentre I and circumcentre O , suppose that M is the mid-point of BC , and that $IO \perp AM$.

$$\text{Prove that } \frac{2}{BC} = \frac{1}{AB} + \frac{1}{AC}.$$

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.

Let $AB = c$, $BC = a$, and $CA = b$ be the sides of the triangle, and let R , r , and s be its circumradius, inradius, and semiperimeter, respectively. Without loss of generality, we may assume that $\angle B > \angle C$. The quadrilateral $AIMO$ has perpendicular diagonals, so that its sides satisfy the relation

$$AI^2 + OM^2 = IM^2 + AO^2.$$

Since

$$\begin{aligned} AI^2 &= r^2 + (s-a)^2, & OM^2 &= R^2 - \frac{a^2}{4}, & IM^2 &= R^2, \\ AO^2 &= r^2 + \left(\frac{a}{2} - (s-b)\right)^2 = r^2 + \frac{(b-c)^2}{4}, \end{aligned}$$

we obtain

$$r^2 + (s-a)^2 + R^2 - \frac{a^2}{4} = R^2 + r^2 + \frac{(b-c)^2}{4},$$

which is equivalent to

$$\frac{2}{a} = \frac{1}{b} + \frac{1}{c}.$$

This completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2877. [2003 : 464] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let O be an interior point of $\triangle ABC$ such that $AB + BO = AC + CO$. Suppose that P is a variable point on the side BC , and that Q and R are points on AB and AC , respectively, such that $PQ \parallel CO$ and $PR \parallel BO$.

Prove that the perimeter of quadrilateral $AQPR$ is constant.

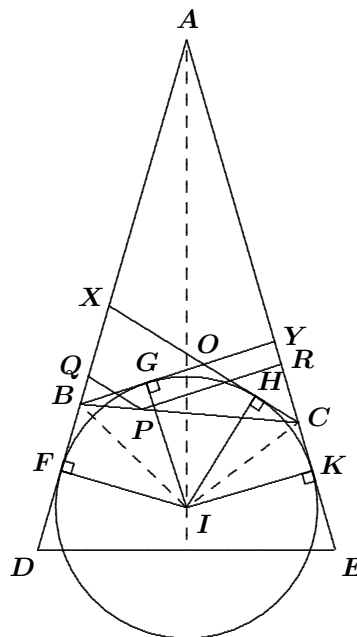
Solution by the proposer.

Let D be a point on AB beyond B such that $BD = BO$, and let E be a point on AC beyond C such that $CE = CO$. Since $AB + BO = AC + CO$, we have $AD = AE$. Since $BD = BO$, $CE = CO$, and $AD = AE$, the bisectors of angles DBO , ECO , and DAE are the perpendicular bisectors of DO , EO , and DE , respectively. These three angle bisectors are concurrent at the circumcentre I of $\triangle ODE$.

Let F , G , H , and K be the feet of the perpendiculars from point I to DB , BO , OC , and CE , respectively. Then $IF = IG = IH = IK$. Let Γ be the circle with centre I and radius IG . Clearly, the circle Γ touches the lines DB , BO , OC , and CE at points F , G , H , and K , respectively. Let X and Y be the intersection points of the lines CO and BO with the sides AB and AC , respectively. Clearly, Γ is the common excircle of triangles ABY and AXC .

It is well known that

$$AB + BY + YA = 2AF \quad \text{and} \quad AX + XC + CA = 2AK.$$



Since $AF = AK$, we have $AB + BY + YA = AX + XC + CA$, from which we get

$$BX + BY = CX + CY \quad (1)$$

We put $\frac{BP}{BC} = x$, so that $\frac{PC}{BC} = 1 - x$. Since $PQ \parallel CX$, we have

$$\frac{XQ}{XB} = \frac{CP}{CB} = 1 - x \quad \text{and} \quad \frac{PQ}{CX} = \frac{BP}{BC} = x.$$

Thus, $XQ = (1 - x)XB$ and $PQ = xCX$. Since $PR \parallel BY$, we have

$$\frac{RY}{CY} = \frac{PB}{CB} = x \quad \text{and} \quad \frac{PR}{BY} = \frac{PC}{BC} = 1 - x,$$

so that $RY = xCY$ and $PR = (1 - x)BY$. Consequently,

$$\begin{aligned} XQ + QP + PR + RY &= (1 - x)XB + xCX + (1 - x)BY + xCY \\ &= XB + BY + x(CX + CY - BX - BY). \end{aligned}$$

Using (1), we get

$$XQ + QP + PR + RY = XB + BY.$$

Adding $AX + AY$ to both sides of the last equation, we obtain

$$\begin{aligned} AQ + QP + PR + RA &= AX + AY + XB + BY \\ &= AB + AY + BY, \end{aligned}$$

which is clearly a constant.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

2878. [2003 : 464] Proposed by Toshio Seimiya, Kawasaki, Japan.

