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# Leopold Kronecker: A Great Gentleman in Science

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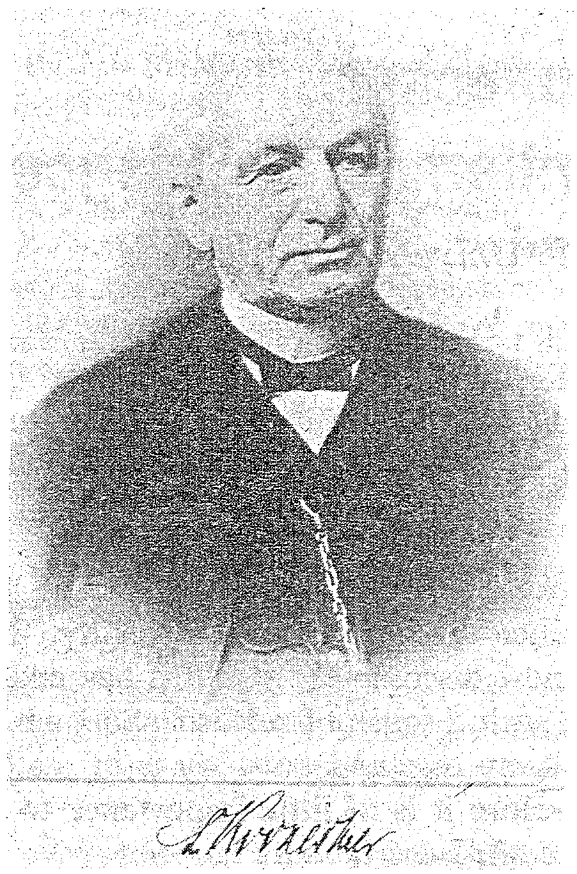
In May 1857, Kronecker, aged 33, wrote to P. G. L. Dirichlet, one of his close friends and former teachers at the university: 'I have made quite a few very interesting new discoveries, and this has enabled me, more than anything else, to feel such a sense of release. So I am not troubled by any trace of ambition and my one and only joy is in really understanding the truth of things. Therefore it is of little importance to me how I spend my time so long as, above all, I use it well.'

Leopold Kronecker was born in December 1823 and died in December 1891, his life thus spanning much of the nineteenth century; and this year, therefore, we commemorate the centenary of his death. The list of Kronecker's mathematical near contemporaries is impressive, and indeed the nineteenth century has been described as 'perhaps the most brilliant era in the long history of mathematics'. It was therefore an exciting and challenging age in which to develop as a mathematician, and Kronecker's own contributions to mathematics, particularly in the fields of number theory, the theory of equations and elliptic functions, were very considerable.

Kronecker's family was Jewish and their home was in Leignitz in Prussia. Leignitz (Legnica), first mentioned in the eleventh century, has seen its share of history. It was the place of residence of the Dukes of Lower Silesia in the later twelfth century, and was the site of a famous battle with the Mongol armies of Genghis Khan in 1241. Now in Poland, it has become one of the most modern industrial centres since the discovery in recent years of deposits of copper in the vicinity. In Kronecker's day Leignitz was a centre of trade and there was an extensive weaving industry there, as well as several small manufacturing firms. Kronecker's father was in commerce, and the family was well-to-do.

The young Leopold was tutored privately in his early years. Later, he went to a preparatory school, and from there to the local Gymnasium (the High School). Kronecker's particular talent for mathematics was soon evident, though he was also an excellent scholar in a wide range of subjects



Leopold Kronecker 1823–1891

including Greek and Latin, Hebrew and philosophy. Indeed, his profound interest in philosophy and his love of the classical languages continued throughout his life. For his future in mathematics, Kronecker was especially fortunate to have, as his teacher at the Gymnasium, E. E. Kummer, later professor of mathematics at the Universities of Breslau and Berlin. Kummer was evidently an inspired and dedicated teacher and, even at this level, he encouraged his students to carry out independent scientific work. Recognizing Kronecker's mathematical bent, Kummer generously gave him extra private instruction.

In the spring of 1841 Kronecker entered the University of Berlin, and was a student there until 1845, with periods spent at the Universities of Breslau and Bonn. He attended lectures by Dirichlet, Jacobi and Steiner and, in Breslau, by his old tutor Kummer. Although the subject of Kronecker's main study was, of course, mathematics, his university education was broadly based and, besides science subjects, he continued to study philosophy as well as Greek and Latin. While still a student, Kronecker gave a simple proof of the following theorem of Gauss: *If $p$ is a prime number, then the polynomial $1+x+x^2+\ldots+x^{p-1}$ is irreducible; that is to say, it cannot be factorized into polynomials of lower degree than $p-1$ with integer coefficients.* Since the mathematics is quite straightforward, we shall describe Kronecker's proof. We begin by recalling some facts about the $p$th roots of unity. These are just $1, \alpha, \alpha^2, \ldots, \alpha^{p-1}$, where $\alpha = e^{2\pi i/p}$;

and, since

$$x^p-1 = (x-1)(x^{p-1}+x^{p-2}+\dots+x+1),$$

those different from 1 satisfy the equation

$$x^{p-1}+x^{p-2}+\dots+x+1 = 0.$$

Also, for fixed $k$ not divisible by $p$, $\alpha^k, \alpha^{2k}, \dots, \alpha^{(p-1)k}$ are all different, and different from 1, and so are just a rearrangement of $\alpha, \alpha^2, \dots, \alpha^{p-1}$; and therefore

$$\alpha^{k(p-1)}+\alpha^{k(p-2)}+\dots+\alpha^k+1 = 0.$$

On the other hand, if $k$ is a multiple of $p$, this sum is equal to $p$. We are now able to show that, for a polynomial

$$f(x) = a_0+a_1x+\dots+a_rx^r,$$

where the $a_i$ are integers, we have

$$f(\alpha)f(\alpha^2)\dots f(\alpha^{p-1}) \equiv f(1)^{p-1} \pmod p. \tag{1}$$

To see this, we write

$$f(x)f(x^2)\dots f(x^{p-1}) = \sum A_n x^n \quad \text{(say)}.$$

We substitute for $x$ each of the values $1, \alpha, \alpha^2, \dots, \alpha^{p-1}$ in turn and sum to give

$$\begin{aligned} f(1)^{p-1}+(p-1)f(\alpha)f(\alpha^2)\dots f(\alpha^{p-1}) &= \sum A_n+\sum A_n\alpha^n+\dots+\sum A_n\alpha^{(p-1)n} \\ &= p(A_0+A_p+A_{2p}+\dots); \end{aligned}$$

from which (1) follows. Now suppose that

$$1+x+x^2+\dots+x^{p-1} = f(x)g(x) \quad \text{(say)},$$

where the degrees $r$ and $s$ of $f(x)$ and $g(x)$ are less than $p-1$ and $f(x)$ and $g(x)$ have integer coefficients. Then, evidently,

$$p = f(1)g(1),$$

and one of these factors of $p$ must be equal to $\pm 1$, say $f(1) = \pm 1$. But, on the other hand, $f(x)$ must vanish for at least one root of unity different from 1, i.e. for one of $\alpha, \alpha^2, \dots, \alpha^{p-1}$, so that

$$f(\alpha)f(\alpha^2)\dots f(\alpha^{p-1}) = 0 \not\equiv f(1)^{p-1} \pmod p;$$

and this contradicts (1).

Kronecker received his doctorate in 1845 for a dissertation 'De Unitatibus Complexis'. Motivated by attempts to resolve the famous 'last theorem' of Fermat, mathematicians—among them Kummer and

Dirichlet— had been trying to carry over the divisibility theory of the familiar integers to more general so-called 'algebraic integers', and in particular to restore unique factorization among these numbers. Kummer finally achieved this breakthrough (but not a complete proof of Fermat's theorem, which is still tantalizing mathematicians today) by his introduction of ideal numbers. It was the work of Kummer on algebraic numbers which inspired Kronecker, whose dissertation represents the beginning of his own systematic researches in this area. Among the ordinary integers, two of them, $\pm 1$, play a special role by virtue of the fact that they divide every integer. Their analogues among the algebraic integers are called units, and in most algebraic number fields there exist infinitely many units. Kronecker set himself the task of determining the (very complex) algebraic structure of the collection of units in an algebraic number field in certain special situations, notably for those fields associated with the problems of constructing regular polygons with $n$ vertices. Dirichlet had solved the same kind of problems, but his work was not known to Kronecker during the writing of his dissertation; and the fact that this was republished 37 years later points to the importance of the ideas it contained.

Kronecker's activities at the University were not exclusively scholarly. The year of his entry to the University of Berlin was just seven years before the Revolution in Germany. This was a time of great intellectual ferment, when academics like the historian F. C. Dahlmann and the poet E. M. Arndt were preaching Liberal views and sounding the call for German unity. As a student, Kronecker himself was an ardent Liberal and, while in Bonn, was notably active in the formation of a student association. Later in life, however, he changed his allegiance and became a supporter of Bismarck.

