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THE OLYMPIAD CORNER: 67

M.S. KLAMKIN

The Twenty-Sixth International Mathematical Olympiad (IMO) was held this year in Finland from June 29 to July 9. Teams from 38 countries took part in the competition. This was again a record number of participating countries, up from last year's record of 34 countries. The maximum team size for each country was 6 students, the same as for the last two years. If the number of participating countries continues to increase, the maximum team size will probably be reduced to 4 students (as occurred in Hungary in 1982). Having a smaller team size should make it easier for countries with relatively small populations to field better teams. Also, the expenses will be reduced and the logistics made easier. The total number of participating students was a new record of 209, up from last year's record of 192. The countries participating for the first time were China, Iceland, Iran, and Turkey.

The 1986, 1987, and 1988 IMO's are to be held in Poland, Cuba, and Australia, respectively. I fully expect that there will be a new record number of participating countries for the 1988 Australian IMO.

The six problems of the competition were assigned equal weights of 7 points each (the same as in the last four IMO's) for a maximum possible individual score of 42 (and a maximum possible team score of 252). I believe that this year's competition was harder than last year's. This is evidenced by the facts that only 10 students had scores of at least 35 (last year there were 24) and only 2 had perfect scores (8 last year). For comparison purposes, see the last four IMO reports in [1981: 220], [1982: 223], [1983: 205], and [1984: 249].

First, Second, and Third Prizes were awarded to students with scores in the respective intervals [34,42], [22,33], and [15,21]. Congratulations to the 14 First-Prize winners:

Name	Country	Score
Geza Kos	Hungary	42
Daniel Taturu	Rumania	42
Gabor Megyesi	Hungary	38
Nikolai I. Chaudarov	Bulgaria	37
Philippe Alphonse	Belgium	36
Olga Leonteva	U.S.S.R.	36
Andrew Hassell	Australia	35
Vasil B. Daskalov	Bulgaria	35
Waldemar Horwat	U.S.A.	35
Nguyen T. Dung	Vietnam	35
Hagen V. Eitzen	West Germany	34
Radu Negulescu	Rumania	34
Gelca Razvan	Rumania	34
Jeremy Kahn	U.S.A.	34

As the IMO competition is an individual event, the results are announced officially only for individual team members. However, team standings are usually compiled unofficially by adding up the scores of individual team members. The team results are given in the following table (where the team size is given if less than 6). Congratulations to the winning team, that of Rumania, the country that originated the IMO in 1959.

Rank	Country	Score (max 252)	Prizes 1st	2nd	3rd	Total Prizes	
1	Rumania	201	3	3	-	6	
2	U.S.A.	180	2	4	-	6	
3	Hungary	168	2	2	2	6	
4	Bulgaria	165	2	3	-	5	
5	Vietnam	144	1	3	1	5	
6	U.S.S.R.	140	1	2	2	5	
7	West Germany	139	1	1	4	6	
8	East Germany	136	-	3	3	6	
9	France	125	-	2	3	5	
10	Great Britain	121	-	2	3	5	
11	Australia	117	1	1	2	4	
12,13	Canada	105	-	1	4	5	
12,13	Czechoslovakia	105	-	3	1	4	
14	Poland	101	-	1	4	5	
15	Brazil	83	-	-	2	2	
16	Israel	81	-	1	-	1	
17	Austria	77	-	-	3	3	
18	Cuba	74	-	-	2	2	
19	The Netherlands	72	-	-	1	1	
20	Greece	69	-	1	1	2	
21	Yugoslavia	68	-	-	2	2	
22	Sweden	65	-	-	1	1	
23	Mongolia	62	-	1	-	1	
24,25	Belgium	60	1	-	1	2	
24,25	Morocco	60	-	-	2	2	
26,27	Colombia	54	-	-	2	2	
26,27	Turkey	54	-	-	2	2	
28	Tunisia	46	-	-	2	2	Team of 4
29	Algeria	36	-	-	-	-	
30	Norway	34	-	-	-	-	
31	Iran	28	-	1	-	1	Team of 1
32,33	China	27	-	1	1	2	Team of 2
32,33	Cyprus	27	-	-	1	1	
34,35	Finland	25	-	-	-	-	
34,35	Spain	25	-	-	-	-	Team of 4
36	Italy	20	-	-	-	-	Team of 5
37	Iceland	13	-	-	-	-	Team of 2
38	Kuwait	7	-	-	-	-	Team of 5

The members, scores, prizes, and leaders of the Canadian and U.S.A. teams were as follows:

Martin Piotte	18	(3rd prize)	Ron Scoins, University of Waterloo
Eric Veach	20	(3rd prize)	Tom Griffiths, University of Western Ontario
Frank D'Ippolito	16	(3rd prize)	
Minh Vo	26	(2nd prize)	
Moses Klein	15	(3rd prize)	
Giuseppe Russo	6		

Bjorn Poonen	31	(2nd prize)	Cecil Rousseau, Memphis State University
David Moews	27	(2nd prize)	Gregg Patruno, Columbia University
Jeremy Kahn	34	(1st prize)	
David Grabiner	29	(2nd prize)	
Joseph Keane	24	(2nd prize)	
Waldemar Horwat	35	(1st prize)	

I was glad to note the improved Canadian performance compared to last year's, when the Canadian team ranked 20th with a score of 83 and only one prize. It is of interest to note that Jeremy Kahn (U.S.A.) has so far participated in three IMO's and won 2nd prizes in the first two. Since he has not yet graduated, he may even participate in his fourth IMO next year (he will be 16 years old in October of this year).

The problems of this year's competition are given below. Solutions to these problems, along with those of the 1985 U.S.A. Mathematical Olympiad, will appear in a booklet, *Olympiads for 1985*, obtainable later this year for a small charge from

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M.A.A. Committee on H.S. Contests
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University of Nebraska
Lincoln, Nebraska 68588

26TH INTERNATIONAL MATHEMATICAL OLYMPIAD

First day: July 4, 1985. Time: 4½ hours

1. A circle has centre on the side AB of the cyclic quadrilateral ABCD. The other three sides are tangent to the circle. Prove that AD + BC = AB. (Great Britain)

2. Let n and k be given relatively prime natural numbers, $0 < k < n$. Each number in the set $M = \{1, 2, \ldots, n-1\}$ is coloured either blue or white. It is given that

(i) for each $i \in M$, both i and $n-i$ have the same colour, and

(ii) for each $i \in M$, $i \neq k$, both i and $|i - k|$ have the same colour.

Prove that all numbers in M must have the same colour. (Australia)

3. For any polynomial $P(x) = a_0 + a_1x + \ldots + a_kx^k$ with integer coefficients, the number of coefficients which are odd is denoted by $w(P)$. For $i = 0,1,2,\ldots$, let $Q_i(x) = (1 + x)^i$. Prove that if $i_1,i_2,\ldots,i_n$ are integers such that $0 \le i_1 < i_2 < \ldots < i_n$, then

$$w(Q_{i_1} + Q_{i_2} + \ldots + Q_{i_n}) \ge w(Q_{i_1}).$$

(The Netherlands)

Second day: July 5, 1985. Time: 4½ hours

4. Given is a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26. Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer. (Mongolia)

5. A circle with centre O passes through the vertices A and C of triangle ABC, and intersects the segments AB and BC again at distinct points K and N,

respectively. The circumscribed circles of the triangles ABC and KBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle. (U.S.S.R.)

6. For every real number x_1, construct the sequence x_1, x_2, ... by setting

$$x_{n+1} = x_n(x_n + \frac{1}{n})$$

for each $n \geq 1$. Prove that there exists exactly one value of x_1 for which $0 < x_n < x_{n+1} < 1$ for every n. (Sweden)

In my report on last year's IMO [1984: 249], I expressed critical views (shared by some others) about certain aspects of the IMO. One of my criticisms had to do with the process of awarding special prizes. I was glad to learn that this year, for the first time, a special committee of some of the jury members was formed to make recommendations for special prizes. The process was previously too cursory and haphazard. I would suggest further that, the next time special prizes are awarded, the solution(s) deemed to deserve the prize should be published *as is* in the four official languages in the IMO proceedings as well as in the corresponding student's Olympiad Journal back in his home country.

My other criticism was that too many prizes are awarded. This year there were

14 First Prizes for scores in the interval [34,42],
36 Second Prizes for scores in the interval [22,33],
52 Third Prizes for scores in the interval [15,21].

Thus there was a total of 102 prizes among the 209 contestants. While it is traditional to award prizes to approximately half the contestants, I and others have been against this policy for some time. I am even more strongly opposed to this policy now that the number of participating countries has increased significantly, resulting in an increasing number of "weaker" students. In my view, this policy has the effect of "cheapening" the value of the prizes. Note that a score of 15, which is approximately 1/3 the maximum score, is now considered worthy of a prize. I suppose this policy was instituted so that most teams would have something they could boast of to the folks back home. (This year, for example, 31 of the 38 teams earned at least one prize.) But is this a good enough reason? I agree with the suggestion of Ed Barbeau, a former Canadian team leader, that a score of at least 21 (half the maximum score) be required for a Third or higher Prize. Under this policy, the number of prizes awarded this year would have been about halved. In the 1981 IMO, many students complained that the test was too easy and not up to the usual IMO standards. In view of this, I would be most interested in learning how the contestants themselves feel about the large number of prizes awarded.

