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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

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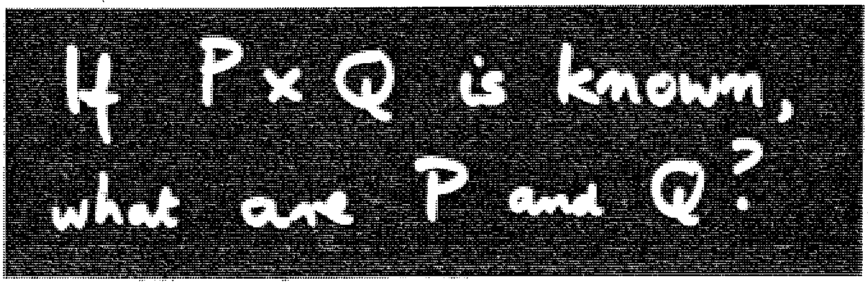
Prime Numbers and Secret Codes

KEITH DEVLIN, *University of Lancaster*

The author continues the discussion of large prime numbers which he began in the last issue of *Mathematical Spectrum*. This article first appeared in the *Guardian* on 18 August 1983.



Certain subjects in mathematics are like distant relatives: you hear nothing from them for years on end and then suddenly they turn up all the time. At the moment it's prime numbers. Within a few months of the discovery of a new largest known prime number comes the publication of a new method to test whether a given number is prime or not.

There seems to be no problem to the layman. A prime number is a whole number which is not divisible (with no remainder) by any whole number other than 1 and itself, so to test if a given number is prime or not you simply look at all smaller numbers (besides 1) and check if any of them divides the number. If none of them does, the number is prime, otherwise it is composite. For small numbers this method is highly feasible, and anyone can quickly see that 2, 3, 5, 7, 11, 13, 17, are prime, whereas 4, 6, 8, 9, 10, 12, 15 are not. Larger numbers can take a little longer. For example, is 221 prime? (Answer, 'No', it is the result of multiplying 13 by 17.) Five-figure numbers already present quite a challenge to the average mind, but the above 'obvious' method works well on a computer, and produces an answer practically instantaneously for any number less than, say, 1 000 000. (Modern computers are capable of performing tens of millions of arithmetic operations in a single second.) But for numbers with, say, in excess of 100 digits, the 'obvious' method is not feasible even on a computer. Computers are still used in order to test the primality of such numbers, but they make use of other methods. The idea is to teach the computer some mathematics.

For some numbers, testing primality is easy, regardless of the size of the number. For example, the largest prime mentioned before is of a special kind: it is a Mersenne number.

Mersenne numbers are of the form $2^N - 1$ (i.e. 2 multiplied by itself N times, minus 1). There is a particularly efficient method for testing whether a Mersenne number is prime or not, known as the Lucas–Lehmer test. For $N = 86243$, a CRAY-1 computer in the United States took just over one hour to discover that the corresponding Mersenne number is prime. To do this by the 'obvious' method would take longer than the lifetime of the universe, even using the CRAY-1, the world's fastest computer. In the space available here, it is not possible to describe how the Lucas–Lehmer test works: it must suffice to say that it makes use of some advanced mathematical ideas.

However, the existence of efficient tests for certain special kinds of numbers does not help much when the problem is to determine whether a randomly chosen large number is prime or not. Moreover this problem is not without relevance. It lies at the heart of what is at the moment the most secure coding technique in the world, the so-called Public Key Cryptography.

Using this method, I could ask you to send me a message in code, explaining the coding method to be used by writing an article in a national newspaper. Using this description, you code the message and send it to me. Suppose now that a bad guy tries to intercept and decode the message. Even though he knows how you coded the message, he is unable to decode it. Knowing the method of encoding does not help you decode.

An extra piece of information is required for this, something known only to the receiver, not the sender. This remarkable method, discovered by the mathematicians Ronald Rivest, Adi Shamir, and Leonard Adleman of the Massachusetts Institute of Technology a few years back, uses large prime numbers, i.e. of the order of 50 to 100 digits.

The potential receiver starts with two (or more) large prime numbers, which he keeps secret. What he sends to the sender is the product of these prime numbers. Knowledge of the product is all that is required for encoding, but decoding requires knowledge of the primes themselves. And whereas it is theoretically possible to factor any number into primes, there is no known method which works for very large numbers. The problem is the time it would take even an extremely fast computer to do the job.

The time required to factor a large number increases exponentially with the size of the number. The rapid growth of exponential functions means that factoring a 100-digit number is practically not feasible. Of course it is always possible that someone will come up with a clever new method for factoring large numbers. Indeed, it may have already been done, but since much of the work on this subject is classified (it is believed), we don't know. In the meantime, the best thing is to use primes as large as possible (though not Mersenne primes as there are only 28 of them known) and hope for the best.

A good, useful method for testing primality should be capable of being executed on a computer in a time that increases linearly with the size of the number. That is, if the size of the number is doubled, so is the running time, and so on. By 'size' is meant the number of digits in the number in this context. (So one can appreciate at once that the 'obvious' primality test does not have this property: if the number of digits in a number is doubled, the time involved in testing primality increases dramatically, using a computer or not.) Until recently, one of the best methods of testing primality in linear time was discovered by Robert Solovay and Volker Strassen in 1976.

The main drawback with their method was that it did not produce a 100 per cent certain result. Typically, if the method came up with the result 'prime' for a given number, then you could be 99 per cent certain that the number was indeed prime, but not absolutely sure. Would you trust your darkest secrets to a coding method which depends upon such a test?

Another method, due to Gary Miller, produces a 100 per cent certain answer in a time which increases with the fourth power of the size of the number (which is feasible for a fast computer), but the method depends upon some mathematics which may itself be wrong. Obviously what was needed was a fast method which gives a 100 per cent answer and which uses mathematics that we know to be correct.

In a recently published article in *Annals of Mathematics* (reference 1), L. M. Adleman, C. Pomerance, and R. Rumely explain such a method. This method uses some of the ideas of the probabilistic Solovay–Strassen test, but is itself not probabilistic. Working from a pre-publication copy of the Adleman paper, H. Cohen and H. W. Lenstra Junior (reference 2) have since managed to speed up the method, and their improvement is due to be published shortly.

The method is not strictly linear in its operating time. It increases with the size of the number raised to the power Cloglog(size), i.e. a constant C times the logarithm of the logarithm of the size. However, because of the extremely slow growth of the logarithm function, the method works well on a computer. Typically, a fast computer requires about 30 seconds to check if a 100-digit number is prime and eight minutes for a 200-digit number. Finding a good method for factoring numbers is now the order of the day.

References

1. L. M. Adleman, C. Pomerance and R. Rumely, On distinguishing prime numbers from composite numbers, *Annals of Mathematics* **117** (1983), 173–206.
2. H. Cohen and H. W. Lenstra Jr, Primality testing and Jacobi sums, *Mathematics of Computation* **42** (1984), 297–330.

All constants are zero!

Dear Editor,

An interesting paradox came about in my calculus class that is new to me. We know $\int 0\,dx = c$, $\int dx = x + c$ and $\int af(x)\,dx = a\int f(x)\,dx$. Thus, $c = \int 0\,dx = 0\int dx = 0(x + c) = 0$. Therefore we have all constants are zero.

Yours sincerely,

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Coin-Tossing

D. J. COLWELL AND J. R. GILLETT, *North Staffordshire Polytechnic*

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Coin-tossing is a subject which is often used to illustrate elementary concepts in probability theory. A practical situation where such tossing regularly takes place is at the start of many sporting fixtures. The coin is used to initiate such fixtures because it is felt, intuitively, that each side will have an equal chance of winning the toss and, therefore, an equal chance of taking advantage of such factors as wind direction or the state of the ground. Such advantages can sometimes be decisive, so it is perhaps worth checking on a sequence of such coin-tossing occasions to see if the intuitive feel about the fairness of coin-tossing is borne out.

The sequence will consist of a succession of runs, and the length of each run is the number of times one of the contestants has won the toss on successive occasions. By assuming that the outcomes of each toss are independent of one another and that the probability of the outcome of each toss is fixed, a model may be constructed for the probabilities of runs of various lengths, and then the expected number of runs of each length can be compared with the actual number of that length.

Cricket is a sport in which the result of the toss can be particularly decisive. Hence the sequence of coin-tossing occasions which will be considered is that provided by the England v. Australia contest.

The first match between these two countries took place in 1876, and since then matches have been played with increasing regularity. Indeed, up to and including the 1981 English cricket season, 248 matches had been scheduled. For the purposes of our coin-tossing investigation, only 247 of these matches can be included, as in one match in one of those fondly remembered dry summers before the last war play was abandoned without a ball being bowled and without the toss ever taking place! In these 247 matches Australia has won the toss 121 times and England has won it 126 times.

To help with our investigation, it is convenient to consider these results as a sequence of A's and E's, where A denotes the event of Australia's winning the toss and E denotes the event of England's winning the toss. We shall assume that the probability of an A or an E is given by $P(\mathrm{A}) = p$ and $P(\mathrm{E}) = q$, where $p + q = 1$.

From the data an estimate of p is $\hat{p} = 121/247 = 0{\cdot}4898785$, which is close to $p = \frac{1}{2}$. We shall assume the A's and E's occur independently of one another, since there appears to be no reason why this should not be the case, and ask whether there is any reason to doubt that $p = q = \frac{1}{2}$.

To help answer this question, it is convenient to introduce random variables X_i

$(i = 1, 2, \ldots)$ defined by

$$X_i = \begin{cases} 1 & \text{if the } i\text{th term of the sequence is A} \\ 0 & \text{if the } i\text{th term of the sequence is E,} \end{cases}$$

and a random variable T defined by

$$T = \sum_{i=1}^{n} X_i.$$

The latter variable represents the number of tosses won by Australia after n matches.

Since $P(X_i = 1) = p$ and $P(X_i = 0) = q$, it may be deduced that

$$E(X_i) = p, \qquad \operatorname{Var}(X_i) = p(1-p)$$
$$E(T) = np, \qquad \operatorname{Var}(T) = np(1-p).$$

Also, for large n, the variable T may be considered to be normally distributed. Hence, with $n = 247$, $p = \frac{1}{2}$ and the observed value of T at 121, the corresponding standard normal distribution value is 0·32. Thus there is no evidence for rejecting the null hypothesis that $p = q = \frac{1}{2}$ at the 74 per cent level of significance!

Having found such strong evidence for accepting $p = q = \frac{1}{2}$, we will now delve further into the data to see if certain statistical predictions, based on $p = q = \frac{1}{2}$, are actually borne out. Our investigation will centre round the lengths of runs of A's and E's in our sequence. Altogether this sequence of A's and E's has 116 runs, of which 58 are of A's and 58 are of E's.

