Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

Volume 43 2010/2011 Number 3

- Dual Rectangles
- Perfect Numbers
- Fibonacci Numbers and Pascal's Triangle

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year and consists of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor

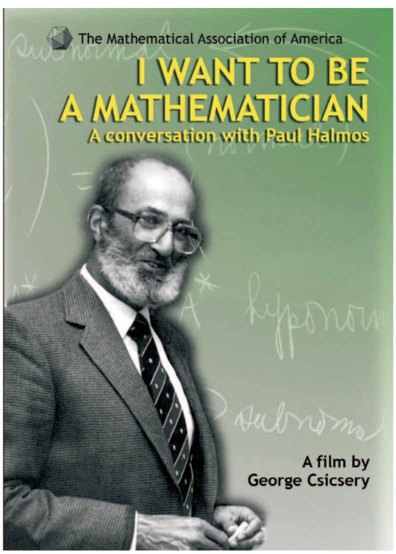
I want to be a mathematician

When I was a kid, the standard reply when asked what you wanted to be when you grew up was: an engine driver. But a mathematician! A DVD has come our way from the Mathematical Association of America of a conversation with one of the best known mathematicians of his age, Paul Halmos (1916–2006). Generations of mathematics students will have been introduced to abstract algebra by Paul Halmos' book *Finite Dimensional Vector Spaces*. In this conversation we catch a glimpse of what made Paul Halmos such an inspiring teacher. He used what he terms 'The Moore method', after the celebrated R. L. Moore but, as he acknowledged, it goes back to Socrates. In a nutshell, you don't say, you ask. Instead of telling his students, or answering the questions they asked, he unnerved them by asking them questions. But it stimulated them to think for themselves. Many teachers at all levels will be familiar with this. Even in a classroom, or a lecture theatre, this technique is one to be aimed for. Paul Halmos lamented the spoonfeeding in schools, colleges, and universities.

Those familiar with the name Paul Halmos will find this conversation of interest. Even those who aren't will gain an insight into what makes a mathematician tick. It may even encourage you, when asked what you want to be, to reply 'I want to be a mathematician'!

Reference

1 G. Csicsery, *I Want to Be a Mathematician: A Conversation with Paul Halmos* (DVD), (MAA, Washington, DC, 2009).



Summing the Reciprocals of Particular Types of Integers

MARTIN GRIFFITHS

1. Introduction

This article arose as a result of a slight coincidence. I recently asked the students in one of my classes each to devise a sequence possessing no obvious formula for the nth term but for which the terms were nonetheless obtained via some strict mathematical rule. A matter of just two or three days later I was reading two related articles (see references 1 and 2) when I realised that they were vaguely connected to a sequence that had been proposed by one of the students. Before going any further, I invite the reader to characterise the numbers in the following sequence:

$$8, 24, 27, 40, 54, 56, 72, 88, 104, 108, 120, 125, \ldots . \tag{1}$$

In references 1 and 2, the authors considered the series of reciprocals of the kth powers of the square-free integers. The numbers in (1) are the positive integers for which 3 is the highest power of any prime appearing in their prime factorisations. Let us denote this sequence by $\{a_n\}$. It occurred to me that some of the ideas in references 1 and 2 could be adapted in order to evaluate

$$S(k) = \sum_{n=1}^{\infty} \frac{1}{a_n^k},$$

where k is a positive integer.

2. Calculating $S(k)$ when $k \geq 2$

It is well known that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(indeed, 14 different proofs of this result can be found in reference 3), where $\zeta(k)$ is the Riemann zeta function given by

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

We find, for certain values of k at least, that $S(k)$ can also be expressed in terms of π, despite the fact that it would appear to be a far more complex beast than the series for $\zeta(k)$.

If the integer n is not divisible by p^3 for any prime p then we shall call n *cube-free* (with *fourth-power-free* integers being defined analogously). Let C and F be the sets of cube-free and fourth-power-free positive integers respectively. We note first that

$$S(k) = \sum_{n \in F} \frac{1}{n^k} - \sum_{n \in C} \frac{1}{n^k}. \tag{2}$$

The next step is to show that $S(k)$ can in fact be expressed in terms of Riemann zeta functions. To this end, we let p be any prime and note that

$$\begin{aligned}\left(1+\frac{1}{p^k}+\frac{1}{p^{2k}}\right)\sum_{n=0}^{\infty}\frac{1}{p^{3kn}} &= \sum_{n=0}^{\infty}\frac{1}{p^{3kn}}+\sum_{n=0}^{\infty}\frac{1}{p^{k(3n+1)}}+\sum_{n=0}^{\infty}\frac{1}{p^{k(3n+2)}} \\ &= \sum_{n=0}^{\infty}\frac{1}{p^{kn}}.\end{aligned}$$

Therefore

$$\prod_p\left(1+\frac{1}{p^k}+\frac{1}{p^{2k}}\right)\prod_p\left(\sum_{n=0}^{\infty}\frac{1}{p^{3kn}}\right)=\prod_p\left(\sum_{n=0}^{\infty}\frac{1}{p^{kn}}\right), \tag{3}$$

where the products are taken over all primes p.

Let us now consider each of the products in (3), with a view to simplifying them. First, any term arising from the expansion of the product on the right-hand side will be of the form $1/m^k$ for some $m \in \mathbb{N}$. The fundamental theorem of arithmetic guarantees that any two such terms will be distinct. Furthermore, for any $n \in \mathbb{N}$, $1/n^k$ does indeed make an appearance in this expansion. It therefore follows that

$$\prod_p\left(\sum_{n=0}^{\infty}\frac{1}{p^{kn}}\right)=\sum_{n=1}^{\infty}\frac{1}{n^k}.$$

A similar argument tells us that

$$\prod_p\left(\sum_{n=0}^{\infty}\frac{1}{p^{3kn}}\right)=\sum_{n=1}^{\infty}\frac{1}{n^{3k}}.$$

Finally, the terms arising from the first product on the left-hand side of (3) will be distinct numbers of the form $1/m^k$ where m is cube-free, and it is clear that, for any $n \in C$, $1/n^k$ does appear in this expansion.

From this we have

$$\sum_{n\in C}\frac{1}{n^k}\sum_{n=1}^{\infty}\frac{1}{n^{3k}}=\sum_{n=1}^{\infty}\frac{1}{n^k}. \tag{4}$$

Similarly,

$$\sum_{n\in F}\frac{1}{n^k}\sum_{n=1}^{\infty}\frac{1}{n^{4k}}=\sum_{n=1}^{\infty}\frac{1}{n^k}. \tag{5}$$

Now, from (2), (4), and (5), we have the result

$$S(k)=\zeta(k)\left(\frac{1}{\zeta(4k)}-\frac{1}{\zeta(3k)}\right). \tag{6}$$

When k is even we may obtain expressions for $S(k)$ in terms of π. This is possible because of the result (see, for example, reference 4, p. 266)

$$\zeta(2m)=\frac{(-1)^{m+1}(2\pi)^{2m}B_{2m}}{2(2m)!}, \tag{7}$$

where B_n is the nth Bernoulli number. The Bernoulli numbers occur as the coefficients of $x^n/n!$ in the infinite series given by

$$\frac{x}{e^x - 1} = B_0 + B_1\frac{x}{1!} + B_2\frac{x^2}{2!} + \cdots,$$

and also satisfy the recurrence relation

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0,$$

for $n \geq 1$, where $B_0 = 1$. They arise in a wide variety of mathematical problems.

Let us, for example, use (6) and (7) to obtain an exact numerical expression for $S(2)$ as follows:

$$\begin{aligned} S(2) &= \zeta(2)\left(\frac{1}{\zeta(8)} - \frac{1}{\zeta(6)}\right) \\ &= \frac{\pi^2}{6}\left(\frac{9450}{\pi^8} - \frac{945}{\pi^6}\right) \\ &= \frac{315(10-\pi^2)}{2\pi^6}. \end{aligned}$$

More generally, we have the result

$$\begin{aligned} S(2m) &= \zeta(2m)\left(\frac{1}{\zeta(8m)} - \frac{1}{\zeta(6m)}\right) \\ &= \frac{B_{2m}}{(2\pi)^{4m}(2m)!}\left(\frac{(-1)^m(8m)!}{(2\pi)^{2m}B_{8m}} - \frac{(6m)!}{B_{6m}}\right). \end{aligned}$$

3. The special case $k = 1$

It is well known that the series of reciprocals of the primes diverges (see, for example, reference 4, pp. 18–19). This implies that the series

$$S(1) = \sum_{n=1}^{\infty} \frac{1}{a_n}$$

also diverges. To see this we may note that $8p \in \{a_n\}$ for all odd primes p. Thus, with p_n denoting the nth prime, for any integer $k \geq 2$ there exists some positive integer, m say, such that

$$\sum_{n=1}^{m} \frac{1}{a_n} \geq \frac{1}{8}\sum_{n=2}^{k} \frac{1}{p_n},$$

as required.

We now consider the possibility of finding an asymptotic formula for

$$S(1, m) = \sum_{n=1}^{m} \frac{1}{a_n}.$$

Although the statement

$$\sum_{\substack{n\in C\\ n\le m}} \frac{1}{n} \sum_{n=1}^{\lfloor \sqrt[3]{m}\rfloor} \frac{1}{n^3} = \sum_{n=1}^{m} \frac{1}{n} \tag{8}$$

is not true in general (indeed the left-hand side is greater in value than the right-hand side when $m \geq 8$), the ratio of the values of the expressions on both sides of (8) does in fact approach 1 as $m \to \infty$, as we now show.

It can be seen that

$$\frac{1}{1^3} \sum_{\substack{n\in C\\ n\le m}} \frac{1}{n} + \frac{1}{2^3} \sum_{\substack{n\in C\\ n\le \lfloor m/2^3\rfloor}} \frac{1}{n} + \frac{1}{3^3} \sum_{\substack{n\in C\\ n\le \lfloor m/3^3\rfloor}} \frac{1}{n} + \cdots + \frac{1}{j^3} \sum_{\substack{n\in C\\ n\le \lfloor m/j^3\rfloor}} \frac{1}{n} = \sum_{n=1}^{m} \frac{1}{n}, \tag{9}$$

where j is the unique positive integer such that $j^3 \leq m < (j+1)^3$. The difference between the left-hand side of (8) and the left-hand side of (9) is certainly no greater than

$$\frac{1}{2^3} \sum_{n=\lfloor m/2^3\rfloor+1}^{m} \frac{1}{n} + \frac{1}{3^3} \sum_{n=\lfloor m/3^3\rfloor+1}^{m} \frac{1}{n} + \cdots + \frac{1}{j^3} \sum_{n=\lfloor m/j^3\rfloor+1}^{m} \frac{1}{n} = f(m) + g(m),$$

where

$$f(m) = \frac{1}{2^3} \frac{1}{\lfloor m/2^3\rfloor + 1} + \frac{1}{3^3} \frac{1}{\lfloor m/3^3\rfloor + 1} + \cdots + \frac{1}{j^3} \frac{1}{\lfloor m/j^3\rfloor + 1} \tag{10}$$

and

$$g(m) = \frac{1}{2^3} \sum_{n=\lfloor m/2^3\rfloor+2}^{m} \frac{1}{n} + \frac{1}{3^3} \sum_{n=\lfloor m/3^3\rfloor+2}^{m} \frac{1}{n} + \cdots + \frac{1}{j^3} \sum_{n=\lfloor m/j^3\rfloor+2}^{m} \frac{1}{n}.$$

Now

$$\begin{aligned} \sum_{n=\lfloor m/2^3\rfloor+2}^{m} \frac{1}{n} &< \int_{\lfloor m/2^3\rfloor+1}^{m} \frac{1}{x}\,\mathrm{d}x \\ &\leq \int_{m/2^3}^{m} \frac{1}{x}\,\mathrm{d}x \\ &= \ln m - \ln\left(\frac{m}{2^3}\right) \\ &= 3\ln 2, \end{aligned}$$

and continuing in this way we obtain

$$g(m) < \frac{3\ln 2}{2^3} + \frac{3\ln 3}{3^3} + \cdots + \frac{3\ln j}{j^3} < 3\left(\frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) < \frac{\pi^2}{2}. \tag{11}$$

On considering (10) and (11) it follows that there exists some positive number, M say, such that $g(m) + f(m) < M$ for all m.

Putting all of the above together gives us

$$\sum_{n=1}^{m}\frac{1}{n}\le\sum_{\substack{n\in C\\ n\le m}}\frac{1}{n}\sum_{n=1}^{\lfloor\sqrt[3]{m}\rfloor}\frac{1}{n^3}\le M+\sum_{n=1}^{m}\frac{1}{n},$$

which, in conjunction with the result $\sum_{k=1}^{m}(1/k)\sim\ln m$, implies that

$$\sum_{\substack{n\in C\\ n\le m}}\frac{1}{n}\sim\frac{\ln m}{\zeta(3)}.$$

Likewise we may obtain

$$\sum_{\substack{n\in F\\ n\le m}}\frac{1}{n}\sim\frac{\ln m}{\zeta(4)},$$

to give

$$S(1,m)\sim\ln m\left(\frac{1}{\zeta(4)}-\frac{1}{\zeta(3)}\right).$$

With regard to the application of this asymptotic formula for $S(1,m)$, we note that although (7) gives us a formula for $\zeta(n)$ for even n, there is no known formula for $\zeta(n)$ when n is odd. In fact, until 1979 it was not even known whether or not $\zeta(3)$ was rational, at which point it was finally proved to be irrational (see reference 5).

4. Further generalisations

Our results can easily be generalised to the sum

$$S_l(k)=\sum_{n=1}^{\infty}\frac{1}{a_{n,l}^k},$$

where $\{a_{n,l}\}$ denotes the sequence of positive integers for which l is the highest power of any prime appearing in their prime factorisations. For $k\ge 2$ this series can be expressed in terms of Riemann zeta functions (and thus in terms of π if k is even), while if $k=1$ it diverges for any $l\in\mathbb{N}$. In the latter case we may obtain an asymptotic formula for the sum of the first m terms of the series.

