Crux Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

The Back Files

The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

ISSN 0705 - 0348

CRUX MATHEMATICORUM

Vol. 6, No. 4

April 1980

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
Publié par le Collège Algonquin

The assistance of the publisher and the support of the Canadian Mathematical Olympiad Committee, the Carleton University Mathematics Department, the Ottawa Valley Education Liaison Council, and the University of Ottawa Mathematics Department are gratefully acknowledged.

CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is $10.00. Back issues: $1.00 each. Bound volumes with index: Vols. 1&2 (combined), $10.00; Vols. 3,4,5, $10.00 each. Cheques and money orders, payable to CRUX MATHEMATICORUM (in US funds from outside Canada), should be sent to the managing editor.

All communications about the content of the magazine (articles, problems, solutions, etc.) should be sent to the editor. All changes of address and inquiries about subscriptions and back issues should be sent to the managing editor.

Editor: Léo Sauvé, Architecture Department, Algonquin College, 281 Echo Drive, Ottawa, Ontario, K1S 1N3.

Managing Editor: F.G.B. Maskell, Mathematics Department, Algonquin College, 200 Lees Ave., Ottawa, Ontario, K1S 0C5.

Typist-compositor: Gorettie C.F. Luk.

* * *

CONTENTS

EQUAL CEVIANS

J.R. POUNDER

1. *Introduction.*

Let ABC be a given triangle with side lengths a,b,c satisfying $a \le b \le c$. (This convention is followed throughout.) The plane configuration formed by adding an equilateral triangle externally [internally] on each side as base and joining each new vertex to the opposite vertex of ABC is a familiar one [1, p. 22; 2, p. 72]. In particular, it has these interrelated properties (Figures 1, 2):

(i) The three lines AA' etc. [AA" etc.] are concurrent at a point F' [F"], and meet at equal angles.

(ii) The segments AA' etc. [AA" etc.] have a common length d' [d''], and $AF' + BF' + CF' = d'$ when F' is inside ABC, in which case $AP + BP + CP > d'$ for $P \neq F'$.

(iii) A rotation of triangle B'CB [C"AC] about B' [C"] through 60° yields an equilateral triangle of side d' [d''] whose vertices lie at distances a,b,c from the point A [B].

A question not obviously related to this "Fermat configuration" is the following: *A triangle* ABC *and a coplanar point* O *(not a vertex) determine three unique points* L,M,N *on lines* BC,CA,AB, *respectively, such that* AL,BM,CN *are concurrent at* O. *For what points* O *have the segments* AL,BM,CN *a common length* l, *say, and how is* l *determined by* a, b, *and* c?

Let us term *normal* any solution of this problem such that O does not lie on any side of triangle ABC, so that the segments of length l are equal cevians. Then we shall prove that a necessary condition for a normal solution is that l should be the larger positive root of the equation

$$\sqrt{4l^2 - 3a^2} = \sqrt{4l^2 - 3b^2} + \sqrt{4l^2 - 3c^2}, \tag{1}$$

and that this root is expressible explicitly as

$$l = \tfrac{1}{2}(d' + d''), \tag{2}$$

where d',d'' are defined in (ii) above, so that O can be located by a Euclidean construction.

Before discussing how far this condition for a normal solution is sufficient, or how many points O there can be, we introduce the notion of *abnormal* solution, i.e. one in which the point O does lie on a side of ABC. Since l must then take

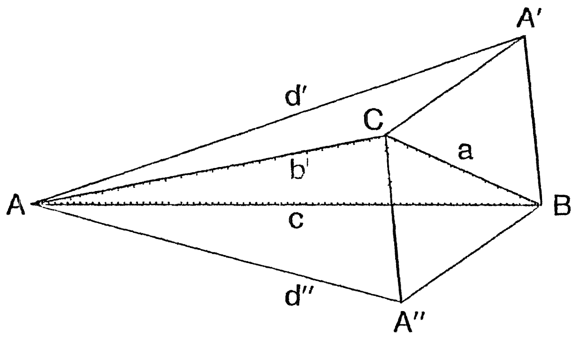


Figure 1

Fermat configuration: d' and d''

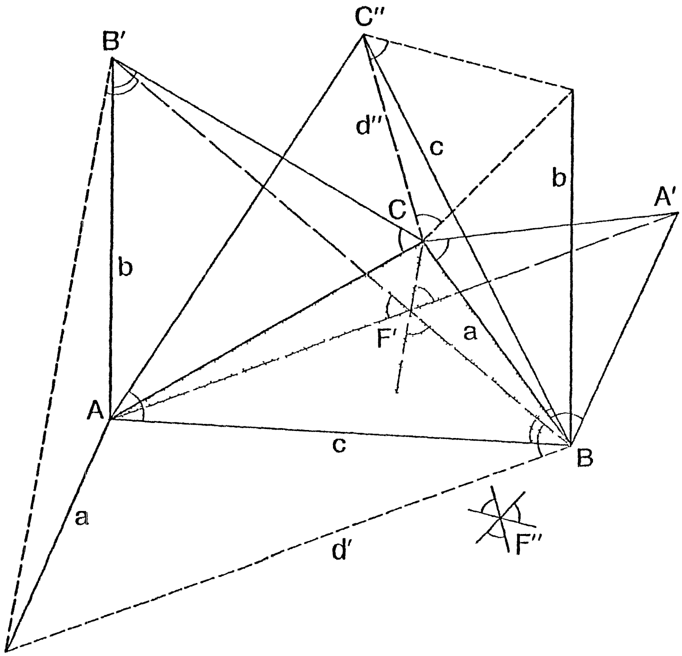


Figure 2

Some properties of the Fermat configuration one of the values a, b, or c, all such points O are easily located; depending on the shape of triangle ABC, the number of abnormal solutions is in general 6, 4, or 2, with 5, 3, and 0 occurring in critical cases (Figure 3). We shall show that, corresponding to the cevian length l given by (2),

(i) there are two points O, distinct except when $a = b = c$;

(ii) if either solution is abnormal, a or b equals l;

(iii) for a scalene triangle, at most one of these solutions is abnormal.

Thus there are normally two points O with $l = \frac{1}{2}(d' + d'')$; but they may coincide, and one or both may lie on a side of ABC. Such abnormal solutions satisfying (2) we may call *special*: they are illustrated in Figure 4.

2. *Derivation.*

The proof of these assertions will follow from the theorems of Menelaus [2, p. 64; 3, p. 66] and Stewart [2, p. 58; 3, p. 6]. AL,BM,CN being cevians of length l concurrent at O, let x,y,z be the signed distances $\overline{OL}$, $\overline{OM}$, $\overline{ON}$. (Note that x is negative if A and O are separated by line BC.) Then, by applying the theorem of Menelaus to triangle LOB with transversal AMC, we find that

$$(l - x)\overline{LC} = y\overline{BC} \tag{3a}$$

and similarly

$$(l - x)\overline{LB} = z\overline{CB}; \tag{3b}$$

subtracting these, we have

$$x + y + z = l. \tag{4}$$

Applying Stewart's theorem to the four points O,B,C,L (or noting that angles OLB, OLC are either equal or supplementary and using the cosine law for triangles), we get

$$OB^2 \cdot \overline{CL} + OC^2 \cdot \overline{LB} + OL^2 \cdot \overline{BC} + \overline{CL} \cdot \overline{LB} \cdot \overline{BC} = 0.$$

On substituting for $\overline{LB}$ and $\overline{CL}$ from (3) and simplifying, this becomes

$$\{4xyz + (y + z)yz\}(x + y) = a^2yz,$$

or, using (4),

$$(l + 3x)(l - x) = a^2,$$

provided $yz \neq 0$. Thus x is given by

$$\pm(l - 3x) = \sqrt{4l^2 - 3a^2} \equiv \alpha. \tag{5}$$

For a normal solution we have $xyz \neq 0$, and thus get two corresponding values for y and z in terms of β and γ, where