I have so far given complete team results for the 22nd IMO (1981), the 23rd (1982), the 24th (1983), the 25th (1984), and the 26th (1985). (For 1981-1984, see the references on page 202 of the present column.) For completeness, I now give complete team results from the 1st IMO (1959) to the 21st (1979). (There was no IMO in 1980.) I am grateful to Jim Williams (Australia) for making these results available to me. He photographed them from large posters at the 1984 IMO in Prague and sent me readable prints. I subsequently found these results in the article "International Mathematical Olympiad and Forms of Work with Talented Pupils", by J. Moravčik, ČSSR, in the booklet *MMO PRAHA (1984) SYMPOSIUM*.

	Rank	Country	Score	Prizes 1st	2nd	3rd	Team size
1st IMO (1959) in Rumania	1	Rumania	249	1	2	2	8
	2	Hungary	233	1	1	2	8
	3	Czechoslovakia	192	1	-	-	8
	4	Bulgaria	131	-	-	-	8
	5	Poland	122	-	-	-	8
	6	U.S.S.R.	111	-	-	1	4
	7	East Germany	40	-	-	-	8
			max 320	3	3	5	52
2nd IMO (1960) in Rumania	1	Czechoslovakia	257	1	1	2	8
	2,3	Hungary	248	2	2	-	8
	2,3	Rumania	248	1	1	1	8
	4	Bulgaria	175	-	-	1	8
	5	East Germany	38	-	-	-	8
			max 360	4	4	4	40
3rd IMO (1961) in Hungary	1	Hungary	270	2	3	1	8
	2	Poland	203	1	-	-	8
	3	Rumania	197	-	1	1	8
	4	Czechoslovakia	159	-	-	1	8
	5	East Germany	146	-	-	1	8
	6	Bulgaria	108	-	-	-	8
			max 320	3	4	4	48
4th IMO (1962) in Czechoslovakia	1	Hungary	289	2	3	2	8
	2	U.S.S.R.	263	2	2	2	8
	3	Rumania	257	-	3	3	8
	4,5	Poland	212	-	1	3	8
	4,5	Czechoslovakia	212	-	1	3	8
	6	Bulgaria	196	-	1	2	8
	7	East Germany	153	-	1	-	8
			max 368	4	12	15	56
5th IMO (1963) in Poland	1	U.S.S.R.	271	4	3	1	8
	2	Hungary	234	-	5	3	8
	3	Rumania	191	1	1	3	8
	4	Yugoslavia	162	1	2	1	8
	5	Czechoslovakia	151	1	-	1	8
	6	Bulgaria	145	-	-	3	8
	7	East Germany	140	-	-	3	8
	8	Poland	134	-	-	2	8
			max 320	7	11	17	64
6th IMO (1964) in U.S.S.R.	1	U.S.S.R.	269	3	1	3	8
	2	Hungary	253	3	1	1	8
	3	Rumania	213	-	2	3	8
	4	Poland	209	1	1	3	8
	5	Bulgaria	198	-	-	3	8
	6	East Germany	196	-	1	2	8

	Rank	Country	Score	Prizes 1st	2nd	3rd	Team size
	7	Czechoslovakia	194	-	2	2	8
	8	Mongolia	169	-	-	1	8
	9	Yugoslavia	155	-	1	1	8
			max 336	7	9	19	72
7th IMO (1965) in East Germany	1	U.S.S.R.	281	5	2	-	8
	2	Hungary	244	3	2	2	8
	3	Rumania	222	-	4	3	8
	4	Poland	178	-	1	3	8
	5	East Germany	175	-	2	3	8
	6	Czechoslovakia	159	-	1	3	8
	7	Yugoslavia	137	-	-	2	8
	8	Bulgaria	93	-	-	1	8
	9	Mongolia	63	-	-	-	8
	10	Finland	62	-	-	-	8
			max 320	8	12	17	80
8th IMO (1966) in Bulgaria	1	U.S.S.R.	293	5	1	1	8
	2	Hungary	281	3	1	2	8
	3	East Germany	280	3	3	-	8
	4	Poland	269	1	4	1	8
	5	Rumania	257	1	2	2	8
	6	Bulgaria	236	-	1	3	8
	7	Yugoslavia	224	-	2	1	8
	8	Czechoslovakia	215	-	1	2	8
	9	Mongolia	90	-	-	-	8
			max 320	13	15	12	72
9th IMO (1967) in Yugoslavia	1	U.S.S.R.	275	3	3	2	8
	2	East Germany	257	3	3	1	8
	3	Hungary	251	2	3	3	8
	4	Great Britain	231	1	2	4	8
	5	Rumania	214	1	1	4	8
	6,7	Bulgaria	159	1	-	1	8
	6,7	Czechoslovakia	159	-	1	3	8
	8	Yugoslavia	136	-	-	3	8
	9	Sweden	135	-	-	2	8
	10	Italy	110	-	1	1	6
	11	Poland	101	-	-	1	8
	12	Mongolia	87	-	-	1	8
	13	France	41	-	-	-	5
			max 336	11	14	26	99
10th IMO (1968) in U.S.S.R.	1	East Germany	304	5	3	-	8
	2	U.S.S.R.	298	5	1	2	8
	3	Hungary	291	3	3	2	8
	4	Great Britain	263	3	2	2	8
	5	Poland	262	2	3	2	8
	6	Sweden	256	1	2	5	8
	7	Czechoslovakia	248	2	4	-	8
	8	Rumania	208	1	1	2	8

	Rank	Country	Score	Prizes 1st	2nd	3rd	Team size
	9	Bulgaria	204	-	3	1	8
	10	Yugoslavia	177	-	-	3	8
	11	Italy	132	-	-	1	8
	12	Mongolia	74	-	-	-	8
			max 320	22	22	20	96
11th IMO (1969) in Rumania	1	Hungary	247	1	4	2	8
	2	East Germany	240	-	4	4	8
	3	U.S.S.R.	231	1	3	3	8
	4	Rumania	219	-	4	2	8
	5	Great Britain	193	1	1	1	8
	6	Bulgaria	189	-	-	3	8
	7	Yugoslavia	181	-	2	2	8
	8	Czechoslovakia	170	-	-	3	8
	9	Mongolia	120	-	-	1	8
	10,11	France	119	-	1	-	8
	10,11	Poland	119	-	1	-	8
	12	Sweden	104	-	-	-	8
	13	Belgium	57	-	-	-	8
	14	Netherlands	51	-	-	-	8
			max 320	3	20	21	112
12th IMO (1970) in Hungary	1	Hungary	233	3	1	3	8
	2,3	East Germany	221	1	2	4	8
	2,3	U.S.S.R.	221	2	1	3	8
	4	Yugoslavia	209	-	3	3	8
	5	Rumania	208	-	3	4	8
	6	Great Britain	180	1	-	6	8
	7,8	Bulgaria	145	-	-	3	8
	7,8	Czechoslovakia	145	-	-	4	8
	9	France	141	-	1	4	8
	10	Sweden	110	-	-	2	8
	11	Poland	105	-	-	1	8
	12	Austria	104	-	-	1	8
	13	Netherlands	87	-	-	1	8
	14	Mongolia	58	-	-	1	8
			max 320	7	11	40	112
13th IMO (1971) in Czechoslovakia	1	Hungary	255	4	4	-	8
	2	U.S.S.R.	205	1	5	2	8
	3	East Germany	142	1	1	4	8
	4	Poland	118	1	-	4	8
	5,6	Great Britain	110	-	1	4	8
	5,6	Rumania	110	-	1	4	8
	7	Austria	82	-	-	4	8
	8	Yugoslavia	71	-	-	2	8
	9	Czechoslovakia	55	-	-	1	8
	10	Netherlands	48	-	-	2	8
	11	Sweden	43	-	-	2	7
	12	Bulgaria	39	-	-	-	8
	13	France	38	-	-	-	8

	Rank	Country	Score	Prizes 1st	2nd	3rd	Team size
	14	Mongolia	26	–	–	–	8
	15	Cuba	9	–	–	–	4
			max 320	7	12	29	115
14th IMO (1972) in Poland	1	U.S.S.R.	270	2	4	2	8
	2	Hungary	263	3	3	2	8
	3	East Germany	239	1	3	4	8
	4	Rumania	206	1	3	1	8
	5	Great Britain	179	–	2	4	8
	6	Poland	160	1	1	1	8
	7,8	Austria	136	–	–	5	8
	7,8	Yugoslavia	136	–	–	3	8
	9	Czechoslovakia	130	–	–	4	8
	10	Bulgaria	120	–	–	2	8
	11	Sweden	60	–	–	2	8
	12	Netherlands	51	–	–	–	8
	13	Mongolia	49	–	–	–	8
	14	Cuba	14	–	–	–	3
			max 320	8	16	30	107
15th IMO (1973) in U.S.S.R.	1	U.S.S.R.	254	3	2	3	8
	2	Hungary	215	1	2	5	8
	3	East Germany	188	–	3	4	8
	4	Poland	174	–	2	4	8
	5	Great Britain	164	1	–	5	8
	6	France	153	–	3	1	8
	7	Czechoslovakia	149	–	1	4	8
	8	Austria	144	–	–	6	8
	9	Rumania	141	–	1	3	8
	10	Yugoslavia	137	–	–	5	8
	11	Sweden	99	–	1	1	8
	12,13	Bulgaria	96	–	–	1	8
	12,13	Netherlands	96	–	–	2	8
	14	Finland	86	–	–	2	8
	15	Mongolia	65	–	–	1	8
	16	Cuba	42	–	–	1	5
			max 320	5	15	48	125
16th IMO (1974) in East Germany	1	U.S.S.R.	256	2	3	2	8
	2	U.S.A.	243	–	5	3	8
	3	Hungary	237	1	3	3	8
	4	East Germany	236	–	5	2	8
	5	Yugoslavia	216	2	1	2	8
	6	Austria	212	1	1	4	8
	7	Rumania	199	1	1	3	8
	8	France	194	1	1	3	8
	9	Great Britain	188	–	1	3	8
	10	Sweden	187	1	1	–	8
	11	Bulgaria	171	–	1	4	8
	12	Czechoslovakia	158	–	–	2	8
	13	Vietnam	146	1	1	2	5
	14	Poland	138	–	–	2	8

