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Articles are normally commissioned by the Editors; the Editorial Committee also welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. All correspondence about the contents should be sent to:

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The Mysterious Atmosphere

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We live in a mysterious physical environment, the most important component of which for man's daily life is a rotating spherical shell of air, whose thickness is extremely small compared to its radius. Most of us live entirely in this shell and so we condition ourselves to its vagaries fairly philosophically, apart from complaining about it in day-to-day conversation! Although our daily lives depend sensitively on its oscillations, we tend to accept, without thinking much about it, that for example, it is sometimes warm in the U.K., and sometimes cold—almost independently of the month of the year. We realise of course that at certain times the atmosphere gathers itself up, particularly in low latitudes, into systems of extreme concentration of kinetic energy with consequent catastrophic damage to human societies, but we soon relapse into apathy; many of us feel that the weather forecast for tomorrow sometimes turns out to be a reasonable prediction for our outdoor activities, and sometimes not! Perhaps after all, the retired gentleman in Newton Abbot with his statistical system, mentioned in the House of Commons a year or so ago, can in some strange way perform just as well as any science-based prediction. At the same time we get excited about the possibilities of exploration of the planets by expensive pieces of equipment in the form of artificial satellites, and few realise that at close hand, in our own thin spinning shell of air, we have a fascinating unsolved scientific problem.

We are all aware that we live in a world of widespread inherent instabilities which are difficult to control, whether we think of human societies and their associated economic systems or of the weather. These systems are unstable in the sense that small disturbances tend to increase at a very rapid exponential rate. Of course there is an upper limit to this growth, at least in the atmosphere if not in human societies! Small disturbances tend to congregate into long waves in the atmosphere (wavelengths of the order of 4,000 to 5,000 km) in approximate horizontal planes, forming at their crests closed circulations which develop into cyclones and depressions (regions of comparatively low pressure). But why should they have the characteristic dimensions of the Atlantic and Pacific systems? Why should they have a lifetime of about two weeks or so before they die by merging into other systems? Why should we in the U.K. complacently assume that the vast amount of ice which is found between latitude 72° and the Pole will always remain as it is? Will this soon melt, as was recently claimed by a well-known Russian geophysicist, because of the exponential increase of man's technology and the associated large-scale pollution of the atmosphere?

The only incisive approach to these questions surely lies in the classical methods of Applied Mathematics. The atmosphere is an excellent subject for mathematical-numerical modelling. We begin of course with the complex reality then extract what we consider to be the main factors in it (this process is partly subjective and naturally controversial in the early stages of model development); the mathematical description of these factors and their relationships then leads to a purely mathematical problem. Generally this problem consists of solving time-dependent, non-linear partial differential equations as these are a natural expression of spatial and temporal atmospheric variations. The solution can be used to reveal new features of the physical structure which the complex details of reality sometimes initially hide from us, and, with the development of new ideas, further component parts are parametrised in mathematical terms; a hierarchy of models, as it were, is developed reaching towards the essential core of the problem.

So the first stage of the Applied Mathematician's work is to study intensively the basic physics of the problem. We all know how complex the flow systems are by looking at the various cloud structures so beautifully pictured by satellite signals.

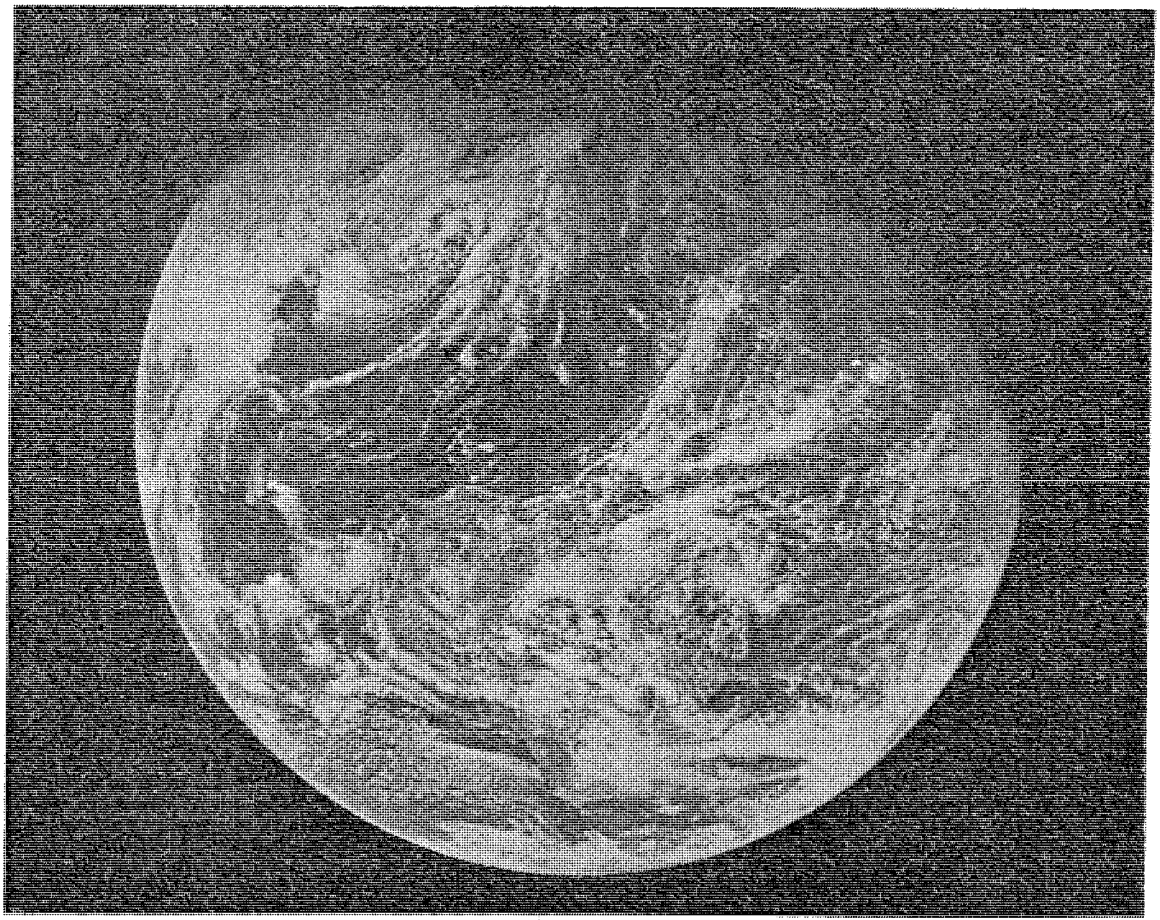

Figure 1. The earth from Apollo 8, 21 December 1968. South America is left of centre. A sub-tropical high pressure belt and inter-tropical convergence zone can be seen, together with a 'large eddy' (depression) in the Atlantic.

Two good examples are shown in Figures 1 and 2. These are of great use in weather forecasting; they track the cyclonic systems over areas such as the Atlantic and Pacific, where there are only a few sea-level observational stations. They are also used in hurricane and typhoon 'dodging' by ships and in giving early warning of

Figure 2. The earth from Space Station Zond 5, altitude 90,000 km, 21 September 1968, 12.08 hours Moscow Time. Africa and Arabia can be seen right of centre.

the impending arrival of these systems to coastal regions. The extreme physical complexity of the global problem is illustrated by a simple statistic; at any one time there are over the earth's surface (on average) 50 depressions, 200,000 showers, and 2,000 thunderstorms! But there is also a clearly visible structure. There is, for example, the well-marked cloudless band, almost encircling the globe (the subtropical high pressure zone) in low latitudes. In the spring and autumn there are two, one in the northern and one in the southern hemisphere; in the winter and summer there is generally only one. There is the clearly marked mid-latitude zone of long cyclonic waves (called baroclinic waves) developing cyclonic centres at the crests of the wave and moving generally from West to East. Indeed, if we could imagine the atmosphere to be marked out by a vast cluster of balloons on the global scale, we would have the impression of a gigantic vortex, centred approximately at the pole, and spinning generally with westerly winds relative to the earth's surface.

There is an excess of heat from solar radiation at the equator; this is being continuously syphoned off to the pole where there is a heat deficit. The wind, thus created, is deflected to the right in the northern hemisphere to form the Westerlies. This flow is, however, unstable if the gradient of temperature between equator and pole exceeds a certain critical value, and the cyclone wave system is formed. An inherent property of the system is that it conveys heat to high latitudes and so reduces the global, pole to equator, temperature contrast. At a stage when the pole has been well injected with warmer air, the large-scale meridional temperature contrast decreases and the baroclinic wave activity subsides until the contrast again builds up past the critical stage. In the vertical direction the temperature variation is generally stable (in that any vertical disturbance is returned to its original position), but in certain singular areas, particularly where warm and cold air meet in the turmoil of what are called 'fronts' (or battlegrounds), then vertical large-scale extensive upward convection takes place. At the equator the converging air from northern and southern hemispheres meets over the very hot sea surface producing a vertical upward breakthrough of heat and vapour along what is called the intertropical convergence zone. The subsequent condensation of water vapour in the upper atmosphere releases enormous amounts of energy, initiating a large-scale poleward movement of heat and momentum which is later taken over by the baroclinic waves.

The initial attempt to formulate and solve the mathematical problem posed by a model of the atmosphere was made during the first World War by L. F. Richardson, a British mathematician. He had been carrying out research on the approximate solution of non-linear partial differential equations by finite difference methods, and he applied his numerical technique to a sample 6-hour period of integration. In those days it all had to be done by hand and he did this during 'rest' periods between carrying the wounded during battles. At one stage he actually lost his working papers, but fortunately they were discovered some months later under a heap of coal. In 1921 his work was published in the form of a book on *Weather Prediction by Numerical Processes*. In this he put forward the then fantastic idea that we needed to collect together 64,000 computing human beings in a large hall in order to be able to compute the equations with sufficient speed to keep ahead of the weather. The equivalent to this Lewis Carroll idea has now of course appeared in the form of the new computer at the British Meteorological Office. Although Richardson's actual example of a computed weather change was quite unlike the real thing in 1921, it has always been the belief of mathematical meteorologists that the mathematical method was the only one likely to succeed in breaking through the problem of understanding large-scale atmospheric flow and of making accurate predictions. Richardson's exercise was not accurate, partly because there were not enough accurate observations, but primarily because numerical techniques were then in their infancy.

