PI MU EPSILON JOURNAL

VOLUME 8 FALL 1988 NUMBER 9

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JOURNAL

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CONTENTS

PI MU EPSILON JOURNAL
THE OFFICIAL PUBLICATION OF THE NATIONAL HONORARY MATHEMATICS SOCIETY

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MAXIMAL POLYGONS FOR EQUITRANSITIVE PERIODIC TILINGS

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Abstract

It has been shown [1, 67-09] that in any periodic equitransitive tiling by convex polygonal tiles, the maximum number of sides of any tile is 66. This maximum is achieved in the periodic symmetry group *p6m*. We extend this result by determining the maximum number of sides in each of the remaining 16 periodic symmetry groups.

I. *Introduction.* A convex *tiling* is a set of closed polygonal regions, known as *tiles*, which cover the plane without gap or overlap. If the vertices of adjacent tiles meet, the tile is edge-to-edge. A tiling is *periodic* if its group of symmetries is one of the 17 periodic groups, often known as wallpaper groups. (See [3] for a full derivation of the 17 periodic groups.) A periodic tiling is characterized by the fact that it has two translative symmetries in nonparallel directions. Suppose we represent these translations by the vectors x and y. The *periodic parallelogram* of a periodic tiling is the parallelogram with sides x and y having the minimal positive area. This parallelogram has the property that by replicating the region inside the parallelogram along the translation vectors, the entire tiling may be reconstructed.

A tiling is *equitransitive*, if for each k, all polygons having k sides are in the same transitivity class; that is, all polygons having the same number of sides can be mapped to each other by a symmetry of the tiling.

In this paper, we will study periodic equitransitive tilings with convex polygons. In any such tiling, the maximum number of sides on any

* Work on this paper was done while the authors were participants in the Research Experiences for Undergraduates program at Oregon State University. Their work was partially supported by NSF Grant DMS-8712402. The authors would like to thank Branko Grünbaum, Robby Robson, and Paul Cull for their kind support.

tile is 66. (See [1].) This maximum is achieved in the periodic symmetry group $p6m$. We will consider bounds for the other 16 periodic groups. To do this, we will prove the following theorem.

Theorem. In any equitransitive tiling with one of the 17 periodic symmetry groups, the maximum number of sides on any tile is given in Table 1.

Table 1 Maximal polygons

	symmetry group	p_k max	maximal polygon
I	*cmm*	4	30
	p6	6	42
	p31m	6	42
	p3	3	24
	pgg	2	18
	p2	2	18
	pm	2	18
	cm	2	18
	p1	1	12
II	*pmg*	4	18
	pg	2	12
III	*p4*	4	28
	p4g	4	28
IV	*p4m*	8	48
	pmm	4	24
	p3m1	6	36
	p6m	12	66

Here p_k *max* is the maximum number of centroids possible in the period parallelogram, as explained below.

The proof of the theorem proceeds in stages. In section 2, we pose a lemma which gives an initial upper bound for the maximal polygon. We will call this initial upper bound m_g, where g is the symmetry group under consideration. For the groups labeled I, above, construction proves that m_g equals the number of sides on the maximal polygon, For the remaining cases, we must revise our initial estimates. This is done in sections 3 through 5.

2. ***The Initial Upper Bound.*** To get the initial upper bound on the maximal polygon for a particular symmetry group, we use the following lemma which is stated without proof. A proof can be found in [1].

Lemma. In any periodic tiling, if the period parallelogram contains the centroids of p_k k-gons, where k is an integer, then

$$3p_3 + 2p_4 + p_5 \geq \sum_{k=7}^{m_g} (k-6)p_k.$$

From the lemma it is possible to get an estimate for m_g, the maximum number of sides on a polygon in a particular symmetry group. As exemplified below, m_g depends entirely upon the maximum value of p_k. Because we require our figures to be equitransitive, the maximum value of pk will equal the maximum number of k-gon centroids in the period parallelogram. The maximum number of centroids depends on the symmetries present in the tiling group, as illustrated by the dots in the right halves of Figures I through 16 [2]. A key to these group diagrams is given in Table 2. Substituting this maximum value of p_k into the lemma inequality yields an estimate for m_g, the maximal polygon,

As an example, we work through this process for the symmetry group *cmm*. An examination of Figure 1, the group diagram for *cmm*, shows that the maximum number of centroid images is achieved when a centroid is placed in "general position," off all lines of symmetry. In *cmm*, this maximum is four, which implies that there are at most four k-gons (for each k) in the period parallelogram. With p_k at most four, the lemma yields the following result:

$$\begin{aligned} 24 &= (3\cdot 4) + (2\cdot 4) + (1\cdot 4) \\ &\geq 3p_3 + 2p_4 + p_5 \\ &\geq p_7 + 2p_8 + \dots + (m_{cmm} - 6)p_{m_{cmm}}. \end{aligned}$$

So, to maintain the inequality, $m_{cmm} = 30$.

To verify that this estimate does in fact correspond to a tiling, we must find a periodic equitransitive tiling with convex polygonal tiles in symmetry group *cmm* which contains 30-sided polygons. Figure I shows an example of such a tiling.

Using the lemma, similar estimates can be made for the groups *p6*, *p31m*, *p3*, *pgg*, *p2*, *pm*, *cm*, and pl. Figures illustrating the maximal p_k for these groups and the corresponding tiles are shown in Figures 2 - 9. Thus, for these first nine symmetry groups, the estimate for the maximal polygon given by the lemma produces an actual tiling.

3. *The Second Set of Groups.* The second set of groups are those in which *two* images of a polygon must always appear in the period parallelogram. There are exactly two symmetry groups in which this occurs: *pmg* and pg.

In *pmg*, for example, each center of symmetry occurs twice in the period parallelogram. This means that the period parallelogram must contain at least two of every polygon type, so $p_k \geq 2$. Additionally, the group symmetries shown in Figure 10 require that $p_k \leq 4$.

Table 2. Key to symmetry group diagrams

Symbol	Meaning
———	Line of reflection
- - - - -	Line of glide reflection
◇ ◆	Center of 2-fold rotation. Black figure indicates that rotation lies on a line of reflection
A ▲	Center of 3-fold rotation
□ ■	Center of 4-fold rotation
⬡ ●	Center of 6-fold rotation

With these constraints, the lemma gives

$$24 = (3\cdot 4) + (2\cdot 4) + (1\cdot 4)$$
$$\geq 3p_3 + 2p_4 + p_5$$
$$\geq p_7 + 2p_8 + \ldots + (m_{pmg} - 6)p_{m_{pmg}}$$
$$24 \geq (m_{pmg} - 6)2$$
$$18 \geq m_{pmg}.$$

So 18-gons are the maximum polygons possible for the symmetry group *pmg*. Figure 10 shows the corresponding tiling with 18-gons. A similar argument for the symmetry group *pg* yields a tiling with 12-gons. (See Figure 11).

4. *The Third Set of Groups.* In the third set of groups we find that m_g must be divisible by four. Two symmetry groups for which this occurs are *p4* and *p4g*. For *p4* and *p4g*, the maximum value of p_k is four. (See Figures 12 and 13.) So, by the lemma, $m_g \leq 30$. We now show that in both of these cases $m_g = 28$.

Suppose 30-gons are possible in *p4*. Since the four-fold center of rotation is the only center which occurs once in the period parallelogram and since $p_{30} = 1$, the 30-gon must be centered on this four-cent&.. But, since 30 is not divisible by four, this is impossible. Placing the center of the 30-gon anywhere else in the period parallelogram would require that $p30 > 1$, so 30-gons are not possible in *p4*. For similar reasons, 29-gons are not possible. Thus, we must take $m_{p4} = 28$. Figure 12 shows an example of an equitransitive *p4* tiling with convex polygons using 28-gons.

Next, suppose $m_{p4g} = 30$ and 30-gons are possible in *p4g*. For $p_{30} = 1$, the 30-gons must be centered on either a two- or a four-center. The 30-gons cannot be on the four-centers since 30 is not divisible by four. Suppose that the 30-gons sit on two-centers. By inspection of the group diagram, one finds that each 30-gon can touch other 30-gons either four or zero times. Assume the 30-gons each touch four other 30-gons. Then the remaining 26 sides form a closed concave figure centered on the four-fold rotation. Now, the number of sides of any polygonal figure centered on the four-fold rotation must, of course, be divisible by four. Since 26 is not divisible by four, we have a contradiction.

The remaining possibility is that the 30-gons touch each other zero times. This possibility is ruled out as follows. The lemma dictates that, with a 30-gon present, only four other types of polygons can exist in the tiling: 3-, 4-, 5-, and 6-gons. By the group symmetries, these four polygons can each compose at most eight sides of the 30-gon. Three of the these polygons contributing eight sides each leaves six sides for the remaining polygon. The group symmetries, however, prohibit a polygon from contributing six sides. So, again, we have a contradiction.

The symmetry group will not permit 29-gons since 29 is an odd number. Thus $m_{p4g} \leq 28$. Figure 13 shows an example of an equitransitive tiling with convex 28-gons for the symmetry group *p4g*.

5. *The Fourth Set of Groups.* The remaining four symmetry groups, *pmm*, *p3m1*, *p4m*, and *p6m*, have the property that all centers of rotation lie on lines of reflection. Because of this symmetry, we can determine the number of distinct tiles which must be in the period parallelogram. By application of the lemma, we are then able to reduce the initial estimate for the maximal polygon.

To illustrate this procedure, we consider the group $p4m$. Examining Figure 14, we see that for any k, there are at most eight k-gons in the period parallelogram. Application of the lemma yields $m_{p4m} \leq 54$. This upper bound assumes that for $k = m_{p4m}$, $p_k = 1$. In $p4m$, this is true only if the maximal polygon lies on a four-fold center of rotation. This requirement forces m_{p4m} to be divisible by four. Hence, $m_{p4m} \leq 52$ and the other possible values are also multiples of four.

A maximal polygon on a four-center can touch an identical polygon at most four times. This leaves $(m_{p4m} - 4)$ edges to be adjacent to other polygons. The lines of reflection passing through the four-centers allow these other polygons to contribute at most eight edges to the maximal polygon. From this we determine that the period parallelogram must include at least $(m_{p4m} - 4)/8$ other polygons besides the one on the four-center. With this fact we show that $m_{p4m} \neq 52$.

Suppose $m_{p4m} = 52$. Then there are at least $(52 - 4)/8 = 6$ polygons other than 52-gons, But, from the lemma, when $p_{52} = 1$, $p_k = 0$ for all $k \geq 9$. This leaves the inequality

$$p_7 + 2p_8 \leq 2.$$

The inequality shows that in addition to the 3-, 4-, 5-, and 6-gons, we can have either two 7-gons or one 8-gon. So, with a 52-gon in the tiling, only five other polygon types are possible. This is not enough. Therefore, we have shown that $p4m$ does not admit 52-gons. Stepping down by four, the next possibility is $m_{p4m} = 48$. The construction in Figure 14 shows an equitransitive $p4m$ tiling with convex 48-gons.

Using similar techniques, it is possible to reduce initial estimates for maximal polygons in the groups pmm, $p3m1$, and $p6m$. The resulting tiles are illustrated in Figures 15-17. The above methods were used in [1] to get an estimate for the maximum polygons possible in the symmetry group $p6m$.

REFERENCES

1. Ludwig Danzer, Branko Grünbaum, and G. C. Shephard, *Equitransitive Tilings, or How to Discover New Mathematics*, Math. Mag., 60 (1987) 67-89.
2. Branko Grünbaum and G. C. Shephard, *Tilings and Patterns*, W. H. Freeman, New York, 1987.
3. George E. Martin, *Transformation Geometry: An Introduction to Symmetry*, Springer-Verlag, New York-Heidelberg-Berlin, 1982.

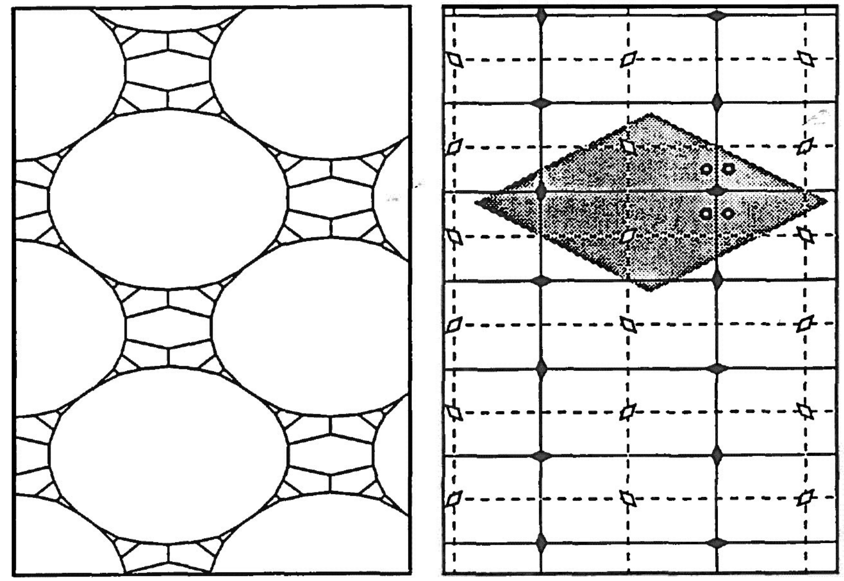


Figure 1. (Left) cmm with 30-gons. In addition, there are 3-, 4-, 5-, 6-, and 7-gons present in the figure. (Right) Group diagram for cmm demonstrating that p_k *max* in cmm is 4.

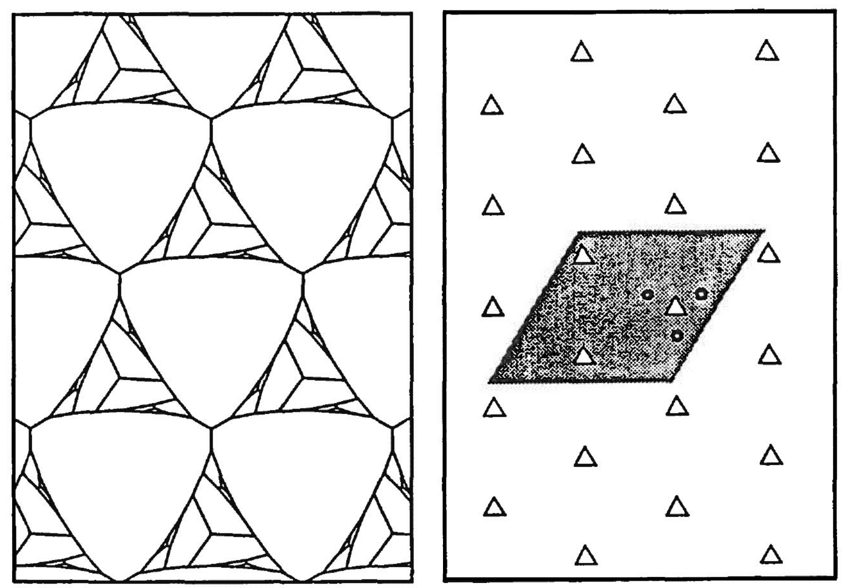


Figure 2. (Left) $p6$ with 42-gons. In addition, there are 3-, 4-, 5-, and 6-gons present in the figure. (Right) Group diagram for $p6$ demonstrating that p_k *max* in $p6$ is 6.

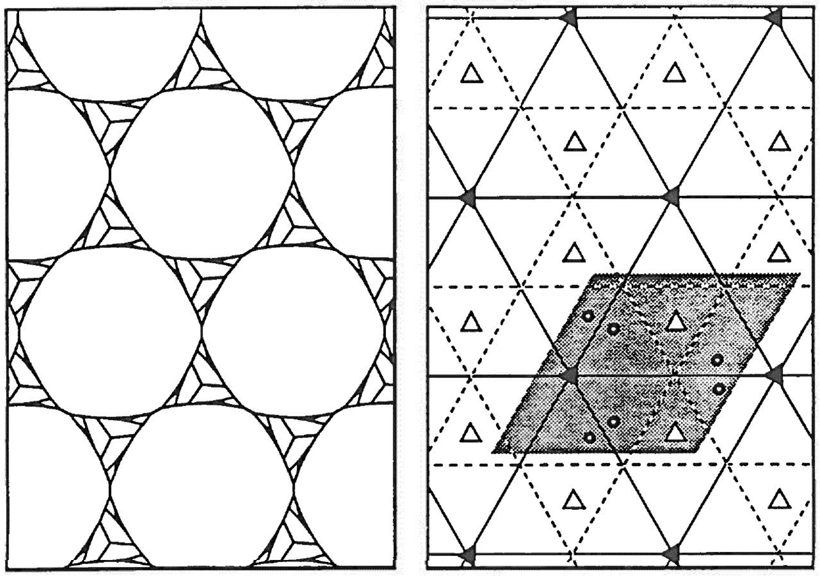


Figure 3. (Left) $p3lm$ with 42-gons. In addition, there are 3-, 4-, 5-, and 6-gons present in the figure. (Right) Group diagram for $p3lm$ demonstrating that p_k *max* in $p3lm$ is 6.

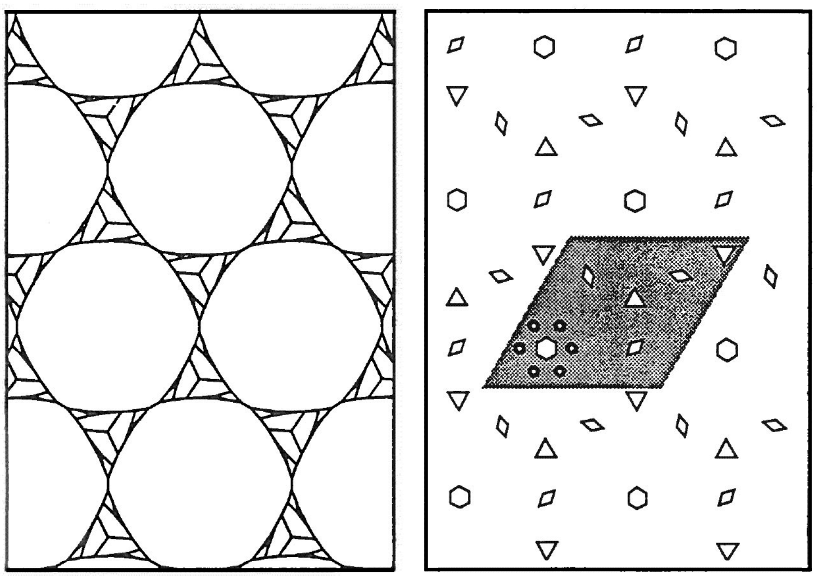


Figure 4. (Left) $p3$ with 24-gons. In addition, there are 3-, 4-, 5-, and 6-gons present in the figure. (Right) Group diagram for $p3$ demonstrating that p_k *max* in $p3$ is 6.

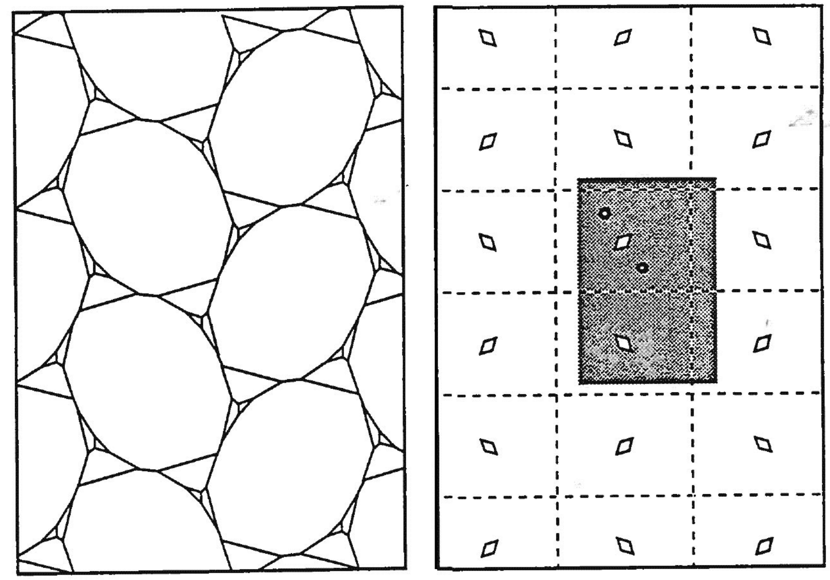


Figure 5. (Left) pgg with 18-gons. In addition, there are 3-, 4-, and 5-gons present in the figure. (Right) Group diagram for pgg demonstrating that p_k *max* in pgg is 2.

