Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
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From the Editor

Sums of squares

In Volume 42, Number 2, I wrote an editorial prompted by Bob Bertuello, one of our readers, entitled 'Difference of squares'. The problem there was to express a number as the difference of two squares. It ended in the air by asking the question: can you express a number as the sum of two squares?

Bob, on the ball as ever, has now sent this problem: find all rectangles with integer sides which have diagonal length 2050. In effect, he is asking for all possible positive integers x, y such that $x^2 + y^2 = 2050^2$, or all *Pythagorean triples* x, y, z with $z = 2050$.

In the previous editorial, we gave formulae for all primitive Pythagorean triples x, y, z, i.e. those with highest common factor 1. One of x, y must be even and the other odd and z must be odd. If we take x to be the even one, then

$$x = 2st, \qquad y = s^2 - t^2, \qquad z = s^2 + t^2,$$

where s, t are integers (with $s > t > 0$), s, t are coprime, and one of s, t is even (so the other must be odd). For example, the smallest primitive Pythagorean triple (4, 3, 5) comes from $(s, t) = (2, 1)$. To get all Pythagorean triples, we multiply through by the highest common factor, d. Thus, in Bob's problem, there is no primitive Pythagorean triple because the third number of a primitive Pythagorean triple is always odd. (I wonder why?) In prime factors,

$$2050 = 2 \times 5^2 \times 41,$$

so the possible values of d are 2, 10, 50, 82, 410. The values $d = 50, 82, 410$ are easily dealt with; they give the Pythagorean triples

$$50 \times (40, 9, 41), \qquad 82 \times (24, 9, 25), \qquad 410 \times (4, 3, 5),$$

so we have found three of Bob's rectangles.

Let us consider the case $d = 2$. We want all (s, t) such that $s^2 + t^2 = 1025$. When we were considering differences of squares, we factorised. Thus, if

$$s^2 - t^2 = 1025,$$

we wrote

$$(s + t)(s - t) = 5^2 \times 41,$$

giving the possibilities

$$(s + t, s - t) = (1025, 1),\ (205, 5),\ (41, 25),$$

so

$$(s, t) = (513, 512),\ (105, 100),\ (33, 8).$$

Can a similar thing be done with a sum of two squares? We can factorise $s^2 + t^2 = 1025$ to

$$(s + \mathrm{i}t)(s - \mathrm{i}t) = 5^2 \times 41 = (2 + \mathrm{i})^2(2 - \mathrm{i})^2(5 + 4\mathrm{i})(5 - \mathrm{i}),$$

where i is our friend $\sqrt{-1}$. Now

$$(2+\mathrm{i})^2(5+4\mathrm{i}) = (3+4\mathrm{i})(5+4\mathrm{i}) = -1+32\mathrm{i}$$

and

$$(2+\mathrm{i})^2(5-4\mathrm{i}) = (3+4\mathrm{i})(5-4\mathrm{i}) = 31+8\mathrm{i}.$$

Without worrying too much about what we are doing, we put $s+\mathrm{i}t = -\mathrm{i}(-1+32\mathrm{i})$, to give $(s,t) = (32,1)$, and $s+\mathrm{i}t = 31+8\mathrm{i}$, to give $(s,t) = (31,8)$. (I sneaked in the $-\mathrm{i}$ to make s and t positive!) This gives the primitive Pythagorean triples

$$(2\times 32\times 1, 32^2-1^2, 32^2+1^2), \qquad (2\times 31\times 8, 31^2-8^2, 31^2+8^2),$$

i.e.

$$(64, 1023, 1025), \qquad (496, 897, 1025).$$

If we now multiply by the highest common factor, 2, we obtain two more of Bob Bertuello's rectangles.

Readers may like to try their hands at finding the remaining two when the highest common factor is 10. Thus, start with

$$s^2+t^2 = 5\times 41,$$

write $5 = (2+\mathrm{i})(2-\mathrm{i})$, $41 = (5+4\mathrm{i})(5-4\mathrm{i})$ and … follow your nose.

Two questions remain. First, why can you do this with so-called Gaussian integers, i.e. complex numbers such as $2+\mathrm{i}$, with integer real and imaginary parts? We seem to be treating them like ordinary integers, factorizing them and using some sort of unique factorization.

The other problem is: how do you express a prime number as the sum of two squares? I can do 2: $2 = 1^2+1^2$. It is not possible for primes such as 7, 11, 19, 23, which are of the form $4k+3$. But what about primes such as 5, 13, 17, 29, which are of the form $4k+1$? In fact, $5 = 2^2+1^2$, $13 = 3^2+2^2$, $17 = 4^2+1^2$, and $29 = 5^2+2^2$. Pierre Fermat proved that this is always possible. But how do you do it? Calculators to the ready!

Triangular numbers

The nth triangular number T_n is defined as

$$\mathrm{T}_n = \tfrac{1}{2}n(n+1).$$

For every triangular number T_a for $a>1$, there are triangular numbers T_b and T_c such that

$$\mathrm{T}_a + \mathrm{T}_b = \mathrm{T}_c.$$

In fact,

$$\mathrm{T}_a + \mathrm{T}_{\mathrm{T}_a - 1} = \mathrm{T}_{\mathrm{T}_a}.$$

Bridgewater State University, USA **Tom Moore**

A Look at Some Noninteger Representations of Numbers

MARTIN GRIFFITHS

All of us are very familiar with the idea of representing real numbers by way of the decimal system. Some may also have experience of using other bases to represent numbers; binary and hexadecimal, for example. In this article we explore the possibility of using noninteger bases, and consider some of the consequences of adopting such representations.

1. Introduction

We all know that, provided an infinite sequence of 9s is not allowed to appear, any positive real number has a unique decimal representation. There is in fact an analogous result for each integer base n, $n \geq 2$. However, I wonder how many of us have considered the possibility of representing numbers using bases other than integers? I must admit that I had not done so until recently encountering a problem in reference 1 (see Exercise 35, p. 85) concerning what the author calls 'a phi number system', where phi is the golden ratio, denoted by ϕ and equal to $(\sqrt{5}+1)/2$. The problem was to show, by considering real numbers written with the digits 0 and 1 using base ϕ, that there are infinitely many ways to represent the number 1.

Before giving a solution, we define some notation that will be used throughout. First, a subscript will be used to indicate the base of any representation that is not written in decimal. So, for example,

$$235.021_b = 2b^2 + 3b + 5 + 2b^{-2} + b^{-3}.$$

Next, the phrase 'base b representation' shall be abbreviated to 'b-rep'. Finally, we let $R_b(x)$ denote the number of distinct b-reps of x. If there are infinitely many such representations we write $R_b(x) = \infty$.

Returning to the problem, we can easily verify that $\phi^n = \phi^{n-1} + \phi^{n-2}$ for any integer n so that, for any positive integer k, we have

$$1 = \phi^{-2k} + \sum_{j=1}^{k} \phi^{1-2j}.$$

Here are the first few representations:

$$1 = 0.11_\phi = 0.1011_\phi = 0.101011_\phi = 0.10101011_\phi = \cdots .$$

The purpose of this article is to explore the situation for some other bases between 1 and 2.

2. Base $\sqrt{2}$

Although, initially at least, an irrational base may seem an irrational starting point for this investigation, it is actually the simplest of the bases between 1 and 2 to consider because of its relationship with 2-reps in terms of structure and rules of arithmetic. Since $(\sqrt{2})^k + (\sqrt{2})^k = (\sqrt{2})^{k+2}$, we see that when adding two numbers both written using base $\sqrt{2}$ the rule of 'carrying' is similar to that for binary arithmetic; the difference being that digits are carried two places to the left rather than one.

Let us first exhibit a simple construction to demonstrate that $R_{\sqrt{2}}(\sqrt{2}) = \infty$. We have, on using the formula for the sum to infinity of a geometric progression, that

$$\begin{aligned} 0.\dot{1}\dot{0}_{\sqrt{2}} &= \frac{1}{\sqrt{2}} + \frac{1}{(\sqrt{2})^3} + \frac{1}{(\sqrt{2})^5} + \cdots \\ &= \frac{1/\sqrt{2}}{1 - 1/(\sqrt{2})^2} \\ &= \sqrt{2}. \end{aligned}$$

Any irrational number has a unique 2-rep (binary representation). In particular,

$$\begin{aligned} 0.1_{\sqrt{2}} &= \frac{1}{\sqrt{2}} \\ &= 0.10110101000001\cdots{}_2. \end{aligned}$$

Since $(\sqrt{2})^{2n} = 2^n$, the above 2-rep can be converted back to a $\sqrt{2}$-rep simply by inserting zeros in the appropriate places to give

$$0.1_{\sqrt{2}} = 0.0100010100010001000000000000010\cdots{}_{\sqrt{2}},$$

where the nth digit after the point in this representation is zero if n is odd, or equal to k if n is even, where k is the $(n/2)$th digit after the point in the 2-rep. It follows that

$$\begin{aligned} \sqrt{2} &= 0.\dot{1}\dot{0}_{\sqrt{2}} \\ &= 0.1_{\sqrt{2}} + 0.0\dot{0}\dot{1}_{\sqrt{2}} \\ &= 0.010001010001\cdots{}_{\sqrt{2}} + 0.001010101010\cdots{}_{\sqrt{2}} \\ &= 0.011011111011\cdots{}_{\sqrt{2}}, \end{aligned}$$

a $\sqrt{2}$-rep of $\sqrt{2}$ for which the first digit after the point is 0 but the nth digit after the point is 1 for all odd n, $n > 1$. Next we obtain the 2-rep of $0.101_{\sqrt{2}}$, convert it back to base $\sqrt{2}$ by inserting zeros as in the previous example, and use the fact that $0.\dot{1}\dot{0}_{\sqrt{2}} = 0.101_{\sqrt{2}} + 0.000\dot{0}\dot{1}_{\sqrt{2}}$ to give a new $\sqrt{2}$-rep in which the first and third digits after the point are both 0 but the nth digit after the point is 1 for all odd n, $n > 3$. Continuing in this way clearly gives us that $R_{\sqrt{2}}(\sqrt{2}) = \infty$.

It is actually true that $R_{\sqrt{2}}(x) = \infty$ for every positive number x. To see this, start by choosing any nonnegative numbers a and b such that $x = a + b\sqrt{2}$ and obtain their binary representations

$$a = a_m a_{m-1} \cdots a_0.a_{-1}a_{-2}\cdots{}_2 \quad \text{and} \quad b = b_n b_{n-1} \cdots b_0.b_{-1}b_{-2}\cdots{}_2.$$

Then

$$\begin{aligned} x &= \sum_{k=0}^{\infty} a_{m-k}2^{m-k} + \sqrt{2}\sum_{k=0}^{\infty} b_{n-k}2^{n-k} \\ &= \sum_{k=0}^{\infty} a_{m-k}(\sqrt{2})^{2(m-k)} + \sum_{k=0}^{\infty} b_{n-k}(\sqrt{2})^{2(n-k)+1}, \end{aligned}$$

giving us a $\sqrt{2}$-rep of x composed by interleaving the digits of the 2-reps of a and b. However, there are infinitely many choices for a and b since if $0 \le a < x$ we then set b equal to $(x-a)/\sqrt{2}$.

Imposing constraints on the way that the representation is constructed can lead to unique representations of numbers using noninteger bases. Suppose that we want a b-rep of x, with $1 < b < 2$. First find the integer n such that $b^{n-1} < x \le b^n$. If $x = b^n$ then put a 1 in the n th position of the representation (abiding by the normal conventions on place-value) and, after including any zeros as may be necessary, the representation has been found. Otherwise, find the largest integer k (this will be always be 1 for the bases we are considering here) for which $kb^{n-1} < x$ and put a k in the $(n-1)$ th position. In the latter case the same procedure is followed for the number $x - kb^{n-1}$, again including zeros as necessary. This is a type of division algorithm, and can be continued indefinitely to obtain a b-rep of x.

Quite clearly the above algorithm can only lead to one possible b-rep, which we denote by $u_b(x)$, but will often give rise to more complicated-looking ones than could be obtained otherwise. For example, while $u_{\sqrt{2}}(5)$ is $10001_{\sqrt{2}}$, $u_{\sqrt{2}}(3)$ is not $101_{\sqrt{2}}$ but rather $1000.000001001\cdots_{\sqrt{2}}$.

Will the addition of two numbers given in this unique representation, using the previously mentioned rules for carrying, give us an answer in unique representation? By considering the addition of 2 and 3 we find that the answer to this is in the negative.

If a number x has a terminating $\sqrt{2}$-rep then this is the only terminating $\sqrt{2}$-rep of x (in contrast to the ϕ-reps considered in Section 1). To see this, note that there exist $m, n \in \mathbb{N}$ and $p, q \in \mathbb{Z}$, with $a_k = 0$ or $a_k = 1$, $1 \le k \le m$, and $b_k = 0$ or $b_k = 1$, $1 \le k \le n$, such that

$$
\begin{aligned}
x &= \sum_{k=1}^{m} a_k(\sqrt{2})^{2(p-k)} + \sum_{k=1}^{n} b_k(\sqrt{2})^{2(q-k)+1} \\
&= \sum_{k=1}^{m} a_k 2^{p-k} + \sqrt{2}\sum_{k=1}^{n} b_k 2^{q-k} \\
&= a + b\sqrt{2},
\end{aligned}
$$

for some $a, b \in \mathbb{Q}$. The uniqueness of the terminating $\sqrt{2}$-rep follows from the uniqueness of terminating 2-reps.

3. Base $\frac{3}{2}$

Let us show that $R_{3/2}(1) = \infty$. First note that $1_{3/2}$ is the only terminating $3/2$-rep of 1. This is because, for any positive integer m, the fraction

$$
\begin{aligned}
0.a_{-1}a_{-2}\cdots a_{-m\,3/2} &= \sum_{k=1}^{m} a_{-k}\left(\tfrac{3}{2}\right)^{-k} \\
&= \frac{2\cdot 3^{m-1}a_{-1} + 2^2\cdot 3^{m-2}a_{-2} + \cdots + 2^m a_{-m}}{3^m}
\end{aligned}
$$

has an even numerator and odd denominator and thus can never be equal to 1. A slightly more general argument shows that, for any rational b with $1 < b < 2$, 1_b is the only terminating b-rep of 1.