Two new factors have changed the position dramatically since about 1950. One is the rapid development of instrumentation. The use of satellites to collect meteorological data is one of the most sensible applications of modern technology. It has recently been developed to a point where from radiation measurements, it is possible to deduce accurate estimates of the variation of temperature with height from the sea surface to satellite height (100 miles, say). With the use of a sophisticated global balloon system it will soon become possible to obtain the instantaneous global distribution of the main relevant variables needed to specify initial conditions; from these, accurate forward time integrations for several days ahead can be made. The other revolutionary development is in the computer technology field. It is now possible to integrate, for long periods, systems of partial differential equations by finite difference methods, and to reproduce the main flow development characteristics associated with the birth of successive depressions—even unto the third generation!

However, the detailed forecast for tomorrow is on occasion still no further advanced than in the past; there is still, for example, the unsolved problem of the exact timing of an incoming system of Atlantic depressions when they are being retarded by a continental high pressure barrier. The interaction of these large-scale systems poses a very interesting dynamical-mathematical problem, of great practical importance. There is also the unsolved problem, posed by the interaction between the surface boundary layer (dominated by the effect of surface friction responsible for the small-scale eddies near the earth's surface) and the 'free' atmosphere (characterised by very much larger eddies) on the scale of the depressions. It may well turn out to be necessary to develop a better formulation of this layer phenomenon before really reliable forecasts of several days become possible. It may also be necessary to arrive at a better understanding of the dynamics of fronts, whose detailed dynamical structure has been recently revealed by some very interesting observational work at the British Meteorological Office. So, despite much recent progress, the atmosphere still presents fascinating unsolved problems; indeed this is its essential attraction to Applied Mathematicians.

I would now like to conclude this article by presenting a problem for discussion; it is not only an interesting problem but one on which the future of man himself may depend. The problem is this: given that man (primarily in mid-latitudes) will continue to increase his industrial activity exponentially (with the associated increase in atmospheric pollution), will this lead to a large-scale melting of the polar ice, or alternatively to an expansion of the present ice limit to lower latitudes by, say, 1984? In other words, are we heading for another Ice Age or a Flood?

Of course a reliable solution can only be based on a sophisticated physical-mathematical model, but the essence of the problem can be expressed simply in qualitative physical terms. Let us first suppose that there is no interference by man and consider the approximate balance as it has existed during the present century.

Suppose there is an increase, $+\delta A$, in the area of the polar ice, leading to an increase, $+\delta\alpha$, in the earth's albedo (i.e., reflecting power) in that area (ice reflects back into space more solar radiation than an ice-free sea surface). This leads to a cooling in high latitudes and an increase, $+\delta Q$, in the heating contrast between equator and pole (which as was mentioned earlier, controls the ebb and flow of the baroclinic waves). The increase in heating contrast leads to an increase in depression wave activity, which results in an increase in the transfer of heat to the pole. This in turn melts the polar ice and produces a reduction in the ice area, i.e., $-\delta A$, to balance out our initial postulated increase $+\delta A$. So a quasi-equilibrium is achieved.

However, suppose that man now interferes with the cycle. By increasing mid-latitude pollution, the heating contrast δQ is reduced, so reducing the heat transfer to the pole, and resulting in an *increase* in δA, which will possibly set in train a new Ice Age, if the effect is cumulative. The most difficult component to express in mathematical form is probably the relationship between atmospheric pollution and consequent changes in atmospheric heating. Given an *exponential* increase in man's industrial activity, this could well become a central problem of the atmosphere, requiring urgent solution by 1984.

The New Geometry

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Many people, including some eminent mathematicians, believe that geometry is no longer an important subject. A few go further and say that it is dead. There are mathematicians who admit, sometimes with pride, that their knowledge of geometry is slight. Yet geometry was once the most important branch of mathematics.

Why has it declined? It is not easy to answer this question. Perhaps modern mathematicians cannot do geometry, or perhaps the abstract theories evolving from geometry, and replacing it, are so general that spatial intuition is no longer an asset.

Another possibility is that geometry as a subject in its own right has suffered because of the invention of coordinates by Descartes in the seventeenth century. The introduction of coordinates made a profound impression on the solution of many geometrical problems, because it changed them into problems in algebra or analysis. This greatly increased the possibilities, but it had the serious disadvantage that it became all too easy for the algebra or analysis to take control and become an end in itself rather than a means to an end. Look at any text-book on coordinate geometry and you will find that very little of it is concerned with geometry. For example, an exercise which requires you to find the equation of the tangent to a circle at a certain point is not an exercise in geometry. To lose sight of this is to forget what geometry is about. Let us use coordinates by all means and let us exploit their invention by incorporating them in systems of great generality, but let us keep them in their proper place.

The decline of geometry, whatever the cause, is to be deplored. However marvellous our modern theories are, we should not cut them off from their roots. Further we have by no means exhausted the possibilities which can arise from spatial intuition and speculations about the nature of physical space; if we abandon geometry we are leaving a task unfinished. To illustrate this I shall try to convince the reader that thinking about space leads naturally to the theory of manifolds, one of the important branches of modern mathematics. If you are a mathematician and manifolds leave you cold, it is probably because your spatial intuition is weak.

The original meaning of the word 'geometry' was 'earth-measurement' and Euclidean geometry was the mathematical theory that grew out of the physical problems encountered in actually measuring. Over the centuries we appear to have led ourselves to believe that three-dimensional Euclidean geometry provides a very good mathematical model of physical space. However, experience suggests that we might well be deceiving ourselves if we think that this is so. After all, people used to think that the earth was flat, and presumably the mathematical model provided by two-dimensional Euclidean geometry was regarded as adequate. We now know better, although we still find that it is useful to regard the earth as at least approximately flat when considering only a small portion of it. It depends on the scale of what we are doing: thus it would be foolish to regard the earth as being approximately flat when considering a flight of 10,000 miles, and it would be absurd to take the curvature of the earth into account when marking out a tennis court.

To move from this two-dimensional illustration of self-deception to its three-dimensional analogue is not easy. Picture, if you can, a two-dimensional creature living on the surface of a cylinder, which you, but not the creature, can readily think of as being in three-dimensional space. The creature observes its surroundings and comes to the perfectly reasonable conclusion that Euclidean geometry of two dimensions is a good mathematical model for the physical space in which it exists. This is because it is very small in relation to the cylinder and has no means of detecting the curvature; nor, for that matter, can it imagine how a two-dimensional space can be curved, even though it is well aware of one-dimensional curves. But, after going through some sort of technological revolution, the creature is able to conduct some experiments which cast doubts in its mind and it finally realises how it has deceived itself when it sets off in what appears to be a straight line but returns to its starting point without changing course.

The experience of mankind with the flat earth idea was not quite the same as this; being three-dimensional creatures ourselves (or so it seems) we should really have been able to think in three dimensions. Our two-dimensional creature does not have this facility. Having demonstrated that its space is not properly represented by two-dimensional Euclidean geometry, it has the very difficult problem of trying to work out a more suitable mathematical model. We in our superior position know that it is going to have trouble if it starts walking along a generator of the cylinder.

It seems almost certainly true that we are in a three-dimensional analogue of this two-dimensional situation, although the analogy is unlikely to be a good one. As yet our experiments have not been on a broad enough scale to confirm this with any certainty and we have no complete idea of what physical space is like. We must remember that all the evidence points to the fact that we ourselves are apparently very insignificant in relation to the universe as a whole. Moreover our measurements of large distances cannot be regarded as being reliable. All that we can say is that our observations and measurements suggest that space is locally Euclidean rather than Euclidean.

What do we mean by this? The answer depends upon our point of view. At this stage, I think that I ought to declare my interest and admit that I wish to put forward the point of view of the pure mathematician. In one sense, this makes things easier for me than if I were trying to be an applied mathematician, for then my aim would be to solve the mysteries of the universe by constructing a mathematical model of it, finding out how the model worked and then interpreting the results. The acid test comes when these are to be checked by experiment, but the pure mathematician is not concerned with this. What he must do is to give precise definitions of the mathematical model and to work out the theory; it is not up to him to decide how close to reality the mathematical model comes. At the same time, he should not cut it off from its roots and he must be aware that inadequacies of the model may lead to a demand for attention to be turned to some alternative.

In an article of this kind, it is not possible to give the precise definitions required for a mathematical theory arising out of the intuitive concept of a locally Euclidean space. That would be too complicated. However I shall try to say more about what the intuitive idea is, in the hope of whetting the reader's appetite.

We suppose that we are dealing with a space of some kind and, as usual in geometry, the basic elements are called points. The space itself is called a manifold, which stems from the fact that we can regard it as obtained by fusing together several different Euclidean spaces. For convenience we restrict ourselves to the case in which these Euclidean spaces are two-dimensional.

We assume that for each point P there is a portion N of the space which contains P and which is in one–one correspondence with a Euclidean plane. Let us call N the basic neighbourhood of P. This depends upon P; different points may have different basic neighbourhoods. To make our calculations possible we use coordinates, but recall that they must be kept in their proper place. Thus we can attach coordinates (x, y) to each point in the basic neighbourhood N of P. We do this by considering our one–one correspondence ϕ between N and the Euclidean plane; we attach to the point Q in N the coordinates of $\phi(Q)$.