In the general case, given that a run in the sequence is of A's, the probability that it is of length r $(r = 1, 2, \ldots)$ is $p^{r-1}q$, whereas, given that a run is of E's, the probability that it is of length r $(r = 1, 2, \ldots)$ is $q^{r-1}p$. When $p = q = \frac{1}{2}$ both these probabilities are equal to $(\frac{1}{2})^r$ $(r = 1, 2, \ldots)$ and, with this probability model, the expected values (rounded to the nearest integer) of runs of various lengths for both Australia and England, separately, may be compared with the actual distribution of runs, as shown in Table 1.

A chi-squared test shows that the model is almost a perfect fit in the case of Australia and that there is no evidence for rejecting the model at the 5 per cent level of significance in the case of England.

We can also consider the total number of runs of length r $(r = 1, 2, \ldots)$ in the sequence, whether they be of A's or E's. In the general case, when we consider a run, the probability that it is of A's and of length r is $p^r q$, whereas the probability that it is

TABLE 1

Length of run (r)	1	2	3	4	5	6	7	8
Number of runs of A's of length r	26	16	8	6	0	0	1	1
Number of runs of E's of length r	31	8	8	5	2	3	1	0
Expected number of runs	29	15	7	4	2	1	0	0

TABLE 2

Length of run (r)	1	2	3	4	5	6	7	8
Total number of runs of length r (both A's and E's)	57	24	16	13	2	3	2	1
Expected number of runs	58	29	15	7	4	2	1	0

of E's and of length r is $q^r p$. Hence the probability of having a run of length r, whether it be of A's or E's, is $p^r q + q^r p$ ($r = 1, 2, \ldots$). In the case $p = q = \frac{1}{2}$, this last probability becomes $(\frac{1}{2})^r$ ($r = 1, 2, \ldots$). Thus the observed and expected values (rounded to the nearest integer) for this model are as shown in Table 2.

Using a chi-squared test, it can be seen that the probability model again leads to an almost perfect fit.

Finally, since Test Matches between England and Australia have usually been played in the form of five-match series, we shall examine the data available for the results of tossing the coin in each series by modelling each series with the binomial probability distribution, based on the terms of $(p+q)^5$, where $p = q = \frac{1}{2}$. Thus, in any five-match series, the probability of Australia's winning u ($u = 0, 1, 2, 3, 4, 5$) tosses, and hence of England's winning $(5-u)$ tosses, is ${}^5C_u p^u q^{5-u}$. The observed and expected results (rounded to the nearest integer) for the 34 five-match series which have been played are as given in Table 3.

TABLE 3

Five-match series		Observed frequency	Expected frequency
Number of tosses won by Australia	Number of tosses won by England		
0	5	1	1
1	4	7	5
2	3	8	11
3	2	12	11
4	1	4	5
5	0	2	1

Yet again, the probability model is a good fit.

Throughout this article, when the estimate $\hat{p} = 0{\cdot}4898785$, provided by the data, is used in place of $p = \frac{1}{2}$, virtually identical results are obtained.

Short Reasons Why the Impossible is Impossible

KEITH AUSTIN, *University of Sheffield*

Keith Austin is a Lecturer in Pure Mathematics at the University of Sheffield. He is interested in the appreciation of mathematics, and, with this in mind, he has adopted a rather unusual style for this article. The article describes four mathematical problems as briefly as possible, and puts them into a story about people and animals. Instead of giving more information about the problems, the article concentrates on the reactions of the characters to the problems and to one another.

1. In which the White Rabbit sets a problem and Alice falls in the river

It was a warm summer afternoon. Alice and the White Rabbit were sitting on a grassy bank by the river.

Rabbit gave Alice a piece of paper with Figure 1 drawn on it, and asked her to redraw it so that no two lines crossed. After a few minutes Alice drew Figure 2. Rabbit then gave her Figure 3 but Alice laughed and said she had met it before and knew it was impossible to draw it without lines crossing. She then drew Figure 4 and said that was also impossible.

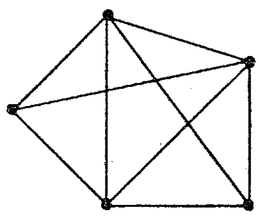
Figure 1

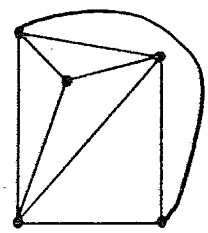
Figure 2

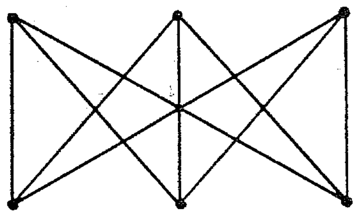
Figure 3

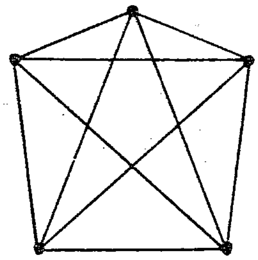
Figure 4

Rabbit was annoyed that Alice had met his problem, so he asked why Figures 3 and 4 were impossible. She explained that as they were so simple you could easily try all the possibilities and see none worked.

Rabbit scribbled away furiously and finally gave Alice Figure 5. After working for quite a time Alice said, 'I don't think it is possible. I think I have tried every possibility but I might have missed one as there are so many'.

'That's not good enough', snapped Rabbit, 'I want a short definite reason why it is impossible'.

Alice studied the diagram for a while and then gleefully shouted, 'I've got it', which made Rabbit jump. 'If I redraw some of the points and lines from Figure 5 I can get Figure 6, which we know is impossible to draw, and so the whole of Figure 5 is impossible'.

'Would you like to try another one?' said Rabbit.

'If it is possible to draw then my drawing is a nice short answer, but if it is impossible then I might not be as lucky as before to find a short reason why', replied Alice.

'No', said Rabbit, 'If it is impossible then you are certain to be able to find either Figure 3 or Figure 4 somewhere in it, possibly with extra points like b, c and i in Figure 6. This was proved by Kazimierz Kuratowski in 1930'.

'He didn't do much', said Alice, 'I can easily show that if Figure 3 or Figure 4 are in a diagram then it is impossible to draw it'.

Rabbit snorted and pushed Alice in disgust. Unfortunately she lost her balance and rolled down the bank into the river. As she climbed out, wet through, she asked Rabbit why he was annoyed.

'What did you just say?'

'I said I could show that if a diagram contained Figure 3 or Figure 4 then it . . .'. Alice's voice trailed off as she realised her mistake and she blushed. As Rabbit looked at her, sopping wet and embarrassed, he felt sorry he had pushed her in and a tear appeared in his eye. Alice saw it and got out her hanky to wipe it away, but the hanky was wet through and so she gave up the idea.

'I can prove that if a diagram contains Figure 3 or Figure 4 then it cannot be drawn, but Mr Kuratowski proved that if a diagram cannot be drawn then it contains Figure 3 or Figure 4, which is a very different matter', said Alice.

'Well done!' said Rabbit, 'It's very easy to make the slip of the tongue that you made, but it really is important to get it the right way round'.

'This means that when you set me a problem, whether it is possible or impossible, I can justify my answer by a single drawing.

'I suppose this problem of avoiding lines crossing is important for wires in electrical circuits, pipes carrying gas and water, and even for roads'.

Just then they heard voices from the other side of the grassy bank. They peered over the top and saw a young boy, a plump teddy bear and a small pig.

2. In which Christopher Robin sets a problem and Piglet says it is like Poohsticks

'Now Pooh and Piglet, I have a problem for you'.

'Oh thank you Christopher Robin', said Piglet.

'Look at this drawing', said Christopher Robin, as he showed them Figure 7.

'I want you to rub out some of the lines so that the final diagram has precisely one line from each dot on the left and one line or no line from each dot on the right'.

'Come on Pooh, let's get started'.

'What is the point?' said Pooh.

'Oh! Why!' said Piglet, 'Don't you think it is a jolly thing to do on a warm summer afternoon?'

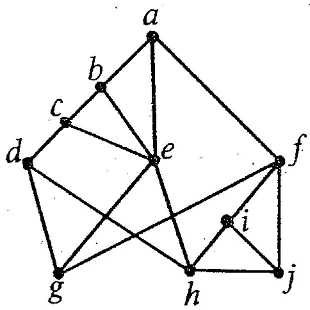

Figure 5

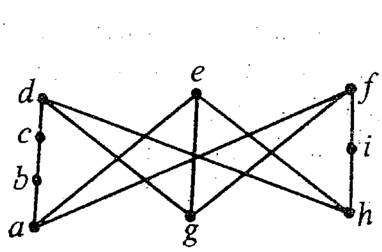

Figure 6

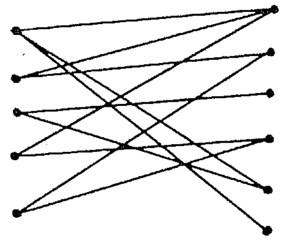

Figure 7

'No', said Pooh.

'Why!' said Piglet, 'It's like Poohsticks, and you like that. It's just a game. The point is that it's a nice thing to do on a sunny day'.

'Well, I do like Poohsticks', agreed Pooh, 'So perhaps I will get to like this game'.

Christopher Robin suggested that they thought of the dots on the left as pots of honey and those on the right as bears. Then a line meant that the bear at one end would eat the pot at the other end. They had to find a different bear to eat each pot of honey.

Pooh said it was ridiculous to suggest there could be a pot of honey that a bear would not eat. Also he did not like the idea of having to share the honey with other bears.

They worked for a while and finally produced Figure 8.

Christopher Robin congratulated them and then asked them to try Figure 9.

After a lot of work, Pooh snorted, 'It's impossible'.

'Right!' said Christopher Robin, 'Then show me why it is impossible'.

'Because we can't do it', said Pooh.

'Don't be silly', said Piglet, 'There must be some special reason in the diagram'. After a while, Piglet shrieked, 'I see why! Take the top three and the bottom dots on the left and draw all the lines from them'. He drew Figure 10.

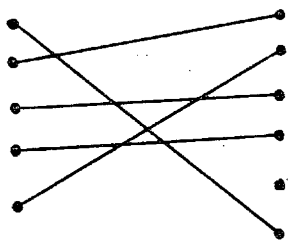

Figure 8

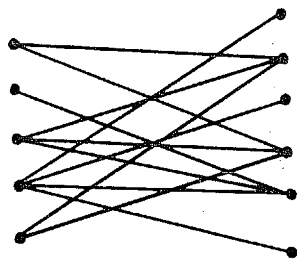

Figure 9

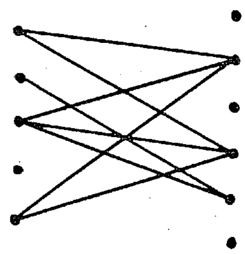

Figure 10

'These four dots on the left are only joined to a total of three dots on the right, so it is impossible'.

'Well done!' said Christopher Robin.

'Piglet was just lucky', said Pooh.

'No he wasn't', said Christopher Robin, 'Whenever the problem is impossible, you can find some dots on the left which are joined, in total, to fewer dots on the right. For example, you found four pots of honey which were liked by a total of only three bears'.

