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A PARABOLA IS NOT AN HYPERBOLA

DAN PEDOE

In recent years two mature persons have separately asked me about the envelope which is obtained when two line segments TP, TQ, not necessarily equal in length, are each divided into equal parts and the divisions numbered, from T towards P in the one case, and from Q towards T in the other case, and the divisions with the same number are joined by lines or threads (see Figure 1).

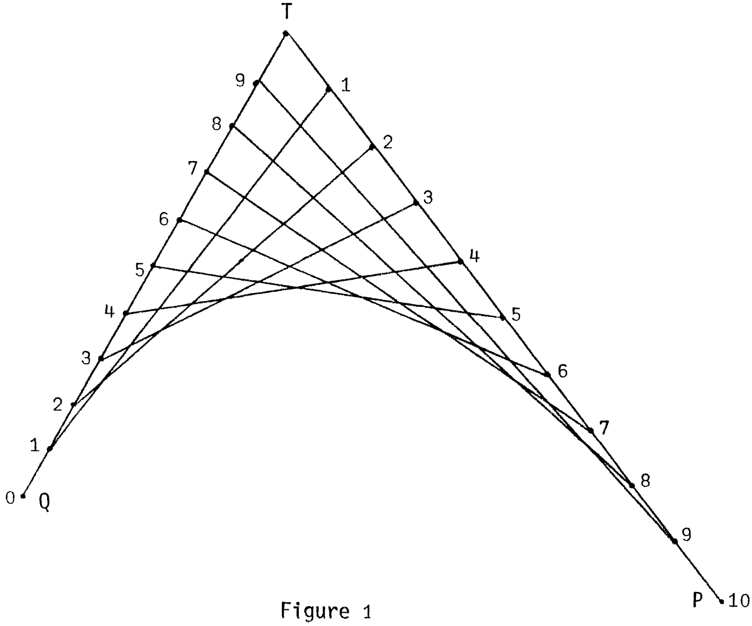

Figure 1

One of my interlocutors thought the envelope was an hyperbola, and the other thought it was either a circle or an ellipse.

The envelope is a *parabola*, and since this decoration is displayed nowadays in kindergartens and at art fairs, perhaps the reasons for this assertion should be given.

If corresponding points (those with the same number) on the lines TP and TQ are P_i, Q_i, then the essence of the construction is that

$$P_iP_j/P_jP_k = Q_iQ_j/Q_jQ_k,$$

or, in other words, the ranges are *similar*.

This statement implies the equality of the cross-ratios $\{P_i P_k, P_j \infty\} = \{Q_i Q_k, Q_j \infty\}$, introducing the notions of projective geometry (Pedoe [3], p. 328), and therefore, by the converse of Chasles' Theorem ([3], p. 330; Durell [1], p. 69), the lines $P_i Q_i$ touch a conic, which is also touched by the *line at infinity* (the join of the points ∞ on the two lines). *The envelope is therefore a parabola.*

It seems worthwhile to recall the properties of the parabola which can be obtained by elementary methods and which lead to the above theorem.

The conic sections can be discussed in various ways, but in my book *Geometry and the Liberal Arts* ([4], p. 202), I obtain them as envelopes by paper-folding, since this is the quickest way of obtaining the focal properties, and the focus-directrix properties follow quite simply. The parabola is obtained (Figure 2) as the envelope of lines YP, where Y lies on a given line l (the tangent at the vertex), S, the focus, is a given point not on l, and the angle SYP is always a right angle.

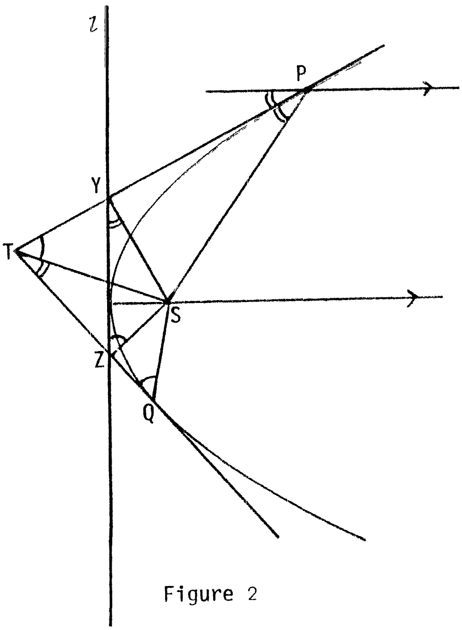

Figure 2

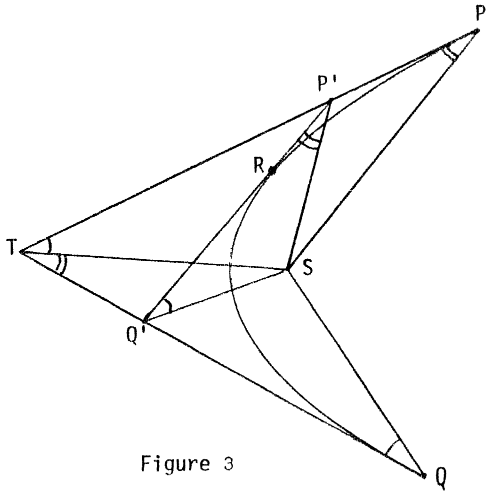

Figure 3

It follows by a simple limit argument that the point of contact P of YP with its envelope is such that angle SPY is equal to the angle between SY and l.

If TP and TQ are tangents at P and Q, respectively, intersecting l at Y and Z, respectively, then, since the points T, Y, S, Z lie on a circle, we have

$$\angle SYZ = \angle STZ \quad \text{and} \quad \angle YTS = \angle YZS.$$

If now P'Q' touches the parabola at R (Figure 3), with P' on TP and Q' on TQ, it follows that triangle SP'Q' has a fixed shape if we keep P and Q fixed and allow R to vary. The triangles SPT, SP'Q', and STQ are similar to each other and, by using triangle SP'Q' as an intermediary, we find that PP'/P'T = TQ'/Q'Q. Hence the variable tangent P'Q' cuts out similar ranges on the fixed tangents TP and TQ. The converse, which is what we require, is given in Macaulay [2, p. 43] as a Corollary and left to the student.

As a final remark, familiar to readers of my book [4, p. 202], one cannot buy stencils of parabolas or hyperbolas in the United States. The one I use is made in Switzerland.

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1. C.V. Durell, *Projective Geometry*, Macmillan, London, 1945.
2. F.S. Macaulay, *Geometrical Conics*, Cambridge University Press, 1921.
3. D. Pedoe, *A Course of Geometry*, Cambridge University Press, 1970.
4. ________, *Geometry and the Liberal Arts*, St. Martin's Press, New York, 1978.

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* * *

ALGORITHMS AND POCKET CALCULATORS: NOMINAL INTEREST RATES

CLAYTON W. DODGE

A concern of great interest is credit interest rates. Although truth-in-lending laws require the disclosure of the nominal interest rate, recent surveys have shown gross violations by lenders including even some well-established banks. It is our purpose to calculate the nominal annual interest rate for common consumer loans.

We shall confine our discussion to the equal monthly payment loan; that is, an *annuity certain* with monthly payments. Its *term* n is the number of monthly payments of $\$P$ each due at the end of the month; hence n is the number of months the loan runs. Its *period* is the time between payments, here taken as 1 month. Interest is compounded monthly; if the *nominal annual rate* is r, then an interest $r/12$ of the outstanding balance is added to that balance each month just before the monthly payment is subtracted. The amount borrowed is the *present value* $\$V$ of the annuity.

