Mathematical Spectrum

Volume 12 1979/80 *Number 2*

A Magazine of *Contemporary Mathematics*

Published by the *Applied Probability Trust*

Mathematical Spectrum is a magazine for the instruction and entertainment of student mathematicians in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

Volume 12 of *Mathematical Spectrum* will consist of three issues, of which this is the second. The first issue was published in September 1979, and the third will appear in May 1980.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

Articles are normally commissioned by the Editors; the Editorial Committee also welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. All correspondence about the contents should be sent to:

The Editor, *Mathematical Spectrum*,
Hicks Building, The University, Sheffield S3 7RH.

Mathematical Spectrum Awards for Volume 11

In 1977 the editors of *Mathematical Spectrum* instituted two annual prizes for contributors who are still at school or are undergraduates in colleges or universities. One prize, to the value of £20, is for an article published in the magazine; another of £10 is for a letter or the solution of a problem. Volume 11 did not contain any articles by authors eligible for the £20 prize. However, the editors have decided to award a prize of £10 to Neil Norman for his letter *Prime numbers to base 7* (Volume 11, Number 3, pp. 95–96) and a further prize of £10 to Bruce Westbury for his solution to Problem 10.5 (Volume 11, Number 1, page 29). We look forward to further contributions from our readers.

The 21st International Mathematical Olympiad

COLIN GOLDSMITH
Marlborough College

Readers of *Mathematical Spectrum* will have seen reports of previous years' contests, which are for teams of up to eight pre-university students. New and old readers will appreciate the tough nature of the Olympiad if they settle down and tackle the questions, allowing about an hour for each.

A record number of countries (23) were represented, including for the first time Brazil, Israel and Luxembourg. The results of the leading nations were as shown in the table.

	Competitor number								Total	Prizes			
	1	2	3	4	5	6	7	8		1st	2nd	3rd	Special
U.S.S.R.	19	27	35	34	36	40	36	40	267	2	4	1	
Romania	35	23	17	33	32	28	39	33	240	1	4	2	
West Germany	21	14	35	33	39	30	34	29	235	1	5	1	
United Kingdom	23	23	34	30	32	34	21	21	216		4	4	
U.S.A.	17	9	27	17	39	24	32	34	199	1	2	2	
Vietnam	32	29	33	40						1	3		1
East Germany	33	13	31	20	19	19	26	19	180		2	2	
Czechoslovakia	16	20	24	17	23	40	12	26	178	1		4	
Hungary	15	20	16	18	34	22	17	34	176		2	2	
Yugoslavia	13	27	13	11	23	26	24	35	172		1	4	

Since this year's Olympiad was held for the first time in Britain, it is appropriate to describe some of the arrangements. The various countries had sent in 80 possible questions altogether. A short list of 26 was made by Dr T. Fletcher, Dr D. Monk and Mr R. Lyness, and these were discussed by the team leaders, with Dr Fletcher as chairman, three days before the competition. This process went exceptionally smoothly, and the final choice of six questions emerged without serious disagreement. Since all the really hard questions were voted against and question 3 was subsequently eased considerably when the detailed wording was considered, the papers were ultimately a little less testing than usual and seven students obtained full marks or only dropped one point.

The competitors were housed in Westfield College, London, and the domestic side, travel and entertainment were all administered by the School Mathematics Project under an organizing committee set up by the National Committee for Mathematical Contests. Special mention must go to Mr J. Hersee who was in charge of all these aspects. Group visits were made to Hampton Court, Greenwich (by boat), Windsor Castle and Stratford, and the interpreter-guides attached to each team took their students on further sightseeing and shopping trips to central London. A most lively concert was put on by the Royal Academy of Music, and an enjoyable reception was held by the Greater London Council on the terrace of County Hall, with a steel band playing. Two nights at the end were spent in Oxford.

All previous Olympiads have been government financed. This one was the responsibility of the School Mathematics Project. The Department of Education and Science made a substantial contribution, and generous amounts were given by the Royal Society, the British Council and various firms and schools. Dr B. Thwaites chaired the organizing committee and was primarily responsible for finance.

The Olympiad was officially opened by Sir James Hamilton, Permanent Secretary of the DES. The prizes were presented by the Duchess of Gloucester, and a message from the Duke of Edinburgh was read at the formal dinner at which the guests of honour were representatives of the sponsors.

We can congratulate ourselves on the relaxed and efficient staging of the 21st IMO, which was patently enjoyed by the teams and their leaders. The British team were admirable hosts, and it was gratifying that they were the only team to secure eight prizes.

The six questions were:

1. Let p and q be natural numbers such that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that p is divisible by 1979.

2. A prism with pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ as top and bottom faces is given. Each side of the two pentagons and each of the line-segments A_iB_j, for

all $i,j = 1,\ldots,5$, is coloured either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been coloured has two sides of a different colour. Show that all 10 sides of the top and bottom faces are the same colour.

3. Two circles in a plane intersect. Let A be one of the points of intersection. Starting simultaneously from A two points move with constant speeds, each point travelling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that, at any time, the distances from P to the moving points are equal.

4. Given a plane π, a point P in this plane and a point Q not in π, find all points R in π such that the ratio $(QP + PR)/QR$ is a maximum.

5. Find all real numbers a for which there exist non-negative real numbers x_1, x_2, x_3, x_4, x_5 satisfying the relations

$$\sum_{k=1}^{5} kx_k = a, \qquad \sum_{k=1}^{5} k^3 x_k = a^2, \qquad \sum_{k=1}^{5} k^5 x_k = a^3.$$

6. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that

$$a_{2n-1} = 0, \qquad a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1}), \qquad n = 1, 2, 3, \ldots;$$

where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$.

Note: A path of n jumps is a sequence of vertices $(P_0, \ldots, P_n)$ such that

(i) $P_0 = A$, $P_n = E$;

(ii) for every i, $0 \le i \le n-1$, P_i is distinct from E;

(iii) for every i, $0 \le i \le n-1$, P_i and P_{i+1} are adjacent.

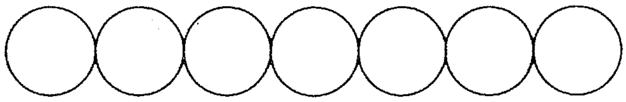
How to Play Games with Trees

ALAN SLOMSON
University of Leeds

Nearly all games are of interest to mathematicians; the nature of the mathematical interest varying according to the type of game. This article deals only with certain games of skill. I can best describe the sort of game I have in mind by giving you an example.

Kayles†

This game is played by two people who start with a row of touching coins:



The rules are very simple.

(1) The players move alternately.

(2) On each move a player can remove either any one coin or any two *touching* coins.

(3) The player who removes the last coin wins. (Equivalently, if there is no coin left when it is your turn to move, you have lost.)

With just these three rules the game is not interesting, because the first player has an easy way to force a win. With an odd number of coins in the starting position he removes the middle coin, and with an even number he removes the middle two coins. In either case the position is now symmetrical with two rows of coins of equal length. All the first player now has to do is to keep the position symmetrical by copying in one half of the position the move the second player has just made in the other half. In this way the position is always symmetrical when it is the second player's turn to move. Thus so long as there is a coin for him to take, there will be another coin for the first player to take. Hence the first player must take the last coin and so win.

The game can be made interesting by adding a rule to prevent this happening. The way we choose to do this is by the additional rule:

(4) On the first move with an odd number of coins the central coin may not be removed by itself, and with an even number of coins the central two coins may not be removed.

† The word 'kayles' originally meant a set of skittles, and later gave its name to the skittle game played with them. The game described here is an intellectual version of skittles, and was suggested by the great puzzle expert H. E. Dudeney in his book *The Canterbury Puzzles and Other Curious Problems*. Removing a coin corresponds to knocking down a skittle. Our rule (2) is based on the assumption that it is only possible to knock down a single skittle, or two adjacent skittles, with each throw.

With this rule added the game becomes quite interesting and I suggest that before reading further you try playing it, beginning with 12, 13 or 14 coins in the starting position.

The special features to notice about Kayles, which it shares with all the games to be considered here, are as follows:

(a) It is a game between two players.

(b) No physical dexterity is required; it is a purely intellectual game (unlike, for example, cricket or darts or snooker).

(c) It is entirely a game of intellectual skill; there is no element of chance (unlike, for example, Snakes and Ladders or Backgammon).

(d) It is a game of 'perfect information', that is, both players know throughout the game exactly what the position is. So there is no element of bluffing involved. (Unlike most card games, for example, Bridge or Poker).

(e) At each stage the players have only a finite number of possible moves to choose from.

(f) Each game must end after a finite number of moves.

This last requirement is obviously necessary for a game played by mortals. Clearly it holds in Kayles since eventually all the coins must be removed. The official Laws of Chess stipulate only that a player may claim a draw if a position is about to be repeated for the third time, not that the game is automatically drawn in this situation. So in theory, if neither player claimed the draw, a game of Chess could go on indefinitely. However, by adopting the more sensible rule that a thrice repeated position automatically makes the game drawn, we can make Chess fit the above pattern. (Since there is only a finite number of different possible positions in Chess, eventually one or other of them must be repeated three times.)

Only mathematicians would ever seriously consider 'infinite games', that is, games which go on for an infinite number of moves. Although such games are not much use for recreational amusement, they do turn out to be mathematically interesting and have applications, for example, in set theory.

The games I am going to deal with here are those which satisfy all the conditions (a)–(f) listed above. This includes well-known games like Noughts-and-Crosses (Tic-Tac-Toe), Draughts (Checkers), Chess and Go, as well as many others not so well known. They can all be analysed mathematically in the same way. I shall describe the method used by explaining how it works in the case of Kayles, and to keep things simple I shall use the game of Kayles with only four coins in the starting position. (Of course with so few coins the game is not at all interesting, but before we analyse it mathematically, you may like to decide on the basis of your intuition alone whether you would rather be the first or the second player in this game.)