'So Goldilocks gets the fourth pot', whispered Pooh.

Ignoring Pooh's remark, Christopher Robin continued, 'This was proved by Philip Hall in 1935'.

'So when you give us a problem we can always find a single diagram which justifies our answer, whether it is possible or impossible'.

'Just like Mr Kuratowski's problem', shouted Alice as she and the White Rabbit ran down the bank to join the other three.

3. In which Christopher Robin takes care of Alice and the White Rabbit sets another problem

After the five had introduced themselves, Christopher Robin noticed that Alice was wet through and shivering. He took off his coat and put it on her, told her to sit down by him and put his arm round her to try and keep her warm. Pooh turned so that his back was to the couple and began to pretend he was playing the violin, but he stopped when Piglet gave him an angry look.

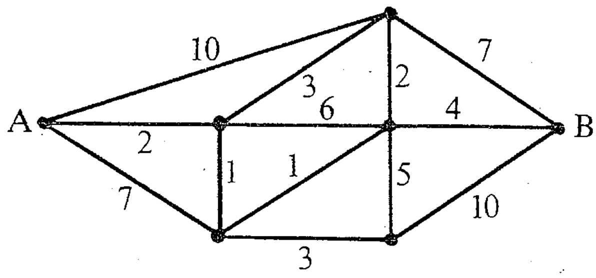

Figure 11

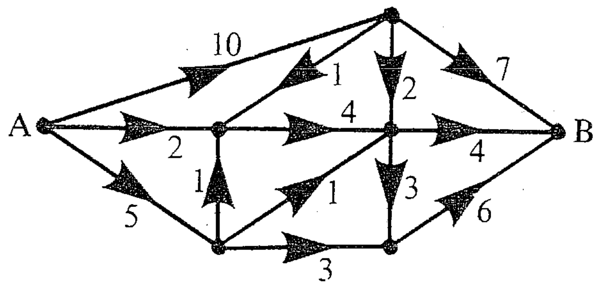

Figure 12

The group then began to discuss their two problems, until Rabbit said, 'I have another problem which is similar to those two'. He showed them Figure 11 and explained that it was a road map, and the number on each road was the maximum number of cars that could go along that road. The problem was to send 17 cars from A to B, indicating how many should go along each road, and in which direction. After a time, the others produced Figure 12.

Rabbit congratulated them and asked them to try and send 19 cars from A to B on Figure 13.

Some time later, the others agreed it was impossible. Rabbit asked for a simple reason why it was impossible. After a few minutes, Alice drew a dotted line as in Figure 14.

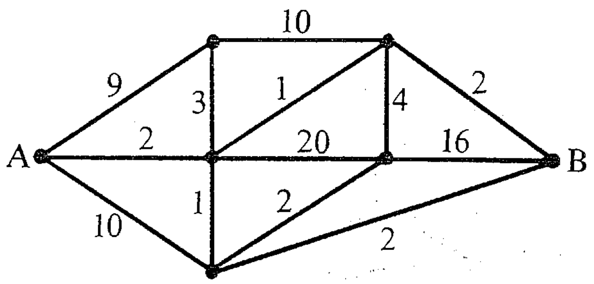

Figure 13

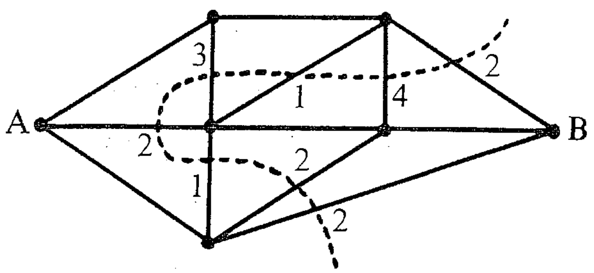

Figure 14

'To get from A to B, all the 19 cars must cross the dotted line. However, if we look at the roads crossing it, the maximum number of cars is $2+4+1+3+2+1+2+2=17$. So the task is impossible', said Alice. 'I expect it wasn't just luck that we could find that dotted line'.

'No', said Rabbit, 'Lester Ford and Delbert Fulkerson showed in 1956 that, if you cannot send the cars through, then you will be able to draw a dotted line which shows it is impossible'.

4. In which Owl tries to carpet his floor and Alice goes home to tea

'Do all problems work out in this way, with a short reason for impossibility?' asked Alice.

'That's a hard question', said Rabbit. 'There will be problems people have not yet considered, and so no one will have shown there is always a simple reason for impossibility'.

'I met a problem just today which we might look at', said Christopher Robin. 'Owl was trying to carpet his floor, and he had divided his floor into six squares. He had five pieces of carpet, each of which would just cover certain squares of the floor. I can show you which carpet pieces covered which squares, in this diagram (Figure 15). The left-hand dots are squares of floor, the right-hand dots are pieces of carpet, and a line means that the square at one end is covered by the carpet at the other end.

'Owl wants to cover his floor with pieces of carpet which do not overlap'.

'Ah!' squeaked Piglet excitedly, 'So that the sets of squares have intersection equal to the empty set'.

Pooh jumped at this remark and threw his gaze heavenward. He did not like sets, particularly the empty set. It made him think that it had originally been a full set—full of honey—and someone, not Pooh, had emptied it.

'So', continued Christopher Robin, 'You have to choose some dots on the right, so that no two of them are joined to the same left dot, but, between them, they are joined to every left dot'.

'I think I can do it', said Alice. 'Take the second, third and fifth pieces of carpet'.

'Correct', said Christopher Robin, 'That is just what Owl did. Unfortunately, while he was getting his hammer and nails, Eeyore came by, and decided to eat the pieces of carpet. Owl flew into a rage at this, but later he calmed down and got five more pieces of carpet, as in this next diagram (Figure 16)'.

'This time it is impossible', said Piglet, after a while.

'Why?' asked Christopher Robin.

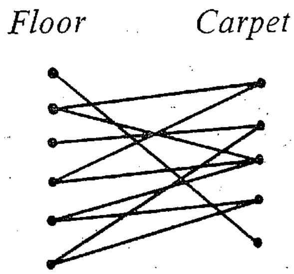

Figure 15

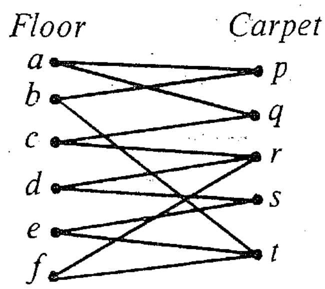

Figure 16

'To cover a we need p or q. If we have p then because of b we cannot have t. So, because of f we have r, and because of e we have s. But r and s overlap at d. So we cannot have p. So we must have q, but then, because of c, we cannot have r. So, because of d we have s, and because of f we have t. But s and t overlap at e. So we cannot have q. So we cannot do it', said Piglet breathlessly.

'That's not a short or simple reason', said Alice.

Rabbit glared at her. 'Don't you be rude, or you will find yourself in the river again'.

'With cement sandals!', murmured Pooh, under his breath.

'Well tried, Piglet', chuckled Christopher Robin. 'I am afraid I don't know a theorem which says that when the carpet problem is impossible, you can find a short reason why. We will all have to take it home and think about it'.

'Good gracious!' said Alice, 'It's nearly tea time. I must be off'.

'May I see you home?' asked Christopher Robin. Alice happily agreed and the two of them walked off hand in hand through the wood.

Piglet watched them go, with a sentimental smile on his face. 'Ah!' he said, 'With love, nothing is impossible'.

'That remark is in good taste', said Pooh.

'Oh!' said Piglet with surprise.

'Yes', said Pooh, 'A syrupy taste, just like honey'.

Powerful Numbers

A reader, M. K. Smithers, has written to *Mathematical Spectrum* about these intriguing numbers. Examples are

$$153 = 1^3 + 5^3 + 3^3,$$

$$4150 = 4^5 + 1^5 + 5^5 + 0^5,$$

$$4679307774 = 4^{10} + 6^{10} + 7^{10} + 9^{10} + 3^{10} + 0^{10} + 7^{10} + 7^{10} + 7^{10} + 7^{10} + 4^{10}.$$

Thus each number is the sum of the nth powers of its digits for some exponent n. The number of powerful numbers for different exponents up to 10 is given by the following table:

$n=$3	4	5	6	7	8	9	10
4	3	6	1	5	3	4	1

This table is from M. R. Mudge, *Computer Bulletin*, II/33 (1982). You might like to try to find the other three powerful numbers for exponent 3. The complete list of powerful numbers up to exponent 10 is given by J. Randle in *The Mathematical Gazette* **52** (1968), p. 383.

Beauty in Mathematics

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Introduction

The aesthetic appeal of mathematics has been a powerful motivating force for pure mathematicians over the ages. 'The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry', proclaimed Bertrand Russell in his essay 'The Study of Mathematics' (see reference 5).

Perhaps surprisingly, aesthetic feeling has also influenced many applied mathematicians. The idea of beauty runs as a constant theme through Einstein's thought, and Maxwell's addition of the vital displacement current term in the equations of electromagnetism was motivated by the aesthetic appeal of the form of the resulting equations.

In my own case, early aesthetic experiences involved the power of the calculus, and the elegant form of Lagrange's equations. The latter give the equations of motion of a dynamical system. They take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i} \qquad (i = 1, 2, 3, \ldots, n),$$

where n is the number of degrees of freedom of the system described by generalised coordinates $q_1, q_2, q_3, \ldots, q_n$, T is the kinetic energy and V is the potential energy of the system (see reference 8). What joy, I thought, to avoid all those tedious insertions of forces and reactions! How imaginative!

However, perhaps the Hamiltonian equations, which perform the same task, have an even greater appeal, in that they have the extra power of forming a bridge from classical mechanics to quantum mechanics. The simplest case occurs when we can write $H = T + V$ and introduce certain generalised momenta p_i $(i = 1, 2, 3, \ldots, n)$ to match the generalised coordinates, so as to obtain the equations

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}.$$

Equations of this form have importance in differential geometry, which further increases their aesthetic appeal.

Not just equations, nor isolated theorems, but whole mathematical theories have considerable beauty. The exposition is important here. For this reason I link the theories with specific books. For example, we have Gauss's treatise *Disquisitiones Arithmeticae* on number theory, *Galois Theory* by Artin, *Quantum Mechanics* by Dirac, *Fundamental Theory* by Eddington, *Principia* by Newton, and so on. Would the latter be more beautiful if written in the language of the calculus? How is the beauty of the physical theories affected by their correctness?

These are works of art on a grand scale like the symphonies and quartets in music. Alas, we cannot discuss them here. But there are also beautiful results on a smaller scale—the 'songs' of mathematics. Let us look at some of these. First it would be useful to have some criteria by which to judge them aesthetically. For this we turn to G. H. Hardy.

Criteria

To most readers, Hardy is not as well known as Einstein or Russell. Since Hardy was also a 'character' in the grand tradition of English eccentrics, I cannot resist the temptation to digress for a moment to write about him.