There is in fact, for each k with $k = 1, 2, \ldots$, a nonterminating $\frac{3}{2}$-rep for 1, $0.a_{-1}a_{-2}\cdots{}_{3/2}$ say, for which $a_{-k} = 0$. To see this, let us consider $u_{3/2}(\frac{1}{3}) = 0.0010000010\cdots{}_{3/2} = 0.b_{-1}b_{-2}\cdots{}_{3/2}$, say. We then have

$$\begin{aligned} 1 &= \tfrac{2}{3} + \tfrac{1}{3} \\ &= 0.1_{3/2} + 0.00b_{-3}b_{-4}\cdots{}_{3/2} \\ &= 0.10b_{-3}b_{-4}\cdots{}_{3/2}\,. \end{aligned}$$

Now say that $b_{-k} = 1$ for some $k \geq 3$. From the construction of $u_{3/2}(\frac{1}{3})$ we know that if $x = \frac{1}{3} - 0.b_{-1}b_{-2}\cdots b_{-(k-1)3/2}$ then $(\frac{2}{3})^k < x < (\frac{2}{3})^{k-1}$. Therefore,

$$\begin{aligned} 0.b_{-1}b_{-2}\cdots b_{-(k-1)}0\dot{1}_{3/2} &= \tfrac{1}{3} - x + \sum_{m=k+1}^{\infty} \left(\tfrac{2}{3}\right)^m \\ &= \frac{1}{3} - x + \frac{(\frac{2}{3})^{k+1}}{1-\frac{2}{3}} \\ &= \tfrac{1}{3} - x + \tfrac{4}{3}\left(\tfrac{2}{3}\right)^{k-1} \\ &> \tfrac{1}{3}. \end{aligned}$$

From this it is clear that

$$0.1b_{-2}\cdots b_{-(k-1)_{3/2}} < 1 < 0.1b_{-2}\cdots b_{-(k-1)}0\dot{1}_{3/2}, \tag{1}$$

showing that there does exist some nonterminating $\frac{3}{2}$-rep for 1 such that $a_{-k} = 0$. To see that this is also true for $k = 1$ we note both that $0.01_{3/2} < 1$ and

$$\begin{aligned} 0.0\dot{1}_{3/2} &= \left(\tfrac{2}{3}\right)^2 + \left(\tfrac{2}{3}\right)^3 + \cdots \\ &= \tfrac{4}{3} \\ &> 1. \end{aligned}$$

As we now explain, it follows from the above that there exist infinitely many nonterminating $\frac{3}{2}$-reps for 1. From (1) we know that for any $k \geq 3$ it is possible to write 1 in the form $0.1b_{-2}\cdots b_{-(k-1)}0c_{1,k}c_{2,k}\cdots{}_{3/2}$ where $u_{3/2}(\frac{1}{3}) = 0.b_{-1}b_{-2}\cdots{}_{3/2}$ and $\{c_{n,k}\}$ is some sequence of 0s and 1s that depends on k. (Note though that this sequence is not necessarily unique.) Thus, in particular, if $b_{-p} = b_{-q} = 1$ for some p and q with $3 \leq p < q$, then there exists a $\frac{3}{2}$-rep for 1 of the form $0.1b_{-2}\cdots b_{-(p-1)}0c_{1,p}c_{2,p}\cdots{}_{3/2}$ and another of the form $0.1b_{-2}\cdots b_{-(q-1)}0c_{1,q}c_{2,q}\cdots{}_{3/2}$. These $\frac{3}{2}$-reps share the first $p-1$ digits after the point yet differ in the pth digit. This is sufficient to show that $R_{3/2}(1) = \infty$.

It is interesting to see for which rational bases b, with $1 < b < 2$, it is possible to use the above reasoning to show that $R_b(1) = \infty$. For the case $b = \frac{3}{2}$ we required that $\sum_{m=k+1}^{\infty}(\frac{2}{3})^m$ be greater than or equal to $(\frac{2}{3})^{k-1}$. This generalises to the requirement that $\sum_{m=k+1}^{\infty}(1/b)^m$ be greater than or equal to $(1/b)^{k-1}$, which, on noting that

$$\sum_{m=k+1}^{\infty}\left(\frac{1}{b}\right)^m = \frac{1}{b-1}\left(\frac{1}{b}\right)^k,$$

leads to the condition

$$\frac{1}{b-1}\left(\frac{1}{b}\right)^k \geq \left(\frac{1}{b}\right)^{k-1}.$$

Some simplification then yields the quadratic inequality $b^2 - b - 1 \leq 0$. Thus the set of values of b between 1 and 2 for which this is satisfied is $(1, \phi]$, and the number that prompted this investigation has resurfaced.

4. Base b with $b \in (\phi, 2)$?

Is it possible to find bases $b \in (\phi, 2)$, arbitrarily close to 2, such that $R_b(1) = \infty$? By considering the equation $f_n(x) = 0$, where

$$f_n(x) = x^n - x^{n-1} - \cdots - x - 1,$$

it can be shown that the answer is in the affirmative. First, the equation $(x-1)f_n(x) = 0$ is rearranged to give $x^n(x-2) = -1$. Next, let $x_0 \in \mathbb{R}$ be any fixed number satisfying $1 < x_0 < 2$. Then there exists some sufficiently large $n \in \mathbb{N}$, n_0 say, such that $x_0^{n_0}(x_0 - 2) < -1$. Note though that $x^{n_0}(x-2) \to 0$ as $x \to 2$ (from below). Thus, by continuity, $x^{n_0}(x-2) = -1$ has a solution $\alpha(n_0)$ such that $x_0 < \alpha(n_0) < 2$. This in turn implies, since x_0 was an arbitrary element of the interval $(1, 2)$, that for any $0 < \varepsilon < 1$ we may find n and $\alpha(n)$ such that $2 - \varepsilon < \alpha(n) < 2$ and $\alpha(n)^n(\alpha(n) - 2) = -1$.

Since $\alpha(n) \neq 1$ it follows from the above that $\alpha(n)$ is a solution to $f_n(x) = 0$. Thus

$$\alpha(n)^n = \alpha(n)^{n-1} + \cdots + \alpha(n) + 1,$$

from which it follows, adopting a similar argument to that used in Section 1 to show that $R_\phi(1) = \infty$ (in fact it should be noted here that $\alpha(2) = \phi$), that

$$1 = \alpha(n)^{-nk} + \sum_{i=1}^{k}\sum_{j=1}^{n-1}\alpha(n)^{-j-n(i-1)},$$

for all $n \geq 2$ and $k \geq 1$. Thus $R_{\alpha(n)}(1) = \infty$. The interested reader might like to go on to consider the situation in greater generality.

Reference

1 D. E. Knuth, *The Art of Computer Programming*, Vol. 1, *Fundamental Algorithms* (Addison-Wesley, Reading, MA, 1968).

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An Example of Minimizing a Function the Optimal Way

RUSSELL EULER and JAWAD SADEK

An optimization problem is posed. A calculus-based solution and an elementary algebra solution are given. The two solutions illustrate the fact that calculus is not always the 'optimal' way to solve an optimization problem.

Optimization problems are often solved using calculus methods. Examples using non-calculus techniques are sometimes useful in complementing, not replacing, calculus-based methods (see Chapter 4 of reference 1 and reference 2, for instance). Within this context, the following problem illustrates the fact that calculus-based methods are not always the best approach to attack some optimization problems.

Problem For $x > 0$, find the absolute minimum value of

$$f(x) = \frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}. \tag{1}$$

Attempting to solve this problem by first calculating $f'(x)$ directly from (1) using the quotient rule can easily lead to a quagmire. We note here that (1) intentionally contains an algebraic red herring. It serves to emphasize the old adage 'look before you leap' which, when translated mathematically, in this context at least, means to simplify before you differentiate.

In fact, simplifying the expression first yields the following elegant non-calculus solution. Algebraically, (1) can be simplified to give

$$f(x) = 3\left(x + \frac{1}{x}\right).$$

Since $(\sqrt{x} - 1/\sqrt{x})^2 \geq 0$ for all $x > 0$, $x + 1/x \geq 2$ with equality when $x = 1$. So,

$$\min_{x>0} f(x) = 6.$$

The purpose of this article is to give a calculus and a non-calculus solution to the following generalization of (1).

Problem If $x > 0$ and n is a positive integer, find the absolute minimum value of

$$f(x) = \frac{(x + 1/x)^{2n} - (x^{2n} + 1/x^{2n}) - 2}{(x + 1/x)^n + (x^n + 1/x^n)}.$$

These solutions, in addition to being instructive in their own right, serve to make the point that some optimization problems are not well suited to elementary calculus.

A calculus solution

We have

$$
\begin{aligned}
f(x) &= \frac{(x+1/x)^{2n} - (x^{2n}+2+1/x^{2n})}{(x+1/x)^n + (x^n+1/x^n)} \\
&= \frac{(x+1/x)^{2n} - (x^n+1/x^n)^2}{(x+1/x)^n + (x^n+1/x^n)} \\
&= \frac{[(x+1/x)^n + (x^n+1/x^n)][(x+1/x)^n - (x^n+1/x^n)]}{(x+1/x)^n + (x^n+1/x^n)} \\
&= \left(x+\frac{1}{x}\right)^n - \left(x^n + \frac{1}{x^n}\right) \\
&= \sum_{i=0}^{n} \binom{n}{i} x^{n-i} \left(\frac{1}{x}\right)^i - x^n - \frac{1}{x^n} \\
&= \sum_{i=1}^{n-1} \binom{n}{i} x^{n-2i}. \qquad (2)
\end{aligned}
$$

If $n = 1$, then $f(x) = 0$ for all $x > 0$. So, $\min_{x>0} f(x) = 0$. In what follows we assume that $n \neq 1$.

The derivative of f is

$$f'(x) = \sum_{i=1}^{n-1} \binom{n}{i} (n-2i) x^{n-2i-1}.$$

This can be rewritten in the form

$$x^{n-1} f'(x) = \binom{n}{1}(n-2)x^{2n-4} + \cdots - \binom{n}{n-1}(n-2).$$

Now we use Descartes' rule of signs (see reference 3, A-118).

Descartes' rule of signs *Given a polynomial $P(x)$ with real coefficients and nonzero constant term, the number of positive zeros of $P(x)$ is never greater than the number of variations in sign in $P(x)$ and, if less, then always by an even number.*

Since there is only one change of sign in the coefficients of the polynomial $x^{n-1}f'(x)$, it can have at most one positive root. Hence $f'(x)$ has at most one positive root. To show that $f'(1) = 0$ we prove the following lemma.

Lemma 1 *We have*

$$\sum_{i=1}^{n-1} i\binom{n}{i} = n(2^{n-1} - 1).$$

Proof Differentiating the identity $(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$ yields

$$n(x+1)^{n-1} = \sum_{i=1}^{n} i\binom{n}{i} x^{i-1}.$$

Letting $x = 1$ gives the equality $n2^{n-1} = \sum_{i=1}^{n} i\binom{n}{i}$, and so

$$\sum_{i=1}^{n-1} i\binom{n}{i} = n2^{n-1} - n = n(2^{n-1} - 1),$$

and the lemma is proved.

Now we write

$$\begin{aligned} f'(1) &= \sum_{i=1}^{n-1} \binom{n}{i}(n - 2i) \\ &= n\sum_{i=1}^{n-1}\binom{n}{i} - 2\sum_{i=1}^{n-1} i\binom{n}{i} \\ &= n(2^n - 2) - 2n(2^{n-1} - 1) \\ &= 0, \end{aligned}$$

where we used lemma 1 and $\sum_{i=1}^{n-1}\binom{n}{i} = 2^n - 2$, a fact that is also a consequence of the binomial theorem applied to $(1+1)^n$.

Now

$$f(x) = \binom{n}{1}x^{n-2} + \cdots + \binom{n}{n-1}x^{2-n},$$

so $f(x) = f(1/x)$ for $x > 0$ and $f(x) \to \infty$ as $x \to \infty$. Hence $f(x)$ has a unique turning point at $x = 1$ and this is a minimum (see figure 1 for $f(x)$ in the case $n = 3$). Thus $f(x)$ has the absolute minimum value $f(1) = (2^{2n} - 4)/(2^n + 2) = 2^n - 2$ for $x > 0$.

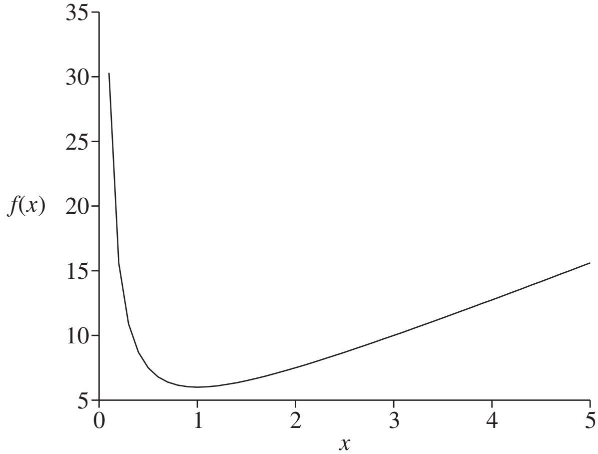

Figure 1

An algebraic solution

From (2), we obtain

$$\begin{aligned} f(x) &= \sum_{i=1}^{n-1} \binom{n}{i} x^{n-2i} \\ &= \binom{n}{1} x^{n-2} + \binom{n}{2} x^{n-4} + \binom{n}{3} x^{n-6} \\ &\quad + \cdots + \binom{n}{n-1} x^{6-n} + \binom{n}{n-2} x^{4-n} + \binom{n}{n-3} x^{2-n}. \end{aligned}$$

If n is odd, then

$$\begin{aligned} f(x) &= \binom{n}{1}\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \binom{n}{2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \binom{n}{3}\left(x^{n-6} + \frac{1}{x^{n-6}}\right) \\ &\quad + \cdots + \binom{n}{(n-1)/2}\left(x + \frac{1}{x}\right) \\ &\geq 2 \sum_{i=1}^{(n-1)/2} \binom{n}{i} \\ &= 2 \frac{\sum_{i=1}^{n-1} \binom{n}{i}}{2} \\ &= 2 \frac{2^n - 2}{2} \\ &= 2^n - 2. \end{aligned}$$

The facts that $x^p + 1/x^p \geq 2$ whenever $x > 0$ and $\sum_{i=1}^{n-1} \binom{n}{i} = 2^n - 2$ were used in the preceding computations.

