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Ramanujan—His Life and Work

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His life

The year of 1987 marks the centenary of the birth of one of the most romantic figures in mathematics—the great self-taught Indian mathematician Srinivasa Ramanujan.

Ramanujan was born on 22 December 1887 of poor parents in the small town of Erode, near Kumbakonam, in southern India. He went to a local elementary school at the age of five, and two years later transferred to the Kumbakonam Primary and High School where he spent the next nine years. Here his extraordinary mathematical powers were recognised and in 1903, at the age of 16, he won a scholarship to the Government College of Kumbakonam to study for a degree from the University of Madras. By this time he was devoting all of his time to mathematics and would not study any other subject. In particular, his failure to study English and physiology caused him to fail his first-year examinations at the college and his scholarship was discontinued.

The next ten years of Ramanujan's life were spent in near obscurity. He continued to work passionately and independently at mathematics, jotting down his results in two large notebooks. Throughout this time he had access to just a handful of mathematics books. By far the most influential of these was G. S. Carr's *Synopsis of Pure and Applied Mathematics*, a collection of some 6000 theorems which had been compiled by the author as an aid in his tutoring of students at the University of Cambridge. Ramanujan had first come across the book at the age of 16 in his final year at High School. It had opened up a whole new world to him and he had immediately set about proving the formulae in it. He had no other books to help him, so each solution was a piece of research as far as he was concerned. Over the next decade, Ramanujan went on to discover many new results of his own. Although he did not know it at the time, some of his results were already well known in the western world. Others were of unprecedented imagination and originality. One or two were incorrect. Following the style of Carr's 'synopsis', Ramanujan wrote down the bare statements of his theorems in his notebooks without indicating any proofs. Eminent mathematicians have since put much effort into providing proofs of these results, and the work is still going on.

For the first few of these 'wilderness years', Ramanujan had no settled occupation. His needs were simple, and he was happy provided he could pursue his mathematical interests. But in 1909 at the age of 22 he married and needed to secure a settled means of livelihood. His bride was just nine years old and, remarkably, is still alive—in her 87th year! It was at about this time that Ramanujan's work came to the attention of some eminent local mathematicians including Mr Ramaswami Aiyar, the founder of the Indian Mathematical Society, and Mr Seshu Aiyar, who was Principal of the Government College of Kumbakonam. They were impressed by the contents of Ramanujan's notebooks and were able to find clerical employment for him, at the same time encouraging him to continue in his mathematical endeavours.

Srinivasa Ramanujan 1887–1920

Ramanujan's first published work was in the form of three questions posed in the 1911 volume of the *Journal of the Indian Mathematical Society*. These (shown in figure 1) gave a fair idea of the sort of mathematics in which he was interested. Three longer articles were published in the same journal during 1911 and 1912.

Ramanujan was urged by Seshu Ayiar and others to bring his work to the attention of mathematicians in the West, and so it was that one morning in January 1913, G. H. Hardy, then Fellow of Trinity College, Cambridge, received a large untidy envelope decorated with Indian stamps.

Hardy (1877–1947), together with his great collaborator J. F. Littlewood, was a dominant influence on research in pure mathematics in England for over a generation. His passion for mathematics was rivalled only by that for the game of cricket. The economist Maynard Keynes, who began his career as a mathematician and who was a friend of Hardy's, once told him that if he had read the stock exchange quotations with the same concentration he brought to the cricket scores, he could not have helped becoming a rich man. For many years Hardy's highest term of praise was 'in the Hobbs class', later to be reluctantly superseded by 'in the Bradman class' in deference to the

Question 260. Show, without using calculus, that

$$\tfrac{3}{2}\log 2 = 1+\frac{2}{4^3-4}+\frac{2}{8^3-8}+\frac{2}{12^3-12}+\dots$$

Question 261. Show that

$$(a)\quad \left(1+\frac{1}{1^3}\right)\left(1+\frac{1}{2^3}\right)\left(1+\frac{1}{3^3}\right)\dots = \frac{1}{\pi}\cosh(\tfrac{1}{2}\pi\sqrt{3})$$

$$(b)\quad \left(1-\frac{1}{2^3}\right)\left(1-\frac{1}{3^3}\right)\left(1-\frac{1}{4^3}\right)\dots = \frac{1}{3\pi}\cosh(\tfrac{1}{2}\pi\sqrt{3})$$

Question 283. Show that it is possible to solve the equations

$$\begin{aligned} x+y+z &= a & p^3x+q^3y+r^3z &= d\\ px+qy+rz &= b & p^4x+q^4y+r^4z &= e\\ p^2x+q^2y+r^2z &= c & p^5x+q^5y+r^5z &= f \end{aligned}$$

where x, y, z, p, q and r are unknowns. Solve the above when $a = 2$, $b = 3$, $c = 4$, $d = 6$, $e = 12$ and $f = 32$.

Figure 1

batsman he described as 'in a whole class above any other batsman who has ever lived'. Only the likes of Archimedes. Newton and Gauss were considered to be in the Bradman class. Hardy's little book, *A Mathematician's Apology* remains one of the best-written accounts of what motivates and justifies the pursuit of pure mathematics.

When Hardy read the letter, written in halting English and containing the bare statements of over a hundred theorems, he might have been tempted to dismiss it as the workings of a crank. It later transpired that two eminent English mathematicians had already rejected the manuscripts out of hand. But on closer inspection, Hardy recognised that it could only be the work of a genius. There followed some months of correspondence during which time Ramanujan was at last able to secure a research studentship at the University of Madras. Hardy was determined that Ramanujan should be brought to England and, after various difficulties had been overcome, the Indian mathematician arrived at Cambridge in April 1914, at the age of 26. Here he pursued a brilliant, though all too brief, career.

During his five years in England, Ramanujan published over thirty research papers, seven of them in collaboration with Hardy. His work was mainly in the fields of infinite series, products and integrals, number theory, the theory of partitions and continued fractions. Much of it was undoubtedly 'in the Bradman class'. Ramanujan became a Fellow of the Royal Society early in 1918 and a Fellow of Trinity College, Cambridge, later in the same year. He was the first Indian to achieve either of these distinctions.

But after three years at Cambridge, in the summer of 1917, Ramanujan became ill. The exact nature of the disease appears never to have been determined, but it was thought to have been some form of tuberculosis. His condition was probably not helped by his severe self-imposed diet—he was a very strict vegetarian. Ramanujan spent much of the next two years in nursing homes and sanatoria in Cambridge, Wells, Matlock, London and Putney. His mathematical output, though more spasmodic, continued to be of the highest quality.

It was while Ramanujan was a patient in Putney, in January 1919, that the famous 'taxi-cab' conversation took place. Ramanujan could remember the idiosyncrasies of numbers in an almost uncanny way; Littlewood had once remarked that 'every positive integer was one of Ramanujan's personal friends'. While visiting Ramanujan in Putney on one occasion, Hardy remarked that he had ridden there in taxi-cab No. 1729 and that this number seemed rather a dull one. 'No, no, Hardy', Ramanujan replied, 'it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways'. ($1729 = 12^3+1^3 = 10^3+9^3$). Hardy then asked Ramanujan whether he could tell him the solution of the corresponding problem for fourth powers. Ramanujan replied, after a moment's thought, that he knew of no obvious example and supposed that the first such number must be very large. (In recounting this story, Hardy remarks that Euler had given the example $158^4+59^4 = 134^4+133^4$.)

Early in 1919, Ramanujan showed some improvement and was able to make the voyage home to India. But he never really recovered and died at Kumbakonam on 26 April 1920. He was just 32 years old. His widow recalls that, despite frequently being in severe pain, he continued his mathematical work right up until his death.

But the story of Ramanujan does not end there.

Ramanujan's published papers make a volume of nearly 400 pages (reference 5), but he also left a mass of unpublished work. After his death, all the papers and notebooks in his possession were handed over to the University of Madras and sent to Hardy on 30 August 1923. The famous early notebooks written before 1913 were later returned to the University of Madras and are now housed there in the Ramanujan Institute. A facsimile edition of the notebooks (reference 9) was published in 1957.

The twenty years or so following Ramanujan's death saw the publication of numerous papers based on his unpublished work, including several by Hardy. In 1940 Hardy published a book (reference 4) based on lectures on Ramanujan's work which he had delivered at the University of Harvard. Another prolific author was G. N. Watson (1886–1965), Professor of Pure Mathematics at Birmingham University. Nearly thirty of Watson's papers owed some of their inspiration to the work of Ramanujan. It was no doubt for this reason that in 1928 Hardy passed on to Watson all of the Ramanujan material in his possession. In 1929 Watson and B. M. Wilson, who was then at Liverpool University, undertook the task of editing Ramanujan's unpublished work. Unfortunately, after making substantial progress, Wilson passed away prematurely in 1935 at the age of 38. Watson's interest in the project evidently waned soon after this and the manuscripts lay neglected in Watson's possession until his death in February 1965.

They were rediscovered, in Watson's home, by Professor Robert Rankin of the University of Glasgow, when he visited Mrs Watson in July 1965 to seek information for Watson's obituary which he was writing for the London Mathematical Society. Rankin had long been interested in the work of Ramanujan. He had assisted Hardy in the latter's writing of his 1940 book of lectures and had written several articles, based on Ramanujan's work, in his own right. Since Watson, Hardy and Ramanujan had all been Fellows of Trinity College, Cambridge, Rankin suggested to Mrs Watson, who gladly agreed, that all the Ramanujan material should be given to Trinity College Library. All this material has now been properly catalogued and is available there to scholars for consultation.