Since the time of Newton, apart from a few isolated figures such as Boole, Cayley and Sylvester, British pure mathematicians have been rather overshadowed by the dominant schools of applied mathematics. Hardy changed all that. He was one of the greatest mathematicians of the twentieth century and the founder, with J. E. Littlewood, of the first British school of mathematics to stand comparison with the great continental schools. He worked in analytic number theory and classical analysis. His approach to mathematics was entirely aesthetic, and he fervently hoped that none of his work would ever be applied to anything. A delightful pen portrait of Hardy by his friend the novelist C. P. Snow is contained in Snow's collection of essays *Variety of Men* (see reference 6), and also occurs as an introduction to Hardy's essay *A Mathematician's Apology* (reference 3), in which his ideas on the nature of mathematics are set forth.

Perhaps the essence of Hardy is conveyed amusingly by the list of six New Year wishes which he sent to a friend in the 1920's (see reference 7):

(1) Prove the Riemann hypothesis (a certain, still unproven, important conjecture in classical analysis).

(2) Make 211 not out in the fourth innings of the last test match at the Oval.

(3) Find an argument for the non-existence of God which shall convince the general public.

(4) Be the first man at the top of Mount Everest.

(5) Be proclaimed the first President of the USSR, of Great Britain and Germany.

(6) Murder Mussolini.

There is also the delightful story of Hardy who, when fearful that his enemy God would take advantage of a storm-swept North Sea to destroy him on his journey home from Scandinavia, sent a postcard to a fellow mathematician before sailing.

On the card he wrote: 'Have proved the Riemann hypothesis. Yours, G. H. Hardy.' This would ensure that God would not allow him to die and receive the credit he did not deserve. At moments of depression this story induces some relieving chuckles, not least on account of the contradiction with wish (3)!

In *A Mathematician's Apology*, Hardy seizes on five qualities that should be possessed by a beautiful piece of mathematics. These are generality, depth, unexpectedness, inevitability, and economy. The first three refer more to the content and the last two more to the presentation of the mathematics. Like any value judgement, the interpretation of these criteria depends very much on the experience, knowledge and prejudices of the individual. Generality and depth are particularly difficult to judge. Both are concerned with the significance of the work to mathematics as a whole, something that may change with time, so that a valid judgement may well require the hindsight of history. But, as with other subjective experiences, examples provide the best guide to what is meant.

Examples

My first two examples are those quoted by Hardy in *A Mathematician's Apology*.

Theorem 1. *There is an infinity of prime numbers.*

Proof. Suppose that $2, 3, 5, \ldots, p$ is the complete sequence of primes. Now consider $q = (2.3.5 \ldots p) + 1$. Every number which is not prime is divisible by at least one prime. Thus either q is prime or q is divisible by some prime which cannot be $2, 3, 5, \ldots, p$. Therefore there is a prime greater than p. Contradiction! Thus the set of primes must be infinite.

Theorem 2. $\sqrt{2}$ *is irrational.*

Proof. Suppose that $\sqrt{2} = p/q$, where the greatest common divisor (p, q) is 1. Then $2q^2 = p^2$. Thus $2|p^2$, where $|$ means divides. But then $2|p$. Thus $p = 2a$. But now $2q^2 = 4a^2$. This means that $q^2 = 2a^2$. Then $2|q^2$ and so $2|q$. But then $(p,q) \geq 2$. Contradiction! Thus $\sqrt{2}$ is irrational.

Both proofs are by *contradiction*, or *reductio ad absurdum* as it is often called (see reference 1). Hardy remarks that this is a far finer gambit than any chess gambit, since a chess player may offer the sacrifice of a pawn or even a major piece, but a mathematician offers the game!

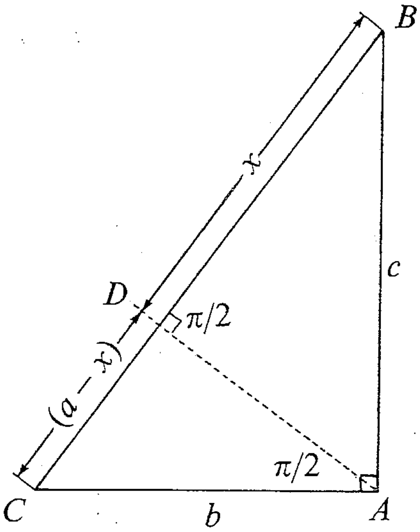
Theorem 1 is not particularly deep, but does lead to deep questions about the distribution of the primes. Here depth involves the difficulty of proof as measured by the sophistication of the machinery required, rather than the significance of the result to mathematics as a whole.

Theorem 2 is a deeper result of greater generality, in that it opens up a whole new region of mathematics, namely the real numbers, and eventually the calculus and classical analysis.

A theorem that has many proofs which lend themselves to comparison is the theorem of Pythagoras, which I give here as follows.

Theorem 3. *$a^2 = b^2 + c^2$, where a, b, c are the appropriate sides of a right-angled triangle.*

I do not wish to enter here into a detailed comparison of the various proofs, but content myself with the remark that, although some of the dissection proofs are very beautiful, my favourite proof is based on the concept of similarity. I choose this because the proof fits more easily into a formal deductive scheme (thereby making the proof 'rigorous') than the essentially more intuitive dissection proofs.



Proof. From the similarity of the triangles ABC and DBA, $a/c = c/x$. Thus $c^2 = ax$. From the similarity of the triangles ABC and DAC, $b/(a-x) = a/b$. Thus $b^2 = a^2 - ax$. Hence $a^2 = b^2 + c^2$.

Before going further we need some definitions. Let A and B be sets and $f: A \to B$ a mapping. Then f is called *injective* if distinct elements of A are always mapped by f into distinct elements of B; f is called *surjective* if every element of B is the image under f of some element of A. If f is both injective and surjective it is called *bijective*. A bijective mapping $f: A \to B$ is therefore one which sets up a 'one–one' correspondence between the elements of A and B. By way of examples consider the mappings

$$f: \{1, 2, 3, \ldots\} \to \{2, 4, 6, \ldots\}, \tag{1}$$

where $f(n) = 2n$,

$$f: \{1, 2, 3, \ldots\} \to \{1, 2, 4, 6, \ldots\}, \tag{2}$$

where $f(n) = 2n$,

$$f: \{1, 2, 3, \ldots\} \to \{2, 4, 6, \ldots\}, \tag{3}$$

where $f(n) = n$ if n is even and $f(n) = n + 1$ if n is odd.

The mapping (1) is both injective and surjective, (2) is injective but not surjective, and (3) is surjective but not injective.

For our next theorem we venture into the infinite. Without going into the (difficult) details, we remark that every set, finite or infinite, has a size—a cardinal number. We shall denote by $|A|$ the cardinal number of the set A. Let us say that, given two sets A and B, *B has a larger size* than A, i.e. $|B| > |A|$, if there exists an injective mapping $A \to B$ but *no* bijective mapping $A \to B$; also let us say that A and B *have the same size*, i.e. $|A| = |B|$ if there exists a bijective mapping $A \to B$. A famous theorem of Georg Cantor implies that, just as there are finite sets of all sizes, so also an infinite set is not just infinite but there is a whole hierarchy of infinities; in fact, given any set finite or infinite, we can find a larger one.

Theorem 4 (Cantor's theorem). *Given a set S, the set of all subsets of S, denoted by $\mathscr{P}(S)$, and called the 'power set' of S, is larger than S, i.e. $|\mathscr{P}(S)| > |S|$.*

Proof. It is easy to find an injective mapping $S \to \mathscr{P}(S)$; we simply associate with each s in S the singleton set $\{s\}$ in $\mathscr{P}(S)$. Now let $f: S \to \mathscr{P}(S)$ be *any* injective mapping. We want to show that it cannot possibly be bijective; so we want to show that it is *not* surjective. Suppose it is and seek a contradiction. For each element s of S, $f(s)$ is a subset of S, so may or may not contain s as an element. Let us call an element s of S an *included* element or a *non-included* element according as $s \in f(s)$ or $s \notin f(s)$; and let T be the subset of S consisting of all the non-included elements of S. (Of course, $T \in \mathscr{P}(S)$.) Since $f: S \to \mathscr{P}(S)$ is assumed to be surjective, given any member S' of $\mathscr{P}(S)$, there is an element s' of S such that $f(s') = S'$. In particular, there exists $t \in S$ such that $f(t) = T$. Now either (a) $t \in T$ or (b) $t \notin T$. But if (a) holds, then, by definition of T, t is a non-included element of S and $t \notin f(t) = T$. Also, if (b) holds, then, by definition of T, t is an included element of S and $t \in f(t) = T$. In each case we obtain a clear contradiction. So we are bound to conclude that f cannot possibly be surjective. Thus $|\mathscr{P}(S)| > |S|$.

Here again our proof is by *reductio ad absurdum*, as in the case of Theorems 1 and 2. Cantor's investigations in the late nineteenth century opened up a whole new realm of infinite numbers. In this realm it is possible to do a kind of arithmetic with some rather strange results. You may like to consult references 1 and 2 for further details.

My next example involves a rather unexpected relationship between numerical size and algebraic structure. An *integral domain D* is a set in which we can do addition, subtraction, and multiplication rather as we can in the set of integers using the usual operations of addition and multiplication. In particular, the rule that, if $ab = 0$ for a, $b \in D$, then $a = 0$ or $b = 0$, is obeyed. A whole class of examples of integral domains is provided by choosing a prime number p and considering the set $\{0, 1, 2, 3, \ldots, p-1\}$ under the operations of addition and multiplication modulo p.

A *field* resembles the set of real numbers, in that we can not only do addition, subtraction, and multiplication, but we can also divide by any non-zero element. The examples of integral domains provided above are also examples of fields. This is true in view of our next result.

Theorem 5. Any integral domain containing only a finite number of elements must be a field.

Proof. Let D be a finite integral domain. Let $a \neq 0$, $a \in D$. Define a mapping $f: D \to D$ by $f(d) = ad$. Now, if $f(d_1) = f(d_2)$, then $ad_1 = ad_2$. Thus $a(d_1 - d_2) = 0$. But $a \neq 0$, and D is an integral domain. Hence $d_1 - d_2 = 0$. Thus $d_1 = d_2$. This means that f is injective. But, since D is finite, this forces f to be surjective. (Think!) Hence, if b is *any* element of D, there exists d such that $f(d) = b$. Thus $ad = b$. But this means that a divides b for any b and any non-zero a. Hence D is a field.

Apart from the simple elegance of the proof, the result is striking in that a numerical quantity, namely the number of elements in the domain, implies an algebraic property, namely divisibility.