We represent the game by means of the diagram shown in Figure 1. The initial position is at the bottom, and the positions connected to it are those that can be reached in just one move, and so on up the diagram until we reach positions where there are no coins left and the game is over. Boxes with nothing in them correspond to these *terminal positions* where the game has ended. The roman numerals are there

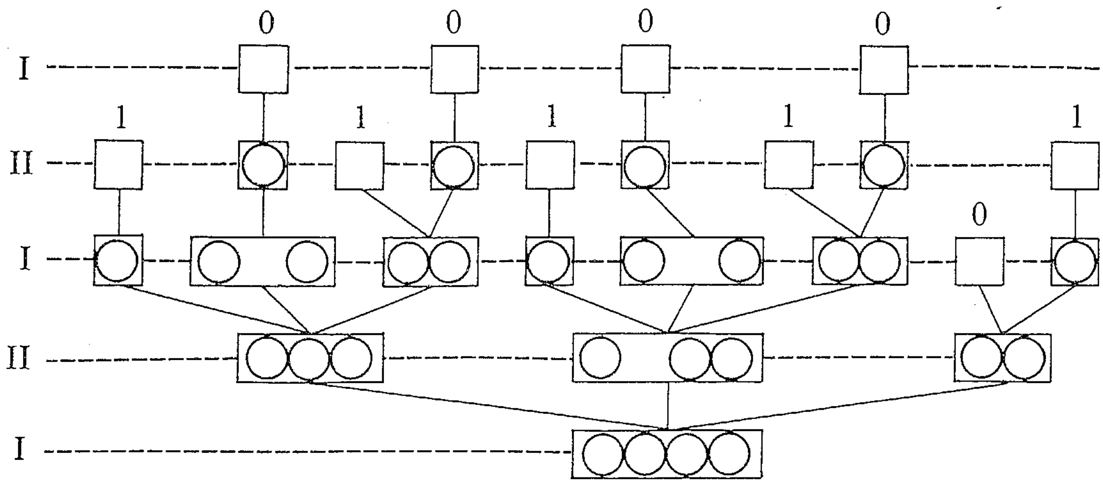


Figure 1

to indicate at each level in the diagram whose turn it is to move. Thus the diagram shows that there are three possible moves in the initial position: (i) the removal of one end coin leaving a row of three coins, (ii) the removal of one non-end coin leaving a row of two coins together with a single coin, and (iii) the removal of two end coins leaving a row of two coins (remember that the removal of the two central coins is prohibited by rule (4)).

The terminal positions are labelled according to the outcome of the game from the point of view of the first player. He scores 0 if he has lost the game, which happens when it is his turn to move and there are no coins left, and 1 if he has won the game which happens when it is the second player's turn to move and there are no coins left.

Once we have labelled the terminal positions with the results the actual positions in the diagram become irrelevant and we can throw them away. In this manner we obtain Figure 2. The structure in this diagram is called a *tree* because it does look rather like one. The points in the tree corresponding to the positions in the game are called *nodes*. Tree structures occur in a number of important mathematical applications apart from games, for example in probability theory and in the analysis of languages (either natural languages or the formal languages of logic and computer programming).

The tree gives us all the information we need to know about the game, and every

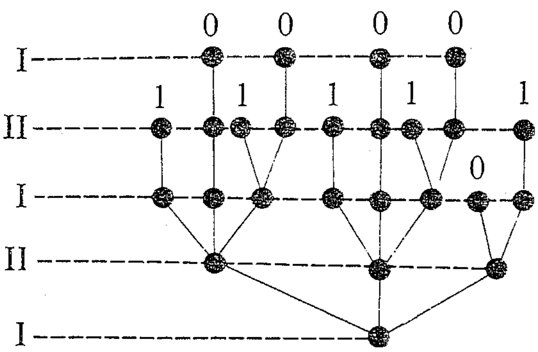


Figure 2

game of the type we are considering may be represented by a tree. From this point of view the only differences between the games lie in the structure of the trees and the ways in which the end positions are labelled. In Kayles each game must be won by one or other of the players, and so the terminal positions are all labelled either 0 or 1. In games where draws are possible some of the final positions would be labelled $\frac{1}{2}$. There is no reason why we should not allow other scores at the end of the game. For example, it has sometimes been argued that to stalemate your opponent in Chess is rather more creditable than obtaining a draw in other ways and so should be rewarded with rather more than half a point, say with three-quarters of a point; the player being stalemated getting the other quarter point. In general we let the scores awarded in the terminal positions to the first player be any numbers between 0 and 1. It is convenient, though not essential, to adopt the convention that if the first player scores x then the second player scores $1 - x$.

How do we play a game when we are given its tree? The game is played by moving up the tree from the starting position, the players taking it in turn to decide which path to follow from each node. The aim of the first player is to reach a terminal position with as high a score, x, as possible. Since the second player's score is $1 - x$, his aim is to reach a position with as low a score as possible. In this way we see that all the games we are considering are really just different tree-climbing games and instead of learning separately the rules of all these games all we need to do is learn how to climb trees. So Noughts-and-Crosses is just a simple version of Chess! (Pause for a moment to think how practical this is; we shall return to this point later.)

Thus I now present to you a new game, as yet untitled, which is completely described by the tree in Figure 3. The nodes have been labelled with letters so that I can refer to them. You will notice that in this game draws are possible as some of the terminal nodes are associated with the score $\frac{1}{2}$.

Once we are given the tree we can easily work out the best moves for each player in each position and thus evaluate each position from the point of view of the first player. Consider for example node K in Figure 3. In this position it is the second player's turn to move, and he has the choice of moving to either the terminal position T or to the terminal position U. Since he is aiming at a final position with as low a score as possible he will choose to move to T when he wins the game and the first player scores 0. Thus the value of position K from the first player's point of view is 0,

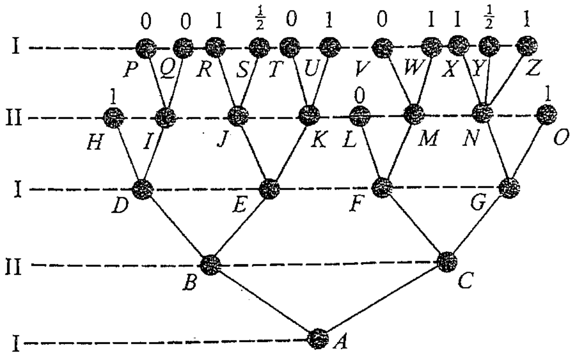


Figure 3

since he is bound to end up with this score from position K provided that the second player moves intelligently. In a similar way the value of node J to the first player is $\frac{1}{2}$ since from this position a sensible second player will choose to move to S when the game ends in a draw. In a like manner we can assign values, from the point of view of the first player, to all nodes in the tree which have only terminal nodes immediately above them.

Now consider node E. Here it is the first player's turn to move. He can choose to move either to J, which we have just seen has value $\frac{1}{2}$ to him, or to K which has value 0 to him. Since he is trying to get as high a score as possible he will choose to move to J. Then with best play the game will end in a draw. Since the first player will choose to move to J from E, the value of node E to him is the same as the value of J, namely $\frac{1}{2}$.

Carrying on in this way we can assign values to all the nodes in the tree. The value of a node tells us the result of the game, from the point of view of the first player, assuming that both players make the best possible moves from that position. The formula for calculating the value of a node will be apparent from the discussion above. When it is the first player's turn to move the value of the node is the *maximum* value of the nodes he can move to and when it is the second player's turn to move, the value is the *minimum* of all the nodes he can move to. The values given to the nodes using this formula are shown in Figure 4.

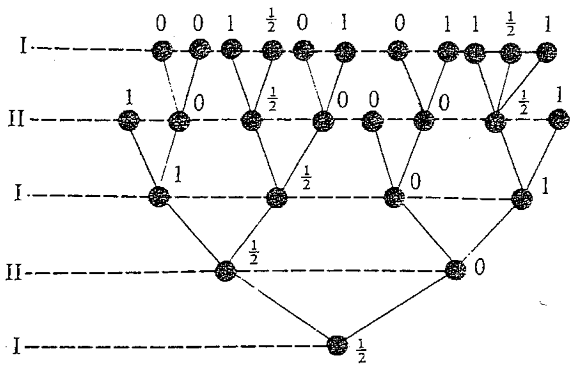


Figure 4

In this case the starting position, node A, gets the value $\frac{1}{2}$. This tells us that with best play on both sides the game will be drawn. It is easy to see that if the players move intelligently they will move through positions B, E and J and will finish at S. In this particular game any alternative choice of move at any stage would be a bad blunder as it would result in a loss for the player concerned. In general, however, there may be more than one 'best move' for a player in certain positions.

We can now easily work out the outcome of the game of Kayles with four coins in the starting position, assuming best play on both sides. All we have to do is to assign values to the nodes on the tree in Figure 2 using the method described above. I shall leave it to you to do this. When you have completed the calculation you will know whether it is better to be the first or the second player in this game.

In Kayles, since draws are not possible, each terminal position gets either the value 0 or 1, and hence every node in the tree gets one or other of these values. In particular, the initial position either gets the value 0, which means that with best play on both sides the first player is bound to lose, or it gets the value 1, which means that with best play he will win. In this latter case we say that there is a *forced win* for the first player since he can move in such a way that, whatever the second player does, the game ends in a win for the first player. The case where the initial position gets the value 0 corresponds to there being a forced win for the second player. We have in fact proved a theorem about games which satisfy our conditions (a)–(f):

If each game must end in a win for one or other player, then there must either be a forced win for the first player or a forced win for the second player.