Now suppose that P' is a point in N different from P. Like P, this point P' has a basic neighbourhood N', which can be regarded as a Euclidean plane and whose points can be given coordinates. Since P' is in both N and N', these two portions of the space overlap. The situation is thus something like that shown rather crudely in Figure 1. Here, of course, we are not representing the whole space, but

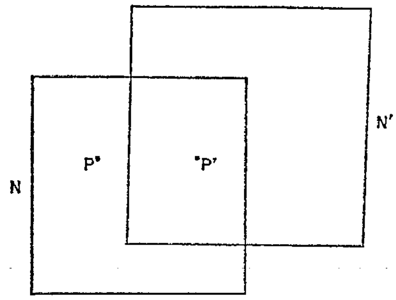

Figure 1

only two portions of it. Since P' is in N, we can attach coordinates to it; suppose these are (a, b). In general, these are not the same as the coordinates (a', b') which we can attach to P' in virtue of its membership of N'. If we now consider all the points which lie in both N and N', we see that they can be given coordinates in two different ways. If we know how the coordinates in N and N' are assigned, then we can work out how the different coordinates for the common points are connected, and we obtain what is known as a coordinate transformation. In Figure 2, we

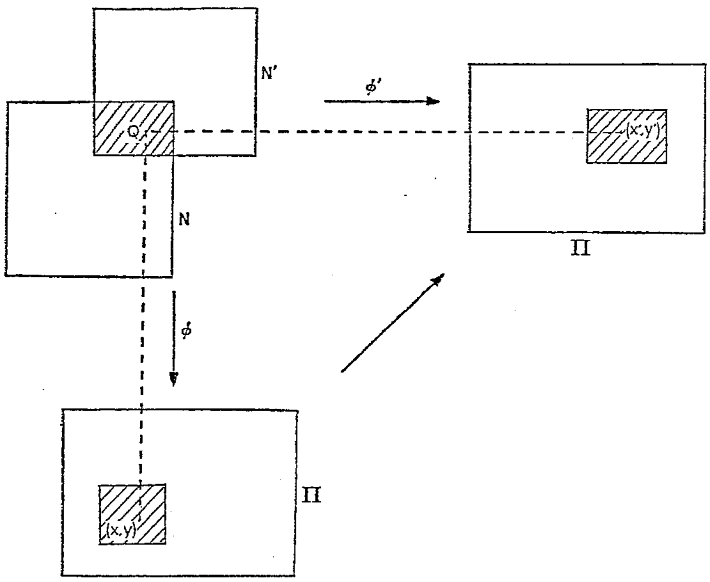

Figure 2

represent this diagrammatically. The vertical arrow represents the one–one correspondence ϕ between N and the Euclidean plane Π; similarly the horizontal arrow represents the one–one correspondence ϕ' between N' and Π, which is drawn twice in the figure to allow more scope. For a point Q in N and N', suppose that $\phi(Q)$ is the point (x, y) in Π and that $\phi'(Q)$ is the point (x', y'). To get from

(x, y) to (x', y'), we first apply ϕ^{-1}, the inverse of ϕ, in order to get Q, and then apply ϕ'. Then we find that

$$(x', y') = \phi'(\phi^{-1}(x, y)).$$

Thus we have the idea of the transformation of coordinates from N to N'. This transformation is just the composition $\phi' \circ \phi^{-1}$ defined on the shaded region in the lower copy of Π and taking it onto the shaded region in the other copy. These shaded regions are the parts corresponding to the intersection of the two basic neighbourhoods, which is also shaded in Figure 2.

In the full definition, the possible transformations of coordinates are subject to certain restrictions, but we shall not discuss the technicalities.

In the theory of manifolds, what interests us most is what we can say about our spaces as a whole. For example, we may consider the problem of constructing vector fields. By a vector field on the Euclidean plane Π we mean a family of vectors, one defined at each point of Π. Such a family can be obtained by taking tangents to a family of curves having the property that through each point of Π there passes precisely one curve of the family. We can transfer such a vector field to a basic neighbourhood N by simply matching the points of Π and N according to the one–one correspondence ϕ. Having done this, let us consider another basic neighbourhood N' which, as above, has some points in common with N. In the common part of N and N', we can express our vector field in terms of the coordinates in N' and then consider the question of extending it to the whole of N'. We may or may not be able to do this in a reasonable manner; if we can, then we ask whether it is possible to continue the process until the whole manifold is covered.

A simple example of a manifold is the surface of a sphere in three-dimensional Euclidean space. This surface as a whole is not in one–one correspondence with the Euclidean plane in a reasonable way as far as our concept of manifold is concerned. However, if we remove just one point, then the rest can be put into one–one correspondence with Π by the process known as stereographic projection. Using this idea, we can cover the sphere with two neighbourhoods. It should be observed that, although these are matched with Π, they do not possess all the properties of Π and in fact are curved versions of it. An interesting property of the sphere is that we cannot put a vector field on it in a continuous manner (that is, having no sudden changes of magnitude or direction) in such a way that every individual vector is non-zero. Thus, if an animal like a porcupine rolls itself up into a ball, and all its quills are then brushed flat, there must be a crown somewhere. If it could manage to contort itself into the shape of a ring, the situation would be different; we can put a continuous vector field on a ring-shaped surface in such a way that every individual vector is non-zero.

We have expressed our ideas loosely and have simplified the situation in several ways. For example, we have confined our attention to the case of a manifold of dimension two, because we have taken our basic neighbourhoods to be planes. However we have given all the necessary ingredients for starting the theory of manifolds. To study this theory properly requires a certain amount of mathematical experience. In particular it is necessary to know some topology.

Whether the physical universe can be regarded as a manifold has not yet been convincingly established. However, many physical problems can be described in terms of manifolds as suitable underlying spaces. Thus applied mathematicians cannot really afford to ignore them. For the pure mathematician there are plenty of unsolved problems to keep the subject well and truly alive. But to be fully conversant with what is going on, he must have a good geometrical background. He must not join the band wagon of those who believe that geometry is in decline.

The Legend of Leonardo of Pisa

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One of the most interesting chapters in the history of mathematics is the almost legendary story which has grown up around Leonardo of Pisa (1175–1225), better known by his nickname Fibonacci (i.e. the son of Bonaccio). Leonardo, the most outstanding mathematician of the Middle Ages, made many valuable contributions to mathematics. In his book *Liber Abaci*, written in 1202, he explained the advantages of a decimal number system over the clumsy Roman system which was then in use in Europe. This book, above all else, was responsible for the adaptation of the decimal system in Europe. However, Leonardo is best remembered today not for his important mathematical work, but for a trivial exercise he included in *Liber Abaci*. Let us see, then, what must be one of the most consequential exercises ever devised to rack a student's brain.

Rabbits are assumed to breed as follows. A pair of rabbits during its first month of life produces no offspring, but at the end of the second and each subsequent month it produces one new pair. Starting with one newly born pair, how many pairs will there be at the beginning of months 1, 2, 3, ... ? It is easily found that the numbers required form a sequence, whose first nineteen terms are listed (for reference purposes) below:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181.

For more than six centuries neither the problem nor its solution received any attention; maybe everyone was too busy containing the rabbits! Then at the beginning of the nineteenth century Edouard Lucas, a French mathematician, began research into the above sequence of numbers, which he christened the 'Fibonacci numbers'. From that time until the present day the amount written about the

Fibonacci numbers has increased at a rate comparable with that of those famous thirteenth-century rabbits themselves. Indeed, in 1963, a Fibonacci association was formed and is now responsible for publishing the *Fibonacci Quarterly*, a journal which contains the latest discoveries about these mystical numbers. Whereas much of what has been written on the subject has been more akin to fiction than to fact, we shall take a brief look at some of the valid reasons why the Fibonacci numbers have attracted so much attention.

The sequence of Fibonacci numbers can be defined without recourse to the rabbit problem by stating the first two numbers and noting that each subsequent number is the sum of the two preceding ones. Thus if we use u_n to denote the nth Fibonacci number, then $u_1 = u_2 = 1$ and

$$u_{n+2} = u_{n+1} + u_n \quad (n \geqslant 1). \tag{*}$$

The relation (*) does not provide a practical means of calculating the Fibonacci numbers, for to calculate a particular Fibonacci number it is first necessary to calculate all of the preceding ones. To overcome this difficulty, we obtain an explicit formula for the nth Fibonacci number. We begin by finding a formula for any u_n which satisfies (*). Note that we do not initially insist that $u_1 = u_2 = 1$, so that the u_n we determine need not be the Fibonacci numbers. As a first guess we assume that $u_n = x^n$, where x is to be determined. If $u_n = x^n$ satisfies (*), then

$$u_{n+2} - u_{n+1} - u_n = x^n(x^2 - x - 1) = 0,$$

whence (ignoring the uninteresting case $x = 0$) we have $x^2 - x - 1 = 0$, i.e. $x = (1 \pm \sqrt{5})/2$. Thus $u_n = ((1+\sqrt{5})/2)^n$ and $u_n = ((1-\sqrt{5})/2)^n$ both satisfy (*) and from this it is seen that, for any constants A and B,

$$u_n = A((1+\sqrt{5})/2)^n + B((1-\sqrt{5})/2)^n$$

also satisfies (*). We now use the disposable constants A and B to arrange that $u_1 = u_2 = 1$. For this to be so, we must have

$$u_1 = A((1+\sqrt{5})/2) + B((1-\sqrt{5})/2) = 1 \quad \text{and}$$
$$u_2 = A((1+\sqrt{5})/2)^2 + B((1-\sqrt{5})/2)^2 = 1.$$

On solving these (rather nasty) simultaneous equations for A and B, we arrive at the remarkable formula for the nth Fibonacci number:

$$u_n = (1/\sqrt{5})(((1+\sqrt{5})/2)^n - ((1-\sqrt{5})/2)^n).$$

We observe that $(1/\sqrt{5})((1-\sqrt{5})/2)^n$ is less than $\frac{1}{2}$ in absolute value and so u_n is the nearest integer to $(1/\sqrt{5})((1+\sqrt{5})/2)^n$. This observation provides a useful method of calculating Fibonacci numbers.

Surely one of the most pleasing features of the Fibonacci numbers is the ease with which it is possible to discover many of their properties. As an example let us look at the list of Fibonacci numbers printed above and ask which Fibonacci numbers are even. A few seconds' thought tells us that each third Fibonacci number is even, and this is not difficult to prove. We leave it to the reader to discover for himself which Fibonacci numbers are divisible by 3, 4, 5, etc. Not all the properties of these numbers are as easily found as the ones we have just indicated, and we now consider a less obvious property. We shall see how the nth Fibonacci number is related to the Fibonacci numbers corresponding to the factors of n. We illustrate our point by examining the 18th Fibonacci number $u_{18} = 2{,}584$. The non-trivial factors of 18 are 2, 3, 6, 9 which have associated Fibonacci numbers $u_2 = 1$, $u_3 = 2$, $u_6 = 8$, $u_9 = 34$ all of which are factors of u_{18}. Nor is this just a happy coincidence for it can be shown that if m is a factor of n, then u_m is a factor of u_n. The reader should test this result for himself by selecting suitable values of m and n. Notice in particular that this result tells us that if n is divisible by 3 then u_n is divisible by $u_3 = 2$, which is the conclusion we came to at the beginning of this paragraph. If n contains a factor m, where $2 < m < n$, then u_n will contain a factor u_m, where $2 \leqslant u_m < u_n$, which shows that u_n is not a prime number. For example, we know that u_{99} cannot be a prime number for it has a factor $u_3 = 2$. Thus, if $n \neq 4$ and u_n is a prime number, then so is n. We might be tempted to think that u_n is a prime number whenever n is one. However, this is not so, for $u_{19} = 4{,}181 = 37 \times 113$ is not a prime number. Euclid showed that there is an infinite number of prime numbers, but no one knows whether or not there is an infinite number of primes in the Fibonacci sequence.