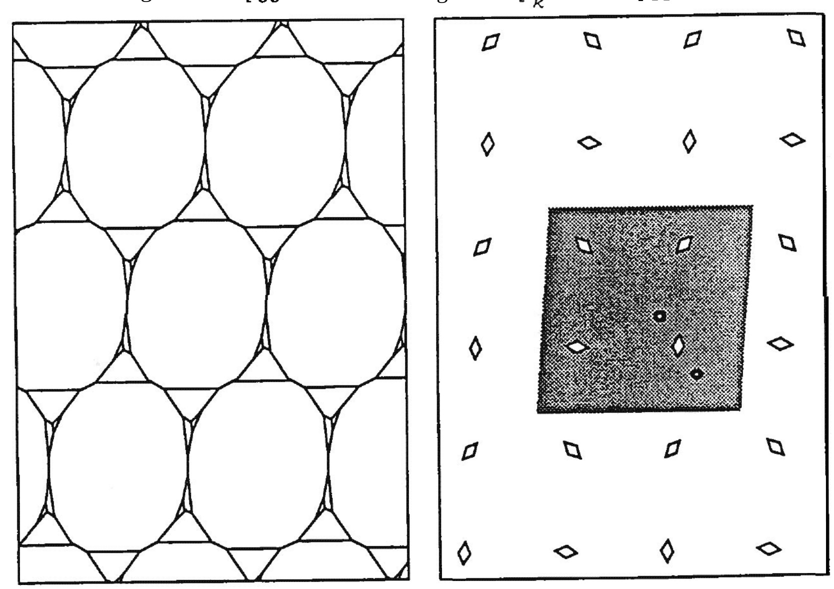


Figure 6. (Left) $p2$ with 18-gons. In addition, there are 3-, 4-, and 5-gons present in the figure. (Right) Group diagram for $p2$ demonstrating that p_k *max* in $p2$ is 2.

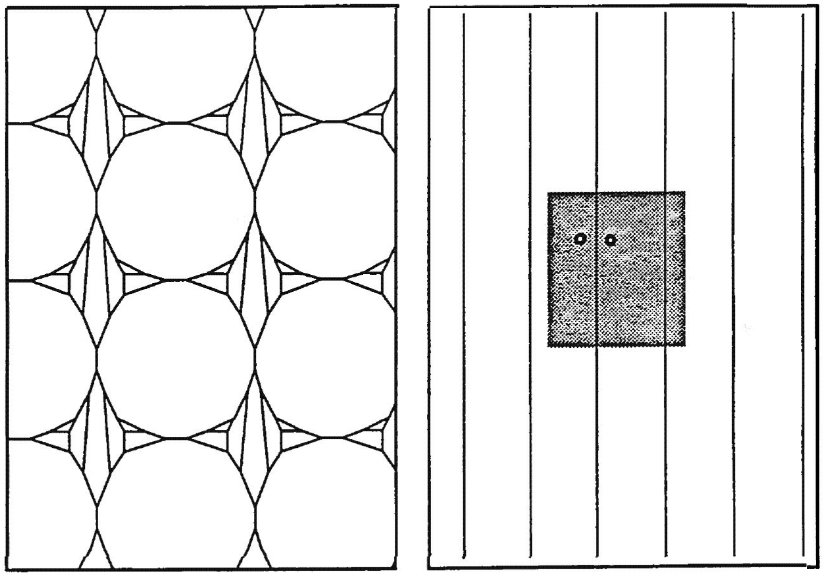


Figure 7. (Left) pm with 18-gons. In addition. there are 3-, 4-, 5-, and 6-gons present in the figure. (Right) Group diagram for *pm* demonstrating that p_k *max* in *pm* is 2.

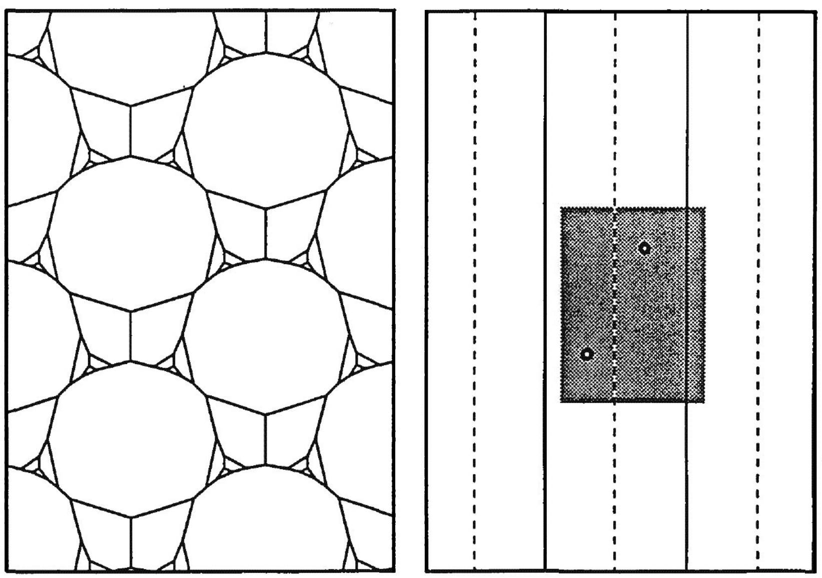


Figure 8. (Left) *cm* with 18-gons. In addition, there are 3-, 4-, 5-, and 6-gons present in the figure. (Right) Group diagram for *cm* demonstrating that p_k *max* in *cm* is 2.

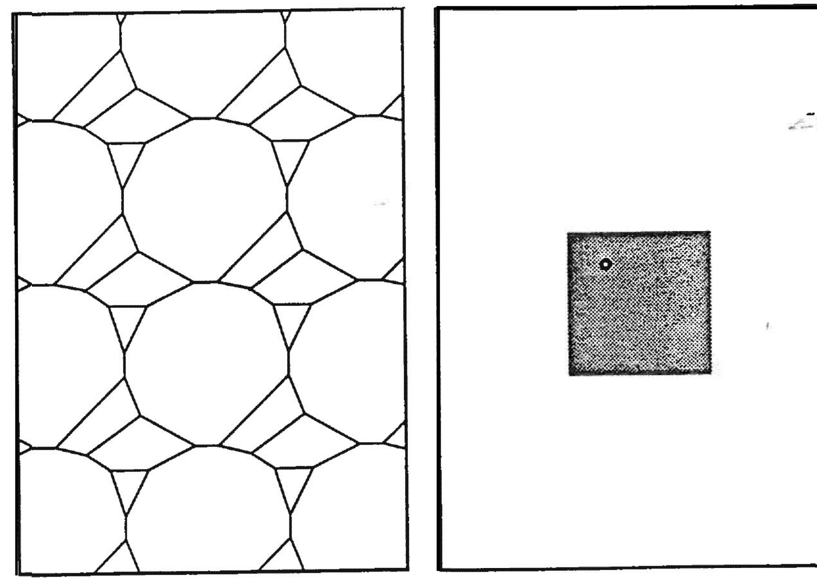


Figure 9. (Left) *p*1 with 12-gons. In addition, there are 3-, 4-, and 5-gons present in the figure. (Right) Group diagram for *p*1 demonstrating that p_k *max* in *p*1 is 1.

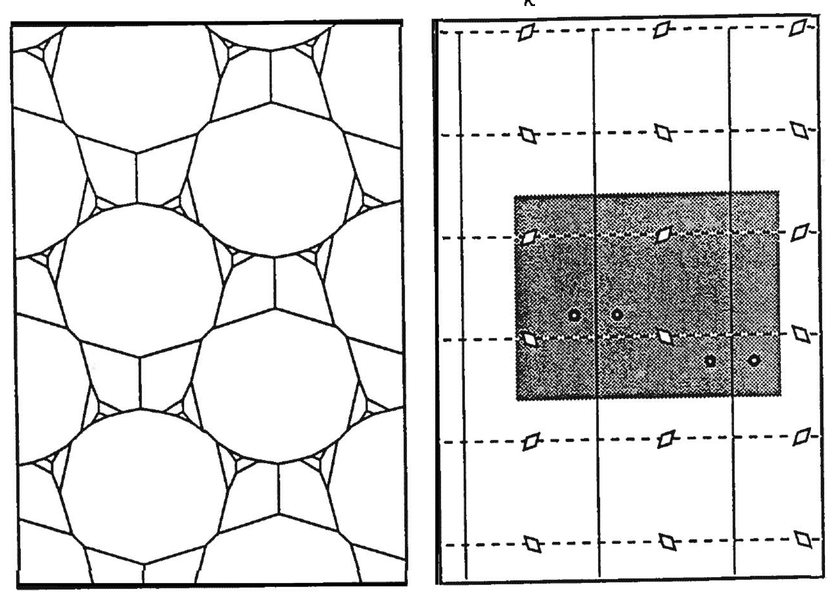


Figure 10. (Left) *pmg* with 18-gons. In addition, there are 3-, 4-, 5-, and 6-gons present in the figure. (Right) **Group** diagram for *pmg* demonstrating that p_k *max* in pmg is 4.

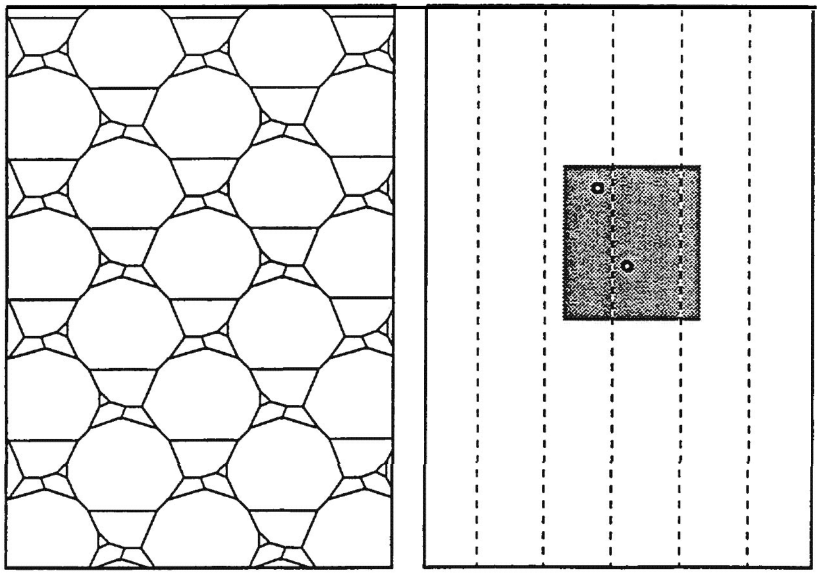


Figure 11. (Left) pg with 12-gons. In addition, there are 3-, 4-, 5-, and 6-gons present in the figure. (Right) Group diagram for pg demonstrating that p_k *max* in *pg* is 2.

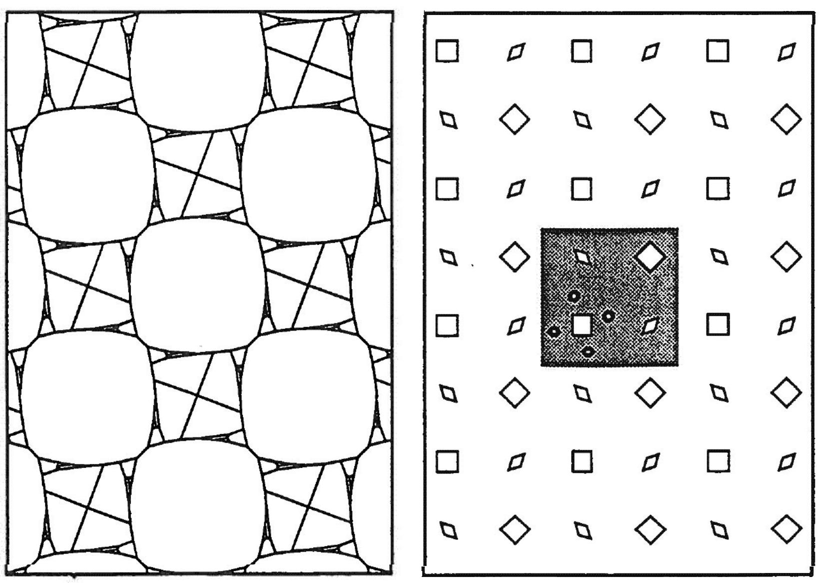


Figure 12. (Left) *p*4 with 28-gons. In addition, there are 3-, 4-, 5-, and 6-gons present in the figure. (Right) Group diagram for *p*4 demonstrating that p_k *max* in *p*4 is 4.

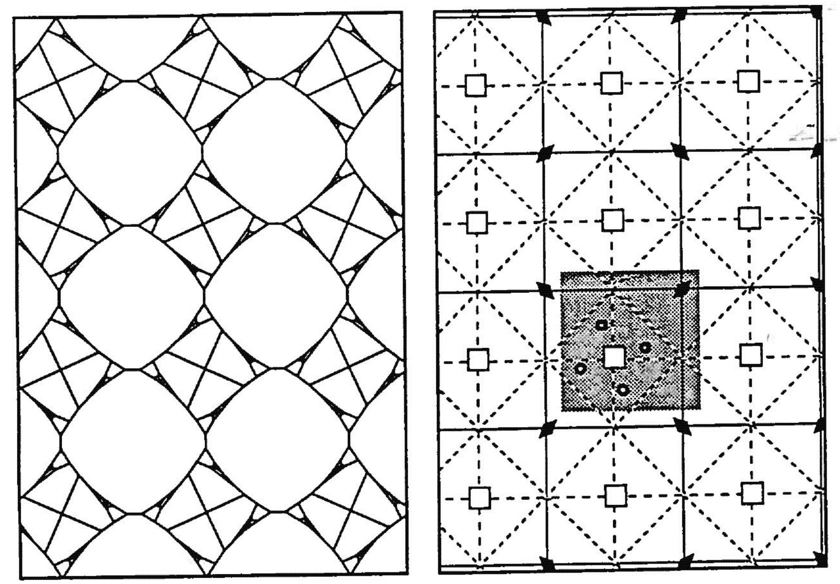


Figure 13. (Left) *p4g* with 28-gons. In addition, there are 3-, 4-, 5-, and 6-gons present in the figure. (Right) Group diagram for p4g demonstrating that p_k *max* in p4g is 4.

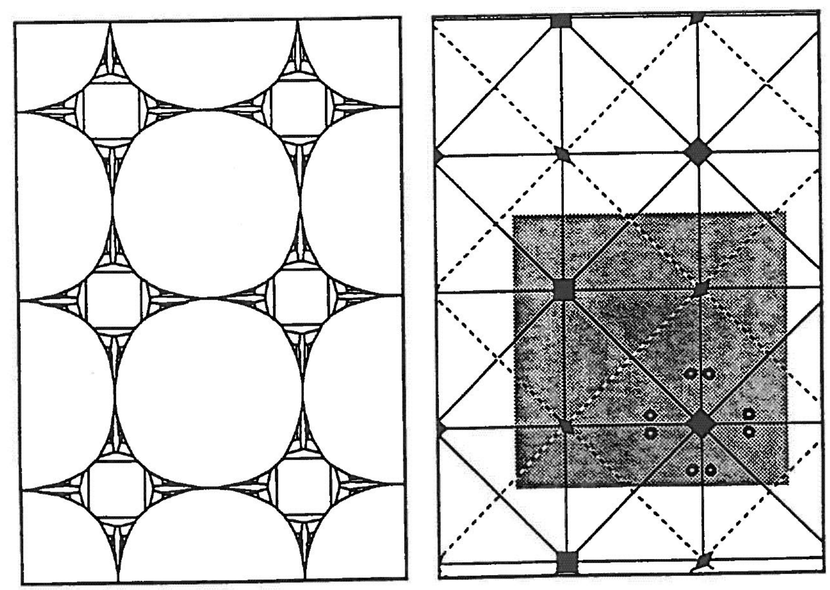


Figure 14. (Left) *p4m* with 48-gons. In addition, there are 3-, 4-, 5-, 6-, 7-, and 8-gons present in the figure. (Right) Group diagram for *p4m* demonstrating that p_k *max* in *p4m* is 4.

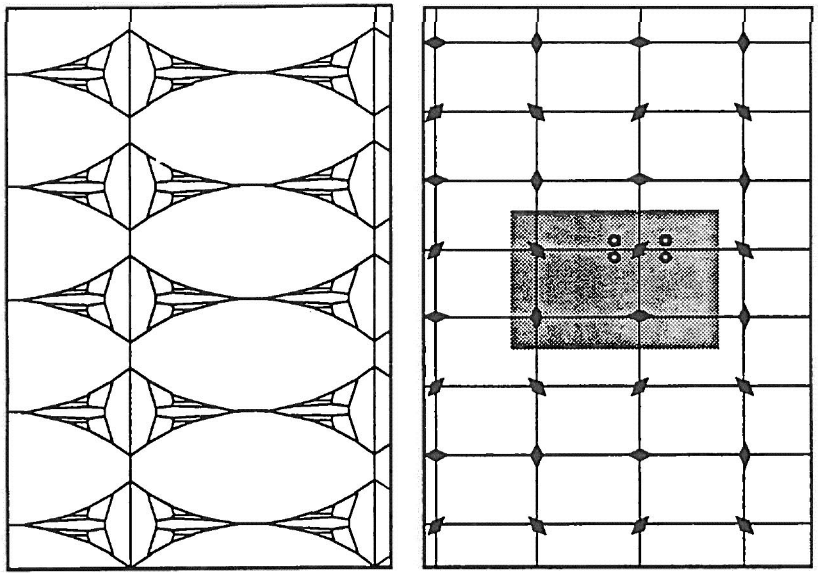


Figure 15. (Left) *pmm* with 24-gons. In addition, there are 3-, 4-, 5-, 6-, and 7-gons present in the figure. (Right) Group diagram for *pmm* demonstrating that p_k *max* in *pmm* is 4.

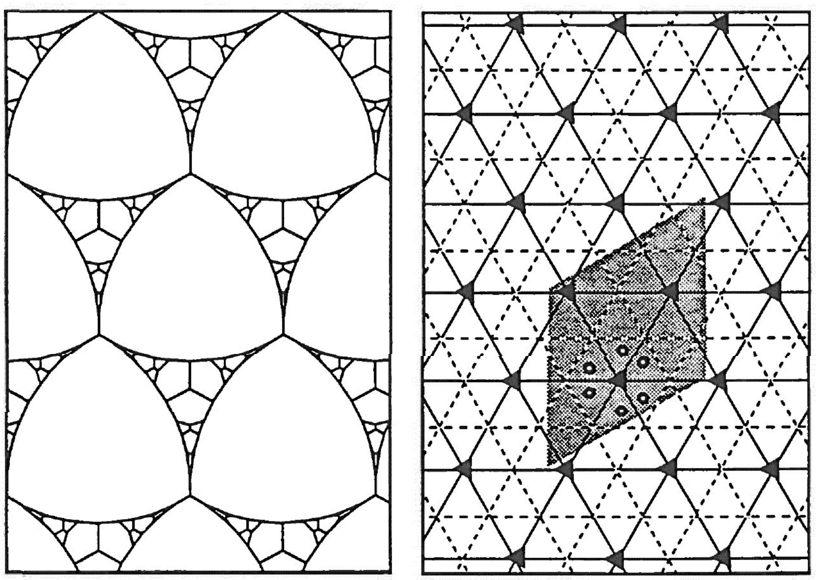


Figure 16. (Left) *p3m1* with 36-gons. In addition, there are 3-, 4-, 5-, 6-, and 7-gons present in the figure. (Right) Group diagram for *p3m1* demonstrating that p_k *max* in *p3m1* is 6.

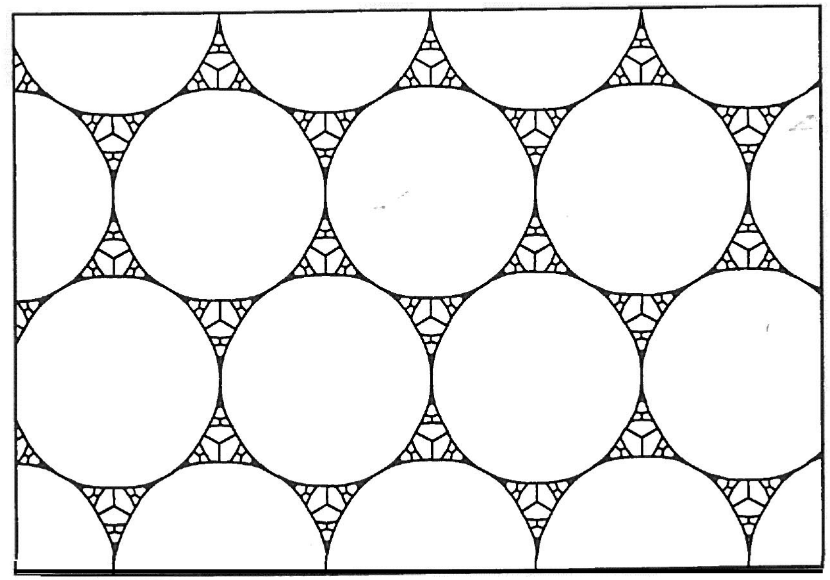


Figure 17. *p6m* with 66-gons. In addition, there are 3-, 4-, 5-, 6-, and 7-gons present in the figure. A full explanation of the derivation of this figure can be found in [1].

PI MU EPSILON PLANS A 75TH BIRTHDAY CELEBRATION

In 1989, the Pi Mu Epsilon, Inc. National Honorary Mathematics Society, incorporated on May 25, 1914, under the laws of the State of New York, will celebrate its 75th anniversary as a national mathematics honorary with over 250 chapters in 46 states and the District of Columbia.

