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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

RENSEIGNEMENTS GÉNÉRAUX

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire. Bien que principalement de nature éducative, elle sert aussi à ceux qui la lisent pour des raisons professionnelles, culturelles ou récréative.

Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

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THE OLYMPIAD CORNER

No. 159

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

Last issue we gave some of the problems selected for consideration by the jury at the 34th I.M.O. at Istanbul, Turkey, but which were not used. This number we give the remaining problems. My thanks go to Georg Gunther, the Canadian Team leader in 1993, for collecting these problems and forwarding them to me. Send me your nice solutions!

PROBLEMS PROPOSED AT ISTANBUL

11. *Proposed by Spain.* Given the triangle ABC, let D, E be points on the side BC such that $\angle BAD = \angle CAE$. If M and N are, respectively, the points of tangency with BC of the incircles of the triangles ABD and ACE, show that

$$\frac{1}{MB} + \frac{1}{MD} = \frac{1}{NC} + \frac{1}{NE}\,.$$

12. *Proposed by India.* Define a sequence $\langle f(n)\rangle_{n=1}^{\infty}$ of positive integers by: $f(1) = 1$, and

$$f(n) = \begin{cases} f(n-1) - n, & \text{if } f(n-1) > n; \\ f(n-1) + n, & \text{if } f(n-1) \le n, \end{cases}$$

for $n \geq 2$. Let $S = \{n \in \mathbb{N} \mid f(n) = 1993\}$.

(i) Prove that S is an infinite set.

(ii) Find the least positive integer in S.

(iii) If all the elements of S are written in ascending order as $n_1 < n_2 < n_3 < \dots$, show that

$$\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = 3.$$

[Here $\mathbb{N}$ denotes the set of natural numbers.]

13. *Proposed by India.* A natural number n is said to have the property P if, whenever n divides $a^n - 1$ for some integer a, n^2 also necessarily divides $a^n - 1$.

(a) Show every prime number n has property P.

(b) Show there are infinitely many composite numbers n that possess property P.

14. *Proposed by Ireland.* Let S be the set of all pairs (m, n) of relatively prime positive integers m, n with n even and $m < n$. For $s = (m, n) \in S$ write $n = 2^k n_0$ where k, n_0 are positive integers with n_0 odd and define $f(s) = (n_0, m + n - n_0)$. Prove that f is a function from S to S and that for each $s = (m, n) \in S$, there exists a positive integer

$t \le (m+n+1)/4$ such that $f^t(s) = s$, where $f^t(s) = (\underbrace{f \circ f \circ \cdots \circ f}_{t \text{ times}})(s)$. If $m+n$ is a prime number which does not divide $2^k - 1$ for $k = 1, 2, \ldots, m+n-2$, prove that the smallest value of t which satisfies the above conditions is $[(m+n+1)/4]$ where $[x]$ denotes the greatest integer $\le x$.

15. *Proposed by Romania.* Let $c_1, \ldots, c_n \in \mathbb{R}$ $(n \ge 2)$ such that $0 \le \sum_{i=1}^n c_i \le n$. Show that we can find integers $k_1, \ldots, k_n$ such that $\sum_{i=1}^n k_i = 0$, $1 - n \le c_i + nk_i \le n$ for every $i = 1, \ldots, n$.

16. *Proposed by the U.K.* A circle S is said to cut a circle Σ *diametrally* if and only if their common chord is a diameter of Σ. Let S_A, S_B, S_C be three circles with distinct centres A, B, C respectively. Prove that A, B, C are collinear if and only if there is no unique circle S which cuts each of S_A, S_B, S_C diametrally. Prove further that if there exists more than one circle S which cuts each of S_A, S_B, S_C diametrally, then all such circles S pass through two fixed points. Locate these points in relation to the circles S_A, S_B, S_C.

17. *Proposed by the U.K.* A finite set of (distinct) positive integers is called a "DS-set" if each of the integers divides the sum of them all. Prove that every finite set of positive integers is a subset of some DS-set.

18. *Proposed by the U.S.A.* Prove that

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \ge \frac{2}{3}$$

for all positive real numbers a, b, c, d.

19. *Proposed by Vietnam.* Solve the following system of equations, in which a is a given number satisfying $|a| > 1$:

$$\begin{cases} x_1^2 = ax_2 + 1 \\ x_2^2 = ax_3 + 1 \\ \quad\vdots \\ x_{999}^2 = ax_{1000} + 1 \\ x_{1000}^2 = ax_1 + 1. \end{cases}$$

20. *Proposed by Vietnam.* Let a, b, c, d be four nonnegative numbers satisfying $a+b+c+d = 1$. Prove the inequality

$$abc + bcd + cda + dab \le \frac{1}{27} + \frac{176}{27}abcd.$$

*   *   *

Next we turn to the results of the 35th I.M.O. which was written in Hong Kong, July 13–14, 1994. My sources this year are Professor Andy Liu, The University of Alberta, whose personal account of the I.M.O. experience appeared last month, and Professor Richard Nowakowski, Dalhousie University, Halifax, Nova Scotia who was the Canadian Team leader. I hope that I have made no serious errors in compiling the information they forwarded.

This year a total of 385 students from 69 countries took part. This number is down somewhat from the previous year, but still represents a high level of interest and involvement. The organizational headaches associated with such a level of participation must be daunting.

The contest is officially an individual competition and the six problems were assigned equal weights of seven points each (the same as at the last 13 I.M.O.'s), for a maximum possible individual score of 42 (and a total possible of 252 for a national team of six students). (This year 13 teams consisted of fewer than six students, with Cuba and Luxembourg sending a team of one student.) For comparisons see the last 13 I.M.O. reports in [1981: 220], [1982: 223], [1983: 205], [1984: 249], [1985: 202], [1986: 169], [1987: 207], [1988: 193], [1989: 193], [1990: 193], [1991: 257], [1992: 263] and [1993: 256].

This year there were 22 perfect scores (including all six members of the winning U.S.A. team.) The median score was 19. The jury awarded a first prize (Gold) to the thirty students who scored 40 or more points on the papers. Second (Silver) prizes went to the sixty-four students with scores from 30 to 39, and third (Bronze) prizes went to the ninety-eight students who scored in the range from 19–29. Any student who did not receive a medal, but who scored 7 on at least one problem, was awarded Honourable Mention.

Congratulations to the Gold Medalists

Name	Country	Score	Name	Country	Score
Eisenkoibl, T.	Austria	42	Maclean, C.	U.K.	42
Dimitrov, M.	Bulgaria	42	Khazanov, A.	U.S.A.	42
Siderov, I.	Bulgaria	42	Bem, J.	U.S.A.	42
Peng, J.	China	42	Lurie, J.	U.S.A.	42
Yao, J.	China	42	Shazeer, N.	U.S.A.	42
Zheny, J.	China	42	Wang, S.	U.S.A.	42
Golle, P.	France	42	Weinstein, J.	U.S.A.	42
Szeidl, A.	Hungary	42	Iordanov, S.	Bulgaria	41
Bobinski, G.	Poland	42	Mirzi Khany, M.	Iran	41
Schreiber, T.	Poland	42	Ramin Rad, M.	Iran	41
Bandarko, M.	Russia	42	Takahasi, S.	Japan	41
Karassev, R.	Russia	42	Dao, H.L.	Vietnam	41
Norine, S.	Russia	42	Leung, E.C.W.	Canada	40
Zhatos, A.	Slovakia	42	Bayer, A.	Germany	40
Sannikov, V.	Ukraine	42	Myers, J.	U.K.	40

Next we give the problems from this year's I.M.O. Competition. Solutions to these problems, along with those of the 1994 U.S.A. Mathematical Olympiad, will appear in a booklet entitled *Mathematical Olympiads* 1994 which may be obtained for a small charge from: Dr. W.E. Mientka, Executive Director, M.A.A. Committee on H.S. Contests, 917 Oldfather Hall, University of Nebraska, Lincoln, Nebraska, U.S.A. 68588.

35th INTERNATIONAL MATHEMATICAL OLYMPIAD

Hong Kong

First Day — July 13, 1994 (4.5 hours)

1. Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \le n$ for some i, j, $1 \le i \le j \le m$, there exists k, $1 \le k \le m$, with $a_i + a_j = a_k$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_m}{m} \ge \frac{n+1}{2} .$$

2. ABC is an isosceles triangle with $AB = AC$. Suppose that

(i) M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB;

(ii) Q is an arbitrary point on the segment BC different from B and C;

(iii) E lies on the line AB and F lies on the line AC such that E, Q and F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE = QF$.

3. For any positive integer k, let $f(k)$ be the number of elements in the set $\{k+1, k+2, \ldots, 2k\}$ whose base 2 representation has precisely three 1's.

(a) Prove that, for each positive integer m, there exists at least one positive integer k such that $f(k) = m$.

(b) Determine all positive integers m for which there exists exactly one k with $f(k) = m$.

Second Day — July 14, 1994 (4.5 hours)

4. Determine all ordered pairs (m, n) of positive integers such that

$$\frac{n^3 + 1}{mn - 1}$$

is an integer.

5. Let S be the set of real numbers greater than -1. Find all functions $f : S \to S$ satisfying the two conditions

(i) $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x and y in S;

(ii) $f(x)/x$ is strictly increasing for $-1 < x < 0$ and for $0 < x$.

6. Show that there exists a set A of positive integers with the following property: for any infinite set S of primes, there exist positive integers $m \in A$ and $n \notin A$, each of which is a product of k distinct elements of S for some $k \geq 2$.

* * *

As the I.M.O. is officially an individual event, the compilation of team scores is unofficial, if inevitable. These totals, and prize rewards are given in the following table.

