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E U R E K A

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LIMERICKEN

Arkimedes ifrån Syracusa
han satt i badet och busa'.
Som en kork flöt han upp,
PRINCIPEN fann upp:
med *Jag har det* han naken utrusa'.

Text: *Einar Lunell*
Bild: *Andrejs Dunkels*

This poem and drawing appeared in the Swedish journal *Elementa* 58, 1975:1, edited by Andrejs Dunkels, who also did the drawing and gave permission to have it reproduced here.

The story of Archimedes of Syracuse sitting in his bath, watching the cork bobbing up, and then running around naked shouting "I have found it!", is too well known to warrant translation.

The editor is pleased to be able to present to readers this artistic depiction of the patron saint of EUREKA.

ON URQUHART'S ELEMENTARY THEOREM OF EUCLIDEAN GEOMETRY

KENNETH S. WILLIAMS, Carleton University

In a recent article Pedoe [1] mentions that he has attempted to find a proof of Urquhart's Theorem (stated below) which does not involve circles (see also [2]). Here is a simple proof which only involves the sine formula for triangles and a few simple trigonometric identities.

LEMMA 1. In ΔABC we have

$$\frac{BC+CA}{AB} = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}.$$

Proof. By the sine formula we have (as $C = \pi - (A+B)$)

$$\frac{BC+CA}{\sin A+\sin B} = \frac{AB}{\sin C} = \frac{AB}{\sin(A+B)}$$

and so

$$\frac{BC+CA}{AB} = \frac{\sin A+\sin B}{\sin(A+B)} = \frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B)} = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}.$$

LEMMA 2. If

$$\cos\tfrac{1}{2}(L-M)\cos\tfrac{1}{2}(N+P) = \cos\tfrac{1}{2}(L+M)\cos\tfrac{1}{2}(N-P) \tag{1}$$

then we have

$$\sin\tfrac{1}{2}(L-P)\sin\tfrac{1}{2}(N+M) = \sin\tfrac{1}{2}(L+P)\sin\tfrac{1}{2}(N-M). \tag{2}$$

Proof. Applying the identity $2\cos R\cos S = \cos(R+S)+\cos(R-S)$ to each side of (1) we obtain

$$\cos\tfrac{1}{2}(L-M+N+P)+\cos\tfrac{1}{2}(L-M-N-P) = \cos\tfrac{1}{2}(L+M+N-P)+\cos\tfrac{1}{2}(L+M-N+P),$$

which gives

$$\cos\tfrac{1}{2}(L-M-N-P)-\cos\tfrac{1}{2}(L+M+N-P) = \cos\tfrac{1}{2}(L+M-N+P)-\cos\tfrac{1}{2}(L-M+N+P). \tag{3}$$

Then applying the identity $\cos R-\cos S = 2\sin\frac{1}{2}(R+S)\sin\frac{1}{2}(S-R)$ to each side of (3) we obtain (2).

URQUHART'S THEOREM. In the figure, if $AB+BF = AD+DF$ then we have $AC+CF = AE+EF$.

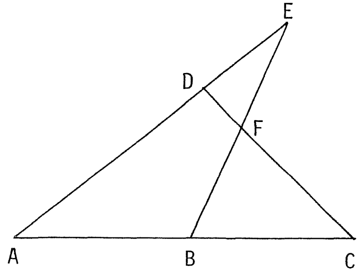
Proof. We join AF and set

$$\angle BAF = \phi, \quad \angle DAF = \theta, \quad \angle AFB = \mu, \quad \angle AFD = \lambda.$$

Applying Lemma 1 to Δs ABF and ADF we obtain

$$\frac{AB+BF}{AF} = \frac{\cos\frac{1}{2}(\mu-\phi)}{\cos\frac{1}{2}(\mu+\phi)}, \quad \frac{AD+DF}{AF} = \frac{\cos\frac{1}{2}(\lambda-\theta)}{\cos\frac{1}{2}(\lambda+\theta)},$$

and as $AB+BF = AD+DF$ we have



$$\frac{\cos\frac{1}{2}(\mu-\phi)}{\cos\frac{1}{2}(\mu+\phi)} = \frac{\cos\frac{1}{2}(\lambda-\theta)}{\cos\frac{1}{2}(\lambda+\theta)}\ .$$

Thus by Lemma 2 (as $0 < \lambda-\phi < \pi$, $0 < \mu-\theta < \pi$) we have

$$\frac{\sin\frac{1}{2}(\lambda+\phi)}{\sin\frac{1}{2}(\lambda-\phi)} = \frac{\sin\frac{1}{2}(\mu+\theta)}{\sin\frac{1}{2}(\mu-\theta)}\ . \tag{4}$$

Finally applying Lemma 1 to Δs ACF and AEF we obtain

$$\frac{AC+CF}{AF} = \frac{\cos\frac{1}{2}(\pi-\lambda-\phi)}{\cos\frac{1}{2}(\pi-\lambda+\phi)} = \frac{\sin\frac{1}{2}(\lambda+\phi)}{\sin\frac{1}{2}(\lambda-\phi)}\ ,$$

$$\frac{AE+EF}{AF} = \frac{\cos\frac{1}{2}(\pi-\mu-\theta)}{\cos\frac{1}{2}(\pi-\mu+\theta)} = \frac{\sin\frac{1}{2}(\mu+\theta)}{\sin\frac{1}{2}(\mu-\theta)}\ ,$$

and the required result $AC+CF = AE+EF$ now follows from (4).

REFERENCES

1. Dan Pedoe, The most elementary theorem of Euclidean geometry, *Mathematics Magazine*, 49 (1976), 40-42.

2. Léo Sauvé, On Circumscribable Quadrilaterals, EUREKA, Vol. 2 (1976), 63-67.

* * *

PROBLEMS--PROBLÈMES

Problem proposals, preferably accompanied by a solution, should be sent to the editor, whose name appears on page 107.

For the problems given below, solutions, if available, will appear in EUREKA *Vol.* 2, *No.* 9, *to be published around Nov.* 15, 1976. *To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than Nov.* 1, 1976.

151. *Proposed by Léo Sauvé, Algonquin College.*

METAPHORS

I'm a riddle in nine syllables,
An elephant, a ponderous house,
A melon strolling on two tendrils.
O red fruit, ivory, fine timbers!
This loaf's big with its yeasty rising.
Money's new-minted in this fat purse.
I'm a means, a stage, a cow in calf.
I've eaten a bag of green apples,
Boarded the train there's no getting off.

SYLVIA PLATH (1932-1963)
From *Crossing the Water*.

Identify the speaker and thereby solve the riddle.

152. *Proposé par Jacques Marion, Université d'Ottawa.*

Si $a > e$, montrer que l'équation $e^z = az^m$ possède m solutions à l'intérieur du cercle $|z| = 1$.

153. *Proposé par Bernard Vanbrugghe, Université de Moncton.*

Montrer que les seuls entiers positifs qui vérifient l'équation

$$a \cdot b = a + b$$

sont $a = b = 2$.

154. *Proposed by Kenneth S. Williams, Carleton University.*

Let p_n denote the nth prime, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, etc. Prove or disprove that the following method finds p_{n+1} given $p_1, p_2, \ldots, p_n$.