Councillors and officers have been making plans for appropriate ways of celebrating the birthday.

The Spring 1989 issue of the Pi Mu Epsilon Journal will contain a history of Pi Mu Epsilon, along with a complete list of past officers and councillors, winners of the annual paper competitions, presenters of papers at the annual meetings, and much more.

The 1989 National Pi Mu Epsilon Meeting will be at the University of Colorado in Boulder from August 7 through August 10.

COUNTING BIT STRINGS WITH A SINGLE OCCURRENCE OF 00

by Thomas E. Moore
Bridgewater State College

Discrete mathematics is filled with problems requiring the idea of recursive problem-solving coupled with mathematical induction. Some of these problems stand out because they interconnect with other problems or they reward us with new insights with each new attack on them. We have seen such a problem in the literature [4] but neither its solution nor its intrinsic value seems well known. This article gives it the exposure it deserves.

__*The*__ __Problem.__ Let a_n denote the number of n-bit strings (also called zero-one sequences or binary sequences of length n) with exactly one occurrence of two consecutive zeros. For example, the 5-bit strings 00110 and 10010 qualify but 10001 and 11010 do not. The sequence $\{a_n\}$, $n \geq 1$, begins 0, 1, 2, 5, 10, 20. What is the rule?

Our solution depends on solving a related problem: how many n-bit strings have no consecutive zeros occurring? Denoting this count by b_n, we imagine forming such an n-bit string. The first (leftmost) bit is either 0 or 1. If the first bit is 0 then the string must begin 01... and may be completed in b_{n-2} ways. If it begins 1... then it may be completed in b_{n-1} ways. Therefore, $b_n = b_{n-1} + b_{n-2}$, $n \geq 3$, with $b_1 = 2$ and $b_2 = 3$. We conclude that $b_n = F_{n+2}$, the $(n+2)$nd Fibonacci number. Recall that this famous sequence is defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 3$.

Returning to our original problem, consider the position of the single 00 in the n-bit string. Three cases suggest themselves:

(1) ....100 n-3 free bits

(2) 001.......... n-3 free bits

(3) ......1001... k plus n-k-4 free bits.

The strings of the first and second cases each may be completed in b_{n-3} ways while those of the third case may be completed in $b_{n-k-4}b_k$ ways. In the latter case k ranges from 0 to n-4.

Thus the total number of n-bit strings with a single occurrence of 00

is

$$a_n = 2b_{n-3} + \sum_{k=0}^{n-4} b_{n-k-4}b_k, \text{ that is,}$$

$$a_n = 2F_{n-1} + \sum_{k=0}^{n-4} F_{n-k-2}F_{k+2}.$$

Expanding, and using Fn-1 = F_1F_{n-1}, we have

$$(*) \quad a_n = F_{n-1}F_1 + F_{n-2}F_2 + F_{n-3}F_3 + \cdots + F_2F_{n-2} + F_1F_{n-1}, \; n \geq 4,$$

with a = 0, a = 1 and a = 2.

Note that this recurrence relation actually holds for $n \geq 3$.

The sequence { a } has appeared in the literature (see [1] and [3]) in various ways unconnected with our current problem. Hoggatt [1] calls it the first Fibonacci convolution sequence because of the form of the recurrence (*).

__*Second*__ __Attack.__ Let's take a look at n-bit strings with a single occurrence of 00 generated in a deliberate but natural way, in the cases n = 2 to 6. This is displayed in the table below.

n	qualifying strings					a_n
2	00					1
3	100	001				1 + 1
4	0100	1001	0010			
	1100		0011			2 + 1 + 2
5	10100	01001	10010	00101		
	01100	11001	10011	00110		
	11100			00111		3 + 2 + 2 + 3
6	010100	101001	010010	100101	001010	
	110100	111001	010011	100110	001011	
	011100	011001	110011	100111	001110	
	111100		110010		001111	
	101100				001101	5 + 3 + 4 + 3 + 5

The organization of this data calls attention to the symmetry in the summands that add to a . An array similar to the usual display of Pascal's triangle is definitely in order here. Therefore, calculating a few more lines of data, we have the array:

```
                 1
               1   1
             2   1   2
           3   2   2   3
         5   3   4   3   5
       8   5   6   6   5   8
    13   8  10   9  10   8  13
  21  13  16  15  15  16  13  21
  ................................
```

Conjecture. One can't help but notice that $21 = 13 + 8$, $13 = 8 + 5$, $16 = 10 + 6$, $15 = 9 + 6$, ... Further inspection of the data in the array raises a suspicion that we turn into a conjecture and a new recursive description of a_n.

(**) $$a_n = a_{n-1} + a_{n-2} + F_{n-1},\ n \geq 3, \text{ with } a_1 = 0,\ a_2 = 1.$$

We may confirm this as fact by proving the equivalence of the recurrence (**) and our earlier one (*), by induction on n.

To enable this we temporarily denote the terms generated by the new recurrence by c_n, so that (**) becomes

$$c_n = c_{n-1} + c_{n-2} + F_{n-1},\ n \geq 3,\ c_1 = 0,\ c_2 = 1.$$

Then we prove $c_n = a_n$ as follows.

First, c_1, c_2, c_3 and a_1, a_2, a_3 equal 0, 1 and 2, respectively.

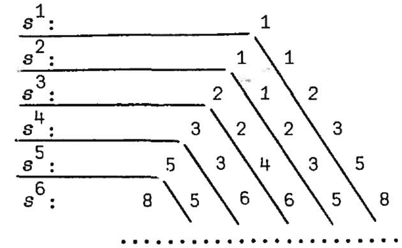
Assume that $c_i = a_i$ for $i = 1, 2, \ldots, n$, for some $n \geq 3$. Then

$$\begin{aligned} c_{n+1} &= c_n + c_{n-1} + F_n \\ &= a_n + a_{n-1} + F_n \\ &= F_{n-1}F_1 + F_{n-2}F_2 + F_{n-3}F_3 + \cdots + F_2F_{n-2} + F_1F_{n-1} \\ &\quad + F_{n-2}F_1 + F_{n-3}F_2 + F_{n-4}F_3 + \cdots + F_1F_{n-2} + F_n \\ &= F_nF_1 + F_{n-1}F_2 + F_{n-2}F_3 + \cdots + F_3F_{n-2} + F_{n-1} + F_n \\ &= F_nF_1 + F_{n-1}F_2 + F_{n-2}F_3 + \cdots + F_3F_{n-2} + F_2F_{n-1} + F_1F_n \\ &= a_{n+1}. \end{aligned}$$

The Fibonacci triangle. Our triangular array has been studied by Hosoya [2], apparently for its own sake, devoid of any motivation or natural context. Our problem provides that context.

We make some observations of our own here.

If we regard the array as sequences $\{s^k_n\}$ of numbers laid down along diagonals that parallel the right side of the triangle,



then each of the sequences is a Fibonacci-type sequence governed by the rule $s^k_n = s^k_{n-1} + s^k_{n-2}$, $n \geq 3$. This is easily deduced from our second recurrence (**).

All the terms of a diagonal sequence may be obtained by adding the corresponding terms of the two preceding sequences, that is, $s^k_n = s^{k-1}_n + s^{k-2}_n$, $k \geq 3$. Equivalently, the terms of each sequence are a constant multiple of the corresponding terms of the classical Fibonacci sequence, that is, $s^k_n = F_kF_n$, $n \geq 1$, fixed k.

The central term in alternate rows of the array is always a square. Indeed, these are the terms $s^n_n = F^2_n$, by our previous observation.

I would like to acknowledge the insights of my students Cindy Steeves, Susan Clark and Anthony D'Arezzo on this problem.

REFERENCES

1. V. E. Hoggatt, Jr., *Convolution Triangles for Generalized Fibonacci Numbers,* The Fibonacci Quarterly, *8* (1970) 158-171.
2. H. Hosoya, *Fibonacci Triangle*, The Fibonacci Quarterly, *14* (1976) 173-179.
3. J. Riordan, *Combinatorial Identities*, John Wiley & Sons, New York, *1968.*
4. A. Tucker, *Applied Combinatorics*, John Wiley & Sons, New York, *1980.*

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? Ed.

ON REDUCTION OF CONIC SECTIONS

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Usually translations and rotations of the coordinate system are used in the reduction of conic sections to standard forms. This involves lengthy algebra. Without suggesting techniques of linear algebra, we would like to study some interesting methods.

The rotation through the angle θ for which

$$(1) \qquad \tan 2\theta = \frac{B}{A - C}, \quad A - C \neq 0$$

eliminates the xy term in

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \ (A, B, C \text{ not all } 0).$$

Since $\tan\theta = m$ is the slope of the x'-axis (Figure 1), (1) is equivalent to

$$(2) \qquad Bm^2 + 2(A - C)m - B = 0.$$

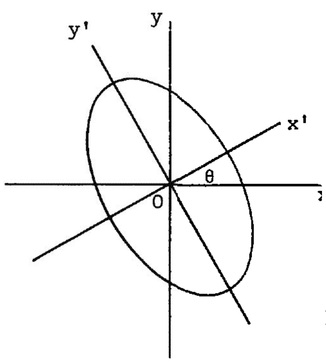

Figure 1

Note that if $A = C$, then the roots of (2) will be ± 1. So (2) is actually a generalization of (1).

1. The Reduction of Central Conics. Consider

$$Ax^2 + Bxy + Cy^2 = Q.$$

The line $y = mx$ intersects this conic in two points obtained from

$$Ax^2 + Bxy + Cy^2 = Q$$
$$y = mx.$$

Solving this set of equations, we obtain

$$(3) \qquad x^2 = \frac{Q}{Cm^2 + Bm + A}, \quad y^2 = \frac{m^2Q}{Cm^2 + Bm + A}.$$

If we substitute the roots of (2) in (3), we get the x' and y' intercepts of the conic section. For example, for the positive root of (2) we get $x^2 + y^2 = a^2$, and for the other root we get $x^2 + y^2 = b^2$. In case of a hyperbola, one of these values will be negative which corresponds to the conjugate axis. Some examples will clarify the idea.

Example 1.1. Consider the ellipse

$$73x^2 - 72xy + 52y^2 = 100.$$

From (2) we have

$$12m^2 - 7m - 12 = 0.$$

So we have $m = 4/3$ or $m = -3/4$. We usually choose the positive slope for the x'-axis. From (3) and $m = 4/3$, we obtain

$$x^2 = \frac{36}{25} \text{ and } y^2 = \frac{64}{25}.$$

Thus $a^2 = x^2 + y^2 = 4$. Similarly, from (3) and $m = -3/4$, we obtain

$$x^2 = \frac{16}{25} \text{ and } y^2 = \frac{9}{25}$$

which implies $b^2 = x^2 + y^2 = 1$. Consequently, the standard form will be

$$\frac{x'^2}{4} + y'^2 = 1.$$

A sketch of the graph is shown in Figure 2.

Example 1.2. Consider the hyperbola

$$3x^2 + 8xy - 3y^2 = 20.$$

From (2) we have

$$8m^2 + 12m - 8 = 0$$

which gives $m = 1/2$, $m = -2$. By (3), to $m = 1/2$ corresponds

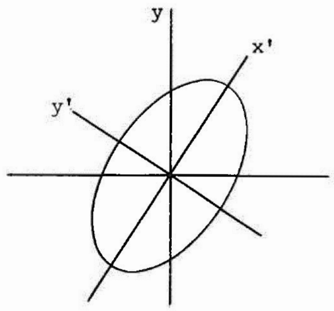

Figure 2

$$x^2 = \frac{16}{5} \text{ and } y^2 = \frac{4}{5}.$$

Therefore $a^2 = x^2 + y^2 = 4$. By (3), to $m = -2$ corresponds

$$x^2 = \frac{-4}{5} \text{ and } y^2 = \frac{-16}{5}.$$

Thus $x^2 + y^2 = -4$. So, the y'-intercept is imaginary, and the reduced equation is

$$\frac{x'^2}{4} - \frac{y'^2}{4} = 1.$$

2. ***The Reduction of Parabolas.*** For the parabola

$$(4) \qquad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \; B^2 - 4AC = 0$$

the quadratic part is a perfect square, that is, (4) can be written as

$$(ax + by)^2 + Dx + Ey + F = 0.$$

Since the line $ax + by = 0$ intersects the parabola in a single point, it must be parallel to the axis of the parabola. Thus we can choose this line for the x'-axis or y'-axis. We shall give an example. Consider the parabola

$$x^2 - 4xy + 4y^2 + 6\sqrt{5}x - 2\sqrt{5}y - 15 = 0.$$

This can be written as

$$(5) \qquad (x - 2y)^2 + 2\sqrt{5}(3x - y) - 15 = 0.$$

The slope of $x - 2y = 0$ is $1/2$, so one can draw the x'-axis (Figure 3). Thus the equations of the rotation will be

$$x = (2x' - y')/\sqrt{5}$$

$$y = (x' + 2y')/\sqrt{5}$$

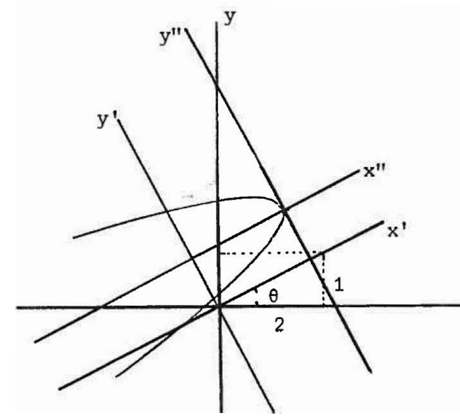

Figure 3

We substitute for x and y in (5) and get

$$[(-5y')/\sqrt{5}]^2 + 10x' - 10y' - 15 = 0$$

or

$$y'^2 - 2y' + 2x' - 3 = 0.$$

One may complete the square, carry the algebra further, and sketch the graph (Figure 3).

3. ***Conjugate Axes.*** Another way of obtaining (2) is the use of conjugate axes.

Consider the central conic

$$Ax^2 + Bxy + Cy^2 = Q,$$

and the line $y = mx$. Consider an arbitrary line $y = mx + b$ parallel to $y = mx$, which intersects the conic in M and N. As b varies, the locus of the midpoint of the line segment MN is a straight line through 0 which is called the axis conjugate to $y = mx$ (Figure 4). Let us obtain this line.

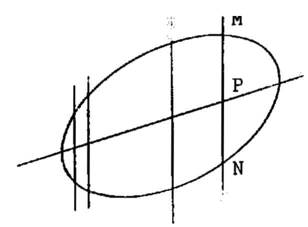

Figure 4

The set of equations

$$(6)\qquad \begin{aligned} Ax^2 + Bxy + Cy^2 &= Q \\ y &= mx + b \end{aligned}$$

gives M and N. Eliminating y, we obtain

$$(7)\qquad (A + Bm + Cm^2)x^2 + (Bb + 2Cmb)x + Cb^2 - Q = 0.$$

Let $P(x,y)$ be the midpoint of MN. Then

$$x = \frac{1}{2}(x_1 + x_2),$$

where x_1 and x_2 are the roots of (7). So, from (6) and (4), we get

$$x = \frac{-b(B + 2Cm)}{2(A + Bm + Cm^2)}$$

$$y = \frac{-bm(B + 2Cm)}{2(A + Bm + Cm^2)} + b$$

Eliminating b, we obtain

$$y = \frac{Bm + 2A}{-B - 2Cm}\,x.$$

In order that $y = mx$ would be the x'-axis, the conjugate axes must be perpendicular to $y = mx$, since the principal axes of a central conic are conjugate and perpendicular. Thus

$$m\left[\frac{Bm + 2A}{-B - 2Cm}\right] = -1.$$

This equation implies (2), that is,

$$Bm^2 + 2(A - C)m - B = 0.$$

4. Suggestions for *further* work.

a. Obtain (2) by finding extrema of the distance from the center of the conic to a point on it.

b. Find the center of a conic section with the use of conjugate axes.

c. Using the idea of conjugate axes, obtain the vertex of a parabola.

THE BEST CYLINDER FOR THE MONEY

by Margaret W. Maxfield
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The "best" cylinder calls for allocating twice as much area resource to the side(s) as to the bottom and top (if any). This result holds, whether the "best" is defined to be the cylinder that maximizes the volume for given cost of materials for the surface, or whether it is defined to be the one that minimizes the cost for given volume.

Theorem I. Let the base of a cylinder have perimeter px and area Ax^2, and let the altitude of the cylinder be h, where x is a convenient linear measure; x and h are to be chosen so as to optimize the cylinder. Then the cylinder with maximum volume for given cost and the cylinder with minimum cost for given volume require twice as much expenditure for the side(s) as for the bottom and the top.

Proof. Let the costs per square unit of materials be S for the side, B for the bottom, and T for the top. Let θ be the angle between any side element of the cylinder and the bases. Then

$$(1)\qquad \begin{aligned} \text{total cost} &= \text{cost of side(s)} + \text{cost of bottom and top} \\ &= Spxh\csc\theta + (T + B)Ax^2, \end{aligned}$$

$$(2)\qquad \text{volume} = Ax^2 h.$$

For maximum volume with given cost, (2) is the objective function, so at the optimum the derivative of the volume is zero. In this case, (1) is the constraint; since the cost is given as a fixed constant, its derivative is also zero. Conversely, (1) may be the objective function with (2) as constraint. In either case, the derivatives of both are zero at the optimum (values at optimum are shown by capital letters):

$$(3)\qquad (Sp\csc\theta)(H + Xh') + 2(T + B)AX = 0$$

$$(4)\qquad A(2XH + X^2h') = 0$$

From (4), $Xh' = -2H$. Substituting this in (3), transposing, and multiplying by X, we have

$$(Sp\csc\theta)XH = 2(T + B)AX^2,$$

which is the desired result.

Example. For a right circular cylinder, let x be the radius of a base. Then the perimeter multiple p is 2π, and the area multiple A is π. Since the angle θ is a right angle, its *cosecant* is 1. For a prism with a regular n-gon for base, let x be half the length of one edge of the n-gon. Then p is $2n$ and A is $n \operatorname{ctn}(\pi/n)$.

The theorem can be used to optimize relative to surface area rather than cost of area materials, if all unit costs are set equal to 1. For open (no top) cylinders, T is set equal to zero.

Cylinders in 2-Space.

What happens if the 3-dimensional cylinder, with its 2-dimensional bases, is replaced by a 2-dimensional "cylinder" (actually a parallelogram), with 1-dimensional bases (line segments of length x)? It turns out that the multiple is not 2, but 1; that is, in optimal 2-dimensional cylinders, the resources spent on the sides amount to 1 times the resources spent on the bottom and the top,

Hypercylinders.

The result can be generalized to higher dimensions.

Theorem 2. For n an integer greater than 1, let the base of a cylinder in n-space have "perimeter" $p x^{n-2}$ and "area" $A x^{n-1}$. The "volume" is proportional to $h x^{n-1}$. Then the cylinder with maximum volume for given cost and the cylinder with minimum cost for given volume require a ratio of n-1 between the cost of the "sides" and the cost of the bases.

Proof. The objective function and the constraint, in either order, are

$$\text{(5)} \qquad \text{cost} = S p x^{n-2} h + (T + B) A x^{n-1},$$

$$\text{(6)} \qquad \text{volume} = A x^{n-1} h.$$

When the derivatives of both functions are set equal to zero at the optimum, the result follows as in Theorem 1.

JOB-SEARCH IDEAS FOR MATH MAJORS (AND THEIR MENTORS, FAMILIES, AND FRIENDS) ©

by Patricia Clark Kenschaft
Montclair State College

If you have a college degree in electrical engineering or accounting or hotel management or elementary education, you know the pigeonhole in which to look for a job. Your education has prepared you for a specific career path. You are set as long as you are content with the work, the available income, and the philosophical implications of your career -- assuming the job market doesn't evaporate in your field. You know what you are prepared for, and college graduation involves merely seeking an acceptable starting niche.

If, however, you have just earned a bachelor's or master's degree in mathematics (or English, or history, or French, or philosophy), you are now faced with overchoice. You have more life-time flexibility than someone with a vocational degree, but your career trail is not so clearly blazed. "Flexibility" and "overchoice" are two sides of the same coin. You have prepared yourself for many careers, and now, because you live in a complex, specialized society, you must choose one.

"I'll choose any that will take me!" is often the quaking inner response. "Who wants me?" Who wants you often depends partly on how you sell yourself, just as it does for an electrical engineering applicant. However, in both cases it also depends partly on luck. Employers might buy your services for any of the following, among other, possibilities.

Teaching. The national need for math teachers is soaring and will continue to do so for the next decade. It helps if you have certification in at least one state, but, if you didn't accomplish this in college, there are several options still open. Private schools as a rule do not expect certification, but they pay less and often expect a larger chunk of your personal life than public schools. They often also hire part-timers (for a pittance).

There are several MAT (Masters of Arts in Teaching) programs available at fine universities if a lifetime in the most desirable teaching positions might be worth a year's preparation.

