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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

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Carmichael Numbers

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The discovery of ever larger prime numbers gets the headlines these days, but for the professional mathematician such announcements evoke barely a yawn. After all, back in 350 BC the Greek mathematician Euclid proved that there are infinitely many prime numbers, so it comes as no surprise when a new, larger one is found.

(A prime number is one that can be divided only by itself and 1. Thus 2, 3, 5, 7, 11, 13, 17 and 19 are the prime numbers less than 20. Prime numbers become rarer the higher you go, though by Euclid's theorem, they never peter out.)

Far more significant was the discovery earlier this year that there are infinitely many Carmichael numbers. Credit for this discovery goes to three mathematicians at the University of Georgia: Englishman Andrew Granville, formerly of Cambridge University, and Americans Carl Pomerance and Red Alford. Their discovery represents the final nail in the coffin for a putative test for prime numbers that goes back to the great French amateur mathematician Pierre de Fermat.

On 18 October 1640, in a letter to his colleague Frenicle, Fermat noted that if p is any prime number, then for any number n that you choose, p exactly divides $n^p - n$. This was a truly remarkable observation of the 'think of a number' variety. For example, take the prime number 3. Then, pick any number n you like; raise n to the power 3; then subtract n. The answer will always be divisible by 3. Suppose you pick $n = 8$. Raising 8 to the power 3, you get 512. Subtract 8 from 512 to give 504. This number 504 is now divisible by 3.

Or take $n = 35$. Then $35^3 = 42\,875$. Subtract 35 and you get 42 840. Again this number can be divided by 3. Known as Fermat's Little Theorem (to distinguish it from the famous Fermat's Last Theorem, which to this day remains unproved), this fact lies behind many common card tricks where a conjurer will 'predict' a chosen card. To mathematicians, however, the main interest lies in its ability to test for primality. Or to be more precise, to test for non-primality.

Primality testing became big business about a decade ago when a highly secure form of data encryption was developed that makes use of large prime numbers, prime numbers having a hundred or so digits. For a prime number p with a hundred digits, testing to see if p is prime by looking for numbers that divide into p could take billions of centuries, even using the world's fastest computers. So mathematicians use other methods.

A particularly simple method uses Fermat's Little Theorem. To test if a number p is prime, see if p divides into 2^p-2. If it does not, you know that p cannot be prime. But what happens if p does divide 2^p-2? Well, the chances are that p is prime, but you cannot be sure. The problem is that, though p always divides 2^p-2 if p is prime, there are also some non-prime values for p that have this property; for example, 341 does, and yet $341 = 11\times31$, so this number is not prime.

How about trying a different number, say 3, working out 3^p-3? Or 4, or 5, or 6, or 7, and so on; maybe one of these will work? Unfortunately, there are some non-prime numbers p for which, whatever number n you take, n^p-n is divisible by p. The smallest such is 561 ($= 3\times11\times17$). These numbers p are called Carmichael numbers, after the man who, in 1910, found 561 had this property.

From Carmichael's time, the question was, just how many such numbers are there? There are none other than 561 below 1000, and only six more below 10 000. There are just 43 less than a million.

Using high-powered computers, a number of mathematicians pushed the search further and further, with Pinch at Cambridge finding 105 212 Carmichael numbers less than 10^{15} earlier this year. With the discovery of so many numbers of this kind, it soon became clear that Fermat's Little Theorem was clearly not suitable to use as a genuine and reliable test for primality, and in fact the smart money was on there being infinitely many such numbers, just as there are infinitely many prime numbers.

Well, the smart money turned out to be very smart. Using some highly sophisticated mathematical techniques, what the University of Georgia team recently discovered is the following strange seeming fact. There is some number K such that for any number x bigger than K, there are more than x raised to the power $\frac{2}{7}$ Carmichael numbers less than x. Though the Georgia three have no idea how big this number K is, their mathematics guarantees that such a number exists. Then, since you can make the number x raised to the power $\frac{2}{7}$ as big as you please by choosing x big enough, it follows that the number of Carmichael numbers must be infinite.

Which might be the end of the story. But remember, Euclid's 2000-year-old proof that there are infinitely many primes turned out to be just the start of a long and of late expensive computational odyssey.

On a Problem of Thwaites

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The author's main research interest is in geometric inequalities, with a particular interest in the application of optimisation techniques. He also enjoys problem solving, especially in the area of Euclidean geometry.

The following problem, submitted by G. N. Thwaites, appeared as Problem 23.5 in the second issue of Volume 23 of *Mathematical Spectrum*.

> ABC is a triangle and ABC', BCA' and CAB' are equilateral triangles drawn on the sides AB, BC and CA, respectively, of ABC, exterior to ABC. Prove that AA', BB' and CC' are concurrent and of equal length.

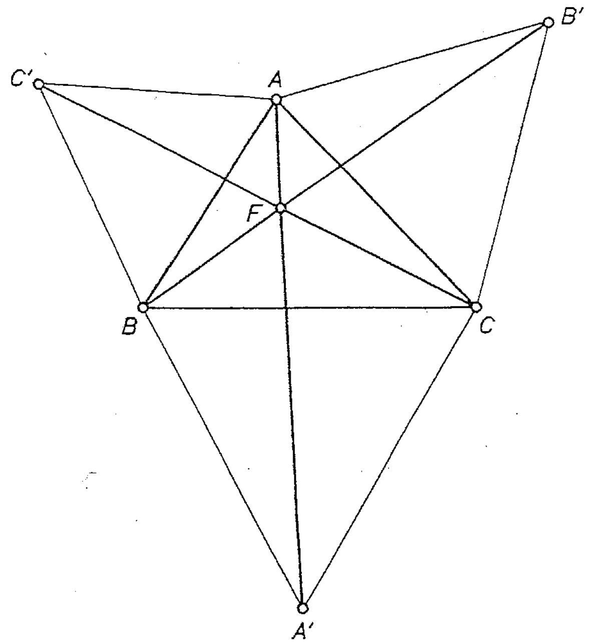

Figure 1

The point in question is the *Fermat point* F of the triangle ABC (reference 2) and is the point P in the plane of ABC such that the sum $AP+BP+CP$ is minimal. One method of proof, for concurrency, uses the theorem of Ceva (reference 2) which is as follows.

> In a triangle ABC, straight-line segments AX, BY and CZ are drawn from the vertices to the opposite sides. Then AX, BY and CZ are concurrent if and only if
>
> $$\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB}=1 \qquad (1)$$
>
> or, alternatively,

$$\frac{\sin\alpha_1}{\sin\alpha_2}\,\frac{\sin\beta_1}{\sin\beta_2}\,\frac{\sin\gamma_1}{\sin\gamma_2} = 1, \tag{2}$$

where $\alpha_1 = \angle XAB$, $\alpha_2 = \angle CAX$, etc.

If we apply the sine law, it is easy to see that, for example,

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{\sin(\beta+60°)}{\sin(\gamma+60°)},$$

where angles ABC and BCA have measures β and γ, respectively. If we write similar quantities for the other vertex angles, equation (2) now becomes

$$\frac{\sin(\beta+60°)}{\sin(\gamma+60°)}\,\frac{\sin(\gamma+60°)}{\sin(\alpha+60°)}\,\frac{\sin(\alpha+60°)}{\sin(\beta+60°)} = 1, \tag{3}$$

where $\angle CAB = \alpha$, and so the lines AA', BB' and CC' are concurrent.

Next we observe that the pair of triangles $C'BC$ and ABA' are congruent and so $AA' = CC'$. Similarly $\Delta AA'C \cong \Delta BCB' \Rightarrow AA' = BB'$ and so $AA' = BB' = CC'$. The congruence property also provides the result

$$\angle CC'B = \angle A'AB, \qquad \angle ABB' = \angle AC'C,$$

from which we obtain

$$\angle A'AB + \angle ABB' = 60°.$$

Consequently $\angle BFA = 120°$. Similarly, $\angle AFC = \angle CFB = 120°$.

A simple but elegant proof that the point F has the aforementioned minimal property comes from an idea of statics.

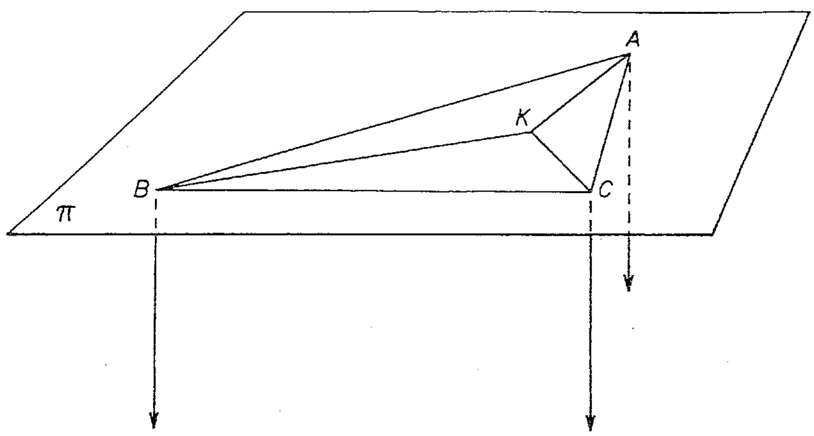

Figure 2

Consider a horizontal plane (table top) with three non-collinear holes A, B and C (see figure 2). Tie together three equal weights and pass one through each of the holes with the knot K above the plane. When the system attains equilibrium, the knot K will be at the Fermat point of the triangle ABC, since, at equilibrium, the three angles at K are equal.

However, at equilibrium, the weights have fallen so as to minimise the potential energy of the weights. Thus the total length of the string below the table is maximised, which results in the sum $AK+BK+CK$ being minimised. Thus the Fermat point has the minimal property claimed. For more details, see reference 3.

