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## GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

## RENSEIGNEMENTS GÉNÉRAUX

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire. Bien que principalement de nature éducative, elle sert aussi à ceux qui la lisent pour des raisons professionnelles, culturelles ou récréative.

Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

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# THE OLYMPIAD CORNER

No. 142

R.E. WOODROW

*All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.*

As a first group of questions we give the 1st U.K. Schools Mathematical Challenge. In this contest calculators are **not** permitted and the entrants are given one hour to complete the test. The contest is for students under 15 years of age. In a future column, we will give the answers only (and we will likely not publish any more detailed solutions). Thanks go to Georg Gunther, Sir Wilfred Grenfell College, for collecting the contest and forwarding it to me.

## 1ST U.K. SCHOOLS MATHEMATICAL CHALLENGE

March 3, 1988 (Time: 1 hour)

**1.** Half of $99\frac{1}{2}$ is

A. $45\dfrac{1}{4}$ B. $45\dfrac{3}{4}$ C. $49\dfrac{1}{4}$ D. $49\dfrac{1}{2}$ E. $49\dfrac{3}{4}$

**2.** In this question (including all five answers) the letters 'o' and 'f' each appear

A. once B. twice C. three times D. four times E. five times

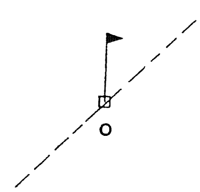
**3.** Harold is 8cm taller than Jack. Jim is 12cm shorter than Harold. Jack is 125cm tall. How tall is Jim (in centimetres)?

A. 129 B. 121 C. 105 D. 113 E. 145

**4.** How many minutes are there between 11.41h and 14.02h?

A. 141 B. 261 C. 241 D. 221 E. 361

**5.** The flag is given a half-turn anti-clockwise about the point O and is then reflected in the dotted line. Which picture shows the correct final position of the flag?




A.  B.  C.  D.  E. 

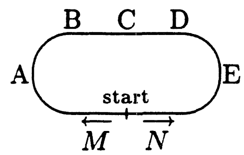
**6.** On my calculator $1/3 = 0.3333333$. What would $1/30$ be?

A. 00.333333 B. 0.3030303 C. 0.3333333 D. 0.0303030 E. 0.0333333

**7.** How many numbers from 1 to 100 have a figure '5' in them?

A. 10 B. 15 C. 19 D. 20 E. None of these

**8.** Malcolm covers any distance in one third of the time it takes Nikki to run the same distance. They set off in opposite directions round the track as shown. Where will they meet for the first time?




A B C D E

**9.** Two cats together catch sixty mice. If Rosie catches three mice for every two that Josie catches, how many does Josie catch?

A. 2 B. 30 C. 24 D. 40 E. 36

**10.** Which of the following is nearest to your own age, expressed in seconds?

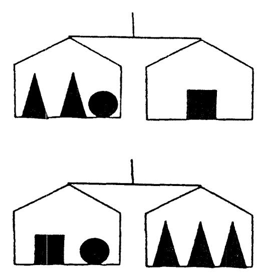
A. 5 000 000 B. 50 000 000 C. 500 000 000 D. 5 000 000 000 E. 50 000 000 000

**11.** Quince, quonce and quance are three types of fruit. If seven quince weigh the same as four quonce, and five quonce weigh the same as six quance, then the order of heaviness of the fruits (heaviest last) is

A. quince, quonce, quance B. quance, quince, quonce C. quonce, quance, quince

D. quonce, quince, quance E. quince, quance, quonce.

**12.** A child's box of bricks contains cubes, cones and spheres. Two cones and a sphere on one side of a pair of scales will just balance a cube on the other side, and a sphere and a cube together will just balance three cones. How many spheres will just balance a single cone?



A. 1 B. 2 C. 3 D. 4 E. 5

**13.** A recipe for eight flapjacks needs 2 oz butter, 3 oz sugar and 4 oz rolled oats. How many flapjacks can I make if I have 14 oz butter, 15 oz sugar and 16 oz rolled oats?

A. 40 B. 32 C. 44 D. 56 E. None of these

**14.** Weighing the baby at the clinic was a problem. The baby would not keep still and caused the scales to wobble. So I held the baby and stood on the scales while the nurse read off 78 kg. Then the nurse held the baby while I read off 69 kg. Finally I held the nurse while the baby read off 137 kg. What would the combined weight of nurse, baby and me be (in kilograms)?

A. 142 B. 147 C. 206 D. 215 E. 284

**15.** Arash runs faster than Betty, and Dovey will always beat Chandra in a race. Betty is never beaten by Edwina. One day all five race against each other. Just one of the following results is possible. Which is it? ($ABCDE$ indicates 'Arash first,...')

A. $ABCDE$ B. $BEDAC$ C. $ABCED$ D. $ADBCE$ E. $ADCEB$

**16.** A quadrilateral can have four right angles. What is the largest number of right angles an octagon (8 sides) can have?

A. 6 B. 4 C. 2 D. 3 E. 8

**17.** Our class decided to raise money for charity using a 'silver line'. We invited people to put 10p pieces edge to edge to make a long line. The completed line was 25 metres long. Roughly how much money did we make?

A. £25 B. £100 C. £500 D. £1000 E. £5000

**18.** How big is the angle between the hour hand and the minute hand of a clock at twenty to five?

A. 100° B. 25° C. 90° D. 105° E. 80°

**19.** How many pairs of numbers of the form $x, 2x+1$ are there in which both numbers are prime numbers less than 100?

A. 3 B. 4 C. 6 D. 7 E. More than 7

**20.** How many triangles have all three angles perfect squares (in degrees)?

A. 0 B. 1 C. 2 D. 3 E. 4

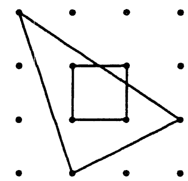
**21.** Which of the following has the greatest value?

A. $2^{32}$ B. $4^{15}$ C. $8^{11}$ D. $16^{8}$ E. $32^{6}$

**22.** We can write '384' as '$4\bar{2}4$', the bar indicating a *negative* digit (so $4\bar{2}4$ means $4 \times 100 - 2 \times 10 + 4$). How could we write 1988?

A. $2\bar{1}0\bar{2}$ B. $200\bar{2}$ C. $2\bar{1}2\bar{2}$ D. $2\bar{1}\bar{1}\bar{2}$ E. $20\bar{1}\bar{2}$

**23.** The dots are one unit apart. The region common to both the triangle and the square has area (in square units)



A. $\frac{9}{10}$ B. $\frac{15}{16}$ C. $\frac{8}{9}$ D. $\frac{11}{12}$ E. $\frac{14}{15}$

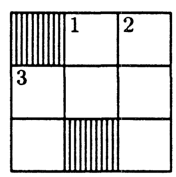
**24.** John, Peter, Rudolf, Susie and Tony decide to set some questions for the Schools Mathematical Challenge. John thinks up twenty-five questions and circulates them to the others. Peter, not wishing to be outdone, then sets twenty-six questions. Rudolf decides he had better get cracking and produces thirty questions. Susie sets to and comes up with thirty-nine posers, only to be beaten by Tony who produces fifty-five stunning problems. If each person takes half as long as the previous person to set each question, and Rudolf takes one hour to set all his questions, then the total time spent setting questions is

A. 7h $10\frac{1}{2}$m B. 11h $34\frac{1}{2}$m C. 7h 45m D. 2h 55m E. 5h 50m

**25.** A crossnumber is like a crossword except that the answers are numbers, with one digit in each square. What is the sum of the two entries in the bottom row of the crossnumber shown here?

CLUES

| Across | Down |
|---|---|
| 1. Prime number. | 1. Prime number. |
| 3. Square of the answer to 3 Down. | 2. Square of the answer to 1 Down. |
| | 3. Square root of the answer to 3 Across. |




* * *

Next we give an Olympiad level contest for which, of course, we solicit your nice solutions. The problems that we give are the 11th Form problems from the XXV Soviet Mathematical Olympiad, Smolensk, Russia, April 17 – 24, 1991. Once again I am indebted to Georg Gunther, Sir Wilfred Grenfell College, for collecting these problems and sending them in to me.

## XXV SOVIET MATHEMATICAL OLYMPIAD, 11TH FORM

### Day 1

**11.1** For any nonnegative integer $n$ the number $a_{n+1}$ is obtained from $a_n$ by the following rule: if the last digit of $a_n$ does not exceed 5, then this digit is removed and the remaining sequence of digits forms a decimal representation of $a_{n+1}$ (if $a_{n+1}$ contains no digits the process stops). Otherwise $a_{n+1} = 9a_n$. Can $a_0$ be chosen so that this process is infinite? (A. Azamov, S. Konjagin)

**11.2** The reals $\alpha$ and $\beta$ are such that

$$\alpha^3 - 3\alpha^2 + 5\alpha = 1, \quad \beta^3 - 3\beta^2 + 5\beta = 5.$$

Find $\alpha + \beta$. (B. Kukushkin)

**11.3** There are points $A$, $B$, $C$, $D$, $E$ on a sphere, such that the chords $AB$ and $CD$ intersect at a point $F$ and $A$, $C$, $F$ are equidistant from the point $E$. Prove that $BD$ and $EF$ are perpendicular. (B. Chinik, I. Sergeev)

**11.4** Does there exist a set containing:

(a) four noncollinear vectors such that the sum of each possible pair is perpendicular to the sum of the other two?

(b) 91 nonzero vectors with the sum of any 19 vectors perpendicular to the sum of all others?

All vectors are supposed to belong to the same plane. (D. Fomin)

Day 2

**11.5** On the sides $AB$ and $AD$ of the square $ABCD$ points $K, N$ respectively are given so that $AK \cdot AN = 2BD \cdot DN$. The lines $CK$ and $CN$ intersect the diagonal $BD$ at the points $L$ and $M$. Prove that the points $K, L, M, N, A$ are concyclic. (D. Tereshin)

**11.6** There are 100 mutually conflicting countries on the planet Xenon. For maintaining peace military alliances were established. No one alliance contains more than 50 countries and each two countries both belong to at least one alliance. What is the least possible number of alliances? What would this number be under the additional condition that no pair of alliances jointly contains more than 80 countries? (D. Flaas)

**11.7** The $2n$ distinct reals $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are given. A table of $100 \times 100$ cells is filled so that in a cell, that is the intersection of the $i$th row with the $j$th column, the number $a_i + b_j$ is written down. Given that the products of all numbers in each column are equal to each other, prove that the products of all numbers in each row are equal to each other as well. (D. Fomin)

**11.8** The numbers $x_1, \ldots, x_{1991}$ are such that

$$|x_1 - x_2| + \cdots + |x_{1990} - x_{1991}| = 1991.$$

What is the greatest possible value of the expression $|y_1 - y_2| + \cdots + |y_{1990} - y_{1991}|$ where $y_k = (x_1 + \cdots + x_k)/k$? (A. Kachurovski)

* * *

Next we look at some comments, corrections, and an alternative solution sent in by our readers in response to recent numbers of the Corner. First an alternate solution.

**5.** [1986: 231; 1991: 202] *1986 Austrian-Polish Mathematical Competition.*

Determine all quadruples $(x, y, u, v)$ of real numbers satisfying the simultaneous equations

$$x^2 + y^2 + u^2 + v^2 = 4,$$
$$xu + yv = -xv - yu,$$
$$xyu + yuv + uvx + vxy = -2,$$
$$xyuv = -1.$$

*Alternative solution by Roger Lee, student, Harvard University.*

The published solution uses all four equations. In fact the second is redundant. From the first and last equations,

$$\sqrt{\frac{x^2 + y^2 + u^2 + v^2}{4}} = 1 = \sqrt[4]{|xyuv|}.$$

But the Root Mean Square–Harmonic Mean inequality applied to $|x|$, $|y|$, $|u|$ and $|v|$ implies that equality holds only if $|x| = |y| = |u| = |v| = 1$. Divide the third equation by the last to find $x + y + z + w = 2$. So three of $x, y, u$ and $v$ are 1 and one is $-1$. These four quadruples do indeed satisfy the equations.

