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CONTENTS

For Vol. 4 (1978), the support of Algonquin College, the Samuel Beatty Fund, and Carleton University is gratefully acknowledged.

A DIRECT PROOF OF PICK'S THEOREM

A. LIU

Pick's Theorem, discovered by George Pick in 1899, asserts that the area α of a simple lattice polygon is given by

$$\alpha = i + \frac{1}{2}b - 1,$$

where i and b denote respectively the number of interior and boundary lattice points of the polygon. Traditional proofs (see [1] - [5], for example) involve so-called *primitive triangles*, which satisfy $i = 0$ and $b = 3$, as well as other mathematical concepts. We give below a simple proof based on a direct counting of lattice points.

We shall use mathematical induction on the number of sides of the simple lattice polygon, and as a basis consider first a lattice triangle (not necessarily primitive). Around the triangle draw the smallest rectangle with edges parallel to the axes of the lattice grid. At least one vertex of the triangle must coincide with one vertex of the rectangle. The five nonequivalent configurations are depicted in Figure 1.

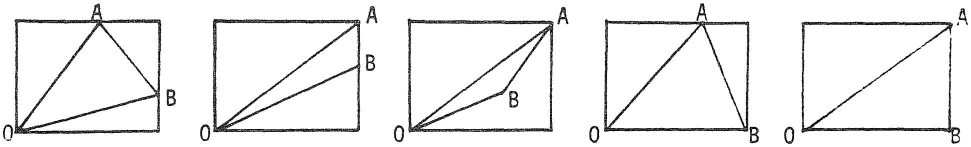

Figure 1

We shall verify Pick's Theorem for only one of these cases, the others being similar.

In Figure 2, let O be the origin of a coordinate system and let the coordinates of A and B be (p,s) and (q,r), respectively. Let x, y, and z denote respectively the number of lattice points on OB, OA, and AB (excluding the endpoints). The number of interior lattice points

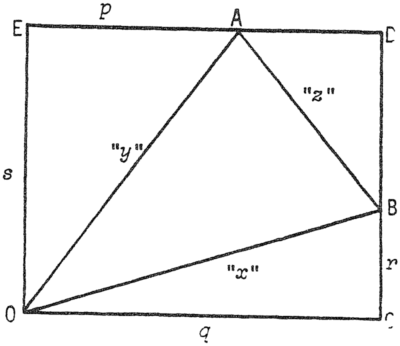

Figure 2

of OCDE $= (q-1)(s-1)$,

of OBC $= \frac{1}{2}((q-1)(r-1)-x)$,

of OAE $= \frac{1}{2}((p-1)(s-1)-y)$,

$$\text{of ABD} = \frac{1}{2}((q-p-1)(s-r-1)-z);$$

hence the number of interior lattice points of OAB is

$$i = (q-1)(s-1) - \frac{1}{2}((q-1)(r-1)-x) - \frac{1}{2}((p-1)(s-1)-y)$$
$$- \frac{1}{2}((q-p-1)(s-r-1)-z) - x - y - z$$

or

$$i = \frac{1}{2}(qs - pr - x - y - z - 1). \tag{1}$$

The number of boundary lattice points of OAB is

$$b = x + y + z + 3 \tag{2}$$

and the area of OAB is

$$\alpha = qs - \frac{1}{2}qr - \frac{1}{2}ps - \frac{1}{2}(q-p)(s-r)$$

or

$$\alpha = \frac{1}{2}(qs - pr). \tag{3}$$

Pick's Theorem for the triangle now follows from (1), (2), and (3).

For the induction step, we need the elementary observation that every simple polygon other than a triangle has an interior diagonal. This is clear if the polygon is convex. If, on the other hand, the interior angle at some vertex, say V, is greater than 180°, then a ray emanating from V and sweeping the interior of the polygon must strike another vertex (otherwise the "polygon" encloses an infinite area), and this determines an interior diagonal with V as one endpoint.

We now assume that Pick's Theorem holds for all simple lattice polygons of $3, 4, \ldots, n-1$ sides, where $n > 3$. Any simple lattice polygon P of n sides can then be partitioned into two simple lattice polygons P_1 and P_2 by an interior diagonal of P. Let d be the number of lattice points on this diagonal (excluding its endpoints). Using self-explanatory notation, we have

$$\alpha = \alpha_1 + \alpha_2, \tag{4}$$

$$i = i_1 + i_2 + d, \tag{5}$$

$$b = b_1 + b_2 - 2d - 2, \tag{6}$$

and, by the induction hypothesis,

$$\alpha_1 = i_1 + \frac{1}{2}b_1 - 1, \tag{7}$$

$$\alpha_2 = i_2 + \frac{1}{2}b_2 - 1. \tag{8}$$

Pick's Theorem now follows since (4) - (8) imply that

$$\alpha = i + \frac{1}{2}b - 1.$$

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* * *

BARBEAU CONSECUTIVE PRODUCTS

J.A.H. HUNTER and LÉO SAUVÉ

E.J. Barbeau noted [1978: 120] the following intriguing pair of consecutive products with consecutive first factors and a two-digit last detached factor:

$$2342 = 2 \cdot 1171,$$
$$2343 = 3 \cdot 11 \cdot 71.$$

We develop here an algorithm that will generate *all* pairs of consecutive products of this and similar nature.

We consider first the simplest type, in which the last detached factor has a single digit. We will find all integers x such that

$$x = a(10b + c), \qquad x + 1 = (a+1)bc, \qquad 0 < c < 9. \tag{1}$$

Eliminating x from (1) gives $(a+1)bc - a(10b+c) = 1$, which is equivalent to

$$t(uv + 1) = 19 - 10u + 10v, \tag{2}$$

where $t = 10 - c$, $u = a + 1$, and $v = b - 1$. Multiplying both sides of (2) by the nonzero number t and rearranging yield the equivalent equation

$$(10 - ut)(10 + vt) = 100 - 19t + t^2$$

which, if we revert back to a, b, c, becomes

$$[c - (10 - c)a][c + (10 - c)b] = 10 - c + c^2. \qquad (3)$$

The right side of (3) and the second factor on the left are both positive; hence the first factor is also positive, so that

$$\frac{c}{10 - c} > a \geq 1$$

and $c > 5$.

For $c = 6$, equation (3) reduces to $(3 - 2a)(3 + 2b) = 10$, and there are no solutions, since the two sides are of different parity.

For $c = 7$, equation (3) becomes

$$(7 - 3a)(7 + 3b) = 52,$$

and all the possibilities are included in the adjoining table, in which the second and third columns contain all pairs of complementary factors of 52.

c	$7 - 3a$	$7 + 3b$	a	b
7	1	52	2	15
7	2	26	-	
7	4	13	1	2

c	$9 - a$	$9 + b$	a	b
9	1	82	8	73
9	2	41	7	32

For $c = 8$, equation (3) reduces to $2(4 - a)(4 + b) = 33$, and it is clear there are no solutions.

For $c = 9$, equation (3) becomes

$$(9 - a)(9 + b) = 82,$$

and all the possibilities are listed in the table.