Rank	Country	Score	Gold	Silver	Bronze	Total
1.	U.S.A.	252	6	-	-	6
2.	China	229	3	3	-	6
3.	Russia	224	3	2	1	6
4.	Bulgaria	223	3	2	1	6
5.	Hungary	221	1	5	-	6
6.	Vietnam	207	1	5	-	6
7.	U.K.	206	2	2	2	6
8.	Iran	203	2	2	2	6
9.	Romania	198	-	5	1	6
10.	Japan	180	1	2	3	6
11.	Germany	175	1	2	3	6
12.	Australia	173	-	2	3	5
13.–15.	Korea	170	-	2	4	6
13.–15.	Poland	170	2	-	3	5
13.–15.	Taiwan	170	-	4	1	5
16.	India	168	-	3	3	6
17.	Ukraine	163	1	1	2	4
18.	Hong Kong	162	-	2	4	6
19.	France	161	1	1	3	5
20.	Argentina	159	-	3	1	4
21.	Czech Republic	154	-	2	2	4
22.	Slovakia	150	1	1	2	4
23.	Byelorussia	144	-	1	4	5
24.–25.	Canada	143	1	-	3	4
24.–25.	Israel	143	-	1	4	5
26.	Colombia	136	-	2	2	4
27.	South Africa	120	-	-	3	3
28.	Turkey	118	-	-	4	4
29.–30.	New Zealand	116	-	-	4	4
29.–30.	Singapore	116	-	-	4	4
31.	Austria	114	1	-	-	1
32. (Team of 5)	Armenia	110	-	-	4	4
33.	Thailand	106	-	-	3	3
34.–35.	Belgium	105	-	-	2	2
34.–35.	Morocco	105	-	-	2	2
36.	Italy	104	-	-	2	2

Rank	Country	Score	Gold	Silver	Bronze	Total
37.	The Netherlands	99	-	-	2	2
38.	Latvia	98	-	-	3	3
39.–40. (Team of 5)	Brazil	95	-	2	-	2
39.–40.	Georgia	95	-	-	2	2
41.	Sweden	92	-	-	1	1
42.	Greece	91	-	-	1	1
43.	Croatia	90	-	-	2	2
44. (Team of 5)	Estonia	82	-	-	1	1
45.	Norway	80	-	1	1	2
46.	Macao	75	-	1	-	1
47.	Lithuania	73	-	-	1	1
48.	Finland	70	-	-	-	-
49.	Ireland	68	-	-	-	-
50. (Team of 4)	Madagascar	67	-	-	1	1
51.	Mongolia	65	-	1	-	1
52.	Trinidad and Tobago	63	-	-	-	-
53.	The Philippines	53	-	-	-	-
54.–56. (Team of 2)	Chile	52	-	1	-	1
54.–56.	Moldavia	52	-	-	1	1
54.–56.	Portugal	52	-	-	-	-
57. (Team of 4)	Denmark	51	-	-	2	2
58.	Cyprus	48	-	-	-	-
59. (Team of 5)	Slovenia	47	-	-	-	-
60.	Indonesia	46	-	-	-	-
61. (Team of 5)	Bosnia	44	-	-	1	1
62.	Spain	41	-	-	-	-
63. (Team of 3)	Switzerland	35	-	-	1	1
64. (Team of 1)	Luxembourg	32	-	1	-	1
65.–66. (Team of 4)	Iceland	29	-	-	-	-
65.–66.	Mexico	29	-	-	-	-
67.	Kirgistan	24	-	-	-	-
68.–69. (Team of 5)	Kuwait	12	-	-	-	-
68.–69. (Team of 1)	Cuba	12	-	-	-	-

This year the Canadian Team slipped from 17th place to 24th place in a tie (again this year) with Israel. The performance was somewhat hindered by an outbreak of food-poisoning which resulted in a team member not having the full time to work on the test. The team members were:

Robert Edward Bridson	24	Bronze
Christopher Hendrie	13	
Cyrus Hsia Chen	17	
Alyssa Ker	20	Bronze
Edward C.W. Leung	40	Gold
Kevin Purbhoo	29	Bronze

The team leader this year was Dr. Richard Nowakowski of Dalhousie University, Halifax, Nova Scotia. The deputy leader was Mr. Ravi Vakil, now at Harvard University.

The United States team placed first this year with a perfect score. Hearty Congratulations! The members were:

Alexandre Khazanov	42	Gold
Jeremy Bem	42	Gold
Jacob Lurie	42	Gold
Naom Shazeer	42	Gold
Stephen Wong	42	Gold
Jonathon Weinstein	42	Gold

The U.S.A. team was led this year by Professor W. Mientka, University of Nebraska.

* * *

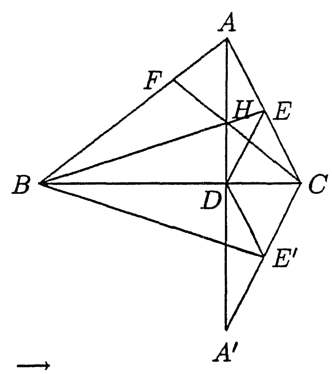
To finish we return to a problem of the Canadian Mathematical Olympiad and give some further background that has come to light.

5. [1994: 189]

Let ABC be an acute angled triangle. Let AD be the altitude on BC, and let H be any interior point on AD. Lines BH and CH, when extended, intersect AC and AB at E and F, respectively. Prove that $\angle EDH = \angle FDH$.

Comment and alternative solution by Andy Liu, University of Alberta.

This problem appeared in *Mathematics Magazine* **37** (1964), p. 338. It was reproduced as problem 135 on page 38 of *Mathematical Quickies*, by Charles W. Trigg, Dover, 1985. The solution given by Nathan Altshiller Court in *M.M.* 37 is also given in *Mathematical Quickies* on page 139.



Yet another solution is the following. Reflect the original diagram about BC. Note that A', D and H are collinear. Applying Ceva's theorem to triangle ACH we have

$$1 = \frac{\overrightarrow{AD}}{\overrightarrow{DH}} \cdot \frac{\overrightarrow{HF}}{\overrightarrow{FC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}} = -\frac{\overrightarrow{A'D}}{\overrightarrow{DH}} \cdot \frac{\overrightarrow{HF}}{\overrightarrow{FC}} \cdot \frac{\overrightarrow{CE'}}{\overrightarrow{E'A'}} .$$

Applying the converse of Menelaus' Theorem to triangle $A'CH$, E', D and F are collinear. Hence $\angle FDA = \angle E'DA' = \angle EDA$.

* * *

That's all the space we have this month. Send me your nice solutions, Olympiads, and pre-Olympiad contest materials.

* * * * *

PETER JOSEPH O'HALLORAN (1931–1994)

The death occurred on 25 September 1994 of Peter O'Halloran, founder of the Australian Mathematics Competition and a range of national and international mathematics enrichment activities.

In 1970, Peter was one of the original appointments at the Canberra College of Advanced Education (later to become the University of Canberra). In 1972/3 he was the first CCAE academic to take study leave. Part of this was taken at the University of Waterloo, Canada, where he gained the idea of a broadly based mathematics competition for high school students. On his return he often enthused to his colleagues about the potential value of such a competition in Australia.

In 1976, while President of the Canberra Mathematical Association, he established a committee to run a mathematics competition in Canberra. This was so successful that the competition became national by 1978 as the Australian Mathematics Competition. It is now history that this competition has grown to the stage of having over 500,000 entries annually, and is probably the biggest mass-participation event in Australia.

In 1984 Peter founded the World Federation of National Mathematics Competitions. For several years the main activity of the WFNMC was a journal, *Mathematics Competitions*, which acted as a vital line of communication for people trying to set up similar activities in other countries. In recent years its activities have expanded to include an international conference, and a set of international awards (the Hilbert and Erdős awards) to recognize mathematicians prominent in enriching mathematics education. Most recently, the WFNMC has become a Special Interest Group of the International Commission on Mathematical Instruction.

One of the highlights of Peter's career was hosting the 1988 IMO in Canberra, which attracted a record number of countries at the time and set new standards in many aspects of organisation.

In 1989 Peter established the Asia Pacific Mathematics Olympiad, providing a regional Olympiad for countries in the dynamic Pacific rim area.

Peter's last main duty was to preside at the WFNMC conference in Bulgaria in July 1994. It was obvious to all who were there that Peter was ill. It was generally thought that he was experiencing another bout of pleuro-pneumonia, from which he had suffered in 1993. On his return home however further tests revealed that Peter's condition was much more serious, and cancer was diagnosed. He spent most of his last month at home.

On 31 August he was presented at home with the David Hilbert Award, which he had declined to accept earlier in the year while still president of the WFNMC. A small party of 30 to 40 of Peter's relatives and local colleagues were in attendance. The David Hilbert Award is the highest international award of the WFNMC and in Peter's case was awarded for "his significant contribution to the enrichment of mathematics learning at an international level".

On 19 September he was awarded the World Cultural Council's "Jose Vasconcelos" World Award for Education at a special ceremony at Chambery, France. This award "is granted to a renowned educator, an authority in the field of teaching or to a legislator of education policies who has a significant influence on the advancement in the scope of culture for mankind". The qualifying jury is formed by several members of the interdisciplinary Committee (of the World Cultural Council) and a group of distinguished educators.

Due to his illness Peter was unable to travel. Instead his eldest daughter Genevieve and son Anthony travelled to Chambery to receive the Award on his behalf. Fortunately Peter was still alive on 23 September on their return to Australia and was able to receive the award in person.

Many mathematicians have made significant individual contributions to the subject itself. Peter's influence was much more direct, bringing mathematics to the world. With his driving energy and the institutions he created he has significantly increased people's awareness of mathematics and what it can do, throughout the world.

Peter saw the main advantage to be derived from the WFNMC was the help it could give to mathematics education in developing countries. I was seated next to him in a debate on the value of competitions at the 1992 International Conference on Mathematical Education in Québec where he was paid the ultimate compliment to which he would have aspired. One delegate gave a well-planned attack on competitions, based on the usual lines, that competitions encouraged elitism, etc. In response, a delegate from the small African country of Malawi, unknown to Peter, responded with an emotional thank you to Peter and the people of the Australian Mathematics Competition for what they had made possible in her country. This was a most moving experience.

Peter of course will be irreplaceable. Fortunately, however, the institutions he established all have the resources, particularly human resources, to ensure that the good work will continue.

Peter Taylor
University of Canberra
Belconnen, ACT 2616
Australia

[A longer version of this obituary will appear in an upcoming issue of the *Australian Mathematical Society Gazette.*]

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **June 1, 1995,** *although solutions received after that date will also be considered until the time when a solution is published.*

1981. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an obtuse triangle with $\angle A > 90°$. Let I and O be the incenter and circumcenter of ΔABC. Suppose that $[IBC] = [OBC]$, where $[XYZ]$ denotes the area of triangle XYZ. Prove that

$$[IAB] + [IOC] = [ICA] + [IBO].$$

1982. *Proposed by Tim Cross, Wolverley High School, Kidderminster, U.K.*

Determine all sequences $a_1 \leq a_2 \leq \cdots \leq a_n$ of positive real numbers such that

$$\sum_{i=1}^{n} a_i = 96, \quad \sum_{i=1}^{n} a_i^2 = 144 \quad \text{and} \quad \sum_{i=1}^{n} a_i^3 = 216.$$

1983. *Proposed by K. R. S. Sastry, Dodballapur, India.*

A convex quadrilateral $ABCD$ has an inscribed circle with center I and also has a circumscribed circle. Let the line parallel to AB through I meet AD in A' and BC in B'. Prove that the length of $A'B'$ is a quarter of the perimeter of $ABCD$.