* * *

Next, two corrections. Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario, points out that in the solution to #1 [1992: 231], $w = (2n)!$ should obviously read $w = (2n + 1)!$. Thanks for spotting the misprint!

Peter de Caux, Eatonton, Georgia, points out an oversight in the generalization by Murray Klamkin of #4 [1992: 303]. The second line should read "$G(x)$ is decreasing and concave up". Somehow concave got left out. My apologies again.

* * *

We now return to the file of solutions sent in by the readers to problems from the December 1991 number of the Corner. These are solutions to problems of the *1991 Japanese I.M.O.* [1991: 290].

**1.** Let $P$, $Q$ and $R$ be three points on the sides $BC$, $CA$ and $AB$ of a triangle $ABC$ respectively, such that

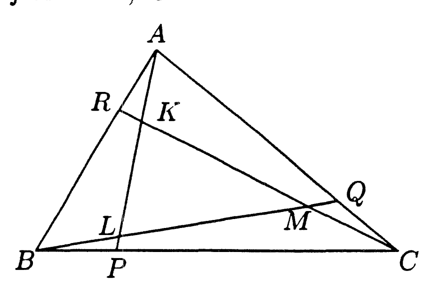
$$BP : PC = CQ : QA = AR : RB = t : 1 - t$$

for some real number $t$. Show that the three line segments $AP$, $BQ$ and $CR$ will form a triangle, $\Delta$ say. Find the ratio of the area of triangle $ABC$ to the area of $\Delta$ in terms of $t$.

*Solution by Pavlos Maragoudakis, student, University of Athens, Greece.*

For a polygon $P$, let $[P]$ denote its area. Then

$$\frac{[BCQ]}{[ABC]} = \frac{BC \cdot CQ}{BC \cdot AC} = \frac{CQ}{AC} = t. \qquad (1)$$




Let $K$ be the point of intersection of $AP$ and $CR$, $L$ that of $AP$ and $BQ$ and $M$ the intersection of $BQ$ and $CR$.

The theorem of Menelaus for triangle $BCQ$ and the line $ALP$ gives

$$\frac{BL}{QL}\cdot\frac{QA}{CA}\cdot\frac{CP}{BP}=1, \qquad \text{so} \qquad \frac{BL}{QL}=\frac{CA}{QA}\cdot\frac{BP}{CP}\ .$$

But $CA/QA = 1/(1-t)$ and $BP/PC = t/(1-t)$. So $BL/QL = t/(1-t)^2$ and

$$\frac{BL}{BQ}=\frac{t}{t+(1-t)^2}\ . \tag{2}$$

Now

$$\frac{[BLP]}{[BQC]}=\frac{BL\cdot BP}{BQ\cdot BC}=\frac{t}{t+(1-t)^2}\cdot t=\frac{t^2}{t^2-t+1}\ .$$

Using (1),

$$[BLP]=\frac{t^3}{t^2-t+1}[ABC]. \tag{3}$$

Cyclically we obtain

$$[BQC]=[CRA]=[APC]=t[ABC],$$

$$[MQC]=[ARK]=[BPL]=\frac{t^3}{t^2-t+1}[ABC].$$

Now

$$\begin{aligned}[\Delta]&=[KLM]=[ABC]-[ABP]-[BQC]-[ARK]+[BLP]+[MQC]+[ARL]\\ &=\left(1-3t+\frac{3t^3}{t^2-t+1}\right)[ABC]=\frac{(2t-1)^2}{t^2-t+1}[ABC].\end{aligned}$$

Finally

$$\frac{[ABC]}{[\Delta]}=\frac{t^2-t+1}{(2t-1)^2}\ .$$

When $t = 1/2$ the above ratio has no meaning since $\Delta$ becomes a single point.

**2.** Let $\mathbb{N}$ be the set of positive integers and let $p$, $q$ be mappings from $\mathbb{N}$ to $\mathbb{N}$ given by:

$$p(1)=2,\ p(2)=3,\ p(3)=4,\ p(4)=1,\ p(n)=n \text{ for } n\geq 5,$$

$$q(1)=3,\ q(2)=4,\ q(3)=2,\ q(4)=1,\ q(n)=n \text{ for } n\geq 5.$$

(i) Find the mapping $f$ from $\mathbb{N}$ to $\mathbb{N}$ such that $f(f(n)) = p(n) + 2$, $n \geq 1$.
(ii) Show that there is no mapping $g$ from $\mathbb{N}$ to $\mathbb{N}$ such that $g(g(n)) = q(n) + 2$, $n \geq 1$.

*Solution by Joseph Ling, The University of Calgary.*

(i) We are asked to find "the" mapping $f$ such that $f(f(1)) = 4$, $f(f(2)) = 5$, $f(f(3)) = 6$, $f(f(4)) = 3$ and $f(f(n)) = n + 2$ for $n \geq 5$. I find at least **two** such functions. (Are there more?)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | $2k,\ k \geq 5$ | $2k+1,\ k \geq 5$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f_1(n)$ | 2 | 4 | 7 | 5 | 3 | 9 | 6 | 11 | 8 | $2k+3$ | $2k$ |
| $f_2(n)$ | 5 | 1 | 9 | 7 | 4 | 11 | 3 | 13 | 6 | $2k+5$ | $2k-2$ |

(ii) We prove that there is no function $g$ from $\mathbb{N}$ to $\mathbb{N}$ such that

$$g(g(1)) = 5, \quad g(g(2)) = 6, \quad g(g(3)) = 4, \quad g(g(4)) = 3,$$

and $g(g(n)) = n + 2$ for $n \geq 5$. Indeed, if $g$ is such a function, consider $g(3)$.

*Case* 1. $g(3) = 1 \Rightarrow g(1) = g(g(3)) = 4$, $g(4) = g(g(1)) = 5$, $g(5) = g(g(4)) = 3$ and $g(3) = g(g(5)) = 7$, a contradiction.

*Case* 2. $g(3) = 2 \Rightarrow g(2) = 4$, $g(4) = 6$, $g(6) = 3$ and $g(3) = g(g(6)) = 8$, a contradiction.

*Case* 3. $g(3) = 3 \Rightarrow g(g(3)) = 4$, a contradiction.

*Case* 4. $g(3) = 4 \Rightarrow g(4) = 4$, a contradiction.

*Case* 5. $g(3) = n \geq 5$. Then $g(n) = g(g(3)) = 4$, and $g(4) = g(g(n)) = n + 2$. Then $g(n+2) = g(g(4)) = 3$ and $g(3) = g(g(n+2)) = n+4$, a contradiction. Therefore, no such $g$ exists.

**3.** Suppose $A$ is a positive 16-digit integer. Show that we can find some consecutive digits of $A$ such that the product of these digits is a perfect square.

*Solution by Gillian Nonay, Wilfrid Laurier University, Waterloo, Ontario.*

Let $A = a_1 a_2 \ldots a_{16}$ be a positive 16-digit integer and let $p_i = \prod_{j=1}^{i} a_j$. We can assume no $a_i = 0$. Clearly we can express each of these products as

$$p_i = 2^{2\alpha_i + w_i} 3^{2\beta_i + x_i} 5^{2\gamma_i + y_i} 7^{2\delta_i + z_i}$$

where $w_i$, $x_i$, $y_i$ and $z_i$ are either 0 or 1. We are only interested in whether the exponents are even or odd and accordingly we define the binary sequence $s_i = (w_i, x_i, y_i, z_i)$. If $s_i = (0,0,0,0)$ for some $i$ then $p_i$ is a perfect square. Now assume $s_i \neq (0,0,0,0)$, $1 \leq i \leq 16$. There are 15 non-zero binary sequences of length 4, and so (by the pigeon hole principle) two of the $s_i$ must be equal. Say $s_i = s_j$ with $i < j$. Then $p_j/p_i = \prod_{k=i+1}^{j} a_k$ must be a perfect square.

*Notes:* 1. Without loss of generality we may assume that none of the digits of $A$ are 0, 1, 4, or 9. (This doesn't affect the proof.)

2. Here, 16 is "best possible". That is, there exist 15-digit integers containing no consecutive digits whose product is a perfect square. One example is 232523272325232 (a palindrome).

**4.** Consider the $10 \times 14$ matrix $(a_{i,j})$, where $a_{i,j} = 0$ or 1 for $1 \leq i \leq 10$, $1 \leq j \leq 14$, and such that each column contains an odd number of ones and each row also contains an odd number of ones. Show that among the numbers $a_{i,j}$ such that $i + j$ is even, there is precisely an even number of ones.

*Solution by Gillian Nonay, Wilfrid Laurier University, Waterloo, Ontario.*

We first generalize the problem. Let $A = (a_{i,j})$ be an $r \times c$ 0-1 matrix in which each row and each column contains an odd number of ones. Then $r$ and $c$ must have the same parity. (If, for example, $r$ was odd and $c$ was even, then counting the number of ones by rows would give an odd number of ones in $A$, whereas counting the number of ones by columns would give an even number of ones in $A$.) Label the rows of $A$ with the elements of $R = \{1, 2, \ldots, r\}$ and label the columns of $A$ with the elements from $C = \{1, 2, \ldots, c\}$. Let $X$ and $Y$ be subsets of $R$ and $C$, respectively, and define $\overline{X} = R - X$, $\overline{Y} = C - Y$, $x = |X|$, and $y = |Y|$. Let $S = \sum_{i\in X}\sum_{j\in\overline{Y}} a_{ij} + \sum_{i\in\overline{X}}\sum_{j\in Y} a_{i,j}$.

We show that $S$ is even iff $x \equiv y \bmod 2$. The proof follows.

$$S = \left(\sum_{i\in X}\sum_{j\in\overline{Y}} a_{i,j} + \sum_{i\in X}\sum_{j\in Y} a_{i,j}\right) + \left(\sum_{i\in\overline{X}}\sum_{j\in Y} a_{i,j} + \sum_{i\in X}\sum_{j\in Y} a_{i,j}\right) - 2\sum_{i\in X}\sum_{j\in Y} a_{i,j}$$
$$= \sum_{i\in X}\sum_{j=1}^{c} a_{i,j} + \sum_{i=1}^{r}\sum_{j\in Y} a_{i,j} - 2\sum_{i\in X}\sum_{j\in Y} a_{i,j}.$$

Since the number of ones in each row is odd, $\sum_{i\in X}\sum_{j=1}^{c} a_{i,j}$ will be even if $x$ is even and odd if $x$ is odd. Similarly, since the number of ones in each column is odd, $\sum_{j=1}^{r}\sum_{j\in Y} a_{i,j}$ will be even if $y$ is even and odd if $y$ is odd.

As a special case we can let $S$ be the number of ones among the entries $a_{i,j}$ such that $i + j$ is even. (In this form the problem can be restated as a chessboard problem. With this interpretation and assuming the upper left square of the chessboard is white, the proposer's problem is to show that the number of ones on all the white squares is even.) We get the following: $S$ is even if and only if either

1. $r$ and $c$ are both even and $r \equiv c \bmod 4$ or
2. $r$ and $c$ are both odd and $r \equiv c + 2 \bmod 4$.

To see this consider the following short proofs.