Similarly, if n is even, then

$$\begin{aligned} f(x) &= \binom{n}{1}\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \binom{n}{2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \binom{n}{3}\left(x^{n-6} + \frac{1}{x^{n-6}}\right) \\ &\quad + \cdots + \binom{n}{n/2-1}\left(x^2 + \frac{1}{x^2}\right) + \binom{n}{n/2} \\ &\geq 2 \sum_{i=1}^{n/2-1} \binom{n}{i} + \binom{n}{n/2} \\ &= 2 \frac{\sum_{i=1}^{n-1} \binom{n}{i} - \binom{n}{n/2}}{2} + \binom{n}{n/2} \\ &= \sum_{i=1}^{n-1} \binom{n}{i} - \binom{n}{n/2} + \binom{n}{n/2} \\ &= 2^n - 2. \end{aligned}$$

Thus, the inequality $f(x) \geq 2^n - 2$ holds for all positive integers n. Since $f(1) = 2^n - 2$, the required minimum is $2^n - 2$.

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Extra-magic squares – part 1

1	823	821	809	811	797	19	29	313	31	23	37
89	83	211	79	641	631	619	709	617	53	43	739
97	227	103	107	193	557	719	727	607	139	757	281
223	653	499	197	109	113	563	479	173	761	587	157
367	379	521	383	241	467	257	263	269	167	601	599
349	359	353	647	389	331	317	311	409	307	293	449
503	523	233	337	547	397	421	17	401	271	431	433
229	491	373	487	461	251	443	463	137	439	457	283
509	199	73	541	347	191	181	569	577	571	163	593
661	101	643	239	691	701	127	131	179	613	277	151
659	673	677	683	71	67	61	47	59	743	733	41
827	3	7	5	13	11	787	769	773	419	149	751

The entries in this 12×12 magic square consist of the first 144 prime numbers except that the prime number 2 is replaced by 1. The row, column, and diagonal sums are 4514.

Reference

1 Kazem Faeghi, *The Mathematical Amusements and Games.*

Students' Investigation House,
Shariati Avenue, Sirjan, Iran

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A Trajectory for Maximum Impact

J. GILDER

Or how to knock things down.

Toys were unavailable during the Second World War, so my father made most of mine. A single soldier was expanded to a platoon of solid lead replicas, and wood and aluminium tubing turned into field guns and tanks. These weapons were spring powered and fired pencil stubs as shells. The results were unfortunately less than realistic, for the shells just bounced off the soldiers!

Considering the matter now, it seems obvious that the shells needed to be heavier; but just what combination of shell weight and gun elevation would be most effective in knocking things over needs to be investigated.

Before the advent of gunpowder, all missiles were launched by means of the release of mechanically stored energy. Trebuchets used the potential energy of a raised weight while ballistas utilised energy stored in stretched fibres. Unfortunately little is known about how these were actually used. It is most probable that they used missiles of uniform weight to ensure consistency and were deployed at maximum range for the safety of their users. However, it would be interesting to know if the operators of these ancient siege engines knew how to maximise their effectiveness.

A spring-powered gun exploits the potential energy of the compressed spring. It therefore seems reasonable to assume that such a weapon always imparts the same amount of energy to the shell. Thus the initial kinetic energy of the shell will be assumed to be constant, whatever its mass.

The ability of a shell to knock things over is measured by its horizontal component of momentum. It is this that we hope to maximise.

Let x and y be coordinates measured horizontally and vertically upwards respectively from an origin O. Let the gun be at O and the target at $(d, 0)$. Let the mass of the shell be m with initial velocity u at an angle of θ to the positive x-axis, as in figure 1.

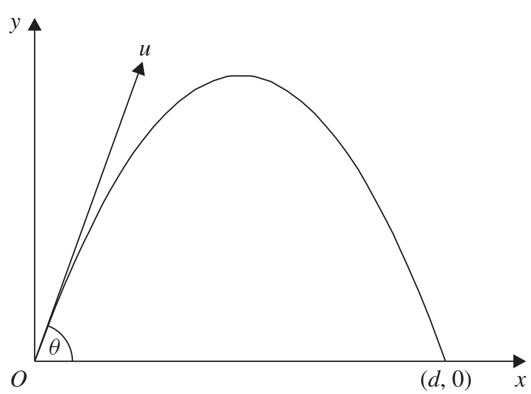

Figure 1

The equations of motion under constant acceleration tell us that, after time t, the coordinates of our shell will be

$$x = ut\cos\theta, \qquad y = ut\sin\theta - \tfrac{1}{2}gt^2,$$

where g is the acceleration due to gravity.

Now $y = 0$ when $t = 0$ and $t = 2u\sin\theta/g$, that is, when $x = 0$ and $x = 2u^2\sin\theta\cos\theta/g$. So, to hit the target, we have

$$d = \frac{2u^2\sin\theta\cos\theta}{g}. \tag{1}$$

The constancy of the initial kinetic energy gives us

$$mu^2 = b, \tag{2}$$

say. Thus

$$md = \frac{2b\sin\theta\cos\theta}{g},$$

that is, for a fixed angle of elevation, *mass* times *range* is a constant.

We can rearrange (1), as

$$u^2\sin\theta\cos\theta = \frac{gd}{2} = c, \tag{3}$$

say. The shell's horizontal momentum I, say, is $mu\cos\theta$ throughout its flight. Using (2) to replace m gives

$$I = \frac{b}{u}\cos\theta.$$

Using (3) to replace u gives

$$I^2 = \frac{b^2}{c}\sin\theta\cos^3\theta.$$

Now I is positive, and so we are interested in the maximum value of z where

$$z^2 = \sin\theta\cos^3\theta, \qquad z \text{ positive}.$$

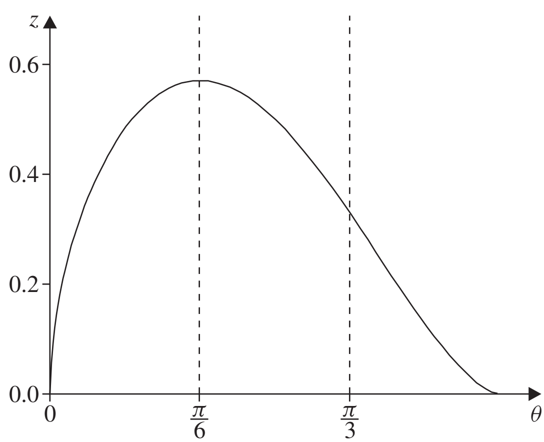

Figure 2

A graph of z plotted against θ is displayed in figure 2. This seems to show a clear maximum when $\theta = \pi/6$.

On differentiating z^2 with respect to θ, we obtain

$$2z\frac{\mathrm{d}z}{\mathrm{d}\theta} = \cos^4\theta - 3\sin^2\theta\cos^2\theta,$$

and the right-hand side is indeed zero when $\theta = \pi/6$. Thus we have obtained maximum impact with a gun elevation of $30°$.

To find the corresponding value of m, we need to know the value of the energy constant b. This could be difficult to ascertain in practice.

However, we have already noted that, for a fixed angle of elevation, *mass* times *range* is a constant. So if, with an elevation of $30°$, we fire a shell of mass w and achieve a range r, then on changing the mass to wr/d we would achieve our desired range d.

It now remains for someone to repeat the analysis for the situation where the target is at an elevation α with respect to the gun. My guess, taking a hint from the well-known results concerning maximum range, is that the elevation of the gun would turn out to be

$$\alpha + \frac{\pi/2 - \alpha}{3}.$$

We will no doubt be informed in due course.

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Mathematical Spectrum Awards for Volume 43

Prizes have been awarded to the following student readers for contributions in Volume 43:

Jonathan Weisbrod

for the article 'An Unusual Look at Primitive Pythagorean Triples';

Abbas Rouholamini Gugheri

for various contributions.

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £50 for letters, solutions to problems and other items.

Catalan Congruences with Interesting Dividends

THOMAS KOSHY

The well-known Catalan numbers, studied for nearly two centuries, are named after the Belgian mathematician E. C. Catalan. Around 1883, he established that the number $X_{a,b} = (2a)!\,(2b)!/a!\,b!\,(a+b)!$, where $a, b \geq 0$, is an integer. This article focuses on applications of this number-theoretic fact to Catalan numbers, including some divisibility properties.

Catalan numbers

Like Fibonacci and Lucas numbers, Catalan numbers are also a delightful playground with boundless opportunities for exploration and experimentation. They have spectacular applications to abstract and linear algebra, combinatorics, graph theory, number theory, sports, and computer science (see references 1 and 2). Catalan numbers, C_n, are defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n \geq 0.$$

They can be extracted from Pascal's triangle in a number of different ways (see reference 3). The first ten Catalan numbers are

$$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862.$$

They are named after the Belgian mathematician Eugène C. Catalan (1814–1894), who found them in his study of well-formed sequences of parentheses in 1838. The great Swiss mathematician Leonhard Euler (1707–1783) found them in 1751 while studying triangulations of convex polygons. But the Chinese mathematician Antu Ming (circa 1692–1763) discovered them in 1730 through his geometric models (see reference 4).

Around 1883, Catalan found that the number

$$X_{a,b} = \frac{(2a)!\,(2b)!}{a!\,b!\,(a+b)!}, \qquad a, b \geq 0,$$

is an integer (see references 5–7). Around thirty-six years later, it re-surfaced in *The American Mathematical Monthly* as a problem by H. C. Feemster of York College, Nebraska, USA (see references 3 and 8). In 1976, R. H. Cobb showed that $X_{a,b}$ is an integer by counting the number of times a prime appears as a factor of $X_{a,b}$ (see reference 6). A few months later, D. R. Breach of the University of Canterbury, New Zealand, established the same result by investigating the coefficient of x^{m+n} in the expansion of $(1+x)^{2m}(1-x)^{2n}$ (see reference 7). In the following year, F. Ayres established the recurrence relation

$$X_{a-1,b+1} = 4X_{a-1,b} - X_{a,b},$$

where

$$X_{k,0} = X_{0,k} = \binom{2k}{k},$$

the central binomial coefficient; he then concluded the result by induction (see reference 5). For the curious-minded, we add that

$$\frac{(2a)!\,(2b)!\,(2c)!}{a!\,b!\,c!\,(a+b+c)!}$$

need *not* be an integer. For example,

$$\frac{6!\,8!\,10!}{3!\,4!\,5!\,12!}$$

is not an integer.

Catalan's discovery that $X_{a,b}$ is an integer implies that

$$\begin{aligned}\left[\binom{2a}{a}\binom{2b}{b}\right]^2 \Big/ \binom{a+b}{a} &= \frac{(2a)!\,(2a)!\,(2b)!\,(2b)!\,a!\,b!}{a!\,a!\,a!\,a!\,b!\,b!\,b!\,b!\,(a+b)!} \\ &= \binom{2a}{a}\binom{2b}{b}\frac{(2a)!\,(2b)!}{a!\,b!\,(a+b)!}\end{aligned}$$

is also an integer (see references 3 and 9).

This result has some interesting applications to Catalan numbers. To see this, we let $a = n-r$ and $b = r$. Then it follows that

$$\left[\binom{2n-2r}{n-r}\binom{2r}{r}\right]^2 \Big/ \binom{n}{r}$$

is an integer (Catalan found in 1885 that this quotient is an integer when r is a prime (see reference 10)); that is,

$$[(n-r+1)C_{n-r}(r+1)C_r]^2 \equiv 0 \left(\bmod \binom{n}{r}\right).$$

So

$$[(r+1)(n-r+1)C_rC_{n-r}]^2 \equiv 0 \left(\bmod \binom{n}{r}\right). \tag{1}$$

For example, $(5 \cdot 6C_4C_5)^2 = (30 \cdot 14 \cdot 42)^2 = 311\,169\,600 \equiv 0 \ (\bmod \binom{9}{4})$.

Interesting byproducts

Congruence (1) has interesting byproducts. To see this, first we let $r = 2$. Then (1) yields

$$\begin{aligned}[6(n-1)C_{n-2}]^2 &\equiv 0 \left(\bmod \binom{n}{2}\right), \\ 6^2(n-1)^2C_{n-2}^2 &= \frac{n(n-1)}{2}K, \\ 2 \cdot 6^2(n-1)C_{n-2}^2 &= nK \\ &\equiv 0 \ (\bmod n), \\ 2 \cdot 6^2C_{n-2}^2 &\equiv 0 \ (\bmod n),\end{aligned} \tag{2}$$

where K is some positive integer, since $(n, n-1) = 1$ and (a, b) denotes the greatest common factor of a and b.

Suppose that $(n, 6) = 1$. Then congruence (2) implies that $C_{n-2}^2 \equiv 0 \pmod{n}$. In particular, let p be a prime greater than 3. Then $C_{p-2}^2 \equiv 0 \pmod{p}$; so $C_{p-2} \equiv 0 \pmod{p}$; that is, $p|C_{p-2}$. Consequently,

$$\frac{1}{p}C_{p-2} = \frac{(2p-4)!}{p!\,(p-2)!}$$

is an integer.

For example, $17|C_{15}$ and $23|C_{21}$, where $C_{15} = 9\,694\,845$ and $C_{21} = 24\,446\,267\,020$.

(Interestingly, Catalan had found in 1888 that

$$\frac{(2n-4)!}{n!\,(n-2)!}$$

is an integer when $(n, 6) = 1$ (see reference 10); so $n|C_{n-2}$ when $(n, 6) = 1$).

Next, we let $r = 3$ in congruence (1). Then

$$[20(n-2)C_{n-3}]^2 \equiv 0 \left(\bmod\ \frac{n(n-1)(n-2)}{6}\right).$$

This implies that

$$2400(n-2)C_{n-3}^2 \equiv 0 \pmod{n(n-1)}. \tag{3}$$

As an illustration,

$$2400 \cdot 7C_6^2 = 2400 \cdot 7 \cdot 132^2 \equiv 0 \pmod{9 \cdot 8}.$$

Finally, we let n be a prime p. Then congruence (1) becomes

$$[(r+1)(p-r+1)C_r C_{p-r}]^2 \equiv 0 \left(\bmod\ \binom{p}{r}\right). \tag{4}$$

But

$$\binom{p}{r} \equiv 0 \pmod{p}$$

when $0 < r < p$ (see reference 2). Consequently, congruence (4) yields

$$[(r+1)(r-1)C_r C_{p-r}]^2 \equiv 0 \pmod{p},$$

where $0 < r < p$. Therefore, $(r+1)(r-1)C_r C_{p-r} = 0 \pmod{p}$. Since $(r-1, p) = 1$, it follows that

$$(r+1)C_r C_{p-r} \equiv 0 \pmod{p}, \tag{5}$$

where $1 < r < p$.