Rank	Country	Score	Prizes 1st	2nd	3rd	Team size
15	Netherlands	112	-	-	1	8
16	Finland	111	-	-	1	8
17	Cuba	65	-	-	-	7
18	Mongolia	60	-	-	-	8
		max 320	10	24	37	140

17th IMO (1975) in Bulgaria

Rank	Country	Score	1st	2nd	3rd	Team size
1	Hungary	258	-	5	3	8
2	East Germany	249	-	4	4	8
3	U.S.A.	247	3	1	3	8
4	U.S.S.R.	246	1	3	4	8
5	Great Britain	239	2	2	3	8
6	Austria	192	1	1	2	8
7	Bulgaria	186	-	1	4	8
8	Rumania	180	-	1	3	8
9	France	176	1	1	1	8
10	Vietnam	175	-	1	3	7
11	Yugoslavia	163	-	1	1	8
12	Czechoslovakia	162	-	-	2	8
13	Sweden	160	-	2	-	8
14	Poland	124	-	1	1	8
15	Greece	95	-	1	-	8
16	Mongolia	75	-	-	1	8
17	Netherlands	67	-	-	1	8
		max 320	8	25	36	135

18th IMO (1976) in Austria

Rank	Country	Score	1st	2nd	3rd	Team size	
1	U.S.S.R.	250	4	3	1	8	
2	Great Britain	214	2	4	1	8	
3	U.S.A.	188	1	4	1	8	
4	Bulgaria	174	-	2	6	8	
5	Austria	167	1	2	5	8	
6	France	165	1	3	1	8	
7	Hungary	160	-	3	4	8	
8	East Germany	142	-	2	3	8	
9	Poland	138	-	-	6	8	
10	Sweden	120	-	1	3	8	
11	Rumania	118	-	1	3	8	
12,13	Czechoslovakia	116	-	1	3	8	
12,13	Yugoslavia	116	-	1	3	8	
14	Vietnam	112	-	1	3	8	
15	Netherlands	78	-	-	1	8	
16	Finland	52	-	-	1	8	
17	Greece	50	-	-	-	8	
18	Cuba	16	-	-	-	3	
19	West Germany					2	unclassified
		max 320	9	28	45	139	

19th IMO (1977) in Yugoslavia

Rank	Country	Score	1st	2nd	3rd	Team size
1	U.S.A.	202	2	3	1	8
2	U.S.S.R.	192	1	2	4	8
3,4	Great Britain	190	1	3	3	8
3,4	Hungary	190	1	3	2	8
5	Netherlands	185	1	2	3	8
6	Bulgaria	172	-	3	3	8

	Rank	Country	Score	Prizes 1st	2nd	3rd	Team size
	7	West Germany	165	1	1	4	8
	8	East Germany	163	2	1	1	8
	9	Czechoslovakia	161	-	3	2	8
	10	Yugoslavia	159	-	3	3	8
	11	Poland	157	1	2	2	8
	12	Austria	151	1	1	2	8
	13	Sweden	137	1	1	2	8
	14	France	126	1	-	-	8
	15	Rumania	122	-	1	2	8
	16	Finland	88	-	-	1	8
	17	Mongolia	49	-	-	-	8
	18	Cuba	41	-	-	-	4
	19	Belgium	33	-	-	-	7
	20	Italy	22	-	-	-	5
	21	Algeria	17	-	-	-	3
			max 320	13	29	35	155
20th IMO (1978) in Rumania	1	Rumania	237	2	3	2	8
	2	U.S.A.	225	1	3	3	8
	3	Great Britain	201	1	2	2	8
	4	Vietnam	200	-	2	6	8
	5	Czechoslovakia	195	-	2	3	8
	6	West Germany	184	1	-	3	8
	7	Bulgaria	182	-	1	3	8
	8	France	179	-	2	4	8
	9	Austria	174	-	3	2	8
	10	Yugoslavia	171	-	1	2	8
	11	Netherlands	157	-	1	1	8
	12	Poland	156	-	-	2	8
	13	Finland	118	-	-	2	8
	14	Sweden	117	-	-	1	8
	15	Cuba	68	-	-	2	4
	16	Turkey	66	-	-	-	8
	17	Mongolia	61	-	-	-	8
			max 320	5	20	38	132
21st IMO (1979) in Great Britain	1	U.S.S.R.	267	2	4	1	8
	2	Rumania	240	1	4	2	8
	3	West Germany	235	1	5	2	8
	4	Great Britain	218	-	4	4	8
	5	U.S.A.	199	1	2	2	8
	6	East Germany	180	-	2	2	8
	7	Czechoslovakia	178	1	-	4	8
	8	Hungary	176	-	2	2	8
	9	Yugoslavia	168	-	1	4	8
	10	Poland	160	-	2	3	8
	11	France	155	1	-	1	8
	12	Austria	152	-	-	4	8
	13	Bulgaria	150	-	-	5	8
	14	Sweden	143	-	2	1	8
	15	Vietnam	134	1	3	-	4
	16	Netherlands	131	-	1	1	8

Rank	Country	Score	Prizes 1st	2nd	3rd	Team size
17	Israel	119	-	-	2	8
18	Finland	89	-	-	1	8
19	Belgium	66	-	-	1	8
20	Greece	57	-	-	1	8
21	Cuba	35	-	-	-	4
22	Brazil	19	-	-	-	5
23	Luxembourg	7	-	-	-	1
		max 320	8	32	43	166

*

The following problem set, for which I solicit elegant solutions from all readers, was provided by Walther Janous, Innsbruck, Austria, to whom my thanks.

16TH AUSTRIAN MATHEMATICAL OLYMPIAD (Final Round)

First day: June 12, 1985. Time: 4½ hours

1. Determine all quadruples (a,b,c,d) of nonnegative integers such that

$$a^2 + b^2 + c^2 + d^2 = a^2b^2c^2.$$

2. For $n = 1,2,3,\ldots,$ let

$$f(n) = 1^n + 2^{n-1} + 3^{n-2} + \ldots + (n-1)^2 + n.$$

Determine

$$\min_{n\geq 1} \frac{f(n+1)}{f(n)}.$$

3. A line intersects the sides (or sides produced) BC,CA,AB of a triangle ABC in the points A_1,B_1,C_1, respectively. The points A_2,B_2,C_2 are symmetric to A_1,B_1,C_1 with respect to the midpoints of BC,CA,AB, respectively. Prove that A_2, B_2, and C_2 are collinear.

Second day: June 13, 1985. Time: 4½ hours

4. Determine all natural numbers n such that the equation

$$a_{n+1}x^2 - 2x\sqrt{a_1^2 + a_2^2 + \ldots + a_{n+1}^2} + a_1 + a_2 + \ldots + a_n = 0$$

has real solutions for all real $a_1,a_2,\ldots,a_{n+1}$.

5. Let $\{a_n\}$ be a sequence of natural numbers satisfying

$$a_n = \sqrt{(a_{n-1}^2 + a_{n+1}^2)/2}$$

for all $n \geq 1$. Prove that the sequence is a constant one.

6. Determine all functions $f:R \to R$ satisfying the functional equation

$$x^2 f(x) + f(1-x) = 2x - x^4$$

for all $x \in R$.

*

Finally, I present solutions to some problems proposed in earlier columns.

7. [1981: 73] *From the 1979 Moscow Olympiad.*

Each of a set of stones has a mass of less than 2 kg, and the total mass of the stones is more than 10 kg. A subset of the stones is chosen whose total mass is as close as possible to 10 kg. Let the (positive) difference between the total mass of the subset and 10 kg be D. If we start out with different sets of stones, what is the largest possible D?

Solution by Andy Liu, University of Alberta.

It is clear that there are at least 6 stones in each set. Consider first a set of stones each of which has a mass of 20/11 kg. The subsets with 5 and 6 stones have respective total masses of 100/11 kg and 120/11 kg, which are closest to 10 kg, yielding $D = 10/11$ for this set. We show that this is the largest possible value of D.

Let S be a set of stones satisfying the hypothesis, with a subset A of stones whose total mass $10+D$ kg is the closest to 10 kg. If A contains a stone of mass less than $2D$ kg, then its removal from A will yield a subset with total mass closer to 10 kg. Since each stone has a mass of less than 2 kg, we have $D < 1$ and there are at least 6 stones in A.

Let the subset B be obtained from A by removing all but 5 stones. The total mass of the stones in B is at least $10D$ kg but less than 10 kg, so its difference from 10 kg is at most $10-10D$ kg. Since the total mass of the stones in A is at least as close to 10 kg, we must have $D \leq 10-10D$, or $D \leq 10/11$.

Now let S be a set of stones satisfying the hypothesis, with a subset A of stones whose total mass $10-D$ kg is the closest to 10 kg, with $D > 10/11$. Since the total mass of the stones in S exceeds 10 kg, $S-A$ is nonempty. Let the minimum mass of any stone in $S-A$ be m kg. If $m < 2D$, then the addition to A of a stone from $S-A$ with minimum mass will yield a subset with total mass closer to 10 kg. Hence we have $m \geq 2D$, and so $D < 1$ since each stone has a mass of less than 2 kg.