1. If $r = 2m$ and $c = 2n$ then we take $X = \{1, 3, \ldots, 2n-1\}$ and $Y = \{2, 4, \ldots, 2n\}$ so that $x = m$, $y = n$. Since $x \equiv y \bmod 2$ we see that $r \equiv c \bmod 4$.

2. If $r = 2m + 1$ and $c = 2n + 1$, then we take $X = \{1, 3, \ldots, 2m + 1\}$, $Y = \{2, 4, \ldots, 2n\}$ so that $x = m + 1$, $y = n$. Since $x \equiv y \bmod 2$ we see that $r \equiv c + 1 \bmod 4$.

Since $10 \equiv 14 \bmod 4$, the question as stated by the proposer has now been solved.

* * *

That completes our file of solutions for the December 1991 number of the Corner. We move now to the January 1992 issue. The first group are solutions to problems of the *XLI Mathematics Olympiad in Poland* [1992: 2–3].

**1.** Determine all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ which satisfy

$$(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2)$$

for all real $x, y$.

*Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by Murray S. Klamkin, University of Alberta; by Joseph Ling, The University of Calgary; by Pavlos Maragoudakis, student, University of Athens, Greece; by Beatriz Margolis, Paris, France; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Margolis's solution.*

We show that the given condition is equivalent to $f(x) = x(\alpha+x^2)$ for all $x$, for some $\alpha \in \mathbb{R}$. A direct calculation shows that every cubic of this form satisfies the conditions, so suppose

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2-y^2). \tag{1}$$

Taking $x = y \neq 0$ we conclude that $f(0) = 0$. Define $a = x+y$, $b = x-y$ so that $a+b = 2x$, $a-b = 2y$ and (1) is equivalent to

$$bf(a) - af(b) = ab(a^2-b^2), \quad a, b \in \mathbb{R}.$$

If $ab \neq 0$

$$\frac{f(a)}{a} - \frac{f(b)}{b} = a^2 - b^2.$$

In other words $f(x)/x - x^2 = \alpha$, a constant, for $x \neq 0$. But then for all $x$, $f(x) = \alpha x + x^3$, as desired.

Note too that $\alpha = \lim_{x\to 0} f(x)/x$ , so that $f'(0) = \alpha$.

[*Editor's note.* Klamkin points out that this method can be used to show that there are no functions satisfying

$$(x-y)f(x+y) - (x+y)f(x-y) = k(xy)^n(x^2-y^2)$$

for $k$ and $n$ given constants with $n \neq 1$.]

**2.** Let $n > 2$ be a natural number and let $x_1, \ldots, x_n$ be positive real numbers. Prove:

$$\frac{x_1^2}{x_1^2 + x_2x_3} + \frac{x_2^2}{x_2^2 + x_3x_4} + \cdots + \frac{x_n^2}{x_n^2 + x_1x_2} \leq n - 1.$$

*Comment by Murray S. Klamkin, University of Alberta.*

This problem appeared previously [1985: 304; 1991: 41]. Also a generalization has already appeared [1988: 3; 1990: 155]. [*Editor's note.* Seung-Jin Bang, Seoul, Republic of Korea, also noted the prior use of this problem.]

**3.** In a tournament of $n$ participants, each pair has played exactly one game (no ties). Show that either

(i) the league splits into two (nonempty) subsets $A$, $B$ so that each player of group $A$ has beaten each player of group $B$, or

(ii) all players can be arranged into a cyclic sequence in which every player has beaten his successor.

*Solution by Kar Keung To (1992 Hong Kong I.M.O. Team Member), King's College, Hong Kong (forwarded by Kin Li, Hong Kong University of Science and Technology).*

Take the longest cyclic sequence $C$. If player $X$ is not in the sequence, then either $X$ beats every player in the sequence or loses to every player in the sequence. (Otherwise, $X$ beats $P_i$ and loses to $P_j$ in the sequence. If $P_i$ and $P_j$ are not adjacent, then by moving from $P_j$ to $P_i$ in the sequence, there will be a pair such that $X$ beats one and loses to its successor. $X$ can then be inserted into the sequence, a contradiction to $C$ being longest.)

For the players not in $C$, let $A_0$ be the set that beats everybody in $C$ and $B_0$ be the set that loses to everybody in $C$. Now nobody in $B_0$ beat anybody in $A_0$ (otherwise we can enlarge $C$ by inserting the player from $B_0$ and the player from $A_0$). If $C$ included every player, then (ii) is true. Otherwise, let $A = A_0$ and $B = B_0 \cup C$, or $A = A_0 \cup C$ and $B = B_0$, to see that (i) is true.

[*Editor's note.* The argument of this paragraph also shows (taking $C$ to be a singleton) that (i) holds unless there *is* a nontrivial cycle to which To's argument applies. C. Wildhagen, Rotterdam, The Netherlands, also points out that the problem is well known from graph theory: each connected digraph either contains a directed Hamilton cycle, or a directed (edge-)cutset.]

**4.** A triangle of all sides $\geq 1$ is placed in a square of side 1. Show that the centre of the square belongs to the triangle.

*Solutions by Pavlos Maragoudakis, student, University of Athens, Greece; and by Kor Keung To (1992 Hong Kong I.M.O. Team Member), King's College, Hong Kong.*

Suppose the centre $O$ of square $ABCD$ with side length one does not belong to $\Delta XYZ$ but that $\Delta XYZ$ lies inside the square and has $XY, YZ, XZ \geq 1$. Let $M$, $N$, $P$ and $Q$ be the midpoints of $AB$, $BC$, $CD$ and $DA$, respectively. It follows that one of $MP$, $NQ$, or the two diagonals $AC$ or $BD$ divide the square in two pieces so that one of the pieces contains no points of $\Delta XYZ$ in its interior. Consider now the line segment through the centre which is perpendicular to this segment. By the pigeon hole principle, two of the vertices of the triangle will belong to the same quarter of the square bordered on two sides by the segments. However, the furthest pair of points in each quarter are the endpoints of the diagonal not containing the centre, in case the segments are parallel to the sides, and the vertices of the square (in case the segments are diagonals). This distance is at most one, and equality is only possible when the two vertices of the triangle are vertices of the square (and the "half-square" is formed by two sides and a diagonal). The same reasoning forces the third vertex to be a side of the square as well, and hence the triangle contains the centre, a contradiction.

**5.** Given a sequence of positive integers $(a_n)$ with $\lim_{n\to\infty}(n/a_n) = 0$, show that, for some $k$, there are not less than 1990 perfect squares between $a_1 + \cdots + a_k$ and $a_1 + \cdots + a_k + a_{k+1}$.

*Solutions by Pavlos Maragoudakis, student, University of Athens, Greece; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Wildhagen's solution.*

We actually prove a slight generalization. Let $(a_n)$ be a sequence of positive real numbers with the property that

$$\lim_{n\to\infty} n/a_n = 0, \tag{1}$$

assuming that $a_n \neq 0$ for all $n$. Put $S_k = \sum_{j=1}^{k} a_j$, for $k \in \mathbb{N}$. We prove the following result:

*For each $l \in \mathbb{N}$ there exist finitely many $k \in \mathbb{N}$ such that there*

$$\text{are at least } l \text{ perfect squares between } S_k \text{ and } S_{k+1}. \tag{2}$$

Suppose that (2) does not hold for some $l \in \mathbb{N}$. Then there exists some $n \in \mathbb{N}$ such that

$$\sqrt{s_{m+1}} - \sqrt{s_m} < l \tag{3}$$

for all $m \geq N$. Summing these relations for $m = N, N+1, \ldots, k-1$ yields $\sqrt{S_k} - \sqrt{S_N} < (k-N)l$, hence $\sqrt{S_k} < c_1 + lk$, for all $k \geq N+1$, where $c_1$ denotes some constant. Choose $N_1 \geq N+1$ such that $k \geq N_1$ implies $lk > |c_1|$. Then $\sqrt{S_k} < 2lk$, i.e.

$$S_k < 4l^2k^2 \quad \text{for all } k \geq N_1. \tag{4}$$

On the other hand (1) implies that for some $m \in \mathbb{N}$, $a_j > 16l^2 j$ for all $j \geq M$. Therefore

$$S_k = \sum_{1 \leq j \leq m-1} a_j + \sum_{j=M}^{k} a_j > c_2 + 8l^2k^2$$

for all $k \geq M$, with $c_2$ some constant. Choose $M_1 \geq M$ such that $c_2 > -4k^2$ for all $k \geq M_1$. Then

$$S_k > 4l^2k^2 \quad \text{for all } k \geq M_1. \tag{5}$$

The contradiction arises by a comparison of (4) and (5). This implies that (2) is indeed valid.

**6.** Prove that for every integer $n > 2$ the number

$$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k \binom{n}{3k}$$

is divisible by 3.

*Solutions by Curtis Cooper, Central Missouri State University, Missouri; by Pavlos Maragoudakis, student, University of Athens, Greece; by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Wang's solution.*

Let $f(n)$ denote the given sum. We show that for all natural numbers $n$

$$f(n) = \frac{2}{3}(\sqrt{3})^n \cos\left(\frac{n\pi}{6}\right).$$

Let $\omega = (-1+\sqrt{3}\,i)/2$ denote a complex third root of unity. Then

$$\sum_{j=0}^{2}(1-\omega^j)^n = \sum_{j=0}^{2}\sum_{q=0}^{n}(-1)^q\binom{n}{q}\omega^{jq} = \sum_{q=0}^{n}\left((-1)^q\binom{n}{q}\sum_{j=0}^{2}\omega^{jq}\right).$$

Since

$$\sum_{j=0}^{2} \omega^{qj} = 1 + \omega^q + \omega^{2q} = \begin{cases} 3 & \text{if } q \equiv 0 \pmod 3 \\ 0 & \text{if } q \equiv 1, 2 \pmod 3), \end{cases}$$

we have

$$\sum_{j=0}^{2}(1-\omega^j)^n = 3 \sum_{q \equiv 0 \bmod 3} (-1)^q \binom{n}{q} = 3 \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^{3k} \binom{n}{3k} = 3 \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k \binom{n}{3k} = 3f(n).$$

Thus

$$f(n) = \frac{1}{3}\sum_{j=0}^{2}(1-\omega^j)^n = \frac{1}{3}((1-\omega)^n + (2+\omega)^n).$$

Now

$$(1-\omega)^n = \left(\frac{3-\sqrt{3}\,i}{2}\right)^n = (\sqrt{3})^n \left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^n$$

and

$$(2+\omega)^n = \left(\frac{3+\sqrt{3}\,i}{2}\right)^n = (\sqrt{3})^n \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n.$$

Substituting, and using De Moivre's formula we obtain

$$f(n) = \frac{2}{3}(\sqrt{3})^n \cos\left(\frac{n\pi}{6}\right)$$

from which we can deduce easily by considering the residues of $n$ modulo 12 that

$$f(n) = \begin{cases} 2 \times 3^{(n-2)/2} & \text{if } n \equiv 0 \bmod 12 \\ 3^{(n-1)/2} & \text{if } n \equiv 1 \bmod 12 \\ 3^{(n-2)/2} & \text{if } n \equiv 2 \bmod 12 \\ 0 & \text{if } n \equiv 3, 9 \bmod 12 \\ -3^{(n-2)/2} & \text{if } n \equiv 4, 8, 10 \bmod 12 \\ -3^{(n-1)/2} & \text{if } n \equiv 5, 7, 11 \bmod 12 \\ -2 \times 3^{(n-2)/2} & \text{if } n \equiv 6 \bmod 12 \end{cases}$$

From this it is evident that 3 divides $f(n)$ for all $n \geq 3$.

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That is all the space we have for this column. Send me your national and regional Olympiad contests. Also please send any interesting pre-Olympiad contest material you may have.