As an illustration,

$$6C_5C_6 = 6 \cdot 42 \cdot 132 \equiv 0 \pmod{11} \quad \text{and} \quad 7C_6C_7 = 7 \cdot 132 \cdot 429 \equiv 0 \pmod{13}.$$

In particular, if $r < p-1$, then $(r+1, p) = 1$; then by (5) $C_r C_{p-r} \equiv 0 \pmod{p}$; so either $p|C_r$ or $p|C_{p-r}$. Thus, we have the following theorem.

Theorem 1 *Let p be a prime and $1 < r < p-1$. Then either $p|C_r$ or $p|C_{p-r}$.*

As an example,

$$C_{10}C_7 = 16\,796 \cdot 429 \equiv 0 \pmod{17}.$$

Theorem 1 has two interesting byproducts, as the following two corollaries reveal.

Corollary 1 *If p is a prime, $p \geq 5$, then $p|C_{(p+1)/2}$.*

Proof By letting $r = (p-1)/2$ in theorem 1, either $p|C_{(p-1)/2}$ or $p|C_{(p+1)/2}$. But

$$\begin{aligned} C_{(p-1)/2} &= \frac{2}{p+1}\binom{p-1}{(p-1)/2} \\ &= \frac{2}{p+1}\,\frac{(p-1)!}{[(p-1)/2]!\,[(p-1)/2]!}. \end{aligned}$$

Clearly, p does not divide the right-hand side, so $p \nmid C_{(p-1)/2}$. Thus $p|C_{(p+1)/2}$.

As an illustration, $C_7 \equiv 0 \pmod{13}$ and $C_9 \equiv 0 \pmod{17}$, where $C_9 = 4\,862$.

Corollary 2 *If q and $p \geq 2q+1$ are primes, then $p|C_{p-q}$.*

Proof Since

$$C_q = \frac{1}{q+1}\binom{2q}{q} = \frac{(2q)!}{(q+1)!\,q!}$$

and $p \geq 2q+1$, it follows that $p \nmid C_q$. Therefore, by theorem 1, $p|C_{p-q}$.

It follows, for example, by corollary 2 that:

- if $p \geq 5$, then $p|C_{p-2}$,
- if $p \geq 7$, then $p|C_{p-3}$ (this follows from congruence (3) also),
- if $p \geq 11$, then $p|C_{p-5}$.

As an illustration, $11|C_8$ and $23|C_{18}$, where $C_8 = 1\,430$ and $C_{18} = 477\,638\,700$.

Interestingly, it also follows from corollary 2 that if $p = 2q+1$, then $p|C_{(p+1)/2}$. For example, $11|C_6$ and $23|C_{12}$, where $C_{12} = 208\,012$.

It is worth noting that theorem 1 does *not* work for $r = 0$ or $p-1$. For instance, let $p = 5$. In each case, since $C_0 = 1 = C_1$, $C_4 = 14$, and $C_5 = 42$, it follows that $5 \nmid C_1$ and $5 \nmid C_4$. But, notice that $C_5 \equiv 2 \pmod{5}$ and $C_4 \equiv -1 \pmod{5}$.

More generally,

$$\begin{aligned} C_p &= \frac{(2p)!}{(p+1)!\,p!} \\ &= 2\frac{(2p-1)\cdots(p+2)}{(p-1)!}, \\ (p-1)!\,C_p &\equiv 2(p-1)! \pmod{p}, \end{aligned}$$

so $C_p \equiv 2 \pmod{p}$.

On the other hand,

$$\begin{aligned} C_{p-1} &= \frac{(2p-2)!}{p!\,(p-1)!} \\ &= \frac{(2p-2)\cdots(p+1)}{(p-1)!}, \\ (p-1)!\,C_{p-1} &\equiv (p-2)! \pmod{p}, \end{aligned}$$

so

$$(p-1)C_{p-1} \equiv 1 \pmod{p}$$

and hence

$$C_{p-1} \equiv -1 \pmod{p}.$$

Acknowledgment

The author would like to thank the Editor for his constructive suggestions for improving the quality of this article.

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Odd and Even Fractions

COLIN FOSTER

This article extends the familiar definitions of *odd* and *even* from the domain of the natural numbers to that of fractions (the rationals) and examines their behaviour under addition, subtraction, multiplication, and division.

Introduction

The terms *odd* and *even* are, of course, well-established for the natural numbers, and the behaviour of pairs of odd and even numbers under the four operations is elementary. But can we have odd and even *fractions*? Since, for fractions composed of integers, both the numerator and the denominator can be either even or odd, there are the following four possibilities (with nonzero denominator):

$$\frac{\text{even}}{\text{even}}, \qquad \frac{\text{odd}}{\text{odd}}, \qquad \frac{\text{even}}{\text{odd}}, \qquad \frac{\text{odd}}{\text{even}}.$$

We immediately have a potential problem with equivalent fractions, since, for example,

$$\frac{4}{6} = \frac{\text{even}}{\text{even}}$$

but cancels down to

$$\frac{2}{3} = \frac{\text{even}}{\text{odd}}.$$

However, this difficulty arises only in connection with the $\frac{\text{even}}{\text{even}}$ case. Any fraction can become $\frac{\text{even}}{\text{even}}$ by multiplying the numerator and denominator by any even number, but the other three types are *stable* and cannot be interconverted. So a fraction of the type $\frac{\text{odd}}{\text{even}}$ can never be equal to another of the type $\frac{\text{odd}}{\text{odd}}$ or $\frac{\text{even}}{\text{odd}}$, and the same is true of the other two types. However, if we restrict our fractions to those in their simplest form, then we will exclude $\frac{\text{even}}{\text{even}}$, since 2, at least, must be a factor of both the numerator and the denominator. This leaves us with the following three separate distinct possibilities:

$$\frac{\text{odd}}{\text{odd}}, \qquad \frac{\text{even}}{\text{odd}}, \qquad \frac{\text{odd}}{\text{even}}.$$

Definitions

We can now choose to call $\frac{\text{odd}}{\text{odd}}$ an *odd fraction*, since in the case where the numerator is a multiple of the denominator, for example $\frac{15}{3}$, the result is an odd natural number. (Writing this natural number as a fraction with a denominator of 1 also replicates the $\frac{\text{odd}}{\text{odd}}$ structure.) For the same reason, it seems sensible to regard $\frac{\text{even}}{\text{odd}}$ as an *even fraction,* since, again when the numerator is a multiple of the denominator, for example $\frac{12}{3}$, an even number is the result. This just leaves $\frac{\text{odd}}{\text{even}}$, which can never give an integer answer. For this reason, I choose to regard this as neither odd nor even. For our purposes, even and odd are defined only if the denominator is odd, in which case the parity of the fraction is the same as that of the numerator. To summarise, we are making the following definitions:

$$\frac{\text{odd}}{\text{odd}} = \text{odd}, \qquad \frac{\text{even}}{\text{odd}} = \text{even}, \qquad \frac{\text{odd}}{\text{even}} = \text{neither}, \qquad \left(\frac{\text{even}}{\text{even}} = \text{unsimplified}\right).$$

Properties

With natural numbers under addition, table 1 applies. The same results obtain for subtraction, provided that we are comfortable with defining odd and even numbers on the negative integers. We now consider how odd and even *fractions* behave under addition and subtraction.

Adding two odd fractions leads to

$$\frac{\text{odd}}{\text{odd}} + \frac{\text{odd}}{\text{odd}} = \frac{\text{odd} \times \text{odd} + \text{odd} \times \text{odd}}{\text{odd} \times \text{odd}} = \frac{\text{odd} + \text{odd}}{\text{odd}} = \frac{\text{even}}{\text{odd}} = \text{even}.$$

With two even fractions, we have

$$\frac{\text{even}}{\text{odd}} + \frac{\text{even}}{\text{odd}} = \frac{\text{even} \times \text{odd} + \text{even} \times \text{odd}}{\text{odd} \times \text{odd}} = \frac{\text{even} + \text{even}}{\text{odd}} = \frac{\text{even}}{\text{odd}} = \text{even}.$$

With one of each, we have

$$\frac{\text{odd}}{\text{odd}} + \frac{\text{even}}{\text{odd}} = \frac{\text{odd} \times \text{odd} + \text{even} \times \text{odd}}{\text{odd} \times \text{odd}} = \frac{\text{odd} + \text{even}}{\text{odd}} = \frac{\text{odd}}{\text{odd}} = \text{odd}.$$

So we see behaviour identical to that displayed by the natural numbers, and subtraction will work similarly, with the same provisos as above.

Multiplication gives us

$$\begin{aligned}
\text{odd} \times \text{odd} &= \frac{\text{odd}}{\text{odd}} \times \frac{\text{odd}}{\text{odd}} = \frac{\text{odd}}{\text{odd}} = \text{odd}, \\
\text{even} \times \text{odd} &= \frac{\text{even}}{\text{odd}} \times \frac{\text{odd}}{\text{odd}} = \frac{\text{even}}{\text{odd}} = \text{even}, \\
\text{even} \times \text{even} &= \frac{\text{even}}{\text{odd}} \times \frac{\text{even}}{\text{odd}} = \frac{\text{even}}{\text{odd}} = \text{even},
\end{aligned}$$

again replicating the behaviour of the natural numbers.

Table 1

+	**odd**	**even**
odd	even	odd
even	odd	even

However, it would be nice to integrate the *neithers*, of the form $\frac{\text{odd}}{\text{even}}$, into the structure, if possible. We find that

$$\begin{aligned}
\text{neither} + \text{odd} &= \frac{\text{odd}}{\text{even}} + \frac{\text{odd}}{\text{odd}} \\
&= \frac{\text{odd} \times \text{odd} + \text{odd} \times \text{even}}{\text{even} \times \text{odd}} \\
&= \frac{\text{odd} + \text{even}}{\text{even}} \\
&= \frac{\text{odd}}{\text{even}} \\
&= \text{neither}, \\
\text{neither} + \text{even} &= \frac{\text{odd}}{\text{even}} + \frac{\text{even}}{\text{odd}} \\
&= \frac{\text{odd} \times \text{odd} + \text{even} \times \text{even}}{\text{even} \times \text{odd}} \\
&= \frac{\text{odd} + \text{even}}{\text{even}} \\
&= \frac{\text{odd}}{\text{even}} \\
&= \text{neither}.
\end{aligned}$$

However,

$$\begin{aligned}
\text{neither} + \text{neither} &= \frac{\text{odd}}{\text{even}} + \frac{\text{odd}}{\text{even}} \\
&= \frac{\text{odd} \times \text{even} + \text{odd} \times \text{even}}{\text{even} \times \text{even}} \\
&= \frac{\text{even} + \text{even}}{\text{even}} \\
&= \frac{\text{even}}{\text{even}} \\
&= \,?
\end{aligned}$$

So with the addition of two neithers, we have an ambiguous case. An example would be

$$\begin{aligned}
\tfrac{3}{8} + \tfrac{1}{2} &= \tfrac{7}{8} \text{ (neither)}, \\
\tfrac{1}{6} + \tfrac{1}{2} &= \tfrac{2}{3} \text{ (even)}, \\
\tfrac{1}{10} + \tfrac{1}{2} &= \tfrac{3}{5} \text{ (odd)}.
\end{aligned}$$

Considering multiplication, we have

$$\text{neither} \times \text{neither} = \frac{\text{odd}}{\text{even}} \times \frac{\text{odd}}{\text{even}} = \frac{\text{odd}}{\text{even}} = \text{neither},$$

$$\text{neither} \times \text{odd} = \frac{\text{odd}}{\text{even}} \times \frac{\text{odd}}{\text{odd}} = \frac{\text{odd}}{\text{even}} = \text{neither}.$$

However, this time

$$\text{neither} \times \text{even} = \frac{\text{odd}}{\text{even}} \times \frac{\text{even}}{\text{odd}} = \frac{\text{even}}{\text{even}} = ?$$

So under multiplication it is with evens that the neithers lead to ambiguity. An example would be

$$\begin{aligned} \tfrac{3}{8} \times \tfrac{2}{5} &= \tfrac{3}{20} \text{ (neither)}, \\ \tfrac{1}{2} \times \tfrac{4}{5} &= \tfrac{2}{5} \text{ (even)}, \\ \tfrac{1}{2} \times \tfrac{2}{5} &= \tfrac{1}{5} \text{ (odd)}. \end{aligned}$$

So the neithers are *dominant*, in the sense that combination by addition with odd or even fractions, or by multiplication with odd fractions, or other neithers, leads to neithers. However, adding two neithers, or multiplying a neither by an even fraction, can lead to odd, even, or neither fractions.

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Extra-magic squares – part 2

5	31	35	60	57	34	8	30
19	9	53	46	47	56	18	12
16	22	42	39	52	61	27	1
63	37	25	24	3	14	44	50
26	4	64	49	38	43	13	23
41	51	15	2	21	28	62	40
54	48	20	11	10	17	55	45
36	58	6	29	32	7	33	59

This magic square does not only have all its row-sums and column-sums as well as the two diagonal sums equal, to 260, but the sums of the squares of the entries in each row, column, and diagonal are also equal, at 11 180.

Reference

1 Kazem Faeghi, *The Mathematical Amusements and Games.*

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The Truel Problem

A. G. SUMMERS and K. L. Q. READ

We were inspired to write this article by a problem posed by Simon Singh in *Fermat's Last Theorem* (see p. 167). The original problem, discussed briefly in Appendix 9 of the book, is one of Game Theory. The appeal of the problem lies in the counter-intuitive answers that arise, and in its solution by use of school-level and early undergraduate-level probability theory.

1. Introduction

The truel (triangular duel) problem first came to the authors' attention through its mention in *Fermat's Last Theorem* by Simon Singh (see reference 1). The purpose of this article is to present some mathematical analysis of the problem, drawing attention to aspects which may appear intuitive or counter-intuitive. The treatment is based on probability. We now state the problem. Three persons, A, B, and C, say, are to fight a triangular duel (truel). We assume that they have constant probabilities of success (i.e. of killing his or her intended targeted opponent), say p, q, and r respectively, where without loss of generality we assume that $0 < p < q < r < 1$, and all attempts, whether by the same or different individuals, are taken to be independent. In the interest of fairness they agree that A will fire first, then B if still alive, then C if still alive, continuing in this order until just one person is left alive (the eventual winner): we call this the *shoot to kill* (STK) strategy. We also assume that, if a person has a choice of opponents to fire at, he or she will aim at the person with the higher hit rate (probability of killing). Finally, we neglect the possibility that anybody's aim is so bad that someone is killed unintentionally.

