Mathematical Spectrum

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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing.

From the Editor

Mathematics, with love

Most of our readers will not need to be reminded of the use of mathematics in many areas of life, such as science, commerce, medicine, you name it. But winning a partner? You must be joking! I kid you not! *Mathematics with Love* is the story of Barnes Wallis, later Sir Barnes Wallis, and his attempts to woo Molly Bloxam. It was the 1920s — no email nor texting, and people didn't phone. It was the age of letter-writing. At the time, Wallis was working on the development of an airship and Molly was a student struggling with basic mathematics. Wallis was twice Molly's age. It was an age of parental authority and Mr Bloxam was against the match. But they were allowed to write to each other. This book is their correspondence over two years. A large part of the earlier letters describe Wallis's attempts to teach Molly the rudiments of trigonometry and calculus. He put many hours into the task — a labour of love! He wrote later that he rarely put less than 10 or 12 hours a week into writing to Molly. And it worked! On her 20th birthday, Molly was allowed to say 'yes'.

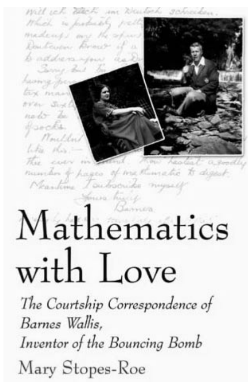
Today's readers may well be bemused by the effort that they put into letter-writing. They were oh so polite to each other! In the later letters mathematics disappears altogether. Some readers may find Wallis's expressions of his love pall somewhat at this stage.

After fatal crashes, the airship that Wallis helped to design was scrapped. But he achieved lasting fame through his design of the bouncing bomb that was used to blow up dams in the Second World War, immortalized in the film *The Dam Busters*. He was voted one of the BBC's Great Britons in 2002. Barnes and Molly were happily married for more than 50 years. And their coming together was aided by mathematics!

I cannot say from this correspondence that I would have liked to have been taught mathematics by Wallis! One last question: How would you explain trigonometry and calculus to your intended?

Reference

1 M. Stopes-Roe, *Mathematics with Love* (Macmillan, New York, 2005).



For something completely different, a reader, James Whiteman, has sent in a problem he saw on the *Mind Bending Puzzles Calendar*, by C. Pickover.

> Captain Kirk's starship leaves Earth for Mars at the same time as Captain Eck's starship leaves Mars for Earth. Each ship travels at a constant speed, but one is faster than the other. After meeting and passing, Kirk requires $22\frac{1}{2}$ hours to reach Mars, while Eck requires only 10 hours to reach Earth. Assuming stationary planets, what total time did each starship require for its interplanetary journey?

I was invigilating an exam recently – not the most exciting of occupations – and spotted this question on a maths paper.

> An old invoice showed that 72 identical office chairs had been purchased for £ $*\,557.7*$, the first and last digits being illegible. What was the cost of one office chair? (There was no discount for bulk orders.)

More interesting than the usual run of exam questions!

Late news

A new Mersenne prime has been found, the 42nd. A Mersenne prime is a prime number of the form $2^n - 1$, so-called because they were first studied by a 17th Century French monk Marin Mersenne. The new one is

$$2^{25\,964\,951} - 1,$$

announced on 26 February 2005 and found by Dr Martin Nowak, an eye specialist in the German town of Michelfeld, using GIMPS software (GIMPS is the Great Internet Mersenne Prime Search). The new Mersenne prime is also the largest known prime number, with 7.8 million digits. Anyone can join the search!

Multiples of 11

A number is a multiple of 11 if and only if the alternating sum of its digits is a multiple of 11, e.g.

$$\begin{aligned} 11 \times 28 &= 308, & 3 - 0 + 8 &= 11, \\ 11 \times 47 &= 517, & 5 - 1 + 7 &= 11, \\ 11 \times 7319 &= 80\,509, & 8 - 0 + 5 - 0 + 9 &= 22. \end{aligned}$$

Can you prove this?

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Generalizing a Partition Problem

PHILIP MAYNARD

It was shown in reference 1 that the set $\{1, 2, \ldots, n\}$ can be partitioned into two sets, such that the sum of the elements in the first set equals that of the second, if and only if $4|n$ or $4|(n+1)$. In this article we are interested in generalizing this result. Let $\Omega \subset \mathbb{N}$ be a finite set. For some natural number k, we say that Ω is *k-separable* if there exist k pairwise disjoint sets, $A_1, A_2, \ldots, A_k$, such that

$$\Omega = \bigcup_{i=1}^{k} A_i,$$

and such that for any i, j we have $\sum_{a\in A_i} a = \sum_{b\in A_j} b$. In essence, we can partition Ω into k subsets, the sum of whose respective elements are equal. We shall almost exclusively be interested in the set $\Omega_n := \{1, 2, \ldots, n\}$ for each natural number n. Now assume that for some natural numbers n, k, the set Ω_n is k-separable. Then it is immediately clear that $k|n(n+1)/2$. Note also that, for any $i \in \{1, 2, \ldots, k\}$, we have $\sum_{a\in A_i} a = n(n+1)/2k$ and, since $\max\{1, 2, \ldots, n\} = n$, we must also have that $n(n+1)/2k \geq n$, that is, $n+1 \geq 2k$. We summarize these conditions in the following lemma.

Lemma 1 *Let n and k be natural numbers. If Ω_n is k-separable, then $k|n(n+1)/2$ and $n+1 \geq 2k$.*

Our aim is to prove the converse of this result. We begin with some simple lemmas; the first is self-evident, so we omit the proof.

Lemma 2 *Let $A, B \subset \mathbb{N}$ be finite, disjoint k-separable sets for some natural number k. Then $A \cup B$ is k-separable.*

Lemma 3 *Let n and k be natural numbers. If $2k|n$ or $2k-1 = n$, then Ω_n is k-separable.*

Proof For the first part, by Lemma 2, it is enough to show that any set of $2k$ consecutive integers is k-separable. If $A = \{a+1, a+2, \ldots, a+2k\}$, for some natural number a, then we obtain

$$\begin{aligned} A_1 &= \{a+1, a+2k\}, \\ A_2 &= \{a+2, a+2k-1\}, \\ \vdots\ &\quad \vdots \\ A_k &= \{a+k, a+k+1\}. \end{aligned}$$

For $\Omega_{2k-1} = \{1, 2, \ldots, 2k-1\}$, we obtain

$$\begin{aligned} B_1 &= \{2k-1\}, \\ B_2 &= \{1, 2k-2\}, \\ \vdots\ &\quad \vdots \\ B_k &= \{k-1, k\}. \end{aligned}$$

This completes the proof.

Lemma 4 *Let n and k be natural numbers. If $k|n(n+1)/2$ and $n+1 \geq 2k$, then Ω_n is k-separable.*

Proof For each natural number n, let $P(n)$ be the statement 'if $k|n(n+1)/2$ and $n+1 \geq 2k$, then Ω_n is k-separable'.

We must prove the validity of $P(n)$ for each natural number n. We proceed by induction on n. The validity of $P(1)$, $P(2)$, $P(3)$ and $P(4)$ is easily checked. Now assume the validity of $P(l)$ for all integers $0 < l < n$. We assume that $k|n(n+1)/2$ and $n+1 \geq 2k$ for some k and must prove that Ω_n is k-separable. Now, if $2k|n$ or $2k-1 = n$, then, by Lemma 3, we are done. Hence we assume that $n > 2k$. Let $n \equiv t \mod 2k$ where $0 < t < 2k$. Then

$$\Omega_n = \{1, 2, \ldots, t, t+1, \ldots, t+2k, \ldots, t+2(x-1)k+1, \ldots, t+2xk = n\},$$

for some natural number x. By Lemma 3, the set $\{1, 2, \ldots, 2(x-1)k\}$ is k-separable since $2k|2(x-1)k$. Hence, if we add $t+2k$ to each term, then the set $\{t+2k+1, \ldots, t+2xk\}$ is k-separable. Hence, by Lemma 2, it is sufficient to show that $\Omega_{t+2k} = \{1, 2, \ldots, t+2k\}$ is k-separable. Now, $n = t+2xk$ so that $n(n+1)/2 = t(t+1)/2 + kx(2t+2xk+1)$. Since, by assumption, $k|n(n+1)/2$, it follows that $k|t(t+1)/2$. Let $m = t(t+1)/2k$. There are two cases depending on the parity of m.

Case 1 (m even). We must partition Ω_{t+2k} into k sets with sums of elements equal to $2t+2k+1+m$. We write Ω_{t+2k} in the following form:

$$\Omega_{t+2k} := \Big\{1, 2, \ldots, t+m, t+m+1, t+m+2, \ldots, \\ t+k+\frac{m}{2}, t+2k, t+2k-1, \ldots, t+k+\frac{m}{2}+1\Big\}.$$

We thus consider the following $k - m/2$ sets:

$$\{t+2k, t+1+m\}, \{t+2k-1, t+2+m\}, \ldots, \left\{t+k+\frac{m}{2}+1, t+k+\frac{m}{2}\right\}.$$

Note that $k > m/2$ since $2k-1 \geq t$ and $m = t(t+1)/2k$. We have thus reduced our problem to showing that Ω_{t+m} is $(m/2)$-separable. Note also that the following three conditions are satisfied:

1. $m|(t+m)(t+m+1)$. (This is since $(t+m)(t+m+1) = t(t+1) + m(2t+m+1)$ and $t(t+1)/m = 2k$.)
2. $t+m+1 \geq m$. (That is, $t+1 \geq 0$, which is clearly satisfied.)
3. $t+m < n$. (This is since $n \geq t+2k > t+m$.)

We can now invoke the induction hypothesis $P(t+m)$ to complete this case.

Case 2 (m odd). We must partition Ω_{t+2k} into k sets with sums of elements equal to $2t+2k+1+m$. Again we write Ω_{t+2k} in the following form:

$$\Omega_{t+2k} := \Big\{1, \ldots, t+m+1, t+m+2, \ldots, \\ t+k+\frac{m-1}{2}, t+k+\frac{m+1}{2}, t+2k, t+2k-1, \ldots, t+k+\frac{m+3}{2}\Big\}.$$

We thus consider the following $k-(m+1)/2$ sets:

$$\{t+2k, t+1+m\}, \{t+2k-1, t+2+m\}, \dots, \left\{t+k+\frac{m+1}{2}+1, t+k+\frac{m+1}{2}-1\right\}.$$

We now pause to show that we can assume that $k-(m+1)/2>0$, i.e. $2k>m+1$. Since $2k>t$ and $m=t(t+1)/2k$, it follows that $2k\ge m+1$. Now assume that $2k=m+1$ and so $m=t$. Then $\Omega_{t+2k}=\Omega_{2t+1}$ and we wish to show that the set is $((t+1)/2)$-separable. Putting $t+1=2r$, this is equivalent to showing that Ω_{4r-1} is r-separable. This is easy since, by Lemma 3, $\{1,\dots,2r-1\}$ is r-separable and, as in the first part of the proof of Lemma 3, we have that $\{2r,\dots,4r-1\}$ is r-separable. Thus, by Lemma 2, so is their union.

Thus we have reduced the problem to showing that the set $\Omega_{t+m}\cup\{t+k+(m+1)/2\}$ can be partitioned into $(m+1)/2$ sets with equal sums of their elements. Or, as is more useful, we have reduced the problem to showing that the set Ω_{t+m} can be partitioned into m sets with equal sums of their elements. (These, along with $t+k+(m+1)/2$, can then be paired off.) Note also that the following three conditions are satisfied:

1. $m|(t+m)(t+m+1)/2$. (This is since $(t+m)(t+m+1)/2m = t(t+1)/2m + (2mt+m^2+m)/2m = k+t+(m+1)/2\in\mathbb{N}$.)
2. $t+m+1\ge 2m$, i.e. $t+1\ge m$. (Since $m=t(t+1)/2k$ and also $t<2k$, it follows that $m<t+1$.)
3. $t+m<n$. (We have already shown that $2k>m+1$. Then $n\ge 2k+t>m+t+1$.)

Again we can invoke the induction hypothesis $P(t+m)$ to complete this case.

Combining Lemmas 1 and 4, we have our main result.

Theorem 1 *Let n and k be natural numbers. Then Ω_n is k-separable if and only if $k|n(n+1)/2$ and $n+1\ge 2k$.*

Note that the proof of Lemma 4 is essentially constructive; we now provide an example.

