Mathematical Spectrum

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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

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A Family of Heron Triangles

K. R. S. SASTRY

A look at triangles with integral sides and area.

1. Introduction

Heron gave the formula

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}\,, \quad s = \tfrac{1}{2}(a+b+c) \quad (1.1)$$

for the area of the triangle with sides (a, b, c). According to historians of mathematics, he lived in Alexandria, Egypt, in the first century. The discovery of the integer-sided *and* the integer-area triangle (13, 14, 15; 84) is attributed to Heron. To honour Heron, the name *Heron triangle* is given to a triangle in which the sides and the area are integers. In Dickson's monumental work (reference 1), we find a number of early attempts to determine Heron triangles. Some modern approaches are available in references 2 and 3–7. Dickson uses the name Heron triangle if a triangle has rational sides and area. However, these rationals can always be rendered integers, so in this article a Heron triangle has integer sides and area. We use the word 'side' also to mean 'the length of a side'.

We determine a countable set of Heron triangles. This particular set exhibits many interesting properties. We list them later on. In particular, given any triangle in the set, we can find another member so that (i) the two can be adjoined to obtain a new Heron triangle that is not similar to members of our family, and (ii) we can generate *Brahmagupta quadrilaterals*, i.e. cyclic quadrilaterals having integer sides, integer diagonals and integer area. But first we recall a few necessary facts about a triangle. The reader may frequently visit figure 1 and reference 8 or a similar geometry text to get a feel of the facts we mention.

2. Necessary facts about a triangle

In a triangle ABC, the three internal angle bisectors concur at a point I called the *incentre*. We may draw a circle with centre I tangential to the sides of the triangle at D, E, F. The radius r of this circle is called the *inradius*. We may use a property of the tangents to a circle to deduce that $\mathrm{AE} = \mathrm{AF} = s - a$, etc. Furthermore, $r = \triangle/s$.

The internal bisector of angle A and the external bisectors of angles B and C also concur at a point $\mathrm{I_A}$ called the *excentre* opposite the vertex A. We may draw another circle with centre $\mathrm{I_A}$ tangential to the sides of the triangle at $\mathrm{D'}, \mathrm{E'}, \mathrm{F'}$. The radius r_A of this excircle is called an *exradius*. This time $\mathrm{AE'} = \mathrm{AF'} = s$ and $r_\mathrm{A} = \triangle/(s-a)$.

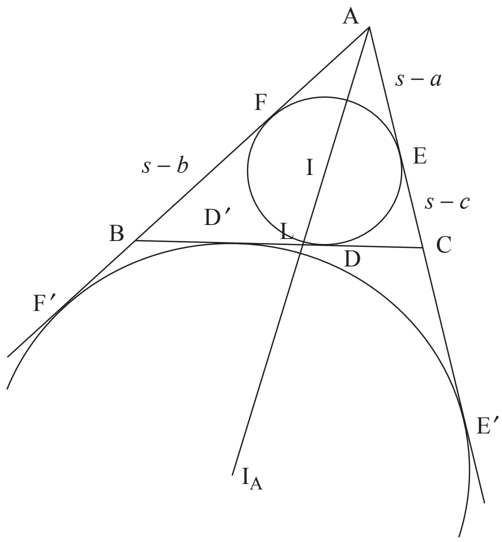

Figure 1. The incircle and an excircle of △ABC.

Theorem 2.1. (Angle bisector.) *An angle bisector of a triangle divides the opposite side into segments that are in the same ratio as the other two sides.*

In figure 1 we have $\mathrm{BL}/\mathrm{LC} = \mathrm{AB}/\mathrm{AC}$, for example.

Theorem 2.2. (A characterizing property of a cyclic quadrilateral.) *A quadrilateral is cyclic, i.e. inscribable in a circle, if and only if a pair of opposite angles sum to* π.

Theorem 2.3. (Ptolemy's theorem.) *In a cyclic quadrilateral* ABCD *with diagonals* AC *and* BD *we have* $\mathrm{AC}\cdot\mathrm{BD} = \mathrm{AB}\cdot\mathrm{CD} + \mathrm{AD}\cdot\mathrm{BC}$.

In fact, Ptolemy's theorem gives another characterizing property of a cyclic quadrilateral. The reader may wish to show this.

We use some other facts when needed, but we assume them to be known to the reader.

3. The determination of the family

We assume that a triangle whose sides have the form $(a, b, c) = (p, p+1, 2p-1)$ is a Heron triangle. Then, from (1.1), $\triangle = p(2(p-1))^{1/2}$ must be a natural number. Therefore, we put $p - 1 = 2k^2$, that is, $p = 2k^2 + 1$, where k is a natural number. This gives

$$T_k = (a_k, b_k, c_k) = \bigl(2k^2+1,\, 2(k^2+1),\, 4k^2+1\bigr) \qquad (3.1)$$

for $k = 1, 2, 3, \ldots$. The triangle T_k in (3.1) has area $\triangle_k = 2k(2k^2+1)$, so it is a Heron triangle. Some initial members of this family are $T_1 = (3, 4, 5)$, $T_2 = (9, 10, 17)$, $T_3 = (19, 20, 37)$, $T_4 = (33, 34, 65)$, and so on. We should remark that the appearance of consecutive integers as sides a and b does not imply that (3.1) yields all such triangles—for instance, the reader may verify that Heron's own (13, 14, 15) triangle cannot be obtained from (3.1).

In the next section, we list 10 properties of the family of triangles just described.

4. Properties of the family

For $k = 1, 2, 3, \ldots$, T_k denotes the kth member of the family, a_k, b_k, c_k denote the sides, $\triangle_k$ the area and $\mathrm{A}_k, \mathrm{B}_k, \mathrm{C}_k$ the vertices of T_k. In what follows, we drop the index k if the property under discussion is independent of k.

Theorem 4.1. *The following properties hold:*

(i) *Let r_k denote the inradius of T_k. Then $r_k = k$.*

(ii) $s - a = a$, *that is,* $\mathrm{AE} = \mathrm{AF} = \mathrm{BC}$.

(iii) $s - c = 1$, *that is,* $\mathrm{CD} = \mathrm{CE} = 1$, *a constant for all k.*

(iv) $r_{\mathrm{A}} = 2r$ *holds for all T_k.*

(v) $\tan(\frac{1}{2}\mathrm{B}) \cdot \tan(\frac{1}{2}\mathrm{C}) = \frac{1}{2}$.

(vi) I *quadrisects the internal angle bisector* AL *such that* $\mathrm{AI} > \mathrm{IL}$.

(vii) I *bisects* $\mathrm{AI_A}$.

(viii) *The altitude (through the vertex* A*)* $h_a = 4r$.

(ix) $b_{2k} = 2c_k$.

(x) $\mathrm{B}_k + \mathrm{C}_{2k} = \pi$.

The property (x) enables us to adjoin the Heron triangles T_k and T_{2k} to generate new Heron triangles, as mentioned earlier. The triangles T_k and T_{2k} will also be used to generate Brahmagupta quadrilaterals.

Proof. The verification of the first four properties is straightforward, and is left to the reader. To establish (v), consider figure 2.

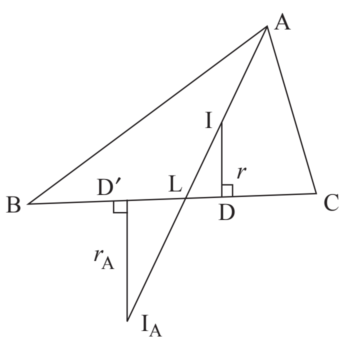

Figure 2. Verification of (v) and (vi).

We see that

$$\tan(\tfrac{1}{2}\mathrm{B}) \cdot \tan(\tfrac{1}{2}\mathrm{C}) = \frac{r}{\mathrm{BD}}\,\frac{r}{\mathrm{DC}} = \frac{r}{s-b}\,\frac{r}{s-c} = \frac{k^2}{2k^2} = \tfrac{1}{2}\,.$$

To establish (vi), we stay with figure 2 and apply theorem (2.1) twice. To do so, we consider $\triangle\mathrm{ABL}$ and $\triangle\mathrm{ACL}$ and note that BI and CI respectively bisect the angles B and C. Hence,

$$\frac{\mathrm{AI}}{\mathrm{IL}} = \frac{\mathrm{AB}}{\mathrm{BL}} = \frac{\mathrm{AC}}{\mathrm{CL}} = \frac{\mathrm{AB}+\mathrm{AC}}{\mathrm{BL}+\mathrm{CL}} = \frac{b+c}{a} = 3\,.$$

In the above, we used a property of equal ratios, namely, if $\lambda = e/f = g/h$, then $\lambda = (e+g)/(f+h)$.

To establish (vii), we use (iv) and the fact that $\triangle\mathrm{ILD}$ is similar to $\triangle\mathrm{I_ALD'}$. This gives $\mathrm{LI_A} = 2(\mathrm{IL})$. Hence, $\mathrm{II_A} = \mathrm{IL} + \mathrm{LI_A} = 3(\mathrm{IL})$. However, from (vi) we have $\mathrm{AI} = 3(\mathrm{IL})$. Thus, $\mathrm{AI} = \mathrm{II_A}$ and I bisects $\mathrm{AI_A}$.

Property (viii) follows from the fact that $h_a/r = \mathrm{AL}/\mathrm{IL} = 4$.

Property (ix) is easy to see, since $T_k = (a_k, b_k, c_k) = (2k^2+1, 2(k^2+1), 4k^2+1)$.

To establish property (x), we use the cosine rule, $\cos\mathrm{A} = (b^2+c^2-a^2)/(2bc)$, for $\triangle\mathrm{ABC}$. This gives

$$\cos\mathrm{B}_k = \frac{4k^2-1}{4k^2+1} \quad \text{and} \quad \cos\mathrm{C}_{2k} = -\frac{4k^2-1}{4k^2+1}\,.$$

Hence

$$\mathrm{B}_k + \mathrm{C}_{2k} = \pi\,.$$

In the next two sections we consider two applications of (x).

5. More general Heron triangles

We can easily generate Heron triangles in which two sides are not consecutive integers in two ways. In the first, we substitute a rational number that is not an integer for k and then remove fractions. For example, let $k = \frac{1}{3}$ in the triangle T_k in (3.1). This gives $(a', b', c') = (11/9, 20/9, 13/9)$. By removing fractions, we obtain the Heron triangle $(a, b, c) = (11, 20, 13)$. This has area $\triangle = 66$. More generally, setting $k = p/q$ in (3.1) gives the Heron triangles

$$(a, b, c; \triangle) = \left(2p^2+q^2, 2(p^2+q^2), 4p^2+q^2; 2pq(2p^2+q^2)\right). \quad (5.1)$$

Here, p and q may be any two relatively prime natural numbers.

In the second method, we use theorem 4.1(x). In this case, we enlarge the sides suitably, if necessary, and adjoin the two triangles T_k and T_{2k} so that B_k and C_{2k} become adjacent angles. This can be done in four ways. We illustrate the technique with T_1 and T_2 in figure 3.

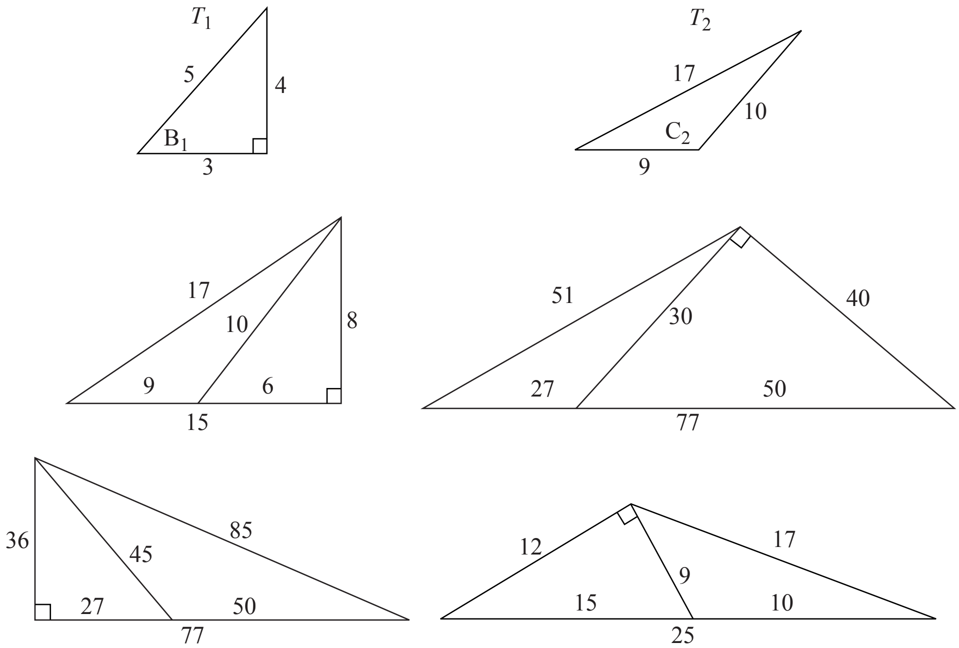

Figure 3. New Heron triangles from $T_1 = (3, 4, 5)$ and $T_2 = (9, 10, 17)$ where $B_1 + C_2 = \pi$.