If we take a second look at equation (3), it becomes clear that the concurrency of AA', BB' and CC' remains true if 60° is replaced by the more general base angle measure ϕ $(-\frac{1}{2}\pi \leqslant \phi \leqslant \frac{1}{2}\pi)$. Since, also, the measure of ϕ can be negative, the three triangles, which are now similar isosceles, may likewise be constructed *inwardly* on the sides of ABC. This leads to the following question:

What is the locus of P as ϕ varies?

In order to answer this question we need some ideas from projective geometry, in particular the notion of a *cross ratio*. We give an outline only. For more details see for example reference 4.

Definition. The *cross ratio* $R(P_1P_2, P_3P_4)$ of four ordered collinear points P_i whose signed distances from a fixed point O of the line are x_i $(i = 1,2,3,4)$ is given by

$$R(P_1P_2, P_3P_4) = \frac{x_1-x_3}{x_1-x_4} \Big/ \frac{x_2-x_3}{x_2-x_4}$$

(the value ∞ is allowed). Although it has been defined in terms of lengths, the important fact about a cross ratio is that it is invariant under projection so that, in the notation of figure 3, $R(P_1P_2, P_3P_4) = R(P_1'P_2', P_3'P_4')$.

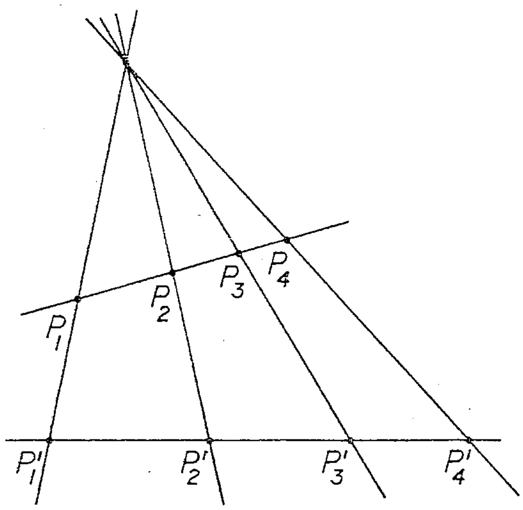

Figure 3

This invariance allows us to define the cross ratio of four concurrent lines (i.e. four lines of a *pencil*) to be equal to the cross ratio cut on any intersecting line. A fundamental theorem of projective geometry states that:

If there exists a one-to-one correspondence between the lines of two pencils which preserves cross ratios, then the locus of the point of intersection of corresponding lines is a conic passing through the vertices of the pencils.

In order to apply this theorem to our situation we refer to figure 4 and simply notice that, by the similarity of the appropriate right-angled triangles,

$$R(A'_{\phi_1}A'_{\phi_2}, A'_{\phi_3}A'_{\phi_4}) = R(B'_{\phi_1}B'_{\phi_2}, B'_{\phi_3}B'_{\phi_4}).$$

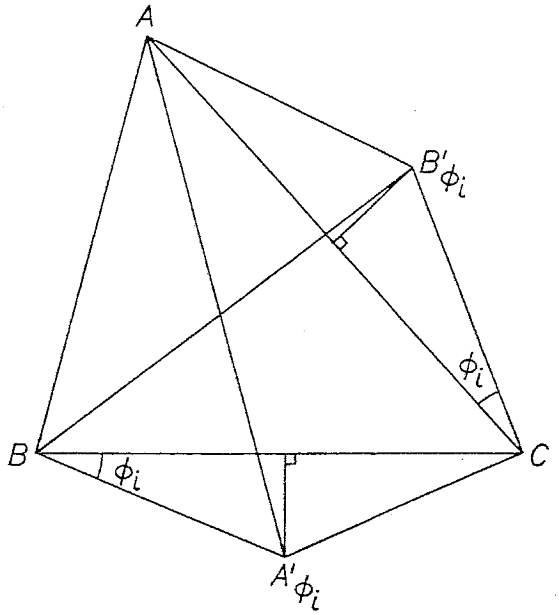

Figure 4

Therefore, as ϕ varies continuously, the intersection of the lines AA'_{ϕ} (through A) and BB'_{ϕ} (through B) traces out a conic passing through A and B.

This conic is, in fact, a hyperbola and has been called *Kiepert's* hyperbola (reference 1) after Ludwig Kiepert (born 1846 in Breslau, died 1934 in Hanover) who discovered it in 1869 while still a student (of Weierstrass) at the University of Berlin. However, it seems that Kiepert did not pursue geometry. History tells us that he did some postgraduate work in the theory of elliptic functions, after which he turned his hand to actuarial mathematics. He also became involved in university administration, having been appointed Dean at the University of Hanover in 1901. He continued there until his retirement in 1921, at which time he was appointed 'Professor Emeritus'. For more information on this interesting mathematician, see reference 5.

Kiepert's hyperbola received considerable attention for the next fifty or so years, but then it seemed to disappear from the geometry of the triangle. This is something of a pity, since this conic has many interesting properties; in particular, it passes through a host of interesting points. Some of the more familiar ones include: the vertex A ($\phi = -\alpha$, if $\alpha \leqslant 90°$, or $180° - \alpha$ if $\alpha > 90°$), vertices B ($\phi = -\beta$) and C ($\phi = -\gamma$), the centroid M ($\phi = 0$), the Fermat point ($\phi = 60°$) and the orthocentre H ($\phi = \pm 90°$). Another interesting case occurs when $\phi = 30°$. In this instance, the triangle $A'B'C'$ is equilateral—a result which is generally attributed to Napoleon Bonaparte, see reference 2. To obtain a rough picture of the hyperbola, assume ABC to be acute angled with $\alpha > \beta > \gamma$. Then one branch contains the points A, C, M, F and H, whereas the other branch passes through the vertex B.

The story does not end there. By Desargues's theorem on triangles in perspective (reference 2), there corresponds to the locus of points $P = AA' \cap BB' \cap CC'$ an envelope of lines

$$p = (AB \cap A'B') \cup (BC \cap B'C') \cup (CA \cap C'A').$$

Since this correspondence is one-to-one, intuitively one can imagine that, as the point P traces out a conic, the corresponding lines p are tangents to a second conic. This is exactly the case; and the conic is in fact a parabola, known as *Kiepert's parabola*. The latter also has interesting features. That it is a parabola is clear from the fact that, when $\phi = 0$, i.e. A', B' and C' are the midpoints, respectively, of the sides BC, CA and AB, the corresponding tangent is precisely the line at infinity. Furthermore it is tangent to all three sides of ABC. To see this, consider the case when, for example, the points A, A' and B' are collinear. It is easy to see that such a condition enables one to show that side AC is a tangent. However, a more difficult exercise is to determine the value (actually there are two values) of ϕ giving this condition. The directrix of the parabola is also interesting. It turns out to be the well-known *Euler line* of ABC, which contains the aforementioned points M and H along with O and N, the centres of the circumcircle and the nine-point circle, respectively, of the given triangle. A good account of both conics is given in the excellent book by Casey, reference 1.

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Bernoulli Numbers

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The author obtained an M.Sc. at the University of Glasgow, after which he was a research assistant in the Department of Computer Science at the University of Strathclyde. He is now an analyst and programmer in the Computer Services Department of Strathclyde Region. His main mathematical interest is in number theory.

In his article 'Patterns and Primes in Bernoulli's Triangle' in *Mathematical Spectrum* Volume 23, Number 4, A. W. F. Edwards defines Bernoulli numbers ${}^nB_r$ for $0 \leqslant r \leqslant n$ as:

$$ {}^nB_r = \sum_{i=0}^{r} {}^nC_i, \quad \text{where } {}^nC_i = \binom{n}{i} = \frac{n!}{(n-i)!\,i!}, $$

from Pascal's triangle. Edwards showed that ${}^nB_r$ (r odd, $r > 1$) is composite for all n such that $n+1 > r!$. It is then easy to check that there are no primes with $n = 3$ or 5. In his proof Edwards showed that

$$ {}^nB_r = \frac{n+1}{r!}\left(r! + \sum_{\substack{i=3\\ i \text{ odd}}}^{r} \frac{r!}{i!} n(n-1)\dots(n-i+2) \right) $$
$$ = \frac{XY}{Z}, $$

with $X = n+1$, $Z = r!$ and Y equal to the term in large brackets. Note that $Y > r!$. Cancelling out $\gcd(X, Z)$ gives xY/z, where $\gcd(x, z) = 1$. Then $Y > r! \geqslant z$ and so $Y/z > 1$ is a non-trivial factor. Thus, if $x \neq 1$, we have a non-trivial factorization of ${}^nB_r$. Therefore, ${}^nB_r$ can be a prime only when $n+1$ divides $r!$. This is a stronger result than that of Edwards.

For $r = 7$, $7! = 5040 = 2^4 \times 3^2 \times 5 \times 7$, which has 60 possible divisors, some of which (namely 1, 2, 3, 4, 5, 6 and 7) can be discounted because $n \geqslant r$. It is then possible to verify that there are no prime Bernoulli numbers for $r = 7$.

We can restrict the choices for n further by the following result.

Proposition. If n and r are both odd then ${}^nB_r$ is even.

Proof.

$$ {}^nB_r = \sum_{i=0}^{r} \binom{n}{i} = \left\{\binom{n}{0} + \binom{n}{1}\right\} + \left\{\binom{n}{2} + \binom{n}{3}\right\} + \dots + \left\{\binom{n}{r-1} + \binom{n}{r}\right\}. $$

We pair off consecutive values and use the fact that

$$\binom{n}{2i}+\binom{n}{2i+1}=\binom{n+1}{2i+1}=\binom{2m}{2i+1}$$

for some m. It is enough to show that this is even to prove the result.