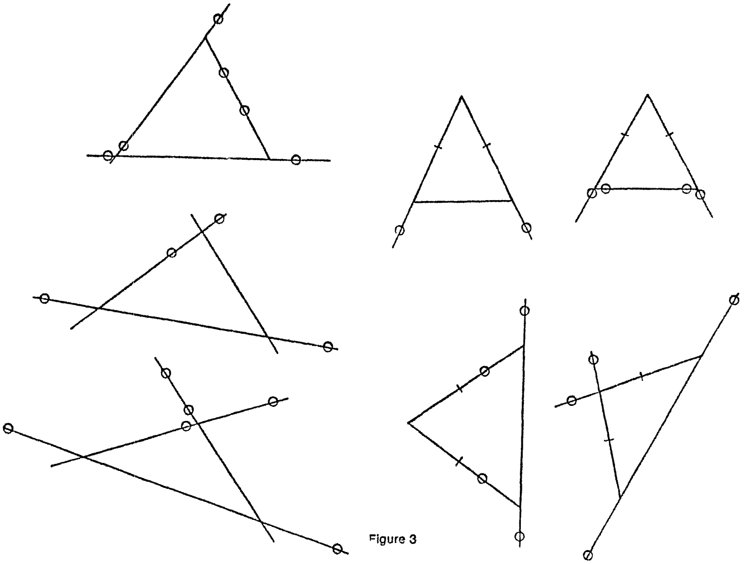


Figure 3

Location of 0: abnormal solutions

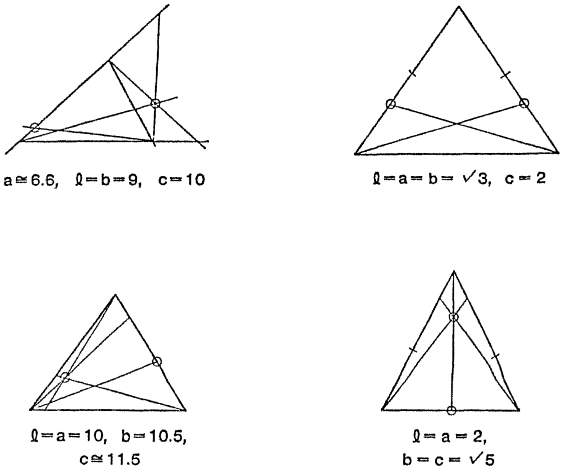


Figure 4

Location of 0: special solutions

$$\beta \equiv \sqrt{4l^2 - 3b^2} \quad \text{and} \quad \gamma \equiv \sqrt{4l^2 - 3c^2}.$$

Noting that $\alpha \geq \beta \geq \gamma$, we conclude from (4) and (5) that $\alpha = \beta + \gamma$, which is equation (1).

To solve for l, we could square both sides, collect terms and square again. Alternatively, from the identity

$$(\alpha - \beta - \gamma)(-\beta + \gamma + \alpha)(-\gamma + \alpha + \beta)(\alpha + \beta + \gamma) = \Sigma\alpha^4 - 2\Sigma\beta^2\gamma^2,$$

where the last three factors on the left side are here all positive, we find on simplifying the right side that (1) is equivalent to

$$l^4 - \tfrac{1}{2}(a^2 + b^2 + c^2)l^2 + 3\Delta^2 = 0, \tag{6}$$

where Δ is the area of triangle ABC. Adding a term in l^2 to "complete the square" gives, since l is positive,

$$l^2 + \sqrt{3}\Delta = l\sqrt{\tfrac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}\Delta}, \tag{7}$$

the larger root of which is

$$l = \tfrac{1}{2}\left\{\sqrt{\tfrac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}\Delta} + \sqrt{\tfrac{1}{2}(a^2 + b^2 + c^2) - 2\sqrt{3}\Delta}\right\};$$

this is equation (2).

(The smaller root of (7) corresponds to a solution of (1) in which each radical is imaginary: the quadratic equation for e.g. α^2, equivalent to (6), has a negative "constant" term.)

The point O is now located by means of (4) and (5); x, y, and z satisfy one or other of the sets

$$l - 3x = \mp\alpha, \qquad l - 3y = \pm\beta, \qquad l - 3z = \pm\gamma, \tag{8}$$

the upper or lower sign being taken throughout. If it turns out that $x = 0$ (i.e. $l = a$) or $y = 0$ (i.e. $l = b$), then the point O gives what we have called a special solution. (It is easily verified from (1) that $l = c$ is not possible.)

3. *Discussion.*

Our results have especially simple consequences for *isosceles* triangles:

(i) If the base is the shorter side, then l measures the altitude (Figure 5a).

(ii) If the base is the longer side, then l is $\sqrt{3}/2$ times the base, O trisects the cevian through the apex, and the locus of O when the altitude is varied while the base is kept fixed is an ellipse of eccentricity $\sqrt{3}/2$ (Figure 5b).

These follow directly from (1), (2), and (8).

It would be natural to seek the points O as the intersections of loci Γ defined

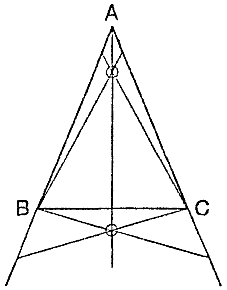


Figure 5a

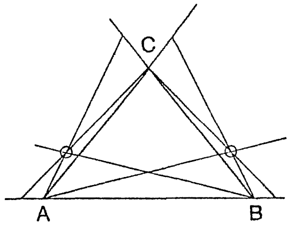


Figure 5b

Equal cevians for isosceles triangles

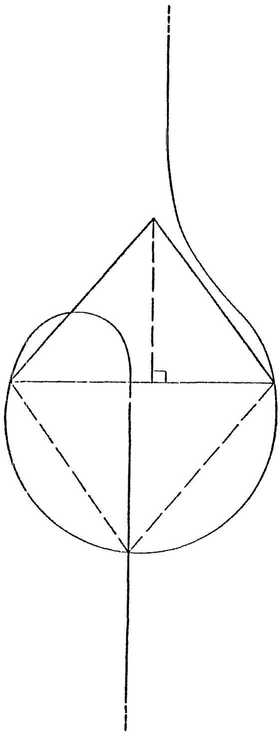


Figure 6a

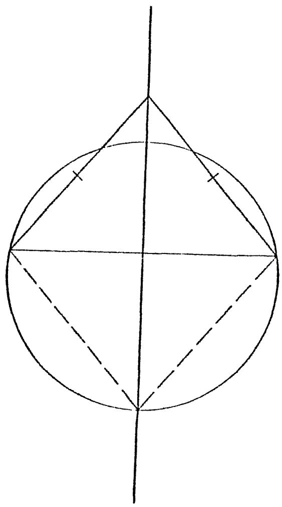


Figure 6b

Γ: Locus of intersection of two equal cevians as follows: Γ is the set of points P for which the cevians through two particular vertices of a given triangle are of equal length. Apart from "abnormal" solutions, for which P lies anywhere in the line joining these vertices, the locus Γ is a cubic curve (strophoid), which reduces to the union of a circle and a line when the sides opposite those vertices are equal (Figure 6).

REFERENCES

1. H.S.M. Coxeter, *Introduction to Geometry*, Second Edition, John Wiley & Sons, New York, 1969.

2. Howard Eves, *A Survey of Geometry*, Revised Edition, Allyn and Bacon, Boston, 1972.

3. H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, New Mathematical Library No. 19, M.A.A., 1967.

Department of Mathematics, University of Alberta, Edmonton, Alberta, T6G 2G1.

* * *

A CURIOSITY OF $PQ(P+Q) = A^N$

HERBERT E. SALZER

The Diophantine equation (in positive integers)

$$pq(p+q) = a^n, \qquad (p,q) = 1, \quad n \geq 3 \tag{1}$$

appears at first glance to be simpler than the Fermat equation

$$A^n + B^n = C^n, \qquad n \geq 3. \tag{2}$$

Actually, (1) and (2) are equivalent. For multiplying (2) by A^nB^n solves (1) and, from (1), since $(p,q)=1$, $p=A^n$, $q=B^n$, $p+q=C^n$, which solves (2).

For $n=3k$, (1) has no solution, even when $(p,q)\neq 1$, because if $(p,q)=D$, $D>1$, for $P=p/D$, $Q=q/D$, we have

$$PQ(P+Q) = (a^k/D)^3, \qquad (P,Q) = 1,$$

which is equivalent to the insoluble Fermat equation for $n=3$.

However, when $n\neq 3k$, and $(p,q)\neq 1$, then (1) is easily seen to have solutions, the most obvious being for $p=q$, such as

$$p = q = 2^k, \qquad a = 2, \qquad n = 3k+1$$

or

$$p = q = 2^k A^{3k+1}, \quad a = 2A^3, \quad n = 3k+1.$$

For the more interesting case $p \neq q$ we have, in marked contrast to the known impossibility of (1) for $n \leq 125000$ [1], the following

THEOREM. The equation $pq(p+q) = a^n$ has solutions $p \neq q$ for every $n \neq 3k$.

The result follows from

$$6^{r+1} \cdot (2 \cdot 6^{r+1}) \cdot (6^{r+1} + 2 \cdot 6^{r+1}) = 6^{3r+4}. \tag{3}$$

For $n = 3k+1$, $k \geq 0$, let $r = k-1$ in (3) to get the solution

$$p = 6^k, \quad q = 2 \cdot 6^k, \quad a = 6;$$

and for $n = 3k+2$, $k \geq 0$, let $r = 3k^2 + 4k$ in (3) to get the solution

$$p = 6^{3k^2+4k+1}, \quad q = 2 \cdot 6^{3k^2+4k+1}, \quad a = 6^{3k+2}. \quad \square$$

For playing with (2) in the form (1), it may be preferable to replace the restriction $(p,q) = 1$ by an equivalent equation which is more explicit, namely

$$pr + qs = 1, \quad r,s \text{ integers}, \quad rs < 0.$$

Then the Fermat equation is seen to be equivalent to the following Diophantine system of two equations in the five unknowns p, q, a, r, s:

$$pq(p+q) = a^n$$

$$pr + qs = 1.$$

REFERENCE

1. S.S. Wagstaff, Jr., "The Irregular Primes to 125000", *Math. Comput.*, 32 (1978) 583-591.

941 Washington Avenue, Brooklyn, New York 11225.

* * *

64 MILLION TRILLION COMBINATIONS

Our title here is in fact the title of a Black Widowers story by Isaac Asimov appearing in the May 5, 1980 issue of *Ellery Queen's Mystery Magazine*. Readers of this journal are sure to enjoy reading it. To quote from Ellery Queen's introduction to the story, "It is a fascinating potpourri of alliterations, Chaldean wise men, a Slovenian ex-waiter, mathematical wizards, Goldbach's conjecture, lemmas, computers, code words, romantic poetry ..."