In a row list the integers from 1 to $p_n - 1$. Corresponding to each r $(1 \le r \le p_n - 1)$ in this list, say $r = p_1^{a_1} \ldots p_{n-1}^{a_{n-1}}$, put $p_2^{a_1} \ldots p_n^{a_{n-1}}$ in a second row. Let ℓ be the smallest odd integer not appearing in the second row. The claim is that $\ell = p_{n+1}$.

Example. Given $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, $p_6 = 13$.

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 3 & 5 & 9 & 7 & 15 & 11 & 27 & 25 & 21 & 13 & 45 \end{array}$$

We observe that $\ell = 17 = p_7$.

155. *Proposed by Steven R. Conrad, B.N. Cardozo High School, Bayside, N.Y., and Ira Ewen, James Madison High School, Brooklyn, N.Y.*

A plane is *tessellated* by regular hexagons when the plane is the union of congruent regular hexagonal closed regions which have disjoint interiors. A *lattice point* of this tessellation is any vertex of any of the hexagons.

Prove that no four lattice points of a regular hexagonal tessellation of a plane can be the vertices of a regular 4-gon (square).

This theorem may be called the *4-gon conclusion*.

(This problem was originally written for the 1976 New York State Math League Meet, held on May 1, 1976.)

156. *Proposé par Léo Sauvé, Collège Algonquin.*

Déterminer tous les entiers n pour lesquels l'implication suivante est vraie:

Pour tous les réels a, b, c non nuls et de somme non nulle,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c} \Longrightarrow \frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{a^n + b^n + c^n}.$$

157. *Proposed by Steven R. Conrad, B.N. Cardozo High School, Bayside, N.Y.*

In base fifty, the integer x is represented by the numeral CC and x^3 is represented by the numeral $ABBA$. If $C > 0$, express all possible values of B in base ten.

158. *Proposed by André Bourbeau, École Secondaire Garneau.*

Devise a Euclidean construction to divide a given line segment into two parts such that the sum of the squares on the whole segment and on one of its parts is equal to twice the square on the other part.

159. *Proposed by Viktors Linis, University of Ottawa.*

Show that

$$x! + y! = z!$$

has only one solution in positive integers, and that

$$x!y! = z!$$

has infinitely many for $x > 1$, $y > 1$, $z > 1$.

160. *Proposed by Viktors Linis, University of Ottawa.*

Find the integral part of $\sum_{n=1}^{10^9} n^{-2/3}$.

This problem is taken from the list submitted for the 1975 Canadian Mathematics Olympiad (but not used on the actual exam).

* * *

SOLUTIONS

93. [1975: 97; 1976: 45] *Proposed by H.G. Dworschak, Algonquin College.*

Is there a convex polyhedron having exactly seven edges?

III. *Solution by Charles W. Trigg, San Diego, California.*

The polyhedron with the fewest possible faces is the tetrahedron which has 4 faces and 6 edges. To increase the number of edges, another face must be added. For minimum increase, this added face must be triangular. If the addition is accomplished by a single truncation, the resulting polyhedron will have 9 edges. If the triangle is added at a vertex of the tetrahedron, there will be 4 edges issuing from the vertex and 4 opposite edges, so $n = 8$. Hence, no polyhedron exists with $n = 7$.

With another approach, for a minimal increase in the number of edges, draw a line from one vertex in a face of the tetrahedron to the opposite edge. This divides that edge into two segments, so the minimal increase is 2, one the line and the other the additional segment, making 8 edges.

111. [1976: 25, 95] *Late solution: Viktors Linis, University of Ottawa.*

115. [1976: 25, 98] *Proposed by Viktors Linis, University of Ottawa.*

Prove the following inequality of Huygens:

$$2\sin\alpha + \tan\alpha \geq 3\alpha, \qquad 0 \leq \alpha < \frac{\pi}{2}.$$

III. *Solution by the proposer.*

For $0 \le x < \frac{\pi}{2}$, we have $\cos x \le \cos\frac{x}{2}$, and hence $\tan x \ge 2\sin\frac{x}{2}$. Squaring gives

$$\tan^2 x = \sec^2 x - 1 \ge 4\sin^2\frac{x}{2} = 2 - 2\cos x,$$

and so

$$2\cos x + \sec^2 x \ge 3.$$

Integrating from 0 to $\alpha < \frac{\pi}{2}$, we obtain

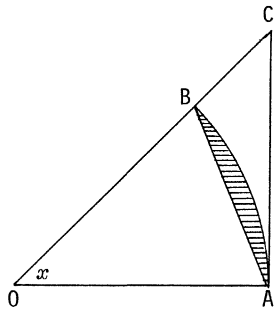
$$2\sin\alpha + \tan\alpha \ge 3\alpha.$$

Comment by the proposer.

Huygens' inequality has a very simple geometric interpretation. It is equivalent to

$$\frac{x - \sin x}{2} < \frac{1}{3}\,\frac{\tan x - \sin x}{2},$$

which relates the area S_1 of the shaded segment to the area S of ΔABC (see figure). With $OA = 1$, the inequality gives $S_1 < \frac{1}{3}S$.



A direct geometric proof in the spirit of Huygens (period 1660-1675) would now appear very cumbersome. The area S_1 needs to be approximated from above, by circumscribing polygons. However without the appropriate limit process the ratio 1/3 of the areas cannot be obtained. In Huygens' time the power series expansions for $\log(1+x)$, $\sin x$, $\tan x$, $\arctan x$ were essentially known although the connection between the coefficients and the derivatives (not yet invented!) was not clearly understood. The series most widely used were those for $\log(1+x)$ [Mercator] and for $\arctan x$ [Gregory]. For these series the geometric derivations were simpler. Also the intuitive notion of "convergence" for the alternating series began to emerge [explicitly demonstrated by Leibniz]. In this framework Huygens' inequality can be expressed as follows. Since

$$\tan x = 2\tan\frac{x}{2}\left(1 - \tan^2\frac{x}{2}\right)^{-1} = 2\tan\frac{x}{2}\left(1 + \tan^2\frac{x}{2} + \tan^4\frac{x}{2} + \ldots\right),$$

$$\sin x = 2\tan\frac{x}{2}\left(1 + \tan^2\frac{x}{2}\right)^{-1} = 2\tan\frac{x}{2}\left(1 - \tan^2\frac{x}{2} + \tan^4\frac{x}{2} - \ldots\right),$$

valid for $|x| < \frac{\pi}{2}$, we need to show that

$$6\tan\frac{x}{2} - 2\tan^3\frac{x}{2} + 6\tan^5\frac{x}{2} - 2\tan^7\frac{x}{2} + \ldots > 3x$$

or, with $\tan\frac{x}{2} = u$,

$$\arctan u < u - \frac{u^3}{3} + u^5 - \frac{u^7}{3} + \ldots,$$

and finally

$$u-\frac{u^3}{3}+\frac{u^5}{5}-\frac{u^7}{7}+\ldots < u-\frac{u^3}{3}+u^5-\frac{u^7}{3}+\ldots .$$

The difference R.H.S. - L.H.S. is an alternating series

$$\frac{4}{5}u^5-\frac{4}{21}u^7+\frac{8}{9}u^9-\frac{8}{33}u^{11}+\ldots$$

with the positive coefficients of the form $\frac{4n}{4n+1}$ and the negative of the form $\frac{4n}{3(4n+3)}$. The condition for the monotone decreasing alternating series is $u^2 < 3$ which is certainly satisfied.

Also solved by the proposer (two additional solutions).