So the only solutions of (3) in positive integers are

$$(a,b,c) = (2,15,7), (1,2,7), (8,73,9), (7,32,9);$$

and the desired consecutive products can now be found from (1). These are, in increasing order,

$27 = 1 \cdot 27$	$314 = 2 \cdot 157$	$2303 = 7 \cdot 329$	$5912 = 8 \cdot 739$
$28 = 2 \cdot 2 \cdot 7$	$315 = 3 \cdot 15 \cdot 7$	$2304 = 8 \cdot 32 \cdot 9$	$5913 = 9 \cdot 73 \cdot 9$

The procedure, which should now be clear, can be applied to the general case. For an n-digit last detached factor, the corresponding equations are

$$x = a(10^n b + c), \qquad x + 1 = (a + 1)bc \qquad (1)'$$

and

$$[c-(10^n-c)a][c+(10^n-c)b] = 10^n - c + c^2, \qquad 5\cdot 10^{n-1} < c < 10^n - 1. \qquad (3)'$$

For $n=2$, it will be found that Diophantine equation (3)' has exactly 24 solutions (a,b,c). The corresponding pairs of Barbeau consecutive products are, in increasing order,

$267 = 1\cdot 267$	$291059 = 41\cdot 7099$
$268 = 2\cdot 2\cdot 67$	$291060 = 42\cdot 70\cdot 99$
$953 = 1\cdot 953$	$571112 = 8\cdot 71389$
$954 = 2\cdot 9\cdot 53$	$571113 = 9\cdot 713\cdot 89$
$1154 = 2\cdot 577$	$587916 = 28\cdot 20997$
$1155 = 3\cdot 5\cdot 77$	$587917 = 29\cdot 209\cdot 97$
$2342 = 2\cdot 1171$	$787709 = 13\cdot 60593$
$2343 = 3\cdot 11\cdot 71$	$787710 = 14\cdot 605\cdot 93$
$2651 = 1\cdot 2651$	$911910 = 10\cdot 91191$
$2652 = 2\cdot 26\cdot 51$	$911911 = 11\cdot 911\cdot 91$
$8931 = 3\cdot 2977$	$1679930 = 70\cdot 23999$
$8932 = 4\cdot 29\cdot 77$	$1679931 = 71\cdot 239\cdot 99$
$26324 = 4\cdot 6581$	$2036627 = 73\cdot 27899$
$26325 = 5\cdot 65\cdot 81$	$2036628 = 74\cdot 278\cdot 99$
$27134 = 2\cdot 13567$	$2334207 = 31\cdot 75297$
$27135 = 3\cdot 135\cdot 67$	$2334208 = 32\cdot 752\cdot 97$
$37023 = 7\cdot 5289$	$5641514 = 86\cdot 65599$
$37024 = 8\cdot 52\cdot 89$	$5641515 = 87\cdot 655\cdot 99$
$43455 = 15\cdot 2897$	$9939104 = 32\cdot 310597$
$43456 = 16\cdot 28\cdot 97$	$9939105 = 33\cdot 3105\cdot 97$
$71423 = 11\cdot 6493$	$46589003 = 97\cdot 480299$
$71424 = 12\cdot 64\cdot 93$	$46589004 = 98\cdot 4802\cdot 99$
$74619 = 9\cdot 8291$	$95099102 = 98\cdot 970399$
$74620 = 10\cdot 82\cdot 91$	$95099103 = 99\cdot 9703\cdot 99$

With 4 solutions for $n=1$ and 24 for $n=2$, it might be conjectured that the number of solutions increases rapidly with n, but such is apparently not the case. The number of solutions *does* increase, but not spectacularly. For $n=3$, for example, there are only 35 solutions. Since these have been found, they may as well be recorded here.

2667 = 1 • 2667
2668 = 2 • 2 • 667

84503 = 1 • 84503
84504 = 2 • 84 • 503

119910 = 10 • 11991
119911 = 11 • 11 • 991

239244 = 4 • 59811
239245 = 5 • 59 • 811

239336 = 8 • 29917
239337 = 9 • 29 • 917

251501 = 1 • 251501
251502 = 2 • 251 • 501

1279340 = 20 • 63967
1279341 = 21 • 63 • 967

1595259 = 19 • 83961
1595260 = 20 • 83 • 961

2567204 = 4 • 641801
2567205 = 5 • 641 • 801

2671334 = 2 • 1335667
2671335 = 3 • 1335 • 667

4640103 = 13 • 356931
4640104 = 14 • 356 • 931

4815188 = 28 • 171971
4815189 = 29 • 171 • 971

5966099 = 17 • 350947
5966100 = 18 • 350 • 947

7307109 = 9 • 811901
7307110 = 10 • 811 • 901

26181077 = 13 • 2013929
26181078 = 14 • 2013 • 929

27752087 = 11 • 2522917
27752088 = 12 • 2522 • 917

56911112 = 8 • 7113889
56911113 = 9 • 7113 • 889

87415118 = 98 • 891991
87415119 = 99 • 891 • 991

116435039 = 31 • 3755969
116435040 = 32 • 3755 • 969

219914271 = 243 • 904997
219914272 = 244 • 904 • 997

651039038 = 26 • 25039963
651039039 = 27 • 25039 • 963

928674395 = 605 • 1534999
928674396 = 606 • 1534 • 999

1177526019 = 109 • 10802991
1177526020 = 110 • 10802 • 991

1261079028 = 36 • 35029973
1261079029 = 37 • 35029 • 973

1430339070 = 310 • 4613997
1430339071 = 311 • 4613 • 997

1648050299 = 701 • 2350999
1648050300 = 702 • 2350 • 999

3262535198 = 802 • 4067999
3262535199 = 803 • 4067 • 999

4844999150 = 850 • 5699999
4844999151 = 851 • 5699 • 999

11991319010 = 110 • 109011991
11991319011 = 111 • 109011 • 991

27308492007 = 331 • 82502997
27308492008 = 332 • 82502 • 997

27362574035 = 965 • 28354999
27362574036 = 966 • 28354 • 999

56669255018 = 982 • 57707999
56669255019 = 983 • 57707 • 999

109893991004 = 332 • 331005997
109893991005 = 333 • 331005 • 997

496508990003 = 997 • 498002999
496508990004 = 998 • 498002 • 999

995009991002 = 998 • 997003999
995009991003 = 999 • 997003 • 999

One last word. All this has been done in base 10; but the same investigations can be carried out in any base r. One need only replace 10 by r in (1)' and (3)'.

Biographical Note

J.A.H. Hunter is the author of FUN WITH FIGURES (FIGURES FOR FUN outside North America), which has been for many years the leading daily newspaper "math feature" in the English-speaking world. The daily feature at present appears in Britain, Eire, Hong Kong, Australia, New Zealand, South Africa, the United States, Canada, and occasionally in India. He is also the author of the monthly magazine PUZZLER series, which appears in the Canadian monthly *Saturday Night*, and of several books on mathematical recreations.

His published output to date runs to more than 7000 problems, far more than the combined total of Sam Loyd and Dudeney. Sam Loyd had about 5000 problems published in his lifetime, but more than 3000 of them were chess problems and items like acrostics—not math problems; and Dudeney had well under 2000 problems published.

Hunter is clearly a candidate for inclusion in the *Guinness Book of World Records*.

* * *

REVIEW

$$\begin{array}{r} \text{HAVE} \\ \text{SOME} \\ \text{SUMS} \\ \underline{\text{TO}} \\ \text{SOLVE} \end{array}$$

The Compleat Alphametic Book. Steven Kahan, Baywood Publishing Company, Inc., 120 Marine Street, Farmingdale, N.Y., 1978, xiv + 114 pp. $4.95.

This delightful collection of alphametics springs from the pen of one of America's experts on the form. Kahan is the editor of the alphametics section of *The Journal of Pecreational Mathematics*. The book contains forty-two alphametics: the title, the dedication, twenty ideal doubly-true alphametics (*idt*), and twenty alphametics in narrative settings.

The preface provides a brief history of this type of problem, a detailed solution of the classic

$$\begin{array}{r} \text{SEND} \\ \underline{\text{MORE}} \\ \text{MONEY} \end{array},$$

and useful tips for finding solutions to most problems. Interspersed throughout the book are some "flabbergasting facts" concerning the positive integers—mathematical tidbits which are well-known to most mathematicians, although they may be new to the uninitiated layman.

Answers to all problems appear at the end of the book as well as, in an earlier section, "directed approaches" which provide clues as to the logic required to find the answer. But the clues in many of the directed approaches are rather skimpy. A few more detailed solutions, like the one in the preface, would have been more helpful to a budding alphametician. Furthermore, the author uses only the bare bones of elementary arithmetic. Some of the solutions could be improved by the use of slightly more sophisticated, but still quite elementary, techniques such as congruences.