In Kayles whether there is a forced win for the first or for the second player depends on the number of coins in the starting position. For example, with 21 coins the first player has a forced win, but with 23 coins the second player has a forced win. In each particular case we can work this out from the tree of the game, just as we did above with the game corresponding to the tree of Figure 3. The tree also tells us what the winning strategy is. If there is a forced win for the first player then the initial node of the tree gets the value 1. The winning strategy for the first player is simply always to move to a node with value 1. The fact that the initial position has the value 1 means that this is always possible. Following this strategy the game will end at a position with value 1, that is, a position in which the first player has won the game. Similarly, when there is a forced win for the second player his winning strategy is always to move to nodes which have the value 0.

(Of course, if you are suddenly challenged to play a game of Kayles, and your opponent won't wait until you have drawn the tree of the game, this method of finding the winning strategy is of no use. In Kayles there is an alternative method of finding the winning strategy which involves being able to remember a few numbers and being able to do some simple calculations in your head, but the details of this method will have to be deferred to a later article.)

In a game, such as Chess, where draws are possible, there is a third possibility, namely that with best play on both sides the game will be drawn. In the case of Chess it is not known which of these possibilities is actually the case, although most Chess players would bet on the game's being a draw with best play. But, you will surely ask, if the method of analysing games using trees is as easy as is made out above, why does no-one yet know the outcome of a perfect game of Chess?

The big snag is that while in principle the tree method for analysing games is fine, in practice the trees involved are usually too large for the calculation to be carried out. Even for a fairly simple game like Kayles I had to restrict myself to a starting position with only four coins to get a tree that would fit easily on a page. If you try to construct the tree for Kayles with 12 coins in the starting position you will see how large it is.

For 'small' games like Kayles and Noughts-and-Crosses, although the trees tend to be too large for hand calculations, they are well within the range of a digital computer. There are 5890 different positions that can arise in a game of Noughts-and-Crosses (if we take symmetries into account, there are only 826 essentially different positions) and for a computer this is a very small number. The number of different positions does not by itself tell us how large the tree of the game is, since, as is evident from Figure 1, the same position may occur at different nodes of the tree. A better measure of the size of the tree is given by the number of terminal nodes. Each terminal node corresponds to a different way of playing the game. So counting the number of terminal nodes is equivalent to counting the number of different ways of playing the game.

A game of Noughts-and-Crosses is determined by the order in which the players choose the nine squares in which to place their noughts and crosses. Hence the number of different games is at most the number of different orders in which these nine squares may be chosen, that is, at most 9! The actual number will be rather less than this because not all games last the full nine moves. $9! = 362880$, and while this number is too large for a hand calculation it is child's play to a modern digital computer.

The number of different games of Chess is vastly greater. A very accurate estimate is not easy to come by, but we can get a rough estimate as follows. The length of an average tournament game is about 40 moves, that is 40 white moves and 40 black moves, making 80 moves in all. In the initial position white has 20 opening moves to choose from and black has 20 possible replies to each of these. The number of possible moves tends to increase in the early part of the game as the pieces are developed and so become more mobile, and then to decrease as the pieces are exchanged off. A reasonable estimate is that there are on average 30 possible moves in each position. Thus a tournament game consists, on average, of 80 choices of moves, each choice being between, on average, 30 different possible moves.

Thus, on this estimate, the number of different games of Chess that can be played is

$$30^{80} = 1{\cdot}5 \times 10^{118}.$$

If anything, this is a gross underestimate, since tournament games rarely end in checkmate or a forced draw. Instead players resign in hopeless positions long before they are checkmated, and agree draws in level positions. Thus the true length of an average game of chess, played out to the end, is almost certainly very much larger than 40 moves. However the estimate above is large enough to make my point, that a calculation involving the complete tree of Chess is quite beyond the range of any conceivable computer, let alone existing machines.

Let us imagine that we had a computer whose working parts were at the sub-atomic level, and which performed calculations by transmitting messages across a distance corresponding to the diameter of the nucleus of a hydrogen atom. According to the theory of relativity the maximum velocity at which messages can be transmitted is the speed of light which is just less than 3×10^{10} centimetres per second. The diameter of the hydrogen nucleus is rather more that 10^{-13} centimetres. Thus our imaginary computer would be able to do at most

$$\frac{3 \times 10^{10}}{10^{-13}} = 3 \times 10^{23}$$

calculations per second. That is extraordinarily fast, but even if we suppose that each of the $1{\cdot}5 \times 10^{118}$ games of Chess involves just one calculation, it would still take the computer

$$\frac{1{\cdot}5 \times 10^{118}}{3 \times 10^{23}} = 5 \times 10^{94}$$

seconds to complete the calculation. There are somewhat less than 4×10^{7} seconds in one year. So our superfast computer would need more than 10^{87} years for its calculation! The current estimated age of the solar system is about 10^{10} years, and it is estimated that the Sun will go on producing enough heat to support life on Earth for at most another 10^{10} years. Thus we can safely say that a complete analysis of Chess is inconceivable in practice.

Despite the impracticability of a complete analysis of Chess using the tree method, this method of analysing positions by evaluating nodes on trees is the basis of almost all computer Chess programmes. Because it is not possible to analyse to the bitter end, the final positions on the tree which the computer uses have to be those in which the game has not yet finished. They therefore cannot be simply scored 0, $\frac{1}{2}$ or 1 according to the result of the game. Instead it is necessary to have a special evaluation function which assigns values to positions in which the game has not finished without analysing further moves. This can be done by using the number of pieces belonging to the two sides, their mobility and so on. The strength of different computer Chess programmes depends partly on the sophistication of its evaluation function, partly on the way the programme decides whether to analyse a position further, or merely to evaluate it, and partly on various programming tricks which enable it to value the nodes on the tree efficiently. The details may be found in David Levy's book, mentioned below.

For some games like Kayles, and the matchstick game Nim, a complete analysis is possible without using trees, but instead a shorter method which uses so-called Grundy functions. I hope to return to this point in a later article.

For further reading

When mathematicians talk about the 'theory of games' they generally mean games without perfect information, that is bluffing games. Games of this type were ruled out by our requirement (d), and thus most mathematics books which have 'games' included in their titles are not about the sort of games discussed here. A delightful exception is John Conway's *On Numbers and Games* published by Academic Press in 1976. You will probably find a good deal of this book difficult reading, but you should find much of interest in the section on Games, which does not require a knowledge of the section on Numbers.

David Levy's *Chess and Computers*, published by B. T. Batsford in 1976, explains how the tree method is used in computer Chess programmes and has much interesting information about Chess-playing by computer.

Is the Fosbury Flop a Mechanical Success?

E. A. TROWBRIDGE
University of Sheffield

Introduction

The aim in high-jumping is to clear a bar set at a predetermined height without dislodging it from its supports. In order to do this the athlete must leave the take-off area with a thrust which will carry his body over the bar. Two objectives have to be achieved. First, the athlete's mass centre must be given an airborne trajectory which reaches its highest point at the bar; and, secondly, his limbs and body must be moved in such a way that they do not knock the bar off. One might think that if the athlete's body passes over the bar, then necessarily his mass centre must also pass over the bar. This is certainly true in most high-jump techniques like the 'scissors', the 'western roll' and the 'straddle', although in the 'straddle' the athlete does drape himself around the bar to keep his mass centre as close to the bar as possible. In the 1960s a jumper called Dick Fosbury pioneered a new technique in which the athlete essentially travels backwards over the bar with his back in an arc. If a picture is taken at the instant of clearance, the athlete is face upwards and his head and arms hang on one side of the bar while his legs are on the other side. Simple mechanics can be used to show that, although the jumper clears the bar (hopefully), his mass centre passes under the bar. We shall see that the higher the mass centre is raised the bigger the thrust that must be given at take-off. These facts suggest that a jumper using the 'Fosbury flop' technique should be able to clear a bar at a given height with less thrust than a jumper using the 'straddle' or 'scissors' techniques. As a corollary, a 'Fosbury flopper' should theoretically be able to jump higher for the same thrust.

Mathematical description of the athlete

It must be recognised that an athlete is a very complicated biomechanical system. However, it may be possible to obtain some useful information about the mechanical aspects of high-jumping by considering a simple model of the athlete which retains enough reality to provide worthwhile predictions. In this article the athlete will be considered as a system of nine hinged rods representing arms, legs and trunk, and a disc which represents the head. Note that the athlete is not a rigid body, since the rods can move relative to one another. Consideration of data presented in Dyson (reference 1) suggests that the dimensions and masses of the disc–rods configuration can be represented as in Figure 1. The mass of the athlete is m; his height is l.