Example 1 Consider $n=44$ and $k=10$ (these are compatible with the conditions of Theorem 1). Firstly,

$$\{25,\dots,44\} = B_1 := \{25,44\}\cup B_2 := \{26,43\}\cup\dots\cup B_{10} := \{34,35\}.$$

We now concentrate on the set Ω_{24}. Then $24\equiv 4 \mod 20$. In the notation of the proof of Lemma 4, we have $t=4$ and thus $m=1$. Then we form the sets: $A_1 := \{24,6\}$, $A_2 := \{23,7\},\dots, A_9 := \{16,14\}$. We are then left with the, in this case, trivial task of partitioning $\{1,2,3,4,5,15\}$ into one set whose elements sum to 30. Thus $A_{10} := \{1,2,3,4,5,15\}$. One suitable partition of Ω_{24} into 10 sets is then $A_i\cup B_i$, $i=1,2,\dots,10$. That is,

$$\{6,24,25,44\},\quad \{7,23,26,43\},\quad \{8,22,27,42\},\quad \{9,21,28,41\},$$
$$\{10,20,29,40\},\quad \{11,19,30,39\},\quad \{12,18,31,38\},\quad \{13,17,32,37\},$$
$$\{14,16,33,36\},\quad \{1,2,3,4,5,15,34,35\}.$$

The above example illustrates that, in general, there will be many different ways to partition the sets Ω_n with the required properties.

Reference

1 H. Shultz, Summation properties of $\{1, 2, \ldots, n\}$, *Math. Spectrum* **23** (1990/1), pp. 8–11.

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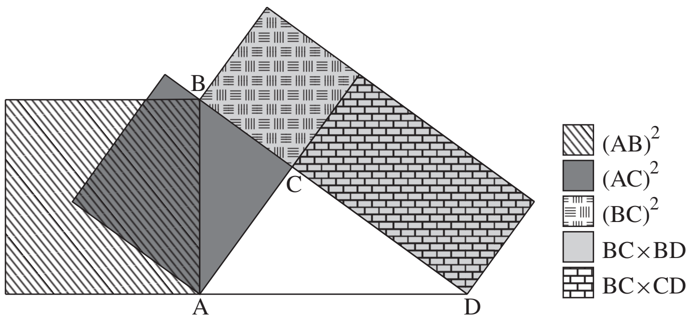
A proof of Pythagoras' theorem using the 'mean proportional'



Triangles ABC, DBA and DAC are similar right-angled triangles, so that

$$\frac{\mathrm{AB}}{\mathrm{BC}} = \frac{\mathrm{BD}}{\mathrm{AB}} \quad \text{and} \quad \frac{\mathrm{BC}}{\mathrm{AC}} = \frac{\mathrm{AC}}{\mathrm{CD}}.$$

Hence

$$(\mathrm{AB})^2 = \mathrm{BC} \times \mathrm{BD} \quad \text{and} \quad (\mathrm{AC})^2 = \mathrm{BC} \times \mathrm{CD},$$

i.e. AB is the 'mean proportional' of BC and BD and AC is the 'mean proportional' of BC and CD. But

$$\mathrm{BD} = \mathrm{BC} + \mathrm{CD},$$

so that

$$\mathrm{BC} \times \mathrm{BD} = (\mathrm{BC})^2 + \mathrm{BC} \times \mathrm{CD},$$

whence

$$(\mathrm{AB})^2 = (\mathrm{BC})^2 + (\mathrm{AC})^2,$$

which is Pythagoras' theorem for triangle ABC. The shaded areas in the figure illustrate the proof.

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When Tweedledum and Tweedledee met Mr Fibonacci

P. GLAISTER

A glimpse into a mathematical wonderland of series

In Wonderland, not so long ago, Tweedledum and Tweedledee, known as Dum and Dee to their friends, were doing nothing much, which they were particularly good at, when a stranger appeared from nowhere. He introduced himself as Mr Fibonacci, Bonaccio's most well-known son, and said he had a few ideas he would like to try out on the twins. He omitted to tell them that he had already approached a rabbit who, on the face of it, seemed a trifle unbalanced. More to the point, the rabbit said that he had heard all about Mr Fibonacci's theory on rabbits and was having nothing to do with it!

Although Dee and Dum liked to give the impression of being at the low end of the Wonderland rainbow of intellect, Mr Fibonacci soon discovered that their creator had bestowed on them a fair degree of mathematical competence, Dum being particularly good at algebra and Dee adept at differential calculus. In order to explain the first problem, Mr Fibonacci set the twins on a walk. Dum was asked to walk forward a distance of one, followed by one half of this distance, and then one half of that distance, and so on. (Of course, in reality it is not possible to take an infinite number of individual walks of ever-decreasing distance, but this is Wonderland! In any case, by taking more and more individual walks, a limiting value in (1) is approached, and this is the distance required.) The total distance Dum travelled was

$$A = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots . \tag{1}$$

(Note that Wonderland is a dimensionless world.)

Dee, on the other hand, was asked to set off walking in a similar manner to Dum, but to multiply each of the individual distances by the natural numbers $n = 1, 2, 3, \ldots$, so that he travels a distance

$$B = 1 \times 1 + \tfrac{1}{2} \times 2 + \tfrac{1}{4} \times 3 + \cdots . \tag{2}$$

It was at precisely this point that Mr Fibonacci became suspicious of the twins' abilities, when he heard them mumbling something about a tortoise.

Mr Fibonacci then asked the twins to try two new walks, both similar to Dee's modification of Dum's original walk, but where the individual distances are first multiplied by the triangle numbers $T_n = \frac{1}{2}n(n+1) = 1, 3, 6, 10, \ldots$, and then multiplied by the square numbers $n^2 = 1, 4, 9, \ldots$. The distances travelled in these walks would therefore be

$$C = 1 \times 1 + \tfrac{1}{2} \times 3 + \tfrac{1}{4} \times 6 + \cdots ,$$
$$D = 1 \times 1 + \tfrac{1}{2} \times 4 + \tfrac{1}{4} \times 9 + \cdots .$$

Having got pretty tired doing their first walks, the twins asked if it was possible to work out the distances on paper. Mr Fibonacci assured them that by combining their strengths they would be able to do so and, while they were at it, they might as well have a look at (1) and (2). He concluded by warning them of the dangers of working with infinite expressions, or series as he called them. Fortunately, the twins' creator was no mean mathematician himself, and would be watching over them to ensure that they did not break any mathematical rules.

Dee and Dum set to, and this is a summary of what they found. Starting with series (1), the best approach is to determine

$$S = 1 + t + t^2 + \cdots\ , \tag{3}$$

and then substitute $t = \frac{1}{2}$, or any other value of t for that matter for which the result is valid. Multiplying (3) by t and then subtracting (3), we obtain

$$\begin{aligned} S - tS &= 1 + t + t^2 + \cdots \\ &\qquad - t - t^2 - \cdots \\ &= 1, \end{aligned}$$

so that

$$S = 1 + t + t^2 + \cdots = \frac{1}{1-t} \quad \text{(valid for } -1 < t < 1\text{)}. \tag{4}$$

Therefore with $t = \frac{1}{2}$ the value of $S = 1/(1 - \frac{1}{2}) = 2$ and the distance for the first walk is

$$A = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots = 2.$$

For the second walk, multiplying (4) by t, i.e.

$$\frac{t}{1-t} = t + t^2 + t^3 + \cdots \quad \text{(valid for } -1 < t < 1\text{)}, \tag{5}$$

and then differentiating each side of (5) and simplifying the left-hand side, we obtain

$$\frac{1}{(1-t)^2} = 1 + 2t + 3t^2 + \cdots \quad \text{(valid for } -1 < t < 1\text{)}. \tag{6}$$

Therefore, substituting $t = \frac{1}{2}$ in (6),

$$B = 1 \times 1 + \tfrac{1}{2} \times 2 + \tfrac{1}{4} \times 3 + \cdots = 4,$$

and so the second walk is exactly twice the distance of the first walk with

$$2\big(1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots\big) = 1 \times 1 + \tfrac{1}{2} \times 2 + \tfrac{1}{4} \times 3 + \cdots\ .$$

Similarly, firstly differentiating (6) and substituting $t = \frac{1}{2}$, and secondly multiplying (6) by t, differentiating and substituting $t = \frac{1}{2}$, and then simplifying, shows that

$$C = 1 \times 1 + \tfrac{1}{2} \times 3 + \tfrac{1}{4} \times 6 + \cdots = 8,$$

and

$$D = 1 \times 1 + \tfrac{1}{2} \times 4 + \tfrac{1}{4} \times 9 + \cdots = 12.$$

These are distances of four times and six times the first walk with $12A = 6B = 3C = 2D$, i.e.

$$\begin{aligned} 12\big(1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots\big) &= 6\big(1 \times 1 + \tfrac{1}{2} \times 2 + \tfrac{1}{4} \times 3 + \cdots\big) \\ &= 3\big(1 \times 1 + \tfrac{1}{2} \times 3 + \tfrac{1}{4} \times 6 + \cdots\big) \\ &= 2\big(1 \times 1 + \tfrac{1}{2} \times 4 + \tfrac{1}{4} \times 9 + \cdots\big). \end{aligned} \tag{7}$$

Therefore multiplying the individual distances in the first walk, which correspond to the terms in the bracket on the left-hand side of (7), by the natural numbers, the triangle numbers and the square numbers, changes the distance of the walk by a factor of 2, 4 and 6 respectively. Moreover, replacing the natural numbers by the triangle numbers and the square numbers in the second bracket of (7), changes the distance by a factor of 2 and 3 respectively.

After all that effort the twins felt very pleased with themselves. Unfortunately, they were in for a bit of a shock as Mr Fibonacci announced that he had some more walks that he would like them to try. He described a new sequence of numbers that they would need, given by the following formula:

$$F_n = F_{n-1} + F_{n-2}, \qquad n = 3, 4, \ldots, \tag{8}$$

where every number beyond the first two is the sum of the previous two, and, starting with $F_1 = F_2 = 1$, the sequence is $1, 1, 2, 3, 5, 8, 13, 21, \ldots$. Not surprisingly, the twins decided to call these the *Fibonacci numbers*. They were asked to work out the distance travelled if the individual distances in the original walk were now multiplied by the Fibonacci numbers, so that the distance travelled is

$$E = 1 \times F_1 + \tfrac{1}{2} \times F_2 + \tfrac{1}{4} \times F_3 + \cdots, \tag{9}$$

and, while they were at it, they might as well try

$$G = 1 \times F_1 + \tfrac{1}{2} \times 2F_2 + \tfrac{1}{4} \times 3F_3 + \cdots,$$

i.e. multiplying each of the new individual distances in (9) by the natural numbers. This was clearly a more difficult challenge for Dee and Dum, as we now see.

Again it is more useful to consider

$$U = F_1 + tF_2 + t^2F_3 + \cdots, \tag{10}$$

and then substitute $t = \frac{1}{2}$. Multiplying (10) by t, and separately by t^2, and then subtracting from (10), we have

$$\begin{aligned} U - tU - t^2U &= F_1 + tF_2 + t^2F_3 + t^3F_4 + t^4F_5 + \cdots \\ &\qquad - tF_1 - t^2F_2 - t^3F_3 - t^4F_4 - \cdots \\ &\qquad\qquad - t^2F_1 - t^3F_2 - t^4F_3 - \cdots \\ &= F_1 + t(F_2 - F_1) + t^2(F_3 - F_2 - F_1) + t^3(F_4 - F_3 - F_2) + \cdots \\ &= 1, \end{aligned}$$

using $F_1 = F_2 = 1$ and the formula for F_n in (8) for $n = 3, 4, \ldots$, so that

$$U = \frac{1}{1 - t - t^2} = F_1 + tF_2 + t^2F_3 + \cdots, \tag{11}$$

which is valid for $-\frac{1}{2}(\sqrt{5} - 1) < t < \frac{1}{2}(\sqrt{5} - 1)$, i.e. $-0.618\ldots < t < 0.618\ldots$. (The value $\frac{1}{2}(\sqrt{5} - 1)$ in the limits on the range of values of t for which the series is valid is given by $\lim_{n\to\infty}(F_n/F_{n+1})$ and is the reciprocal of the well-known golden ratio $\frac{1}{2}(\sqrt{5} + 1) = 1.618\ldots$.)

Now, substituting $t = \frac{1}{2}$ in (11) gives $U = 1/(1 - \frac{1}{2} - \frac{1}{4}) = 4$, so that

$$E = F_1 + \tfrac{1}{2}F_2 + \tfrac{1}{4}F_3 + \cdots = 4,$$

which is the same distance as the second walk the twins tried, with

$$1 \times 1 + \tfrac{1}{2} \times 2 + \tfrac{1}{4} \times 3 + \cdots = 1 \times F_1 + \tfrac{1}{2} \times F_2 + \tfrac{1}{4} \times F_3 + \cdots . \qquad (12)$$

This means that replacing the natural numbers on the left-hand side of (12) by the Fibonacci numbers leaves the distance unchanged, which quite surprised the twins.