Consider, for example, *idt* 17 on page 63, which is equivalent to

$$2(\text{NINETEEN}) + 3(\text{FOUR}) + \text{TWO} = \text{FIFTYTWO},$$

with the additional condition that TWO be even. The appearance of TWO both as an addend and at the end of the sum suggests that we first solve the easier problem

$$2(\text{NINETEEN}) + 3(\text{FOUR}) = \text{FIFTY} \cdot 10^3, \tag{1}$$

where it is clear that

$$2(\text{EEN}) + 3(\text{OUR}) \equiv 0 \pmod{10^3} \tag{2}$$

and

$$2\text{N} + 3\text{R} \equiv 0 \pmod{10}. \tag{3}$$

Since $2\text{N} < 10$ and R is even, (3) soon yields $(\text{N},\text{R}) = (1,6)$ or $(3,8)$. Substituting these in (2) gives six values of (EEN,OUR), three of which are eliminated by the requirement that TWO be even, leaving (771,486), (113,258), and (553,298). The unique solution of (1) is now quickly found. It is

$$2(30314113) + 3(6258) = 60647000,$$

from which it follows that $\text{TWO} = 492$.

Finally, the subtitle of the book is a compleat misnomer, for the alphametics in the collection involve only addition. A few examples involving other operations would be needed for completeness, and a few hints regarding the *construction* of alphametics would also have been useful.

KENNETH M. WILKE,
Topeka, Kansas.

* * *

HOW'S THAT AGAIN?

The following quotation is from H. Lenz, *Grundlagen der Elementarmathematik*, Berlin, 1961, p. 11:

„Dieses Buch ist nicht für Anfänger geschrieben, denn es setzt keine Vorkenntnisse voraus." (This book is not written for beginners, for it presupposes no previous knowledge.)

L.F.M.

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before February 1, 1979, although solutions received after that date will also be considered until the time when a solution is published.

381. *Proposé par Sidney Kravitz, Dover, N.J.*

Résoudre l'addition légumatique décimale suivante:

$$\begin{array}{r} \text{BETTE} \\ \underline{\text{TOMATE}} \\ \text{OIGNON} \end{array}.$$

382*. *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Let a, b, c, d be positive integers. Evaluate

$$\lim_{n\to\infty} \frac{a(a+b)(a+2b)\ldots(a+(n-1)b)}{c(c+d)(c+2d)\ldots(c+(n-1)d)}.$$

383. *Proposed by Daniel Sokolowsky, Antioch College, Yellow Springs, Ohio.*

Let m_a, m_b, m_c be respectively the medians AD, BE, CF of a triangle ABC with centroid G. Prove that

(a) if $m_a : m_b : m_c = a : b : c$; then ΔABC is equilateral;

(b) if $m_b/m_c = c/b$, then either (i) $b = c$ or (ii) quadrilateral AEGF is cyclic;

(c) if both (i) and (ii) hold in (b), then ΔABC is equilateral.

384*. *Proposé par Hippolyte Charles, Waterloo, Québec.*

Résoudre le système d'équations suivant pour x et y:

$$\frac{(ab+1)(x^2+1)}{x+1} = \frac{(a^2+1)(xy+1)}{y+1}$$

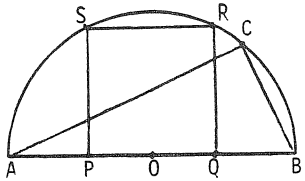
$$\frac{(ab+1)(y^2+1)}{y+1} = \frac{(b^2+1)(xy+1)}{x+1}$$

385. *Proposed by Charles W. Trigg, San Diego, California.*

In the decimal system, there is a 12-digit cube with a digit sum of 37. Each of the four successive triads into which it can be sectioned is a power of 3. Find the cube and show it to be unique.

386. *Proposed by Francine Bankoff, Beverly Hills, California.*

A square PQRS is inscribed in a semicircle (O) with PQ falling along diameter AB (see figure). A right triangle ABC, equivalent to the square, is inscribed in the same semicircle with C lying on the arc RB. Show that the incenter I of triangle ABC lies at the intersection of SB and RQ, and that



$$\frac{RI}{IQ} = \frac{SI}{IB} = \frac{1+\sqrt{5}}{2}, \text{ the golden ratio.}$$

387*. *Proposed by Harry D. Ruderman, Hunter College Campus School, New York.*

N persons lock arms to dance in a circle the traditional Israeli Hora. After a break they lock arms to dance a second round. Let $P(N)$ be the probability that for the second round no dancer locks arms with a dancer previously locked to in the first round. Find $\lim\limits_{N\to\infty} P(N)$.

388. *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

Prove that the line containing the circumcenter and the incenter of a triangle is parallel to a side of the triangle if and only if (in the usual notation)

$$s^2 = \frac{(2R-r)^2(R+r)}{R-r}.$$

389. *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

Prove that all the numbers in the sequence

$$100001,\ 10000100001,\ 1000010000100001, \ldots$$

are composite.

390. *Proposed by Gali Salvatore, Ottawa, Ontario.*

Show how to find the complete factorization of $2^{38}+1$ using only pencil and paper (no computers), having given that it consists of four distinct prime factors, none repeated, one of which is 229.

* * *

The inventor of π was not Thagoras, but an Englishman called Cumference, a relative of Falstaff. He was subsequently knighted.

C.F. MOPPERT, in *James Cook Mathematical Notes*, No. 14 (January 1978).

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

321. [1978: 65] *Proposed by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.*

For some we loved, the loveliest and the best
That from his Vintage rolling Time has prest,
Have drunk their Cup a Round or two before,
And one by one crept silently to rest.

OMAR KHAYYÁM

$$\text{ONE} \times \text{ONE} = \text{BYGONE}$$

Regard the above equality as an arithmetical multiplication in the decimal system. Each digit has been replaced by one and only one letter. Different digits have been replaced by different letters. Restore the digits.

I. *Solution by Jeremy Primer, student, Columbia H.S., Maplewood, N.J.*

Since

$$\text{ONE}(\text{ONE} - 1) = \text{BYG} \cdot 1000 = \text{BYG} \cdot 2^3 \cdot 5^3,$$

we seek two consecutive integers (hence relatively prime), one of which is of the form $8j$ with j not a multiple of 5, and the other of the form $125k$ with k odd. As

$$316 < \sqrt{100000} < \text{ONE} < 1000,$$

the only possibilities for ONE are soon found to be 376 and 625. But ONE = 376 implies B = G, so the only solution, which is satisfactory in all respects, is

$$625 \times 625 = 390625.$$

II. *Adapted from a comment by Leroy F. Meyers, The Ohio State University.*

Just as Euclid never wrote "Parallel straight lines are straight lines which..." [1975: 70], so Omar Khayyám never wrote the epigraphic verse in the proposal. It is due to Edward FitzGerald (1809-1883), who transmuted mathematician Omar Khayyám's *Rubáiyát* from the Persian into an English poetic necklace every quatrain of which is a pearl. FitzGerald, in fact, published four versions of his *Rubáiyát* during his lifetime, and the pearl quoted above first appeared in this form as quatrain No. 22 in the second version (1868).

But I will let bygones be bygones and come to the problem at hand. Numbers such as 376 and 625, which reappear at the end of their squares,

$$376^2 = 141\underline{376}, \qquad 625^2 = 390\underline{625},$$

are called *automorphic numbers*. For each $n > 1$, there are at most two n-digit automorphic numbers (if numbers with initial digit 0 are excluded). If x is one of them, the other is

$$10^n + 1 - x. \tag{1}$$

If $x > 9 \cdot 10^{n-1} + 1$, then that solution is unique, for then (1) gives a number with initial digit 0. If x^2 is required to have $2n$ digits, as in the present problem, then $x > 10^{n-\frac{1}{2}}$. All n-digit automorphic numbers for $1 \le n \le 100$ can be obtained, with the help of (1), from a table given in Gardner [1]. More information on automorphic numbers can be found in Kraitchik [2].