The athlete is to be thought of as a system of n particles. If $\boldsymbol{r}_i$ is the position vector of the ith particle and $\bar{\boldsymbol{r}}$ is the position vector of the mass centre of the n particles, we define

$$\bar{\boldsymbol{r}} = \sum_{i=1}^{n} m_i \boldsymbol{r}_i \Big/ \sum_{i=1}^{n} m_i, \qquad \sum_{i=1}^{n} m_i = m. \tag{1}$$

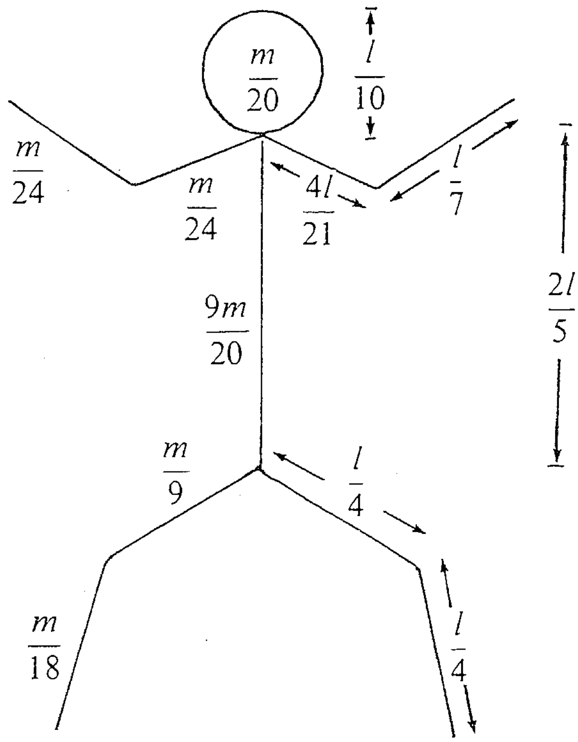


Figure 1. Masses and lengths of rods and disc which constitute the athlete.

It can be shown (see reference 2) that, as far as the rate of change with respect to time of the linear momentum of the system is concerned, the whole system can be replaced by a single particle of mass equivalent to the total mass of the system situated at the mass centre. The resultant force acting on the system is then equal to the product of the total mass and the acceleration of the mass centre. Mathematically, this is written as

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}, \tag{2}$$

where $\boldsymbol{F}$ is the resultant external force.

Integrating equation (2) with respect to time gives

$$m[\dot{\boldsymbol{r}}(t + \delta t) - \dot{\boldsymbol{r}}(t)] = \int_t^{t+\delta t} \boldsymbol{F}(\tau)\, d\tau = \boldsymbol{I}, \tag{3}$$

where $\boldsymbol{I}$ is an impulse.

Consequently, in order to investigate the thrust needed to raise the mass centre of the athlete to a given height and the resulting trajectory through the air, we can replace the athlete by a single particle of mass m.

The motion of the athlete's mass centre

Equation (3) can be used to obtain the impulsive force needed to produce a jump of a given height. Suppose a particle, mass m, which is moving with horizontal speed U is struck a blow I in the vertical direction. If the horizontal and vertical components of the velocity after the impulse are u and v respectively, then the impulse equations are

horizontally: $$m(u - U) = 0, \tag{4}$$

vertically: $$mv - 0 = I. \tag{5}$$

The components u, v are the initial horizontal and vertical speeds in the subsequent two-dimensional projectile problem. If air resistance is neglected the mass centre travels in a parabolic trajectory. Referred to a rectangular cartesian coordinate system Oxy at the point of take-off, the differential equations governing the motion are

horizontally: $$m\ddot{x} = 0,$$

vertically: $$m\ddot{y} = -mg.$$

It is to be assumed that the mass centre is a height h above the ground at the instant of take-off. So, at $t = 0$, $x = 0$ and $y = h$. Integration with respect to time, and application of the prescribed initial conditions, give

$$x = ut, \qquad y = -\tfrac{1}{2}gt^2 + vt + h.$$

The maximum height is attained when $\dot{y} = 0$, so that

$$t = v/g,$$

and

$$y_{\max} = \tfrac{1}{2}v^2g + h. \tag{6}$$

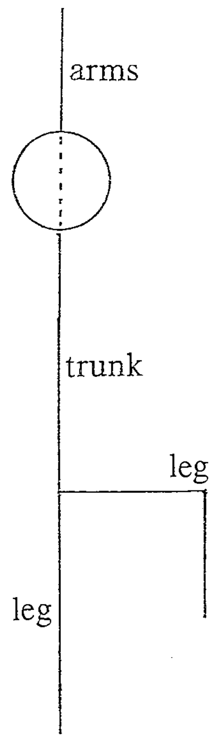


Figure 2. Position of arms and legs at take-off.

Comparison of different high-jump techniques

Equation (6) shows that y_{max} depends linearly on height h. As a consequence, if the high-jumper raises both arms and his leading leg just before thrusting with his take-off leg he will gain some advantage. Also, tall high-jumpers will have a physical advantage over small high-jumpers.

Taking the idealised positions of the athlete's limbs at take-off as represented in Figure 2, we can calculate the y coordinate of the mass centre. For the data given in Figure 1,

$$m\bar{y} = \underbrace{\frac{m}{20}\left[\frac{l}{2}+\frac{2l}{5}+\frac{l}{20}\right]}_{\text{head}} + \underbrace{\frac{9m}{20}\left[\frac{l}{5}+\frac{l}{2}\right]}_{\text{trunk}} + \underbrace{\frac{2m}{24}\left[\frac{l}{2}+\frac{2l}{5}+\frac{4l}{21}+\frac{l}{14}\right]}_{\text{upper arms}}$$

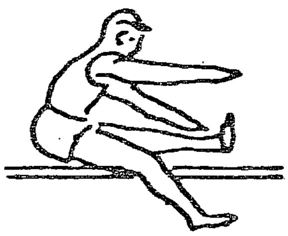
$$+ \underbrace{\frac{2m}{24}\left[\frac{l}{2}+\frac{2l}{5}+\frac{2l}{21}\right]}_{\text{lower arms}} + \underbrace{\frac{m}{9}\left[\frac{l}{2}\right]}_{\text{upper bent leg}} + \underbrace{\frac{m}{18}\left[\frac{l}{2}-\frac{l}{8}\right]}_{\text{lower bent leg}}$$

$$+ \underbrace{\frac{m}{9}\left[\frac{l}{4}+\frac{l}{8}\right]}_{\text{upper straight leg}} \qquad + \underbrace{\frac{m}{18}\left[\frac{l}{8}\right]}_{\text{lower straight leg}},$$

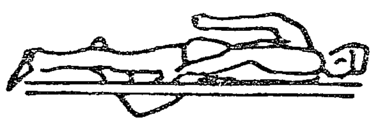
which gives, at the instant of take-off,

$$\bar{y} = 0{\cdot}667l.$$

So for a six-foot-tall athlete, $\bar{y} = 4{\cdot}002$ feet, and for a five-foot-tall athlete $\bar{y} = 3{\cdot}335$ feet. This gives a difference of 8·0 inches in the maximum height obtainable for the same thrust.



scissors



straddle

Figure 3. The 'scissors' and 'straddle' high jump.

Equation (6) also shows that y_{max} depends quadratically on v and so, from equation (5), on the thrust I. Increasing the thrust increases the height to which the mass centre can be raised. It is clear that, if the jumper adopts the very early style of high-jumping called the 'scissors' (see Figure 3), his mass centre will be much higher when he passes over the bar (so requiring a greater thrust) than if he drapes himself around the bar in a horizontal line as in the 'straddle'. The matchstick idealisation

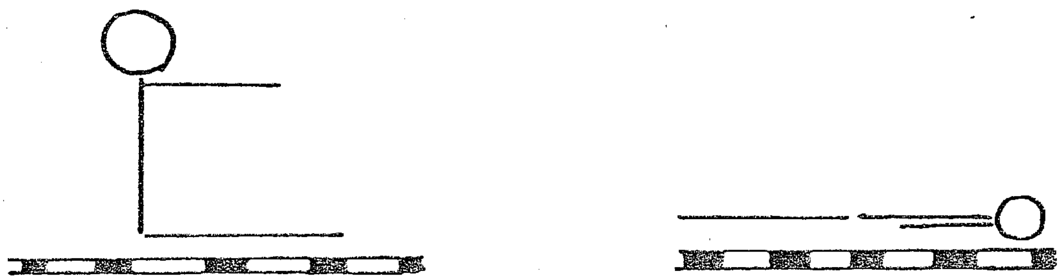


Figure 4. Matchstick idealisation of the 'scissors' and 'straddle'.

(see Figure 4) adopted here indicates that, for a six-foot-tall athlete using the 'scissors' technique, the mass centre is 1·075 feet above the bar at the instant of clearance, while in the 'straddle' it is just above the bar. Suppose the bar is set at a height of 6 feet. Then equations (5) and (6) show that the ratio of the thrusts required to raise the mass centres the necessary height to clear the bar is

$$I_{\text{straddle}}/I_{\text{scissors}} = 0{\cdot}79.$$

This shows that only 79 % of the thrust needed to clear the bar with the 'scissors' technique is required to clear with the 'straddle', provided the arms and legs are manipulated correctly.

In the 1968 Olympic Games, Dick Fosbury perfected a technique which developed from the 'modified scissors', in that he crossed the bar at approximately right angles (see Figure 5). We shall see that this technique is even more efficient mechanically than the 'straddle', since the highest point reached by the athlete's mass centre can be below the bar even though the athlete passes over the bar. Figure 6 is an idealisation of the high-jumper drawn at the instant that the mass centre reaches its highest point.

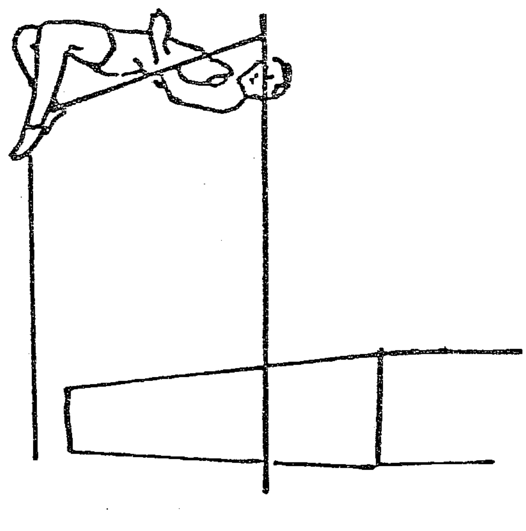


Figure 5. The 'Fosbury flop' high jump.