1984. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U.K.*

Find an integer $n > 1$ so that there exist n consecutive integer squares having an average of n^2.

1985. *Proposed by Murray S. Klamkin and Andy Liu, University of Alberta.*

Let $A_1A_2 \ldots A_{2n}$ be a regular $2n$-gon, $n > 1$. Translate every even-numbered vertex $A_2, A_4, \ldots, A_{2n}$ by an equal (nonzero) amount to get new vertices $A_2', A_4', \ldots, A_{2n}'$, and so that the new $2n$-gon $A_1A_2'A_3A_4' \ldots, A_{2n-1}A_{2n}'$ is still convex. Prove that the perimeter of $A_1A_2' \ldots A_{2n-1}A_{2n}'$ is greater than the perimeter of $A_1A_2 \ldots A_{2n}$.

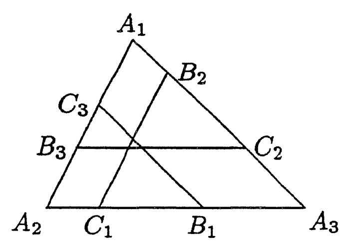
1986. *Proposed by Jisho Kotani, Akita, Japan.*

Suppose that the 4th-degree polynomial $p(x)$ has three local extrema, at $x = x_0$, x_1 and x_2, so that $p(x_0) = p(x_2) = m$ and $p(x_1) = M$, where $m < M$. Let A be the area of the region bounded by $y = m$ and $y = p(x)$, and let B be the area of the region bounded by $y = p(x)$ and $y = M$. Find B/A.

1987. *Proposed by Herbert Gülicher, Westfalische Wilhelms-Universität, Münster, Germany.*

In the figure, $B_2C_1 \| A_1A_2$, $B_3C_2 \| A_2A_3$ and $B_1C_3 \| A_3A_1$. Prove that B_2C_1, B_3C_2 and B_1C_3 are concurrent if and only if

$$\frac{A_1C_3}{C_3B_3} \cdot \frac{A_2C_1}{C_1B_1} \cdot \frac{A_3C_2}{C_2B_2} = 1.$$



1988. *Proposed by Peter Hurthig, Columbia College, Burnaby, B.C.*

Show that any triangle can be dissected into 19 or fewer convex pentagons of equal area.

1989*. *Proposed by Ignotus, Engelberg, Switzerland.*

The sequence of non-negative integers $0, 1, 3, 0, 4, 9, 3, 10, \ldots$ is defined as follows: $a_0 = 0$ and

$$a_n = \begin{cases} a_{n-1} - n & \text{if } a_{n-1} \geq n \\ a_{n-1} + n & \text{otherwise} \end{cases}$$

for $n \geq 1$. Does every non-negative integer occur in the sequence? [*Editor's note.* This problem is closely related to problem 12, proposed but not used at the 1993 IMO; see this issue's Olympiad Corner.]

1990. *Proposed by Leng Gangsong, Hunan Educational Institute, Changsha, China.*

Let r be the inradius of a tetrahedron $A_1A_2A_3A_4$, and let r_1, r_2, r_3, r_4 be the inradii of triangles $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$ respectively. Prove that

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} \leq \frac{2}{r^2},$$

with equality if the tetrahedron is regular.

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1761. [1992: 205; 1993: 175] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

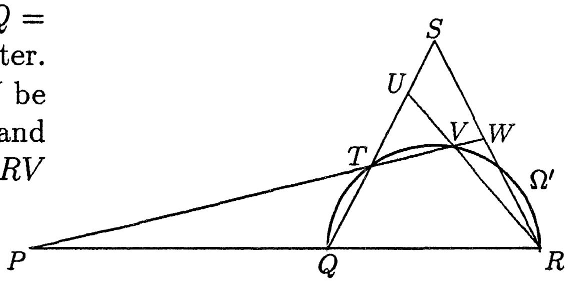
ABC is an isosceles triangle with $AB = AC$. Let D be the foot of the perpendicular from C to AB, and let M be the midpoint of CD. Let E be the foot of the perpendicular from A to BM, and let F be the foot of the perpendicular from A to CE. Prove that $AF \leq AB/3$.

II. *Comment by Shailesh Shirali, Rishi Valley School, India.*

There is an additional feature about the configuration that makes the result $AF \leq AB/3$ very transparent. I refer to the diagram on [1993: 175] in Jordi Dou's solution. Let the line CE meet AB at K. Then, rather surprisingly, K turns out to be a point of trisection of AB; that is, $AK = AB/3$. The inequality to be proved is an immediate corollary to this statement, because clearly $AF \leq AK$.

We shall prove, more generally, the following.

LEMMA. ΔSQR is isosceles with $SQ = SR$, and Ω' is the semicircle on QR as diameter. Let $T = SQ \cap \Omega'$ and let a transversal $PTVW$ be as shown, with P on line QR, W on side SR, and V the other point of intersection with Ω'. Let RV meet SQ at U. Then



$$\frac{QU}{US} = 2\,\frac{PQ}{QR}\,.$$

Proof. Put $\angle TPR = \alpha$ and $\angle SQR = \angle SRQ = \theta$. Then $\angle PTQ = \theta - \alpha$ and

$$\angle SRU = \angle SRQ - \angle VRQ = \angle SQR - (180^\circ - \angle VTQ) = \angle VTQ - \angle PQT = \alpha.$$

Using the law of sines,

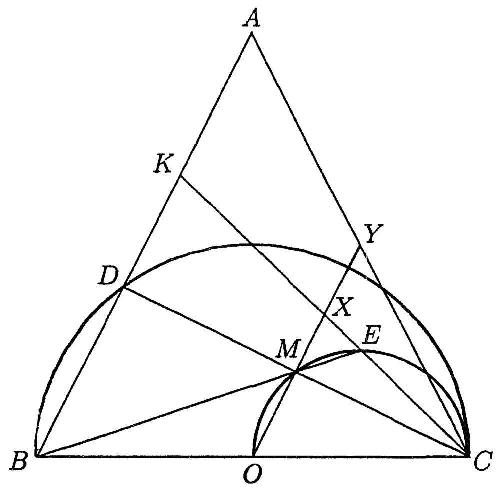
$$\frac{PQ}{QR} = \frac{PQ/QT}{QR/QT} = \frac{\sin(\theta-\alpha)/\sin\alpha}{1/\cos\theta} = \frac{\sin(\theta-\alpha)}{\sin\alpha}\cdot\cos\theta$$

and

$$\frac{QU}{US} = \frac{QU/UR}{US/UR} = \frac{\sin(\theta-\alpha)/\sin\theta}{\sin\alpha/\sin(180-2\theta)} = \frac{\sin(\theta-\alpha)}{\sin\alpha}\cdot 2\cos\theta,$$

and the result follows. □

Now, turning to the problem at hand, let the line OM meet the lines CA and CF at Y and X respectively. Then ΔYOC is isosceles and the conditions of the lemma apply. It follows that



$$\frac{YX}{XO} = \frac{1}{2}\frac{BO}{OC} = \frac{1}{2}\,,$$

so X is a point of trisection of YO. It follows that K is a point of trisection of AB, as claimed.

[*Editor's comment.* Looking back at the solutions received for this problem, the editor notes that solver L.J. Hut also noticed the property $AK = AB/3$, though it was not mentioned prominently in his solution.]

* * * * *

1883. [1993: 264] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle and construct the circles with the sides AB, BC, CA as diameters. $A'B'C'$ is the triangle containing these three circles and whose sides are each tangent to two of the circles. Prove that

$$[A'B'C'] \geq \left(\frac{13}{4} + \sqrt{3}\right)[ABC],$$

where $[XYZ]$ denotes the area of triangle XYZ.

Solution by the proposer.

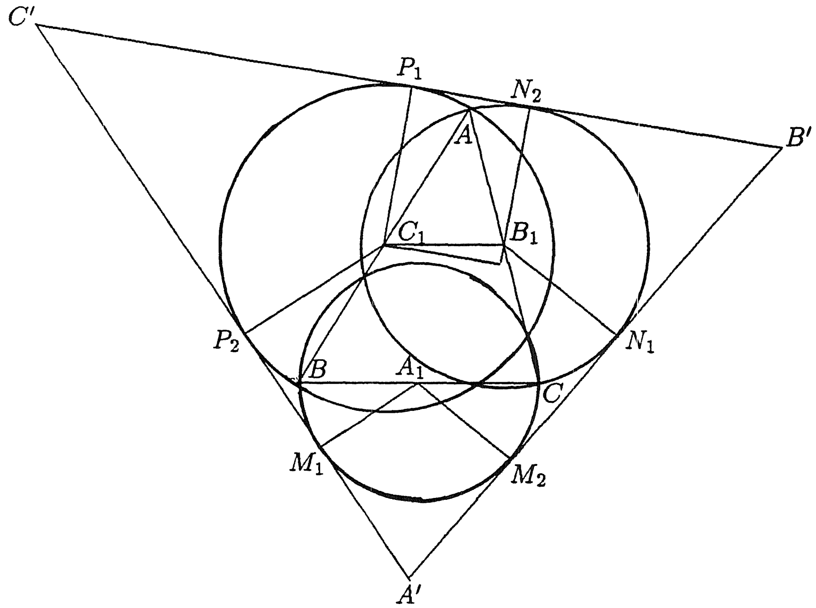


Figure 1

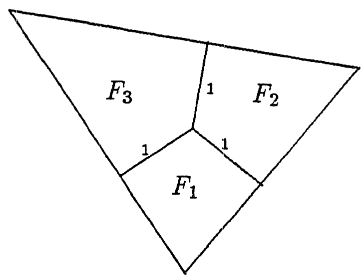


Figure 2

In Figure 1, A_1, B_1, C_1 are the midpoints of the sides of ΔABC. We easily see that

$$(P_1N_2)^2 = \left(\frac{a}{2}\right)^2 - \left(\frac{c}{2} - \frac{b}{2}\right)^2 = (s-b)(s-c)$$

where a, b, c are the sides of ABC and s is the semiperimeter. Thus

$$[P_1C_1B_1N_2] = \frac{b+c}{2}\sqrt{(s-b)(s-c)}\ ,$$

where $[X]$ denotes the area of figure X. Similarly,

$$[M_1A_1C_1P_2] = \frac{c+a}{2}\sqrt{(s-c)(s-a)}, \quad [N_1B_1A_1M_2] = \frac{a+b}{2}\sqrt{(s-a)(s-b)}\ .$$

The triangle in Figure 2 is similar to $\Delta A'B'C'$, has inradius 1, and is divided into three regions of areas F_1, F_2, F_3 and similar to the quadrilaterals $A_1M_1A'M_2$, $B_1N_1B'N_2$, $C_1P_1C'P_2$ respectively. Then

$$[A_1M_1A'M_2] = \frac{a^2}{4}F_1, \quad [B_1N_1B'N_2] = \frac{b^2}{4}F_2, \quad [C_1P_1C'P_2] = \frac{c^2}{4}F_3.$$

Therefore we have

$$[A'B'C'] = \sum \frac{a^2}{4}F_1 + \sum \frac{b+c}{4}\sqrt{(s-b)(s-c)} + \frac{F}{4} \tag{1}$$

where $F = [ABC]$, and the sums are cyclic.