The most exciting discovery amongst Watson's possessions was a collection of 87 loose sheets which had been written by Ramanujan during the last year of his life after his return from England to India. This manuscript is now widely referred to as Ramanujan's 'lost notebook', though strictly speaking it was not a notebook nor was it lost, probably just filed away and forgotten. In typical Ramanujan style, this 'notebook' contains over six hundred mathematical formulae listed one after the other without proof. This then was the result of Ramanujan's labour shortly before his death. The American number theorist Professor George Andrews has set himself the task of proving these formulae and he has already published several papers reporting progress to date.

The task of editing the earlier notebooks, begun by Watson and Wilson, has been taken up by Professor Bruce Berndt of the University of Illinois. The first of three volumes (reference 1) devoted to this project has now appeared. Its author believes that these volumes will yet further enhance the reputation of Ramanujan.

It is now evident that Hardy's comment of 1940 (reference 3) that 'Ramanujan left a mass of unpublished work which has never been analysed properly until the last few years' was at least fifty years premature!

There are only two known photographs of Ramanujan as an adult, one in a cap and gown which Hardy thought 'made him look rather ridiculous' and his famous passport photograph taken in 1919. In this, according to Hardy, 'he looks rather ill (and no doubt was very ill): but he looks all over the genius he was'. This passport photograph provided the portrait for the

Indian postage stamp (illustrated on page 2) which was issued on 22 December 1962, the 75th anniversary of Ramanujan's death.

In 1983, the American sculptor Paul Granlund was commissioned to make a bust of Ramanujan, based on the 1919 passport photograph. Eight copies were cast. One of these was presented to Ramanujan's widow and another purchased and sent to the Indian Academy of Sciences in Bangalore. Five others were acquired by various American institutions or individuals. On the initiative of Robert Rankin, money was raised by British mathematicians to purchase the remaining bust. This was unveiled on 27 May 1986 in the Library of the Department of Pure Mathematics at the University of Cambridge.

In closing this biographical sketch, it may be mentioned that reference 10 provides a full biography, while in reference 2 Bruce Berndt gives an interesting account of a visit to India to see various people and places associated with Ramanujan.

Some examples of his work

To describe the best of Ramanujan's mathematical work would take us outside the scope of this present article. In any case several of the references admirably cover this ground already. I shall therefore limit my attention to just three papers whose contents may be described in simple terms.

1. *Highly composite numbers.* A highly composite number is, in a sense, the very opposite of a prime number. A prime number has only two divisors—itself and unity. Ramanujan calls a number 'highly composite' if it has more divisors than any preceding number. For example, the first few highly composite numbers are 2, 4, 6, 12, 24, 36, 48, 60 and 120. In a paper in 1915, Ramanujan proved many properties of highly composite numbers. He also gave a list of 103 such numbers up to the number 6746 328 388 800. This represented a staggering amount of calculation, for he knew of no formula for generating them. A readable introduction to Ramanujan's work on this topic is given in Chapter 14 of reference 7.

2. *On the expression of a number in the form* $ax^2+by^2+cz^2+du^2$. One of the great accomplishments of Joseph Lagrange (1736–1813) was to prove that all positive integers can be expressed as the sum of four squares. In an article of 1917, Ramanujan considered the question: for which positive integral values of a, b, c and d can all positive integers be expressed in the form $ax^2+by^2+cz^2+du^2$? He proved that there are exactly 54 sets of values of a, b, c and d for which this is true.

3. *On a set of simultaneous equations.* The reader is no doubt familiar with systems of *linear* simultaneous equations. There are straightforward methods of solving any such system, for example by eliminating variables one at a time.

But consider the system of six equations in six unknowns given in Ramanujan's Question 283 posed in figure 1. These equations are non-linear and their solution is not so easy—attempts to eliminate variables can lead one into a hopeless mess.

In 1912, Ramanujan gave an ingenious method of solving such a system of $2n$ equations in $2n$ unknowns:

$$\begin{aligned}
x_1 \quad &+ x_2 \quad &+ \ldots + x_n \quad &= a_1 \\
x_1 y_1 \quad &+ x_2 y_2 \quad &+ \ldots + x_n y_n \quad &= a_2 \\
x_1 y_1^2 \quad &+ x_2 y_2^2 \quad &+ \ldots + x_n y_n^2 \quad &= a_3 \\
&&\vdots \\
x_1 y_1^{2n-1} &+ x_2 y_2^{2n-1} &+ \ldots + x_n y_n^{2n-1} &= a_{2n}
\end{aligned}$$

where $a_1, a_2, \ldots, a_{2n}$ are given and $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are to be found.

This paper, Ramanujan's third, was published in the *Journal of the Indian Mathematical Society* shortly before he first wrote to Hardy. It was just two pages long and gave neither references nor any motivation for solving the given system. Ramanujan was presumably interested in the problem only for its intrinsic attractiveness (as Hardy remarks in reference 3, 'Beauty is the first test: there is no permanent place in the world for ugly mathematics').

The solving of such systems of equations now plays a prominent part in the theory of error-correcting codes, a rapidly-growing branch of mathematics with important practical applications.

Coding theorists, unaware of Ramanujan's papers, rediscovered his method of solution over fifty years later, thus reversing the usual roles whereby Ramanujan was making the rediscovery.

The connection with coding theory, together with Ramanujan's method of solution, are discussed in Chapter 11 of reference 6. They will also be described in a forthcoming *Mathematical Spectrum* article on error-correcting codes. In the meantime, readers may enjoy trying for themselves the simple case of four equations

$$\begin{aligned}
x_1 \quad + x_2 \quad &= a_1 \\
x_1 y_1 + x_2 y_2 &= a_2 \\
x_1 y_1^2 + x_2 y_2^2 &= a_3 \\
x_1 y_1^3 + x_2 y_2^3 &= a_4
\end{aligned}$$

to be solved for x_1, x_2, y_1 and y_2.

In 1970, N. Levinson (reference 8) wrote an expository article on coding theory in which he showed how theorems from number theory play a central role. This, he claimed, was contrary to G. H. Hardy's view that number theory could not have any useful application. It is of interest, therefore, to see another result from pure mathematics finding an application in coding theory, especially when that result is due to Hardy's great protegé, Ramanujan.

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1987 and all that

Guy Willard, of Corpus Christi College, Cambridge, has sent us the following intriguing identities.

$$\begin{vmatrix} 1 & 9 \\ 8 & 7 \end{vmatrix} - \begin{vmatrix} 1 & 9 \\ 8 & 6 \end{vmatrix} = (19-86)-(19-87)$$

$$1.9.8-7 = 1.9+8.7$$

$$\int_1^8 \int_9^7 \mathrm{d}x\,\mathrm{d}y = 1^9-8-7$$

$$\int_{1+9}^{8+7} \mathrm{d}x = \sqrt{(1+9+8+7)}$$

$$-1^3-9^3+8^3+7^3 = (\sqrt{1+9+8+7})^3 = (\sqrt{1+9+8+7})!+\sqrt{1+9+8+7}$$

$$(-1^1+9^2-8^3+7^4)+(19-8+7) = 1987$$

$$1+98+7 = 19+87$$

Evaluating NBA Basketball Statistics

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With the possible exception of cricket, the game of basketball is perhaps unsurpassed in terms of the statistical data recorded. The National Basketball Association (NBA), for example, records points scored, rebounds, assists, blocked shots, steals, fouls and turnovers among others. These categories reflect both positive and negative contributions to the team, and the positive contributions can be both offensive and defensive. Whereas scoring points and assists belong to the offence and blocked shots and steals contribute to the defence, rebounding can be either offensive or defensive in nature. We now give brief definitions of these terms.

Scoring. A basket from the field counts for 2 points, unless it is shot from beyond the 3-point line, in which case it is awarded 3 points. A basket from the free throw attempt counts 1 point. Free throws are awarded as a result of fouls by the opposing team.

Rebounding. A rebound is the possession of the ball after a field goal attempt is unsuccessful. A rebound is offensive or defensive according to whether the offending or defending team gains possession of the ball.

Assist. The assist is defined as a pass which leads directly to a successful field goal attempt by a teammate.

Steal. The steal consists of a direct snatching of the ball from the possession of the opposing team.

Blocks. A blocked shot consists of the thwarting of a field goal attempt by making contact with the ball, without fouling.

Turnover. A turnover is throwing the ball away, resulting in losing the possession of the ball to the opposing team.