THE OLYMPIAD CORNER: 14

MURRAY S. KLAMKIN

We will give later on in this column one new Practice Set, No. 12, and solutions to Practice Set 11. But first we wish to reopen Problem 10-3, for which a solution was published here last month [1980: 75]. We have received a different solution from V.N. Murty, Pennsylvania State University. His solution is given below, in somewhat modified form, to which we then add our own comment pointing out a couple of generalizations. Murty wondered if part (b) of the problem can be proved without using calculus. We would appreciate receiving such a proof if one can be found.

10-3. For $a \geq b \geq c \geq 0$, establish the inequality

$$b^m c + c^m a + a^m b \geq bc^m + ca^m + ab^m$$

(a) when m is a positive integer;

(b) find a proof valid for all real $m \geq 1$.

Solution by V.N. Murty.

The inequality to be established is equivalent to

$$F(a,b,c) \equiv (a^m b - ab^m) + (b^m c - bc^m) + (c^m a - ca^m) \geq 0. \tag{1}$$

It is easy to verify that

$$F(a,b,c) = -\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^m & b^m & c^m \end{vmatrix}. \tag{2}$$

The determinant in (2) is an example of an *alternant* (when $m=2$ it is called a *Vandermonde determinant*).

(a) It is clear that (1) holds if $m=1$, so we assume that the integer $m \geq 2$. It follows from the theory of alternants (see [1]) that

$$F(a,b,c) = -\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \cdot H(a,b,c), \tag{3}$$

where $H(a,b,c)$ is the sum of all the homogeneous products of degree $m-2$ in the 3 variables a,b,c; that is,

$$H(a,b,c) = \sum_{r+s+t=m-2} a^r b^s c^t. \tag{4}$$

The Vandermonde determinant in (3) has the value

$$V(a,b,c) \equiv -(a-b)(a-c)(b-c).$$

Since $a \ge b \ge c \ge 0$, we have $V \le 0$ and $H \ge 0$, so $F = -VH \ge 0$ and (1) is established.

(b) We will now show that (1) holds for arbitrary real $m \ge 1$. We will need the following well-known inequality [2]: if $b \ge 0$, $c \ge 0$, and $m \le 0$ or $m \ge 1$, then

$$mb^{m-1}(b-c) \ge b^m - c^m \ge mc^{m-1}(b-c). \tag{5}$$

Consider the function of 3 variables $F(a,b,c)$ in the region of 3-dimensional space defined by $a \ge b \ge c \ge 0$. It follows from (5) that

$$\frac{\partial F}{\partial a} = ma^{m-1}(b-c) - (b^m - c^m) \ge mb^{m-1}(b-c) - (b^m - c^m) \ge 0;$$

hence $F(a,b,c) \ge F(b,b,c) = 0$.

Comment by M.S.K.

Both the proof of part (b) given above and the one we gave last month show that (1) holds also for $m \le 0$, provided only that we assume $a \ge b \ge c > 0$. The inequalities (5) are reversed when $0 < m < 1$. From this it follows that the inequality (1) is also reversed when $0 < m < 1$.

We give two generalizations. The first is suggested by (1):

If $a_1 \ge a_2 \ge \ldots \ge a_n \ge 0$ *and* $m \le 0$ *or* $m \ge 1$ *(with* $a_n > 0$ *when* $m \le 0$*), then*

$$F(a_1,a_2,\ldots,a_n) \equiv (a_1^m a_2 - a_1 a_2^m) + (a_2^m a_3 - a_2 a_3^m) + \ldots + (a_n^m a_1 - a_n a_1^m) \ge 0, \tag{6}$$

the inequality being reversed if $0 < m < 1$.

First observe that equality holds in (6) when $n = 2$, and that setting $a_1 = a_2$ reduces (6) to a similar sum involving the n-1 variables $a_2, a_3, \ldots, a_n$. The proof can now proceed by induction. It almost exactly parallels the one given above for part (b).

Our second generalization is suggested by (2) and is restricted to positive integers m:

If $a_1 \ge a_2 \ge \ldots \ge a_n \ge 0$ *and the integer* $m \ge n-1$, *then*

$$F(a_1,a_2,\ldots,a_n) \equiv (-1)^{n(n-1)/2} \cdot A(a_1,a_2,\ldots,a_n) \ge 0, \tag{7}$$

where $A(a_1,a_2,\ldots,a_n)$ *is the alternant*

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_n^{n-2} \\ a_1^m & a_2^m & \cdots & a_n^m \end{vmatrix}.$$

It follows from the theory of alternants that

$$A = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_n^{n-2} \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} \cdot H(a_1,a_2,\ldots,a_n), \tag{8}$$

where $H(a_1,a_2,\ldots,a_n)$ is the sum of all the homogeneous products of degree $m-n+1$ (defined as in (4)) and is therefore nonnegative. The Vandermonde determinant in (8) has the value

$$V(a_1,a_2,\ldots,a_n) \equiv (-1)^{n(n-1)/2} \prod_{1\le i<j\le n} (a_i - a_j).$$

Hence we have (7):

$$F = (-1)^{n(n-1)/2} VH \ge 0.$$

REFERENCES

1. T. Muir, W.H. Metzler, *A Treatise on the Theory of Determinants*, Dover, New York, 1960, Chapter XI.

2. G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, 1952, p. 39.

*

PRACTICE SET 12

12-1. (a) Show that one root of the equation

$$x^4 + 5x^2 + 5 = 0$$

is $x = \omega - \omega^4$, where ω is a complex fifth root of unity (i.e., $\omega^5 = 1$).

(b) Determine the other three roots as polynomials in ω.

12-2. Prove that the midpoints of the six edges of a tetrahedron are cospherical (i.e. lie on a common sphere) if and only if the four altitudes of the tetrahedron are concurrent.

12-3. If Alice and Bob toss 11 and 9 fair coins, respectively, show that the probability that Alice gets more heads than Bob is

$$\frac{1}{2} + \frac{1}{2^{21}}\binom{20}{10} \approx 0.588.$$

*

SOLUTIONS TO PRACTICE SET 11

11-1. If z_1, z_2, z_3, z_4 are complex numbers of unit modulus, prove that

$$|z_1-z_2|^2|z_3-z_4|^2 + |z_1+z_4|^2|z_3-z_2|^2 = |z_1(z_2-z_3)+z_3(z_2-z_1)+z_4(z_1-z_3)|^2.$$

Solution.

Let

$$a = (z_1 - z_2)(z_3 - z_4), \qquad b = (z_1 + z_4)(z_3 - z_2),$$

and

$$c = z_1(z_2 - z_3) + z_3(z_2 - z_1) + z_4(z_1 - z_3),$$

so that what we have to show is

$$|a|^2 + |b|^2 = |c|^2. \tag{1}$$

It is easily verified that $a + b = -c$; hence

$$(a+b)(\overline{a}+\overline{b}) = c\overline{c},$$

and it follows that (1) holds if and only if

$$a\overline{b} + b\overline{a} = 0. \tag{2}$$

Now $|z_i| = 1$ implies $\overline{z}_i = 1/z_i$, so (2) can be written

$$(z_1-z_2)(z_3-z_4)\left(\frac{1}{z_1}+\frac{1}{z_4}\right)\left(\frac{1}{z_3}-\frac{1}{z_2}\right) + (z_1+z_4)(z_3-z_2)\left(\frac{1}{z_1}-\frac{1}{z_2}\right)\left(\frac{1}{z_3}-\frac{1}{z_4}\right) = 0,$$

and the truth of this last equation becomes obvious when we multiply both sides by $z_1z_2z_3z_4$.

11-2. Sum the series

$$S = \frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \ldots + \frac{a(a+1)\ldots(a+m-1)}{b(b+1)\ldots(b+m-1)}.$$

Solution.