It was evident that Kronecker was destined for an academic career, but family circumstances influenced the course of this. Following the death of his uncle (the father of his wife-to-be), Kronecker was called upon to manage this uncle's estate and to run his business. This occupied Kronecker during the 10 years immediately following his years as a student, and thus postponed his re-entry into university life. In mitigation of this, and on account of his considerable managerial success, Kronecker became financially independent and was thus able, in due course, to take up his mathematical career as a freelance with no imposed duties except those which were self-imposed. It was during those years away from the University that Kronecker fell in love with, and in 1848 married, his cousin Fanny Prausnitzer. We read of a blissfully happy married life, and it was a cultured and loving home in which they brought up their large family. Fanny herself was both charming and intellectually distinguished. In a tribute to Kronecker's life and work, Heinrich Weber speaks of the warmth and welcome of the Kronecker home, now in Berlin, and of stimulating evenings spent there, where the conversation would turn from penetrating scientific discussions to reflections on literature and the arts. Kronecker loved music all his life and, through Dirichlet, he had been introduced to Felix Mendelssohn; and so we may imagine music forming part of the entertainment during those social evenings.

Kronecker's 10-year period away from academic life was no time of intellectual idleness, for he corresponded regularly and enthusiastically with Kummer throughout those years. In 1853 he was ready to submit a fundamental paper on the algebraic solution of equations. Another of his most noted contributions, a memoir on the general quintic equation, appeared in 1858.

In 1861, on Kummer's nomination, Kronecker became a member of the Berlin Academy of Sciences. This gave him the right to lecture in the University of Berlin, and this he did, unpaid, for over 20 years until, in 1883 when Kummer retired, he became a salaried professor. Kronecker always liked to lecture on the problems which were occupying him at the time, and so his students were led to the frontiers of current research—provided only that they could follow his expositions! This was by no means always so, and evidently the numbers in his audience would quickly dwindle as the lectures progressed. With those who stayed the course, however, Kronecker loved to communicate his thoughts, and the keenest of his students would often accompany him on his way home after the formal session was over. He showed a warm personal interest in their progress, and in return they were devoted to him, and many of them independently furthered his ideas in their own later researches.

We have already indicated some of the areas of Kronecker's mathematical interests. It has been claimed that his most profound contributions are to be found in his efforts to unify arithmetic, algebra and analysis, and in his work on elliptic functions. Kronecker was also one of the few mathematicians who, in the early years, properly understood the Galois theory of equations. His reconstruction of Hermite's solution of the general equation of degree 5 by means of this theory made the solution seem more natural and comprehensible. Kronecker's paper of 1870, in which he introduced the notion of an Abelian group by means of a set of axioms, is almost in the spirit of the modern 'conceptual' approach to algebraic structures. In this paper he also established the fundamental structural theorem for Abelian groups. In school and undergraduate mathematics, we use two pieces of notation introduced by Kronecker—the 'Kronecker $\delta$' and the upright bars for determinants.

But Kronecker was a philosopher as well as a practising mathematician, and he had a deep interest in the foundations of mathematics. He was an algorist: all non-constructive reasoning was regarded by him as illegitimate and he disallowed any existence proof which provided no actual method for finding the object whose existence was claimed. His views were strongly held and vehemently proclaimed, and they led to disputes with his fellow mathematicians in later years, in particular with his colleague at the the University of Berlin, Kurt Weierstrass, and with the founder of the theory of transfinite numbers, Georg Cantor, one of his former students in Berlin. Weierstrass was the leading analyst of his day, dedicated to completing the work of his great predecessors Abel and Jacobi. His work on limits and convergence led him to develop a theory of irrational numbers based on convergent sequences of rationals. Though generally acceptable then, it left certain questions open to logical objection. However, out of Weierstrass's work, analysis as we know it today, with its precise standards of rigour, has gradually developed. Cantor was initially a classical analyst in the Weierstrass tradition, who made important contributions to the foundations of the subject. But it was his work with transfinite numbers which heralded a revolution in mathematical thought. Before Cantor, mathematicians had accepted the notion of the infinite in the situation of a sequence 'tending to infinity', but were not prepared to accept an actual infinity *per se*. Cantor's great achievement was to introduce precisely this concept, in fact a whole class of different infinities—that corresponding to the countable sets such as the natural numbers, that of the continuum of real numbers, and so on. For Kronecker, the type of non-constructive reasoning used by Weierstrass and Cantor was deeply suspect. His famous dictum, 'God made the integers, all the rest is the work of man', pronounced at the Berlin Congress in 1886, encapsulated his philosophy, and he envisaged the inclusion of essentially the whole of mathematics within the terms of arithmetic. Kronecker's objections to the work of Weierstrass, and of Cantor more particularly, were not of course without justification. Much time and energy during the intervening years, between his time and ours, has been spent on attempts to come to terms with the problems which the theory of transfinite numbers has brought to light. Paradoxes at the very heart of the subject have been exposed and dealt with in different ways. Since the beginning of our own century, mathematicians engaged in laying these new foundations for analysis have come to be grouped into different schools of thought, according to their philosophy of the foundations of mathematics: formalists and logicists, and the intuitionists whose philosophy was anticipated by Kronecker himself. Though his views were perhaps extreme, yet we no doubt owe much to Kronecker, and his school of thought, for his caution and scepticism, and his refusal to allow certain modes of reasoning to go unchallenged; the foundations of our subject are now being laid all the more surely, and its limitations all the more readily understood.

It has often been claimed that these mathematical disputes turned into personal quarrels. As regards the extent to which this was true we can only surmise. Towards the end of the 1880s, Weierstrass evidently told those in his confidence that his long friendship with Kronecker was over, though Kronecker himself seemed unaware of this. Weierstrass even considered the possibility of leaving Berlin for Switzerland to avoid the continuing conflict; but, since he did not wish his successor at the university to be chosen by Kronecker, he decided to stay. Cantor's career was dogged by depressive illnesses, and some have attributed these, at least in part, to his strained relations with Kronecker and the latter's vociferous opposition to his work. Cantor himself certainly believed that it was on account of Kronecker that he was never appointed to the prestigious University of Berlin. On the other hand, neither of Kronecker's official biographers made much of the personal aspect of the dispute with Cantor; indeed one of them has stated that 'no personal considerations influenced him as some have supposed'.

This same biographer wrote that 'everyone knew that [Kronecker] was a great gentleman in science as well as in ordinary life'. He was, indeed, as an older man a much respected and even formidable figure in the University of Berlin. That he was a good administrator had already been proved earlier during his years in business, and now in his later life his considerable talents in this direction were put to the service of both the University and the Academy of Sciences. Kronecker was small in stature but continued to have considerable vigour; he had been a good gymnast and swimmer in his youth. He enjoyed travelling extensively in Europe to maintain his connections with scientific colleagues abroad, and in 1884 he became a Foreign Associate of the Royal Society of London. Until the last year of his life, when he became a Christian, Kronecker continued in the Jewish faith, though he had his children brought up as Protestant Christians. In the summer of 1891, Kronecker's wife was seriously injured while they were mountain climbing, and it was her death soon afterwards which finally broke Kronecker's spirit. He succumbed to bronchitis and died only four months later.

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**Three 1's make 32**

Using three 1's and any of the usual mathematical symbols ($+$, $-$, $\times$, $\div$, $\sqrt{\ }$, $\cdot$, !, etc.), make the number 32.

ALAN FEARNEHOUGH
(Portsmouth Sixth Form College)

# I Love a Mystery, or Never Mind the Answer Feel the Question

**KEITH AUSTIN,** *University of Sheffield*

The author is interested in people's reactions to mathematics; for example, his own response to a mathematical problem can range from no interest at all to 'I should like to think about that'. Recently he has been training to be a Methodist local preacher. He points out that 'I Love a Mystery' was the title of a radio and film detective series of the 1930s and 1940s.

Question: In what film by Alfred Hitchcock does a windmill play a vital role?

If you have seen the film 'Foreign Correspondent' then you will have had no difficulty in answering the question, for the windmill, set in the bleak Dutch landscape, is the point of the film that impresses itself most on the mind of the viewer. You may also remember American reporter Huntley Haverstock had reached the windmill while in pursuit of the man who had assassinated the diplomat Van Meer amidst a sea of umbrellas and that, when the assassin's car had vanished into thin air, Huntley Haverstock had investigated the windmill that, curiously, was turning against the wind, only to discover Van Meer alive and well. However, do you remember *why* these spies and counterspies were chasing around a Europe which was on the brink of war in 1939?

The reason for all this activity that makes up the film is called the McGuffin. In this film it is a secret clause in a treaty which Van Meer has memorised. This is explained briefly in the film, but otherwise very little is made of the McGuffin. So we have the paradoxical situation that the film is all about the McGuffin and yet it is not about the McGuffin.

The point of this article is to help you see that the answers to mathematical problems, or mysteries as we should call them, can play the role of a McGuffin. The desire to find the answer can lead to a hive of mental activity. The important thing to remember is that it is the activity that matters and not the answer—the mystery is the thing, not its resolution.