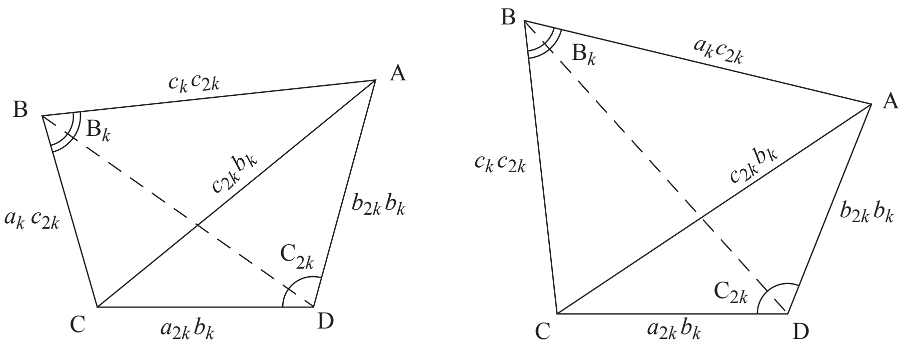

Figure 4. A pair of Brahmagupta quadrilaterals from T_k and T_{2k}.

The four general Heron triangles obtained are (15, 8, 17), (77, 40, 51), (36, 77, 85) and (12, 25, 17).

6. Brahmagupta quadrilaterals

The Indian mathematician Brahmagupta used the idea of adjoining two right-angled triangles to obtain a Heron triangle. He used a similar principle to generate a cyclic quadrilateral in which the sides, the diagonals and the area are all integers. He also gave the formula

$$\triangle = \sqrt{(s-a)(s-b)(s-c)(s-d)}\,, \quad s = \tfrac{1}{2}(a+b+c+d)$$

for the area of a cyclic quadrilateral with sides a, b, c, d. See reference 9 for more information on Brahmagupta. Then came Kummer's complex method to generate more generally a Heron quadrilateral in which the sides, the diagonals and the area are all integers. This method is outlined in reference 1. We use the triangles T_k and T_{2k} to form a quadrilateral in which a pair of opposite angles will have measures B_k and C_{2k}. Then theorem 4.1(x) ensures that this is a cyclic quadrilateral in view of theorem 2.2. The following algorithm gives the complete construction:

1. $T_k = (a_k, b_k, c_k)$, $T_{2k} = (a_{2k}, b_{2k}, c_{2k})$ and $B_k + C_{2k} = \pi$.
2. $c_{2k} \cdot T_k = (a_k c_{2k}, b_k c_{2k}, c_k c_{2k})$, and $b_k \cdot T_{2k} = (a_{2k} b_k, b_{2k} b_k, c_{2k} b_k)$.
3. Place the triangles $c_{2k} T_k$ and $b_k T_{2k}$ along the common side in one of the two ways shown in figure 4.
4. Calculate the diagonal BD by theorem 2.3. This gives
$$\mathrm{BD} = a_k b_{2k} + c_k a_{2k} \quad \text{or} \quad a_{2k} a_k + b_{2k} c_k\,.$$
5. The area of (either) Brahmagupta quadrilateral is $\triangle = 6k(4k^2+1)(6k^2+1)(8k^2+3)$.

Here is a numerical example: $T_1 = (3, 4, 5)$, $T_2 = (9, 10, 17)$ gives the Brahmagupta quadrilaterals having sides

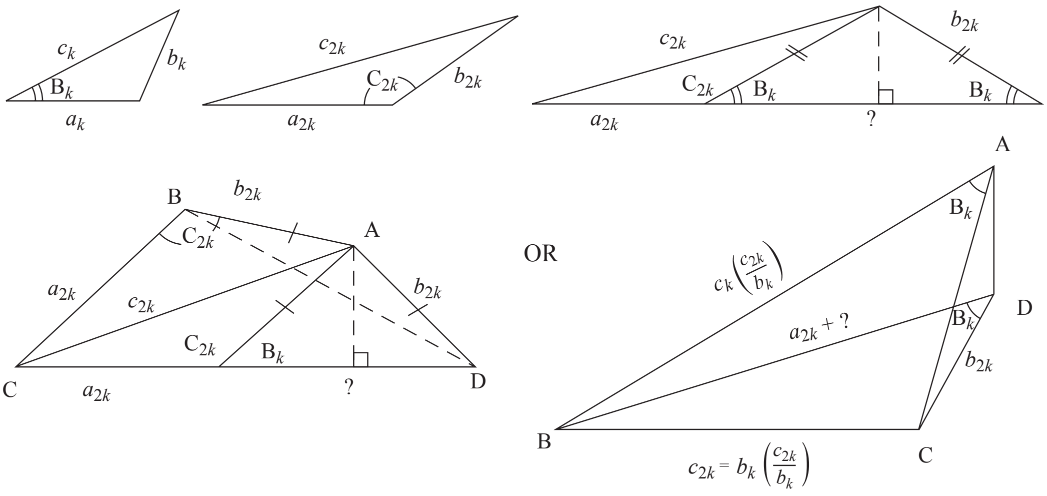

Figure 5. A new pair of Brahmagupta quadrilaterals from the same T_k and T_{2k}.

85, 51, 36, 40 with diagonals 68, 75 and 68, 77 and area 2310.

As a matter of fact, we have actually produced a cyclic Heron *pentagon*! We invite the reader to figure this out. Here is a hint: consider the two Brahmagupta quadrilaterals of figure 4 together, and use theorem 2.3 repeatedly to calculate various lengths.

We also invite the reader to figure out other ways of using the triangles T_k and T_{2k} to generate further Brahmagupta quadrilaterals. The diagrams in figure 5 provide hints.

In the last diagram of figure 5, $\triangle$ABC is T_k enlarged by the factor c_{2k}/b_k. On the same side of BC we have $\angle$BAC $=$ $\angle$BDC $=$ B_k, so a corollary of theorem 2.2 proves that ABCD must be cyclic. Since AC $= a_k(c_{2k}/b_k)$, Ptolemy's theorem then shows that AD must have rational length. The entire quadrilateral may then be suitably enlarged to obtain integral values for sides and diagonals, when the area will be integral too.

7. Conclusion

The rich harvest that we have reaped by this indepth study of a particular countable subset of Heron triangles may encourage the reader to undertake similar study. Remarkably, the triangles (5.1) have a rational median when $p = m^2 - n^2,\ q = 3mn,\ m > n$. The reader may recall that a median of a triangle is the line segment between a vertex and the mid-point of the opposite side. Incidentally, this observation suggests the following open problem:

Question 7.1. Find, if any, pairs of natural numbers p and q such that both $9p^2 + 4q^2$ and $16p^2 + 9q^2$ are simultaneously squares.

For such solutions p and q, the triangle (5.1) will have two rational medians. The subject of Heron triangles offers a large variety of challenges to the reader interested in problem-solving; see references 1, 4–7 and 10. We pose another Heron problem to the reader:

Question 7.2. Determine pairs of (non-isosceles) Heron triangles such that each pair has equal perimeter and equal area.

In references 9 and 4, such pairs of *isosceles* Heron triangles have been determined. A partial solution to the above general problem appears in reference 5.

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Tiling with Penrose Rhomb Clusters

ELLEN PERSTEIN

A recursive formulation gives a straightforward method for producing arbitrarily large tilings.

1. Introduction

For centuries man has used rectangles, triangles, and regular hexagons to cover or 'tile' a plane surface with repetitive or periodic patterns. These shapes have twofold, threefold, fourfold, or sixfold symmetry. Examples of such tilings range from Greek mosaics to brick walls to bathroom floors. Regular pentagons with their fivefold symmetry do not tile a plane; there are always gaps or overlaps.

In the 1970s, Roger Penrose, a mathematical physicist and cosmologist who is also a devotee of geometric puzzles and recreational mathematics, found sets of prototiles related to a regular pentagon that can tile a plane. One set of two quadrilaterals became known as 'kite' and 'dart'. Another set of two rhombs or rhombuses, a fat one with internal angles of 72° and 108° and a skinny one with internal angles of 36° and 144°, became known as the Penrose rhombs.

Penrose required that his rhombs be placed in accordance with edge-matching rules to tile a plane with patterns that never exactly repeat themselves. Grünbaum and Shephard (reference 1) prove that, when the placement of tiles satisfies the matching rules, these patterns are aperiodic. The patterns display fivefold symmetry; whole regions can be rotated by a multiple of 72° so that the arrangement looks just the same. These tilings have another intriguing characteristic; in a sufficiently large tiling the ratio of fat rhombs to skinny rhombs approaches $(1 + \sqrt{5})/2$, the golden ratio used throughout the ages by painters, sculptors, architects and even nature itself to create pleasing proportions.

In the January 1977 issue of *Scientific American*, Martin Gardner devoted his Mathematical Games column (reference 2) to the question of covering a plane with tiles. This sparked a flurry of research that brought tiling to the forefront of modern physics. Two paragraphs from reference 3 (pp. 86–87) give a very brief indication of the importance of Penrose's tiles:

> Initially just a playful creation, Penrose's tilings took on added significance with the unexpected discovery of crystalline materials that showed fivefold symmetry. Such a symmetry is physically impossible for a crystalline material in which groups of atoms fall into a regularly repeating, or periodic, pattern. The rules of geometry for a regular latticework in three-dimensional space allow only twofold, threefold, fourfold, and sixfold symmetries.
>
> Penrose's tilings, with their intriguing blend of order and disorder, provide simple geometric models for how groups of atoms may be arranged within these novel materials, now known as *quasiperiodic crystals*, or more simply, *quasicrystals*. A quasicrystal doesn't have the traditional lattice structure of an ordinary crystal. A view from inside a quasicrystal shows that its latticework looks somewhat different from place to place, even though its overall structure has a kind of long-range order.

The February 1990 issue of *Discover* magazine had a very interesting article by Hans C. von Bayer that gave more information about Penrose tiles and quasicrystals. This was the article that triggered my addiction to Penrose tiles.

Martin Gardner gave a more recent update on the significance of Penrose tiles in reference 4: 'Since then [1977], physicists have written hundreds of research papers on quasicrystals and their unique thermal and vibrational properties. Although Penrose's idea started as a strictly recreational pursuit, it paved the way for an entirely new branch of solid-state physics.'

2. Early approaches to Penrose tilings

People have found various methods for producing Penrose tilings:

- Grünbaum and Shephard (reference 1) define *composition* to mean 'the process of taking unions of tiles so as to build up larger tiles which are basically the same shapes as those from which we started and whose edge modifications and labels (induced by those of the original tiles) specify a matching condition equivalent to the original one'. *Decomposition* means the inverse of the process of composition—the cutting up of large tiles into smaller ones. The process of increasing the size and complexity of a patch of tiles (even a single tile) by expansion and then decomposing them into tiles of the original size is called *inflation*. Repetitions of this process with an inflation factor of 2.168034 (the square of the golden ratio) give arbitrarily large and complex patches of tiles implying that the whole plane can be tiled. Successive inflations appear in reference 1, Figure 10.3.20.