To find a formula relating the constants of our annuity, consider the portion of the original debt retired by each payment. If V_k is the portion retired by the kth payment, then

$$V_k(1+r/12)^k = P \quad \text{and} \quad V_k = P(1+r/12)^{-k},$$

since V_k earns interest for k months. Thus, for all payments we have

$$V = P(1+r/12)^{-1} + P(1+r/12)^{-2} + \ldots + P(1+r/12)^{-n}.$$

The right side is simply a geometric series whose sum is

$$V = P \cdot \frac{1-(1+r/12)^{-n}}{r/12}.$$

We let $v = V/P$ be the present value of an annuity of monthly payment \$1, so that

$$v = \frac{1-(1+r/12)^{-n}}{r/12}. \tag{1}$$

Thus, knowing the amount financed (the present value $\$V$) and the monthly payment $\$P$, we readily find v and can then use (1) to calculate r. Unfortunately, it is not possible to solve explicitly for r when n is large.

Before continuing our development, we define one interest rate we shall not use, but which deserves mention. If rate r is compounded monthly, then the *true annual rate* t is given by

$$t = (1+r/12)^{12} - 1.$$

That is, the true annual rate is the rate charged once a year that would be equivalent to the nominal annual rate compounded monthly if no payments were made during the year. It is common for banks to advertise prominently the true rate for their savings accounts, since $t > r$, but not for their loans.

There is one further rate we shall need, the *add-on rate* s defined by

$$nP = V(1+ns/12). \tag{2}$$

Thus, on a \$4000 auto loan with a term of 48 months, an add-on rate of .06 (or 6%) adds to the principal $\$V$, 6% of the principal times the number of years (not the number of months) the loan runs, in this case 24%, so the total repaid will be

$$4000(1 + \frac{48}{12} \times .06) = \$4960$$

and each monthly payment will be

$$P = \frac{4960}{48} = \$103.33.$$

For comparison, a loan with a nominal annual rate of 6% ($r = .06$) and running for 48 months with payments of \$103.33 each would retire a debt of

$$V = 103.33 \, \frac{1 - (1 + .005)^{-48}}{.005} = \$4399.82,$$

considerably more than the \$4000 borrowed with the 6% add-on rate. Hence, when $s = .06$, r is much greater. To find r we utilize formula (1),

$$\frac{4000}{103.33} = 38.7109 = \frac{1 - (1 + r/12)^{-48}}{r/12},$$

and guess values for r to make $v = 38.7109$, interpolating to find new guesses for r. Thus

$$\text{for } r = .10, \qquad v = 39.4282,$$

$$\text{for } r = .11, \qquad v = 38.6914,$$

so $.10 < r < .11$. By interpolation we try

$$r = .10973 \quad \text{and find} \quad v = 38.7111,$$

a value very close to the desired 38.7109. This value is correct as far as calculated (to the nearest .00001). Therefore an add-on rate of 6% on a 4-year loan produces a nominal annual rate of 10.97%, almost double the add-on rate.

It is no wonder that lending institutions used to advertise their add-on rates for such loans. Many lenders would not mention the nominal rate even when pressed. Because some institutions still quote the add-on rate, it is quite appropriate that we discuss how to calculate the nominal rate. The method of the example above is reasonable and produces whatever degree of accuracy is desired. So it probably is the best method to use when more than just a rough estimate for r is sought.

We turn our attention now to rough approximations to r when n and s are known. From formula (2) we find

$$s = \frac{12}{n} \cdot \frac{nP - V}{V} = \frac{12}{n}\left(\frac{n}{v} - 1\right),$$

so the add-on rate s is always easily found.

The author has examined loans running from 6 to 72 months and having add-on rates from .05 to .24. It is doubtful one could obtain an add-on rate less than .05 and only a very poor risk would be forced to accept one higher than .15. Terms under 12 months are short enough so precise calculation is generally not necessary, and loans with terms of more than 5 or 6 years probably should have their rates calculated very carefully (by the interpolation method above).

If we let $r = ks$, then k is greatest for n about 12 to 30 months and for small s. Thus, for $n = 30$ and $s = .05$, $k = 1.87$; for $n = 60$ and $s = .15$, $k = 1.64$. As n and s increase further, k drops sharply; for example, $k = 1.48$ for $n = 72$ months and $s = .24$.

A simple formula for r in terms of s is

$$r = \frac{7}{4}s \approx \sqrt{3}s, \tag{3}$$

being accurate to within

$$5\% \text{ when } 6 \le n \le 60 \text{ and } .07 \le s \le .14$$

and

$$8\% \text{ when } 6 \le n \le 72 \text{ and } .05 \le s \le .15.$$

An alternative formula, accurate to within 4% for $.05 \le s \le .09$ and $12 \le n \le 72$, the range in which most bank loans lie, is

$$r = \frac{9}{5}s = 1.8s. \tag{4}$$

All but extreme cases should lie in this range.

Naturally, either formula (3) or (4) can be used as a first estimate for r and then refined by the interpolation process.

Our last formula gives less than 1% error for $12 \le n \le 60$ and $.05 \le s \le .15$. It is

$$r = s\left\{-13\left(\frac{n}{12}\right)^2 + 92\left(\frac{n}{12}\right) + 1385\right\}/800 + \frac{s^2}{12}\left\{\left(\frac{n}{12}\right)^2 - \frac{10n}{12} + 5\right\}. \tag{5}$$

When n is increased to 72 months, relative error increases to as much as 3%. For $.15 \le s \le .24$ and $12 \le n \le 72$, error is less than 2%.

Formula (5) is easily programmed for a programmable calculator and gives results that should be more than sufficient for practical purposes. (How much does it matter whether the nominal rate is 10.1% as against 10.2%?) Of course, if greater precision is required, the interpolation technique is available.

For another approach to nominal interest rates by Newton's method, see the article by J.E. Morrill [1]. Next time around, the last prepared article in this series[1] will present formulas for approximating $\Gamma(x)$, the generalization of the factorial function, for real x. Readers who have suggestions for further topics in this series, either pet functions to approximate or calculator problems to discuss,

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[1]For the earlier articles in the series, see [1978: 96, 154, 217, 279]. (Editor)

are invited to write to the author at the address given below.

REFERENCE

1. John E. Morrill, Finding Truth in Lending, *Mathematics Magazine*, 50 (January 1977) 30-32.

Mathematics Department, University of Maine, Orono, Maine 04469.

* * *

THE OLYMPIAD CORNER: 5

MURRAY S. KLAMKIN

This month, in addition to the solutions to Practice Set 4, we give the problems posed at the Eighth U.S.A. Mathematical Olympiad, which took place on May 1, 1979. These problems, with solutions (together with those of XXI International Mathematical Olympiad, to be held on July 2-3, 1979, in England), will appear later this year in a pamphlet compiled by Samuel L. Greitzer. These pamphlets (50¢ per copy) will be obtainable from Dr. Walter E. Mientka, Executive Director, MAA Committee on High School Contests, 917 Oldfather Hall, University of Nebraska, Lincoln, Neb. 68588. The problems were prepared by the Examination Committee of the U.S.A. Mathematical Olympiad, consisting of Murray S. Klamkin (Chairman), T.J. Griffiths, and Cecil C. Rousseau.

THE EIGHTH U.S.A. MATHEMATICAL OLYMPIAD (May 1979)

(5 *questions* - 3 *hours*)

1. Determine all nonnegative integral solutions $(n_1, n_2, \ldots, n_{14})$, if any, apart from permutations, of the Diophantine equation

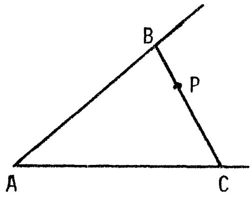
$$n_1^4 + n_2^4 + \ldots + n_{14}^4 = 1599.$$

2. A *great circle* $\mathcal{E}$ on a sphere is one whose center is the center O of the sphere. A *pole* P of the great circle $\mathcal{E}$ is a point on the sphere such that OP is perpendicular to the plane of $\mathcal{E}$. On any great circle through P, two points A and B are chosen equidistant from P. For any *spherical triangle* ABC (the sides are great circle arcs), where C is on $\mathcal{E}$, prove that the great circle arc CP is the angle bisector of angle C.

3. Given are three identical n-faced dice whose corresponding faces are identically numbered with arbitrary integers. Prove that, if they are tossed at

random, the probability that the sum of the top three face numbers is divisible by 3 is greater than or equal to 1/4.