Let

$$u_0 = 1, \quad u_r = \frac{a(a+1)\ldots(a+r-1)}{b(b+1)\ldots(b+r-1)}, \qquad r = 1,2,\ldots,m; \tag{1}$$

then the required sum is

$$S = \sum_{r=1}^{m} u_r.$$

We will use the *method of differences* to find S (see [1] and [2] for additional information). For this we need a sequence v such that, for some constant k,

$$ku_r = \Delta v_{r-1} = v_r - v_{r-1}, \quad r = 1,2,\ldots,m;$$

and then we will have

$$kS = \sum_{r=1}^{m} ku_r = \sum_{r=1}^{m} (v_r - v_{r-1}) = v_m - v_0. \tag{2}$$

First observe that, from (1),

$$(b+r-1)u_r = (a+r-1)u_{r-1}, \qquad r = 1,2,\ldots,m;$$

hence

$$\begin{aligned}(a+r)u_r - (a+r-1)u_{r-1} &= (a+r)u_r - (b+r-1)u_r \\ &= (a-b+1)u_r. \end{aligned} \tag{3}$$

Thus if we set $v_r = (a+r)u_r$ then (3) becomes

$$\Delta v_{r-1} = v_r - v_{r-1} = (a-b+1)u_r.$$

Now, from (2)

$$(a-b+1)S = v_m - v_0 = (a+m)u_m - a.$$

If $b \neq a+1$, we therefore have

$$S = \frac{a}{a-b+1}\left\{\frac{(a+1)(a+2)\ldots(a+m)}{b(b+1)\ldots(b+m-1)} - 1\right\}.$$

But if $b = a+1$, then $u_r = a/(a+r)$ from (1) and all we can say is that

$$S = a\left\{\frac{1}{a+1} + \frac{1}{a+2} + \ldots + \frac{1}{a+m}\right\},$$

which is not much more than simply a restatement of the problem.

REFERENCES

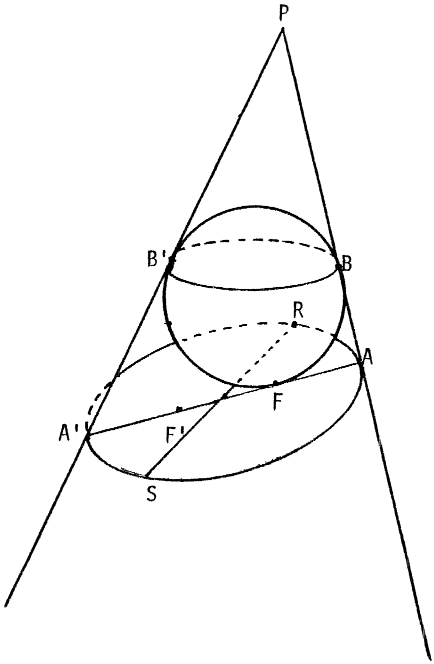
1. H.S. Hall and S.R. Knight, *Higher Algebra*, Macmillan, London, Fourth Edition, 1891 (reprinted dozens of times since), Chapter 29.

2. S. Barnard and J.M. Child, *Higher Algebra*, Macmillan, London, 1936 (reprinted many times), Chapter 8.

11-3. If RS denotes any diameter of a given elliptical cross section of a right circular cone whose vertex is P, prove that PR + PS is constant.

Synthetic solution.

Let AA' be the major axis of the elliptical cross section which forms the base of the conical cap. It is known that in a right circular cone the *Dandelin sphere* (named after Germinal Dandelin (1794-1847)) which is inscribed in the conical cap touches the elliptical base at a focus F (see figure). Let F' be the other focus and B,B' the points where the elements PA,PA' touch the sphere.



Let RS be any diameter of the elliptical base. PR is the sum of tangents from P and R to the sphere, and a similar statement can be made about PS. Since FS = RF', we have

$$\begin{aligned}
PR + PS &= (PB + RF) + (PB' + FS)\\
&= (PB + PB') + (RF + RF')\\
&= (PB + PB') + AA'\\
&= (PB + FA) + (PB' + A'F)\\
&= PA + PA'\\
&= \text{constant.}
\end{aligned}$$

Analytic solution.

We can take $x^2 + y^2 = k^2z^2$ as the Cartesian equation of a right circular cone with vertex P at the origin. We consider only the nappe for which $z \geq 0$.

Let (a,b,c) be any point inside the nappe and consider any chord RS of the nappe which has (a,b,c) as its midpoint. If $R=(x_1,y_1,z_1)$ and $S=(x_2,y_2,z_2)$, we have

$$\begin{aligned} PR+PS &= \sqrt{x_1^2+y_1^2+z_1^2} + \sqrt{x_2^2+y_2^2+z_2^2} \\ &= (z_1+z_2)\sqrt{k^2+1} = 2c\sqrt{k^2+1} \\ &= \text{constant.} \end{aligned}$$

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before September 1, 1980, although solutions received after that date will also be considered until the time when a solution is published.

531. *Proposé par Allan Wm. Johnson Jr., Washington, D.C.*

Résoudre la cryptarithmie multiplicative suivante:

```
  CINQ
   SIX
 -----
 ----
----6
---9
------
TRENTE .
```

532. *Proposed by Arun Sanyal, Indian Institute of Technology, Kharagpur, India.*

Let triangles ABP, CDQ be directly similar to a triangle α; triangles ACR, BDS directly similar to a triangle β; and triangle PQT directly similar to β. Prove that RST is directly similar to α.

533. *Proposed by James Gary Propp, Harvard College, Cambridge, Massachusetts.*

Consider the following products over the complex field:

$$\prod_{k=1}^{n}(x+2k-1) \quad \text{and} \quad \prod_{k=1}^{n}(x-2k+1).$$

(a) For $n=1,2,3,4,5$, find all complex x such that each product is real and rational.

(b)* Are there, for any $n>5$, any real irrational x such that each product is rational?

534. *Proposed by Leroy F. Meyers, The Ohio State University.*

Some time ago I noticed that the exponent of 2 in the prime factorization of $n!$ seems to be approximately twice the exponent of 3 in the same factorization, at least for small values of n, say up to 100. Is this true in general? What about the exponents of other primes? More precisely, if n is any positive integer and p is prime, let $e_p(n)$ be the exponent of p in the prime factorization of $n!$. Is it true that $\lim_{n\uparrow} e_2(n)/e_3(n)=2$? What about $\lim_{n\uparrow} e_p(n)/e_q(n)$ for primes p and q?

535. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Given a triangle ABC with sides a,b,c, let T_a,T_b,T_c denote the angle bisectors extended to the circumcircle of the triangle. Prove that

$$T_aT_bT_c \geq \frac{8}{9}\sqrt{3}\,abc,$$

with equality attained in the equilateral triangle.

536. *Proposed by B. Leeds, Forest Hills, New York.*

Through each of the midpoints of the sides of a triangle ABC, lines are drawn making an acute angle θ with the sides. These lines intersect to form a triangle A'B'C'. Prove that A'B'C' is similar to ABC and find the ratio of similitude.

537. *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*

Find all pairs of integers (k,p) such that

$$(k-p)(2^{p+1}-2) = (p+1)(2^p-2).$$

538. *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

Find

$$\lim_{n\to\infty} \sqrt{(5+1\sqrt{(6+2\sqrt{(7+3\sqrt{\ldots\sqrt{((n+3)+(n-1)\sqrt{(n+4+ne^n)})\ldots)))}}}}}.$$

539. *Proposed by Charles W. Trigg, San Diego, California.*

From among the three-digit primes less than 500, form four four-term arithmetic progressions in which the first and last terms contain the same decimal digits.

540. *Proposed by Leon Bankoff, Los Angeles, California.*

Professor Euclide Paracelso Bombasto Umbugio has once again retired to his *tour d'ivoire* where he is now delving into the supersophisticated intricacies of the works of Grassmann, as elucidated by Forder's *Calculus of Extension.* His goal is to prove Neuberg's Theorem:

If D, E, F *are the centers of squares described externally on the sides of a triangle* ABC, *then the midpoints of these sides are the centers of squares described internally on the sides of triangle* DEF.

Help the dedicated professor emerge from his self-imposed confinement and enjoy the thrill of hyperventilation by showing how to solve his problem using only high-school, synthetic, Euclidean, "plain" geometry.

* * *

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

387. [1978: 251; 1979: 201; 1980: 46] *Proposed by Harry D. Ruderman, Hunter College Campus School, New York.*

N persons lock arms to dance in a circle the traditional Israeli Hora. After a break they lock arms to dance a second round. Let $P(N)$ be the probability that for the second round no dancer locks arms with a dancer previously locked to in the first round. Find $\lim_{N\to\infty} P(N)$.

II. *Comment by Charles M. Grinstead, Dartmouth College, Hanover, New Hampshire.*

It is easy to show that $\lim_{N\to\infty} P(N)$ is the same as $\lim_{N\to\infty} Q(N)$, where $Q(N)$ is the probability for the corresponding problem where the dancers are in a straight line rather than in a circle.

This limit exists, and does in fact equal e^{-2}. This was first shown by Kaplansky [1]. In fact, he gives the asymptotic probability that exactly r pairs will be holding hands both before and after they split up for the second round, where r is any fixed integer. This probability is

$$\frac{2^r e^{-2}}{r!}.$$

This result is also given in a recent paper by Robbins [2].

REFERENCES

1. Irving Kaplansky, "The Asymptotic Distribution of Runs of Consecutive Elements," *Annals of Mathematical Statistics*, 16 (1945) 200.

2. David P. Robbins, "The Probability That Neighbors Remain Neighbors After Random Rearrangements," *American Mathematical Monthly*, 87 (February 1980) 122.

* * *

447. [1979: 132] *Proposed by Viktors Linis, University of Ottawa.*

The number $\sum_{k=1}^{n} \frac{2^k}{k}$ is represented as an irreducible fraction $\frac{p_n}{q_n}$.