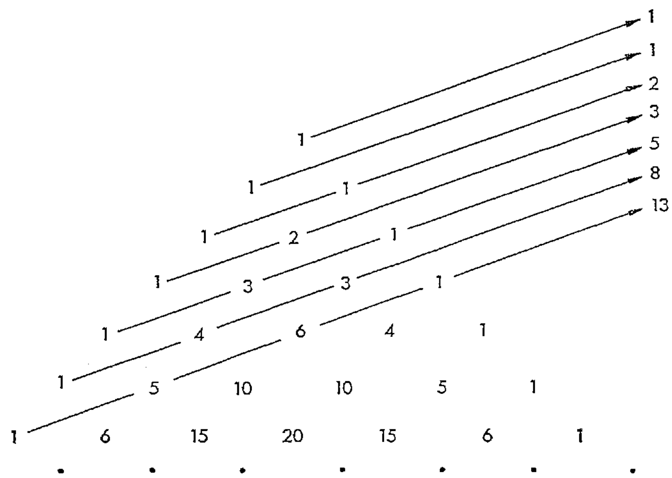
We now mention two problems which remained unsolved for over a century and whose solutions have been given in recent times. These problems concern the occurrence of squares and cubes in the Fibonacci sequence. The only squares in the above list of Fibonacci numbers are 1 and 144, the only cubes are 1 and 8. For many years it was conjectured that 1 and 144 were the only Fibonacci squares and that 1 and 8 were the only Fibonacci cubes. However, it was not until 1953 that the conjecture concerning the squares was proved true and not until 1968 that the conjecture concerning the cubes was established. Many unsolved problems still remain and perhaps in a few years' time some of them will be solved by readers of this article.

As a further easily proved result we shall find a formula for the sum of the first n Fibonacci numbers. Let $S_n = u_1 + \ldots + u_n$ denote the sum of the first n Fibonacci numbers so that $S_1 = 1$, $S_2 = 2$, $S_3 = 4$, $S_4 = 7$, $S_5 = 12$, etc. We note that each of S_1, S_2, S_3, S_4, S_5 is one less than each of u_3, u_4, u_5, u_6, u_7 and we are led to conjecture that $S_n = u_{n+2} - 1$. This conjecture is true as we now show. To do this we invoke the 'method of differences', which is frequently used in the summation of series, and the relation $u_n = -u_{n+1} + u_{n+2}$. Now

$$\begin{aligned} S_n &= u_1 + u_2 + \ldots + u_{n-1} + u_n \\ &= (-u_2 + u_3) + (-u_3 + u_4) + \ldots + (-u_n + u_{n+1}) + (-u_{n+1} + u_{n+2}) \\ &= u_{n+2} - u_2 = u_{n+2} - 1. \end{aligned}$$

We invite the reader to prove that the sum of the squares of the first n Fibonacci numbers is $u_n u_{n+1}$.

The Fibonacci numbers have the habit of cropping up in the most unlikely places, both within and outside mathematics itself. As one of their appearances within mathematics, we mention their occurrence as diagonal sums in the Pascal triangle. The figure below shows how the sums of the elements in certain diagonals of the Pascal triangle give the Fibonacci sequence.



The ancestry of the bee provides an amusing example of how the Fibonacci numbers occur in non-mathematical situations. When two bees mate only some of the female's eggs are fertilised by the male. The fertilised eggs hatch into female bees, either queens or workers, and the unfertilised eggs hatch (surprisingly) into male bees, called drones. Thus a male bee has one parent and a female bee has two parents. In addition to its two parents a female bee has three grandparents, five great-grandparents and eight great-great-grandparents. If the family tree of a female bee is traced backwards it will be found that in any particular generation, the number of male ancestors, the number of female ancestors, and the total number of ancestors are consecutive Fibonacci numbers.

Whenever we look at a pineapple, a pine cone, or a sunflower we are brought face to face with the Fibonacci numbers' most surprising appearance. If we examine the surface of a pineapple, we will see that it is composed of hexagonal cells which are arranged in two families of spirals around the pineapple: there will be one family of thirteen spirals sloping steeply up in one direction and a second family of eight spirals sloping less steeply up in the opposite direction. A trained eye will also detect a family of five spirals which circle a pineapple horizontally. A similar manifestation is found in the two families of spirals which emanate from the top of a pine cone. The numbers of spirals in the two families are consecutive Fibonacci numbers, (3, 5), (5, 8), (8, 13), (13, 21) being frequently occurring combinations. The particular combination which occurs depends upon the species of tree which produced the cone. To convince even the most doubting reader that some *real* connection between the Fibonacci numbers and plant life does exist we consider the sunflower. The face of a sunflower consists of tightly packed disc florets which form two families of intersecting spirals originating from the centre. For an average size sunflower the numbers of spirals present are 34 and 55, whereas for a giant sunflower the numbers become 55 and 89. In the early 1950's the Russians announced that they had grown a giant giant sunflower, whose spirals numbered 89 and 144. Then in 1968 the Americans, presumably not wishing to be outgrown by the Russians, reported that they had cultivated a giant giant giant sunflower with 144 and 233 spirals. The world now eagerly awaits the emergence of a giant giant giant giant sunflower with 233 and 377 spirals!

A Method of Solving the General Quartic Equation[1]

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Any quartic equation

$$x^4+a_1x^3+a_2x^2+a_3x+a_4=0 \tag{1}$$

can be transformed into an equation for which the condition

$$a_1^2a_4=a_3^2 \tag{2}$$

is valid if we assume as known some method of solving cubic equations. For the substitution $y=x+\lambda$ turns the quartic equation

$$y^4+b_1y^3+b_2y^2+b_3y+b_4=0 \tag{3}$$

into (1), with

$$\begin{aligned} a_1 &= 4\lambda+b_1, \\ a_2 &= 6\lambda^2+3b_1\lambda+b_2, \\ a_3 &= 4\lambda^3+3b_1\lambda^2+2b_2\lambda+b_3, \\ a_4 &= \lambda^4+b_1\lambda^3+b_2\lambda^2+b_3\lambda+b_4. \end{aligned}$$

Thus (2) holds precisely if

$$(4\lambda+b_1)^2(\lambda^4+b_1\lambda^3+b_2\lambda^2+b_3\lambda+b_4)=(4\lambda^3+3b_1\lambda^2+2b_2\lambda+b_3)^2,$$

i.e. when λ is a root of the cubic equation

$$\begin{aligned}(b_1^3-4b_1b_2+8b_3)\lambda^3+(b_1^2b_2+2b_1b_3-4b_2^2+16b_4)\lambda^2 \\ +(b_1^2b_3+8b_1b_4-4b_2b_3)\lambda+b_1^2b_4-b_3^2=0. \end{aligned} \tag{4}$$

We may therefore concentrate on the equation (1) for which condition (2) is satisfied.

[1] The method outlined in this article was enunciated originally by S. S. Greatheed in *Cambridge Math. Journal,* Vol. 1, 1838/9, and was re-discovered by the present writer when he was a fifth-form pupil.

If $a_1 = 0$, then, by (2), $a_3 = 0$ and (1) reduces to the form

$$x^4 + a_2 x^2 + a_4 = 0,$$

which is a quadratic in x^2 and so is immediately soluble.

Suppose, then, that $a_1 \neq 0$. Then (1) may be rewritten as

$$x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_3^2 a_1^{-2} = 0,$$

or

$$x^2 + a_3^2(a_1 x)^{-2} + a_1(x + a_3(a_1 x)^{-1}) + a_2 = 0.$$

Writing

$$w = x + a_3(a_1 x)^{-1}, \qquad (5)$$

we therefore have

$$w^2 + a_1 w + (a_2 - 2a_3 a_1^{-1}) = 0. \qquad (6)$$

The solution of (1), when (2) is satisfied, thus reduces to the solution of the pair of quadratic equations (5) and (6).

One final remark: we could first transform (3) so that $b_1 = 0$; and in that case (4) would assume a much simpler form.

The Theory and Practice of Sample Surveys

Sample surveys are at the basis of much of the demographic work carried out by the U.K. Central Statistical Office, the U.S.A. Bureau of Census, and Government Statistical Departments all over the world. They also form an important part of the techniques of modern market research.

The opinion polls held just before the 1970 British General Election stimulated a great deal of interest in sampling methods; the public were particularly curious about the errors which led to the pollsters' incorrect predictions. The Editors of *Mathematical Spectrum* felt that the subject was of such relevance as to warrant the commissioning of two complementary articles on sampling surveys. The first, by Dr. T. J. Rao, outlines the general principles of sampling theory, and considers some of its most important applications. The second, by Professor P. G. Moore, is more specifically concerned with opinion polls, and contains illustrations from the 1970 British elections. After perusing these two articles, readers will have a clearer appreciation of the value of sample surveys and of the errors which may arise in carrying them out.

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Sample Survey Techniques and their Applications

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1. Planning of sample surveys

The need for statistical information is rapidly growing in our society; huge amounts of data are being collected and used by the State to plan its economic and social activities, the effect of sales and production in the business and industrial sectors and the development of research and social projects in scientific institutions, public organizations and international agencies. While *experimental data* are usually collected by laboratory techniques, *survey data*, the location and source of which are known, are collected and recorded by field observation or enquiry. Survey data may thus be obtained by a *complete enumeration survey* where data are collected on each unit (such as person, household, farm, school, factory) of the *population*, this being the collection of all units of the type being studied. Alternatively, a *sample survey* may be carried out where a part (*sample*) of the population is surveyed, and inference is drawn about the population as a whole on the basis of this sample.

The obvious advantages of a sample survey over a complete enumeration survey are a reduction in the cost of the survey and the greater speed at which the data are obtained. By using specially qualified personnel, carefully supervising the field work, and thoroughly scrutinising the data one can aim at a greatly increased accuracy of the results obtained through a sample survey. In any sample survey there is always a *sampling error* which arises from drawing inferences about the whole population on the basis of observation of only a part of it (the sample). This the statistician tries to reduce by choosing a 'good' technique of sampling. However, it should be pointed out that, though the complete enumeration survey is void of sampling error, other types of errors creep in at the stages of the survey designs and collection and processing of data. These errors are termed *non-sampling errors* and are common to both complete enumeration and sample surveys.