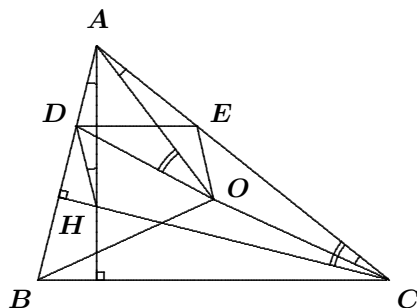
Given an acute triangle ABC with orthocentre H and circumcentre O , suppose that the perpendicular bisector of AH meets AB and AC at D and E , respectively.

Prove that A is an excentre of $\triangle ODE$.

Solution by the proposer.

Since $AH \perp BC$, we have $\angle BAH = 90^\circ - \angle ABC$. Since DE is the perpendicular bisector of AH , we have $DA = DH$, so that

$$\begin{aligned} \angle DHA &= \angle DAH \\ &= \angle BAH \\ &= 90^\circ - \angle ABC. \end{aligned} \quad (1)$$



Since O is the circumcentre of $\triangle ABC$, we have $OA = OC$. Since $\angle AOC = 2\angle ABC$, we have

$$\begin{aligned}\angle OAC &= \angle OCA = \frac{1}{2}(180^\circ - \angle AOC) = \frac{1}{2}(180^\circ - 2\angle ABC) \\ &= 90^\circ - \angle ABC.\end{aligned}\quad (2)$$

From (1) and (2), we have $\triangle ADH \sim \triangle AOC$. Since $\triangle ADH$ and $\triangle AOC$ are directly similar, we have $\triangle ADO \sim \triangle AHC$. Hence,

$$\angle AOD = \angle ACH = 90^\circ - \angle BAC.$$

Since O is the circumcentre of $\triangle ABC$, we have

$$\begin{aligned}\angle OAB &= \angle OBA = \frac{1}{2}(180^\circ - \angle AOB) = \frac{1}{2}(180^\circ - 2\angle ACB) \\ &= 90^\circ - \angle ACB.\end{aligned}$$

Thus,

$$\begin{aligned}\angle ODB &= \angle AOD + \angle OAB = (90^\circ - \angle BAC) + (90^\circ - \angle ACB) \\ &= 180^\circ - (\angle BAC + \angle ACB) = \angle ABC.\end{aligned}\quad (3)$$

Since $DE \perp AH$ and $BC \perp AH$, we have $DE \parallel BC$. Therefore,

$$\angle ADE = \angle ABC.\quad (4)$$

From (3) and (4), we have $\angle ODB = \angle ADE$. Thus, AD is the exterior bisector of $\angle ODE$. Similarly, AE is the exterior bisector of $\angle OED$. Thus, A is an excentre of $\triangle ODE$.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU, Bucharest, Romania.

Only the proposer gave a purely classical geometric solution. Bataille's solution was the only other purely geometric solution. Janous made use of coordinates and the others made use of trigonometry.

2879. [2003 : 465] *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

Let D , E , F be arbitrary points on the sides BC , CA , AB , respectively, of triangle ABC . By the theorem of Miquel, the circles AEF , BFD , CDE are concurrent at O , say.

Let P be an arbitrary point in the plane of $\triangle ABC$, and let A' , B' , C' be the second points of intersection of PA , PB , PC with the circles AEF , BFD , CDE , respectively.

Prove that O , P , A' , B' , and C' are concyclic.

Solution by Toshio Seimiya, Kawasaki, Japan.