The result is trivially true for $2m = 2$ and 4, and we use induction on m. If $i = m-1$ or 0, then

$$\binom{2m}{2m-1}=\binom{2m}{1}=2m,$$

which is even. If $0 < i < m-1$, then

$$\begin{aligned}\binom{2m}{2i+1} &= \binom{2m-1}{2i}+\binom{2m-1}{2i+1}\\ &= \binom{2m-2}{2i-1}+\binom{2m-2}{2i}+\binom{2m-2}{2i}+\binom{2m-2}{2i+1}.\end{aligned}$$

The middle two terms obviously add up to an even number. By induction, the outer two are also even, and so the whole expression is even.

This result means that, instead of looking at all possible divisors of $r!$ to locate possible n, we only have to look at those for which n is even, i.e. $n+1$ is odd, and so we can ignore all even divisors of $r!$. With this restriction, the number of possibilities for $r = 7$ reduces from 53 to 8, for $r = 9$ reduces from 151 to 15 and for $r = 11$ reduces from 529 to 54.

Using this method, I have verified that for $r = 9$ there is only one occurrence of a prime Bernoulli number, namely for $n = 314$, and that for $r = 11$ no possible value of n produces a prime except possibly $n = 155\,924$, which is too large for me to check. For higher r, possible values for n pass beyond my ability to check completely. However, I have found the following prime Bernoulli numbers:

$$^{62}B_{13},\ ^{64}B_{13}; \qquad ^{62}B_{17},\ ^{64}B_{17},\ ^{90}B_{17},\ ^{1364}B_{17},\ ^{1682}B_{17};$$

$$^{56}B_{19},\ ^{152}B_{19},\ ^{1728}B_{19},\ ^{9944}B_{19}; \qquad ^{246}B_{21}; \qquad ^{56}B_{23},\ ^{64}B_{23},\ ^{1784}B_{23};$$

$$^{64}B_{25},\ ^{90}B_{25},\ ^{220}B_{25},\ ^{224}B_{25},\ ^{1274}B_{25},\ ^{2430}B_{25},\ ^{57\,056}B_{25};$$

$$^{1310}B_{29},\ ^{3552}B_{29}$$

with a search limit of $n = 65\,535$. No primes were found in this range for $r = 15$, 27 or 31. The last of the above list has 94 digits. It is possible to proceed much further. Suffice it to say that, as r increases, there are more values of n to check, even over a restricted range, which may or may not lead to an increased frequency of primes.

If we approach from another angle, for a given value of n we can search exhaustively through all odd $r < n$ such that the largest prime factor of $n+1$ is less than or equal to r. Note that, if $n+1$ is prime, there cannot be any solutions, since we require $r \leqslant n$. The smallest n for which there exists a prime Bernoulli number is $n = 56$, with $r = 19$ or 23. Of all even $n \leqslant 256$, there is a total of 31 prime Bernoulli numbers with odd r. For even r, there is a much greater frequency.

As an aside, I have also found the 'adjacent triples'

$$^{13}B_{12},\ ^{14}B_{12},\ ^{15}B_{12}; \qquad ^{147}B_{16},\ ^{148}B_{16},\ ^{149}B_{16}; \qquad ^{57}B_{24},\ ^{58}B_{24},\ ^{59}B_{24}$$

to be prime, as well as many adjacent pairs.

Cyclic Quadrilaterals

J. H. LITTLEWOOD

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In any cyclic quadrilateral in which circles have been drawn, each having one of the diagonals and two of the sides as tangents, then

(a) the circle centres are at the corners of a rectangle;
(b) each of the sides of the quadrilateral is parallel to one of the common tangents.

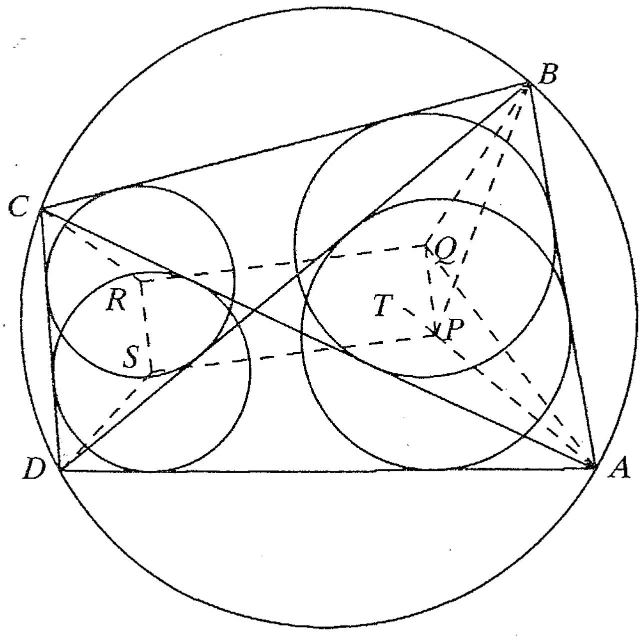

Figure 1

In figure 1, $ABCD$ is cyclic and P, Q, R and S are the circle centres. Broken lines join the circle centres to the corners of the quadrilateral and to each other. Also AP is produced to T and A is joined to Q. The first step is to show that $PABQ$ is cyclic, and this will be done by showing that the angles PAQ and PBQ are equal. Thus:

$$PAQ = PAB - QAB = \tfrac{1}{2}(BAD - BAC) = \tfrac{1}{2}CAD,$$

$$PBQ = QBA - PBA = \tfrac{1}{2}(ABC - ABD) = \tfrac{1}{2}CBD = \tfrac{1}{2}CAD.$$

Hence $PABQ$ is cyclic, and so are $QBCR$, $RCDS$ and $SDAP$.

The exterior angle TPQ of the cyclic quadrilateral $PABQ$ is equal to the interior opposite angle $ABQ = \frac{1}{2}ABC$. Similarly angle $TPS = \frac{1}{2}ADC$, so that $QPS = \frac{1}{2}ABC + \frac{1}{2}ADC = 90°$. Similarly the angles at Q, R and S are also 90°. Hence $PQRS$ is a rectangle.

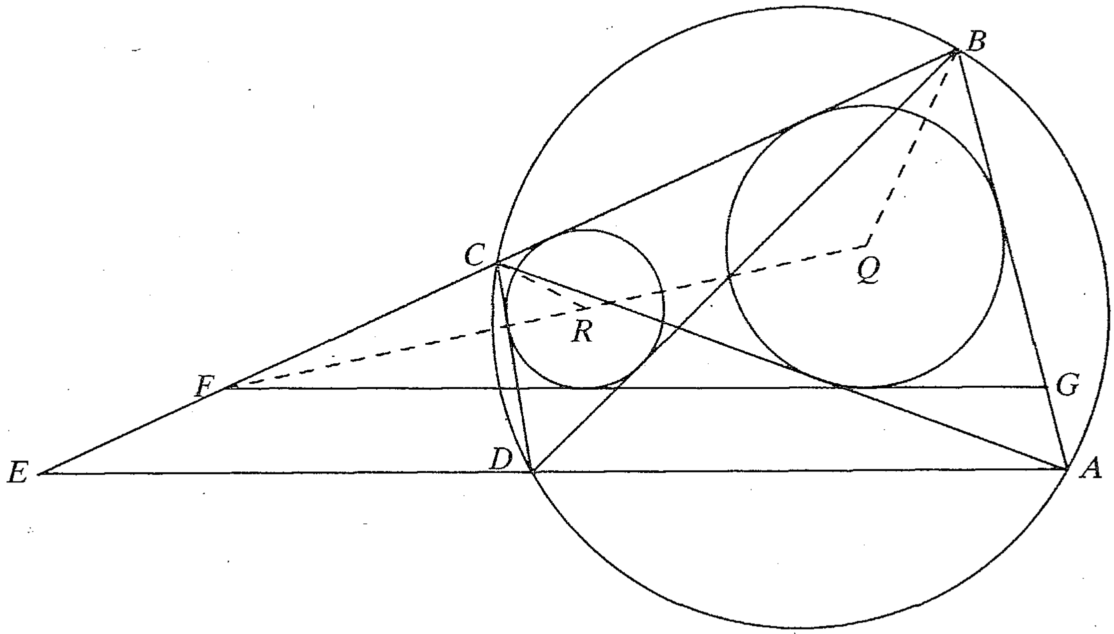

Figure 2

Figure 2 is similar to figure 1, but with only two circles (centres Q and R). Opposite sides AD and BC are produced to meet at E; F is the meeting point of the common tangents FG and FB and the line of centres FRQ, which bisects the angle BFG. Considering the cyclic quadrilateral $CRQB$, the exterior angle CRF is equal to the interior opposite angle $QBC = \frac{1}{2}B$. Angle $BCR = \frac{1}{2}C$. Hence angle $BFR = \frac{1}{2}(C-B)$, and therefore angle $BFG = C-B$. But angle BEA is also equal to $C-B$. Thus FG is parallel to EA. Similarly it may be shown that each of the other three sides of the quadrilateral is parallel to one of the common tangents.

APR Made Difficult

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In some GCSE syllabuses, students are required to estimate the annual percentage rate (APR) from a flat annual percentage for a loan taken out over a number of years that is repaid monthly. APR is the annual interest rate that would be charged for a loan when the sum was paid back at the end of the period of the loan. When paying back by instalments, the outstanding debt is reduced each month so that the flat rate of interest is somewhat less than the true APR.

Example

The cash price for a new car is £8000. Sara decides to buy it on credit terms which are 30% deposit and 36 monthly repayments of £205. Find the flat interest rate per annum of the loan and use this answer to estimate the true annual rate (APR).