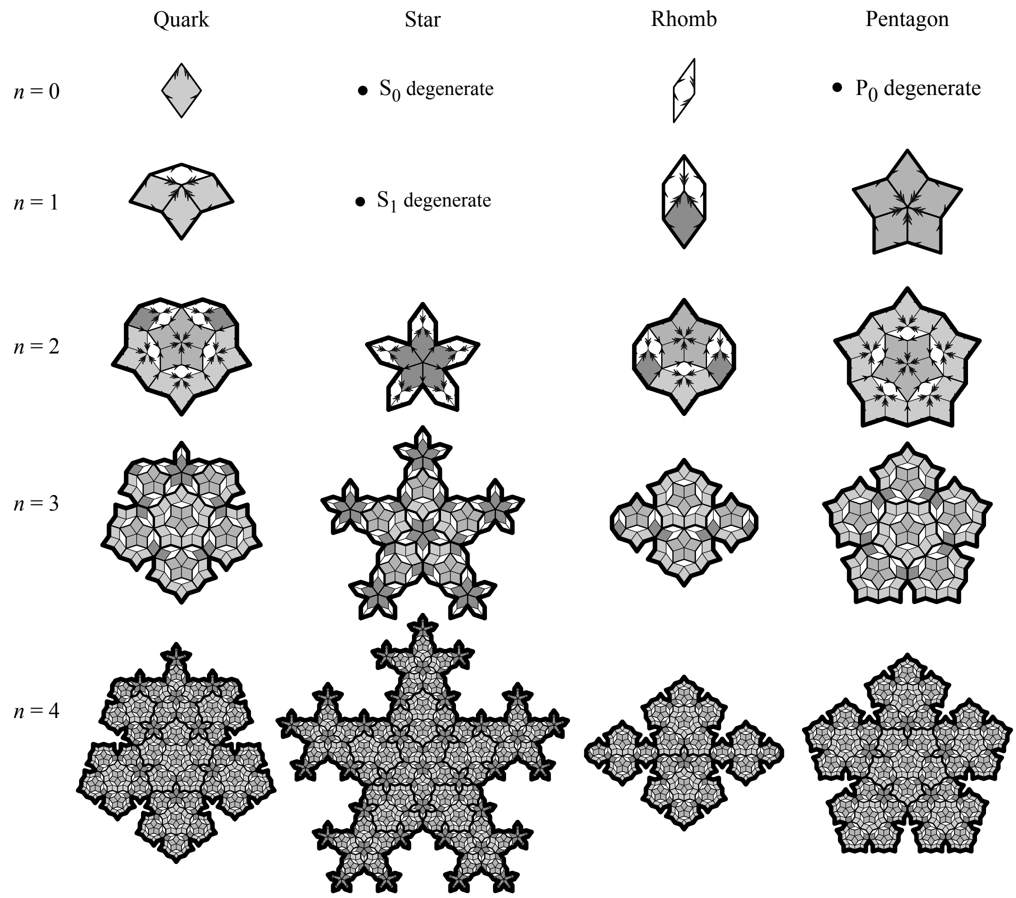

Figure 1. Recursive formulation levels 0–4.

- Peterson (reference 3) describes a method of growing larger tilings of Penrose rhombs from a seed consisting of a single tile. Tiles are added according to matching rules until there are no more forced moves, producing a faceted 'dead' shape. Then a tile is placed at a corner of the dead shape and growth continues until the next dead shape appears. A large tiling is shown in reference 3, Figure 3.16.
- In reference 5, Wagon gives a MATHEMATICA® program that performs repeated dissections to produce tilings with arbitrarily large numbers of small Penrose rhombs that can be enlarged to make a tiling of the plane as shown in reference 5, Figure 4.14(b).

3. The clusters and local tiling rules

Penrose required that the sides of the fat and skinny rhombs must be joined according to edge-matching rules. One way of identifying the sides, all of which are the same length, is by using single and double arrows as shown in the level $n = 0, 1, 2$ clusters of figure 1. Single arrows must match with single arrows and double arrows with double arrows. Penrose's edge-matching rules alone do not guarantee that any number of tiles will fit to create a flawless pattern. It is possible to place a tile properly and run into trouble many moves later with a gap that neither shape can fill.

I created my early rhomb tilings mostly by trial and error. As I gained experience, I noted that large tilings showed many repetitions of clusters shaped like stars and pentagons that appeared in more and more complex similar clusters. Examination showed that only two more clusters were needed as fillers around the stars and pentagons. The four clusters are pentagon (P), quark (Q), rhomb (R), and star (S). All except quark take their name from the shapes they exhibit at higher levels. Rhombs and quarks have reflection symmetry about one axis. Pentagons and stars have fivefold rotational and reflection symmetry. P_n, Q_n, R_n and S_n represent levels of these four clusters where n in this article takes the values from 0 to 5. Figure 1 shows levels 0–4 of these clusters. Note that the centres of Q_2 and S_2 are not the same even though their shapes are the same and there are five Q_0s in each centre. Rhombs and quarks at every level above 1 always have exactly two extemal rhombs of level 1. For tiling to continue, these R_1s require careful placement of the rhombs and quarks.

In tiling a plane with any one level of these four clusters, a wide variety of tilings is possible. Creating a tiling is somewhat like making your own jigsaw puzzle with no picture to go by. Using higher-level clusters permits faster tiling, and the distinctive shapes of the four clusters help in making decisions. I developed the local rules given below and used them in choosing a next cluster from the possible ones that fit.

Rule 1. The arrows on all external edges must match. Above level 0 only single arrows remain on external edges. Double arrows match on internal edges only. Above level 2, the figures in this paper omit all arrows. The matching of internal arrows has been taken care of in lower levels.

Rule 2. When two R_1s meet at a point of a star, their Q_0 subparts, which are fat rhombs, must match at the star point. This avoids violation of the arrow matching rules.

Rule 3. Rhombs occur only between stars, and may thus force a new star.

Rule 4. When rule 3 forces a new star, the three configurations shown in the left column of figure 2 permit a continuation of tiling. Sometimes the clusters already in place constrain the choice; a free choice of one of the three configurations permits the variety that is possible in these tilings. Remember to obey rule 2.

Rule 5. The exterior star of a quark always fits into a pentagon and may thus force a new pentagon.

Rule 6. The right column of figure 2 shows the three configurations that fit around a pentagon. Be aware of the orientation of the rhombs. Remember to obey rule 2.

Rule 7. A tip vertex of a skinny rhomb must not touch a middle vertex of another skinny rhomb.

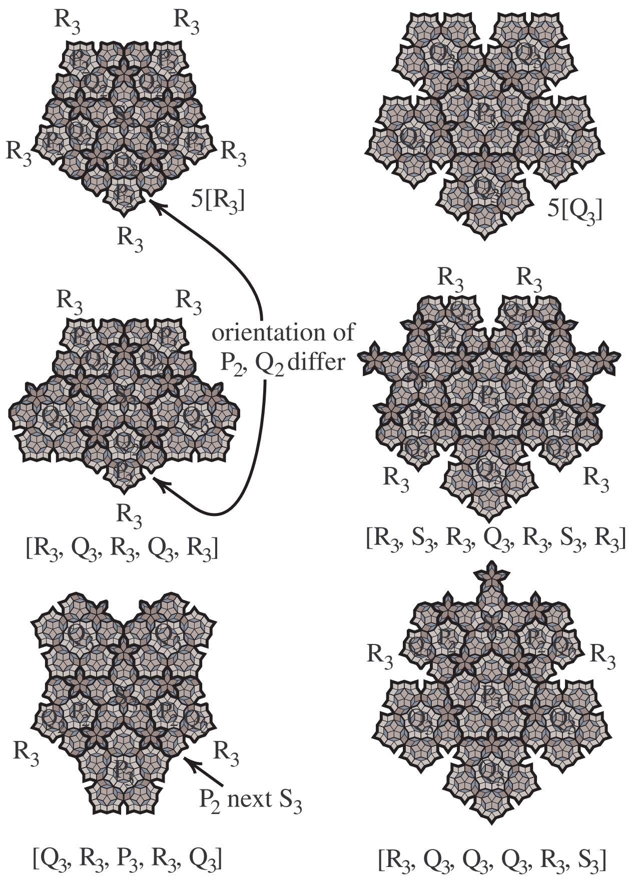

Figure 2. Clusters around S_3 and P_3.

Figure 3 shows an aperiodic tiling made of level 3 clusters placed according to the local rules. It contains over 14 000 Penrose rhombs, and shows levels 1–4 of pentagons, quarks, rhombs and stars. It even contains one R_5 nestled in two arms of an S_5. The arrows are lost at this scale, but the way R_1s meet at the points of stars demonstrates proper edge-matching.

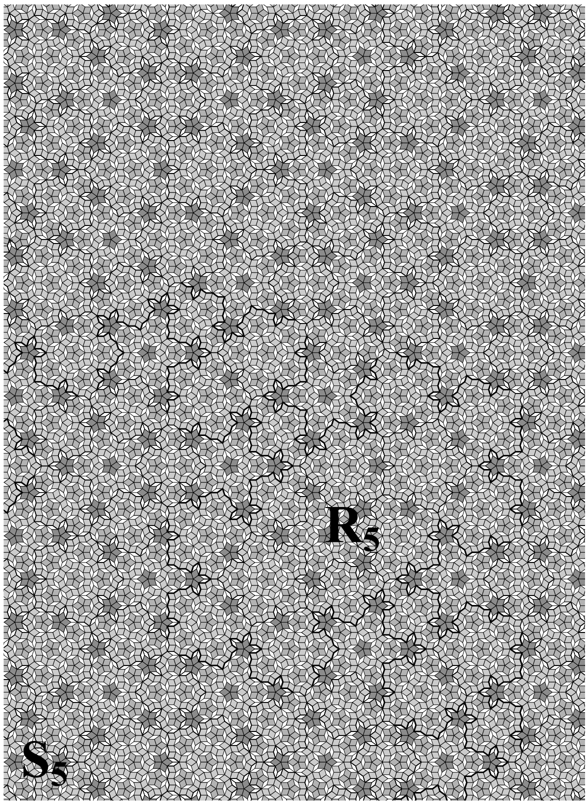

Figure 3. A tiling of about 14 000 tiles.

All three of the figures mentioned in references 1, 3 and 5 show rhomb tilings made up of various levels of pentagons, quarks, rhombs and stars. Obviously, others have produced these clusters for several years, but perhaps did not recognize their usefulness for producing arbitrarily large tilings more readily.

In Figure 10.3.20 of reference 1, iterations 2, 6 and 10 show pentagons P_1, P_2 and P_3, whereas iterations 4, 8 and 12 show stars S_2, S_3 and S_4. Iterations 1, 5, 9 and 13 show developments of a kite and dart pattern that Gardner calls 'infinite sun' in reference 6; iterations 3, 7 and 11 show developments of a kite and dart pattern that Gardner calls 'infinite star'. The fact that these successive inflations show increasingly complex patterns of infinite sun, pentagon, infinite star and star is, to me, a truly beautiful and remarkable aspect of Penrose prototiles.

Table 1. Ratios of fat to skinny rhombs.

	One each of P, Q, R, S		Golden Ratio (GR) = 1.618034	
Level	Fat rhombs	Skinny rhombs	Fat/Skinny	F/S–GR
1	9	3	3.000000	1.381966
2	51	27	1.888889	0.270855
3	329	193	1.704663	0.086629
4	2 201	1 327	1.658628	0.040594
5	14 899	9 083	1.640317	0.022283
6	101 401	62 177	1.630844	0.012810
7	692 169	425 823	1.625485	0.007451
8	4 732 851	2 917 227	1.622380	0.004346
9	32 394 089	19 989 313	1.620570	0.002536
10	221 851 001	136 986 127	1.619514	0.001480
11	1 519 863 859	938 826 203	1.618898	0.000864
12	10 414 399 801	6 434 447 777	1.618538	0.000504
13	71 369 749 929	44 100 910 143	1.618328	0.000294

4. The recursive formulation

After producing figure 3, I realized that the levels of clusters were recursive, that it was possible to produce arbitrarily large Penrose rhomb tilings without the need for inflations or searching. The recursive formulation gives straightforward rules for constructing ever-larger patterns from an initial one. Defining R_0 as the skinny rhomb, Q_0 as the fat rhomb and P_0, S_0, and the special pairing $[\mathrm{R}_0, \mathrm{S}_0]$ as points, the general recursive rules are:

$$\begin{aligned}
\mathrm{P}_{n+1} &= \mathrm{P}_n + 5[\mathrm{Q}_n],\\
\mathrm{Q}_{n+1} &= \mathrm{P}_n + \big[3[\mathrm{Q}_n], [\mathrm{R}_n, \mathrm{S}_n], \mathrm{R}_n\big],\\
\mathrm{R}_{n+1} &= \mathrm{R}_n + [\mathrm{Q}_n, \mathrm{R}_n, \mathrm{P}_n],\\
\mathrm{S}_{n+1} &= \mathrm{S}_n + 5[\mathrm{R}_n, \mathrm{S}_n] \quad \text{(note that } \mathrm{S}_1 \text{ is also a point).}
\end{aligned}$$

In this formulation, the symbol $+$ means ‘surrounded by’.