(a) Show that p_n is even.

(b) Show that if $n > 3$ then p_n is divisible by 8.

(c) Show that for every natural number k there exists an n such that all the numbers $p_n, p_{n+1}, \ldots$ are divisible by 2^k.

Editor's comment.

Our proposer found this problem in the Russian journal *Kvant* (No. 12, 1977, pp. 36-37, Problem M434), where it appeared with a solution by D. Fadéev. The original proposer, if not Fadéev, was not identified and the earlier issue where the problem was first proposed was not available. The Russian solution given below was edited from a translation supplied by V. Linis.

Solution by D. Fadéev.

To prove (a) and (b) we will use the following obvious fact: if an irreducible fraction p/q is the sum of several irreducible fractions whose numerators are all divisible by 2^m, then p is also divisible by 2^m. For (a), we note that the irreducible equivalent of each fraction $2^k/k$, $k = 1,2,3,\ldots$, has an even numerator since $2^{k-1} \geq k$ for all $k \geq 1$; hence $2|p_n$ for all $n \geq 1$. Similarly, for (b) we note that

$$\frac{2}{1} + \frac{2^2}{2} + \frac{2^3}{3} + \frac{2^4}{4} = \frac{8 \cdot 4}{3},$$

$$\frac{2}{1} + \frac{2^2}{2} + \frac{2^3}{3} + \frac{2^4}{4} + \frac{2^5}{5} = \frac{8 \cdot 32}{15},$$

and $2^{k-3} > k$ for all $k \geq 6$; hence $2^3|p_n$ for all $n \geq 4$.

To prove (c), consider the function defined by

$$L_n(x) \equiv \frac{x}{1} + \frac{x^2}{2} + \ldots + \frac{x^n}{n}, \quad x \neq 1, \quad n \geq 4. \tag{1}$$

We show that the polynomial

$$f(x) \equiv L_n(2x - x^2) - 2L_n(x)$$

has the form

$$f(x) = x^{n+1}\left\{\frac{a_0}{n+1} + \frac{a_1 x}{n+2} + \ldots + \frac{a_{n-1}x^{n-1}}{2n}\right\}, \tag{2}$$

where the a_i are integers. Observe that

$$L_n'(x) = 1 + x + \ldots + x^{n-1} = \frac{1 - x^n}{1 - x};$$

hence

$$f'(x) = \frac{1 - (2x - x^2)^n}{1 - (2x - x^2)} \cdot (2 - 2x) - 2 \cdot \frac{1 - x^n}{1 - x}$$

$$= x^n \cdot \frac{2\{1 - (2 - x)^n\}}{1 - x}$$

$$= x^n(a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}), \tag{3}$$

where the a_i are integers, since $(1 - x)|\{1 - (2 - x)^n\}$. Now (2) follows by integration from (3) and the fact that $f(0) = 0$. Since

$$f(2) = L_n(0) - 2L_n(2) = -2L_n(2),$$

we get, on putting $x = 2$ in (2),

$$L_n(2) = -2^n\left\{\frac{a_0}{n+1} + \frac{2a_1}{n+2} + \ldots + \frac{2^{n-1}a_{n-1}}{2n}\right\}. \tag{4}$$

From the inequality $n < 2^{(n+1)/2}$ (which holds for all $n \geq 1$) we get

$$n + k \leq 2n < 2^{(n+3)/2}, \qquad k = 1, 2, \ldots, n;$$

hence all denominators on the right in (4) are less than $2^{(n+3)/2}$ and none is divisible by $2^{[(n+3)/2]+1}$, where the square brackets denote the greatest integer function. Since $n \geq 4$, it now follows from (1) and (4) that in $L_n(2) = p_n/q_n$, the numerator p_n is divisible by $2^{n-1-[(n+3)/2]}$. Observe here that

$$n - 1 - \left[\frac{n+3}{2}\right] \geq n - 1 - \frac{n+3}{2} = \frac{n-5}{2},$$

so it increases without bound with n. Thus, given any natural number k, it suffices to take $n \geq 2k+5$ to be sure that p_n, p_{n+1}, ... are all divisible by 2^k.

Partial solutions (parts (a) and (b) only) were received from LEROY F. MEYERS, The Ohio State University; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; and KENNETH M. WILKE, Topeka, Kansas.

* * *

448. [1979: 133] *Proposed by G. Ramanaiah, Madras Institute of Technology, Madras, India.*

A function f is said to be an *inverse point function* if $f(k) = f(1/k)$ for all $k > 0$. Show that the functions g and h defined below are inverse point functions:

$$g(k) = \frac{1}{k}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(1-\operatorname{sech}\lambda_n k)}{\lambda_n^3},$$

$$h(k) = \frac{1}{k^2}\sum_{n=1}^{\infty}\frac{\lambda_n k - \tanh\lambda_n k}{\lambda_n^5},$$

where $\lambda_n = (2n-1)\pi/2$.

Solution by the proposer.

The boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -1, \qquad |x| < a,\ |y| < b,$$

with $u = 0$ at $|x| = a$ and at $|y| = b$, has the unique solution

$$u(x,y) = 2a^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\lambda_n^3}\left\{1 - \frac{\cosh\lambda_n(y/a)}{\cosh\lambda_n(b/a)}\right\}\cosh\lambda_n(x/a) \tag{1}$$

or, equivalently,

$$u(x,y) = 2b^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\lambda_n^3}\left\{1 - \frac{\cosh\lambda_n(x/b)}{\cosh\lambda_n(a/b)}\right\}\cosh\lambda_n(y/b). \tag{2}$$

Equating the values of $u(0,0)$ obtained from (1) and (2) and setting $k = b/a$, we obtain

$$g(k) = g(1/k).$$

Integrating (1) and (2) over the rectangle $|x| = a$, $|y| = b$ and equating the resulting expressions yields

$$h(k) = h(1/k).$$

* * *

449. [1979: 133] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Let p be a prime $\equiv 3 \pmod 8$ and let each of the numbers α, β, γ have one of the values ± 1. Prove that the number $N_p(\alpha,\beta,\gamma)$ of consecutive triples x, $x+1$, $x+2$ $(x=1,2,\ldots,p-3)$ with

$$\left(\frac{x}{p}\right)=\alpha,\quad \left(\frac{x+1}{p}\right)=\beta,\quad \left(\frac{x+2}{p}\right)=\gamma,\quad \left(\frac{\ }{p}\right)=\text{ the Legendre symbol}$$

is the same no matter what values are assigned to α, β, γ.

For example, when $p=19$ we have the table

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\left(\frac{x}{p}\right)$	+1	-1	-1	+1	+1	+1	+1	-1	+1	-1	+1	-1	-1	-1	-1	+1	+1	-1

from which it is easily seen that $N_p(\alpha,\beta,\gamma)=2$ for all eight values of the triple (α,β,γ).

Solution by the proposer.

First we note that, as $p \equiv 3 \pmod 8$, we have

$$\left(\frac{-1}{p}\right)=\left(\frac{2}{p}\right)=-1,\qquad \left(\frac{-2}{p}\right)=+1.$$

Since, for $x \not\equiv 0 \pmod p$,

$$\frac{1}{2}\left\{1+\alpha\left(\frac{x}{p}\right)\right\}=\begin{cases}1, & \text{if}\left(\frac{x}{p}\right)=\alpha\\ 0, & \text{if}\left(\frac{x}{p}\right)=-\alpha\end{cases},$$

we have

$$\begin{aligned}N_p(\alpha,\beta,\gamma) &= \frac{1}{8}\sum_{x=1}^{p-3}\left\{1+\alpha\left(\frac{x}{p}\right)\right\}\left\{1+\beta\left(\frac{x+1}{p}\right)\right\}\left\{1+\gamma\left(\frac{x+2}{p}\right)\right\}\\ &= \frac{1}{8}\{p-3+\alpha A+\beta B+\gamma C+\alpha\beta D+\beta\gamma E+\gamma\alpha F+\alpha\beta\gamma G\},\end{aligned}$$

where

$$A=\sum_{x=1}^{p-3}\left(\frac{x}{p}\right),\qquad B=\sum_{x=1}^{p-3}\left(\frac{x+1}{p}\right),\qquad C=\sum_{x=1}^{p-3}\left(\frac{x+2}{p}\right),$$

$$D=\sum_{x=1}^{p-3}\left(\frac{x(x+1)}{p}\right),\qquad E=\sum_{x=1}^{p-3}\left(\frac{(x+1)(x+2)}{p}\right),\qquad F=\sum_{x=1}^{p-3}\left(\frac{x(x+2)}{p}\right),$$

$$G = \sum_{x=1}^{p-3} \left(\frac{x(x+1)(x+2)}{p}\right).$$

Since $\sum_{x=0}^{p-1} \left(\frac{x}{p}\right) = 0$, we have

$$A = -\left(\frac{-2}{p}\right) - \left(\frac{-1}{p}\right) = -1 + 1 = 0,$$

$$B = -1 - \left(\frac{-1}{p}\right) = -1 + 1 = 0,$$

$$C = -1 - \left(\frac{2}{p}\right) = -1 + 1 = 0.$$

Furthermore, since $\sum_{x=0}^{p-1} \left(\frac{x(x+k)}{p}\right) = -1$ when $p \nmid k$ (see, e.g., Vinogradov [1]), we have