Solution

The amount borrowed is $0.7\times£8000 = £5600$. The total credit price $= 0.3\times£8000+36\times£205 = £9780$. So the extra amount paid for the credit is $£9780-£8000 = £1780$. Expressing this extra charge as a percentage of the amount borrowed, we obtain

$$\frac{1780\times100}{5600} = 31.8\%$$

to three significant figures. Dividing this answer by 3, we obtain the flat interest rate per annum of the loan, i.e. $31.8/3 = 10.6\%$.

To estimate the APR from this figure, examiners recommend multiplying by some number in the range 1.8 to 2.0. This would produce an APR in the range 19.08% to 21.2%.

I regard this as very unsatisfactory without some justification for this method. To produce a justification, it would seem necessary to develop a way of calculating APR 'exactly' and then see if approximations can be made to obtain the procedure described above.

Finding an exact formula for APR

Let us define the following variables.

C = sum borrowed,
P = monthly repayment,
n = number of years of repayment,
D_k = debt at end of kth month.

$100A$ = annual percentage rate,
$100F$ = flat percentage rate,
$100M$ = monthly percentage rate,

(The rates A, M and F are decimals and so will need to be multiplied by 100 to be converted into percentages.)

The flat percentage rate is found by dividing the total interest paid by the sum borrowed and averaging over the number of years of repayment. Since $12n$ monthly repayments are made over n years, the total interest paid is $12nP-C$. The interest paid over n years (expressed as a proportion of the cost price) is therefore

$$\frac{12nP-C}{C}.$$

Dividing by n we obtain the flat percentage rate, i.e.

$$F = \frac{12nP-C}{nC}. \tag{1}$$

Now interest is charged on a monthly basis using the monthly percentage which is related to the APR by

$$(1+M)^{12} = 1+A. \tag{2}$$

Now the debt at the end of the kth month is given by the difference equation

$$\begin{aligned} D_{k+1} &= D_k + MD_k - P \\ &= (1+M)D_k - P, \end{aligned}$$

with $D_0 = C$. Applying this equation for $k = 0, 1, 2, \ldots, 12n$ gives the following set of equations:

$$\begin{aligned} D_1 &= (1+M)C-P \\ D_2 &= (1+M)^2C-P\{1+(1+M)\} \\ D_3 &= (1+M)^3C-P\{1+(1+M)+(1+M)^2\} \\ \ldots &= \ldots \qquad \ldots \qquad \ldots \\ D_{12n} &= (1+M)^{12n}C-P\{1+(1+M)+(1+M)^2+\ldots+(1+M)^{12n-1}\}. \end{aligned}$$

Since there is no debt at the end of the period of repayment, $D_{12n} = 0$ and so the last equation becomes

$$(1+M)^{12n}C = P\{1+(1+M)+(1+M)^2+\ldots+(1+M)^{12n-1}\}.$$

The right-hand side contains a geometric series which can be summed and so

$$(1+M)^{12n}C = \frac{P\{(1+M)^{12n}-1\}}{M}.$$

Hence

$$P = \frac{MC}{1-(1+M)^{-12n}}. \tag{3}$$

From (1),

$$P = \frac{C(nF+1)}{12n}.$$

Substituting in (3) and rearranging for F, we obtain

$$F = \frac{12M}{1-(1+M)^{-12n}} - \frac{1}{n}.$$

Also, from (2), $M = (1+A)^{1/12}-1$ and so

$$F = \frac{12\{(1+A)^{1/12}-1\}}{1-(1+A)^{-n}} - \frac{1}{n}. \tag{4}$$

This is the exact formula relating F, A and n. The problem is that we need to calculate A given F and n, but A cannot be made the subject of the formula. However, we can rearrange the formula to obtain the following iterative formula:

$$A = \left\{\frac{1}{12}\left(F+\frac{1}{n}\right)\{1-(1+A)^{-n}\}+1\right\}^{12}-1,$$

which appears to converge when the first estimate of A is set to $2F$. Perhaps the reader may care to prove this.

The following program for the CASIO fx-7000G will calculate A using this iterative formula:

```
"FLAT RATE"?->F:"NO OF YEARS"?->N:2F->A:Lbl 1:
((F+1÷N)(1-(1+A)xʸ(-N))÷12+1)xʸ12-1->A
Goto 1
```

This program was used to complete the following table for A.

		n				
		1	2	3	4	5
	5%	9.49	9.73	9.72	9.64	9.55
	10%	19.5	19.7	19.5	19.1	18.7
$100F\%$	15%	30.1	30.1	29.3	28.5	27.7
	20%	41.3	40.7	39.3	37.9	36.6
	25%	53.1	51.7	49.4	47.3	45.5

Linear approximation of the formula

Using the binomial theorem we can write

$$\begin{aligned}12\{(1+A)^{1/12}-1\} &= 12(1+\tfrac{1}{12}A-\tfrac{11}{288}A^2+\ldots-1)\\ &= A-\tfrac{11}{24}A^2,\end{aligned}$$

ignoring terms in A^3 and higher. Also,

$$\begin{aligned}\frac{1}{1-(1+A)^{-n}} &= \frac{1}{1-(1-nA+\frac{1}{2}n(n+1)A^2)}\\ &= \frac{1}{nA}\{1+\tfrac{1}{2}(n+1)A+\text{terms of higher degree in } A\}.\end{aligned}$$

So

$$\begin{aligned}F &= \frac{1}{nA}\{1+\tfrac{1}{2}(n+1)A\}(A-\tfrac{11}{24}A^2)-\frac{1}{n}\\ &= \frac{12n+1}{24n}A,\end{aligned}$$

ignoring terms in A^2 and higher. So

$$\frac{A}{F}\approx\frac{24n}{12n+1}=\begin{cases}1.846 & (\text{if } n=1),\\ 1.92 & (\text{if } n=2),\\ 1.946 & (\text{if } n=3).\end{cases}$$

This shows that, if A is small, A can be estimated by multiplying F by some number in the range 1.8 to 2, which justifies the method mentioned at the beginning of the article. A better estimate could be obtained by forming a quadratic approximation of the formula and solving for A by use of the quadratic formula. This is left as an exercise for the reader.

Cylindrical and toroidal noughts and crosses

1	2	3
4	5	6
7	8	9

Cylindrical noughts and crosses is the ordinary game but with the board wrapped around a cylinder. If we join the vertical sides, then this means that 2–6–7, 3–4–8, 2–4–9 and 1–6–8 are also winning lines. Can one player always win? Can there be any tie games? What happens when the board is wrapped around a torus?

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Figured Tours

GEORGE JELLISS

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In the first ever mathematical paper on Knight's tours (reference 1), Leonhard Euler gave (among much else) the eight geometrically distinct symmetric Knight's tours of the 5×5 board. Figure 1 (a) shows the first of these tours. The numbers are assigned to the successive squares visited by a chess Knight in covering the board in a single path. Euler did not mention the fact that the numbers along one diagonal in this tour happen to form an arithmetical progression with common difference 6. This is the only one of the eight tours to have this property which makes it a 'figured tour'. An arithmetical progression from 1 to n^2 with common difference $n+1$ has n members for any n, so a figured tour with numbers $1, n+2, 2n+3, 3n+4, \ldots, n^2$ can be considered on any size of square board. Figures 1 (b) and 1 (c) show similar symmetric tours I have constructed for $n = 7$ and 9.

23	18	5	10	**25**
6	11	24	**19**	14
17	22	**13**	4	9
12	**7**	2	15	20
1	16	21	8	3

(a)

23	38	47	40	35	32	**49**
46	7	22	37	48	**41**	34
21	24	39	6	**33**	36	31
8	45	20	**25**	30	5	42
19	14	**17**	44	11	26	29
16	**9**	2	13	28	43	4
1	18	15	10	3	12	27

(b)

77	46	27	16	79	72	49	60	**81**
14	17	78	47	28	59	80	**71**	62
45	76	15	26	73	48	**61**	50	39
18	13	74	29	58	**51**	40	63	70
75	44	25	52	**41**	30	57	38	7
12	19	42	**31**	24	53	8	69	64
43	32	**21**	34	9	56	67	6	37
20	**11**	2	23	54	35	4	65	68
1	22	33	10	3	66	55	36	5

(c)

Figure 1

For n even the numbers have to be arranged along a row or column instead of diagonally, since they are alternately odd and even. No solution is possible, however, in the 6×6 case, even if we drop the requirement for the numbers to be in order of magnitude. The 8×8 (standard chessboard) case was solved over 50 years ago by T. R. Dawson (reference 2) as shown in figure 2 (a). The 10×10 case is easy to do—try it as an exercise. Other arithmetical progressions can also be used to make figured tours. Figure 2 (b) by S. H. Hall and T. R. Dawson (reference 3) uses the multiples of 7. Figure 2 (c), one of my own (reference 4), shows the multiples of 8. These two tours differ from the previous examples in being closed tours (i.e. the squares numbered 1 and 64 are a Knight's move apart).