Figure 1 shows pentagons, quarks, rhombs and stars of levels 0–4. The scale gets smaller at higher levels because the clusters grow rapidly in size and complexity and must lose inner detail to appear at printable size. There are no arrows above level 2, but proper positioning of external copies of R_1, whether part of a rhomb or quark, insures matching. Using the recursive rules, table 1 shows the number of fat and skinny rhombs in one set of clusters for levels 1–18. The ratio of fat to skinny rhombs quickly approaches the golden ratio.

5. Illustrations

Repeated application of the recursive rules made it easy to produce the figures in this paper using Adobe® ILLUSTRATOR®. Successively bolder outlines attached at each level of recursion reveal the inner structure.

6. Concluding remarks

Use of the recursive rules offers a constructive method for Penrose rhomb tilings of arbitrary complexity. It would seem that four tilings are possible, depending upon the choice of initial cluster. In some sense they are all the same, because developing level $n+1$ of one cluster requires producing level n of all the clusters.

Recursion enables one to take any tiling of level n clusters and replace each cluster by its corresponding cluster of level m (m greater than n) to produce another tiling of greater size and complexity. One can produce ever-larger tilings without hunting for the next cluster that fits.

Acknowledgements

Martin Gardner, Doris Schattschneider and Ivan Sutherland provided me with much-needed help and encouragement. Ivan, in particular, has been of invaluable assistance in providing the illustrations used here.

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Spreading a Virus Among Intravenous Drug Users

JOE GANI

A cautionary tale!

We know that epidemics spread from infectives, that is, individuals who have an infectious disease and can transmit it, to susceptibles, or individuals who do not have the disease. But the exact mechanism of infection varies from one disease to another. In some cases, direct contact between an infective and a susceptible is necessary, while in others, droplets of fluid carried in the air from an infective with a viral disease are enough to cause infection among susceptibles. In parasitic diseases, the infection is spread through a parasitic carrier of the noxious bacterium or virus. A method of transmission which is particularly dangerous for intravenous drug users is the result of re-using infected hypodermic needles, for example at parties where such needles may be exchanged among the users.

How can we work out the probability that $k \le \min\{i, n\}$ new infectives will be created if i infectives and n susceptibles exchange needles randomly at a party? Each infective is assumed to target one susceptible, but a susceptible may be targeted by j infectives, where $j = 0, 1, 2, \dots, i$. In the case where $j = i$, all the infectives target a single susceptible. The model used in this case is called an occupancy model, and can be represented by a set of n boxes (susceptibles), in which i balls (infected needles from infectives) are to be placed at random. Needles from the susceptibles cannot cause infection, and will therefore not be taken into account. Such occupancy problems have a long history, of which Section 5 of Chapter II in Feller's classic book, reference 1, provides a useful outline. In this article, we give a brief account of how to tackle this occupancy problem.

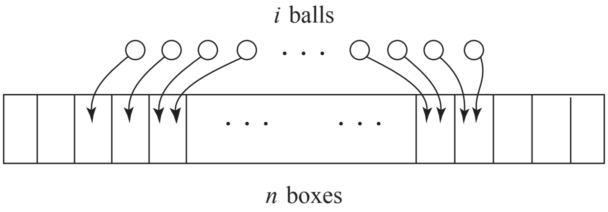


Figure 1. i balls (infected needles) to be placed in n boxes (susceptibles).

Let us consider figure 1, in which i balls (infected needles) are to be placed at random in n boxes (susceptibles), so that each ball has the probability $1/n$ of being allocated to any particular box. Let us assume that, after all the balls have been allocated, the number of boxes with one or more balls in them is $k \le \min(i, n)$, and let us define the probability

$$p_k(i, n) = \mathrm{P}\{k \text{ boxes with balls} \mid i \text{ balls and } n \text{ boxes initially}\}.$$

If the number of balls were to be increased by 1, so that we would have $i + 1$ balls to place in the n boxes, we can derive the recursive relation

$$p_k(i + 1, n) = \frac{p_k(i, n)k}{n} + p_{k-1}(i, n)\Big(1 - \frac{k-1}{n}\Big). \quad (1)$$

What this tells us is that if k boxes are to be occupied by $i + 1$ balls, then this can happen in one of two ways: either k boxes were already occupied by i balls, and the additional ball falls into one of these k boxes with probability k/n, or only $k - 1$ boxes were occupied by i balls, and the new ball falls into a kth box, bringing the total up to k, with probability $1 - (k - 1)/n$.

Starting with $p_0(0, n) = 1$, which states that the probability $i = 0$ is 1, we can then use (1) to build up the sequence of the sequence of probabilities

$$\begin{aligned} &p_1(1, n) = 1\,, \\ &p_1(2, n) = \frac{1}{n}\,, \quad p_2(2, n) = 1 - \frac{1}{n}\,, \\ &p_1(3, n) = \frac{1}{n^2}\,, \quad p_2(3, n) = \frac{3}{n}\Big(1 - \frac{1}{n}\Big)\,, \\ &\qquad\qquad p_3(3, n) = \Big(1 - \frac{1}{n}\Big)\Big(1 - \frac{2}{n}\Big)\,, \quad (2) \end{aligned}$$

and so on. What the last line of (2) tells us is that, in a party with $n = 5$ susceptibles and 3 infectives at which needles are exchanged at random, the probabilities of 1, 2, or 3 new infectives are 0.04, 0.48, 0.48 respectively, so that 2 or 3 new infectives are very likely to be created at a single party. The procedure (1) can become rather tedious, and a simpler method using the probability generating function of the probabilities $p_k(i, n)$ commends itself.

Let us write

$$f_{in}(u) = \sum_{k=1}^{\min(i,n)} u^k p_k(i, n)\,, \qquad 0 < u \le 1\,,$$

for the probability generating function of the probabilities we have been considering, with $f_{0n}(u) = p_0(0, n) = 1$. Then from (1), it is readily shown that

$$f_{i+1,n}(u) = \frac{u(1-u)}{n}\,\frac{\mathrm{d}}{\mathrm{d}u}\big(f_{in}(u)\big) + u f_{in}(u)\,, \quad (3)$$

for all $i = 0, 1, 2, \dots$. Starting from $f_{0n}(u) = 1$, we can easily derive from (3) the generating functions

$$f_{1n}(u) = u\,, \qquad f_{2n}(u) = \frac{u}{n} + u^2\Big(1 - \frac{1}{n}\Big)\,,$$

$$f_{3n}(u)=\frac{u}{n^2}+\frac{3}{n}\Big(1-\frac{1}{n}\Big)u^2+\Big(1-\frac{1}{n}\Big)\Big(1-\frac{2}{n}\Big)u^3\,,$$

and all $f_{in}(u)$ for higher values of i. We assume for simplicity that $n \geq i$, but if i happens to be larger than n, then all coefficients of powers higher than u^n will disappear automatically. As an exercise, why not derive $f_{4n}(u)$ and $f_{5n}(u)$ when $n=3$, and tabulate the probabilities of 1, 2 or 3 new infectives?

This occupancy model has shown that, if a group of, say, $n = 10$ susceptibles and a single infective $i = 1$, meet regularly at parties and exchange needles, it takes only a few such parties before the entire group becomes infected. If we need a moral to the story, it is that intravenous drug use is an extremely dangerous pastime, to be avoided at all costs!

References

1. W. Feller, *An Introduction to Probability Theory and its Applications* (third edition, John Wiley, New York, 1968).

***Joe Gani** is a retired applied probabilist, who lives in Canberra, Australia. His main interest is the modelling of epidemics; he has recently published a book jointly with his colleague Daryl Daley, entitled* Epidemic Modelling: An Introduction *(Cambridge University Press, 1999).*

Corrections

Arithmetic mean–geometric mean inequality

Norman Routledge has pointed out an error in my review of *Random Walks of George Pólya* by Gerald Alexanderson (*Mathematical Spectrum*, Vol. 33, No. 2, p. 48). In the proof that $A \geq G$, we need $e^x \geq 1+x$ for all x, not just for $x \geq 0$, since $(a_i/A)-1$ may well be negative. The proof then goes through unaltered. Mr Routledge points out that the proof can be pushed further, since $e^x > 1+x$ if $x \neq 0$, so the argument proves that $A > G$ unless all the a_i are equal.

David Sharpe
Editor

Niels Abel

Harry Burkill has asked us to point out that the first formula on page 3 of his article 'Niels Henrik Abel: 1802–1829' (*Mathematical Spectrum*, Vol. 33, No. 1, p. 3) contains an error. It should read

$$(1-x)^\alpha=\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n\,.$$

Jens Carstensen

The author of the article 'About Hexagons' (*Mathematical Spectrum*, Vol. 33, No. 2, pp. 37–40) is Jens Carstensen. We apologise for the misprint.

Consistently wrong clocks

'Even a stopped clock is right twice a day. After some years, it can boast a long series of successes.' So wrote the Austrian baroness Marie von Ebner-Eschenbach (1830–1916), who is famous for her novels and for being quoted.

Suppose that the time displayed by a defective analogue clock changes at a constant rate which is k times the correct rate. A perfect clock would have $k=1$, which case we disregard. The quotation deals with the case $k=0$, and the clock runs backwards if k if negative.

In terms of k, how many times per day on average does the defective clock show the correct time?

Now consider the corresponding problem for a digital clock. Because the time displayed changes in a stepwise fashion rather than continuously, we have to ask a different question:

In terms of k, for how many seconds per day on average does the defective clock show the same time as a perfect clock of the same type?

Assume that the time displayed changes instantly and with a perfect regularity governed by the value of k. There are four cases, depending on the type of display: 12 hour or 24 hour, hours and minutes, or hours, minutes and seconds.

John MacNeill

Mathematics in the Classroom
What's New in DERIVE™ 5?

P. SCHOFIELD

A selection of the new facilities available in DERIVE™ 5, and how these can be used to support mathematical teaching and learning activities.

I have been acquainted with DERIVE for over eight years, working my way up through versions 2, 3 and 4. I have made use of DERIVE in many aspects of my own mathematical development and (extensively) in working with college mathematics students. Why should I be so enthusiastic about this new issue of the program? Because the more I get to know about DERIVE 5, the more it is changing my whole approach to using the program as an IT tool to support my teaching. At best, in a short article, I can only give you a small selection of some of the new things in DERIVE 5. If you require more details, then I suggest that you consult the DERIVE website (reference 1).

1. 3D-plots with DERIVE 5

In DERIVE 5, the authors have made considerable improvements to the 3D-plotting facilities. It is now possible to plot several 3D-surfaces and lines and carry out 'real time' manipulations in DERIVE's 3D-plot window. This can now be used to provide visual support for a simple example like the following.

Example 1. For the cone, with equation $2x^2+2y^2-z^2=0$, find the equation of the normal line and tangent plane at the point P with coordinates $(1, 1, 2)$.

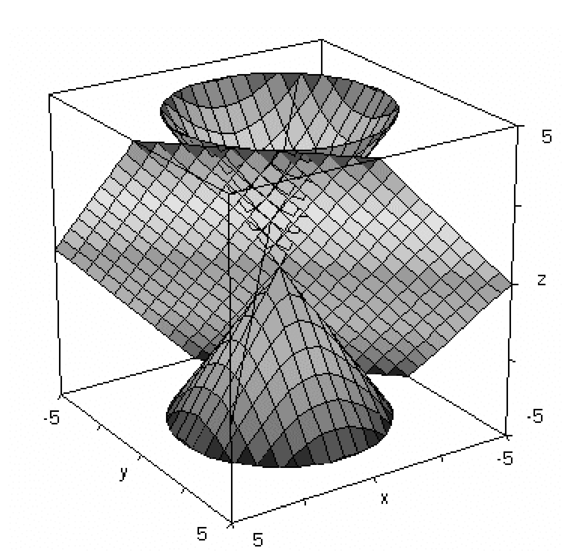


Figure 1. A cone, tangent plane and normal.