**Mystery 1**

Draw a $3\times3$ array of squares and in each square write three different letters. Is it always possible to choose one letter in each square, and cross out all the other letters, so that when we look at the chosen letters we find no repeats in any vertical column or horizontal row? For example,

| | | |
|---|---|---|
| ABC | BCD | ACD |
| CDE | ACD | ABC |
| CDE | BCE | ABE |

which we can solve by crossing out the letters to leave

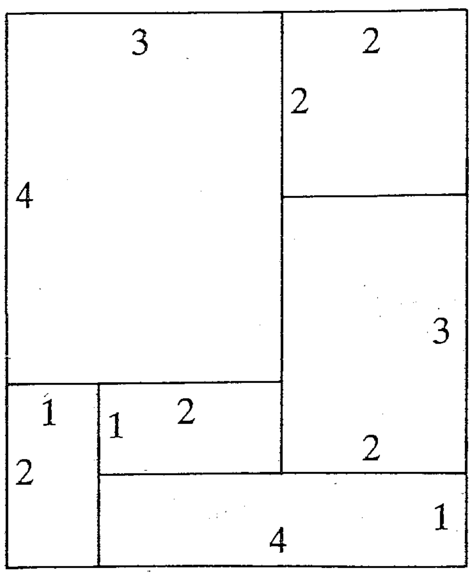
A B C
C D A .
D C E

There are other answers.

Many people when faced with one of these mysteries say that they do not know where to start. The answer is that that is why it is a mystery. Remember Rubik's cube? No one knew how to solve it when they first got it; so what did they do? They played around with it and saw what happened. So play around with the mystery—try things and see what happens, like making your own squares of letters and looking for an answer.

**Mystery 2**

I have a large rectangle divided into a number of smaller rectangles. Each small rectangle has sides which are integers, i.e. whole numbers. In addition, at least one side of each small rectangle is even. Is it always the case that my large rectangle has at least one of its sides which is even? For example,




The vertical side of the large rectangle is even.

I live close to the Pennine Way, a long-distance footpath from the centre of England to the Scottish border, and, although I have no intention of walking the whole length, I am happy to walk the odd mile or two in my home area. Similarly with these mysteries, I have a little dabble with them, now and then, trying the odd idea and perhaps finding the answer for a group of cases with a special property.

**Mystery 3**

I have a map showing a number of villages and the roads that join them. I choose a village and draw a diagram for that village as follows.

> I mark a dot for each village except my chosen village; I put the dots in a row, in any order I want. Then I draw a line between any two dots if their villages are joined by a road.

I repeat this until I have drawn a diagram for every village. For example:

Map

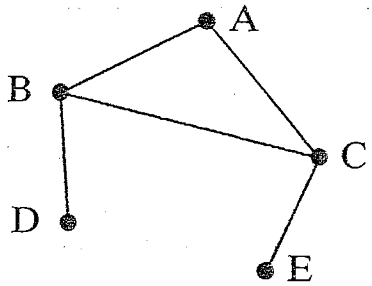


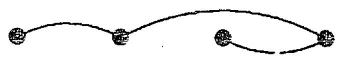

Diagram for A 

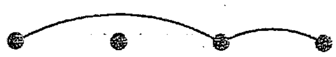
Diagram for B 

Diagram for C 

Diagram for D 

Diagram for E 

Suppose now that I gave you just the five diagrams, could you work out the road layout of the original map, not necessarily putting all the names to the villages?

In this case you could; however, the mystery is, could you do it whatever map I started with?

A speaker recently said that the important thing is not what you have done but what you have become. So with these mysteries, the important thing is not that you find the answer but that you sharpen up your mind by the mental activity involved in the search.

Lewis Carroll published a book of 'pillow problems' for the reader to tackle while lying awake at night. The mysteries can be used in a similar way, although my experience is that they usually send me to sleep; you might find the mysteries a help if you have difficulty in getting to sleep.

**Mystery 4**

Write down a row of positive integers; the row can be as long or as short as you like. Below it write a similar row of the same length. Repeat with a 3rd row, 4th row, .... However, no two rows that you write can be related in the following way: each number in one row is less than or equal to the corresponding number in the other row, e.g. you cannot have

$$\begin{array}{lrrrrr} & 2, & 5, & 4, & 3, & 3, \\ \text{and} & 3, & 10, & 8, & 3, & 10. \end{array}$$

The mystery is, if you write down any rows you wish, will you eventually find yourself unable to continue? For example, if we write down

$$\begin{array}{rrr} 2, & 3, & 1 \\ 1, & 10, & 1 \\ 2, & 1, & 2 \\ 1, & 1, & 3 \\ 3, & 1, & 1 \\ 1, & 7, & 2 \end{array}$$

then we cannot add another row

It is said that we do not enjoy or find interesting any mathematics unless 'we make it our own'. If our teacher or a book spells out the whole mathematical story then it feels it belongs to the teacher or the author. But with a mystery, we are only given the setting, all the rest is up to us—we develop the search in our own way, we become the Miss Marple or the Philip Marlowe.

**Mystery 5**

Write out a rectangular array of integers, positive, negative or zero. Work out the sum of each horizontal row and each vertical column, e.g.

$$\begin{array}{rrrrcc} -2 & -3 & -2 & 1 & & -6 \\ 1 & -4 & 3 & 2 & & 2 \\ 0 & 2 & -3 & 1 & & 0 \\ & & & & & \uparrow \\ -1 & -5 & -2 & 4 & \leftarrow & \text{sums} \end{array}$$

You are allowed to select some rows and change the sign of all the numbers in those rows and then select some columns and change all the signs in those columns. Your target is to make all the sums positive or zero. For example, changing the signs in rows 1 and 3 from the top and column 2 from the left gives the following.

$$\begin{array}{rrrrcc}
2 & -3 & 2 & -1 & & 0 \\
1 & 4 & 3 & 2 & & 10 \\
0 & 2 & 3 & -1 & & 4 \\
 & & & & & \uparrow \\
3 & 3 & 8 & 0 & \leftarrow & \text{sums}
\end{array}$$

The mystery is, is it always possible to reach the target?

Have you noticed how the posters for video cassettes or the descriptions in the *Radio Times* or *TV Times* are much more exciting than the actual films to which they refer? Yet it is not that they are misleading, rather that they hint at things we have not met—they rely on the attraction of the unknown. Similarly, advertisements are said to tap into the emotions and *curiosity* of their viewers and readers. So it can be with these mysteries—you can become so intrigued by the search for the unknown that you find 'you cannot put the mystery down', but work at it every scrap of spare time you have.

**Mystery 6**

In a youth club, every girl knows at least one boy, but no boy knows every girl. Is it always possible to find girls $G_1$ and $G_2$ and boys $B_1$ and $B_2$ so that $G_1$ knows $B_1$ but not $B_2$, while $G_2$ knows $B_2$ but not $B_1$? For example, what about a club in which

Anne knows Harry, Ian and Ken;
Barbara knows Harry and Ian;
Christine knows Harry, Jack and Ken;
Diana knows Ken.

One answer is that Anne knows Ian but not Jack, while Christine knows Jack but not Ian. There are other answers.

Finally, there is the temptation at times to become depressed by these mysteries, particularly if you think you have made a great step forward and then it crumbles to nothing. The going can be tough, but remember not to put the mystery on a high pedestal. Put the mystery in its proper place—it is just a few words written with the aim of producing mental activity.

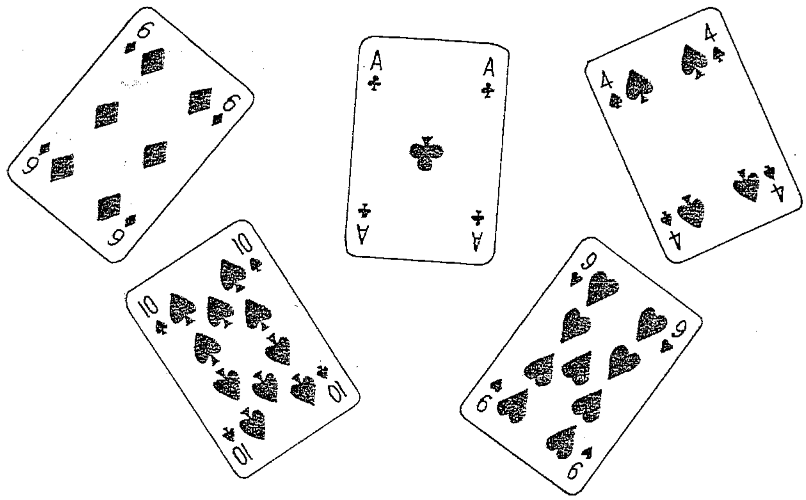
Never feel in any way the mysteries have overcome you. Remember the words of Ma Joad in *The Grapes of Wrath*:

'They can't lick us, Pa, we're the people!'