4. Show how to construct a chord BPC of a given angle A, through a given point P within the angle A, such that $1/BP + 1/PC$ is a maximum.



5. A certain organization has n members ($n \geq 5$) and it has $n+1$ three-member committees, no two of which have identical membership. Prove that there are two committees which share *exactly one* member.

SOLUTIONS TO PRACTICE SET 4

4-1. What is the probability of an odd number of sixes turning up in a random toss of n fair dice?

Solution.

For $0 \leq k \leq n$, the probability of k sixes turning up in a random toss of n fair dice is

$$\binom{n}{k}\left(\frac{5}{6}\right)^{n-k}\left(\frac{1}{6}\right)^{k};$$

hence, with $a = 5/6$ and $b = 1/6$, the required probability is

$$\begin{aligned} P &= \binom{n}{1}a^{n-1}b + \binom{n}{3}a^{n-3}b^{3} + \binom{n}{5}a^{n-5}b^{5} + \ldots \\ &= \text{sum of the even-ranked terms in the expansion of } (a+b)^{n} \\ &= \frac{1}{2}\{(a+b)^{n} - (a-b)^{n}\} \\ &= \frac{1}{2}\left\{1 - \left(\frac{2}{3}\right)^{n}\right\}. \end{aligned}$$

4-2. If a, b, c, d are real, prove that

$$\left\{\begin{aligned} a^2+b^2 &= 2, \\ c^2+d^2 &= 2, \\ ac &= bd, \end{aligned}\right\} \quad \text{if and only if} \quad \left\{\begin{aligned} a^2+c^2 &= 2, \\ b^2+d^2 &= 2, \\ ab &= cd. \end{aligned}\right\}$$

Solution.

The result follows immediately from the identity

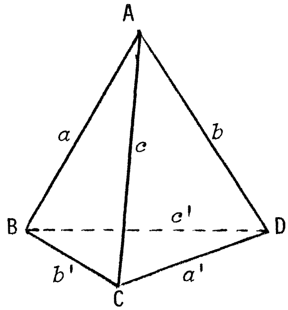
$$(a^2+b^2-2)^2+(c^2+d^2-2)^2+2(ac-bd)^2 = (a^2+c^2-2)^2+(b^2+d^2-2)^2+2(ab-cd)^2.$$

The result can be generalized. It is then equivalent to stating that if the row vectors of an $n\times n$ matrix are orthonormal, so are the column vectors.

4-3. If a, a'; b, b'; and c, c' are the lengths of the three pairs of opposite edges of an arbitrary tetrahedron, prove that

(i) there exists a triangle whose sides have lengths $a+a'$, $b+b'$, and $c+c'$;

(ii) the triangle in (i) is acute.



Solution.

It will be seen that the proof of (ii) implies (i); nevertheless we include a direct proof of (i).

(i) Let $p = a+a'$, $q = b+b'$, $r = c+c'$. We assume that the edges of the tetrahedron have been labeled so that $p\ge q\ge r$. Then (see figure)

$$b+c > a', \quad b'+c' > a', \quad b+c' > a, \quad b'+c > a,$$

and adding these inequalities yields $q+r > p$. Thus there is a triangle PQR with sides of lengths p, q, r.

(ii) From the law of cosines involving the largest angle P,

$$p^2 = q^2 + r^2 - 2qr\cos P,$$

we see that triangle PQR is acute-angled if and only if $\cos P > 0$ or

$$q^2+r^2-p^2 > 0. \tag{1}$$

We will establish the inequalities

$$b^2+b'^2+c^2+c'^2-a^2-a'^2 \ge 0 \tag{2}$$

and

$$2bb'+2cc'-2aa' > 0, \tag{3}$$

from which (1) follows by addition.

To prove (2), let $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$ denote vectors from some origin to the vertices A, B, C, D of the tetrahedron (see figure). With the usual notation $\vec{v}^2 = \vec{v}\cdot\vec{v} = |\vec{v}|^2 \ge 0$ and other properties of the dot product of two vectors, the left side of (2) becomes

$$(\vec{A}-\vec{D})^2+(\vec{B}-\vec{C})^2+(\vec{A}-\vec{C})^2+(\vec{B}-\vec{D})^2-(\vec{A}-\vec{B})^2-(\vec{C}-\vec{D})^2 = (\vec{A}+\vec{B}-\vec{C}-\vec{D})^2 \ge 0.$$

To prove (3), suppose the framework of the tetrahedron in our figure is considered to be hinged along CD and flattened to form a convex quadrilateral ADBC. The lengths of all edges remain the same except that of AB which is increased. For the quadrilateral, the *Ptolemaic inequality* assures us that, in magnitude only,

$$AD \cdot BC + AC \cdot BD \geq AB \cdot CD, \tag{4}$$

with equality if and only if ADBC is a cyclic quadrilateral (Ptolemy's Theorem). Hence (4) holds *a fortiori*, with strict inequality, for the tetrahedron, where AB is shorter, and thus (3) is established.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before October 1, 1979, although solutions received after that date will also be considered until the time when a solution is published.

441. *Proposed by Sunder Lal, Institute of Advanced Studies, Meerut University, Meerut, India (part (a)); and the editor (part (b)*).*

(a) Solve the decimal alphametic

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 ASHA
  GOT
  THE .
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MEDAL
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(b)* Who was ASHA and what did he or she do to deserve THE MEDAL?

442*. *Proposed by Sahib Ram Mandan, Indian Institute of Technology, Kharagpur, India.*

Prove that the equation of *any* quartic (a plane curve of order 4) may, in an infinity of ways, be thrown into the form

$$aU^2 + bV^2 + cW^2 + 2fVW + 2gWU + 2hUV = 0,$$

where $U = 0$, $V = 0$, $W = 0$ represent three conics.

443. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

(a) Here are seven consecutive squares for each of which its decimal digits sum to a square:

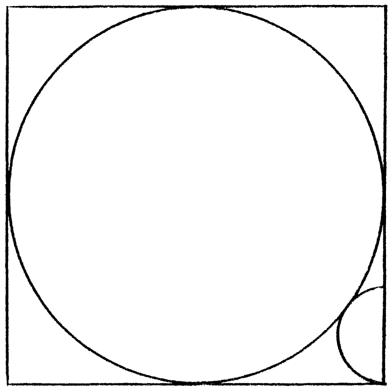
81, 100, 121, 144, 169, 196, 225.

Find another set of seven consecutive squares with the same property.

(b)* Does there exist a set of more than seven consecutive squares with the same property?

444. *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*

A circle is inscribed in a square ABCD. A second circle on diameter BE touches the first circle, as shown in the figure. Show that AB = 4BE.



445. *Proposed by Jordi Dou, Escola Tecnica Superior Arquitectura de Barcelona, Spain.*

Consider a family of parabolas escribed to a given triangle. To each parabola corresponds a focus F and a point S of intersection of the lines joining the vertices of the triangle to the points of contact with the opposite sides. Prove that all lines FS are concurrent.

446. *Proposed by the late R. Robinson Rowe, Sacramento, California.*

An errant knight stabled at one corner of an $N \times N$ chessboard is "lost", but happens to be at the diagonally opposite corner. If he moves at random, what is the probable number of moves he will need to get home (a) if $N = 3$ and (b) if $N = 4$?

447. *Proposed by Viktors Linis, University of Ottawa.*

The number $\sum_{k=1}^{n} \frac{2^k}{k}$ is represented as an irreducible fraction $\frac{p_n}{q_n}$.

(a) Show that p_n is even.

(b) Show that if $n > 3$ then p_n is divisible by 8.

(c) Show that for every natural number k there exists an n such that all the numbers p_n, p_{n+1},... are divisible by 2^k.