121. [1976: 41] *Proposed by Léo Sauvé, Algonquin College.*

For which n is there a convex polyhedron having exactly n edges?

I. *Solution by Leroy F. Meyers, The Ohio State University.*

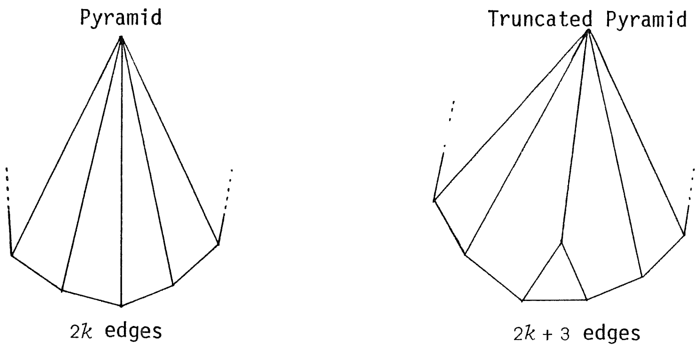


Figure 1

In both cases in Figure 1, we have $k \geq 3$. The only omitted number of edges (greater than 5) is 7. See Problem 93 [1975: 97; 1976: 45, 111].

II. *Solution by Charles W. Trigg, San Diego, California.*

A semi-regular pyramid with a regular k-gon base ($k > 2$) is a convex polyhedron with $2k$ edges. A similar pyramid having a tetrahedron cap on one isosceles triangle face, such that the sum of the dihedral angles of the pyramid and tetrahedron at each edge common to both is $< 180°$, is a convex polyhedron with $2k+3$ edges. (Such a cap is always possible since the vertex of the cap can be taken inside the tetrahedron formed by extending the planes of the base and of two faces adjacent to the same triangular face of the pyramid.)

Together these two families of polyhedrons contain convex polyhedrons with n edges for all even $n \geq 6$ and odd $n \geq 9$. Thus they contain convex polyhedrons with n edges for all $n > 5$ except 7. For $n = 7$, see Problem 93.

III. *Solution by the proposer.*

A tetrahedron is a convex polyhedron having exactly 6 edges. It was proved in Problem 93 that there is no convex polyhedron having exactly 7 edges. We will show that, in addition to $n = 6$, there is a convex polyhedron having exactly n edges for every $n \geq 8$. We will restrict ourselves to the family F of convex polyhedra each of which has at least one vertex of *degree* 3 (incident with exactly 3 edges).

The proof is by induction. It is clear that F contains polyhedra with exactly n edges for $n = 8$ (pyramid on a square base), $n = 9$ (triangular prism), and $n = 10$ (pyramid on a convex pentagonal base). For $n \geq 8$, let P_n be a polyhedron in F having exactly n edges. P_n has at least one vertex V of degree 3. If we truncate a tetrahedron at V (see Figure 2), the resulting polyhedron P_{n+3} has $n+3$ edges, it is convex, and each of the three new vertices is of degree 3. Hence P_{n+3} belongs to F, and the induction is complete.

Also solved by G.D. KAYE, *Department of National Defence; and* F.G.B. MASKELL, *Algonquin College.*

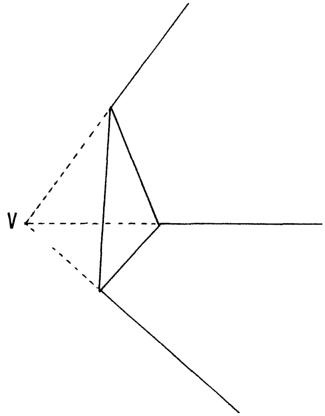


Figure 2

122. [1976: 41] *Proposed by Jeremy Wheeler, British Railways, Melbourne, Derby, England.*

I had leant my ladder up against the side of the house to paint my bedroom window and found that it just reached the bottom of the window. My son was pushing a box around and was just able to get it under the ladder. The box was a 1-metre cube and the ladder was 4 metres long. How high was the bedroom window off the ground?

I. *Solution by Leroy F. Meyers, The Ohio State University.*

If we place a 1-metre box under an s-metre ladder, then, from Figure 1,

$$\begin{aligned} s^2 &= (x+1)^2 + (\frac{1}{x}+1)^2 \\ &= x^2 + 2x + 1 + \frac{1}{x^2} + \frac{2}{x} + 1 \\ &= (x+\frac{1}{x})^2 + 2(x+\frac{1}{x}). \end{aligned}$$

Solving this equation for $x + \frac{1}{x}$ yields

$$x + \frac{1}{x} = -1 + \sqrt{s^2+1},$$

where only the positive sign is possible, since $x + \frac{1}{x}$ is positive. Solving for x now yields

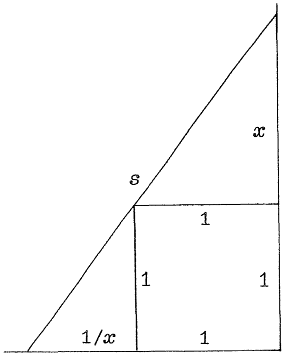


Figure 1

$$x = \frac{\sqrt{s^2+1} - 1 \pm \sqrt{s^2 - 2 - 2\sqrt{s^2+1}}}{2}$$

and so

$$x + 1 = \frac{\sqrt{s^2+1} + 1 \pm \sqrt{s^2 - 2 - 2\sqrt{s^2+1}}}{2}.$$

There are two solutions if $s > 2\sqrt{2}$ (as in the problem statement, where $s = 4$), there is one solution if $s = 2\sqrt{2}$, and there is no solution if $s < 2\sqrt{2}$.

Thus the solutions are

$$x + 1 = \frac{\sqrt{17} + 1 \pm \sqrt{14 - 2\sqrt{17}}}{2} \approx 3.7609 \text{ or } 1.3622 \text{ metres.}$$

II. *Solution by F.G.B. Maskell, Algonquin College.*

We have from Figure 2

$$\sec\theta + \csc\theta = 4,$$

and squaring yields

$$1 + \sin 2\theta = 4\sin^2 2\theta,$$

so that, since $\sin 2\theta > 0$,

$$\sin 2\theta = \frac{1 + \sqrt{17}}{8}.$$

Squaring again and expressing everything in terms of $\sin\theta$ gives

$$256\sin^4\theta - 256\sin^2\theta + 18 + 2\sqrt{17} = 0,$$

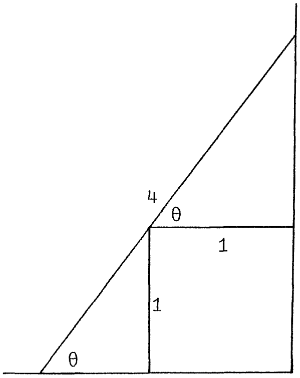


Figure 2

giving for the height of the window

$$4\sin\theta = \sqrt{8 \pm \sqrt{46 - 2\sqrt{17}}} \approx 3.7609 \text{ or } 1.3622 \text{ metres.}$$

III. *Comment by Charles W. Trigg, San Diego, California.*

The same configuration with different dimensions has previously appeared in *School Science and Mathematics* as solutions to Problems:

1508, Vol. 37 (December 1937), pp. 1125-1126;
1751, Vol. 42 (April 1942), p. 386;
1882, Vol. 44 (November 1944), pp. 772-773;
1958, Vol. 46 (April 1946), p. 387;
2213, Vol. 51 (January 1951), p. 74.