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# BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

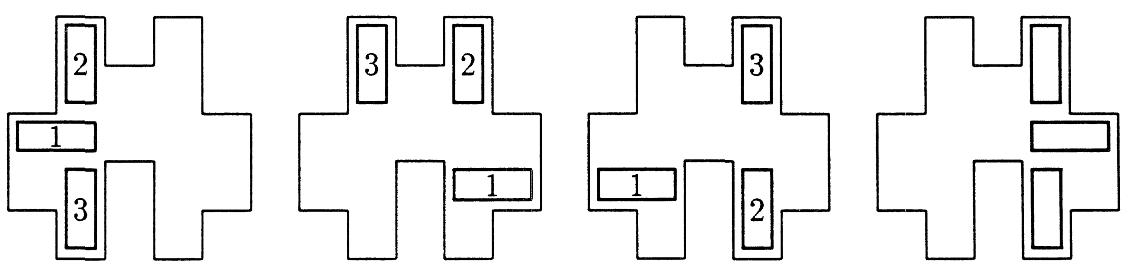
*Another Fine Math You've Got Me Into ...* by Ian Stewart. Published by W.H. Freeman & Company, New York, 1992, ISBN 0-7167-2342-5 (hardcover), -2341-7 (paperback), xii+269 pages, US$21.95/$13.95. *Reviewed by Richard Guy.*

I've always thought that Martin Gardner, who, appropriately enough, is presenting this book, was an impossible act to follow, but Ian Stewart bids fair to be doing so. We've already had *Concepts of Modern Mathematics*, Penguin, 1975; *The Problems of Mathematics*, Oxford Univ. Press, 1987; *Does God Play Dice?*, Penguin, 1989; and *Game, Set and Math*, Oxford & Cambridge, 1989, Penguin, 1991; and we can look forward to *Fearful Symmetry: Is God a Geometer?* and perhaps also to translations (from French) of *Into the Two-and-a-halfth Dimension*, *The Groups of Wrath* and *Oh! Catastrophe.*

Martin Gardner has always claimed that his success is because he is not a mathematician. Ian Stewart doesn't have this advantage. Is there too much mathematics in his books for the general reader? Maybe, but I hope that his material is sufficiently cunningly arranged that the nonmathematician can skip the more technical parts, at least on a first reading, and yet still be drawn into the subject so that she wants to learn more. I say "hope" because, as a mathematician, I'm unable to judge this.

For recreational math buffs, there are many old familiars, but always with a new twist. We will all have our favorites: here are some of mine.

River crossing problems (Ch. 1) can be traced back [**9**] for more than a millenium, but I haven't earlier seen them related via the Tower of Hanoï to Sierpiński's gasket, and to the following sliding block puzzle, to which I give a solution, since Stewart doesn't (quite rightly, the reader must be made to work). Move the three dominoes from the left to the right of the diagram. The numbers indicate the order of sliding. Number 1 corresponds to the llama, which must be left in the company of neither the lion nor the lettuce.




Sol Golomb's polyominoes (Ch. 2) have been around for quite a while [**3**]; about the same time as Sol himself, and we celebrated a Fiesta del Sol recently. But there are still many unsolved problems: two of them were settled by blind mathematician Karl Dahlke in 1989 [**2** — the 76 in the second title should be 78].

Chris Zeeman's application of catastrophe theory to the Origin of Species is explored in Chapter 3 and his catastrophe machine illustrates the breaking of symmetry (Ch. 10).

Why can a spotted animal have a striped tail, but not the other way round?

Chapter 9 discusses classical game theory, while combinatorial games appear in Chapter 11. Here the chocolate bar game should have been identified as Chomp and attributed to David Gale, and the arithmetical form mentioned [**8**].

In Ch. 12 the general gas law is described as Boyle's law, whereas Dalton, Gay-Lussac or Charles should get equal billing. The thermodynamics of curlicues concerns exponential sums and their limits as fractals. The Lehmers' beautiful papers [**7**] should have received a mention here.

Other topics are drawing Venn diagrams (Ch. 4); Pick's theorem on lattice areas (Ch. 5); touring the sights (or sites) and incidence matrices of graphs (Ch. 6); a knight's tour on a torus revealed by a Hamilton circuit on a tesseract (Ch. 7); the group theory of bell-ringing (Ch. 13) where a Cayley graph [**5**] is used as an illustration but not mentioned by name; packs and clusters (points in the plane at integer distances apart) (Ch. 14); and (Ch. 15) the mathematics of musical scales, including Barbour's vindication of Strähle's construction for placing the frets on a guitar — continued fractions might have appeared here.

It's sobering to find one's self becoming part of history. The sofa problem (Ch. 16) might have been discussed at the 1965 Copenhagen convexity conference, as suggested by Albert Wormstein, but I doubt it, since Leo Moser wasn't there. He was proud of the fact that he only crossed the Atlantic once in his lifetime; at an early age and from East to West. It certainly was discussed a year later in Michigan, but in East Lansing rather than Ann Arbor. There Leo circulated his historic *Fifty Poorly Posed Problems in Combinatorial Geometry*, which became a major part of the basis of [**1**]. The seven people most likely to have been involved are John Conway, Hallard Croft, Leo Moser, Jonathan Schaer, John Selfridge, Geoffrey Shephard and this reviewer. I'm fairly sure that the Conway Car and Shephard Piano were conceived at that meeting.

Projective planes are discussed in Chapter 8, leading to a discussion of the proof [**6** — what is a proof?] of the non-existence of a projective plane of order 10. Euler's officers problem [**4**] might have had a mention here. Hopefully the author will return to this fertile topic on future occasions and show us the remarkable connexions with Hamming codes, Steiner systems, difference sets, map-coloring problems, nim-like games, tournaments, Zarankiewicz's problem and Langford–Skolem sequences. There are two different significant ways of labelling the Fano configuration; why is PSL(2,7) $\cong$ GL(3,2)? Come to Vancouver in August and hear the M.A.A. Student Chapter lecture on the unity of combinatorics. But buy a copy of the book and bring it along to read, just in case the lecture gets boring.

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**1.** Hallard T. Croft, Kenneth J. Falconer & Richard K. Guy, *Unsolved Problems in Geometry*, Springer, 1991.

**2.** Karl A. Dahlke, The Y-hexomino has order 92; A heptomino of order 76; erratum, *J. Combin. Theory Ser. A*, **51**(1989) 125–126, 127–128; **52**(1989) 321; *MR* **90b**:05041, **90b**:05042, **90j**:05053.

**3.** T. R. Dawson & W. E. Lester, A notation for dissection problems, *Fairy Chess Review*, **3**(Apr 1937) 5, 46–47.

**4.** L. Euler, *Verhandelingen Zeeuwsch Genootschap der Wetenschappen*, **9**(1782) 85–239.

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**6.** Clement W. H. Lam, Larry Henry Thiel & S. Swiercz, A computer search for a projective plane of order 10, in *Algebraic, Extremal and Metric Combinatorics*, Cambridge Univ. Press, 1988, 155–165.

**7.** D. H. & Emma Lehmer, Picturesque exponential sums I, *Amer. Math. Monthly*, **86**(1979) 725–733; II, *J. reine angew. Math.*, **318**(1980) 1–19; *MR* **80m**:10030, **81j**:10056.

**8.** Fred. Schuh, The game of divisors, *Nieuw Tijdschrift voor Wiskunde*, **39**(1952) 299–304.

**9.** David Singmaster, *Sources in Recreational Mathematics*, 1986, 1987, ... .

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## PROBLEMS

*Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.*

*Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.*

*To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before* **September 1, 1993**, *although solutions received after that date will also be considered until the time when a solution is published.*

**1811**. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $d$ and $k$ be natural numbers with $d|k$. Let $X_k$ be the set of all $k$-tuples $(x_1, \ldots, x_k)$ of integers such that $0 \le x_1 \le \cdots \le x_k \le k$ and $x_1 + \cdots + x_k$ is divisible by $d$. Furthermore let $Y_k$ be the set of all elements $(x_1, \ldots, x_k)$ of $X_k$ such that $x_k = k$. What is the relationship between the sizes of $X_k$ and $Y_k$?

**1812**. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABC$ is a right-angled triangle with the right angle at $C$. Let $D$ be a point on side $AB$, and let $M$ be the midpoint of $CD$. Suppose that $\angle AMD = \angle BMD$. Prove that

$$\angle ACD : \angle BCD = \angle CDA : \angle CDB.$$

**1813***. *Proposed by D.N. Verma, Bombay, India.*

Suppose that $a_1 > a_2 > a_3$ and $r_1 > r_2 > r_3$ are positive real numbers. Prove that the determinant

$$\begin{vmatrix} a_1^{r_1} & a_1^{r_2} & a_1^{r_3} \\ a_2^{r_1} & a_2^{r_2} & a_2^{r_3} \\ a_3^{r_1} & a_3^{r_2} & a_3^{r_3} \end{vmatrix}$$

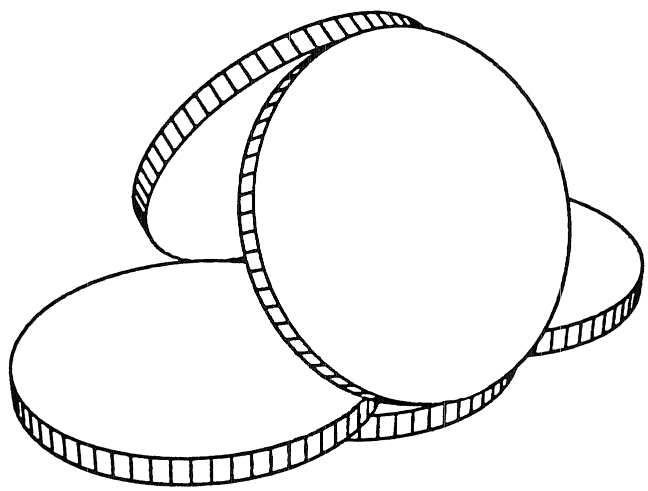
is positive.

**1814**. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Given are the fixed line $\ell$ with two fixed points $A$ and $B$ on it, and a fixed angle $\varphi$. Determine the locus of the point $C$ with the following property: the angle between $\ell$ and the Euler line of $\Delta ABC$ equals $\varphi$.

**1815**. *Proposed by Stan Wagon, Macalester College, St. Paul, Minnesota.*

An old puzzle (see *Mathematical Puzzles and Diversions*, Martin Gardner, Simon & Schuster, New York, 1959, p. 114, or *Puzzlegrams*, Simon & Schuster/Fireside, New York, 1989, p. 171) asks that five congruent coins be placed in space so that each touches the other four. The solution often given is as illustrated below: one coin supports two others, which meet over the center of the bottom coin, with two tilted coins forming the sides of the tent-like figure.



Show that this solution is *invalid* if the coins are nickels. Assume (despite the picture) that the nickels are ordinary circular cylinders with diameter to height ratio of 11 to 1.

**1816**. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Given a finite set $\mathcal{S}$ of $n+1$ points in the plane, with two distinguished points $B$ and $E$ in $\mathcal{S}$, consider all polygonal paths $\mathcal{P} = P_0P_1\dots P_n$ whose vertices are all points of $\mathcal{S}$, in any order except that $P_0 = B$ and $P_n = E$. For such a path $\mathcal{P}$ define $l(\mathcal{P})$ to be the length of $\mathcal{P}$ and

$$a(\mathcal{P}) = \sum_{i=1}^{n-1} \theta(\overrightarrow{P_{i-1}P_i}, \overrightarrow{P_iP_{i+1}}),$$

where $\theta(\mathbf{v},\mathbf{w})$ is the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$, $0 \le \theta(\mathbf{v},\mathbf{w}) \le \pi$. Prove or disprove that the minimum values of $l(\mathcal{P})$ and of $a(\mathcal{P})$ are attained for the same path $\mathcal{P}$.