It is easy to calculate the vector equation of the normal to be $\boldsymbol{r} = (1, 1, 2) + t(1, 1, -1)$ and the equation of the tangent plane to be $x + y - z = 0$. The top and bottom surfaces of the cone, together with the normal line and tangent plane can be entered in DERIVE's Algebra window using the single 'list' instruction:

[√(2x^2+2y^2), − √(2x^2+2y^2), [1+t, 1+t, 2−t],x+y]

Plotting this in the 3D-plot window yields the composite display in figure 1.

This display comes to life when manipulated using the new 'real time' dynamic 3D-plot facilities of DERIVE 5. Figure 2 shows three possible 'stills' illustrating the composite nature of the plots.

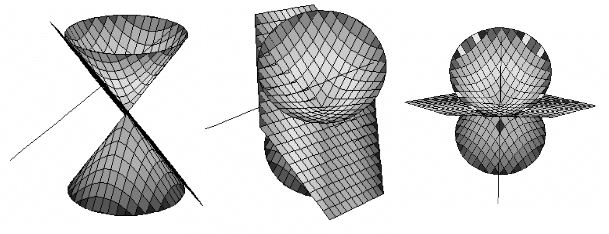


Figure 2. Some 'stills' of the cone plot.

Another interesting surface to model is a sphericon, featured in Ian Stewart's article in reference 2. This solid can be constructed from two congruent 90° circular cones (having a common axis) joined by their bases. These are then cut in half by a plane through the common axis and one half is rotated through a right angle until it matches up again with the other half. The resulting solid has a single face and many curious properties. The half cut cone surfaces can be combined using the instruction:

[IF(x < 0, √ ((x+5)^2−y^2), √((x−5)^2−y^2)),
IF(y < 0, − √ ((y+5)^2−x^2), − √((y−5)^2−x^2))]

and 3D-plotting will display figure 3, which can be manipulated in real time as before.

2. Simplify/approximate before plotting in the plot windows

In DERIVE 5, there are now option settings 'Simplify Before Plotting' and 'Approximate Before Plotting' in both the 2D- and 3D-Plot windows. Their main advantage is that simplification or approximation of algebraic expressions before plotting is no longer necessary.

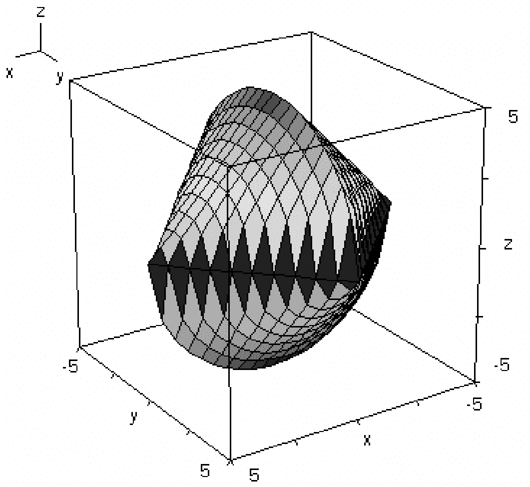


Figure 3. The sphericon.

Example 2. (*2D-plotting a fractal C-curve.*) This can be constructed using a single straight line segment joining two points p and q as an initiator and, as a generator, distorting each line segment into two orthogonal equal length sub-segments joining p and q. See figure 4.

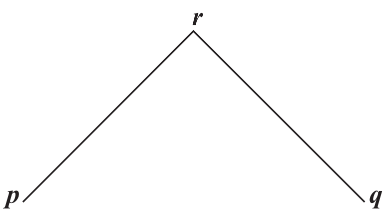


Figure 4. Generator for a C-curve.

By straightforward vector geometry,

$$r = \tfrac{1}{2}\big(p + q + \big((p-q)_2, (q-p)_1\big)\big).$$

The final ingredient is an iteration to generate a sequence of points, starting with p and, at each new level, distorting each line segment joining a pair of successive points using the generator. Therefore, in DERIVE's algebra window enter the following three assignment expressions:

CGEN(p,q) :=
[(p + q + [(p − q) ↓ 2, (q − p) ↓ 1])/2, q]
CGENS(x) :=
APPENDVECTOR(IF(i=0,[x ↓1],□
CGEN(x ↓ i, x ↓ (i + 1))), i, 0, DIM(x) −1)
C(p, q, n) :=
ITERATE(CGENS(x), x, [p, q], n)

Then 2D-plotting **C([–1.8,–1],[1.8,–1],12)** (with 'Simplify Before Plotting' on, 'Change Plot Colours' off and 'Options > Points' set to Connect, Small) yields figure 5.

To obtain the same plot in DERIVE 4 you would first have to simplify (or approximate) **C([–1.8,–1],[1.8,–1],12)** and then plot the resulting vector containing $2^{12} + 1$ points! If you wish, you can still do this in DERIVE 5 (although I can't see much point in it).

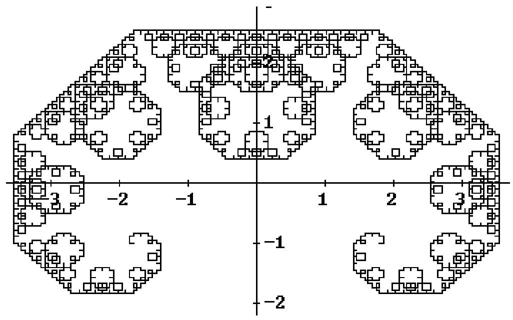


Figure 5. A fractal C-curve.

Remark. Using DERIVE to generate simple fractals is an interesting and challenging exercise to set for college students. Although none of the mathematics involved is terribly difficult, there are several ingredients. For example: 2D-vector geometry; iteration; self-similarity; infinity; calculating associated lengths and areas (involving summing geometric progressions). In DERIVE 5, the Simplify/Approximate Before Plotting options provide facilities for smoother IT modelling of this activity.

3. The Boolean area plotter

This new feature plots areas in the 2D-plot window specified by 'inequations' (in two variables) and logical connectives. Amongst other things, it has been incorporated into a new instruction **PlotInt** for plotting definite integrals. With 2D-plot options Simplify Before Plotting and Change Plot Colours both on, 2D-plotting

PlotInt(2cos(x^2−3),x,−3,3)

yields figure 6.

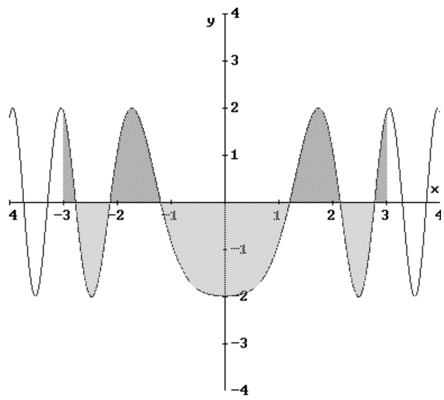


Figure 6. A 2D-plot of a definite integral.

There is a dynamic dimension to this plot. The positive part of the integral is plotted (first) in one colour followed by the negative part plotted in a different colour. There are also associated instructions: **AreaBetweenCurves**; **AreaUnderCurve**; **AreaAboveCurve**, which are more or less self-explanatory.

Another application of the Boolean area plotter illuminates concepts relating to 2D-transformations. For example, entering and 2D-plotting:

squ(x,y) := (MOD(x,2)−1)(MOD(y,2)−1) > 0

will cover the whole plane in a pattern of unit squares as in figure 7.

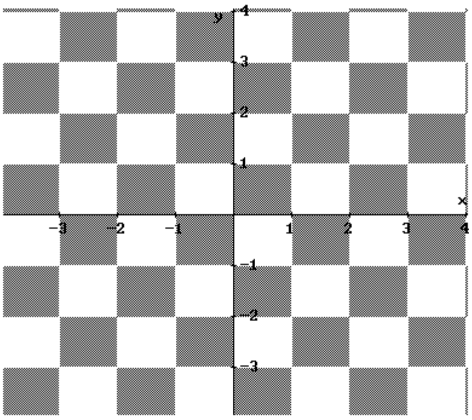


Figure 7. The basic pattern of unit squares.

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ can be thought of as acting on (i) a point, (ii) a shape or (iii) the whole plane ($\mathbb{R}^2$). The Boolean area plotter provides an excellent means of visualizing part (iii) of this concept.

Example 3. Calculate the change of axes required to induce the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$

and use this with DERIVE 5 to illustrate that T acts on the whole of $\mathbb{R}^2$. The change of axes required depends upon the inverse of the matrix of T. This can be easily calculated to be

$$\tfrac{1}{3}\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix},$$

suggesting the change of coordinates $(-x + 2y)/3$ for x and $(2x - y)/3$ for y. Entering and 2D-plotting

squ((−x+2y)/3,(2x−y)/3)

generates figure 8. This is clearly the required transformation, and it is also clear that the transformation is acting upon the unit square tessellation across the whole of $\mathbb{R}^2$. Simply clicking the mouse on the 'zoom out' button of DERIVE's 2D-plot window emphasizes this point further. Using **squ** with non-linear changes of coodinates can generate interesting results. For example, figure 9 shows the 2D-plots of

squ(2x^2−y^2,2y^2−x^2),

and

squ(2sinx+cosy,2siny+cosx),

respectively.

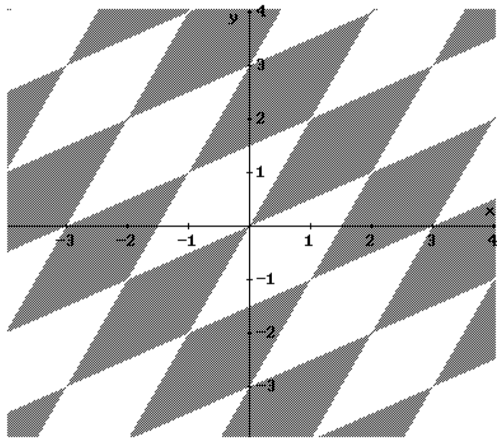


Figure 8. The transformed pattern of squares.

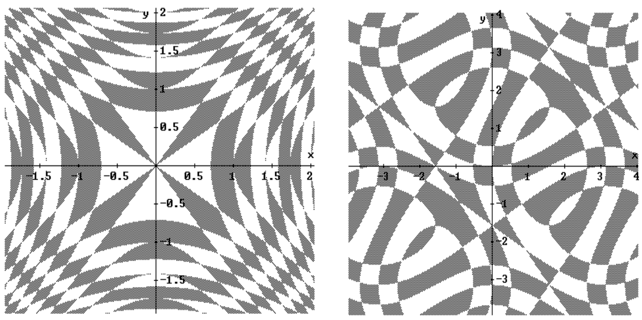


Figure 9. Non-linear transformations.

4. The new format of .dfw files

One of the most important new facilities of DERIVE 5 is the introduction of the .dfw file format. In a .dfw file you can have almost anything. You can insert and edit text using (a fairly basic) text editor, embed 2D-plot and 3D-plot windows, copy and paste pictures, text, etc., from other Windows applications. In addition, you can hide any bit of algebraic code you don't want the user to see, but the instructions and operations defined by this code will be retained when the file is saved under the new .dfw format. The next example illustrates this.

Example 4. (*Drawing cobweb diagrams for fixed-point iterations.*) It is now possible in DERIVE 5 to provide students with an IT tool to construct cobweb diagrams (it will, of course, do students no harm to construct one or two of these by hand). See figure 10. This .dfw file explains how to configure the 2D-plot window and gives an example of the cobweb instruction. It does not show the background coding.

Potentially, .dfw files have many applications. If you are interested in seeing some of these, you can browse through the Users' folder of DERIVE 5.03. This folder contains .dfw files of 'one off' applications like the one above, as well as .dfw files for setting up extensive systems for drawing network graphs and designing 2D-symmetry group patterns. See figures 11 and 12.

Drawing Cobweb Diagrams for Fixed-Point Iterations

To draw a cobweb diagram for a fixed-point iteration to find a solution of **f(x) = x**, starting with first guess **x0**, enter and plot **fixedp(f(x),x0,n,s)** (**n** = number of iterations ≤ 8 and **s** = scale for size for arrows).

Select: 2D-plot window: Options>Approximate Before Plotting; Options>Display>Points>Connect(Yes), Medium. For Parametric Plot Parameters set 'Apply parameters to rest of plot list' (On).