Another theorem of this type, due to the Scottish mathematician J. H. M. Wedderburn in 1905, involves the idea of a division ring. A *division ring* S is a set in which we can add, subtract, multiply and divide, as in a field, but the multiplication is not necessarily commutative, i.e. if $a, b \in S$, then we may *not* assume $ab = ba$. A famous example of a non-commutative division ring was invented by Hamilton in 1843, final illumination coming it appears when taking his wife for a walk along the Royal Canal in Dublin. Perhaps the practice is to be recommended! We define

$$H = \{a + bi + cj + dk | a, b, c, d \text{ real numbers}\},$$

where i, j, k, may be regarded as algebraic symbols subject to certain rules of manipulation as given below. An element $a + bi + cj + dk$ of this set H is called a *quaternion*. Quaternions are added and multiplied as if they were expressions in 'ordinary' algebra but bearing in mind the following rules for the multiplication of the symbols i, j, k:

$$ij = k, \qquad jk = i, \qquad ki = j,$$

$$ji = -k, \qquad kj = -i, \qquad ik = -j,$$

$$i^2 = j^2 = k^2 = -1.$$

The quaternions are like four-dimensional complex numbers. We can add, multiply, subtract, and divide (by non-zero quaternions), but, in general, multiplication is not commutative. In fact, taking $x = i$, $y = j$, we have $xy \neq yx$. For division we note that, if the quaternion $q = a + bi + cj + dk \neq 0$, then $a^2 + b^2 + c^2 + d^2 \neq 0$, and

$$1/q = (a - bi - cj - dk)/(a^2 + b^2 + c^2 + d^2).$$

Wedderburn's theorem may now be stated.

Theorem 6. Every finite division ring is a field.

Here again, the number of elements determines a purely algebraic property, namely commutativity.

Unfortunately a proof of this theorem is too difficult to give here. For those with sufficient background knowledge an elegant proof due to Witt is given in reference 9, pp. 104–105. But many proofs exist, and it is instructive to compare them, particularly from the aesthetic point of view; see reference 4.

I should have liked to include here Gödel's proof of the incompleteness of axiomatic number theory. But this glorious result and proof is more a symphonic poem than a song; for details see almost any book on mathematical logic.

Epilogue

At a time of increasing philistinism, when the utility of mathematics is emphasised, it seems timely to reassert the message of such mathematicians as Einstein, Hardy, and Russell. Certainly for me a major attraction of mathematics is its beauty. As Hardy remarked, the mathematician is a maker of patterns like the poet, the artist and the musician. The mathematician makes patterns with ideas. Beauty is the first test. Ugly mathematics has no permanent place in the world.

Many years ago, O'Shaughnessy wrote an ode, 'We are the music-makers'. This beautiful poem has found its way into many anthologies. I quote its first verse:

> We are the music-makers,
> And we are the dreamers of dreams,
> Wandering by lone sea-breakers,
> And sitting by desolate streams;
> World-losers and world-forsakers,
> On whom the pale moon gleams:
> Yet we are the movers and shakers
> Of the world for ever, it seems.

I like to think that O'Shaughnessy is speaking for the mathematician as well as for the poet. It would not be inappropriate.

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Minimum-Area Surfaces, Soap Films and Soap Bubbles

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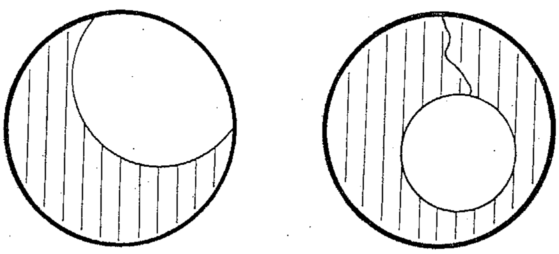
Cyril Isenberg obtained his B.Sc. and Ph.D. degrees at King's College London. From 1965 to 1967 he was a Resident Research Associate at the Argonne National Laboratory, Chicago, Illinois, U.S.A. In 1967 he returned to Britain as a lecturer in theoretical physics at the University of Kent at Canterbury. His research interests are in the fields of solid-state and liquid-state physics. Since 1973 he has popularized the science and mathematics of soap films by lecturing on this subject. His article on minimum roadway problems was published in Volume 15 Number 2 of *Mathematical Spectrum.*

1. Introduction

Soap films have the property that their energy is proportional to their surface area. A soap film contained by a closed boundary of wire will come to equilibrium with a minimum-energy configuration, and this will also be a minimum-area configuration (reference 3). For example, a soap film bounded by a circular wire ring will not come to rest with a configuration that bulges out of the ring, as this is not a minimum-area surface, but will form a surface in the form of a plane disc. A further example of this minimum-area property is provided by a film contained by the circular ring with a length of thread joining two points on the circumference. If the soap film above the thread is broken, the remaining planar film, bounded by the thread and an arc of the ring, will take up its minimum area configuration by pulling the thread into the arc of a circle (Figure 1(a)). A disc of soap film with a closed loop of thread, attached by a single thread to the wire ring, provides another example. On breaking the soap film inside the loop of thread, the resulting minimum surface, bounded by the ring and the loop of thread, is formed after the thread has been pulled into a circle by the soap film (Figure 1(b)). This configuration maximizes the area of the hole bounded by the loop of thread and consequently minimizes the area of the soap film. This minimum-area property of soap films was used in an earlier article in this magazine (reference 4) to solve problems concerned with the determination of minimum-length roadway systems.



(a) (b)

Figure 1. Planar minimum-area surfaces contained by a circular ring and thread.

In 1744, the famous Swiss mathematician Leonard Euler studied the problem of determining the minimum-area surface bounded by two equal coaxial, parallel, circular rings. One might reasonably guess that it is the cylindrical surface joining the two rings. He showed that it is a catenoid surface, a surface with axial symmetry that dips into the central axis (Figure 2(a)). The upper cross-section of the surface has the same shape as a hanging chain. This result can be obtained by dipping two rings into soap solution and forming the surface joining them. The catenoid surface results if the parallel rings are sufficiently close together. As the distance between the rings is increased the surface dips further towards the central axis. Eventually a critical separation is reached at which the surface breaks up into two discs, each contained by a ring. For greater separations of the rings the two discs provide the only minimum surface.

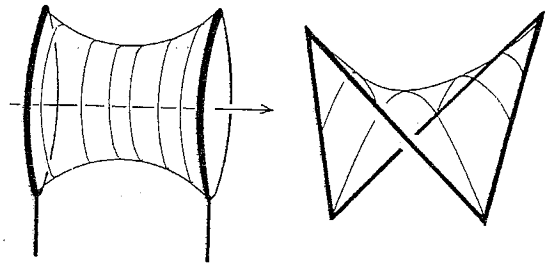

(a) (b)

Figure 2. Minimum-area surfaces contained by (a) two coaxial rings and (b) a skew quadrilateral.

H. A. Schwartz in 1890 produced an analytic solution to the minimum-area surface contained by a skew quadrilateral, which is a quadrilateral that is not planar. The surface is saddle-like (Figure 2(b)), and can be obtained by dipping a wire skew quadrilateral into soap solution.

An appropriate simple soap solution for all these soap film demonstrations consists of 99 per cent cold water with 1 per cent of domestic liquid detergent added. The final solution should be thoroughly stirred and any bubbles removed from the surface before use.

2. Minimum surfaces contained by frames

What is the minimum-area surface bounded by the six edges of a wire equilateral tetrahedral framework? One might conjecture that this surface is formed by the four planar faces. However, further thought will indicate that only three faces are necessary to form a surface contained by the six edges. If this is the minimum-area surface, the soap-film surface formed by dipping the framework into soap solution will form on three faces. The soap-film minimum-area surface produced by immersing the equilateral tetrahedron in soap solution is shown in Figure 3. It consists of lines of soap films that begin at each of the four vertices of the tetrahedron and meet at the centre. Each line is formed by the intersection of three planar triangular surfaces. These surfaces intersect each other at 120° at any point along the line and any two lines intersect at $\cos^{-1}(-1/3) \simeq 109°28'$.

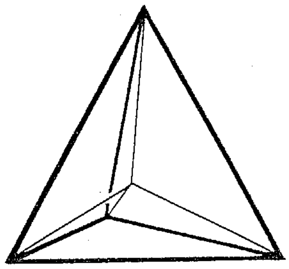
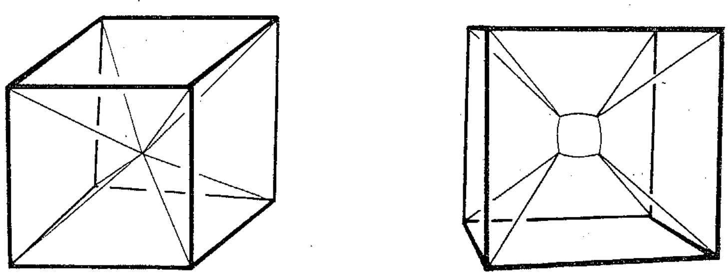


Figure 3. The minimum-area surface contained by a wire tetrahedron.

The Belgian physicist Joseph Plateau (see reference 5) was the first scientist to show experimentally that these geometrical results were general properties of minimum-area surfaces. If more than three surfaces intersect along a line, or if the surfaces do not meet at 120 °, the total surface area is increased. Likewise if the four lines of intersection do not meet at the required angle or if more, or fewer lines, meet at a point, the overall surface area will increase. In general the surfaces can be curved, but these geometric properties remain unchanged. A simple calculation will show that the ratio of the minimum surface area to the surface area of three faces of the tetrahedron is $\sqrt{2/3}$, which confirms that it is smaller than the area of the sum of the three faces.

Having obtained the minimum surface contained by the tetrahedron, which consisted of a number of surfaces meeting at a point in the centre, one might be tempted to guess that the minimum surface contained by a wire cube also consists of a number of planar surfaces meeting at a point at the centre, Figure 4(a). However, this surface does not satisfy the geometrical conditions associated with minimum surfaces. The central point is formed by the intersection of eight lines, not four, and along any line the surfaces do not meet at 120 °. The minimum surface contained by the cube can be simply obtained by dipping it into some soap solution. The surface that results is shown in Figure 4(b). It has a plane square-like surface at the centre with four lines of soap film meeting at each vertex of the 'square'. This minimum surface area is seen to satisfy all the geometrical conditions. There are four triangular sections of surface perpendicular to the 'square' film and eight slightly curved trapezoidal surfaces. The central 'square' is parallel to one set of cube faces. Consequently there are three possible minimum surfaces, all with equal area, with



(a) (b)

Figure 4. Surfaces contained by a cubic framework: (a) the surface formed by the body diagonals, (b) the minimum-area surface.

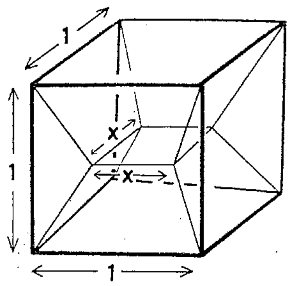


Figure 5. The surface with planar sections that approximates the minimum-area surface contained by the cubic framework.

the central square-like surface parallel to each set of faces of the cube. The square-like surface consists of sides that are slightly curved in order that the vertices of the 'square', which are formed by lines of soap film, have the required angle of 109°28′.