448. *Proposed by G. Ramanaiah, Madras Institute of Technology, Madras, India.*

A function f is said to be an *inverse point function* if $f(k) = f(1/k)$ for all $k > 0$. Show that the functions g and h defined below are inverse point functions:

$$g(k) = \frac{1}{k}\sum_{n=1}^{\infty} (-1)^{n-1}\frac{(1 - \operatorname{sech}\lambda_n k)}{\lambda_n^3},$$

$$h(k) = \frac{1}{k^2}\sum_{n=1}^{\infty} \frac{\lambda_n k - \tanh\lambda_n k}{\lambda_n^5},$$

where $\lambda_n = (2n-1)\pi/2$.

449. *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Let p be a prime $\equiv 3 \pmod 8$ and let each of the numbers α, β, γ have one of the values ± 1. Prove that the number $N_p(\alpha,\beta,\gamma)$ of consecutive triples x, $x+1$, $x+2$ $(x = 1,2,\ldots,p-3)$ with

$$\left(\frac{x}{p}\right) = \alpha, \quad \left(\frac{x+1}{p}\right) = \beta, \quad \left(\frac{x+2}{p}\right) = \gamma, \qquad \left(\frac{-}{p}\right) = \text{the Legendre symbol}$$

is the same no matter what values are assigned to α, β, γ.

For example, when $p = 19$ we have the table

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\left(\frac{x}{p}\right)$	+1	-1	-1	+1	+1	+1	+1	-1	+1	-1	+1	-1	-1	-1	-1	+1	+1	-1

from which it is easily seen that $N_p(\alpha,\beta,\gamma) = 2$ for all eight values of the triple (α,β,γ).

450*. *Proposed by A. Liu, University of Alberta.*

Triangle ABC has a fixed base BC and a fixed inradius. Describe the locus of A as the incircle rolls along BC. When is AB of minimal length (geometric characterization desired)?

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

368. [1978: 192] *Proposed by Lai Lane Luey, Willowdale, Ontario.*

Let a, n be integers with $a \geq n \geq 0$, c any constant, and

$$f(a) = \sum_{k=0}^{a} (-1)^k \binom{a}{k} (a-k+c)^n.$$

Prove that $f(a) = 0$ if $a > n$ and $f(n) = n!$.

Solution by Gali Salvatore, Perkins, Québec.

It is a well-known fact of the calculus of finite differences that, for a polynomial

$$g(x) = b_0x^n + b_1x^{n-1} + \ldots + b_n$$

and difference interval h, we have

$$\Delta^n g(x) = n!b_0h^n \quad \text{and} \quad \Delta^{n+1}g(x) = \Delta^{n+2}g(x) = \ldots = 0.$$

The shifting operator E is related to Δ by $\Delta = E-1$; hence for $g(x) = x^n$ and $h=1$ we get

$$\Delta^a x^n = (E-1)^a x^n = \sum_{k=0}^{a} (-1)^k \binom{a}{k} (x+a-k)^n = \begin{cases} n! & \text{if } a=n \\ 0 & \text{if } a>n \end{cases}$$

and the desired result follows upon setting $x=c$.

Also solved by PAUL J. CAMPBELL for the Beloit College Solvers, Beloit, Wisconsin; STANLEY COLLINGS, The Open University, Milton Keynes, England; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; G.P. HENDERSON, Campbellcroft, Ontario; ALLAN Wm. JOHNSON Jr., Washington, D.C.; M.S. KLAMKIN, University of Alberta; BOB PRIELIPP, The University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; HAROLD N. SHAPIRO, Courant Institute of Mathematical Sciences, New York University; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; and KENNETH S. WILLIAMS, Carleton University, Ottawa. A comment was received from BASIL C. RENNIE, James Cook University of North Queensland, Australia.

Editor's comment.

Several solvers referred to Gould's excellent expository article [1], which contains a wealth of information about and references to related problems.

REFERENCE

1. H.W. Gould, "Euler's formula for nth differences of powers", *American Mathematical Monthly*, 85 (1978) 450-467.

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369. [1978: 192] *Proposé par Hippolyte Charles, Waterloo, Québec.*

Trouver toutes les solutions réelles de l'équation

$$\sin(\pi \cos x) = \cos(\pi \sin x).$$

Solution de Leroy F. Meyers, The Ohio State University.

L'équation donnée équivaut à chacune des équations suivantes, où k désigne un entier quelconque:

$$\cos\left(\frac{\pi}{2} - \pi\cos x\right) = \cos(\pi\sin x),$$

$$\frac{\pi}{2} - \pi\cos x = -2k\pi \pm \pi\sin x,$$

$$\cos x \pm \sin x = 2k + \frac{1}{2},$$

$$\sqrt{2}\cos\left(x \mp \frac{\pi}{4}\right) = 2k + \frac{1}{2}.$$

Or la dernière entraîne $|2k + \frac{1}{2}| \le \sqrt{2}$, qui ne peut subsister que pour $k = 0$. Il résulte donc

$$\cos\left(x \mp \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}},$$

d'où

$$x = 2m\pi \pm \frac{\pi}{4} \pm \text{Arccos}\,\frac{1}{2\sqrt{2}},$$

m étant un entier quelconque et toutes les combinaisons de signes étant possibles.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; H.G. DWORSCHAK, Algonquin College, Ottawa; G.C. GIRI, Research Scholar, Indian Institute of Technology, Kharagpur, India; G.P. HENDERSON, Campbellcroft, Ontario; F.G.B. MASKELL, Collège Algonquin, Ottawa; BOB PRIELIPP, The University of Wisconsin-Oshkosh; G. RAMANAIAH, Madras Institute of Technology, India; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; et par le proposeur.

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370. [1978: 193] *Proposed by O. Bottema, Delft, The Netherlands.*

If K is an inscribed or escribed conic of the given triangle $A_1A_2A_3$, and if the points of contact on A_2A_3, A_3A_1, A_1A_2 are T_1, T_2, T_3, respectively, then it is well-known that A_1T_1, A_2T_2, A_3T_3 are concurrent in a point S. Determine the locus of S if K is a parabola.

Solution by the proposer.

Let (x_1,x_2,x_3) be the homogeneous barycentric point coordinates with respect to the triangle. The coordinates of the line with equation

$$u_1x_1 + u_2x_2 + u_3x_3 = 0 \quad \text{are} \quad (u_1,u_2,u_3).$$

Thus, if l is the line at infinity, we have

$$l = (1,1,1) \quad \text{and} \quad A_2A_3 = (1,0,0),\ A_3A_1 = (0,1,0),\ A_1A_2 = (0,0,1).$$

A conic K tangent to the last three lines has, in line coordinates, an equation of the type

$$\alpha_1 u_2 u_3 + \alpha_2 u_3 u_1 + \alpha_3 u_1 u_2 = 0. \qquad (1)$$

The pole of $A_2A_3 = (1,0,0)$ with respect to (1) has the equation $\alpha_3 u_2 + \alpha_2 u_3 = 0$; hence $T_1 = (0,\alpha_3,\alpha_2)$ and similarly $T_2 = (\alpha_3,0,\alpha_1)$, $T_3 = (\alpha_2,\alpha_1,0)$. If $S = (x_1,x_2,x_3)$ where $x_1x_2x_3 \neq 0$, we have $T_1 = (0,x_2,x_3)$, $T_2 = (x_1,0,x_3)$, $T_3 = (x_1,x_2,0)$. Hence

$$\alpha_1 : \alpha_2 : \alpha_3 = x_2x_3 : x_3x_1 : x_1x_2.$$

The conic (1) is a parabola if l is one of its tangents, that is, when $\alpha_1 + \alpha_2 + \alpha_3 = 0$. The locus of S therefore has the equation

$$x_2x_3 + x_3x_1 + x_1x_2 = 0,$$

which represents the *circumscribed Steiner ellipse* of the triangle $A_1A_2A_3$. (It passes through the vertices, the tangent at any vertex is parallel to the opposite side, and its centre is the centroid of the triangle.) Strictly speaking, the vertices should be excluded.