Fouls. Fouls are generally caused by physical contact with opposing players in possession of the ball. If the opposing player is in the act of shooting during the foul, one or two free throws are awarded depending upon whether the field goal attempt is successful or not. If the opposing player is not in the act of shooting, no free throws are awarded until the fifth foul in each half, whereupon two free throws are awarded for each foul. A *technical foul* is awarded for misconduct of a player or a coach or from other infractions of the rule not involving play. One free throw is awarded to the opposing team for each technical foul, but the possession of the ball is

Table 1

NBA Team Statistics

Definition	Abbreviation	1981–2	1982–3	1983–4	1984–5	1985–6	Mean
Field goals attempted	*FGA*	88.2	89.7	88.4	89.1	88.7	88.8
Field goals made	*FGM*	43.3	43.5	43.5	43.8	43.2	43.5
Field goal ratio	*FGR*	0.491	0.485	0.492	0.491	0.487	0.489
Free throws attempted	*FTA*	28.6	28.3	29.7	29.4	30.3	29.3
Free throws made	*FTM*	21.3	20.9	22.6	22.4	22.9	22.0
Free throw ratio	*FTR*	0.746	0.740	0.760	0.762	0.756	0.753
Defensive rebounds	*DR*	29.1	29.6	28.8	29.2	29.4	29.2
Offensive rebounds	*OR*	14.3	14.8	14.2	14.3	14.1	14.3
Total rebounds	*REB*	43.5	44.5	43.0	43.5	43.6	43.6
Assists	*AST*	25.2	25.9	26.2	26.2	26.0	25.9
Steals	*STL*	8.5	8.9	8.5	8.5	8.8	8.6
Blocks	*BLK*	5.4	5.6	5.3	5.3	5.3	5.4
Team fouls	*FLS*	26.2	25.6	25.8	24.9	25.2	25.5
Turnovers	*TO*	17.7	19.1	17.9	17.9	17.8	18.1
Points	*PTS*	108.6	108.5	110.1	110.8	110.2	109.6

unaffected. The *offensive foul* is a foul committed by a player in possession of the ball. It results in a turnover and is registered as such.

Duncan and Litwiller (reference 1) first suggested that the various contributions of individual players can be combined to give a meaningful scheme for determining the most productive player in basketball. Specifically, they devised a scheme which adds the contributions of individual players from the main categories of points, rebounds and assists only. This scheme leaves out the lesser categories of blocked shots and steals and also tacitly assumes that the rebound and the assist are equivalent to one point each. In this article, we re-examine the scheme of Duncan and Litwiller by actually determining the values of the various categories from available data and apply the results to determine the most productive player by a new scheme which includes blocked shots and steals also.

We begin with the average team statistics of the NBA for the past few seasons. They are compiled from the Official NBA guide (reference 2) and displayed in table 1 along with their definitions and abbreviations. From definitions, we have the following immediate relations:

$$FGR = FGM/FGA, \tag{1}$$

$$FTR = FTM/FTA, \tag{2}$$

$$REB = DR+OR. \tag{3}$$

One important statistic that is not recorded is the number of possessions of the ball (*POS*) by each team. We calculate *POS* as follows. Each possession of the ball terminates in one of the three possible ways: (1) leading to a field goal attempt, (2) leading to a turnover and (3) leading to a 2-shot free throw. If α and β are the numbers of fouls leading to 2-shot and 1-shot free throws, respectively, then

$$POS = FGA + TO + \alpha. \tag{4}$$

Since eight fouls (first four in each half) do not result in penalty, we have

$$FLS = \alpha + \beta + 8. \tag{5}$$

Now, if ϵ is the number of technical fouls by each team, we have the identity

$$FTA = 2\alpha + \beta + \epsilon. \tag{6}$$

Solving equations (5) and (6) simultaneously, we get

$$\alpha = FTA - FLS + 8 - \epsilon \tag{7}$$

and

$$\beta = 2FLS - FTA - 16 + \epsilon. \tag{8}$$

Substituting the mean values of FTA, FLS, FGA and TO from table 1 and setting $\epsilon = 0.8$ (data collected by the author), we get $\alpha = 11.0$ and $\beta = 6.5$, whence $POS = 117.9$ for the data of table 1.

We next proceed to find the average value of a possession. For this, we make a distinction between an initial backcourt possession and a mature frontcourt possession on the verge of a field goal attempt or a 2-shot free throw. Since the ball was not turned over, the latter has a higher potential of producing points than the former. Denoting the values of the initial and mature possessions by P_i and P_f respectively, we get

$$P_i = \frac{PTS}{POS} \tag{9}$$

and

$$P_f = \frac{PTS}{POS - TO}. \tag{10}$$

The average value of a possession anywhere on the court is therefore

$$P_{av} = \tfrac{1}{2}(P_i + P_f). \tag{11}$$

Substituting the mean values from table 1, we get $P_i = 0.93$, $P_f = 1.10$ and $P_{av} = 1.01$.

Having determined the average values of the possession of the ball, we now proceed to evaluate the various NBA categories.

Scoring. Scoring is the most important statistic, since the outcome of the game is decided solely on points. A point is a point, whether scored from the field or from the foul line.

Rebounding. A defensive rebound is equivalent to an initial backcourt possession whereas an offensive rebound is equivalent to a mature frontcourt possession which offers an immediate chance of a field goal attempt. Hence their respective values are given by P_i and P_f, i.e., 0.93 and 1.10.

Assist. There are generally two opposing views about the assist. Some experts consider the assist to be the equivalent of two points (cf. reference 3), but others argue that it is merely a pass and the two points are already credited to the scorer. It may be noted that the Basketball Association of America awarded assists only to outstanding passes which led to scoring (reference 4). We here take a compromise view. Since the assist results in a basket worth 2 points, whereas, without the assist, the possession would have still been worth P_f, the assist has the value $2 - P_f$ or 0.90.

Steal. The steal can take place anywhere on the court and therefore neutralises the opponents' possession valued P_{av}, which is 1.01. Frequently, the opponents' defence is not in place following a steal, which enhances the chance of scoring. The last effect is difficult to determine. Thus the value of the steal is placed at slightly greater than 1.01.

Blocked shot. The blocked shot nullifies a field goal attempt by the opponent which neutralises the opponents' frontcourt possession worth P_f. Thus a block is equivalent to 1.10 points.

Turnover. The turnover is a negative statistic which is nearly the reverse of a steal. Since the turnover can take place anywhere on the court, it is equivalent to $-P_{av}$, i.e., -1.01 points.

Fouls. The foul is the other negative statistic whose value depends upon the nature of the foul. The first four team fouls in each half do not result in penalty. Normally, 2-shot and 1-shot fouls are equivalent to $-2FTR = -1.51$ and $-FTR = -0.75$ respectively. Technical fouls result in 1-shot free throws. Since the best free-throw shooter on the field usually takes the free throw, its value is less than $-FTR = -0.75$.

In a nutshell, the values of all the positive NBA categories are approximately equal to one point. Therefore, the scheme of Duncan and Litwiller (reference 1) of adding points, rebounds and assists to determine the total performance of a player is reasonable. However, this scheme can be improved if one includes blocked shots and steals also. To illustrate this point, we now proceed to determine the 10 most productive players in a recent regular season of the NBA according to the scheme of Duncan and Litwiller and according to our new scheme. Table 2 shows the contributions made by the leading 10 players. The average points are obtained by adding the total numbers of positive contributions according to each scheme and dividing by the number of games played. According to either scheme, Bird was the clear winner. It was the all-round game of Bird (fourth in scoring,

Table 2

NBA individual statistics

Player	Games	Points	Rebounds	Assists	Steals	Blocks	Average (D–L)	Average (TAN)	Rank (D–L)	Rank (TAN)
Bird	82	2115	805	557	166	51	42.40	45.05	1	1
Wilkins	78	2366	618	206	138	49	40.90	43.29	2	2
Olajuwon	68	1597	781	137	134	231	36.99	42.35	6	3
Barkley	80	1603	1026	312	173	125	36.76	40.49	8	4
English	81	2414	405	320	73	29	38.75	40.01	3	5
Dantley	76	2267	395	264	64	4	38.50	39.39	4	6
Johnson	72	1354	426	907	113	16	37.32	39.11	5	7
Malone	74	1759	872	90	67	71	36.77	38.64	7	8
Thomas	77	1609	277	830	171	20	35.27	37.75	9	9
Sampson	79	1491	879	283	99	129	33.58	36.47	10	10

seventh in rebounding and ninth in steals) that made him the most productive player, as neither the scoring champion (Wilkins) nor the leaders in rebounding (Laimbeer), assists (Johnson), blocked shots (Bol) or steals (Robinson) matched this effort. Indeed, Bird was unanimously voted the 'most valuable player' by a panel of basketball writers. Table 2 also shows that the ranking for the six middle positions are all different according to the two schemes, which illustrates the importance of including the blocked shots and steals in assessing the total perfomance of a basketball player. Developing similar schemes for other games like soccer and hockey would be difficult as defensive efforts do not register in the scoreboard at all in such games.

References

1. D. R. Duncan and B. E. Litwiller. *Math. Teach.*, **66** (1973), 201–206.
2. *Official NBA Guide 1986* (The Sporting News Publ. Co., St. Louis, 1986).
3. J. West. *Basketball My Way* (Prentice-Hall, Englewood Cliffs, 1973), p.15.
4. *The Sports Encyclopedia: Pro Basketball* (Grosset and Dunlap, New York, 1975), p.368.

How not to cancel?

We know that $16/64 = 1/4$ by cancelling the 6's, i.e.

$$\frac{16}{64} = \frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}.$$

Are there other examples of $ab/bc = a/c$? Can you find all of them? Are there examples with three-digit numbers? What about using non-decimal bases, e.g. $13/32 = 1/2$ in base 4?