In Section 2 we find in terms of p, q, and r the respective probabilities, P, Q, and R, say, of A, B, and C winning the truel if A uses the STK strategy. In Section 3, we analyse the apparently counter-intuitive possibility (raised by Singh) that, so long as both of his opponents survive, A, the weakest shot of the three, may improve his or her chance of winning by *firing into the air* (in effect forfeiting a turn) as long as B and C both survive. We call this the FIA strategy. Assuming that the first player (A) is sufficiently experienced to know p, q, and r accurately, the conditions under which the FIA strategy improves his chance of winning are established in Section 4. Section 5 is devoted to the possibility of a 'fair' truel, i.e. one in which all the participants have the same probability of winning ($\frac{1}{3}$). Section 6 deals with numerical examples and special cases.

2. Solution for *P*, *Q*, and *R* for the STK strategy

For general use later, let X and Y stand for any two of A, B, and C, and let their respective hit rates be θ and φ. Also let $f(\theta, \varphi)$ be the probability that X wins a duel with Y, given that Y fires first (so that $f(\varphi, \theta)$ is the probability that Y wins a duel with X, given that X fires first). By considering all possible outcome sequences and using independence, we see that

$$f(\theta, \varphi) = (1-\varphi)\theta + (1-\varphi)^2(1-\theta)\theta + (1-\varphi)^3(1-\theta)^2\theta + \cdots .$$

Noting that successive terms on the right-hand side form a geometric progression with common ratio $(1-\theta)(1-\varphi)$, we have

$$f(\theta,\varphi)=\frac{(1-\varphi)\theta}{1-(1-\theta)(1-\varphi)}=\frac{(1-\varphi)\theta}{\theta+\varphi-\theta\varphi}.$$

Applying this result with X as A and Y as B, we have

$$\text{P(A wins a duel with B, B firing first)}=f(p,q)=\frac{(1-q)p}{p+q-pq},$$
$$\text{P(B wins a duel with A, A firing first)}=f(q,p)=\frac{(1-p)q}{p+q-pq};$$

while by the law of total probability, we have

$$\text{P(A wins a duel with B, A firing first)}=1-f(q,p)=\frac{p}{p+q-pq},$$
$$\text{P(B wins a duel with A, B firing first)}=1-f(p,q)=\frac{q}{p+q-pq}.$$

Clearly, similar formulae may be written down for duels involving C.

The ground rules we have set out for a truel imply the following sequence of events.

- A first fires at C; if he is successful (with probability p) he goes on to fight a duel with B (with B firing first).
- If A fails to hit C, B next fires at C and if he is successful (with probability q) he goes on to fight a duel with A (with A firing first).
- If A and B both fail to hit C, C next fires at B and if he is successful (with probability r) he goes on to fight a duel with A (with A firing first).
- Otherwise, if A, B, and C all fail with their first shots we revert to the starting situation.

On this basis we have the recursive equation for P

$$P=pf(p,q)+(1-p)q[1-f(q,p)]+(1-p)(1-q)r[1-f(r,p)]+(1-p)(1-q)(1-r)P,$$

where the last term on the right-hand side represents the event that A, B, and C all miss their first shots, so that we revert to the starting situation. Hence,

$$P=\left[\frac{p^2(1-q)+(1-p)pq}{p+q-pq}+\frac{(1-p)(1-q)rp}{p+r-pr}\right]\Bigg/\,1-(1-p)(1-q)(1-r). \quad (1)$$

Considering B and C similarly, we find

$$Q=\left[\frac{pq+(1-p)^2q^2}{p+q-pq}\right]\Bigg/\,1-(1-p)(1-q)(1-r), \quad (2)$$
$$R=\left[\frac{(1-p)^2(1-q)r^2}{p+r-pr}\right]\Bigg/\,1-(1-p)(1-q)(1-r). \quad (3)$$

3. Analysis of the FIA strategy for participant A

Participant A is indeed a bad shot, but he is not stupid. He knows that B and C are better shots than he is. As long as both B and C survive, each in their turn will fire at the other rather than at A because A is the less dangerous opponent. By *firing into the air* to start with (and as long as both B and C are alive), A can ensure that he will eventually end up with the first shot in a *duel* with the survivor of B and C. We now let P', Q', and R' be the respective probabilities of A, B, and C winning on the basis of this modified strategy. Since by initially firing into the air A in effect gives up his turn, B in effect shoots first, aiming at C. Considering the possible outcomes under these new rules, we have

$$P' = q[1 - f(q, p)] + (1-q)r[1 - f(r, p)] + (1-q)(1-r)P',$$

giving

$$P' = \left[\frac{pq}{p+q-pq} + \frac{(1-q)rp}{p+r-pr}\right] \bigg/ 1 - (1-q)(1-r). \tag{4}$$

Considering B and C similarly, we find

$$Q' = \left[\frac{(1-p)q^2}{p+q-pq}\right] \bigg/ 1 - (1-q)(1-r), \tag{5}$$

$$R' = \left[\frac{(1-p)(1-q)r^2}{p+r-pr}\right] \bigg/ 1 - (1-q)(1-r). \tag{6}$$

4. Which strategy is better for A?

We now analyse the condition under which P' given by (4) exceeds P given by (1), i.e.

$$\begin{aligned}&\left[\frac{pq}{p+q-pq} + \frac{(1-q)rp}{p+r-pr}\right] \bigg/ 1 - (1-q)(1-r)\\ &\quad > \left[\frac{p^2(1-q)+(1-p)pq}{p+q-pq} + \frac{(1-p)(1-q)rp}{p+r-pr}\right] \bigg/ 1 - (1-p)(1-q)(1-r).\end{aligned}$$

To simplify this we note the following identities:

$$\begin{aligned}&pq[1-(1-p)(1-q)(1-r)] - [p^2(1-q) + pq(1-p)][1-(1-q)(1-r)]\\ &\quad \equiv p^2q^2 - p^2r(1-q)^2,\\ &(1-q)pr[1-(1-p)(1-q)(1-r)] - (1-p)(1-q)rp[1-(1-q)(1-r)]\\ &\quad \equiv (1-q)rp^2.\end{aligned}$$

Hence,

$$[p^2q^2 - p^2r(1-q)^2][p + r(1-p)] + (1-q)rp^2(p+q-pq) > 0.$$

Dividing by p^2 and simplifying, we arrive at

$$r^2(1-p)(1-q)^2 - rq(1-pq) - pq^2 < 0. \tag{7}$$

Since $r > 0$, this will be true for $r < r^*$, where r^* is the positive root of

$$r^2(1-p)(1-q)^2 - rq(1-pq) - pq^2 = 0.$$

We now consider for what range of values of p and q it is *always* best for A to fire into the air. This is equivalent to asking when $r^* \geq 1$. Clearly, by putting $r = 1$ in (7), we obtain

$$\begin{aligned} r^* \geq 1 &\iff (1-q)^2(1-p) - q(1-pq) - pq^2 \leq 0 \\ &\iff p \geq 1 - \frac{q}{(1-q)^2}. \end{aligned} \tag{8}$$

As we must have $p < q$, for a useful solution we also require

$$q > 1 - \frac{q}{(1-q)^2}, \quad \text{or } q > (1-q)^3, \quad \text{i.e. } q > 0.317\,672\ldots \text{ or } 0.318 \text{ to 3 decimal places.} \tag{9}$$

Finally, the right-hand side of (8) is negative or zero (making the condition automatic) if

$$q \geq (1-q)^2 \quad \text{or } q \geq \frac{3-\sqrt{5}}{2} = 0.381\,966\ldots \text{ or } 0.382 \text{ to 3 decimal places.}$$

Hence, using these approximate values, A should fire into the air until either B or C is eliminated if and only if

$$q \geq 0.382,$$

or

$$0.318 < q < 0.382 \textbf{ and either } p \geq 1 - \frac{q}{(1-q)^2} \textbf{ or } \left\{ p < 1 - \frac{q}{(1-q)^2} \text{ and } r < r^* \right\}, \tag{10}$$

or

$$\{q \leq 0.318 \text{ and } r < r^*\}. \tag{11}$$

5. Conditions for a fair truel

Given the sequential nature of the firing in a truel, an intriguing question is whether or not it is possible for a truel to be fair in the sense of equal probabilities of success for all three participants. Intuitively, if A, B, and C are all very poor shots, so that p, q, and r are all close to zero, then relatively small increments $q = p + \varepsilon$, $r = q + \eta$ would suffice to compensate B and C for firing later. This is confirmed graphically below.

5.1. Assuming that A follows the STK strategy

From (2), the condition $Q = \frac{1}{3}$ may be solved for r in terms of p and q as follows:

$$r = \frac{3[pq + (1-p)^2q^2] - (p+q-pq)^2}{(p+q-pq)(1-p)(1-q)}. \tag{12}$$

Using the further condition that $P = \frac{1}{3}$ (and hence $R = \frac{1}{3}$), yields another expression for r, and hence a quartic equation for q in terms of p:

$$(1-p)^3(2-p)q^4 + p^2(1-p)(9-8p)q^3 - p^2(5-12p)q^2 - 2p^3(1+2p)q + p^4 = 0. \tag{13}$$

We solved for q, given p, by substituting for r from (12) into (3), taking $R = \frac{1}{3}$, and solving numerically using the *Solver* facility in MICROSOFT EXCEL®. It appears that the maximum possible value of p is $0.263120\ldots$, and the graphs of q, r, and r^* as defined by (7) are shown in figure 1. Analysis of (7) showed that if $r^* > r$ then FIA is better for A. The graphs show that for a fair STK truel, A does better with FIA if $p > 0.12$ (approximately).

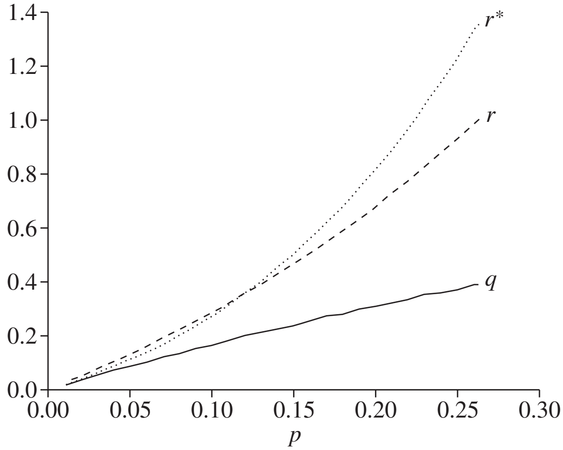

Figure 1 Shoot to kill strategy.

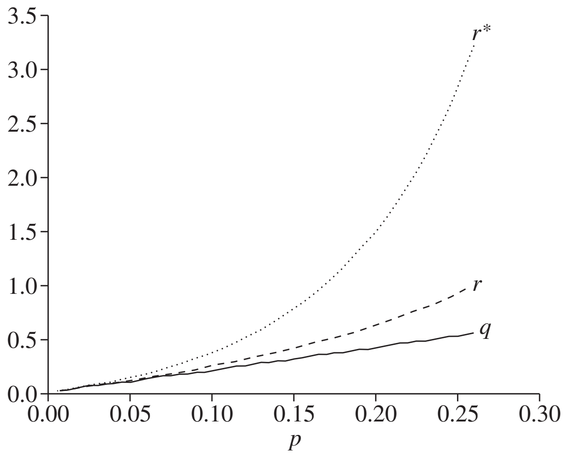

Figure 2 Fire into air strategy.

5.2. Assuming that A uses the FIA strategy

We now require p, q, and r such that $P' = Q' = R' = \frac{1}{3}$. From (5), the condition $Q' = \frac{1}{3}$ is easily solved for r in terms of p and q. Substituting for r, and using (6) and $R' = \frac{1}{3}$, we find

$$q = \frac{p(4-5p)}{(1-p)(2-p)}; \tag{14}$$

hence,

$$r = \frac{p(4-5p)}{2(1-p)(1-2p)}. \tag{15}$$

Thus, subject to $p < q < r < 1$, a fair truel may be generated by choosing an arbitrary p and then setting q and r as prescribed by (14) and (15). It is easily seen from (14) that $q > p$

provided that $p < \sqrt{3}-1$; also $p > 0$ and $q > 0 \Rightarrow p < \frac{4}{5}$ and $r < 1 \Rightarrow p < (5-\sqrt{7})/9 = 0.261\,583\ldots$. As this last, most restrictive condition on p implies the rest, a fair truel may be generated for any $p < 0.261\ldots$. These results are displayed in figure 2 (again with the r^* graph), which resembles that for the STK strategy. It appears that, for all admissible p, $r^* > r$, and so for no fair FIA truel A will do better to use STK.

6. Examples

We present some illustrative results in two tables. Table 1 involves fractional p, q, and r, whereas table 2 shows a selection of truels that are approximately fair (being based on numerical solutions) on the basis that A uses STK.

Row 1 of table 1 shows a truel that is fair if A uses FIA (the optimal choice). Rows 2 and 3 are examples of truels that are not fair on either strategy, but again FIA is best for A. In rows 4 and 5 we have $p = 0.1$ and $q = 0.2$: for FIA to be best for A we require (by (8))

$$r < r^* = 0.3596.$$

Hence, choosing $r = 0.36$ in row 4 makes P just greater than P' (STK is just better), whilst choosing $r = 0.35$ in row 5 makes P' just greater than P (FIA is just better). Row 6 retains $p = 0.1$ and $q = 0.2$, but now the value of r is set by (12) to make $Q = \frac{1}{3}$. As $0.391 > 0.3596$ we now find $P > P'$.

The truels in table 2 achieve $P = Q = R = \frac{1}{3}$ to at least five decimal places. Row 1 has a 'baseline' value of 0.1 for p: as r narrowly exceeds r^*, STK is slightly better for A here. In row 2 $P = Q = R = P' = \frac{1}{3}$ to five decimal places, so that for A STK and FIA are roughly equivalent.

The value of p in row 3 is chosen by solving (13) with $q = 0.317\,673$. This value is just greater than the sole real root of the equation $q = (1-q)^3$ (the minimum value for q in condition (9)). In this case $r < r^*$, and so the 'or…or' part of condition (10) holds (for A to use FIA), so P' just exceeds P.

Table 1 Examples with fractional p, q, and r.