This is in no wise a complete bibliography.

IV. *Comment by Steven R. Conrad, Benjamin N. Cardozo High School, Bayside, N.Y.*

The numbers in this proposal make an otherwise elegant solution appear not so elegant. If the cube were 12 meters wide and the ladder 35 meters long, the problem becomes much nicer. These numbers appeared in my version of this problem which appeared in *The Mathematics Student Journal* of October 1973.

Also solved by STEVEN R. CONRAD, *Benjamin N. Cardozo High School, Bayside, N.Y. (solution as well)*; G.D. KAYE, *Department of National Defence;* ANDRÉ LADOUCEUR, *École Secondaire De La Salle; and* CHARLES W. TRIGG, *San Diego, California (solution as well). The proposer provided an answer only.*

Editor's comment.

It is an interesting elementary exercise to show that the different radical expressions arrived at in solutions I and II represent the same numbers.

With the exception of one, all the "other solvers" gave a unique answer to the problem: 3.7609 meters. The exceptional one gave two answers, both wrong (although his reasoning was correct). The others apparently felt that it would not be necessary to use a ladder to paint a window that was only 1.3622 meters from the ground. It seems to me both answers should be given for the proposer, who speaks in the problem, might be a midget. (How tall are you, Jeremy?)

On the other hand if, with the dimensions proposed by Conrad, the proposer's son was pushing a 12-meter cube under a 35-meter ladder, then it would be some ladder! some cube! *some son!*

123. [1976: 41] *Proposed by Walter Bluger, Department of National Health and Welfare.*

By means of only three weighings on a two-pan balance, you are to find among 13 dimes the one counterfeit coin and be able to tell whether it is heavier or lighter than a true coin. You are given the 13 coins and a balance, and you may bring anything you like with you that may help you in solving the problem.

I. *Solution by Léo Sauvé, Algonquin College.*

The most elegant solution that I have seen for the corresponding problem involving 12 coins is due to F.J. Dyson (*Math. Gazette*, XXX (1946), 231). His method can easily be adapted to the present problem involving 13 coins, and I proceed to do so.

I	II	III	IV
0	$\underline{000}$	222	000
1	$\underline{001}$	221	001
2	002	$\underline{220}$	220
3	$\underline{010}$	212	010
4	$\underline{011}$	211	011
5	$\underline{012}$	210	012
6	020	$\underline{202}$	202
7	021	$\underline{201}$	201
8	022	$\underline{200}$	200
9	100	$\underline{122}$	122
10	101	$\underline{121}$	121
11	102	$\underline{120}$	120
12	110	$\underline{112}$	112

I am given 13 coins which I clearly identify by numbering them 0, 1, 2, ..., 12; and I have brought with me a good coin which I identify as g. In column I of the table I enter the identifying numbers of the 13 coins. For each n in column I, I enter in column II the base three representation of n, and in column III the base three representation of $26-n$, using three digits in each case. In the first row, I underline $\underline{000}$. In each succeeding row I underline whichever number, in column II or III, is such that the first change of digit is cyclic in the order 012.

Finally, I transfer the underlined numbers to column IV.

Let us represent the three weighings by W_i, $i = 1, 2, 3$. For each W_i, place in the left pan L of the balance all the coins for which the digit in the ith column of IV is 0; and place in the right pan R the good coin g and all the coins for which the digit in the ith column of IV is 2. Thus we get the following weighings:

	L					R				
W_1:	(0	1	3	4	5)	(2	6	7	8	g)
W_2:	(0	1	6	7	8)	(2	9	10	11	g)
W_3:	(0	2	3	8	11)	(5	6	9	12	g)

For $i = 1, 2, 3$, let $a_i = 0$ if the left pan sinks in W_i, $a_i = 2$ if the right pan sinks, and $a_i = 1$ if there is equilibrium. The base three number $a_1a_2a_3$ lies, in column II or III, in the row corresponding to the false coin. If $a_1a_2a_3$ is underlined in column II or III, the false coin is heavier; otherwise it is lighter.

For example, if $a_1a_2a_3 = 021$, then the false coin is No. 7 and it is lighter; if $a_1a_2a_3 = 112$, the false coin is No. 12 and it is heavier.

A careful study of the mechanism clearly shows why the method works. Furthermore it can obviously be generalized to n weighings of $\frac{1}{2}(3^n - 1)$ coins.

II. *Solution by Clayton W. Dodge, University of Maine at Orono.*

We label the coins 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, a, b, and c. We also require one coin g known to be good. We list the coins of a weighing by an ordered pair (L,R) and we consider whether the left pan is heavier H or lighter L than the right pan or perhaps they balance B.

First weigh (12345, 6789g). If L (the left pan is light), then one of 1, 2, 3, 4, or 5 is light, or one of 6, 7, 8, or 9 is heavy. Then weigh (126, 348). If L again, then 1 or 2 is light or 8 is heavy, and weighing (1,2) will determine which. If H in the second weighing, then one of 3 or 4 is light or 6 is heavy, and weighing (3,4) will suffice. If, in the second weighing, B occurs, then 7 or 9 is heavy or 5 is light, so weigh (7,9). If H occurs in the first weighing, the situation is similar, the second weighing is exactly the same, and the third weighing is adjusted appropriately. If B occurs in the first weighing, then one of the coins 0, a, b, c is defective, so weigh (0a, bg). If L or H occurs, weigh (0,a); if B occurs, weigh (c,g).

III. *"Solution" by Steven R. Conrad, Benjamin N. Cardozo High School, Bayside, N.Y.; and friend.*

Bringing with me my swami, I need zero weighings to determine the counterfeit. Unfortunately, I now need two weighings to determine whether it is lighter or heavier than a true coin, as I am most forgetful.

IV. *Comment by Dan Eustice, The Ohio State University.*

The following tables are self-explanatory. In teaching induction to undergraduates, one can start building up the entries in the tables and the various induction steps usually become clear to the students.

WEIGHING COINS ON TWO-PAN BALANCE
One false coin in the pile

Table 1

False coin known to be Heavier (Lighter)

Number of weighings	1	2	3	4	5	...	n
Max. number of coins	3	9	27	81	243	...	3^n

Table 2

False coin can be either Lighter or Heavier with some good coins

Number of weighings	1	2	3	...	n
Number of good coins	1	3	9	...	3^{n-1}
Max. number of coins*	2	5	14	...	$(3^n + 1)/2$

Table 3

False coin can be either Lighter or Heavier

Number of weighings	1	2	3	...	n
Max. number of coins*	1	4	13	...	$(3^n - 1)/2$

Table 4

Suspected coins: If false Heavy or if false Light with some good coins

Number of weighings	1	2	3	...	n
Number of susp. H coins	1	4	13	...	$(3^n - 1)/2$
Number of susp. L coins	1	4	13	...	$(3^n - 1)/2$
Number of good coins	1	3	9	...	3^{n-1}

The * in Tables 2 and 3 refer to the fact that it is not necessary to determine whether the false coin is lighter or heavier. If it's desired to know lighter or heavier, then each entry is one less. Thus, Problem 123 is the third entry in Table 2 decreased by one: 13.

Table 4 only lists the results needed in the induction to fill out Tables 2 and 3.