References

1 M. D. Hirschhorn, Sums involving square-free integers, *Math. Gazette* **87** (2003), pp. 527–528.

2 J. A. Scott, Square-freedom revisited, *Math. Gazette* **90** (2006), pp. 112–113.

3 R. Chapman, Evaluating $\zeta(2)$, available at http://www.secamlocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf.

4 T. M. Apostol, *Introduction to Analytic Number Theory* (Springer, New York, 1976).

5 R. Apéry, Irrationalité de $\zeta(2)$ et $\zeta(3)$, *Astérisque* **61** (1979), pp. 11–13.

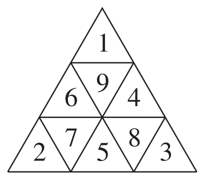
The author is Head of Mathematics at Colchester County High School and a part-time Lecturer in Mathematics at the University of Essex. His mathematical interests are wide-ranging, and he has published articles in The Mathematical Gazette, the Journal of Mathematical Biology, and The Fibonacci Quarterly. Many of his articles arise as a consequence of ideas that originate in the classroom or lecture theatre. He is the author of a book about the central binomial coefficients, published by the United Kingdom Mathematics Trust and aimed at able 16–20-year-old students and their teachers. He is currently Reviews Editor of The Mathematical Gazette.

Magic triangles

$2+5+6+7=20 \qquad 2+7+5+8+3=25$

$3+4+5+8=20 \qquad 3+8+4+9+1=25$

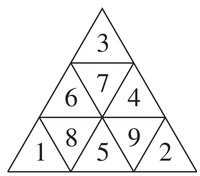
$1+6+4+9=20 \qquad 1+9+6+7+2=25$



$1+5+6+8=20 \qquad 1+8+5+9+2=25$

$2+4+5+9=20 \qquad 2+9+4+7+3=25$

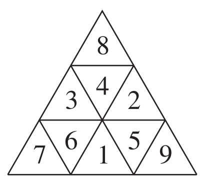
$3+6+4+7=20 \qquad 3+7+6+8+1=25$



$7+1+3+6=17 \qquad 7+6+1+5+9=28$

$9+2+1+5=17 \qquad 9+5+2+4+8=28$

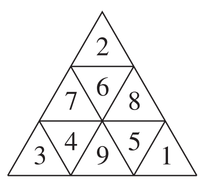
$8+3+2+4=17 \qquad 8+4+3+6+7=28$



$3+9+7+4=23 \qquad 3+4+9+5+1=22$

$1+8+9+5=23 \qquad 1+5+8+6+2=22$

$2+7+8+6=23 \qquad 2+6+7+4+3=22$



Students' Investigation House,
Before Farhang Crossroads,
Shariati Avenue, Sirjan, Iran

Abbas Rooholamini Gugheri

Complex Iteration and Roots of Unity

CAMILLA JORDAN and DAVID JORDAN

1. Introduction

Possibly the best known instances of sequences obtained by recurrence or iteration are the Fibonacci sequence u_n determined by the rule

$$u_0 = 1, u_1 = 1, \ldots, u_{n+1} = u_n + u_{n-1} \quad (n > 0) \tag{1}$$

and sequences arising from the quadratic iteration

$$z_{n+1} = z_n^2 + c \quad (n \geq 0), \tag{2}$$

where c and the initial value z_0 are arbitrary complex numbers. Although (2) is of some interest in the real case, it is the use of complex numbers that leads to the well-known dramatic pictures of sets such as the Mandlebrot set and Julia sets. We propose to study sequences similar to (1) in the complex case, allowing an arbitrary pair of initial values and replacing $u_n + u_{n-1}$ by a linear combination of these values. Thus we study recurrence sequences of the form $z_0, z_1, z_2, \ldots, z_n, \ldots$, where, for $n \geq 1$,

$$z_{n+1} = az_n + bz_{n-1} \tag{3}$$

and a and b are arbitrary complex numbers. For example, if $a = \mathrm{i}$ and $b = -1$ the first few terms of the sequence starting with $z_0 = \mathrm{i}$ and $z_1 = 1$ are i, 1, 0, −1, −i, 2, 3i.

Not surprisingly we often obtain sequences which tend to 0 or to ∞. However, we can also obtain periodic sequences, annuli and striking pictures such as those shown in figure 3, below, where the points in the sequence appear to plot a number of closed curves, apparently

Figure 1 Deformations of annuli and ellipses.

ellipses. In this article we will describe some of the pictures that can be obtained and explain the properties of the parameters that are used for the different types of pictures.

We will also see how these can be regarded as linear approximations of quadratic iterations related to (2) but with z_{n+1} depending on z_{n-1} as well as z_n. In the quadratic iterations, the annuli become deformed into weird shapes resembling animal skulls, jellyfish and necklaces, whereas the ellipses are deformed into curious shapes which may resemble fish or boots – see figure 1.

2. What types of pictures are generated?

Given $a, b \in \mathbb{C}$ in (3), let λ_1 and λ_2 be the roots of the *characteristic equation*

$$\lambda^2 - a\lambda - b = 0.$$

Thus $a = \lambda_1 + \lambda_2$ and $b = -\lambda_1\lambda_2$. It is possible to show that the sequences will neither tend to 0 nor to ∞ when $|\lambda_1| = 1 = |\lambda_2|$ and $\lambda_1 \neq \lambda_2$ and that this is equivalent to the following pair of conditions on a and b:

$$b = \cos\theta + \mathrm{i}\sin\theta, \tag{4}$$

$$a = 2s(\cos\rho + \mathrm{i}\sin\rho), \quad \text{where } \rho = \frac{\pi + \theta}{2}, \tag{5}$$

for some real numbers θ and s with $|s| < 1$. It is in this case, which is called *elliptic*, that the recurrence sequences give striking pictures such as those in the diagrams. For readers familiar with eigenvalues and their role in the analysis of the Fibonacci sequence, λ_1 and λ_2 are the eigenvalues of the matrix

$$\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}.$$

The elliptic case occurs when these eigenvalues are different but both have modulus 1.

2.1. Periodic sequences

Choosing $\theta = 0$ in (4) and $s = \frac{1}{2}$ in (5), so that $a = \mathrm{i}$ and $b = 1$, and taking arbitrary starting points z_0, z_1 results in a sequence in which the block consisting of

$$z_0, z_1, \mathrm{i}z_1 + z_0, \mathrm{i}z_0, \mathrm{i}z_1, -z_1 + \mathrm{i}z_0, -z_0, -z_1, -\mathrm{i}z_1 - z_0, -\mathrm{i}z_0, -\mathrm{i}z_1, z_1 - \mathrm{i}z_0$$

is repeated periodically. These twelve numbers need not be distinct. Notice that choosing a starting point and then taking every third term gives a subsequence of the form $z, \mathrm{i}z, -z, -\mathrm{i}z$. Thus the block of twelve is made up of three of these subsequences. If $z_1 = z_0$, $\mathrm{i}z_0, -z_0$ or $-\mathrm{i}z_0$ then two of these subsequences share the same terms and if $z_1 = \mathrm{i}z_0$ then $z_2 = z_5 = z_8 = z_{11} = 0$. So we can obtain 12, 8 or 5 distinct numbers of distinct terms. Figure 2 shows some examples.

Such periodic sequences occur when λ_1 and λ_2 are both roots of unity. If λ_1 is a primitive m th root of unity and λ_2 is a primitive n th root of unity, we get a periodic sequence of period ℓ where ℓ is the least common multiple of m and n. In the example above,

$$\lambda_1 = \cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right) \quad \text{and} \quad \lambda_2 = \cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right)$$

and so $m = n = 12$.

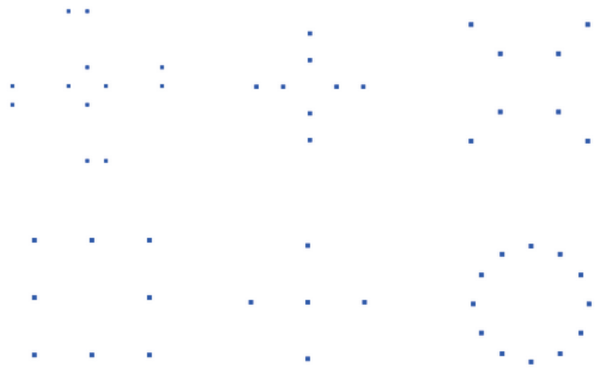


Figure 2 Starting values are $-0.25,\ 1;\ 1,\ -\mathrm{i};\ -1+\mathrm{i},\ 1+\mathrm{i};\ 1,\ 1;\ 1,\ \mathrm{i};\ \frac{\sqrt{3}}{2}-\frac{\mathrm{i}}{2},\ 1$.

2.2. Ellipses

Pictures such as those in figure 3 are obtained when, in (4), $\theta = n\pi/d$ where n/d is a rational number in its lowest terms but λ_1 and λ_2 are not roots of unity. The number, m, of ellipses is closely related to the sum, $n+d$, of the numerator and denominator. If $n+d$ is even then $m = d$ and if $n+d$ is odd then $m = 2d$. In all the examples shown in figure 3 the value of s is -0.4 and the starting values are $z_0 = \mathrm{i}, z_1 = 1$.

It can be shown that the shapes are ellipses centred on the origin. Each ellipse is obtained from the first by successive rotations, through ρ in (5), returning to the first after m steps. Successive points lie on successive ellipses. The recurrence sequence appears to give all the points on the ellipses but does not in fact do so. For readers familiar with the term, the points form a *dense* subset of the ellipses.

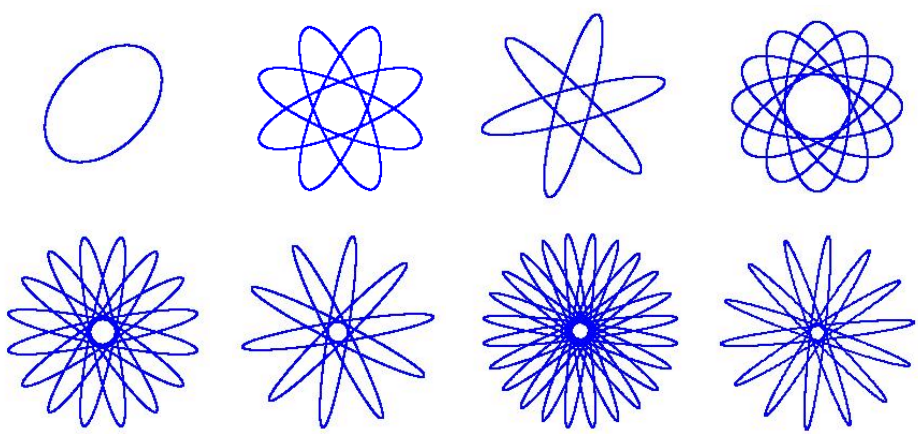


Figure 3 Recurrence sequences for $\theta = \pi,\ \frac{\pi}{2},\ \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{\pi}{4},\ \frac{\pi}{5},\ \frac{\pi}{6},\ \frac{\pi}{7}$.

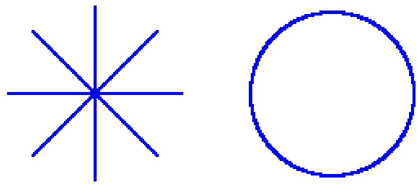


Figure 4 Two recurrence sequences.

Figure 5 A recurrence sequence where θ is not a rational multiple of π.

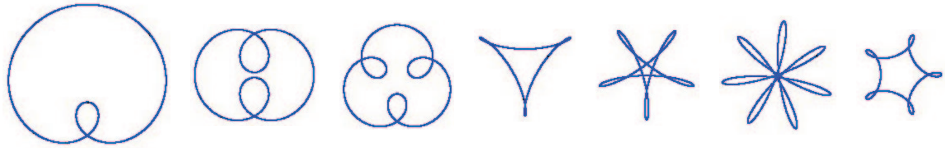


Figure 6 Recurrence sequences where θ is not a rational multiple of π but the result is not an annulus.

The value of s in (5) together with the starting values control the size of the ellipses. Indeed the starting values can cause the ellipses to degenerate into lines or merge into a circle as shown in figure 4. The lines were generated from the starting values $z_0 = \mathrm{i}$, $z_1 = 1$ and the circle from the starting values $z_0 = \frac{1}{4}((\sqrt{7}-1) - \mathrm{i}(\sqrt{7}+1))$, $z_1 = 1$. In the latter case, $\binom{z_1}{z_0}$ is an eigenvector of the matrix

$$\begin{pmatrix} \frac{1}{2}(-1+\mathrm{i}) & \mathrm{i} \\ 1 & 0 \end{pmatrix}.$$

2.3. Further possibilities

Figure 5 shows an example with $s = -0.2$, $z_0 = \mathrm{i}$, $z_1 = 1$ and $\theta = \pi/\sqrt{2}$, which is not a rational multiple of π. Because no integer multiple of θ is equal to π, the rotation of the ellipses gives infinitely many ellipses, and the recurrence sequence appears to fill an annulus. This will normally be the result when, in (4), θ is not a rational multiple of π. However, exceptional cases can occur. In figure 6, $\theta = \pi/\sqrt{2}$ and s has been chosen so that $\lambda_2^m = \lambda_1^n$

for integers n and m. (From left to right the pictures have $n = 1$, $m = 2$; $n = 1$, $m = 3$; $n = 1$, $m = 4$; $n = 1$, $m = -2$; $n = 2$, $m = -3$; $n = -1$, $m = 4$; $n = 3$, $m = -4$ respectively. The starting values are $z_0 = z_1 = \mathrm{i}$ for all but the rightmost picture. This has starting values $z_0 = 0.5\mathrm{i}$, $z_1 = 1$.)

3. Hénon recurrence sequences

There is a class of nonlinear recurrence sequences known as *Hénon recurrence sequences*. In general these are defined by the equation

$$w_{n+1} = \alpha - \beta w_{n-1} - w_n^2, \tag{6}$$

where $\alpha, \beta \in \mathbb{C}$.

Setting $\beta = 0$ and $\alpha = -c$, this gives a formula very similar to (2). Indeed the two are equivalent in the sense that starting them both with w_0 gives sequences that are negatives of each other. Real and complex Hénon recurrence sequences, and their history, are well-documented, see, for example, reference 1 for the real case.