The area of the surface in Figure 4(a), which is joined to all twelve edges of the unit cube, is $3\sqrt{2} = 4{\cdot}242641$, so the area of the minimum-area surface will be less than this area. A simple approximation to this minimum surface will confirm that the area is indeed smaller than $3\sqrt{2}$. Consider the simple approximation to the minimum area surface consisting of a horizontal square surface of side x (Figure 5) joined by planar vertical triangular surfaces to the vertical edges of the framework and *planar* trapezoidal surfaces joined to the horizontal edges. These sections of surface are curved in the case of the true minimum surface. Table 1 gives the area of

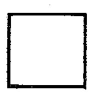
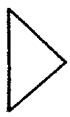
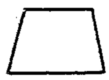
NUMBER	TYPE	AREA
1		x^2
4		$\sqrt{2}(1-x)/4$
8		$(1+x)\sqrt{1+(1-x)^2}/4$

Table 1

these sections of surface for a unit cube and the number of sections forming the full surface area $A(x)$. Thus $A(x)$ is given by

$$A(x) = x^2 + 4(\sqrt{2}(1-x)/4) + 8(1+x)\sqrt{1+(1-x)^2}/4$$
$$= x^2 + \sqrt{2}(1-x) + 2(1+x)(2-2x+x^2)^{1/2}.$$

To minimize $A(x)$ with respect to x, we put

$$\frac{dA(x)}{dx} = 0.$$

This gives

$$2x - \sqrt{2}x + 2(2-2x+x^2)^{1/2} - 2(1-x^2)(2-2x+x^2)^{-1/2} = 0.$$

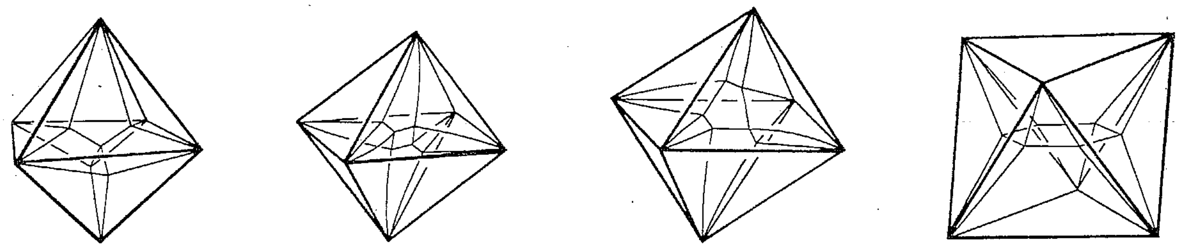
If we solve this equation numerically, we obtain x_m, the value of x that minimizes $A(x)$. This value is given by

$$x_m = 0{\cdot}0729103,$$

and the associated area $A(x_m)$ is given by

$$A(x_m) = 4{\cdot}242531.$$

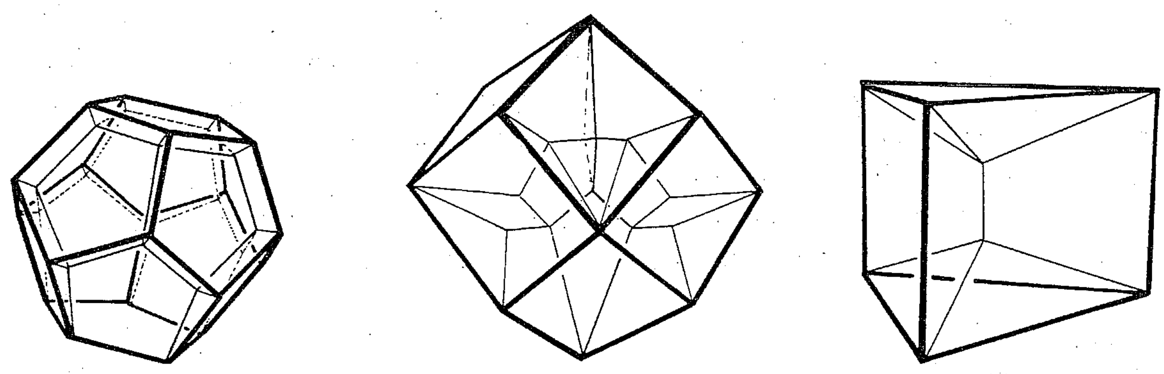
This is seen to be less than $A(0) = 3\sqrt{2} = 4{\cdot}242641$, the surface area of the configuration shown in Figure 4(a). The true minimum-area surface will have an area that is less than $A(x_m)$.



(a) (b) (c) (d)

Figure 6. Minimum surfaces contained by an octahedral framework.

More complex frameworks like the octahedron will have several minimum-area surfaces. Some of these surfaces are shown in Figure 6. The surface with the greatest symmetry is shown in Figure 6(a). It consists of planar sections of surface, in the form of kites, which meet at a point at the centre of the octahedron. The sections of surface, and lines of intersection of the surface, satisfy the geometric conditions of Plateau. The other minimum surfaces (Figures 6(b), 6(c) and 6(d)) have a planar square-like surface, a pentagonal-like surface and a hexagonal-like surface respectively, in the central region of the framework. These 'polygons' do not have straight edges, as the lines forming their boundaries must form vertices with angles of 109°28′. Consequently, they are slightly curved. One can obtain all of these minima from one of the minimum soap film surfaces by perturbing it (blowing onto it) so that it jumps into another minimum-area configuration.



(a) (b) (c)

Figure 7. The minimum surface contained by (a) a regular dodecahedron, (b) a rhombic dodecahedron and (c) a triangular prism.

The minimum-area surfaces produced by a regular dodecahedron, a rhombic dodecahedron and a triangular prism framework are shown in Figure 7. The regular dodecahedron contains a surface that appears to be pushed to one side of the framework. The surface in the rhombic dodecahedron is one of the most complicated, with a central point formed by the intersection of pentagonal planar surfaces. The triangular prism has three vertical planar surfaces that are bounded by the edges of the framework and meet along a central vertical axis. At the top and bottom of this axis other surfaces, bounded by the horizontal edges, intersect it.

3. Bubbles contained by frames

A bubble can be trapped in a wire framework. This can be done by first forming a minimum-area surface and then partially redipping it so that air is trapped to form a bubble. When the frame is withdrawn from the solution the bubble, which is formed in the central region of the frame, tends to take up the symmetry of the frame. The bubble is constrained by surfaces joining it to the edges of the framework. The energy of the system, which now consists of contributions from the surface area and the air in the bubble, is no longer proportional to the surface area. Consequently the equilibrium configuration is not a minimum-area configuration, but depends on a more general thermodynamic quantity, the *free energy*. However, the surfaces still retain the geometrical properties associated with minimum-area surfaces.

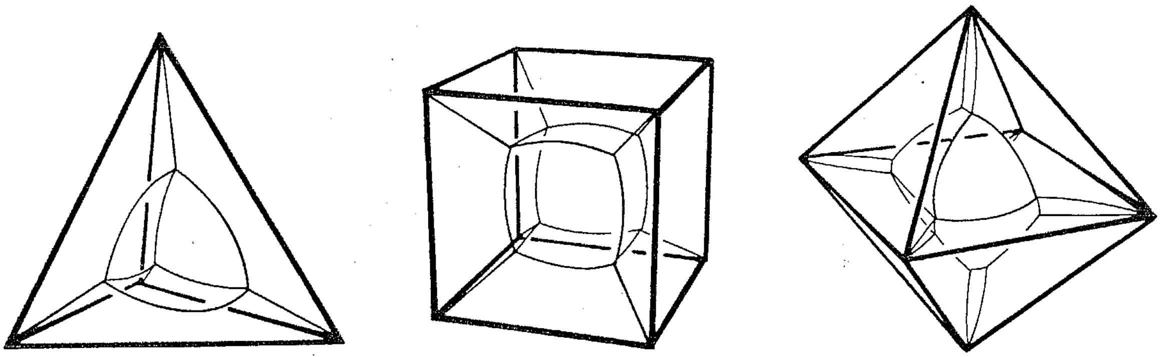
A tetrahedron, cube and octahedron with a trapped bubble are shown in Figure 8. The size of the bubbles depends on the quantity of air enclosed. In the case of the tetrahedron (Figure 8(a)) and the cube (Figure 8(b)) the bubbles have the full symmetry of the frameworks. The bubble inside the octahedron (Figure 8(c)) does not have full symmetry of the framework.

The dodecahedron has faces which are regular pentagons. The angles at the corners of the pentagons are 108°, which is not very different from 109°28′, the angle between lines of soap film. So, if one forms a bubble inside a dodecahedral framework, the bubble looks like a smaller copy of the framework. However, the edges of the bubble will be slightly curved in order that the lines of soap film meet at 109°28′ (Figure 9). Figures 10(a) and 10(b) show bubbles trapped in the rhombic dodecahedron and triangular prism frameworks.

Small skeletons are to be found on the ocean floors that are formed from bubbles which have solidified along the direction of the lines of intersection of the surfaces. These are known as *radiolarian skeletons:* they are typically 0·1 mm in length, and so can only be seen under a microscope. The arms of the skeletons will intersect in fours at angles of 109°28′. Skeletons similar to bubbles trapped in tetrahedral and triangular prism frameworks are shown in Figures 11(a) and 11(b). Figure 11(c) shows a more complicated radiolarian with numerous arms.

4. Foams

Foams produced by soap solutions are basically surfaces containing large numbers of bubbles. All these bubbles will satisfy the geometrical properties associated with minimum surfaces; the lines of soap film are formed by three



(a) (b) (c)

Figure 8. Bubbles trapped in (a) tetrahedral, (b) cubic and (c) octahedral frameworks.