New Jersey is pioneering an "Alternative Certification" program that accepts people with degrees in mathematics (and other subject areas) and requires courses and supervision while you work full time as a teacher. You need a school system that will hire you and supervise you to enter the program, and the first year is grueling, but the teacher training is meaningful because you are using it as you get it.

Programming. Programing jobs are the entry to many other computer careers. You do a year or two of programming and then move into systems analysis, management, in-house teaching or some other computer-related work that uses your mathematical background. You may not use the actual facts that you learned in your undergraduate major, but you employ the discipline of problem analysis and solving. "Engineer" is a title often given to someone in computers, many of whom were once mathematics majors.

Several undergraduate computer courses are essential for entering this career path, one with many variations. If you have had only a few, don't undersell their value. It is your ability to think mathematically that will serve your employer in both the short run and the long run. You probably will have to learn a new computer language, at the least. It is your ability to learn and to teach yourself that is your strong point. Sell it! Don't be afraid to say that you want to program for a year or two and then move up in the hierarchy, either in a specific direction or as needed by the company.

Insurance. Insurance companies employ many mathematicians. The traditional route has been the actuarial ladder, and it helps if you took the first one or two of the ten actuarial exams in college. They are given in May and November of each year, and it is rumored that the pass-rate is better in November.

However, the exams are not essential, now or later. If you like working with numbers and patterns, you can inquire about entry level jobs in insurance companies. It is wise not to close off the possibility of taking the exams, especially at first. However, you should be aware that failing an exam several times -- and, therefore, being closed out of the later ones -- need not ruin your life, There are satisfying alternative jobs in insurance.

Statistics. Statisticians are plentiful in the insurance, telephone, and pharmaceutical industries. Their judgements are also crucial in four steps of environmental protection, (1) the process of making laws, as well as (2) obeying them, (3) enforcing them, and (4) prosecuting violators. The highest salaries are available in the second group because private industry is the employer.

Applied Mathematics. Other applied mathematics jobs are available supporting scientists, engineers, computer scientists, biologists, and economists if you take a strong minor in another field, but these do not seem to be as plentiful as the other categories.

Accounting. Accounting jobs are available for math majors both in accounting firms and in the accounting departments of many corporations. Your employer may want you to have had or to take some accounting courses to supplement your math background, but having completed a math major proves you aren't afraid of demanding work. If you like numbers and are accurate, this may be your niche.

One math graduate wrote that getting an MBA is repetitive after a math major. Others comment on how easy it was. Your math background will stand you in good stead if you take this route.

Operations Research. Operations research is a burgeoning field. This is the mathematical study of efficiency. Math graduates may plan efficient inventory management, assignment of employees, space usage, or portfolio allotment. They can look for ways to make workers more efficient, more comfortable and more motivated.

It helps to have had a course or courses in operations research, math modelling, linear programming, statistics, and/or data analysis. However, these can be graduate courses, or the material can be self-taught. You have to convince your prospective employer, however, that you really want to do the work.

Business. Businesses and banks often perceive math graduates as people who have proved themselves smart and hardworking. Indeed, you are! If you want a job in sales or entry level administration, don't let a peon in the personnel department de-select you because you aren't a business major. Many company executives would prefer your qualifications. You need to reach them.

How to get started? Ask! Ask! Ask! Ask people what is available, both specifically near them and in general. Ask anyone you know and then ask them to give you suggestions of others with whom you can talk. Find some career paths that seem tolerable, and learn all you can about them. Go to a library and read professional journals. Show yourself you can educate yourself.

Which do you want? Maybe they all sound acceptable. Then make yourself seem enthusiastic about each as you approach acceptable employers in that field. Don't be afraid to use more than one vita, each emphasizing an aspect of you. It is important, however, as you approach any prospective employer to appear genuinely enthusiastic about that particular job. This need not involve dishonesty. You may well be enthusiastic about more than one career path. Just make sure you know something about any one you pursue.

"Getting a job is a job in itself." It usually takes several months. Since you will have other distractions in your last semester in college, it helps to begin earlier than that if you want to have a professional job the summer after graduation. If you didn't do that, and can't turn back the clock, resign yourself to the fact that this is not going to be a quick process. It takes time, but it's worth it. Your future career is in the balance.

"Eighty percent of job openings are never advertised," goes another saying. The grapevine is important. Hone your telephone skills. Some college placement services are extremely helpful to their students and alumni. Employment agencies also can help. Consult your telephone book. However, only those professionals who have some comprehension of what mathematics is will be helpful in finding you suitable openings.

It is common, perhaps inevitable, to feel useless, depressed and altogether discouraged when searching for a job. The emotional crunch is serious. Unless you have friends, family, and mentors who keep reassuring you, it is hard to remember that "This too shall pass" and that math majors almost always end up with satisfying careers. In the crucible, one always wonders, "Will I be the exception?"

Probably not! You are now facing a major disadvantage of capitalism. If you lived in a Communist country, you would be assigned a job, and whether or not you liked it, whether or not your boss liked you or your work, it would be yours. This too has its disadvantages. Communist leadership tells its people that Americans have the "freedom to be unemployed." This is true, and unpleasant. You also have, however, the freedom to choose a job and to be individually chosen. It takes time. There is no powerful mechanism helping you, except the intrinsic need of the world for diligent, competent workers. You want to be, and can be, one of those workers. If you don't lose hope, you will be.

ABBREVIATED BIBLIOGRAPHY

1. *The CPC Annual, A Directory of Employment Opportunities for College Graduates in Engineering, Science, The Computer Field, and Other Technical Options*, published by the College Placement Career Council, Inc., 62 Highland Avenue, Bethlehem, PA 18017 (telephone: 215-868-1421) is a 500+-page compendium of American employers actively seeking technically educated college graduates, listing current job openings and the people to contact for each.

2. *What Color is Your Parachute?* by Richard Nelson Bolles (Ten-speed Press, 1987) is a best-selling "practical manual for job-hunters and career-changers" that has been successfully used by millions of readers. It describes in detail networking techniques, briefly mentioned above.

3. "Mathematics Multiplies Career Options," Summer, 1988, *Journal of Career Planning & Employment*, is written for college career counselors. It summarizes my research about the graduates of my own department and challenges widespread myths about mathematics.

Editor's Note

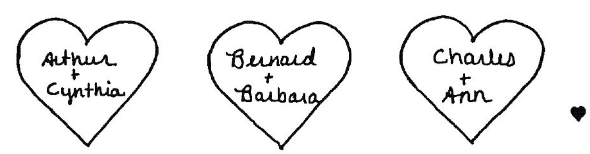
For information on career opportunities in mathematics and computing with the National Security Agency see the advertisement on page 632.

* *

During the 1988 Spring Semester, I gave one of my classes this very old chestnut from Maurice Kraitchik's* MATHEMATICAL RECREATIONS, *W. W. Norton & Co., Inc., New York, 1942:*

> *Three men - Arthur, Bernard and Charles - with their wives - Ann, Barbara and Cynthia - make some purchases. When their shopping is finished each finds that the average cost in dollars of the articles he or she has purchased is equal to the number of his or her purchases. Arthur has bought 23 articles more than Barbara, and Bernard has bought 11 more than Ann. Each husband has spent $63 more than his wife. Who is the husband of whom?*

After presenting a carefully reasoned solution, Rebecca Fee drew the following charming summary. Note the variant of the Halmos-phomofed tombstone.



R. Fee

* *The problem appears in the LADIE'S DIARY for 1739-40 and might even be older.*

Editor

DIVERGENCE IS NOT THE FAULT OF THE SERIES

by F. C. *Leary*
St. Bonaventure University

$$\sum_{k=0}^{\infty} 2^k = -1.$$

Nonsense, right? Everyone knows that the formula $\sum_{k=0}^{\infty} ar^k = a/(1 - r)$ is valid if and only if $|r| < 1$. Unfortunately, this conditioned response to a series of real numbers fails to consider the possibility that we may not be talking about series the way we did in elementary calculus. Consider the following situation. In an n-bit machine using two's complement notation for integer arithmetic (essentially, arithmetic mod 2^n), -1 is represented as a string of n ones. This string represents the decimal integer $1 + 2 + 2^2 + \ldots + 2^{n-1}$. Since n is arbitrary, we seem to have evidence to support convergence of the series. Which conclusion is correct? Both, it turns out, because we are playing by a different set of rules in each case.

In calculus, we study the infinite series $\sum_{k=0}^{\infty} a_k$ via its sequence of partial sums $s = \{s_n\}$ where $s_n = \sum_{k=0}^{n-1} a_k$. If this sequence has limit A, then A is called the sum of the series and we say the series converges; otherwise the series diverges. Our notion of convergence relies on the absolute value to measure the "closeness" of a partial sum s_n to X. In fact, we say that the sequence converges to λ if given any $\epsilon > 0$, there exists a positive integer N such that $|s_n - \lambda| < \epsilon$ whenever $n \geq N$. Absolute value measures the physical closeness of s_n and λ on the number line, so in this setting the mathematical notion of close corresponds to the physical notion, and $\sum_{k=0}^{\infty} 2^k$ diverges.

If we want this series to converge, we're going to need a different way to measure "close." Observe that each partial sum of our series is an integer. The Fundamental Theorem of Arithmetic assures us that any integer n $\neq 0, \pm 1$ can be written as a product of primes $n = \pm p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, where the distinct positive primes p_i, their exponents, and the sign of the product are uniquely determined by n. Let p be a positive prime and define the p-order of n, denoted by $ord_p(n)$, to be the exponent of p in the prime factorization of n. Thus, $ord_p(n)$ is the greatest exponent for which p^a divides n and is zero if p does not divide n. Define the *p-adic* absolute value of n as $|n|_p = e^{-ord_p(n)}$. We set $|0|_p = 0$. It is not difficult to show that $|\ |_p$ has the following properties.

1. $|n|_p \geq 0$ for all integers
2. $|n|_p = 0$ if and only if n = 0
3. $|nm|_p = |n|_p |m|_p$
4. $|n + m|_p \leq |n|_p + |m|_p$.

Thus, $|\ |_p$ behaves much like the ordinary absolute value. In fact, we can define the p-adic distance between two integers n and m by $|n - m|_p$. Using this distance to measure closeness, we mimic our previous definition of convergence and say that the sequence of integers $s = \{s_n\}$ converges to λ in the p-adic sense if given $\epsilon > 0$ there exists a positive integer N such that $|s_n - \lambda|_p < \epsilon$ whenever $n \geq N$.

For our purpose, take $p = 2$. The n-th partial sum of the series is

$$s_n = \sum_{k=0}^{n-1} 2^k = 2^n - 1.$$

Thus,

$$ord_2(s_n - (-1)) = ord_2(2^n) = n$$

and

$$|s_n - (-1)|_2 = e^{-n}.$$

Since e^{n} has limit 0 (in the usual sense) as n tends to infinity, the sequence of partial sums converges to -1 and so $\sum_{k=0} 2^k = -1$.

Both ord_p and $|\ |_p$ can be defined over the rational numbers Q and, ultimately, over the field of p-adic numbers Q_p, the completion of Q with respect to the p-adic metric. In any event, restricting to the integers provides a simple and convenient example to illustrate the fact that changing the distance measuring tool (metric, in the vernacular) may very well change the collection of sequences and series that converge. So, the moral of the story is this: when presented with a sequence or series, make no assumptions about convergence or divergence until the measuring tool is known.

MULTIPLIER PROBLEM REVISITED

by Oliver D. Anderson
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William M. Perel [1] stated the following problem:

> "Given a single digit, is it always possible to find a natural number such that the product of the number and the single digit will have the same digits as the original number, in the same order, except that the first digit of the product will be the last digit of the original number?"

Perel interpreted this as allowing the number to have a leading zero; and he proceeded, via linear congruence theory, to identify appropriate numbers for all the digits (zero and unity being trivial cases).

We believe that an alternative approach, based on the knowledge of repeating decimals, gives such numbers slightly more quickly - although, pedagogically, Perel's approach may provide greater insight, if only one method is to be considered. (Compare Anderson [2].) However, we think the alternative outlined below is highly instructive, and (coupled with Perel's method) will yield increased understanding of such problems.

It is well known that the reciprocal of a prime number greater than five, p say, gives a repeating decimal with cycle length $c < p$; and, moreover, for integers r, $1 < r < p$, r/p will yield a repeating decimal which frequently has the same cycle as $1/p$, but starting at a different point. Thus, the problem posed by Perel immediately suggests that we look for numbers from among the cycles of digits which occur in such repeating decimals.

Common knowledge of the cycle for p = 7, "142857", and how those for $r/7$ relate to it, tells us we should immediately find a multiplier there. And, indeed,

$$5 \times 142857 = 714285 = 142857.$$

Of course, for any $p > 7$, $1/p < .1$, so an appropriate cycle will need to lead off with a zero, and its next digit will need to be at least as great as its last digit. For instance, the cycle for p = 13 is

"076923", and clearly this gives another multiplier, since

$$4 \times 076923 = 307692 = 076923.$$

But, looking at the next prime 17, we get the cycle as "0588235294117647" which is not appropriate, as 5 < 7. However, when $p = 19$, we get

$$2 \times 052631576947366421 = 105263157894736842 = \underline{052631578947368421}.$$

Continuing like this, by scanning down a table of prime repeating cycles, we obtain the following table:

Digit	Prime	Multiplier
2	19	052631578947368421
3	29	0344627586206696551724137931
4	13	076923
5	7	142857
6	59	0169491525423728813559322033898305084745762711864406779661
7	23	0434782608695652173913
8	79	0126562278481
9	89	01123595505617977526089867640449436202247191.

The form of the table immediately suggests that, for the digits 4, 5 and 7, we also try their respective multiples 39, 49 and 69; and these indeed give alternative multipliers of, respectively, 025641, 020408163265306122448979591836734693877551 and 0144927536231864057971.

Replacing the three rows in our table with these values effectively retrieves Perel's table (which omits the leading zeros and trailing unities), apart from a typographic error at the end of his digit 6 multiplier. But note a mistake with Perel's table. Perel was looking for the smallest (positive) multiplier in each case, and clearly our table gives a smaller number for the digit 5 (142857, as opposed to 0204081632653061224489795918367346936775 51). We leave it for the interested reader to find the slip in Perel's argument.

In fact, it is clearly unnecessary to restrict the problem to single digits. Given any integer i, a satisfactory multiplier would be the repeating cycle corresponding to $1/(10i - 1)$. For instance, when $i = 10$, the multiplier would be 01 giving 10 x 01 = 10 = 01. But, if we are looking for the smallest multipliers, then we should also consider trying the repeating cycles for the reciprocals of any proper divisors of $(10i - 1)$, when this is composite.

REFERENCES

1. Perel, William M., *A Multiplier Problem*, Pi Mu Epsilon Journal, Vol. 8, No. 8 (1988) 518-519.
2. Anderson, Oliver D., *On Littlewoods's Little Puzzle*, Teaching Mathematics and its Applications 7, to appear.

LETTERS TO THE EDITOR

Dear Editor,

In the article "A Multiplier Problem" by William M. Perel, pages 518-519 of the Spring 1988 Pi Mu Epsilon Journal, there is a special case which has fewer digits than given in this article.

Solving (1) for N: $N = \dfrac{h(10^n - m)}{10m - 1}$, which for $m = 5$, $h = 7$, $n = 5$ gives $N = 14285$ and $5(142857) = 714285$.

It is possible to find fractional values of m which yield similar results. For example, if $m = .8$, $h = 5$, $n = 5$, then $N = 71428$ and $.8(714285) = 571428$.

It is also possible to move a digit from first to last instead of vice-versa. For example, $3(142857) = 428571$ and $3(285714) = 857142$. Here, too, fractional values of m are possible. For example, $1.5(3529411764705882) = 5294117647058823$.

And, some fractional values are ambidextrous: $1.2(45) = 54$, $1.2(4545) = 5454$, etc., $3.4(15) = 51$, $3.4(1515) = 5151$, etc.

John M. Howell
Littlerock, CA

Dear Editor,

In response to "Lies, Spies, AIDS, and Drugs" in the Fall 1987 issue, I would like to comment on California's program of testing for TB. For many years California has required testing of all teachers. This is done by giving a patch test on the arm, a simple and inexpensive test. I have no figures on the results, but if we assume that only .1% of those tested actually have TB, that these people will all test positive on the patch test, and that 5% of those without TB will test positive, then for each 1000 tested, we will have positive results for 50 people who do not have TB and one who does. This is then followed by x-rays of all who test positive, a relatively expensive test and one which subjects people to radiation.

The patch test saves having to x-ray 95% of the teachers, clearly a desirable result. In general, I would expect that a positive test in any of the cases mentioned by Mr. Brunson would be a preliminary result and would be followed by further testing before any decision was made.

Yours truly,

Henry J. Osner
Mathematics Instructor
Modesto Junior College
Modesto, CA 95350

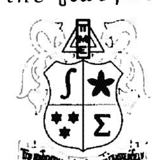
Dear Editor,

One of Good's laws is that if I don't see the page proofs of an article there is nearly always a misprint, the average number per article being 2 1/2 (see J. Statist. Planning and Inference 18, 1988, p. 134 for more details). This law was again verified in my article "A common misuse of 'denoted'", in the Journal, Vol. 8 (Spring 1988), p. 520, line 2 of the text where "large majority of scientists" should read "large minority of scientists." It is somewhat embarrassing to have misprints in articles concerning English style!

Yours sincerely,

I. J. Good
University Distinguished Professor of Statistics
Adjunct Professor of Philosophy
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061

Here is a copy of the Pi Mu Epsilon shield. What is the motto of the Society? Its colors? What do the four parts of the shield represent?



Pi Mu Epsilon, Inc.

NATIONAL HONORARY MATHEMATICS SOCIETY

The answers to these questions and much more will appear in the special Spring 1989 issue of this journal.

PUZZLE SECTION

Edited by

Joseph D. E. Konhauser

*The PUZZLE SECTION is for the enjoyment of those readers who are addicted to working doublecrostics or who find an occasional mathematical puzzle attractive. We consider **mathematical** puzzles to be problems whose solutions consist of answers **immediately** recognizable as correct by simple observation and requiring little formal proof. **Material submitted** and not used here **will** be sent to the Problem Editor if deemed appropriate for the **PROBLEM DEPARTMENT**.*

*Address **all** proposed puzzles and puzzle solutions to Professor Joseph D. E. **Konhauser, Mathematics and** Computer Science **Department,** Macalester College, St. **Paul,** Minnesota 55105. **Deadlines** for puzzles **appearing in** the Fall Issue will be the next February 15, and for the puzzles appearing in the **Spring** Issue **will** be the next September 15.*

PUZZLES FOR SOLUTION

1. *Proposed bq John M. Howell, Littlerock, CA.*

In different bases, reciprocals of numbers repeat after different numbers of "decimal" places. For example, 1/7 repeats after six places in base 10, after one place in base 8 and after three places in bases 2, 4 and 16. Find a prime number (less than 100) which repeats after the same number of places in bases 10, 2, 4, 8 and 16.

2. *Proposed by the Editor.*

For which isosceles triangles ABC, with AB = AC, is there a line segment MN, with M on AB and N on AC and MN parallel to BC, which separates the interior of triangle ABC into two parts of equal area and separates the triangle ABC into two pieces of equal length (that is, MA + AN = MB + BC + CN)? See Figure I.

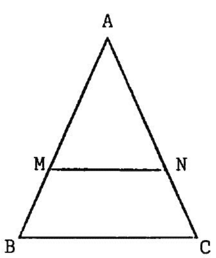


Figure I

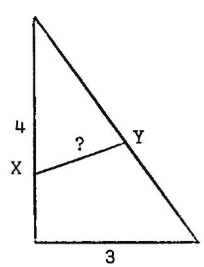


Figure 2

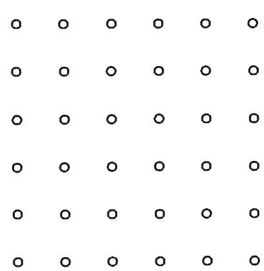


Figure 3

3. *Proposed by the Editor.*

On the 3-4-5 triangle in Figure 2, locate points X and Y so that the line segment XY separates the interior of the triangle into two parts of equal area and so that the line segment is as short as possible.

4. *Contributed.*

Given the 6 x 6 square array of points in Figure 3, is it possible to color 18 of the points red and the other 18 blue so that no four points of the same color are vertices of a square? (Don't forget to consider "tilted" squares.)

5. *Contributed.*

If five x's and four o's are placed at random in a 3 × 3 arrangement of nine squares, what is the probability that some row, column or diagonal will contain only squares marked o?

6. *Proposed by the Editor.*

Find a rule of formulation for the 3 × 3 square array in Figure 4.

1	6	2
5	4	9
8	7	3

Figure 4

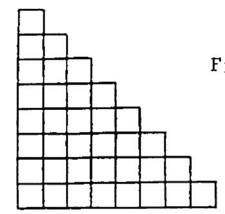


Figure 5

7. *Proposed by the Editor.*

Dissect the staircase-like piece in Figure 5 into three pieces which can be reassembled to form a square.