David Singmaster
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Interference in Printing and in Textile Manufacture

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Dr Firby is a lecturer in pure mathematics at the University of Exeter. Currently his interests lie in various aspects of the geometry of graphics.

1. Introduction

When two or more periodic phenomena coincide, interference often appears. The strength of this interference may spoil effects we are attempting to produce and so it is necessary to understand why and when it appears and how to avoid it.

The type of interference considered here is that which emerges when two regular screens of dots are overlaid. Combinations of this type are common in printing and in textile manufacture, where the interference we shall describe is a constant problem.

2. Regular tessellations

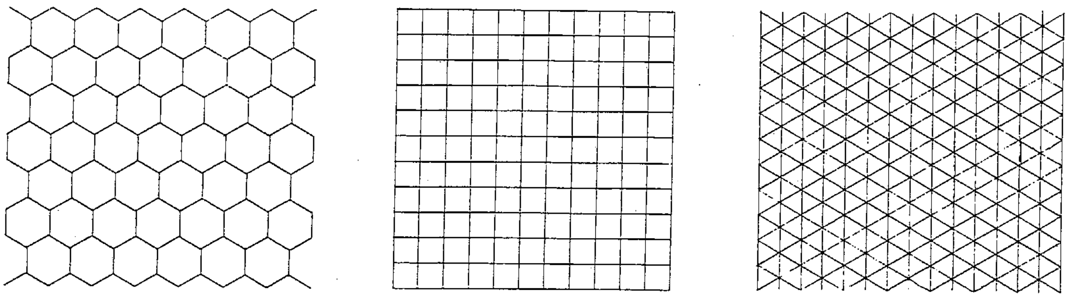
A polygon with p vertices, p edges of equal length, and with all internal angles equal is called a regular p-gon. If the plane is filled by regular p-gons so that q surround each vertex we get the regular tessellation denoted by the Schläfli symbol $\{p,q\}$. Since the internal angle of a regular p-gon is $(1-2/p)\pi$, if q of them come together at a vertex then

$$\left(1-\frac{2}{p}\right)\pi = \frac{2\pi}{q}$$

and so

$$(p-2)(q-2) = 4.$$

The three possible ways of factorizing 4, namely 4×1, 2×2 and 1×4, yield the only three possible regular tessellations of the plane, $\{6,3\}$, $\{4,4\}$ and $\{3,6\}$. These are shown in figure 1.

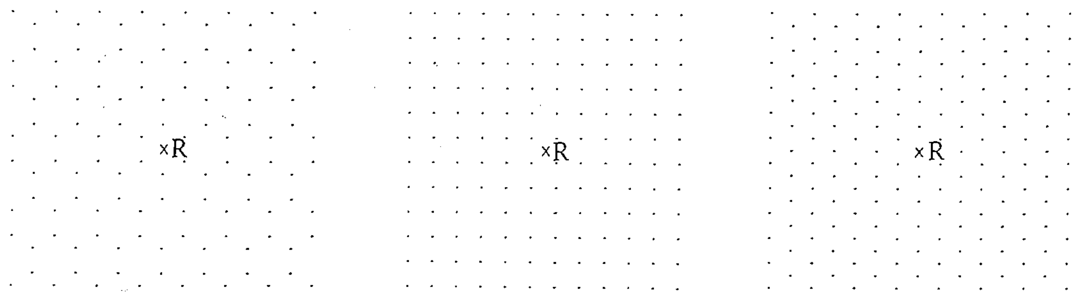


(*a*) (*b*) (*c*)

Figure 1. Regular plane tessellations: (*a*) $\{6,3\}$, (*b*) $\{4,4\}$, (*c*) $\{3,6\}$.

3. Regular screens of dots

With each of the regular plane tessellations can be associated a regular screen of dots as illustrated in figure 2.



(a) (b) (c)

Figure 2. The regular screen of dots associated with: (a) {6, 3}, (b) {4, 4}, (c) {3, 6}.

The screens (b) and (c) frequently occur in applications in industry. They are known as the quadratic (or Levy) screen and the hexagonal (or Schulz) screen, respectively. Notice the symmetry of these screens. The quadratic screen in figure 2(b) is said to have four-fold symmetry since it regains its initial configuration four times in a rotation of 360° about the point R. The screens in figures 2(a) and 2(c) have six-fold symmetry about the marked point R.

4. Overlaying regular screens of dots

When two regular screens are overlaid, the two types of symmetry combine to produce a picture which also has rotational symmetry. Sometimes this symmetric pattern will be particularly pronounced, its cells being much larger than the cells of each of the underlying screens. This periodic pattern which we shall call the interference pattern is caused by areas in which there is a high density of dots alternating with areas in which few dots have settled. The exact nature and size of the interference pattern can be determined using the measurements of the underlying screens and a little elementary geometry. Here we simply use symmetry arguments to determine the nature of the patterns.

Suppose first that a screen with four-fold symmetry is overlaid on a screen with six-fold symmetry. Then at any point in the picture there are two groupings of dots. The one originating from the quadratic screen regains its position after a 90° rotation, while the one originating from the hexagonal screen regains its position after a 60° rotation. It follows that after a 180° rotation both screens will regain their original position. Thus the whole picture has two-fold symmetry. So, for a hexagonal screen printed over a quadratic screen any interference takes the form of parallel bands of densely grouped dots and lightly grouped dots. This is illustrated in figure 3.

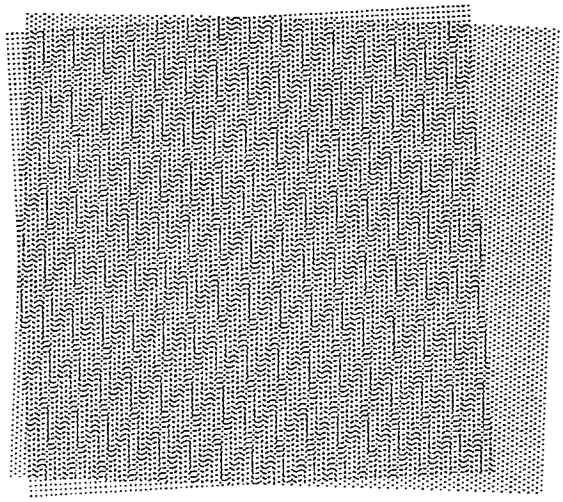


Figure 3. A hexagonal screen overlaid on a quadratic screen.

The reasoning above suggests the following result. If two screens have m-fold and n-fold symmetry then the interference formed when they are overlaid has l-fold symmetry where l is the greatest common divisor of m and n. From this it follows that when two quadratic screens are overlaid the interference will have four-fold symmetry and therefore will take the form of the tessellation $\{4,4\}$. When two hexagonal screens are overlaid the interference will have six-fold symmetry. These cases are illustrated in figures 4 and 5, respectively.

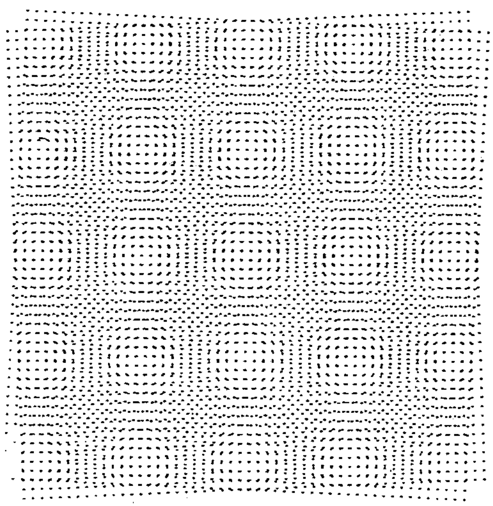


Figure 4.
Two quadratic screens overlaid.

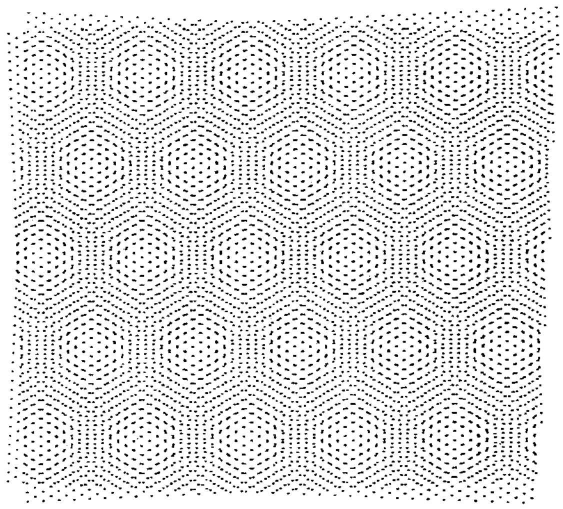


Figure 5.
Two hexagonal screens overlaid.

5. Conclusions

Of the three possible regular screens of dots, two occur commonly in practice. These are the quadratic screen, corresponding to the tessellation $\{4,4\}$, and the hexagonal screen, corresponding to the tessellation $\{3,6\}$.

The quadratic screen is commonly used in printing where each of the different colours in a picture is placed on a different screen. When the screens are overlaid to complete the picture, care must be taken to ensure that the interference pattern shown in figure 4 does not emerge.