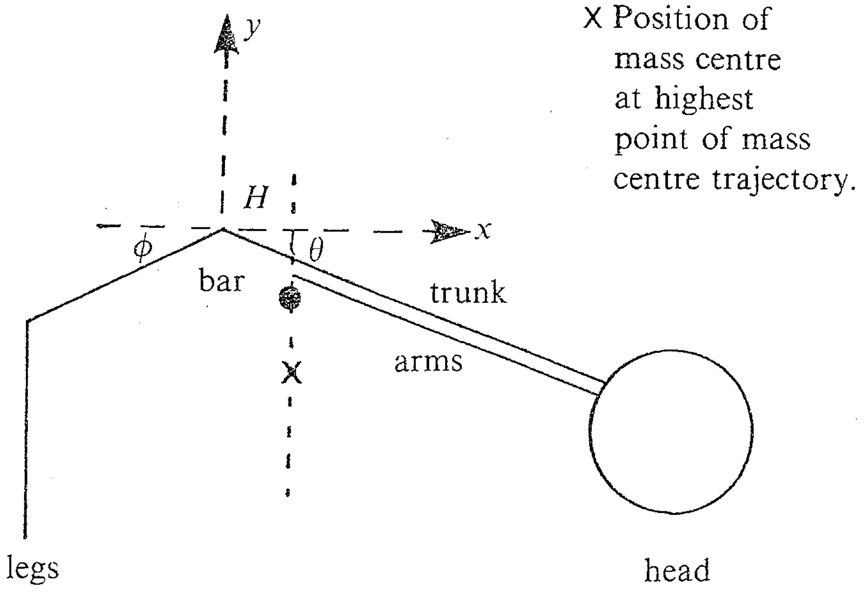


Figure 6. Matchstick idealisation of the 'Fosbury flop'.

The position of the mass centre relative to the hips H which are above the bar is

$$m(\bar{x},\bar{y}) = \overset{\text{head}}{\frac{m}{20}\left[\frac{9l}{20}\cos\theta, \frac{-9l}{20}\sin\theta\right]} + \overset{\text{lower arms}}{\frac{m}{12}\left[\frac{29l}{210}\cos\theta, \frac{-29l}{210}\sin\theta\right]}$$

$$+ \overset{\text{upper arms}}{\frac{m}{12}\left[\frac{32l}{105}\cos\theta, \frac{-32l}{105}\sin\theta\right]} + \overset{\text{trunk}}{\frac{9m}{20}\left[\frac{l}{5}\cos\theta, \frac{-l}{5}\sin\theta\right]}$$

$$+ \overset{\text{upper legs}}{\frac{2m}{9}\left[\frac{-l}{8}\cos\phi, \frac{-l}{8}\sin\phi\right]} + \overset{\text{lower legs}}{\frac{m}{9}\left[\frac{-l}{4}\cos\phi, \frac{-l}{4}\sin\phi - \frac{l}{8}\right]}.$$

So

$$(\bar{x},\bar{y}) = \left[\left\{l\cos\theta\,.\,\frac{251}{1680} - \frac{l\cos\phi}{18}\right\}, \left\{-l\sin\theta\,.\,\frac{251}{1680} - \frac{l\sin\phi}{18} - \frac{l}{72}\right\}\right].$$

As an example, let us consider $\theta = \phi = 20°$ and $l = 6$ feet; then, relative to H,

$$(\bar{x},\bar{y}) = (6{\cdot}349, -6{\cdot}047) \quad \text{inches.} \tag{7}$$

Equation (7) shows that the mass centre is approximately 6·35 inches to the right of the hips and 6 inches below the hips at the highest point of the mass centre trajectory. We wish this high point to be on a vertical line drawn through the bar, and we require the trunk just to clear the bar at this instant. This will happen with the above data when the hip joint is 2·31 inches above the bar, with the hips 6·35 inches behind the bar. The y component of equation (7) shows the mass centre will be

3·74 inches below the bar at this instant. Consequently, the mass centre need only be raised to approximately 5 feet $8\frac{1}{4}$ inches if the bar is set at 6 feet, and less thrust is required than in the 'straddle' or 'scissors' technique. In fact, for a six-foot-tall athlete jumping a bar set at 6 feet,

$$\frac{I_{\text{flop}}}{I_{\text{scissors}}} = 0{\cdot}72, \qquad \frac{I_{\text{flop}}}{I_{\text{straddle}}} = 0{\cdot}91.$$

This shows that only 72 % of the thrust used in the 'scissors' is needed to jump the bar using the 'Fosbury flop' and 91 % of the thrust used in the 'straddle' is needed for success using the 'Fosbury flop'.

As the bar is raised higher and higher, the thrust ratio approaches unity. The interested reader might like to calculate $I_{\text{flop}}/I_{\text{straddle}}$ when the bar is set at 7 feet. At these bar heights, a single inch may be the difference between success and failure, and so any possible mechanical advantages, even though they may be small, are important.

Obviously there are substantially more facets to high-jumping than just raising the mass centre to a certain height. However, athletics coaches agree that this feature is the dominant one; no matter how the arms and legs are manipulated, if the mass centre is not high enough at the bar, then the result will be failure. The preceding discussion gives some insight into the mechanical reasons for adopting a particular style of high-jumping, and it shows why the 'Fosbury flop' has been so successful in the past decade.

References

1. G. Dyson, *The Mechanics of Athletics* (University of London Press, London, 1973).
2. B. H. Chirgwin and C. Plumpton, *A Course of Mathematics for Engineers and Scientists*, Volume 3 (Pergamon Press, Oxford, 1961).

Numerical Integration

H. V. SMITH
Leeds Polytechnic

The whole subject of classical mathematical analysis is founded on integration and differentiation. For example, the Taylor coefficients are usually expressed as derivatives. Without doubt differentiation and integration are two of the earliest tools aquired by any would-be scientist and are indeed of fundamental importance.

The subject of numerical analysis, on the other hand, concerns itself with analysing methods for calculating the numerical values of quantities defined in classical analysis.

In this article we shall discuss some of the more well-known formulae of numerical integration or, as it is more commonly known, numerical quadrature. It is unfortunate that much will have to be omitted and much will have to be condensed, but the story will still be seen to be one of fascination culminating in the evaluation of the error term in some of the most important quadrature formulae.

The classical theory of numerical quadrature is by now well documented in many textbooks (see for example reference 2) where it is seen that families of linear formulae of the following structure are displayed:

$$\int_{-1}^{1} w(x)f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i) + E_n(f).$$

The factor $w(x)$ is referred to as a weight-function; it is a function which occurs so often as part of an integrand that it is more convenient to write it as a separate quantity. The numbers w_i are said to be the weights of the quadrature rule, the x_i's the abscissae; and $E_n(f)$ is referred to as the truncation error of the formula.

An example of such a formula would be the trapezoidal rule (reference 2)

$$\int_{-1}^{1} f(x)\,dx = h(\tfrac{1}{2}f_0 + f_1 + \cdots + f_{n-1} + \tfrac{1}{2}f_n) - nh^3 f''(r)/12,$$

where $w(x) = 1$, $h = 2/n$, $f_0 = f(-1)$, $f_k = f(-1 + hk)$ and $f_n = f(1)$, and r is some point in $(-1, 1)$. This rule, which is destined to play a central role in our story, is an example of a class of formulae known as Newton–Cotes formulae and has the advantage that the weights are nearly equal to each other.

The trapezoidal rule is not of high accuracy, the magnitude of the error term being (reference 2)

$$|E_n| = nh^3|f''(r)|/12.$$

And so we see the truncation error E_n will tend to zero like $1/n^2$ as $n \to \infty$ if $f''(r)$ is continuous and hence bounded in $(-1, 1)$. However, it is rather unfortunate that in practice the accuracy of this rule cannot be improved indefinitely by increasing the number of intervals, a point being reached beyond which roundoff error becomes more significant than truncation error.

Another example of a Newton–Cotes formula would be the so-called parabolic rule or Simpson's rule (reference 2)

$$\int_{-1}^{1} f(x)\,dx = \tfrac{1}{3}h(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots$$
$$\cdots + 4f_{n-3} + 2f_{n-2} + 4f_{n-1} + f_n) - nh^5 f^{(iv)}(r)/180,$$

where again $w(x) = 1$, $f_0 = f(-1)$, $f_k = f(-1 + hk)$ and $f_n = f(1)$. When n is sufficiently large, this formula is usually more accurate than the trapezoidal rule because if $f^{(iv)}(x)$ is continuous in $(-1, 1)$ the truncation error tends to zero like $1/n^4$ as $n \to \infty$.

In the past Simpson's rule was probably the most widely used of all formulae in numerical quadrature since the weights are simple and do not fluctuate unduly. However, it must be stressed that it is only possible to use this formula when the interval $(-1, 1)$ is divided into an even number of intervals of length h.

As well as developing Newton–Cotes type formulae the early researchers also developed what have become known as Gaussian formulae (reference 2). These formulae differ from those of Newton–Cotes by sacrificing equal spacing of the abscissae in the interests of higher accuracy.

One of the main drawbacks against the use of Gaussian quadrature was the fact that the weights and the abscissae, which are the zeros of certain orthogonal polynomials (see below), are not as amenable to manual calculation as those of Newton–Cotes formulae.