30	17	12	15	34	53	48	51
11	14	31	36	47	50	33	54
18	29	16	13	32	35	52	49
1	**10**	**19**	**28**	**37**	**46**	**55**	**64**
20	27	2	9	56	63	38	45
5	8	23	26	41	44	57	60
24	21	6	3	62	59	42	39
7	4	25	22	43	40	61	58

(a)

26	5	34	37	24	55	8	39
33	36	25	6	43	38	23	54
4	27	44	**35**	**56**	**7**	40	9
45	32	57	**28**	**63**	**42**	53	22
58	3	62	**49**	**14**	**21**	10	41
31	46	29	64	61	18	13	52
2	59	48	15	50	11	20	17
47	30	1	60	19	16	51	12

(b)

45	20	43	60	47	58	37	**40**
2	61	46	19	38	41	**48**	57
21	44	1	42	59	**56**	39	36
62	3	18	23	**64**	31	54	49
17	22	63	**32**	55	50	35	30
4	7	**24**	27	10	33	12	53
25	**16**	9	6	51	14	29	34
8	5	26	15	28	11	52	13

(c)

Figure 2

Any symmetrical tour has numerical properties corresponding to this geometrical property. For example, in the symmetric open tours of figure 1 the diametrically opposite pairs of numbers add to a constant value n^2+1, while in the symmetric closed tour (figure 2(c)) they differ by a constant value $\frac{1}{2}n^2$. Properties of this type, affecting all numbers, do not make it a 'figured tour'. I reserve this term for tours in which certain highlighted numbers are arranged in a pattern. If the tour can at the same time be symmetrical, this is a bonus. The name is appropriate since it combines in one concept both senses of the ambiguous term 'figure', which can mean numerical symbol or geometrical shape.

The numbers chosen to feature in a figured tour do not have to be in arithmetical progression. Another sequence of numbers that has n members for any n is that of the squares $1, 4, 9, 16, \ldots, n^2$. The problem of finding a Knight's tour with these numbers forming a straight line was first proposed by G. E. Carpenter and solved by S. Hertzsprung (reference 5) in 1881. A solution to this problem for the case of the 6×6 board (reference 4) is shown in figure 3(a). The convoluted path of the Knight here is, perhaps surprisingly, uniquely determined by the numbers along the first row. Some tour problems have very many solutions. A case such as this where the whole route is determined is particularly notable.

It intrigues me to wonder how Carpenter got this idea. Did it come to him out of the blue, or was it inspired by some result published in a mathematical paper? Logically one would expect it to have evolved from some simpler idea. For example, in the trivial Rook tour of a 2×2 board the square numbers, 1 and 4, are in the same row or column. This might have suggested to someone to try to get a similar result on a larger board. If so, they would have found the uniquely defined 6×6 single-step Rook tour shown in figure 3(b). But as far as I know this was not discovered until more than 100 years later! Similar Rook tours are possible on square boards of side $4k+2$ for any non-negative integer k (try the next case, 10×10, for yourself) (see reference 4). Figure 3 (c) shows that a Rook can easily accomplish the 6×6 task mentioned above that the normally more manoeuvrable Knight fails at. This is also uniquely determined. Figure 3 (d) is a single-step Rook tour with the square numbers in a Knight-path, to compare with figure 3 (a) which is a Knight tour with the squares in a Rook-path.

12	19	22	33	28	31
21	6	13	30	23	34
18	11	20	27	32	29
5	2	7	14	35	24
10	17	26	3	8	15
1	**4**	**9**	**16**	**25**	**36**

(a)

18	19	20	21	22	23
17	8	7	6	5	24
16	**9**	**36**	**1**	**4**	**25**
15	10	35	2	3	26
14	11	34	31	30	27
13	12	33	32	29	28

(b)

4	5	18	19	32	33
3	6	17	20	31	34
2	7	16	21	30	35
1	**8**	**15**	**22**	**29**	**36**
10	9	14	23	28	27
11	12	13	24	25	26

(c)

2	**1**	10	11	20	21
3	8	**9**	12	19	22
4	7	14	13	18	23
5	6	15	**16**	17	24
32	31	30	29	28	**25**
33	34	35	**36**	27	26

(d)

Figure 3

The term 'figured tour' was first used by T. R. Dawson in connection with his Knight's tours showing the square numbers in closed, symmetrical circuits of Knight moves (reference 6). Dawson constructed a collection of 100 of these tours, showing all possible circuits of this type. (Actually there are 106 such circuits, but three require a 9-row board and three others cannot be shown in a Knight's tour.) Examples by Dawson are given in figure 4.

8	55	6	33	10	15	44	31
5	60	**9**	56	43	32	11	14
54	7	34	59	**16**	13	30	45
61	**4**	17	42	57	46	**25**	12
18	53	58	35	24	29	40	47
1	62	3	50	41	**36**	23	26
52	19	**64**	37	28	21	48	39
63	2	51	20	**49**	38	27	22

(a)

52	11	38	59	40	13	46	61
37	58	51	12	45	60	41	14
10	53	**64**	39	50	43	62	47
57	**36**	**9**	44	63	48	15	42
54	**1**	56	**49**	**16**	29	22	31
35	8	**25**	**4**	23	32	19	28
2	55	6	33	26	17	30	21
7	34	3	24	5	20	27	18

(b)

32	3	44	57	30	59	46	11
43	56	31	34	45	10	29	60
2	33	**4**	55	58	35	12	47
53	42	23	**64**	**9**	48	61	28
22	**1**	54	5	24	63	**36**	13
41	52	19	**16**	**49**	8	27	62
18	21	50	39	6	**25**	14	37
51	40	17	20	15	38	7	26

(c)

Figure 4

Although the squares have proved to be the most popular set of numbers to be placed in formation, many others can be considered. Figure 5 shows some examples: (a) the triangular numbers in a triangle, by Dawson (reference 2), (b) the squares *and* the cubes (1, 8, 27, 64) in symmetrical Knight circuits, by F. Hansson (reference 7) and (c) the odd primes forming a rectangle, by Hall and Dawson (reference 3).

4	11	56	9	54	51	34	47
57	8	**3**	12	35	48	53	50
2	5	**10**	**55**	52	13	46	33
7	58	**1**	**36**	**45**	20	49	14
64	43	**6**	**21**	**28**	**15**	32	19
59	40	61	44	37	18	27	24
42	63	38	29	22	25	16	31
39	60	41	62	17	30	23	26

(a)

5	24	**9**	54	7	56	21	18
10	53	6	23	**16**	19	28	57
37	**4**	**25**	**8**	55	22	17	20
52	11	38	15	40	**27**	58	29
3	**36**	**1**	26	59	30	41	46
12	51	14	39	**64**	45	60	31
35	2	**49**	44	33	62	47	42
50	13	34	63	48	43	32	61

(b)

50	15	62	27	60	**17**	42	39
63	28	49	16	**41**	38	**59**	18
14	51	26	**61**	48	**19**	40	**43**
25	64	**29**	12	**37**	44	**31**	58
52	**13**	24	**7**	30	**47**	20	45
1	8	**3**	36	**11**	34	57	32
4	**53**	10	**23**	6	55	46	21
9	2	**5**	54	35	22	33	56

(c)

Figure 5

Finally, here are some more original results. In figure 6 (a) the squares are in a closed path of (0, 1) moves, while in figure 6 (b) they follow a closed path of (2, 3) moves. You may have noticed how often pairs of repeated digit numbers like 22, 33, or pairs that are reversals of each other, like 23, 32, occur on adjacent squares in tours. Do they really occur more often than one would expect from chance—or is it just that we notice such coincidences? In the tour of figure 6 (c) I have deliberately set out to have as many such coincidences as possible in the central area. If you draw the geometrical diagram for this tour you will find it an attractive, near-symmetric pattern.

62	59	42	53	38	57	44	47
41	52	61	58	43	46	37	56
60	63	50	39	54	23	48	45
51	40	**1**	**64**	**49**	**36**	55	24
2	31	**4**	**9**	**16**	**25**	22	35
13	10	15	32	21	8	17	26
30	3	12	5	28	19	34	7
11	14	29	20	33	6	27	18

(a)

1	58	51	32	63	18	**49**	30
52	33	56	59	50	31	62	19
57	2	35	**64**	17	60	29	48
34	53	**4**	55	**36**	15	20	61
3	8	37	**16**	41	24	47	28
38	5	54	11	14	27	44	21
9	12	7	40	23	42	**25**	46
6	39	10	13	26	45	22	43

(b)

38	59	40	57	36	9	26	29
41	56	37	10	31	28	35	8
60	39	58	**43**	**34**	25	30	27
55	42	**45**	**22**	**11**	**32**	7	24
46	61	**54**	**33**	**44**	**23**	20	1
49	52	47	**12**	**21**	4	17	6
62	13	50	53	64	15	2	19
51	48	63	14	3	18	5	16

(c)

Figure 6

The construction of these figured tours provides an interesting mathematical recreation. Some configurations are quite easy to do, or prove impossible, others very difficult. All the examples given here were found 'by hand', without the assistance of a computer. The art lies in selecting the numbers to be displayed and the formations in which to arrange them. I hope readers will be able to compose some figured tours of their own devising. I should be interested to hear of any results for possible inclusion in a book on tours that I am compiling.

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Do You Say Ar or Arc?

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For the inverse function $\sin^{-1}$ a common notation is arcsin, whereas for the inverse function $\sinh^{-1}$ some texts use the notation arcsinh and others arsinh. Let us look for reasons to decide which to use.

With trigonometrical functions, consider the point $(x, y) = (\cos\theta, \sin\theta)$ on the unit circle $x^2+y^2 = 1$, with θ measured in radians (see figure 1). Then θ is the arc length from $(1, 0)$ to (x, y) and $\theta = \arccos x = \arcsin y$. Hence there is logic in using the prefix arc.