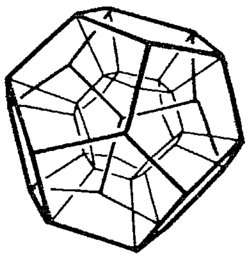
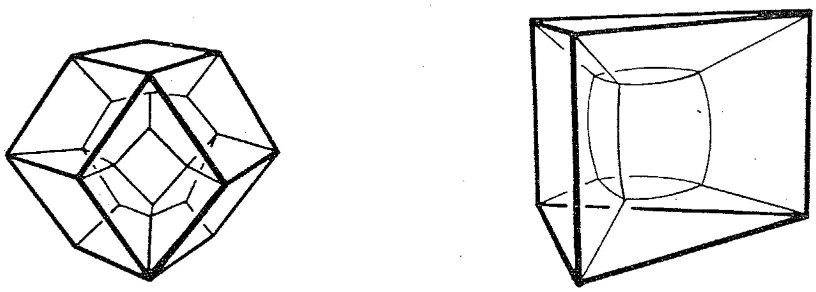
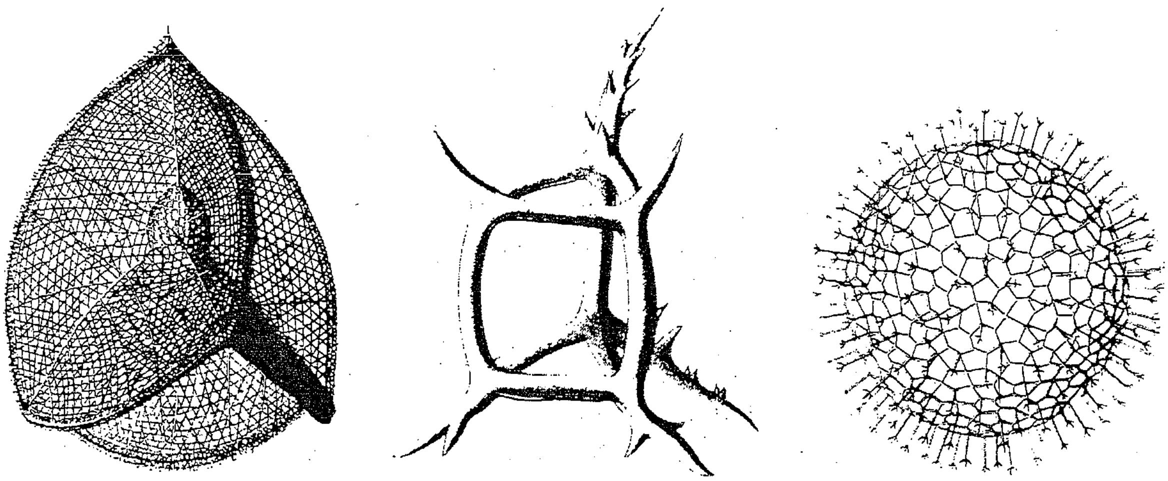


Figure 9. A bubble in a dodecahedral framework.



(a) (b)

Figure 10. Bubbles caught in (a) a rhombic dodecahedral framework and (b) a triangular prism framework.



(a) (b) (c)

Figure 11. Radiolarian skeletons.

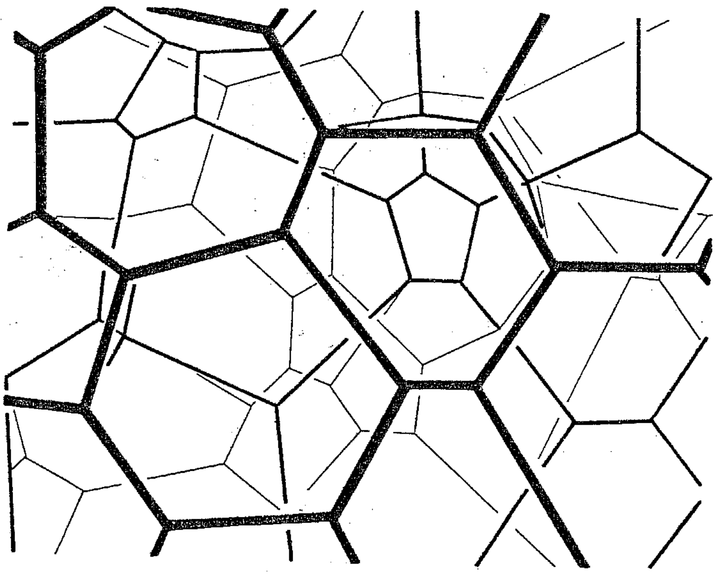


Figure 12. A foam.

surfaces intersecting at 120 °C, and four lines intersect at 109 °28′. These geometrical results are also true for all foams; for example, detergent foams, beer foams, soufflés and cavity-wall insulating foams. It is possible to calculate some simple statistical properties of foams based on the general geometrical result concerning intersecting lines of foam.

Each bubble in a foam (Figure 12) can be approximated by a polyhedron with plane faces. If a face of the polyhedron has n sides and it is assumed that the angle between the edges $\theta \simeq 109°28'$, then the sum of the external angles and the polygon, $(180 - \theta)°$, must add up to 360°. Thus

$$n(180 - \theta) = 360, \tag{4.1}$$

and so

$$n = \frac{360}{180 - \theta}.$$

Substituting $\theta = 109°28'$, we have

$$n = 5{\cdot}104.$$

The assumption that the internal angles of the polygon are 109 °28′ cannot be satisfied if the edges of the polygon are straight. However, it might be reasoned that the average number of sides for a large number of polygons, forming the faces of planar polyhedral cells associated with foams, will approximate to the value of 5·104.

An approximate value for the average number of vertices v and average number of faces f for a polyhedral foam cell can also be obtained. The total sum of all the internal angles of all the faces of a foam cell in the polyhedron approximation is $fn\theta$. As there are three edges meeting at each vertex this is equal to $3\theta v$, the angle associated with the faces surrounding a vertex being 3θ. Thus

$$3\theta v = fn\theta.$$

However, from (4.1) we have

$$3\theta v = f(180n - 360). \tag{4.2}$$

The total number of edges e of the polyhedron is half the product of the number of faces f and the number of edges per face n, so

$$2e = fn.$$

Substituting this into (4.2), we have

$$3\theta v = 360(e - f). \tag{4.3}$$

Now Euler's famous result relating e, f and v for a convex polyhedron (see reference 2) states that

$$e - f = v - 2.$$

Substituting this result into (4.3), we have

$$3\theta v = 360(v - 2).$$

If we now put $\theta = 109{\cdot}47°$, we get

$$v = 22{\cdot}79.$$

If we now substitute for v, θ and n in (4.2), we obtain

$$f = 13{\cdot}39.$$

Finally,

$$e = 34{\cdot}18.$$

These values of n, v, f and e provide approximations to the average number of edges per face, vertices per polyhedral cell, faces per cell and edges per cell. Good agreement between experiment and theory, for soap foam, has been obtained by D. A. Aboav (reference 1).

References

1. D. A. Aboav, The arrangement of cells in a net, *Metallography* **13** (1980), 43–58.
2. R. Courant and H. Robbins, *What is Mathematics?* (Oxford University Press, New York, 1941).
3. C. Isenberg, *The Science of Soap Films and Soap Bubbles* (Tieto, Clevedon, 1978).
4. C. Isenberg, Minimum roadway problems, *Mathematical Spectrum* **15** (1983), 45–53.
5. J. A. F. Plateau, *Statique expérimentale et théorique des liquides soumis aux seules forces moléculaires* (2 volumes, Gauthier-Villars, Paris, 1873).

Kits for demonstrating the minimum surfaces and bubbles are available from Advanced Educational Toys, 6 Abbots Place, Knotts Lane, Canterbury, Kent, CT1 2AH, England, and Cochranes of Oxford Ltd, Leafield, Oxford, OX8 5NT, England. Advanced Educational Toys produce wire sets of frameworks: Cochranes sell a construction kit in the form of plastic rods and corner connectors.

Computer Column

MICHAEL PIFF, *University of Sheffield*

The following program was sent in to us by Liu Zhi-qing, who is a student in China. Your task is to:

(a) Find out what the program is doing;
(b) Write your own program to solve the same problem;
(c) Post it to the Computer Column, *Mathematical Spectrum*, not later than the end of August 1984.

First, the program:

```
 10 LET A = 0
 20 FOR N = 50 TO 99
 30 GOSUB 200
 40 IF K <= A THEN 60
 50 LET A = K
 60 NEXT N
 70 FOR N = 50 TO 99
 80 GOSUB 200
 90 IF K <> A THEN 130
100 PRINT 'N' , 'K'
110 PRINT N , K
120 PRINT
130 NEXT N
140 END
200 LET K = 0
210 FOR I = 1 TO 99
220 IF ( N/I ) <> INT( N/I ) THEN 240
230 LET K = K + 1
240 NEXT I
250 RETURN
```

But, before you all rush off to your micros....

The object of the exercise is to try to make your program as readable and general as possible, whilst keeping to standard BASIC plus long identifiers. You should also think of the problem of efficiency when the number 99 (M say; what would *you* call it?) in the above is replaced by a much larger number. You may, if you wish, like to consider the difference that using some array storage has on the running of your program, and include an alternative version which stores the results of a loop for future use. However, this might not be a very good idea when M is large. Notice also that the first loop only goes from 50 to 99. Why 50? Because this is more efficient! Can you think of some other time-saving ideas along similar lines?

Keep to the same fundamental method of attack for solving the problem rather than some different mathematical approach; treat it as a programming exercise rather than a mathematical one. Your solution should include a description of the problem, and should also include copious REM statements explaining what you are doing, or, alternatively, should be so well written that it needs neither! I reserve the right not to read any unintelligible programs!

Letter to the Editor

Dear Editor,

The proportions of quadratic equations with real and non-real roots

I was particularly interested in John Hey's article in Volume 16 Number 2 as it closely approximated to my article in the December 1971 *Mathematical Gazette* on 'Frequency of types of quadratic numbers'.

My main interest, with merely a statistical answer, was in rational roots, but I also calculated $P(\text{complex root}) = 0{\cdot}5 - \frac{1}{72}(5 + 6\ln 2)$, which agrees with John Hey's article, and I was sent some computed results up to $n = 31$.

I should also like to see a calculated answer to the cubic case.

Yours sincerely,

J. L. G. Pinhey

(44 Netherhill Way, Cambridge CB1 4NY)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and home address and also the postal address of your school, college or university.

Problems

16.7. Determine all distinct pairs of positive integers m, n for which $m^n = n^m$.

16.8. (From *Középiskolai Matematikai Lapok*, used with permission)
Does there exist a positive integer such that we obtain its (a) 57th, (b) 58th part when we delete its first digit?

16.9. (Submitted by J. N. MacNeill, The Royal Wolverhampton School)
Let triangle ABC have circumcentre O, centroid G, orthocentre H and incentre I. Then it is known that O, G and H are collinear (Euler's line) with $OG/GH = \frac{1}{2}$.

But let W be the 'centroid of the perimeter' of triangle ABC, i.e. W is the centre of mass of a uniform wire in the shape of the perimeter of the triangle. Then prove that W, G and I are collinear with $WG/GI = \frac{1}{2}$.

Solutions to Problems in Volume 16, Number 1

16.1. Evaluate $(9 + 4\sqrt{5})^{1/3} + (9 - 4\sqrt{5})^{1/3}$.
Solution by Gary Slater (Université de Sherbrooke, Canada)
Let

$$s = (9 + 4\sqrt{5})^{1/3} + (9 - 4\sqrt{5})^{1/3}.$$

Then

$$\begin{aligned} s^3 &= 18 + 3(9+4\sqrt{5})^{2/3}(9-4\sqrt{5})^{1/3} + 3(9+4\sqrt{5})^{1/3}(9-4\sqrt{5})^{2/3}, \\ &= 18 + 3s(9+4\sqrt{5})^{1/3}(9-4\sqrt{5})^{1/3} \\ &= 18+3s. \end{aligned}$$

Thus

$$s^3 - 3s - 18 = 0$$

or

$$(s-3)(s^2+3s+6) = 0.$$

Since $s^2 + 3s + 6$ does not have real roots, we have $s = 3$.