COMMENTS ON PUZZLES 1 - 7, SPRING 1988

No responses were received for *Puzzle # 1.* The key to the solution is to interpret the given table (see Figure 6) as a "mileage" chart, then its entries are the straight-line distances between the points which are labelled ■ through 6 on the 2 × 1 × 1 rectangular solid in Figure 7.

	1	2	3	4	5
2	1				
3	1	$\sqrt{2}$			
4	2	1	$\sqrt{5}$		
5	1	$\sqrt{2}$	$\sqrt{2}$		
6	$\sqrt{2}$	1		$\sqrt{2}$	

Figure 6

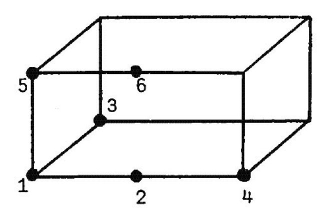
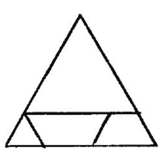


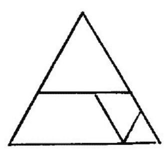
Figure 7

Puzzle # 2 drew eighteen correct responses. The unique solution is $\{-3, 3, 9, 14, 23\}$. Here is **Thomas Mitchell's** argument: Let $S = \{a, b, c, d, e\}$, with $a < b < c < d < e$, be the set in question. If we add the pairwise sums we obtain a total of 184 and this total includes each element of S four times. Hence $a + b + c + d + e = 184/4 = 46$. Since zero is the smallest of the pairwise sums, it must be the sum of the two smallest elements of S: $a + b = 0$. Similarly, 37 is the largest of the pairwise sums so it must be the sum of the two largest elements of S: $d + e = 37$. If $a + b = 0$, $d + e = 37$, and $a + b + c + d + e = 46$, then $c = 9$. The second smallest of the pairwise sums is 6 and it must be the sum of the first and third smallest elements of S: $a + c = a + 9 = 6$; i.e., $a = -3$, from which $b = 3$ since $a + b = 0$. Similarly, the second largest of the pairwise sums is 32 and it must be the sum of the first and third largest elements of S: $c + e = 9 + e = 32$; i.e., $e = 23$, from which $d = 14$ since $d + e = 37$. Collecting the results: $S = \{-3, 3, 9, 14, 23\}$.

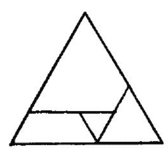
Seven readers submitted the four different solutions to *Puzzle # 3* which are shown below.



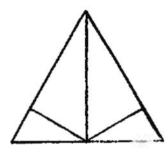
Duisenberg
Howell
Sipka



Duisenberg
Feser
Hess



Slowinski



Tang

For *Puzzle # 4*, **Bill Boulger** and **Mark Evans**, respectively, submitted as "shortest" sequences containing each of the 31 non-empty subsets of $\{a, b, c, d, e\}$ as consecutive elements at least one time *badeabcdeacdbec* and *ebceacdeabcdeabdead*. A shorter sequence satisfying the conditions of the puzzle is *abcdeabdacebd*.

Seven readers, responding to *Puzzle # 5*, showed that 200 is the smallest positive composite integer which cannot be changed into a prime by changing exactly one digit. Here is the argument of **Charles Ashbacher**: Obviously, the number cannot be one digit in length. If we look at all numbers two digits in length, we can make a prime by changing one digit since there are primes that have as leading digit any possible choice for that digit. If we then consider the three-digit numbers, we need a composite number such that no prime has its two leading digits. The first such choice is 20_, as there are no primes that begin with 20. From this it follows that 200 cannot be made into a prime by changing the unit's digit. **David Ehren** remarked that after 200 the next three numbers with the same property are 320, 510 and 530.

For *Puzzle # 6* nine readers submitted 1234759680 as the smallest positive integer consisting of 0 through 9, each used once, which is divisible by each of the digits 2 through 9. The argument of **Bill Boulger** goes as follows: The last digit must be 0 or 5 so that the number is divisible by 5. The last digit must also be even so that the number is divisible by 2. This makes the last digit 0. The last two digits must be divisible by 4 and the last three by 8. The largest three-digit number for which all of the above conditions will be true is 680. These are the last three digits of the number we seek. Since the sum of all ten digits 0 through 9 is 45, any arrangement of digits will be divisible by both 3 and 9. In addition, an even number which is divisible by 3 is divisible by 6. The remaining divisor is 7. Experimenting with positions for the remaining digits which place smaller digits to the left and larger ones to the right gives 1234759680 as the required number.

No correct responses were received for *Puzzle # 7*. To obtain a solution, on each edge of an equilateral triangle of edge length $2\cos 15^\circ$, describe inward and outward isosceles triangles with angles 15°-150°-15°, then the three vertices of the equilateral triangle and the six 150°-vertices of the isosceles triangles comprise a set of nine points such that each point of the set is at a unit distance from exactly four other points in the set.

List of respondents: **Valerie Albano** *(2)*, **Dr. Steve Ascher** *(2)*, **Charles Ashbacher** *(2,5,6)*, **Jeanette Bickley** *(2)*, **Bill Boulger** *(2,4,5,6)*, **Ken Duisenberg** *(2,3,6)*, **David Ehren** *(2,5)*, **Mark Evans** *(2,4,6)*, **Victor G. Feser** *(2,4,5)*, **Richard I. Hess** *(2,3,5,6)*, **Diane L. Howard** *(5)*, **John M. Howell** *(2,3)*, **Michael J. Lenart** *(2)*, **Patrick P. T. Leong** *(2,6)*, **Thomas Mitchell** *(2,6)*, **Jason Pinkney** *(2)*, **Bob Prielipp** *(5)*, **Timothy Sipka** *(2,3,6)*, **Emil Slowinski** *(2,3,6)*, and **Chun Tang** *(2,3)*.

Solution to Mathacrostic No. 26. (See Spring 1988 Issue.)

Words:

A. jutty	K. oddsmen	U. episcope
B. gets it	L. stretto	V. white dwarf
C. lattice	M. mother wit	W. starshaped
D. elbow	N. Avebury Rings	X. crows foot
E. idiophone	O. Karnaugh map	Y. intortion
F. cottier	P. isthmus	Z. Eudoxus
G. knotted	Q. nitid	a. noddy
H. Carnot engine	R. Gudermann	b. cornuted
I. Horsehead	S. adjustment	c. enstatite
J. Ananta	T. nuts	

Quotation: *The new geometry mirrors a universe that is rough, not rounded, scabrous, not smooth. It is a geometry of the pitted, pocked , and broken up, the twisted, tangled, and intertwined. The understanding of nature's complexity awaited a suspicion that it [the complexity] was not just random, not just accident.*

Solved by: **Jeanette Bickley**, Webster Groves High School, MO; **Betsy Curtis**, Saegertown, PA; **Charles R. Diminnie**, St. Bonaventure University, NY; **Victor G. Feser**, University of Mary, Bismarck, ND; **Robert Forsberg**, Lexington, MA; **Meta Harrsen**, Georgian Court College, Lakewood, NJ; **Joan Jordan**, Indianapolis, IN; **Dr. Theodor Kaufman**, Brooklyn, NY; **Henry S. Lieberman**, Waban, MA; **Charlotte Maines**, Rochester, NY; **Don Pfaff**, University of Nevada, Reno, NV; **Stephanie Sloyan**, Georgian court College, Lakewood, NJ; **Michael J. Taylor**, Indianapolis Power and Light Co., IN; **Steven H. Weintraub**, Mathematisches Institut Universitat Bayreuth, Fed. Rep. Germany; **H. J. Michiel Wijers**, The Netherlands; **Barbara Zeeberg**, Denver, CO.

Definitions	*Words*
A. one category of Escher's work relating to infinity (2 wds.)	198 156 69 146 95 21 51 4 218 193 130 36 137
B. borderline	100 67 35 211 61 28 5 52 142 151
C. twist together	113 144 171 131 166 94 23 37 65
D. one of two equal parts	63 8 77 33 188 147
E. a source of terror	56 205 16 41 192 32 210 1
F. energy	204 124 111 64 161
G. a basic tool for testing our notions about the universe (appears as one word, as one word hyphenated, and as two words)	98 183 105 214 90 30 107 92
H. transverse	169 2 87 186 47 173
I. what a snap film spanning a given configuration assumes (2 wds.)	34 216 116 168 24 17 57 200 62 129 74 12
J. a very small quantity	91 15 160 178
K. a target of Bishop George Berkeley in his 1734 essay <u>The Analyst</u> (2 wds.)	136 42 125 123 106 81 31 165 212 155 195 25 68 109 83
L. a morphism of graphs	150 133 164 40 29 110 58
M. also known as ducks, stickers, dibs, hoodles and nibs	20 138 6 189 149 208
N. large above and small or slender below	9 46 181 158 153 202 38 128
O. an adjective used to describe a planar set of non-intersecting polygons with edges parallel and interiors disjoint	55 140 121 89 11 75 174 39 101
P. the four Hebrew letters יהוה, usually transliterated into YHWH, that form the Biblical proper name of God	191 175 141 102 27 43 152 99 184 86 53 207 115 167
Q. along with barium and copper oxide, an ingredient of certain high temperature superconductors	50 119 10 66 206 139 134
R. pointed	135 26 44 13 163 118 85 78 103
S. a renal calculus	96 170 196 132 72 213 18 82 54 126
T. the name of a theorem in plane projective geometry whose converse is its dual	157 73 93 108 187 120 143 3 199
U. ______/Bach = mathematics/music	145 162 176 159 127 180 45 84 201
V. to blanch, as by exclusion of sunlight	185 49 79 215 19 172 97 117
W. the primordial mix from which the elements were supposed to be formed	60 22 194 7
X. entrancing	80 197 114 217 88 71
Y. inventor of an electric lamp superseded by Edison's carbon filament lamp (1864 - 1941)	177 14 182 203 190 76
Z. an irregular tetrahedron a model of which is obtained by folding an acute-angled triangle along the line segments joining the midpoints of its sides	122 48 179 209 59 104 112 154 70 148

Editor's Note - Mathacrostic No. 27 was prepared by Carl E. Gadow on a Sun 3/50 Workstation using MetaForm Professional by Intran Corporation.

Mathacrostic No 27

Proposed by Joseph D. E. Konhauser

The 218 letters to be entered in the numbered spaces in the grid will be identical to those in the 26 keyed Words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters of the Words will give the name(s) of the author(s) and the title of a book; the completed grid will be a quotation from that book.

The solution to *Mathacrostic No.* 26 is given elsewhere in the PUZZLE SECTION.

1 E	2 H	3 T		4 A	5 B	6 M	7 W	8 D	9 N		10 Q	11 O	12 I
13 R	14 Y		15 J	16 E		17 I	18 S	19 V		20 M	21 A	22 W	23 C
24 I	25 K	26 R		27 P	28 B	29 L		30 G	31 K		32 E	33 D	34 I
35 B	36 A	37 C	38 N	39 O	40 L		41 E	42 K	43 P	44 R	45 U	46 N	47 H
48 Z	49 V	50 Q		51 A	52 B	53 P	54 S	55 O	56 E	57 I		58 L	59 Z
60 W	61 B	62 I	63 D		64 F	65 C	66 Q	67 B	68 K	69 A	70 Z	71 X	
72 S	73 T	74 I	75 O	76 Y	77 D	78 R	79 V	80 X	81 K		82 S	83 K	84 U
85 R	86 P		87 H	88 X	89 O	90 G		91 J	92 G	93 T		94 C	95 A
96 S	97 V	98 G	99 P	100 B		101 O	102 P	103 R	104 Z	105 G		106 K	107 G
	108 T	109 K		110 L	111 F	112 Z	113 C	114 X	115 P	116 I	117 V	118 R	119 Q
	120 T	121 O	122 Z		123 K	124 F		125 K	126 S	127 U	128 N		129 I
130 A	131 C		132 S	133 L	134 Q	135 R	136 K	137 A		138 M	139 Q	140 O	141 P
	142 B	143 T	144 C	145 U	146 A	147 D		148 Z	149 M	150 L	151 B	152 P	
153 N	154 Z	155 K	156 A	157 T		158 N	159 U		160 J	161 F	162 U		163 R
164 L	165 K	166 C	167 P	168 I	169 H	170 S		171 C	172 V	173 H	174 O	175 P	176 U
177 Y		178 J		179 Z	180 U	181 N	182 Y	183 G	184 P	185 V		186 H	187 T
188 D	189 M	190 Y	191 P	192 E	193 A		194 W	195 K	196 S	197 X	198 A	199 T	200 I
201 U	202 N	203 Y		204 F	205 E		206 Q	207 P	208 M		209 Z	210 E	211 B
212 K	213 S	214 G	215 V	216 I	217 X	218 A							

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math. Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by July 1, 1989.

Problems for Solution

678. *Proposed by Brian Conrad, Centereach High School, Centereach, New York.*

Find all solutions to this base ten multiplication alphametric in honor of my Soviet mathematician and theoretical physicist pen pal who also is a regular contributor to this department:

$$DMITRI = P \cdot MAVLO.$$

679. *Proposed by Dmitry P. Mavlo, Moscow, U. S. S. R.*

a) Prove this inequality for positive real numbers U, S, and A, dedicated to 100 years of American mathematics, as evidenced by the 100th anniversary of the American Mathematical Society:

$$\frac{U}{(1+U)(1+S)} + \frac{S}{(1+S)(1+A)} + \frac{A}{(1+A)(1+U)} \geq \frac{3USA}{(1+USA)^2},$$

with equality if and only if $U = S = A = 1$.

b) Which inequality, if either, is more general, the *USA* inequality of part a) or the $\pi\mu\varepsilon$ inequality of Problem 642 [Spring 1987, Spring 1988]:

$$(1+\pi\mu\varepsilon)\left[\frac{1}{\pi(1+\mu)} + \frac{1}{\mu(1+\varepsilon)} + \frac{1}{\varepsilon(1+\pi)}\right] \geq 3$$

for positive numbers π, μ, and ε, with equality if and only if $\pi = \mu = \varepsilon = 1$?

680. *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

A regular heptagon (seven-sided polygon) is randomly placed far from an observer. Find the probability that the observer can see four sides of the heptagon.

681. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Professor E. P. B. Umbugio is in the midst of writing his thirteen-volume treatise on analytic geometry. He would like to use the following theorem in Volume 9, but is having difficulty with it. Help the poor old professor by supplying a proof for him.

For $i = 1, 2, \ldots, n$, let P_i represent the plane

$$\frac{x}{a_i} + \frac{y}{b_i} + \frac{z}{c_i} = 1, \text{ where } 3a_ib_i + 3b_ic_i + 3c_ia_i = a_ib_ic_i.$$

Then the intersection of all the planes is nonempty.

682. *Proposed by Brian Conrad, Centereach High School, Centereach, New York.*

Find all the ordered pairs of nonzero integers a and b with b prime such that

$$a^3 - b^3 = a.$$

*683. *Proposed by Jack Garfunkel, Flushing, New York.*

a) Given three concentric circles, construct an isosceles right triangle so that its vertices lie one on each circle.

b) Is the construction always possible?

684. ***Proposed** by **Dmitry** P. **Mavlo, Moscow, U. S. S. R.***

This problem is dedicated to Paul Erdös on his 75th birthday. Erdös and Hans Debrunner published *(El. Math.* 11(1956)20) the following theorem: Let D, E, F be points on the interiors of sides BC, CA, AB of triangle ABC. Then the area $[DEF]$ of triangle DEF cannot be less than the smallest of the three other triangles formed:

$$[DEF] \geq \min\{[AEF], [CDE], [BFD]\}.$$

a) Prove this generalization of the Erdös-Debrunner inequality: Assuming the configuration of the Erdos-Debrunner inequality, for some fixed real number a^*, if $-\infty < a \leq a^*$, then

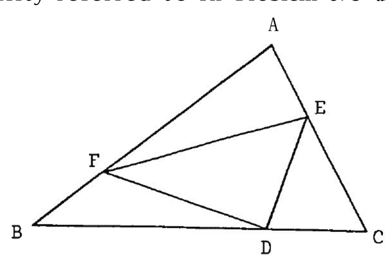
$$[DEF] \geq M^{(\alpha)}, \text{ where } M^{(\alpha)} = \left\{\frac{[AEF]^{\alpha} + [CDE]^{\alpha} + [BFD]^{\alpha}}{3}\right\}^{1/\alpha}$$

is the power mean of order a of the three positive areas $[AEF]$, $[CDE]$, and $[BFD]$.

b) Determine the maximum value of a^* for which the inequality holds.

c) Find all the cases where equality holds.

d) Prove that, for $a = -1$, the inequality of part a) is equivalent to the πμε inequality referred to in Problem 679 b) above.



685. ***Proposed** by R. S. **Luthar, University of Wisconsin Center, Janesville, Wisconsin.***

In any triangle ABC with $C < 45^{\circ}$ and given any other angle D with $0^{\circ} < D < 45^{\circ}$, prove that

$$b\cos D - c\cos(A - D) < a.$$

686. ***Proposed** by **Murray** S. **Klamkin, University of Alberta,** Edmonton, **Alberta,** Canada.*

Determine the matrix $[A - A + I]$, where A is an n by n matrix such that $A^5 + A = 5nI$ and I is the identity matrix.

687. *Proposed, by **Basil Rennie, Burnside,** South **Australia.***

For positive real numbers x and y, prove the "quaint little inequality"

$$4xy \leq (x + y)(xy + 1).$$

688. ***Proposed** by **Willie** Yong, **Singapore, Republic of Singapore.***

A row of n chairs is to be occupied by n boys and girls taken from a group of more than n boys and more than n girls. If the boys do not want to sit next to one another, in how many ways can the children occupy the chairs? *(This problem is taken from the Malaysian Math. Bulletin.)*

689. ***Proposed** by **Willie** Yong, **Singapore, Republic of Singapore.***

Show that for any three infinite sequences of natural numbers

$$a_1, a_2, a_3, \ldots, \quad b_1, b_2, b_3, \ldots, \quad c_1, c_2, c_3, \ldots$$

there can be found numbers p and q such that $a_p \geq a_q$, $b_p \geq b_q$, and $c_p \geq c_q$.

690. ***Proposed** by **David Iny, Rensselaer Polytechnic Institute, Troy,** N w **York.***

A unit square is covered by five circles of equal radius. Find the minimum necessary radius. See Problem 507 *[Fall 1982]*.

Solutions

633. *[Fall 1986, Fall* 1987] ***Proposed** by **Dmitry** P. **Mavlo, Moscow,** U. S. S. R.*

Let $a, b, c > 0$, $a + b + c = 1$, and $n \in N$. Prove that

$$\left[\frac{1}{a^n} - 1\right]\left[\frac{1}{b^n} - 1\right]\left[\frac{1}{c^n} - 1\right] \geq (3^n - 1)^3,$$

with equality if and only if $a = b = c = 1/3$.

II. ***Solution** by **Chris** Long, **Rutgers University, Nw Brunswick, New Jersey.***

Since $a + b + c = 1$, the left side of the stated inequality is equivalent to

$$(1) \quad (abc)^{-n}[(a + b + c)^n - a^n][(a + b + c)^n - b^n][(a + b + c)^n - c^n].$$

Upon expanding $(a + b + c)^n$ into 3 terms and applying the arithmetic-geometric mean inequality to each of $(a + b + c)^n - a^n$, and so on, in Expression (1), we get that

$$\frac{[(a+b+c)^n - a^n]}{3^n - 1}\cdot\frac{[(a+b+c)^n - b^n]}{3^n - 1}\cdot\frac{[(a+b+c)^n - c^n]}{3^n - 1} \geq (abc)^n$$

with equality if and only if $a = b = c$. The desired inequality now follows.

652. [Fall 1987] *Proposed by John M. Howell, Littlerock, California.*

Most people get their news from radio and television. Hence, solve this base 8 alphametric for the greatest *NEWS:*

$$\begin{array}{r} ABC \\ NBC \\ \underline{CBS} \\ NEWS \end{array}$$

I. *Solution by Tak Lee., James Madison High School, Brooklyn, New York.*

Since A, N, and C are nonzero and $C + C + S$ ends in S, then $C = 4$. To maximize SEWS, we should take N as large as possible. If $N = 2$, then

$$7 + 2 + 4 + 2 \geq A + N + C + (\text{carry}) \geq 20 \quad (\text{base } 8),$$

which is impossible. Hence, $N = 1$.

From the inequality above we also see that $E = 7$ requires that $A = 7$, too. Thus E cannot exceed 6. Taking $E = 6$, we get that $A = 7$ and we must carry 2 into the A-N-C column. Hence $B > 4$. Since 6 and 7 are already used, $B = 5$ and $W = 0$. Finally, we take S to be the largest value not yet used: $S = 3$.

The alphametric becomes 754 + 154 + 453 = 1603, so the greatest possible *NEWS* is 1603.

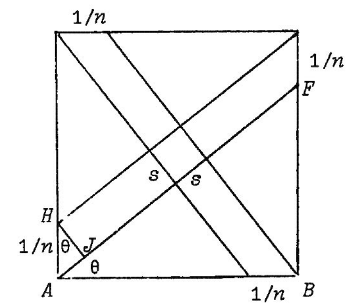
II. *Comment by Alan Wayne, Holiday, Florida.*

If *NEWS* is not restricted in size, there are twelve solutions in base eight, including the one above; no solutions in any odd base; and 48 solutions in base ten, of which the greatest is *NEWS* = 1638.