Suppose A has a subset of stones with total mass strictly between $m-2D$ kg and m kg. Then its replacement in A by a stone in $S-A$ with minimum mass will yield a subset with total mass strictly between $10-D$ kg and $10+D$ kg, contrary to the definition of A. Hence each stone in A can be classified as either *heavy* (with mass at least m kg) or *light* (with mass at most m-$2D$ kg). Moreover, the total mass of the light stones is also at most $m-2D < 2-20/11 = 2/11$ kg.

If A has at least 5 heavy stones, then the total mass of all the stones in A is at least $5m \geq 10D$ kg. Since $D > 10/11$, we have $10D > 10-D$. On the other hand, if A has at most 4 heavy stones, then the total mass of all the stones in A is less than $8+2/11 < 9 < 10-D$ kg since $D < 1$. However, the total mass of all the stones in A is exactly $10-D$ kg. Hence $D > 10/11$ is also impossible in this case.

It now follows that the maximum value of D is indeed 10/11.

*

10. [1981: 74] (Corrected) *From the 1979 Moscow Olympiad.*

A number of pairwise disjoint intervals are chosen along the line segment $[0,1]$. The distance between two points belonging to any one interval, or even belonging to two different intervals, is never equal to 1/10. Show that the sum of the lengths of the chosen intervals is not greater than $\frac{1}{2}$.

Solution by Andy Liu, University of Alberta.

Note that we have added the words "pairwise disjoint" at the beginning of the proposal, for otherwise the theorem is false. To see this, let the chosen intervals be

$$[0, \frac{1}{11}],\ [0, \frac{1}{12}],\ [0, \frac{1}{13}],\ \ldots,\ [0, \frac{1}{n}].$$

Since the harmonic series diverges, n can be chosen large enough so that the sum of the lengths

$$\frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \ldots + \frac{1}{n} > \frac{1}{2}.$$

(In fact, $n = 17$ will do.)

To prove the corrected theorem, let $\cup I$ denote the union of all the chosen intervals. Divide $[0, 1]$ into five subintervals each of length 1/5, and let J be any one of them. If $(\cup I)\cap J$ exceeds 1/10 in length, then two of its points will be exactly 1/10 apart. Hence the length of $(\cup I)\cap J$ is at most 1/10, and the total length of $\cup I$ is at most 1/2.

*

H-3. [1981: 114] *From Középiskolai Matematikai Lapok 60 (1979) 140.*

Given that $a_i = \pm 1$ for $i = 1,2,\ldots,n$, prove that

$$2\sin\frac{\pi}{4}(a_1 + a_1a_2/2 + \ldots + a_1a_2\ldots a_n/2^{n-1}) = a_1\sqrt{2 + a_2\sqrt{2 + \ldots + a_n\sqrt{2}}}.$$

Solution by Noam Elkies, Harvard University.

We use induction. For $n = 1$, $2\sin(\pm\pi/4) = \pm\sqrt{2}$. We now assume that the result is valid for some $n = k-1$, where $k \geq 2$, and prove that it is valid for $n = k$. To do so, we use the half-angle formula

$$2\sin\frac{\theta}{2} = \operatorname{sgn}\theta\cdot\sqrt{2 - 2\cos\theta} = \operatorname{sgn}\theta\cdot\sqrt{2 + 2\sin(\theta-\pi/2)}, \qquad |\theta| < 2\pi \qquad (1)$$

and the fact that

$$a_1 + \frac{a_1a_2}{2} + \ldots + \frac{a_1a_2\ldots a_n}{2^{n-1}}$$

has the same sign as a_1. Thus

$$2\sin\frac{\pi}{4}(a_1 + \frac{a_1a_2}{2} + \ldots + \frac{a_1a_2\ldots a_k}{2^{k-1}})$$

$$= 2\sin\frac{a_1}{2}\{\frac{\pi}{2} + \frac{\pi}{4}(a_2 + \frac{a_2a_3}{2} + \ldots + \frac{a_2a_3\ldots a_k}{2^{k-2}})\}$$

$$= a_1\sqrt{2 + 2\sin\frac{\pi}{4}(a_2 + \frac{a_2a_3}{2} + \ldots + \frac{a_2a_3\ldots a_k}{2^{k-2}})}, \quad \text{by (1)},$$

$$= a_1\sqrt{2 + a_2\sqrt{2 + \ldots + a_k\sqrt{2}}}, \quad \text{by the inductive assumption.}$$

So the stated result is valid for $n = 1,2,3,\ldots$.

*

11. [1981: 237] *Proposed by the United Kingdom (but unused) at the 1981 I.M.O.*

A sequence $\{u_n\}$ of real numbers is given by u_1 and, for $n \geq 1$, by the recurrence relation

$$4u_{n+1} = \sqrt[3]{64u_n + 15}.$$

Describe, with proof, the behavior of u_n as $n \to \infty$.

Solution by Noam Elkies, Harvard University.

Let

$$f(x) = \frac{\sqrt[3]{64x + 15}}{4} \qquad \text{and} \qquad g(x) = f(x) - x.$$

Clearly $f(x)$ increases with x, $g(x) \to -\infty$ as $x \to +\infty$, and $g(x) = 0$ just when

$$64x^3 - 64x - 15 \equiv (4x + 1)(16x^2 - 4x - 15) = 0,$$

that is, just when x has one of the values

$$x_1 = \frac{1 - \sqrt{61}}{8}, \qquad x_2 = -\frac{1}{4}, \qquad x_3 = \frac{1 + \sqrt{61}}{8}, \qquad (1)$$

which are the fixed points of f. Thus the sequence $\{u_n\}$ of iterates of f applied to any u_1 is constant if u_1 is one of the fixed points (1), and otherwise never escapes from the one of the intervals

$$(-\infty,\ x_1), \qquad (x_1,\ x_2), \qquad (x_2,\ x_3), \qquad (x_3,\ +\infty) \tag{2}$$

in which u_1 lies. Since the fixed points are all simple, $g(x)$ is positive in the first and third of the intervals (2), and negative in the second and fourth. Thus $\{u_n\}$ is increasing if u_1 lies in the first or third of the intervals (2), and decreasing if u_1 lies in the second or fourth. In either case, the sequence is bounded and hence converges. By the continuity of f,

$$u_\infty = \lim_{n\to\infty} u_n = \lim_{n\to\infty} f(u_{n-1}) = f(\lim_{n\to\infty} u_{n-1}) = f(u_\infty).$$

Hence u_∞ is one of the fixed points (1). In fact,

$$u_\infty = \begin{cases} x_1 = \dfrac{1-\sqrt{61}}{8}, & \text{if } u_1 < x_2, \\ x_3 = \dfrac{1+\sqrt{61}}{8}, & \text{if } u_1 > x_2. \end{cases}$$

Comment by M.S.K.

To see what is happening geometrically for a sequence defined by $u_{n+1} = f(u_n)$, one should plot the graphs of $y = f(x)$ and $y = x$ (see M.S.K., "Geometric Convergence", *American Mathematical Monthly*, 60 (1953) 256-259).

*

3. [1981: 298; 1982: 167] (Corrected) *From Középiskolai Matematikai Lapok, 62 (1981) 208-209.*

Determine the pairs (m,n) of natural numbers for which the equation

$$\frac{1+\sin^2 nx}{1+\sin^2 mx} = \sin nx$$

has real solutions.

Solution by Noam Elkies, Harvard University.

The given equation is never satisfied if $\sin nx = 0$, so we may consider instead the equivalent equation

$$1 + \sin^2 mx = \sin nx + \frac{1}{\sin nx}. \tag{1}$$

The left side of (1) is always positive and is at most 2, with equality when $\sin^2 mx = 1$; and since $\sin nx$ must be positive, the right side is at least 2, with equality when $\sin nx = 1$. Thus (1) is satisfied only if both

$$\sin^2 mx = 1 \qquad \text{and} \qquad \sin nx = 1,$$

that is, only if

$$mx = m_1 \cdot \frac{\pi}{2}, \qquad m_1 \equiv 1 \pmod 2,$$

and

$$nx = n_1 \cdot \frac{\pi}{2}, \qquad n_1 \equiv 1 \pmod 4.$$

Since clearly $x = 0$ is never a solution, $m/n = mx/nx$ must equal the ratio of two odd numbers. More precisely, m and n must be of the form

$$m = 2^r(2p+1), \qquad n = 2^r(4q+1), \tag{2}$$

where p,q,r are nonnegative integers.

Conversely, if m and n are of the form (2) then $x = \pi/2^{r+1}$ is a real solution of (1).

*

3. [1983: 269] *From the 1983 Austrian Mathematical Olympiad.*

Let P be any point in the plane of a triangle ABC, and let A'B'C' be the cevian triangle of the point P for the triangle ABC (with A' = AP ∩ BC, etc.). If the vertices of triangle A"B"C" are defined by

$$\vec{AA'} = \vec{A'A''}, \qquad \vec{BB'} = \vec{B'B''}, \qquad \vec{CC'} = \vec{C'C''},$$

show that

$$[A''B''C''] = 3[ABC] + 4[A'B'C'],$$

where the square brackets denote the signed area of a triangle.

Solution by Noam Elkies, Harvard University.

More generally, we show that if

$$\vec{A'A''} = k\vec{AA'}, \qquad \vec{B'B''} = k\vec{BB'}, \qquad \vec{C'C''} = k\vec{CC'},$$

where k is a real number, then

$$[A''B''C''] = (2k^2+k)[ABC] + (k+1)^2[A'B'C'].$$

The given problem then corresponds to the case $k = 1$.