With hyperbolic functions, consider the point $P = (X, Y) = (\cosh T, \sinh T)$ on the hyperbola $y^2 = x^2-1$. Let Q be the point $(X, -Y)$ and let F be the point where the line PQ crosses the x-axis. Let us join P to the origin O with a straight line and Q to O with a straight line and consider the area of the region $OPRQ$ (where R is the point $(1, 0)$), which is shaded in figure 2. The area of the triangle OPF is

$$\tfrac{1}{2}\cosh T \sinh T.$$

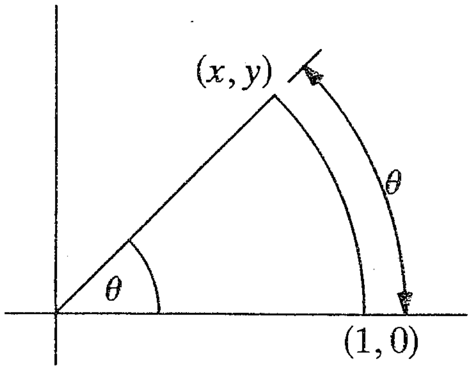

Figure 1

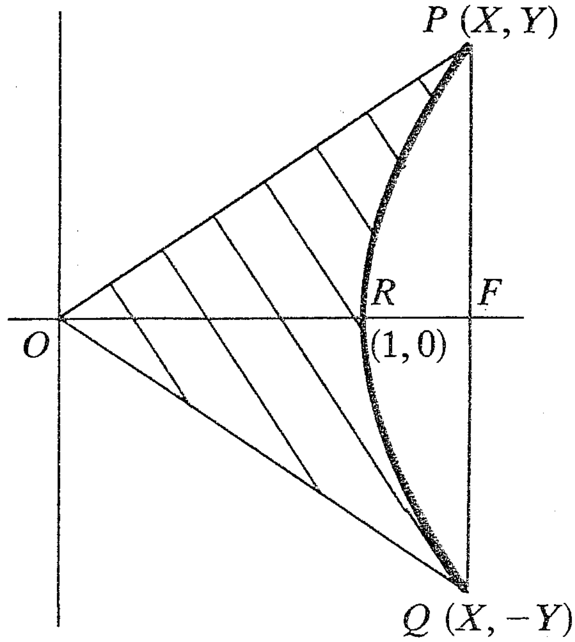

Figure 2

The area under the hyperbola from R to F is

$$\int_1^X \sqrt{x^2-1}\ \mathrm{d}x.$$

Using the substitution $x = \cosh t$, this is

$$\begin{aligned}\int_0^T \sinh^2 t\ \mathrm{d}t &= \tfrac{1}{2}\int_0^T (\cosh 2t - 1)\ \mathrm{d}t \\ &= \tfrac{1}{2}\left[\tfrac{1}{2}\sinh 2t - t\right]_0^T \\ &= \tfrac{1}{4}\sinh 2T - \tfrac{1}{2}T \\ &= \tfrac{1}{2}\cosh T \sinh T - \tfrac{1}{2}T.\end{aligned}$$

Hence the area of the whole shaded region $OPRQ$ is T, where

$$T = \operatorname{arcosh} X = \operatorname{arsinh} Y.$$

So, if we think of the prefix ar as being short for the word area, there is logic in using ar to describe inverse hyperbolic functions.

In both trigonometrical and hyperbolic cases we also have a picture of what the inverse functions represent.

A ticket machine accepts £1, 20p and 10p coins, but does not give change. In how many different ways can a passenger pay exactly £1.20 for a ticket?

The Effect of Dissipation on Flight Times

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When someone takes a catch in a cricket match it is usual for them to throw the ball up in the air again and, in order not to look silly, catch it again! For those unfamiliar with this game and its rather quaint customs, this behaviour is usually a result of the relief and pleasure that one feels at such times, together with the desire to demonstrate to all those watching, especially the opposition, that the catch was actually made. It is only the mathematician who would contemplate thinking any more of this event. Being a mathematician *and* a cricket enthusiast, I cannot help but read more into this than meets the eye! And so … .

It is well known that, if a ball is projected vertically upwards with speed U in a uniform gravitational field (constant gravity), then the total time of flight T is given by

$$T = \frac{2U}{g}, \tag{1}$$

where g is the acceleration due to gravity and we have assumed no air resistance. Since the speed of return is the same as that of projection, we could interpret (1) as

$$T = \frac{\text{initial speed} + \text{final speed}}{g}. \tag{2}$$

Unfortunately, in practice there will always be some air resistance (as all cricketing mathematicians know), and (1) no longer holds. An interesting problem worth considering, however, is to see whether (2) does still hold in this more general case. If so, then the (actual) total time elapsed will be less than that predicted by assuming no resistance. This is because a loss in energy means that the final speed will be less than the initial speed. We discuss the general case when the resistance is $f(v)$ per unit mass, with v representing speed, and where $f(v) \geqslant 0$ for $v \geqslant 0$, and then look at two familiar cases.

From Newton's second law, for the motion on the way up we have

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g - f(v) = v\frac{\mathrm{d}v}{\mathrm{d}x},$$

where x represents the displacement measured vertically upwards, and similarly on the way down

$$\frac{dv}{dt} = g - f(v) = v\frac{dv}{dy},$$

where y is now the displacement measured vertically downwards. Therefore, the total time of flight is given by

$$T = -\int_U^0 \frac{1}{g+f(v)}\,dv + \int_0^V \frac{1}{g-f(v)}\,dv, \tag{3}$$

where V is the final speed of return and is given by

$$-\int_U^0 \frac{v}{g+f(v)}\,dv = \int_0^V \frac{v}{g-f(v)}\,dv. \tag{4}$$

We leave the justification of these as an exercise for the reader, but note that the expressions on the right-hand side of (3) are the times up and down, and that (4) is equivalent to saying that the distances travelled up and down are the same. Equation (4) can be interpreted as $V = V(U)$, and substituting this into (3) gives $T = T(U)$.

Our aim now is to see whether (2) holds in this more general case. To achieve this we form

$$\begin{aligned} F(U) &= \frac{U+V(U)}{g} - T(U) \\ &= \frac{U+V(U)}{g} - \left(\int_0^U \frac{1}{g+f(v)}\,dv + \int_0^{V(U)} \frac{1}{g-f(v)}\,dv\right), \end{aligned} \tag{5}$$

where

$$\int_0^U \frac{v}{g+f(v)}\,dv = \int_0^{V(U)} \frac{v}{g-f(v)}\,dv, \tag{6}$$

using (3) and (4), and investigate the sign of $F(U)$ for $U > 0$. Differentiating (5) and (6) with respect to U we have, respectively,

$$\frac{dF}{dU} = \frac{1}{g}\left(1+\frac{dV}{dU}\right) - \left(\frac{1}{g+f(U)} + \frac{1}{g-f(V)}\frac{dV}{dU}\right) \tag{7}$$

and

$$\frac{U}{g+f(U)} = \frac{V}{g-f(V)}\frac{dV}{dU}. \tag{8}$$

Substituting for dV/dU from (8) into (7), and rearranging, gives

$$\frac{dF}{dU} = \frac{Vf(U) - Uf(V)}{gV(g+f(U))}. \tag{9}$$

From (6), if $U = 0$ then $V = 0$ so $F(0) = 0$, and thus if $dF/dU \geqslant 0$ then

$$T \leqslant \frac{U+V}{g}, \tag{10}$$

which means that the total time will be less than or equal to that predicted by assuming that there is no air resistance. (N.B. A function which is zero at the origin and has a positive slope for positive arguments is necessarily positive for all such arguments.)

If $f(v) = kv^n$ $(n \geqslant 0)$, then

$$\frac{dF}{dU} = \frac{k(VU^n - UV^n)}{gV(g+kU^n)}, \tag{11}$$

and so

$$\frac{dF}{dU} \begin{cases} > 0 & (n > 1), \\ = 0 & (n = 1), \\ < 0 & (0 \leqslant n < 1). \end{cases} \tag{12}$$

Also, $dF/dU = 0$ when $k = 0$. Thus property (2) which holds for no air resistance ($k = 0$) *also* holds for a linear law of resistance ($n = 1$), and the equality (10) is valid in both these cases. However, since $V < U$ for the case with a linear law of resistance, the total time in this case will be less than that with no resistance. For stronger resistance ($n > 1$, e.g., quadratic with $n = 2$) the inequality (10) is strict and

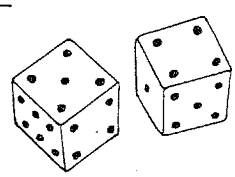
$$T < \frac{\text{initial speed} + \text{final speed}}{g},$$

which *still* means that the (actual) total time of flight is less than that predicted by assuming no resistance.

Readers may like to find explicitly the integrals in (5) and (6) and investigate the nature of $F(U)$ for the two cases mentioned above, as well as any others they can think of. Needless to say, they will find a graphics calculator invaluable. One minor alteration in the analysis above (changing a plus sign in (3) to a minus sign) allows another problem to be tackled, that of determining the *difference* in the time of fall and the time of rise. We leave the reader to follow this up.

Safe catching!

Are you more likely to throw 9 or 10 with two dice? What about three dice?



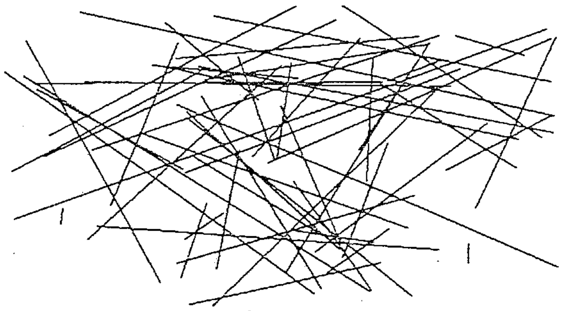
Computer Column

MIKE PIFF

Bresenham's line-drawing algorithm

How do we draw a straight line on a video display? The answer might appear obvious, until we experiment with several possibilities and realize that some give better results than others. The problems we face are jaggedness and variations in thickness of lines, because of the finite resolution of the screen, and lack of speed.