In triangle PAB the circles AFA' , BFB' , and $PA'B'$ are concurrent at O by Miquel's Theorem; thus P , A' , B' , and O are concyclic. Similarly, in triangle PBC , the circles BDB' , CDC' , and $PB'C'$ are concyclic at O . Therefore, O , P , A' , B' , and C' are concyclic.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Most solvers made repeated use of Euclid III.21 and 22: A quadrangle $WXYZ$ is cyclic if and only if the directed angles WXZ and WYZ are equal. (These solutions implicitly proved Miquel's Theorem twice; thus, they essentially reduce to the featured solution.) Bradley included a nice corollary that follows easily from the more detailed approach: If D , E , and F are the mid-points of the sides of $\triangle ABC$ (so that O is the circumcentre), then for any point P , we see that OP is a diameter of the circle $A'B'C'$.

2880. [2003 : 465] *Proposed by Mihály Bencze, Brasov, Romania.*

1. If $x, y, z > 1$, prove that

$$(a) \quad (\log_{yz} x^4 yz) (\log_{zx} xy^4 z) (\log_{xy} xyz^4) > 25,$$

$$(b) \star (\log_{yz} x^4 yz) (\log_{zx} xy^4 z) (\log_{xy} xyz^4) \geq 27.$$

2. \star If $x_k > 1$ ($k = 1, 2, \dots, n$) and $\alpha \geq -1$, prove that

$$\prod_{k=1}^n \log_{b_k} b_k x_k^{\alpha+1} \geq \left(\frac{n+\alpha}{n-1} \right)^n,$$

where $b_k = x_1 \cdots x_{k-1} x_{k+1} \cdots x_n$.

I. Solution by Murray S. Klamkin, University of Alberta, Edmonton, AB.

First we note that $\log_b x = \ln x / \ln b$.

Let $x = e^a$, $y = e^b$, and $z = e^c$, where $a, b, c > 0$. Some algebra reduces the inequality in 1(a) to

$$(T_1 + 3a)(T_1 + 3b)(T_1 + 3c) > \lambda(b+c)(c+a)(a+b),$$

where $T_1 = a + b + c$ and $\lambda = 25$. Letting $T_2 = bc + ca + ab$ and $T_3 = abc$, the inequality becomes

$$4T_1^3 > (\lambda - 9)T_1T_2 - (\lambda + 27)T_3. \quad (1)$$

Schur's Inequality $T_1^3 \geq 4T_1T_2 - 9T_3$ is the best of this class of inequalities. Therefore, in order for inequality (1) to be valid, we must have

$$16T_1T_2 - 36T_3 > (\lambda - 9)T_1T_2 - (\lambda + 27)T_3;$$

that is, $(25 - \lambda)T_1T_2 + (\lambda - 9)T_3 > 0$. It follows that the best possible λ is 25.

To show directly that λ is at most 25 in 1(a) above, just let $a = 1$ and $b = 1$, and let c be arbitrarily small. This shows that 1(b) is not valid. Then the inequality in 2 is not valid in general, since the special case where $n = 3$ and $\alpha = 3$ is the same as 1(b).

II. *Solution by Vasile Cîrtoaje, University of Ploiesti, Romania.*

The inequality in 1(a) is true, but 1(b) is false. The inequality in 2 is true only in the case $n = 2$. For $n \geq 3$, it is true only for $-1 \leq \alpha \leq \alpha_1$, where α_1 is the positive real root of the following equation in α :

$$\left(1 + \frac{\alpha + 1}{n - 2}\right)^{n-1} = \left(1 + \frac{\alpha + 1}{n - 1}\right)^n.$$

We have $\alpha_1 = \sqrt{5}$ for $n = 3$ and $\alpha_1 < 2$ for $n \geq 4$. Also, $\lim_{n \rightarrow \infty} \alpha_1(n) = 1$.

Letting $a_k = \ln x_k$, the inequality in 2 may be transformed into

$$\prod_{k=1}^n \left(1 + \frac{\beta a_k}{s - a_k}\right) \geq \left(1 + \frac{\beta}{n - 1}\right)^n,$$

where $s = a_1 + a_2 + \dots + a_n$ and $\beta = \alpha + 1 \geq 0$. Without loss of generality, we may assume that $s = 1$ (since the right side of the inequality does not involve a_k , while the left side is homogeneous in a_k).

In place of the above inequality, we give the following theorem.