Let (x,y,z) be the barycentric coordinates of P with respect to triangle ABC. The barycentric coordinates of A,B,C are

$$(1,\ 0,\ 0), \qquad (0,\ 1,\ 0), \qquad (0,\ 0,\ 1);$$

those of A',B',C' are

$$\left(0,\ \frac{y}{y+z},\ \frac{z}{y+z}\right), \qquad \left(\frac{x}{z+x},\ 0,\ \frac{z}{z+x}\right), \qquad \left(\frac{x}{x+y},\ \frac{y}{x+y},\ 0\right); \tag{1}$$

and, since $\vec{OA''} = (k+1)\vec{OA'} - k\vec{OA}$, etc., where O is the common origin of vectors, those of A",B",C" are

$$(-k, \frac{(k+1)y}{y+z}, \frac{(k+1)z}{y+z}), \quad (\frac{(k+1)x}{z+x}, -k, \frac{(k+1)z}{z+x}), \quad (\frac{(k+1)x}{x+y}, \frac{(k+1)y}{x+y}, -k). \qquad (2)$$

If Δ' is the determinant of order 3 whose successive rows are the coordinates of A',B',C', as given in (1), then we find that

$$[A'B'C'] = \Delta' \cdot [ABC] = \frac{2xyz}{(y+z)(z+x)(x+y)} \cdot [ABC].$$

And if Δ'' is the determinant whose successive rows are the coordinates of A",B",C", as given in (2), then we find that

$$[A''B''C''] = \Delta'' \cdot [ABC] = \{(2k^2+k) + \frac{2(k+1)^2xyz}{(y+z)(z+x)(x+y)}\} \cdot [ABC]$$

$$= (2k^2+k)[ABC] + (k+1)^2[A'B'C'].$$

*

4, [1984: 214; 1985: 115] *From the 1983 Swedish Mathematical Contest.*

Two concentric circles have radii r and R. A rectangle has two adjacent vertices on one of the circles. The two other vertices are on the other circle. Determine the length of the sides of the rectangle when its area is maximal.

II. *Solution by Dan Sokolowsky, Brooklyn, N.Y.*

Let ABCD be a rectangle with vertices A,B on the inner circle (radius r) and vertices C,D on the outer circle (radius R). Let a line through the common center O perpendicular to AB meet AB in E and CD in F. Then, with the square brackets denoting area, we have

$$[ABCD] = 2[EBCF] = 4[OBC].$$

Now [OBC] will be a maximum when its sides OB = r and OC = R are perpendicular, in which case BC = $\sqrt{R^2+r^2}$ and max [ABCD] = $2Rr$. The other side of the maximal rectangle is then

$$AB = \frac{[ABCD]}{BC} = \frac{2Rr}{\sqrt{R^2+r^2}}.$$

Comment by M.S.K.

The above geometric solution is more elegant than the previous calculus solution [1985: 115] . Moreover, by considering both solutions, we now have an elementary (noncalculus) solution to the extremum problem of finding

$$\max\ 2x(\sqrt{R^2-x^2} + \sqrt{r^2-x^2}), \qquad 0 \le x \le r < R.$$

Editor's note. All communications about this column should be sent directly to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

PROBLEMS - - PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before March 1, 1986, although solutions received after that date will also be considered until the time when a solution is published.

1061. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Later this year, Halley's comet will visit the earth for the first time since 1910. The following paragraph is taken from an article by Thomas O'Toole in *The Washington Post*, 8 August 1985.

Mark Twain, the humorist and author, was born Samuel Langhorne Clemens in the year the comet arrived in 1835 and died when it reappeared in 1910. A year before he died, Twain said, "I came into this world with Halley's comet. It's coming again pretty soon, and I expect to go out with it. It will be the greatest disappointment of my life if I don't go out with Halley's comet."

Commemorate the sesquicentennial of Mark Twain's birth and the diamond anniversary of his death by solving independently the alphametical sums

```
 MARK        TWAIN
 BORN         DIED
 1835,        1910,
COMET        COMET
```

where the 1835 COMET is to be maximized (since it was bigger then) while the 1910 COMET is to be minimized (since it lost weight on its circuit through space during Mark Twain's lifetime).

1062. *Proposed by M.S. Klamkin, University of Alberta.*

(a) Let Q be a convex quadrilateral inscribed in a circle with center O. Prove:

(i) If the distance of any side of Q from O is half the length of the opposite side, then the diagonals of Q are orthogonal.

(ii) Conversely, if the diagonals of Q are orthogonal, then the distance of any side of Q from O is half the length of the opposite side.

(b)* Suppose a convex quadrilateral Q inscribed in a centrosymmetric region with center O satisfies either (i) or (ii). Prove or disprove that the region must be a circle.

1063. *Proposed by Andy Liu, University of Alberta.*

The following is an excerpt from U.I. Lydna's *Medieval Medicine*:

The Chief of a village on Pagan Island was seriously ill. The Oracle revealed that he could only be cured by a potion containing exactly *five* herbs, at least four of which must be of *quint*essential nature. Unfortunately, the Oracle did not reveal what a quintessential herb was, and nobody on Pagan Island knew.

The Grand Alpharmist gathered a number of herbs and concocted sixty-eight potions, each containing exactly five herbs. In an effort to include as many combinations as possible, each trio of herbs was used in exactly one potion. The Oracle was consulted again, but it revealed only that each of the potions contained at least one quintessential herb.

The Chief's condition had deteriorated so much that further delay would prove fatal. The Grand Alpharmist therefore administered one dose of each potion, hoping that one of them would contain the necessary four quintessential herbs.

What was the fate of the Chief?

1064. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Triangles ABC and DEF are similar, with angles $A = D$, $B = E$, $C = F$ and ratio of similitude $\lambda = EF/BC$. Triangle DEF is inscribed in triangle ABC, with D,E,F on the *lines* BC,CA,AB, not necessarily respectively. Three cases can be considered:

Case 1: $D \in BC$, $E \in CA$, $F \in AB$;

Case 2: $D \in CA$, $E \in AB$, $F \in BC$;

Case 3: $D \in AB$, $E \in BC$, $F \in CA$.

For Case 1, it is known that $\lambda \ge 1/2$ (see Crux 606 [1982: 24, 108]). Prove that, for each of Cases 2 and 3,

$$\lambda \ge \sin \omega,$$

where ω is the Brocard angle of triangle ABC. (This inequality also holds *a fortiori* for Case 1, since $\omega \le 30^\circ$.)

1065. *Proposed by Jordan B. Tabov, Sofia, Bulgaria.*

The orthocenter H of an orthocentric tetrahedron ABCD lies inside the tetrahedron. If X ranges over all the points of space, find the minimum value of

$$f(X) = \{BCD\}\cdot AX + \{CDA\}\cdot BX + \{DAB\}\cdot CX + \{ABC\}\cdot DX,$$

where the braces denote the (unsigned) area of a triangle.

(This is an extension to 3 dimensions of Crux 866 [1984: 327].)

1066.* *Proposed by D.S. Mitrinović, University of Belgrade, Yugoslavia.*

Consider the inequality

$$(y^p + z^p - x^p)(z^p + x^p - y^p)(x^p + y^p - z^p)$$
$$\le (y^q + z^q - x^q)^r(z^q + x^q - y^q)^r(x^q + y^q - z^q)^r.$$

(a) Prove that the inequality holds for all real x,y,z if $(p,q,r) = (2,1,2)$.

(b) Determine all triples (p,q,r) of natural numbers for each of which the inequality holds for all real x,y,z.

1067. *Proposed by Jack Garfunkel, Flushing, N.Y.*

(a)* If $x,y,z > 0$, prove that

$$\frac{xyz(x + y + z + \sqrt{x^2 + y^2 + z^2})}{(x^2 + y^2 + z^2)(yz + zx + xy)} \le \frac{3 + \sqrt{3}}{9}.$$

(b) Let r be the inradius of a triangle and r_1,r_2,r_3 the radii of its three Malfatti circles (see Crux 618 [1982: 82]). Deduce from (a) that

$$r \le (r_1 + r_2 + r_3)\cdot\frac{(3 + \sqrt{3})}{9}.$$

1068. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

A triangle ABC has sides a,b,c in the usual order. Prove that

$$\ln c = \ln a - \{\frac{b}{a}\cos C + \frac{b^2}{2a^2}\cos 2C + \frac{b^3}{3a^3}\cos 3C + \ldots\}.$$

(This problem is not new. A reference will be given with the solution.)

1069. *Proposed by Clark Kimberling, University of Evansville, Indiana.*

The point P lies in the plane of but outside a scalene triangle ABC. Show that there exist exactly two pairs (s,t), with $0 \le s,t < \pi$, such that the distances from P to sides BC,CA,AB, respectively, are proportional to

$$\sin(A-s)\sin(A-t) : \sin(B-s)\sin(B-t) : \sin(C-s)\sin(C-t), \qquad (1)$$

that is, such that (1) is a set of trilinear coordinates for P.

1070. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Let O be the center of an n-dimensional sphere. An $(n-1)$-dimensional hyperplane, H, intersects the sphere (O) forming two segments. Another n-dimensional sphere, with center C, is inscribed in one of these segments, touching sphere (O) at point B and touching hyperplane H at point Q. Let AD be the diameter of sphere (O) that is perpendicular to hyperplane H, the points A and B being on opposite sides of H. Prove that A, Q, and B colline.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

290. [1977: 251; 1978: 142] *Proposed by R. Robinson Rowe, Sacramento, California.*

Find a 9-digit integer A representing the area of a triangle of which the three sides are consecutive integers.

II. *Comment by David Singmaster, Polytechnic of the South Bank, London, England.*

An editor's comment contained the following statement [1978: 145]: "The *Heronian problem* is to find a set of expressions for the sides which will yield *all* (integral) Heronian triangles. A complete solution of the Heronian problem does not exist as yet." (To avoid ambiguity, the adjective *rational* will be used below for triangles with rational sides and area, the adjective *Heronian* being reserved for triangles with integral sides and area.)