Similarly, in textile production a hexagonal screen is often printed on fabric which shows the characteristic pattern of the quadratic screen and so efforts must be made to avoid the emergence of the interference pattern shown in figure 3. Interference used to be a persistent problem in the textile industry because the shifting patterns caused vertigo among operatives.

To avoid interference it is necessary to know why and when it occurs and to know how to evaluate its strength and extent under varying conditions. All of this can be done using only elementary geometry and the details may be found in reference 2. Essentially the results indicate that, while some control may be gained through adjustment of the distances between dots on the screens, the main factor is the angle between the overlaid screens. Interference is more pronounced for small angles (for the figures in this article the screens cross at 5°). Since a quadratic screen returns to its original position after 90° it is clear that there is a severe limitation on the number of screens which may be overlaid to produce an interference-free picture.

For a description of another type of commonly occurring interference which can be analysed using only elementary geometry see reference 1.

References

1. P. A. Firby, Controlling interference in graphics, *Math. Gazette*, June 1987.
2. D. Tollenaar, *Moiré Pattern in Printing*, Research and Engineering Council of the Graphic Arts Industry, Washington D.C., 1960.

More on 1987

$$
\begin{aligned}
&\{-1+(\sqrt{9}\times 8)+7\}\times\{(1+9)\div(8-7)\}\\
&\qquad = \{1+(9!\div 8)\div 7!\}\times\{(-1+\sqrt{9})\times(8+7)\}\\
&\qquad = \{1+(98\div 7)\}\times\{19+8-7\}\\
&\qquad = \{1\times 9\times 8\times 7\}-\{1+98+7\}-\{19+87\}+\{(-1+9)\times(8-7)\}\\
&\qquad = \{1^2+9^2+8^2+7^2\}+\{(1\times 98)+7\}\\
&\qquad = \{1+9+8+7\}\times[\{1.9+8.7\}+\{0.1\times 9.8\div 0.7\}]\\
&\qquad = 300.
\end{aligned}
$$

and the 300th prime number is 1987.

KEITH GORDON
The Haberdashers' Aske's
School, Elstree.

Prime Arithmetic Progressions

DAVID BARROW, ISOBEL BUSH and PAUL TAYLOR,
Queen Elizabeth's Grammar School, Blackburn

When they wrote this article, the authors were in the upper sixth studying mathematics, further mathematics and physics with a view to going on to university. David plans to read aeronautical engineering and hopes to be a fighter pilot or engineer. Isobel is thinking of a career as an actuary after taking a degree in mathematics. Paul's intention is to read computer science and electronics and then work in the motor industry.

A standard topic at A-level is the study of arithmetic progressions. An especially interesting example of these progressions is the case in which all the terms are prime numbers. These 'prime arithmetic progressions' (PAPs for short) are not easily found. For any first term a and common difference d, a composite number usually appears quite soon, as the following examples show:

$$\begin{array}{rrrrrr} 2 & 5 & 8 \\ 3 & 5 & 7 & 9 \\ 3 & 7 & 11 & 15 \\ 5 & 11 & 17 & 23 & 29 & 35 \\ 5 & 13 & 21 \\ 7 & 13 & 19 & 25 \end{array}$$

If $a = 2$, only two prime terms are possible; the third term is $2+2d$ which obviously has a factor of 2 and so is not prime. And, in general, if we start with the prime p then at most p prime terms are possible, since the $(p+1)$th term is $p+pd$, which is divisible by p. Thus the first term imposes a restriction on how many terms are possible. But, if we start with a fairly large prime, the possibility of long PAPs remains.

A computer search revealed that PAPs (other than very short ones) have a limited range of common differences: $6, 30, 210, \ldots$ (or multiples of these numbers) and that to find PAPs of lengths even up to 10, 11 or 12 involves very large prime numbers. There are many PAPs of any given length, and it is interesting to find the one which involves the smallest numbers. Our own computer search produced the results given in Table 1. The form of the common differences in this table is significant:

$$6 = 2\times3, \qquad 30 = 2\times3\times5, \qquad 210 = 2\times3\times5\times7,$$

$$2310 = 2\times3\times5\times7\times11, \qquad 13\,860 = 2^2\times3^2\times5\times7\times11,$$

$$60\,060 = 2^2\times3\times5\times7\times11\times13.$$

TABLE 1

Length	Common differences	PAP							
5	6	5	11	17	23	29			
6	30	7	37	67	97	127	157		
7	210	47	257	467	677	887	1097	1307	
8	210	199	...	...	...	...	...	...	1 669
9	210	199	...	...	...	...	...	...	1 879
10	210	199	...	...	...	...	...	...	2 089
11	2 310	110 437	...	...	...	...	...	...	249 037
12	13 860	110 437	...	...	...	...	...	...	262 897
13	60 060	4 943	...	...	...	...	...	...	725 663

Thus these common differences involve all primes up to a certain point in their prime factorization. With $d = 6$, no PAP was found of length greater than 5; with $d = 30$, no PAP was found of length greater than 6; with $d = 210$, no PAP was found of length greater than 10.

It is easy to see why the common differences giving PAPs of lengths $5, 6, \ldots, 13$ in Table 1 are of this form. For consider the arithmetic progression

$$a, \quad a+d, \quad a+2d, \quad \ldots,$$

and let p be a prime number not dividing d. Then

$$\begin{aligned} & a+rd \equiv a+sd \pmod p \\ \Leftrightarrow \quad & rd \equiv sd \pmod p \\ \Leftrightarrow \quad & r \equiv s \pmod p. \end{aligned}$$

Thus, any p consecutive terms in this arithmetic progression have different remainders on division by p. One of them will therefore have remainder 0 and so will be divisible by p, and the terms divisible by p are equally spaced in the sequence, p terms apart. Thus a PAP with common difference d cannot be of length greater than p. (In fact, unless the first term is p, it will be of length strictly less than p.) And this is true for all primes p not dividing d. This explains why the common differences supplying long PAPs involve all the first few primes. For example, take $d = 210 = 2\times3\times5\times7$. Now $11 \nmid d$, so no PAP with $d = 210$ can have more than 11 terms.

Unfortunately, all this gives is an upper bound for the length of a PAP in terms of its first term and its common difference; it does not tell us how to find one of a given length. For example, suppose we wish to find one of length 22. The first term must be a prime greater than 22. The common difference must be such that the first prime not dividing it is greater than 22, such as

$$d = 2\times3\times5\times7\times11\times13\times17\times19.$$

Clearly this will involve some pretty large numbers. We leave readers to hunt for such a PAP!

Computer Column

MIKE PIFF

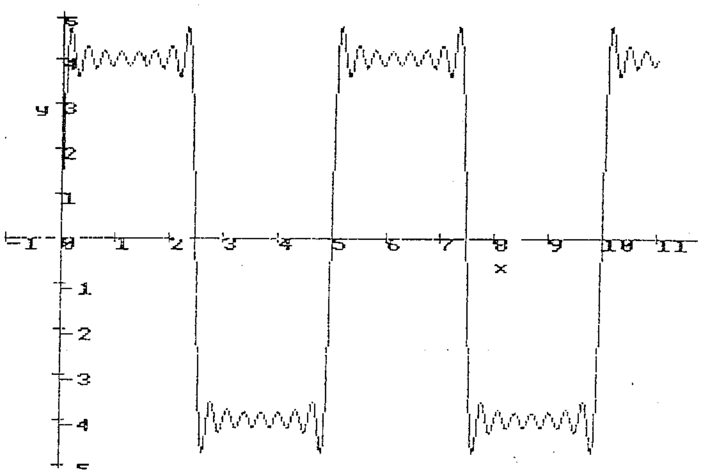
The function

$$f(x) = \begin{cases} 4 & \text{for } 0 < x < 2.5, \\ 0 & \text{for } x = 0, 2.5, 5, \\ -4 & \text{for } 2.5 < x < 5, \end{cases}$$

repeating periodically for all values of x, can be approximated to any degree of accuracy by the *Fourier series*

$$\sum_{j=0}^{\infty} \frac{16}{(2j+1)\pi} \sin \tfrac{2}{5}(2j+1)\pi x.$$

Thus, the 'square wave' $f(x)$ has a decomposition into extremely smooth sine waves. The following BBC BASIC program plots the successive approximations to $f(x)$. You will see the smooth approximations becoming more and more angular. Notice, also, the famous *Gibbs phenomenon*, that the approximations overshoot $f(x)$ at its discontinuities, by finite, non-vanishing amounts.



```
10 MODE128
20 PROCINIT
30 DIM F(550)
40 GCOL3,1
50 PROCNUM(1)
60 P125=PI/125
70 P1600=1600/PI
80 FOR X%=0 TO 550:F(X%)=P1600*SIN(X%*P125):PLOT69,2*X%,F(X%):NEXT X%
90 PROCNUM(1)
100 I%=3
110 REPEAT
120   PROCNUM(I%)
130   K=I%*P125
140   K2=P1600/I%
150   FOR X%=0 TO 550
160   F=F(X%)
170   PLOT 69,2*X%,F
180   F=F+K2*SIN(K*X%)
190   PLOT 69,2*X%,F
200   F(X%)=F
210   NEXT X%
220   PROCNUM(I%)
230   I%=I%+2
240   UNTIL FALSE
250 END
260 DEF PROCINIT
270 LOCAL I,I$
280 REM  VDU19,0,4,0,0,0
290 VDU19,1,1,0,0,0
300 VDU23,1,0;0;0;0;
310 VDU5
320 CLG
330 VDU29,100;510;
340 GCOL0,1
350 MOVE -100,0
360 DRAW 1200,0
370 MOVE 0,-600
380 DRAW 0,600
390 FOR I=-600 TO 1100 STEP 100
400 I$=STR$(I/100)
410 MOVE I,-10
420 DRAW I,10
430 MOVE I,0
440 PRINT I$
450 MOVE -10,I
460 DRAW 10,I
470 MOVE 0,I
480 PRINT I$
490 NEXT I
500 MOVE 800,-50
510 PRINT "x"
520 MOVE -50,300
530 PRINT "y"
540 ENDPROC
550 DEF PROCNUM(I%)
560 LOCAL I$
570 I$=STR$(I%)
580 MOVE -100,500
590 PRINT I$
600 ENDPROC
```

Divisibility by 7

I. Choose a whole number smaller than 7000. Let's take 4410 and 946 as examples. Multiply by 143. This gives 630 630 and 135 278 in our examples. The first is of the form *abcabc*, the second isn't. Here's the interesting thing: 4410 is divisible by 7 and 946 isn't. And this is always the case. Can you see why? Mind you, it's probably easier simply to divide by 7!