Our linear recurrence sequences are related to certain Hénon recurrence sequences. We do this by first finding f such that $f = \alpha - \beta f - f^2$ so that the recurrence sequence beginning f, f is constant. Of course the linear recurrence sequence beginning $0, 0$ is constant at 0 so we need to change variables in (6) to move f to the origin. We do this by writing $z_i = w_i - f$ where $f = \alpha - \beta f - f^2$. The recurrence sequence (6) then becomes $z_{n+1} = -2z_n f - \beta z_{n-1} - z_n^2$ and this leaves the starting values $0, 0$ fixed. We 'linearise' by simply ignoring the quadratic term. It can be shown that, in many circumstances, the behaviour of the Hénon recurrence sequence with starting values close to f, f is related to the behaviour of the linearised sequence with starting values $0, 0$.

Figure 7 shows the points of a Hénon recurrence sequence where $\alpha = \beta = \mathrm{i}$ on the right and a corresponding recurrence sequence of the linearisation on the left. Apart from the degenerate cases shown in figure 4 the linear recurrence sequence produces similar pictures whatever the initial term.

The Hénon recurrence, however, is much less stable and is only bounded when the initial two terms are very close to f. Figure 8 shows some examples of Hénon recurrence sequences ($|\alpha| = 0.675$, $\beta = \mathrm{e}^{\pi/3}$) as the initial two terms are moved away from f. (The initial two terms are 0.98, 0.93, 0.9, 0.83 multiplied by f.)

Where $-\beta$ and $-2f$ satisfy the conditions (4) and (5) with $-\beta = b$ and $-2f = a$ respectively, for the linear recurrence relation to give elliptic sequences, the Hénon recurrence

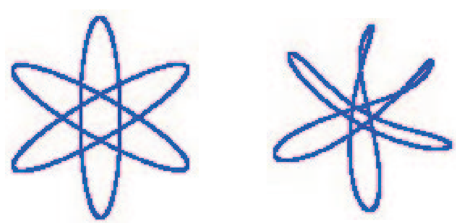


Figure 7 Comparing the Hénon map with its linearisation.

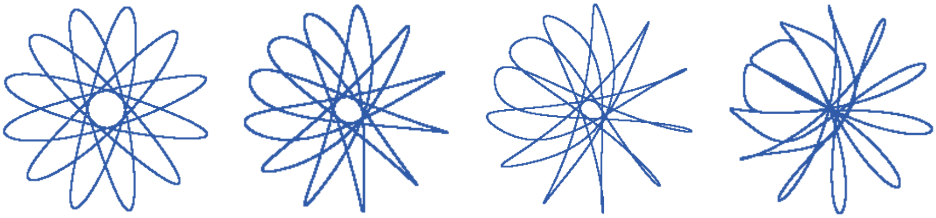


Figure 8 Some Hénon recurrence sequences.

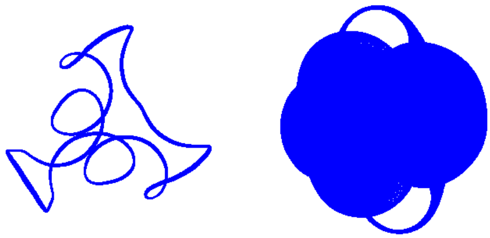


Figure 9 The first components of two orbits of a Hénon map.

sequence can give rise to some striking pictures as shown in figure 9 ($|\alpha| = 0.495$, $\beta = \mathrm{e}^{0.368\pi}$ and $|\alpha| = 0.8625$, $\beta = \mathrm{e}^{0.09\pi}$).

The symmetry present in all these pictures is a feature of Hénon recurrence sequences of this type. The reason for it is beyond the scope of this article but is discussed in reference 2.

There is a Java applet at `http://mcs.open.ac.uk/crj3/Henon/Henon.html` where readers can explore the recurrence sequences of reversible Hénon maps. More information on complex iteration can be found in reference 3.

References

1 R. Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd edn. (Addison Wesley, Redwood City, CA, 1989).

2 C. R. Jordan, D. A. Jordan and J. H. Jordan, Reversible complex Hénon maps, *Experimental Math.* **11** (2002), pp. 339–347.

3 J. Smillie and G. T. Buzzard, Complex dynamics in several variables, in *Flavors of Geometry,* ed. S. Levy (Cambridge University Press, 1997), pp. 117–150.

***David Jordan** is a Professor of Pure Mathematics at the University of Sheffield. **Camilla Jordan** is a Staff Tutor in Mathematics, Computing and Technology in the Yorkshire Region of the Open University. They are co-authors of a text book on Group Theory. They met as PhD students at Leeds University. At weekends they enjoy walking in the Peak District.*

Dual Rectangles

GRAHAM EVEREST and JONNY GRIFFITHS

This article is dedicated to Graham Everest, who died before it appeared in print.

1. Duality

The concept of *duality* in Mathematics is incredibly rich and powerful, occurring in many places. Even so, it is not easy to define in general terms. However, the essential properties entail that the dual of the dual is the starting object and that an object can be self-dual.

In this article, we will say that a pair of rectangles, with sides (a, b) and (c, d), are called *dual* if the area of each is the perimeter of the other. We will assume throughout that the rectangles are lying down: in other words, that the pairs are written with $a \geq b$ and $c \geq d$.

Theorem 1 *There are precisely seven pairs of dual rectangles with integral sides. Two are self-dual:* $(4, 4)$ *and* $(6, 3)$. *The remaining five are*

$$(6, 4)(10, 2), \quad (10, 3)(13, 2), \quad (10, 7)(34, 1), \quad (13, 6)(38, 1), \quad (22, 5)(54, 1).$$

The duality property translates into the following pair of simultaneous equations:

$$\begin{aligned} ab &= 2c + 2d, \\ cd &= 2a + 2b. \end{aligned} \tag{1}$$

There are infinitely many rational solutions to the equations in (1) as we now show. Once b and d are fixed there remain two simultaneous equations in the two variables a and c. It turns out that these have a unique solution provided that $bd \neq 4$. In this case a and c are given explicitly by

$$a = \frac{2d^2 + 4b}{bd - 4}, \qquad c = \frac{4d + 2b^2}{bd - 4}. \tag{2}$$

Thus any pair of rational values for b and d with $bd \neq 4$ yields a pair of rational numbers a and c by solving the simultaneous equations. (Exercise: what happens when $bd = 4$?) What is more, provided that b and d are positive and $bd > 4$, the solutions are guaranteed to consist of positive rational numbers. It follows that there exist infinitely many pairs of dual rectangles with rational sides. Theorem 1 asserts that only finitely many of these pairs have integral sides.

Proof Eliminating d from the equations in (1) shows that c satisfies a quadratic equation

$$2c^2 - cab + 4(a + b) = 0. \tag{3}$$

Solving this equation shows that

$$c = \frac{ab \pm \sqrt{a^2b^2 - 32(a + b)}}{4}.$$

Therefore, a necessary (but not sufficient) condition for c to be integral is

$$a^2b^2 - 32(a+b) = t^2, \tag{4}$$

for some nonnegative integer t. Rearranging (4) and factorizing yields

$$(ab+t)(ab-t) = 32(a+b).$$

Since $ab+t$ divides $32(a+b)$, it follows that

$$ab + t \leq 32(a+b). \tag{5}$$

The hypothesis that $b \leq a$ implies that the right-hand side of (5) is bounded above by $64a$. Since $0 \leq t$, the left-hand side of (5) is bounded below by ab. Therefore

$$ab \leq 64a,$$

and cancelling a shows that $b \leq 64$. By an entirely symmetrical argument, we obtain $d \leq 64$. Hence there are a finite number of integer pairs b and d. Each one yields at most one integer pair a and c using the same method as before. For each value of b and d with $1 \leq b$, $d \leq 64$, and $bd \neq 4$, the corresponding values of a and c in (2) can be checked for integrality, using a computer (which we did). The cases when $bd = 4$ do not lead to consistent equations.

Exercise Using the formulae in (2) turns up quite a few examples of pairs of dual rectangles where three out of the four sides are integers. For example,

$$(7,3)\big(8, \tfrac{5}{2}\big), \quad (7,5)\big(16, \tfrac{3}{2}\big), \quad (33,3)\big(48, \tfrac{3}{2}\big), \quad (89,1)\big(40, \tfrac{9}{2}\big).$$

Adapt the method of proof above to show there are only finitely many pairs of dual rectangles with three integral sides, and list them all.

2. Constructing self-dual rectangles

A rational self-dual rectangle corresponds to a solution of the equation

$$xy = 2x + 2y \quad \text{or} \quad (x-2)(y-2) = 4, \tag{6}$$

in positive rational numbers. Note that, by factorizing, the only solutions to (6) with $x \geq y$ positive integers are $(x, y) = (4, 4)$ or $(6, 3)$. Equation (6) describes a rectangular hyperbola. Suppose that we are given two distinct rational points on the hyperbola

$$P = \left(p, \frac{2p}{p-2}\right) \quad \text{and} \quad Q = \left(q, \frac{2q}{q-2}\right), \qquad \text{with } p, q > 2.$$

Consider the triangle OPQ. The orthocentre of the triangle has rational coefficients. Explicitly,

$$\left(2 + \frac{8}{(p-2)(q-2)}, 2 + \frac{(p-2)(q-2)}{2}\right),$$

and this point lies on the hyperbola too (see reference 1) as figure 1 illustrates.

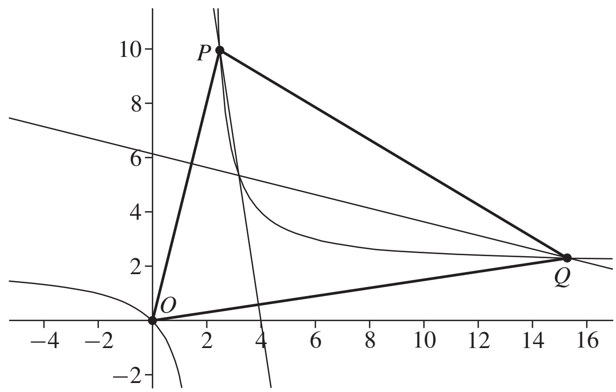


Figure 1

Now define $P + Q$ as the reflection of the orthocentre of the triangle OPQ in the line $y = x$. The point so defined has rational coordinates

$$\left(p, \frac{2p}{p-2}\right) + \left(q, \frac{2q}{q-2}\right) = \left(2 + \frac{(p-2)(q-2)}{2}, 2 + \frac{8}{(p-2)(q-2)}\right). \tag{7}$$

Our remarks confirm that the map defined by (7) is closed; thus it is a binary operation. What is more, we can take (7) to define the map even when $p = q$. In other words, we can also ‘double’ points.

Theorem 2 *The operation defined by (7) is commutative and associative having the point* $(4, 4)$ *as the identity. Thus the set of rational self-dual rectangles forms an abelian group.*

Exercise Prove Theorem 2.

3. From dual rectangles to cubic surfaces

In this section, we show how the theory of Diophantine equations (polynomial equations with integer coefficients) can be used to construct new solutions to (1) from known solutions. Our motivation comes from a recent successful attack, by Gerry Tunnell, upon a classical problem about triangles with rational sides having an integral area (see Koblitz’ excellent book reference 2).

The right-angled triangle with sides 3, 4, and 5 has area equal to 6. The triangle with sides

$$\tfrac{7}{10}, \qquad \tfrac{120}{7}, \qquad \tfrac{1201}{70}$$

is also right-angled and has area equal to 6. In a sense which can be made precise, it is the next simplest such triangle. In reference 3, Section 5.2, the authors used Tunnell’s ideas to point out that there are in fact infinitely many right-angled triangles with rational sides having area equal to 6. The technique uses the theory of certain cubic curves called *elliptic curves* (reference 4 is an excellent reference for background reading). The way it turns out, a geometric binary operation can be defined on the curve

$$y^2 = x^3 - 36x,$$

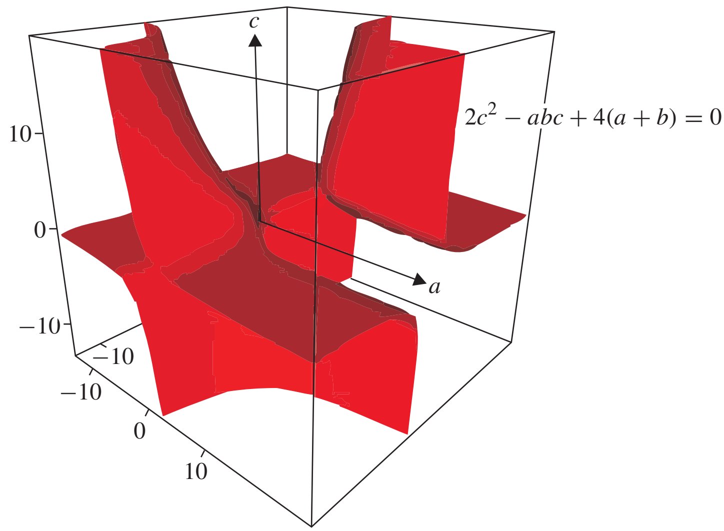


Figure 2

which allows known rational solutions to be used to construct other rational solutions. A dialogue between this curve and triangles then allows other triangles to be constructed.

One of the fascinating aspects of elliptic curves is the way that combining quite simple solutions can yield very complicated solutions. We will model this idea now using (3) and show that a similar phenomenon is at work. This equation defines a cubic surface in 3-space so straight lines should meet it in three points (see figure 2).

Define an operation as follows. Given two rational points on (3), join them by a straight line. The third point of intersection will have rational coordinates. Thus a new solution of (3) will be constructed. Hopefully this will yield a new pair of rational dual rectangles.