**1817**. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

For a natural number $n$, $d(n)$ denotes the number of positive integer divisors of $n$ (including $n$ itself) and $\phi(n)$ denotes the number of positive integers less than $n$ and relatively prime to $n$. Find all positive integers $n$ so that $d(n) + \phi(n) = n$.

**1818**. *Proposed by Ed Barbeau, University of Toronto.*

Prove that, for $0 \le x \le 1$ and a positive integer $k$,

$$(1+x)^k[x + (1-x)^{k+1}] \ge 1.$$

**1819**. *Proposed by Joaquín Gómez Rey, I.B. Luis Bunuel, Alcorcón, Madrid, Spain.*

An urn contains $n$ balls numbered from 1 to $n$. We draw $n$ balls at random, *with replacement* after each ball is drawn. What is the probability that ball 1 will be drawn an odd number of times, and what is the limit of this probability as $n \to \infty$?

**1820***. *Proposed by Juan Bosco Romero Márquez, Universidad de Valladolid, Spain.*

Let $O$ be the point of intersection of the diagonals $AC$ and $BD$ of the quadrangle $ABCD$. Prove that the orthocenters of the four triangles $OAB$, $OBC$, $OCD$, $ODA$ are the vertices of a parallelogram that is similar to the figure formed by the centroids of these four triangles. What if "centroids" is replaced by "circumcenters"?

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## SOLUTIONS

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

**1510***. [1990: 20; 1991: 91] *Proposed by Jack Garfunkel, Flushing, New York.*

$P$ is any point inside a triangle $ABC$. Lines $PA, PB, PC$ are drawn and angles $PAC, PBA, PCB$ are denoted by $\alpha, \beta, \gamma$ respectively. Prove or disprove that

$$\cot\alpha + \cot\beta + \cot\gamma \ge \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2},$$

with equality when $P$ is the incenter of $\Delta ABC$.

*Comment by Jia-Zhi Lian and Ji Chen, Ningbo University, China.*

We generalize Kuczma's inequality (1) on [1991: 92], which he gives in his published solution.

*Theorem.* Let $P$ be any point inside a convex $n$-gon $A_1A_2\ldots A_n$, and put

$$\alpha_1 = \angle PA_1A_n, \quad \alpha_2 = \angle PA_2A_1, \quad \ldots, \quad \alpha_n = \angle PA_nA_{n-1}.$$

Then

$$\sum_{i=1}^{n} \cot\alpha_i \ge \sum_{i=1}^{n} \cot A_i + n\left(\prod_{i=1}^{n} \sin A_i\right)^{-1/n}.$$

*Proof.* In the triangle $PA_iA_{i+1}$, by the sine law we have

$$\frac{\sin(A_i-\alpha_i)}{\sin\alpha_{i+1}}=\frac{PA_{i+1}}{PA_i}$$

for $i=1,2,\ldots,n$, where $A_{n+1}=A_1$ and $\alpha_{n+1}=\alpha_1$. Then

$$\prod_{i=1}^{n}\frac{\sin(A_i-\alpha_i)}{\sin\alpha_i}=1.$$

But

$$\frac{\sin(A_i-\alpha_i)}{\sin\alpha_i}=\sin A_i(\cot\alpha_i-\cot A_i),$$

hence

$$\prod_{i=1}^{n}(\cot\alpha_i-\cot A_i)=\left(\prod_{i=1}^{n}\sin A_i\right)^{-1}.$$

By the A.M.–G.M. inequality,

$$\sum_{i=1}^{n}(\cot\alpha_i-\cot A_i)\geq n\left[\prod_{i=1}^{n}(\cot\alpha_i-\cot A_i)\right]^{1/n}=n\left(\prod_{i=1}^{n}\sin A_i\right)^{-1/n},$$

which proves the theorem.

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**1598***. [1990: 299; 1992: 27] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $\lambda>0$. Determine the maximum constant $C=C(\lambda)$ such that for all non-negative real numbers $x_1,x_2$ there holds

$$x_1^2+x_2^2+\lambda x_1x_2\geq C(x_1+x_2)^2.$$

III. *Generalization (a proposal of Janous and Klamkin) by Marcin E. Kuczma, Warszawa, Poland.*

According to [1992: 29], the original version of the problem asked to find, for a given positive integer $n$ and real $\lambda$, the greatest $C=C(\lambda,n)$ such that

$$\sum x_i^n+\lambda\prod x_i\geq C\left(\sum x_i\right)^n$$

holds for every $x_1,\ldots,x_n\geq 0$; i.e., to minimize, over the set $(\mathbb{R}^+)^n$ of all $n$-tuples $(x_1,\ldots,x_n)$ of positive reals, the value of

$$F_{\lambda,n}(x_1,\ldots,x_n):=\left(\sum x_i\right)^{-n}\left(\sum x_i^n+\lambda\prod x_i\right).$$

(Since $F_{\lambda,n}$ is continuous and homogeneous of order 0, the bound is attained at some points of the closure of $(\mathbb{R}^+)^n$.) Klamkin's conjecture [1992: 28] says

$$C(\lambda,n)=\min\{(n-1)^{1-n},n^{-n}(n+\lambda)\},\tag{1}$$

and we show that this is correct.

On every hyperplane $x_{i_0} = 0$ the product $\prod x_i$ is zero, so the formula defining $F_{\lambda,n}$ is just $(\sum x_i)^{-n}(\sum x_i^n)$, with minimum value $(n-1)^{1-n}$ when all $x_i$'s other than $x_{i_0}$ are equal. Now assume that $F_{\lambda,n}$ is minimized at some point $(x_1, \dots, x_n)$ with all $x_i$'s positive. For each $j$,

$$\frac{\partial F_{\lambda,n}}{\partial x_j} = \left(\sum x_i\right)^{-n}\left(-n\left(\sum x_i\right)^{-1}\left(\sum x_i^n + \lambda \prod x_i\right) + n x_j^{n-1} + \lambda\left(\prod x_i\right)x_j^{-1}\right) = 0.$$

The function $t \mapsto nt^{n-1} + \lambda(\prod x_i)t^{-1}$ is strictly increasing on $(0,\infty)$ if $\lambda \leq 0$ and strictly convex if $\lambda > 0$, and hence it can assume a value not more than twice. Thus, at a minimum point $(x_1, \dots, x_n)$ inside $(\mathbb{R}^+)^n$, the $x_i$'s can have at most two distinct values. By homogeneity and symmetry, it will be enough to consider

$$x_1 = \cdots = x_k =: x \geq 1, \qquad x_{k+1} = \cdots = x_n = 1; \qquad k \in \{1, \dots, n-1\}.$$

At such a point:

$$F_{\lambda,n}(x, \dots, x, 1, \dots, 1) = (kx + (n-k))^{-n}(kx^n + (n-k) + \lambda x^k) =: f_{\lambda,n,k}(x). \tag{2}$$

The choice for minimum in (1) depends on whether $\lambda$ is greater or smaller than the border value

$$\lambda_n := n\left(\left(\frac{n}{n-1}\right)^{n-1} - 1\right)$$

(note that $\lambda_n/n < e - 1 < 2$). The claim becomes:

$$\text{if } \lambda \geq \lambda_n \quad \text{then } f_{\lambda,n,k}(x) \geq (n-1)^{1-n} \quad \text{for every } x \geq 1; \tag{3}$$
$$\text{if } \lambda \leq \lambda_n \quad \text{then } f_{\lambda,n,k}(x) \geq n^{-n}(n+\lambda) \quad \text{for every } x \geq 1. \tag{4}$$

Note that (3) is a consequence of (4). For, having settled (4) in particular for $\lambda = \lambda_n$, we get (3) for $\lambda = \lambda_n$ (the right sides of (3) and (4) are then equal). Thus (3) follows for all $\lambda \geq \lambda_n$ because $f_{\lambda,n,k}(x)$ grows with $\lambda$.

So we are left with (4). Thus assume $\lambda \leq \lambda_n$. Differentiating,

$$f'_{\lambda,n,k}(x) = nk(n-k)(kx + (n-k))^{-n-1} g_{\lambda,n,k}(x)$$

where

$$g_{\lambda,n,k}(x) := x^{n-1} - 1 - (\lambda/n)x^{k-1}(x-1).$$

If $k \leq n-2$, we use $x^{k-1} \leq x^{n-3}$ and $\lambda/n \leq \lambda_n/n < 2$ to obtain

$$\begin{aligned} g_{\lambda,n,k}(x) &\geq x^{n-1} - 1 - 2x^{n-3}(x-1) \\ &= (x-1)((x^{n-2} + x^{n-3} + x^{n-4} + \cdots + 1) - 2x^{n-3}) \\ &= (x-1)((x^{n-2} - x^{n-3}) + (x^{n-4} + \cdots + 1)) > 0 \quad \text{for } x > 1, \end{aligned}$$

showing that $f_{\lambda,n,k}$ is an increasing function in $x \in [1,\infty)$, hence minimized at $x = 1$.

For $k = n-1$,

$$g_{\lambda,n,n-1}(x) = \Big(1 - \frac{\lambda}{n}\Big)x^{n-1} + \frac{\lambda}{n}\,x^{n-2} - 1,$$

with derivative

$$g'_{\lambda,n,n-1}(x) = x^{n-3}\Big(\Big(1 - \frac{\lambda}{n}\Big)(n-1)x + \frac{\lambda}{n}\,(n-2)\Big).$$

The linear expression in the largest parentheses can change sign at most once. At $x = 1$ it is positive. So, either $g_{\lambda,n,n-1}$ is increasing over $[1,\infty)$ or it increases first and then decreases. As $g_{\lambda,n,n-1}(1) = 0$, we see that either $g_{\lambda,n,n-1} > 0$ on $(1,\infty)$ or we have $g_{\lambda,n,n-1} > 0$ in some interval $(1, 1+\delta)$ and $g_{\lambda,n,n-1} < 0$ in $(1+\delta,\infty)$. Recall that the sign of $g_{\lambda,n,k}$ determines the sign of $f'_{\lambda,n,k}$. Accordingly, either $f_{\lambda,n,n-1}$ is increasing in $[1,\infty)$ or it is increasing in $[1, 1+\delta]$ and decreasing in $[1+\delta,\infty)$. The values of $f_{\lambda,n,n-1}$ at $x = 1$ and (the limit value) at $x = \infty$ (cf. (2) with $k = n-1$) are just the two numbers of (1); in the case under consideration ($\lambda \le \lambda_n$) the value at $x = 1$ is smaller.

Summing up, we have shown that if $\lambda \le \lambda_n$ then $n^{-n}(n+\lambda)$ is the global minimum of $f_{\lambda,n,k}$ for every $k \in \{1,\dots,n-1\}$. This ends the proof of (4), hence of (3) and (1).

*Remark.* Here is a step toward a more elegant proof. Reduction to condition (4) is done as above. Setting $t := (x-1)/n \ge 0$ we rewrite the inequality of (4) in the form

$$k(1+nt)^n + (n-k) + \lambda(1+nt)^k \ge (n+\lambda)(1+kt)^n,$$

i.e.,

$$k[(1+nt)^n - 1] - n[(1+kt)^n - 1] - \lambda[(1+kt)^n - (1+nt)^k] \ge 0, \qquad t \ge 0. \tag{5}$$

Here $\lambda$ is multiplied by a nonnegative expression. Thus if we are able to prove (5) for $\lambda = \lambda_n$, we are done, the inequality will hold for any $\lambda \le \lambda_n$ as well.