Also solved by STANLEY COLLINGS, The Open University, Milton Keynes, England; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; G.P. HENDERSON, Campbellcroft, Ontario; SAHIB RAM MANDAN, Indian Institute of Technology, Kharagpur, India; F.G.B. MASKELL, Collège Algonquin, Ottawa; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India. A comment was received from DAN PEDOE, University of Minnesota.

Editor's comment.

Pedoe found this problem as a worked exercise in Sommerville [1], and Maskell located it as Problem 23 in Milne [2].

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1. D.M.Y. Sommerville, *Analytical Conics*, Bell, London, 1924, p. 180.
2. W.P. Milne, *Homogeneous Coordinates*, Edward Arnold & Co., London, 1924, p. 55.

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371. [1978: 224] *Proposed by Charles W. Trigg, San Diego, California.*

In the following skeleton multiplication (see next page), each letter uniquely replaces a digit in the decimal system. Reconstruct the multiplication.

Solution by A. Liu, University of Alberta.

A one-line solution: $1 = Y < A < C < H < I < M < S < T < E = 9$.

Also solved by HAYO AHLBURG, Benidorm, Spain; W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; DANIALL GARBALLA,

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                      E A S Y
        M A T H E M A T I C S
        ---------------------
                    M * * * S
                  A * * * C
                H * * * I
              S * * * T
            Y * * * A
          I * * * M
        T * * * E
      C * * * H
    S * * * T
  Y * * * A
I * * * M
-----------------------------
I * * * * * * * * * * * * * S
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student, Eastview Secondary School, Barrie, Ontario; ALLAN Wm. JOHNSON Jr., Washington, D.C.; ROBERT S. JOHNSON, Montréal, Québec; G.D. KAYE, Department of National Defence, Ottawa; GILBERT W. KESSLER, Canarsie H.S., Brooklyn, N.Y.; F.G.B. MASKELL, Collège Algonquin, Ottawa; HERMAN NYON, Paramaribo, Surinam; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, N.J.; CHARLES SILVERSTEIN, James Monroe H.S., Bronx, N.Y.; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

The editor did not want to deprive Liu of the glory attendant upon the rare feat of a one-line solution, but for lesser mortals a brief explanation may be in order. It is clear that $Y=1$, $E=9$, and that in each five-digit partial product the left digit is 1 less than the right digit, so... . Rendered in numbers, the multiplication is

$$9271 \times 62849628537 = 582678906166527.$$

This problem is truly EASY MATHEMATICS; so one might be able to call it a doubly-true alphametic if only the letters of the product (with ? replacing zero),

ITAMSTE?MYMMIAS,

made sense in any language known to man. A good try was made by Johnson (Robert S.), who found it to be an anagram of

MAYTIME MIST, SAM?

Apparently inspired by this problem, Liu asked students for a well-known name beginning with CRU and ending with RUM, but all he got were crummy answers.

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372. [1978: 224] *Proposed jointly by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.; and Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

A triangle ABC has area 1. Point P is on side a, α units from B; point Q is on b, β units from C; and point R is on c, γ units from A. Prove that, if α/a, β/b, and γ/c are the zeros of a cubic polynomial f whose leading coefficient is unity, then the area of ΔPQR is given by $f(1) - f(0)$.

Solution by Jordi Dou, Escola Tecnica Superior Arquitectura de Barcelona, Spain.

Define α', β', γ' by $\alpha' = \alpha/a$, etc. If the bars denote area, $|ABC| = 1$ implies $|AQR| = (1-\beta')\gamma'$, etc. Hence

$$\begin{aligned} f(1) - f(0) &= \left[(x-\alpha')(x-\beta')(x-\gamma')\right]_0^1 \\ &= 1 - (1-\beta')\gamma' - (1-\gamma')\alpha' - (1-\alpha')\beta' \\ &= |ABC| - |AQR| - |BRP| - |CPQ| \\ &= |PQR|. \end{aligned}$$

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; ROLAND H. EDDY, Memorial University of Newfoundland; HERTA T. FREITAG, Roanoke, Virginia; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; G.P. HENDERSON, Campbellcroft, Ontario; ALLAN Wm. JOHNSON Jr., Washington, D.C.; G.D. KAYE, Department of National Defence, Ottawa; F.G.B. MASKELL, Collège Algonquin, Ottawa; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, N.J.; BASIL C. RENNIE, James Cook University of North Queensland, Australia; ETHEL RUBINSTEIN, Freehold Township High School, New Jersey; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; CHARLES SILVERSTEIN, James Monroe H.S., Bronx, N.Y.; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; KENNETH M. WILKE, Topeka, Kansas; and the proposers.

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373. [1978: 225] *Proposed by Leroy F. Meyers, The Ohio State University.*

Suppose that the human population of the Earth is increasing exponentially at a constant relative rate k, that the average volume of a person stays at V_0, and that the present population is N_0. If people are assumed packed solidly into a sphere, how long will it be until the radius of that sphere is increasing at the speed of light, c, and what will the radius of the sphere be then?

The following approximate data may be used: $N_0 = 4\times 10^9$, $k = 1\%/\text{yr}$, 1 yr = 365.25 da, 1 da = 24·60·60 sec; and, in English units, $V_0 = 4\ \text{ft}^3$ and $c = 186300$ mi/sec, whereas in metric units $V_0 = 0.1\ \text{m}^3$ and $c = 3\times 10^8$ m/sec.

(I heard of this problem several years ago. It must be a well-known bit of mathematical folklore.)

Solution by the proposer.

Let $N(t)$, $V(t)$, and $r(t)$ be the population, the total volume, and the radius of a sphere containing all people, at time t years after now. Then

$$N(0) = N_0 \quad \text{and} \quad V(t) = V_0 N(t) = \frac{4\pi}{3} r^3(t), \qquad t \geq 0.$$

Let $r(0) = r_0$, t^* the time of blowup, and $r(t^*) = r^*$. We are given that $N'(t) = kN(t)$, so

$$N(t) = N_0 e^{kt}, \quad V(t) = V_0 N_0 e^{kt}, \quad \text{and} \quad r(t) = \sqrt[3]{\frac{3V_0N_0}{4\pi}}\, e^{kt/3} = r_0 e^{kt/3}.$$

Since $r'(t) = (k/3)r(t)$, we have $r'(t^*) = c$ (at blowup) just when

$$r^* = r(t^*) = \frac{3c}{k}$$

(which is independent of V_0 and N_0) and

$$t^* = \frac{3}{k} \ln \frac{r^*}{r_0} = \frac{3}{k} \ln \frac{3c}{kr_0}.$$

With our data, we find that $r^* = 300$ light-years. In English units, we get

$$r^* \approx 1.76 \times 10^{15} \text{ mi}, \quad r_0 \approx 1560 \text{ ft} \approx 0.3 \text{ mi}, \quad t^* \approx 10900 \text{ yr};$$

while in metric units

$$r^* \approx 2.84 \times 10^{18} \text{ m}, \quad r_0 \approx 460 \text{ m}, \qquad t^* \approx 10900 \text{ yr}.$$

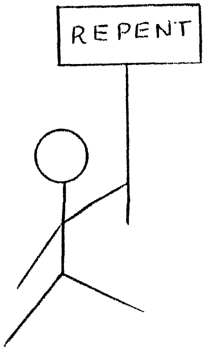
Even a 1% yearly increase in population cannot be kept up for 11 millennia!

Also solved by HAYO AHLBURG, Benidorm, Spain; HERMAN NYON, Paramaribo, Surinam; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, N.J.; and DONALD P. SKOW, McAllen H.S., McAllen, Texas.

Editor's comment.

Ahlburg commented: "The implications are frightening, because the rate k already seems to be bigger in fact than in this horror fiction!"

A historical survey of various models of population growth can be found in Smith [1]. According to one of the models [2], Doomsday will occur on Friday, 13 November, A.D. 2026. Most of our younger readers will live to see at least the dawn of that fateful day, so... (see figure).



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1. David A. Smith, Human Population Growth: Stability or Explosion, *Mathematics*

Magazine, 50 (September 1977) 186-197.