Also solved by CHARLES ASHBACHER, *Mount Mmcy College, Cedar Rapids, IA,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* JAMES E. CAMPBELL, *Indiana University at Bloomington,* MARK EVANS, *Louisville, KY,* VICTOR G. FESER, *University of Mary, Bismarck, ND,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* CARL LIBIS, *Granada Hills, CA,* MIKE PINTER and MARK C. SPRAKER, *Middle Tennessee State University, Murfreesboro,* WADE H. SHERARD, *Furman University, Greenville, SC,* KENNETH M. WILKE, *Topeka, KS,* ALAN WAYNE, *Holiday, FL,* and the PROPOSER. *Partial solutions were received from* DAVID EHREN, *University of Wisconsin, Milwaukee,* and THOMAS M. MITCHELL, *Southern Illinois University at Carbondale.*

*653. [Fall 1987] *Proposed independently by Robert C. Gebhardt, County College of Morris, Randolph, New Jersey, and Clifford H. Singer, Great Neck, New York.*

A small square is constructed inside a square of area 1 by marking off segments of length $1/n$ along each side as shown in the figure below. For $n = 4$ the side s of the small square is $1/5$. For what other positive integral n is s the reciprocal of an integer? (*This proposal is based on a 1985 AIME problem.*)



Solution by William H. Peirce, Stonington, Connecticut.

Let θ denote angle FAB, hence also angle AHJ in the figure. Then in the two right triangles FAB and AHJ we have that

$$\cos\theta = 1/((1 - 1/n)^2 + 1)^{1/2} = n/((n-1)^2 + n^2)^{1/2}$$

and

$$\cos\theta = \frac{s}{1/n} = ns.$$

Hence

$$s = 1/((n-1)^2 + n^2)^{1/2}.$$

Now s must be the reciprocal of an integer, say z. Then we have

$$(n-1)^2 + n^2 = z^2 \quad \text{for integral } z = 1/s.$$

By letting $x = 2n - 1$, we reduce this equation to

$$(1) \qquad x^2 - 2z^2 = -1,$$

which must be solved in positive integers x and z. By the theory of continued fractions, solutions can always be found, and two such solutions are $(x, z) = (1, 1)$ and $(7, 5)$. The general solution to Equation (1) is

$$x = \frac{(1+\sqrt{2})^r + (1-\sqrt{2})^r}{2}, \qquad z = \frac{(1+\sqrt{2})^r - (1-\sqrt{2})^r}{2\sqrt{2}}$$

where r is an odd positive integer. (The choice of $1 + \sqrt{2}$ as a generator is determined from $x + z\sqrt{2}$ where (x, z) is that solution of Equation (1) for which $x + z\sqrt{2}$ has no square root in the set of algebraic integers $a + b\sqrt{2}$, a and b integers.)

Since $n = (x + 1)/2$, the solution to the given problem becomes

$$n = \frac{(1+\sqrt{2})^r + (1-\sqrt{2})^r + 2}{4}, \qquad z = \frac{(1+\sqrt{2})^r - (1-\sqrt{2})^r}{2\sqrt{2}}$$

with r an odd positive integer. The first seven solutions are:

r	n	$1/s = z$
1	1	1
3	4	5
5	21	29
7	120	169
9	697	985
11	4060	5741
13	23661	33461.

Note that both x and z satisfy the linear homogeneous difference equation $f(r) - 6f(r-2) + f(r-4) = 0$ and also that n satisfies $n(r) - 6n(r-2) + n(r-4) + 2 = 0$, which, when used with the given initial conditions, will also produce the solutions listed above.

Also solved by SEUNG-JIN BANG, *Seoul, Korea*, WILLIAM BOULGER, *St. Paul Academy, MN*, JOHN DALBEC, *Youngstown, OH*, CHARLES R. DIMINNIE and HARRY SEDINGER, *St. Bonaventure University, NY*, RUSSELL EULER, *Northwest Missouri State University, Maryville*, RICHARD I. HESS, *Rancho Palos Verdes, CA*, J. C. LINDERS, *Eindhoven University of Technology, The Netherlands*, PETER A. LINDSTROM, *North Lake College, Irving, TX*, TOM MOORE (2 *solutions*), *Bridgewater State College, MA*, L. J. UPTON, *Mississauga, Ontario, Canada*, ALAN WAYNE, *Holiday, FL*, and KENNETH M. WILKE, *Topeka, KS*. *Partial solutions were submitted by* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay*, JAMES E. CAMPBELL, *Indiana University at Bloomington*, DAVID EHREN, *University of Wisconsin, Milwaukee*, MARK EVANS, *Louisville, KY*, JOHN M. HOWELL, *Littlerock, CA*, and WILLIAM S. ROSS, *University of Maine, Orono*.

Moore noted that Pell equations such as $x^2 - 2z^2 = -1$ *are treated* in LeVeque, *Fundamentals of Number Theory*, Addison-Wesley, 1977, p. 202. *Diminnie and Sedinger cited* Niven and Zuckerman, *An Introduction to the Theory of Numbers*, p. 159. *Howell, Upton, and Wayne each found* Albert H. Beiler, *Recreations in the Theory of Numbers*, Dover, 1966, p. 328, *where* 100 *sets of Pythagorean right triangles with legs differing by* 1 *are given. Lindstrom noted that if* (x_i, y_i, z_i) *is a Pythagorean right triangle with legs differing by* 1, *then the next such triple is given by*

$$x_{i+1} = 3x_i + 2z_i + 1, \quad y_{i+1} = 3x_i + 2z_i + 2, \quad z_{i+1} = 4x_i + 3z_i + 2.$$

Hess developed the formula

$$n_i = \frac{(-1+\sqrt{2})(3+2\sqrt{2})^i + (-1-\sqrt{2})(3-2\sqrt{2})^i + 2}{4}$$

for any positive integer i, *which is equivalent to the formula for* n *given in the featured solution.*

654. [Fall 1987] *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

In the game of *Rouge et Noir*, cards are dealt one at a time from a large number of well-shuffled decks until the total pip count is in the range 31 to 40. (Face cards each count 10.) *Boyle Complete* (by Foster, 1916) gives the relative probabilities of arriving at the sums 31, 32, . . . , 40 as 13, 12, ... , 4, respectively. Find a more accurate set of probabilities.

Solution by Mark Evans, Louisville, Kentucky.

Using a Markov chain approach on a computer, I found the probabilities:

Count	Probability	85 x (prob.)
31	0.1480609	12.585
32	0.1379052	11.722
33	0.1275127	10.839
34	0.1168911	9.936
35	0.1060495	9.014
36	0.09499837	8.075
37	0.08374979	7.119
38	0.07231733	6.147
39	0.06071615	5.161
40	0.05179912	4.403
Total	1.00000016	

The solution can be realized by writing a 41 by 41 matrix M of probabilities. Number the rows and columns 0 through 40 to indicate the possible counts. The entry M_{ij} is the probability that the next draw will bring you to count j, given that the count is now i. Thus row 0 is 0, nine entries of 1/13, one entry of 4/13, and thirty entries of 0. That is, starting with a count of 0 and drawing a card, you have probability 0 that the new count will be either 0 or greater than 10, probability 1/13 that the count will become 1, 2, 3, ... , 9, and probability 4/13 of count 10. Each successive row through the 30th row starts with 0 and then has the first 40 elements of the preceding row, forming an upper triangular matrix. In row 7, for example, the first 8 elements are 0 because, if you have a count of 7, then you cannot draw a card and obtain a new count less than 8. If you draw an ace (with probability 1/13), the new count is 8, a draw of 2 gives a count of 9, and so forth, a draw of 10 or J or Q or K (with probability 4/13) gives a count of 17. All higher counts are not possible and hence have probability 0. When you reach a count of 31 to 40, further draws do not change your score, so in those rows each main diagonal element is 1 and all the other elements are 0.

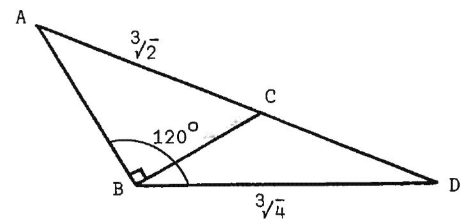
Now M^2 is seen to be the matrix of probabilities after 2 draws; that is, element $(M^2)_{ij}$ is the probability that, if you start with a count of i and draw two cards, then the new count will be j. Since the game can take as many as 31 draws, find M^{31}. Then row 0 of this matrix will consist of 31 zeros and then the ten probabilities listed above, indicating the probabilities of starting with a count of 0 and finishing with counts of 31 to 40.

Although there are theorems that simplify the computations somewhat, a computer is still highly recommended to find the solution.

Also solved by CHARLES ASHBACHER, *Mount Mercy College, Cedar Rapids, IA, who used a computer and tree structures,* JOHN M. HOWELL, *Littlerock, CA, who used a computer simulation with 10000 random trials, and the* PROPOSER, *whose method was not disclosed.*

655. [Fall 1987] *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

In triangle ABD, $\measuredangle B = 120^\circ$. Furthermore, there is a point C on side AD such that $\measuredangle ABC = 90^\circ$, $AC = \sqrt[3]{2}$, and $BD = 2/AC$. Find the lengths of AB and CD.



Solution by Frank P. Battles and Laura L. Kelleher, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts.

From triangle BCD, as seen in the figure, the law of sines gives

$$\frac{CD}{\sin 30^\circ} = \frac{2^{2/3}}{\sin(90^\circ + A)} \quad \text{or} \quad CD = 1/(2^{1/3}\cos A)$$

From triangle ABC we have that $AB = 2^{1/3}\cos A$. Hence $CD = 1/AB$.

Next, apply the law of cosines to triangle BCD to get

$$2^{4/3} = CD^2 + BC^2 - 2\cdot CD\cdot BC\cdot\cos(90^\circ + A).$$

Let $x = AB$ and substitute $\cos(90^\circ + A) = -\sin A = -BC/2^{1/3}$ and $BC^2 = 2^{2/3} - x^2$ to get

$$2^{4/3} = 1/x^2 + 2^{2/3} - x^2 + 2^{2/3}(1/x)(2^{2/3} - x^2).$$

Multiply through by x^2 and rearrange terms to obtain

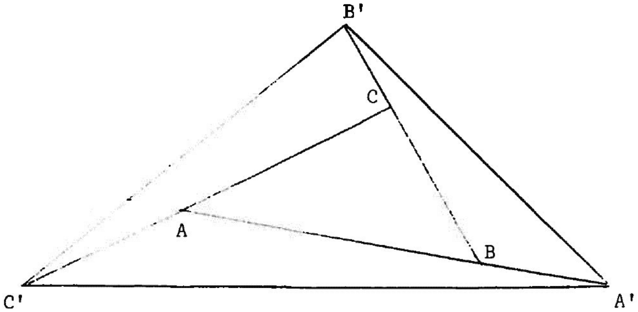
$$2^{2/3}(x^2 - x^3) + 2^{4/3}(x - x^2) + (1 - x^4) = 0,$$

which clearly has the root $x = 1$. Factoring out $1 - x$ produces a polynomial with all positive coefficients, so there are no other positive roots. Thus $AB = 1$ and $CD = 1/AB = 1$ also.

Also solved by GEORGE P. EVANOVICH, *Saint Peter's College, Jersey City, NJ,* MARK EVANS, *Louisville, KY,* JACK GARFUNKEL, *Flushing, NY,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* JOE HOWARD, *New Mexico Highlands University, La Vegas,* RALPH E. KING, *St. Bonaventure University, NY,* HENRY S. LIEBERMAN, *Waban, MA,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* WADE H. SHERARD, *Furman University, Greenville, SC,* ARTHUR H. SIMONSON, *East Texas State University at Texarkana, and the* PROPOSER.

656. [Fall 1987] *Proposed by Jack Garfunkel, Flushing, New York.*

Let ABC be any triangle and extend side AB to A', side BC to B', and side CA to C' so that B lies between A and A', etc., and $BA' = A\cdot AB$, $AC' = \lambda\cdot CA$, and $CB' = \lambda\cdot BC$. Find the value of A so that the area of triangle $A'B'C'$ is four times the area of triangle ABC. See the figure below.



1. *Solution by John P. Holcomb, Jr., St. Bonaventure University, St. Bonaventure, New York.*

Let $[ABC]$ denote the area of triangle ABC. Then

$$[ABC] = \frac{1}{2} AB\cdot BC \sin\sphericalangle CBA \quad \text{and} \quad [BB'A'] = \frac{1}{2} BA'\cdot BB' \sin\sphericalangle B'BA'.$$

Angles CBA and $B'BA'$ are supplementary and hence have the same sines. Since also $BA' = \lambda\cdot AB$ and $BB' = (1 + \lambda)BC$, we have that

$$[BB'A'] = \lambda(\lambda + 1)[ABC].$$

Similar equations hold for triangles $CC'B'$ and $AA'C'$. Then we have

$$3\lambda(\lambda + 1)[ABC] + [ABC] = 4[ABC].$$

Then $A^2 + A - 1 = 0$, which has the one positive root

$$\lambda = \frac{\sqrt{5} - 1}{2} \approx 0.618,$$

the reciprocal of the golden ratio.

II. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

More generally, if $BA' = x\cdot AB$, $AC' = y\cdot CA$, and $CB' = z\cdot BC$, it is known [1] that the ratio of the areas of the two triangles is given by

$$\frac{[A'B'C']}{[ABC]} = (x + 1)(y + 1)(z + 1) - xyz.$$

Consequently, A must satisfy

$$(\lambda + 1)^3 - \lambda^3 = 4, \qquad \text{or} \qquad \lambda = \frac{-1 \pm \sqrt{5}}{2}.$$

A related Diophantine problem is to find all natural numbers n such that there exist positive integer triples (x, y, z) with

$$(x + 1)(y + 1)(z + 1) - xyz = n.$$

Then, for each such n, determine all the integral triples $(x, y, 2)$. For example, if $n = 3m + 4$, then $(x, y, 2) = (m, 1, 1)$ and permutations thereof.

References

1. M. S. Klamkin and A. Liu, Three more proofs of Routh's theorem, *Crux Mathematicorum* 7(1981) 199 - 203.

Also solved by WILLIAM BOULGER, St. Pant Academy, MN, GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, NJ, RICHARD I. HESS, Rancho Palos Verdes, CA, RALPH E. KING, St. Bonaventure University, HENRY S. LIEBERMAN, Waban, MA, WILLIAM H. PEIRCE, Stonington, CT, BOB PRIELIPP, University of Wisconsin-Oshkosh, WADE H. SHERARD, Furman University, Greenville, SC, ARTHUR H. SIMONSON, East Texas State University at Texarkana., ALAN WAYNE, Holiday, FL, KENNETH M. WILKE, Topeka, KS, and the PROPOSER.

657. [Fall 1987] *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Evaluate the trigonometric sum

$$\sin^6\frac{\pi}{8} + \sin^6\frac{3\pi}{8} + \sin^6\frac{5\pi}{8} + \sin^6\frac{7\pi}{8}.$$

I. *Solution by Oxford Running Club, University of Mississippi, University, Mississippi.*

Since $\sin x = \sin(\pi - x)$ and $\sin(\pi/2 - x) = \cos x$, we have

$$S = \sin^6\frac{\pi}{8} + \sin^6\frac{3\pi}{8} + \sin^6\frac{5\pi}{8} + \sin^6\frac{7\pi}{8}$$
$$= 2\left(\sin^6\frac{\pi}{8} + \sin^6\frac{3\pi}{8}\right) = 2\left(\sin^6\frac{\pi}{8} + \cos^6\frac{\pi}{8}\right).$$

Now factor this sum of two cubes to get

$$S = 2\left(\sin^2\frac{\pi}{8} + \cos^2\frac{\pi}{8}\right)\left(\sin^4\frac{\pi}{8} - \sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8} + \cos^4\frac{\pi}{8}\right)$$
$$= 2.1\cdot\left[\left(\sin^2\frac{\pi}{8} - \cos^2\frac{\pi}{8}\right)^2 + \sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right]$$
$$= 2\left[\left(-\cos\frac{\pi}{4}\right)^2 + \left(\frac{1}{2}\sin\frac{\pi}{4}\right)^2\right] = \frac{5}{4}.$$

II. Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.

More generally, to calculate

$$S_n = \sin^n a + \cos^n a,$$

we can just replace $\cos a$ and $\sin a$ by $\{(2 \pm \sqrt{2})/2\}^{1/2}$ or else we can use the recursive relation

$$S_{n+2} = S_n - \frac{1}{4}(\sin^2 2a)S_{n-2}.$$

Also solved by SEUNG-JIN BANG, Seoul, Korea, FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, WILLIAM BOULGER, St. Paul Academy, MN, JAMES E. CAMPBELL, Indiana University at Bloomington, RUSSELL EULER, Northwest Missouri State University, Maryville, GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, NJ, MARK EVANS, Louisville, KY, JACK GARFUNKEL, Flushing, NY, ROBERT C. GEBHARDT, Hopatcong, NJ, RICHARD I. HESS, Rancho Palos Verdes, CA, BRUCE KING, Western Connecticut Connecticut State. University, Danbury, MURRAY S. KLAMKIN, University of Alberta, Canada, PETER A. LINDSTROM, North Lake College, Irving, TX, TX, BOB PRIELIPP, University of Wisconsin-Oshkosh, GEORGE W. RAINEY, Los Angeles, CA, WADE H. SHERARD, Furman University, Greenville, SC, ARTHUR H. SIMONSON, East Texas State University at Texarkana, ALAN WAYNE WAYNE, Holiday, FL, and the PROPOSER.

658. *[Fall 1987] Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Factor $(x + y + z)^7 - x^7 - y^7 - z^7$ into a product of real polynomials, each having degree not to exceed four.

Solution by Bob Prielipp and John Oman, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Let $P(x, y, z) = (x + y + z)^7 - x^7 - y^7 - z^7$. Since $P(x, -x, z) = P(x, y, -x) = P(x, -z, z) = 0$, then $x + y$, $z + x$, and $y + z$ are factors of $P(x, y, z)$. Thus we have

(1) $\quad P(x, y, z) = (x + y)(z + x)(y + z)\cdot Q(x, y, z),$

where $Q(x, y, z)$ is a homogeneous polynomial of degree four. Because $Q(x, y, z)$ must be symmetric in x, y, and z, it must be of the form

$$A(x^4 + y^4 + z^4) + B(x^3y + x^3z + xy^3 + y^3z + xz^3 + yz^3)$$
$$+ C(x^2yz + xy^2z + xyz^2) + D(x^2y^2 + x^2z^2 + y^2z^2).$$

Let $x = y = z = 1$; $x = 2$, $y = 1$, $z = 0$; $x = 3$, $y = 1$, $z = 0$; $x = 2$, $y = 2$, $z = 0$ in (1) to yield these four equations:

$$\begin{aligned} 3A + 6B + 3C + 3D &= 91 \\ 17A + 10B + 4D &= 343 \\ 82A + 30B + 9D &= 1183 \\ 2A + 2B + D &= 63. \end{aligned}$$

It follows that $A = 7$, $B = 14$, $C = 35$, and $D = 21$. Hence

$$\begin{aligned} P(x, y, z) &= 7(x + y)(z + x)(y + z)[(x^4 + y^4 + z^4) \\ &\quad + 2(x^3y + x^3z + xy^3 + y^3z + xz^3 + yz^3) \\ &\quad + 5(x^2yz + xy^2z + xyz^2) + 3(x^2y^2 + x^2z^2 + y^2z^2)] \\ &= 7(x + y)(z + x)(y + z)\cdot[(x^2 + y^2 + z^2 \\ &\quad + xy + xz + yz)^2 + xyz(x + y + z)]. \end{aligned}$$

Also solved by SEUNG-JIN BANG, Seoul, Korea, GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, NJ, RICHARD I. HESS, Rancho Palos Verdes, CA, KENNETH M. WILKE, Topeka, KS, and the PROPOSER.

659. *[Fall 1987] Proposed by Harry Sadinger and Albert White, St. Bonaventure University, St. Bonaventure, New York.*

If $0 < x < 1$, $p > 1$, and $q = p/(p - 1)$, then prove that

$$2^{p-2}(x^p + 1) \le (x^q + 1)^{p-1}.$$

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.

The inequality is not valid as stated. Note that it can be rewritten as

(1) $$(x^p + 1)/2 \le [\{(x^p)^{1/(p-1)} + 1\}/2]^{p-1}.$$

By the power mean inequality, Inequality (1) is valid for all $x \ge 0$ if $1 \le p \le 2$. For $p \ge 2$, the reverse inequality holds.

The correct inequality was also discerned by BARRY BRUNSON, Western Kentucky University, Bowling Green, BOB PRIELIPP, University of Wisconsin-Oshkosh, and the PROPOSERS. *The stated inequality was shown to be incorrect also by* SEUNG-JIN BANG, Seoul, Kohea, GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, NJ, MARK EVANS, Louisville, KY, and RICHARD I. HESS, Rancho Palos Verdes, CA.