Similarly, multiplying (11) by t, then differentiating and simplifying, shows that

$$\frac{1+t^2}{(1-t-t^2)^2} = F_1 + 2tF_2 + 3t^2F_3 + \cdots \quad \text{(valid for } -0.618\ldots < t < 0.618\ldots), \qquad (13)$$

and, with $t = \frac{1}{2}$,

$$G = 1 \times F_1 + \tfrac{1}{2} \times 2F_2 + \tfrac{1}{4} \times 3F_3 + \cdots = 20.$$

Therefore $10A = 5B = 5E = G$, i.e.

$$\begin{aligned} 10\big(1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots\big) &= 5\big(1 \times 1 + \tfrac{1}{2} \times 2 + \tfrac{1}{4} \times 3 + \cdots\big) \\ &= 5\big(1 \times F_1 + \tfrac{1}{2} \times F_2 + \tfrac{1}{4} \times F_3 + \cdots\big) \\ &= 1 \times F_1 + \tfrac{1}{2} \times 2F_2 + \tfrac{1}{4} \times 3F_3 + \cdots . \end{aligned} \qquad (14)$$

In other words, multiplying the individual distances in the original walk, which correspond to the terms in the bracket on the left-hand side of (14), by the natural numbers, the Fibonacci numbers, and the product of these numbers, changes the distance of the walk by a factor of 2, 2 and 10 respectively.

We leave the following two exercises to readers, as they were left for Dee and Dum to do. Firstly, show that

$$1 \times F_1 + \tfrac{1}{2} \times 3F_2 + \tfrac{1}{4} \times 6F_3 + \cdots = 104,$$

(hint: multiply (13) by t^2 and differentiate once, or multiply (11) by t^2 and differentiate twice). Secondly, show that

$$1 \times F_1 + \tfrac{1}{2} \times 4F_2 + \tfrac{1}{4} \times 9F_3 + \cdots = 188,$$

(hint: multiply (13) by t and differentiate once). Readers may also like to consider multiplying the individual distances by other combinations of the numbers discussed.

Mr Fibonacci said that he would have to be on his way again very soon, but before doing so he would like the twins to try one last variation on their walks. This time they would walk forwards and backwards, in an alternating pattern, which seemed more interesting, so they actually did the walking again this time. Dee was asked to walk forward a distance F_1, and then back a distance $\frac{1}{2}F_2$, forward $\frac{1}{4}F_3$, back $\frac{1}{8}F_4$, and so on, so that he was at a displacement

$$P = F_1 - \tfrac{1}{2}F_2 + \tfrac{1}{4}F_3 - \tfrac{1}{8}F_4 + \cdots \qquad (15)$$

from the starting point. Dum, on the other hand, walked forward a distance $1 \times F_1$, and then back a distance $\frac{1}{2} \times 2F_2$, forward $\frac{1}{4} \times 3F_3$, back $\frac{1}{8} \times 4F_4$, and so on, so that he was at a displacement

$$Q = F_1 - \tfrac{1}{2}2F_2 + \tfrac{1}{4}3F_3 - \tfrac{1}{8}4F_4 + \cdots \qquad (16)$$

from the starting point. Again, what they found came as a complete surprise.

To share their surprise, we use (11) with $t = -\frac{1}{2}$, giving

$$P = \frac{1}{1 + \frac{1}{2} - \frac{1}{4}} = \frac{4}{5},$$

whereas from (13) with $t = -\frac{1}{2}$

$$Q = \frac{1 + \frac{1}{4}}{\left(1 + \frac{1}{2} - \frac{1}{4}\right)^2} = \frac{4}{5},$$

meaning that Dee and Dum ended up at exactly the same point, a distance of $\frac{4}{5}$ from the starting point, with

$$F_1 - \tfrac{1}{2}F_2 + \tfrac{1}{4}F_3 - \tfrac{1}{8}F_4 + \cdots = F_1 - \tfrac{1}{2}2F_2 + \tfrac{1}{4}3F_3 - \tfrac{1}{8}4F_4 + \cdots . \tag{17}$$

Therefore, multiplying the individual distances on the left-hand side of (17) by the natural numbers leaves the displacement unchanged. This so intrigued the twins that they thought they would try some of the variations used earlier.

Firstly they multiplied the individual distances in the walk described by (15) by the triangle numbers and, while they were about it, did the same with the walk described by (16). We leave readers to check that by multiplying (13) by t^2 and differentiating once (or by multiplying (11) by t^2 and differentiating twice), then we obtain

$$F_1 + 3tF_2 + 6t^2F_3 + \cdots = \frac{1 + 3t^2 - t^3}{(1 - t - t^2)^3}, \tag{18}$$

also by multiplying (18) by t and differentiating once (or multiplying (13) by t^2 and differentiating twice), then we obtain

$$F_1 + 2 \times 3tF_2 + 3 \times 6t^2F_3 + \cdots = \frac{1 + 2t + 14t^2 + 10t^4 - 2t^5}{(1 - t - t^2)^4}. \tag{19}$$

(Both (18) and (19) are valid for the same values of t as (11).) Substituting $t = -\frac{1}{2}$ in (18) gives

$$F_1 - \tfrac{1}{2} \times 3F_2 + \tfrac{1}{4} \times 6F_3 + \cdots = \tfrac{24}{25} \tag{20}$$

as the displacement from the starting point for the first of these walks, whereas for the second walk we have, from (19) with $t = -\frac{1}{2}$,

$$F_1 - \tfrac{1}{2} \times 2 \times 3F_2 + \tfrac{1}{4} \times 3 \times 6F_3 - \tfrac{1}{8} \times 4 \times 10F_4 + \cdots = \tfrac{48}{25}, \tag{21}$$

which is exactly twice as far ahead as for the walk in (20), with

$$F_1 - \tfrac{1}{2} \times 2 \times 3F_2 + \tfrac{1}{4} \times 3 \times 6F_3 - \cdots = 2\left(F_1 - \tfrac{1}{2} \times 3F_2 + \tfrac{1}{4} \times 6F_3 - \cdots\right). \tag{22}$$

Therefore, multiplying the individual distances on the right-hand side of (22) by the natural numbers changes the displacement by a factor of 2.

Mr Fibonacci had one last challenge. He set Dee and Dum off on an alternating walk based on the natural numbers and the triangle numbers, so that the displacement from the starting point would be

$$1 \times 1 \times 1 - \tfrac{1}{2} \times 2 \times 3 + \tfrac{1}{4} \times 3 \times 6 - \tfrac{1}{8} \times 4 \times 10 + \tfrac{1}{16} \times 5 \times 15 - \cdots . \tag{23}$$

Now the twins knew the consequence of removing the Fibonacci numbers from (20) by differentiating (6) (or differentiating (4) twice) and substituting $t = -\frac{1}{2}$, but (23) is clearly (21) with the Fibonacci numbers removed. As they continued on their walk they realised, after a while, that they seemed to be getting nowhere. They turned round to ask Mr Fibonacci to confirm their observation but, as if by some strange coincidence, he was nowhere to be seen, apparently disappearing as quickly as he had appeared, and they were left wondering if it had all been a dream!

Readers may like to experiment with variations on the Wonderland theme, but should start by proving (multiplying (6) by t^2 and differentiating twice ought to do the trick) what became known as the twins last conjecture (after all, they were pretty worn out by now!)

$$1 \times 1 \times 1 - \tfrac{1}{2} \times 2 \times 3 + \tfrac{1}{4} \times 3 \times 6 - \tfrac{1}{8} \times 4 \times 10 + \tfrac{1}{16} \times 5 \times 15 - \cdots = 0.$$

The author lectures in mathematics at Reading University, with interests in computational fluid dynamics, numerical analysis, perturbation methods as well as mathematics and science education. When running the title of this article past his two children, Anna (13) just gave him 'one of her looks', while Mark (10) thought it better suited to a 'rap'!

To square a number ending in 5

$$115^2 = (11 \times 12)25 \qquad = 13\,225,$$
$$1005^2 = (100 \times 101)25 = 1\,010\,025,$$

and, in general,

$$(a5)^2 = (a \times (a+1))25.$$

Can you prove this?

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Sebastian Hayes

A Simple Geometric Construction of the Harmonic Mean of *n* Variables

JIM ZENG and THOMAS J. OSLER

The harmonic mean of n numbers $a_1, a_2, \ldots, a_n$, is defined to be the number h such that

$$\frac{1}{n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)=\frac{1}{h}.$$

The harmonic mean is the reciprocal of the mean of the reciprocals of

$$a_1, a_2, \ldots, a_n.$$

In this article, we show a simple way to construct the harmonic mean of n positive numbers geometrically, and we give an application to finding the resistance of n resistors in parallel. A construction of the harmonic mean of two numbers is given in reference 1, and a geometrical interpretation of several different means is presented in reference 2.

Firstly we construct the line segment of length y_2, such that

$$\frac{1}{a_1}+\frac{1}{a_2}=\frac{1}{y_2}, \tag{1}$$

see figure 1. Begin by locating the points $(0, a_2)$ and (x_1, a_1), where x_1 is any positive real number. Construct the line segment joining $(0, a_2)$ and $(x_1, 0)$ and the line segment joining the origin and (x_1, a_1). Call the intersection of these two lines (x_2, y_2). Using similar triangles, we see that

$$\frac{y_2}{a_1}=\frac{x_2}{x_1} \quad \text{and} \quad \frac{y_2}{a_2}=\frac{x_1-x_2}{x_1}.$$

Adding these two equations, we obtain

$$\frac{y_2}{a_1}+\frac{y_2}{a_2}=1,$$

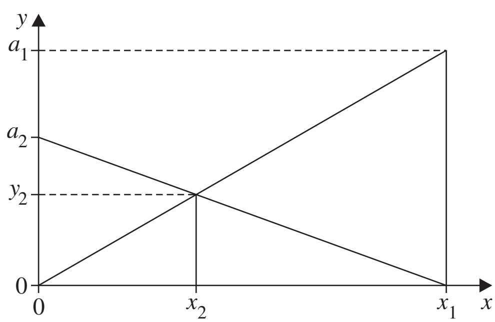

Figure 1

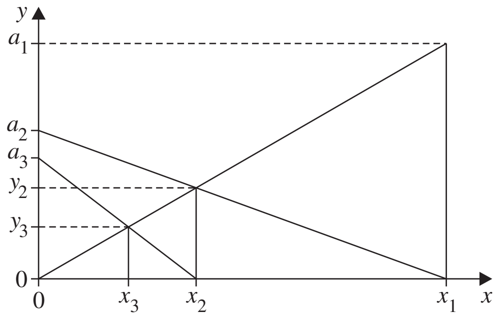

Figure 2

and so y_2 satisfies (1). Thus we have constructed the line segment whose length is the solution of (1).

Next we construct the line segment of length y_3 that is the solution of the following equation:

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = \frac{1}{y_3}. \tag{2}$$

As before, we construct the line segment joining the points $(0, a_3)$ and $(x_2, 0)$, see figure 2. Let (x_3, y_3) be the intersection of this line with the previously drawn line from the origin to the point (x_1, a_1). Repeating the previous argument with the smaller triangles, we obtain

$$\frac{1}{y_2} + \frac{1}{a_3} = \frac{1}{y_3}.$$

Replacing $1/y_2$ by (1), we immediately have (2). Thus we have constructed a line segment of length y_3 that satisfies (2). It is easy to see how this process can be repeated to construct a solution y_n of the general equation

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = \frac{1}{y_n}. \tag{3}$$

We will now show how to construct the harmonic mean of the set of numbers $a_1, a_2, \ldots, a_n$. By the definition given at the beginning, the harmonic mean is $h = ny_n$. (Here y_n is the number given in (3).) A simple modification of the above construction will give us this number h. Recall that x_1 in our construction was arbitrary. Now suppose that we take $x_1 = na_1$. Then we see that

$$\frac{1}{n} = \frac{a_1}{x_1} = \frac{y_n}{x_n}.$$

Thus the harmonic mean is simply $x_n = ny_n = h$, and we have constructed a line segment with this length.

As an application, we recall that when n resistors with resistances $R_1, R_2, \ldots, R_n$ are connected in parallel, then the equivalent resistance R is given by

$$\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n} = \frac{1}{R}.$$

Thus we have given a graphical method of constructing the equivalent resistance of n resistors in parallel.