	Probabilities of hits				Probabilities of winning: STK			Probabilities of winning: FIA		
	p	q	r	r^*	P	Q	R	P'	Q'	R'
1	$\frac{1}{5}$	$\frac{5}{12}$	$\frac{5}{8}$	1.4887	$\frac{11}{36} = 0.306$	$\frac{175}{396} = 442$	$\frac{25}{99} = 0.252$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
2	$\frac{1}{5}$	$\frac{1}{2}$	1	2.3561	$\frac{37}{150} = 0.247$	$\frac{13}{30} = 0.433$	$\frac{8}{25} = 0.32$	$\frac{4}{15} = 0.267$	$\frac{1}{3}$	$\frac{2}{5}$
3	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{2}{3}$	2.3561	$\frac{79}{286} = 0.276$	$\frac{1}{2}$	$\frac{32}{143} = 0.224$	$\frac{17}{55} = 0.309$	$\frac{2}{5}$	$\frac{16}{55} = 0.291$
4	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{9}{25} = 0.36$	0.3596	$\frac{142825}{500108} = 0.28559$	$\frac{3275}{9436} = 0.347$	$\frac{6561}{17861} = 0.367$	$\frac{12925}{45262} = 0.28556$	$\frac{225}{854} = 0.263$	$\frac{1458}{3233} = 0.451$
5	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{7}{20} = 0.35$	0.3596	$\frac{44615}{154546} = 0.2887$	$\frac{655}{1862} = 0.352$	$\frac{3969}{11039} = 0.3595$	$\frac{1345}{4648} = 0.2894$	$\frac{15}{56} = 0.268$	$\frac{147}{332} = 0.443$
6	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{197}{504} = 0.391$	0.3596	$\frac{82431}{298287} = 0.276$	$\frac{1}{3}$	$\frac{116427}{298287} = 0.390$	$\frac{22417}{81719} = 0.274$	$\frac{20493}{81719} = 0.251$	$\frac{38809}{81719} = 0.475$

Table 2 Examples involving numerical solution: fair truels using STK.

	Probabilities of hits				Probabilities of winning: STK			Probabilities of winning: FIA		
	p	q	r	r^*	P	Q	R	P'	Q'	R'
1	0.1	0.163228	0.28369	0.27043	0.33333	0.33333	0.33333	0.33179	0.24243	0.42579
2	0.11883	0.192022	0.34808	0.34807	0.33333	0.33333	0.33333	0.33333	0.23835	0.42832
3	0.20775	0.317673	0.71245	0.86976	0.33333	0.33333	0.33333	0.34142	0.21650	0.44208
4	0.25956	0.381966	0.97991	1.31823	0.33333	0.33333	0.33333	0.34667	0.20168	0.45165
5	0.2	0.307463	0.67627	0.81270	0.33333	0.33333	0.33333	0.34067	0.21858	0.44075

Similarly, in row 4, p is chosen using (13) with $q = 0.381\,966$, just less than the second critical value $(3-\sqrt{5})/2$ so that the ‘either’ part of condition (10) holds, and again P' exceeds P. In row 5, $p = 0.2$ and condition (11) holds for A to use FIA, since $q < 0.317\,672$ and $r < r^*$. The results show the gradually increasing benefit to A of FIA over STK as p is increased, but if A uses FIA then B’s chances are markedly worse and those of C distinctly better.

These results confirm that for a fair STK truel FIA is better for A if $p > 0.12$ (approximately), as noted at the end of Section 5.1.

Reference

1 Simon Singh, *Fermat’s Last Theorem* (Fourth Estate, London, 1998).

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Number coincidences

$$153 = 1^3 + 5^3 + 3^3 = 1 + 2 + 3 + \cdots + 17 = 1! + 2! + 3! + 4! + 5!,$$
$$3^3 + 4^3 + 5^3 = 6^3,$$
$$666 = 1^6 - 2^6 + 3^6 = 6 + 6 + 6 + 6^3 + 6^3 + 6^3$$
$$666 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 5^3 + 4^3 + 3^3 + 2^3 + 1^3,$$
$$3435 = 3^3 + 4^4 + 3^3 + 5^5, \qquad 2^5 \times 9^2 = 2592, \qquad 3^4 \times 425 = 34425,$$
$$998001 = (998 + 001)^2, \qquad 40585 = 4! + 0! + 5! + 8! + 5!,$$
$$145 = 1! + 4! + 5!, \qquad 234256 = (2 + 3 + 4 + 2 + 5 + 6)^4,$$
$$1634 = 1^4 + 6^4 + 3^4 + 4^4, \qquad 8208 = 8^4 + 2^4 + 0^4 + 8^4,$$
$$9474 = 9^4 + 4^4 + 7^4 + 4^4, \qquad 92727 = 9^5 + 2^5 + 7^5 + 2^5 + 7^5,$$
$$93084 = 9^5 + 3^5 + 0^5 + 8^5 + 4^5, \qquad 548834 = 5^6 + 4^6 + 8^6 + 8^6 + 3^6 + 4^6.$$

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Height of Difficulty

PRITHWIJIT DE

Cubic equations are seldom encountered in school mathematics. This article presents a problem in mensuration which gives rise to a cubic equation and discusses a method by which it can be solved. It also illustrates a way of determining the nature of its roots.

1. Genesis

Let us begin with a question that was asked in the 2001 Fermat Contest (see reference 1).

Problem 1 Consider a bottle (see figure 1) which consists of two concentric cylinders of different diameters, d_1 and d_2, where $d_1 < d_2$. The height of the bottle is $u + v$. It is filled with a liquid not to its capacity. The height of the liquid is x units ($v < x < u + v$) when the bottle stands on the wider base and it is y units ($u < y < u + v$) when it stands on the narrower base. What is the height of the bottle in terms of the known quantities?

We can solve this in the following two ways:

(a) by equating the volumes of the liquid in both orientations,

(b) by equating the volumes of the air gap in both orientations.

Of these, the second method is less involved. To see this, we just have to solve the following equation in h (where $h = u + v$):

$$\frac{\pi d_1^2(h - x)}{4} = \frac{\pi d_2^2(h - y)}{4}.$$

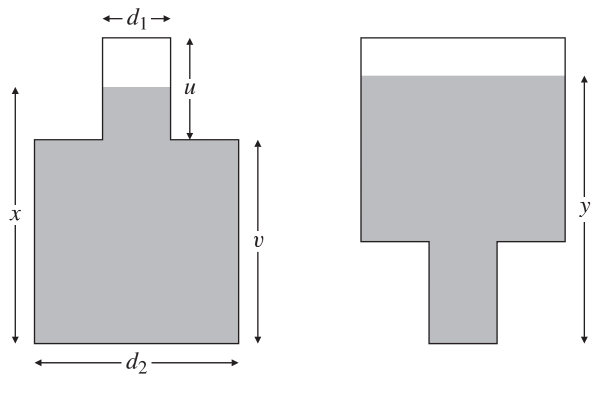

Figure 1 Cylindrical bottle in two different orientations: base-down (left) and head-down (right). The shaded area represents liquid.

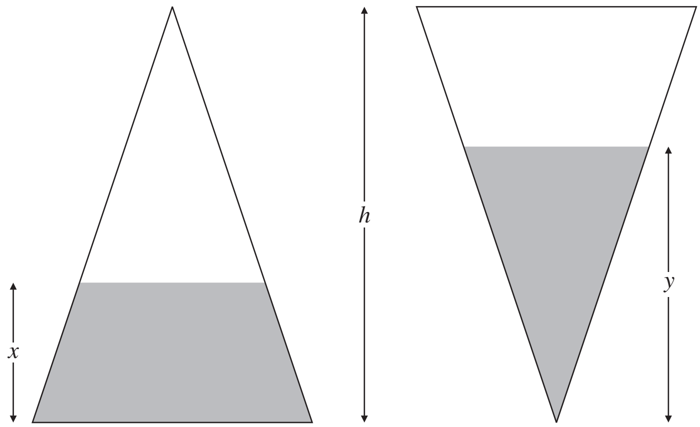

Figure 2 Conical bottle in two different orientations: base-down (left) and head-down (right). The shaded area represents liquid.

Having answered this question it is quite natural to consider bottles of different shapes and ask a similar question. For instance, can we solve the above problem if we have a bottle in the shape of a right circular cone of height h and base radius r, partially filled with a liquid, and the heights of the liquid are x units and y units in the two orientations shown in figure 2? Here we are assuming that $0 < x < h$ and $0 < y < h$.

In this case, it does not matter whether we equate the volumes of the liquid or that of the air gaps in order to find the height h because in either case we have to compare the volume of a frustum of a cone with the volume of a right circular cone. Let x be the vertical height of the liquid when the cone is standing on its base (see the left-hand part of figure 2) and y be the vertical height when it is balanced on its tip (see the right-hand part of figure 2). The volume of the air gap in the first orientation is

$$\frac{\pi r^2 (h-x)^3}{3h^2}$$

and that in the second orientation is

$$\frac{\pi r^2 (h^3 - y^3)}{3h^2}.$$

Equating these two expressions leads us to

$$(h-x)^3 = h^3 - y^3. \tag{1}$$

This reduces to a quadratic equation

$$3xh^2 - 3x^2h + x^3 - y^3 = 0, \tag{2}$$

which yields

$$h = \frac{x}{2} + \frac{\sqrt{12xy^3 - 3x^4}}{6x}.$$

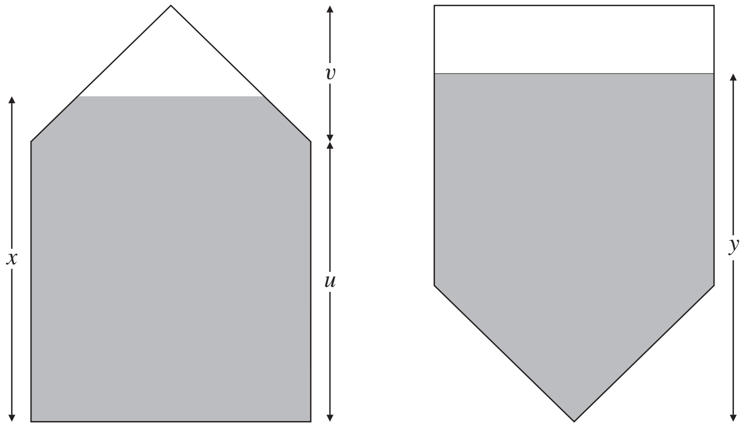

Figure 3 Cone–cylinder bottle in two different orientations: base-down (left) and head-down (right). The shaded area represents liquid.

Here we have taken the positive root because $h > x/2$. If (2) is written as $3xh(h-x)+x^3 = y^3$ then we can see at once that $y^3 > x^3$; hence, $y > x$. Thus the discriminant of the quadratic equation in (2) is always positive and we always get a pair of real roots by solving (2), one of which is positive. This positive root is the height of the bottle. Also notice that if x and y are integers then h cannot be an integer. For if it were an integer then (1) will have positive integer solutions thereby contradicting Fermat's last theorem.

Now we merge the two problems discussed above to obtain a new problem.

Problem 2 Suppose that we have a bottle consisting of a right circular cylinder of base radius r and a right circular cone of identical base radius surmounting the cylinder (see figure 3). What is the height of the bottle?

The bottle is filled with a liquid so that it fills up the conical part partially when base-down. Let the height of the liquid be x units when the bottle stands base-down (see the left-hand part of figure 3) and let the height of the liquid be y units when it stands head-down (see the right-hand part of figure 3). Let the heights of the cylindrical and conical parts be u units and v units respectively and let $h = u + v$. We assume that the liquid level comes above the division between the cone and cylinder in both cases and $0 < x,\ y < h$. The volume of the air gap in the base-down position is

$$\frac{\pi r^2(h-x)^3}{3v^2}$$

and that in the head-down position is

$$\pi r^2(h-y).$$

Set $v = \alpha h$, where α is a number between 0 and 1. Since the volumes are equal, by equating them we obtain

$$\frac{\pi r^2(h-x)^3}{3\alpha^2 h^2} = \pi r^2(h-y). \tag{3}$$

Simplifying (3), we have

$$(3\alpha^2 - 1)h^3 - 3(y\alpha^2 - x)h^2 - (3x^2)h + x^3 = 0. \tag{4}$$

It should be noted that $y > x$. For if it is not the case then $h - y \geq h - x$ will hold and, by virtue of (3), we will have $(h - x)^2 \geq 3v^2$; but $v^2 \geq (h - x)^2$. Therefore, $v^2 \geq 3v^2$, i.e. a contradiction. Equation (4) is a cubic equation in h with real coefficients if $\alpha \neq 1/\sqrt{3}$. If $\alpha = 1/\sqrt{3}$ then (4) reduces to the following quadratic:

$$(3x - y)h^2 - 3x^2h + x^3 = 0. \tag{5}$$

If $y \geq 3x$ then, from (5),

$$(3x - y)h^2 = (3h - x)x^2,$$

implying $x \geq 3h$, which is not so. Therefore, $y < 3x$ and the roots of (5) are

$$\frac{3x^2 \pm \sqrt{4x^3y - 3x^4}}{2(3x - y)}.$$

Both roots are positive. Let

$$f(h) = (3x - y)h^2 - 3x^2h + x^3,$$

then $f(0) = x^3 > 0$, $f(x) = x^2(x - y) < 0$, and $f(y) = -(y - x)^3 < 0$. Therefore one of the roots is less than x and cannot be the height of the bottle. The larger root, which is greater than y, is the required height. Thus the height is given by

$$\frac{3x^2 + \sqrt{4x^3y - 3x^4}}{2(3x - y)}.$$

Now we turn our attention to the case $\alpha \neq 1/\sqrt{3}$. In this case (4) has either one real root or three real roots. The sign of the real root will depend on the sign and magnitude of the coefficients. As we do not have the values of x, y, and α it is not possible to determine the exact values of the roots. But conditions on the coefficients under which the equation has at least one positive real root may be obtained by computing the discriminant of the cubic given by (4). As cubic equations are unfamiliar to many of us, it is an appropriate juncture to visit their unexplored world.

2. The world of the cubic equation

The general form of a cubic equation with real coefficients in an unknown q is

$$aq^3 + 3bq^2 + 3cq + d = 0,$$

where $a \neq 0$ (see reference 2, p. 179). The procedure for extracting the roots of a cubic equation was proposed by the Italian mathematician Gerolamo Cardano (1501–1576). A brief description of the method is given below.