# Patience!

J. C. C. LEACH, *Sheffield City Polytechnic*

The author was a senior metallurgist with British Steel and was involved in a number of important process developments in the steel industry in the Sheffield area. He took early retirement in 1983, and since then he has lectured on metals and materials at Sheffield City Polytechnic, while retaining links with the steel industry.



The rules of a version of patience I occasionally play on rainy days on holiday are as follows:

In a pack of 52 playing cards (without joker) an ace counts as 1, each number as itself, a jack as 11, a queen as 12 and a king as 13. After shuffling the cards, start dealing them one at a time face upwards into a pile in front of you, counting 1, 2, 3, etc. as you do so. If the third card, say, is a 3, put this aside in a special place, because it will be used for scoring at the end of the game. In general, if the $n$th card is an $n$ ($n \leq 13$), save the card for scoring. If 13 cards are dealt without a match, discard them all, and work through another 13. If these and the two subsequent 13s fail to produce a match, all the cards have been used up and the score in this case is zero. Usually, however, several matches (i.e. between the count and the card dealt) will occur. When this happens with the $n$th card, as well as saving separately the matched card as already indicated, return (unshuffled) the dealt cards (all $n-1$ of them) to the bottom of the pack being dealt, and start again from the top counting 1, 2, 3, etc. until another match occurs, or until a full 13 cards have been dealt. Return or discard as appropriate. Continuing this process naturally reduces the pack. When fewer than 13, say $m$, cards are left, count out cards as before, using the same rules for saving cards and returning cards to the pack, or discarding all $m$ cards if no match has occurred. The game ends when all cards have been discarded or saved, and the final score is the sum of the saved cards.

After playing this game a great many times (and getting the children involved) an interesting distribution of scores is revealed. I have found that the frequency with which particular scores occur seems to have peaks at 13 and possibly at 26. I could see no reason why this should be so and tended to ascribe it to too small a population. However, playing a simplified form of the patience using only the cards ace to 5 (and therefore only 20 cards in all), and counting no further than 5 at a time, revealed a more distinct peak, this time at 5, and suspicions of subsequent peaks. This and other oddities pose the question of how to predict theoretically the distribution of scores. Further than this, can one predict mathematically what mean score might be expected, can one determine a score such that the chances of obtaining values above it or below it are equal or approximately equal, and can one determine the initial pack distribution to yield an optimum score (and indeed, what is this optimum score?)? No doubt a computer could be programmed to provide a great deal of experimental evidence, but I have not yet got round to doing this.

More generally, there is the problem of $p$ suits each with $q$ cards. Theoretical results even for quite small values of $p$ and $q$ would be interesting.

# A Class of Constant-Period Paths

**G. DE VISME**

Dr De Visme graduated in physics in 1942. After army service he joined the scientific staff of the GEC Research Laboratories. He subsequently moved to the RAF Education Branch, then to the Department of Electrical Engineering at UMIST, and finally became head of the Electrical Engineering Department at the North Staffordshire Polytechnic. He retired in 1981 and has since indulged his interest in mathematics.

In this article we derive from dimensional considerations the path of a particle oscillating in a vertical plane under a class of vertical fields of force so that the period of its oscillation is independent of the amplitude of oscillation. Figure 1 shows a particle $P$ of mass $m$ travelling along a smooth curve $C$ under a downward vertical force $Ky^{n-1}$, where $n$ and $K$ are constants ($0 < n \leqslant 2$), having been projected horizontally from $O$ with speed $u$. The particle oscillates to and fro along $C$ between points $A$ and $B$. We wish to derive the shape of $C$ so that the time $T$ between the instant of projection and the instant when the particle first comes to rest at $A$ is independent of $u$.

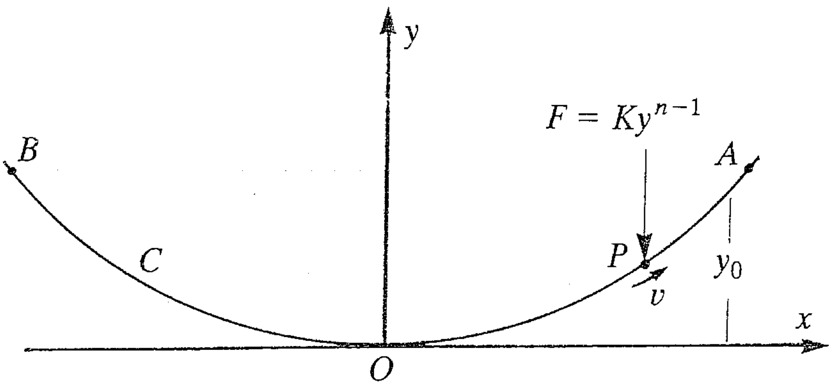



Figure 1

Now the loss of kinetic energy of the particle from $O$ to $P$ is equal to the gain in potential energy,

$$\int_0^y K\eta^{n-1}\,\mathrm{d}\eta = \frac{Ky^n}{n},$$

where $y$ is the height of the particle above $O$. Thus

$$\tfrac{1}{2}mu^2 - \tfrac{1}{2}mv^2 = \frac{Ky^n}{n},$$

where $v$ is the speed of the particle at $P$. Therefore

$$v = u\sqrt{1 - \frac{2K}{mnu^2}y^n}.$$

Now

$$T = \int_0^{s_0} \frac{1}{v}\,\mathrm{d}s,$$

where $s_0$ is the length of the arc $OA$, so that

$$T = \frac{1}{u}\int_0^{y_0} \frac{1}{\sin\theta\sqrt{1 - \frac{2K}{mnu^2}y^n}}\,\mathrm{d}y,$$

where $y_0 = (mnu^2/2K)^{1/n}$, the height of $A$ above $O$, and $\sin\theta = \mathrm{d}y/\mathrm{d}s$.

Writing $y = y_0 z$ and $\sin\theta = f(y) = f(y_0 z)$, we have

$$T = \frac{1}{u}\int_0^1 \frac{y_0}{f(y)\sqrt{1-z^n}}\,\mathrm{d}z.$$

Now

$$u = \sqrt{\frac{2Ky_0^n}{mn}} = \sqrt{\frac{2K}{mn}}\,y_0^{\frac{1}{2}n},$$

so that

$$T = \sqrt{\frac{mn}{2K}}\int_0^1 \frac{y_0^{1-\frac{1}{2}n}}{f(y)\sqrt{1-z^n}}\,\mathrm{d}z$$

$$= \sqrt{\frac{mn}{2K}}\int_0^1 \left[\frac{y^{1-\frac{1}{2}n}}{f(y)}\right]\frac{z^{\frac{1}{2}n-1}}{\sqrt{1-z^n}}\,\mathrm{d}z.$$

We want $T$ to be independent of $y_0$, so the expression $[y^{1-\frac{1}{2}n}/f(y)]$ must be constant, i.e. $f(y) = ay^{1-\frac{1}{2}n}$ for some constant $a$. Then

$$T = \sqrt{\frac{mn}{2K}}\int_0^1 \frac{z^{\frac{1}{2}n-1}}{a\sqrt{1-z^n}}\,\mathrm{d}z.$$

Substituting $z^{\frac{1}{2}n} = \sin\phi$, so that $\frac{1}{2}nz^{\frac{1}{2}n-1}\,\mathrm{d}z = \cos\phi\,\mathrm{d}\phi$, we have

$$T = \sqrt{\frac{mn}{2K}}\,\frac{2}{n}\int_0^{\frac{1}{2}\pi} \frac{\cos\phi}{a\cos\phi}\,\mathrm{d}\phi = \sqrt{\frac{m}{2Kn}}\,\frac{\pi}{a},$$

with the curve being $\sin\theta = \mathrm{d}y/\mathrm{d}s = ay^{1-\frac{1}{2}n}$.

Two well-known cases arise:

1. Normal gravity, for which $n = 1$ and $K = mg$. The equation of $C$ is $\sin\theta = a\sqrt{y}$, or

$$y = \frac{\sin^2\theta}{a^2} = \frac{1-\cos 2\theta}{2a^2}.$$

Now

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta = \frac{2\sin\theta\cos\theta}{a^2}\,\frac{\mathrm{d}\theta}{\mathrm{d}x}.$$

Therefore

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{2\cos^2\theta}{a^2} = \frac{1+\cos 2\theta}{a^2},$$

so that

$$x = \frac{2\theta+\sin 2\theta}{2a^2},$$

measuring from the lowest point $\theta = 0$. So we have a cycloid of generating radius $R = \frac{1}{2}a^2$ and parameter $\phi = 2\theta$:

$$x = R(\phi+\sin\phi), \qquad y = R(1-\cos\phi).$$

The period is

$$\sqrt{\frac{mn}{2K}}\,\frac{\pi}{a} = \sqrt{\frac{m}{2mg}}\,\pi\sqrt{2R} = \pi\sqrt{\frac{R}{g}}.$$