Bresenham devised the following algorithm to draw a straight line from (x_1, y_1) to (x_2, y_2). It can be included in the module `graphics` for use in any of your own modules.



```
PROCEDURE sign(i:INTEGER):INTEGER;
BEGIN
 IF i>0 THEN
  RETURN 1
 ELSIF i<0 THEN
  RETURN -1
 ELSE
  RETURN 0
 END;
END sign;
PROCEDURE Line(x1, y1, x2, y2,
 colour:INTEGER);
VAR
 x, y, i, deltax, deltay,
 s1, s2, temp, error:INTEGER;
 swap:BOOLEAN;
BEGIN
 IF (x1=x2) AND (y1=y2) THEN
  PutPixel(x1, y1, colour);
  RETURN;
 END;
 x:=x1; y:=y1;
 deltax:=ABS(x2-x1);
 deltay:=ABS(y2-y1);
 s1:=sign(x2-x1);
 s2:=sign(y2-y1);
 swap:=deltay>deltax;
 IF swap THEN
  temp:=deltay; deltay:=deltax;
  deltax:=temp;
 END;
 error:=deltay×2-deltax;
 FOR i:=0 TO deltax DO
  PutPixel(x, y, colour);
  WHILE error>=0 DO
   IF swap THEN
    x:=x+s1
   ELSE
    y:=y+s2;
   END;
   error:=error-2×deltax;
  END;
  IF swap THEN
   y:=y+s2;
  ELSE
   x:=x+s1;
  END;
  error:=error+2×deltay;
 END;
END Line;
```

Letters to the Editor

Dear Editor,

Factorising polynomial pairs

Since I submitted my article about factorising polynomial pairs (Volume 23 Number 3, pages 71 to 78), I have seen an article which may be of interest to readers. This is 'Factoring twins and Pythagorean triplets' by Lee H. Minor, *Mathematics and Computer Education*, Volume 23, Number 1 (1989), pages 19 to 29. It deals with, amongst other things, factorisation of quadratics of the type $ax^2+bx\pm c$, an idea that complements the theme of my article.

Yours sincerely,

K. R. S. SASTRY
(Box 21862, Addis Ababa,
Ethiopia)

Dear Editor,

The Ramanujan problem

I came across the Ramanujan Problem: $\sqrt{x}+y = 7$, $\sqrt{y}+x = 11$ (see Volume 24 Number 3, page 79) last year. I decided that there must be other answers—and in some sense there are.

If one eliminates the radicals, one has the two equations: $x = (7-y)^2$ and $y = (11-x)^2$. Elimination then gives a fourth-degree equation, one of whose roots is that given by Ramanujan, so the problem reduces to a cubic. This cubic is not solvable directly, so some numerical method must be used; I used the Newton–Raphson method. However, when one examines the solutions, they only work when one allows + or − for the radicals. The resulting solutions are as follows:

$$\sqrt{x} > 0, \sqrt{y} > 0: \quad x = 9, \quad y = 4,$$

$$\sqrt{x} > 0, \sqrt{y} < 0: \quad x = 12.84813, \quad y = 3.41557,$$

$$\sqrt{x} < 0, \sqrt{y} > 0: \quad x = 7.86869, \quad y = 9.80512,$$

$$\sqrt{x} < 0, \sqrt{y} < 0: \quad x = 14.28319, \quad y = 10.77931.$$

My colleague Mogens Esrom Larsen pointed out that the equations are much easier if one lets $X = \pm\sqrt{x}$ and $Y = \pm\sqrt{y}$, giving $\pm X+Y^2 = 7$, $\pm Y+X^2 = 11$.

Yours sincerely,

DAVID SINGMASTER
(Department of Computing
and Mathematics,
South Bank University,
London SE1 0AA)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

25.1 (Submitted by Alan Fearnehough, Portsmouth Sixth Form College)
Find a formula for the finite product

$$\prod_{r=1}^{n} (4r^2+4r-3).$$

25.2 (Submitted by David Yates, Preston)
$(10^n)^3$ written in full ends with $3n$ 0's. Thus a cube can end with an unlimited run of 0's. What is the case for the other digits 1–9?

25.3 (Submitted by Juan Márquez, Universidad de Valladolid, Spain)
Let T and T' be right-angled triangles and denote by R, r and R', r' the radii of the circles circumscribing and inscribing T and T', respectively. If $R/R' = r/r'$, prove that T and T' are similar triangles.

Solutions to Problems in Volume 24 Number 3

24.7 What is the maximum number of regions into which a circle can be divided by n straight lines?

Solution by Mark Blyth (Gresham's School, Holt)

Consider a circle in which n straight lines are drawn. In order to maximise the number of regions created, each line must cross every other line, resulting in $n-1$ cross-overs. This yields $P(n)$ regions, say. If we draw another line, we have $n+1$ lines and n cross-overs. Each region through which the new line passes is divided by it into two. It will pass through $n+1$ regions, hence creating $n+1$ new regions. Thus

$$P(n+1) = P(n)+n+1,$$

so that

$$P(n) = P(n-1)+n = P(n-2)+(n-1)+n$$
$$= \ldots = P(0)+1+2+\ldots+n.$$

Now $P(0) = 1$, so that

$$P(n) = 1+\tfrac{1}{2}n(n+1) = \tfrac{1}{2}(n^2+n+2).$$

This is the maximum number of regions.

Also solved by Thomas Womack (Winchester College), Julie Stainer (Allegheny College, Pennsylvania) and Harjoat Singh Bhamra (Queen Elizabeth's Grammar School, Blackburn).

24.8 Show that $16n^2+8n(-1)^n-3$ is of the form $k(k+4)$ for some integer k.

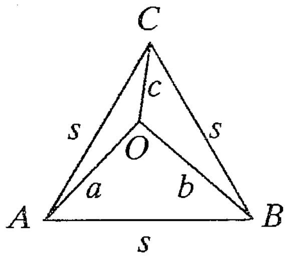
Solution by Matthew Phillips (Richard Hale School, Hertford)

$$16n^2 \pm 8n-3 = (4n \mp 1)(4n \pm 3),$$

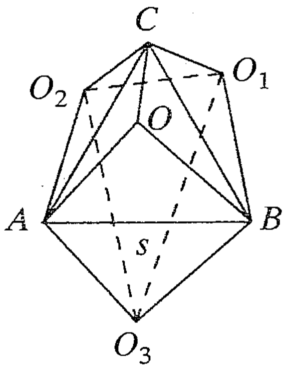
which is of the form $k(k+4)$ for some integer k. This proves more than is asked.

Also solved by Colin Lindsay (Open University), Christopher Carroll (Allegheny College, Pennsylvania), Peter Mason (University of Warwick), Thomas Womack, Sumita Kumar (National University of Lesotho) and Mark Blyth.

24.9 Determine s in terms of a, b and c.



Solution by Jie Ye Zhang (Allegheny College, Pennsylvania)



In the diagram, points O_1, O_2 and O_3 are such that triangles BOC and BO_1C are congruent, COA and CO_2A are congruent, and AOB and AO_3B are congruent. Now $\angle O_2AO_3 = 2\angle BAC = 120°$, so that, from ΔO_2AO_3,

$$O_2O_3 = \sqrt{a^2+a^2-2a\times a\cos 120°} = a\sqrt{3}$$

and

$$\text{area}\,\Delta O_2AO_3 = \tfrac{1}{2}a\times a\sin 120° = \tfrac{1}{4}\sqrt{3}a^2.$$

Similarly, $O_3O_1 = b\sqrt{3}$, $O_1O_2 = c\sqrt{3}$ and triangles O_3BO_1 and O_1CO_2 have areas $\frac{1}{4}\sqrt{3}b^2$ and $\frac{1}{4}\sqrt{3}c^2$, respectively. Now $AO_3BO_1CO_2A$ has area $2\times\text{area}\,\Delta ABC = 2\times\frac{1}{2}s^2\sin 60° = \frac{1}{2}\sqrt{3}s^2$. Hence

$$\tfrac{1}{2}\sqrt{3}s^2 = \tfrac{1}{4}\sqrt{3}(a^2+b^2+c^2)+\text{area}\,\Delta O_1O_2O_3.$$

This gives

$$s^2 = \tfrac{1}{2}(a^2+b^2+c^2)+\tfrac{1}{2}\sqrt{3}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}.$$

Reviews

The Man Who Knew Infinity—A Life of the Genius Ramanujan. By ROBERT KANIGEL. Scribners, London, 1991. Pp. ix+438. £16.95 (ISBN 0-356-20330-1).

This is a compelling biography of one of the greatest geniuses of any age, the Indian mathematician Srinivasa Ramanujan (1887–1920). But it is more than that. It portrays vividly the contrast between the largely self-taught intuitive genius Ramanujan and the master of formal mathematics G. H. Hardy, who had the insight and persistence to spot Ramanujan's genius when he received a letter from the unknown Indian containing many of his early discoveries. It contrasts, too, the Hindu culture that Ramanujan left behind with the academic world of Trinity College, Cambridge, with which Ramanujan was confronted when he came to work with Hardy in England.

The author is American and writes for American readers. He is not a mathematician, nor does he write for mathematicians. So readers might be amused by his attempt to explain an integral to a non-mathematician on page 166, or cricket (Hardy's other passion after mathematics) to readers who know only baseball on page 122, or Cambridge college life at the beginning of the century on page 127.

When Hardy, one of the greatest mathematicians of his day, was asked about his greatest contribution to mathematics, he unhesitatingly replied: 'The discovery of Ramanujan'. So Kanigel tells us, and his biography makes a very good case for the truth of Hardy's reply. Once you start reading this biography, you will find it hard to put down.