ROBIN COOPER
Mathematics teacher and electric keyboard player,
Beaumont School,
St. Albans.

II. I have noticed the following conditions under which a number is divisible by 7. Readers may like to try to prove these results.

A two-digit number is divisible by 7 if one of the following conditions holds:

(1) the tens digit is twice the units digit, e.g. 21:

(2) the difference between twice the units digit and the tens digit is divisible by 7, e.g. 49;

(3) five times the units digit plus the tens digit is a multiple of 7, e.g. 35.

A three-digit number is divisible by 7 if one of the following conditions holds:

(4) the two-digit number formed by the hundreds and tens digit is twice the units digit, e.g. 126;

(5) the tens digit is the sum of half the hundreds digit and twice the units digit, e.g. 882.

A four-digit number is divisible by 7 if one of the following conditions holds:

(6) the thousands digit and the units digit are the same and the remaining two digits form a number divisible by 7, e.g. 9359;

(7) the number formed by the tens and units digits is five times the number formed by the thousands and hundreds digits, e.g. 1995.

A five-digit number is divisible by 7 if the following condition holds:

(8) the units digit is three times the 10^4th digit and the three remaining digits form a number divisible by 7, e.g. 21 266.

A six-digit number is divisible by 7 if one of the following conditions holds:

(9) all the digits are the same;

(10) the units digit is twice the 10^5th digit and the remaining digits form a number divisible by 7, e.g. 211 904.

One way of testing whether a number is divisible by 7 is to remove its units digit and to subtract from this number twice the removed units digit. Repeat this process until you end up with a one-digit number (you can ignore any negative signs). If you end up with 0 or 7, the original number is divisible by 7, otherwise it is not. For example, start with 64 372. The process gives the sequence $6437-4 = 6433$, $643-6 = 637$, $63-14 = 49$, $4-18 = -14$, $1-8 = -7$, so 64 372 is divisible by 7.

L. B. DUTTA
Maguradanga,
Keshabpur,
Jessore,
Bangladesh.

$$p\,|\,n \Leftrightarrow (p-1)\,|\,n$$

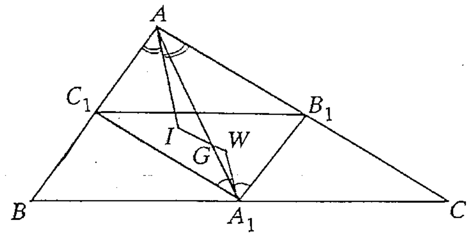
(p prime, n a product of distinct primes).
What is n ?

Letters to the Editor

Dear Editor,

Problem 16.9

The following proof for Problem 16.9, simpler than the one I supplied with the problem or the one given by the solver Ruth Lawrence, recently forced its way into my head.



The centroid W of a uniform wire in the shape of triangle ABC is at the centroid of particles of mass a, b and c respectively at A_1, B_1 and C_1, the respective midpoints of BC, CA and AB. Taking moments about the bisector of angle $B_1A_1C_1$ quickly shows that W lies on this bisector. Similarly W lies on the bisector of angle $A_1B_1C_1$ and hence is the incentre of triangle $A_1B_1C_1$.

Now consider the enlargement with centre G, the centroid of triangle ABC, and scale factor -2. Under this enlargement, $A_1B_1C_1 \to ABC$, so the incentre W of triangle $A_1B_1C_1$ maps to the incentre I of triangle ABC.

Thus $\overrightarrow{IG} = 2\overrightarrow{GW}$ as required.

Yours sincerely,

JOHN MACNEILL
Royal Wolverhampton School.

Dear Editor,

Calendar Mathematicus

According to Boyer's *A History of Mathematics* (Wiley, 1968), Teubner published a 'calendar for mathematicians' during the early part of this century.

I have compiled my own calendar, which at present contains 215 entries. For example, my entry for Christmas day is that Sir Isaac Newton was born on this day in 1642; and my entry for 20 March is that he died in 1727.

I shall be pleased to send interested readers a copy of my calendar if they will write to me, enclosing a stamped addressed envelope. They may be able to help me to fill in the missing dates!

Yours sincerely,

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Dear Editor,

On sums of two powers II

In my letter 'On sums of two powers' (*Mathematical Spectrum* Volume 19 Number 2), I defined $P(n, m, y)$ to be the number of positive integers p for which there are at least m different expressions of the form $p = a_i^n + b_i^n$ for $i = 1, \ldots, m$, where $a_1, b_1, \ldots, a_m, b_m$ are positive integers less than or equal to y whose highest common factor is 1, and gave some particular values. I have now succeeded in pushing y as far as 220 for $n = 2$, and have that

$$P(2,2,220) = 3936, \qquad P(2,5,220) = 77,$$
$$P(2,3,220) = 925, \qquad P(2,6,220) = 59,$$
$$P(2,4,220) = 519, \qquad P(2,7,220) = P(2,8,220) = 5.$$

With $n = 3$, since there is a much smaller number of possible solutions to be verified, I have pushed y to 600 and have that $P(3,2,600) = 303$.

We will now analyse $P(n, m, y)$ in more detail. A graph of $p(y) = P(2,2,y)$ against y reveals an exponential-type curve, prompting us to construct a graph of $\log p(y)$ against $\log y$. The resultant 'curve' is very close to being a straight line with gradient 'near' 2. By drawing another graph, this time of $\sqrt{p(y)}$ against y, we obtain the approximation

$$\sqrt{p(y)} = 0.29y - 1.00,$$

that is,

$$p(y) = 0.0841y^2 - 0.58y + 1.00.$$

Substituting y for known values of $p(y)$ shows a good correspondence (see table 1). Extrapolating the estimation to $y = 1000$, we obtain the approximation $p(1000) = 83\,521$, which would be impossible for me to verify exactly at present.

Table 1

y	actual $p(y)$	estimated $p(y)$
50	178	182
100	767	784
150	1792	1806
200	3263	3249

Proceeding analogously for $q(y) = P(2,3,y)$, we find that

$$q(y) = 0.02\,164y^2 - 0.63\,253y + 4.6225,$$

which also gives a good correspondence for known values. We estimate $q(1000)$ as 21 012.

From the above two approximations it can be conjectured that

$$P(2, m, y) = O(y^2), \quad \text{for all } m \geqslant 2,$$

i.e.

$$\lim_{y\to\infty}\{P(2,m,y)/y^2\} = c, \quad \text{for some constant } c.$$

This is backed up when the same treatment is carried through for $m = 4$ and $m = 5$ to obtain

$$P(2,4,y) = 0.01\,476y^2 - 0.96\,228y + 15.6816,$$

$$P(2,5,y) = 0.0031\,472y^2 - 0.378\,675y + 11.3906,$$

giving estimated $P(2,4,1000) = 13\,813$ and $P(2,5,1000) = 2780$.

With $P(3,2,y)$, the log graph indicates a linear relationship between $P(3,2,y)^{3/4}$ and y given by

$$P(3,2,y)^{3/4} = 0.121y - 1.5.$$

Table 2 exhibits the correspondence of this equation with actual values.

Table 2

y	actual $P(3,2,y)$	estimated $P(3,2,y)$
100	27	23
200	63	64
300	116	114
400	167	169
500	226	230
600	303	295

We can now ask a more general question:

Is it true that $P(n,m,y) = O(y^{4/n})$ whenever $m \geqslant 2$?

If so, then we can define, for $n \geqslant 2$,

$$c(n,m) = \lim_{y\to\infty}\{P(n,m,y)/y^{4/n}\}.$$

Hence $c(3,2) = (0.121)^{4/3} = 0.05\,985$.