V. *Comment by Clayton W. Dodge, University of Maine at Orono.*

The *American Mathematical Monthly* some years ago printed the original problem involving 12 coins. More recently I had the pleasure of refereeing a generalization problem (in the *Monthly*) involving a triple-pan balance. Bluger's generalization to 13 coins is not difficult and not new, although I have not seen it in print earlier. One of the solvers to the triple-pan problem wrote Problem 123 in this form.

A king sent his son to seek his fortune, giving him a gold coin with the admonition that he should not part with it since it would someday save his life. The prince was captured by a rival monarch who said he would execute him unless he could solve a problem. He gave the prince 13 gold coins and a pan balance, stating that one of the 13 coins was counterfeit and weighed either more or less than each of the other 12, and that the prince must find the defective coin and tell whether it was heavier or lighter with just three weighings of the balance. While puzzling over the problem, the prince remembered the words of his father and brought forth the gold coin he had given him. It was identical to the coins the monarch had given him to test. He then easily solved the problem.

Also solved by CLAYTON W. DODGE, *University of Maine at Orono (second solution);* G.D. KAYE, *Department of National Defence;* F.G.B. MASKELL, *Algonquin College (two solutions); and the proposer.*

Editor's comment.

The solutions submitted were of two kinds (not to speak of the "solution" III, about which more later).

One kind, of which solution II is a good example, consisted of an exhaustive examination of all the possible outcomes. This is not difficult to do, but it is tricky; great care must be exercised if one is to be sure that no possible outcome has been overlooked. But if one tries to determine, in n weighings, the one false coin among $\frac{1}{2}(3^n - 1)$ coins, then this method is very difficult or well-nigh impossible to use for $n > 3$.

The other kind consisted in a more or less successful groping for the base three method used so incisively by Dyson (see solution I), which is suggested by the three possible outcomes of each weighing.

Finally, a Bronx cheer for the solver and his fictitious friend in the cop-out "solution" III. Him and his swami! Because of his lack of respect for our common mistress, mathematics, let us all get together and *lean* on him a little bit, Godfather style: *weigh down upon the Swami Ribber.*

124. [1976: 41] *Proposé par Bernard Vanbrugghe, Université de Moncton.*

Calculer: $\lim\limits_{x\to\infty} x\int_0^x e^{t^2-x^2}\,dt.$

Solution de Leroy F. Meyers, The Ohio State University.

$$\lim_{x\to\infty} x\int_0^x e^{t^2-x^2}\,dt = \lim_{x\to\infty}\frac{\int_0^x e^{t^2}\,dt}{\dfrac{e^{x^2}}{x}} = \lim_{x\to\infty}\frac{e^{x^2}}{\dfrac{xe^{x^2}\cdot 2x - e^{x^2}}{x^2}} = \lim_{x\to\infty}\frac{x^2}{2x^2-1} = \frac{1}{2}\,.$$

L'emploi de la règle de l'Hôpital est permis, car

$$\lim_{x\to\infty}\int_0^x e^{t^2}\,dt = \lim_{x\to\infty}\frac{e^{x^2}}{x} = \infty\,.$$

Aussi résolu par G.D. KAYE, *Département de la Défense Nationale; et par le proposeur.*

125. [1976: 41] *Proposé par Bernard Vanbrugghe, Université de Moncton.*

A l'aide d'un compas seulement, déterminer le centre inconnu d'un cercle donné.

(Ce problème est bien connu sons le nom de *problème de Napoléon.*)

I. *Solution by Viktors Linis, University of Ottawa.*

To determine the centre of the circle Γ (see Figure 1):

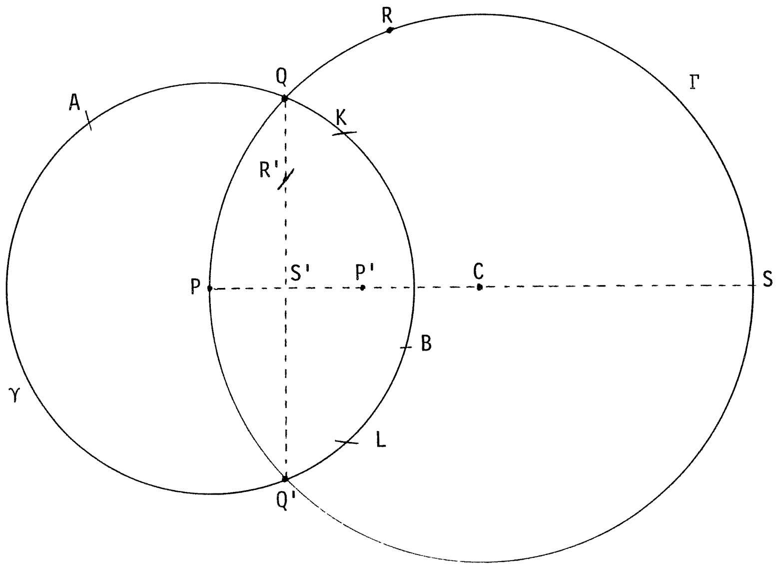


Figure 1

1) choose three points P, Q, R on Γ;

2) draw the circle γ with centre P, radius PQ;

3) find R', the inverse of R with respect to γ, by drawing a circle with centre R, radius RP, marking off A and B, then drawing circles with centres A and B, radii $AP = BP$, intersecting at R';

4) find the reflection P' of P with respect to the line QR'Q' by drawing circles with centres Q and Q', radii $QP = Q'P$, intersecting at P';

5) find the inverse C of P' with respect to γ by drawing a circle with centre P', radius P'P, intersecting γ at K and L, and then drawing circles with centres K and L, radii $KP = LP$, intersecting at C.

Then C is the required centre.

Proof. PS is a diameter of Γ; inversion of Γ with respect to γ gives the straight line QQ', and the inverse of S is S' on QQ' which is the midpoint of PP'. The point C is the midpoint of PS, and hence the centre of Γ, since we have

$$PC \cdot PP' = \tfrac{1}{2} PS \cdot 2PS' = PS \cdot PS' = PQ^2,$$

that is, P' and C are inverses with respect to γ.

II. *Solution du proposeur.*

D'un point A quelconque sur le cercle donné traçons un cercle γ de centre A qui coupe le cercle donné en deux points B et C (voir la figure 2). Soit

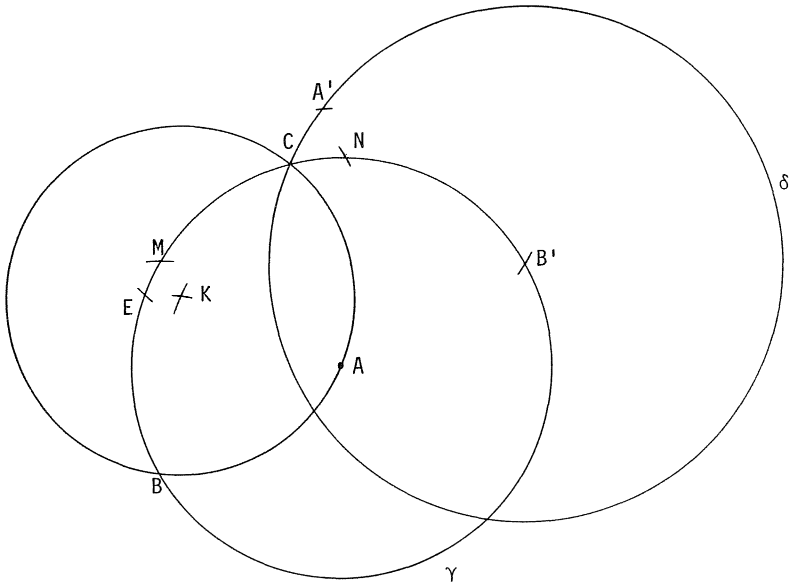


Figure 2

B' le point diamétralement opposé à B sur le cercle γ. B' est obtenu en rapportant $BM = MN = NB' = AB$. Traçons le cercle δ de centre B' et de rayon B'C. De A comme centre, traçons un arc de cercle de rayon B'C qui coupe δ en A'. De A' comme centre avec B'C comme rayon, on trace A'E. BE est le rayon r du cercle donné. De B et A on trace deux cercles de rayon r qui se coupent au centre K recherché.