2. H. von Foerster, P.M. Mora and L.W. Amiot, Doomsday, Friday, 13 November, A.D. 2026, *Science*, 132 (1960) 1292-1295.

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374. [1978: 225] *Proposed by Sidney Penner, Bronx Community College, New York.*

Prove or disprove the following

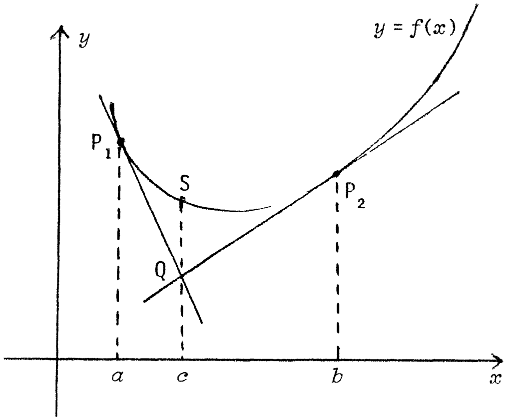
THEOREM. Let R *be the set of real numbers and let the function* $f: R \to R$ *be such that* $f''(x)$ *exists, is continuous and is positive for every* x *in* R*. Let* P_1 *and* P_2 *be two distinct points on the graph of* f*, let* L_1 *be the line tangent to* f *at* P_1 *and define* L_2 *analogously. Let* Q *be the intersection of* L_1 *and* L_2 *and let* S *be the intersection of the graph of* f *with the vertical line through* Q*. Finally, let* R_1 *be the region bounded by segment* P_1Q*, segment* SQ *and arc* P_1S*, and define* R_2 *analogously. If, for each choice of* P_1 *and* P_2*, the areas of* R_1 *and* R_2 *are equal, then the graph of* f *is a parabola with vertical axis.*

Solution by L.F. Meyers, The Ohio State University.

For all real numbers x and a, set

$$h_a(x) = f(x) - f(a) - (x-a)f'(a).$$

Thus h_a specifies the height of the graph of f above the line tangent to the graph at the point $P_1 = (a, f(a))$.



If $b \neq a$, the tangents at the points P_1 and $P_2 = (b, f(b))$ intersect at a point Q (see figure) whose abscissa c satisfies the equation $h_a(c) = h_b(c)$, and a straightforward calculation yields

$$c = b - \frac{h_a(b)}{h_a'(b)}.$$

(This is Newton's formula for finding a second approximation c to a root of the equation $h_a(x) = 0$, given a first approximation b. Note that $h_a'(b) \neq 0$, since we are given that $h_a''(x) = f''(x) > 0$ for all x, whereas $h_a'(a) = 0$ and $b \neq a$.)

The condition of equality of areas becomes

$$\int_a^c h_a(x)dx = \int_c^b h_b(x)dx.$$

Since this is to hold for all a and b (remembering that c depends on both a and b), we may differentiate partially with respect to b, keeping a fixed. For the left side, we get

$$\frac{\partial}{\partial b}\int_a^c h_a(x)dx = h_a(c)\frac{\partial c}{\partial b}.$$

For the right side, we differentiate under the integral sign (since all the conditions are satisfied here) and use, when needed, the following facts:

$$h_b(b) = 0, \qquad \frac{\partial}{\partial b}h_b(x) = (b-x)h_a''(b), \qquad \frac{\partial c}{\partial b} = \frac{h_a(b)h_a''(b)}{(h_a'(b))^2}.$$

We obtain

$$\frac{\partial}{\partial b}\int_c^b h_b(x)dx = h_b(b) - h_b(c)\frac{\partial c}{\partial b} + \int_c^b \frac{\partial}{\partial b}h_b(x)dx$$

$$= -h_b(c)\frac{\partial c}{\partial b} + h_a''(b)\int_c^b (b-x)dx$$

$$= -h_b(c)\frac{\partial c}{\partial b} + h_a''(b)\cdot\frac{(b-c)^2}{2}.$$

Thus

$$(h_a(c) + h_b(c))\frac{\partial c}{\partial b} = h_a''(b)\cdot\frac{(b-c)^2}{2}$$

or

$$(h_a(c) + h_b(c))\cdot\frac{h_a(b)h_a''(b)}{(h_a'(b))^2} = h_a''(b)\cdot\frac{1}{2}\cdot\left(\frac{h_a(b)}{h_a'(b)}\right)^2$$

and simplification (allowed because $h_a(b)$ and $h_a''(b)$ are positive) reduces this to $2(h_a(c)+h_b(c)) = h_a(b)$. A similar argument yields $2(h_a(c)+h_b(c)) = h_b(a)$. Hence

$$h_a(b) = h_b(a).$$

From the definitions of h_a and h_b, this equation is equivalent to

$$f(b) - f(a) = \frac{b-a}{2}\cdot(f'(b)+f'(a)),$$

and this holds for all real a and b, distinct or not. If we let $a = x-t$ and $b = x+t$, where x and t are real, the last relation becomes

$$f(x+t)-f(x-t) = t(f'(x+t)+f'(x-t)),$$

and partial differentiation with respect to t yields

$$f'(x+t)+f'(x-t) = f'(x+t)+f'(x-t)+t(f''(x+t)-f''(x-t)),$$

whence $f''(b)=f''(a)$ for all a and b. Thus f'' is a constant function and so f is a quadratic polynomial, as desired.

Note that the continuity of f'' was not used.

Also solved by JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; and BASIL C. RENNIE, James Cook University of North Queensland, Australia.

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375. [1978: 225] *Proposed by M.S. Klamkin, University of Alberta.*

A convex n-gon P of cardboard is such that if lines are drawn parallel to all the sides at distances x from them so as to form within P another polygon P', then P' is similar to P. Now let the corresponding consecutive vertices of P and P' be $A_1, A_2, \ldots, A_n$ and $A_1', A_2', \ldots, A_n'$, respectively. From A_2', perpendiculars $A_2'B_1$, $A_2'B_2$ are drawn to A_1A_2, A_2A_3, respectively, and the quadrilateral $A_2'B_1A_2B_2$ is cut away. Then quadrilaterals formed in a similar way are cut away from all the other corners. The remainder is folded along $A_1'A_2'$, $A_2'A_3', \ldots, A_n'A_1'$ so as to form an open polygonal box of base $A_1'A_2'\ldots A_n'$ and of height x. Determine the maximum volume of the box and the corresponding value of x.

Solution by the proposer.

P and P' are homothetic polygons; hence all lines A_iA_i' are concurrent in a point O, the *homothetic center* (easy to prove, or see Court [1]). Moreover, A_i' is equidistant from the sides of angle A_i, so A_i' (and hence O) lies on the bisector of angle A_i and O is equidistant from every side of P. Thus there is an inscribed circle with center O and radius r (say) touching every side of P. If K and K' are the areas of P and P', we now have

$$\frac{K'}{K} = \frac{(r-x)^2}{r^2}, \quad 0 \le x \le r$$

and the volume V of the box is given by

$$V = K'x = \frac{K}{r^2}\cdot x(r-x)^2.$$

By the A.M.-G.M. inequality,

$$\frac{r}{3} = \frac{x + \dfrac{r-x}{2} + \dfrac{r-x}{2}}{3} \geq \sqrt[3]{\frac{x(r-x)^2}{4}}$$

with equality iff $x = (r-x)/2$ or $x = r/3$.

Thus, when $x = r/3$, we have

$$V_{\max} = \frac{K}{r^2} \cdot \frac{4r^3}{27} = \frac{8K^2}{27Pe},$$

where Pe denotes the perimeter of P.

This problem is an extension of one given in Stewart [2], which corresponds to the case $n = 3$.

Also solved by JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; G.P. HENDERSON, Campbellcroft, Ontario; G.D. KAYE, Department of National Defence, Ottawa; F.G.B. MASKELL, Algonquin College, Ottawa; L.F. MEYERS, The Ohio State University; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, N.J.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio. One incorrect solution was received.