Also solved by MYOUNG HEE AN, Long Island City H.S., New York; GIUSEPPINA AUDISIO, student, Long Island City H.S., New York; LEON BANKOFF, Los Angeles, California; RICHARD BURNS and KRISTIN DIETSCHE, East Longmeadow H.S., East Longmeadow, Massachusetts (jointly); LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; CECILE M. COHEN, John F. Kennedy H.S., New York; CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, Pennsylvania State University; J.A.H. HUNTER, Toronto, Ontario; ALLAN Wm. JOHNSON Jr., Washington, D.C.; the following students of JACK LeSAGE, Eastview Secondary School, Barrie, Ontario: HARRY BINNENDYK, SCOTT BRUMWELL, JACKIE FERGUSON, STEVE HEAMAN, ROB HIGGINSON, MIKE HODGINS, JANE KOREVAAR, DOUG MacLEOD, SUSAN McARTHUR, PATTI MENEZES, CARYL ONYSCHUK, DEB PHILLIPS, SUSAN REYNOLDS, JIM ROBB, ALAN SMITH, KELLY SWIFT, DAVID THANASSE, BEV WALLBANK, PAT WALSH, and NANCY WHETHAM; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; the late R. ROBINSON ROWE, Sacramento, California; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas (two solutions); KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.[3]

Editor's comment.

I announced recently [1978: 205] that I would henceforth publish only answers to alphametic problems, but no full-blown solution. And no sooner said than not done, for I have here published a solution and a comment. Well-informed readers are advised to keep in mind that *Nemo doctus unquam mutationem consili inconstantiam dixit esse.*[1] And it was Emerson who said:[2] "A foolish consistency is the hobgoblin of little minds, adored by little statesmen and philosophers and divines. With consistency a great soul has simply nothing to do."

Many solvers (and even some proposers) send in only an answer; and for some problems an answer can only be obtained by brute force, which makes a detailed solution unattractive. But this great-souled editor now recognizes that elegance

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[1]No well-informed person ever imputed inconsistency to another for changing his mind. (Cicero, *Epistolae Ad Atticum*, Bk. xvi, epis. 7.)

[2]In *Essays, First Series: Self-Reliance*.

[3]Inadvertently omitted: MARK KLEIMAN, Staten Island, N.Y.

makes its own demands. An elegant solution, when such exists, will always be considered for publication.

Hunter noted that the two 100-digit automorphic numbers mentioned in Gardner [1] can also be found in Madachy [3]. They were discovered in 1964 by R.A. Fairbairn and Hunter himself.

Cohen appended to her solution: "Regards to Alan Wayne, the proposer. We miss you in New York!"

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2. Maurice Kraitchik, *Mathematical Recreations*, Dover, New York, 1953, pp. 77-78.

3. Joseph S. Madachy, *Mathematics on Vacation*, Scribner's, New York, 1966, pp. 175-176.

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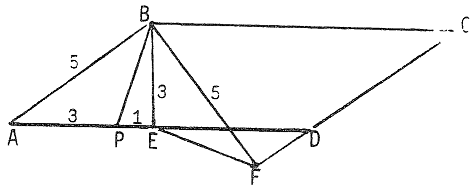
322. [1978: 65] *Proposed by Harry Sitomer, Huntington, N.Y.*

In parallelogram ABCD, $\angle A$ is acute and $AB = 5$. Point E is on AD with $AE = 4$ and $BE = 3$. A line through B, perpendicular to CD, intersects CD at F. If $BF = 5$, find EF. A geometric solution (no trigonometry) is desired.

Solution by Al Goldman, Benjamin N. Cardozo H.S., Bayside, N.Y.

Choose P on AE so that $AP = 3$ (see figure). Since $BE \perp AD$, angles BAP and FBE have corresponding sides perpendicular and so are equal. Thus $\Delta BAP \cong \Delta FBE$ (*SAS*) and

$$EF = PB = \sqrt{1^2 + 3^2} = \sqrt{10}.$$



Also solved by SOKYOUNG AN, student, Long Island City H.S., New York; LEON BANKOFF, Los Angeles, California; RICHARD BURNS and KRISTIN DIETSCHE, East Longmeadow H.S., East Longmeadow, Massachusetts; CECILE M. COHEN, John F. Kennedy H.S., New York; CLAYTON W. DODGE, University of Maine at Orono (three solutions); ROLAND H. EDDY, Memorial University of Newfoundland; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; ALLAN Wm. JOHNSON Jr., Washington, D.C.; JOE KONHAUSER, Macalester College, St. Paul, Minnesota; LAI LANE LUEY, Willowdale, Ontario; F.G.B. MASKELL, Collège Algonquin, Ottawa; HERMAN NYON, Paramaribo, Surinam; HYMAN ROSEN, Yeshiva University High School, Brooklyn, N.Y.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; CHARLES W. TRIGG, San Diego, California (two solutions); KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; JOHN A.

WINTERINK, Albuquerque Technical Vocational Institute, Albuquerque, New Mexico; and the proposer. Two incorrect solutions were received.

Editor's comment.

This problem was first devised by our proposer for the Fall 1977 Contest of the New York City Senior Interscholastic Mathematics League.

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323. [1978: 65] *Proposed by Jack Garfunkel, Forest Hills H.S., Flushing, N.Y., and M.S. Klamkin, University of Alberta.*

If $xyz = (1-x)(1-y)(1-z)$ where $0 \le x,y,z \le 1$, show that

$$x(1-z)+y(1-x)+z(1-y) \ge 3/4.$$

I. *Solution by Basil C. Rennie, James Cook University of North Queensland, Australia.*

From $0 \le x \le 1$ and $(x-\frac{1}{2})^2 \ge 0$, we get $0 \le x(1-x) \le \frac{1}{4}$. From this and two similar results,

$$xyz(1-x)(1-y)(1-z) \le \frac{1}{64}. \qquad (1)$$

By the given relation, the left side of (1) is $(xyz)^2$; so $xyz \le \frac{1}{8}$. Using the relation again yields

$$x(1-z)+y(1-x)+z(1-y) = 1-2xyz \ge \frac{3}{4}.$$

Equality occurs when $x=y=z=\frac{1}{2}$.

II. *Solution and comment by the proposers.*

We wish to minimize $\Sigma x - \Sigma yz$ subject to the constraint

$$\Sigma x - \Sigma yz = 1-2xyz.$$

Equivalently, we wish to maximize xyz subject to

$$xyz = (1-x)(1-y)(1-z).$$

More generally, we will seek to maximize $x_1x_2\ldots x_n$ subject to

$$x_1x_2\ldots x_n = (1-x_1)(1-x_2)\ldots(1-x_n), \qquad 0 \le x_i \le 1.$$

Now some $x_i = 1$ if and only if some $x_j = 0$, and then the maximum is not attained; so we assume $0 < x_i < 1$. Let

$$\frac{x_i}{1-x_i} = t_i, \quad \text{so that} \quad x_i = \frac{t_i}{1+t_i}, \quad t_i > 0.$$

Our problem now is to maximize

$$\Pi \frac{t_i}{1+t_i} \quad \text{subject to} \quad \Pi t_i = 1;$$

that is, we have to minimize

$$\Pi(1+t_i) = 1 + T_1 + T_2 + \ldots + T_n,$$

where T_r denotes the rth elementary symmetric function of $t_1, t_2, \ldots, t_n$ (e.g. $T_2 = \Sigma t_1 t_2$). By Maclaurin's inequality (see [1], for example),

$$\left\{T_r \Big/ \binom{n}{r}\right\}^{1/r} \geq T_n^{1/n},$$

with equality if and only if all $t_i = 1$ (and $x_i = \frac{1}{2}$). Thus

$$x_1 x_2 \ldots x_n \leq 2^{-n}.$$

Consquently, in the original problem, the minimum $\frac{3}{4}$ is taken on for $x = y = z = \frac{1}{2}$.

The proposed inequality was shown by the first proposer to be equivalent to showing that if AD, BE, CF are three interior concurrent cevians of a given ΔABC, then the area of ΔDEF is a maximum if and only if the three cevians are the three medians of ΔABC. The analogous geometric result for simplexes had already been established by the second proposer [2].