En considérant des propriétés géométriques élémentaires, il est facile de démontrer que cette construction nous mène bien au but désiré.

Editor's comments.

1. This problem first came to the attention of the world in 1797 when a book [7] by Lorenzo Mascheroni (1750-1800) was published in Pavia, Italy. In this book, Mascheroni showed that all Euclidean constructions can be carried out with the compasses alone, provided it is understood that a straight line is considered to be known once two distinct points on it are known. Around 1928, it was discovered that an obscure writer named Georg Mohr (1640-1697) had anticipated Mascheroni's discovery

by one hundred twenty-five years in a book called *Euclides Danicus*, published in 1672.

In his constructions, Mascheroni frequently uses the compasses as *dividers*, or modern compasses, which can be used to transport distances from one place to another. But since it is known (see [3], for example) that Euclidean (collapsible) compasses and modern compasses are equivalent tools, every Mascheroni construction can be carried out with Euclidean compasses.

Solution II above is done in the spirit of Mascheroni with compasses occasionally used as dividers. To do this construction with Euclicean compasses would greatly increase the number of circles requiring to be drawn, since transporting a distance with dividers is equivalent to drawing five circles with Euclidean compasses. Solution II requires the drawing of nine circles, and for three of those circles the compasses are used as dividers. Restricted to Euclidean compasses, the construction would thus require the drawing of twenty-one circles.

At the end of his solution II, the proposer says that the proof of the construction is easily found from elementary geometry. Well, easy is as easy does, and he doesn't. Any reader who finds it necessary to look up the "easy" proof of this construction can find it in [6, pp. 24-26].

2. In 1890, A. Adler [1] proved that Mascheroni constructions with Euclidean compasses could be greatly shortened by using inversion. Thus solution I, which uses this method, only requires the drawing of nine circles with Euclidean compasses.

At the outset, in solution I, the radius is unknown. But if one concedes that it is possible to visually choose the points P and Q on the given circle so that the length of segment PQ is greater than half the unknown radius, then the number of circles requiring to be drawn can be reduced to six. The method and proof of this construction can be found in [6, pp. 40-42]. Furthermore, this construction is simpler and more exact than the usual construction with a ruler and compasses.

3. It is known (see [8], for example) that it is impossible to find the centre of a given circle by means of a ruler alone.

4. The proposer credits this problem not to Mascheroni but to Napoleon. When asked for a reference to substantiate this, he could only refer to a note he found in an old notebook, dating back to his high-school days, so he feels he must have got the information (or misinformation) from his teacher at the time.

I have looked into this matter, and this is what I have found. On page 268 of his *A History of Mathematics* [2], Cajori says:

> Napoleon proposed to the French mathematicians the problem, to divide the circumference of a circle into four equal parts by the compasses only. Mascheroni does this by applying the radius three times to the circumference; he obtains the arcs AB, BC, CD; then AD is a diameter; the rest is obvious.

However, before applying the radius to the circumference, the radius, and hence the centre, must be found. Thus, even though Mascheroni's problem must be used to solve a problem proposed by Napoleon, the problem of finding the centre must remain credited to Mascheroni.

But wait... In 1797, when Mascheroni's book was published in Pavia, Napoleon was *général-en-chef de l'armée d'Italie*. During most of the year 1797, Napoleon was in Italy. The battle of Rivoli took place in January 1797; the Treaty of Campoformio was signed in October 1797; and Napoleon returned to Paris in December 1797 [4]. So possibly Mascheroni's problem should now be credited to Napoleon: but if so, it is only by right of conquest.

REFERENCES

1. A. Adler, *Zur Theorie der Mascheronischen Constructionen*, Wiener Sitzungsber, 1890. I obtained this reference from [5].

2. Florian Cajori, *A History of Mathematics*, 2nd edition, Macmillan, N.Y., 1919.

3. Howard Eves, *An Introduction to the History of Mathematics*, Revised Edition, Holt, Rinehart and Winston, 1964, pp. 108-109.

4. J. Christopher Herold, *The Age of Napoleon*, American Heritage Publishing Co., 1963, pp. 62, 64, 65.

5. Hilda P. Hudson, Ruler and Compasses, in *Squaring the Circle and Other Monographs*, Chelsea, 1953. This is probably the most complete reference for Mascheroni's and related problems.

6. A.N. Kostovskii, *Geometrical Constructions Using Compasses Only*, Blaisdell, 1961 (translated from the Russian).

7. Lorenzo Mascheroni, *Geometria del Compasso*, Pavia, 1797.

8. A.S. Smogorzhevskii, *The Ruler in Geometrical Constructions*, Blaisdell, 1961, pp. 75-77 (translated from the Russian).

126. [1976: 41] *Proposed by Viktors Linis, University of Ottawa.*

Show that, for any triangle ABC,

$$|OA|^2 \sin A + |OB|^2 \sin B + |OC|^2 \sin C = 2K,$$

where O is the centre of the inscribed circle and K is the area of ΔABC.

Solution d'André Ladouceur, École Secondaire De La Salle.

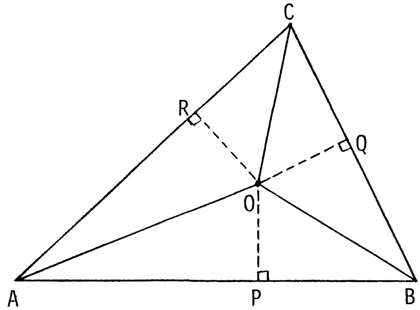
En référant à la figure, on trouve

$$\begin{aligned} 2|\text{Quad. } APOR| &= 4|\Delta APO| \\ &= 2\cdot|OP|\cdot|AP| \\ &= 2\cdot|OA|\sin\frac{A}{2}\cdot|OA|\cos\frac{A}{2} \\ &= |OA|^2 \sin A. \end{aligned}$$

En additionnant ce résultat avec deux autres semblables obtenus en remplaçant A successivement par B et C, on obtient

$$2K = |OA|^2 \sin A + |OB|^2 \sin B + |OC|^2 \sin C.$$

Also solved by G.D. KAYE, *Department of National Defence;* F.G.B. MASKELL, *Algonquin College;* LEROY F. MEYERS, *The Ohio State University;* LÉO SAUVÉ, *Algonquin College; and* CHARLES W. TRIGG, *San Diego, California.*

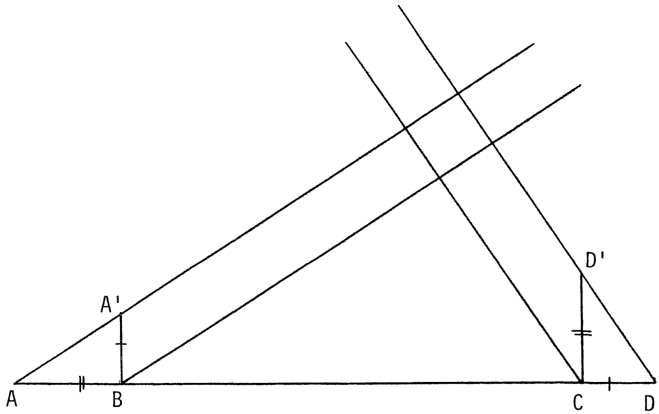


127. [1976: 41] *Proposed by Viktors Linis, University of Ottawa.*

A, B, C, D are four distinct points on a line. Construct a square by drawing two pairs of parallel lines through the four points.