At the design stage it is important to define clearly the objectives of the survey, the population under study and the type of data to be collected. The method of collection of data—whether it is a mail enquiry, personal interview or physical observation must be suitably decided. The type of questionnaire or schedule to be used in the survey must be properly checked for any inconsistencies or ambiguities in its questions, and for this purpose a *pilot study* is always advisable. After deciding upon the method of sample selection, the field stage of the survey involves supervision and checks on the returns; finally the processing stage deals with computations and tabulations of data with interpretation of the findings and a report on the survey.

2. Some techniques of sampling

A *probability* or *random* sample is a sample selected in such a way that each unit in the population has a known probability of selection. Unlike a non-random or haphazard sample, a probability sample lends itself to nice mathematical treatment and a rigorous theory of sampling can be developed. The simplest technique of selection is one in which all units of the population listed in a *sampling frame* are given equal chance of selection. This method is called *equal probability sampling* or *simple random sampling* (SRS). When repetition of units in the sample is allowed we have a *with replacement* sample; otherwise we have a *without replacement* sample. If in addition one has some auxiliary information about the size of the units, the units can be selected with varying probabilities of selection, giving smaller probability of selection to a unit of smaller size and larger probability of selection to a larger unit. One such scheme is the *probability proportional to size* (PPS) scheme. As a simple illustration consider the example of a population of $N = 4$ schools. Under the SRS scheme, when no further information on the schools is available, we would sample each school with probability 0·25. Suppose that we have auxiliary information that there are 30, 20, 10 and 40 teachers in schools 1, 2, 3 and 4 respectively. Here the schools are not of the same size. Hence, we would sample schools with varying probabilities of selection, such that a bigger school is given a larger probability of selection. Thus, under the PPS scheme we would select schools 1, 2, 3 and 4 with probabilities 0·3, 0·2, 0·1 and 0·4 respectively.

The required sample size n may be selected using the *random number tables*. Another operationally convenient and simple technique often used in large-scale surveys consists of selecting the first unit at random in a predetermined *sampling interval*, say $(1, k)$ and then choosing every kth unit. This is called *systematic sampling* and in the simple example above, a systematic sample of size $n = 2$, with $k = N/n = 4/2 = 2$ would comprise either the pair (1, 3) or (2, 4). It is, however, not difficult to consider more complicated cases.

Sometimes, as in a large-scale survey on schools in a country, when a proper sampling frame is not available, it is easy to prepare a sampling frame of the counties and select a sample from this list. Then, after obtaining the sampling frames of schools in these few selected counties, the ultimate units of interest, namely the schools, can be selected from each sampled county. This is called a *multi-stage sampling* technique and is of immense use in large-scale surveys.

3. Estimation and efficiency

After selecting the sample, data on the 'item of interest' are collected on all the units of the sample; it is then that one encounters the problem of drawing inference about the population values of the item of interest. A simple problem is the *estimation* of the *population total* or the *population mean* (*average*) on the basis of the sample. Conditions such as the 'validity' of the estimate can be imposed; this demands that on average the sample estimates should give the population value. It is easy to see that the *sample mean* is such a valid estimate of the population mean for a SRS scheme, while it is not so for a PPS scheme. In the latter case, the probabilities of selection also appear in the estimate. The next important problem is that of estimation of the sampling error of the estimate, which is once again computed from the sample itself.

When the estimates are needed for sub-populations and when it is administratively convenient to set up zonal field offices, it is advantageous to divide the whole population into non-overlapping groups called *strata* and samples drawn independently from each stratum would constitute what is known as *stratified sampling*. The more homogeneous the strata are within themselves, the more precise would be the estimates from each stratum; a weighted combination of these would then give the overall estimate for the population. Other methods of utilisation of auxiliary information for increased efficiency include the *ratio* and *regression* methods of estimation. The statistician always tries to choose a sampling and estimation procedure which is 'efficient' subject to the inevitable 'budget' restrictions which limit the amount which can be spent on his survey.

4. Non-sampling errors

In any survey, non-sampling errors due to the omission or duplication of units, faulty enumeration methods, inaccurate questionnaires and schedules, lack of trained personnel, inadequate supervision and scrutiny, non-response of some of the units sampled, errors in data processing and presentation, all occur at the stages of designing, field work and tabulation. Such non-sampling errors can be assessed and controlled by certain checks of consistency, post-census and post-survey checks, and a study of non-response. The well-known technique of *interpenetrating network of sub-samples* (IPNS) which consists of drawing the sample in the form of two or more sub-samples, selected in an identical manner which are then surveyed and processed by different groups of persons is often used. Agreement in the sub-sample estimates clearly provides a check on the quality of the data.

5. Applications

In this section we briefly outline some important fields of application of sampling techniques.

(a) *Demographic surveys*. Besides the population census, normally undertaken every ten years, the government is interested in the total population, births and deaths, immigration and emigration, family planning and fertility surveys during a particular period of time.

(b) *Socio-economic enquiries*. Perspective planning programmes in any country depend on regular surveys of consumer expenditure and family budgets, levels and conditions of living, transport facilities, community health and educational requirements, unemployment figures and labour enquiries.

(c) *Crop surveys and forest surveys*. Special techniques are devised to estimate the area under a particular crop. In many *crop-cutting experiments*, 'sample cuts' are selected at random from a farm and the estimates of total yield are obtained on the basis of the data collected from these. Using multi-stage sampling methods, estimates for the whole country can be built up. Agricultural surveys on various other crops (sugar cane, fruit, coffee) and studies on the effect of fertilizers have all successfully used these sampling techniques. Systematic sampling plays a prominent role in forest surveys and here interest centres on the estimation of standing timber.

(d) *Business, trade and industrial surveys.* Sampling techniques are useful in investigations of the accounts in company ledgers, where each volume sometimes contains 400–500 pages. In a large manufacturing plant, it is very important that the company auditors should have accurate figures to estimate the value of the materials used in the process of manufacture. Without shutting down the plant for inventory, the company can use sampling methods over a weekend, and take an adequate inventory of its resources with considerable speed. Sample surveys of retail trade flow, consumer surveys on financial holdings, and sales forecasting are also frequently carried out. Estimation of industrial production, transportation requirements and proper man-power planning are of the greatest use in the smooth running of the industrial process. Perhaps it is relevant to mention here that the 'sampling inspection' techniques for industrial quality control are different from the sample survey methods used in industrial establishments.

(e) *Opinion polls and market research surveys.* Public opinion polls, attitude surveys and market research enquiries have become increasingly popular in recent years. In general, when one is aiming at a large-scale survey, a multi-stage stratified sampling design is used. After selecting the ultimate stage units (such as persons), they are interviewed by specially trained investigators. Questionnaires are filled in very carefully, taking precautions to avoid any leading questions. The answers, which are mostly in the form 'yes', 'no' or 'don't know', are processed and the final estimates of proportions obtained. The technique of IPNS previously mentioned may also be used for check on internal consistency as well as for the estimation of sampling errors.

In this context, we might briefly mention the recent predictions for the 1970 election results in Great Britain. Only one opinion poll, conducted two days before the polling date, showed a 1 per cent Conservative party lead, very close to the party's actual 2·4 per cent lead. The other polls, which were taken several days before, many of which did not make suitable adjustments for the failure of some electors to vote, made erroneous predictions in favour of the Labour party. It should, however, be mentioned that some of these polls were in fact very successful in local forecasts. Supervision on field work and careful scrutiny of returns is very essential in a project of this kind, but was probably hard to carry out. It has, however, been suggested that post-election interviews should be conducted to detect any late swings. Should such swings be revealed, not only would pollsters have a check on their techniques, but the public might also become more convinced of the essential correctness and value of polls. Instead of interviewing a fixed sample of people on different occasions, it may for example be advantageous to keep a proportion of them fixed and supplement this number by a new sample. Such *multi-phase sampling* would not only estimate the required quantities but also measure the changes in the attitudes of people over a time period.

Other important surveys are those on the opinions of radio listeners and television viewers; these are very helpful to the broadcasting agencies in planning their programmes to suit the tastes of the public. In fact, these come un broader category of market research surveys, where again sampling designs similar to those mentioned above are used. Since the production of a particular marketed item depends greatly on the reaction of the consumer, it is important that these surveys should be carefully designed and executed. Apart from the probability sampling methods, another technique used in market research surveys is the *quota sampling* system where units are selected from each stratum till the predetermined quota of sample units is obtained.

(f) *Wild life population surveys.* It is very interesting to note the importance and use of sampling techniques in the estimation of wild life populations and related problems. Past surveys include the estimation of the size of a mobile population, such as fish in a lake, using what are known as 'capture–recapture methods', estimation of the number of insects in a field (such as the larvae of Japanese beetle), estimation of the population of aphids in a potato field, soil sampling for potato root eel-worm cysts, estimation of wire-worm populations and the study of traffic flow along major roads.

We have mentioned only briefly the major fields of application of sampling techniques, without going into details, interested readers will find these described in Cochran [1], Murthy [2] and Yates [3].

References

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Sampling and Opinion Polls

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The 1970 British General Election proved to be a graveyard for the reputations of many of the opinion pollsters, and a great deal of recrimination about them was let loose in the British national press. Some of the pollsters maintain that their results were not so inconsistent as has been made out, bearing in mind the limitations of polling techniques. What should we be able to learn from such polls?

An opinion poll is a form of sample survey, and surveys themselves can be concerned with one of two basic types of information. They can be concerned with opinions, for example: 'For whom are you going to vote at the forthcoming election?' amongst factory workers in Derby. Alternatively they can be concerned with facts such as the height of 11-year-old schoolboys in Rugby, or the amount spent by housewives in Guildford on coffee last week. A full survey will nominally cover every individual concerned, i.e. all factory workers in Derby, or all 11-year-old schoolboys in Rugby, or all housewives in Guildford. A sample survey will only cover a fraction of the individuals concerned in the group, and will use the information from this fraction of individuals to estimate the required characteristic for the group (sometimes referred to as a population or universe) as a whole.

From this it follows that two distinct types of possible discrepancy exist, both of which may affect the accuracy with which a sample can be used to estimate the appropriate population characteristic, namely:

(a) a 'bias' of measurement, and
(b) an 'error' of sampling, due to the fact that only a fraction of the population is considered.