Also solved by LEON BANKOFF, Los Angeles, California; W.J. BLUNDON, Memorial University of Newfoundland; G.C. GIRI, Research Scholar, Indian Institute of Technology, Kharagpur, India; ALLAN Wm. JOHNSON Jr., Washington, D.C.; MARK KLEIMAN, Staten Island, N.Y.; JOE KONHAUSER, Macalester College, St. Paul, Minnesota; N. KRISHNASWAMY, student, Indian Institute of Technology, Kharagpur, India; VIKTORS LINIS, University of Ottawa; ANDY LIU, University of Alberta; F.G.B. MASKELL, Collège Algonquin, Ottawa; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam; JAMES PROPP, student, Harvard College, Cambridge, Massachusetts; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; KENNETH M. WILKE, Topeka, Kansas; and KENNETH S. WILLIAMS, Carleton University, Ottawa.

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1. G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, 1952, p. 52.

2. M.S. Klamkin, A volume inequality for simplexes, *Publ. Fac. D'Electrotechn.*, Univ. of Belgrade, No. 357 - No. 380 (1971), pp. 3-5.

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324. [1978: 66] *Proposed by Gali Salvatore, Ottawa, Ontario.*

In the determinant

$$\Delta = \begin{vmatrix} 6 & a & 6 & b \\ c & 8 & d & 2 \\ 1 & e & 5 & f \\ g & 1 & h & 1 \end{vmatrix}$$

replace the letters $a, b, \ldots, h$ by eight different digits so as to make the value of the determinant a multiple of the prime 757.

Solution by Basil C. Rennie, James Cook University of North Queensland, Australia.

There is no problem in getting $\Delta = 0$ by making two rows or columns identical (for example, make col. $3 \equiv$ col. 4), so I assume the proposer wants a nonzero multiple of 757.

Write Δ with rows and columns interchanged,

$$\Delta = \begin{vmatrix} 6 & c & 1 & g \\ a & 8 & e & 1 \\ 6 & d & 5 & h \\ b & 2 & f & 1 \end{vmatrix}, \tag{1}$$

and then compare it with a list of the 12 four-digit multiples of 757. The following are found to contain the known digits of (1) in the proper order:

$$\begin{aligned} 9 \times 757 &= 6813, \\ 13 \times 757 &= 9841, \\ 8 \times 757 &= 6056, \\ 3 \times 757 &= 2271; \end{aligned}$$

so we complete the rows of (1) accordingly. Now if c_i denotes the ith column of (1) we can replace c_4 by $1000c_1 + 100c_2 + 10c_3 + c_4$ to obtain

$$\Delta = \begin{vmatrix} 6 & 8 & 1 & 3 \\ 9 & 8 & 4 & 1 \\ 6 & 0 & 5 & 6 \\ 2 & 2 & 7 & 1 \end{vmatrix} = \begin{vmatrix} 6 & 8 & 1 & 6813 \\ 9 & 8 & 4 & 9841 \\ 6 & 0 & 5 & 6056 \\ 2 & 2 & 7 & 2271 \end{vmatrix} = 757 \times \begin{vmatrix} 6 & 8 & 1 & 9 \\ 9 & 8 & 4 & 13 \\ 6 & 0 & 5 & 8 \\ 2 & 2 & 7 & 3 \end{vmatrix} = 757 \times 2.$$

Also solved by RICHARD BURNS and KRISTIN DIETSCHE, East Longmeadow H.S., East Longmeadow, Massachusetts (jointly); HIPPOLYTE CHARLES, Waterloo, Québec; H.G. DWORSCHAK, Algonquin College, Ottawa; JOE KONHAUSER, Macalester College, St. Paul, Minnesota; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

A glance at the proposer's solution shows that he did indeed want Δ to be a nonzero multiple of 757, but he (and the editor) forgot to mention it. Three solvers saw this omission as an opportunity to get their names on the list of also-rans on the cheap: they sent in solutions for which $\Delta = 0$.

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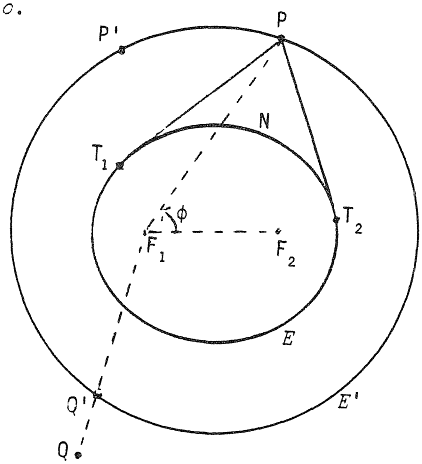
325. [1978: 66] *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

It is well-known that if you put two pins (thumb-tacks) in a drawing board and a loop of string around them you can draw an ellipse by pulling the string tight with a pencil. Now suppose that instead of the two pins you use an ellipse cut out from plywood. Will the pencil in the loop of string trace out another ellipse?

Solution by Gali Salvatore, Ottawa, Ontario.



Let E be the given (plywood) ellipse with foci F_1 and F_2 (see figure). For any point P outside E, let the focal angle of P be

$$\phi \equiv (F_1F_2, F_1P), \quad 0 \le \phi < 2\pi;$$

and, if PT_1 and PT_2 are the tangents from P to E, define

$$\delta P \equiv PT_1 + PT_2 - \text{arc}\, T_1NT_2.$$

It is clear that a point on the required locus is uniquely determined by its focal angle ϕ; and that a point P lies on the locus if and only if $\delta P = l - p$, where l is the length of the string and p the perimeter of E.

Now assume that P is a point on the required locus, and let E' be the unique confocal ellipse through P. We show that E' is the required locus, so the answer to our problem is YES.

Le P' be any point of E'. By a theorem that Salmon [5] attributes to Bishop Graves, $\delta P'$ is constant for all P' on E'; thus

$$\delta P' = \delta P = l - p,$$

and P' is on the locus. Conversely, if Q is any point on the locus, the ray F_1Q meets E' in a locus point Q', and uniqueness then requires $Q = Q'$ since Q and Q' have the same focal angle.

Also solved by the proposer. Comments (consisting mostly of references) were submitted by O. BOTTEMA, H.G. DWORSCHAK, GABRIEL KLAMBAUER (via MARCEL DÉRUAZ), J.D.E. KONHAUSER, and VIKTORS LINIS.

Editor's comment.

Nearly all references given below were included in readers' comments. Several comments claimed that our problem was first solved by Bishop Graves in 1850. This, it seems to me, is not strictly accurate. Graves's Theorem states that if E' is a confocal *ellipse*, then δP is constant for all P on E'; whereas our problem states that if δP is constant ($= l - p$) for all points P on the *curve* E', then E' is a (confocal) ellipse. Our problem is thus an exact *converse* of the Graves Theorem. Our solver proves this converse with a modest degree of rigour, after implicitly making some reasonable assumptions about the mathematical properties of plywood and string.

Graves's Theorem is proved in Edwards [1], Goursat [2] or [3], and Salmon [5]. Of the three proofs, as Déruaz noted, that of Goursat is by far the most convincing. Our problem is mentioned in Zwikker [6], but the "proof" given there is only a heuristic argument which does not utilize Graves's Theorem. Linis mentions a proof by elliptic integrals given by Klein [4]; but this is the only reference I have not actually seen, so I am unable to say whether what Klein proves is Graves's Theorem or its converse.

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4. Felix Klein, *Vorlesungen über höhere Geometrie*, Springer's Grundlehren, Bd. 22, 3rd edition, 1968, pp. 32-35.

5. George Salmon, *A Treatise on Conic Sections*, Chelsea, New York, (no date), p. 377.

6. C. Zwikker, *The Advanced Geometry of Plane Curves and their Applications*, Dover, New York, 1963, pp. 113-114.

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326. [1978: 66] This problem will be discussed in the Vol. 5 No. 1 (January 1979) issue. Solutions are still being sought.

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327. ⌜1978: 66⌝ *Proposé par F.G.B. Maskell, Collège Algonquin, Ottawa.*

Soit p_n le $n$ième nombre premier. Pour quelle(s) valeur(s) de n le nombre p_n^2+2 est-il premier?

Solution de Mark Kleiman, Staten Island, N.Y.