I. *Solution by F.G.B. Maskell, Algonquin College.*

We select the pair of segments (AB, CD) and erect perpendiculars BA' and CD' equal to CD and AB respectively (see figure). If we now join AA', DD',



and complete the construction with the necessary parallels, it is evident that the resulting parallelogram is a square with side

$$\frac{AB \cdot BA'}{AA'} = \frac{AB \cdot CD}{\sqrt{AB^2 + CD^2}} = \frac{DC \cdot CD'}{DD'} .$$

If the perpendiculars at B and C had been dropped below the line instead, the resulting square would be symmetric to the first with respect to line AD.

The same construction applied to the pairs of segments (AC, BD) and (AD, BC) would yield four more solutions, so that there are six solutions in all.

II. *Comment by Steven R. Conrad, Benjamin N. Cardozo High School, Bayside, N.Y.*

Perhaps the most complete treatment of this problem may be found in

Mathematics Magazine, September 1965, pp. 225-228. It has also appeared as Problem 3417 in the January 1931 issue of *The American Mathematical Monthly*, as Problem 3 on the 1898 Eötvös Competition, and as Problem 2776 in the 1961 volume of *School Science and Mathematics*, pp. 555-556. It also occurs in Altshiller Court's *College Geometry* as well as in numerous less well-known texts.

Also solved by G.D. KAYE, *Department of National Defence;* ANDRÉ LADOUCEUR, *École Secondaire De La Salle;* CHARLES W. TRIGG, *San Diego, California; and* BERNARD VANBRUGGHE, *Université de Moncton.*

Editor's comment.

Some of the references mentioned by Conrad refer to the more general problem of constructing a square whose sides (or sides produced) go through four given points *in the plane*. What we were looking for here is a particularly simple construction valid in the special case when the four given points are *on a line*.

In the 1898 Eötvös Competition, however (which can be found in *Hungarian Problem Book I* (Random House, 1963), pp. 11, 60-61), the problem considered is the same as our special case, but it seems to the fond fatherly eyes of this editor that the solution given there is not quite as perspicuous as the one given here.

Clayton W. Dodge, University of Maine at Orono, also sent a brief note pointing out that the problem considered here was a special case of a well-known problem.

128. [1976: 41] *Proposé par Paul Khoury, Collège Algonquin.*

Déterminer les nombres réels a, b, c, ayant donné que l'équation $az^2+bz+c=0$ admet comme une de ses racines $v+v^2+v^4$, où v est une racine imaginaire de $z^7-1=0$.

Solution de Léo Sauvé, Collège Algonquin.

Puisque dans l'équation donnée $az^2+bz+c=0$, on obtient une équation équivalente en divisant les deux membres par a, il n'y a aucune perte de généralité à supposer que $a=1$, et il ne reste alors que b et c à déterminer dans l'équation

$$z^2+bz+c=0. \tag{1}$$

Puisque v est une septième racine imaginaire de l'unité, on a

$$v = \cos\frac{2k\pi}{7}+i\sin\frac{2k\pi}{7}, \quad k\in\{1,2,\ldots,6\}. \tag{2}$$

Mais, quel que soit k, on a

$$\overline{v}=v^6, \quad \overline{v^2}=v^5, \quad \overline{v^4}=v^3,$$

de sorte que si l'on pose

$$p = v+v^2+v^4, \quad q = v^3+v^5+v^6,$$

il vient $\overline{p}=q$. Or p étant racine imaginaire de l'équation (1) dont les coefficients sont réels, l'autre racine doit être $\overline{p}=q$. La somme des racines de l'équation

$z^7 - 1 = 0$ étant nulle, c'est-à-dire $p+q+1 = 0$, on a donc

$$p+q = -b = -1,$$

d'où $b=1$. De plus,

$$\begin{aligned} pq &= \Sigma v^i \quad \text{pour } i = 4,6,7,5,7,8,7,9,10 \\ &= \Sigma v^i \quad \text{pour } i = 1,2,3,4,5,6,7,7,7 \\ &= 2v^7 \\ &= 2, \end{aligned}$$

de sorte que $c = pq = 2$, et l'équation (1) devient

$$z^2 + z + 2 = 0. \tag{3}$$

Also solved by G.D. KAYE, *Department of National Defence; and* F.G.B. MASKELL, *Algonquin College.*

Editor's comment.

Several interesting relations are easy consequences of this problem. For example, if we take $k=1$ in (2), we get

$$\begin{aligned} p &= \operatorname{cis}\frac{2\pi}{7} + \operatorname{cis}\frac{4\pi}{7} + \operatorname{cis}\frac{8\pi}{7} \\ &= (\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}) + i(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}). \end{aligned}$$

But p is a root of (3), whose roots are $-\frac{1}{2} \pm \frac{1}{2} i\sqrt{7}$; hence

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\tfrac{1}{2} \tag{4}$$

and

$$\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \tfrac{1}{2}\sqrt{7},$$

where the positive sign is taken because

$$\sin\frac{2\pi}{7} + \sin\frac{8\pi}{7} = \sin\frac{2\pi}{7} - \sin\frac{\pi}{7} > 0.$$

Now using $\cos 2\theta = 2\cos^2\theta - 1$ in (4), and noting that $\cos\frac{4\pi}{7} = -\cos\frac{3\pi}{7}$, we get

$$\cos^2\frac{\pi}{7} + \cos^2\frac{2\pi}{7} + \cos^2\frac{3\pi}{7} = \frac{5}{4},$$

from which it follows that

$$\sin^2\frac{\pi}{7} + \sin^2\frac{2\pi}{7} + \sin^2\frac{3\pi}{7} = \frac{7}{4}.$$

129. [1976: 42] *Proposed by Léo Sauvé, Algonquin College.*

It has been known since Weierstrass that there exist functions continuous over the whole real axis but differentiable nowhere. Describe a function which is continuous over the whole real axis but differentiable only at (a) $x=0$; (b) a finite number of points; (c) a countable number of points.

Solution by Leroy F. Meyers, The Ohio State University.

Let W be a Weierstrass function, that is, let W be continuous on the entire real line but differentiable nowhere; let g be differentiable on the entire real line; and let $f = gW$. If $g(x)$ is chosen to be

(a) x;

(b) $(x-x_1)(x-x_2)\dots(x-x_n)$, where $x_1,\dots,x_n$ are selected points; or

(c) $\sin x$,

then use of the lemma proved below shows that f is continuous everywhere but differentiable only at

(a) 0;

(b) at the points $x_1,\dots,x_n$; or

(c) the denumerably many integral multiples of π, respectively.