Theorem. If a_1, a_2, \dots, a_n are non-negative numbers such that $\sum_{k=1}^n a_k = 1$, then, for any $\beta \geq 0$,

$$\prod_{k=1}^n \left(1 + \frac{\beta a_k}{1 - a_k}\right) \geq \min \left\{ \left(1 + \frac{\beta}{n - 1}\right)^n, \left(1 + \frac{\beta}{n - 2}\right)^{n-1}, \dots, (1 + \beta)^2 \right\}.$$

Proof. Without loss of generality, assume that $a_1 \leq a_2 \leq \dots \leq a_n$. Let $F(a_1, a_2, \dots, a_n)$ be the left side of the inequality in the theorem. The set of allowed values of (a_1, a_2, \dots, a_n) is a compact set in \mathbb{R}^n , and the function F is continuous on this set. Consequently, F attains a minimum value at some point of the set. We will show that there is some $i \in \{0, 1, \dots, n - 2\}$ such that F attains its minimum when

$$a_k = \begin{cases} 0 & \text{if } 1 \leq k \leq i, \\ \frac{1}{n - i} & \text{if } i < k \leq n. \end{cases}$$

To prove this, we suppose that there is some $j \in \{1, 2, \dots, n - 1\}$ such that $0 < a_j < a_n$. We will show that F is not minimal at (a_1, a_2, \dots, a_n) by showing that F decreases either when a_j and a_n are replaced by 0 and

$a_j + a_n$, or else when a_j and a_n are both replaced by $\frac{1}{2}(a_j + a_n)$. For this, we need to show that at least one of the following inequalities holds:

$$\left(1 + \frac{\beta a_j}{1 - a_j}\right) \left(1 + \frac{\beta a_n}{1 - a_n}\right) > 1 + \frac{\beta(a_j + a_n)}{1 - a_j - a_n}, \quad (2)$$

$$\left(1 + \frac{\beta a_j}{1 - a_j}\right) \left(1 + \frac{\beta a_n}{1 - a_n}\right) > \left(1 + \frac{\beta(a_j + a_n)}{2 - a_j - a_n}\right)^2. \quad (3)$$

Using the notations $a_j + a_n = 2t$ and $\sqrt{a_j a_n} = p$, we have

$$\begin{aligned} & \left(1 + \frac{\beta a_j}{1 - a_j}\right) \left(1 + \frac{\beta a_n}{1 - a_n}\right) - \left(1 + \frac{\beta(a_j + a_n)}{1 - a_j - a_n}\right) \\ &= \frac{1 + 2(\beta - 1)t + (\beta - 1)^2 p^2}{1 - 2t + p^2} - \frac{1 + 2(\beta - 1)t}{1 - 2t} \\ &= \frac{\beta p^2 (\beta - 2 - 2(\beta - 1)t)}{(1 - 2t + p^2)(1 - 2t)}. \end{aligned}$$

Hence, if $\beta - 2 > 2(\beta - 1)t$, then inequality (2) holds.

Similarly, we have

$$\begin{aligned} & \left(1 + \frac{\beta a_j}{1 - a_j}\right) \left(1 + \frac{\beta a_n}{1 - a_n}\right) - \left(1 + \frac{\beta(a_j + a_n)}{2 - a_j - a_n}\right)^2 \\ &= \frac{1 + 2(\beta - 1)t + (\beta - 1)^2 p^2}{1 - 2t + p^2} - \frac{(1 + (\beta - 1)t)^2}{(1 - t)^2} \\ &= \frac{\beta[2(\beta - 1)t - \beta + 2](t^2 - p^2)}{(1 - 2t + p^2)(1 - t)^2}. \end{aligned}$$

Hence, if $\beta - 2 < 2(\beta - 1)t$, then inequality (3) holds.

Thus, either (2) or (3) is true, except when $\beta - 2 = 2(\beta - 1)t$; that is, when $a_j + a_n = (\beta - 2)/(\beta - 1)$. Then equality holds in both (2) and (3). To handle this case, we first replace both a_j and a_n by $\frac{1}{2}(a_j + a_n)$, and then we apply the above argument to $\frac{1}{2}(a_j + a_n)$ and another $a_i > 0$. ■

Now we will see how the minimum that occurs on the right side of the inequality in the theorem may be simplified in various special cases. We will state our results as corollaries.

Corollary 1. Let a, b, c be non-negative numbers such that $a + b + c = 1$.