Assuming $A \leq B \leq C$, we next claim that $A' \geq B' \geq C'$, i.e.,

$$\textit{the angles of triangles } ABC \textit{ and } A'B'C' \textit{ are oppositely ordered.} \tag{2}$$

Proof of claim (2). First we will prove that $C' \leq B'$. Draw $A_1X \parallel A'B'$ and $A_1Y \parallel A'C'$, where X, B_1, Y are collinear and $XY \parallel B'C'$. Thus $\Delta A_1XY \sim \Delta A'B'C'$ (see Figure 3). Then find point Q on A_1C_1 such that $A_1Q = A_1B_1$, and let S be the midpoint of QB_1, so that $QB_1 \perp A_1S$ and A_1S is the bisector of $\angle C_1A_1B_1$ (see Figure 4).

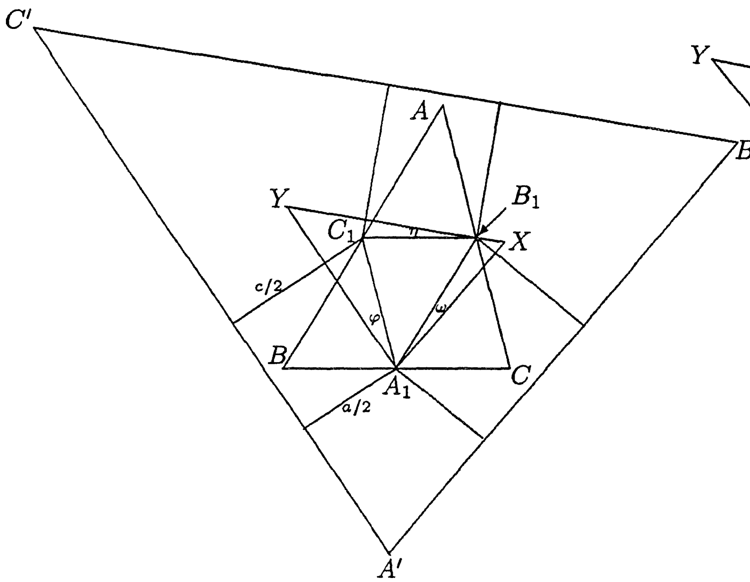


Figure 3

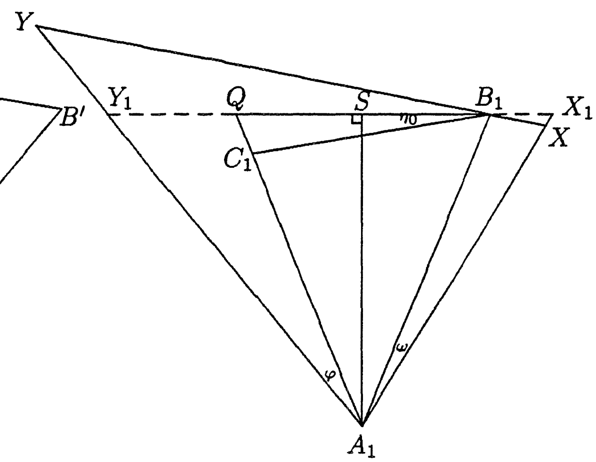


Figure 4

Put $\eta_0 = \angle QB_1C_1$. From ΔB_1C_1Q follows

$$\frac{\sin \angle C_1QB_1}{B_1C_1} = \frac{\sin \eta_0}{C_1Q} = \frac{\sin \eta_0}{A_1Q - A_1C_1} = \frac{\sin \eta_0}{A_1B_1 - A_1C_1},$$

so

$$\sin \eta_0 = \frac{2A_1B_1 - 2A_1C_1}{2B_1C_1} \sin \angle C_1QB_1 = \frac{c-b}{a} \cos \frac{A}{2} \leq \frac{c-b}{a}. \tag{3}$$

Letting $\varphi = \angle YA_1C_1$, $\omega = \angle XA_1B_1$, $\eta = \angle YB_1C_1$, we easily find (Figure 3) that

$$\sin \varphi = \frac{c-a}{b}, \quad \sin \omega = \frac{b-a}{c}, \quad \sin \eta = \frac{c-b}{a},$$

and all three are ≥ 0 since $a \leq b \leq c$. Also

$$\frac{c-a}{b} \geq \frac{b-a}{c},$$

that is, $\varphi \geq \omega$ (since $\varphi, \omega, \eta < 90°$). Therefore $\angle YA_1S \geq \angle SA_1X$. Also, from (3) follows $\eta \geq \eta_0$, and thus (Figure 4)

$$C' = \angle A_1YX \leq \angle A_1Y_1X_1 \leq \angle A_1X_1Y_1 \leq \angle A_1XY = B'.$$

Next we will prove that $B' \leq A'$ in a similar way. Draw $C_1Y \parallel C'A'$ and $C_1Z \parallel C'B'$, where A_1, Y, Z are collinear and $YZ \parallel A'B'$. Thus $\Delta YZC_1 \sim \Delta A'B'C'$ (Figure 5). Then find point Q on C_1B_1 such that $C_1Q = C_1A_1$, and let S be the midpoint of QA_1, so that $QA_1 \perp C_1S$ and C_1S is the bisector of $\angle A_1C_1B_1$ (Figure 6).

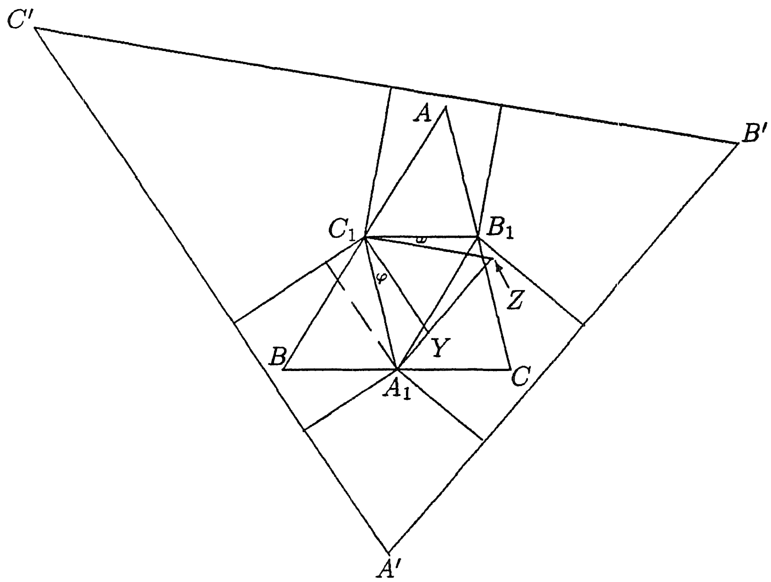


Figure 5

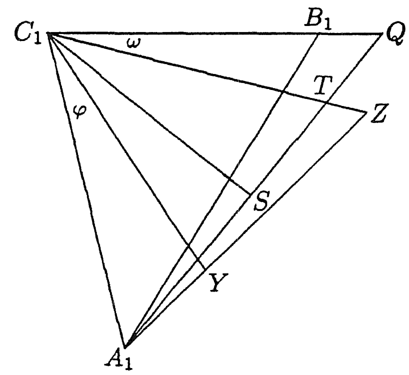


Figure 6

As before, putting $\eta_0 = \angle QA_1B_1$ and considering ΔQA_1B_1 we have

$$\frac{\sin \eta_0}{QB_1} = \frac{\sin \angle C_1QA_1}{A_1B_1}$$

from which follows

$$\sin \eta_0 = \frac{QC_1 - B_1C_1}{A_1B_1} \sin \angle C_1QA_1 = \frac{b-a}{c} \cos \frac{C}{2} \leq \frac{b-a}{c} \, .$$

Letting $\varphi = \angle YC_1A_1$, $\omega = \angle ZC_1B_1$, $\eta = \angle ZA_1B_1$, we have

$$\sin \varphi = \frac{c-a}{b} \, , \quad \sin \omega = \frac{c-b}{a} \, , \quad \sin \eta = \frac{b-a}{c} \, ,$$

so $\eta \geq \eta_0$ and, since

$$\frac{c-a}{b} \geq \frac{c-b}{a} \quad \Longleftrightarrow \quad b^2 - a^2 \geq bc - ac \quad \Longleftrightarrow \quad b + a \geq c$$

(the last inequality being true), also $\varphi \geq \omega$. Thus (Figure 6)

$$B' = \angle YZC_1 \leq \angle STC_1 = \omega + \angle C_1QS \leq \varphi + \angle C_1A_1S \leq \varphi + \angle C_1A_1Z = \angle C_1YZ = A'.$$

This completes the proof of (2).