Example 1 From the list of pairs of integral dual rectangles in Theorem 1, select the rectangle $(6, 4)$ which yields the point $(6, 4, 10)$ on (3), as well as the rectangle $(22, 5)$ which yields the point $(22, 5, 54)$ on (3). The straight line joining these points has equation

$$(a, b, c) = \theta(6, 4, 10) + (1 - \theta)(22, 5, 54),$$

as θ runs over the real numbers. Substitute into (3) and the following cubic equation for θ emerges:

$$88\theta^3 - 185\theta^2 + 97\theta = 0. \tag{8}$$

The three roots of (8) are 0, 1, and $\frac{97}{88}$. The first two correspond to the points we know about. The last one yields the point $(\frac{48}{11}, \frac{343}{88}, \frac{11}{2})$, and this gives rise to the following pair of dual rectangles with rational sides:

$$\left(\tfrac{48}{11}, \tfrac{343}{88}\right)\left(\tfrac{11}{2}, \tfrac{727}{242}\right).$$

Notice the way that combining two integral points produced quite a complicated rational point. This resonates strongly with the theory of elliptic curves. The next example gives another illustration of the phenomenon.

Example 2 Using the rectangles $(10, 3)$ and $(13, 6)$ yields the points $(10, 3, 13)$ and $(13, 6, 38)$ on (3). The straight line joining these points has equation

$$(a, b, c) = \theta(10, 3, 13) + (1 - \theta)(13, 6, 38).$$

Substituting into (3) yields the following cubic equation for θ:

$$225\theta^3 - 517\theta^2 + 292\theta = 0. \tag{9}$$

The three roots of (9) are 0, 1, and $\frac{292}{225}$. The last one yields the point $(\frac{683}{75}, \frac{158}{75}, \frac{50}{9})$ and this gives rise to the following pair of dual rectangles with rational sides:

$$\left(\tfrac{683}{75}, \tfrac{158}{75}\right)\left(\tfrac{50}{9}, \tfrac{2523}{625}\right).$$

Warning This method is not guaranteed to work in a straightforward manner. The geometry of the cubic surface (3) means that the third point of intersection will not necessarily produce a pair of dual rectangles. If the third point of intersection exists and has a positive value of c then the corresponding value of d exists and is positive; hence, a pair of dual rectangles is created. However, a negative or even a zero c value could emerge. In this case the equations make sense but will not allow an interpretation in terms of dual rectangles.

Non-example Suppose that we try to 'add' the two points $(6, 4, 10)$ and $(10, 3, 13)$ which come from Theorem 1. The straight line joining these points has equation

$$(a, b, c) = \theta(6, 4, 10) + (1 - \theta)(10, 3, 13),$$

as θ runs over the real numbers. Substitute into (3) and a cubic equation for θ emerges with roots 0, 1, and $\frac{13}{3}$. The third root yields the point $(-\frac{22}{3}, \frac{22}{3}, 0)$ and hence a value for $d = -\frac{242}{9}$. These values satisfy the equations in (1) but they plainly do not produce a pair of dual rectangles.

References

1 J. A. Scott, Problem 41.8, *Math. Spectrum* **41** (2008/2009), p. 90.
2 N. Koblitz, *An Introduction to Elliptic Curves and Modular Forms* (Springer, New York, 1984).
3 G. Everest and T. Ward, *An Introduction to Number Theory* (Springer, New York, 2005).
4 J. H. Silverman and J. Tate, *Rational Points on Elliptic Curves* (Springer, New York, 1992).

Graham Everest *is a Professor of Pure Mathematics at the University of East Anglia. For many years he has taught Number Theory to all different levels. He maintains a lively interest in the interface between school and college mathematics and university mathematics.*

Jonny Griffiths *teaches at Paston College in Norfolk, where he has been for the last nineteen years. He is currently studying for an MSc in mathematics at the University of East Anglia. Possible claims to fame include being an ex-member of 'Harvey and the Wallbangers', a popular band in the 1980s, and playing the character 'Stringfellow' on the childrens' television programme 'Playdays'.*

Perfect Numbers

SANDRA PULVER

According to Pythagoras (who was born between 580 BC and 572 BC and died between 500 BC and 490 BC), a *perfect number* is a number that is equal to the sum of its proper, positive divisors, excluding itself. Early mathematicians defined perfect numbers to be equal to the sum of their *aliquot parts*. Aliquot parts of a number are its proper divisors (not including the number). This concept is deceptively simple. However, the rarity of perfect numbers reveals their significance in number theory. Perfect numbers were studied by Pythagoras and his followers more for their mystical properties than for their number-theoretic properties, however.

Euclid of Alexandria (born circa 300 BC), author of the *Elements* and father of Euclidean geometry, made the initial number-theoretic discoveries about perfect numbers. He created a simple way to find even perfect numbers. Euclid proved that the formula $2^{p-1}(2^p - 1)$ gives an even perfect number whenever $2^p - 1$ is prime (see reference 1, Book IX, Proposition 36). Euclid gave a rigorous proof of the proposition in *Elements*.

Nicomachus of Gerasa (circa 60–120), in his famous text, *Introduction to Arithmetic* (see reference 2), classified numbers into three groups: superabundant, perfect, and deficient. *Superabundant numbers* are numbers such that the sum of their divisors is greater than the number (i.e. $12, 24, 36, 48, 60, 120, \dots$). *Deficient numbers* are numbers the sum of whose divisors is less than the number (i.e. $2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, \dots$). Nicomachus applied this concept of perfect numbers to morals and also to biology. More than one mouth is too much, or one eye is too little, and animals having these would be at a disadvantage. The perfect number of body parts, in his writings, is the number that humans have.

The first perfect numbers, known to early Greek mathematicians, are as follows:

$$\begin{aligned}
6 &= 2^1(2^2 - 1) = 1 + 2 + 3,\\
28 &= 2^2(2^3 - 1) = 1 + 2 + 3 + 7 + 14,\\
496 &= 2^4(2^5 - 1) = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248,\\
8128 &= 2^6(2^7 - 1) = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508\\
&\qquad + 1016 + 2032 + 4064.
\end{aligned}$$

These numbers have been known since ancient times but the fifth perfect number was not commonly known until 1536, when Hudalrichus Regius published it in his *Ultriusque Arithmetices*. It was found earlier, however, in anonymous, unpublished documents by Arab mathematicians.

Before the discovery of the fifth perfect number, mathematicians assumed that properties that the first four had were true for all perfect numbers. One such assumption, made by Nicomachus, was that the nth perfect number would have n digits. This is true for 6, 28, 496, and 8128, but the fifth perfect number, $33\,550\,336 = 2^{12}(2^{13} - 1)$, has eight digits. Another false assumption was that the last digit of perfect numbers repeated in a pattern: $6, 8, 6, 8, \dots$. All perfect numbers do end in a 6 or an 8, but they do not continue to alternate between these two numbers. In 1603, this was shown when the sixth perfect number was found to be 8 589 869 056; it ended in a 6, not an 8.

Other assumptions made by Nicomachus were that all perfect numbers are even and that there is an infinite number of them. These statements have not been proved or disproved.

Nicomachus' description of the algorithm to generate perfect numbers was precisely that given by Euclid in the *Elements*. Nicomachus offered no justification for any of his assertions, however, but they were taken as fact in Europe even into the 1500s.

Perfect numbers also gained meaning beyond a mathematical context when they were incorporated into religion, as the geocentric theory was used by the church to justify God's positioning of humans. It was believed that God had created the universe in six days and made the moon orbit the earth in twenty-eight days because these numbers are perfect.

Arab mathematicians were also fascinated by perfect numbers. Over a millennium after Euclid, Ibn Al-Haytham (Alhazen) (born circa 100) realized that every even perfect number is of the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is prime, a converse to Euclid's proposition, but he was not able to prove it. In his unpublished work *Treatise on Analysis and Synthesis*, he showed that perfect numbers satisfying certain conditions had to be of the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is prime, but he was not able to prove it in general.

In 1603, Pietro Antonio Cataldi (1548–1626) showed that Nicomachus' assertion that perfect numbers ended in 6 and 8 alternately is false, since he found the sixth (and also the seventh) perfect number. But, despite having made the major advance of finding two new perfect numbers, he also made some false claims. He wrote in *Utriusque Arithmetices* that the exponents

$$p = 2, 3, 5, 7, 13, 17, 19, 23, 29, 31, 37$$

give perfect numbers. He is right for $p = 2, 3, 5, 7, 13, 17, 19$, but only one of his further four claims is correct, that of 31.

The history of perfect numbers has been littered with false starts, incorrect paths, and errors. In 1640, Pierre de Fermat (1601–1665) disproved two of Cataldi's claims. He showed that $2^{23} - 1$ is composite (it is equal to $47 \times 178\,481$) and that $2^{37} - 1$ is composite (it is equal to $223 \times 616\,318\,177$), so that $p = 23$ and $p = 37$ do not give perfect numbers.

Among the many Arab mathematicians to take up the Greek investigation of perfect numbers with great enthusiasm was Ismail ibn Ibrahim ibn Fallus (1194–1239), who wrote a treatise based on the *Introduction to Arithmetic* by Nicomachus. He gave in his treatise a table of ten numbers which he claimed are perfect. The first seven are correct and are, in fact, the first seven perfect numbers, but the remaining three are incorrect.

It was not until the 18th century that Leonhard Euler (1707–1783) proved that the formula $2^{p-1}(2^p - 1)$ will yield all the even perfect numbers. This result is often referred to as the Euclid–Euler theorem. The fifth perfect number was discovered again (after the unknown and unpublished results of the Arabs) and written in a manuscript dated 1461. It is also in a manuscript written by Rigiomontanus on or about that date.

In 1732, Euler proved that the eighth perfect number is

$$2^{30}(2^{31} - 1) = 2\,305\,843\,008\,139\,952\,128.$$

It was the first new perfect number discovered for 125 years, and remained the largest known for over 150 years. Then in 1738, Euler disproved one more of Cataldi's claims when he proved that $2^{29} - 1$ is not prime.

In order for $2^p - 1$ to be prime, it is necessary but not sufficient that p should be prime. Thus, $2^{11} - 1 = 23 \times 89$. So 11 is prime, but $2^{11} - 1$ is not. Prime numbers in the form $2^p - 1$ are called *Mersenne primes* after the French monk Marin Mersenne (1588–1648). Mersenne stated that $2^p - 1$ is prime when

$$p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, \text{ or } 257$$

(67, according to reference 3, p. 210). He was incorrect about 67 and 257, and did not include 61, 89, and 107. The correct values are

$$p = 2, 3, 5, 13, 17, 19, 31, 61, 89, 107, \text{ and } 127.$$

There is a one-to-one correspondence between even perfect numbers and Mersenne primes. Further Mersenne primes were found with the aid of computers. Two high school students discovered the twenty-fifth and twenty-sixth using the *Lucas–Lehmer primality test*. It was found by the Great Internet Mersenne Prime Search (GIMPS); see reference 4.

When the even perfect numbers are written in binary system they follow a pattern:

$$\begin{aligned} 6 &= 110, \\ 28 &= 11100, \\ 496 &= 111110000, \\ 8128 &= 1111111000000, \\ &\vdots \end{aligned}$$

Another property of all even perfect numbers, except 6, is that their *digital sum* is 1. The digital sum of a number is the one-digit sum obtained after repeatedly adding its digits. For example,

$$\begin{gathered} 2+8=10, \qquad 1+0=1, \\ 4+9+6=19, \qquad 1+9=10, \qquad 1+0=1, \\ 8+1+2+8=19, \qquad 1+9=10, \qquad 1+0=1. \end{gathered}$$

The digital sum of a number is its remainder on division by 9. Now, every even perfect number can be written as $2^{p-1}(2^p - 1)$, where $2^p - 1$ and also p is prime. Two is the only even prime number, so p must be odd for all perfect numbers other than 6. Therefore, $p - 1$ is even, and

$$2^{p-1} \equiv (-1)^{p-1} \equiv 1 \pmod 3,$$

so we can write $2^{p-1} = 3x + 1$, for some positive integer x. Hence,

$$\begin{aligned} 2^{p-1}(2^p - 1) &= (3x+1)(6x+1) \\ &= 18x^2 + 9x + 1, \\ &= 9(2x^2 + x) + 1, \end{aligned}$$

so that its digital sum is 1.

All even perfect numbers are triangular, i.e. when they are made into a diagram they look like triangles. *Triangular numbers* are the sum of consecutive whole numbers, beginning with one (and up to a certain point). Thus, the triangular numbers are given by the formula

$$1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1),$$

where n is a positive integer.

The sequence of triangular numbers begins

$$1, 3, 6, 10, 15, 21, 28, 36, 45, \ldots.$$

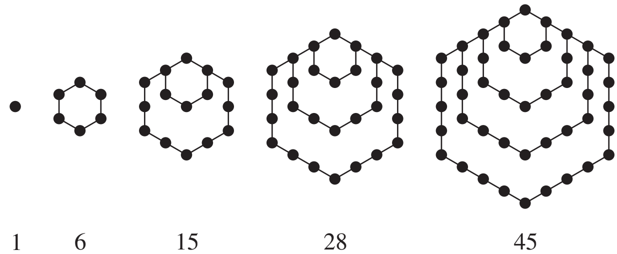


Figure 1 Hexagonal numbers.

Notice that the first two perfect numbers are listed here as well, 6 and 28. In 1509, the French mathematician Charles de Bouvelles (1471–1553) proved that all even perfect numbers are triangular. An even perfect number is of the form

$$2^{p-1}(2^p - 1),$$

where $2^p - 1$ is prime, and this is equal to

$$\tfrac{1}{2}(2^p - 1)2^p,$$

so it is the $(2^p - 1)$th triangular number.

A *hexagonal number* is a positive integer that can be drawn in the shape of a hexagon (see figure 1) using points to represent the numbers. Some hexagonal numbers are 1, 6, 15, 28, and 45; the nth hexagonal number is $n(2n - 1)$. Every even perfect number is a hexagonal number. The Italian mathematician Franciscus Maurolycus (1494–1575) proved this fact in the mid-sixteenth century.

From the days of Euclid and Pythagoras to the days of super computers, perfect numbers have been a constant source of mathematical curiosity. They have eluded mathematicians for millennia and continue to do so. It is still unknown whether there are infinitely many Mersenne primes and therefore infinitely many perfect numbers. But computers have led to a revival of interest in the discovery of Mersenne primes, and therefore of perfect numbers. No one has ever found an odd perfect number; however, no one has proved that no odd perfect numbers exist. There is no single expression that has been discovered that will generate all perfect numbers. Many discoveries about perfect numbers have still been left for us to make.