(a) If $0 \leq \beta \leq 1 + \sqrt{5}$, then

$$\left(1 + \frac{\beta a}{1 - a}\right) \left(1 + \frac{\beta b}{1 - b}\right) \left(1 + \frac{\beta c}{1 - c}\right) \geq \left(1 + \frac{1}{2}\beta\right)^3.$$

(b) If $\beta \geq 1 + \sqrt{5}$, then

$$\left(1 + \frac{\beta a}{1 - a}\right) \left(1 + \frac{\beta b}{1 - b}\right) \left(1 + \frac{\beta c}{1 - c}\right) \geq (1 + \beta)^2.$$

Proof. For $0 \leq \beta \leq 1 + \sqrt{5}$, we have $(1 + \frac{1}{2}\beta)^3 \leq (1 + \beta)^2$; for $\beta \geq 1 + \sqrt{5}$, we have $(1 + \frac{1}{2}\beta)^3 \geq (1 + \beta)^2$. ■

Corollary 2. If a_1, a_2, \dots, a_n are non-negative numbers such that $\sum_{k=1}^n a_k = 1$, then, for $0 \leq \beta \leq 2$,

$$\prod_{k=1}^n \left(1 + \frac{\beta a_k}{1 - a_k}\right) \geq \left(1 + \frac{\beta}{n-1}\right)^n.$$

Proof. The result is trivial for $\beta = 0$. Assuming that $0 < \beta \leq 2$, we will show that

$$\left(1 + \frac{\beta}{n-1}\right)^n \leq \left(1 + \frac{\beta}{n-2}\right)^{n-1} \leq \dots \leq (1 + \beta)^2,$$

or equivalently,

$$n \ln \left(1 + \frac{\beta}{n-1}\right) \leq (n-1) \ln \left(1 + \frac{\beta}{n-2}\right) \leq \dots \leq 2 \ln(1 + \beta).$$

To show this, it suffices to prove that the function $f(x) = (x+1) \ln\left(1 + \frac{\beta}{x}\right)$ is decreasing for $x \geq 1$. We have

$$f'(x) = \ln\left(1 + \frac{\beta}{x}\right) - \frac{\beta(x+1)}{x(x+\beta)} \quad \text{and} \quad f''(x) = \frac{\beta(\beta + (2-\beta)x)}{x^2(x+\beta)^2}.$$

For $x > 0$, we have $f''(x) > 0$, which implies that $f'(x)$ is strictly increasing. Since $f'(x) \rightarrow 0$ as $x \rightarrow \infty$, it follows that $f'(x) < 0$, and then $f(x)$ is strictly decreasing. ■

We could also prove Corollary 2 by applying Jensen's Inequality to the function $g(x) = \ln\left(1 + \frac{\beta x}{1-x}\right)$. We have

$$g''(x) = \frac{\beta[(2-\beta)(1-x) + \beta x]}{(1-x)^2[1 + (\beta-1)x]^2},$$

which is positive for $0 < x < 1$. Therefore, g is convex on the interval $[0, 1]$.

For $n \geq 4$, the inequality in Corollary 2 is valid for $0 \leq \beta \leq \beta_2$, where $\beta_2 \in (2, 3)$ is the positive real root of the following equation in β :

$$\left(1 + \frac{\beta}{n-2}\right)^{n-1} = \left(1 + \frac{\beta}{n-1}\right)^n,$$

and has the property $\lim_{n \rightarrow \infty} \beta_2(n) = 2$.

Corollary 3. If $n \geq 3$ and a_1, a_2, \dots, a_n are non-negative numbers such that $\sum_{k=1}^n a_k = 1$, then, for $\beta \geq 1 + \sqrt{5}$,

$$\prod_{k=1}^n \left(1 + \frac{\beta a_k}{1 - a_k}\right) \geq (1 + \beta)^2.$$

Proof. Assuming that $\beta \geq 1 + \sqrt{5}$, we will show that

$$\left(1 + \frac{\beta}{n-1}\right)^n \geq \left(1 + \frac{\beta}{n-2}\right)^{n-1} \geq \cdots \geq \left(1 + \frac{\beta}{2}\right)^3 \geq (1 + \beta)^2.$$

We can show the last of these inequalities simply by noting that

$$\left(1 + \frac{\beta}{2}\right)^3 - (1 + \beta)^2 = \frac{1}{8}\beta(\beta - 1 - \sqrt{5})(\beta - 1 + \sqrt{5}) \geq 0.$$

The remaining inequalities may be written equivalently as

$$n \ln\left(1 + \frac{\beta}{n-1}\right) \geq (n-1) \ln\left(1 + \frac{\beta}{n-2}\right) \geq \cdots \geq 3 \ln\left(1 + \frac{\beta}{2}\right).$$

To prove these inequalities, we consider the same function as in the proof of Corollary 2; that is, $f(x) = (x+1) \ln\left(1 + \frac{\beta}{x}\right)$. The derivatives $f'(x)$ and $f''(x)$ were given in the proof of Corollary 2. It will suffice to prove that $f(x)$ is increasing for $x \geq 2$.