$$D = -1 - \left(\frac{2}{p}\right) = -1 + 1 = 0,$$

$$E = -1 - \left(\frac{2}{p}\right) = -1 + 1 = 0,$$

$$F = -1 - \left(\frac{-1}{p}\right) = -1 + 1 = 0.$$

Finally

$$G = \sum_{x=1}^{p-3} \left(\frac{x(x+1)(x+2)}{p}\right) = \sum_{x=0}^{p-1} \left(\frac{x(x+1)(x+2)}{p}\right)$$

$$= \sum_{x=0}^{p-1} \left(\frac{(x-1)x(x+1)}{p}\right) = \sum_{x=0}^{p-1} \left(\frac{x(x^2-1)}{p}\right)$$

$$= \sum_{x=0}^{p-1} \left(\frac{(-x)((-x^2)-1)}{p}\right) = \left(\frac{-1}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x(x^2-1)}{p}\right)$$

$$= -G,$$

so $G = 0$. Hence

$$N_p(\alpha,\beta,\gamma) = \frac{1}{8}(p-3),$$

which is independent of α, β, and γ.

Also solved by KENNETH M. WILKE, Topeka, Kansas.

REFERENCE

1. I.M. Vinogradov, *Elements of Number Theory*, Dover, New York, 1954, p. 97.

* * *

450. [1979: 133] *Proposed by A. Liu, University of Alberta.*

Triangle ABC has a fixed base BC and a fixed inradius. Describe the locus of A as the incircle rolls along BC. When is AB of minimal length (geometric characterization desired)?

Solution by Howard Eves, University of Maine.

Let r be the fixed radius of the rolling circle and $2a$ the length of the fixed base segment BC. We will obtain a more general result by allowing the circle to roll over the entire *line* BC. Introduce a rectangular Cartesian coordinate system with x-axis along BC and origin at the midpoint of BC, as shown in the figure. Let the circle, in some arbitrary position, touch BC at D ($\neq$ B or C) and denote the directed distances $\overline{\mathrm{BD}}$ by $m \neq 0$ and $\overline{\mathrm{DC}}$ by $n \neq 0$. We assume for now that $|m| \neq r$. Since tan (B/2) $= r/m$, we have

$$\tan B = \frac{2(r/m)}{1-(r/m)^2} = \frac{2rm}{m^2-r^2}.$$

If A is the point (x,y), it follows that the line BA has the equation

$$(m^2-r^2)y = 2rm(x+a), \tag{1}$$

and (1) remains valid if $|m| = r$. Similarly, line CA has the equation

$$(n^2-r^2)y = -2rn(x-a). \tag{2}$$

After solving (1) and (2) as quadratic equations in m and n, respectively (note that this requires $y \neq 0$), we find

$$my = r(x+a) \pm r\sqrt{(x+a)^2+y^2},$$

$$ny = -r(x-a) \pm r\sqrt{(x-a)^2+y^2}.$$

It follows that

$$2ay = (m+n)y = 2ra \pm r\sqrt{(x+a)^2+y^2} \pm r\sqrt{(x-a)^2+y^2}$$

or

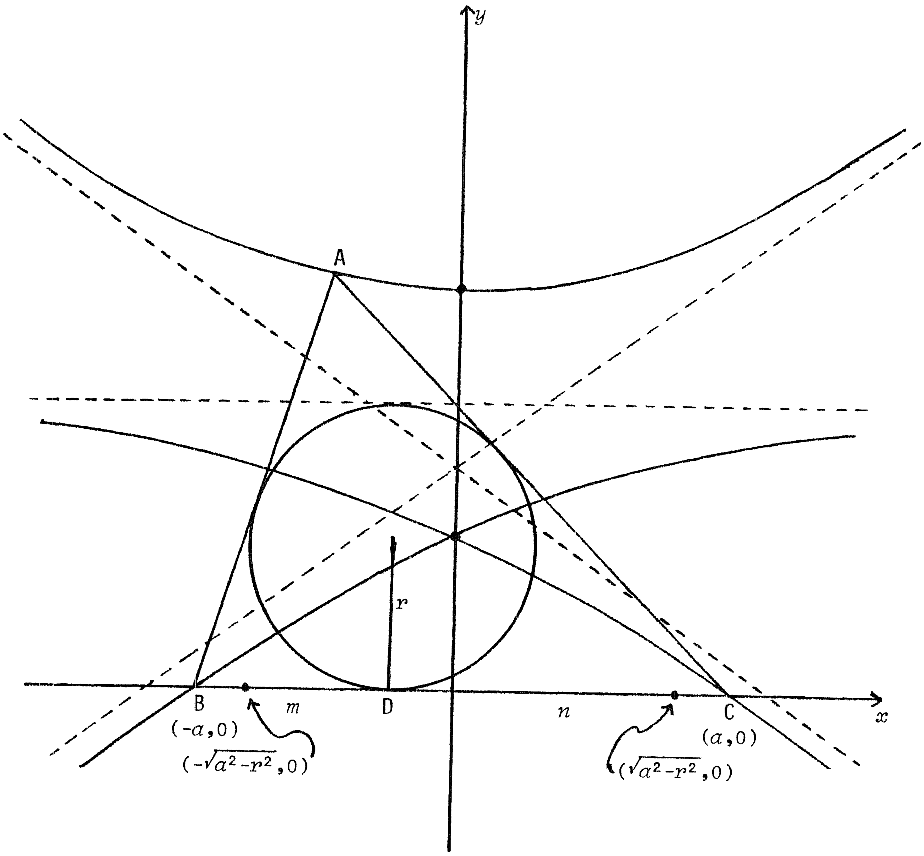
$$2a(y-r) = r\{\pm\sqrt{(x+a)^2+y^2} \pm \sqrt{(x-a)^2+y^2}\}.$$

This equation can be rationalized in the usual way by squaring, rearranging, and squaring again. The result is

$$\{2a^2(y-r)^2 - r^2(x^2+y^2+a^2)\}^2 = r^4\{(x^2-a^2)^2 + 2y^2(x^2+a^2) + y^4\},$$

a seemingly quartic equation in x and y which, since $y \neq 0$, reduces upon expansion to the cubic equation

$$(a^2-r^2)y^3 - 2r(2a^2-r^2)y^2 + r^2(5a^2-r^2-x^2)y - 2r^3(a^2-x^2) = 0. \tag{3}$$



It is easily verified that the graph of (3) is symmetric in the y-axis and passes through the four points

$$(-a,0), \quad (a,0), \quad (0,r), \quad (0, \frac{2a^2r}{a^2-r^2}) \ .$$

(Strictly speaking, since $y \neq 0$ the points $(-a,0)$ and $(a,0)$ are deleted points of the curve.) The graph has the three branches shown in the figure. The horizontal asymptote is the line $y = 2r$; the oblique asymptotes have slopes of $\pm r/\sqrt{a^2 - r^2}$. As the circle rolls along the x-axis from $-\infty$ to $-\sqrt{a^2 - r^2}$, vertex A generates the lower descending branch; as it rolls from $-\sqrt{a^2 - r^2}$ to $\sqrt{a^2 - r^2}$, A generates the upper branch; and as it rolls from $\sqrt{a^2 - r^2}$ to $+\infty$, A generates the lower ascending branch. It is only the upper branch that represents the required locus of the proposed problem; for the other two branches, the circle is an escribed, rather than the inscribed, circle of triangle ABC.

The sought minimum length of AB occurs when AB is the normal from B to the upper branch of the cubic curve.

Also solved by VIKTORS LINIS, University of Ottawa; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

Editor's comment.

It seems to the editor that the last sentence in our featured solution is merely an analytic truism and not the sort of "geometric characterization" the proposer was seeking. The other solvers did not do any better in this respect. The analytic nature of the locus is such that it is unlikely that a proper geometric characterization of the *triangle* ABC with minimum AB can be found. Still, if anyone cares to try again ...

* * *

451. [1979: 166] *Proposed by Herman Nyon, Paramaribo, Surinam.*

Solve the doubly-true alphametic

$$\text{TWENTY} + \text{TWENTY} + \text{THIRTY} = \text{SEVENTY},$$

in which THIRTY is divisible by 30.

Solution by Michael Abramson, 14, student, Benjamin N. Cardozo H.S., Bayside, N.Y.