University of Sheffield — DAVID SHARPE

Guide to Mathematical Methods By JOHN GILBERT. Macmillan Education Ltd, Basingstoke, 1991. Pp. xiv+309. Paperback £10.50 (ISBN 0-333-492099).

This book is designed to provide guidance for first-year engineering and science undergraduates in their mathematics courses. It is considered that many students face problems because of 'a more rigorous and abstract approach in undergraduate studies, the greater expectation of independent study, and the increased pace at which material is presented'.

My main quibble is on the question of the relationship between this book and the single mathematics A-level. Time after time as I worked through it, I found myself thinking 'well surely this must be familiar'. After all, to choose a mathematics component at university implies a knowledge of the A-level syllabus. There may perhaps be a need for a list of standard integrals and differentiations from A-level, but is it really necessary to re-introduce differentiation from first principles and painstakingly derive each standard method and result?

One interesting feature of the book is that the examples and exercises given during each chapter almost exclusively deal with pure mathematics, whereas the miscellaneous exercises seek to apply the principles of each chapter to 'real life' situations. John Gilbert also deserves congratulation for including methods in the answers section, thereby effectively trebling the number of worked examples (which are often the best help to understanding unfamiliar material) in the book.

The chapter on complex numbers is probably the least helpful, even leaving aside the decision to say that 'they arise in connection with the solution of equations', and only give one (trivial) example. And it is certainly not 'obvious' that '$-z$ is the reflection of z in the x-axis [of the Argand diagram]'!

Overall then, this book is unlikely to inspire interest in pure mathematics in any engineers or scientists who have missed their true vocation, but it would deal efficiently and reliably with any problems encountered with a mathematics course. For example, I taught myself directional derivatives and line integrals (two topics entirely new to me) in a couple of hours using the book, and work crises and examination panics could probably have reduced that time still further.

King Edward's School, Birmingham — OLIVER JOHNSON

Mathematica in Action. By STAN WAGON. W. H. Freeman, New York, 1991. Pp. xiv+419. Paperback £21.95 (ISBN 0-7167-2202-X).

Programming in Mathematica. By ROMAN E. MAEDER. Addison-Wesley, Redwood City, California, 1991. Pp. xiv+279. Hardback £38.95 (ISBN 0-201-54877-1), paperback £25.95 (ISBN 0-201-54878-X).

Recent years have witnessed a rapid development of computer algebra, and in particular the introduction of systems based on personal computers. This must inevitably affect mathematics teaching. In undergraduate courses such systems already occur both as tools and as the focus of an entire course. They will undoubtedly move further down the age range. The recent Royal Institution lectures by Richard Dawkins on genetics included computer simulations operated by members of the audience. Young schoolchildren drawn from the audience and asked if they knew how to operate a computer with a mouse invariably answered yes.

One of the most popular systems is Mathematica, from Wolfram Research, which has numerical, symbolic and graphical tools and includes a programming language. The system comes with the Mathematica book by Stephen Wolfram and both these books are designed to complement this, but from different viewpoints.

Stan Wagon is a mathematician and *Mathematica in Action* is designed around the mathematics, mainly number theory and geometry. The techniques available in Mathematica are introduced as tools for mathematical problems. The book's purpose is to show that Mathematica is an easy-to-use tool for mathematical exploration.

Roman Maeder is a computer scientist and *Programming in Mathematica* is designed around building packages and programming. Mathematical examples such as the Runge–Kutta method for solving differential equations and designing a graphics package are introduced to illustrate the programming techniques. The purpose is to develop a good programming style in Mathematica, although the examples are chosen to be useful in their own right.

The difference in outlook is highlighted by the specifics of the program examples given. Wagon's examples are designed for Mathematica 1.2 on a Macintosh, Maeder's examples are for Mathematica 2.0 on a Sun Sparcstation. It also highlights the difference in the equipment grants of such departments. Both books are clearly written and can be recommended to anyone working with Mathematica.

University of Sheffield — ROGER COOK

Discrete Mathematics: An Introduction for Software Engineers. By MIKE PIFF. Cambridge University Press, 1991. Pp. xi+317. Hardback £30.00 (ISBN 0-521-38475-3), paperback £10.95 (ISBN 0-521-38622-5).

This book is intended for first-year computer scientists and software engineers. The topics covered are: basics of mathematical logic, elementary set theory, relations and mappings, graph theory, groups and monoids, and formal languages and grammars. In relation to some of these topics, some algorithms for certain problems are presented, for example Kruskal's and Prim's spanning tree algorithms and Dijkstra's shortest and longest path algorithms.

In the section on formal languages, subjects covered include Backus–Naur form, context-free languages, regular grammars, finite-state machines applied to recognition of words in a language, elimination of left recursive productions from a grammar, and transformation of grammars into Greibach normal form.

There are many exercises throughout; some quite thought-provoking and/or entertaining (although I found one rather careless algebraic mistake!). Sometimes it is not entirely clear what the exercises mean you to do, but generally I should think they would help the reader quite considerably to understand the material.

At the end of the book some example programs are given, related to the themes in the book; for example a program to carry out Prim's algorithm, a graph-colouring program, and a 'lexical analyser'.

The programming language used is Modula-2, although there are some references to Pascal. In fact it is claimed that a knowledge of Pascal or Ada will suffice to understand all the Modula-2 used, although I would have thought that in what is not a text on programming it would have been more helpful to stick to a more widely-known language. However, I do not think it will cause any serious difficulties.

In general I found this book a very easy-to-follow presentation of the material, with not too much abstract mathematics. I would recommend it to anyone beginning a course in computer science or software engineering; the subject matter will certainly be very valuable. Students of mathematics will also find it interesting, and relevant to some courses (though more advanced texts would be required for continued serious study of these areas of mathematics).

University of Oxford JOHN CATLOW

Programs in BBC Basic for Young Mathematicians. By S. G. BREWER. Edinburgh University Press, 1991. Pp. xvii+132. Paperback £12.95 (ISBN 0-7486-0254-2).

The title acts as a reminder to those of us fortunate enough to have moved on to MS-DOS machines that BBC Basic is alive and kicking. The availability of BBC Basic on RM Nimbus machines ensures that there is potentially a very wide audience, although I suspect that the mention of mathematicians in the title will help placate computer scientists who would prefer to see a properly structured language being used. Having said that, there is little to complain about with the programming used which is well documented and written in clear and precise style.

The introduction contains a helpful explanation of BBC Basic which would be useful to users of other dialects. The material is aimed at sixth formers and although the programs become longer and more complex in the later chapters we are introduced to the concept of chaotic behaviour in Chapter 1. Successive chapters cover the old favourites of polar curves, space-filling functions, the Tower of Hanoi and the newer favourite of Mandelbrot. The concluding chapters lead from linear transformations to simple flight simulation and step-by-step methods in mechanics and solutions of differential equations.

The mathematics involved and its relation to the programming is carefully explained at all stages and there should be no excuse for any lack of understanding such as used to be engendered by trying to come to terms with Japanese printer manuals! The book would be a useful addition to a school library. It belongs in the mathematics rather than the computing section and is just the sort of book to give to the student with time to explore the ideas. The author's intention is that the programs stimulate a student's imagination to go on to greater refinements. Inevitably the language means that some of the programs will run fairly slowly, but the transparent nature of the programming should more than compensate for this.

A similar book which is worth looking at is *132 Short Programs for the Mathematics Classroom*, published by the Mathematical Association and Stanley Thorne, Ltd, which has some programs for the BBC, 380Z and Spectrum.

United World College of the Atlantic — ROGER FLETCHER

Other books received

Order Stars. By A. ISERLES AND S. P. NØRSETT. Chapman and Hall, London, 1991. Pp. xi + 248. £25.00 (ISBN 0-412-35260-5).

Rings, Fields and Groups: An Introduction to Abstract Algebra. By R. B. J. T. ALLENBY. Edward Arnold, London, 1991. Pp. xxvi + 383. £16.95 (ISBN 0-7131-3476-3).

This is the second edition of a well-known textbook, including a new chapter on Galois theory. The first edition was reviewed in Volume 16 Number 3, page 100.

Management Mathematics: A User-friendly Approach. By PETER SPRENT. Penguin Books, London, 1991. Pp. viii + 426. Paperback £8.99 (ISBN 0-14-009153-X).

A First Course in Business Mathematics and Statistics. By R. N. ROWE. DP Publications, London, 1991. Pp. v + 145. £3.95 (ISBN 1-870941-78-0).

Calculator Puzzles, Tricks and Games. By NORVIN PALLAS. Dover Publications, New York, 1991. Pp. 96. Paperback £3.55 (ISBN 0-486-26670-2).

Statistical Methods for Business and Economics, fourth edition. By DONALD L. HARNETT AND ASHOK K. SONI. Addison-Wesley, Wokingham, 1991. Pp. xvi + 807. £18.95 (ISBN 0-201-51395-1).

Introductory Statistics, third edition. By NEIL A. WEISS AND MATTHEW J. HASSETT. Addison-Wesley, Wokingham, 1991. Pp. xviii + 940. £18.95 (ISBN 0-201-17833-8).

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	Early gambling	**R. C. Bell**
12.30	Lunch is available at nearby pubs and restaurants	
1.30–3.30	*C18 recreational mathematics*	**John Beasley**
	Puzzles in history	**Edward Hordern**
	Ancient games	**Irving Finkel**
	The centenary of Rouse Ball's 'Mathematical Recreations and Essays'	**David Singmaster**
3.30	Tea	
3.30–6.00	*An exhibition of historical books and puzzles. Anyone else is welcome to bring along items of historical interest*	

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CONTENTS

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