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Curious cubes

$$1^3 = 1,$$
$$8^3 = 512, \qquad 5+1+2 = 8,$$
$$9^3 = 729, \qquad 7+2+9 = 18, \qquad 1+8 = 9,$$
$$17^3 = 4\,913, \qquad 4+9+1+3 = 17,$$
$$18^3 = 5\,832, \qquad 5+8+3+2 = 18.$$

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The Probability of Scoring a Century in Cricket

GRAEME COHEN

In the game of cricket, under suitable assumptions, the probability distribution of the number of scoring strokes made by a given batsman in any innings is geometric. With the further assumption that the ratio of runs made to number of scoring strokes is a constant, the expression $(A/(A+2))^{c/2}$ is an approximate probability of the batsman scoring at least c runs, $c \geq 1$, where A is the batsman's average score over all past innings. These results were obtained by the author in a previous paper, and are compared here with those of a recent paper by Tan and Zhang in *Mathematical Spectrum*.

1. Introduction

In an excellent survey of papers written on statistics (the more mathematical kind) applied to cricket, Clarke, see reference 1, wrote that cricket 'has the distinction of being the first sport used for the illustration of statistics', but: 'in contrast to baseball, few papers in the professional literature analyse cricket, and two rarely analyse the same topic'.

There are, however, a number of papers concerned with the distribution of batting scores, generally with the intention of using published cricket statistics such as the batting average to predict, say, the likelihood of scoring a century. For background to this, see reference 1. The latest journal publication in the area is that of Tan and Zhang, see reference 2. Before that, there were various other papers, see references 3–6. Recently, I provided a different approach (see reference 7) in which I concentrated instead on the distribution of the number of scoring strokes. Scoring strokes are related to the number of runs scored by assuming the ratio of these quantities to be (approximately) constant. Having a probability distribution for the number of scoring strokes then allows the probability of a batsman making any particular score to be determined. I deduced that the expression

$$\left(\frac{A}{A+2}\right)^{c/2}$$

is an approximate probability of the batsman scoring at least c runs, $c \geq 1$, where A is the batsman's average score over all past innings.

Tan and Zhang (see reference 2) gave a more complicated formula based on a least-squares fit and tested it with the batting scores of Jack Hobbs and Herbert Sutcliffe. We will see that the numerical results using my approach agree very closely with theirs and we will show that asymptotically the two approaches, although totally different, give essentially the same conclusion. For the approaches adopted in the earlier papers, see the summary given in reference 7.

We denote the ratio of runs made to number of scoring strokes the *strike constant*. An investigation of this ratio for a large number of Sydney grade cricketers (see reference 8) came up with the value 2.16, with standard deviation 0.25, for traditional cricket, and 1.82, with standard deviation 0.43, for limited-overs cricket. Indoor cricket (ignoring runs subtracted for loss of wicket) was also investigated, the mean strike constant was 2.08 with standard deviation 0.41. Results of a further investigation were presented at the Seventh Australasian Conference on Mathematics and Computers in Sport in September 2004.

Section 2 begins by detailing the reasoning behind the claim that, subject to certain assumptions, the number of scoring strokes follows a geometric distribution and then introduces the strike constant to give the distribution of runs scored. In his discussion of the earlier papers in the area, Clarke, see reference 1, stated 'if a batsman scores only singles and his probability of dismissal is constant, his scores should follow a geometric distribution, the discrete equivalent of the negative exponential'. This observation was based on a viewing of the data, but proves to be a direct consequence of the work below.

2. The probability of scoring a century (or a duck)

Take data pertaining to a particular batsman over a particular period, such as the overall career, or the previous season, or in a particular position, or against a particular team. Let i, n, w, and r be the number of innings, the number of not-out innings, the number of dismissals, and the number of runs scored respectively. Then $w = i - n$. The batsman's traditional average, as used constantly when cricket is discussed, and 'true' average are

$$B = \frac{r}{w} \quad \text{and} \quad A = \frac{r}{i}$$

respectively. Note that

$$A = \frac{w}{i} B = \frac{i-n}{i} B, \tag{1}$$

so that, for overall career results, say, the true average may be determined from the usual published batting statistics. Of course, $A = B$ when $n = 0$.

Furthermore, let b and s be the number of balls faced and the number of scoring strokes made respectively. We must have $r \geq s \geq 0$ and we will assume that $b > i > n \geq 0$, then $w > 0$.

We define p_w as the probability that the batsman's innings ends, out or not out, with each ball faced, and p_s as the probability that the batsman makes a scoring stroke with each ball faced, given that the batsman's innings does not end with that ball. Now,

$$\begin{aligned} p_w &= \frac{i}{b}, & p_s &= \frac{s}{b-i}, \\ q_w &= 1 - p_w, & q_s &= 1 - p_s. \end{aligned}$$

These probabilities are considered to be constant throughout a subsequent innings.

We have made an assumption that no scoring stroke is made from the ball on which the batsman's innings ends (so that $s \leq b - i$). Therefore, we do not take into account the rare instance in which the batsman makes at least one run and is then run out on the same ball while attempting a further run, or the admittedly more common instance in which a captain declares an innings closed following the batsman's final scoring stroke.

In *any* period, the ratio of the number of runs obtained to the number of scoring strokes made is considered to be constant. This is the simplification described above. The ratio is the strike constant, denoted by κ, i.e.

$$\kappa = \frac{r}{s}.$$

Let the random variable X be the number of scoring strokes made by the batsman in a subsequent innings, and let R be the score (number of runs) obtained. In order that $X = k$, for an integer $k \geq 0$, the batsman must face $j+1$ balls, for some $j \geq k$, scoring on k of these and ending the innings on the $(j+1)$th ball. This is still effectively the case when the batsman remains at the wicket but without facing another ball, or if the team's innings then ends or the batsman is then run out. Whether or not the batsman scores off any of the first j balls bowled, these are considered to be independent events and so the distribution of the k scoring strokes among the j balls will be binomial (j, p_s). Write $\mathrm{P}(X = k)$ for the probability that the batsman makes k scoring strokes, $k \geq 0$, before being dismissed. (Later notation will have a corresponding meaning.) Then we obtain

$$\begin{aligned}
\mathrm{P}(X = k) &= \sum_{j=k}^{\infty} q_w^j p_w \binom{j}{k} p_s^k q_s^{j-k} \\
&= \frac{p_s^k p_w q_w^k}{k!} \sum_{j=k}^{\infty} \frac{j!}{(j-k)!} (q_s q_w)^{j-k} \\
&= \frac{p_s^k p_w q_w^k}{(1 - q_s q_w)^{k+1}} \\
&= \frac{p_w}{1 - q_s q_w} \left(\frac{p_s q_w}{1 - q_s q_w} \right)^k \\
&= P Q^k,
\end{aligned}$$

where

$$\begin{aligned}
P &= \frac{p_w}{1 - q_s q_w}, \\
Q &= \frac{p_s q_w}{1 - q_s q_w} = 1 - P.
\end{aligned}$$

Thus the number of scoring strokes made follows a geometric distribution. Using the definitions of p_w and p_s, we find that

$$\begin{aligned}
P &= \frac{i}{i+s} = \frac{\kappa}{A+\kappa}, \\
Q &= \frac{A}{A+\kappa}.
\end{aligned}$$

The expected number of scoring strokes is then easily determined as follows:

$$\mathrm{E}(X) = \sum_{k=0}^{\infty} k\, \mathrm{P}(X = k) = P Q \sum_{k=1}^{\infty} k Q^{k-1} = \frac{Q}{P} = \frac{A}{\kappa}.$$

Then, since $X = 0$ if and only if $R = 0$,

$$\begin{aligned} \mathrm{E}(R) &= \mathrm{E}(R \mid R > 0) \\ &= \mathrm{E}\left(X \frac{R}{X} \,\middle|\, X > 0 \right) \\ &= \mathrm{E}(\kappa X \mid X > 0) \\ &= \kappa\, \mathrm{E}(X) \\ &= A. \end{aligned}$$

That is, sensibly, the expected number of runs is the true average in this model. For a batsman with no not-out innings ($n = 0$), the expected number of runs will be the traditional average B.

We can now obtain the probability of scoring c runs, by which we mean the usual notion in cricket of in fact making c or more runs, except for the case below when we give the probability of a duck (exactly zero runs).

Writing

$$d = \left[\frac{c-1}{\kappa} \right],$$

where $[\cdot]$ denotes the greatest integer not exceeding its argument, we have, for $c \geq 1$,

$$\begin{aligned} \mathrm{P}(R \geq c) &= 1 - \mathrm{P}(R \leq c-1) \\ &= 1 - \mathrm{P}(\kappa X \leq c-1) \\ &= 1 - \mathrm{P}(X \leq d) \\ &= \sum_{k=d+1}^{\infty} \mathrm{P}(X = k) \\ &= Q^{d+1} \\ &= \left(\frac{A}{A+\kappa} \right)^{d+1}. \end{aligned}$$

If $\kappa = 2$, say, then these probabilities are equal for $c = 2m$ and for $c = 2m-1$, for any positive integer m. In practice that should be considered as acceptable, especially for large c, and in this theory it is unavoidable.

Table 4 in reference 7 gave the number of century scores and over-50 scores by the all-time top twenty Test batsmen who had batted in at least 20 innings, as at 7 February 2002, and the estimates of these values resulting from our formula, using the following simplification. Since it is not easy to have access to original score sheets, nor desirable for widest applicability, it is most unlikely that actual values of κ, the ratio over the past number of runs made to number of scoring strokes, can be obtained. The easy approach is to set $\kappa = 2$, although in fact the expected values are not appreciably different whether we take $\kappa = 1.9, 2$ or 2.1. Then, by assuming that c is even, as in the common cases $c = 100$ or $c = 50$, we obtain

$$\mathrm{P}(R \geq c) \approx \left(\frac{A}{A+2} \right)^{c/2}, \tag{2}$$

and this may be adopted as a useful approximation for all $c \geq 1$.

The corresponding results for Hobbs and Sutcliffe, the examples chosen by Tan and Zhang (see table 1 of reference 2), are included in our table 1 below, and give an application of our model with an accuracy similar to that in table 4 of reference 7.

As an application of (2) and, allowing more comparisons with Tan and Zhang's table 1 (see reference 2), we will next find an estimate of a batsman's median score. From (2), for a given probability p, the score c required to ensure that $\mathrm{P}(R \geq c) = p$ is obtained approximately by solving

$$\left(\frac{A}{A+2}\right)^{c/2} = p.$$

We obtain

$$c = \frac{2\log(1/p)}{\log(1+2/A)},$$

where the logarithms may be of any suitable base. Taking $p = 0.5$, and using the fact that

$$\ln\left(1+\frac{2}{A}\right) \approx \frac{2}{A}$$

when $A \geq 20$, say, the median score is approximately $A\ln 2$. This is precisely the result obtained by Tan and Zhang (see reference 2), except that they use a heuristic estimate of A. Thus $0.7A$ would be an easy approximate formula for the median. Using (1), the median score based on published statistics is thus approximately

$$\frac{7(i-n)B}{10i}.$$

Finally, we consider the probability of a batsman getting a duck. From our early work, the probability of a batsman making no scoring stroke is

$$\mathrm{P}(X=0) = P = \frac{\kappa}{A+\kappa}. \tag{3}$$

But this includes the probability of scoring 0, not out. It is necessary here to use a batsman's *wicket average* A_w in place of A. This is the average of only those innings in which the batsman was out; for international players, it may be calculated directly from information on the CricInfo web site (http://www.cricinfo.com/). In most cases, A_w turns out to be very close to the true average A, as would be expected if a batsman averaged much the same in completed innings as in not-out innings. The earlier theory needs to be adjusted in a minor way to allow for the different sample space, i.e. b, r and s now relate only to completed innings, and i in the definitions of p_w and p_s must be replaced by w. Then, in particular, p_w is the probability with each ball faced that the batsman is dismissed. The subsequent analysis would then refer only to completed innings.

Our earlier discussion now suggests that, in place of (3), we take

$$\mathrm{P}(\text{duck}) \approx \frac{2}{A_w+2}. \tag{4}$$

The corresponding estimates for Hobbs and Sutcliffe are included in table 1.

In table 1, the rows are mostly those of table 1 of Tan and Zhang (see reference 2), and the column headed TZ is from their table 1. The column headed C is based on (2) and (4). Note that

Table 1 Batting statistics of Hobbs and Sutcliffe.