Put $z = aq + b$. The equation

$$aq^3 + 3bq^2 + 3cq + d = 0 \tag{6}$$

becomes

$$z^3 + 3Hz + G = 0, \tag{7}$$

where $H = ac - b^2$ and $G = a^2d - 3abc + 2b^3$. Let $z = m^{1/3} + n^{1/3}$. Then

$$z^3 - 3m^{1/3}n^{1/3}z - (m+n) = 0.$$

We put $m^{1/3}n^{1/3} = -H$ and $m + n = -G$. Therefore m and n are roots of

$$t^2 + Gt - H^3 = 0,$$

and we may take $m = \frac{1}{2}(-G + \sqrt{G^2 + 4H^3})$. If Q denotes any one of the three values of

$$\sqrt[3]{\tfrac{1}{2}(-G + \sqrt{G^2 + 4H^3})},$$

the three values of $m^{1/3}$ are Q, ωQ, $\omega^2 Q$, where ω is a complex cube root of unity. Also, because $m^{1/3}n^{1/3} = -H$, the corresponding values of $n^{1/3}$ are $-H/Q$, $-\omega^2 H/Q$, $-\omega H/Q$. Hence the values of z, that is of $aq + b$, are $Q - H/Q$, $\omega Q - \omega^2 H/Q$, $\omega^2 Q - \omega H/Q$. The roots of (6) are

$$\frac{1}{a}\left(-b + Q - \frac{H}{Q}\right), \qquad \frac{1}{a}\left(-b + \omega Q - \frac{\omega^2 H}{Q}\right), \qquad \frac{1}{a}\left(-b + \omega^2 Q - \frac{\omega H}{Q}\right).$$

Let Δ be defined as

$$\Delta = G^2 + 4H^3 = a^2(a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2).$$

We now have the following statements.

1. If $\Delta = 0$, $m = n = -G/2$ and the roots of (7) are $2(-G/2)^{1/3}$, $-(-G/2)^{1/3}$ (repeated).

2. If $G^2 + 4H^3 > 0$, Q is real and (7) (and (6)) has one real and two complex roots.

3. If $G^2 + 4H^3 < 0$, $m = \bar{n}$ and the roots of (7) are $m^{1/3} + n^{1/3}$, $\omega m + \omega^2 n$ and $\omega^2 m + \omega n$, where ω is a complex cube root of unity and $\bar{\omega} = \omega^2$. All roots are real in this case.

3. The final act

Now we are in a position to analyse the nature of the roots of (4). Comparing the general cubic expression with (4) we get $a = 3\alpha^2 - 1$, $b = (x - y\alpha^2)$, $c = -x^2$, and $d = x^3$. After some calculation, we have

$$\Delta = 9a^2\alpha^4(xy)^3\left(t(t-1)^2 - \frac{4\alpha^2}{9}\right),$$

where $t = x/y$ and $\frac{1}{3} < t < 1$ since $x < y < 3x$. Let

$$g(t) = t(t-1)^2 - \frac{4\alpha^2}{9}, \qquad 0 < t < 1.$$

Then

$$g'(t) = 3\big(t - \tfrac{1}{3}\big)(t-1)$$

and

$$g''\big(\tfrac{1}{3}\big) = -2.$$

Thus $g(t)$ has a local maximum at $t = \frac{1}{3}$ and

$$g\big(\tfrac{1}{3}\big) = \tfrac{4}{9}\big(\tfrac{1}{3} - \alpha^2\big).$$

Observe that $g(0) < 0$ and $g(1) < 0$. Now, if $g(\frac{1}{3}) < 0$, i.e. if $\alpha > 1/\sqrt{3}$, then $g(t) < 0$ on $(\frac{1}{3}, 1)$. Therefore, $\Delta < 0$ on $(\frac{1}{3}, 1)$ if $\alpha > 1/\sqrt{3}$ and the equation in (4) has three real and unequal roots. The product of the roots of (4) is $x^3/(1-3\alpha^2)$ which is negative in this case. Thus there can be one negative and two positive roots or three negative roots. The case of three negative roots may be ruled out by the fact that the height h is a positive quantity and so the cubic equation must have a positve root. In the other case, we will have one positive root greater than y and the other positive root less than x. To see this, let

$$f(h) = (3\alpha^2 - 1)h^3 - 3(y\alpha^2 - x)h^2 - (3x^2)h + x^3,$$

and observe that $f(0) = x^3 > 0$, $f(x) = 3\alpha^2x^2(x-y) < 0$, $f(y) = -(y-x)^3$, and $f(h)$ tends to $+\infty$ as $h \to +\infty$. As $f(h)$ is a continuous function on the set of all real numbers it must have a root between 0 and x and another root greater than y, and this root gives the height of the bottle.

If $g(\frac{1}{3}) > 0$, i.e. if $\alpha < 1/\sqrt{3}$, then $g(t)$ must vanish once in $(\frac{1}{3}, 1)$ because $g(1) < 0$. Let $t_0 \in (\frac{1}{3}, 1)$ be such that $g(t_0) = 0$. Then $g(t)$ is positive in $(\frac{1}{3}, t_0)$ and is negative in $(t_0, 1)$. Therefore, Δ is positive in $(\frac{1}{3}, t_0)$ and consequently the cubic in (4) ends up with one real and two nonreal complex conjugate roots. The real root is positive as the product of the roots, $x^3/(1-3\alpha^2)$, is positive. This real root is the height of the bottle.

References

1 I. VanderBurgh, Problem of the Month (Problem 2), *Crux Mathematicorum* **36** (2010), pp. 79–80.

2 S. Barnard and J. M. Child, *Higher Algebra* (Macmillan, London, 1936).

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Letters to the Editor

Dear Editor,

Probability of an acute-angled triangle

Here we give an alternative proof of the following theorem to that given in reference 1.

Theorem *For randomly chosen triangles in a plane, it is likely that there are three times as many obtuse-angled triangles as there are acute-angled ones and hardly any right-angled triangles.*

We first prove the following result.

Lemma *If three distinct points A, B, C are chosen at random on a fixed circle (call it the base circle), the probabilities that $\triangle ABC$ is (1) acute-angled, (2) obtuse-angled, and (3) right-angled, are $\frac{1}{4}$, $\frac{3}{4}$, and 0 respectively.*

Proof Let A be selected anywhere on the base circle (see figure 1(a)) and let A' be diametrically opposite to A. Now B can be selected on either of the semicircles so formed with equal probabilities. For definiteness, we choose the left one (see figure 1(b)). Let B' be diametrically opposite to B. This splits the circle into four open arcs: 1, 2, 3, 4 (see figure 1(c)). C can be chosen on any of these arcs or at A' or B', to obtain a triangle. In the last two cases the triangle will be right-angled. It is clear that there will be no triangle if C is taken at A or B. Let the third point C be chosen on any one of the four open arcs. The situation will be as in table 1. Our lemma seems to hold as there is only one case out of four in which the triangle is

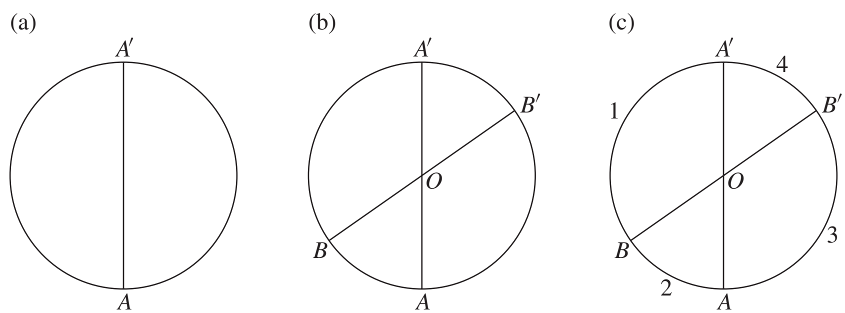

Figure 1

Table 1

If C is on arc	Nature of the triangle
1	Angle B will be obtuse
2	Angle C will be obtuse
3	Angle A will be obtuse
4	All angles will be acute

acute. However, in order to find the probabilities, we use the theorem that

$$p(P.Q) = p(P).p(Q \mid P),$$

where $P.Q$ is the joint occurrence of P and Q and $p(Q \mid P)$ is the conditional probability of Q given that P has occurred.

Now let P mean choosing B and $Q \mid P$ mean choosing C, B having been chosen. Let $\angle AOB = z$. By the random choice of B, it is equally distributed on the semi-circle, so that $0 < z < \pi$. In order to be able to integrate with respect to z over $(0, \pi)$, we assume that the radius OB lies in $(z, z + \mathrm{d}z)$. Now for this choice of B (say the event P), the probability of which is $\mathrm{d}z/\pi$, we can choose C on arc 4, so as to get an acute-angled triangle (the event $Q \mid P$), with probability $p(Q \mid P) = z/2\pi$, z being equally distributed over the semicircle. If the triangle were to be obtuse, event $Q \mid P$ would be that C is on any one of the arcs 1, 2, 3. Then $p(Q \mid P) = (2\pi - z)/2\pi$. Hence, if we integrate with respect to z over $(0, \pi)$, we get (1): for an acute-angled triangle,

$$p(P.Q) = \int_0^\pi \frac{z\,\mathrm{d}z}{2\pi^2} = \frac{1}{4},$$

(2): for an obtuse-angled triangle,

$$p(P.Q) = \int_0^\pi \frac{(2\pi - z)\,\mathrm{d}z}{2\pi^2} = \frac{3}{4}.$$

Hence it follows that (3): the probability for a right-angled triangle is zero. This proves the lemma.

An alternative argument for a right-angled triangle can be given as follows. If $B = A'$, the triangle will be right-angled, then $p(P) = 0$. If B is anywhere else, then for a right-angled triangle, $C = A'$ or $C = B'$, so that $p(Q \mid P) = 0$. In any case, $p(P.Q) = 0$.

Now to prove the theorem, choose any triangle at random in the plane. There will exist a similar and similarly placed triangle on the base circle. All triangles in the plane will thus be partitioned into equivalence classes, the equivalence relation being that the triangle is similar and similarly placed relative to some triangle on the base circle. Thus every triangle $\triangle$ in the plane will be equivalent to a *unique* triangle, say $E(\triangle)$, on the base circle and $E(\triangle)$ can be taken to be the representative of the class of $\triangle$. Hence it is enough to consider randomly chosen triangles on the base circle. Hence our theorem follows from the lemma above.

Yours sincerely,

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Reference

1 R. K. Guy, There are three times as many obtuse-angled triangles as there are acute-angled ones, *Math. Magazine* **66** (1993), pp. 175–179.

Dear Editor,

Integer replacement problems

The following result is quite well known.

Result Let n be a positive integer greater than 2. Then, in the identity

$$(1+2+\cdots+n-1+n)^2 = 1^3+2^3+\cdots+(n-1)^3+n^3, \tag{1}$$

if $(n-1)$ is replaced by 2 on both sides, the equality remains valid, i.e.

$$(1+2+\cdots+n-(n-1)+2)^2 = 1^3+2^3+\cdots+n^3-(n-1)^3+2^3$$

holds.

We were motivated to think of a generalisation of this. We thought of the following problem.

Problem Find a triple of positive integers (n, k, m) such that $\max(k, m) < n$, $k \neq m$, and if k is replaced by m in (1) then the equality remains valid.

Thus one solution of the problem is the triple $(n, n-1, 2)$ for $n > 3$. In short we need

$$\left(\frac{n(n+1)}{2} - k + m\right)^2 = \left(\frac{n(n+1)}{2}\right)^2 - k^3 + m^3,$$

i.e.

$$n(n+1) = m^2 + k^2 + mk - m + k. \tag{2}$$

Below we give three sets of (n, m, k) which satisfy (2). Let a and x be positive integers.

$$\begin{aligned}
\text{Set 1}\qquad n &= (a^2+3a+3)x + a + 1,\\
k &= a(a+2)x + a,\\
m &= (2a+3)x + 2.\\
\text{Set 2}\qquad n &= (a^2+3a+3)x - a - 2,\\
k &= a(a+2)x - a - 2,\\
m &= (2a+3)x.\\
\text{Set 3}\qquad n &= (3a^2+3a+1)x + 3a + 1,\\
k &= (3a^2+2a)x + 3a,\\
m &= (2a+1)x + 2.
\end{aligned}$$

In Set 1, if we put $x = 0$ and $a = n-1$ we get the case given at the beginning.

It is conjectured that the above three sets exhaust the solution set of equation (2).

Yours sincerely,

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Dear Editor,

The equation $x^2 + y^2 = 2$

In Volume 43, Number 2, pp. 89, 90–91, I found a family of integer solutions of the equation $x^2 + y^2 = z^3$ and the integer solutions of the equation $x^2 + y^2 = 2z^2$. We can use these to find rational numbers x, y with three decimal digits such that $x^2 + y^2 = 2$, or integers X, Y such that $X^2 + Y^2 = 2(1000)^2$. We need integers s, t such that $s^2 + t^2 = 10^3$ so we put $s = a^3 - 3a$, $t = 3a^2 - 1$, $10 = a^2 + 1$. Thus $a = 3$, $s = 18$, and $t = 26$, giving

$$X = s^2 - t^2 + 2st = 584, \qquad Y = |s^2 - t^2 - 2st| = 1288.$$

Hence, $(0.584)^2 + (1.288)^2 = 2$.

Yours sincerely,

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Dear Editor,

Curious consecutive numbers

Abbas Rouholamini Gugheri in Volume 43, Number 3, p. 132, gave some examples of consecutive numbers divisible by perfect squares. The chain of five consecutive numbers 844, 845, 846, 847, 848 is an addition to his list. The successive entries of this chain are divisible by 2^2, 13^2, 3^2, 11^2, and 4^2 respectively.

However, we found that the following general formula for N meets the objective easily:

$$N = (150(6n - 5) + 7)^2 \tag{1}$$
$$= 22\,500(6n - 5)^2 + 12\,600n - 10\,451. \tag{2}$$

It is clear from the form (2) of the formula that the numbers $N - 1$, N, $N + 1$, $N + 2$, $N + 3$ are divisible by 2^2, N, 5^2, 3^2, 2^2 respectively for all natural numbers n. Further it was interesting to find that for infinitely many values of n, such as $n = 99, 687, 1067$, and so on, the numbers $N - 2$ and $N - 3$ are divisible by 7^2 and 11^2 respectively.

Thus we can get an infinite chain of seven consecutive numbers, for instance putting $n = 99$ in (1), we get $N = 7\,806\,959\,449$. This yields

$$\begin{aligned}
N - 3 &= 7\,806\,959\,446 = 64\,520\,326 \times 11^2,\\
N - 2 &= 7\,806\,959\,447 = 159\,325\,703 \times 7^2,\\
N - 1 &= 7\,806\,959\,448 = 1\,951\,739\,862 \times 2^2,\\
N &= 7\,806\,959\,449 = 1 \times 88\,357^2,\\
N + 1 &= 7\,806\,959\,450 = 312\,278\,378 \times 5^2,\\
N + 2 &= 7\,806\,959\,451 = 867\,439\,939 \times 3^2,\\
N + 3 &= 7\,806\,959\,452 = 1\,951\,739\,863 \times 2^2.
\end{aligned}$$

Further,

$$
\begin{aligned}
6\,187\,374 &= 229\,162 \times 3^3,\\
6\,187\,375 &= 49\,499 \times 5^3,\\
6\,187\,376 &= 773\,422 \times 2^3,\\
6\,187\,377 &= 18\,039 \times 7^3.
\end{aligned}
$$

Readers are invited to investigate further.