Let us examine these two causes of discrepancy in turn. First, bias: this arises when the measuring method itself is wrong for the purpose in hand, and in this case a survey can clearly give an erroneous result. Suppose the heights of the schoolboys were measured by a tape measure that had stretched, with the consequence that all heights were unknowingly 5 per cent below the true value. Then, even if every 11-year-old schoolboy in Rugby had his height measured, the result would still be wrong by some 5 per cent. The same phenomenon applies to opinion polls. In a political poll taken, say, a fortnight before an election, a bias may occur because the question cannot be linked to what may actually happen in the polling booth, but only to what people think they will do at the time they are polled. The poll could elicit an honest answer from the entire population and still be wrong in the subsequent event when people did or did not vote. It is important to emphasise that this bias has nothing to do with sampling as such: it is entirely connected with the measuring instrument that has been used for the problem in hand, and would occur whether the sample used were 1 per cent, 10 per cent or 100 per cent of the population concerned.

We turn now to errors or discrepancies that occur because we have only looked at a sample and not at the complete population. For present purposes we will assume that there is no bias of measurement as such and hence that a 100 per cent sample would give precisely the answer we seek. In considering these sampling errors we must first distinguish between alternative ways in which a sample can be selected from a defined population. Whilst there are many variations, the choice of sample method used basically rests upon either the *quota* method or the *random* method, and these will now be discussed in turn.

The sample: quota selection

First, let us consider the quota method. Here the sample is selected so as to fill a previously defined quota of individuals. For example, suppose a quota sample of 2,000 individuals is to be selected from the whole of the United Kingdom, the precise purpose for which the sample is required being unspecified. If no restrictions were placed on the quota, then it could be selected in any manner we please: the first 2,000 people coming out of Waterloo Station at 8.30 in the morning, for example. For some purposes this might be a relevant frame of reference (perhaps for asking on which day of the week a person's birthday fell), but not if we were trying to find out the voting intentions of the United Kingdom population, or its travelling habits. Hence some form of quota restrictions could be put on to try to remedy the possible bias. For example, it might be specified that 1,000 of the individuals had to be male and 1,000 female. This would almost certainly improve the situation, but not necessarily completely. For example, suppose one were asking a question about what a person ate for breakfast. By asking both men and women in fixed proportions from among the individuals leaving Waterloo Station, one would undoubtedly get a better answer than one could with an unrestricted sample, which might well contain too many men. But there are still problems, e.g. age distribution or geographical variations which may need to be taken into account to get a better picture of breakfast habits. This could lead to further quota restrictions, e.g. that half the sample should be under 35 years of age, that there should be some specified geographical break-down, and possibly also a social class restriction as well.

Taking all the restrictions into account, the interviewer may now need to find 90 men for the sample, each over 35 years of age, living in the South-East, and in a particular social class, and so on. Within the quota controls, the individuals can be found in any way and it would still be permissible to go back to Waterloo Station for the appropriate portion of the sample. Clearly, the more restrictions that are placed on the sample, the more representative it becomes, but the more difficult and costly it is to find the individuals to fill the various portions of the quota. Quota sampling completely eliminates the common problem of non-response: if a person is not available, you choose another one to fit the quota. There is, however, no way beyond the quota restrictions of guaranteeing freedom from any personal preferences that interviewers may have. Thus one particular interviewer may have a predilection for neatly dressed, congenial or conveniently available respondents. Quota sampling involves a conscious choice by the interviewer which means that the sample is not random. One consequence of this lack of randomness is that an evaluation of the error due to sampling is, in general, not possible from the results of the sample itself. One can examine, when possible, over time and in a number of similar surveys, the discrepancies that arise between the survey result and what subsequently turns out to be the true result from the population as a whole, and then use these to help one to estimate the limits of sample error. We conclude that the reliability of quota samples is a very difficult quantity to evaluate.

The sample: random selection

The alternative method of selecting the sample is through some random choice procedure, whereby each individual in the population concerned has a known chance of appearing in the sample. In simple random sampling there would be an equal chance. Suppose, for illustration, that we wanted a simple random sample of 1,000 voters to be selected from a constituency which has a total of 50,000 voters.

One straightforward but tedious way in which the selection could be made would be to put all the names of the 50,000 individual voters on separate slips of paper, shuffle them all up and draw out, blindfold, 1,000 of the slips. The names appearing on the slips drawn out would then constitute the sample. Those voters, and only those voters, who had been selected must be tracked down and the relevant information obtained from them.

This method of selection is clearly more time-consuming and costly than quota sampling; why, then, should we be put to all this trouble and expense to get the 1,000 persons required? The answer lies in the way in which we can, from the sample itself, estimate in probabilistic terms the possible range of error in using our sample result to generate information about the population. An example will illustrate the point.

Suppose we had a constituency of 50,000 voters exactly 50 per cent of whom were Conservative and 50 per cent Labour. (Notice that we are assuming just two categories and that every voter belongs to one or the other.) Assume that we now draw a simple random sample of 1,000. We would not necessarily expect to find exactly 50 per cent of the sample Conservative and 50 per cent Labour. There is only a finite (albeit very large) number of possible samples of 1,000 that could be drawn and each of these is equally likely to occur. Hence we could form a distribution of these possible samples according to the proportion of Conservatives in each sample. Luckily we need not actually carry out the detailed computations for each possible sample (there are about $10^{2,000}$ of them). Mathematics comes to our help, and the distribution for samples of 1,000 and 2,000 voters respectively is given in Table 1.

TABLE 1
Distribution of results

Percentage of Conservatives in sample	Probability of getting the sample results	
	Sample of 1,000	Sample of 2,000
Under 46·5	·013	·001
46·5–	·044	·012
47·5–	·114	·077
48·5–	·205	·238
49·5–	·248	·345
50·5–	·205	·238
51·5–	·114	·077
52·5–	·004	·012
53·5–	·013	·001

(46·5– indicates at least 46·5 but less than 47·5, etc.)

We see that for samples of 1,000, the probability of being outside the range $50 \pm 3{\cdot}5$ per cent is 0·026; while the probability of being outside the range $50 \pm 2{\cdot}5$ per cent for samples of 2,000 is also 0·026. Thus, given the actual true percentage of Conservatives in the population, we can work out the range of variation in a simple random sample; what is more important, we can apply this reasoning in reverse. Given, for example, a simple random sample of 2,000 with 50 per cent Conservatives, we can be pretty sure that the true *unknown* percentage lies in the range $50 \pm 2{\cdot}5$ per cent. This range of variation, or precision, associated with a given probability value, is clearly a good measure for noting any improvement due to an increase in sample size. As expected, we find that an increase in the sample size gives a better precision, but there is a law of diminishing returns here, for the precision is proportional to $\sqrt{(p(1-p)/n)}$ where $100p$ is the percentage of Conservatives and n is the sample size.

What happens to this precision if the population size is different? It turns out that population size is irrelevant as long as the sample is not too large in relation to the population, say not larger than 10 per cent of the population.

The practical conclusions we can draw about the precision of random sampling are that:

(a) it can be estimated from the internal evidence of the sample;
(b) it depends on the size of the sample;
(c) it does not depend on the size of the population.

Opinion polls

Having said this, what about the opinion polls? Some of these polls used random samples selected in a rather more complicated way than simple random samples, and some used quota samples. Table 2 gives the five main polls whose results were published regularly in the British national press, and the latest results that they published prior to polling day on 18 June 1970. In the month prior to polling day, the polls had shown a diminishing Labour lead, reaching a consistent lead of about 4·0 per cent about two and a half weeks before polling day. The polls had then started to diverge in their results, and this is shown in Table 2.

TABLE 2
Poll results published just before the election

	Gallup	National Opinion Polls	Opinion Research Centre	Marplan	Harris
Normal size of sample used	2,000	2,000	1,100	1,500	2,500
Quota (Q) or Random (R)	Q	R	Q	R	R
Labour lead† (per cent)	7·0	4·1	−1·0	8·7	2·0
Mid-date of fieldwork	15 June	13 June	14/18 June	12 June	17 June

† This was defined as the difference between the percentage of Labour voters and Conservative voters amongst those who said they were going to vote.

From the earlier discussion, a random sample of approximately 2,000 individuals should have a practical level of precision of about $\pm 2{\cdot}5$ per cent. Hence the actual result, a Conservative lead of about 2·4 per cent is not within any of the random sample results, and only within one of the quota results. The O.R.C.

(Opinion Research Centre) result is itself a little different from the others in character, both because it was the latest, and because it involved re-interviewing a number of individuals previously interviewed a few days earlier without taking an entirely fresh sample.

There are three points which should be noted. First, using an opinion poll we can only project opinions forward from what people say they will do on polling day. Even if we had taken a 100 per cent sample, we might still not get an accurate prediction of what would happen on a polling day some way ahead. Secondly, the number who actually vote is smaller than the number who say how they will vote, and any differential turn-out will affect the result. Finally, there is a time lag between the sampling of the opinions and the polling day, during which some people may quite genuinely change their views. In the past these 'errors' have been small in the event and, indeed, probably smaller than the true sampling error as discussed earlier. In the 1970 General Election it seems that these non-sampling errors were of more importance than usual, and this points to the need to devise more sophisticated methods for controlling and measuring them. Nevertheless, it is worth emphasising that we are really dealing with relatively small differences in support for the two main parties, and attempts to measure such small differences must always be fraught with danger unless a great deal of time and money can be spent on the survey methods.

There are special reasons why the polls may not have achieved their aims in the 1970 British General Election. But their failure on this occasion must not be allowed to obscure the general utility and economy of sampling as a technique for obtaining information at minimum cost. In business, in research, in everyday life, sampling is in constant use and, properly handled, it is an extremely powerful tool.

Letter to the Editor

Dear Editor,

L. Råde's article 'A Probabilistic Triangle Problem' in *Mathematical Spectrum*, Vol. 2, No. 2, 1969/70, 57–64, has reminded me of two extensions of the problem which I considered some years ago.

Given that a stick broken at two random points forms a triangle, what is the probability of the resulting triangle being acute? I obtained $P = 12\log 2 - 8 = 0{\cdot}316$.

If we break a stick at $n-1$ random points, what is the probability of forming an n-gon? I obtained $P = 1 - n/2^{n-1}$. For $n = 2, 3, 4, \ldots$, these values are 0, 1/4, 1/2, 11/16, 13/16,..., and converge to 1 rather rapidly.