	Hobbs			Sutcliffe		
	Actual	C	TZ	Actual	C	TZ
True average A	53.04		54.05	54.23		53.48
Wicket average A_w	53.33			54.64		
Median score	40	37.13	37.47	38	37.96	37.07
Number of ducks	4	3.43	1.91	2	2.65	1.61
Number of 50s	43	40.43	41.24	39	33.97	34.15
Number of 100s	15	16.03	16.35	16	13.74	13.41
Number of 200s	1	2.52	2.57	0	2.25	2.07
Number of 300s	0	0.40	0.40	0	0.37	0.32
Probability of making a duck	0.039	0.036	0.018	0.024	0.035	0.019
Probability of reaching 48	0.461	0.411	0.412	0.476	0.419	0.401
Probability of reaching 96	0.157	0.169	0.169	0.202	0.176	0.166
Probability of making 100	0.147	0.157	0.157	0.191	0.164	0.154

'Number of 50s' refers to scores of 50 or more. For this, in column C, the probability obtained from (2) is multiplied by the total number of innings played (102 for Hobbs and 84 for Sutcliffe, in which they were not out seven and nine times respectively). There are corresponding remarks for the numbers of ducks and other scores.

3. Conclusion

The analysis given here is based on a batsman's true average A, i.e. simply the average of all scores, out or not out. The wicket average A_w, which refers specifically to completed innings, is approximately the same as A but should be used for questions concerning completed innings; otherwise use A. Among other things, we have justified the simple formula

$$\left(\frac{A}{A+2}\right)^{c/2}$$

as an approximate probability of scoring at least c runs, and the formula

$$\frac{2}{A_w+2}$$

as an approximate probability of a duck. Both of these have been compared favourably with results from the history of cricket. The work depends crucially on the concept of the strike constant, although less crucially on the value chosen for it. As a theoretical construct, its worth seems clear, and further investigation of the notion is being carried out.

We now show that it is not surprising that table 1 gives a strong agreement between our approach and that of Tan and Zhang, see reference 2. They fitted a curve of the form $y = a\mathrm{e}^{-\alpha x}$ to a graph of all Hobbs' scores, and then Sutcliffe's. For Hobbs, for example, they plotted (x_j, j) where x_j was the jth greatest score, for $j = 1, \dots, 102$. The least-squares approach gave the following estimates of the parameters: $a = 104$ and $\alpha = 0.0185$. The aggregate score r

is then given by

$$r = \int_0^\infty y \, \mathrm{d}x = \frac{a}{\alpha}$$

and their estimate of the true mean score follows as $r/a = 1/\alpha$. This value and the corresponding value for Sutcliffe are included in table 1. Using A for the true mean, as above, Tan and Zhang would have been justified in taking $\alpha = 1/A$. It follows also that the probability of a score of c or more is given by

$$\left(\frac{\alpha}{a}\right) \int_c^\infty y \, \mathrm{d}x,$$

so that, for Tan and Zhang,

$$\begin{aligned} \mathrm{P}(R \geq c) &= \frac{\alpha}{a} a \int_c^\infty \mathrm{e}^{-\alpha x} \, \mathrm{d}x \\ &= \mathrm{e}^{-c\alpha} \\ &= \mathrm{e}^{-c/A}. \end{aligned}$$

We can relate this to our estimate as follows:

$$\left(\frac{A}{A+2}\right)^{c/2} = \left(\left(1 + \frac{2}{A}\right)^{A/2}\right)^{-c/A} \sim \mathrm{e}^{-c/A},$$

as $A \to \infty$.

By totally different methods, we have thus arrived at much the same result. Furthermore, this asymptotic result does not depend on the value of κ.

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Square Roots by Subtraction

FRAZER JARVIS

Introduction

When I was at school, my mathematics teacher showed me the following very strange method to work out square roots, using only subtraction, which is reputedly an old Japanese method. I'll start by writing down the algorithm in a fairly formal way, which may, for example, make it easier to implement on a computer.

Although this method converges much more slowly than, for example, Newton's method for finding square roots, this method also has its advantages. Perhaps the main advantage from a computational point of view is that, when finding square roots of integers, no infinite decimals are involved at any step, which can cause loss of precision due to rounding errors.

Algorithm to compute the square root of an integer n

Initial step

Let $a = 5n$ (this multiplication by 5 is the only time when an operation other than addition and subtraction is involved), and put $b = 5$.

Repeated steps

R1. If $a \geq b$, replace a with $a - b$, and add 10 to b.

R2. If $a < b$, add two zeroes to the end of a, and add a zero to b just before the final digit (which will always be 5).

As a result, the digits of b approach the digits of $\sqrt{n}$.

Examples

Example 1 To illustrate the algorithm, we start to work out the value of $\sqrt{2}$. The initial step sets $a = 5 \times 2 = 10$ and $b = 5$. So we start with the pair $(a, b) = (10, 5)$.

Now we perform the repeated steps. As $10 > 5$, we must apply R1. We therefore replace a with $a - b$, so that now $a = 10 - 5 = 5$, and add 10 to b, i.e. $b = 15$. The new pair is therefore $(5, 15)$.

We apply the rule again. This time, however, $a = 5 < 15 = b$, and so we use R2. This rule says: add two zeroes to the end of a, to make $a = 500$, and put a zero before the final digit of b, so that $b = 105$. The pair becomes $(500, 105)$.

Now $500 > 105$, so we apply R1, and we replace a by $a - b = 500 - 105 = 395$, and add 10 to b, so that b becomes 115. Our pair is then $(395, 115)$.

Repeatedly applying the algorithm gives:

$$
\begin{aligned}
&(10, 5) \xrightarrow{\text{R1}} (5, 15) \xrightarrow{\text{R2}} (500, 105) \xrightarrow{\text{R1}} (395, 115) \xrightarrow{\text{R1}} (280, 125) \xrightarrow{\text{R1}} (155, 135)\\
&\xrightarrow{\text{R1}} (20, 145) \xrightarrow{\text{R2}} (2\,000, 1\,405) \xrightarrow{\text{R1}} (595, 1\,415) \xrightarrow{\text{R2}} (59\,500, 14\,105)\\
&\xrightarrow{\text{R1}} (45\,395, 14\,115) \xrightarrow{\text{R1}} (31\,280, 14\,125) \xrightarrow{\text{R1}} (17\,155, 14\,135)\\
&\xrightarrow{\text{R1}} (3\,020, 14\,145) \xrightarrow{\text{R2}} (302\,000, 141\,405) \xrightarrow{\text{R1}} (160\,595, 141\,415)\\
&\xrightarrow{\text{R1}} (19\,180, 141\,425) \xrightarrow{\text{R2}} (1\,918\,000, 1\,414\,205) \longrightarrow \cdots .
\end{aligned}
$$

It is clear that the digits of b are settling down to start with $14142\ldots$ ($\sqrt{2} = 1.414\,213\,56\ldots$).

In fact, the algorithm does not only work for positive integers, it works for all positive numbers, including decimals such as 2.71, and even the irrational number π. In these cases, rather than add two zeroes in R2, we have to multiply by 100 to shift the decimal point two places to the right (which is the same thing for integers).

Example 2 We calculate $\sqrt{2.345}$. Initially, $a = 5 \times 2.345 = 11.725$ and $b = 5$. Now apply the algorithm as before to get:

$$
\begin{aligned}
&(11.725, 5) \xrightarrow{\text{R1}} (6.725, 15) \xrightarrow{\text{R2}} (672.5, 105) \xrightarrow{\text{R1}} (567.5, 115) \xrightarrow{\text{R1}} (452.5, 125)\\
&\xrightarrow{\text{R1}} (327.5, 135) \xrightarrow{\text{R1}} (192.5, 145) \xrightarrow{\text{R1}} (47.5, 155) \xrightarrow{\text{R2}} (4\,750, 1\,505)\\
&\xrightarrow{\text{R1}} (3\,245, 1\,515) \xrightarrow{\text{R1}} (1\,730, 1\,525) \xrightarrow{\text{R1}} (205, 1\,535)\\
&\xrightarrow{\text{R2}} (20\,500, 15\,305) \xrightarrow{\text{R1}} (5\,195, 15\,315) \xrightarrow{\text{R2}} (519\,500, 153\,105) \longrightarrow \cdots ,
\end{aligned}
$$

the correct value for $\sqrt{2.345}$ is $1.531\,339\,28\ldots$.

Although the method works perfectly well for any positive number, it is usually easiest to begin by multiplying or dividing the given number by 100 as often as is necessary until it lies between 1 and 100. For example, given the number $23\,450$, we would begin by dividing it by 100 twice to get 2.345, and then applying the algorithm as above. Since we have divided it by 100 twice, we must multiply by 10 twice to get the correct square root. Thus $\sqrt{23\,450} = 153.133\,928\ldots$.

Example 3 We now observe how the algorithm copes with a perfect square, let $n = 16$. First, multiply by 5, and set $a = 80$. As usual, $b = 5$. Then, applying the algorithm gives:

$$
\begin{aligned}
&(80, 5) \xrightarrow{\text{R1}} (75, 15) \xrightarrow{\text{R1}} (60, 25) \xrightarrow{\text{R1}} (35, 35)\\
&\xrightarrow{\text{R1}} (0, 45) \xrightarrow{\text{R2}} (0, 405) \xrightarrow{\text{R2}} (0, 4\,005) \xrightarrow{\text{R2}} (0, 40\,005),
\end{aligned}
$$

since multiplying 0 by 100 leaves it unchanged. In practice, we can stop as soon as $a = 0$, and identify the square root by removing the final digit from b.

It is an amusing exercise to program a computer to do this algorithm, at least if n is an integer, and because only integer additions and subtractions are used, there are no errors due to floating-point arithmetic. On the other hand, it is necessary to use more complicated storage techniques (strings or arrays) to store the values of a and b as they get larger and larger, and the algorithm will get slower and slower at producing successive digits.

Explanation of the algorithm

It is a bit tricky to explain that the algorithm does indeed produce the correct answer. Before we embark on the demonstration, let's make a couple of observations.

Note that the number of times R1 is applied in between the applications of R2 ought to give the sequence of digits in b. We can check that all these digits lie between 0 and 9. Indeed, let us choose a point at which R2 is applied. So we begin with $a < b$, we change a to $100a$, and then subtract successively $10b - 45$ (this is what you get by putting 0 before the final 5 of b), $10b - 35$, $10b - 25$, and so on. We can do no more than nine of these subtractions (to subtract ten times would take away ten numbers averaging exactly $10b$ from $100a$, i.e. to subtract $100b$ from $100a$, but, since $a < b$, the first number of the pair would become negative, which is not allowed). It follows that we can apply R1 at most nine times between any two applications of R2, and so the digits appearing in b really are exactly the number of times R1 is applied in between applications of R2.

As remarked above, we will assume that our initial number lies between 1 and 100, so that its square root lies between 1 and 10. Thus we suppose that $b = b_0.b_1b_2b_3\ldots$, where all of the b_i lie between 0 and 9. Now, let us show that $n = b^2$. We shall consider a modified version of our algorithm which incorporates the decimal point.

Algorithm to compute the square root of *n*

Initial step

Let $a = n/2$, and put $b = 0.5$.

Repeated steps

R1$'$. If $a \geq b$, replace a with $a - b$, and increase the next-to-last decimal digit of b by 1.

R2$'$. If $a < b$, add two zeroes to the end of a, divide b by 10, and add a zero to b just before the final decimal digit (which will always be 5)

Example 1$'$ Let $n = 2$ (as in Example 1). We begin by setting $a = n/2 = 1$, and $b = 0.5$, and apply the algorithm as follows:

$$\begin{gathered}
(1, 0.5) \xrightarrow{\mathrm{R1}'} (0.5, 1.5) \xrightarrow{\mathrm{R2}'} (0.5, 0.105) \xrightarrow{\mathrm{R1}'} (0.395, 0.115) \xrightarrow{\mathrm{R1}'} (0.280, 0.125) \\
\xrightarrow{\mathrm{R1}'} (0.155, 0.135) \xrightarrow{\mathrm{R1}'} (0.020, 0.145) \xrightarrow{\mathrm{R2}'} (0.020\,00, 0.014\,05) \xrightarrow{\mathrm{R1}'} (0.005\,95, 0.014\,15) \\
\xrightarrow{\mathrm{R2}'} (0.005\,950\,0, 0.001\,410\,5) \xrightarrow{\mathrm{R1}'} (0.004\,539\,5, 0.001\,411\,5) \\
\xrightarrow{\mathrm{R1}'} (0.003\,128\,0, 0.001\,412\,5) \xrightarrow{\mathrm{R1}'} (0.001\,715\,5, 0.001\,413\,5) \\
\xrightarrow{\mathrm{R1}'} (0.000\,302\,0, 0.001\,414\,5) \xrightarrow{\mathrm{R2}'} (0.000\,302\,000, 0.000\,141\,405) \\
\xrightarrow{\mathrm{R1}'} (0.000\,160\,595, 0.000\,141\,415) \xrightarrow{\mathrm{R1}'} (0.000\,019\,180, 0.000\,141\,425) \\
\xrightarrow{\mathrm{R2}'} (0.000\,019\,180\,00, 0.000\,014\,142\,05) \longrightarrow \cdots.
\end{gathered}$$

(Note that the numbers which appear are exactly the same as in Example 1, we have merely multiplied all of them by a power of 10.)