George Woltman has distributed a search program via the internet known as GIMPS (see reference 4) in which hundreds of volunteers use their personal computers to perform pieces of the search. The efforts of GIMPS volunteers make this distributed computing project the discoverer of all 13 of the largest known Mersenne primes. In fact, as of September 2008, GIMPS participants have tested and double-checked all values of p below 17 001 247 and tested all exponents below 21 842 101 at least once.

As of June 2009, 47 Mersenne primes are known, the largest from $p = 43\,112\,609$, but the region after the 40th known Mersenne prime has not been completely searched, so identification of the nth Mersenne prime is tentative for $n \geq 41$.

References

1 T. L. Heath, *The Thirteen Books of Euclid's Elements* (Dover, New York, 1956).

2 Nicomichus of Gerasa (translated by M. L. D'Ooge), *Introduction to Arithmetic* (Macmillan, New York, 1926).

3 D. M. Burton, *Elementary Number Theory*, 3rd edn. (Brown, Dubuque, IA, 1994).
4 http://www.mersenne.org

***Sandra Pulver** is a professor of mathematics at Pace University, New York, where she has taught for 45 years. Active in professional organizations at the city, state, and national levels, she is a coordinator and host of the annual Greater Metropolitan New York Math Fair, which serves as a forum for outstanding high school students to display original work in mathematics. She also has served as a regional chairperson of the international Mathematics Associations of America (MAA) contest. For the past 18 years, she has served as a grand judge in the mathematics category of the International Science and Engineering Fair, which showcases the creative and innovative research projects of the world's brightest young scholars from the US and more than 25 other countries.*

Conjuring tricks with numbers

1. Choose a domino with a and b dots, and hide it. Add 6 to twice a, multiply by 5, add b, and subtract 30. Give me this number and I will tell you the number of dots on the domino. What are they?

2. Start with your date of birth, multiply the day of the month by 25 and add 12, multiply by 4, subtract 28, add the month, multiply by 5, subtract 10, multiply by 20, subtract 1800, and add the last two digits of the year. The result is your date of birth.

3. Choose three digits a, b, and c. Use your calculator to check that $abcabc$ is divisible by 7, 11, and 13.

Faculty of Applied Engineering Sciences,
University College, Ghent, Belgium
Tanja Van Hecke

Curious triplets

The numbers 3, 4, 6 have the property that the sum of any two of them when divided by the third leaves remainder 1. Can you find other triplets with this property?

Lucknow, India
M. A. Khan

Divisibility and Periodicity in the Fibonacci Sequence

JAY L. SCHIFFMAN

A look at divisibility properties of Fibonacci numbers.

The Fibonacci sequence

The *Fibonacci sequence* F_n is defined recursively as follows:

$$\mathrm{F}_1 = \mathrm{F}_2 = 1,$$
$$\mathrm{F}_n = \mathrm{F}_{n-2} + \mathrm{F}_{n-1}, \quad \text{for } n \geq 3.$$

When it is required, we define $\mathrm{F}_0 = 0$. It is well known that every third Fibonacci number is even and every fifth Fibonacci number is divisible by 5. In contrast, some easily derived results may be less well known including that no odd Fibonacci number is divisible by certain primes including 17 and 61 and no Fibonacci number is congruent to either 4 modulo 8 or 6 modulo 8. Another classical result asserts that one of the first n^2 terms is divisible by n. Our goal is to illustrate how modular arithmetic can be used in discovering signs of divisibility and periodicity ideas in the Fibonacci sequence. As a postscript, we will briefly view the Lucas sequence as a foil to the Fibonacci sequence.

For example, the following properties hold.

1. Every third Fibonacci number is even.
2. Every fourth Fibonacci number is divisible by 3.
3. Every fifth Fibonacci number is divisible by 5.
4. Every sixth Fibonacci number is divisible by 4.
5. Every seventh Fibonacci number is divisible by 13.
6. Every eighth Fibonacci number is divisible by 7.
7. Every ninth Fibonacci number is divisible by 17.
8. Every tenth Fibonacci number is divisible by 11.
9. Every twelfth Fibonacci number is divisible by 6, 9, 12, and 16.
10. Every fourteenth Fibonacci number is divisible by 29.
11. Every fifteenth Fibonacci number is divisible by 10 and 61.
12. Every twentieth Fibonacci number is divisible by 15.

Table 1 The periodicity of the Fibonacci sequence modulo n for $n = 1, \ldots, 100$.

n	Periodicity of F_n	n	Periodicity of F_n	n	Periodicity of F_n	n	Periodicity of F_n
1	1	26	84	51	72	76	18
2	3	27	72	52	84	77	80
3	8	28	48	53	108	78	168
4	6	29	14	54	72	79	78
5	20	30	120	55	20	80	120
6	24	31	30	56	48	81	216
7	16	32	48	57	72	82	120
8	12	33	40	58	42	83	168
9	24	34	36	59	58	84	48
10	60	35	80	60	120	85	180
11	10	36	24	61	60	86	264
12	24	37	76	62	30	87	56
13	28	38	18	63	48	88	60
14	48	39	56	64	96	89	44
15	40	40	60	65	140	90	120
16	24	41	40	66	120	91	112
17	36	42	48	67	136	92	48
18	24	43	88	68	36	93	120
19	18	44	30	69	48	94	96
20	60	45	120	70	240	95	180
21	16	46	48	71	70	96	48
22	30	47	32	72	24	97	196
23	48	48	24	73	148	98	336
24	24	49	112	74	228	99	120
25	100	50	300	75	200	100	300

Consider, for example, divisibility by 5. The Fibonacci sequence modulo 5 is given by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-2} + F_{n-1}$, for $n \geq 2$, and is thus

$$0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, \ldots,$$

and hence repeats from the twentieth term. It follows that every fifth term is congruent to $0 \pmod 5$ and so is divisible by 5. Similarly the Fibonacci sequence modulo 7 is

$$0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0, 1, 1, \ldots,$$

so we have repeats from the sixteenth term. Thus every eighth term is congruent to $0 \pmod 7$ and so is divisible by 7.

Next consider divisibility by n. Since there are n^2 pairs a, b such that $0 \leq a$ and $b \leq n-1$, the Fibonacci sequence modulo n necessarily repeats by the n^2 term. Since repetition of consecutive pairs works backwards as well as forwards, it must repeat from the beginning by the n^2 term. Table 1 gives the number of terms in the repeating block of the Fibonacci sequence modulo n for $n = 1, \ldots, 100$.

Table 2 The first Fibonacci number divisible by n for $n = 1, \ldots, 100$.

n	First Fibonacci number divisible by n	n	First Fibonacci number divisible by n	n	First Fibonacci number divisible by n	n	First Fibonacci number divisible by n
1	F_1	26	F_{21}	51	F_{36}	76	F_{18}
2	F_3	27	F_{36}	52	F_{42}	77	F_{40}
3	F_4	28	F_{24}	53	F_{27}	78	F_{84}
4	F_6	29	F_{14}	54	F_{36}	79	F_{78}
5	F_5	30	F_{60}	55	F_{10}	80	F_{60}
6	F_{12}	31	F_{30}	56	F_{24}	81	F_{108}
7	F_8	32	F_{24}	57	F_{36}	82	F_{60}
8	F_6	33	F_{20}	58	F_{42}	83	F_{84}
9	F_{12}	34	F_9	59	F_{58}	84	F_{24}
10	F_{15}	35	F_{40}	60	F_{60}	85	F_{45}
11	F_{10}	36	F_{12}	61	F_{15}	86	F_{132}
12	F_{12}	37	F_{19}	62	F_{30}	87	F_{28}
13	F_7	38	F_{18}	63	F_{24}	88	F_{30}
14	F_{24}	39	F_{28}	64	F_{48}	89	F_{11}
15	F_{20}	40	F_{30}	65	F_{35}	90	F_{60}
16	F_{12}	41	F_{20}	66	F_{60}	91	F_{56}
17	F_9	42	F_{24}	67	F_{68}	92	F_{24}
18	F_{12}	43	F_{44}	68	F_{18}	93	F_{60}
19	F_{18}	44	F_{30}	69	F_{24}	94	F_{48}
20	F_{30}	45	F_{60}	70	F_{120}	95	F_{90}
21	F_8	46	F_{24}	71	F_{70}	96	F_{24}
22	F_{30}	47	F_{16}	72	F_{12}	97	F_{49}
23	F_{24}	48	F_{12}	73	F_{37}	98	F_{168}
24	F_{12}	49	F_{56}	74	F_{57}	99	F_{60}
25	F_{25}	50	F_{75}	75	F_{100}	100	F_{150}

Since this sequence repeats modulo n and $F_0 = 0$ there must be $m > 0$ such that F_m is divisible by n. Let m be the smallest such positive integer. Thus the Fibonacci sequence modulo n is

$$
\begin{aligned}
&0 = F_0, F_1, F_2, \ldots, F_{m-1}, F_m \equiv 0,\\
&F_{m-1}, -F_{m-2}, F_{m-3}, -F_{m-4}, \ldots, (-1)^m F_1, F_{2m} \equiv (-1)^{m+1} F_0 = 0,\\
&(-1)^m F_1, (-1)^m F_2, \ldots, (-1)^m F_{m-1}, F_{3m} = (-1)^m F_m \equiv 0,\\
&(-1)^m F_{m-1}, -(-1)^m F_{m-2}, (-1)^m F_{m-3}, \ldots, F_{4m} = 0, \ldots.
\end{aligned}
$$

Since $F_1, F_2, \ldots, F_{m-1} \not\equiv 0 \pmod{n}$, it follows that the terms which are divisible by n are precisely $F_m, F_{2m}, F_{3m}, \ldots$.

Table 2 gives the first Fibonacci number divisible by n for $n = 1, \ldots, 100$.

In order to secure the period of the Fibonacci sequence for composite indices, the algebra of residue classes and the lowest common multiple are both key. For example, the period of the Fibonacci sequence modulo 10 is 60. The period of the sequence modulo 2 is 3 (the sequence of remainders is $1, 1, 0, 1, 1, 0, \ldots$) and the period of the sequence modulo 5 is 20 (the sequence of remainders is $1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, 2, 3, \ldots$). Since the highest common factor of 3 and 20 is 1, the period of the Fibonacci sequence modulo 10 is the lowest common multiple of 3 and 20, which is 60.

The period of the Fibonacci sequence modulo 98 is 336, which represents the longest period in table 1.

While the period of the unit's digit in the Fibonacci sequence is of length 60, the period of the last two digits is of length 300 and of the last three digits is of length 1500. The periods thereafter increase ten-fold. Thus the period of the last four digits is 15 000, the period of the last five digits is 150 000, and so on.

In examining table 2, we notice that the composite integer 98 is not a divisor of the Fibonacci sequence initially until the 168th term. To find the initial Fibonacci number which is divisible by a given integer, we first factorize that integer into its prime factors. For example, the entry point for the composite integer 90 can be secured as follows. First note that $90 = 2 \times 3^2 \times 5$. The entry point for 2 is F_3, the entry point for $9 = 3^2$ is F_{12}, and the entry point for 5 is F_5. The lowest common multiple of 3, 12, and 5 is 60. Hence F_{60} is the first Fibonacci number to be divisible by 90.

It can be difficult to find the initial entry points of primes into the sequence. On the other hand, there are some established results that enable us to obtain a foothold on the possible entry points into the sequence for primes and entail the following.

1. For any prime p, the initial Fibonacci number to be divisible by p must occur by the time we reach F_{p^2} terms by the above.

2. The prime p will enter the Fibonacci sequence at the latest at either the position F_{p-1} or F_{p+1} while if earlier then at a divisor of these. The only exception is the prime 5 which enters the sequence as the fifth term. (We are asserting that $F_5 = 5$.) Utilizing mathematical parlance, if $p \equiv 1 \pmod 5$ or if $p \equiv 4 \pmod 5$, then $p \mid F_{p-1}$, while if $p \equiv 2 \pmod 5$ or if $p \equiv 3 \pmod 5$, then $p \mid F_{p+1}$. Moreover, $5 \mid F_5$.

To cite some examples, $19 \mid F_{18}$ and $7 \mid F_8$. On the other hand, we note that although $13 \mid F_{14}$ from this result, the first Fibonacci number to be divisible by 13 is $F_7 = 13$. Similarly, $61 \mid F_{60}$, but $F_{15} = 610$ is the first Fibonacci number to be divisible by 61.

Some integers entering the Fibonacci sequence for the initial time coincide with their term numbers in the sequence. This is true with the integers 5 and 25 (the entry point for 5 is the fifth term) as well as the integers 12 and 60, and is not a coincidence. We can prove that the entry point for integers of the form 5^n and 12×5^n, for $n \geq 0$, have this property (reference 1). I checked the entry points into the Fibonacci sequence for each of the primes below 1000 using MATHEMATICA® as well as the smaller ones with the Texas Instruments TI VOYAGE 200 graphics calculator. Observe from table 2 that since the prime 17 initially enters the Fibonacci sequence at the ninth term, the Fibonacci numbers which are divisible by 17 are $F_9, F_{18}, F_{27}, \ldots$, and these are all even integers as the subscripts are multiples of 3. Hence no odd Fibonacci number is divisible by 17. This is likewise true for the primes 19, 23, 31, 53, 61, 79, and 83. No Fibonacci number has a remainder of either four or six upon division by eight and that the length of the period of the Fibonacci numbers modulo 8 is twelve. The Fibonacci sequence modulo 8 is 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, after which it repeats, so no Fibonacci number is congruent to either $4 \pmod 8$ or $6 \pmod 8$.

The Lucas sequence

The Lucas sequence L_n obeys the identical recursion rule as the Fibonacci sequence. The difference is that the initial two terms are 1 and 3 respectively. Thus $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-2} + L_{n-1}$ for $n \geq 3$. When it is required, we define $L_0 = 2$. The initial two values indeed play a role in changes with regards to divisibility. For example, in contrast to the Fibonacci numbers, no Lucas number is divisible by any of the following primes less than one hundred: 5, 13, 17, 37, 53, 61, 73, 89, and 97. On average, one-third of the prime numbers never enter the Lucas sequence (see reference 3). For example, to show that no Lucas number is divisible by 5, examine the sequence of remainders modulo 5 and note that 0 never occurs. On the other hand, the period of the Lucas numbers modulo n is always finite and repeats in a manner similar to the Fibonacci sequence.