Le nombre $p_n^2 + 2$ est premier si et seulement si $n=2$. En effet, $p_2^2 + 2 = 3^2 + 2 = 11$ qui est premier et

$$n \neq 2 \implies p_n \neq 3 \implies p_n = 3k \pm 1 \implies p_n^2 + 2 = 3(3k^2 \pm 2k + 1).$$

Also solved by LEON BANKOFF, Los Angeles, California; RICHARD BURNS AND KRISTIN DIETSCHE, East Longmeadow H.S., East Longmeadow, Massachusetts (jointly); LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; J.A.H. HUNTER, Toronto, Ontario; J.D.E. KONHAUSER, Macalester College, St. Paul, Minnesota; PETER A. LINDSTROM, Genesee Community College, Batavia, N.Y. (two solutions); ANDY LIU, University of Alberta, Edmonton; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; JEREMY PRIMER, sophomore, Columbia H.S., Maplewood, N.J.; the late R. ROBINSON ROWE, Sacramento, California; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

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328. [1978: 66] *Proposed by Charles W. Trigg, San Diego, California.*

$2k(k+1)$ dominoes, each $2" \times 1"$, can be arranged to form a square with an empty $1" \times 1"$ space in the center.

(a) Show that for all k there is an arrangement such that no straight line can divide the ensemble into two parts without cutting a domino.

(b) Is it always possible to arrange the dominoes so that the ensemble can be separated into two parts by a straight line that cuts no domino?

Solution by Andy Liu, University of Alberta, Edmonton.

We first point out that a holed square composed of $2k(k+1)$ dominoes has side $2k+1$.

(a) Figure 1 shows the arrangement for $k=1$ and Figure 2 shows an inductive expansion from $2k+1$ to $2k+3$. It is easy to check that they have the desired property.

(b) The task is possible except when $k=1$. Figure 3 shows the arrangement for even k and Figure 4 for odd $k \neq 1$. In both figures, there is no problem in packing the blank rectangles with dominoes as one side of each retangle is even.

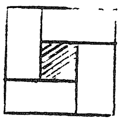

Figure 1

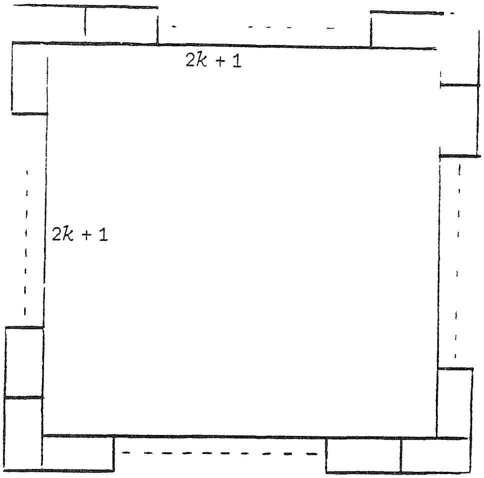

Figure 2

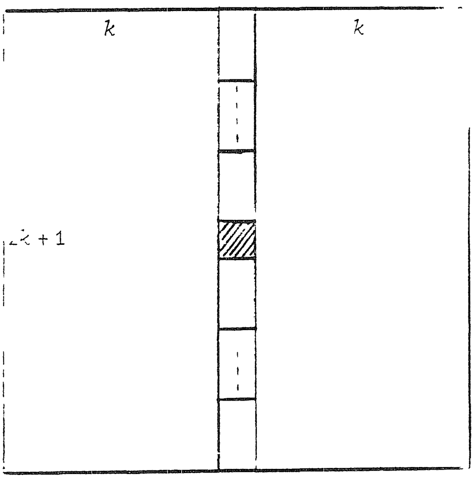

Figure 3

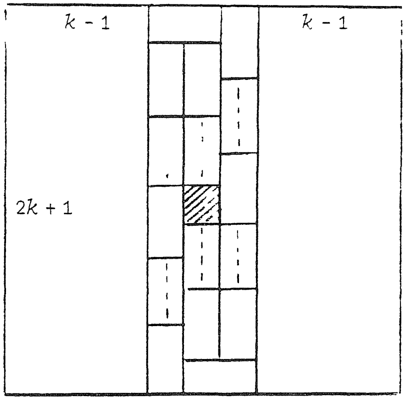

Figure 4

Also solved by CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montréal, Québec; LEROY F. MEYERS, The Ohio State University; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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329. [1978: 66] *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

"The product of the ages of my three children is less than 100," said Bill, "but even if I told you the exact product, and even told you the sum of their ages, you still couldn't figure out each child's age."

"I would have trouble if different ages are very close," said John as he looked at the children, "but tell me the product anyway."

Bill told him, and John confidently told each child his age.

If *you* can now also tell the three ages, what are they?

Solution by the proposer.

Since the knowledge of both their product and sum does not determine them uniquely, the ages must form one of a set of triples all of which have the same product and all of which have the same sum. With the product less than 100, the only possibilities are given in the following table:

Product	Sum	Set of triples	
36	13	{2,2,9}	{1,6,6}
40	14	{2,2,10}	{1,5,8}
72	14	{3,3,8}	{2,6,6}
90	20	{2,3,15}*	{1,9,10}*
90	16	{2,5,9}	{3,3,10}
96	21	{1,8,12}	{2,3,16}*

Since John has trouble with close ages, yet was confidently able to associate an age with each child, the correct triple cannot be one with consecutive ages. This eliminates the starred triples in the table.

It is also given that *you* can now tell the ages (even though *you* don't see the children or know the product that Bill gave). Had the given product been 36, for example, John could have chosen the correct set by seeing the children, but *you* could not. The given product must therefore have been 96, for which there is only one acceptable triple, {1,8,12}, which furnishes the correct ages.

Also solved by MYOUNG HEE AN, Long Island City H.S., New York; ANDY LIU, University of Alberta, Edmonton; LEROY F. MEYERS, The Ohio State University; SIDNEY PENNER, Bronx Community College, New York; KENNETH M. WILKE, Topeka, Kansas; and KENNETH S. WILLIAMS, Carleton University, Ottawa.

Editor's comment.

All but two of the above solvers arrived at the answer {1,8,12}, but the analysis by which the two different answers were arrived at appeared less convincing to the editor.

This problem is an example of the type called *Census-Taker Problems*. Such problems were one of the few amusing things to come out of World War II. For more examples see [1] and [2], especially [1]. Both references were sent by Wilke.

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1. M.H. Greenblatt, *Mathematical Entertainments*, Thomas Y. Crowell Co., New York, 1965, pp. 1-7.

2. James F. Hurley, *Litton's Problematical Recreations*, Van Nostrand Reinhold Co., New York, 1971, pp. 212, 324.

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330. [1978: 67] *Proposed by M.S. Klamkin, University of Alberta.*

It is known that if any one of the following three conditions holds for a given tetrahedron, then the four faces of the tetrahedron are mutually congruent (i.e., the tetrahedron is isosceles):

1. The perimeters of the four faces are mutually equal.
2. The areas of the four faces are mutually equal.
3. The circumcircles of the four faces are mutually congruent.

Does the condition that the incircles of the four faces be mutually congruent, also, imply that the tetrahedron be isosceles?

Solution by Leon Bankoff, Los Angeles, California.

The answer is NO. A counterexample will show that a tetrahedron may have faces with equal inradii without being isosceles.

Consider a semicircle (A) of unit radius and an inscribed isosceles ΔABC with $BC \parallel DE$, as shown in Figure 1. If $\angle BAC = \theta$, then the inradius r of ΔABC is given by

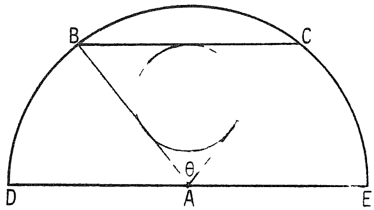

Figure 1

$$r(\theta) = \frac{\sin\theta}{2(1+\sin\frac{1}{2}\theta)}, \qquad 0 \le \theta \le 180^\circ. \tag{1}$$

Here $r(0) = r(180^\circ) = 0$, and it is an easy calculus problem to show that the maximum value of r occurs when

$$\theta = \theta_0 = 2\,\mathrm{Arcsin}\frac{\sqrt{5}-1}{2} \approx 76^\circ 20' 43''.$$

The continuity of (1) shows that for every θ_1 such that $0 < \theta_1 < \theta_0$ there is a θ_2 such that $\theta_0 < \theta_2 < 180°$ for which $r(\theta_1) = r(\theta_2)$. For example, if $\theta_1 = 60°$, we find from (1) that

$$\theta_2 = 2\,\mathrm{Arcsin}\frac{3+\sqrt{33}}{12} \approx 93°33'26'',$$

and it is easy to verify that then $r(\theta_1) = r(\theta_2) = \sqrt{3}/6$.