For example: with the above 2D-plot window settings enter and plot:

$$\#1: \quad \mathrm{fixedp}\left(4 \cdot \cos\left(\frac{x+2}{4}\right) - 2,\ 4,\ 6,\ 1\right)$$

for Figure 1 (or double click on Figure 1 to go directly to the 2D-plot window):

Figure 1

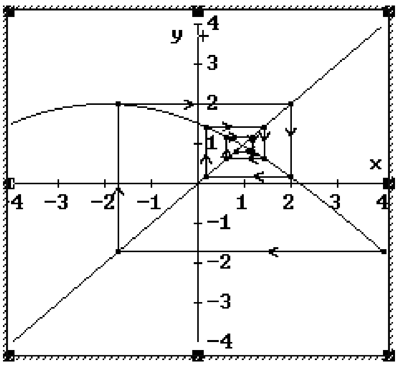
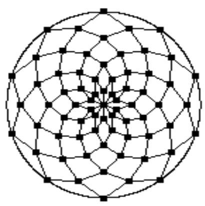
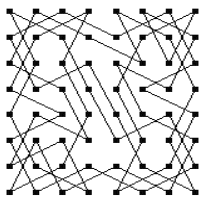


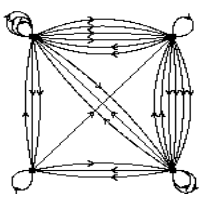
Figure 10. An example of a .dfw file.



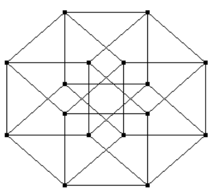
Dream catcher



Knight's tour

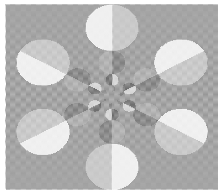


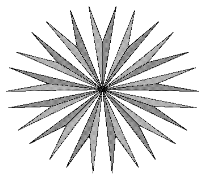
Digraph with loops



Hypercube

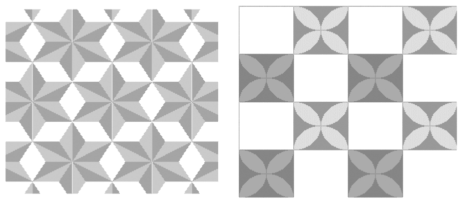
Figure 11. Examples of the graphs drawn by the Users' folder file Ngraphs5.dfw.





Cyclic and dihedral symmetry group patterns

A strip-repeating symmetry group pattern



Plane-filling symmetry group patterns

Figure 12. Examples of patterns generated by the Users' folder file Patts2D.dfw.

5. Conclusions

The co-authors of DERIVE 5, Albert Rich and Theresa Selby, are to be congratulated on putting together an excellent all-round package for carrying out mathematics and interpreting mathematical ideas. They have followed the well-tried DERIVE policy of consulting with many working mathematicians and mathematics teachers (from lots of different countries), listening carefully to their ideas and suggestions, and implementing many of these in skillful and innovative ways into DERIVE 5. If you would like to try more activities with DERIVE 5, work through reference 3, which comes as part of the basic package when you order a copy of the program. The product is usable and sensibly priced (a full list of prices can be found on the DERIVE website, reference 1), and it will continue to be popular with mathematicians and mathematics teachers worldwide.

References

1. DERIVE Web Site, available at http://www.derive.com.
2. I. Stewart, Cone with a twist, *Scientific American,* Oct. 1999, 116–117.
3. B. Kutzler and V. Kokol-Voljc, *DERIVE 5. The Mathematical Assistant for Your PC,* Texas Instruments, 2000.

*The author is Head of Mathematics at Trinity and All Saints College, University of Leeds. College students follow degrees in education with subject studies in mathematics. Over the past eight years **Dr Schofield** has made extensive use of DERIVE in working with college students.*

Computer Column

Dabbling in genetics

A little while ago, I had to fill in a life insurance form that (as such forms must these days) asked about all sorts of horrible diseases I might have inherited. Apart from making me depressingly aware of my own mortality, it also started me thinking. Many of the diseases on the list, for one reason or another, made it less likely that their sufferers would have children, or at least less advisable. Did that mean that, eventually, they would all die out, or would they manage to hang on somehow? There was, of course, only one way to find out, and I sat down at my computer to try and model the situation.

An advantage such horrible diseases often have is that they are carried by so-called 'recessive' genes. To take a less morbid example, consider the gene for hair colour. The gene for dark hair is a 'dominant' gene, but the gene for auburn hair is recessive. As with all genes, every child receives one copy of a hair colour gene from each parent, possibly for different colours. The colour choice is then determined by the most dominant gene, so that they will have dark hair if *either* parent passes on a dark hair gene, but auburn hair only if it is passed on by *both* parents.

A similar thing happens in many diseases: the disease is caused by a defective copy of a particular gene which is recessive relative to the undamaged version. Thus, if a person receives one healthy copy and one defective copy, they will not get the disease, but can potentially pass on the defective gene to their children. For this reason, such people are often called 'carriers' of the disease.

How could I model this easily? Firstly, I decided to consider a population where the number of people in each generation was constant, and where the chances of anyone having a child were only determined by whether or not they had the disease. I also assumed, for the sake of simplicity, that people could only partner up within their own generation, so that children could only be born to parents in the previous generation. The final assumption was that the chances of someone who had inherited the disease having children was a fixed percentage less likely than for someone who had not. The program I eventually came up with is listed in the appendix, written in BASIC.

First of all, the program sets up a population of size `sample`, described by a two-dimensional array, `genes`. This gives each person two entries, one for each copy of the problem gene. The entries are then given random values of either 0 or 1, where 0 implies that the gene is healthy, and 1 that it is defective. The program also sets up a second array, `genesng`, to store the genes of this generation's offspring.

The next step is to create a new generation. To do this, the program creates `sample` children, each receiving two random genes from random parents. If either parent turns out to have the disease (by having two defective copies of the gene) then they are only accepted as a parent `chances` per cent of the time. Otherwise, they are rejected and a new selection is made. (The code for this may appear to be more complicated than it needs to be, since a simple `GOTO` could replace the `DO` loop. The problem with `GOTO`s is that they need markers to reference, which could cause confusion in bigger programs, and that they can make programs harder to follow.)

The program then moves the new array, `genesng`, into `genes` to indicate that the focus of the next iteration will be the children, and their attempts to have children of their own. Rather than have the program print out results at each generation, it then repeats the iteration `stepsize` times before it looks at the results. Having done so, it then iterates on, up to generation `generations`.

At present, the only number that is calculated is the percentage of carriers in the population (who have only one defective gene), but clearly I could also have looked at the number for people infected, for example. The number of generations so far and the carrier percentage are then written to a file. The results of six runs, with different values for `chances`, are shown in figure 1.

The results show that, even if the disease only gives its sufferers a 10 per cent chance of having children, it still manages to survive for more than sixty generations (or about 1500 years). It would be very interesting to see how well this matches the results for real diseases.

Another exercise in modelling that came to me as I was writing this article might appeal to the gamblers among you. How effective are different betting strategies for games of chance? For example, if you repeatedly toss a coin, is there any way of betting on the outcomes that is likely to make you a profit? How would you write a program to test it out? If you find a good strategy, write in to me and we can share the profits!

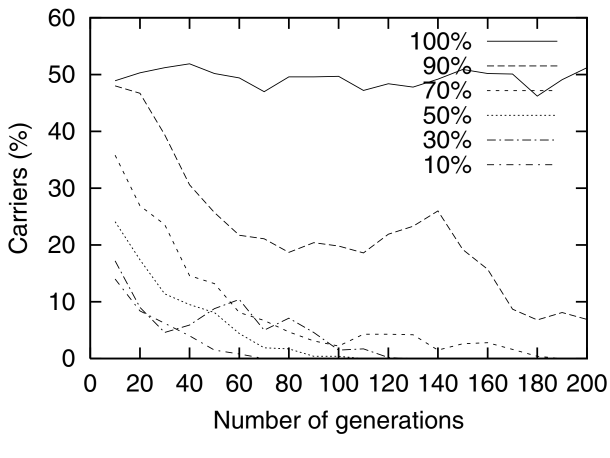


Figure 1. Sample output, for a sample size of 1000 and various chances of survival.

Appendix. Program listing

```
REM Genetics example

REM Set up variables
RANDOMIZE TIMER
sample = 1000
generations = 200
stepsize = 10
chances = 90
DIM genes(sample, 2) AS INTEGER
DIM genesng(sample, 2) AS INTEGER

REM Initialise array
FOR i = 1 TO sample
 genes(i, 1) = INT(RND * 2)
 genes(i, 2) = INT(RND * 2)
NEXT

REM Open file to contain output
OPEN "results.dat" FOR OUTPUT AS #1

REM Now start iterating
FOR k = 1 TO generations / stepsize
 FOR j = 1 TO stepsize
  FOR i = 1 TO sample

REM Choose parents and genes
   FOR h = 1 TO 2
    parentok = 0
    DO WHILE parentok = 0
     parent = INT(RND * sample) + 1
     IF (genes(parent, 1) + genes(parent, 2) = 2) THEN
      IF RND * 100 < chances THEN
       parentok = 1
      ELSE
       parentok = 0
      END IF
     ELSE
      parentok = 1
     END IF
    LOOP
    gene = INT(RND * 2) + 1
    genesng(i, h) = genes(parent, gene)
   NEXT h
  NEXT i

REM Move focus to children
  FOR i = 1 TO sample
   genes(i, 1) = genesng(i, 1)
   genes(i, 2) = genesng(i, 2)
  NEXT i
 NEXT j

REM Analyse the results
 finalcarrier = 0
 FOR i = 1 TO sample
  IF genes(i, 1) + genes(i, 2) = 1 THEN
   finalcarrier = finalcarrier + 1
  END IF
 NEXT i
 PRINT #1, k * stepsize, finalcarrier * 100 / sample

REM Go back and do more iterations
NEXT k

REM Now we're finished!
CLOSE
END
```

Peter Mattsson

Letters to the Editor

Dear Editor,

A problem about rational numbers

Most readers will probably have come across the following type of problem. Find two numbers whose:

(a) difference equals the difference of their squares;

(b) sum equals the difference of their squares;

(c) difference equals their product;

(d) sum equals their product.

These are relatively simple to solve in rational numbers.

Following on from these, I propose the following:

Find two rational numbers whose sum equals the sum of their squares. Thus, $x^2+y^2 = x+y$, or $y^2-y+(x^2-x) = 0$. Thus, solving for y,

$$y = \tfrac{1}{2}\big(1 \pm \sqrt{1-4(x^2-x)}\,\big) = \tfrac{1}{2}\big(1 \pm \sqrt{1+4x(1-x)}\,\big).$$

This is real when $1+4x-4x^2 \geq 0$, i.e. when $\frac{1}{2}(1-\sqrt{2}\,) \leq x \leq \frac{1}{2}(1+\sqrt{2})$. For y to be rational, we must have

$$\sqrt{1+4x(1-x)} = \pm a \qquad (a \in \mathbb{Q})\,.$$

Squaring, we obtain $4x(1-x) = a^2-1 = (a+1)(a-1)$. If we put $4x = a+1$ and $1-x = a-1$, we obtain $x = 3/5$. If we put $4x = a-1$ and $1-x = a+1$, we obtain $x = -1/5$. When $x = 3/5$, $y = 6/5$ or $-1/5$. When $x = -1/5$, $y = 3/5$ or $2/5$. Thus, $(\frac{3}{5}, -\frac{1}{5})$, $(\frac{3}{5}, \frac{6}{5})$, $(\frac{2}{5}, -\frac{1}{5})$, $(\frac{2}{5}, \frac{6}{5})$ are solutions for x and y.

From the above results and further trials, it appears that the most likely values of x and y are those fractions with denominators of 5^n ($n \geq 0$). The following further solutions (apart from the trivial solutions $(0, 0)$, $(0, 1)$, $(1, 1)$) have been found:

$$\big(-\tfrac{3}{25}, \tfrac{4}{25}\big), \big(-\tfrac{3}{25}, \tfrac{21}{25}\big), \big(\tfrac{28}{25}, \tfrac{4}{25}\big), \big(\tfrac{28}{25}, \tfrac{21}{25}\big), \big(\tfrac{26}{125}, -\tfrac{18}{125}\big),$$
$$\big(\tfrac{26}{125}, \tfrac{143}{125}\big), \big(\tfrac{99}{125}, -\tfrac{18}{125}\big), \big(\tfrac{99}{125}, \tfrac{143}{125}\big), \big(\tfrac{217}{625}, -\tfrac{119}{625}\big),$$
$$\big(\tfrac{217}{625}, \tfrac{744}{625}\big), \big(\tfrac{408}{625}, -\tfrac{119}{625}\big), \big(\tfrac{408}{625}, \tfrac{744}{625}\big), \big(\tfrac{123}{3125}, -\tfrac{114}{3125}\big),$$
$$\big(\tfrac{123}{3125}, \tfrac{3239}{3125}\big), \big(\tfrac{3002}{3125}, \tfrac{114}{3125}\big), \big(\tfrac{3002}{3125}, -\tfrac{3239}{3125}\big).$$

Similar solutions to the above are found for $x^2+y^2 = x-y$. Is there a general solution?