We begin with $a = n/2$, and, using R1$'$, we have to subtract various numbers (which we occasionally make smaller using R2$'$). Eventually, the first number in the pair approaches 0. It follows that a is the sum of all the numbers which we take away from it, to get 0.

Before the first application of R2$'$, we make b_0 applications of R1$'$, subtracting $b_0/2$ on average each time. The total amount subtracted in these steps is therefore $b_0^2/2$.

Between the first and second applications of R2$'$, we make b_1 applications of R1$'$, subtracting $b_0/10 + b_1/200$ on average each time. The total amount subtracted in these steps is therefore $b_0b_1.10^{-1} + b_1^2.10^{-2}/2$.

Between the second and third applications of R2$'$, we make b_2 applications of R1$'$, subtracting $b_0/100 + b_1/1\,000 + b_2/20\,000$ on average each time. The total amount subtracted in these steps is therefore $b_0b_2.10^{-2} + b_1b_2.10^{-3} + b_2^2.10^{-4}/2$.

In general, between the kth and $(k+1)$th applications of R2$'$, we make b_k applications of R1$'$, subtracting $b_0.10^{-k} + b_1.10^{-k-1} + \cdots + b_{k-1}.10^{1-2k} + b_k.10^{-2k}/2$ on average each time. The total amount subtracted in these steps is therefore $b_0b_k.10^{-k} + b_1b_k.10^{-k-1} + \cdots + b_{k-1}b_k.10^{1-2k} + b_k^2.10^{-2k}/2$.

But remember that $n/2$ must be the sum of all the terms we are subtracting, thus

$$
\begin{aligned}
n/2 = b_0^2/2 & \\
&+ b_0b_1.10^{-1} + b_1^2.10^{-2}/2 \\
&+ b_0b_2.10^{-2} + b_1b_2.10^{-3} + b_2^2.10^{-4}/2 \\
&+ \cdots \\
&+ b_0b_k.10^{-k} + b_1b_k.10^{-k-1} + \cdots + b_{k-1}b_k.10^{1-2k} + b_k^2.10^{-2k}/2 \\
&+ \cdots,
\end{aligned}
$$

i.e. the total subtracted before the first application of R2$'$, between the first and second applications, between the second and third applications, and so on. Note that the coefficient of b_ib_j is 10^{-i-j} if i and j are different, and is $10^{-i-j}/2$ if $i = j$. However, if we expand

$$(b_0 + b_1.10^{-1} + b_2.10^{-2} + b_3.10^{-3} + \cdots)^2,$$

then the coefficient of b_ib_j is 2.10^{-i-j} if i and j are different, and is 10^{-i-j} if $i = j$. It follows that the sum equal to $n/2$ above is exactly

$$\frac{b_0 + b_1.10^{-1} + b_2.10^{-2} + b_3.10^{-3} + \cdots)^2}{2},$$

and so

$$n = (b_0 + b_1.10^{-1} + b_2.10^{-2} + b_3.10^{-3} + \cdots)^2,$$

so that

$$\sqrt{n} = b_0 + b_1.10^{-1} + b_2.10^{-2} + b_3.10^{-3} + \cdots,$$

and the digits b_i coming out of the algorithm are therefore exactly the digits in the decimal expansion of $\sqrt{n}$, as required.

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The Poisson and Exponential Distributions

JOHN C. B. COOPER

1. Introduction

The Poisson distribution is a discrete distribution with probability mass function

$$P(x) = \frac{e^{-\mu}\mu^x}{x!},$$

where $x = 0, 1, 2, \ldots$, the mean of the distribution is denoted by μ, and e is the exponential. The variance of this distribution is also equal to μ.

The exponential distribution is a continuous distribution with probability density function

$$f(t) = \lambda e^{-\lambda t},$$

where $t \geq 0$ and the parameter $\lambda > 0$. The mean and standard deviation of this distribution are both equal to $1/\lambda$.

The cumulative exponential distribution is

$$F(t) = \int_0^{\infty} \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda t}. \tag{1}$$

2. Relation between the Poisson and exponential distributions

An interesting feature of these two distributions is that, if the Poisson provides an appropriate description of the number of occurrences per interval of time, then the exponential will provide a description of the length of time between occurrences. To understand this, consider that, in a Poisson process, if events occur on average at the rate of λ per unit of time, then there will be on average λt occurrences per t units of time. The Poisson distribution describing this process is therefore $P(x) = e^{-\lambda t}(\lambda t)^x/x!$, from which $P(x = 0) = e^{-\lambda t}$ is the probability of no occurrences in t units of time.

Another interpretation of $P(x = 0) = e^{-\lambda t}$ is that this is the probability that the time, T, to the first occurrence is greater than t, i.e.

$$P(T > t) = P(x = 0 \mid \mu = \lambda t) = e^{-\lambda t}.$$

Conversely, the probability that an event does occur during t units of time is given by

$$P(T \leq t) = 1 - P(x = 0 \mid \mu = \lambda t) = 1 - e^{-\lambda t}.$$

Note that this is the cumulative exponential distribution which, when differentiated with respect to t, produces the probability density function of the exponential distribution $f(t) = \lambda e^{-\lambda t}$.

Table 1 Goals scored by Wolves FC, 2003–2004 season.

Number of goals (x)	Observed frequency of x	Theoretical frequency of x
0	15	14
1	12	14
2	8	7
3	2	2
4	1	1
Total goals	38	38

3. An example from soccer

During the 2003–2004 soccer season, Wolves scored 38 goals over 38 league games and finished bottom of the English Premier League. The actual distribution of these goals is shown in table 1, from which it can be readily verified that the mean number of goals per 90 minute game is 1 with a variance of 1.08. Thus mean and variance are approximately equal. Fitting a Poisson distribution with $\mu = 1$ to this data, we may easily calculate the theoretical or expected frequencies of $x = 0, 1, 2, 3, 4$ goals, which is are also recorded in table 1. The Poisson distribution clearly provides a good fit to Wolves' goal scoring record for the season under study.

Suppose now that we wish to model the times between goals rather than the number of goals scored. We know that goals are scored on average at the rate of 1 per 90 minutes or, alternatively, at the rate of $\frac{1}{90} = 0.0111$ goals per minute or $t/90$ goals per t minutes, etc. Now, to say that no goal is scored during the first minute of play is the same as saying that the first goal is scored sometime after one minute. The probability that no goals are scored during any one minute period may be obtained from the Poisson distribution with $\mu = 0.0111$, i.e.

$$P(x = 0) = \frac{\mathrm{e}^{-0.0111} 0.0111^0}{0!} = \mathrm{e}^{-0.0111} = 0.99 \quad \text{(rounded to the second decimal place).}$$

Thus the probability that the time between goals exceeds one minute is 0.99 and, conversely, the probability that the time between goals is one minute or less is 0.01.

Generalising, for the time between goals to exceed t minutes implies that no goals are scored during the first t minutes. This time, the probability of $x = 0$ goals in any t minute period is found from the Poisson distribution with $\mu = t/90$ which in turn reduces to $\mathrm{e}^{-t/90}$. Thus, the probability that a goal is scored before t minutes have elapsed is $1 - \mathrm{e}^{-t/90}$. Note that this expression is of the same form as (1), and is therefore a cumulative exponential distribution. Differentiating with respect to t, we obtain $\frac{1}{90}\mathrm{e}^{-t/90}$, which is the exponential distribution with $\lambda = \frac{1}{90}$.

Returning to our soccer example, Wolves scored their first goal of the season 71 minutes into their first game. They did not, however, score again until 30 minutes into their seventh game, i.e. $19 + 90 + 90 + 90 + 90 + 90 + 30 = 499$ minutes later! Then, their third goal of the season occurred 75 minutes into their eighth game, or $60 + 75 = 135$ minutes after their second goal. This procedure for computing inter-goal times was repeated up to their last goal of the season, which occurred in the 70th minute of their 37th game. The actual distribution

Table 2 Inter-goal times, Wolves FC, 2003–2004 season.

Inter-goal time (minutes)	Occurrences	Frequency (%)	Actual cumulative frequency	Theoretical cumulative frequency
1–100	26	0.70	0.70	0.67
101–200	9	0.24	0.94	0.89
201–300	0	0.00	0.94	0.96
301–400	1	0.03	0.97	0.99
401–500	1	0.03	1.00	1.00

of these times in class intervals of 100 minutes is shown in table 2. The mean and standard deviation of this data are 87.54 and 93.81 respectively, which are sufficiently close to suggest that inter-goal times may be modelled by the exponential distribution.

To fit an exponential distribution to these times requires an estimate of the parameter λ. In the absence of the information that the mean inter-goal time is 87.54, we may proceed as follows. A mean scoring rate of 1 goal per 90 minutes implies an average time of 90 minutes between goals (note that this is very close to 87.54). Thus, $1/\lambda$ is estimated to be equal to 90 from which we calculate that $\lambda = 0.0111$ and $f(t) = 0.0111\mathrm{e}^{-0.0111t}$. We may now estimate, for example, that goals are expected to occur within 100 minutes of one another or less with a probability of

$$\int_0^{100} 0.0111\mathrm{e}^{-0.0111t}\,\mathrm{d}t = [-\mathrm{e}^{-0.0111t}]_0^{100} = 1 - \mathrm{e}^{-1.11} = 0.67.$$

Any such cumulative probability may be estimated in precisely the same way. For example, if we wanted to estimate the probability of an inter-goal time of 45 minutes or less, it could be readily obtained from $1 - \mathrm{e}^{-0.0111\times 45} = 0.39$ (the actual value is in fact 0.41). Table 2 shows that the theoretical exponential distribution of inter-goal times is a reasonable fit.

Finally, one important property of the exponential distribution is that it has no memory. This means that the probability that a goal will be scored in the next time period is totally independent of what has happened in previous time periods. For example, at the start of their seventh game of the season, Wolves had not scored since the 71st minute of their first game, i.e. 469 minutes ago. We might be tempted to surmise that the probability of a goal in the next 100 minutes must be extremely high. In fact, however, the probability is precisely the same regardless of this recent goal famine. To demonstrate this, we must calculate the conditional probability of a goal between the 469th minute and the 569th minute given that there were no goals between the 71st minute and the 469th minute. This is obtained as follows:

$$P(469 \le t \le 569 \mid t \ge 469) = \frac{P(469 \le t \le 569)}{P(t \ge 469)},$$

which reduces to

$$\frac{0.9982 - 0.9945}{0.0055} = 0.67.$$

This is precisely the same as the unconditional probability calculated earlier.

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Estimating the Eccentricity of the Earth's Orbit

A. TAN

Introduction

Planets possess two types of angular motions in space. First, the motion of a planet around the Sun is its revolution. According to Kepler's first law of planetary motion, the orbit of a planet is an ellipse with the Sun at one focus. The ellipticity of the orbit is given by its *eccentricity* which is defined as the ratio of the distance between the two foci and the length of its major axis. The Earth's orbit is slightly elliptical having an eccentricity of 0.017 (see website 1). The two apsidal points of interest on the orbit are the *perihelion* (the point of closest approach of the planet to the Sun), and the *aphelion* (the point of farthest retreat of the planet from the Sun). For the Earth's orbit, the perihelion falls most frequently around January 3 and the aphelion falls most frequently around July 5 (see website 2).

The second type of angular motion of a planet is its rotation or the spin about its own axis which is entirely independent of its revolution. The tilt of the axis from the normal to the orbital plane is mainly responsible for the seasons. The *Winter Solstice* marks the day when the north pole of the Earth is tilted farthest away from the Sun and occurs most frequently on December 21 (see website 2). The *Summer Solstice* falls most frequently on June 21 (see website 2). On that day, the south pole is tilted farthest away from the Sun. Two other days of interest are the *Vernal Equinox* and the *Autumnal Equinox* on which days and nights are equal at every location on the globe. The Vernal Equinox occurs most frequently on March 20 while the Autumnal Equinox occurs most frequently on September 22 and 23 (see website 2).

Figure 1 shows the Earth's orbit as viewed from the northern sky. The perihelion, aphelion, Winter Solstice, Summer Solstice, Vernal Equinox, and Autumnal Equinox are denoted by

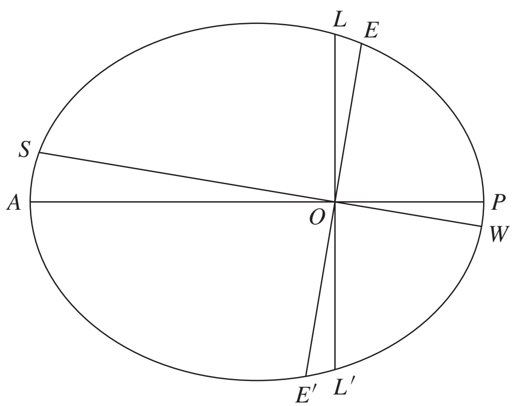

Figure 1 The Earth's orbit (O is the location of the Sun).