Yours sincerely,

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Dear Editor,

Prime rationals

A *prime rational number* is a quotient of two prime numbers, for example $\frac{47}{53}$. We know that the set of all rational numbers is dense in the set of all real numbers, i.e. every real number can be approximated to by a rational number to any given degree of accuracy. Is the same true of the set of all prime rationals and their negatives?

Yours sincerely,

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Dear Editor,

A Fibonacci similarity

Start with any two given numbers m, n. They could be natural numbers, integers, rational, real, complex numbers, whatever. Now construct the Fibonacci-like sequence (x_r) using the rule

$$x_1 = m, \qquad x_2 = n, \qquad x_r = x_{r-1} + x_{r-2} \quad \text{for } r > 2.$$

Thus the sequence (x_r) starts

$$x_1 = m, \qquad x_2 = n, \qquad x_3 = m + n, \qquad x_4 = m + 2n, \qquad x_5 = 2m + 3n,$$

and evidently $x_r = m\mathrm{F}_{r-2} + n\mathrm{F}_{r-1}$ for $r > 2$, where (F_r) is the Fibonacci sequence defined by

$$\mathrm{F}_1 = \mathrm{F}_2 = 1, \qquad \mathrm{F}_r = \mathrm{F}_{r-1} + \mathrm{F}_{r-2} \quad \text{for } r > 2.$$

The sum of the first r terms of the sequence (x_r) is

$$m(1 + \mathrm{F}_1 + \mathrm{F}_2 + \cdots + \mathrm{F}_{r-2}) + n(\mathrm{F}_1 + \mathrm{F}_2 + \cdots + \mathrm{F}_{r-1}) = m\mathrm{F}_r + n(\mathrm{F}_{r+1} - 1).$$

Notice that

$$
\begin{aligned}
&\text{the sum of the first 2 terms } = m + n = x_3,\\
&\text{the sum of the first 6 terms } = 8m + 12n = 4x_5,\\
&\text{the sum of the first 10 terms } = 55m + 88n = 11x_7,\\
&\text{the sum of the first 14 terms } = 377m + 609n = 29x_9,\\
&\text{the sum of the first 18 terms } = 2584m + 4180n = 76x_{11},\\
&\text{the sum of the first 22 terms } = 17\,711m + 28\,656n = 199x_{13}.
\end{aligned}
$$

This suggests that

$$\text{the sum of the first } 4k-2 \text{ terms } = r_k x_{2k+1},$$

where

$$r_1 = 1, \qquad r_2 = 4, \qquad r_k = 3r_{k-1} - r_{k-2} \quad \text{for } k \geq 3.$$

Yours sincerely,

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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st July will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

44.5 Let P_1 be the point $(x_1, 0)$ on the x-axis. The straight line through P_1 perpendicular to the x-axis meets the curve $y = a^x$ (where $a > 0$, $a \neq 1$) at $Q_1 = (x_1, y_1)$. The tangent to the curve at Q_1 meets the x-axis at $P_2(x_2, 0)$. This is repeated for P_2, and so on, to obtain the sequences $\{x_n\}$ and $\{y_n\}$. Show that the sequence $\{x_n\}$ is an arithmetic progression and $\{y_n\}$ is a geometric progression. Carry out a similar procedure using the parabola $y = ax^2$, where $a > 0$.

(Submitted by Zhang Yun, Sunshine High School, Xi'an Jiaotong University, China)

44.6 The triangular number 120 is an sum of four consecutive powers of 2 ($120 = 8 + 16 + 32 + 64$). Prove that, for every positive integer n, there is a triangular number which is the sum of n consecutive powers of 2.

(Submitted by Tom Moore, Bridgewater State University, USA)

44.7 The function f satisfies $|f(x)| \leq \frac{1}{2}$ and $|f'(x)| \leq 1/b^2$ for all $x \in [a, b]$, where $0 < a < b$. If $|f(a)| \geq 1/\sqrt{a}$, prove that $|f(b)| \geq 1/\sqrt{b}$.

(Submitted by Spiros Andriopoulos, Third High School of Amaliada, Eleia, Greece)

44.8 For a given positive integer n, find the minimum value of

$$\left(1 + \frac{1}{\sin^n \alpha}\right)\left(1 + \frac{1}{\cos^n \alpha}\right),$$

for $0 < \alpha < \pi/2$.

(Submitted by Abbas Rouholamini Gugheri, Sirjan, Iran)

Solutions to Problems in Volume 43 Number 3

43.9 The polynomial $ax^2 + bx + c$, with a, b, c real numbers and $a > 0$, has nonreal roots. When does one of its roots lie on the parabola with equation $y = ax^2 + bx + c$ viewed as lying in the Argand diagram?

Solution by Jonny Griffiths, who proposed the problem

The roots of the polynomial are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \mathrm{i}\sqrt{4ac - b^2}}{2a},$$

with $4ac > b^2$ since the roots are not real. The point $(-b/2a, -\sqrt{4ac - b^2}/2a)$ cannot lie on the parabola because the parabola lies above the x-axis. The point $(-b/2a, -\sqrt{4a - b^2}/2a)$ lies on the parabola if and only if

$$\begin{aligned} a\left(\frac{-b}{2a}\right)^2 + b\frac{-b}{2a} + c &= \frac{\sqrt{4ac - b^2}}{2a} \\ \iff \quad \frac{-b^2}{4a} + c &= \frac{\sqrt{4ac - b^2}}{2a} \\ \iff \quad 4ac - b^2 &= 2\sqrt{4ac - b^2} \\ \iff \sqrt{4ac - b^2} &= 2 \\ \iff 4ac - b^2 &= 4. \end{aligned}$$

43.10 The nonnegative real numbers x, y, z satisfy $yz + zx + xy = 1$. Find lower bounds which can be attained for each of the expressions

$$x + y + z \quad \text{and} \quad 2x + y + z.$$

Solution by J. A. Scott, who proposed the problem

Put $s = x + y + z$. We have

$$\left(x - \frac{1}{\sqrt{3}}\right)^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 + \left(z - \frac{1}{\sqrt{3}}\right)^2 \geq 0,$$

so

$$x^2 + y^2 + z^2 - \frac{2}{\sqrt{3}}(x + y + z) + 1 \geq 0,$$

so

$$x^2 + y^2 + z^2 \geq \frac{2s}{\sqrt{3}} - 1.$$

Also,

$$\begin{aligned} s^2 &= x^2 + y^2 + z^2 + 2(yz + zx + xy) \\ &= x^2 + y^2 + z^2 + 2. \end{aligned}$$

Hence,

$$s^2 - 2 \geq \frac{2s}{\sqrt{3}} - 1,$$

so

$$s^2 - \frac{2\sqrt{3}}{3}s - 1 \geq 0,$$

so

$$(s - \sqrt{3})\left(s + \frac{\sqrt{3}}{3}\right) \geq 0,$$

so $s \geq 3$ since $s + \sqrt{3}/3 > 0$. Moreover, $s = \sqrt{3}$ when $x = y = z = \sqrt{3}/3$.

Next,

$$\begin{aligned} (2x + y + z)^2 &= 4x^2 + 4x(y + z) + (y + z)^2 \\ &= 4x^2 + 4(1 - yz) + (y + z)^2 \\ &= 4x^2 + 4 + (y - z)^2 \\ &\geq 4, \end{aligned}$$

so $2x + y + z \geq 2$. Moreover, $2x + y + z = 2$ when $x = 0,\ y = z = 1$.

43.11 Sum the finite series

$$F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_nF_{n+1},$$

where F_r denotes the rth Fibonacci number.

Solution 1 by M. A. Khan, who proposed the problem

When n is odd,

$$\begin{aligned} &F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_nF_{n+1} \\ &\quad = F_2F_2 + F_2(F_4 - F_2) + (F_4 - F_2)F_4 + \cdots + F_{n-1}(F_{n+1} - F_{n-1}) \\ &\qquad + (F_{n+1} - F_{n-1})F_{n+1} \\ &\quad = F_{n+1}^2. \end{aligned}$$

When n is even,

$$\begin{aligned}
&\mathrm{F}_1\mathrm{F}_2 + \mathrm{F}_2\mathrm{F}_3 + \mathrm{F}_3\mathrm{F}_4 + \cdots + \mathrm{F}_n\mathrm{F}_{n+1} \\
&\quad = \mathrm{F}_1(\mathrm{F}_3 - \mathrm{F}_1) + (\mathrm{F}_3 - \mathrm{F}_1)\mathrm{F}_3 + \mathrm{F}_3(\mathrm{F}_5 - \mathrm{F}_3) + \cdots + \mathrm{F}_{n-1}(\mathrm{F}_{n+1} - \mathrm{F}_{n-1}) \\
&\qquad + (\mathrm{F}_{n+1} - \mathrm{F}_{n-1})\mathrm{F}_{n+1} \\
&\quad = \mathrm{F}_{n+1}^2 - 1.
\end{aligned}$$

Solution 2 by Samuel Davies (George Spencer Academy, Nottingham, UK)

When $n = 1$, $\mathrm{F}_1\mathrm{F}_2 = 1 = \mathrm{F}_2^2$. Assume that

$$\sum_{k=1}^{r} \mathrm{F}_k\mathrm{F}_{k+1} = \mathrm{F}_{r+1}^2,$$

where r is odd. Then

$$\begin{aligned}
\sum_{k=1}^{r+2} \mathrm{F}_k\mathrm{F}_{k+1} &= \mathrm{F}_{r+1}^2 + \mathrm{F}_{r+1}\mathrm{F}_{r+2} + \mathrm{F}_{r+2}\mathrm{F}_{r+3} \\
&= \mathrm{F}_{r+1}(\mathrm{F}_{r+1} + \mathrm{F}_{r+2}) + \mathrm{F}_{r+2}\mathrm{F}_{r+3} \\
&= \mathrm{F}_{r+1}\mathrm{F}_{r+3} + \mathrm{F}_{r+2}\mathrm{F}_{r+3} \\
&= (\mathrm{F}_{r+1} + \mathrm{F}_{r+2})\mathrm{F}_{r+3} \\
&= \mathrm{F}_{r+3}^2.
\end{aligned}$$

Hence, by induction,

$$\sum_{k=1}^{n} \mathrm{F}_k\mathrm{F}_{k+1} = \mathrm{F}_{n+1}^2$$

when n is odd.

When n is even,

$$\mathrm{F}_1\mathrm{F}_2 + \mathrm{F}_2\mathrm{F}_3 = 1 \times 1 + 1 \times 2 = 3 = \mathrm{F}_3^2 - 1.$$

Assume that $\sum_{k=1}^{r} \mathrm{F}_k\mathrm{F}_{k+1} = \mathrm{F}_{r+1}^2 - 1$, when r is even. It follows as in the case of n odd that

$$\sum_{k=1}^{r+2} \mathrm{F}_k\mathrm{F}_{k+1} = \mathrm{F}_{r+3}^2 - 1.$$

Hence, by induction,

$$\sum_{k=1}^{n} \mathrm{F}_k\mathrm{F}_{k+1} = \mathrm{F}_{n+1}^2 - 1$$

when n is even.

Also solved by Henry Ricardo (Tappan, New York, USA), Gian Paulo Almirante (Milan, Italy), and Abbas Rouholamini Gugheri (Sirjan, Iran).

43.12 The point of intersection of a directrix and the major axis of a conic (an ellipse, a hyperbola, or a parabola) is denoted by K, and A, B are two points on the conic such that A, B, K are collinear. The reflection of A in the major axis is denoted by D, and F is the focus corresponding to the directrix. Show that B, D, F are collinear.

Solution by Zhang Yun, who proposed the problem

Consider the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b > 0,$$

the focus $F(ae, 0)$ and the corresponding directrix $x = a/e$, where $e = (a^2 - b^2)/a^2$ is the eccentricity of the ellipse. Then K is the point $(a/e, 0)$. Denote the coordinates of A by (x_1, y_1) and of B by (x_2, y_2) and denote the slope of the straight line ABK by k, so that it has equation

$$y = k\left(x - \frac{a}{e}\right).$$

Then x_1 and x_2 are the roots of the equation

$$\frac{x^2}{a^2} + \frac{k^2(x - a/e)^2}{b^2} = 1,$$

i.e.

$$\left(\frac{1}{a^2} = \frac{k^2}{b^2}\right)x^2 - \frac{2ak^2}{eb^2}x + \left(\frac{k^2a^2}{e^2b^2} - 1\right) = 0,$$

so that

$$x_1 + x_2 = \frac{2ak^2}{eb^2} \Big/ \left(\frac{1}{a^2} + \frac{k^2}{b^2}\right), \qquad x_1x_2 = \left(\frac{k^2a^2}{e^2b^2} - 1\right) \Big/ \left(\frac{1}{a^2} + \frac{k^2}{b^2}\right).$$

For $D(x_1, -y_1)$, $B(x_2, y_2)$, $F(ae, 0)$ to be collinear, we require

$$\begin{vmatrix} x_1 & -y_1 & 1 \\ x_2 & y_2 & 1 \\ ae & 0 & 1 \end{vmatrix} = 0.$$

This determinant is equal to

$$\begin{aligned}
&(x_1 - ae)y_2 + (x_2 - ae)y_1 \\
&\quad = (x_1 - ae)k\left(x_2 - \frac{a}{e}\right) + (x_2 - ae)k\left(x_1 - \frac{a}{e}\right) \\
&\quad = k\left(2x_1x_2 - \left(ae + \frac{a}{e}\right)(x_1 + x_2) + 2a^2\right) \\
&\quad = \frac{k}{1/a^2 + k^2/b^2}\left[\frac{2k^2a^2}{e^2b^2} - 2 - a\left(e + \frac{1}{e}\right)\frac{2ak^2}{eb^2} + 2a^2\left(\frac{1}{a^2} + \frac{k^2}{b^2}\right)\right] \\
&\quad = 0.
\end{aligned}$$

This argument can be adapted for a hyperbola and for a parabola.

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