We see that $f''(x) > 0$ for $0 < x < x_1$, and $f''(x) < 0$ for $x > x_1$, where $x_1 = \beta/(\beta - 2)$. Hence, $f'(x)$ is strictly increasing for $x \in (0, x_1]$ and strictly decreasing for $x \in [x_1, \infty)$. Moreover, $\lim_{x \rightarrow 0^+} f'(x) = -\infty$ and $\lim_{x \rightarrow \infty} f'(x) = 0$. Therefore, there exists $x_2 \in (0, x_1)$ such that $f'(x_2) = 0$, $f'(x) < 0$ for $x \in (0, x_2)$, and $f'(x) > 0$ for $x \in (x_2, \infty)$. If we can show that $f'(2) > 0$, then we will have $f'(x) > 0$ for $x \geq 2$, which implies that $f(x)$ is strictly increasing for $x \geq 2$.

We calculate $f'(2) = \ln\left(1 + \frac{\beta}{2}\right) - \frac{3\beta}{2(\beta+2)}$. This expression is easily shown to be a strictly increasing function of β for $\beta \geq 1$. Since we have $\beta \geq 1 + \sqrt{5} > 3$, we may take $\beta = 3$ in the expression to deduce that

$$f'(2) > \ln\left(1 + \frac{3}{2}\right) - \frac{3(3)}{2(3+2)} = \ln\left(\frac{5}{2}\right) - \frac{9}{10} \approx 0.016.$$

We conclude that $f'(2) > 0$. ■

Corollary 4. If $n \geq 3$ and a_1, a_2, \dots, a_n are non-negative numbers such that

$$\sum_{k=1}^n a_k = 1, \text{ then}$$

$$\prod_{k=1}^n \left(1 + \frac{\beta a_k}{1 - a_k}\right) \geq \left(1 + \frac{\beta}{2}\right)^3$$

for $\beta_1 \leq \beta \leq 1 + \sqrt{5}$, where β_1 is the positive real root of the equation $8\beta^3 + 15\beta^2 - 54\beta - 108 = 0$.

Proof. This proof is like the proof of Corollary 3. Assuming $\beta \leq 1 + \sqrt{5}$, we have $\left(1 + \frac{\beta}{2}\right)^3 \leq (1 + \beta)^2$. Assuming further that $\beta \geq \beta_1$, we have

$8\beta^3 + 15\beta^2 - 54\beta - 108 \geq 0$; that is, $\left(1 + \frac{\beta}{3}\right)^4 \geq \left(1 + \frac{\beta}{2}\right)^3$. It remains to prove that

$$\left(1 + \frac{\beta}{n-1}\right)^n \geq \left(1 + \frac{\beta}{n-2}\right)^{n-1} \geq \cdots \geq \left(1 + \frac{\beta}{3}\right)^4.$$

As in the proof of Corollary 3, we consider $f(x) = (x+1)\ln\left(1 + \frac{\beta}{x}\right)$. We find that $f'(3) = \ln\left(1 + \frac{\beta}{3}\right) - \frac{4\beta}{3(\beta+3)}$ and that we want $f'(3) > 0$ in order to complete the proof. The expression for $f'(3)$ can be shown to be a strictly increasing function of β for $\beta \geq 1$. Since we have $\beta \geq \beta_1 > \frac{5}{2}$, we may take $\beta = \frac{5}{2}$ in the expression to deduce that

$$f'(3) > \ln\left(1 + \frac{5/2}{3}\right) - \frac{4(5/2)}{3(5/2+3)} = \ln\left(\frac{11}{6}\right) - \frac{20}{33} \approx 7.5 \times 10^{-5}.$$

We conclude that $f'(3) > 0$. ■

—Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; PIERRE BORNSZTEIN, Maisons-Laffitte, France; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution.

Guersenzvaig discovered and proved results very similar to those of Cîrtoaje given above. Unfortunately, we could only present one of these excellent solutions.

Crux Mathematicorum with Mathematical Mayhem

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