From reference 2 we see that a set of polynomials $q_0(x), q_1(x), \ldots, q_n(x) \ldots$ is said to be orthogonal with respect to a non-negative weight-function $w(x)$ over an interval (a, b) if

$$\int_a^b w(x) q_i(x) q_j(x)\, dx = 0 \qquad \text{when } i \neq j.$$

Thus an example of Gaussian quadrature, namely Gauss–Legendre quadrature (reference 2), would be

$$\int_{-1}^{1} f(x)\, dx = \sum_{i=1}^{n} w_i f(x_i) + E_n(f)$$

where

(i) $w(x) = 1$,

(ii) the abscissae x_i are the roots of the Legendre polynomial $P_n(x)$ (reference 2),

(iii) $$w_i = -\frac{2}{(n+1)P_{n+1}(x_i)P_n'(x_i)},$$

(iv) $$E_n(f) = \frac{2^{2n+1}(n!)^4 f^{(2n)}(r)}{(2n+1)((2n)!)^3}, \qquad \text{where } r \in (-1, 1).$$

Hence when $n = 2, 3$ or 4 we obtain the following table (reference 2).

TABLE 1

n	Abscissae x_i	Weights w_i
2	$\pm 0{\cdot}577350$	1
3	0	8/9
	$\pm 0{\cdot}774597$	5/9
4	$\pm 0{\cdot}339981$	0·652145
	$\pm 0{\cdot}861136$	0·347855

Similarly, Gauss–Hermite quadrature (reference 2) may be summarised in the following manner:

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i) + E_n(f),$$

where

(i) $w(x) = \exp(-x^2)$,

(ii) the abscissae x_i are the zeros of the Hermite polynomial $H_n(x)$ (reference 2),

(iii) $$w_i = -\frac{2^{n+1} n! \sqrt{\pi}}{H_n'(x_i) H_{n+1}(x_i)},$$

(iv) $$E_n(f) = \frac{n! \sqrt{\pi} f^{(2n)}(r)}{2^n (2n)!}, \qquad \text{where } r \in (0, \infty).$$

In this case we obtain Table 2.

TABLE 2

n	Abscissae x_i	Weights w_i
2	$\pm 0{\cdot}707107$	0·886227
3	0	1·181636
	$\pm 1{\cdot}224745$	0·295409
4	$\pm 0{\cdot}524648$	0·804914
	$\pm 1{\cdot}650680$	0·081313

From Tables 1 and 2 it is now obvious that quadrature rules such as these would not be employed to any great extent by a person performing many calculations manually. With the advent of the computer, however, things changed dramatically.

It is interesting to note that Gaussian quadrature will integrate numerically an infinite number of non-polynomial functions for which the truncation error involved is zero. Indeed it is stated in all textbooks that numerical quadrature formulae are exact for polynomials of a high enough degree. This is stressed so often that one could be forgiven for believing them to be exact only for polynomials, which is not the case as the following shows.

Let us consider some Gaussian quadrature formulae of the type

$$I = \int_{-a}^{a} w(x) g(x)\,dx \simeq \sum_{i=1}^{n} w_i g(x_i),$$

where

(i) $w(x)$ is an even weight-function (for some examples see Table 3),

(ii) $x_i = -x_{n-i}$, $w_i = w_{n-i}$,

(iii) $a = 1$ or ∞.

TABLE 3

Weight-function $w(x)$	Interval $[-a, a]$	Value of integral I
$(1-x^2)^{-1/2}$	$[-1, 1]$	$\frac{1}{2}\pi$
$(1-x^2)^{1/2}$	$[-1, 1]$	$\frac{1}{4}\pi$
$\exp(-x^2)$	$(-\infty, \infty)$	$\frac{1}{2}\sqrt{\pi}$

In order to illustrate our idea we restrict our attention to Gauss–Legendre quadrature.

Theorem. Suppose that $w(x) = 1$ and $g(x) = f(x)/(f(x) + f(-x))$, where $f(x)$ is not odd. Then $I = 1$ and this value is obtained without truncation error by any Gauss–Legendre quadrature formula.

Proof. Making the substitution $y = -x$ in the integral

$$I = \int_{-1}^{1} \frac{f(x)}{f(x) + f(-x)}\, dx$$

we obtain

$$I = \int_{-1}^{1} \frac{f(-y)}{f(y) + f(-y)}\, dy.$$

Hence

$$2I = \int_{-1}^{1} 1\, dx = 2,$$

i.e.

$$I = 1.$$

But the x_i, namely the roots of $P_n(x)$, are symmetrically placed about the origin and the same weights correspond to x_i and to $-x_i$. (See Table 1.) Moreover $\sum_{i=1}^{n} w_i = 2$ for Gauss–Legendre quadrature. We therefore have

$$\sum_{i=1}^{n} w_i g(x_i) = \tfrac{1}{2} \sum_{i=1}^{n} w_i (g(x_i) + g(-x_i)) = \tfrac{1}{2} \sum_{i=1}^{n} w_i = 1 = I,$$

exactly.

We now list some important weight-functions for which a corresponding version of the above theorem may be proved.

Numerical examples

1. Consider the example

$$J = \int_{0}^{(1/2)\pi} \frac{\sin x}{\sin(x + \frac{1}{4}\pi)}\, dx$$

$$= \frac{\pi}{2\sqrt{2}} \int_{-1}^{1} \frac{\sin[\frac{1}{4}\pi(1 + x)]}{\sin[\frac{1}{4}\pi(1 + x)] + \sin[\frac{1}{4}\pi(1 - x)]}\, dx.$$

The integrand is of the type covered by Theorem 1. Thus, an application of any Gauss–Legendre quadrature formula will lead to $J = \pi/2\sqrt{2}$.

2.

$$J = \int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+e^{-2x}} dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2} \frac{e^x}{e^x + e^{-x}} dx.$$

Once again the integrand is of the type covered by our theorem, and so applying any Gauss–Hermite quadrature formula leads to $J = \frac{1}{2}\sqrt{\pi}$.

We stress that, in each example, if the calculations were done by using tables, there might be a rounding error.

Results such as these and many others have naturally had the effect of focussing one's attention on Gaussian quadrature and away from the trapezoidal rule in particular, and Newton–Cotes quadrature in general. However, over the past fifteen years or so there have been some notable successes for this class of formulae. For example, in 1975 those very results which have just been deduced for Gaussian quadrature had already been anticipated by Churchhouse (reference 1) for Newton–Cotes formulae. And in 1971 Hunter (reference 3) considered the numerical evaluation of integrals by the trapezoidal rule in which the integrand was assumed periodic with period equal to the range of integration.

TABLE 4

Number of intervals in trapezoidal rule	Value of error term
4	$-0{\cdot}0147687$
8	$-0{\cdot}0355361$
16	$-0{\cdot}0338575$
32	$-0{\cdot}0338528$

At that time this paper probably received little attention, but it does allow us to evaluate the truncation error. Unfortunately we do not have more space in which to elaborate on this method (an account of which may be obtained from the author), but to convey the spirit of the method we quote the results obtained when it is applied to the integral

$$\int_{-1}^{1} \sin(x^2)\, dx = 0{\cdot}6205366.$$

From Table 1, with $n = 2$, the approximate value of the integral is $0{\cdot}6543894$, the error thus being $-0{\cdot}0338528$. Applying this new method to the evaluation of the error term leads to the results quoted in Table 4.

Acknowledgement

I am indebted to Dr D. B. Hunter for his assistance in the preparation of this article.

References

1. R. F. Churchhouse, Perfect numerical integration by Simpson's rule, *Math. Gaz.* **59** (1975), 159–162.
2. F. Hildebrand, *Introduction to Numerical Analysis* (McGraw-Hill, New York, 1956).
3. D. B. Hunter, The evaluation of integrals of periodic functions, *BIT* **11** (1971), 175–180.
4. H. V. Smith, The evaluation of the error term in some numerical quadrature formulae, Technical Report Number 1, School of Mathematics and Computing, The Polytechnic, Leeds, September 1978.

Hats Off

HARRIS S. SHULTZ
California State University, Fullerton

1. The classical hat-check problem

A large number of elementary problems in the calculus of probabilities depend on combinatorics. One such well-known combinatorial problem has been formulated in the following practical form:

> If n men have their hats randomly returned, what is the probability that none of the men winds up with his own hat?

Alternative forms of this problem found in the literature are:

> If each of n people at a table orders a different sandwich and if the orders are distributed at random, what is the probability that no one receives the correct sandwich?

or

> If n letters are delivered at random to n mail boxes, what is the probability that no one receives the correct letter?

We shall consider two different solutions to this problem, and then, in the third section, state and solve a variation of it.

By a *permutation* of the set $S_n = \{1, 2, \ldots, n\}$ we mean a one-to-one function whose domain and range is this set. For example, $(1\,3\,2)$ is the permutation of S_3 under which $1 \to 3, 3 \to 2$, and $2 \to 1$. Note that $(1\,3\,2)$ could also represent a permutation of S_5; here it is understood that $4 \to 4$ and $5 \to 5$. Another permutation of S_5 is $(1\,3\,2)(4\,5)$, under which $1 \to 3$, $3 \to 2$, $2 \to 1$, $4 \to 5$, and $5 \to 4$. For each n there are a total of $n!$ permutations of S_n.

Our hat-check problem or its alternative forms may be stated, mathematically, as follows:

> For a random permutation σ of S_n, what is the probability that $\sigma(i) \neq i$ for all $i \in S_n$?