As it happens, a complete solution of the Heronian problem appeared at just about the time (1978) that the editor made his comment. The following result is given in [2] (for a complete understanding of which a look at [1] will be useful):

A triangle is Heronian if and only if its sides can be represented as

(i) $a(u^2+v^2)$, $b(r^2+s^2)$, $a(u^2-v^2) + b(r^2-s^2)$, *where* $auv = brs$;

or

(ii) a reduction by a common factor of a triangle given by (i).

This states, in effect, that *all* Heronian triangles and *only* Heronian triangles, are obtained by the juxtaposition of two Pythagorean triangles or by the reduction of such a juxtaposition by a common factor.

Given any Heronian triangle, drop an altitude. This altitude must be rational, and multiplying all lengths by its denominator yields the juxtaposition of two Pythagorean triangles. So every Heronian triangle satisfies (i) or (ii). This is old, possibly ancient.

Conversely, suppose we have a triangle satisfying (i) or (ii). If it satisfies (i), then it is the juxtaposition of two Pythagorean triangles and constitutes a rational triangle; and if it satisfies (ii), then it is likewise a rational triangle. That the rational triangle is actually Heronian in both cases is guaranteed by the following theorem, which is stated and proved in [2]:

Let k,a,b,c *be positive integers. Then* a,b,c *are the sides of a Heronian triangle if and only if* ka,kb,kc *form a Heronian triangle.*

REFERENCES

1. David Singmaster, "Some Corrections to Carlson's 'Determination of Heronian Triangles'", *The Fibonacci Quarterly*, 11 (1973) 157-158.

2. __________, Letter to the Editor, *Mathematical Spectrum*, 11 (1978/79) 58-59.

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879. [1983: 242; 1985: 23] *Proposed by Leroy F. Meyers, The Ohio State University.*

The U.S. Social Security numbers consist of 9 digits (with initial zeros permitted). How many such numbers are there which do not contain any digit three or more times consecutively?

IV. *Solution by Dag Jonsson, Uppsala, Sweden.*

A nine-digit number may contain 0, 1, 2, 3, or 4 blocks of two equal digits. For k blocks in given positions, there are $10\cdot 9^{8-k}$ different numbers if contiguous digits are different except inside blocks. The number of ways of ordering k blocks and $9-2k$ single positions ($9-k$ objects in all) is $\binom{9-k}{k}$. Thus the total number of nine-digit numbers which do not contain any digit three or more times consecutively is

$$10\cdot\sum_{k=0}^{4}\binom{9-k}{k}\cdot 9^{8-k} = 936\,845\,190. \quad \square$$

In the same way one can show, more generally, that for n-digit numbers in base b there are

$$b\cdot\sum_{k=0}^{[n/2]}\binom{n-k}{k}(b-1)^{n-1-k}$$

different numbers with the given restriction. In particular, the number of different n-digit binary sequences with the given restriction is

$$2\cdot\sum_{k=0}^{[n/2]}\binom{n-k}{k}.$$

* * *

898. [1983: 313; 1985: 64] A solution was received from DAG JONSSON, Uppsala, Sweden.

* * *

926. [1984: 89; 1985: 155] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Let P be a fixed point inside an ellipse, L a variable chord through P, and L' the chord through P that is perpendicular to L. If P divides L into two segments of lengths m and n, and if P divides L' into two segments of lengths r and s, prove that $1/mn + 1/rs$ is a constant.

III. *Comment by Jordan B. Tabov, Sofia, Bulgaria.*

I recently found this problem and some related results in M. Barbarin, "Puissance d'un point par rapport à une conique à centre", *Mathesis*, Tome premier, 1881, pp. 85-87.

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930. [1984: 89; 1985: 162] *Proposed by the Cops of Ottawa.*

Does there exist a tetrahedron such that all its edge lengths, all its face areas, and its volume are integers? If so, give a numerical example.

VI. *Comment by J.G. Flatman, M.D., Timmins, Ontario.*

In response to the editor's request [1985: 166], I can confirm that there is indeed an arrangement of the edge lengths 6,7,8,9,10,11 for which the Tabov tetrahedron has integral volume. If OABC is the tetrahedron, then the arrangement

$$OA = 6, \quad OB = 10, \quad OC = 11,$$
$$BC = 9, \quad CA = 7, \quad AB = 8$$

yields volume 48. Moreover, the same volume results if opposite sides are interchanged in two (but not one or three) of the pairs (6,9), (10,7), (11,8).

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939. [1984: 115] (Corrected) *Proposed by George Tsintsifas, Thessaloniki, Greece.*

ABC is an acute triangle with AB < AC, altitude AD, and orthocenter H. M being an interior point of segment DH, lines BM and CM intersect sides CA and AB in B' and C', respectively. Prove that BB' < CC'.

Solution by the proposer (revised by the editor).

The problem was incorrect as originally proposed. We prove the corrected version given above and then extend the problem.

We get DB/DA = DH/DC from similar triangles ABD and CHD (see Figure 1); so

$$DB \cdot DC = DH \cdot DA > DM \cdot DA.$$

Let B_1 be symmetric to B with respect to D. Then $DB_1 \cdot DC > DM \cdot DA$, and it follows that B_1 is an interior point of circle AMC. Therefore $\angle MB_1A > \angle MCA$, and hence

$$\angle MBA > \angle MCA. \qquad (1)$$

Now (1) successively implies each of the following: $\angle BB'C > \angle BC'C$; circle $BB'C$ intersects segment CC' at an interior point, C_1, of that segment; $\angle C_1BC > \angle B'CB$; $BB' < CC_1$; and finally

$$BB' < CC'. \quad \square \tag{2}$$

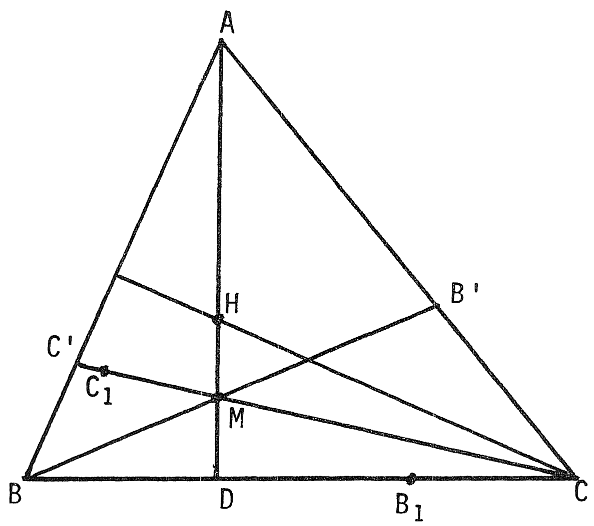

Figure 1

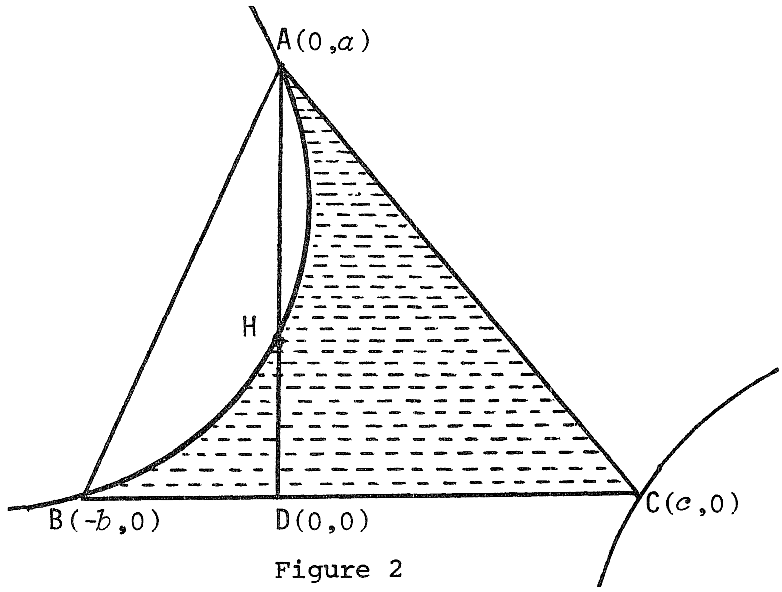

Figure 2

Observe that (2) is a consequence of (1) alone, that is, (2) holds whenever M is a point satisfying (1), *whether or not* M *lies on the segment* DH. We will find the locus of the points M interior to triangle ABC for which (1), and hence (2), holds.

We introduce a coordinate system with origin at D and axes along BC and AD, as shown in Figure 2, and use the coordinates

$$A(0,a), \quad B(-b,0), \quad C(c,0), \quad M(x,y),$$

so that a, b, c, y, $x+b$, $c-x$, and $c-b$ are all positive quantities. Then

$$\begin{aligned}
\angle MBA > \angle MCA &\iff \tan\angle MBA > \tan\angle MCA \\
&\iff \frac{ax - by + ab}{b(x+b) + ay} > \frac{ac - ax - cy}{c(c-x) + ay} \\
&\iff (ax-by+ab)(c^2-cx+ay) > (ac-ax-cy)(bx+b^2+ay) \\
&\iff f(M) < 0,
\end{aligned}$$

where

$$f(M) \equiv ax^2 + \frac{2(a^2+bc)}{b-c}xy - ay^2 + a(b-c)x + (a^2+bc)y - abc.$$

The equation $f(M) = 0$ is that of a rectangular hyperbola. One branch goes through $A(0,a)$, $B(-b,0)$, and $H(0,\frac{bc}{a})$; the other branch goes through $C(c,0)$. Since we already know that $f(M) < 0$ when M is an interior point of segment DH, it follows that $f(M) < 0$, and (2) holds, if and only if M lies in the shaded region of Figure 2.