The proposers stated that the error in the statement of this problem is a copying mistake for which they apologize. Your editor accepts 50 *lashes with a wet noodle for not spotting it prior to publishing it.*

660. [Fall 1987] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

Recently, the elderly numerologist E. P. B. Umbugio read the life of Leonardo Fibonacci and became interested in the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, ..., where each number after the second one is the sum of the two preceding numbers. He is trying to find a 3×3 magic square of distinct Fibonacci numbers (but $F_1 = 1$ and $F_2 = 1$ can both be used), but has not yet been successful. Help the professor by finding such a magic square or by proving that none exists.

Solution by David Ehren, University of Wisconsin, Milwaukee, Wisconsin.

Let the magic square be

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i. \end{matrix}$$

Then $b + e + h = d + e + f$, so that $b + h = d + f$. We can, without loss of generality, assume that $1 \le b \le d < f < h$. But since they are Fibonacci numbers, then $d + f \le h < h + b$, a contradiction. So, unfortunately for the professor, the only magic squares with just Fibonacci numbers must have the trivial patterns with repeats.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, THOMAS E. MOORE, Bridgewater State College, MA, BOB PRIELIPP, University of Wisconsin-Oshkosh, and the PROPOSER.

Moore and Prielipp each found the generalization to all magic squares of distinct Fibonacci numbers by John L. 'Known, Jr., Reply to exploring Fibonacci magic squares, The Fibonacci Quarterly 3(1965)146. Permitting both F_1 and F_2 to be used does not invalidate the proof given there.

661. [Fall 1987] *Proposed by John M. Howell, Littlerock, California.*

a) How close to a cubical box can you get if the sides and the diagonal of a rectangular parallelepiped are all integral?

*b) How close can you get to a cube if all the face diagonals must be integral, too?

I. *Solution to part a) by Richard I. Hess, Rancho Palos Verdes, California.*

You can get arbitrarily close to a cube if the sides and space diagonal are integers. Let the sides be n, n, and $n \pm 1$. Then the diagonal is given by $d = (2n^2 + (n \pm 1)^2)^{1/2}$, which reduces to

$$3d^2 = 9n^2 \pm 6n + 3 = (3n \pm 1)^2 + 2 = p^2 + 2.$$

This is a Fermat-Pell equation with solution

p	1	5	19	71	265	989	3691	...
d	1	3	11	41	153	571	2131	...
n	0	2	6	24	88	330	1230	...
$n \pm 1$	1	1	7	23	89	329	1231	... ,

where $n = (p \pm 1)/3$, whichever sign makes n integral. The sign, in fact, alternates and is opposite that used for the third side $n \pm 1$. We can write $n = [(p + 1)/3]$, where the brackets indicate the greatest integer function. Also, it is readily seen that the smaller leg (either n or $n - 1$) is $[d/\sqrt{3}]$. Now each of p and d satisfies the recursion formula $f(k + 2) = 4f(k + 1) - f(k)$. Explicitly,

$$p_r = \frac{(1 + \sqrt{3})(2 + \sqrt{3})^r + (1 - \sqrt{3})(2 - \sqrt{3})^r}{2}$$

and

$$d_r = \frac{(1 + \sqrt{3})(2 + \sqrt{3})^r - (1 - \sqrt{3})(2 - \sqrt{3})^r}{2\sqrt{3}}.$$

These formulas produce all the stated near cubes for r a positive integer. Note that $r = 0$ gives the first set of values in the table above, values that do not form a real box.

II. *Solution to part b) by Robert C. Gebhardt, Hopatcong, New Jersey.*

In the July 1970 issue of *Scientific American* in the "Mathematical Games" column, pp. 117-119, Martin Gardner discussed rectangular parallelepipeds with integer values for edge lengths, face diagonal lengths and interior diagonal lengths. There, he states that the smallest brick with integral edges and face diagonals (and nonintegral space diagonal) has edges 44, 117, and 240. The brick of edges 104, 153, and 672 has all diagonals integral except one face diagonal. No brick with all these measurements integral has been found as yet. Whether one exists is unanswered.

III. *Solution by the Proposer.*

Albert H. Beiler, Recreations in the Theory of *Numbers*, on page 146 gives formulas for the sides x, y, z of boxes with integral face diagonals:

$$x = |2mn(3m^2 - n^2)(3n^2 - m^2)|, \quad y = |8mn(m^4 - n^4)|,$$

$$z = |(m^2 - n^2)(m^2 + n^2 + 4mn)(m^2 + n^2 - 4mn)|.$$

It is readily but tediously checked that

$$x^2 + y^2 = 4m^2n^2(5m^4 - 6m^2n^2 + 5n^4)^2,$$

$$y^2 + z^2 = (m^2 - n^2)^2(m^4 + 18m^2n^2 + n^4)^2,$$

and

$$z^2 + x^2 = (m^6 + 3m^4n^2 + 3m^2n^4 + n^6)^2 = (m^2 + n^2)^6;$$

i.e., all three face diagonals are integers when the sides are integral.

662. *[Fall 1987] Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Solve the equation

$$10^{2y-4} - 2\cdot 10^{y-2} - 10^{y-1} + 20 = 0.$$

Solution by Man Wayne, Holiday, Florida.

Let $x = 10^{y-2}$. Then $0 = x^2 - 12x + 20 = (x - 2)(x - 10)$, so $x = 2$ and $x = 10$. Hence $y = 2 + \log 2$ and $y = 3$.

Essentially this same solution was also submitted by SEUNG-JIN BANG, *Seoul, Korea,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* BARRY BRUNSON, *Western Kentucky University, Bowling Green,* JAMES E. CAMPBELL, *Indiana University at Bloomington,* DAVID DELESTO, *North Scituate, Rhode Island,* DAVID EHREN, *University of Wisconsin, Milwaukee,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* GEORGE P. EVANOVICH, *Saint Peter's College, Jersey City, NJ,* MARK EVANS, *Louisville, KY,* VICTOR G. FESER, *University of Mary, SLA-marck, W,* JACK GARFUNKEL, *Flushing, NY,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* JOHN P. HOLCOMB, JR., *St. Bonaventure University, NY,* JOE HOWARD, *New Mexico Highlands University, Las Vegas,* BRUCE KING, *Western Connecticut State University, Danbury,* RALPH E. KING, *St. Bonaventure University, NY,* PETER A. LINDSTROM, *North Lake College, Irving, TX,* YOSHINOBU MURAYOSHI, *Portland, OR,* OXFORD RUNNING CLUB, *University of Mississippi, University,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* WADE H. SHERARD, *Furman University, Greenville, SC,* ARTHUR H. SIMONSON, *East Texas State University at Texarkana,* TIMOTHY SIPKA, *Anderson University, IN,* THOMAS F. SWEENEY, *Russell Sage College, Troy, NY,* STEPHANIE M. TYLER, *Sulphur Springs, TX,* W. R. UTZ, *University of Missouri, Columbia,* LIEN VUONG, *Lamar University, Beaumont, TX,* KENNETH M. WILKE, *Topeka, KS,* CHARLES ZIEGENFUS, *James Madison University, Harrisonburg, VA, and the* PROPOSER. *A partial solution was submitted by* CHARLES ASHBACHER, *Mount Mercy College, Cedar Rapids, IA.*

663. *[Fall 1987] Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Find a series expansion for the integral

$$\int_0^{\pi/2} \frac{x\,dx}{\sin x}$$

I. *Solution by Richard I. Hess, Rancho Palos Verdes, California.*

By multiplying by x the series for $\csc x$ found in standard books of tables or by dividing x by the well-known series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots,$$

we obtain the power series, convergent on the interval $(-\pi, \pi)$,

$$x \csc x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \frac{31x^6}{15120} + \dots + \frac{(-1)^{n-1}2(2^{2n-1} - 1)B_{2n}x^{2n}}{(2n)!} + \dots,$$

where B_{2n} is a Bernoulli number. The desired integral is now found by term by term integration from 0 to $\pi/2$. We get

$$\int_0^{\pi/2} \frac{x\,dx}{\sin x} = \frac{\pi}{2} + \frac{(\pi/2)^3}{18} + \frac{7(\pi/2)^5}{1800} + \dots + \frac{(-1)^{n-1}2(2^{2n-1} - 1)B_{2n}(\pi/2)^{2n+1}}{(2n+1)!} + \dots,$$

which is twice Catalan's constant.

II. *Solution by the Proposer.*

Consider

$$I(k) = \int_0^{\pi/2} \frac{\sin^{-1}(k \sin x)}{\sin x}\,dx.$$

Then

$$\frac{dI}{dk} = \int_0^{\pi/2} \frac{dx}{(1 - k^2\sin^2 x)^{1/2}} = K(k),$$

where $K(k)$ is a complete elliptic function of the first kind. Integrating between 0 and 1 gives

$$I(1) = \int_0^{\pi/2} \frac{x\,dx}{\sin x} = \int_0^1 K(k)\,dx.$$

Finally, expanding out the integrand in the latter integral as a power series in k^2 and integrating, we obtain

$$I(1) = 2\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \dots\right) = 2G,$$

where $G \approx 0.915965$ is Catalan's constant.

III. *Solution by Robert C. Gebhardt, Hopatcong, New Jersey.*

A Taylor's series about $\pi/2$ for $\csc x$ is

$$\frac{1}{\sin x} = \csc x = 1 + \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{5}{24}\left(x - \frac{\pi}{2}\right)^4 + \dots$$

Now multiply by x, expand each term, and integrate from 0 to $\pi/2$ to get

$$\int_0^{\pi/2} \frac{x\,dx}{\sin x} = \left[\frac{x^2}{2} + \frac{x^4}{8} - \frac{x^3\pi}{6} + \frac{x^2\pi^2}{16} + \dots\right]_{x=0}^{\pi/2} \approx 1.83.$$

IV. *Solution by Barry Brunson, Western Kentucky University, Bowling Green, Kentucky.*

Knopp ([1], p. 156) refers to the "partial fraction decomposition" (and its proof in [2]) of the cotangent function:

$$\pi\cot\pi z = 1 + \sum_{n=1}^{\infty} \frac{2z^2}{z^2 - n^2} \quad \text{for } z \text{ not an integer.}$$

As in [1], we can use this series, together with the relations

$$\tan x = \cot x - 2\cot 2x \quad\text{and}\quad \csc x = \cot x + \tan x/2,$$

to obtain

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2}{n^2\pi^2 - x^2},$$

which we multiply by x to get a series for $x\csc x$ that converges for all x except integral multiples of π. Term-by-term integration then yields

$$\int_0^{\pi/2} \frac{x\,dx}{\sin x} = \frac{\pi}{2} + 2\sum_{n=1}^{\infty}(-1)^n\left[x - \frac{n\pi}{2}\ln\frac{n\pi + x}{n\pi - x}\right]_{x=0}^{\pi/2}$$

$$= \frac{\pi}{2} + \pi\sum_{n=1}^{\infty}(-1)^n\left\{1 - n\ln\frac{2n+1}{2n-1}\right\} \approx 1.8319.$$

References

1. K. Knopp, *Infinite Sequences and Series*, Dover, 1956.
2. ...., *Theory of Functions*, v. II, Dover, 1947.

V. *Solution by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Since $x\csc x = {}_2F_1(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \sin^2 x)$, a generalized hypergeometric function (see [1], p. 224), we have

$$\int_0^{\pi/2} x\csc x\,dx = \sum_{k=0}^{\infty} \frac{(1/2)_k(1/2)_k}{(3/2)_k\,k!}\int_0^{\pi/2} \sin^{2k} x\,dx$$

where $(r)_k = r(r+1)\dots(r+k-1)$ is the generalized factorial function. The integral on the right side of this equation can be evaluated by Wallis' Formula (see [2], p. 223)

$$\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{(2k-1)(2k-3)\dots 1}{2k(2k-2)\dots 2}\,\frac{\pi}{2}$$

to give

$$\int_0^{\pi/2} \frac{x\,dx}{\sin x} = \frac{\pi}{2}\sum_{k=0}^{\infty} \frac{(2k-1)(2k-3)\dots(3)(1)(1/2)_k(1/2)_k}{(2k)(2k-2)\dots(4)(2)(3/2)_k\,k!}$$

References

1. M. E. Goldstein and W. H. Braun, *Advanced Methods for the Solution of Differential Equations*, National Aeronautics and Space Administration, 1973.
2. H. W. Reddick and F. W. Miller, *Advanced Mathematics for Engineers, 3rd ed.*, Wiley, 1955.

Also solved by BARRY BRUNSON (second solution), GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, NJ, and ALAN WAYNE, Holiday, FL. Brunson's second solution was of the form of Solution I. Evanovich found the series of Solution IV in Hobson, Plane Trigonometry, Cambridge University Press, 1891, p. 335, Equation 72. Wayne found the value 2G for the integral in A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, Integrals and Series, Gordon and Breach, 1986, vol. 1, p. 388, Section 2.5.4, Formula 5, where G is given by the series of Solution II,

$$G = 1/1^2 - 1/3^2 + 1/5^2 - 1/7^2 + \dots + (-1)^k/(2k+1)^2 + \dots,$$

as stated in I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products, Academic Press, 1980, p. 417, Section 3.747, Formula 2.

664. [Fall 1987] *Proposed by William M. Snyder, Jr., University of Maine, Orono, Maine.*

In this sentence the number of occurrences of the digit 0 is ____, of 1 is ____, 2 is ____, 3 is ____, 4 is ____, 5 is ____, 6 is ____, 7 is ____, 8 is ____, and of the digit 9 is ____.

a) Fill in the blanks to make the sentence true.

*b) How many solutions are there?

(This problem appeared on the bulletin board of a community college in Maryland.)

Composite of all solutions, by Elizabeth Andy, Limerick, Maine.

Assuming each blank is filled with a digit or perhaps one blank is filled with a two-digit number, then there are a total of 20 or 21 digits, so the sum of the numbers filling the blanks is 20 or 21. Since each blank is filled with a positive integer, then only one "large" number blank can be filled with a number greater than 1. If the digit 2 does not appear again, then blank number 2 can be filled with either a 1 or a 2. If 2 appears exactly twice, then blank number 2 can be filled with a 3 provided blank 3 is filled with a 2. Juggling numbers then produces one or more solutions.

The two most common solutions were

Blank no.:	0	1	2	3	4	5	6	7	8	9
Solution 1:	1	7	3	2	1	1	1	2	1	1
Solution 2:	1	11	2	1	1	1	1	1	1	1

with 7 and 6 solvers, respectively. Two solvers submitted Solution 3 and one sent in Solution 4, where t = ten in any base greater than ten. These two solutions were

Blank no.:	0	1	2	3	4	5	6	7	8	9
Solution 3:	10	11	02	01	01	01	01	01	01	01
Solution 4:	1	t	1	1	1	1	1	1	1	1.

Those who found Solution 3 noted that more zeros can be appended to give infinitely many different solutions of this form. The proposer also found Solution 5:

Blank no.:	0	1	2	3	4	5	6	7	8	9
Solution 5:	one	one	one	one	one	one	one	one	one	one.

Five solvers "proved" that their solutions were unique. Assumptions made to arrive at unique solutions varied, and not all assumptions were stated explicitly.

Solution 5 can be frustrating to solvers looking for "legitimate" solutions. It is like the problem of dividing 14 lumps of sugar among 3 tea cups in such a way that each cup has an odd number of lumps. There are three solutions. Try to find them before reading further. The first is to break one lump into two smaller ones so each cup can have 5 lumps. The next solution is to put 7 lumps in the first cup, 7 in the second, and then put the second cup in the third cup. The last solution is to put 1 lump in the first cup of tea, 1 in the second cup of tea, and 12 lumps of sugar in the third cup of tea, since, after all, 12 lumps of sugar is certainly an odd number for a cup of tea!

Regarding Solution 5 in particular and this type of problem in general, we state

A problem that's alphanumerical
Can make many people hysterical.
But don't call the sheriff;
You can grin and bear, if,
Your mind has a bent that is clerical.

In the listing below of solvers, each name is followed by the solution number or numbers that the solver found and by the letter p if a uniqueness proof was submitted.

Also solved by CHARLES ASHBACHER (2), *Mount Mercy College, Cedar Rapids, IA,* FRANK P. BATTLES *and* LAURA L. KELLEHER (1, 2, 3, 4), *Massachusetts Maritime Academy, Buzzards Bay,* JAMES E. CAMPBELL (1, 2, p), *Indiana University at Bloomington,* VICTOR G. FESER (2, p), *University of Mary, Bismarck, ND,* RICHARD I. HESS (1, p), *Rancho Palos Verdes, CA,* TIMOTHY SIPKA (2, 3), *Anderson University, IN,* STEPHANIE M. TYLER (1), *Sulphur Springs, TX,* LIEN VUONG (1, p), *Lamar University, Beaumont, TX,* MICHAEL D. WILLIAMS (1, p), *Wake Forest University, Winston-Salem, NC, and the* PROPOSER (1, 2, 5).

Correction

Following the solution to Problem 644 [Spring 1987, Spring 1988], *I commented that "almost no one saw the immediate relation that since the area of the large circle equals that of the four small circles, then* A = B *immediately." Somehow I overlooked one excellent solution that I had even marked because it did do just that. That delightful solution was by* FRANCIS C. LEARY, *St. Bonaventure University, NY.*

Preliminary Announcement

PI MU EPSILON STUDENT CONFERENCE

Saint John's University
Collegeville, Minnesota
March 31 and April 1, 1988

The principal speaker will be

Richard Askey

University of Wisconsin at Madison

The meeting is open to all mathematicians and mathematics students, not just members of Pi Mu Epsilon. The conference provides an excellent forum for students who have been working on independent study or research projects. Each college in the area is invited to send at least one student to the conference. Additional information will be mailed to all colleges in the area next January.

Jennifer Galovich **612-363-3192**
Jim Wilmesmeier **612-363-3092**

1988 NATIONAL Pi MU EPSILON MEETING

The Annuai Meeting of the Pi Mu Epsilon National Honorary Mathematics Society was held in Providence, Rhode Island, August 8 through August 11. Highlights **included** an opening reception at the Rhode Island State House, the Pi Mu Epsilon Council Luncheon and Business meeting, a Pizza Party Reception for student, alumni and faculty members of Pi Mu **Epsilon,** the Student Paper Sessions, the Annual Banquet, informal student parties, and the 14th Annual J. Sutherland Frame Lecture.

At the Annuai Banquet, Past-president Milton D. Cox received Pi Mu Epsilon's highest honor, the C. C. **MacDuffee** Award for Distinguished Service.

This year, the Frame Lecturer was Professor Doris W. Schattschneider, Moravian College. Professor Schattschneider is immediate past-editor of Mathematics Magazine and is widely known for her contributions to geometry in the area of **tilings** of the plane and for her studies of the mathematics of the works of Dutch artist M. C. **Escher.**

Professor Schattschneider entertained her audience with "You. Too. Can Tile the **Conway** Way." In the course of her lecture, she created a plane-filling tile in honor of Pi Mu Epsilon. The tile is reproduced on page 625.

PROGRAM - STUDENT PAPER SESSIONS

maximal Polygons for Convex. Periodic **Tilings**	Annette M. Matthews *Oregon* Gamma *Portland,* State University
An Introduction to **Equitransitive Tilings**	Gerry Wuchter Ohio Delta Miami *University*
Finding a Generator of a Finitely Generated **Abelian** Group	David L. Jakes Ohio Delta *Miami University*
The Cross Product in **n-Space**	Joel Atkins Indiana Gamma Rose-Hulman *Institute of* Technology

A Continued Fraction Approach for Factoring Large Numbers
Robert Coary
Washington Beta
University of Washington

Perfect [lumbers. Abundants numbers, and Deficient Numbers
Sarah Taylor
North Carolina Delta
Cost Carolina University

The Problem of the Traveling Salesman - A study in Computational Complexity and Heuristic Algorithms
Douglas Galarus
Montana Alpha
The University of Montana

Using Portable Intermediate Code in Compiler Construction
Paul Stodguill
Pennsylvania Rho
Dickinson College

numerical Solutions for the Three-Dimensional Heat Conduction Equation
Charles Jabbour
Texas Nu
University of Houston - Downtown

The Effect of Intra-Cluster Correlations on the Regression Estimation in the Finite Population Inference
Dewi T. Saleh
Kansas Gamma
Wichita State University

QR Revisited -- A Parallel flpproach to the QR - Decomposition
Lara Aist
Maryland Gamma
University of Maryland - Baltimore County

R Brute Force flpproach to Solving a Putnam Problem
Summer Quimby
Wisconsin Delta
St. Norbert College

The Sturm-Liouville mathematical System
Lorie Ceremuga
Ohio Xi
Youngstown State University

The Annihilation Operator
Marc Ahrendt
Illinois Iota
Elmhurst College

mathematical Measures in the Analysis of Image
Clifford D. Krumvieda
Texas Eta
Texas A&M University

Squaring a Square
Terry Henderschott
Ohio Delta
Miami University

The Area of Wasted Space in a Rosette Design
George Anderson
Ohio Delta
Miami University

Some Results for Chromatic and Associated Polynomials of Graphs
Laura Cuthbertson
Arkansas Beta
Hendrix College

The Greedy Spy
Janelle Beefier
Minnesota Delta
St. John's University/ College of St. Benedict

Heuristic Arguments in mathematics
David Petry
Oregon Alpha
University of Oregon

Removing the Absolute from Relativity
Katie Coenen
Wisconsin Delta
St. Norbert College

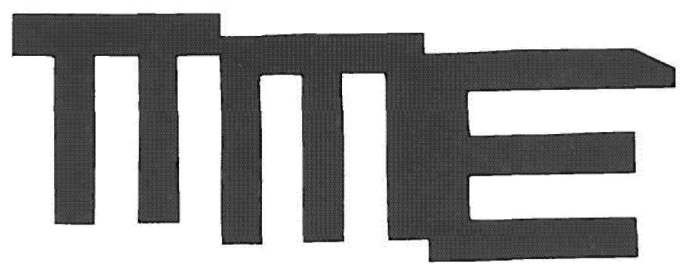
Dynamics of PDP Models
George Ashline
New York Epsilon
St. Lawrence University

A Venture into Chaos
Erlan Wheeler
Virginia Beta
Virginia Tech

Fractals -- mathematical Chaos
Michael F. Marcon
Louisiana Epsilon
McNeese State University

* *

Given an unlimited supply of copies of the polygon below the plane can be filled without gaps and without overlap. This tile was created by Professor Doris Schattschneider and was discussed in her J. S. Frame Lecture in Providence, RI in August 1988.