Now, since $F_1 = \cot(A'/2)$, etc., we see that

$$F_1 \leq F_2 \leq F_3.$$

Thus from Chebyshev's inequality we have

$$\sum \frac{a^2}{4} F_1 \geq \frac{a^2+b^2+c^2}{3 \cdot 4}(F_1+F_2+F_3). \tag{4}$$

The minimum area of the triangle of Figure 2 is $3\sqrt{3}$ (when the triangle is equilateral), so

$$F_1 + F_2 + F_3 \geq 3\sqrt{3}. \tag{5}$$

Also from the obvious inequality $b + c \geq 2\sqrt{bc}$ we have

$$\sum \frac{b+c}{4}\sqrt{(s-b)(s-c)} \geq \sum \frac{1}{2}\sqrt{bc(s-b)(s-c)}\,. \tag{6}$$

From the A.M.–G.M. inequality we have

$$\sum \frac{1}{2}\sqrt{bc(s-b)(s-c)} \geq \frac{3}{2}\sqrt[3]{abc(s-a)(s-b)(s-c)} = \frac{3}{2}\sqrt[3]{\frac{4RF \cdot F^2}{s}} = \frac{3F}{2}\sqrt[3]{\frac{4R}{s}}\,. \tag{7}$$

But it is known that $2s \leq 3\sqrt{3}\,R$, or

$$\frac{R}{s} \geq \frac{2}{3\sqrt{3}} \tag{8}$$

(see item 5.3 of [1]). From (6)–(8) we conclude

$$\sum \frac{b+c}{4}\sqrt{(s-b)(s-c)} \geq \sqrt{3}\,F. \tag{9}$$

Therefore, from (1), (4), (5) and (9) we have

$$[A'B'C'] \geq (a^2+b^2+c^2)\frac{\sqrt{3}}{4} + \sqrt{3}\,F + \frac{F}{4}\,.$$

But $a^2 + b^2 + c^2 \geq 4\sqrt{3}\,F$ (see item 4.4 of [1]), so

$$[A'B'C'] \geq 3F + \sqrt{3}\,F + \frac{F}{4} = \left(\frac{13}{4} + \sqrt{3}\right)F$$

as claimed. Equality holds when ABC is equilateral.

Reference:

[1] O. Bottema et al, *Geometric Inequalities.*

One incorrect solution to this problem was received. The editor would like to see an easier proof of this problem, or at least an easier demonstration of (2).

Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain, has read that a reference to the configuration of the problem occurs in an article by the Spanish geometer Juan Durán Loriga on p. 16 *of* Journal de Mathématiques Elémentaires, 1897, *where the three circles of the problem are called potential circles. Can anyone supply the editor with a copy of this article?*

* * * * *

1888. [1993: 265] *Proposed by Erich Friedman, Stetson University, DeLand, Florida.*

Prove that for every finite set A of positive integers, there exists a finite set B of positive integers so that $B \supseteq A$ and

$$\prod_{x\in B} x = \sum_{x\in B} x^2.$$

Solution by Waldemar Pompe, student, University of Warsaw, Poland.

Clearly, we can find a positive integer m such that $A \subseteq \{1,2,\ldots,m\}$. So it is enough to prove that there is a finite set B of positive integers such that $\{1,2,\ldots,m\} \subseteq B$ and

$$\prod_{x\in B} x = \sum_{x\in B} x^2.$$

We can assume that $m \geq 5$. Let the sequence (x_n) of positive integers be defined as follows: $x_0 = 1$ and

$$x_n = m!\,x_0\ldots x_{n-1} - 1 \qquad \text{for } n \geq 1.$$

It is easy to see that $m < x_1 < x_2 < \ldots$. Let $B_0 = \{1,2,\ldots,m\}$ and

$$B_i = \{1,2,\ldots,m,x_1,x_2,\ldots,x_i\}$$

for $i \geq 1$. Set $k = m! - (1^2+2^2+\cdots+m^2) > 0$ (for $m \geq 5$). We will prove using induction on i that

$$\prod_{x\in B_i} x - \sum_{x\in B_i} x^2 = k - i. \tag{1}$$

Indeed, if $i = 0$ then (1) is true by the definition of k. If (1) holds for some i, then we get

$$\begin{aligned}\prod_{x\in B_{i+1}} x - \sum_{x\in B_{i+1}} x^2 &= (m!\,x_0\ldots x_i - 1)\prod_{x\in B_i} x - (m!\,x_0\ldots x_i - 1)^2 - \sum_{x\in B_i} x^2 \\ &= (m!\,x_0\ldots x_i - 1)^2 + (m!\,x_0\ldots x_i - 1) - (m!\,x_0\ldots x_i - 1)^2 - \sum_{x\in B_i} x^2 \\ &= \prod_{x\in B_i} x - \sum_{x\in B_i} x^2 - 1 = k - (i+1),\end{aligned}$$

and the induction proof is complete. Therefore, take $B = B_k$, and according to (1), the problem is solved.

Also solved by the proposer.

Pompe also mentioned that if multisets are allowed then the problem becomes easier: just add enough 1's and 2's to A to get B. This version of the problem was solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; and FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia.

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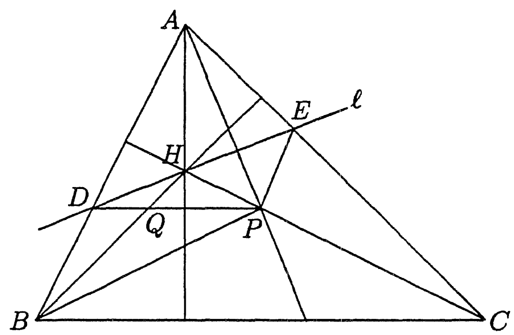
1891. [1993: 294] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a non-right-angled triangle with orthocenter H. A line ℓ through H meets AB and AC at $D \neq B$ and $E \neq C$ respectively. Let P be a point such that $AP \perp \ell$. Prove that $[PBD] : [PCE] = DH : HE$, where $[XYZ]$ denotes the area of triangle XYZ.

Solution by Waldemar Pompe, student, University of Warsaw, Poland.

Fix the line ℓ. If $d(X, k)$ denotes the distance from the point X to the line k, we get

$$\frac{[PBD]}{[PCE]} = \frac{BD \cdot d(P, AB)}{CE \cdot d(P, AC)} .$$



BD and CE are constant (if ℓ is fixed) and because of the homothety the ratio $d(P, AB)/d(P, AC)$ is constant too. Therefore the ratio $[PBD] : [PCE]$ is constant while ℓ is constant and P moves.

Thus it suffices to prove the claim when P lies, for example, on the line CH (see the figure). Let $PD \cap BH = Q$. Since $AP \perp \ell$, H is the orthocenter of ΔADP, so $AH \perp PQ$, which gives $PQ \parallel BC$. Hence

$$\frac{[PHD]}{[PBD]} = \frac{HQ}{QB} = \frac{HP}{PC} = \frac{[PHE]}{[PCE]} ,$$

whence

$$\frac{[PBD]}{[PCE]} = \frac{[PHD]}{[PHE]} = \frac{HD}{HE} .$$

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; HIMADRI CHOUDHURY, student, Hunter H.S., New York; TIM CROSS, Wolverley High School, Kidderminster, U.K.; MARCIN E. KUCZMA, Warszawa, Poland; PAVLOS MARAGOUDAKIS, Pireas, Greece; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

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1892. [1993: 294] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Let $n \geq 4$ be an integer. Find the exact upper and lower bounds for the cyclic sum

$$\sum_{i=1}^{n} \frac{x_i}{x_{i-1}+x_i+x_{i+1}}$$

(where of course $x_0 = x_n$, $x_{n+1} = x_1$), over all n-tuples of nonnegative numbers $(x_1, \ldots, x_n)$ without three zeros in cyclic succession. Characterize all cases in which either one of these bounds is attained.

Solution by Murray S. Klamkin, University of Alberta.

This problem for the case $n = 6$ was proposed by me as a Quickie problem (*Mathematics Magazine* **64** (1991), pp. 198, 206) and the solution for general n follows similarly from the special case $n = 6$ with a slight addition when n is odd. We show that

$$1 \leq \sum_{i=1}^{n} \frac{x_i}{x_{i-1}+x_i+x_{i+1}} \leq \left[\frac{n}{2}\right]$$

where these bounds are best possible.

Let S be the given sum and $T = \sum x_i$. Then

$$S > \frac{x_1}{T} + \frac{x_2}{T} + \cdots + \frac{x_n}{T} = 1.$$

That 1 is the best possible lower bound follows by choosing $x_i = 1/\varepsilon^{n-i}$ where $\varepsilon \ll 1$. [*Editor's note.* Then we get

$$\begin{aligned} S &= \frac{\frac{1}{\varepsilon^{n-1}}}{1+\frac{1}{\varepsilon^{n-1}}+\frac{1}{\varepsilon^{n-2}}} + \sum_{i=2}^{n-1} \frac{\frac{1}{\varepsilon^{n-i}}}{\frac{1}{\varepsilon^{n-i+1}}+\frac{1}{\varepsilon^{n-i}}+\frac{1}{\varepsilon^{n-i-1}}} + \frac{1}{\frac{1}{\varepsilon}+1+\frac{1}{\varepsilon^{n-1}}} \\ &= \frac{1}{\varepsilon^{n-1}+1+\varepsilon} + \sum_{i=2}^{n-1} \frac{1}{\varepsilon^{-1}+1+\varepsilon} + \frac{1}{\varepsilon^{-1}+1+\varepsilon^{1-n}} \end{aligned}$$

which approaches 1 as $\varepsilon \to 0$.]

Now we show that the least upper bound is m for $n = 2m$ and for $n = 2m+1$.

Case (i): $n = 2m$. Note that

$$\frac{x_1}{x_{2m}+x_1+x_2} + \frac{x_2}{x_1+x_2+x_3} \leq \frac{x_1}{x_1+x_2} + \frac{x_2}{x_1+x_2} = 1 \tag{1}$$

[this works if $x_1 + x_2 > 0$, and the case $x_1 = x_2 = 0$ is obvious.—*Ed.*], and the same is true for all the remaining consecutive pairs. Hence the least upper bound is m and it is achieved if $x_2 = x_4 = \cdots = x_{2m} = 0$ or if $x_1 = x_3 = \cdots = x_{2m-1} = 0$ [from (1) we see that exactly one of every consecutive pair of x_i's is 0.—*Ed.*].