Numerical examination shows that for $1 \le k < n \le 12$ the left side of (5) with $\lambda = \lambda_n$ is a polynomial *with all coefficients nonnegative*; the optimistic belief in "good behavior of mathematical beings" says that this ought to be true for *every* $n$ and $k$. Is it?

* * * * *

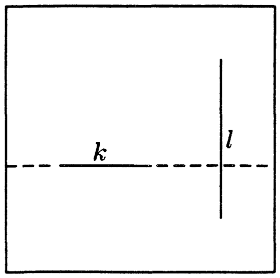
**1719***. [1992: 45] *Proposed by E. Lindros, Le Colisée, Québec.*

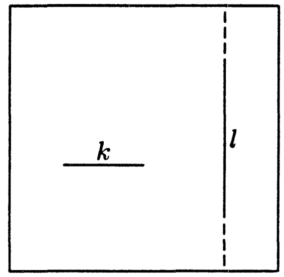
Let $k$ and $l$ be two positive real numbers less than 1, and consider a square of side 1 whose sides are horizontal and vertical. A horizontal line of length $k$ is drawn at random inside the square (the line cannot stick outside the square). Independently, a vertical line of length $l$ is drawn at random inside the square. What is the probability that the two lines intersect?

*Solution by Ian Goldberg, student, University of Waterloo.*

If we extend the horizontal segment to span the square, the probability that the vertical segment intersects it is $l$. [*Editor's note:* think of the vertical segment as fixed and the horizontal segment as varying, not the other way around!] If we extend the vertical segment to span the square, the probability that the horizontal segment intersects it is $k$.

Since the two segments intersect if and only if each intersects with the extension of the other [and since these two probabilities are independent — *Ed.*], the required probability is $kl$.




*Also solved (not usually this nicely!) by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; and CHRIS WILDHAGEN, Rotterdam, The Netherlands.*

*Klamkin solved the more general problem where the two line segments are replaced by two randomly placed rectangles of given dimensions; the probability that they intersect is not as neat, it seems.*

*Konečný's proof used a related* Mathematics Magazine *problem, #868 (solution, by Konečný in fact, on pp.* 110–111 *of the March–April* 1974 *issue).*

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**1720**. [1992: 45] *Proposed by P. Penning, Delft, The Netherlands.*

The osculating circle at point $P$ (not a vertex) of a conic intersects the conic in one other point $Q$. Find a simple construction for $Q$, given the conic itself, its axes and the tangent at $P$.

*Solution by Dan Pedoe, Minneapolis, Minnesota.*

If a circle intersects a conic in the points $A, B, C, D$ (in any order), the chords $AB$ and $CD$ are equally inclined to the principal axes of the conic. This theorem can be found in every textbook on conics. The simplest proof: if the conic referred to its principal axes is $aX^2 + bY^2 - 1 = 0$, and the circle is $X^2 + Y^2 + 2gX + 2fY + c = 0$, then the chords of intersection are given (in pairs) by

$$aX^2 + bY^2 - 1 + k(X^2 + Y^2 + 2gX + 2fY + c) = 0,$$

for special values of $k$, and since there is no term in $XY$ in this equation, the equation of the chords must be of the form

$$(pX + qY + r)(pX - qY + s) = 0.$$

If the points $A, B, C$ coincide, as they do if the circle is an osculating circle, then $AD$ and the tangent to the conic at $A$ make equal angles with the principal axes of the conic, so that the construction asked for could hardly be simpler.

*Comments by Chris Fisher.*

1) Pedoe's proof is valid for central conics. It can be modified to work also for parabolas by letting the equation of the conic be $aX^2 + bY^2 + dY - 1 = 0$.

2) The solver's claim that "the theorem can be found in every textbook on conics" should be taken with a grain of salt. In his textbook *Geometry, A Comprehensive Course* (Dover, 1988), Pedoe has a nice treatment of pencils of conics through four points, which supplies all the tools needed for the proof, but he seems to have left out this one theorem!

*Also solved by JORDI DOU, Barcelona, Spain; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.*

*More information on constructions in conics can be found in two earlier* Crux *articles by Pedoe, on* [1979: 254–258, 281–287]. *For a neat "paradox" related to Pedoe's solution, see p.* 63 *of the January* 1993 College Math. Journal.

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**1721**. [1992: 74] *Proposed by Murray S. Klamkin, University of Alberta.*

Verify the vector identity

$$[\mathbf{A} \times (\mathbf{B} - \mathbf{C})]^2 + [\mathbf{B} \times (\mathbf{C} - \mathbf{A})]^2 + [\mathbf{C} \times (\mathbf{A} - \mathbf{B})]^2$$

$$= (\mathbf{B} \times \mathbf{C})^2 + (\mathbf{C} \times \mathbf{A})^2 + (\mathbf{A} \times \mathbf{B})^2 + (\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{A} \times \mathbf{B})^2.$$

*Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

The verification is straightforward. First, by the distributive property of the cross product with respect to the sum, we have

$$[\mathbf{A} \times (\mathbf{B} - \mathbf{C})]^2 = [(\mathbf{A} \times \mathbf{B}) - (\mathbf{A} \times \mathbf{C})]^2 = (\mathbf{A} \times \mathbf{B})^2 + (\mathbf{A} \times \mathbf{C})^2 - 2(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{C}),$$

$$[\mathbf{B} \times (\mathbf{C} - \mathbf{A})]^2 = [(\mathbf{B} \times \mathbf{C}) - (\mathbf{B} \times \mathbf{A})]^2 = (\mathbf{B} \times \mathbf{C})^2 + (\mathbf{B} \times \mathbf{A})^2 - 2(\mathbf{B} \times \mathbf{C}) \cdot (\mathbf{B} \times \mathbf{A}),$$

$$[\mathbf{C} \times (\mathbf{A} - \mathbf{B})]^2 = [(\mathbf{C} \times \mathbf{A}) - (\mathbf{C} \times \mathbf{B})]^2 = (\mathbf{C} \times \mathbf{A})^2 + (\mathbf{C} \times \mathbf{B})^2 - 2(\mathbf{C} \times \mathbf{A}) \cdot (\mathbf{C} \times \mathbf{B}).$$

By addition, taking account that $(\mathbf{V} \times \mathbf{W})^2 = (\mathbf{W} \times \mathbf{V})^2$, we obtain as left hand of the proposed identity,

$$2(\mathbf{B} \times \mathbf{C})^2 + 2(\mathbf{C} \times \mathbf{A})^2 + 2(\mathbf{A} \times \mathbf{B})^2 - 2(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{C}) - 2(\mathbf{B} \times \mathbf{C}) \cdot (\mathbf{B} \times \mathbf{A}) - 2(\mathbf{C} \times \mathbf{A}) \cdot (\mathbf{C} \times \mathbf{B}).$$

On the other hand, developing $(\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{A} \times \mathbf{B})^2$ results in

$$(\mathbf{B} \times \mathbf{C})^2 + (\mathbf{C} \times \mathbf{A})^2 + (\mathbf{A} \times \mathbf{B})^2 + 2(\mathbf{B} \times \mathbf{C}) \cdot (\mathbf{C} \times \mathbf{A}) + 2(\mathbf{C} \times \mathbf{A}) \cdot (\mathbf{C} \times \mathbf{B}) + 2(\mathbf{B} \times \mathbf{C}) \cdot (\mathbf{A} \times \mathbf{B}),$$

and as $\mathbf{V} \times \mathbf{W} = -(\mathbf{W} \times \mathbf{V})$ the result follows immediately.

*Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; JORDI DOU, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium,* *Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; JEAN-MARIE MONIER, Lyon, France; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.*

*The proposer comments that the given identity is equivalent to*

$$\frac{1}{d_1^2}+\frac{1}{d_2^2}+\frac{1}{d_3^2}=\frac{1}{h_1^2}+\frac{1}{h_2^2}+\frac{1}{h_3^2}+\frac{1}{h_4^2}, \tag{1}$$

*where $d_1$ is the shortest distance between edges $PA$ and $BC$ of a tetrahedron $PABC$, and $h_1$ is the altitude of $PABC$ from $A$, and $d_2$, $d_3$, $h_2$, $h_3$, $h_4$ are defined analogously. He wonders if any reader can supply a reference for the (known) relation* (1).

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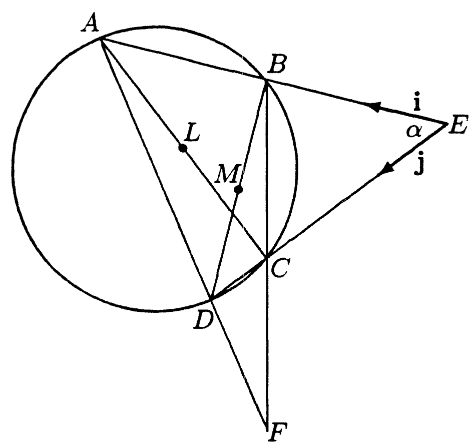
**1722.** [1992: 75] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a cyclic quadrilateral with $BD < AC$. Let $E$ and $F$ be the intersections of $AB, CD$ and of $BC, AD$, respectively, and let $L$ and $M$ be the midpoints of $AC$ and $BD$. Prove that

$$\frac{LM}{EF}=\frac{1}{2}\left(\frac{AC}{BD}-\frac{BD}{AC}\right).$$

*Solution by Christopher J. Bradley, Clifton College, Bristol, U.K.*




Let $E$ be the origin and $\mathbf{i}$, $\mathbf{j}$ unit vectors in the directions $\overrightarrow{EB}$ and $\overrightarrow{EC}$. Let $\mathbf{i}\cdot\mathbf{j}=\cos\alpha$. Since $ABCD$ is a cyclic quadrilateral, $EB\cdot EA = EC\cdot ED$, so we may choose coordinates so that $\overrightarrow{EB}=\lambda\mathbf{i}$, $\overrightarrow{EA}=\mu k\mathbf{i}$, $\overrightarrow{EC}=\mu\mathbf{j}$ and $\overrightarrow{ED}=\lambda k\mathbf{j}$ $(\mu>\lambda)$. Then

$$\begin{aligned}\overrightarrow{LM} &= \overrightarrow{EM}-\overrightarrow{EL}\\ &= \frac{1}{2}(\overrightarrow{EB}+\overrightarrow{ED}-\overrightarrow{EA}-\overrightarrow{EC})\\ &= \frac{1}{2}[(\lambda-\mu k)\mathbf{i}+(\lambda k-\mu)\mathbf{j}].\end{aligned}$$

Also

$$(AC)^2=(\overrightarrow{EC}-\overrightarrow{EA})^2=\mu^2(\mathbf{j}-k\mathbf{i})^2=\mu^2(k^2+1-2k\cos\alpha)$$

and

$$(BD)^2=(\overrightarrow{ED}-\overrightarrow{EB})^2=\lambda^2(k\mathbf{j}-\mathbf{i})^2=\lambda^2(k^2+1-2k\cos\alpha),$$

from which

$$\frac{1}{2}\left(\frac{AC}{BD}-\frac{BD}{AC}\right)=\frac{1}{2}\left(\frac{(AC)^2-(BD)^2}{BD\cdot AC}\right)=\frac{1}{2}\left(\frac{\mu^2-\lambda^2}{\mu\lambda}\right).$$

Now $F$ lies on both $AD$ and $BC$, so may be represented vectorially in two ways:

$$\begin{aligned}\overrightarrow{EF} &= t\,\overrightarrow{EA}+(1-t)\,\overrightarrow{ED}=t\mu k\mathbf{i}+(1-t)\lambda k\mathbf{j},\\ \overrightarrow{EF} &= s\,\overrightarrow{EB}+(1-s)\,\overrightarrow{EC}=s\lambda\mathbf{i}+(1-s)\mu\mathbf{j},\end{aligned}$$

where $0 < t, s < 1$. Hence

$$s\lambda = t\mu k \qquad \text{and} \qquad (1-s)\mu = (1-t)\lambda k.$$

The solution to these simultaneous equations is

$$s = \frac{\mu(\mu - \lambda k)}{\mu^2 - \lambda^2},$$

from which

$$\overrightarrow{EF} = \frac{\lambda\mu(\mu - \lambda k)}{\mu^2 - \lambda^2}\mathbf{i} + \frac{\lambda\mu(\mu k - \lambda)}{\mu^2 - \lambda^2}\mathbf{j} = \frac{\lambda\mu}{\mu^2 - \lambda^2}[(\mu - \lambda k)\mathbf{i} + (\mu k - \lambda)\mathbf{j}].$$

It follows that

$$(EF)^2 = \frac{4\lambda^2\mu^2}{(\mu^2 - \lambda^2)^2}(LM)^2.$$

Thus

$$\frac{LM}{EF} = \frac{1}{2}\left(\frac{\mu^2 - \lambda^2}{\mu\lambda}\right) = \frac{1}{2}\left(\frac{AC}{BD} - \frac{BD}{AC}\right).$$

*Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; and the proposer.*

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**1723**. [1992: 75] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $n \geq 3$ be an integer. Determine all continuous functions $f : [0,1] \to \mathbb{R}$ such that

$$f(x_1) + \cdots + f(x_n) = 1$$

for all $x_1, \ldots, x_n \in [0,1]$ satisfying $x_1 + \cdots + x_n = 1$.