Yours sincerely,

DAVID SINGMASTER

(Department of Mathematics, Polytechnic of the South Bank, London S.E.1)

Problems and Solutions

An Interesting Problem in Division

P. H. Harris

ICI Fibres Limited, Pontypool, Monmouthshire

Statement of the problem

A 17-digit number containing all the digits 0 to 9 is exactly divisible by a 2-digit number to give a 15-digit quotient which also contains all the digits 0 to 9. The division is shown below, each dot replacing a digit normally appearing in the working. Restore all the digits.

```
          . . . . . . . . . . . . . . .      (1)
      -----------------------------------
. . ) . . . . . . . . . . . . . . . . .      (2)
        . .                                  (3)
        ---
          . .                                (4)
          . .                                (5)
          ---
              . .                            (6)
              . .                            (7)
              ---
              . . .                          (8)
                . .                          (9)
                ---
                  . .                        (10)
                  . .                        (11)
                  ---
                  . . .                      (12)
                  . . .                      (13)
                  -----
                        . . .                (14)
                          . .                (15)
                        -----
                            . .              (16)
                            . .              (17)
                            ---
                              . . .          (18)
                              . . .          (19)
                              -----
                                . . .        (20)
                                . . .        (21)
                                -----
                                  . . .      (22)
                                  . . .      (23)
                                  -----
```

I composed this problem some years ago during an attack of 'flu, after failing to devise one despite many earlier efforts. Interested readers may wish to find the solution for themselves before reading further.

Solution

This can be derived in the following way. The layout shows 0 to occupy the 3rd, 8th, 9th and 12th positions in the quotient. The highest 2-digit product of the divisor occurs in lines (3), (9) and (15). Therefore, the remaining eight products in the working must be all different since we are told that all the nine possible products appear. The highest 2-digit product must then be $5 \times$ divisor since there are four other 2-digit products. The subtraction shows that the highest 2-digit product must also be greater than 90 and therefore the only possible divisor is 19.

Since (4), (10) and (16) must be at least 50, the product 19 must be at (7). The product 38 can only be at (11) since it must give a 2-digit number on subtraction. The remaining 2-digit products, 57 and 76, cannot yet be assigned. We can now write:

$$\begin{array}{r} 5\,.\,0\,1\,5\,2\,.\,0\,0\,5\,.\,0\,.\,.\,. \\ 19\overline{)\,1\,0\,.\,.\,2\,9\,0\,.\,.\,1\,0\,.\,.\,.\,.\,.\,.} \end{array}$$

If we now insert the possible numbers for the missing lines of working down to (17), the possible values of the missing digits in the dividend as far as the 13th can be found. There are two possibilities for some and three for others. Using this information, together with the given fact that all the digits 0 to 9 occur in the dividend, we obtain the following conditions which must be satisfied by the last four digits of the dividend:

1. there must be at least one 4;
2. there must be at least three different digits in the range 3 to 8;
3. if none of them is a 7 then they must be all different and in the range 3 to 8;
4. if 171 is one of the last three products then there must be at least one 5.

Remembering that 114 must be one of the last three products we can list the eighteen possible solutions for the last six lines of working. However, the only one which satisfies the four conditions above is:

$$\begin{array}{lr} 1\,6\,4 & (18) \\ \underline{1\,5\,2} & (19) \\ 1\,2\,7 & (20) \\ \underline{1\,1\,4} & (21) \\ 1\,3\,3 & (22) \\ \underline{1\,3\,3} & (23) \end{array}$$

After inserting the remaining 3-digit product 171 at (12) and (13), 8 is the only digit missing from the dividend. 57 must then be at (17) and the remaining digits can immediately be inserted to give:

$$\begin{array}{r} 5\,4\,0\,1\,5\,2\,9\,0\,0\,5\,3\,0\,8\,6\,7 \\ 19\overline{)\,1\,0\,2\,6\,2\,9\,0\,5\,1\,1\,0\,0\,8\,6\,4\,7\,3} \end{array}$$

* * * * *

Readers who have not yet reached the age of 20 on 1 April 1971 are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

Problems

3.4. Let u_n denote the nth Fibonacci number. (See the article in this issue by R. J. Webster.) Show that (i) when m divides n, then u_m divides u_n, (ii) when $n \geqslant 8$, then u_n has not more than $n/4$ digits in its decimal expansion.

3.5. There are n sisters who live in n different suburbs of a town. Each wishes to hear daily the news of all the others. They agree that, to reduce their overall telephone account, each will pass on all the news she knows at every telephone call made to, or received from, another sister. Show that, for $n > 3$, all the news can be circulated in $2n-4$ telephone calls. (This problem was submitted by A. V. Boyd, Department of Statistics, University of the Witwatersrand, Johannesburg, South Africa.)

3.6. Show that, for all integers $n \geqslant 6$, a square can be subdivided into n non-overlapping squares.

3.7. Let $z_1, z_2, \ldots, z_n$ be any complex numbers. Show that there exists a unique complex number z which minimizes the sum

$$\sum_{k=1}^{n} |z-z_k|^2.$$

Solutions to Problems in Volume 3, Number 1

3.1. (i) Prove that, if a, b are positive integers, the difference between $(a+2b)/(a+b)$ and $\sqrt{2}$ is less than the difference between a/b and $\sqrt{2}$, so that

$$\tfrac{1}{1}, \tfrac{3}{2}, \tfrac{7}{5}, \tfrac{17}{12}, \tfrac{41}{29}, \ldots$$

is a sequence of better and better rational approximations to $\sqrt{2}$.

(ii) Obtain the first seven terms of the sequence, beginning with 1/1, in which successive terms are generated by the transformation $a/b \to (a+b)/a$. Show that this sequence provides steadily improving rational approximations to $\frac{1}{2}(1+\sqrt{5})$.

Solution by A. H. Rodgers (Royal Belfast Academical Institution)

(i) Denote the nth term of the sequence shown by x_n. Then $x_1 = 1$ and $x_{n+1} = (x_n+2)/(x_n+1)$. Further,

$$x_{n+1}-\sqrt{2} = \frac{x_n+2-\sqrt{2}x_n-\sqrt{2}}{x_n+1} = \frac{(\sqrt{2}-1)(\sqrt{2}-x_n)}{x_n+1}.$$

Thus

$$|x_{n+1}-\sqrt{2}| < |x_n-\sqrt{2}|.$$

(ii) Denote the nth term of the sequence shown by y_n. Then $y_1 = 1$ and $y_{n+1} = 1+1/y_n$. Put $\gamma = \frac{1}{2}(1+\sqrt{5})$. Then $\gamma = 1+1/\gamma$ and

$$y_{n+1}-\gamma = \frac{1}{y_n}-\frac{1}{\gamma} = \frac{\gamma-y_n}{\gamma y_n}.$$

Hence

$$|y_{n+1}-\gamma| < |y_n-\gamma|.$$

Also solved by J. D. Hoddy (Hulme Grammar School for Boys, Oldham).

3.2. Let n be a positive integer. Show that among any $n+1$ different integers chosen from $1, 2, 3, \ldots, 2n-1, 2n$ there are always two such that one divides the other.

Solution

Let $a_1, a_2, \ldots, a_{n+1}$ be any $n+1$ different integers chosen from $1, 2, \ldots, 2n$. Write $a_k = 2^{r_k}u_k$ for $1 \leqslant k \leqslant n+1$, where $r_k \geqslant 0$ and u_k is odd. Then each of the $n+1$ u_k's is equal to one or other of the n numbers 1, 3, 5,..., $2n-1$. Hence two of the u_k's must be equal, say $u_s = u_t$. Hence either a_s divides a_t or a_t divides a_s.

Also solved by J. D. Hoddy (Hulme Grammar School for Boys, Oldham), and A. H. Rodgers (Royal Belfast Academical Institution).

3.3. Prove that $\log x$ cannot be expressed in the form $f(x)/g(x)$, where $f(x)$, $g(x)$ are polynomials in x.

Solution by A. H. Rodgers (Royal Belfast Academical Institution)

Suppose that $\log x = f(x)/g(x)$ and let n be any positive integer. Then

$$\frac{f(x^n)}{g(x^n)} = \log x^n = n \log x = \frac{nf(x)}{g(x)}.$$

Hence $f(x^n)g(x) = nf(x)g(x^n)$. By comparing leading coefficients, we see that this is impossible.

Also solved by J. D. Hoddy (Hulme Grammar School for Boys, Oldham) and G. Mungo (George Watson's College, Edinburgh).

Book Reviews

Modern Applied Algebra. By G. Birkhoff and T. C. Bartee. McGraw-Hill, New York, 1970. Pp. 431. £5·50.

This book is aimed at a university rather than a school audience, but a great deal of it is at a level which can be reached quickly by sixth-formers, particularly those who have followed a School Mathematics Project type course. Since the ideas contained in it are likely to become of considerable importance in the application of modern algebra, it is worth while for sixth-formers to dip into it.

To appreciate the importance of this work, it is necessary to go back some 30 years to the publication of *A Survey of Modern Algebra* by one of the present authors, G. Birkhoff, with Saunders Maclane as co-author. That book, more than any other, introduced the post-war undergraduate to what is termed abstract algebra; this was the generation that gave rise to the group of young mathematics masters who introduced into school curricula set theory, matrix theory and the other ideas which make up the elementary part of abstract algebra. These developments have been criticised on the grounds that they have little relevance to the outside world. While some of these criticisms are justified in part, this book shows that the ideas of abstract algebra have applications and places these applications in the proper algebraic context.

The second author of the present book is an electrical engineer, with a particular interest in large-scale computers; this has influenced the types of applications considered. The book appears to be split up into sections, not always coinciding with chapters, alternately written by Birkhoff who deals with general theory, and Bartee with applications. This alternation of style is, in fact, quite pleasing.

The first two chapters deal with the basic ideas of sets, functions and relations; these usually begin any book on abstract algebra, although there is here a side-glance at graph theory with an up-to-date flavour. The third chapter is on finite-state sequential machines, motivated by a brief account of modern computers. This chapter affords a nice application of the ideas introduced in the first two chapters. The next chapter is somewhat of a surprise, being a brief introduction to ALGOL, followed by a very short account of the problem of computer program compilation. In the course of the next chapter, which is concerned with Boolean algebra, ALGOL is used to give some illustrations of its application; others are to the theory of switching circuits and, in the following chapter, to computer design. The next chapter is on groups and monoids, which are immediately applied in the following chapter to the construction of error correcting or detecting codes. Then follow three solid chapters of algebra dealing with lattices, rings, ideals and polynomials, and ending with the theory of finite fields. These are directly applied to the development of more error-detecting codes. The twelfth chapter is again concerned with an application of finite fields, this time to the problem of producing very long recurrent sequences used in satellite tracking systems. The final chapter is entitled 'Computability' and deals both with the limitations of computers and the very modern theory of mathematical linguistics. This has arisen from the linguistic work of Noam Chomsky, and has become a keypoint in computer science, particularly in the design of computer languages.