It will be easier to follow the solution if the alphametic is written in column form, as shown on the right.

$$\begin{array}{r} \text{TWENTY} \\ \text{TWENTY} \\ \underline{\text{THIRTY}} \\ \text{SEVENTY} \end{array}$$

Since $30|\text{THIRTY}$, we must have $Y = 0$, and then $T = 5$, $S = 1$, and $E = 6$ or 7. Now

$$N + R = 9 \quad \text{and} \quad E + I = 9,$$

so $(E,I) = (6,3)$ or $(7,2)$, and $\{N,R\} = \{7,2\}$ or $\{6,3\}$, leaving $\{W,H,V\} = \{4,8,9\}$. The equation

$$2W + H + 1 \equiv V \pmod{10}$$

is now seen to have the unique solution $(W,H,V) = (4,9,8)$, from which it follows that $(E,I) = (6,3)$. If $(N,R) = (2,7)$, the resulting THIRTY is not divisible by 30; hence $(N,R) = (7,2)$ and we have the unique solution

$$\begin{array}{r} 546750 \\ 546750 \\ \underline{593250} \\ 1686750 \end{array} \quad (= 30 \cdot 19775).$$

Also solved by GÖRAN ÅBERG, Själevad, Sweden; LOUIS H. CAIROLI, student, Kansas State University, Manhattan, Kansas; CLAYTON W. DODGE, University of Maine at Orono; MILTON P. EISNER, J. Sargeant Reynolds Community College, Richmond, Virginia; HOWARD EVES, University of Maine; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; EDGAR LACHANCE, Ottawa, Ontario; J.A. McCALLUM, Medicine Hat, Alberta; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, New Jersey; HYMAN ROSEN, Cooper Union, Brooklyn, N.Y.; CHARLES W. TRIGG, San Diego, California; FERRELL WHEELER, student, Forest Park H.S., Beaumont, Texas; KENNETH M. WILKE, Topeka, Kansas; JOHN A. WINTERINK, Albuquerque Technical Vocational Institute, Albuquerque, New Mexico; and the proposer.

* * *

452. [1979: 166] *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

Precocious Percy wrote a polynomial on the blackboard and told his mathematics professor: "This polynomial has my age as one of its zeros." The professor looked at the blackboard and thought to himself: "This polynomial is monic, quintic, has integral coefficients, and is truly an odd function. If I try 10, I get -29670."

Find Percy's age and the polynomial.

I. *Solution by Richard Burns, East Longmeadow H.S., East Longmeadow, Massachusetts.*

Let λ be Percy's age and P the polynomial. From the nature of P, we have

$$P(x) = x(x^4 + ax^2 + b) \equiv xQ(x),$$

where a and b are integers. From the given data, we get $Q(\lambda) = 0$ and $Q(10) = -2967$; hence

$$\begin{aligned} Q(\lambda) - Q(10) &= (\lambda^4 + a\lambda^2 + b) - (10^4 + a \cdot 10^2 + b) \\ &= (\lambda - 10)(\lambda + 10)(\lambda^2 + 100 + a) = 2967 = 3 \cdot 23 \cdot 43. \qquad (1) \end{aligned}$$

Thus $\lambda - 10 = 3$ or 23, but only the first possibility gives Percy a precocious age, $\lambda = 13$, and then equating the third factors in (1) gives $a = -226$. Finally, from $Q(\lambda) = 0$ or $Q(10) = -2967$ we get $b = 9633$.

Percy is 13 years old and the polynomial is

$$P(x) = x^5 - 226x^3 + 9633x.$$

II. *Comment by Hyman Rosen, Cooper Union, Brooklyn, N.Y.*

Percy is in the prime of life!

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; MILTON P. EISNER, J. Sargeant Reynolds Community College, Richmond, Virginia; HOWARD EVES, University of Maine; M.S. KLAMKIN, University of Alberta; M. PARMENTER, Memorial University of Newfoundland; HYMAN ROSEN, Cooper Union, Brooklyn, N.Y.; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; and the proposer. One incorrect solution was received.

Editor's comment.

The proposer mentioned that variants of this problem can be found in Graham [1] and Trigg [2]. Our incorrect solver gave 10.6711 years as Percy's age. Apparently his Percy celebrates his birthday every 52.56 minutes.

REFERENCES

1. L.A. Graham, *The Surprise Attack in Mathematical Problems*, Dover, New York, pp. 69-70.

2. Charles W. Trigg, *Mathematical Quickies*, McGraw-Hill, New York, 1967, pp. 64, 188 (pp. 54, 170 in the Russian edition published in Moscow in 1975).

* * *

453. [1979: 166] *Proposed by Viktors Linis, University of Ottawa.*

In a convex polyhedron each vertex is of degree 3 (i.e. is incident with exactly 3 edges) and each face is a polygon which can be inscribed in a circle. Prove that the polyhedron can be inscribed in a sphere.

Solution by Leroy F. Meyers, The Ohio State University.

Four noncoplanar points determine a unique sphere containing them. With each vertex X of the given polyhedron, associate the sphere S_X containing that vertex and its three adjacent vertices. (Since the four vertices do not all belong to the same face, they are noncoplanar.) Since each face of the polyhedron is inscribable in a circle, each face has a single boundary curve, and so the vertices and edges of the polyhedron form a connected network. Suppose that the given polyhedron cannot be inscribed in a sphere. Then the spheres associated with some two vertices of the polyhedron must be distinct. These two vertices can be joined by a path of edges and vertices. Beginning at one of the given points, trace out the path until for the first time a vertex B is reached whose associated sphere S_B is distinct from the sphere S_A associated with the preceding vertex A. Then AB is a common side of two faces $\ldots P_1ABQ_1\ldots$ and $\ldots P_2ABQ_2\ldots$ (where P_i may coincide with Q_i). Now the circumcircles of these faces are determined by any three of their points and lie on all spheres containing those three points. Thus P_1 is on S_B since it is on the circle ABQ_1 on that sphere; similarly, Q_2 is on the circle P_2AB on S_A. Hence the four noncoplanar points P_1, A, B, Q_2 are common to S_A and S_B, and so $S_A = S_B$, which contradicts the assumption about A and B. Hence all associated spheres must coincide, so that the given polyhedron can be inscribed in a sphere.

Also solved by JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; HOWARD EVES, University of Maine; M.S. KLAMKIN, University of Alberta; and SAHIB RAM MANDAN, Bombay, India.

Editor's comment.

Our proposer found this problem in the Russian Journal *Kvant*. It appeared as Problem M456 in No. 8, 1977, page 42, proposed by V. Proizvolov, and a solution by I. Bernštein was published in No. 6, 1978, page 44.

Some of the other solutions (including the Russian one) went roughly as follows: They first proved directly (in one way or another) that if A and B are adjacent vertices then $S_A = S_B$, from which they concluded immediately, or nearly so, that $S_A = S_X$ for all vertices X and that the polyhedron can be inscribed in a sphere.

Well, of course, one can "see" that this is so. But, to actually *see* it, the nature of the problem would seem to require a proof by contradiction, such as the one given in our featured solution. As we said on an earlier occasion [1980: 26], a proof is a proof is a proof.

* * *

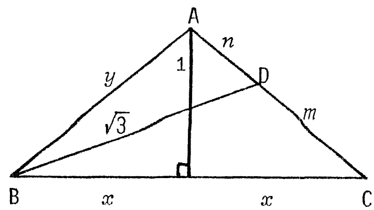
454. [1979: 166] *Proposed by Ram Rekha Tiwari, The Belsund Sugar Co., P.O. Riga, Bihar, India.*

(a) Is there a Euclidean construction for a triangle ABC given the lengths of its internal angle bisectors t_a, t_b, t_c?

(b) Find formulas for the sides a, b, c in terms of t_a, t_b, t_c.

Solution to part (a) *by Howard Eves, University of Maine.*

Consider the special isosceles triangle ABC with $t_a = 1$ and $t_b = t_c = \sqrt{3}$. Let BC $= 2x$, AB $=$ AC $= y$, and let the angle bisector BD divide CA into parts CD $= m$ and DA $= n$, as shown in the figure. Then



$$m + n = \sqrt{x^2+1} = y, \qquad \frac{m}{n} = \frac{2x}{y},$$

and eliminating n yields

$$m = \frac{2xy}{2x+y}. \qquad (1)$$

Applying the law of cosines to triangle BCD, we find

$$3 = 4x^2 + m^2 - 4mx(x/y),$$

and replacing m by its value from (1) gives an equation equivalent to

$$y(12x - 8x^3) = (8x^2 - 3)y^2 - 12x^2.$$

Squaring both sides and replacing y^2 by its value $x^2 + 1$, we get, after reduction,

$$16x^6 + 49x^4 - 102x^2 + 9 = 0.$$

Finally, setting $x^2 = z$, we have

$$16z^3 + 49z^2 - 102z + 9 = 0. \tag{2}$$

Now if there exists a Euclidean construction of a triangle given t_a, t_b, t_c, then that construction applied to our special isosceles triangle would yield, starting from a given unit segment, a segment of length x, and thence a segment of length z satisfying (2). One can easily (though somewhat tediously) show that (2) has no rational root. But it is known that it is impossible, starting with a unit segment, to construct with Euclidean tools a root of a cubic equation having no rational root. It follows that z, and hence our special isosceles triangle, cannot be constructed, and that therefore there is no general Euclidean construction of a triangle given t_a, t_b, t_c. □

It is worth noting that *some* nonequilateral triangles can be constructed with Euclidean tools given the three internal bisectors (e.g., an isosceles triangle with $t_a = 1$ and $t_b = t_c = 2$), and that a Euclidean construction exists for *any* triangle given its three medians or its three altitudes.