2. Simple harmonic motion, for which $n = 2$ and for which we know that the period is independent of the amplitude. In this case $\sin\theta$ is a constant $a$ and the curve $C$ degenerates into two straight lines originating from $O$ with gradients $\pm a/\sqrt{1-a^2}$. In this case

$$T = \frac{\pi}{2a}\sqrt{\frac{m}{K}}.$$

For the general case

$$\frac{dy}{ds} = ay^{1-\frac{1}{2}n},$$

so

$$\frac{ds}{dy} = \frac{y^{\frac{1}{2}n-1}}{a} \quad \text{and} \quad s = \frac{2}{na}y^{\frac{1}{2}n},$$

(measuring from the lowest point). Now

$$\begin{aligned}\left(\frac{dx}{ds}\right)^2 = 1-\left(\frac{dy}{ds}\right)^2 &= 1-a^2y^{2-n}\\ &= 1-a^2\left(\frac{nas}{2}\right)^{(2/n)(2-n)}\\ &= 1-a^2\left(\frac{nas}{2}\right)^{4/n-2}\\ &= 1-a^{4/n}\left(\frac{ns}{2}\right)^{4/n-2}.\end{aligned}$$

Therefore

$$x = \int_0^s \sqrt{1-a^{4/n}\left(\frac{n\sigma}{2}\right)^{4/n-2}}\,d\sigma, \qquad y = \left(\frac{nas}{2}\right)^{2/n},$$

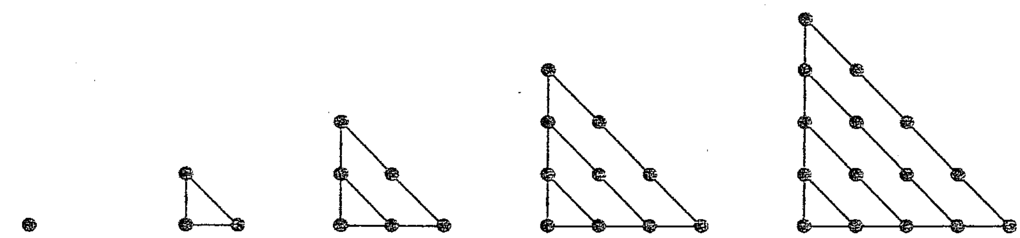
where the integral for $x$ is left for the reader to evaluate.

# Polygonal Numbers

JOSEPH MCLEAN, *Computer Services Department, Strathclyde Region*

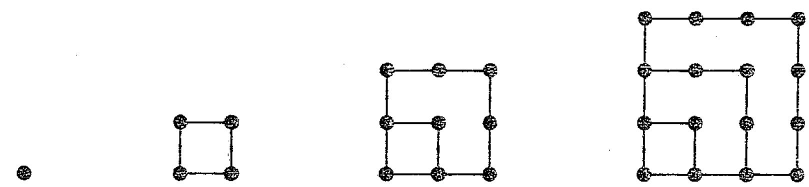
The author obtained an M.Sc. at the University of Glasgow, after which he was a research assistant in the Department of Computer Science at the University of Strathclyde. He is now an analyst and programmer in the Computer Services Department of Strathclyde Region. His main mathematical interest is number theory.

The well-known triangular numbers $t(n) = \frac{1}{2}n(n+1)$ are often depicted graphically as in figure 1, since $t(n)$ is just the sum of the first $n$ natural numbers. Similarly, square and pentagonal numbers can be obtained from the diagrams in figures 2 and 3. The sequences of numbers formed in this way are called *polygonal numbers of degree m*, where $m = 3$ for triangular numbers, $m = 4$ for the squares in figure 2, and the numbering continues logically to higher values.



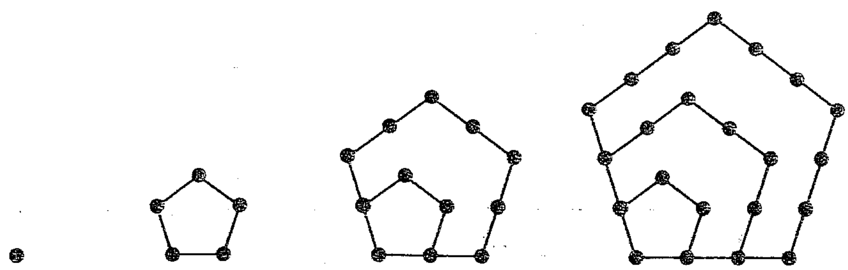
$n = 1$ $n = 2$ $n = 3$ $n = 4$ $n = 5$

Figure 1. Triangular numbers



$n = 1$ $n = 2$ $n = 3$ $n = 4$

Figure 2. Square numbers



$n = 1$ $n = 2$ $n = 3$ $n = 4$

Figure 3. Pentagonal numbers

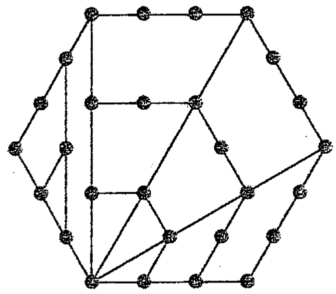


Figure 4

If we denote the polygonal numbers by $S(m,n)$, then it can be seen from figure 4, where a hexagon of degree 4 is shown as a combination of a pentagon of degree 4 and a triangle of degree 3, that, for $m > 3$ and $n > 0$,

$$S(m,n) = S(m-1,n) + S(3,n-1).$$

Hence, by induction on $m$,

$$S(m,n) = S(3,n) + (m-3)\,S(3,n-1)$$

for $m \geqslant 3$, where we define $S(m,0) = 0$ for all $m$ so that the formula is true for all $n \geqslant 1$. If we expand this formula using the fact that $S(3,n) = t(n) = \frac{1}{2}n(n+1)$, we obtain

$$S(m,n) = \tfrac{1}{2}n(n+1) + \tfrac{1}{2}(m-3)\,n(n-1)$$

or

$$S(m,n) = \tfrac{1}{2}(m-2)\,n^2 + \tfrac{1}{2}(4-m)\,n.$$

Now to the problem at hand. Some time ago I discovered the formula

$$\prod_{k=2}^{\infty}\left(1-\frac{2}{k(k+1)}\right) = \tfrac{1}{3}.$$

Since the product is just

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{S(3,k)}\right),$$

this leads to the more general product

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{S(m,k)}\right).$$

Hence the problem: find a general formula for the product

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{S(m,k)}\right)$$

for $m \geqslant 3$ and $n \geqslant 2$ in terms of $m$ and $n$, and prove it.

*Hint* 1. For each $m$, start by getting an estimate for the $n \to \infty$ case and work from there.

*Hint* 2. The solution does not require anything higher than squares of $n$ and $m$.

In looking for an answer to the problem, I supposed that

$$\prod_{k=2}^{n}\left(1-\frac{1}{S(m,k)}\right)$$

was of the form $f(m,n)/g(m,n)$, where $f$ and $g$ were quadratic functions in $n$. I then calculated the actual values for the first few $m$ values and $n$ from 2 to 7 in order to produce enough equations to be able to find the coefficients by reducing a matrix. These first few solutions enabled me to conjecture the general solution in terms of $m$ and $n$, which I subsequently proved by induction, as follows.

*Theorem.*

$$\prod_{k=2}^{n}\left(1-\frac{1}{S(m,k)}\right)=\frac{m-2}{m}+\frac{2}{mn}.$$

*Proof.* For $n=2$, the left-hand side equals

$$1-\frac{1}{S(m,2)}=1-\frac{1}{S(3,2)+m-3}=1-\frac{1}{m}=\frac{m-1}{m}$$

and the right-hand equals

$$\frac{m-2}{m}+\frac{2}{m\times 2}=\frac{m-2}{m}+\frac{1}{m}=\frac{m-1}{m}.$$

We now suppose that the result is true for some $n \geqslant 2$. Then

$$\begin{aligned}\prod_{k=2}^{n+1}\left(1-\frac{1}{S(m,k)}\right)&=\left(\frac{m-2}{m}+\frac{2}{mn}\right)\left(1-\frac{2}{(m-2)(n+1)^2+(4-m)(n+1)}\right)\\&=\frac{m-2}{m}+\frac{2}{mn}-\frac{2n(m-2)+4}{mn(n+1)(mn-2n+2)}\\&=\frac{m-2}{m}+\frac{2(n+1)(mn-2n+2)-2n(m-2)-4}{mn(n+1)(mn-2n+2)}\\&=\frac{m-2}{m}+\frac{2mn^2-4n^2+4n+2mn-4n+4-2mn+4n-4}{mn(n+1)(mn-2n+2)}\end{aligned}$$

$$= \frac{m-2}{m} + \frac{2n(mn-2n+2)}{mn(n+1)(mn-2n+2)}$$
$$= \frac{m-2}{m} + \frac{2}{m(n+1)} \, .$$

The result follows by induction.