P, A, W, S, E, and E' respectively. Also shown on figure 1 are the points L and L' at the ends of the *latus rectum*, which is the chord passing through the Sun and perpendicular to the major axis AP. It is evident that the angles WOP, EOL, SOA, and $E'OL'$ are all equal (to α, say). It is also apparent that α is quite small. The interval from W to P is only 13 days, while the interval from S to A is just 14 days. Since the Earth sweeps 360 degrees around the Sun in 365.25 days, these intervals subtend angles of approximately 12.8 and 13.8 degrees at the centre of the ellipse. Note, however, that the corresponding angles subtended by the two intervals at the Sun are necessarily the same and equal to approximately 13.3 degrees.

The closeness between the days related to revolution and axial tilt and the consequent smallness of the angles discussed are purely accidental. But this accidental closeness presents us with the opportunity to estimate the eccentricity of the Earth's orbit. In this article, we shall describe two methods of doing so.

Method 1a

Firstly, we notice that the time interval of passage of the Earth from the Autumnal Equinox E' (between September 22 and September 23, i.e. September 22.5) to the Vernal Equinox E (March 20) is $T = 178.5$ days. The ratio of this interval to Earth's year Y (365.25 days) is $R = T/Y = \frac{714}{1461}$. The cointerval from E to E' is $T' = Y - T = 186.75$ days. The difference between T and T' can be exploited to determine the eccentricity e of the Earth's orbit. We can do so by calculating the intervals and comparing them with observed values. There are three options: firstly compare T and T', secondly compare T' and Y, or thirdly compare T and Y. Here we choose the third option.

The problem of planetary motion is most conveniently described in polar coordinates (r, θ), with the Sun at the origin, the major axis AOP on the abscissa and the latus rectum LOL' on the ordinate (see figure 1). The area swept by the planet in traversing an angle $\mathrm{d}\theta$ is $\frac{1}{2}r^2\,\mathrm{d}\theta$. If $\mathrm{d}t$ is the time interval during which this happens, the areal velocity of the planet is given by $\frac{1}{2}r^2\,\mathrm{d}\theta/\mathrm{d}t$. According to Kepler's second law of planetary motion, the areal velocity of the planet is constant, i.e.

$$\frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = c.$$

By separating the variables, we obtain the integral of time

$$\int \mathrm{d}t = \frac{1}{2c}\int r^2\,\mathrm{d}\theta. \tag{1}$$

The general equation of a conic (including the ellipse) in polar coordinates is given in terms of the semi latus rectum p $(=OL)$ as follows:

$$r = \frac{p}{1 + e\cos\theta}. \tag{2}$$

Substituting (2) into (1), we obtain

$$\int \mathrm{d}t = \frac{p^2}{2c}\int \frac{\mathrm{d}\theta}{(1 + e\cos\theta)^2}.$$

If we ignore the constant $p^2/2c$, then the time integral assumes the simple form

$$I = \int \frac{\mathrm{d}\theta}{(1 + e\cos\theta)^2}. \tag{3}$$

The evaluation of this integral is not an easy matter and requires successive trigonometric substitutions. However, it is readily found in mathematical handbooks. Also, mathematical software, for example MATHEMATICA®, can give the result instantly

$$I = -\frac{e\sin\theta}{(1-e^2)(1+e\cos\theta)^2} + \frac{2}{(1-e^2)^{3/2}}\tan^{-1}\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}.$$

Next, we compute the relative magnitudes of the intervals T and Y. We have

$$Y \propto 2[I]_0^{\pi} = \frac{2\pi}{(1-e^2)^{3/2}}. \tag{4}$$

For the interval T, we can write

$$T \propto [I]_{3\pi/2-\alpha}^{3\pi/2} + 2[I]_0^{\pi/2} - [I]_{\pi/2-\alpha}^{\pi/2}.$$

It is apparent that $I_1 = [I]_{3\pi/2-\alpha}^{3\pi/2}$ and $I_2 = [I]_{\pi/2-\alpha}^{\pi/2}$ are very nearly equal, with I_1 fractionally greater that I_2. In the case of the Earth's orbit ($e = 0.017$), $I_1 = 1.088 I_2$. Making the assumption that $I_1 \approx I_2$, we obtain

$$T \propto 2[I]_0^{\pi/2} \approx -\frac{2e}{1-e^2} + \frac{4}{(1-e^2)^{3/2}}\tan^{-1}\sqrt{\frac{1-e}{1+e}}. \tag{5}$$

Setting $T/Y = R$ from (4) and (5), we arrive at a transcendental equation

$$f(e) = g(e),$$

with

$$f(e) = \tan^{-1}\sqrt{\frac{1-e}{1+e}}$$

and

$$g(e) = \frac{357\pi}{1461} + \frac{e}{2}\sqrt{1-e^2}. \tag{6}$$

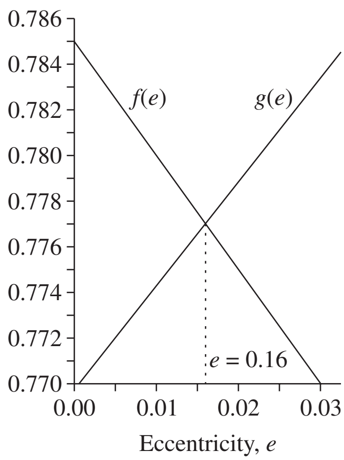

Figure 2 Graphical solution for the eccentricity of the Earth given by the intersection of $f(e)$ and $g(e)$.

Equation (6) is customarily solved by the graphical method (see figure 2). The intersection of the two graphs $f(e)$ and $g(e)$ gives the solution $e = 0.016$. The result compares favourably with the actual value of 0.017 and only marginally underestimates it.

Method 1b

For orbits having small eccentricities, we can proceed with an alternative approach to the integral (3). We can use an approximation to the integral I in (3) but keep the limits intact. Upon series expansion of the denominator and retention of terms up to the first order in e only, we obtain

$$I \approx \int (1 - 2e\cos\theta)\,\mathrm{d}\theta = \theta - 2e\sin\theta.$$

In this case,

$$Y \propto 2[I]_0^{\pi} = 2\pi$$

and

$$T \propto [I]_{3\pi/2-\alpha}^{\pi/2-\alpha} = \pi - 4e\cos\alpha.$$

Once again, setting $T/Y = R$ and solving, we arrive at

$$e = \frac{\pi}{4\cos\alpha}(1 - 2R).$$

Putting $R = \frac{714}{1461}$ and $\alpha = 13.3$ degrees, we get $e = 0.018$. This alternative approach slightly overestimates the eccentricity.

Method 2

This method uses a differential technique instead of the integral approach of the earlier methods. It utilises the difference between the interval of passage from W to P of 14 days and that from S to A of 13 days. The difference is due to the fact that the planet is fastest at its perihelion and slowest at the aphelion. In accordance with Kepler's second law of planetary motion, we can write

$$\frac{1}{2}r_P^2\frac{\mathrm{d}\theta_P}{\mathrm{d}t_P} = \frac{1}{2}r_A^2\frac{\mathrm{d}\theta_A}{\mathrm{d}t_A}.$$

From elementary properties of the ellipse, $r_P = a(1-e)$ and $r_A = a(1+e)$, where a is the semi-major axis. Upon setting $\mathrm{d}\theta_P = \mathrm{d}\theta_A = \alpha$, $\mathrm{d}t_P = 13$ days and $\mathrm{d}t_A = 14$ days, we obtain

$$\frac{a^2(1-e)^2}{a^2(1+e)^2} = \frac{13}{14},$$

which gives

$$e = \frac{1 - \sqrt{\frac{13}{14}}}{1 + \sqrt{\frac{13}{14}}} \approx 0.0185.$$

This method slightly overestimates the eccentricity. It should be noted that the dates of the solstices, especially the apsidal points, vary from year to year. The intervals $\mathrm{d}t_P$ and $\mathrm{d}t_A$ reflect the average values of the intervals.

Remarks

The accidental closeness of the dates of the winter solstice and the perihelion provides us with two methods for estimating the eccentricity of the Earth's orbit. These methods, though necessarily approximate, nevertheless gave fairly accurate estimates of the eccentricity. It should be stated that, due to various perturbation effects, the perihelion regresses slowly at the rate of about one revolution in 21 000 years (see website 3). In the foreseeable future, the perihelion date will actually edge closer to the winter solstice, thus rendering these methods even more accurate.

Acknowledgements This study received support from a NASA Minority University Initiative Grant. Method 1b is due to Dermot Roaf, who reviewed this article.

Websites

1 http://science.nasa.gov/headlines/y2001/ast04jan_1.htm
2 http://aa.usno.navy.mil/data/docs/EarthSeasons.html
3 http://aa.usno.navy.mil/faq/docs/seasons_orbit.html

A. Tan *is a professor of physics at Alabama Agricultural and Mechanical University. He has special interest in applied mathematics and has frequently published articles in Mathematical Spectrum.*

Mathematics in the Classroom

Ordered or unordered?

I asked my students to solve the following problem.

> Two numbers are drawn at random, one at a time and without replacement, from the set $\{1, 2, 3, 4, 5, 6\}$. What is the probability that the smaller of the chosen numbers is less than 4?

A student argued that the number of ways of selecting two numbers is ${}^6C_2 = 15$, the number of ways of selecting two numbers both at least 4 is ${}^3C_2 = 3$, so the number of ways of selecting two numbers so that the smaller of them is less than 4 is $15 - 3 = 12$. Hence the required probability is $\frac{12}{15} = \frac{4}{5}$.

The student used unordered selections rather than ordered ones. The correct argument is that the number of ways of selecting two from six is $6 \times 5 = 30$, the number of ways of selecting two numbers in which both are at least 4 is $3 \times 2 = 6$, so the number of ways in which the smaller number is less than 4 is $30 - 6 = 24$. Hence the required probability is $\frac{24}{30} = \frac{4}{5}$.

So the student arrived at the correct answer by false reasoning, a salutary reminder not just to check the answer! The curious thing is that, no matter how many numbers you start with, how many are chosen, say r, and whatever bound replaces 4, unordered and ordered selections will give the same probability. This is because each unordered selection is counted $r!$ times in an ordered selection. So it really doesn't matter whether the selection is ordered or unordered, and I can just check the answer after all!

Ramanujan School of Mathematics,
Patna, India

Anand Kumar

Letters to the Editor

Dear Editor,

Super π-day

With reference to Raphael Rosen's letter in Volume 36, Number 3, the people of San Francisco's Exploratorium must have a future date circled in red for 649 years hence, when there will be great rejoicing and all sorts of odd chanting and prancing at 1.59pm on Monday, March 14th 2653. Superpies will certainly be the order of the day.

Yours sincerely,

Bob Bertuello
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UK)

PS $\pi = 3.1415926535\ldots.$

Dear Editor,

Maybe quantum mechanics is not so strange after all

Peter Mattsson in his interesting article *Quantum computers* (*Math. Spectrum*, Volume 37, Number 1) says that 'in the quantum world, nothing is certain; rather than allowing us to predict exactly what will happen, the laws of quantum mechanics can only assign probabilities'. This is certainly the official view but for this very reason many scientists, notably Einstein, believed that there must be something wrong with quantum mechanics since they could not accept that there was radical indeterminacy at the very heart of physics.

One way out is to suppose that the indeterminacy is only the result of our ignorance and that, at a deeper level, there is determinacy. In the prevailing positivistic climate of the early twentieth century, such an appeal to 'hidden variables' was unacceptable and von Neumann actually published a 'proof' (subsequently shown to be defective) that no 'hidden variable' theory which accounted for the observed facts would ever be forthcoming. For all that, one or two mavericks like David Bohm kept the 'hidden-variables' interpretation alive and, currently, it seems to be making something of a come back. Ian Stewart, writing in *New Scientist* (25 September 2004), concedes that 'the door is still open for a deterministic explanation'. A certain Tim Palmer has recently published a paper in Volume 451 of *Transactions of the Royal Society*, which claims to show that 'the observed properties of the quantum world are consistent with deterministic hidden-variable theories that allow only "local" influence, rather than an ability to influence systems from the other side of the universe'.