To sum up, this book introduces the reader to some of the uses to which abstract algebra has been put during the last 20 years. There are omissions: there is no mention of the theory of linear programming, for example. But there is enough here to suggest that abstract algebra, like other branches of mathematics before it, has found its applications.

University of Exeter D. Rees

Metrication. Revised edition, edited by F. W. KELLAWAY. Penguin Books, Harmondsworth, Middlesex, 1970. Pp. 124. £0·30, paperback.

This is an extremely useful short book. In his Foreword, the Editor explains that 'Metrication is the word chosen to represent the various processes involved in changing from one system of weights, measures and money to another in which powers of ten are the key.' The book appears at a time when the U.K. is changing over to decimal currency (15 February 1971) and industry is gradually moving towards the metric system; its timing could not be better. It consists of five articles: a 'General Survey' by Norman Clarke, Secretary of the Institute of Mathematics and its Applications, 'Metrication in Science and Technology' by Professor M. J. Lighthill of Cambridge, 'The Industrial Scene' by Dr. W. S. Hollis of the Institution of Production Engineers, 'Metrication and Commerce' by Mr. J. D. Buchanan of the Consumer Council, and finally 'Metrication and the Teacher' by Dr. E. D. Tagg of the University of Lancaster.

Each author has considered the advantages to be derived from metrication mainly, but not exclusively, in his own area of interest. The difficulties likely to arise are outlined in considerable detail, and ways in which they can be overcome are recommended. As the Editor points out in a final short Note, it will be the joint responsibility of teachers, the Council of Technical Examining Bodies, publishers and other parties interested in education to help smooth over the transition period in Britain.

There are six helpful Appendices on SI (standard international) units, multiples and submultiples of decimal units, Imperial–metric conversion, the use of SI units in scientific journals, British decimal currency recommendations, and a Bibliography. A brief index concludes the book.

As a schoolchild, I began my education in French schools, and later (almost accidentally) switched to English schools. I can still recall the immense difficulty which I experienced in learning the Imperial system of weights and measures. I never mastered it; to this day, I cannot readily recall how many quarts there are in a gallon or how many acres in a square mile. Thus I needed no convincing that the metric system is easier to understand and to apply; what this book has clarified for me is the immense difficulties which conversion will entail in science, in technology and industry, in the simplest commercial transactions, and in schools where text-books will largely have to be rewritten. Metric measurements and SI units are clearly simpler, more rational, and in the long term time-saving; but it will be quite a while before British practice is able to conform to them.

I had not fully realised the long history of attempts at metrication in Britain. These date from the Second Report of the Standards Commissioners 'On the question of the introduction of the metric system of weights and measures into the United Kingdom' published in 1869. Despite their recommendation that the Government should provide facilities for the introduction and use of metric weights and measures in the U.K., it was not until Parliament passed a Weights and Measures Act in 1897 that the metric system was legalised for use on a permissive basis in this country. As the chart provided on the inside back cover would seem to indicate, it is unlikely that metrication will have been completed in industry, schools and scientific work in the United Kingdom much before 1975–1980.

I believe that this book will be read with great interest by all of those involved in metrication, and that is everyone of us; it is quite clear that once transition has begun, the commercial benefits of metrication will reinforce the value in logic and time-saving which the new system of units offers.

University of Sheffield J. GANI

Basic Numerical Analysis. By J. RIBBANS. International Textbook Co. Ltd, London. Book 1: 1969. Pp. 116. £1·10. Book 2: 1970. Pp. 124. £1·60.

Numerical methods now form an important part of many sixth-form courses, and should also appeal to those following a more traditional programme. These books may help to spread the gospel.

After an initial chapter about errors, Book 1 deals with Differences and Interpolation (Chapters 2 and 3), Differentiation and Integration (Chapter 4), Linear Equations by elimination and relaxation (Chapter 5), Polynomial and Transcendental Equations (Chapter 6), Curve Fitting (Chapter 7), Differential Equations (Chapter 8). The author is a university lecturer and clearly has in mind first-year undergraduates, though the book is claimed to be suitable for sixth-form students. The list of contents indicates that a knowledge of much of the 'A' level syllabus is a necessary prerequisite, and some readers may be put off by the early use of the binomial series for $(1+\Delta)^p$ where p is rational and Δ is an operator. It should be noted that Chapters 5, 6 and 7 are independent of the rest and more elementary, but nevertheless I would not recommend this as a text-book for an 'A' level numerical methods course.

There are exercises at the end of each chapter, with answers where appropriate. The questions are to be tackled with the aid of a desk calculator, and the text is written with this in mind. Computers get only an occasional brief mention. One snag about a book of this kind is that it cannot be used casually for enrichment as can others in a school mathematics library; accompanying practical work is essential, and this is time consuming. Nevertheless, there is satisfaction to be obtained from a lengthy calculation leading to an accurate answer guaranteed by all the accompanying checks.

Sixth-formers, with or without computing facilities available, may be more interested in numerical analysis for computing. If so, this is not the book for them. For others who would enjoy settling down with a desk calculator and need guidance on how to tackle problems, there is the choice (at an elementary level) between this book and

(i) *Numerical Mathematics* by A. J. Moakes (Macmillan),

(ii) *Numerical Analysis* by Watson, Phillipson and Oates (Arnold).

The former is brief and easy to use. The latter (in two books) covers rather less ground than the books under review, at a slower pace.

There are a number of misprints and errors, but nothing which would trouble an alert reader. An amusing example is that $1/(2x)$ appears twice on page 69 as $1/2x$ and (inevitably?) on page 70 as $\frac{1}{2}x$.

For use at school, Book 2 has little appeal. It extends most of the topics in Book 1; matrix methods, partial differential equations and Fourier series are introduced, but only cursorily.

Marlborough College C. C. GOLDSMITH

Analogue Computers. By TIMOTHY and MICHAEL BRAND. Edward Arnold, Maidenhead, Berks, 1970. Pp. 138. £1·20.

Books on analogue computers at this level are rare. The authors set out to give an outline of different analogue devices and describe the mathematics of the electronic analogue computer. The book has a wealth of diagrams and exercises (with answers, a major asset).

After a brief introductory chapter, there is an analysis of analogue computer circuitry and methods. This section is logically developed and well laid out (although the cross-referencing is not always easy to follow). I found the final brief section on applications rather limited. It is followed by a comprehensive glossary.

The main drawback is that the book assumes a well-founded knowledge of differential equations, but only the most rudimentary grounding in physics. It is not, therefore, accessible to 'O' level students, although they can cope with the physics. However it will be irksome to the more senior student, who will surely know enough physics to make much of the glossary and some of the text unnecessary. I think that it is rather dry as an introduction to its subject, unless the reader takes a perverse delight in such arid tomes. However, its admirable clarity and simplicity make it most valuable both as a reference book and as a text-book—its price £1·20 is not prohibitive.

Mathematical VIIth, Westminster School P. W. K. RUNDELL

Elementary Differential Equations. By R. L. E. SCHWARZENBERGER. Chapman and Hall, London, 1969. Pp. xii, 98. £1·00.

This book provides an introduction to ordinary differential equations with emphasis on qualitative theory. Qualitative theory aims to discover useful facts about the shape of the graphs of the unknown functions which satisfy the differential equation. This theory is a valuable tool for studying the numerous equations which cannot be solved by formal methods. The first two chapters of the book discuss the meaning of a differential equation and its geometrical interpretation, with many examples. Chapter 3 discusses some first-order equations which are solvable by formal methods. Chapter 4 discusses pairs of autonomous equations and Chapter 5 gives a brief sketch of nth-order linear equations with constant coefficients. Throughout, the subject is treated as pure mathematics with no mention of applications. Designed primarily for first-year college and university students, it is simply written and could be read with profit by an able sixth-former. A good grasp of calculus and coordinate geometry is all that is required.

University of Durham R. A. SMITH

Excursions in Geometry. By C. STANLEY OGILVY. Oxford University Press, New York, 1969. Pp. 186. £2·60.

'It is regrettable that so few non-trivial theorems can be proved within the framework of the traditional geometry course, when so many startlingly good ones lie just around the corner.' So writes the author of this little book before sallying forth with his periscope to give the reader a glimpse of such topics as harmonic division, Apollonian circles, inversive geometry, the sphere hexlet, golden section, angle trisection, conic sections and projective geometry. Although the book is written primarily for the layman, few students of mathematics will read it without finding at least something to interest and amuse.

Excellent material for the school library.

King's College, London J. A. TYRRELL

Notes on Contributors

D. R. Davies has taught in several universities and colleges in England and Wales, and is now Professor of Applied Mathematics in the University of Exeter. He is the author of numerous papers on various aspects of fluid dynamics and heat transfer, but his principal interest is in the mathematics and physics of weather prediction.

E. M. Patterson has taught in Leeds, Sheffield and St. Andrews and is now a Professor of Pure Mathematics in the University of Aberdeen. His interests include algebra, geometry and topology; and he has published books and papers in each of these fields. A book that may be of particular interest to our readers is *Elementary Abstract Algebra* (Oliver & Boyd) written jointly with the late D. E. Rutherford.

R. J. Webster, who is a Lecturer in Pure Mathematics in the University of Sheffield, is at present spending a year at Dalhousie University, Nova Scotia, Canada. His research is, in the main, centred on the study of convexity; he has given many invited addresses to schools and to mathematical societies in universities.

J. Macey was a member of the Vth Form at Nottingham High School at the time when he wrote the article printed in this issue.

T. J. Rao is a Lecturer in the Statistical Laboratory, Department of Mathematics, University of Manchester. He graduated with the degree of M.A. in Mathematics from Andhra University, and obtained the degrees of M.Stat. and Ph.D. at the Indian Statistical Institute, Calcutta, where he was later a staff member. His main interests are sample surveys and statistical methods, on which he has published several papers.

P. G. Moore is Professor of Statistics and Operational Research, and also Academic Dean, at the London Graduate School of Business Studies. He is also a Director of Shell U.K. Ltd and a part-time member of the Post Office Board for Data Processing.

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