For $n = 4$ we must turn to the previously published results of Lander, Parkin and Selfridge (see references), which give all equations leading to the calculation of $P(4,2,15\,000) = 45$. Applying the treatment we obtain a *very rough* estimate that

$$P(4,2,y) = 0.0028y + 3.$$

We now have

$$c(2,2) = 0.0841, \qquad c(3,2) = 0.059\,85, \qquad c(4,2) = 0.0028.$$

A graph of $c(n,2)$ against n is a very vague object, being of only 3 points, but a smooth curve passing through these points seems to be slightly convex and cuts the n-axis at about the value 4.7. I put this forward as a rough justification that $c(5,2)$, and indeed $c(n,2)$ for $n \geqslant 5$, does not exist, and consequently that the equation

$$x^n + y^n = u^n + v^n$$

has no non-trivial solutions in positive integers for $n \geqslant 5$.

Let us now consider $Q(n,m)$, defined as the smallest y such that $P(n,m,y) \neq 0$. A special program was run, and produced the results shown in table 3.

Table 3

m	$Q(2,m)$	m	$Q(2,m)$
7	164	10	371
8	165	11	399
9	268	12	400

Some elementary number theory which I do not include here (but have available for those interested) provides the existence of $Q(2,m)$ for all $m \geqslant 2$ and an algorithm which gives very good upper bounds, in particular that

$$Q(2,13) < Q(2,14) < Q(2,15) < Q(2,16) \leqslant 1088.$$

More complex theory also guarantees that $Q(3,m)$ exists for all $m \geqslant 2$, although I know of no convenient method for obtaining upper bounds in this case. However, I can announce finally that $Q(3,3) = 436$, with the equations

$$436^3+167^3 = 423^3+228^3 = 414^3+255^3,$$

and that $P(3,3,600) = 3$.

The next step, I hope, is to obtain values for $Q(3,4)$ and $Q(4,3)$. I wait in anticipation.

References

1. L. J. Lander and T. R. Parkin, Equal sums of biquadrates, *Math. Comput.* **20** (1966), 450–451.
2. L. J. Lander, T. R. Parkin and J. L. Selfridge, A survey of equal sums of like powers, *Math. Comput.* **21** (1967), 446–459.

JOSEPH MCLEAN
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Editorial Note. Joseph McLean's letter in Volume 19 Number 2, of which the present letter is a sequel, was wrongly edited in lines 22–25 on page 59, which should be disregarded. In fact $Q(2,7) = 164$ as above, and $P(3,3,550) = 3$ (not 0).

How would you work out the remainder when

$$1^5+2^5+\ldots+100^5$$

is divided by 4?

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

20.1. (Submitted by Chris Hillman, a student at Cornell University, USA)
Three primes of the form p, $p+2$, $p+4$ are said to form a 'prime triple'. How many prime triples are there?

20.2 (Submitted by Seung Bang, Seoul, Korea)
Evaluate the integral

$$\int_0^x \frac{t^{n-1}}{\sqrt{t^{2n}+a}}\,dt,$$

where a is a positive real number and n is a positive integer.

20.3. (Submitted by J. N. MacNeill, Royal Wolverhampton School)
A curve lies in the first quadrant; the origin lies on the curve, as does the point P. R_1 is the region enclosed by the arc OP, the vertical through P and the x-axis; R_2 is the region enclosed by the arc OP, the horizontal through P and the y-axis, and n is a positive constant. Find an equation for the curve if, for all positions of P,

(a) the area of R_2 is n times the area of R_1;

(b) the volume swept out when R_2 is rotated through one revolution about the y-axis is n times the volume swept out when R_1 is rotated through one revolution about the x-axis.

20.4. (Submitted by D. J. Roaf, Exeter College, Oxford)
Five raffle tickets are drawn at random without replacement from a set of 149 numbered from 1 to 149 consecutively. Find the mean and the variance of their sum.

Solutions to Problems in Volume 19 Number 2

19.4 How many triangles are there whose vertices are vertices of a given $(2n+1)$-sided regular convex polygon and which contain the centre of the circle circumscribing the polygon?

Solution by Adrian Hill (Trinity College, Cambridge)

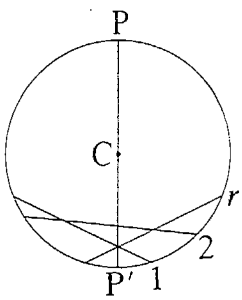


Figure 1

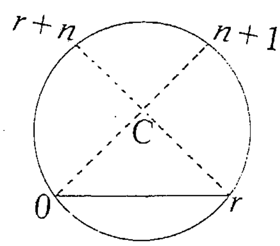


Figure 2

Denote by C the centre of the circumscribing circle, by P one of the vertices of the polygon and by P′ and the point of the circle diametrically opposite P. Any triangle enclosing C must have an edge which crosses the line segment CP′. Suppose this edge has end-points which are r places apart among the vertices of the polygon. There are r possibilities for this edge (see figure 1). Having fixed this edge, there are now $(r+n)-(n+1)+1 = r$ possibilities for the third vertex of the triangle (see figure 2), giving r^2 such triangles. The possible values of r are $1, 2, \ldots, n$, so the total number of such triangles is

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

An alternative way of tackling this problem, used by János Sütö of Kelvin Hall Comprehensive School, Kingston upon Hull, who submitted it, is to determine the number of triangles which do not contain the centre of the circle and subtract this number from the total number of triangles on the $2n+1$ vertices.

19.5 Evaluate the infinite product

$$\sqrt{\tfrac{1}{2}} \times \sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}} \times \sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}} \times \ldots$$

Solution by Amites Sarkar (Winchester College)

We have the rules

$$\sin a = 2\sin\tfrac{1}{2}a\cos\tfrac{1}{2}a, \qquad \cos\tfrac{1}{2}a = \sqrt{\tfrac{1}{2}+\tfrac{1}{2}\cos a},$$

so that

$$\begin{aligned}
1 &= \sin\tfrac{1}{2}\pi \\
&= 2\sin\tfrac{1}{4}\pi\cos\tfrac{1}{4}\pi \\
&= 2^2\sin\tfrac{1}{8}\pi\cos\tfrac{1}{8}\pi\cos\tfrac{1}{4}\pi \\
&= 2^3\sin\tfrac{1}{16}\pi\cos\tfrac{1}{16}\pi\cos\tfrac{1}{8}\pi\cos\tfrac{1}{4}\pi \\
&= \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
&= 2^n\sin\frac{\pi}{2^{n+1}}\cos\frac{\pi}{2^{n+1}}\cos\frac{\pi}{2^n}\ldots\cos\frac{\pi}{8}\cos\frac{\pi}{4} \\
&= 2^n\sin\frac{\pi}{2^{n+1}}\sqrt{\tfrac{1}{2}} \times \sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}} \times \sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}} \times \ldots \quad (n \text{ terms}).
\end{aligned}$$

Thus the infinite product

$$\begin{aligned}
\sqrt{\tfrac{1}{2}} \times \sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}} \times \sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}} \times \ldots &= \lim_{n\to\infty}\frac{1}{2^n\sin\dfrac{\pi}{2^{n+1}}} \\
&= \lim_{n\to\infty}\frac{\dfrac{\pi}{2^{n+1}}}{\sin\dfrac{\pi}{2^{n+1}}} \times \frac{2}{\pi} = \frac{2}{\pi}.
\end{aligned}$$

I believe that this is a famous result of Vieta.

19.6 Find all positive integers x for which $d(8x) = 4p$, where p is a prime and $d(n)$ denotes the number of positive divisors of the positive integer n.

Solution by Adrian Hill (Trinity College, Cambridge)

Write $x = 2^{\alpha_1}3^{\alpha_2}\ldots p_n^{\alpha_n}$ in its prime factorization, so that $8x = 2^{\alpha_1+3}3^{\alpha_2}\ldots p_n^{\alpha_n}$. A positive divisor of $8x$ will be of the form $2^{\beta_1}3^{\beta_2}\ldots p_n^{\beta_n}$, where $0 \leqslant \beta_1 \leqslant \alpha_1+3$ and $0 \leqslant \beta_i \leqslant \alpha_i$ for $2 \leqslant i \leqslant n$. Thus the number of positive divisors of $8x$ is $(\alpha_1+4)(\alpha_2+1)\ldots(\alpha_n+1) = 4p$. There are five possibilities:

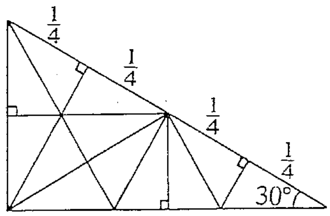
(i) $\alpha_r = p-1$ for some $r > 1$ and $\alpha_i = 0$ for all other i, giving $x = p_r^{p-1}$;
(ii) $\alpha_1 = 4p-4$ and $\alpha_i = 0$ for $i > 1$, giving $x = 2^{4(p-1)}$;
(iii) $\alpha_1 = 2p-4$, $\alpha_r = 1$ for some $r > 1$ and all other $\alpha_i = 0$, giving $x = 2^{2(p-2)}p_r$;
(iv) $\alpha_1 = p-4$, $\alpha_r = 3$ for some $r > 1$ and all other $\alpha_i = 0$, giving $x = 2^{p-4}p_r^3$;
(v) $\alpha_1 = p-4$, $\alpha_r = \alpha_s = 1$ for some r and s $(2 \leqslant r < s)$ and all other $\alpha_i = 0$, giving $x = 2^{p-4}p_r p_s$.