---

**Squares and cubes**

Here are some curious results concerning squares and cubes:

$$16^2+72^2+162^2 = 178^2$$
$$45^2+81^2+97^2+117^2 = 178^2$$
$$12^2+20^2+36^2+60^2+162^2 = 178^2$$
$$27^2+36^2+45^2+65^2+72^2+135^2 = 178^2$$
$$18^2+36^2+45^2+45^2+65^2+90^2+117^2 = 178^2$$
$$18^2+18^2+45^2+45^2+65^2+81^2+90^2+90^2 = 178^2$$
$$4^2+17^2+27^2+51^2-8^2-9^2-25^2 -55^2+5^3+8^3+25^3+28^3-4^3-9^3-26^3-27^3 = 2$$
$$9^2+55^2+64^2+99^2-19^2-27^2-49^2 -129^2+10^3+27^3+49^3+65^3-9^3-28^3-50^3-64^3 = 3$$
$$16^2+125^2+129^2+163^3-33^2-64^2-81^2 -251^2+17^3+64^3+81^3+126^3-16^3-65^3-82^3-125^3 = 4$$

**References**

Mortezza Jammi, Mysteries of numbers. *Reconciliation with Mathematics* Vol. X No. 1 (1987).

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# Computer Column

MIKE PIFF

**Insertion sorting**

In this column, we look at the problem of sorting a real array. There are several ways of accomplishing this task, but the simplest and most efficient for $n \leqslant 50$ items is probably *insertion* sort. This looks at item $i$ for $i$ ranging from 2 to $n$ successively, and inserts it in its correct position amongst items 1 to $i-1$.

We create a definition module `sort.def` containing the following.

```
DEFINITION MODULE Sort;
PROCEDURE InsertionSort(VAR a:ARRAY OF REAL; lo, hi, step:INTEGER);
END Sort.
```

To access insertion sort we simply `IMPORT` it from this module. There are three redundant parameters to this procedure. Parameters *lo* and *hi* should be set to 0 and $n-1$, respectively, and *step* to 1. Their significance will become clear in the next column.

Here is the implementation of module `sort.mod`.

```
IMPLEMENTATION MODULE Sort;
PROCEDURE InsertionSort
(VAR a:ARRAY OF REAL;
lo,hi,step:INTEGER);
VAR i, j:INTEGER; active:BOOLEAN;
temp:REAL;
BEGIN
 i:=lo;
 WHILE (i+step)≤hi DO
  j:=i; i:=i+step;
  temp:=a[i]; active:=TRUE;
  WHILE active DO
   IF j<lo THEN active:=FALSE;
   ELSIF temp<a[j] THEN
    a[j+step]:=a[j];
    j:=j-step;
   ELSE active:=FALSE;
   END;
  END;
  a[j+step]:=temp;
 END;
END InsertionSort;
BEGIN
END Sort.
```

---

**Then and now ...**

From *The History of Mathematics in Europe*, by J. W. N. Sullivan (1925):

> It is interesting to know the standard of University mathematics during the latter part of the fourteenth century. ...At Vienna was taught the theory of proportions, perspective, the first five books of Euclid and the measurement of areas. The teacher read through the works on these subjects with the student, but there were no examinations. Degrees were granted upon the student taking an oath that he had listened to the required course of exposition.

T. G. DALE
(20 Pitcullen Crescent, Perth PH2 7HT)

# Letters to the Editor

Dear Editor,

*Weed v. reed*

In Volume 23 Number 1, page 7, David Singmaster posed the following problem.

> A water weed grows 3 feet on the first day, and its growth on each succeeding day is half that on the preceding day. A reed grows 1 foot on the first day, and its growth on each succeeding day is twice that on the preceding day. When are they of equal size?

I was expecting someone to have sent an answer to be published in a subsequent issue. As this has not happened, here is the answer I worked out when I first saw the problem.

After $d$ days, the height of the water weed is $3(1.5)^{d-1}$ feet and that of the reed is $3^{d-1}$ feet. These will be equal when $d = \log 6/\log 2$, i.e. after 2.585 days. The height when they are equal is 5.705 feet.

Yours sincerely,
BARRY CHRISTIAN
(Chemistry Department,
Sheffield City Polytechnic)

Dear Editor,

*Powers of numbers equal to the sum of consecutive integers*

On page 94 of Volume 23 Number 3, I listed general formulae for powers of any odd number. On page 57 of Volume 18 Number 2, Joseph McLean suggested that readers tackle the harder problem associated with powers of even numbers. My progress so far is as follows.

If the even number is a power of 2, then there is no solution.

*Proof.* Let

$$m^n = a+(a+1)+\ldots+b \quad (n \geqslant 1),$$

where $m = 2^r$ $(r \geqslant 1)$. Then

$$(2^r)^n = \tfrac{1}{2}(b+a)(b-a+1).$$

Therefore both $(b+a)$ and $(b-a+1)$ must be powers of 2, and so we have $b+a = 2^i$, $b-a = 2^j-1$, for some $i$ and $j$. Adding, we have $2b = 2^i+2^j-1$. This is an odd number, and therefore a solution in integers is impossible.

I next consider the case of

$$m^n = a_n+(a_n+1)+\ldots+b_n \quad (n \geqslant 1),$$

where $m = 2\times$(an odd number greater than 1). One solution to this case, giving the sequences of integers $(a_n)$ and $(b_n)$, is as follows:

$$a_1 = \tfrac{1}{4}(m-6), \qquad a_n = \tfrac{1}{2}ma_{n-1}+c_n \qquad (n \geqslant 2);$$
$$b_1 = \tfrac{1}{4}(m+6), \qquad b_n = \tfrac{1}{2}mb_{n-1}-c_n \qquad (n \geqslant 2);$$
$$c_2 = \tfrac{1}{4}(3m-14), \quad c_n = 2c_{n-1}+\tfrac{1}{4}(m-2) \quad (n \geqslant 3).$$

For example:

$$18^1 = 3+\ldots+6 \qquad (a_1 = 3,\ b_1 = 6,\ c_2 = 10);$$
$$18^2 = 37+\ldots+44 \qquad (a_2 = 9\times 3+10,\ b_2 = 9\times 6-10,\ c_3 = 2\times 10+4 = 24);$$
$$18^3 = 357+\ldots+372 \qquad (a_3 = 9\times 37+24,\ b_3 = 9\times 44-24,\ c_4 = 2\times 24+4 = 52);$$
$$18^4 = 3265+\ldots+3296 \quad (a_4 = 9\times 357+52,\ b_4 = 9\times 372-52).$$

Note that the number of integers ($s_n$) required to represent $m^n$ is $s_n = 2^{n+1}$. Also,

$$a_n+b_n = \tfrac{1}{2}m(a_{n-1}+b_{n-1}) = (\tfrac{1}{2}m)^n.$$

The final case considered is for

$$m^n = a_n+\ldots+b_n,$$

where $m$ is an even number of the form $m = 2^r d$ $(r \geqslant 2)$ and where $d$ $(d > 1)$ is an odd number. Although I am able to find solutions for any $m$ of this form, I am unable to discover a general formula. For example, if $m = 24$, the recurrence relations are

$$a_1 = 7, \qquad a_n = 8a_{n-1}+c_n,$$
$$b_1 = 9, \qquad b_n = 8b_{n-1}-c_n,$$
$$c_2 = 4, \qquad c_n = 3c_{n-1}+7,$$

which give us the following:

$$24^1 = 7+8+9, \qquad 24^3 = 499+\ldots+525,$$
$$24^2 = 60+\ldots+68, \qquad 24^4 = 4056+\ldots+4136.$$

Readers are invited to search for general formulae.

We note finally that the list of consecutive integers in a particular case may not be unique, e.g.

$$6^5 = 56+\ldots+136 = 90+\ldots+153.$$

Yours sincerely,

BOB BERTUELLO
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Dear Editor,

*The Gudermannian function*

On reading Terry S. Grigg's letter on page 94 of Volume 23 Number 3 of *Mathematical Spectrum*, it occurred to me that readers may be interested to know that the Gudermannian function is the inverse of a more familiar function.

Starting from the definition $\mathrm{gd}(x) = \tan^{-1}\sinh x$ and letting $y = \mathrm{gd}(x)$, we can write $\sinh x = \tan y$. We now solve for $x$ using the definition of $\sinh x$, i.e. $\frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}) = \tan y$, which leads to the quadratic equation in $\mathrm{e}^x$,

$$\mathrm{e}^{2x} - 2\tan y\,\mathrm{e}^x - 1 = 0.$$

Hence

$$\mathrm{e}^x = \tan y \pm \sqrt{\tan^2 y + 1} = \tan y \pm \sec y.$$

Since $\sec y > \tan y$ for $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$ and $\mathrm{e}^x > 0$ we can ignore the negative solution, so

$$\mathrm{e}^x = \tan y + \sec y \quad \text{or} \quad x = \ln(\tan y + \sec y).$$

In other words,

$$\mathrm{gd}^{-1}(x) = \ln(\tan x + \sec x).$$

This function will undoubtedly be familiar to teachers of A-level mathematics as the integral of $\sec x$.