Editor's comment.

The incorrect solution wrongly assumed that P and P' are similar only if the two polygons are regular.

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1. Nathan Altshiller Court, *College Geometry*, Barnes & Noble, New York, 1952, p. 38.

2. C.A. Stewart, *Advanced Caluclus*, Methuen, London, 1946.

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376. [1978: 225] *Proposed by V.G. Hobbes, Westmount, Québec.*

Isosceles triangles can be divided into two types: those with equal sides longer than the base and those with equal sides shorter than the base. Of all possible isosceles triangles what proportion are long-legged?

Solution by G. Ramanaiah, Madras Institute of Technology, Madras, India.

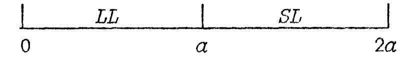
The problem is ill-posed: the answer depends upon a probability distribution, several of which are possible, but none is given. So different answers are possible, and they correspond to essentially different problems. Several will be given below.

Let a be the length of the equal sides, b that of the third side, θ the vertex angle, LL = long-legged triangle, SL = short-legged triangle, and p the proportion of of long-legged triangles.

(a) For given a, let the values of θ be uniformly distributed over the interval $(0,\pi)$:

$$p = \frac{1}{3}.$$

(b) For given a, let the values of b be uniformly distributed over the interval $(0,2a)$:



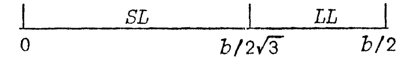
$$p = \frac{1}{2}.$$

(c) For given b, let the values of a be uniformly distributed over the interval $(b/2,\infty)$:

$$p = 1.$$

Here almost every isosceles triangle is long-legged!

(d) For given b, let the radii of the incircles be uniformly distributed over the interval $(0,b/2)$:



$$p = \frac{3-\sqrt{3}}{3}.$$

(e) For given b, let the radii of the excircles opposite the vertex angle be uniformly distributed over the interval $(b/2,\infty)$:

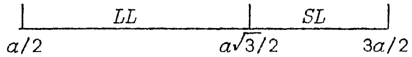
$$p = 0.$$

Here almost every isosceles triangle is short-legged!

(f) For given b, let the altitudes to side a be uniformly distributed over the interval $(0,b)$:

$$p = \frac{2-\sqrt{3}}{2}.$$

(g) For given a, let the medians to side a be uniformly distributed over the interval $(a/2,3a/2)$:



$$p = \frac{\sqrt{3}-1}{2}.$$

(h) For given b, let the bisectors of one of the equal angles be uniformly distributed over the interval $(2b/3, b\sqrt{2})$:

SL	LL
$2b/3$ — $b\sqrt{3}/2$	$b\sqrt{3}/2$ — $b\sqrt{2}$

$$p = \frac{6\sqrt{2} - 3\sqrt{3}}{6\sqrt{2} - 4}.$$

Also solved by HAYO AHLBURG, Benidorm, Spain; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; ROLAND H. EDDY, Memorial University of Newfoundland; HERTA T. FREITAG, Roanoke, Virgina; T.J. GRIFFITHS, A.B. Lucas Secondary School, London, Ontario; ROBERT S. JOHNSON, Montréal, Québec; G.D. KAYE, Department of National Defence, Ottawa; GILBERT W. KESSLER, Canarsie H.S., Brooklyn, N.Y.; FRIEND H. KIERSTEAD Jr., Cuyahoga Falls, Ohio; ANDY LIU, University of Alberta; F.G.B. MASKELL, Algonquin College, Ottawa; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Ten of the above solvers proclaimed their faith in one particular probability distribution, to the exclusion of all others, and gave a unique answer to the problem (which varied from solver to solver). But, except for one who gave a lengthy exegesis, they did not disclose on what Revelation, or on the word of what Ayatollah, their faith was based.

Problems on probabilities in continuum (most problems on geometric probability are of this nature) can, if not described unambiguously, lead to apparent paradoxes. One such that is historically important is known as *Bertrand's Paradox: Consider a circle of unit radius. The length of the side of an equilateral triangle inscribed in the circle is* $\sqrt{3}$. *What is the probability that a random chord in the circle is longer than* $\sqrt{3}$? Different answers are obtained, depending upon the randomization procedure utilized. It is not really a paradox since, as our solver noted, the different answers correspond to essentially different problems.

For Bertrand's Paradox see Neuts [1], and for a careful treatment of probabilities in continuum see Uspensky [2].

REFERENCES

1. Marcel F. Neuts, *Probability*, Allyn and Bacon, Boston, 1973, pp. 48-50.

2. J.V. Uspensky, *Introduction to Mathematical Probability*, McGraw-Hill, New York, 1937, pp. 235-259.

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377. [1978: 226] *Proposed by Michael W. Ecker, Pennsylvania State University.*

For $n = 1, 2, 3, \ldots$, let $f(n)$ be the number of zeros in the decimal representation of n, and let

$$F(p) = \sum_{n=1}^{\infty} \frac{f(n)}{n^p}.$$

Find the domain of F, that is, the real values of p for which the series $F(p)$ converges.

(This problem was suggested by Problem E 2675, *American Mathematical Monthly*, 84 (October 1977) 652.)

Solution by Steve Curran for the Beloit College Solvers, Beloit, Wisconsin.

We have $f(n) \geq 1$ if $10|n$ and $f(n) \geq 0$ if $10 \nmid n$; hence

$$F(p) = \sum_{n=1}^{\infty} \frac{f(n)}{n^p} \geq \sum_{10|n} \frac{1}{n^p} = \sum_{m=1}^{\infty} \frac{1}{(10m)^p} = \frac{1}{10^p} \sum_{m=1}^{\infty} \frac{1}{m^p}.$$

The series on the right diverges for $p \leq 1$, and so does $F(p)$ by the comparison test.

The number of digits in the decimal representation of n is $1 + [\log_{10} n]$; hence $f(n) < 1 + \log_{10} n$ and

$$F(p) = \sum_{n=1}^{\infty} \frac{f(n)}{n^p} \leq \sum_{n=1}^{\infty} \frac{1 + \log_{10} n}{n^p}.$$

By the integral test, the series on the right converges for $p > 1$, and so does $F(p)$ by comparison.

So the domain of F is $\{p | p > 1\}$.

Also solved by ALLAN Wm. JOHNSON Jr., Washington, D.C.; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio (two solutions); DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; and the proposer.

Editor's comment.

Johnson referred to Hardy and Wright [1], where it is mentioned that the series in this problem is one example of a *Dirichlet series* and that the sum function, $F(p)$, is called the *generating function* of $f(n)$.

The *Monthly* problem mentioned in our proposal was proposed by R.P. Boas. It asked for the values of $a > 0$ for which the following series converges:

$$\sum_{n \geq 1} \frac{a^{f(n)}}{n^2}.$$

The surprising answer, $0 < a < 91$, can be found in [2].

Stone referred to an interesting note by Wadhwa [3], which is related to our $F(1)$. Let Z_i denote the set of integers having exactly i zeros in their decimal representation and let

$$s_i = \sum_{n \in Z_i} \frac{1}{n}.$$

Wadhwa shows that (a) every s_i converges and the sequence (s_i) is strictly decreasing; (b) for every i, $s_i > 19.28$. It now comes as no suprise that

$$F(1) = s_1 + 2s_2 + 3s_3 + \dots$$

is divergent.

REFERENCES

1. G.H. Hardy & E.M. Wright, *An Introduction to the Theory of Numbers*, Fourth Edition, Oxford University Press, London, 1960, pp. 244-246.

2. Allen Stenger, Solution to Problem E 2675, *American Mathematical Monthly*, 86 (January 1979) 58.

3. A.D. Wadhwa, Some convergent subseries of the harmonic series, *American Mathematical Monthly*, 85 (October 1978) 661-663.

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378. [1978: 226] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

(a) Find four positive decimal integers in arithmetic progression, each having the property that if any digit is changed to any other digit, the resulting number is always composite.