Concluding remarks

The use of technology enables us to discover new insights, seek patterns, and form conjectures based upon the analysis of these patterns into the fascinating world of the Fibonacci sequence. Vehicles such as the excellent *On-Line Encyclopedia of Integer Sequences*, managed by Neil A. J. Sloane at AT&T Labs (see reference 7), and the website *Wolfram MathWorld*, managed by Eric Weisstein (see reference 6), and the computer algebra system MATHEMATICA promise additional stimulating results and lend themselves well for further research.

Acknowledgements

The author would like to thank his colleagues Abdul Hassen, Thomas Osler, and Marcus Wright as well as the Editor for their helpful suggestions and comments which served to improve the manuscript.

References

1 B. Cloitre, Private communication.

2 R. Honsberger, A second look at the Fibonacci and Lucas numbers, in *Mathematical Gems III* (MAA, Washington, DC, 1985).

3 L. C. Lagarias, The set of primes dividing the Lucas numbers has density $\frac{2}{3}$, *Pacific J. Math.* **118** (1985), pp. 449–461.

4 M. Renault, The period of F(mod m) for 1 < m < 2002, available at http://www.math.temple.edu/~renault/fibonacci/fiblist.html

5 N. N. Vorob′ev, *Fibonacci Numbers* (Blaisdell, New York, 1961).

6 http://mathworld.wolfram.com/

7 http://oeis.org/

Jay L. Schiffman *has taught mathematics at Rowan University for the past seventeen years. His research interests are number theory, discrete mathematics, and the interface of mathematics with technology in the classroom as manifested via the use of graphics calculators, computer algebra systems, and mathematical websites.*

Fibonacci, Lucas, and Pell Numbers, and Pascal's Triangle

THOMAS KOSHY

Fibonacci, Lucas, Pell, and Pell–Lucas numbers belong to a large family of positive integers. Using Lockwood's identity, developed from the binomial theorem, we show how they can be computed from Pascal's triangle, the well-known triangular array of the binomial coefficients $\binom{n}{k}$, where $0 \leq k \leq n$. This close link with Pascal's triangle brings these numbers within reach of many mathematical amateurs.

Introduction

Fibonacci numbers F_n and Lucas numbers L_n continue to provide invaluable opportunities for exploration, and contribute handsomely to the beauty of mathematics, especially number theory. They are often defined recursively as follows:

$$F_1 = 1 = F_2, \qquad\qquad L_1 = 1,\ L_2 = 3,$$
$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 3; \qquad L_n = L_{n-1} + L_{n-2}, \quad n \geq 3.$$

(Both Fibonacci and Lucas numbers satisfy the same recurrence relation.) These recursive definitions can be used to develop Binet's explicit formulas (see reference 1)

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2}$$

are solutions of the quadratic equation $x^2 = x + 1$ and $n \geq 1$. Notice that $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$, and $\alpha\beta = -1$. (These facts will come in handy a bit later.) They can be confirmed using induction, or generating functions as the French mathematician Abraham De Moivre (1667–1754) did in 1718. The first five Fibonacci numbers are 1, 1, 2, 3, and 5; the first five Lucas numbers are 1, 3, 4, 7, and 11.

Lucas' formula

In 1876, the French mathematician François Edouard Anatole Lucas (1842–1891) discovered another explicit formula for F_n:

$$F_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{k}, \tag{1}$$

where $\lfloor x \rfloor$ denotes the *floor* of the real number x, that is, the greatest integer less than or equal to x.

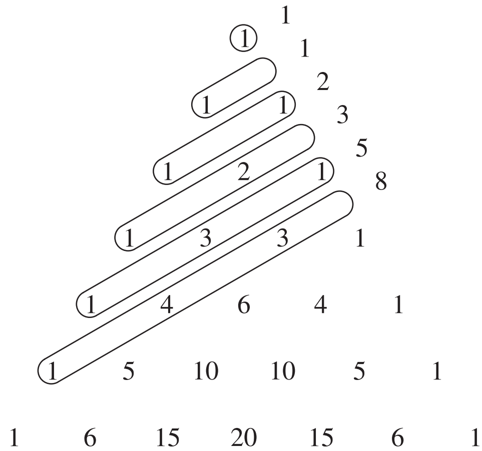


Figure 1 Pascal's triangle.

This formula can be established using the strong version of induction. Since

$$\sum_{k=0}^{0} \binom{1-k-1}{k} = \binom{0}{0} = 1 = \mathrm{F}_1$$

and

$$\sum_{k=0}^{0} \binom{2-k-1}{k} = \binom{0}{0} = 1 = \mathrm{F}_2,$$

(1) holds when $n = 1$ and $n = 2$.

Now assume that (1) holds for all positive integers less than or equal to m, where m is an arbitrary positive integer, i.e.

$$\mathrm{F}_m = \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-k-1}{k}.$$

There are two cases we need to consider: m is odd and m is even. In each case, using Pascal's identity and a lot of algebra, it can be shown that the formula works for $n = m+1$ (see reference 1). Thus, by the strong version of induction, Lucas' formula holds for every positive integer n.

It follows by Lucas' formula that Fibonacci numbers can be obtained as sums of the binomial coefficients along the rising-diagonals in Pascal's triangle; see figure 1. We will develop another method for computing them from Pascal's triangle. But before we do, we will show how Lucas, Pell, and Pell–Lucas numbers (defined later) can be extracted from the array. To this end, we need an identity which can be obtained from the binomial theorem.

Lockwood's identity

Let x and y be arbitrary real numbers. Then, by the binomial theorem, we have

$$\begin{aligned}
x + y &= (x + y), \\
x^2 + y^2 &= (x + y)^2 - 2xy, \\
x^3 + y^3 &= (x + y)^3 - 3(xy)(x + y), \\
x^4 + y^4 &= (x + y)^4 - 4(xy)(x + y)^2 + 2(xy)^2, \\
x^5 + y^5 &= (x + y)^5 - 5(xy)(x + y)^3 + 5(xy)^2(x + y).
\end{aligned}$$

In each case, the expression $x^n + y^n$ is expressed as a sum of $\lfloor n/2 \rfloor + 1$ terms in xy and $x + y$.

More generally, we have the following identity, developed by E. H. Lockwood in 1967 (see reference 2):

$$x^n + y^n = (x + y)^n + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x + y)^{n-2k},$$

where $n \geq 1$. This identity also can be confirmed using the strong version of induction, Pascal's identity, and a lot of messy algebra. It can be rewritten as follows:

$$x^n + y^n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x + y)^{n-2k}, \tag{2}$$

where

$$\binom{r}{-1} = 0.$$

It follows from (2), for example, that

$$x^7 + y^7 = (x + y)^7 - 7(xy)(x + y)^5 + 14(xy)^2(x + y)^3 - 7(xy)^3(x + y).$$

Lucas numbers and Pascal's triangle

Lockwood's identity yields several interesting dividends. To begin with, we now can extract Lucas numbers from Pascal's triangle. To see this, we let $x = \alpha$ and $y = \beta$ in (2). This yields

$$\begin{aligned}
\mathrm{L}_n &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (-1)^k \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right].
\end{aligned} \tag{3}$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad \mathbf{3} \quad 1 \\
1 \quad \mathbf{4} \quad 6 \quad \mathbf{4} \quad 1 \\
\mathbf{1} \quad 5 \quad \mathbf{10} \quad 10 \quad 5 \quad 1 \\
1 \quad \mathbf{6} \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
\mathbf{1} \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}
$$

Figure 2

Consequently, L_n can be computed by adding up the elements along two alternate rising-diagonals. For example,

$$
\begin{aligned}
\mathrm{L}_7 &= \sum_{k=0}^{3}\left[\binom{7-k}{k}+\binom{6-k}{k-1}\right] \\
&= \left[\binom{7}{0}+\binom{6}{-1}\right]+\left[\binom{6}{1}+\binom{5}{0}\right]+\left[\binom{5}{2}+\binom{4}{1}\right]+\left[\binom{4}{3}+\binom{3}{2}\right] \\
&= (1+0)+(6+1)+(10+4)+(4+3) \\
&= (0+1+4+3)+(1+6+10+4) \\
&= 29
\end{aligned}
$$

(see the bold-faced numbers in figure 2).

Notice that (2) can also be written as follows:

$$x^n + y^n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(xy)^k(x+y)^{n-2k}.$$

Consequently,

$$\mathrm{L}_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k}.$$

Thus L_n can be computed using the elements on the rising-diagonal beginning at $\binom{n}{0}$ with weights $n/(n-k)$. For example,

$$\begin{aligned}
L_7 &= \sum_{k=0}^{3} \frac{7}{7-k}\binom{7-k}{k} \\
&= \frac{7}{7}\binom{7}{0} + \frac{7}{6}\binom{6}{1} + \frac{7}{5}\binom{5}{2} + \frac{7}{4}\binom{4}{3} \\
&= 1 + 7 + 14 + 7 \\
&= 29,
\end{aligned}$$

as expected.

Equation (3), coupled with (1), yields the following well-known formula connecting Fibonacci and Lucas numbers:

$$\begin{aligned}
L_n &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k-1} + \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \\
&= \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-i-2}{i} + \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \\
&= F_{n-1} + F_{n+1}.
\end{aligned}$$

This can also be established using Binet's formulas, which is a lot simpler (see reference 1).

Next we turn to Pell and Pell–Lucas numbers.

Pell and Pell–Lucas families

Pell numbers P_n and Pell–Lucas numbers Q_n are also often defined recursively as follows:

$$\begin{aligned}
&P_1 = 1,\ P_2 = 2, && Q_1 = 1,\ Q_2 = 3, \\
&P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 3; && Q_n = 2Q_{n-1} + Q_{n-2}, \quad n \geq 3.
\end{aligned}$$

They can also be defined by Binet-like formulas as follows:

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}, \qquad Q_n = \frac{\gamma^n + \delta^n}{2},$$

where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ are solutions of the quadratic equation $x^2 = 2x + 1$, and $n \geq 1$. Notice that $\gamma + \delta = 2$, $\gamma - \delta = 2\sqrt{2}$, and $\gamma\delta = -1$. The first five Pell numbers are 1, 2, 5, 12, and 29; the first five Pell–Lucas numbers are 1, 3, 7, 17, and 41.

Pell–Lucas numbers and Pascal's triangle

We can also extract Pell–Lucas numbers from Pascal's triangle with proper weights. To see this, we let $x = \gamma$ and $y = \delta$ in (2). Then

$$Q_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \left[\binom{n-k}{k} + \binom{n-k-1}{k-1}\right] 2^{n-2k-1}.$$

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

Figure 3

For example,

$$\begin{aligned}
Q_7 &= \sum_{k=0}^{3}\left[\binom{7-k}{k}+\binom{6-k}{k-1}\right]2^{6-2k} \\
&= \left[\binom{7}{0}+\binom{6}{-1}\right]2^6+\left[\binom{6}{1}+\binom{5}{0}\right]2^4+\left[\binom{5}{2}+\binom{4}{1}\right]2^2+\left[\binom{4}{3}+\binom{3}{2}\right]2^0 \\
&= (1+0)\cdot 2^6+(6+1)\cdot 2^4+(10+4)\cdot 2^2+(4+3)\cdot 2^0 \\
&= 239.
\end{aligned}$$

Consequently, Q_7 can be computed by multiplying the sums of the entries inside the solid loops beginning at $\binom{7}{0}$ in figure 3 by the weights 2^6, 2^4, 2^2, and 2^0 respectively, and then adding up the products.

Likewise, $Q_6 = (1+0)\cdot 2^5 + (5+1)\cdot 2^3 + (6+3)\cdot 2^1 + (1+1)\cdot 2^{-1} = 99$; see the dotted loops in figure 3.

Odd-numbered Fibonacci numbers and Pascal's triangle

Next we will show that odd-numbered Fibonacci numbers can be computed from Pascal's triangle in a different way. To this end, we let n be odd and change y to $-y$ in (2). Then we obtain

$$x^n - y^n = \sum_{k=0}^{(n-1)/2}(-1)^k\left[\binom{n-k}{k}+\binom{n-k-1}{k-1}\right](-xy)^k(x-y)^{n-2k}. \tag{4}$$

Letting $x=\alpha$ and $y=\beta$, this yields

$$(\alpha-\beta)F_n = \sum_{k=0}^{(n-1)/2}(-1)^k\left[\binom{n-k}{k}+\binom{n-k-1}{k-1}\right](\alpha-\beta)^{n-2k}.$$

So

$$\mathrm{F}_n = \sum_{k=0}^{(n-1)/2} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] 5^{(n-2k-1)/2},$$

where n is odd.

For example,

$$\begin{aligned}
\mathrm{F}_7 &= \sum_{k=0}^{3} (-1)^k \left[\binom{7-k}{k} + \binom{6-k}{k-1} \right] 5^{3-k} \\
&= \left[\binom{7}{0} + \binom{6}{-1} \right] 5^3 - \left[\binom{6}{1} + \binom{5}{0} \right] 5^2 + \left[\binom{5}{2} + \binom{4}{1} \right] 5^1 - \left[\binom{4}{3} + \binom{3}{2} \right] 5^0 \\
&= (1+0) \cdot 5^3 - (6+1) \cdot 5^2 + (10+4) \cdot 5^1 - (4+3) \cdot 5^0 \\
&= 13
\end{aligned}$$

(see the solid loops in figure 3).

Odd-numbered Pell numbers and Pascal's triangle

Using (4), odd-numbered Pell numbers also can be computed from Pascal's triangle. To see this, letting $x = \gamma$ and $y = \delta$, (4) yields

$$\begin{aligned}
(\gamma - \delta)\mathrm{P}_n &= \sum_{k=0}^{(n-1)/2} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (-\gamma\delta)^k (\gamma - \delta)^{n-2k}, \\
\mathrm{P}_n &= \sum_{k=0}^{(n-1)/2} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (2\sqrt{2})^{n-2k-1} \\
&= \sum_{k=0}^{(n-1)/2} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] 8^{(n-2k-1)/2},
\end{aligned}$$

where n is odd. Thus, P_n can be computed using the same loops for F_n, but with different weights, where n is odd.