The answer is, of course, $N/n!$, where N is the number of permutations of S_n such that $\sigma(i) \neq i$ for all $i \in S_n$. We shall use the following notation: $n(A)$ denotes the cardinality of the set A, AB denotes the intersection of the sets A and B, and A' denotes the complement of A. Consider the case $n = 4$. If we let U be the set of all permutations of S_4 then $n(U) = 4!$. If we define A_i to be the set of all σ in U such that $\sigma(i) = i$, then the probability we seek is

$$P_4 = \frac{n(A'_1A'_2A'_3A'_4)}{4!}.$$

The formula

$$n(A'_1A'_2A'_3A'_4) = n(U) - [n(A_1) + \cdots + n(A_4)] + [n(A_1A_2) + \cdots + n(A_3A_4)] \\ - [n(A_1A_2A_3) + \cdots + n(A_2A_3A_4)] + n(A_1A_2A_3A_4) \qquad (1)$$

is a generalization of $n(A'B') = n(U) - [n(A) + n(B)] + n(AB)$ and is called the 'inclusion–exclusion' principle (see reference 1). Since $n(A_i) = 3!$, $n(A_iA_j) = 2!$, $n(A_iA_jA_k) = 1$ and $n(A_iA_jA_kA_l) = 1$ if i, j, k, l are distinct, it follows from (1) that

$$n(A'_1A'_2A'_3A'_4) = 4! - \binom{4}{1}3! + \binom{4}{2}2! - \binom{4}{3} + 1 = 4!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right].$$

Therefore, the probability is

$$P_4 = 1 - 1 + 1/2! - 1/3! + 1/4!.$$

For general n, the analogous probability is

$$P_n = 1 - 1 + 1/2! - 1/3! + \cdots (-1)^n/n!.$$

Recalling that

$$e^{-1} = 1 - 1 + 1/2! - 1/3! + \cdots$$

we see that the probability P_n is approximately $1/e = 0{\cdot}368$. It is rather surprising that the probability is almost independent of the number of men whose hats are being permuted.

2. Derangements

Let us now approach the same problem somewhat differently. A permutation σ of S_n for which $\sigma(i) \neq i$ for each i in S_n is called a *derangement*. For $n = 5$ there are two types of derangements. Examples (one of each permutation type) are (12345) and (123)(45). The former is called a *5-cycle* while the latter is the product of a 3-cycle and a 2-cycle which are disjoint. There are 4! permutations of S_5 which are 5-cycles while there are $\binom{5}{3}2!$ permutations of S_5 which are products of a 3-cycle and a 2-cycle (disjoint). Thus, for $n = 5$, the probability of a derangement is

$$\frac{1}{5!}\left[4! + \frac{5!}{3!2!}2!\right] = \frac{1}{5} + \frac{1}{(3)(2)}. \tag{2}$$

For $n = 7$ there are four types of derangements. Examples (one of each permutation type) and the number of each such type are:

Type	Number
$(1\,2\,3\,4\,5\,6\,7)$	$6!$
$(1\,2\,3\,4\,5)(6\,7)$	$\binom{7}{5}4!$
$(1\,2\,3\,4)(5\,6\,7)$	$\binom{7}{4}3!2!$
$(1\,2\,3)(4\,5)(6\,7)$	$\dfrac{\frac{7!}{3!2!2!}}{2!}2!$

Thus, for $n = 7$, the probability of a derangement is

$$\frac{1}{7!}\left[6! + \frac{7!}{(5)(2)} + \frac{7!}{(4)(3)} + \frac{7!}{(3)(2)^2 2!}\right] = \frac{1}{7} + \frac{1}{(5)(2)} + \frac{1}{(4)(3)} + \frac{1}{(3)(2)^2 2!}. \tag{3}$$

A pattern is now emerging and which can be clearly identified. By a *partition* of a natural number n we mean a representation of n as a sum, without regard to order, of natural numbers. In (2) we notice the partitions 5 and $3 + 2$ of 5 while in (3) we notice the partitions 7, $5 + 2$, $4 + 3$ and $3 + 2 + 2$ of 7. Generalizing these results, one can deduce that the probability that a randomly selected permutation of S_n is a derangement is

$$\sum \frac{1}{b_1^{k_1} b_2^{k_2} \cdots b_r^{k_r} k_1! k_2! \cdots k_r!} \tag{4}$$

summed over all partitions $k_1 b_1 + k_2 b_2 + \cdots + k_r b_r$ of n for which the distinct summands $b_i \geq 2$. It is, of course, rather surprising that (4) is approximately $1/e$.

3. A related problem

We now pose the following question:

> If n men have their hats randomly returned, what is the probability that exactly two men wind up with each other's hat while every other man winds up with neither his own hat nor that of the man on whose head his own hat rests?

For $n = 2$ the probability is 1/2 while for $n = 3$ and $n = 4$ it is 0. Let us restate the problem as follows:

> For a random permutation σ of S_n find the probability that there exist distinct integers j and k such that $\sigma(j) = k$ and $\sigma(k) = j$ and, if $i \neq j$ and $i \neq k$, then $\sigma(\sigma(i)) \neq i$.

Our problem is to determine the number of permutations of S_n which leaves no number fixed and which can be expressed as a product of disjoint cycles precisely one of which is a 2-cycle. We shall call such a permutation *admissible*. For $n = 8$ there are two types of admissible permutations. Examples (one of each permutation type) and the number of each such type are:

Type	Number
(1 2 3 4 5 6)(7 8)	$\binom{8}{2}5!$
(1 2 3)(4 5 6)(7 8)	$\dfrac{\dfrac{8!}{3!3!2!}}{2!}2!2!$

Dividing the sum of the second column by 8! we obtain a probability of an admissible permutation as

$$\frac{1}{2}\left[\frac{1}{6} + \frac{1}{(3)^2 2!}\right].$$

For $n = 11$ we can distinguish four types of admissible permutations. Examples (one of each permutation type) and the number of each such type are:

Type	Number
(1 2 3 4 5 6 7 8 9)(10 11)	$\binom{11}{2}8!$
(1 2 3 4 5 6)(7 8 9)(10 11)	$\dfrac{11!}{6!3!2!}5!2!$
(1 2 3 4 5)(6 7 8 9)(10 11)	$\dfrac{11!}{5!4!2!}4!3!$
(1 2 3)(4 5 6)(7 8 9)(10 11)	$\dfrac{\dfrac{11!}{3!3!3!2!}}{3!}2!2!2!$

Dividing the sum of the second column by 11! we obtain a probability of an admissible permutation for $n = 11$ as

$$\frac{1}{2}\left[\frac{1}{9} + \frac{1}{(5)(4)} + \frac{1}{(6)(3)} + \frac{1}{(3)^3 3!}\right].$$

With relatively little difficulty one could now prove that, for $n > 2$, the probability that exactly two men wind up with each other's hat while every other man winds up with neither his own hat nor that of the man on whose head his own hat rests is given by

$$\frac{1}{2}\sum\frac{1}{b_1^{k_1}b_2^{k_2}\cdots b_r^{k_r}k_1!\,k_2!\cdots k_r!}$$

where we sum over all partitions $k_1b_1 + k_2b_2 + \cdots + k_rb_r$ of $n - 2$ in which the distinct summands $b_i \geq 3$. Some specific probabilities of this event for increasing n are given below:

n	Probability	n	Probability
3	0·00000	8	0·11111
4	0·00000	9	0·11310
5	0·16667	10	0·11146
6	0·12500	11	0·11142
7	0·10000	12	0·11159

At this point the reader should be able to verify the following: If we sum over all partitions $k_1b_1 + k_2b_2 + \cdots + k_rb_r$ of n in which $b_1, b_2, \ldots, b_r$ are the distinct summands, then

$$\sum\frac{1}{b_1^{k_1}b_2^{k_2}\cdots b_r^{k_r}k_1!\,k_2!\cdots k_r!} = 1.$$

Readers interested in similar combinatorial problems of probability may refer to the two books listed below.

References

1. G. Berman and K. D. Fryer, *Introduction to Combinatorics* (Academic Press, London, 1972).
2. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1, 3rd edn. (Wiley, New York, 1968).

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

Problems

12.4. (Submitted by G. N. Copley, Templecombe, Somerset.) Find the number of digits required to print simultaneously all the integers from zero to a million inclusive in (a) decimal form (base ten), (b) binary form (base two), (c) duodecimal form (base twelve).

12.5. (Submitted by J. Ramsden, Birmingham.) (i) In a sequence of real numbers, the sum of every N consecutive terms is negative, whereas the sum of every M consecutive terms is positive. Show that the sequence must have fewer than $M + N - D$ terms, where D is the highest common factor of M and N. (ii) In a sequence of positive real numbers, the product of every N consecutive terms is less than one, whereas the product of every M consecutive terms is greater than one. Show that, again, the sequence must have fewer than $M + N - D$ terms.

12.6. There are six events, and in each year from AD 1 to 1979 exactly one of these events occurs. Show that there exist years x, y, z, with $x = y + z$, in which the same event occurs.

Solutions to Problems in Volume 11, Number 3

11.7. Show that $e^{kx} + k(1 - e^x) \geq 1$ for every real number x and every integer k.

Solution

When $k = 0$ or 1, the result is clear, so assume $k \neq 0, 1$, and put

$$y = e^{kx} + k(1 - e^x).$$

Then

$$\frac{dy}{dx} = ke^{kx} - ke^x,$$

which is zero if and only if $x = 0$. Also,

$$\begin{aligned}\frac{d^2y}{dx^2} &= k^2e^{kx} - ke^x \\ &= k(k-1) \qquad \text{when } x = 0,\end{aligned}$$

which is positive for all integers $k \neq 0, 1$. Thus the function y has a minimum at $x = 0$. Also $y(0) = 1$. Hence $y \geq 1$ for all x.

11.8. Arrange the digits 0 to 9 to form five two-digit numbers in such a way that the product of these five numbers is maximal.

Solution by Gary Slater (University of Sherbrooke, Canada.)