(b) Can the four integers be consecutive?

(This problem was suggested by Problem 1029* in *Mathematics Magazine*, 51 (January 1978) 69.)

I. *Solution of part (a) by Murray S. Klamkin, University of Alberta.*

The smallest arithmetic progression with the desired property, which is quickly found in a table of primes, can be stretched out to five terms:

$$200, \quad 202, \quad 204, \quad 206, \quad 208.$$

The only way of possibly obtaining a prime by changing only one digit in any of these numbers is by changing a units' digit to 1, 3, 7, or 9. But here

$$3|201, \quad 7|203, \quad 3|207, \quad 11|209.$$

Many more such arithmetic progressions are easily found by scanning the table of primes further.

There are even infinite arithmetic progressions with the same property. For example,

$$n_k = 2310k - 210, \qquad k = 1, 2, 3, \dots .$$

Changing the units' digit 0 of n_k to 1, 3, 7, or 9 never yields a prime since

$$11 \mid (n_k + 1), \quad 3 \mid (n_k + 3), \quad 7 \mid (n_k + 7), \quad 3 \mid (n_k + 9).$$

II. *Solution of part (b) by Harry L. Nelson, Livermore, California; and the proposer (independently).*

With the help of a computer, we found that the following five consecutive integers have the desired property:

$$872894, \quad 872895, \quad 872896, \quad 872897, \quad 872898. \tag{1}$$

Here it is clearly sufficient to verify that 872897 and all its single-digit mutations are composite. We first find $872897 = 263 \cdot 3319$ and the rest of the verification is in the following table, in which the second column contains the smallest prime factor of the number on the left.

072897	3	802897	53	870897	3	872097	3	872807	13	872891	271
172897	41	812897	733	871897	13	872197	59	872817	3	872893	7
272897	19	822897	3	873897	3	872297	191	872827	23	872899	17
372897	3	832897	13	874897	23	872397	3	872837	7		
472897	37	842897	11	875897	11	872497	37	872847	3		
572897	13	852897	3	876897	3	872597	11	872857	43		
672897	3	862897	7	877897	17	872697	3	872867	31		
772897	757	882897	3	878897	139	872797	17	872877	3		
972897	3	892897	607	879897	3	872997	3	872887	523		

Part (a) was also solved by HAYO AHLBURG, Benidorm, Spain; HARRY L. NELSON, Livermore, California; HERMAN NYON, Paramaribo, Surinam; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Answers to part (a) are plentiful. In fact, Ahlburg and Wilke each gave more than a dozen, all found from a table of primes, in which the largest number does not go much beyond 2000. There is here an *embarras de richesses* which paradoxically impoverishes this part of the problem.

For part (b), the proposer wrote that sets of three consecutive integers with the desired property are not hard to find and he gave several examples. But he did not give the smallest such set, which is {324,325,326}. Neither he nor Nelson guaranteed that their answer (1) was the smallest possible set of four or more consecutive integers with the property, although Nelson considered it unlikely that a smaller set exists.

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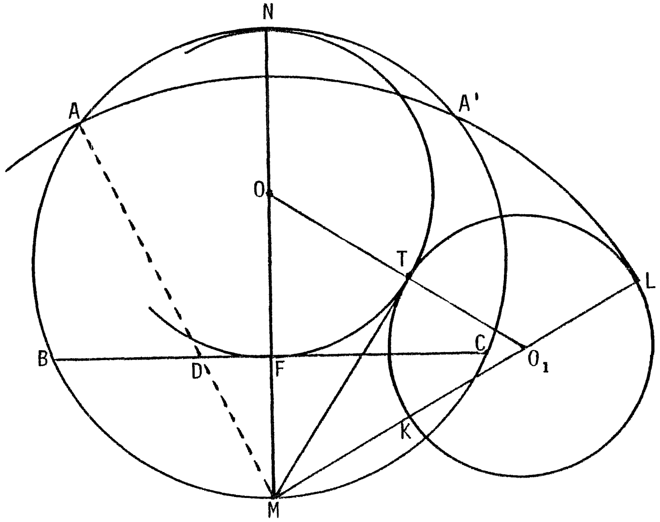
379, [1978: 226] *Proposed by Peter Arends, Algonquin College, Ottawa.*

Construct a triangle ABC, given angle A and the lengths of side a and t_a (the internal bisector of angle A).

Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.

Construction. Draw segment BC = a. By a classical construction, describe circle BMCN, in which arc BNC contains an angle equal to A, and diameter MN $\perp$ BC meets BC in F (see figure). O being the midpoint of NF, describe circle O(ON) (centre O and radius ON) and one of its tangents MT. Produce OT to O_1 so that $TO_1 = \frac{1}{2}t_a$ and describe circle $O_1(O_1T)$ to meet MO_1 in K and L, as shown in the figure. Finally, describe circle M(ML) to meet arc BNC in A and A'. Then ΔABC is one solution to the problem (the other being ΔA'BC).



Proof. Let AM cut BC in D. Since M is the midpoint of arc BMC, AD bisects angle A and we have only to show that AD = t_a. Since D, A, N, F are concyclic, we have

$$MD \cdot MA = MF \cdot MN = MT^2 = MK \cdot ML = MK \cdot MA;$$

hence MD = MK and AD = KL = t_a.

With slight modifications, the method used here works equally well if the external bisector t_a' is given instead of t_a.

Also solved by HAYO AHLBURG, Benidorm, Spain; LEON BANKOFF, Los Angeles, California; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain;

JACK GARFUNKEL, Forest Hills H.S., Flushing, N.Y.; ALLAN Wm. JOHNSON Jr., Washington, D.C.; F.G.B. MASKELL, Algonquin College, Ottawa; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam; G.C. PARANJPE and N.K. SINHA, both from the Zambia Institute of Technology, Kitwe, Zambia (independently); JEREMY D. PRIMER, student, Columbia H.S., Maplewood, N.J.; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; and JOHN A. WINTERINK, Albuquerque Technical Vocational Institute, Albuquerque, New Mexico. Comments were received from M.S. KLAMKIN, University of Alberta; and from BOB PRIELIPP, The University of Wisconsin-Oshkosh.

Editor's comment.

Ce problème a de la barbe; in other words, this is a real oldie. Several readers submitted the historical information which follows. Court gives and solves the problem [1, pp. 63-64] and says [1, p. 297] that it was considered by Pappus in his *Mathematical Collection*. The problem is also given and solved in Casey [2]. It appeared three times in the problem section of the *American Mathematical Monthly*, in 1906, 1931, and 1974 (see [3] for details), and several other unspecified published occurrences are reported in [3]. Where and when will it surface next?

Three solvers showed that the condition for solutions to exist is $a \geq 2t_a \tan(A/2)$, an inequality that is mistakenly reversed in [3]. Some solutions, both in the literature and among those recorded here, merely show that the triangle is *constructible*, but do not give details of the construction. The construction and proof presented here are, in the editor's opinion, as good as any of the others he has seen in the literature, and better than most.

REFERENCES

1. Nathan Altshiller Court, *College Geometry*, Barnes and Noble, New York, 1952.

2. John Casey, *A Sequel to Euclid*, Dublin, 1892, p. 80.

3. Editor's comment for Problem E 2499, *American Mathematical Monthly*, 82 (December 1975) 1015.

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A BORDERED PRIME MAGIC SQUARE FOR 1979

2777	1409	2339	1481	1061	2699	2087
2531	1889	2237	2459	1229	2081	1427
1367	2357	2399	1511	2027	1601	2591
2909	1031	1607	1979	2351	2927	1049
1301	2741	1931	2447	1559	1217	2657
1097	1877	1721	1499	2729	2069	2861
1871	2549	1619	2477	2897	1259	1181

Here is an array of 49 distinct four-digit primes in which the 3×3 array, the 5×5 array, and the 7×7 array, all centered at 1979, are magic squares.

ALLAN Wm. JOHNSON JR.,
Washington, D.C.

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