GLEANINGS FROM THE CHAPTER REPORTS

ARKANSAS BETA (Hendrix College). The Hendrix-Rhodes-Sewanee Symposium was held at Rhodes in April. Student participants from Hendrix were Carol Parker, Laura Cuthbertson, Pat Dyer, Julie Honeycutt and Sheri 'Jordan. In April, Hendrix successfully hosted the 50th annual Arkansas/Oklahoma MAA Section Meeting. Hendrix students giving talks were Joe Francic, Julie Honeycutt, Laura Cuthbertson, Cheri Holden and Cord Parker. Enough money was raised to meet Hendrix's goal for the N. A. Court Fund. Pi Mu Epsilon members receiving awards at the Honors Convocation were Mark Lancaster and Gary Patterson (The McHenry-Lane Mathematics Award), Carol Parker (The Hogan Mathematics Prize) and Laura Cuthbertson (The Parker Undergraduate Research Award). Chapter President, Carol Parker, and Chapter Secretary, Marla Beggs, graduated with Departmental Distinction. Program Calendar highlights during the school year included "Statistics" by Katherine Bennet Ensor (Rice University), "Glimpses of mathematics in Moscow" by Zeev Barel (Hendrix), "4-5 Hole Problem" by Marvin Keenor (Oklahoma State University), "Crossword Compilation" by Hal Berghel (University of Arkansas at Fayettville), "The Axiom of Choice and Its Relatives" by Dwayne Coffins (Hendrix), 'Lattice-Ordered Groups" by Mona Cherri (University of Central Arkansas), "Algebraic Geometry" by Cassandra Cox (University of Arkansas at Little Rock) and "The Calculus of Variation" by Richard Rolleigh (Physics Department, Hendrix). A picnic and the induction of new members closed out the year's activities.

CONNECTICUT GAMMA (Fairfield University). During the fall semester the chapter sponsored two lectures. Julius Zelmanowitz (University of California at Santa Barbara) gave the first talk, "Structure of matrix Subrings." In the second, Joan Wyzkoski (Fairfield University), described her summer research experience with "Parallel Processing and numerical Linear Algebra." In the spring, the annual initiation ceremony culminated a week-long celebration of the Centennial of the American Mathematical Society. Two speakers from the Courant Institute of Mathematical Sciences, New York University, began the celebration. Sylvain Cappell discussed "Non-Linear Similarities." "Daniel Bernoulli: The Smallpox Controversy and the Rise of mathematical Epidemiology" was the topic of Warren Hirsch's talk. Clayton Dodge (University of Maine), the Problem Editor of the Pi Mu Epsilon Journal, gave his "Reflections of a Problem Editor" at the induction ceremony. During the Annual Arts and Sciences Awards Ceremony, three members, Jill Christensen, Christine Kolar and Jean-Marie Matthews received recognition for their outstanding performance in mathematics. Each was given a Pi Mu Epsilon certificate of achievement, one of Martin Gardner's books, and one-year memberships in the Mathematical Association of America.

GEORGIA BETA (Georgia Institute of Technology). At the Honors program, Jon. M Jenkins and Randall Avery Shealey each received a mathematics book of his choice. The recipients received the degree B.S. in Applied Mathematics, having earned grade point averages of at least 3.7 (A = 4.0) in all mathematics courses taken.

GEORGIA DELTA (Spelman College). In October, the college hosted a group of representatives from AT&T in Chicago. The seminar was held at Morehouse College and the representatives gave a presentation on "Digital Images Processing." In January, Dr. Lee Lorch, Emeritus Professor of Mathematics at York University in Ontario, Canada, and Visiting Professor of Mathematics at Spelman, lectured on the "History of mathematics." At a ceremony in April, 18 students and 6 faculty members from the Atlanta University Center were initiated into Pi Mu Epsilon. Guest speaker, Dr. Henry Gore, Chairman of the Mathematics Department at Morehouse College, spoke on "The Philosophical Foundations of mathematics."

ILLINOIS IOTA (Elmhurst College). On November 6, 1986, the Illinois Iota Chapter was installed by National President, Milton Cox. Dr. Cox led a discussion on "Discrete Mathematics in the Curriculum." A joint Christmas bake sale with the Elmhurst College Mathematics Club netted $49.40. In April, the Associated Colleges of the Chicago Area Spring Student Symposium was held at North Central College. Five members of Pi Mu Epsilon presented papers and nine new members were initiated. Mathematics Awareness week was celebrated with a display and puzzle contest. In May, the chapter sponsored a Career Night at which three alumni talked about their careers in teaching, computers/business, and the actuarial field.

KANSAS GAMMA (Wichita State University). Speakers during the school year included Phyliss McNickle on "Preparing for the Job Search Success," Laurie Frisch on "An Investigation into the Smoothing of β–Curves to Predict the Accretion of Ice on an Airfoil," Linda D. Casey on "Cooperative Education: Learning that Works," Dr. Gonzalo Mendieta on "Historical notes on Probability," Glenn fox on "Pseudoprimes," Dr. Stephen Brady on "Some Remarks on N. Bourbaki," Dr. David Mentor on "mother nature -- The Greatest mathematician," 'Jeanne Daharsh on "Working as an Actuary," Michael Cardenas on "Numerical Odyssey," and Dr. William Perel on "Topics in the "History of mathematics." At the joint meeting of the Mathematical Association of America and Kansas Association of Teachers in Mathematics, student Laurie Frisch spoke on "An Investigation into the Smoothing of β-Curves to Predict the Accretion of Ice on an Airfoil," Student Mukul Patel spoke on "Working Habits of Ramanujan," and student Dewi Saleh spoke on "The Effect of Intra-Cluster Correlations on the Regression Estimation in Finite Population Inference."

MICHIGAN DELTA (Hope College). In addition to a full schedule of departmental colloquia which members of Pi Mu Epsilon are encouraged to attend, the chapter organized and supervised a sale of used mathematics books, sponsored a dessert get-together at the home of Chapter Advisor, Professor Elliot A. Tanis, and provided proctors and graders for a Hope College-sponsored mathematics competition for 300 area high school students.

MINNESOTA GAMMA (Macalester College). Student James E. Colliander lectured on "A Brachistochrone Through the Earth." Jim repeated his lecture at the Annual Student Pi Mu Epsilon Conference in April at St. John's University in Collegeville, MN. The Annual Initiation Program featured Professor Georgia Benkart (University of Wisconsin), who lectured on "What is Lie Algebra, Anyway?" Ann C. Decker and Queen. Lee foo received the Ezra Camp Awards. Rebecca M. fee was awarded The Mathematics Achievement Prize. A fall picnic and an end-of-classes picnic rounded out the chapter's social activities. A T-shirt sale raised funds to support chapter programs. A very successful Career Night was held at the home of Professor Allan Kirch. Macalester graduates John Kim, Janet Nelson F.C.A.S., Kathryn Gretler, George Letter and Leonard Volovets, respectively talked about their experiences in investing/banking, insurance, industry (3M), teaching (St. Paul Academy) and graduate school (University of Minnesota).

MINNESOTA ZETA (Saint Mary's College). The chapter conducted a broad spectrum of business either through special committee work or regular chapter-wide business meetings. Dr. Richard Jarvinen (St. Mary's) lectured on "Mathematics of Star Wars." At the initiation ceremony, Dr. Paul Froeschl talked about "Lewis Carroll's method of Condensation," A puzzle contest was part of February activities.

NEW MEXICO BETA (New Mexico Institute of Mining and Technology). Together with the Math Club and the Mathematics Department, the chapter sponsored campus-wide picnics during the fall and spring semesters. The chapter assisted in the promotion of Mathematics Awareness week. During the year, eight lectures were co-sponsored by Pi Mu Epsilon and the Mathematics Club. These included "Testing Surface Antennas" by Clyde Dubbs (NM Tech), "Intractability" by Dr. Jack fink (Technical Vocational Institute, Albuquerque), "Why Statistics" by Dr. Robert Easterling (Sandia), "Clifford Algebras and Physical Reality" by Dr. Lawrence Werbelow (Chemistry, NM Tech), "When to Solve, When to Approximate" by Clay Williams (MIT Lincoln Labs), "Introduction to Multigrid methods" by Dr. Steve Shaffer (NM Tech), "Leonhard Euler: A mathematician for all Seasons" by Dr. Alan Sharples and "Fractals" by Dr. Alan Hurd (Sandia National Labs). The chapter and math club jointly sponsored a weekly problem-solving contest which was administered by Professor Clyde Dubbs (NM Tech).

NEW YORK OMEGA (St. Bonaventure University). Problem solving has become part of the honors option available for several upper division courses. The annual Pi Mu Epsilon Award was presented to Heather Danahy. Honorable mention was received by Christina Malack. Chapter-sponsored lectures during the school year included "Strange Attractors" by Professor Erik Hemmingsen (Syracuse University), "Optimization without Calculus" by Dr. Harry Sedinger (St. Bonaventure) and "The Byzantine General's Problem" by Dr. Steven Andrianoff (st. Bonaventure). "A Mathematical Mystery Tour" from the NOVA series on PBS was shown in January. Dr. Myra J. Reed, active supporter of the chapter since 1979 and faculty correspondent, passed away last fall following heart surgery. In recognition of her devotion to New York Omega, the Pi Mu Epsilon Award has been renamed the Myra J. Reed Award.

NEW YORK PHI (Potsdam College). The chapter sponsored a very successful faculty/student mixer. In the fall, Professor L. C. Kappe (SUNY at Binghamton) lectured on "mysterious Transcendental Numbers." The fall induction dinner was held at Sunset Lodge in Norwood, NY. In the spring, the chapter sponsored another successful mixer. Dr. Pat Roberts gave a talk regarding her findings on recent research on mathematics majors. The spring induction meeting was at Uncle Max's in Potsdam, NY.

NORTH CAROLINA LAMBDA (Wake Forest University). A full program of speakers included Professors James Kuzmanovich (WFU) and Stan Thomas (WFU) on "Professional Opportunities in Computer Science and mathematics," Dr. John Baxley (WFU) on "A NASA Summer," Professors Fred Howard (WFU) and Elmer Hayashi on "mathematics Related to the Work of Ramanujan," Professor Irl C. Bivens (Davidson college) on "When Derivatives Count, Coordinates Don't Count, and We Count the Derivatives: Discovering Formulas Through A Priori Resumptions," professor David C. Wilson (WFU) on "Problem Solving in mathematics," Provost (and James B. Duke Professor of Mathematics at Duke University) Phillip A. Griffiths on "The Unity of mathematics: Poncelet's Porism," and Professor Daniel Canas (WFU) on "GraphOS: A Graphic Operating System." WFU student, David McLean, spoke on his honors paper "Plane Symmetry Groups," and Duke University graduate student, Salman Azhar, spoke "On Learning Permutation Groups by Examples." Video tapes 'Artificial Intelligence Techniques at Kennedy Space Center" and "The Life and Work of Srinivasa Ramanujan" were shown in March. A Mathematics and Computer Science Department Picnic completed the school year activities.

OHIO NU (University of Akron). Students Beth A. Moore, Sheryl M. Patrick, Christine M. Sullivan and 'Jon Zeigler were awarded one-year memberships in the American Mathematical Society. Darrel Ulm received a one-year membership in the Association for Computing Machinery, Robert Bodi, Michael R. Henry, Phillip M. Lovalenti, Yu Ping, Jeffrey S. Umlauf and Paul Wilson were given one-year memberships in the Society of Industrial and Applied Mathematics. The Samuel Selby Scholarships were awarded to Sheryl M. Patrick and 'Jeffrey S. Umlauf. Andrew T. Christian was the Northeastern Ohio Science Fair Winner in the Mathematical Sciences Category.

OHIO OMICRON (Mount Union College). In October, the chapter sponsored a trip to the 14th annual Pi Mu Epsilon Conference at Miami University in Oxford, Ohio. Chapter President Jim Kirklin spoke at the Conference. In March, former Mount Union student and Pi Mu Epsilon member, currently a graduate student in mathematics at Indiana University, John Nedel, gave a talk.

OHIO ZETA (University of Dayton). In August, 'Jeff Diller delivered a paper "The Isoperimetric Inequality" at the National Pi Mu Epsilon Conference at the University of Utah. In September, Dr. Sheldon Davis (Miami University) lectured on "The normal Moore Space Problem - An Introduction to Topology." Several chapter members presented papers at the Annual Pi Mu Epsilon Conference at Miami University including Paul W. Kollner on "HoB to Win at Nim," Mark Liatti on "Don't Let it Bug You," Margie Mascolino on "What's the Difference?", Matt Davison on "Ramsey Numbers and Complete Graphs," Rosemary A. Secoda on "Interpolating Polynomials and Their Applications," Julie Anderson on "Boolean Algebras as Vector Spaces," 'Jeff Diller on "Curves and Geodesics," and Greg Scanlon on 'Relating Operations by Means of a Function." Matt Davison, Paul Kollner and Margie Mascolino presented their talks again at a Pi Mu Epsilon Conference at St. Norbert's College in November. Greg Scanlon, Rosemary Secoda, Mary Kazcynski, 'Jeff Diller, Mark Liatti, Viki Steinlage, Michelle Arkony and Matt Davison presented talks at regular Pi Mu Epsilon meetings during the school year.

PENNSYLVANIA NU (University of Scranton). At the Fall initiation, guest speaker was *Martin Hopeman*, General Electric, who spoke on "Engineering and mathematics." At the Spring Initiation, guest speaker was *Dr. James C. LoPresto*, Professor of the Physical Sciences, who lectured on "Undergraduate mathematics Applied to Scientific Research." Math movies during the school year included "Donald in Mathmagicland" and "Music of the Spheres." Visiting Professor *Janusz Kaptor* (Poland) lectured on "*Topological Degree.*" Several chapter members attended the Mathematical Association of America Meetings at Bethany College in April.

PENNSYLVANIA OMICRON (Moravian University). During the fall semester the chapter hosted a talk on "Combinatorics" by Chester *Salwach* of Lafayette College. The major event sponsored by the chapter was the second annual Moravian College Student Mathematics Conference. Seventeen colleges and universities in eastern Pennsylvania, New Jersey and Delaware were represented among the 104 participants. The keynote speaker was *Neil Sloane*, AT&T Bell Laboratories who gave an "Introduction to Coding Theory." Eleven student papers completed the program. The Third Annual Moravian College Student Mathematics Conference is planned for February, 1989.

SOUTH CAROLINA DELTA (Furman University). Included in the year's activities were support of a math tournament for high school students, a career information lecture on actuarial science, and several social events including the annual banquet and initiation of new members. Lectures were given by Sayre Associates affiliates *Kristie Sayre* and *Joey Nichols* on "A Career in Rctuarial Science," *Terri Lindquester* on "Hamilton Properties in Graphs," *Sheila Waggoner* on "Applications of maximum Principles and Partial Differential Equations," *Robert Jamison* on "Counting the Forests in a Tree," *C.H. Edwards, Jr.* on "Isaac newton's Nosecone Problem Revisited," *Peter Braza* on "Euler's Theorem and an Unbreakable Secret Code." and *Craig Guilbault* on "Topology and Knot Theory."

TENNESSEE GAMMA (Middle Tennessee State University). Eleven chapter meetings included the presentations "Is it True or Ain't it?" by *Dr. Harold S. Spraker*, "numerical Analysis" by Dr. *Paul Hutcheson*, "Jobs in mathematics" by Ms. *Lora Clark*, "A Revolution in Algebra - A Tribute to Evariste Galois" by *Dr. Vatsala Krishnamani*, "Who Cares about Statistics?" by *Dr. Curtis Church*, and "How to The the G.R.E. in mathematics" and "The Fibonacci Sequence" by *Mike Pinter*. The chapter created a new organization named the Mathematics Organization of Middle Tennessee State University, designed to promote interest and scholarships in mathematics for non-majors and for freshmen and sophomore mathematics majors. Also instituted was the Middle Tennessee State University Problem Solving Group. *Joey Peay* won the chapter's contest to design a T-shirt to celebrate Mathematics Awareness Week. During that week the chapter co-sponsored a mathematics film festival with the new Mathematics Organization. Chapter members worked as proctors and graders for the annual junior high mathematics contest which this year tested 400 area students.

TEXAS IOTA (The University of Texas at Arlington). in the fall semester, the chapter sponsored a talk on "Careers in mathematics" by *Dr. Kent Nagle* of the University of South Florida. A most exciting success came in the annual UTA Science Fair Competition in April. The chapter presented a video, slide, and computer display on fractals and fractal geometry at which fractal image postcards were sold. The presentation won the "Blinded Me with Science" Award, The fractal video and postcards were obtained from Art Matrixx of Ithaca, New York, a private company working with the National Supercomputing Facility at Cornell. The inspiration for a fractal display came from *Jennifer Zobitz's* survey article on fractals in the Fall 1987 Pi Mu Epsilon Journal. During National Mathematics Awareness Week lectures were presented by graduate advisor *Theresa Kelly*, department chairman *Dr. George Fixx* and departmental Ph.D. students on their areas of research. The annual department barbeque/picnic brought the successful school year to a close.

VIRGINIA GAMMA (James Madison University). The year's activities began with a welcome back meeting/social. October began with the annual fall picnic with ACM and math club, a fund-raising book sale, and an interesting and informative talk on polyhedra by *Dr. William M. Sanders*, a member of the faculty. In February, *Dr. Mullinex* lectured on the decimal representation of fractions. National Mathematics Awareness Week was celebrated with films, lectures, a puzzle and a faculty reception. April began with the Annual Spring Banquet for Pi Mu Epsilon and the Math Club. The year ended with the Annual Spring Picnic with ACM and the Math Club.

WISCONSIN (St. Norbert College). *Shelly Braatz, Katie Coenen, Mary Ehle* and *Summer Quimby* were student attendees of the 1987 Pi Mu Epsilon Conference in Salt Lake City, Utah in August. *Mary* and *Summer* presented papers. The students were accompanied by chapter advisor, *Dr. Rick Puss.* In October, *Shelly Braatz, Katie Coenen, Kandi Kilkelly, Summer Quimby, Becky Vande Hey* and *Colleen Weyers* attended the regional Conference at Miami University where *Summer* presented a paper. In March, *Brian Augustian, Mary Ehle, Dave* "Otis" *fanner*, and *Steve Setterlun* attended the Regional Conference at St. John's University, with papers by *Brian, Mary*, and Otis. In April, *Katie, Coenen, Chris Ferriter*, and *Summer Quimby* attended the conference at Rose-Hulman in Indiana, with *Summer* giving a paper. Talks during the academic year included "Actuaries: Technicians or Leaders?" by *Mrs. Lynn Debbink*, AAL Insurance, "An Introduction to Integer Programming" by *Dr. Rick* 'Ross, St. Norbert College, "The Legacy of Leonardo of Pisa: A mathematical Gem from the middle Ages" by *Dr. Forrest Baulieu*, University of Wisconsin - Green Bay, "An Introduction to Chinese mathematics Education" by *Mr. Huang Too*, St. Norbert College, and "How to Ask Sensitive Questions Without Getting Punched in the Nose" by *Dr. frank Hannick*, Mankato State University. The highlights of the academic year included running the sixth annual High School Math Competitions (in conjunction with Sigma Nu Delta Math Club) and hosting the Second Annual Pi MU Epsilon Regional Conference at which the invited speaker, *Dr. Joseph Gallian*, University of Minnesota - Duluth, talked on "Modular Arithmetic in the marketplace" and "Traversing a Grid on a Torus."

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