Now we can construct our counterexample. Let θ_1, θ_2 be angles such that

$$0 < \theta_1 < \theta_0 < \theta_2 < 180° \quad \text{and} \quad r(\theta_1) = r(\theta_2).$$

Take two congruent isosceles triangles RPS and QPS, drawn on opposite sides of their common base PS, having legs of unit length and vertical angle θ_1 at R and O; and take also two congruent isosceles triangles PQR and SOR, drawn on opposite sides of their common base QR, having legs of unit length and vertical angle θ_2 at P and S. If the first pair is hinged at the common base PS to make $\angle QPR = \theta_2$, and the second pair is hinged at the common base QR to make $\angle PRS = \theta_1$, then the two pairs can be fitted together to make the tetrahedron P-QRS shown in Figure 2. It is a nonisosceles tetrahedron whose faces are congruent only in pairs, but the inradii of the faces are all equal since $r(\theta_1) = r(\theta_2)$. Such a tetrahedron might well be called *semi-isosceles*.

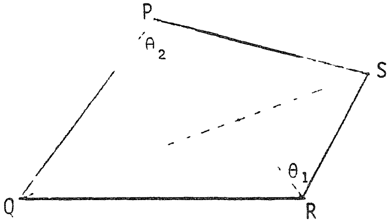

Figure 2

Editor's comment.

The proposal mentions three conditions that imply that a tetrahedron is isosceles. The first (perimeters) is easy to prove. It appears as an exercise in Altshiller-Court [1, p. 112] and a proof can be found in Honsberger [2, p. 91]. The second (areas) is proved in [1, p. 108] and in [2, p. 94]. For the third, it is easy to show first that the congruence of the circumcircles implies that the circumcentre and the incentre of the tetrahedron coincide, and then the tetrahedron is isosceles by a theorem proved in [1, p. 107] and [2, p. 96].

REFERENCES

1. Nathan Altshiller-Court, *Modern Pure Solid Geometry*, Chelsea, 1964.

2. Ross Honsberger, *Mathematical Gems II*, Mathematical Association of America, 1976.

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331. [1978: 100] *Proposed by J.A.H. Hunter, Toronto, Ontario.*

Here is a problem to welcome CRUX MATHEMATICORUM.

Each distinct letter in this alphametic stands for a particular but different digit. It is certainly appropriate, but this unique TITLE is truly odd! So what do you make of it?

$$\begin{array}{r} \text{WELL} \\ \text{WELL} \\ \text{A} \\ \underline{\text{NEW}} \\ \text{TITLE} \end{array}$$

Solution by Allan Wm. Johnson Jr., Washington, D.C.

The problem involves seven distinct letters, and no number base is specified; so we explore various bases ≥ seven. The following results will be established:

1. The alphametic has no solution in base seven or base eight.
2. It has a unique solution in base nine:

$$8711 + 8711 + 6 + 578 = 20217 \equiv 0 \pmod 2). \tag{1}$$

3. It has two solutions in base ten:

$$\begin{aligned} 4722 + 4722 + 9 + 674 &= 10127 \equiv 1 \pmod 2), \\ 9811 + 9811 + 7 + 589 &= 20218 \equiv 0 \pmod 2). \end{aligned} \tag{2}$$

4. In each base $B \geq$ eleven, it has at least two solutions in which TITLE is odd. Hence ten is the only base in which the alphametic has a unique odd TITLE.

5. In each base $B \geq$ nine, the alphametic has at least one solution in which TITLE is even. Thus ten is the only base in which the alphametic has exactly two solutions.

Proof of 5. An examination of digit patterns in (1) and (2) suggests the following letter substitutions for any base $B \geq$ nine:

$$W = B-1, \quad E = B-2, \quad L = 1, \quad A = B-3, \quad N = 5, \quad T = 2, \quad I = 0.$$

To verify that these substitutions lead to a solution we add column by column from right to left:

$L + L + A + W = 2B - 2$ or E with a carry of 1;
$L + L + E + \text{carry} = B + 1$ or L with a carry of 1;
$E + E + N + \text{carry} = 2B + 2$ or T with a carry of 2;
$W + W + \text{carry} = 2B$ which is 20 or TI in base B.

Here we have $\text{TITLE} = 2B^4 + 2B^2 + 2B - 2$, which is clearly even.

Proof of 4. Here we will use when necessary $a =$ ten, $b =$ eleven, $c =$ twelve, etc.

One solution with an odd TITLE is given by

(base eleven) $9455 + 9455 + 7 + 349 = 18154$ $(= 25469_{10})$,
(base twelve) $9744 + 9744 + 2 + \alpha 79 = 18147$ $(= 34759_{10})$,
(base sixteen) $8966 + 8966 + 5 + e98 = 12169$ $(= 74089_{10})$,

and, for other bases $B \geq$ ten, by the substitutions

$$W = B-6,\quad E = B-3,\quad L = 2,\quad A = B-1,\quad N = 6,\quad T = 1,\quad I = B-10.$$

A second solution with an odd TITLE is given by

(base eleven) $9366 + 9366 + 4 + 539 = 18163$ $(= 25479_{10})$,
(base fourteen) $7944 + 7944 + 8 + \alpha 97 = 12149$ $(= 44165_{10})$,

and, for other bases $B \geq$ eleven, by the substitutions

$$W = B-5,\quad E = B-3,\quad L = 2,\quad A = B-2,\quad N = 6,\quad T = 1,\quad I = B-8.$$

That these are solutions can be verified by column by column addition, as in the proof of 5. For the first solution we get

$$\text{TITLE} = 2B^4 - 10B^3 + B(B+1) + 2B - 3,$$

and for the second

$$\text{TITLE} = 2B^4 - 8B^3 + B(B+1) + 2B - 3,$$

both of which are clearly odd.

Proof of 1, 2, *and* 3. These can be verified without too much difficulty with pencil and paper and a modest amount of brute force; but to ease the editor's task I attach a computer print-out which attests to their veracity.

Also solved by LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montréal, Québec; PETER A. LINDSTROM, Genesee Community College, Batavia, N.Y.; J. WALTER LYNCH, Georgia Southern College, Statesboro, Georgia; F.G.B. MASKELL, Collège Algonquin, Ottawa; HERMAN NYON, Paramaribo, Surinam; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

There was indeed an authentic-looking computer print-out which attested to the veracity of 1, 2, and 3. It now reposes in the archives of this journal.

Johnson (Robert S., not our featured solver) appended to his answer: "And a tip of my hat to my friend the proposer in Toronto!"

* * *

332. [1978: 100] *Proposed by Leroy F. Meyers, The Ohio State University.*

In the quadratic equation

$$A(\sqrt{3}-\sqrt{2})x^2 + \frac{B}{\sqrt{2}+\sqrt{3}}x + C = 0,$$

we are given:

$A = \sqrt[4]{49+20\sqrt{6}}$;

B = the sum of the geometric series

$$8\sqrt{3} + (8\sqrt{6})(3^{-\frac{1}{2}}) + 16(3^{-\frac{1}{2}}) + \ldots;$$

and the difference of the roots is

$$(6\sqrt{6})^{\log 10 \,-\, 2\log\sqrt{5} \,+\, \log\sqrt{\log 18 \,+\, \log 72}},$$

where the base of the logarithms is 6. Find the value of C.

(This was the first of three problems in a final examination, 3 hours long, for Dutch high school students in 1916.)