Case (ii): $n = 2m+1$. We can assume without loss of generality that the smallest denominator is $x_1 + x_2 + x_3$. Then,

$$\frac{x_1}{x_{2m+1}+x_1+x_2} + \frac{x_2}{x_1+x_2+x_3} + \frac{x_3}{x_2+x_3+x_4} \leq 1.$$

As above, each of the remaining consecutive pairs of terms is bounded by 1. The upper bound m is achieved as above;

$$x_2 = x_4 = \cdots = x_{2m} = 0 \quad \text{or} \quad x_1 = x_3 = \cdots = x_{2m+1} = 0$$

(or symmetrically).

Comment: In a similar fashion we can extend the problem for the sum

$$S = \sum_{i=1}^{n} \frac{x_i}{x_{i-1} + x_i + \cdots + x_{i+k}}$$

where k is a fixed positive integer $< n-2$. Again the greatest lower bound is 1 and one can get arbitrarily close to it by choosing as before $x_i = 1/\varepsilon^{n-i}$. The least upper bound is more involved. As an example, let $n = 6$ and $k = 2$. Assuming without loss of generality that the smallest denominator is $x_1 + x_2 + x_3 + x_4$, then the sum of the first four terms of S is ≤ 1. Similarly, the sum of the remaining pair of terms is also ≤ 1. That one can get arbitrarily close to 2 for the sum follows by choosing $x_1 = 1/\varepsilon^2$, $x_3 = 1/\varepsilon$ and the remaining terms each equal to 1.

Also solved by the proposer. One partial solution was received.

* * * * *

1893. [1993: 294] *Proposed by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*

For a non-obtuse triangle ABC, D is the foot of the perpendicular from A to BC, E is the point of intersection of BC and the internal bisector of $\angle A$, and M is the midpoint of the segment BC. Suppose that the lengths of the segments BD, DE, EM, MC (in that order) are in arithmetic progression.

(a) Describe all such triangles ABC.

(b) Let F be the point of intersection of AC and the line passing through E that is perpendicular to AE. Show that triangle EMF is isosceles.

Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.

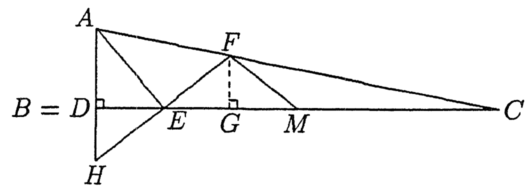
(a) We may let

$$BD = u, \quad DE = u + d, \quad EM = u + 2d, \quad MC = u + 3d;$$

then from $BD + DE + EM = BM = MC$ we have $3u + 3d = u + 3d$, so $u = 0$, and hence $B = D$ and $\angle B = 90°$. Furthermore $BE = d$ and $EC = 5d$. Therefore (with a, b, c the sides of the triangle) ΔABC satisfies $\angle B = 90°$ and, since AE bisects $\angle A$,

$$\frac{c}{b} = \frac{DE}{EC} = \frac{1}{5}\,.$$

(b) Let H be the point of intersection of AB with EF, and let G be the foot of the perpendicular from F to BC. Then $EH = EF$ [because $AE \perp HF$ and AE bisects $\angle A$], $\angle HBE = \angle FGE$ and $\angle BEH = \angle GEF$, so ΔBEH is congruent to ΔGEF. Thus $BE = EG$. Since $EM = 2BE$, G is the midpoint of EM and therefore $EF = MF$.



Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. One partial solution was received.

* * * * *

1894. [1993: 294] *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.* (Dedicated in memoriam to K.V.R. Sastry.)

Evaluate

$$\sum_{k=0}^{n} \frac{\binom{n+1}{k}\binom{n}{k}}{\binom{2n}{2k}}$$

where n is a nonnegative integer.

I. *Solution by Chris Wildhagen, Rotterdam, The Netherlands.*

Let s_n be the given sum. Then

$$s_n = \frac{(n+1)!n!}{(2n)!} \cdot t_n,$$

where

$$t_n = \sum_{k=0}^{n} \frac{(2n-2k)!(2k)!}{(n+1-k)!(n-k)!(k!)^2} = \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{n+1-k}.$$

We shall derive a closed form for t_n by the method of generating functions.

$$\begin{aligned}
\sum_{n\geq 0} t_n x^n &= \sum_{k\geq 0}\binom{2k}{k}x^k \cdot \sum_{n\geq k}\binom{2n-2k}{n-k}\frac{x^{n-k}}{n+1-k} \\
&= \sum_{k\geq 0}\binom{2k}{k}x^k \cdot \sum_{l\geq 0}\binom{2l}{l}\frac{x^l}{l+1} \\
&= \frac{1}{\sqrt{1-4x}} \cdot \frac{1}{2x}\left(1-\sqrt{1-4x}\right) = \frac{1}{2x}\left(\frac{1}{\sqrt{1-4x}} - 1\right) \\
&= \sum_{n\geq 1} \frac{1}{2}\binom{2n}{n} x^{n-1},
\end{aligned}$$

hence $t_n = \frac{1}{2}\binom{2n+2}{n+1}$ and

$$s_n = \frac{1}{2}\binom{2n+2}{n+1} \cdot \frac{(n+1)!n!}{(2n)!} = \mathbf{2n+1}.$$

[*Editor's note by Edward Wang.* The fact that

$$\sum_{k\geq 0}\binom{2k}{k}x^k = \frac{1}{\sqrt{1-4x}}\,, \tag{1}$$

i.e., $1/\sqrt{1-4x}$ is the generating function for the sequence $\{\binom{2k}{k}\}_{k=0}^{\infty}$, is well known and follows from the Binomial Theorem $(1+x)^\alpha = \sum_{k\geq 0}\binom{\alpha}{k}x^k$ (for all reals α) and the easily verified fact that $(-4)^k\binom{-1/2}{k} = \binom{2k}{k}$ (see e.g. Exercise 32 on p. 213 of *Applied Combinatorics*, 2nd edition, by Alan Tucker). Integrating (1) and dividing by x, one then gets

$$\sum_{k\geq 0}\binom{2k}{k}\frac{x^k}{k+1} = \frac{1}{2x}(1-\sqrt{1-4x}).$$

That is, $\frac{1}{2x}(1-\sqrt{1-4x})$ is the generating function for the sequence $\{\frac{1}{k+1}\binom{2k}{k}\}_{k=0}^{\infty}$ of the Catalan numbers, a fact which is also well known (see e.g. p. 233 of *Applied Combinatorics* by Fred S. Roberts).]

II. *Solution by Peter L. Montgomery, Centrum voor Wiskunde en Informatica, Amsterdam, The Netherlands.*

By induction on m it is easy to show that

$$\sum_{k=0}^{m}\frac{\binom{n}{k}\binom{n+1}{k}}{\binom{2n}{2k}} = (2m+1)\frac{\binom{n}{m}^2}{\binom{2n}{2m}}\,, \qquad 0 \leq m \leq n.$$

When $m = n$ the right side equals $2n+1$.

Editor's comments by Edward Wang.

It is quite amazing that an inductive proof is possible when the extra "parameter" m is introduced! However, despite the solver's claim that the induction is easy, the editor believes that many readers would find it challenging to work out the details.

Montgomery sent in a second solution, which is similar to Solution I. Both of his solutions were forwarded by Richard McIntosh, University of Regina.

Also solved by H.L. ABBOTT, University of Alberta; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; and the proposer.

* * * * *

1895. [1993: 295] *Proposed by Ji Chen and Gang Yu, Ningbo University, China.*

Let P be an interior point of a triangle $A_1A_2A_3$; R_1, R_2, R_3 the distances from P to A_1, A_2, A_3; and R the circumradius of $\Delta A_1A_2A_3$. Prove that

$$R_1R_2R_3 \le \frac{32}{27}R^3,$$

with equality when $A_2 = A_3$ and $PA_2 = 2PA_1$.

Comment by the editor.

Murray Klamkin has informed the editor that this problem is the same as problem 10282 in the February 1993 *American Mathematical Monthly* (although the proposers are different). Since the *Monthly* published the problem before us, we'll wait until they publish their solution, and then give the list of solvers of *Crux* 1895 and perhaps print a solution too.

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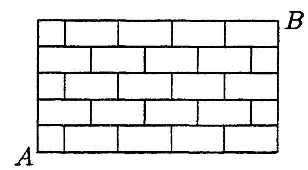
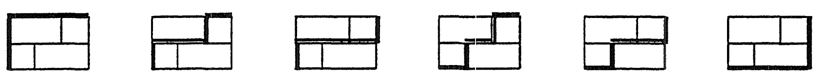
1896. [1993: 295] *Proposed by N. Kildonan, Winnipeg, Manitoba.*



Consider an $m \times n$ "brick wall" grid of m rows and n columns, made up of 1×2 bricks with 1×1 bricks at the ends of rows where needed, and so that we always have a 1×1 brick in the lower left corner. The diagram shows the case $m = 5$, $n = 9$. Let $f(m,n)$ denote the number of walks of minimum length (using the grid lines) from A to B, so for example $f(2,3) = 6$:



Prove that

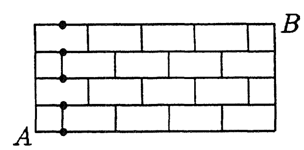
$$f(m,n) = f(m,n-2) + f(m-1,n-1)$$

for all $m > 1$ and $n > 2$.

Solution by Douglas E. Jackson, Eastern New Mexico University, Portales.

Assume that $n+m$ is odd and $n, m > 0$. In this case the $m \times n$ wall has a 1×1 brick in the top right corner. Let P be a grid point that is l units below the top and k units to the left of the right edge of the wall, where $l \le m$ and $k < n$. In other words, P is a grid point that is not on the left edge. $g(l,k)$ will denote the number of shortest walks from P to B. Then

$$f(m,n) = \sum_{i=0}^{m} g(i, n-1)$$



for $m, n \ge 1$, since $g(i, n-1)$ is the number of walks from A to B making a right move for the first time at step $m - i + 1$. [*Editor's note*: e.g., in the diagram at right, each $g(i, n-1)$ corresponds to the number of walks from one of the dots to B.] Now, if P is at the lower left corner of a brick then the first two moves are either (up, right) or (right, right), else the first two moves are either (right, up) or (right, right). In either case, this leads to the observation that

$$g(l,k) = g(l,k-2) + g(l-1,k-1),$$

for $l \geq 1$ and $k \geq 2$. If $m > 1$ and $n > 2$ then

$$\begin{aligned} f(m,n) &= \sum_{i=0}^{m} g(i,n-1) = g(0,n-1) + \sum_{i=1}^{m}[g(i,n-3) + g(i-1,n-2)] \\ &= \sum_{i=0}^{m} g(i,n-3) + \sum_{i=0}^{m-1} g(i,n-2) = f(m,n-2) + f(m-1,n-1), \end{aligned}$$

since $g(0,n-1) = g(0,n-3) = 1$. If $n+m$ is even, the argument is the same although g will be a different function [only because the wall will have a 1×2 brick instead of a 1×1 brick in the top right corner — *Ed.*].