Suppose that we have found the solution, and that it is

$$(10a_1 + a_2)(10a_3 + a_4)(10a_5 + a_6)(10a_7 + a_8)(10a_9 + a_{10}),$$

where $a_1 > a_3 > a_5 > a_7 > a_9$. Then $a_9 > a_{10}$, for otherwise we could interchange a_9 and a_{10} and so increase the product. We also claim that $a_4 > a_2$. To see this, we must have

$$\begin{aligned}(10a_1 + a_2)(10a_3 + a_4) &\geq (10a_1 + a_4)(10a_3 + a_2)\\ &\Rightarrow (a_1 - a_3)(a_4 - a_2) \geq 0\\ &\Rightarrow a_4 \geq a_2 \qquad \text{because } a_1 > a_3.\end{aligned}$$

Hence $a_4 > a_2$. Similarly, $a_6 > a_4$, $a_8 > a_6$, $a_{10} > a_8$.

Thus

$$a_1 > a_3 > a_5 > a_7 > a_9 > a_{10} > a_8 > a_6 > a_4 > a_2,$$

from which it follows that the five numbers must be 90, 81, 72, 63, 54.

11.9. Let $Z_1Z_2Z_3Z_4$ be a convex quadrilateral in the plane, denote by W_1, W_2, W_3, W_4 the midpoints of the squares, drawn externally to the quadrilateral, with sides Z_1Z_2, Z_2Z_3, Z_3Z_4, Z_4Z_1 respectively, and let U_1, U_2, U_3, U_4 be the midpoints of the squares with sides W_1W_2, W_2W_3, W_3W_4, W_4W_1 respectively. Show that (a) W_1W_3, W_2W_4 are equal and perpendicular, (b) U_1Z_2 and U_3Z_4 are perpendicular to Z_1Z_3, and

$$U_1Z_2 = U_3Z_4 = \tfrac{1}{2}Z_1Z_3.$$

Solution

We consider the configuration in the Argand diagram, denoting the point Z_1 by the complex number z_1 etc. Then

$$\begin{aligned}
w_1 &= \tfrac{1}{2}(z_1 + z_2) + \tfrac{1}{2}i(z_1 - z_2), & u_1 &= \tfrac{1}{2}(w_1 + w_2) + \tfrac{1}{2}i(w_1 - w_2),\\
w_2 &= \tfrac{1}{2}(z_2 + z_3) + \tfrac{1}{2}i(z_2 - z_3), & u_2 &= \tfrac{1}{2}(w_2 + w_3) + \tfrac{1}{2}i(w_2 - w_3),\\
w_3 &= \tfrac{1}{2}(z_3 + z_4) + \tfrac{1}{2}i(z_3 - z_4), & u_3 &= \tfrac{1}{2}(w_3 + w_4) + \tfrac{1}{2}i(w_3 - w_4),\\
w_4 &= \tfrac{1}{2}(z_4 + z_1) + \tfrac{1}{2}i(z_4 - z_1), & u_4 &= \tfrac{1}{2}(w_4 + w_1) + \tfrac{1}{2}i(w_4 - w_1),
\end{aligned}$$

from which it follows that

$$w_4 - w_2 = i(w_3 - w_1),$$

which gives (a). It also follows that

$$z_2 - u_1 = u_3 - z_4 = \tfrac{1}{2}i(z_3 - z_1),$$

which proves (b).

Book Reviews

Introductory Mathematical Analysis. By I. J. MADDOX. Adam Hilger Ltd, Bristol, 1977. Pp. xii + 327. £7·50 (hard cover); £5·00 (soft cover).

Where do you start a course in analysis? If, like me, you begin with the real numbers, this book is not for you. After reading fourteen preliminary pages on symbolic logic, sets, relations and functions, the reader is given the definitions of group, ring, field, linear space and algebra, all in the first 25 pages. Professor Maddox then begins his analysis course by indicating how the Peano axioms are used to deduce properties of the natural numbers. From these the rationals, reals and complex numbers are constructed in the usual way, and many of their analytical and topological properties are deduced.

All the above is accomplished in 93 pages. This seems to be very difficult work for the beginning analyst. Having surmounted this formidable obstacle, the reader will find that the remainder of the book is a fairly fast-moving standard first course in analysis.

Chapters on the convergence of sequences and series (40 pages) are followed by one on the continuity of complex-valued functions defined on subsets of the complex numbers (26 pages). The next chapter (37 pages) considers the differentiability of such functions, and after a discussion of maxima and minima of differentiable real-valued functions of a real variable, several mean value theorems and applications are obtained. Complex power series are then covered (14 pages) and these are used to define the elementary functions (exp, sin, cos, etc.) (29 pages). The concluding chapters deal with Newton and Riemann integrals (47 pages) and the continuity and differentiability of functions of two real variables (15 pages).

The book moves at rather a fast pace, and I think a reader would find it hard going if this is his first encounter with analysis. However, it obtains most of the standard results of a first analysis course and has lots of good exercises with answers and hints to more than half of them.

University of Durham J. BOLTON

Principles of Statistical Techniques. By P. G. MOORE, Cambridge University Press, London, 1979. Pp. viii + 228. £4·50 (first paperback edition).

The first edition of this book was published over twenty years ago and the fact that it has survived so long and a paperback edition has just been printed speaks for itself.

It is suggested that it is suitable for schools and universities and indeed this comment is correct. The author introduces most of the topics by discussing a practical problem, after which the statistical concept under consideration is explained and illustrated by means of several examples. Students meeting these concepts for the first time should find this approach very acceptable, particularly if they are not too concerned with the underlying mathematics. This approach makes the book very easy to read without at any time giving the impression of being a book of recipes—an impression often gained when reading introductory texts which try to avoid detailed use of mathematics.

Most of the material required at this level is included—tabulation, pictorial representation, frequency distributions, averages, dispersion, elementary probability, binomial, Poisson and normal distributions, significance tests, sampling, time series and correlation and regression—and in addition a short but useful chapter is included on simulation. Each topic is covered thoroughly without too much emphasis on proofs and the book contains many worked examples and exercises. It contains sufficient to satisfy those students following introductory courses and should whet the appetites of those wishing to study statistics to a greater depth, perhaps at some later date.

Teesside Polytechnic G. E. SKIPWORTH

Aha! Insight. By MARTIN GARDNER. W. H. Freeman and Company Limited, Reading, 1978. Pp. viii + 179. £4·00 (soft cover).

Sudden flashes of inspiration or creative leaps of the mind are called 'aha! reactions'. Martin Gardner in his usual lively style poses for the reader, through a series of problems, situations where an 'aha!' (or perhaps 'Eureka!') insight might occur. The problems are classified into sections: Combinatorial, Geometry, Number, Logic, Procedural and Word, each illustrated with amusing cartoon-like pictures. Many of the problems turn out to be old favourites in attractive disguises, others are completely new.

A fascinating book to put into a school or college library. It will provide challenges for all who dip into it, from the first-former to the sixth-former. At the same time it will give the teacher scope for many useful ideas, and indeed some 'aha!' situations for himself. If mathematics was always taught in the Martin Gardner style, it would be hard to imagine its ever being dull.

University of Durham — M. L. CORNELIUS

Notes on Contributors

Colin Goldsmith is a graduate of Cambridge. After three years as an Instructor Officer in the Royal Navy he went to teach at Marlborough College, where he has remained ever since. He became head of the Mathematics Department, and is now a housemaster. Part author and editor of many SMP books, he is also vice-chairman of the National Committee for Mathematical Contests. He was leader of the British Team at the 1979 International Mathematical Olympiad.

Alan Slomson, a graduate of Oxford, is a Lecturer in Pure Mathematics at the University of Leeds and a Tutor for the Open University. He is one of the three compilers of *Chez Angélique: The Bumper Book of Late Night Problems*, which arose from the activities of a mathematical nightclub at Open University summer schools. His research work is in the field of mathematical logic. Outside mathematics his main interests are chess, walking and politics.

E. A. Trowbridge is a Lecturer in the Applied Mathematics and Computing Science Department at the University of Sheffield. Having taught in many areas of secondary and tertiary education, including school, technical college, polytechnic and university, he feels that more emphasis should be placed on the development of mathematical models in teaching. Although not a high jumper himself, he considers the practical aspects of his mechanical theories while running marathons and long-distance fell races.

Harry V. Smith contributed also to Volume 11 of *Mathematical Spectrum*. He took his first degree at Sir John Cass College, London, and obtained an M.Sc. by research into complex analysis from the University of Kent at Canterbury. After a period of school teaching he became a lecturer at a college of education. He is now a Senior Lecturer in the School of Mathematics and Computing at Leeds Polytechnic. Since his postgraduate student days his research interest has shifted to numerical analysis. For relaxation he indulges in judo and rambling.

Harris Shultz received his B.A. at Cornell University, his M.S. and Ph.D. at Purdue University. He is a Professor of Mathematics at California State University, Fullerton, and his research interests lie in operational calculus and generalized functions.

Contents

ISSN 0025–5653

PRICES (*postage included*)

Prices for Volume 12 (*Issues Nos.* 1, 2, *and* 3):

Subscribers in Britain and Europe: £1·60
Subscribers overseas: £3·20 (US\$6.40; \$A5.50)
(These prices apply even if the order is placed by an agent in Britain.)
A discount of 10% is allowed on all orders for five or more copies.

Back issues:

All back issues are available; information concerning prices may be obtained from the Editor.

Enquiries about rates, subscriptions and advertisements should be directed to:

Editor—*Mathematical Spectrum*,
Hicks Building,
The University,
Sheffield S3 7RH, England.

Printed in England by Galliard (Printers) Ltd, Great Yarmouth