Yours sincerely,
ALAN FEARNEHOUGH
(Portsmouth Sixth Form College)

Dear Editor,

*Summing powers of integers*

In his article in Volume 23 Number 4, O. D. Anderson comments on an error commonly found in tables of the sums of powers of integers. The origin of the erroneous coefficient is none other than James Bernoulli's (1713) *Ars Conjectandi* itself. His table is reproduced on page 127 of my *Pascal's Arithmetical Triangle* (Griffin, London, 1987), where you will find further references and, on pages 82 to 84, a description of how the coefficients may be simply found by inverting a matrix of binomial coefficients.

Yours sincerely,
A. F. W. EDWARDS
(Gonville and Caius College,
Cambridge)

# Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

# Problems

24.1 (Submitted by D. J. Yates)
A 'sad' number is a number such that, if its digits are squared and added, and this is repeated, the result is not 1. A 'happy' number is a number which is not sad. All the digits but 5 appear in the first twelve happy numbers; the twelfth happy number is 68.
(i) Verify that all sad numbers iterate to 4.
(ii) Which is the first happy number in which the digit 5 appears?

24.2 (Submitted by John MacNeill, The Royal School, Wolverhampton—see his book review in this issue)
Investigate the shape of the face of a rhombic dodecahedron and discover what this has to do with a regular tetrahedron and a sheet of A4 paper.

24.3 (Submitted by Alan Fearnehough, Portsmouth Sixth Form College)
Determine the exact value of $\cot 80° + \operatorname{cosec} 40°$.

# Solutions to Problems in Volume 23 Number 3

23.7 Show that

$$\pi = 4 \sum_{r=1}^{\infty} \cot^{-1} 2r^2.$$

*Solution* by Nicholas Saltmarsh (Gresham's School, Holt)

$$\begin{aligned}\sum_{r=1}^{n} \cot^{-1} 2r^2 &= \sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{2r^2}\right)\\ &= \sum_{r=1}^{n} \tan^{-1}\left(\frac{(2r+1)-(2r-1)}{(2r+1)(2r-1)+1}\right)\\ &= \sum_{r=1}^{n} \left\{\tan^{-1}\left(\frac{1}{2r-1}\right) - \tan^{-1}\left(\frac{1}{2r+1}\right)\right\}\\ &= \tan^{-1} 1 - \tan^{-1}\left(\frac{1}{2n+1}\right).\end{aligned}$$

Now let $n \to \infty$ to give

$$\sum_{r=1}^{\infty} \cot^{-1} 2r^2 = \tfrac{1}{4}\pi.$$

Also solved by Oliver Johnson (King Edward's School, Birmingham), Gregory Economides (University of Newcastle upon Tyne Medical School) and M. Movahhedian (Isfahan University of Technology, Iran).

23.8 Factorise the $n \times n$ determinant

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{(n-1)2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{vmatrix}.$$

*Solution* by Nicholas Saltmarsh

Denote the given determinant by $D_n$. If we subtract each column from the preceding column and expand along the top row, we obtain

$$D_n = (-1)^{n-1} \begin{vmatrix} 1-\omega & \omega(1-\omega) & \cdots & \omega^{n-2}(1-\omega) \\ 1-\omega^2 & \omega^2(1-\omega^2) & \cdots & \omega^{(n-2)2}(1-\omega^2) \\ \vdots & \vdots & \ddots & \vdots \\ 1-\omega^{n-1} & \omega^{n-1}(1-\omega^{n-1}) & \cdots & \omega^{(n-2)(n-1)}(1-\omega^{n-1}) \end{vmatrix}.$$

If, for $1 \leqslant r \leqslant n-1$, we take out factors $1-\omega^r$ from row $r$ and $\omega^r$ from column $r+1$, we obtain

$$D_n = (-1)^{n-1} \omega^{\frac{1}{2}(n-2)(n-1)} D_{n-1} \prod_{r=1}^{n-1} (1-\omega^r).$$

Since $D_1 = 1$, we can apply this repeatedly to give

$$D_n = (-1)^{\frac{1}{2}n(n-1)} \omega^{\frac{1}{6}n(n-1)(n-2)} \prod_{r=1}^{n-1} (1-\omega^r)^{n-r}.$$

Also solved by Oliver Johnson, Gregory Economides, M. Movahhedian and Nick Shea (University College, Oxford).

23.9 A ball is hit in the air with speed $U$ and caught at the same height. Determine the maximum length of the path travelled by the ball over all possible angles of projection.

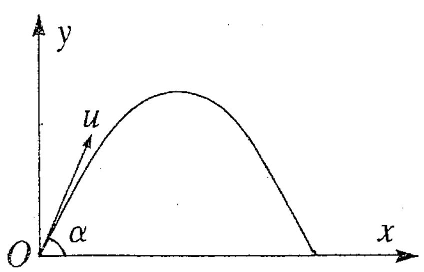
*Solution* by Nicholas Saltmarsh

We have

$$x = (U\cos\alpha)t, \qquad y = (U\sin\alpha)t - \tfrac{1}{2}gt^2,$$

where $t$ denotes time. Thus

$$\frac{dx}{dt} = U\cos\alpha, \qquad \frac{dy}{dt} = U\sin\alpha - gt.$$




If $T$ is the time when the ball reaches the highest point of its flight,

$$0 = U\sin\alpha - gT,$$

so that $T = (U\sin\alpha)/g$. The length of the path is now $s$, where

$$\begin{aligned} s &= 2\int_0^T \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t \\ &= 2\int_0^T \sqrt{U^2\cos^2\alpha + (U\sin\alpha - gt)^2}\ \mathrm{d}t. \end{aligned}$$

If we substitute $U\sin\alpha - gt = U\cos\alpha\sinh\theta$, we obtain

$$\begin{aligned} s &= -\frac{2U^2\cos^2\alpha}{g}\int_{\sinh^{-1}(\tan\alpha)}^{0} \cosh^2\theta\ \mathrm{d}\theta \\ &= \frac{U^2\cos^2\alpha}{g}\int_0^{\sinh^{-1}(\tan\alpha)} (\cosh 2\theta + 1)\ \mathrm{d}\theta \\ &= \frac{U^2\cos^2\alpha}{g}\left[\tfrac{1}{2}\sinh 2\theta + \theta\right]_0^{\sinh^{-1}(\tan\alpha)} \\ &= \frac{U^2\sin\alpha}{g} + \frac{U^2\cos^2\alpha}{g}\ln(\sec\alpha + \tan\alpha). \end{aligned}$$

To find out when this is maximum, we have

$$\begin{aligned} \frac{g}{U^2}\frac{\mathrm{d}s}{\mathrm{d}\alpha} &= \cos\alpha + \cos\alpha - 2\sin\alpha\cos\alpha\ln(\sec\alpha + \tan\alpha) \\ &= 0, \end{aligned}$$

so that

$$1 - \sin\alpha\ln(\sec\alpha + \tan\alpha) = 0,$$

since $\cos\alpha \neq 0$. This is satisfied when $\alpha = \alpha_0 \approx 56.5°$, and the maximum length of path is

$$\frac{U^2}{g\sin\alpha_0} \approx \frac{6U^2}{5g}.$$

Also solved by Gregory Economides and Nick Shea.

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A train travels from $A$ to $B$ at an average speed of 30 miles per hour, and returns at an average speed of 45 miles per hour. What is the average speed over the whole journey?

# Reviews

**The Oxford Minidictionary of Mathematics.** By MICHAEL WARDLE. Oxford University Press, 1990. Pp. xiii + 285. Paperback £2.50 (ISBN 0-19-212276-2).

As it claims, this 'minidictionary' uses simple, everyday language and examples in its brief characterisations of the standard mathematical terms met in the National Curriculum, GCSE, and Scottish Standard Grade courses.

Many entries, presumably included for completeness, nevertheless seem superfluous: for example 'divisibility—the property of being divisible' or 'spherical—any object which is shaped like a sphere is spherical'.

An amusing entry: 'Viewed from the inside the surface of [a] football would appear to be concave'.

An inaccurate entry: 'One of the classic problems, which has not yet been solved, is how to trisect an angle using only a pair of compasses and a straight edge'.

Curious entries: 'happy number—if, when the digits of a number are squared and then added, and this is repeated, the result is 1, the number is called a happy number'. Also 'sad number—the name given to a number which is not a happy number'.

I doubt if students will make much use of this dictionary. They might be advised rather to buy a notebook and jot in it such things as they feel personally likely to need to refer to.

67 Wilbraham St, Preston, Lancs, PR1 5NN DAVID J. YATES

**The Sacred Beetle and Other Great Essays in Science.** Essays selected by MARTIN GARDNER. Oxford University Press, republished 1990. Pp. 432. £6.95 (ISBN 0-19-286047-X).

This is a collection of 35 essays, not, for the most part, in science, but on science. Martin Gardner has chosen extracts from various sources. He writes, 'There are "essays" here that are chapters, not written to be read apart from the book in which they first appeared. Several were originally lectures. Two are from works of fiction. Some are brief enough to be called sketches, others long enough to be called treatises. Some are heavy with scientific erudition, others glance at science casually over a shoulder.' The list of authors is exceptional, including Bertrand Russell, Charles Darwin, Albert Einstein, Francis Bacon and H. G. Wells. However, the standard of the contributions is mixed. Some are genuinely outstanding but every reader will think that there are some items that do not merit inclusion. Yet the collection overcomes the faults of some of its parts. If one essay does not satisfy, then there are immediately available many alternatives which will provide something attractive to the reader with a general interest in science. This is a good book to have, but I think one would not read it through long hours; one would read it a little at a time, returning to it often.