Yours sincerely,

BOB BERTUELLO
(12 Pinewood Road,
Midsomer Norton,
Bath BA3 2RG.)

Dear Editor,

A Tale of Two Series—A Dickens of an Integral

Thanks to P. Glaister for his interesting article in *Math. Spectrum*, Vol. 33, No. 2. The sum

$$V = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots$$

is simply represented by $\int_0^1 x^{-x}\,\mathrm{d}x$ as this changes the signs of alternate terms in the sequence of integrals Mr Glaister evaluates. The answer to 12 decimal places is 1.291285997063 (worked out on a Texas TI-83 calculator). It is interesting that

$$\int_0^1 x^{-x}\,\mathrm{d}x = \sum_{x=1}^{\infty} x^{-x}.$$

Are there any other functions such that $\int_0^1 f(x)\,\mathrm{d}x = \sum_{x=1}^{\infty} f(x)$?

Yours sincerely,
A. G. Summers
(57 Conduit Road,
Stamford,
Lincs PE9, 1QL.)

Dear Editor,

A series expansion of $1/\pi$

The binary expansion of $1/\pi$ can be expressed in the form

$$\frac{1}{\pi} = \sum_{n=1}^{\infty} u\left(\sin 2^n\right) 2^{-n},$$

where

$$u(x) = \begin{cases} 1 & \text{if } x \le 0, \\ 0 & \text{if } x > 0\,. \end{cases}$$

Readers are invited to prove this in the problems section of this issue.

Reverting to the more usual decimal expansion, I have verified using MATHEMATICA® that the decimal expansion of the series

$$\sum_{n=1}^{\infty} l(x) 10^{-n}$$

agrees with $1/\pi$ in the first 300 places, where

$$\begin{aligned} l(x) = {} & c(x) - b(x)c(x) + 4b(x)d(x) - c(x)d(x) \\ & + 2\bigl(-1 + c(x)\bigr)\bigl(-1 + 2b(x)d(x)\bigr)e(x) \\ & + a(x)\bigl(-c(x) + 2b(x)\bigr)\bigl(4 + c(x) - 9d(x)\bigr) \\ & + 6d(x) - 2e(x) + \bigl(4b(x)d(x) + c(x) \\ & \times \bigl(9 - 9b(x) - 5d(x) + 10b(x)d(x)\bigr)\bigr)e(x), \end{aligned}$$

with

$$\begin{aligned} a(x) &= u\bigl(\sin(2 \cdot 10^{x-1})\bigr), \\ b(x) &= u\bigl(\sin(4 \cdot 10^{x-1})\bigr), \\ c(x) &= u(\sin 10^x), \\ d(x) &= u\bigl(\cos(2 \cdot 10^{x-1})\bigr), \\ e(x) &= u\bigl(\cos(4 \cdot 10^{x-1})\bigr). \end{aligned}$$

I have not checked this beyond the 300th decimal place.

Yours sincerely,
Milton M. Chowdhury
(Student, Dept of Mathematics,
UMIST, Manchester.)

Dear Editor,

Spiralling to a limit

Starting with the vertices P_1 (0, 1), P_2 (1, 1), P_3 (1, 0), P_4 (0, 0) of a square, construct the point P_5 as the midpoint of P_1P_2, P_6 as the midpoint of P_2P_3, P_7 as the midpoint of P_3P_4, and so on. The polygonal spiral path $P_1P_2P_3P_4P_5P_6\ldots$ appears to tend to a point P inside the square. What are the coordinates of P?

If P_n has coordinates (x_n, y_n), then

$$x_n = \tfrac{1}{2}(x_{n-4} + x_{n-3}) \quad \text{for} \quad n \ge 5,$$

and similarly for y_n. It is easy to prove by induction that

$$\tfrac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2$$

and that

$$\tfrac{1}{2}y_n + y_{n+1} + y_{n+2} + y_{n+3} = \tfrac{3}{2}$$

for $n \ge 1$. If we now let n tend to infinity, we see that the coordinates (x, y) of P satisfy

$$\tfrac{7}{2}x = 2, \qquad \tfrac{7}{2}y = \tfrac{3}{2}\,,$$

so that the limiting point is $(\frac{4}{7}, \frac{3}{7})$. But how do you prove that there *is* a limiting point? Could any reader help?

Yours sincerely,
Farshid Arjomandi
(Student,
University of Santa Cruz,
California,
USA.)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

33.9 Prove that

$$\frac{1}{\pi} = \sum_{n=1}^{\infty} \frac{u(\sin 2^n)}{2^n}\,,$$

where

$$u(x) = \begin{cases} 1 & \text{if } x \le 0, \\ 0 & \text{if } x > 0. \end{cases}$$

(See the letter by Milton Chowdhury in this issue)

33.10 Colin is playing a computer game and his latest win has just brought his life-time success rate, as displayed by the computer, up to 95%, his highest so far. How many further consecutive wins would be needed to bring his displayed success rate up to 96%, given that the success rate is displayed rounded to the nearest 1% and Colin has lost 5 times?

(Submitted by John MacNeill, University of Warwick)

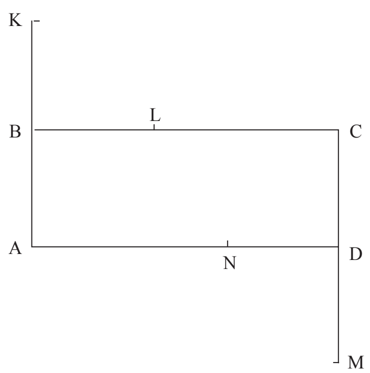
33.11 Solve the simultaneous equations

$$\sum_{i=1}^{n} x_i = 1, \qquad \sum_{i=1}^{n} y_i = 1, \qquad \sum_{i=1}^{n} (x_i - y_i)^2 = 2\,,$$

where $n \ge 2$ and the x_i and y_i are non-negative real unknowns.

(Submitted by Hassan Shah Ali, Tehran)

33.12



In the diagram, ABCD is a rectangle with AB $= a$, BC $= b$ ($a < b$) and K, L, M, N lie on sides AB, BC, CD, DA (possibly produced) as shown, such that AK $= b$, BL $= a$, CM $= b$, DN $= a$. When are K, L, M, N collinear?

(Submitted by Ahmet Özban, Kirikkale University, Turkey)

Solutions to Problems in Volume 33 Number 1

33.1 Find a formula for the product

$$\prod_{r=1}^{n} \cos\frac{x}{2^r}\,,$$

and use it to sum the infinite series

$$\sum_{r=1}^{\infty} \frac{1}{2^r}\tan\frac{x}{2^r}\,, \qquad \sum_{r=1}^{\infty} \frac{1}{4^r}\tan^2\frac{x}{2^r}\,.$$

Solution by Daniel Lamy (Nottingham High School)

Write

$$N = \prod_{r=0}^{n-1} \sin\frac{x}{2^r}\,.$$

Then

$$\begin{aligned} N &= \prod_{r=0}^{n-1} 2\sin\frac{x}{2^{r+1}}\cos\frac{x}{2^{r+1}} \\ &= 2^n N \frac{\sin(x/2^n)}{\sin x} \prod_{r=1}^{n} \cos\frac{x}{2^r}\,, \end{aligned}$$

so

$$\begin{aligned} \prod_{r=1}^{n} \cos\frac{x}{2^r} &= \frac{\sin x}{2^n \sin(x/2^n)} \\ &= \frac{\sin x}{x}\,\frac{x/2^n}{\sin(x/2^n)} \\ &\to \frac{\sin x}{x} \quad \text{as } n \to \infty\,, \end{aligned}$$

and so

$$\prod_{r=1}^{\infty} \cos\frac{x}{2^r} = \frac{\sin x}{x}\,.$$

Now

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left|\sec\frac{x}{2^r}\right|\right) = \frac{1}{2^r}\tan\frac{x}{2^r}\,,$$

so

$$\begin{aligned} \sum_{r=1}^{n} \frac{1}{2^r}\tan\frac{x}{2^r} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{r=1}^{n} \ln\left|\sec\frac{x}{2^r}\right|\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\ln\left|\sec\frac{x}{2}\sec\frac{x}{2^2}\cdots\sec\frac{x}{2^n}\right|. \end{aligned}$$

Let $n \to \infty$ to give

$$\sum_{r=1}^{\infty} \frac{1}{2^r} \tan \frac{x}{2^r} = \frac{\mathrm{d}}{\mathrm{d}x} \ln \left| \frac{x}{\sin x} \right|$$
$$= \frac{\sin x}{x} \frac{\sin x - x \cos x}{\sin^2 x}$$
$$= \frac{1}{x} - \cot x \,.$$

Now differentiate to give

$$\sum_{r=1}^{\infty} \frac{1}{4^r} \sec^2 \frac{x}{2^r} = -\frac{1}{x^2} + \operatorname{cosec}^2 x \,,$$

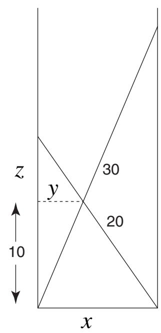
so

$$\sum_{r=1}^{\infty} \frac{1}{4^r} \left(1 + \tan^2 \frac{x}{2^r} \right) = \operatorname{cosec}^2 x - \frac{1}{x^2} \,,$$

and so

$$\sum_{r=1}^{\infty} \frac{1}{4^r} \tan^2 \frac{x}{2^r} = \operatorname{cosec}^2 x - \frac{1}{x^2} - \tfrac{1}{4} \left(\frac{1}{1 - \frac{1}{4}} \right)$$
$$= \operatorname{cosec}^2 x - \frac{1}{x^2} - \tfrac{1}{3} \,.$$

32.2 Two old, imperial ladders, 20 ft and 30 ft long, cross 10 ft above the ground when leaning against opposite walls in a passageway, as shown. What is the width of the passageway?



Solution by Scott Brown, Faulkner University, Montgomery, Alabama, USA.

Let x, y, z be the distances in feet shown in the figure. Then

$$\frac{x}{y} = \frac{30}{\sqrt{10^2 + y^2}} = \frac{10 + z}{z} \,,$$

and

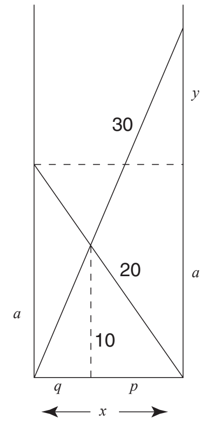
$$20^2 = x^2 + (10 + z)^2.$$

From these we can eliminate y, z to obtain the equation

$$x^8 - 2200x^6 + 163 \times 10^4 x^4 - 454 \times 10^6 x^2 + 385 \times 10^8 = 0 \,.$$

This equation has two real positive roots which can be found to be approximately 12.3119 and 18.7316. The second of these gives a negative value of z, so the width of the passageway is approximately 12.3119 ft.