Solutions and/or comments were received from J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and RICHARD KATZ and DAN SOKOLOWSKY, California State University at Los Angeles (jointly).

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940. [1984: 115] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Show that, for any triangle ABC,

$$\sin B\sin C + \sin C\sin A + \sin A\sin B \le \frac{7}{4} + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{9}{4}.$$

Solution by M.S. Klamkin, University of Alberta.

More generally, we show that

$$\Sigma\sin B\sin C \le k + (18-8k)\Pi\sin\frac{A}{2}, \qquad k \ge 1, \tag{1}$$

and

$$k + (18-8k)\Pi\sin\frac{A}{2} \le \frac{9}{4}, \qquad k \le \frac{9}{4}, \tag{2}$$

where the sum and product are cyclic over A,B,C. The proposed result is then a combination of (1) and (2) for $k = 7/4$.

We first establish (1) for $k = 1$, that is,

$$\Sigma\sin B\sin C \le 1 + 10\Pi\sin\frac{A}{2}. \tag{3}$$

The first of the inequalities

$$s^2 \le 4R^2 + 4Rr + 3r^2 \le 4R^2 + 6Rr - r^2$$

can be found in [1984: 157] and the second is equivalent to Euler's inequality $R \ge 2r$. Hence

$$s^2 \le 4R^2 + 6Rr - r^2,$$

and this is found to be equivalent to (3) when we use the well-known R,r,s representations

$$\Sigma\sin B\sin C = \frac{s^2 + 4Rr + r^2}{4R^2} \quad\text{and}\quad \Pi\sin\frac{A}{2} = \frac{r}{4R}.$$

To complete the proof of (1), we need only show that

$$1 + 10\Pi\sin\frac{A}{2} \le k + (18-8k)\Pi\sin\frac{A}{2}, \quad k > 1.$$

But this inequality is equivalent to the well-known

$$\Pi\sin\frac{A}{2} \le \frac{1}{8} \tag{4}$$

for all $k > 1$ and to the negation of (4) for all $k < 1$. Therefore (1) holds for all triangles if and only if $k \ge 1$, and equality holds for all k when triangle ABC is equilateral.

As for (2), it is equivalent to (4) for all $k \le 9/4$ and to the negation of (4) for all $k > 9/4$. Therefore (2) holds for all triangles if and only if $k \le 9/4$, and equality holds for all k when triangle ABC is equilateral.

We have shown, in particular, that the double inequality

$$\Sigma \sin B \sin C \le k + (18-8k)\Pi \sin \frac{A}{2} \le \frac{9}{4}$$

holds for all triangles if and only if $1 \le k \le 9/4$, and equality holds throughout for all k when triangle ABC is equilateral.

Also solved by LEON BANKOFF, Los Angeles, California; W.J. BLUNDON, Memorial University of Newfoundland; CURTIS COOPER, Central Missouri State University; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh (two solutions); and the proposer.

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941. [1984: 155] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Independently solve each of the following alphametics in base ten:

$$6 \cdot \text{GEESE} = \text{FLOCK},$$
$$7 \cdot \text{GEESE} = \text{FLOCK},$$
$$8 \cdot \text{GEESE} = \text{FLOCK}.$$

Solution by Stewart Metchette, Culver City, California.

In all three alphametics, G = 1 and E ≠ 0.

(a) 6•GEESE = FLOCK. Here E must be odd, since otherwise E = K. Since 98765/6 < 16461, we must have

$$\text{GEESE} = 133\text{S}3 \quad \text{or} \quad 155\text{S}5.$$

Testing all possible values of S gives no solution in the first case, and in the second case we get the unique solution

$$6 \cdot 15545 = 93270.$$

(b) 7•GEESE = FLOCK. Since 98765/7 < 14110, we must have

$$\text{GEESE} = 122\text{S}2 \quad \text{or} \quad 133\text{S}3.$$

Testing all possible values of S in both cases yields the unique solution

$$7 \cdot 12272 = 85904.$$

(c) 8•GEESE = FLOCK. Here 98765/8 < 12346, so we must have

$$\text{GEESE} = 122\text{S}2,$$

and testing all possible values of S gives the unique solution

$$8 \cdot 12232 = 97856. \quad \square$$

It takes at least 2 GEESE to make a FLOCK, and there is no room for more than 9. For

$$n \cdot \text{GEESE} = \text{FLOCK}, \qquad 2 \le n \le 9,$$

there is no solution for $n = 2,5,9$, one solution for $n = 3,6,7,8$, and five solutions for $n = 4$.

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; JACK LESAGE, Eastview Secondary School, Barrie, Ontario; J.A. McCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Pensacola Junior College, Florida; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer.

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942. [1984: 155] *Proposed by Freeman Dyson, University of California, Davis (permanent address, Institute for Advanced Study, Princeton).*

The infinite tree T is defined as the unique connected graph having three edges at every vertex and no closed cycles. We consider functions $f(P)$ defined on the vertices P of T. The neighbor-averaging operator A is defined by

$$Af(P) = \frac{1}{3}\{f(Q) + f(R) + f(S)\},$$

where Q,R,S are the neighbors of P on T. A is a linear bounded operator on the function f. The eigen-set E of A is the set of complex numbers λ for which a non-zero function f_λ exists with

$$Af_\lambda(P) = \lambda f_\lambda(P).$$

The problem: What is E?

Partial solution by the proposer.

This problem is a good pedagogical example to illustrate the pitfalls of loose thinking about operators. The operator A looks so simple as to be almost trivial, but its behavior depends crucially on the space U of functions f on which A is allowed to act. The two most natural choices for U give completely different answers to the problem.

Case 1. U is the space of bounded functions f with

$$|f(P)| < k < \infty.$$

Then I believe (but have not proved) that E is the ellipse

$$x^2 + 9y^2 \le 1, \qquad \lambda = x + iy. \tag{1}$$

Case 2. U is the space of square-summable functions f with

$$\sum_P |f(P)|^2 < \infty.$$

Then E is the empty set. □

For experts in function theory, it is natural to consider instead of the eigenset E the "spectrum" S of A. S is defined as the set of points λ for which the operator $(\lambda - A)^{-1}$ fails to be analytic in λ. In Case 1, I believe (but have not proved) that S and E are identical. In Case 2, S is the real line-segment joining the foci of the ellipse (1). The fact that E is empty means that the spectrum is purely continuous.

A partial solution was received from WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.

Editor's comment.

We have heard that this problem was completely solved in 1984 by someone at Cornell University, who probably got the problem directly from our proposer. If so, we hope that someone will persuade the solver to send us his solution, or a reference if the solution has been or will be published elsewhere.

Our proposer, a physicist, is the author of *Weapons and Hope*, a study of the development of nuclear arms with proposals for avoiding their use, published in 1984 by Harper & Row. Nearly all the contents originally appeared in *The New Yorker*.

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943. [1984: 155] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

For positive integer n, the numbers $u_k(n)$, $k = 0,1,2,\ldots$, are defined by

$$\frac{x^n}{(1-x)(1-x^n)} = \sum_{k=0}^{\infty} u_k(n)x^k, \qquad |x| < 1.$$

Find a simple expression for $u_k(n)$.

Solution by Leroy F. Meyers, The Ohio State University.

Since $|x| < 1$, the geometric series

$$\sum_{r=0}^{\infty} x^r = \frac{1}{1-x} \qquad \text{and} \qquad \sum_{j=1}^{\infty} x^{jn} = \frac{x^n}{1-x^n}$$

both converge absolutely, and so their Cauchy product converges to the product of their sums. Thus

$$\frac{x^n}{(1-x)(1-x^n)} = \Big(\sum_{r=0}^{\infty} x^r\Big)\Big(\sum_{j=1}^{\infty} x^{jn}\Big) = \sum_{k=0}^{\infty} u_k(n)x^k,$$

where

$$u_k(n) = \left[\frac{k}{n}\right],$$

the greatest integer in k/n, since x^k can come from only a product $x^r x^{jn}$ with $1 \le j \le \lceil k/n \rceil$ and $r = k-jn$.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; NGAI-YING WONG, Hong Kong; and the proposer.

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944. [1984: 155] *Proposed by the Cops of Ottawa.*

Find all primes p such that $2^p + p^2$ is also a prime.

I. *Composite of the solutions of Malcolm A. Smith, Georgia Southern College, Statesboro; and J. Suck, Essen, West Germany.*

More generally, we consider the function

$$f(n) = 2^n + n^2, \qquad n \text{ natural number.}$$

Clearly, $f(n)$ is even and greater than 2, and hence composite, for all even n. For odd n, we have

$$f(1) = 3, \qquad f(3) = 17,$$

both primes; and for all odd $n > 3$ that are *not* multiples of 3,

$$f(n) = (2^n+1) + (n+1)(n-1)$$

is composite, being a multiple of 3. For the first term is divisible by $2+1 = 3$; and so is the second term since one of $n-1$, n, $n+1$ is divisible by 3 and n is not. We conclude that a necessary condition for $f(n)$ to be a prime is that $n = 1$ or a multiple of 3. This condition, however, is not sufficient since

$$f(27) = 134\,218\,457 = 73\cdot 521\cdot 3529.$$

If n is restricted to primes p, then $p = 3$ is the unique answer to our problem.