Also solved by Malcolm Smithers (Open University), Patrick Chapel (Wymondham College), R. F. C. Dobbs (Winchester College), Martin Green (Royal Wolverhampton School), Chung-Lin Hsu (Royal Wolverhampton School), Ruth Lawrence (St Hugh's College, Oxford).

16.2. A lady enters a supermarket with her calculator, which she uses to add up her bill. She buys four items, but unfortunately she presses the multiplication button instead of the addition button. When she gets to the checkout, the cashier tells her that her bill is £7·11. 'Yes, that is what I got', says the customer. How much did each item cost?

Solution

We denote the prices of the four items by a, b, c, d so that

$$a+b+c+d = abcd = 7{\cdot}11.$$

We now put $A = 200a$, $B = 200b$, $C = 200C$, $D = 200d$, so that A, B, C, D are the prices of the items in halfpence, and so are integers. We now have

$$\begin{aligned} A+B+C+D &= 1422 \\ ABCD &= 711 \times 10^6 \times 2^4 = 79 \times 2^{10} \times 3^2 \times 5^6, \end{aligned}$$

and we have to solve these equations in integers. We are now reduced to a case-by-case analysis. One of A, B, C, D is a multiple of 79. Suppose, for example, that $D = 79$. Then $A + B + C = 1343$, $ABC = 2^{10} \times 3^2 \times 5^6$. But then the arithmetic mean of A, B, C is $447\frac{2}{3}$, whereas their geometric mean is $\sqrt[3]{(2^{10} \times 3^2 \times 5^6)} > 447\frac{2}{3}$, so this contradicts the well-known inequality 'geometric mean $\leq$ arithmetic mean' (GM $\leq$ AM for short). Having found the possible values of D by this method (namely $2 \times 79, 3 \times 79, 4 \times 79, 5 \times 79, 6 \times 79, 8 \times 79$), we now see that one of A, B, C, say C, is divisible by 25. Consider, for example, $D = 2 \times 79$, $C = 25$. Then $A + B = 1239$, $AB = 2^9 \times 3^2 \times 5^4$, but this violates GM $\leq$ AM. This will leave various possibilities for (C, D), with corresponding values for $A + B, AB$, from which we can obtain the values of A, B, which must be integers. This gives three solutions, namely

$$\begin{gathered} 256, 625, 225, 316, \\ 240, 300, 250, 632, \\ 240, 320, 625, 237. \end{gathered}$$

If we now go back to pounds, we get the three solutions

$$\begin{gathered} 1{\cdot}28, 3{\cdot}125, 1{\cdot}125, 1{\cdot}58, \\ 1{\cdot}20, 1{\cdot}50, \ 1{\cdot}25 \ \ 3{\cdot}16 \\ 1{\cdot}20, 1{\cdot}60, \ 3{\cdot}125, 1{\cdot}185. \end{gathered}$$

The problem is somewhat easier to solve if we insist that none of the prices involves halfpence, in which case the solution is unique.

Also solved by Malcolm Smithers (Open University).

16.3. The trump suit in a game of bridge is given as shown, and South leads. How can North and South guarantee to take at least three tricks no matter how the other five trumps are distributed among the two hands?

North
K J 3 2

West East

A 9 5 4
South

Solution

For the first trick, South leads the 4 and North plays the King.

For the second trick, North leads the 2.

(a) If either East cannot follow suit or plays the 10 or Queen, then South plays the Ace. In the former case, all the missing cards lie with West and the lead of the 5 from South must promote the Jack for a third winning trick. In the latter case, the lead of the 9 from South will force out the remaining high card, again leaving the Jack to take a third winning trick.

(b) If East follows suit with a card smaller than the 10, then South plays the 9. If West can beat this, then there is only one more card of the suit with East or West and this will fall when the Ace is played. Again, the Jack will provide the third winner. If West cannot beat the 9, then this takes the trick and the Ace will win a third trick.

Also solved by Patrick Chappel (Wymondham College).

The following problem also appeared in Volume 16 Number 1 by permission of the magazine *Acorn User*: Use the digits 1 to 9 once and once only to make a number divisible by 9 such that the first eight digits give a number divisible by 8, the first seven give a number divisible by 7, etc. How many such numbers are there?

A. R. Poole of Broadstairs supplied the unique solution 381654729; his Sinclair ZX81 16K came up in just under five minutes with 381654730 due to rounding errors. We have since learned that the problem first appeared as a *Sunday Times* Brainteaser. Dr R. Shail, of the University of Surrey, has written drawing our attention to an attractive theoretical solution by J. V. and R. Shail in the December 1983 issue of the *Mathematical Gazette*, Note 67.33.

Book Reviews

Mathematics: The Loss of Certainty. By Morris Kline. Oxford University Press, 1983. Pp. 366. £4·95.

The hard-cover edition of this book was reviewed in *Mathematical Spectrum* Volume 14 Number 1. The author begins his introduction by quoting Henri Poincaré: 'To foresee the future of mathematics, the true method is to study its history and its present state'. His review of the development of mathematics, both intuitive and rigorous, from its earliest beginning to modern times, shows its connections with nature, religion and science, and provides fascinating background reading as an aid to understanding an intricate subject. The publication of this inexpensive paperback edition is to be welcomed.

University of Sheffield MAVIS HITCHCOCK

The Lady or the Tiger? and other Logical Puzzles. By RAYMOND SMULLYAN. Penguin Books Ltd, London, 1983. Pp. 203. £2·50 paperback.

This is a puzzle book, and is a sequel to *What is the Name of This Book?* (also published by Penguin Books, and reviewed in Volume 14 Number 3). Like the earlier book, it is not really one that can be dipped into, and it also has a purpose other than pure entertainment. The author sets out to explore Gödel's work on indeterminacy. He goes further into it than in the previous book, and I think his explanation is easier to understand. However, I feel that there is less pure fun in this book than in the previous one: the reader will look in vain for absent-minded professor jokes, for example.

The book is divided into four parts. The first two contain the same kind of puzzle as *What is the Name of This Book?* For example, from the third chapter, 'The Asylum of Doctor Tarr and Professor Fether': in a particular asylum, all the inhabitants are either doctors or patients, and are either sane, and thus believe all true statements to be true and all false statements to be false, or insane, thus believing all true statements to be false and all false statements to be true. One of the inhabitants makes a statement from which it is possible to deduce that he is an insane doctor. Can you supply such a statement? The third part is what Smullyan describes as a mathematical novel. It is like a detective story. A bank in Monte Carlo has a safe with a combination lock, and the combination has been lost. However, certain facts are known about the mechanism, and from these our detective must find a combination which unlocks the safe. Smullyan does not succeed in getting me on the edge of my seat, but what this 'mathematical novel' lacks in suspense it makes up, I feel, in interest, in the problems encountered on the way. This leads to part four, which investigates provability.

The format is similar to that of *What is the Name of This Book?*, that is, in each chapter, the puzzles are presented, not in the standard specimen-case manner, but in a logical order, often embedded in a narrative, as for instance the above-mentioned chapter describes an investigation of some asylums, and the answers are given at the end of the chapter. Each answer discusses the problem in a helpful manner.

Altogether, this book is readable and entertaining, well worth the money. Smullyan is intending some sequels, which I look forward to reading.

The Sixth Form, Tapton School, Sheffield — R. KNIGHT

Quotes, damned quotes, and . . . Compiled by JOHN BIBBY, 1983. Available from 33 Haugh Shaw Road, Halifax, W. Yorks HX1 3AH. Pp. 52. £2·20 (or $4) including postage.

John Bibby, in a published letter, asked readers to send him their favourite statistical quotes. This little book is the result. It contains both wit ('Counting in octal is just like counting in decimal if you don't use your thumbs') and wisdom ('If a man will begin with certainties he shall end in doubts; but if he will be content to begin with doubts he shall end in certainties'), and more besides.

Prelude to Mathematics. By W. W. SAWYER. Dover Publications, Inc., New York (distributed by Constable and Company Limited, London), 1982. (Originally published by Pelican Books, 1955.) Pp. 219. £4·05.

Books by W. W. Sawyer make excellent reading, and a reprint of his 1955 *Prelude to Mathematics* is to be welcomed. In his own words, he is writing 'for the person who is interested in getting inside the mind of a mathematician'.

Maths at Work. By G. HOWSON and R. MCCLONE, Heinemann Ltd, London, 1983. Pp. 200. £7·50 paperback.

Mathematics and mathematicians are beginning to get a better press than they have had for some time. A recent article in *Business Weekly* was entitled 'Industry's Hot New Find: The Mathematician'. Amongst other things it indicated how, by some simple applications of elementary statistical sampling, one particular company had cut down on its costs from $24 000 to $3 000. Mathematics and mathematicians are much more clearly seen to be useful.

This book continues in the growing tradition of showing how mathematics is useful in a number of different contexts. The title is ambiguous and this may well be deliberate. Chapters show how mathematics is used in various parts of the world of work (interpreted liberally); to a certain extent they all show mathematics working.

In their introduction the editors say that the level of mathematics used is within the range of senior sixth-formers. This is probably right, though at times some would find the going hard, largely because of the emphasis on model-building which is not yet very well established at this level. Such a student may well be surprised at the relatively large number of references to probability and statistics compared with calculus, though there are examples of each. Even more prominent is the use of graphs and diagrams which are essential to the development of the argument, and not just to decorate the text.

One important aspect of applied mathematics that does show up is how often it is difficult to get numerical values for the variables in the formulae developed, yet at the same time the analysis can still be very useful since it has identified the important variables and the relationships between them.

With examples from nursing, retail stores, biology, shipping in the Suez Canal, traffic flow at roundabouts, archaeology and insurance (amongst others) there is at least one chapter to interest nearly everyone, and at the same time the overall picture is one of mathematics being widely useful and used.

University of Sheffield PETER HOLMES

Rings, Fields and Groups. By R. B. J. T. ALLENBY. Edward Arnold, London, 1983. Pp. 294. £9·95.

At first glance, this textbook contains most of the topics that usually appear in an undergraduate course of abstract algebra. But the author has made a real attempt to show his readers how exciting the subject is, by his informal style and, in particular, by the fascinating potted biographies of the pioneers of the subject which appear at appropriate points. This is mathematics with a human face. He has also taken pains to take his readers behind the subject to see what is really happening.

Herein lies the danger. On the one hand, the author has perhaps somewhat overplayed the intention to start everything right at the beginning. Your reviewer was amused by the somewhat elaborate proof on page 3 that there is only one empty set; and there is an unfortunate error in the proof of the equivalence of the various statements of the Principle of Mathematical Induction on page 17. On the other hand, the author does assume a considerable degree of sophistication on the part of his readers when he goes deeper into the subject, particularly in his comments, and one wonders whether sections of the books will be understood by anyone not already reasonably familiar with the subject matter.

But here is no dry-as-dust textbook, and it will pay rich dividends to those who give time and effort to delve into its pages.

University of Sheffield D. W. SHARPE

CONTENTS

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