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; MARCIN E. KUCZMA, Warszawa, Poland; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP, Tucson; and the proposer.

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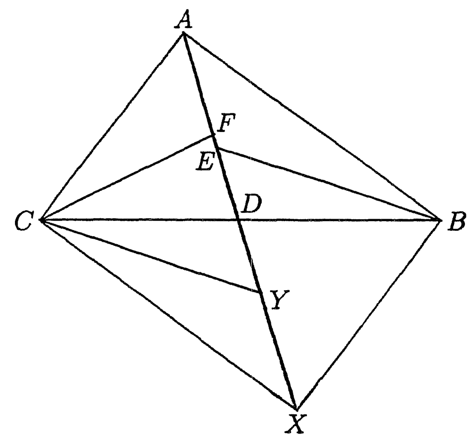
1897. [1993: 295] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

In triangle ABC the angle bisectors of angles B and C meet the median AD at points E and F respectively. If $BE = CF$ then prove that ABC is isosceles.

Solution by Esther Szekeres, Turramurra, Australia.

Assume the opposite, take $\angle C$ larger than $\angle B$. Extend AD to X such that $AD = DX$. Then $ABXC$ is a parallelogram with $CX > AC$, therefore

$$\angle CAX > \angle CXA = \angle DAB. \qquad (1)$$



Let Y be a point on DX such that CY is parallel to BE; then, by the congruence of ΔCYD and ΔBED, $CY = BE = CF$ (by assumption), so ΔCFY is isosceles, and therefore

$$\angle CFY = \angle CYF = \angle YEB.$$

But

$$\angle CFY = \angle CAF + \frac{1}{2}\angle C \quad \text{and} \quad \angle YEB = \angle YAB + \frac{1}{2}\angle B,$$

and considering (1) and the assumption that $\angle C > \angle B$, $\angle CFY$ must be larger than $\angle YEB$. Contradiction!

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER BRADLEY, Clifton College, Bristol, U.K.; HIMADRI CHOUDHURY, student, Hunter H.S., New York; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, Connecticut; MURRAY KLAMKIN and ANDY LIU, University *of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; PAVLOS MARAGOUDAKIS, Pireas, Greece (two solutions); VEDULA N. MURTY, Maharanipeta, India; PAUL PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; PAUL YIU, Florida Atlantic University, Boca Raton; and the proposer. One incorrect solution was received.*

One of Maragoudakis's solutions was very similar to Professor Szekeres's neat solution above. As this solution shows, this problem is a lot easier to prove than the similar (and famous) Steiner–Lehmus Theorem!

* * * * *

1898. [1993: 295] *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

The faces of a tetrahedron $ABCD$ are cut by a line in four distinct points A', B', C', D' (with A' opposite A, etc.). Prove that the midpoints of the segments AA', BB', CC', DD' are coplanar.

Comment by Waldemar Pompe, student, University of Warsaw, Poland.

This problem is a special case ($n = 3$) of *Crux* 587 [1981: 305–306].

Editor's comment by Chris Fisher. Since Pompe is still a student we can only speculate that in 1981 he was reading *Crux* while his classmates were struggling with fractions! The problem he refers to goes as follows.

Let $\sigma = A_0A_1\dots A_n$ be an n-simplex in $\mathbb{R}^n$. A straight line cuts the $(n-1)$-dimensional faces σ_i opposite A_i in the points B_i. If M_i is the midpoint of the line segment A_iB_i, show that all the points M_i lie in the same $(n-1)$-dimensional hyperplane.

The proof did not appear in *Crux*. Instead, a reference was given to a simple solution by I. Paasche in *Elemente der Mathematik* **31** (1976) 14–15, where the same problem had been posed by Murray Klamkin. Since the proof is so short, we will reproduce a translation of it here.

Choose coordinates so that A_0 is the origin $(0,0,\dots,0)$ and, for $1 \le i \le n$, A_i is a vertex of the reference simplex (all zeros except for a 1 in the ith position). σ_0 therefore satisfies $x_1+\dots+x_n = 1$, while the plane σ_i is $x_i = 0$. Let the parametric form of the given line be $\{(u_1,\dots,u_n) + t(v_1,\dots,v_n) \mid t \in \mathbb{R}\}$ with $B_0 = (u_1,\dots,u_n)$ in σ_0 (i.e. $\sum u_i = 1$) and the vector $(v_1,\dots,v_n)$ "normalized" so that $\sum v_i = 1$ (which is possible since the line is not parallel to σ_0). It follows that

$$B_i = (u_1,\dots,u_n) - \frac{u_i}{v_i}(v_1,\dots,v_n), \quad i = 1,\dots,n$$

(which is well defined — $v_i \neq 0$ since the line is not parallel to σ_i). Consequently,

$$v_1(A_1 + B_1) + \dots + v_n(A_n + B_n) = A_0 + B_0.$$

Since $M_i = (A_i + B_i)/2$, $i \ge 0$, and $\sum v_i = 1$, M_0 is a convex combination of the other M_i and therefore their span has dimension at most $n-1$, so they lie in a hyperplane.

The case $n = 2$ is the familiar result: *the midpoints of the diagonals of a complete quadrilateral are collinear.* It can be found in books on analytic or projective geometry; a pair of proofs is given by Dan Pedoe in his *Geometry, A Comprehensive Course*, sections 9.4 and 11.1; another pair can be found in R.A. Johnson, *Advanced Euclidean Geometry*, pages 62 and 172. Moreover, it is an immediate consequence of "Bodenmiller's Theorem": *the three circles with the diagonals of a complete quadrilateral as diameters intersect in the same two points* (Rudolf Fritsch, Remarks on Bodenmiller's Theorem, *Journal of Geometry* **47** (1993) 23–31). This leads to the obvious question: is there an n-dimensional version of Bodenmiller's Theorem? More precisely, do the hyperspheres having the segments A_iB_i as diameters intersect in the same $(n-2)$-sphere (contained in a hyperplane)?

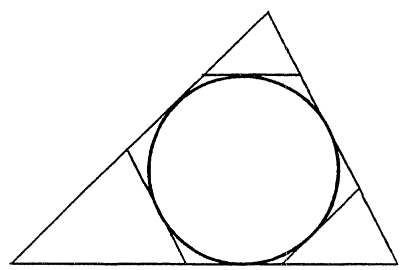
Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; MURRAY S. KLAMKIN, University of Alberta; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Penning's solution is quite similar to that of Paasche, but he did not generalize the result; only Bradley and Klamkin treated the n-dimensional version.

*   *   *   *   *

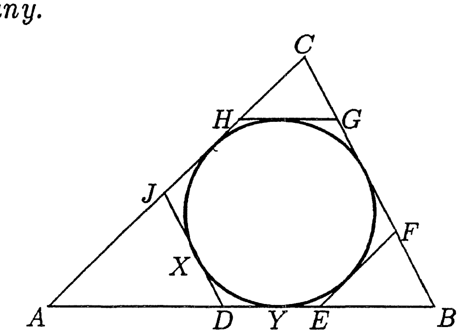
1899. [1993: 295] *Proposed by Neven Jurić, Zagreb, Croatia.*

In a triangle with circumradius R, three lines are drawn tangent to the inscribed circle and parallel to the sides, cutting three small triangles off the corners of the given triangle as shown. Let the circumradii of these triangles be R_a, R_b, R_c. Show that $R_a + R_b + R_c = R$.



Solution by Šefket Arslanagić, Berlin, Germany.

Label the diagram as shown. Because of equality of tangent lengths [$DX = DY$, etc.], we have



$$p(ADJ) + p(BEF) + p(CHG) = p(ABC), \quad (1)$$

where $p(ADJ)$ denotes the perimeter of $\triangle ADJ$, etc. We have also

$$\Delta ADJ \sim \Delta ABC, \quad \Delta EBF \sim \Delta ABC, \quad \Delta HGC \sim \Delta ABC,$$

with similarity coefficients k_1, k_2, k_3 respectively [e.g., $AD = k_1 AB$, etc.]. Letting $p = p(ABC)$, we have by (1) that $k_1p + k_2p + k_3p = p$, i.e.,

$$k_1 + k_2 + k_3 = 1.$$

Thus $k_1R + k_2R + k_3R = R$, i.e., $R_a + R_b + R_c = R$.

Remark: similar relations are valid for inradii, altitudes, etc.

[Arslanagić gave two other solutions as well!—*Ed.*]

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; SAM BAETHGE, Science Academy, Austin, Texas; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; HIMADRI CHOUDHURY, student, Hunter H. S., New York; JULIO CESAR DE LA YNCERA, Gaithersburg, Maryland; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; MURRAY S. KLAMKIN, University of Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; MARCIN E. KUCZMA, Warszawa, Poland; MARCUS LANG, student, Bundesrealgymnasium, Graz, Austria; PAVLOS MARAGOUDAKIS, Pireas, Greece; VEDULA MURTY, Maharanipeta, India; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; ESTHER SZEKERES, Turramurra, Australia; ALBERT W. WALKER, Toronto, Ontario; EDWARD T. H. WANG, Wilfrid Laurier University, and SIMING ZHAN, University of Waterloo, Waterloo, Ontario; and the proposer.

Very similar proofs were received from Penning, Pompe and Szekeres. The fact that the same relation holds for inradii etc. was pointed out by several solvers.

The same problem appeared as Gy. 2587 *in the Hungarian journal* KOMAL *(a solution, along the same lines as the above, is on pp.* 211–212 *of the* 1990 *volume). Perz notes that almost the same problem was proposed but not used for the* 1982 *IMO. Bellot found related problems, using the same configuration, on page* 349 *of Mitrinović, Pečarić and Volenec,* Recent Advances in Geometric Inequalities, *Kluwer,* 1989*; as problem* 3 *of the* 1964 *IMO (see pp.* 6, 66–67 *of Greitzer's* International Mathematical Olympiads 1959–1977, *MAA,* 1978*); in D. Kontogiannis's* 1981 *book (in Greek)* Mathematikes Olympiades*; and in* Crux 1245 [1988: 189].