For example,

$$\begin{aligned}
\mathrm{P}_7 &= \sum_{k=0}^{3} (-1)^k \left[\binom{7-k}{k} + \binom{6-k}{k-1} \right] 8^{3-k} \\
&= (1+0) \cdot 8^3 - (6+1) \cdot 8^2 + (10+4) \cdot 8^1 - (4+3) \cdot 8^0 \\
&= 169
\end{aligned}$$

(see the solid loops in figure 3).

Acknowledgments

The author would like to thank Angelo Di Domenico and the Editor for their helpful comments and suggestions for improving an earlier version of this article.

References

1 T. Koshy, *Fibonacci and Lucas Numbers with Applications* (John Wiley, New York, 2001).
2 E. H. Lockwood, A side-light on Pascal's triangle, *Math. Gazette* **51** (1967), pp. 243–244.

Thomas Koshy *received his PhD in algebraic coding theory. He is an author of seven books, including 'Fibonacci and Lucas Numbers with Applications' and 'Catalan Numbers with Applications', and a recipient of a number of awards, including the Faculty of the Year Award in 2007. He retired in 2010 after forty years at Framingham State College, Massachusetts.*

Consecutive numbers divisible by perfect squares

$$48 = 3 \times 4^2, \qquad 49 = 1 \times 7^2, \qquad 50 = 2 \times 5^2,$$
$$98 = 2 \times 7^2, \qquad 99 = 11 \times 3^2, \qquad 100 = 1 \times 10^2,$$
$$1680 = 105 \times 4^2, \qquad 1681 = 1 \times 41^2, \qquad 1682 = 2 \times 29^2,$$
$$1683 = 187 \times 3^2, \qquad 1684 = 421 \times 2^2.$$

Can readers find other examples?

Students' Investigation House,
Before Farhang Crossroads,
Shariati Avenue, Sirjan, Iran

Abbas Rooholamini Gugheri

Sums of cubes

The equation $x^3 + y^3 = 7$ has a solution in rational numbers

$$\left(x = \tfrac{5}{3},\ y = \tfrac{4}{3}\right).$$

The equation $x^3 + y^3 = z^3$ has no solution in positive integers by Fermat's last theorem, but

$$x^3 + y^3 = z^3 - 1$$

has a solution $x = 6,\ y = 8,\ z = 9$.

Students' Investigation House,
Before Farhang Crossroads,
Shariati Avenue, Sirjan, Iran

Abbas Rooholamini Gugheri

Letters to the Editor

Dear Editor,

Amicable chains and half-amicable pairs

In their article 'Friends in high places' (see Volume 42, Number 2, pp. 54–58), Roger Webster and Gareth Williams considered *amicable pairs* of numbers such as 220, 284, each of which is the sum of the proper divisors of the other. Now I acquaint readers with *amicable chains*. If we denote the sum of the proper divisors of n by $S(n)$, then

$$\begin{aligned} &S(12496) = 14288, \qquad &&S(14288) = 15472,\\ &S(15472) = 14536, &&S(14536) = 14264,\\ &S(14264) = 12496. \end{aligned}$$

Similar amicable chains are

$$2115324,\ 3317740,\ 3649556,\ 2797612$$

and

$$1264460,\ 1547860,\ 1727636,\ 1305184.$$

There is an amicable chain of 28 numbers beginning with 14316.

If we exclude the divisor 1, then the pair 48, 75 form a *half-amicable pair*; so do 140, 195.

Yours sincerely,

Abbas Rooholamini Gugheri
(Students' Investigation House
Before Farhang Crossroads
Shariati Avenue
Sirjan
Iran)

Dear Editor,

Message in a bottle

I enjoyed Prithwijit De's article 'Message in a bottle' (see Volume 43, Number 2, pp. 50–52), but thought readers might be interested that the minimisation of the expression for the surface area,

$$A = r^2\pi\left(1 + \frac{3 - 2\cos\alpha}{3\sin\alpha}\right) + \frac{2V}{r},$$

for $r > 0$ and $0 < \alpha < \pi/2$, can be achieved algebraically rather than by using multivariable calculus. For, by the arithmetic–geometric mean inequality, we have

$$\begin{aligned} \frac{A^3}{27} &= \left[\frac{1}{3}\left(r^2\pi\left(1 + \frac{3 - 2\cos\alpha}{3\sin\alpha}\right) + \frac{V}{r} + \frac{V}{r}\right)\right]^3\\ &\geq \pi V^2\left(1 + \frac{3 - 2\cos\alpha}{3\sin\alpha}\right)\\ &= \pi V^2\left(1 + \frac{5t}{6} + \frac{1}{6t}\right), \end{aligned}$$

on substituting $t = \tan(\alpha/2)$ and using $\cos\alpha = (1-t^2)/(1+t^2)$ and $\sin\alpha = 2t/(1+t^2)$. Applying the arithmetic–geometric mean inequality again for $5t/6 + 1/6t$ then shows that

$$\pi V^2\left(1 + \frac{5t}{6} + \frac{1}{6t}\right) \geq \pi V^2\left(1 + \frac{\sqrt{5}}{3}\right).$$

Taken together, these two inequalities identify the minimising area as $\sqrt[3]{9(3+\sqrt{5})\pi V^2}$, which is attained when we have equality throughout. This occurs when

$$\frac{5t}{6} = \frac{1}{6t} \quad \text{and} \quad \frac{V}{r} = r^2\pi\left(1 + \frac{3 - 2\cos\alpha}{3\sin\alpha}\right)$$

so that $\tan^2(\alpha/2) = \frac{1}{5}$, i.e. $\cos\alpha = \frac{2}{3}$, $\sin\alpha = \sqrt{5}/3$, and

$$\frac{V}{r} = r^2\pi\left(1 + \frac{\sqrt{5}}{3}\right) \quad \text{or} \quad r = \sqrt[3]{\frac{3V}{(3+\sqrt{5})\pi}}.$$

Yours sincerely,

Nick Lord
(Tonbridge School
Tonbridge
Kent TN9 1JP
UK)

Dear Editor,

First 100 numbers

If the first 100 numbers are placed in alphabetical order, the following placings are noticed:

1 is placed 50th,	5 is placed 25th,	9 is placed 38th,
2 is placed 100th,	6 is placed 64th,	10 is placed 76th,
3 is placed 88th,	7 is placed 52nd,	100 is placed 51st.
4 is placed 36th,	8 is placed 1st,	

But, if the first 1000 numbers are placed in alphabetical order, the following placings are noticed:

1 is placed 450th,	5 is placed 125th,	9 is placed 338th,
2 is placed 900th,	6 is placed 664th,	10 is placed 776th,
3 is placed 788th,	7 is placed 552nd,	100 is placed 551st.
4 is placed 236th,	8 is placed 1st,	

It is interesting to note that the placings of most numbers in the 1000 series consist of a digit placed at the front of the placing for the same number in the 100 series.

Why is this?

Yours sincerely,

Bob Bertuello
(12 Pinewood Road
Midsomer Norton
Bath BA3 2RG
UK)

Dear Editor,

Primes with a pattern

I have found the following primes which may interest readers:

12345678910987654321,
135791113151719,
1357911131517191715131197531,
19171513119753135791113151719,
9109, 91009, 9100009, 91000009, 9100000009,
91000000000000000009,
910000000000000000000009,
181, 188888881, 188888888888881,
131, 133331, 1333331,
191, 19991, 199999991,
313, 919, 27127, 81181,
131, 191, 1811, 17291, 165611, 115943231.

Yours sincerely,
Bablu Chandra Dey
(25K Christopher Road
Kolkata, 700046
India)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st November will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

43.9 The polynomial $ax^2 + bx + c$, with a, b, c real numbers and $a > 0$, has nonreal roots. When does one of its roots lie on the parabola with equation $y = ax^2 + bx + c$ viewed as lying in the Argand diagram?

(Submitted by Jonny Griffiths, Paston College, Norfolk, UK)

43.10 The nonnegative real numbers satisfy $yz + zx + xy = 1$. Find lower bounds which can be attained for each of the expressions

$$x + y + z \quad \text{and} \quad 2x + y + z.$$

(Submitted by J. A. Scott, Chippenham, UK)

43.11 Sum the finite series

$$\mathrm{F}_1\mathrm{F}_2 + \mathrm{F}_2\mathrm{F}_3 + \mathrm{F}_3\mathrm{F}_4 + \cdots + \mathrm{F}_n\mathrm{F}_{n+1},$$

where F_r denotes the rth Fibonacci number.

(Submitted by M. A. Khan, Lucknow, India)

43.12 The point of intersection of a directrix and the major axis of a conic (an ellipse, hyperbola, or parabola) is denoted by K, and A, B are two points on the conic such that A, B, K are collinear. The reflection of A in the major axis is denoted by D, and F is the focus corresponding to the directrix. Show that B, D, F are collinear.

(Submitted by Zhang Yun, China)

Solutions to Problems in Volume 43 Number 1

43.1 *Numerical partial fractions* Given two fractions, it is easy to compute their sum as a single fraction. For example,

$$\frac{3}{5} + \frac{7}{13} = \frac{(3 \times 13) + (7 \times 5)}{5 \times 13} = \frac{74}{65}.$$

However, the converse problem is not so easy. Find integers a, b, c, d, f such that

$$\frac{325}{1357} = \frac{a}{59} + \frac{b}{23}, \qquad \frac{257}{1001} = \frac{c}{7} + \frac{d}{11} + \frac{f}{13}.$$

Solution by Henry Ricardo, Tappan, NY, USA and by Gian Paolo Almirante, Milano, Italy

We use Euclid's algorithm on 59 and 23:

$$\begin{aligned}
59 &= 2 \times 23 + 13,\\
23 &= 1 \times 13 + 10,\\
13 &= 1 \times 10 + 3,\\
10 &= 3 \times 3 + 1,
\end{aligned}$$

so that

$$\begin{aligned}
1 &= 10 - 3 \times 3\\
&= 10 - 3(13 - 10)\\
&= -3 \times 13 + 4 \times 10\\
&= -3 \times 13 + 4(23 - 13)\\
&= 4 \times 23 - 7 \times 13\\
&= 4 \times 23 - 7(59 - 2 \times 23)\\
&= 18 \times 23 - 7 \times 59,
\end{aligned}$$

so that

$$325 = 5850 \times 23 - 2275 \times 59$$

and

$$\tfrac{325}{1357} = \tfrac{5850}{59} - \tfrac{2275}{23}.$$

For simpler values of a, b, we write

$$325 = a \times 23 + b \times 59,$$

so that

$$(5850 - a)23 = (2275 + b)59$$

and

$$5850 - a = 59k, \qquad 2275 + b = 23k,$$

for some integer k. Hence

$$(a, b) = (5850 - 59k, -2275 + 23k).$$

If we put $k = 99$, we obtain $(a, b) = (9, 2)$ which gives

$$\tfrac{325}{1357} = \tfrac{9}{59} + \tfrac{2}{23}.$$

The second numerical partial fraction is equivalent to finding integers c, d, f such that

$$143c + 91d + 77f = 257.$$

The highest common factor of 143 and 91 is 13, and

$$2 \times 143 - 3 \times 91 = 13.$$

(This can be obtained by using Euclid's algorithm on 143 and 91.) Also

$$2 \times 13 + 3 \times 77 = 257,$$

so that

$$257 = 4 \times 143 - 6 \times 91 + 3 \times 77,$$

and one solution is

$$\tfrac{257}{1001} = \tfrac{4}{7} - \tfrac{6}{11} + \tfrac{3}{13}.$$

Also solved by Abbas Rooholamini Gugheri (Sirjan, Iran).

43.2 Given a unit circle, angles x, y measured in radians and arc length z as shown, how are x, y, and z related? (See figure 1.)

Solution by Abbas Rooholamini Gugheri

The angle subtended by arc BC at the centre of the circle is z radians, so that $\angle BDA = \angle CEA = z/2$. Similarly, $\angle DCE = w$, where w is the arc length DE. From triangles CDF and ACE, we have

$$x = \frac{z}{2} + \frac{w}{2} \quad \text{and} \quad \frac{w}{2} = y + \frac{z}{2}.$$

Hence $x = y + z$.