Penwith College, Penzance I. M. RICHARDS

**The Puzzling World of Polyhedral Dissections.** By STEWART T. COFFIN. Recreations in Mathematics, Number 6. Oxford University Press, 1990. Pp. 196. Hardback £ 17.50 (ISBN 0-19-853207-5).

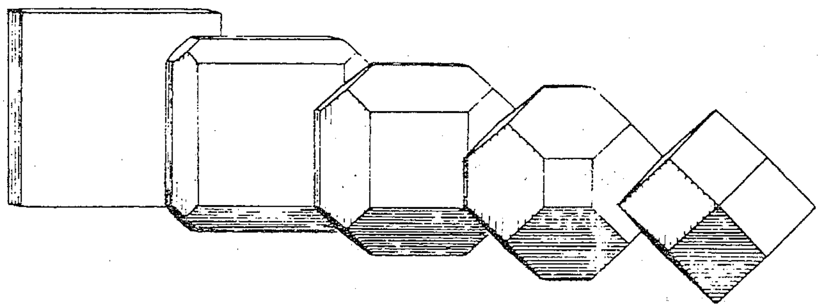
Few people could have written such a book as this; the American author has decades of experience in designing and producing geometrical puzzles. There are 212 figures, each consisting typically of several diagrams (line drawings or retouched photographs), with just enough detail to convey the necessary information to the reader. Indeed all the material in this book is presented in a pleasingly concise style which allows the discussion of a very large number of puzzles.

An ideal polyhedral dissection puzzle would have a relatively small number of pieces (all different, none symmetrical) which arrange in one way only into a highly symmetrical solid; the pieces would be variations on a theme, perhaps different arrangements of the same basic elements, would interlock and would assemble in one order only! Of course, there are many satisfying puzzles in which these conditions are relaxed, such as when the pieces are to be packed into a given box.

Reading this book, I found it frustrating that recourse to pencil and paper would rarely help me solve a puzzle and that I lacked the time or skill to make the puzzle. For those interested in woodworking, there is a chapter which deals with tools, materials and techniques.

Stewart Coffin tends to discount the mathematical content of his book. Certainly there are few proofs or enumerations of possibilities, but results are quoted, some unsolved problems are mentioned and the reader is advised to have a reference book to hand for looking up 'rhombic triacontahedron' or the like. Also, mathematical questions are likely to be raised in the reader's mind ... .

For instance, we are told that a rhombic dodecahedron, one of the few symmetric solids that pack to fill space, can be obtained from a cube by a process of bevelling all its edges at 45° to the faces until none of the cube's surfaces remains except the centre-points of its faces (see diagram). So what happens if we start



with, say, a regular tetrahedron and bevel its edges in the most symmetrical way until none of the tetrahedron's surface remains except the centre-points of its faces? We are told that the rhombic dodecahedron is the same as the cube in that it has three fourfold axes of symmetry, four threefold axes, six twofold axes, appears square in outline when viewed along a fourfold axis and appears hexagonal in outline when viewed along a threefold axis. Does this bevelling process preserve such properties? If you can see the silhouette of a randomly rotating convex solid, or see its silhouette from viewpoints of your choice, how can you tell the shape of the solid? Don't ask me.

This book deserves to find readers who will share Stewart Coffin's pleasure in the designing and making of polyhedral puzzles. Finally, I would challenge the readers of this review to investigate the shape of a face of a rhombic dodecahedron and to discover what this has to do with a regular tetrahedron and a sheet of A4 paper.

The Royal School, Woolverhampton JOHN MACNEILL

**Galois Theory** (second edition). By Ian Stewart. Chapman and Hall, London, 1989. Pp. xxx + 202. Hardback £24.00 (ISBN 0-412-34540-4), paperback £12.95 (ISBN 0-412-34550-1).

Ian Stewart has written a book so good that it poses a challenge to every lecturer in every university in this country. Galois theory was created to solve a concrete problem: which polynomial equations have solutions that can be expressed in terms of combinations of $n$th roots. For example, the real solution of $x^3+3x-2 = 0$ is $(1+\sqrt{2})^{1/3}+(1-\sqrt{2})^{1/3}$, which is a combination of cube roots and square roots. Galois's answer was to associate with a polynomial a group, its Galois group, whose internal structure indicated whether the polynomial's roots were expressible as such a combination of $n$th roots. I cannot begin to communicate the beauty of Galois theory in a review. Suffice it to say that an answer to the question 'Why am I learning all this abstract algebra?' might well be 'So that you can appreciate Galois theory'. So let us take it as read that Galois theory is worth studying. What is so marvellous about Stewart's book? First of all it is very clear. Each new idea is accompanied with a concrete example, and there is a little picture on page 97 which I have never thought of drawing, which makes a crucial and difficult idea utterly transparent. I am not exaggerating when I say that I never really understood what an automorphism was until I saw this picture—I just thought I did! Secondly, the material in the book is well motivated: a number of well-known problems are solved *en route* (as well as the central problem of Galois theory mentioned above). Proofs that the angle cannot be trisected nor the cube duplicated nor the circle squared by ruler and compasses are given; and the regular heptadecagon is constructed before the reader's eyes. More importantly the more technical bits are well motivated too. Constantly Stewart anticipates the reader's questions: 'Why do I want Theorem 10.1?', 'Isn't this irreducibility stuff a bit haphazard?', 'How do I calculate a Galois group?' Thirdly, the book is reasonably self-contained and contains precisely as much as the well-intentioned undergraduate wants to know about Galois theory.

Any lecturer giving a course on Galois theory should use Stewart's book. I would put this more strongly: the lecturer should deviate not one iota from Stewart's order and notation. Indeed the lecturer is excused from lecturing, and the students are excused from being mere stenographers. Instead, lectures can be devoted to the tricky bits, to the exercises, to working through some examples, to giving yet more motivating material. A lecturer who gives his own course in Galois theory without backing it up with published notes of the quality of Stewart is wasting his undergraduates' time *and they have better things to do*. Not least, they have to spend time getting to the bottom of courses that have no comparable textbooks.

Winchester College NICK MACKINNON

**Does God Play Dice?** By IAN STEWART. Penguin Books, 1990. Pp. 317. Paperback £6.99 (ISBN 0-14-012501-9).

Chaos theory is concerned with the discovery that systems which obey possibly very simple rules may exhibit wholly unpredictable behaviour. This is because two initial states which are distinct but indistinguishable by practical measurement can result in wildly different solutions after a surprisingly short time.

This discovery has led to a deeper understanding of many natural phenomena, for example the structure of the asteroid belt and the unpredictable nature of the weather.

Dr Stewart has written a fascinating book which illuminates this area. He introduces it with the use of simple examples accessible to people with quite elementary mathematical backgrounds and he describes the more complex mathematics extremely well. I certainly learned a great deal, even being able to say to myself at one point, 'that looks like a recipe for chaos', before the book told me it was.

Don't be put off by the unexplained terms used on page 70: they will become clear later. Don't be put off, either, by the topologists' contempt for straight lines; they are used for the axes of the graphs which illustrate and explain the principles involved.

For me one of the most important aspects of the book is the historical and philosophical perspective which the author introduces. This helps one to understand how attitudes change and how discoveries are made and eventually come to light. I would recommend this book to anyone with an interest in science or mathematics.

Oakham School G. N. THWAITES

**The Magic of Number.** By K. R. IMESON. 1989. Distributed by The Mathematical Association, 259 London Road, Leicester LE2 3RE. Pp. 96. Paperback. £7.00 (ISBN 0-9514716-0-0)

This book wasn't meant to be difficult. It isn't. Sadly, it's not all that interesting either. The foreword explains that the eleven chapters are an introduction to the beauty of numbers for 10 to 16 year olds, while the introduction suggests that GCSE teachers may find ideas for projects within. Fair enough. At times, however, these restrictions seem too much. A lot of the first half of the book spirals around the golden ratio, but I recall Isaac Asimov managing the same feat faster and better in *X Stands for Unknown*. The first few anecdotes are funny (especially the one about crows), but many of the rest are obscure and irrelevant. Nevertheless, the details of the golden ratio in music are interesting, as is the family history of George Bidder (of calculating fame). The bit about $1+\sqrt{2}$ was new; this number doesn't even seem to feature in *The Penguin Book of Curious and Interesting Numbers*. Occasionally the author exaggerates to make a point; for example, it wouldn't take me a lifetime to divide $13^{41}$ by 53 as stated on page 79.

Perhaps this book will inspire a few people. One of its good features is its attention to history, especially with regard to Greek mathematics. There is nothing about the Babylonians, though. When was the last time you counted in base 60?

Trinity College, Cambridge AMITES SARKAR

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