I. *Solution by Leon Bankoff, Los Angeles, California.*

The contrived obfuscation in this problem is easily eliminated. First, note that

$$A = \sqrt[4]{49+20\sqrt{6}} = \sqrt{5+2\sqrt{6}} = \sqrt{3}+\sqrt{2}.$$

Next, observe that the geometric series has common ratio $\sqrt{2}/\sqrt{3} < 1$, so it converges to

$$B = \frac{8\sqrt{3}}{1-\dfrac{\sqrt{2}}{\sqrt{3}}} = 24(\sqrt{2}+\sqrt{3}).$$

Thus the given quadratic can be written

$$x^2 + 24x + C = 0,$$

and the sum and product of its roots are respectively -24 and C. Since $18 \cdot 72 = 6^4$, the exponent in the expression for the difference of the roots is

$$\log 2 + \log 5 - \log 5 + \log 2 = 2\log 2, \quad \text{giving} \quad \left(6^{3/2}\right)^{2\log 2} = 6^{\log 8} = 8.$$

With a sum of -24 and a difference of 8, the roots must be -8 and -16, so

$$C = (-8)(-16) = 128.$$

II. *Comment by the proposer.*

The reader's first question is likely to be an approximate translation of the title of the article [1] where I found this problem: "And what were the other two problems?" The author (the article was unsigned, so I presume he was the editor) lists all the prerequisite knowledge needed to solve the problem, and indicates that it would have been better pedagogy to give separate questions for each technique. He gives as examples of better examinations one from the Cape of Good Hope (South Africa) and one for entrance to Brown University in 1925. Would the reader like to give (or take!) that Dutch examination from 1916 with no partial credit allowed? I think that Hall and Knight would have rather enjoyed it (see [1977: 58-59]).

Also solved by MYOUNG HEE AN, 1978 graduate, Long Island City H.S., New York; CECILE M. COHEN, J.F. Kennedy H.S., New York; CLAYTON W. DODGE, University of Maine at Orono; G.C. GIRI, Research Scholar, Indian Institute of Technology, Kharagpur, India; ALLAN Wm. JOHNSON Jr., Washington, D.C.; J.D.E. KONHAUSER, Macalester College, Saint Paul, Minnesota; N. KRISHNASWAMY, student, Indian Institute of Technology, Kharagpur, India; LAI LANE LUEY, Willowdale, Ontario; F.G.B. MASKELL, Collège Algonquin, Ottawa; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; JEREMY PRIMER, student, Columbia H.S., Maplewood, N.J.; HYMAN ROSEN, Yeshiva University H.S., Brooklyn, N.Y.; the late R. ROBINSON ROWE, Sacramento, California; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; KENNETH M. WILKE, Topeka, Kansas; and the proposer. Three incorrect solutions were received.

Editor's comment.

All correct solutions submitted were essentially alike, and practically any one of them could have been selected to be the featured one. The one I finally chose was one of only three which indicated an awareness on the part of the solver that the geometric series *did* converge because its common ratio was less than 1.

Of the three incorrect solvers, one admitted being unfamiliar with logarithms, particularly in an unusual base, so it figures. The other two were professional mathematicians who obviously did not graduate from a Dutch high school.

REFERENCE

1. P. Wijdenes, Voor het laatst twee vraagstukken?, *Euclides*, 6 (1929-1930) 263-276.

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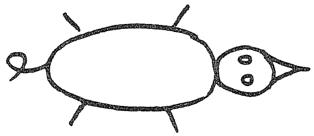
333. [1978: 101] *Proposed by the late R. Robinson Rowe, Sacramento, California.*

The World War I COOTIE, lousy vector of trench fever, popularized a simple but hilarious game by that name in the early 1920's. Five or more players, each with pad and pencil, cast a single die in turn. Rolling a 6, a player sketched a "body" on the pad (see figure) and on later turns added a head with a 5, four legs with a 4, the tail with a 3. Having the head, he could add two eyes with a 2 and

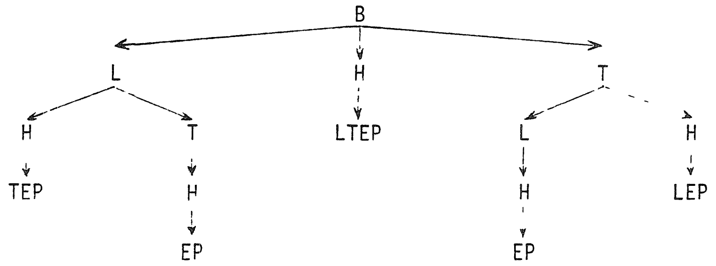


a proboscis or nose with a 1. Having all six, he yelled "COOOOOTIEEEEE!" and raked in the pot.

What was the probability of capturing a COOTIE in just six turns?

I. *Solution by Vince Streif for the Beloit College Solvers, Beloit, Wisconsin.*

Let B, L, H, T, E, P represent respectively the addition of a body, legs, head, tail, eyes, and proboscis. You must first have a body, and you must have a head before you add either eyes or a proboscis, but otherwise the order is flexible.



$$\left(\frac{1}{6}\right)^6(3!) \qquad \left(\frac{1}{6}\right)^6(2!) \qquad \left(\frac{1}{6}\right)^6(4!) \qquad \left(\frac{1}{6}\right)^6(2!) \qquad \left(\frac{1}{6}\right)^6(3!)$$

The diagram shows all possible ways of forming a COOTIE, together with the probability of occurrence of each of the five mutually exclusive ways. The probability of capturing a COOTIE in just six turns is thus

$$\left(\frac{1}{6}\right)^6(3! + 2! + 4! + 2! + 3!) = \frac{5}{5832} = 0.000857\ldots .$$

II. *Comment by the proposer.*

The chance seems small, but with 5 players and perhaps 10 games in an evening —with many parties and many evenings—the chance becomes realistic. The chance increases rapidly for 7, 8, or 9 turns, so that it is not surprising that a player was often "loused", with someone shouting COOTIE! before he even had a body on his pad. This was one cause of hilarity; the other was the weird diversity of sketches by sub-artistic players. This feature was lacking in patented games with attachable parts (and modified critical paths) which soon appeared on the market.

Also solved by LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; ROBERT S. JOHNSON, Montréal, Québec; JACK LeSAGE, Eastview Secondary School, Barrie, Ontario; LAI LANE LUEY, Willowdale, Ontario; J. WALTER LYNCH, Georgia Southern College, Statesboro, Georgia; JEREMY

PRIMER, student, Columbia H.S., Maplewood, N.J.; HYMAN ROSEN, Yeshiva University H.S., Brooklyn, N.Y.; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

Editor's comment.

Four of the above solvers arrived at a different answer by interpreting the rules of the game differently and using combinations of bodiless tails and legs, and headless eyes and proboscides.

Well, if you can have a grin without a cat... .

* * *

LETTER TO THE EDITOR

Dear Editor:

References [1], [2], and [11] in [1978: 84] are incomplete. They should read:

1. J.S. Mackay, Early History of the Symmedian Point, *Proceedings of the Edinburgh Mathematical Society*, XI (1892-1893), pp. 92-103.

2. J.S. Mackay, Symmedians of a Triangle and their Concomitant Circles, *Proceedings of the Edinburgh Mathematical Society*, XIV (1896), pp. 37-103.

11. Émile Lemoine, Note sur un point remarquable du plan d'un triangle, *Nouvelles Annales de Mathématiques*, 2^{e} série, t. XII (1873), pp. 364-366.

LEROY F. MEYERS,
The Ohio State University.

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A PANMAGIC SQUARE FOR 1978

This panmagic square of order 4 and constant 1978 gives this constant in 32 ways: as the sum of its elements in 4 rows, 4 columns, 2 main diagonals, 3 broken upward diagonals, 3 broken downward diagonals, and 16 groups of 4 adjoining cells (the sides marked *a* and those marked *b* being considered to be identified).

a

487	494	496	501
498	499	489	492
493	488	502	495
500	497	491	490

b *b*

a

RAM REKHA TIWARI,
The Belsund Sugar Co.,
P.O. Riga, Bihar, India.

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