Note that (iv) and (v) do not apply when $p = 2$ or 3 and that (iii) reduces to (i) when $p = 2$.

Also solved by Martin Anthony, University of Glasgow.

19.7 Twenty-five points are given in a right-angled triangle whose smallest angle is 30° and whose hypotenuse has length 1. Prove that three of the points can be chosen so that they lie in a semicircle, the diameter of the whole circle being no larger than 0.29.

Solution by Adrian Hill (Trinity College, Cambridge)



The triangle can be split up into twelve congruent right-angled triangles as shown. Thus at least one of the smaller triangles must contain three of the given points, either within it or on its boundary. The length of the longest side of each small triangle is $\frac{1}{6}\sqrt{3} < 0.29$, so three of the points must lie in a semicircle, the diameter of the whole being no larger than 0.29.

The 1987 problem

In Volume 19 Number 2 page 36, we posed our annual puzzle, to express the numbers 1 to 100 in terms of the digits of the year in order, using only the operations $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, !, brackets and concatenation (i.e. forming 19 from 1 and 9, for example). We failed with nine of the numbers. Many readers have sent in solutions, sometimes breaking the rules of the game. For example $88 = 1^9+87$ uses powers, and so is not allowed, some readers used the digits out of order or failed to use them all, and we did not intend concatenation to include $100 = 1\sqrt{9}+87$, for example! A number of readers sent in

$$29 = (1\div\sqrt{9})\times 87$$

which we at the editorial office did not get. Readers who supplied this were Sacha Cole (Cherwell School, Oxford), Andrew Weir, Alex Newman and John Harrison (Winchester College) and Anna-Marie Gerrard and Anne Keast-Butler (The Perse School for Girls, Cambridge).

Book Reviews

Probability: an Introduction. By GEOFFREY GRIMMETT AND DOMINIC WELSH. Clarendon Press, Oxford, 1986. Pp. ix + 211. £9·95 paperback, £20 hardback.

Probability, which impinges on our lives in so many ways, can be developed as applied mathematics or as pure mathematics. In the first case there is a difficulty of definition, as we have to avoid the circular one of considering combinations of 'equally likely' events. In the second case one can easily define probability as a measure attached to elements of an event space; this is the approach the authors of this book adopt, and they go on, very successfully, to develop the theory through random variables, generating functions, distribution and density functions, multivariate distributions, moments and limit theorems. They get as far as branching processes, random walks and random processes.

The difficulty with the pure mathematics approach is making the connection with probability in real life, as we estimate it and are affected by it; if this is not done then a lot of so-called probability is really combinatorics in disguise. Books such as Feller's devote considerable space to making the connection, but Grimmett and Welsh almost entirely avoid the issue, although many of their examples are given realistic settings in terms of card games, bus queues, etc. On page 3 they mention 'assessment of the likelihoods of these events' but they do not define likelihood (and obviously are using it in its non-technical sense). On page 124 they mention throwing a fair die a million times, but add that they actually used a computer. What they are reporting in fact is the behaviour of a particular pseudo-random-number generator. This lapse apart, the book is very honest; when a result can be proved easily it is proved, and when it cannot references are given. There is no glossing over difficulties.

There are many excellent examples in the book but answers are given to only some of them; Eddington's Controversy (problem 1.16) surely deserved an answer. Also the comment 'this is rather horrible' to problem 6.15(iii) is not very helpful.

A good undergraduate will find this book a very useful supplement to lecture courses; a weaker one will need considerable tutorial help with it.

Royal Holloway and Bedford New College H. J. GODWIN

The Fascination of Statistics. Edited by R. J. BROOK, G. C. ARNOLD, T. H. HASSARD and R. M. PRINGLE. Marcel Dekker, Inc. 1986. Pp. xi + 456. \$29·50.

This book sets out to show the reader a wide range of applications of probability and statistics; not just statistics as the title implies. It comprises a collection of 30 papers under the following headings: probability, condensing complex data, hypothesis testing, estimation, experimental, prediction and modelling. The 35 contributors come from a variety of disciplines, e.g. statistics, psychology, medicine, city planning and law, and the topics of the papers reflect this broad base.

Most papers describe a practical problem (e.g., Is a new milk drink finding the market for which it was created?) requiring techniques in probability or statistics for its solution, and provide a discussion of the solution. Some problems are solved using procedures familiar to students with a limited experience of probability and statistics. Other problems, however, require more advanced procedures. Generally the concepts of these advanced techniques are presented clearly and without cumbersome detail, revealing to readers the technique's power for problem-solving.

Readers will find the papers covering familiar techniques interesting and easy to read. I believe, however, that even readers with a limited knowledge of the subject will find something of interest in the papers reporting, in simple terms, the application of the unfamiliar techniques, giving them some insight into the more powerful tools available to statisticians.

The remaining papers are more discursive, revealing something of the history, development and current state of particular topics. These papers will almost certainly appeal less to those recently introduced to the subject. However, they provide an interesting discussion of the development and evolution of these selected topics.

This book is nicely produced with a good number of cartoon-type illustrations adding a 'further dimension' (from the preface) to the papers. I was aware of a scattering of typographical errors, e.g. page 82 the summation sign in the definition of $d(x, y)$ is incorrectly placed, and in the third paragraph on page 85 a $\varnothing$ should be a θ. Additionally, I am unhappy about the confidence interval for the regression line presented on page 284. However, overall, this book belongs on the library shelf of educational establishments teaching probability and statistics, so that students can dip into the book and see where their studies may lead. Many papers also provide ideas and material which might well be adopted for class or tutorial practical sessions or discussion.

University of Sheffield G. M. CONSTABLE

Thirty Years That Shook Physics—The Story of Quantum Mechanics. By GEORGE GAMOW. Dover 1985. Pp. xiv+224. £4·45.

This book is a Dover reprint of a book first published in 1966. It comes from a well-respected author of many books, including *Mr Tompkins in Wonderland*: those who have looked at any of the Mr Tompkins books will appreciate the very characteristic style. The author has mixed a series of amusing anecdotes about well-known physicists and mathematicians with his serious intent of showing the development of quantum mechanics from about 1900 to 1930. He was himself a part of this exciting development so he was in an excellent position to comment on the ideas and how they came to fruition. Although Professor Gamow has tried to make the technical arguments simple, some of the concepts are not easy and parts of the book need very careful thought and study.

If I were looking for a semi-historical book on quantum mechanics with a flavour of the ideas rather than the detailed technical knowledge, this book would be perfect. It is thoroughly enjoyable to read and gives an insight into the characters of the leading actors on the stage in the development of quantum mechanics.

University of Sheffield D. M. BURLEY

A Number for Your Thoughts. By MALCOLM E. LINES. Adam Hilger Ltd, Bristol, 1984. Pp. 214. £4·95 paperback.

You do not need to know very much mathematics to be able to enjoy this book; only an elementary knowledge of our number system is required. You will rapidly discover how easy it is to read, unlike many mathematics books, and I am sure that a quick glance among its pages will soon convince you that they contain many amazing and unlikely patterns which have intrigued people for many centuries.

The book starts by taking us right back in the past to the actual formation of number systems, and even includes an explanation of how to count in base -10! It then goes on to look at a variety of properties of prime numbers. Included is a section telling you how, in the 1970s, two 18-year-old students became the world record holders for finding the (then) largest known prime number, and we are encouraged to investigate various ideas ourselves. We are then introduced to the 'Baffling Law of Benford' and to *self*, *perfect*, *friendly* and even *weird* numbers. If you had never realised that we had such names for the numbers in our number system then you should make a point of reading the appropriate chapters in this book to find out what they are. Our attention is drawn to some very unusual series, which you are unlikely to have met at school. They are very simple to form, but still present today's mathematicians with some unsolved problems. You will be familiar with Pythagoras' Theorem and will be able to find some whole numbers a, b and c which satisfy the equation $a^2+b^2 = c^2$; but how many of you have considered trying to find solutions to $a^n+b^n = c^n$, where $n > 2$? It is a problem which has intrigued mathematicians for several centuries, ever since Fermat claimed to have proved that no such solutions with $a, b, c \neq 0$ exist (Fermat's Last Theorem).

Although there is a chapter on magic squares and cubes, it serves as no more than a brief introduction as so much has already been written on this topic elsewhere. However, it does give an indication of how difficult it was to find a magic cube at all. This theme is carried on in the next few chapters. They show that, in spite of the initial difficulties in finding numbers satisfying certain criteria, once a small breakthrough has been made in the right direction there are often far more numbers satisfying the condition than not.

You may by now be under the impression that this book is concerned only with whole numbers, but this is not the case. The following chapters bring into use fractions, irrationals and even complex numbers. The final chapter takes us even further and talks about infinity. As you read this chapter, you will realise that today's mathematicians are not content with just one infinity. They have several, some of which are bigger than others. This leads to some unexpected results that the average sixth former may find hard to accept, although the mathematics involved is not difficult.

The book concludes with a short update about the work done by Gerd Faltings on the Mordell conjecture, which is related to Fermat's Last Theorem. Gerd Faltings has, since this book was published, received the Field's Medal (an honour equivalent to the Nobel Prize). You will find other unsolved problems in this book. Who knows—you may, one day, achieve similar fame!

Supply Teacher, Sheffield LEA JOYCE PORTEOUS

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