Klamkin gives the n-dimensional generalization $\sum_{i=0}^{n} R_i = (n-1)R$, *where the* R_i *are the circumradii of the small n-dimensional simplexes cut off from an n-dimensional simplex of circumradius R by planes parallel to the faces of the simplex and tangent to its inscribed sphere. (In fact he notes that the circumradii could again be replaced by inradii, etc.) His proof is similar to the above one but uses altitudes instead of perimeters, with the known relation* $\sum h_i^{-1} = r^{-1}$, *where the* h_i *are the altitudes of the simplex and* r *is the inradius (e.g., see p.* 463 *of* Recent Advances in Geometric Inequalities*).*

* * * * *

1900. [1993: 295] *Proposed by Joaquín Gómez Rey, I.B. Luis Buñuel, Alcorcón, Madrid, Spain.*

Given a cube of edge 1, choose four of its vertices forming a regular tetrahedron of edge $\sqrt{2}$ (the face diagonal of the cube). The other four vertices form another such tetrahedron. Find the volume of the union of these two tetrahedra.

Solution by Johannes Kepler (with comments by Robert Geretschläger, Bundesrealgymnasium, Graz, Austria).

Dear Sirs and Madams,

It is with great interest that I note your choice of my *stella octangula* as the subject of problem 1900 in your fine journal.

Many things have changed since my *Harmonices Mundi* was first published, not least of which is the language in which the international scientific community communicates. Only the title of your august publication is left as a remnant of the Latin I was formerly used to corresponding in. Nevertheless, I will attempt to "give it my best shot" (as I understand to be the correct jargon), and respond in English, as befits this age of the nearing of the third millenium *anno domini.*

The volume of the *stella octangula* is one-half the volume of the cube whose vertices it shares. Please allow me to quote from my *Harmonices Mundi.*

> "Among [the Platonic solids] there are two notable pairs, which are composed from different classes. The males are the cube and dodecahedron from the class of primary solids, the females are the octahedron and the icosahedron, the secondary solids. In addition to these comes a loner or androgyne, the tetrahedron, as it can be inscribed in itself, as the females can be inscribed in the males, subordinate to them, and having female sexual characteristics opposite the corresponding male ones, i.e. vertices opposite faces.
>
> Also, as the tetrahedron is element, bowels and skeleton of the male cube, so in another manner is the female octahedron element and part of the tetrahedron. The tetrahedron therefore stands between the pair.
>
> The main difference between these two groups or families is the fact that the relationship in the cube family is speakable. For the tetrahedron is one-third of the cube, the octahedron one-half of the tetrahedron, or thus one-sixth of the cube. In the dodecahedron family however, the relationship is unspeakable, but divine."

As I understand it, in modern times, more extensive intermediate calculations are included in mathematical publications than was the case in my day. I would thus like to explicate the origins of the fractions quoted herein.

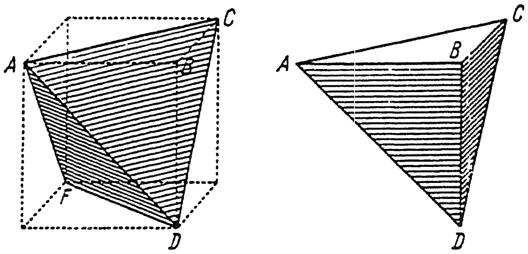
The tetrahedron results from slicing four right-angled tetrahedra (with three orthogonal sides of unit length) from the unit cube. Each of these has a volume one-sixth of the cube's. Thus, one-third of the cube's volume remains for the regular tetrahedron. (I have included a photocopy of my original etchings of these polyhedra to facilitate the understanding of my explanations. A marvelous invention, the photocopier.)

The volume of the octahedron [the intersection of the two tetrahedra] is, of course, one-third the product of the area of the center square and the altitude. As the area of the square is half the area of a face of the cube, and the altitude equal to the length of the cube's edge, the volume of the octahedron is indeed one-sixth the volume of the cube.

The fact that the volume of the *stella octangula* is one-half the volume of the cube now follows, since twice one-third less one-sixth is one-half. *Quod erat demonstrandum.*

In priori figurâ apparet Tetraedron ACDF latens in Cubo, sic ut quodlibet Tetraëdri planum ut ACD, tegatur ab uno cubi angulo ACDB. In secunda figura apparet Cubus AED latens intus in Dodecaëdro, sic ut quodlibet Cubi planum, ut AED, tegatur à duobus Dodecaëdri angulis seu Pentaëdro ABCDE quod est sectile in tria Tetraëdra dissimilia per duo plana, DCA, et ABD.

Intra Tetraëdron est Dodecaëdron 3. Vltima primariarum, similis scilicet *Compositae* ex Cubi partibus, Tetraëdri similibus, id est, ex Tetraëdris irregularibus, quibus tegitur Cubus intus. Huic succedit *181*
Icosaëdron 4. ob similitudinem, ultima secundariarum, angulo solido plurilineari utentium. Intimum est Octoëdron 5. Cubi simile, et prima



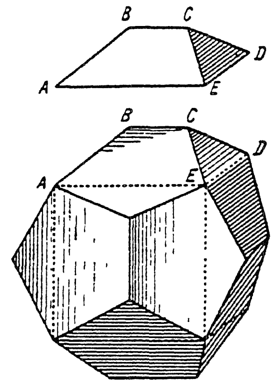
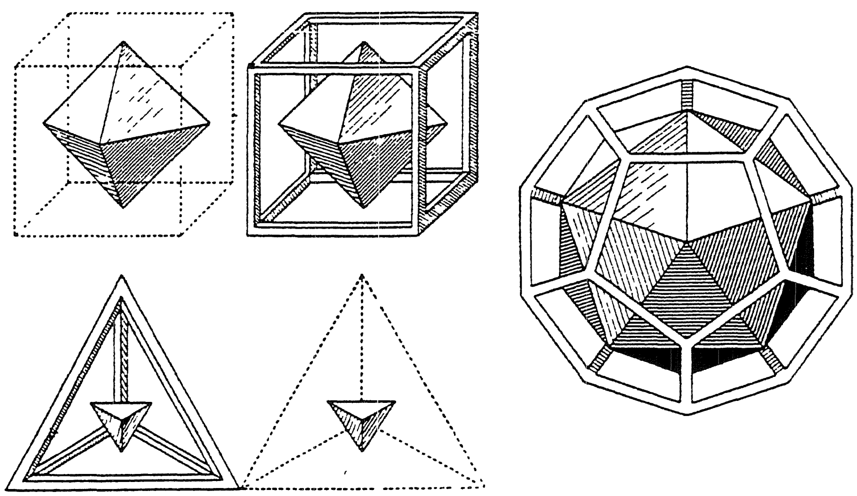


figura secundariarum, cui ideò primus locus interiorum debetur, quippe inscriptile: uti cubo circumscriptili primus exteriorum.

Sunt autem notabilia duo veluti conjugia harum figurarum, ex diversis combinata classibus: Mares, Cubus et Dodecaëdron ex primarijs; foeminae, Octoëdron et Icosiëdron ex secundarijs; quibus accedit una veluti coelebs aut Androgynos, Tetraëdron; quia sibi ipsi inscribitur,

Hic vides Octaëdron inscriptum Cubo; Icosiedron Dodecaèdro, Tetraëdron Tetraèdro.



ut illae foemellae maribus inscribuntur et veluti subjiciuntur, et signa sexus foeminina masculinis opposita habent, angulos scilicet planiciebus.

Praetereà ut Tetraëdron est elementum, viscera et veluti costa Cubi Maris; sic Octaëdron foemina, est elementum et pars Tetraëdri, aliâ ratione: ita mediat Tetraëdron in hoc conjugio.

7) inscriptili

It has been very interesting writing once again on this subject, especially in a language that was just beginning to gain a bit of respect during my lifetime, not least due to the toils of a Mr. Shakespeare. I would very much like to correspond with modern-day mathematicians on related subjects, but since I have been dead and buried for nearly 400 years, and my grave long obliterated, this may prove somewhat cumbersome. One could try writing my esteemed colleague Dr. Geretschläger at the school bearing my name in Graz, and see what happens.

Comments by R. Geretschläger. 1994 marks the 400th anniversary of Kepler's arrival in Graz, where he began his career teaching at the local Lutheran high school (until he was forced to leave a few years later due to the effects of the encroaching counter-reformation). In honor of this, our school is installing a Kepler-"museum" (actually just two rooms) with various objects relating to his life and times and scientific work on display. I am personally involved in building a three meter high mirrored icosahedron, which will symbolize both his work on Platonic solids and the mystic aspects of his writing (or rather those aspects of his world-view we would consider mystic today). Maybe my involvement in this project is the reason for his using me to channel this morsel of his knowledge into the here-and-now.

There is, by the way, another easy way to see why the volume of the *stella octangula* is half the volume of the cube, without referring to the inscribed octahedron. If the cube is cut up into eight smaller cubes, whose edges are of length one-half, the eighth of the *stella octangula* in each cubic part is obtained by slicing off three small right-angled tetrahedra, each of which has three orthogonal edges of length one-half. The volume of each of these is one-sixth the volume of the small cube, and so one-half the volume is removed in total.

Also solved by SAM BAETHGE, Science Academy, Austin, Texas; C. J. BRADLEY, Clifton College, Bristol, U.K.; RICHARD K. GUY, University of Calgary; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; MARCIN E. KUCZMA, Warszawa, Poland; MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, and FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain (three solutions); UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP, Tucson; and the proposer. Two other readers found the volume of the intersection of the tetrahedra rather than the union. There was also one incorrect solution sent in.

For any readers who wish to try "communicating" with Kepler, Dr. Geretschläger's school address is: Keplerstrasse 1, A-8020 Graz, Austria!

By the way, readers will have noted that the first paragraph of the quote from Harmonices Mundi *contains views that are (or should be) no longer taken seriously in today's world, in particular those pertaining to females being "subordinate" to males; with all due respect to the great Kepler (and to use another example of English jargon), he was "out to lunch" in this instance! The editor feels confident that no reader would be offended by the reprinting of a 400-year-old quotation. Still, the point should be made.*

* * * * *

Short articles intended for publication should be sent to Dr. Hanson, contest problem sets and solutions to Olympiad Corner problems should be sent to Dr. Woodrow and other problems and solutions to Dr. Sands.

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