II. *Solution by Curtis Cooper, Central Missouri State University, Warrensburg.*

Consider the function

$$g(p,q) = (q-1)^p + p^{q-1}, \qquad p \text{ and } q \text{ primes.}$$

We prove the theorem: *If $p \ne q$, then $g(p,q)$ is composite.*

Proof. This is clearly true if $p = 2$, for then $g(p,q)$ is even and greater than 2. If $p \ne 2$, then p is odd and

$$(q-1)^p \equiv -1 \pmod{q};$$

and, since p and q are relatively prime, Fermat's Theorem gives

$$p^{q-1} \equiv 1 \pmod{q}.$$

Therefore

$$g(p,q) = (q-1)^p + p^{q-1} \equiv 0 \pmod{q},$$

and $g(p,q)$ is composite, being greater than q. □

It follows from our theorem that a necessary condition for $g(p,q)$ to be prime is that $p = q$. The condition, however, is not sufficient since

$$g(5,5) = 1649 = 17 \cdot 97.$$

In particular, if $q = 3$, then $g(p,3) = 2^p + p^2$ is a prime only if $p = 3$; and since $g(3,3) = 17$, a prime, $p = 3$ is the unique answer to our problem.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes; WALTHER JANOUS, Innsbruck, Austria; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; EDWIN M. KLEIN, University of Wisconsin-Whitewater; KEE-WAI LAU, Hong Kong; J.A. McCALLUM, Medicine Hat, Alberta; STEWART METCHETTE, Culver City, California; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; DANIEL ROPP, student, Stillman Valley High School, Illinois; D.J. SMEENK, Zaltbommel, The Netherlands; KENNETH M. WILKE, Topeka, Kansas (two solutions); and the proposers.

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945. [1984: 156] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Solve the system

$$\begin{aligned} x + y + z + t &= 2, \\ x^2 + y^2 + z^2 + t^2 &= 118, \\ x^3 + y^3 + z^3 + t^3 &= 176, \\ x^4 + y^4 + z^4 + t^4 &= 6514. \end{aligned}$$

Solution by Jordan B. Tabov, Sofia, Bulgaria.

The given system is of the type

$$\Sigma x = a, \quad \Sigma x^2 = b, \quad \Sigma x^3 = c, \quad \Sigma x^4 = d. \tag{1}$$

A quadruple (x,y,z,t) is a solution of (1) if and only if it is a permutation of the four zeros of the function

$$f(\lambda) = (\lambda-x)(\lambda-y)(\lambda-z)(\lambda-t) \equiv \lambda^4 - p\lambda^3 + q\lambda^2 - r\lambda + s, \tag{2}$$

where $p = \Sigma x$, $q = \Sigma xy$, $r = \Sigma xyz$, and $r = xyzt$ are the elementary symmetric functions of the roots. The values of p,q,r,s can easily be found in terms of a,b,c,d by using Newton's identities. However, it will be simpler for present use (and helpful for possible future use) to record here the following results, which are given in [1]:

$$p = a, \quad q = \frac{1}{2}(a^2-b), \quad r = \frac{1}{6}(a^3-3ab+2c), \quad s = \frac{1}{24}(a^4+8ac-6a^2b+3b^2-6d).$$

With the given values of a,b,c,d, we find $p = 2$, $q = -57$, $r = -58$, $s = 112$, and (2) becomes

$$f(\lambda) = \lambda^4 - 2\lambda^3 - 57\lambda^2 + 58\lambda + 112 \equiv (\lambda+7)(\lambda+1)(\lambda-2)(\lambda-8).$$

The solutions of the given system are therefore the 24 permutations of $(-7,-1,2,8)$.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; W.J. BLUNDON, Memorial University of Newfoundland; CURTIS COOPER, Central Missouri State University, Warrensburg; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; J.A. McCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Pensacola Junior College, Florida; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; DANIEL ROPP, student, Stillman Valley High School, Illinois; D.J. SMEENK, Zaltbommel, The Netherlands; JORDAN B. TABOV, Sofia, Bulgaria (second solution); KENNETH M. WILKE, Topeka, Kansas; NGAI-YING WONG, Hong Kong; and the proposer.

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1. H. Dörrie, *Kubische und Biquadratische Gleichungen*, Leibniz Verlag, München, 1948.

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946. [1984: 156] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

The nth differences of a funtion f at r are defined as usual by $\Delta^0 f(r) = f(r)$ and

$$\Delta^1 f(r) = \Delta f(r) = f(r+1)-f(r), \qquad \Delta^n f(r) = \Delta(\Delta^{n-1} f(r)), \qquad n = 1,2,3,\ldots .$$

Prove or disprove that, if $\Delta^n f(1) = n$ for $n = 0,1,2,\ldots$, then

$$f(n) = (n-1)\cdot 2^{n-2}.$$

I. *Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.*

We will prove a more general result, namely

$$\Delta^n f(r) = (r + 2n - 1)\cdot 2^{r-2}, \quad r = 1,2,3,\ldots; \quad n = 0,1,2,\ldots, \tag{1}$$

by induction on r, and then the required result will be obtained by setting $n = 0$.

If $r = 1$, then (1) holds for all n since it reduces to the hypothesis $\Delta^n f(1) = n$. Suppose (1) holds for all n for some $r \geq 1$; then

$$\begin{aligned}\Delta^n f(r+1) &= \Delta^n f(r) + \Delta^{n+1} f(r)\\ &= (r+2n-1)\cdot 2^{r-2} + (r+2n+1)\cdot 2^{r-2}\\ &= (r+2n)\cdot 2^{r-1}\\ &= \{(r+1)+2n-1\}\cdot 2^{(r+1)-2}.\end{aligned}$$

Therefore (1) holds (for all n) for all $r = 1,2,3,\ldots$. Now setting $n = 0$ in (1) gives

$$f(r) = \Delta^0 f(r) = (r - 1)\cdot 2^{r-2},$$

which is the result required by the problem statement.

II. *Solution by M.S. Klamkin, University of Alberta.*

The two dual problems

(a) given $f(r)$, $r = 0,1,2,\ldots,$ determine $\Delta^n f(1)$; and

(b) given $\Delta^r f(1)$, $r = 0,1,2,\ldots,$ determine $f(n)$

are well-known results in the Calculus of Finite Differences. Their solutions are given by the two fundamental theorems [1]:

$$\Delta^r f(x) = f(x+r) - \binom{r}{1} f(x+r-1) + \binom{r}{2} f(x+r-2) - \ldots + (-1)^r f(x),$$

$$f(x+r) = f(x) + \binom{r}{1}\Delta^1 f(x) + \binom{r}{2}\Delta^2 f(x) + \ldots + \binom{r}{r}\Delta^r f(x). \qquad (1)$$

Setting $x = 1$ and $r = n-1$ in (1) gives, when $\Delta^n f(1) = n$ for $n = 0,1,2,\ldots,$

$$f(n) = \binom{n-1}{1}\cdot 1 + \binom{n-1}{2}\cdot 2 + \ldots + \binom{n-1}{n-1}\cdot (n-1).$$

This known sum can be obtained by differentiating $(1+x)^{n-1}$ with respect to x and then setting $x = 1$. The result is the proposer's formula

$$f(n) = (n-1)\cdot 2^{n-2}.$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JORDAN B. TABOV, Sofia, Bulgaria; and the proposer.

REFERENCE

1. C. Jordan, *Calculus of Finite Differences*, Chelsea, New York, 1947.

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947. [1984: 156] *Proposed by Jordi Dou, Barcelona, Spain.*

Let ABCD be a quadrilateral (not necessarily convex) with AB = BC, CD = DA, and AB $\perp$ BC. The midpoint of CD being M, points K and L are found on line BC such that AK = AL = AM. If P,Q,R are the midpoints of BD,MK,ML, respectively, prove that PQ $\perp$ PR.

Solution by Jordan B. Tabov, Sofia, Bulgaria.

We have

$$\vec{PQ} = \vec{BQ} - \vec{BP} \qquad \text{and} \qquad \vec{PR} = \vec{BR} - \vec{BP}$$

$$= \frac{1}{2}(\vec{BM} + \vec{BK} - \vec{BD}) \qquad = \frac{1}{2}(\vec{BM} + \vec{BL} - \vec{BD})$$

$$= \frac{1}{2}(\vec{DM} + \vec{BK}) \qquad = \frac{1}{2}(\vec{DM} - \vec{BK}).$$

Hence $PQ \perp PR$ if and only if

$$(\vec{DM} + \vec{BK})\cdot(\vec{DM} - \vec{BK}) = DM^2 - BK^2 = 0,$$

or, since $AB \perp BK$, if and only if

$$AK^2 - AB^2 = DM^2. \tag{1}$$

Now AM is a median of isosceles triangle ACD, so

$$AK^2 = AM^2 = \frac{1}{4}(2AC^2 + CD^2) = \frac{1}{4}(2AC^2 + 4DM^2); \tag{2}$$

and

$$AB^2 = \frac{1}{2}AC^2 \tag{3}$$

in isosceles right triangle ABC. Finally, (1) follows from (2) and (3).

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; B.M. SALER, Agincourt, Ontario; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Brooklyn, N.Y.; and the proposer.

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MESSAGE FROM THE MANAGING EDITOR

I am pleased to inform subscribers that, effective October 1, 1985, *Crux Mathematicorum* will become an official publication of the Canadian Mathematical Society and will therefore no longer be sponsored by the Carleton-Ottawa Mathematics Association nor published by Algonquin College.

The office of the Managing Editor will move from Algonquin College to the offices of the Canadian Mathematical Society on September 10, 1985. All communications intended for the Managing Editor should henceforth be addressed as follows:

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Léo Sauvé will continue as Editor until a new one is appointed by the C.M.S. His address remains unchanged (see front page).

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