LEMMA. f is differentiable precisely at the zeros of g.

Proof. If $g(a) = 0$, then

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{g(x)W(x) - g(a)W(a)}{x - a} = \lim_{x\to a} \frac{g(x)W(x) - 0}{x - a} \\ &= \lim_{x\to a} \frac{g(x) - g(a)}{x - a} \cdot W(x) = g'(a)W(a). \end{aligned}$$

If $g(a) \neq 0$, then there is an open interval I containing a such that, for all $x \in I$, we have $g(x) \neq 0$. Suppose that f is diffentiable at a. Now $W(x) = f(x)/g(x)$ for all $x \in I$; hence W is the quotient of two functions differentiable at a, and so must itself be differentiable at a, contrary to the definition.

Also solved by the proposer.

Editor's comment.

In case some curious readers would like to know what a Weierstrass function looks like, here is the original one discovered by Weierstrass. It was published by Du Bois-Reymond in 1875 with Weierstrass' own proof [1]. It is

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

where b is an odd integer, and a is such that $0 < a < 1$, and $ab > 1 + \frac{3\pi}{2}$.

Other examples, and a full discussion of the subject, can be found, for example, in [2,3].

REFERENCES

1. Du Bois-Reymond, *Crelle's Journal*, vol. LXXIX (1875), pp. 21-37. I found this reference in [2].

2. E.W. Hobson, *The Theory of Functions of a Real Variable*, Dover, 1957, vol. 2, pp. 401-412.

3. A.N. Singh, The Theory and Construction of Non-Differentiable Functions, in *Squaring the Circle and Other Monographs*, Chelsea, 1953.

130. [1976: 42] *Proposé par Jacques Marion, Université d'Ottawa.*

Soit A l'anneau $\{z : r \le |z| \le R\}$. Montrer que la fonction $f(z) = \frac{1}{z}$ n'est pas limite uniforme de polynômes sur A.

Solution du proposeur.

Soit $p(z)$ un polynôme quelconque. D'après le théorème du module maximum, il existe un $z_0 \in A$ tel que $|z_0| = R$ et $|p(z)| \le |p(z_0)|$ pour tout $z \in \overline{B}(0,R)$. Choisissons $z_1 \in A$ tel que $|z_1| = r$. On a alors

$$|p(z_0) - \frac{1}{z_0}| \ge \left||p(z_0)| - \frac{1}{|z_0|}\right| = \left||p(z_0)| - \frac{1}{R}\right|$$

et

$$|p(z_1) - \frac{1}{z_1}| \ge \left||p(z_1)| - \frac{1}{|z_1|}\right| = \left||p(z_1)| - \frac{1}{r}\right|,$$

avec

$$|p(z_1)| \le |p(z_0)| = \max_{z \in A} |p(z)| \tag{1}$$

et $\frac{1}{R} < \frac{1}{r}$.

Choisissons arbitrairement un $\varepsilon > 0$ tel que $\varepsilon < \frac{1}{2}(\frac{1}{r} - \frac{1}{R})$, de sorte que

$$\frac{1}{r} - \varepsilon > \frac{1}{R} + \varepsilon.$$

Si

$$\left||p(z_0)| - \frac{1}{R}\right| < \varepsilon \quad \text{et} \quad \left||p(z_1)| - \frac{1}{r}\right| < \varepsilon,$$

alors

$$|p(z_1)| > \frac{1}{r} - \varepsilon > \frac{1}{R} + \varepsilon > |p(z_0)|,$$

en contradiction avec (1). Donc, pour $z = z_0$ ou z_1, on doit avoir

$$|p(z) - \frac{1}{z}| \ge \left||p(z)| - \frac{1}{|z|}\right| \ge \varepsilon,$$

et la conclusion du théorème en découle.

* * *

— Maître Andry, reprit Jehan, toujours pendu à son chapiteau, tais-toi ou je te tombe sur la tête!

Maître Andry leva les yeux, parut mesurer un instant la hauteur du pilier, la pesanteur du drôle, multiplia mentalement cette pesanteur par le carré de la vitesse, et se tut.

VICTOR HUGO, Notre-Dame de Paris

The Möbius strip has interested mathematicians since its description by A.F. Möbius in 1865. A Roman mosaic of the 3rd century A.D. recently found near Arles, France, shows that Möbius bands were known in the ancient world.

Encyclopaedia Britannica 1975 *Yearbook of Science and the Future*, p. 264

MATH HOMEWORK

A challenge is something I've always sought;
   I rarely decline a bet,
But there's one test of skill and thought
   That I've not conquered yet.

On Tuesday is writ upon the board
   The problems on Friday due,
And my spirits have often soared
   At the thought of staging a coup
   (by getting everything right.)

This time my work will be correct;
   I'll not a detail miss.
With red my paper will not be flecked
   —Oh, what a state of bliss!

But obstacles so soon arise,
   And quickly do loom bigger.
I surely did not realize
   I'd have so much to figure.

What *are* the limits of this function?
   How should I integrate it?
I'm rather puzzled at this junction,
   So far I can't evaluate it.

Should I integrate by parts?
   Or maybe substitute?
"If seven men owned nineteen carts,
   ...And what's their speed enroute?"

As the day begins to break,
   A thought upon me dawns:
A perfect paper I must forsake;
   I've got to stop these yawns!

At last I'm finished with my work,
   But I've made mistakes aplenty,
Although my task I didn't shirk.
   Only three are right in twenty.

(After all)
To change my major'd be no sin;
   ASC I'd best be leaving
—Perhaps I'd better major in
   Underwater basket weaving.

LEYLA WOODS, student
The Ohio State University

THERE'S NO POLICE LIKE HOLMES

From *The Adventure of the Lion's Mane*, by Sir Arthur Conan Doyle; in *The Annotated Sherlock Holmes*, edited by William S. Baring-Gould (Clarkson N. Potter, 1967), Vol. II, p. 778.

Murdock was the mathematical coach at the establishment, a tall, dark thin man, so taciturn and aloof that none can be said to have been his friend. He seemed to live in some high, abstract region of surds and conic sections with little to connect him with ordinary life.

Marginal note on the same page:

"Now these words were written by Holmes himself," Mr. A.S. Galbraith wrote in "The Real Moriarty," "and if he thought of 'surds' as typifying the sort of thing a mathematician would have uppermost in his mind, obviously he knew little of [mathematics]."

op. cit., p. 780.

...he [Murdock] would insist upon some algebraic demonstration before breakfast.

Marginal note on the same page:

This "shows [Murdock] sadistic or mad," Mr. Galbraith wrote. "In the writer's moderately wide experience in mathematical circles, [it is] unheard of."

* * *

LETTERS TO THE EDITOR

Dear editor:

...I've talked very highly about your publication. And, strangely (ha) enough, I'm not the only one of your 400 or so "subscribers" who has been spreading the good word. Sam Greitzer mentioned to me that I should subscribe! I told him that I was just about to tell him the same thing. You are performing a service to the mathematics community that is not being performed elsewhere.

STEVEN R. CONRAD,
Benjamin N. Cardozo High School,
Bayside, N.Y.

Dear editor:

...I have just received the first four Volume 2 copies of EUREKA and am fascinated with them! It is very difficult for me to put them down...

Thanks again for EUREKA. It is absolutely delightful!

CLAYTON W. DODGE,
University of Maine at Orono