I. *Solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut.*

The required functions are exactly those of the form $f(x) = (1 - nc)x + c$, $c$ an arbitrary real number.

Assume $f : [0,1] \to \mathbb{R}$ is a continuous function that satisfies the stated condition. Let $x, y \in [0,1]$, and let

$$x_1 = x_2 = \frac{x+y}{2}, \quad x_3 = 1 - x - y, \quad x_4 = \cdots = x_n = 0.$$

Then

$$2f\left(\frac{x+y}{2}\right) + f(1-x-y) + (n-3)f(0) = 1. \tag{1}$$

Now let $x_1 = x$ and $x_2 = y$, with all the other $x_j$'s as above. Then

$$f(x) + f(y) + f(1-x-y) + (n-3)f(0) = 1. \tag{2}$$

From (1) and (2) it follows that, for all $x, y \in [0,1]$,

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y). \tag{3}$$

Since $f$ is continuous, it is both convex and concave, according to (3) [and hence is a linear function]. Thus $f(x) = (f(1) - f(0))x + f(0)$. But $f(1) + (n-1)f(0) = 1$, so that $f(1) - f(0) = 1 - nf(0)$. Therefore

$$f(x) = (1 - nf(0))x + f(0)$$

where $f(0)$ is arbitrary. The check that such an $f$ satisfies the stated condition is immediate.

II. *"Solution" by N. Withheld.*

We show that the only function satisfying the given condition is the identity $f(x) = x$. To see this, choose $x_1 = \cdots = x_n = 1/n$. Then $nf(1/n) = 1$, so $f(1/n) = 1/n$. Letting $n \to \infty$, we get by continuity of $f$ that $f(0) = 0$. Now let $p/q$ (where $q - p \geq 2$) be any rational in the interval $(0,1)$. Then $1 = p/q + (q-p) \cdot 1/q$ implies that

$$1 = f\left(\frac{p}{q}\right) + (q-p)f\left(\frac{1}{q}\right) = f\left(\frac{p}{q}\right) + (q-p) \cdot \frac{1}{q} = f\left(\frac{p}{q}\right) + 1 - \frac{p}{q},$$

so $f(p/q) = p/q$. Again, continuity implies that $f(x) = x$ for all $x$ in $[0,1]$.

*Also solved by H.L. ABBOTT, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; JEAN-MARIE MONIER, Lyon, France; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.*

"Solution" II *is a combination of two solutions, both making the same elementary but amusing error, which readers may like to find for themselves!*

*Wildhagen notes that the solution consists precisely of all linear functions from* $[0,1]$ *to* $\mathbb{R}$ *passing through the point* $(1/n, 1/n)$.

*The proposer also considered the case* $n = 2$, *which is much different: any continuous function* $f : [0,1] \to \mathbb{R}$ *such that* $f(1/2) = 1/2$ *and whose graph is symmetric about the point* $(1/2, 1/2)$ *satisfies the condition.*

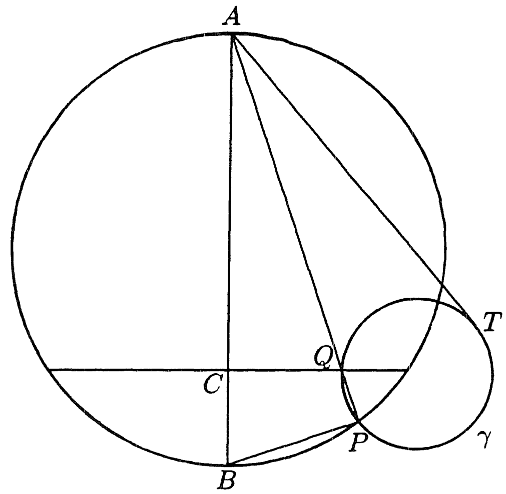
* * * * *

**1724**. [1992: 75] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

A fixed plane intersects a fixed sphere forming two spherical segments. (Each segment is a region bounded by the plane and one of the spherical caps it cuts from the sphere.) Let $S$ be one of these segments and let $A$ be the point on the sphere furthest from $S$. A variable chord of the sphere through $A$ meets the boundary of $S$ in two points $P$ and $Q$. Let $\gamma$ be a variable sphere whose only constraint is that it passes through $P$ and $Q$. Prove that the length of the tangent from $A$ to $\gamma$ is a constant. (*Note*: in view of *Crux* 1070 [1987: 31], this generalizes problem 1155 of *Mathematics Magazine*, solution on p. 47 of the 1984 volume.)

*Solution by P. Penning, Delft, The Netherlands.*

We assume that $P$ is on the fixed sphere and $Q$ is on the fixed plane. Let $T$ be the point in the plane of $ABP$ where the tangent from $A$ touches $\gamma$. The plane through $AT$ and $AP$ intersects $\gamma$ in a circle to which $AT$ is also tangent. So




$$(AT)^2 = AP \cdot AQ.$$

Let the fixed sphere have radius $R$, let $C$ be the projection of $A$ on the fixed plane, and let $B$ be the other intersection of $AC$ with the fixed sphere. Then the right triangles $APB$ and $ACQ$ are similar, so $CA/AQ = AP/AB$. Thus

$$(AT)^2 = AP \cdot AQ = CA \cdot AB = 2R \cdot CA,$$

independent of the choice of chord and variable sphere.

*Also solved by JORDI DOU, Barcelona, Spain; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; MARCIN E. KUCZMA, Warszawa, Poland; DAN PEDOE, Minneapolis, Minnesota; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.*

*Dou, Wildhagen and the proposer observe that the constant distance AT is equal to the distance from A to any point on the intersection of the fixed sphere and fixed plane.*

* * * * *

**1725.** [1992: 75] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Note that the Pythagorean triangle with sides 3,4,5 has the following property: one leg (3) is a triangular number; the other leg (4) is a square number; and the hypotenuse (5) is a pentagonal number, that is, a number of the form $n(3n-1)/2$. Find another Pythagorean triangle with the same property.

*Solution by Charles Ashbacher, Cedar Rapids, Iowa.*

With a simple computer program to conduct a search, the following solution was found:

$$x = \mathbf{100} = 10^2, \quad y = \mathbf{105} = \frac{14 \cdot 15}{2}, \quad z = \mathbf{145} = \frac{10(3 \cdot 10 - 1)}{2},$$

$$x^2 + y^2 = 10000 + 11025 = 21025 = z^2.$$

*The same solution was also found by RICHARD I. HESS, Rancho Palos Verdes, California; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; and the proposer. One incorrect solution (due perhaps to misreading the problem) was sent in.*

*Nobody was able to find any other solutions to this problem. Are there any? The proposer found Pythagorean triples $(x, y, z)$ in which $y$ is triangular and $z$ is pentagonal, but $x$ is not square.*

*     *     *     *     *

**1726**. [1992: 75] *Proposed by Hidetosi Fukagawa, Aichi, Japan.*

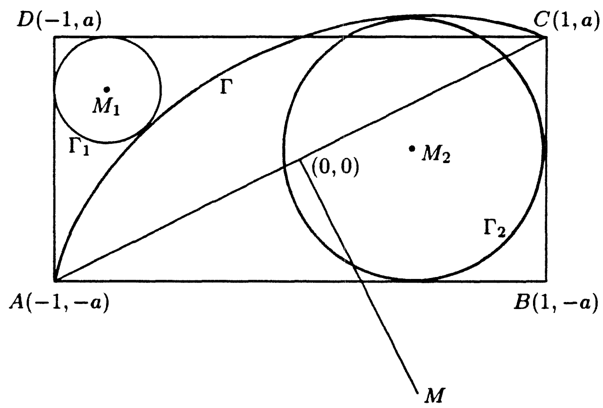
Circular arc $AC$ lies inside rectangle $ABCD$, dividing it into two parts with inscribed circles of radii $r_1$ and $r_2$. Show that $r_1 + r_2 = 2r$, where $r$ is the inradius of the right-angled triangle $ADC$.

*Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We use rectangular coordinates and put

$$A(-1,-a), B(1,-a), C(1,a), D(-1,a),$$

where $a > 0$. Note that the line perpendicular to $AC$ and passing through the midpoint $(0,0)$ of $AC$ has the equation $(x, y) = t(a, -1)$. Then the circle $\Gamma$ containing arc $AC$ and having centre $M(ta, -t)$ will have radius




$$R = \sqrt{(ta+1)^2 + (t-a)^2} = \sqrt{(1+a^2)(1+t^2)} \qquad (1)$$

and thus equation

$$(x - ta)^2 + (y + t)^2 = (1 + a^2)(1 + t^2).$$

In order to get $r_1$ we start from the circle $\Gamma_1$ with centre $M_1(-1 + \lambda, a - \lambda)$ and radius $r_1 = \lambda$. Then $M_1M = R + r_1$ yields

$$(ta + 1 - \lambda)^2 + (t + a - \lambda)^2 = R^2 + 2R\lambda + \lambda^2,$$

i.e. from (1)

$$\begin{aligned} 0 &= \lambda^2 - 2[(1+a)(1+t) + R]\lambda + 4ta \\ &= [\lambda - (1+a)(1+t) - R]^2 + 4ta - [(1+a)(1+t) + R]^2 \\ &= [\lambda - (1+a)(1+t) - R]^2 + [(1+t)^2 - (1+t^2)][(1+a)^2 - (1+a^2)] \\ &\qquad -(1+a)^2(1+t)^2 - 2(1+a)(1+t)\sqrt{(1+a^2)(1+t^2)} - (1+a^2)(1+t^2) \\ &= [\lambda - (1+a)(1+t) - R]^2 - [(1+t)\sqrt{1+a^2} + (1+a)\sqrt{1+t^2}\,]^2, \end{aligned}$$

i.e.

$$\lambda = (1+a)(1+t) + R - [(1+t)\sqrt{1+a^2} + (1+a)\sqrt{1+t^2}\,].$$

Similarly, for $r_2$ we start from circle $\Gamma_2$ with centre $M_2(1 - \mu, -a + \mu)$ and radius $r_2 = \mu$, and $M_2M = R - r_2$, and get

$$(ta - 1 + \mu)^2 + (t - a + \mu)^2 = R^2 - 2R\mu + \mu^2,$$

i.e.

$$\mu^2 - 2[(1+a)(1-t) - \sqrt{(1+a^2)(1+t^2)}\,]\mu - 4ta = 0,$$

i.e.

$$\mu = (1+a)(1-t) - R + [(1+a)\sqrt{1+t^2} - (1-t)\sqrt{1+a^2}\,].$$

Thus

$$r_1 + r_2 = \lambda + \mu = 2(1+a) - 2\sqrt{1+a^2} = 2r,$$

as

$$r = \frac{1}{2}(AB + BC - AC) = \frac{1}{2}(2 + 2a - 2\sqrt{1+a^2}\,).$$

The proof shows that arc $AC$ does not have to lie entirely inside of $ABCD$. It only has to separate $\Gamma_1$ and $\Gamma_2$.

*Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. Three other solutions only handled special cases. There was also one incorrect solution sent in.*

*The problem was taken from a 1786 sangaku, now lost.*

*P. Penning, Delft, The Netherlands, remarks that one of the common tangents to the circles $\Gamma_1$ and $\Gamma_2$ appears to be parallel to $AC$. Can someone supply a proof?*

*It would be nice to see a simpler, more geometric solution to this problem!*

* * * * *

**1727.** [1992: 75] *Proposed by Joaquín Gómez Rey, I.B. Luis Buñuel, Alcorcón, Madrid, Spain.*

For $n$ a positive integer, evaluate $A/B$ where

$$A = \prod_{k=1}^{n}(2k-1) \quad \text{and} \quad B = \sum_{j=0}^{[n/2]} \frac{1}{(n-2j)!4^j(j!)^2}\,.$$

*Solution by Chris Wildhagen, Rotterdam, The Netherlands (slightly modified by the editor).*

Let

$$f(x) = \left(\frac{1}{2}x^4 + x^2 + \frac{1}{2}\right)^n.$$

The coefficient of $x^{2n}$ in $f(x)$ is

$$\sum_{j=0}^{[n/2]} \frac{n!}{j!2^j(n-2j)!(n-j-(n-2j))!2^{n-j-(n-2j)}} = \sum_{j=0}^{[n/2]} \frac{n!}{(j!)^2(n-2j)!4^j} = n!B.$$

Since

$$f(x) = \frac{(x^2+1)^{2n}}{2^n}\,,$$

the coefficient of $x^{2n}$ in $f(x)$ is also

$$\frac{1}{2^n}\binom{2n}{n} = \frac{1}{2^n}\frac{(2n)!}{(n!)^2} = \frac{1}{n!}A.$$

Thus $n!B = A/n!$, so

$$\frac{A}{B} = (n!)^2.$$

*Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; and the proposer. One incorrect solution was sent in.*

* * * * *

**1728.** [1992: 75] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

$AD, BE$ and $CF$ are the altitudes of an acute triangle $ABC$, and $H$ is its orthocentre. Points $K, L, M$ lie on $EF, FD, DE$ respectively. Show that $AK, BL$ and $CM$ meet in a point of the Euler line of $\Delta ABC$ if

(i) $K, L, M$ are the points where the incircle of $\Delta DEF$ touches the sides of $\Delta DEF$;

(ii) $K, L, M$ are the feet of the altitudes of $\Delta DEF$.

I. *Solution to part* (i) *by Jordi Dou, Barcelona, Spain.*

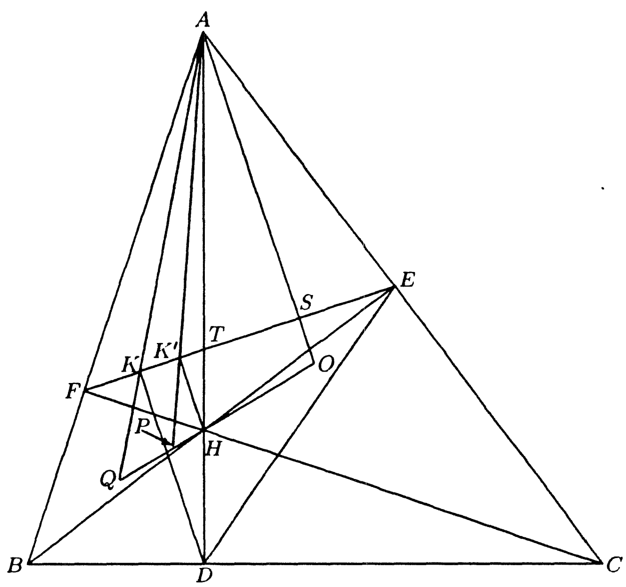
The Euler line of $\Delta ABC$ is $OH$, where $O$ and $H$ are the triangle's circumcentre and orthocentre; the point $H$ is also the *incentre* of $\Delta DEF$ and the *circumcentre* of $\Delta KLM$. Since $LM$ and $BC$ are both perpendicular to $HD$, we have $LM \| BC$; similarly $MK \| CA$ and $KL \| AB$. Consequently there exists a dilatation taking $\Delta ABC$ and its circumcentre $O$ to $\Delta KLM$ and its circumcentre $H$. It follows that $AK$, $BL$, $CM$, and $OH$ concur in the centre of the dilatation; that is, $AK$, $BL$, and $CM$ meet in a point of $OH$, as desired.

II. *Solution to part* (ii) *by Toshio Seimiya, Kawasaki, Japan.*

Here $K$, $L$, $M$ are the feet of the altitudes of $\Delta DEF$, so let $K'$, $L'$, $M'$ be the feet of the perpendiculars from $H$ to the sides of $\Delta DEF$ — the points of tangency of the incircle from part (i), so that $AK'$, $BL'$, $CM'$ meet in a point $P$ of $OH$. Define $Q = AK \cap OH$, $S = AO \cap EF$, and $T = AH \cap EF$. Note that $AO \perp EF$ from (i) (because $AO$ and $K'H$ are related by the dilatation).

Because $Q, H, P, O$ (on $OH$) and $K, T, K', S$ (on $EF$) are perspective from $A$, their cross ratios are equal; that is,

$$\{QH, PO\} = \{KT, K'S\}. \qquad (1)$$



Because $DK$, $HK'$, and $AS$ are parallel we get

$$\{KT, K'S\} = \{DT, HA\}. \tag{2}$$

Since $H$ and $A$ are the points where the internal and external bisectors of $\angle TED$ meet the line $TD$, we get finally

$$\{DT, HA\} = -1. \tag{3}$$

From (1), (2) and (3) we conclude that $\{QH, PO\} = -1$. Similarly, the points where $BL$ and $CM$ intersect $OH$ make with $H$, $P$, and $O$ a cross ratio of $-1$, so that the three lines meet in the point $Q$ of the Euler line $OH$, as desired.

*Comment.* Smeenk points out that if (iii) $K$, $L$, $M$ are the points where the angle bisectors of $\Delta DEF$ meet the opposite sides, then $AK$, $BL$, $CM$ once again meet in a point of the Euler line of $\Delta ABC$ (namely at $H$). Happily, not *every* point of the plane lies on $OH$! For example, if (iv) $K$, $L$, $M$ are the points where the medians of $\Delta DEF$ meet the opposite sides, then $AK$, $BL$, $CM$ meet in the Lemoine point of $\Delta ABC$, which generally does not lie on the Euler line.

*Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; and the proposer. (Dou and Seimiya also submitted complete solutions to both parts.)*

*Bellot's solution to part* (ii) *made use of trilinear coordinates.*

* * * * *

**1729**. [1992: 76] *Proposed by Susan Gyd, Nose Hill, Alberta.*

The story is well known that when Hardy visited Ramanujan in hospital and suggested that the number of his taxi, 1729, was not very interesting, Ramanujan immediately observed that, on the contrary, it was the least number that was expressible as the sum of two cubes in two different ways:

$$12^3 + 1^3 = 1728 + 1 = 1729 = 1000 + 729 = 10^3 + 9^3.$$

These sums are easy to see because 1 is one of the cubes, and because 729 is the last segment of 1729. Are there any numbers whose *squares* behave in the same way? That is, can we have $x^2 + 1 = y^2 + z^2$ with $x, y, z$ distinct positive integers and $z^2$ a final segment of $x^2 + 1$?

*Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We have to have

$$x^2 + 1 = (10^n w)^2 + z^2,$$

where $y = 10^n w$ and $z < 10^n$; i.e.,

$$(x - z)(x + z) = (10^n w - 1)(10^n w + 1). \tag{1}$$

In order to get an infinity of solutions of (1) we put $n = 3k$ and $w = t^3$, and thus (1) becomes

$$(x - z)(x + z) = (10^k t - 1)(10^{2k} t^2 + 10^k t + 1)(10^k t + 1)(10^{2k} t^2 - 10^k t + 1).$$

We assign

$$x - z = (10^k t - 1)(10^{2k}t^2 - 10^k t + 1) = 10^{3k}t^3 - 2\cdot 10^{2k}t^2 + 2\cdot 10^k t - 1,$$

$$x + z = (10^k t + 1)(10^{2k}t^2 + 10^k t + 1) = 10^{3k}t^3 + 2\cdot 10^{2k}t^2 + 2\cdot 10^k t + 1,$$

whence

$$z = 2\cdot 10^{2k}t^2 + 1$$

($< 10^{3k}$ for $t < \sqrt{(10^k - 10^{-2k})/2}$ ), and

$$x = 10^{3k}t^3 + 2\cdot 10^k t, \quad y = 10^{3k}t^3.$$

For example, $k = t = 1$ yields the values

$$x = 1020, \qquad y = 1000, \qquad z = 201, \qquad x^2 + 1 = 1040401 = y^2 + z^2;$$

$k = 1$, $t = 2$ yields the values

$$x = 8040, \qquad y = 8000, \qquad z = 801, \qquad x^2 + 1 = 64641601 = y^2 + z^2.$$

*Also solved by CHARLES ASHBACHER, Cedar Rapids, Iowa; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; MARCIN E. KUCZMA, Warszawa, Poland; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; J.A. MCCALLUM, Medicine Hat, Alberta; KENNETH M. WILKE, Topeka, Kansas; and the proposer. Two incorrect solutions were sent in.*

*Maragoudakis, Wilke, and the proposer also gave infinitely many solutions, the other solvers various finite numbers of solutions. Some "small" ones given by solvers (and not derivable by the above solution) are:*

$$\begin{array}{llll}
x = 1032, & y = 1000, & z = 255, & x^2 + 1 = 1065025 = y^2 + z^2,\\
x = 1068, & y = 1000, & z = 375, & x^2 + 1 = 1140625 = y^2 + z^2,\\
x = 1280, & y = 1000, & z = 799, & x^2 + 1 = 1638401 = y^2 + z^2,\\
x = 1380, & y = 1000, & z = 951, & x^2 + 1 = 1904401 = y^2 + z^2,\\
x = 10468, & y = 10000, & z = 3095, & x^2 + 1 = 109579025 = y^2 + z^2.
\end{array}$$

*Are there any further solutions with* cubes *beyond the original Ramanujan number? That is, are there distinct positive integers $x$, $y$, $z$ such that $x^3 + 1 = y^3 + z^3 > 1729$ and $z^3$ is a final segment of $x^3 + 1$?*

*　　　*　　　*　　　*　　　*

Short articles intended for publication should be sent to Dr. Hanson, contest problem sets and solutions to Olympiad Corner problems should be sent to Dr. Woodrow and other problems and solutions to Dr. Sands.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