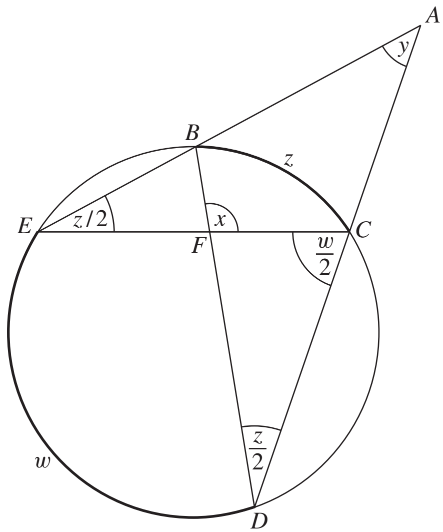


Figure 1

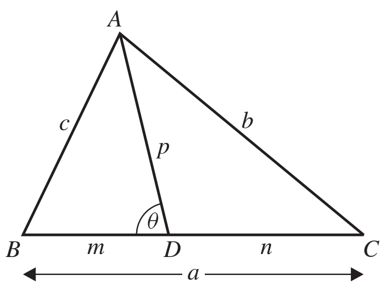


Figure 2

43.3 How are a, b, c, m, n, p related? (See figure 2.)

Solution by Abbas Rooholamini Gugheri, who proposed the problem

Apply the cosine formula to triangles ABD and ACD to give

$$c^2 = m^2 + p^2 - 2mp\cos\theta, \qquad b^2 = n^2 + p^2 - 2np\cos(\pi - \theta),$$

so that

$$mb^2 + nc^2 = m(n^2 + p^2) + n(m^2 + p^2),$$

or

$$mb^2 + nc^2 = (mn + p^2)(m + n).$$

Indika Shameera Amarasinghe (University of Kelaniya, Sri Lanka) sent two further solutions. One is to drop the perpendicular AX from A to BC and use Pythagoras' theorem on triangles ABX and ACX to give

$$c^2 = AX^2 + (m - DX)^2 = p^2 - DX^2 + (m - DX)^2 = p^2 + m^2 - 2mDX$$

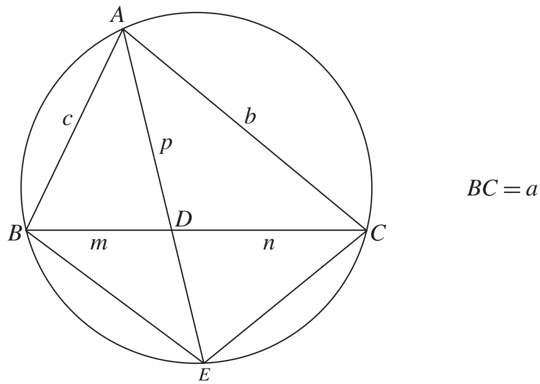


Figure 3

and

$$b^2 = AX^2 + (n + DX)^2 = p^2 + n^2 + 2nDX.$$

Hence

$$mb^2 + nc^2 = m(p^2 + n^2) + n(p^2 + m^2) = (mn + p^2)(m + n).$$

The second solution uses Ptolemy's theorem for the circumcircle of triangle ABC (see figure 3), giving

$$b \times BE + c \times CE = a(p + DE).$$

Triangles ACD and BDE are similar, so that

$$\frac{BE}{b} = \frac{DE}{n} = \frac{m}{p}.$$

Triangles ABD and CDE are similar, so that

$$\frac{CE}{c} = \frac{n}{p}.$$

Hence

$$b\left(\frac{bm}{p}\right) + c\left(\frac{cn}{p}\right) = a(p + DE) = (m + n)\left(p + \frac{mn}{p}\right),$$

whence

$$mb^2 + nc^2 = (mn + p^2)(m + n).$$

43.4 (i) For a fixed positive integer m, determine

$$\sum_{k=1}^{\infty} \frac{1}{k(m + k)}.$$

(ii) For fixed positive integers m, p with $m > p$, determine

$$\sum_{k=1}^{\infty} \frac{1}{(p + k)(p + 1 + k) \cdots (m + k)}.$$

Solution by Abbas Rooholamini Gugheri

(i) We have

$$\frac{1}{k(m+k)} = \left(\frac{1}{k} - \frac{1}{m+k}\right)\frac{1}{m},$$

so that, for $N > m$,

$$\begin{aligned}\sum_{k=1}^{N} \frac{1}{k(m+k)} &= \left(\frac{1}{1} - \frac{1}{m+1}\right)\frac{1}{m} + \left(\frac{1}{2} - \frac{1}{m+2}\right)\frac{1}{m} + \cdots + \left(\frac{1}{N} - \frac{1}{m+N}\right)\frac{1}{m} \\ &= \frac{1}{m}\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{m}\right) - \frac{1}{m}\left(\frac{1}{N+1} + \cdots + \frac{1}{N+m}\right) \\ &\to \frac{1}{m}\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{m}\right) \quad \text{as } N \to \infty.\end{aligned}$$

(ii) We have

$$\begin{aligned}&\frac{1}{(p+k)(p+1+k)\cdots(m+k)} \\ &\quad = \left[\frac{1}{(p+k)(p+1+k)\cdots(m+k-1)} \right. \\ &\qquad \left. - \frac{1}{(p+1+k)(p+2+k)\cdots(m+k)}\right]\frac{1}{m-p},\end{aligned}$$

so

$$\begin{aligned}&\sum_{k=1}^{N} \frac{1}{(p+k)(p+1+k)\cdots(m+k)} \\ &\quad = \frac{1}{m-p}\left[\frac{1}{(p+1)(p+2)\cdots m} - \frac{1}{(p+1+N)(p+2+N)\cdots(m+N)}\right] \\ &\quad \to \frac{1}{(m-p)(p+1)(p+2)\cdots m} \quad \text{as } N \to \infty.\end{aligned}$$

Also solved by Henry Ricardo.

Reviews

Alex's Adventures in Numberland. By Alex Bellos. Bloomsbury, London, 2010. Hardback, 448 pages, £18.99 (ISBN 978-0747597162).

Alex Bellos subtitles this fascinating book *Dispatches from the Wonderful World of Mathematics*, the title itself obviously being inspired by Lewis Carroll's *Alice* books. After taking a degree in mathematics and philosophy Bellos became a journalist, first in Brighton and later as a foreign correspondent including for *The Guardian* in Brazil, one result of which was that he ghost wrote the autobiography of the footballer Pelé.

Bellos writes in a very lively and engaging way, the dispatches he sends from Numberland coming from several areas of mathematics, beginning with the emergence of ideas of number and ending with Cantor and the Hilbert Hotel.

Since we are in Numberland, we begin with Chapter Zero before coming to Chapter One. Chapter Zero is of course appropriate to the time before mathematics really got going when number ideas were slowly emerging. Bellos points out that, like other primitive peoples, Indian tribes in the Amazon Basin have no real sense of numbers beyond three or four. One tribe could distinguish between small numbers of objects but saw larger clusters as being much the same and close together. He describes recent research on children, which suggests that they see numbers in a similar way, essentially on a logarithmic rather than a linear scale as they do when they are older.

Chapter One is a dispatch from the beginnings of mathematics, describing how different peoples have counted and why the decimal system with base 10, reflecting the number of our fingers, has proved so successful even though mathematically a duodecimal system with base 12 has a lot to be said for it, 12 having more factors than 10. (Yes, there is a Dozenal Society trying to convert us to the duodecimal system!) One common way of counting and doing elementary arithmetic was to use an abacus, first introduced by the Romans and certainly used in Victorian schools in Britain. In Japan, however, after-school abacus clubs for children are very popular and Bellos takes us to one, where in a competition one boy was able to multiply two six-digit numbers together in a few seconds. The great 17th century mathematician, Gottfried Leibnitz, favoured the binary system which uses only 0s and 1s and which to him symbolised the cosmos as composed of *being* and *non-being* or nothingness. In wanting this system he was centuries ahead of his time as modern computer technology relies at a basic level on a language composed of 0s and 1s.

In his next dispatch Bellos encounters two of the all-time greats of ancient mathematics, Pythagoras and Euclid. Pythagoras' theorem, that the square on the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides, is the most famous theorem in geometry and Bellos gives a variety of proofs of it, mentioning a book published in 1440 which gave 371 of them. Euclid's *Elements* contained 465 theorems, which summarized the extent of Greek knowledge of geometry around 300 BC. Euclid also investigated the three-dimensional shapes that could be formed by joining identical regular polygons together and found that there were only five. Geometrical shapes have intrigued others besides the Greeks and we are told about their use in Islamic art as well as tessellations and origami.

Zero is now seen as just as much part of the real number system as the positive and negative numbers but this was not always so. Bellos describes the steps by which zero came to be seen as a *bona fide* real number, the first of which were due to Indian mathematicians in the 7th century.

Any dispatches from Numberland must include one from the Land of Pi, the ratio of the circumference of a circle to its diameter. Bellos describes how early attempts to calculate pi fitted regular polygons inside and outside a circle so as to give lower and upper bounds on pi. Two mathematicians in 5th century China used polygons of 12 288 sides to get pi as 3.141592. In the 17th century Leibnitz and Gregory came up with an infinite series for pi, other infinite series coming later, so that pi could then be calculated longhand. The last of the longhand calculators was D. F. Ferguson, who found pi to 620 places by July 1946, but then became the first of the mechanical calculators when he used a desk calculator to add another 200 digits by September 1947. Computers then took over and by September 1949 pi was known to 2037 digits, the present number of known digits far exceeding this. Bellos meets the Chudnovsky brothers, who built a supercomputer out of mail-order parts in their Manhattan apartment and, using this and their own formula, calculated pi to two billion decimal places. Pi was proved irrational in 1767 and transcendental, that is, not the solution of a polynomial equation, much later. Bellos points out that British 50p and 20p coins are examples of seven-sided curves which have constant width, so that, if a 50p piece rolls along the ground, the distance from the ground to the top of the coin is always the same. A circle of course has this property and can be used as a wheel since an axle at its centre is always the same distance above the ground. If circular wheels are used for heavy loads, the axle may not be able to take the strain, whereas rollers the shape of a 50p coin are much more able to do this

Algebra and coordinate geometry too get a dispatch from Numberland in which we meet Descartes, who gave us Cartesian coordinates and hence a method of solving geometric problems algebraically, as well as Tartaglia, who in the 1530's first solved a cubic equation, and Piet Hein, a 20th century Dane, who found a shape halfway between an ellipse and a rectangle, which proved just right for a roundabout in a rectangular plaza in Stockholm.

I like doing sudoko puzzles and so particularly enjoyed reading Bellos' dispatch from Mathematical Puzzle Land to learn that sudoko was devised by a Japanese puzzle maker, Maki Kaji, and launched in his puzzle magazine in the early 1980s. They did not attract much attention at first but spread like wildfire after a New Zealander, Wayne Gould, wrote a program that generated them and persuaded an American newspaper to publish one of his puzzles. Other puzzles Bellos reports on are magic squares, tangrams, the Fifteen puzzle, and Rubik's cube. He also visited Martin Gardner, who sadly died recently, but who for 24 years wrote a monthly mathematics column in *Scientific American* covering a vast range of mathematics in lively and lucid prose as well as writing a best-selling book, *The Annotated Alice*, in which he supplied a compendium of footnotes to Lewis Carroll's *Alice* books.

In his next dispatch Bellos meets Neil Sloane, who has been collecting sequences for nearly fifty years and founded *The On-Line Encyclopedia of Integer Sequences* (http://oeis.org) with more than 16 000 entries. One sequence is of course the prime numbers, the record since 1952 for the largest known prime nearly always being a Mersenne prime of the form $2^n - 1$, where n is an integer. Infinite sequences lead into Zeno's paradox of Achilles and the tortoise as well as how to stack rulers with maximum overhang so that they do not topple. The Greeks were intrigued by *perfect numbers*, numbers whose factors add up to the number itself, the first two of which are 6 and 28, and *amicable numbers*, two numbers such that the factors of each add up to the other, the first such pair being 220 and 284. Bellos tells us that more recently a French mathematician, Rene Poulet, has called a chain of numbers *sociable* if the factors of the first add up to the second, of the second to the third, and so on, until the factors of the last number in the chain add up to the first. Currently 175 chains of sociable numbers are known, almost all chains of four though none of three. These are only some of the fascinating pieces of number lore in this dispatch.

Bellos saves one of the most famous sequences, the *Fibonacci sequence*, for his next dispatch, in which he relates it to the golden ratio, which arises when a line is cut into two sections in such a way that the ratio of the length of the line to that of the longer section equals that of the length of the longer to that of the shorter section, this ratio also being the limit of the ratio of successive terms in the Fibonacci sequence. Besides giving examples of the Fibonacci sequence in nature, Bellos tells us about the *golden rectangle* and the *logarithmic spiral* and how and why the latter appears in nature. He also meets a dentist, who discovered that the most attractive false teeth which make the wearer's smile look right are based on the golden ratio.

Bellos has two dispatches from the lands of probability and statistics. The 17th century French mathematician Blaise Pascal was asked two questions on the likelihood of certain results in the throwing of two dice by a gambler interested in the mathematics of dicing as well as making money. With the help of Fermat, he of the Last Theorem, Pascal was able to answer the questions and in doing so they laid the foundations of modern probability theory. Bellos meets the man who sets odds for more than half the world's slot machines and looks at the casino games of roulette and craps as well as explaining why in a randomly selected group of 23 people it is more likely than not that two people in it will have the same birthday, but it needs 253 people for it to be more likely than not that someone shares your own birthday date.

Bellos describes an experiment he did, inspired by a similar one by the great French mathematician, Henri Poincaré, in which he makes a daily trip to his local baker's to buy a baguette which he then weighs. He finds the weights over quite a few days are distributed as a bell curve or normal distribution with a cluster of weights around one value, falling away to fewer and fewer greater and smaller weights. The bell curve is of course very familiar to statisticians and Bellos describes how its significance for more and more statistical data came to be recognised.

In his last dispatch Bellos reaches the end of the line. In *Elements* Euclid gave five axioms from which he could prove his 465 theorems. Later mathematicians were happy enough with his first four axioms but the fifth bothered them. It later became known as the parallel postulate, being stated in the form that 'given a line and a point not on it, then there is at most one line that goes through the point and is parallel to the original line'. Was this really an axiom? Could it not be proved from the others? In the 19th century a Hungarian, Janos, and a Russian, Lobachevsky, overturned the parallel postulate and showed that there were geometries in which it did not hold. This led to Riemann establishing an all-embracing theory which included Euclidean and non-Euclidean geometries within it. At the same time, Cantor came up with a new way to think about infinity in terms of sets of numbers and their *cardinality*, the number of members of the set, showing that the integers and the rational numbers are countable and so have the same cardinality but the real numbers, not being countable, have a greater cardinality. He pays a visit to the Hilbert Hotel with its infinite number of rooms, whose manager can solve all his problems of accommodating guests.

I very much enjoyed reading this book and can thoroughly recommend it. As I have tried to show in this review, it covers a considerable amount of mathematics and, whilst quite a lot of the material is well-known and in other books, it is presented in lively and interesting ways. I found that even as a professional mathematician I learnt many new and fascinating things. I was also intrigued by the people Bellos meets, who have found new and sometimes unusual applications of mathematics. This is a book which deserves to be in every senior school and university library. If you have not already got a copy, persuade someone to give you it as a present or buy it yourself. It is worth it!

University of Sheffield **Derek Collins**

Other books received

Calculus Deconstructed: A Second Course in First-Year Calculus. By Zbigniew H. Nitecki. The Mathematical Association of America, USA, 2009. Hardback, 491 pages, \$72.50 (ISBN 978-0-883-85756-4).

Applied Data Mining for Business and Industry. By Paolo Giudici and Silva Figini. John Wiley, Chichester, 2nd edition, 2009. Paperback, 249 pages, £34.50 (ISBN 978-0-470-05887-9).

Mathematical Spectrum

Volume 43 2010/2011 Number 3

ISSN 0025-5653

http://ms.appliedprobability.org
Published by the Applied Probability Trust
Printed by Berforts South East Ltd, Stevenage, Hertfordshire, UK