Comments were received from LEON BANKOFF, Los Angeles, California; and SAHIB RAM MANDAN, Bombay, India.

Editor's comment.

This problem has a long history. The few facts given below were taken from a note by Nathan Altshiller Court [1], a copy of which was submitted by Bankoff, and from an article by O. Bottema [2], brief extracts of which were sent by Mandan. See those references for additional or more precise information.

The problem of constructing a triangle given the three internal bisectors was first proposed by H. Brocard in 1875 in the *Nouvelle Correspondance Mathématique*. The challenge found no takers during the lifetime of the journal (1875-1880). In 1889, the Dutch mathematician F. J. Van Den Berg published a lengthy article in *Nieuw Archief Voor Wiskunde* wherein he showed that the problem requires solving an equation of degree 16. In 1896, P. Barbarin showed in *Mathesis* that the requisite equation is of degree 14 when the three angle bisectors (internal or external) are concurrent, of degree 16 when the three bisectors form a triangle, and that these equations are, in general, irreducible. The first simple proof of the impossibility of the construction in general was given by Korselt in 1897. He showed that in the special case when two bisectors are equal the equation reduces to a generally irreducible cubic, from which he concluded, as in our featured solution, the impossibility of the general case. The problem has the distinct honor of having been the object of two doctoral dissertations, one in Bern (1889) and one in Chicago (1911).

REFERENCES

1. Nathan Altshiller Court, "The Problem of the Three Bisectors," *Scripta Mathematica*, XIX (June-September 1953) 218-219.

2. O. Bottema, "A Theorem of F.J. Van Den Berg (1833-92)," *Nieuw Archief Voor Wiskunde* (3), XXVI (1978) 161-171.

*　　　　*　　　　*

455. [1979: 167] *Proposé par Hippolyte Charles, Waterloo, Québec.*

Calculer l'intégrale

$$I = \int_0^{\frac{\pi}{2}} \frac{x\cos x\ \sin x}{\cos^4 x + \sin^4 x}\,dx.$$

Solution by Jeremy D. Primer, student, Columbia H.S., Maplewood, New Jersey.

We will use the well-known fact that, for any integrand f and constant a,

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx. \tag{1}$$

Using elementary trigonometric identities, the integrand in our problem can be rewritten as

$$f(x) = \frac{x\sin 2x}{1+\cos^2 2x},$$

from which it is seen that

$$f(\tfrac{\pi}{2} - x) = \frac{\pi}{2}\cdot\frac{\sin 2x}{1+\cos^2 2x} - f(x).$$

Now, using (1), we have

$$I = \frac{1}{2}\left\{\int_0^{\frac{\pi}{2}} f(x)dx + \int_0^{\frac{\pi}{2}} f(\tfrac{\pi}{2} - x)dx\right\} = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1+\cos^2 2x}\ dx$$

$$= -\frac{\pi}{8}\arctan\ (\cos 2x)\Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{16}\ .$$

Also solved by PAUL BRACKEN, student, University of Toronto; G.C. GIRI, Midnapore College, West Bengal, India; M.S. KLAMKIN, University of Alberta; VIKTORS LINIS, University of Ottawa; ANDERS LÖNNBERG, Mockfjärd, Sweden; F.G.B. MASKELL, Algonquin College, Ottawa; LEROY F. MEYERS, The Ohio State University; DONALD L. MUENCH, St. John Fisher College, Rochester, N.Y.; BOB PRIELIPP, The University of Wisconsin-Oshkosh; HYMAN ROSEN, Cooper Union, Brooklyn, N.Y.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

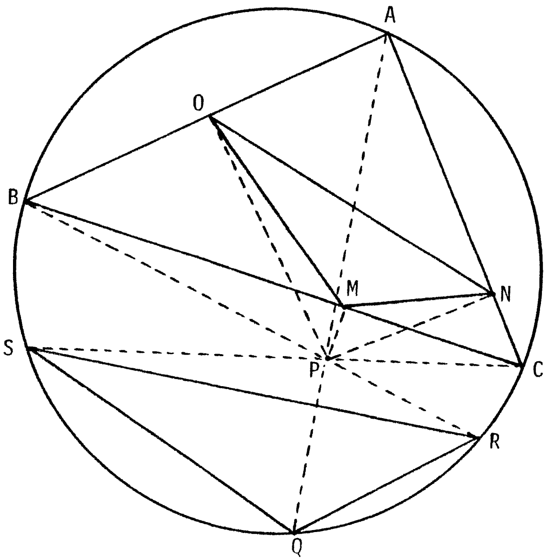
* * *

456. [1979: 167] *Proposed by Orlando Ramos, Havana, Cuba.*

Let ABC be a triangle and P any point in the plane. Triangle MNO is determined by the feet of the perpendiculars from P to the sides, and triangle QRS is determined by the cevians through P and the circumcircle of triangle ABC. Prove that triangles MNO and QRS are similar.

Solution by Howard Eves, University of Maine.

The proof given below assumes that the point P is located as in the figure. Slight modifications may be needed if P is located elsewhere in the plane (but not on the circumcircle of triangle ABC).



We have

$$\angle OMN = (\pi - \angle OMP) + (\pi - \angle PMN)$$
$$= \angle RBA + \angle SCA$$

(since PMOB and PMNC are cyclic)

$$= \tfrac{1}{2}(\text{arc } AR + \text{arc } SA)$$
$$= \tfrac{1}{2}\,\text{arc } SR$$
$$= \angle SQR.$$

Similarly $\angle MNO = \angle QRS$ and $\angle NOM = \angle RSQ$. It follows that triangles MNO and QRS are similar.

In the limiting case when P lies on the circumcircle of triangle ABC, the points Q,R,S all coincide with P and the points M,N,O are on a line, the *Simson line* of triangle ABC with respect to point P.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands, M.S. KLAMKIN, University of Alberta; SAHIB RAM MANDAN, Bombay, India; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; JOHN A. WINTERINK, Albuquerque Technical Vocational Institute, Albuquerque, New Mexico; and the proposer.

*　　　　*　　　　*

ADDENDUM TO "THE OLYMPIAD CORNER: 14"

Here is additional information relating to Problem 10-3 (pp. 106-108). Inequality (5) on page 107 also follows from the solution to part (b) given last month (p. 75). It can be shown that inequality (7) on page 107 is also valid for any real $m \geq n-1$. One can use induction on n to obtain

$$\frac{\partial^{n-1}F}{\partial a_1 \ldots \partial a_{n-1}} \geq 0, \qquad \text{which then implies that} \qquad \frac{\partial^{n-2}F}{\partial a_1 \ldots \partial a_{n-2}} \geq 0,$$

etc. These inequalities can be traced back to Segar [1]-[3].

For Problem 11-2 (pp. 110-111), the two references for the method of differences given on page 111 contain only examples of an algebraic nature. We give below one example involving transcendental functions. More can be found in Glasser and Klamkin [4] and in Durell and Robson [5].

To find

$$S = \sum_{r=1}^{m} u_r, \qquad \text{where } u_r = \arctan \frac{1}{r^2+r+1},$$

observe that

$$\arctan \frac{1}{r+1} - \arctan \frac{1}{r} = \arctan \frac{\frac{1}{r+1} - \frac{1}{r}}{1 + \frac{1}{r+1} \cdot \frac{1}{r}} = -\arctan \frac{1}{r^2+r+1} = -u_r;$$

so with $v_r = \arctan \frac{1}{r+1}$ we have

$$-S = \sum_{r=1}^{m} -u_r = \sum_{r=1}^{m} (v_r - v_{r-1}) = v_m - v_0 = \arctan\frac{1}{m+1} - \arctan 1 = -\arctan\frac{m}{m+2}.$$

MURRAY S. KLAMKIN

REFERENCES

1. H.W. Segar, "Some Inequalities," *Messenger of Mathematics*, 19 (1889) 189-192.

2. ________, "Some Inequalities," *ibid.*, 20 (1890) 54-59.

3. ________, "On an Inequality," *ibid.*, 22 (1892) 47-51.

4. M.L. Glasser and M.S. Klamkin, "On Some Inverse Tangent Summations," *Fibonacci Quarterly*, 14 (1976) 385-388.

5. C.V. Durell and A. Robson, *Advanced Trigonometry*, G. Bell and Sons, London, 1959, pp. 127-133.

* * *

A LOVER'S (EUCLIDEAN) VALEDICTION

Our two souls therefore, which are one,
  Though I must go, endure not yet
A breach, but an expansion,
  Like gold to airy thinness beat.

If they be two, they are two so
  As stiff twin compasses are two;
Thy soul, the fixed foot, makes no show
  To move, but doth, if th' other do.

And though it in the center sit,
  Yet when the other far doth roam,
It leans and hearkens after it,
  And grows erect, as it comes home.

Such wilt thou be to me, who must
  Like th' other foot, obliquely run;
Thy firmness makes my circle just,
  And makes me end where I begun.

JOHN DONNE, from
A Valediction: Forbidding Mourning

* * *