Note. This problem has been going the rounds recently. See, for example, our Mathematics in the Classroom column in Volume 33, Number 2, and Scott Brown points out that it has also appeared in Pimuepsilon, Problem 623. Bob Bertuello, who proposed the problem, gave an alternative solution which led to a somewhat simpler quartic equation. From the diagram,



we have

$$\frac{q}{10} = \frac{x}{a + y} \quad \text{and} \quad \frac{p}{10} = \frac{x}{a} \,,$$

whence

$$p + q = x = \frac{10x}{a} + \frac{10x}{a + y} \,,$$

giving

$$a^2 - 20a + ay - 10y = 0 \,. \tag{4}$$

Also

$$x^2 = 30^2 - (a + y)^2 = 20^2 - a^2 \,,$$

whence

$$a = \frac{500 - y^2}{2y} \,. \tag{5}$$

If we substitute for a in (1) and simplify, we obtain the quartic equation

$$y^4 + 20\,000y - 250\,000 = 0 \,. \tag{3}$$

Using Newton's method with $y_0 = 250\,000/20\,000 = 12.5$ (since y^4 is small compared to 250 000), we soon find

$$y_3 = y_4 = 11.59594541 \ldots \,.$$

Now, by (2), $a = 15.76128712\ldots$ and

$$x = \sqrt{20^2 - a^2} = 12.311857 \ldots \,.$$

Bob Bertuello points out that, if the ladders are of lengths r and s units and cross at height h, (3) becomes

$$y^4 + 4bhy - b^2 = 0 \quad \text{with} \quad b = r^2 - s^2 \,.$$

Manfred Ritter, of Sydney, Australia, has sent the following intriguing formulae which he claims fit together to give a specific solution. In Bob Bertuello's notation,

$$S = \tfrac{1}{2}\sqrt{r^2 - s^2} \,, \qquad K = \frac{2h}{S} \,, \qquad G = \frac{\sqrt{27}\,K^2}{8} \,,$$
$$Y = 2 \sinh \left(\tfrac{1}{3} \sinh^{-1} \frac{G}{2} \right) G^{-1/3} \,,$$

$$X = \tfrac{1}{2}K^{1/3}\Big[\Big(\frac{2}{\sqrt{Y}} - Y\Big)^{1/2} - \sqrt{Y}\Big],$$
$$P = S\Big(X + \frac{1}{X}\Big), \qquad x = \sqrt{r^2 - P^2}.$$

33.3 Let $x_1, \ldots, x_n$ ($n > 2$) be real numbers such that $x_1 > x_2 > \cdots > x_n$. Prove that

$$\frac{1}{x_1 - x_2} + \frac{1}{x_2 - x_3} + \cdots + \frac{1}{x_{n-1} - x_n} + \frac{1}{x_n - x_1} > 0.$$

Solution by H. A. Shah Ali, who proposed the problem

The problem turned out to be much easier than we originally realized! Since $n > 2$ (the original problem mistakenly had $n \geq 2$), $x_1 - x_n > x_1 - x_2 > 0$, so

$$\frac{1}{x_1 - x_2} > \frac{1}{x_1 - x_n}.$$

Hence

$$\frac{1}{x_1 - x_2} + \frac{1}{x_2 - x_3} + \cdots + \frac{1}{x_{n-1} - x_n} > \frac{1}{x_1 - x_2} > \frac{1}{x_1 - x_n},$$

so

$$\frac{1}{x_1 - x_2} + \frac{1}{x_2 - x_3} + \cdots + \frac{1}{x_{n-1} - x_n} + \frac{1}{x_n - x_1} > 0.$$

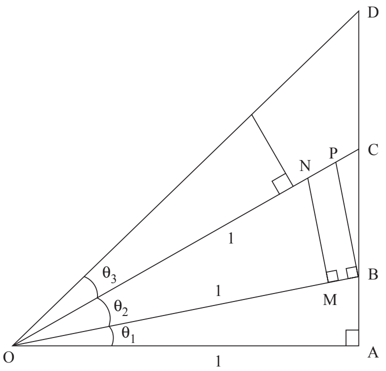
Also solved by Daniel Lamy.

33.4 Let $\theta_1, \ldots, \theta_n$ ($n \geq 2$) be positive real numbers such that $\theta_1 + \cdots + \theta_n \leq \frac{1}{2}\pi$. Prove that

$$\prod_{r=1}^{n} \tan\theta_r \leq 1.$$

Solution by Daniel Lamy

We may suppose that $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_n$. Then $2\theta_{n-1} \leq \theta_{n-1} + \theta_n \leq \frac{1}{2}\pi$, so $\theta_1 \leq \cdots \leq \theta_{n-1} \leq \frac{1}{4}\pi$.



From the diagram,

$$\tan\theta_2 = \mathrm{MN} < \mathrm{PB} < \mathrm{BC},$$

and $\tan\theta_3 < \mathrm{CD}$ etc., so

$$\tan\theta_1 + \cdots + \tan\theta_{n-1} < \mathrm{AB} + \mathrm{BC} + \cdots = \tan(\theta_1 + \cdots + \theta_{n-1}).$$

Since $\theta_1 + \cdots + \theta_{n-1} \leq \frac{1}{2}\pi - \theta_n$,

$$\tan(\theta_1 + \cdots + \theta_{n-1}) \leq \tan(\tfrac{1}{2}\pi - \theta_n) = \cot\theta_n,$$

so

$$\tan(\theta_1 + \cdots + \theta_{n-1})\tan\theta_n \leq 1.$$

Hence

$$(\tan\theta_1 + \cdots + \tan\theta_{n-1})\tan\theta_n \leq 1.$$

But, for $1 \leq i \leq n-1$, $0 < \theta_i \leq \frac{1}{4}\pi$, so $0 < \tan\theta_i \leq 1$. Hence,

$$\prod_{i=1}^{n-1} \tan\theta_i \leq \tan\theta_{n-1} \leq \tan\theta_1 + \cdots + \tan\theta_{n-1},$$

so

$$\prod_{i=1}^{n} \tan\theta_i \leq (\tan\theta_1 + \cdots + \tan\theta_{n-1})\tan\theta_n \leq 1.$$

An alternative solution by Hassan Shah Ali, who proposed the problem

Put $\theta_{n+1} = \theta_1$. Then, for $r = 1, \ldots, n$,

$$0 < \theta_r + \theta_{r+1} \leq \theta_1 + \cdots + \theta_n \leq \tfrac{1}{2}\pi,$$

so

$$\cos(\theta_r + \theta_{r+1}) \geq 0,$$

whence

$$\cos\theta_r \cos\theta_{r+1} - \sin\theta_r \sin\theta_{r+1} \geq 0,$$

and thus

$$\tan\theta_r \tan\theta_{r+1} \leq 1.$$

(Note that $\cos\theta_r > 0$ for each r.) Hence,

$$(\tan\theta_1 \tan\theta_2)(\tan\theta_2 \tan\theta_3)\cdots(\tan\theta_n \tan\theta_1) \leq 1.$$

so

$$\Big(\prod_{r=1}^{n} \tan\theta_r\Big)^2 \leq 1.$$

Now square root.

Medians

A 'median' of a polygon is a straight line which joins a vertex to the mean of the other vertices. Then the medians of a polygon are concurrent.

GUIDO LASTERS
Tienen, Belgium

Reviews

Zero: the Biography of a Dangerous Idea. By Charles Seife. Souvenir Press Ltd, London, 2000. Pp. 248. Hardback £18.99 (ISBN 0-285-63586-7).

Zero invites us to consider a concept that we take for granted, the idea of nothingness.

The early part of this book is an historical account of how ancient civilizations refused to include zero as a number because it could not be represented in geometry, nor was it used in counting. From here we progress through time, discovering how opinion on zero changed with advances in maths, ending up with the duality between zero and infinity, applying this to the realms of particle physics and astrophysics.

Zero tells of the influence of religion and philosophy in banishing nought from the number line until recent times, and how it was believed to be an evil. The main theme of the book is the fact that chaos can be created out of nothing—division by zero can destroy logic and order. The book is very factual and easy to understand with even a limited knowledge of maths and physics; however it is repetitive in many areas, as Seife overstates the key points repeatedly. This book is not suited to any serious mathematician—there is little maths, limited only to a basic introduction to the golden ratio, paradoxes, complex numbers and the concept of calculus. Those with an interest in the history of counting and numerical representation, such as Mayan and Babylonian numbers, would probably enjoy this book. Another nice touch was instructions for making a wormhole time machine, and the 'proof' that Winston Churchill was a carrot!

Seife covers a large period of time in his biography and presents many relevant aspects of science and maths, but unfortunately not at great enough depth, and I was left disappointed when the focus turned almost instantly from one topic to another. For £18.99 it might well be worth choosing a book completely devoted to a specific interest, rather than *Zero*, which was fascinating in parts, but indifferent and tedious in others, and rather short.

Nottingham High School — Daniel Lamy

Introducing Mechanics. By Brian Jefferson and Tony Beadsworth. Oxford University Press, 2000. Pp. 554. Paperback £18.50 (ISBN 0-19-914710-8).

This book is aimed at producing a text delivering all the mechanics content for single maths 'A'-level. The book covers everything in the 'A'-level syllabus from simple modelling to complex differential equations, and from simple theories such as Newton's laws of motion to more complex mechanics such as damped oscillations.

The first few chapters open up the ideas and concepts involved in modelling vectors and forces. Then the book proceeds to progress alongside the 'A'-level course, into friction, moments and centres of mass. You can work through the book from beginning to end, building on and strengthening your previous learning, or you can look at it by selection of specific chapters which are good enough to be understood on their own. I did the latter, as I was using it as a simple revision aid for MEI Mechanics 2, and later Mechanics 3 and 4 (differential equations) and was thus dipping into chapters to search for what I needed.

Rather differently from other books, this one has a website associated with it. Downloadable spreadsheets are thus provided, referenced throughout the text, which offer readers opportunities to explore mechanical events by inserting their own data and assessing their results.

Each chapter begins with a quote, as well as an experiment, with data collected (if applicable) which would help put the chapter into a real-life context. The chapters then continue by explaining the basic theories behind the situation, with good use of diagrams and important points helpfully highlighted. Usually three or so examples follow, which summarise the explanation if you did not fully understand it the first time. You are also guided through the thought processes behind the theory, and shown how to lay out your working when solving a problem. The book continues by introducing a new topic, building upon the work of the previous few pages and the diagrams and examples follow.

One thing that I particularly liked about this book was that it had lots of questions and worked examples, so that you end up feeling really confident about the topic. I found that the book was a pleasant part of my revision—it quickly summarised things for me and provided a number of questions to start off with, which allowed me to get into the swing of things quickly. It succeeded in covering everything that I needed, and in some sections it included topics which were part of my Mechanics 4 module, which is a Further Maths MEI module!

The only book that I am sufficiently familiar with to make a sensible and fair comparison is the MEI Mechanics book. I think that these two both have their strengths: the MEI book is in some cases better at explaining aspects, but most of the time *Introducing Mechanics* is much better with its questions, and of course its answers. (I only found one mistake, confirmed by a teacher, whereas the MEI books normally have a healthy selection of mistakes in their answers.) This is a book definitely worthy of borrowing from the college library.

Student, Solihull Sixth Form College — Edward Bateman

The Math Chat Book. By Frank Morgan. MAA, Washington, 2000. Pp. xiv+113. Paperback $19.95 (ISBN 0-88385-530-5).

The Math Chat Book is definitely an entertaining, quick read for the general reader. Frank Morgan has transformed his live call-in TV show and biweekly column into a book full of interesting questions (both mathematical and non-mathematical) and discussions on related subjects. The book covers a wide range of topics, from geometry to physics, from the history of calendars to capacity on motorways, showing part of the fun nature of mathematics.

The book is written in plain English, with anecdotes and flashes of humour. Indeed, much of the material was contributed by the readers of the Math Chat column. Readers may find some of the puzzles familiar, such as 'Why do mirrors reverse left and right but not up and down?' However, there are plenty of new problems to engage them.

Answers to questions are presented in a lively way, frequently with additional pieces of information. For example, the solutions to optimisation problems are followed by a brief account on the breakthrough in double soap bubble. A website on double soap bubble is also given for keen readers.

A few chapters are rather technical compared to the rest. For example, quaternions are mentioned in the chapter on digits, but the chapters are independent of each other, so any one of them can be skipped without affecting the enjoyment of the others.

Student, Impington Village College CHUN CHUNG TANG

Other books received

Using History to Teach Mathematics. Edited by VICTOR KATZ. MAA, Washington, 2000. Pp. 300. Paperback $32.95 (ISBN 0-88385-163-6).
This book is a collection of articles by international specialists in the history of mathematics and its use in teaching, based on presentations given at an international conference in 1996.

Geometry at Work: Papers in Applied Geometry. Edited by CATHERINE A. GORINI. MAA, Washington, 2000. Pp. 300. Paperback $25.95 (ISBN 0-88385-164-4).
This is a collection of papers by various authors to show how geometry appears in areas such as ancient civilizations, architecture, engineering, decision making, number theory, optimization, graph theory, quantum mechanics and crystallography.

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