Yours sincerely,

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Dear Editor,

Pascal's triangle

I would like to comment on Note (a) on Pascal's triangle by S. Jafari (see Volume 36, Number 2, p. 26). He observed that the determinants obtained by choosing the first three entries from rows 4, 5, 6 and by choosing the first four entries from rows 5, 6, 7, 8 of Pascal's triangle are equal to 1. He then asked if this can be generalised.

The following generalisation holds true. We may construct a determinant $D(n, k)$ of order k by k, by choosing the first k terms from any k successive rows, starting say at row $n + 1$, in Pascal's triangle. Then, for all k and n, this determinant has value 1.

Successively in this determinant we subtract row $k - 1$ from row k, row $k - 2$ from row $k - 1, \ldots,$ row 1 from row 2. By the defining property of Pascal's triangle we have that, for $r \geq 2$ and $s \geq 2$,

$$\text{entry } (r, s) = \text{ entry } (r - 1, s - 1) + \text{ entry } (r - 1, s).$$

So, when row $r - 1$ is subtracted from row r, entry (r, s) is replaced by entry $(r - 1, s - 1)$. So the last $k - 1$ rows and columns of the new determinant are equal to the first $k - 1$ rows and columns of $D(n, k)$, i.e. equal to $D(n, k - 1)$. Furthermore, the first column of this determinant is $(1, 0, \ldots, 0)^{\top}$. It follows that

$$D(n, k) = D(n, k - 1).$$

By iterating this result we can see that

$$D(n, k) = D(n, 1) = 1.$$

We note that we have two generalisations here. If we had restricted ourselves to the case $k = n$ then the above proof could not be used. So the second generalisation on the order of the determinant was needed. This then is another example of a more general result being apparently easier to prove than a special case.

We also note that we may choose $k > n$, where the extra terms to the right of Pascal's triangle are taken to be equal to 0, since r objects cannot be chosen from n if $r > n$.

Yours sincerely,

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Dear Editor,

Three squares in arithmetic progression

In his letter in Volume 37, Number 1, Muneer Jebreel provided formulae which give three perfect squares in arithmetic progression and concluded 'I do not know whether all triples of perfect squares which are in arithmetic progression can be obtained in this way'. A slight change in his formula will ensure this.

Let x^2, y^2, z^2 be three perfect squares in arithmetic progression with $x < y < z$. Then

$$y^2 - x^2 = z^2 - y^2,$$

so that

$$z^2 + x^2 = 2y^2.$$

Now either z and x are both even or both odd, so that $(z+x)/2$ and $(z-x)/2$ are positive integers and

$$\left(\frac{z+x}{2}\right)^2 + \left(\frac{z-x}{2}\right)^2 = y^2.$$

Hence

$$\frac{z+x}{2},\ \frac{z-x}{2},\ y$$

is a Pythagorean triple, for which there are well-known formulae (see Theorem 11.1 of reference 1). We can write

$$\left\{\frac{z+x}{2}, \frac{z-x}{2}\right\} = \{2kmn, k(m^2 - n^2)\},$$
$$y = k(m^2 + n^2),$$

for some $k, m, n \in \mathbb{N}$ with $m > n$, m and n coprime, and one of m and n even and the other odd. Thus

$$x = k|2mn - m^2 + n^2|,$$
$$y = k(m^2 + n^2),$$
$$z = k(2mn + m^2 - n^2).$$

It is a simple matter to check that these formulae do give three perfect squares in arithmetic progression.

Reference

1 D. M. Burton, *Elementary Number Theory,* 3rd edn. (McGraw-Hill, New York, 1997).

Yours sincerely,

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[A contribution along similar lines has also been sent in by Philip Maynard — Ed.]

Dear Editor,

Three squares in arithmetic progression

Referring to the letter of Muneer Jebreel in Volume 37, Number 1, it is possible to produce a chain of triplets of the type (a, b, c) of positive integers so that a^2, b^2, c^2 are in arithmetic progression, i.e. satisfying

$$b^2 - a^2 = c^2 - b^2.$$

A triplet (a, b, c) having this property will be called an *arithemtical progression triplet* (APT).

We give an algorithm to obtain a chain of triplets (a, b_n, c_n), $n = 0, \pm 1, \pm 2, \ldots$, whose first member a has some specified value. Suppose that (a, b_0, c_0) is a given triplet with a^2, b_0^2, c_0^2 in arithmetic progression. Then

$$(a, b_1, c_1) = (a, 3b_0 + 2c_0, 4b_0 + 3c_0)$$

is also an APT. It can easily be verified that

$$(4b_0 + 3c_0)^2 - (3b_0 + 2c_0)^2 = (3b_0 + 2c_0)^2 - a^2,$$

using $b_0^2 - a^2 = c_0^2 - b_0^2$. Similarly, from (a, b_1, c_1), we can produce a second APT, and so on. The general recurrence relation can be written as

$$(a, b_n, c_n) \to (a, b_{n+1}, c_{n+1}) = (a, 3b_n + 2c_n, 4b_n + 3c_n). \tag{1}$$

Similarly it can be shown that, for a given APT (a, b_0, c_0), we can go backwards to obtain the APT $(a, 3b_0 - 2c_0, -4b_0 + 3c_0)$. The general recurrence relation in this instance can be written as

$$(a, b_{-n}, c_{-n}) \to (a, b_{-n-1}, c_{-n-1}) = (a, 3b_{-n} - 2c_{-n}, -4b_{-n} + 3c_{-n}). \tag{2}$$

For instance, if we choose $a = 1$, then we can start with the trivial APT $(1, 1, 1)$. Then $(a, b_1, c_1) = (1, 5, 7)$, so that $1^2, 5^2, 7^2$ are in arithmetic progression. Now, $(1, 5, 7)$ can be used to produce the APT $(1, 29, 41)$.

The APT $(1, 5, 7)$ can be used as a generating APT when the first member is 7, provided that it is used in reverse order, i.e. $(7, 5, 1)$. Then if $(a, b_0, c_0) = (7, 5, 1)$, we have

$$(a, b_1, c_1) = (7, 17, 23)$$

and

$$(a, b_{-1}, c_{-1}) = (7, (3 \times 5 - 2 \times 1), (-4 \times 5 + 3 \times 1)) = (7, 13, -17).$$

Here -17 can be replaced by 17, since we are dealing with the squares of the numbers. By proceeding further we can get two distinct chains, one forward and the other backward. The

Table 1

n	a	b_n	c_n	a	b_n	c_n
−4	1	169	−239	7	2477	−3503
−3	1	29	−41	7	425	−601
−2	1	5	−7	7	73	−103
−1	1	1	−1	7	13	−17
0	1	1	1	7	5	1
1	1	5	7	7	17	23
2	1	29	41	7	97	137
3	1	169	239	7	565	799
4	1	985	1393	7	3293	4657

matrix solution (3), below, of (1) and (2) furnishes a quick method of generating the forward and backward chains if the parent APT (a, b_0, c_0) is known. We have

$$\begin{bmatrix} a \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 4 & 3 \end{bmatrix}^n \begin{bmatrix} a \\ b_0 \\ c_0 \end{bmatrix}, \qquad n = 0, \pm 1, \pm 2, \ldots . \tag{3}$$

Table 1 shows the chains for $a = 1$ and $a = 7$.

The last member c_n of any APT (a, b_n, c_n) can serve as the first member of a new chain of APTs when the parent APT becomes (c_n, b_n, a).

Muneer Jebreel gave the formula

$$a = m^2 - n^2 - 2mn, \qquad b = m^2 + n^2, \qquad c = m^2 - n^2 + 2mn, \tag{4}$$

for an APT (a, b, c), where m and n are positive integers. Then, for some specified number a, on putting $a = m^2 - n^2 - 2mn$, we obtain $m = n + \sqrt{2n^2 + a}$. Therefore a must be such that, for some integer n, $2n^2 + a$ is a perfect square. Once this condition is satisfied, we can obtain m and, consequently, from (4), b and c can also be determined. Thus the complete APT (a, b, c) is determined.

Yours sincerely,

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Dear Editor,

Solutions to Diophantine equations

I have found solutions to three Diophantine equations as follows:

$$\begin{aligned} x^3 + y^3 + z^3 = k^2 &\quad \text{has solution} \quad 23^3 + 24^3 + 25^3 = 204^2, \\ x^3 + y^3 = k^2 &\quad \text{has solution} \quad 21^3 + 7^3 = 98^2, \\ x^5 - y^5 = k^2 &\quad \text{has solution} \quad 24^5 - 8^5 = 2816^2. \end{aligned}$$

It would be interesting to know of any other solutions of these equations.

Yours sincerely,

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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st March will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

37.9 Prove that

$$\sqrt{(a+b)(c+d)(e+f)} + \sqrt{ace} \geq \sqrt{adf} + \sqrt{bcf} + \sqrt{bde},$$

for positive real numbers $a, \ldots, f$.

(Submitted by James West, student at Hills Road Sixth Form College, Cambridge)

37.10 There are mn houses at the junction of an $m \times n$ rectangular grid of roads. For which values of m and n is it possible for one of the householders to visit every other house just once and return to his own house?

(Submitted by H. A. Shah Ali, Tehran, Iran)

37.11 The $n \times n$ matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are such that $\boldsymbol{A}+\boldsymbol{B} = \boldsymbol{AB}$, $\boldsymbol{B}+\boldsymbol{C} = \boldsymbol{BC}$, $\boldsymbol{C}+\boldsymbol{A} = \boldsymbol{CA}$. Prove that

$$(\boldsymbol{A} - \boldsymbol{I}_n)^2 + (\boldsymbol{B} - \boldsymbol{I}_n)^2 + (\boldsymbol{C} - \boldsymbol{I}_n)^2 = 3\boldsymbol{I}_n.$$

(Submitted by Mihály Bencze, Romania)

37.12 A triangle ABC has largest angle less than $\frac{2}{3}\pi$ radians. Prove that the angles subtended by the sides of the triangle at an interior point P are equal when the sum $\mathrm{AP} + \mathrm{BP} + \mathrm{CP}$ is a minimum. If $\mathrm{AP} : \mathrm{BP} : \mathrm{CP} = 1 : 2 : 4$ and $\mathrm{AP} + \mathrm{BP} + \mathrm{CP}$ is a minimum, find the angles A, B, C.

(Submitted by J. A. Scott, Chippenham)

Solutions to Problems in Volume 37 Number 1

37.1 Is there a multiple of 2^{666} with 666 digits consisting of only 6s and 7s?

Solution

We prove by induction on n that, for all $n \geq 1$, there is a number N with n digits consisting only of 6s and 7s, such that $N \equiv 0 (\bmod 2^n)$. This is true when $n = 1$ (e.g. take $N = 6$). Now assume that the result is true for some $n \geq 1$, say

$$a_{n-1} \ldots a_0 \equiv 0 (\bmod 2^n),$$

Consider $6a_{n-1}\dots a_0$ and $7a_{n-1}\dots a_0$. If $a_{n-1}\dots a_0 \equiv 0 (\text{mod } 2^{n+1})$, then

$$6a_{n-1}\dots a_0 = 6\times 10^n + (a_{n-1}\dots a_0) \equiv 0(\text{mod } 2^{n+1}).$$

If $a_{n-1}\dots a_0 \not\equiv 0(\text{mod } 2^{n+1})$, then

$$a_{n-1}\dots a_0 = 2^n x,$$

where $x = 1+2k$ for some nonnegative integer k. Then

$$a_{n-1}\dots a_0 \equiv 2^n (\text{mod } 2^{n+1})$$

and

$$\begin{aligned} 7a_{n-1}\dots a_0 &= 7\times 10^n + (a_{n-1}\dots a_0)\\ &\equiv 7\times 5^n \times 2^n + 2^n(\text{mod } 2^{n+1})\\ &\equiv 2^n(7\times 5^n + 1)(\text{mod } 2^{n+1})\\ &\equiv 2^{n+1}\times 2y(\text{mod } 2^{n+1}) \quad \text{for some integer } y\\ &\equiv 0(\text{mod } 2^{n+1}). \end{aligned}$$

This proves the inductive step. Now put $n = 666$.

A solution was received from Norman Routledge, who proved that, for any integer r such that $0 \le r < 666$, there is a 666-digit number consisting only of 6s and 7s which leaves remainder r on division by 2^{666}. Further, 6 and 7 can be replaced by m_1 even and m_2 odd, and 666 can be replaced by any positive integer.

37.2 Let $x_i > 0$ for $i = 1, \dots, n$. Prove that

$$\prod_{i=1}^{n} \frac{1}{n-1}\sum_{j=1}^{n-1} x_{i+j-1} \ge \prod_{i=1}^{n} x_i,$$

where $x_{n+i} = x_i$ for $i = 1, 2, \dots, n-1$.

Solution by James West, Hills Road Sixth Form College, Cambridge

Using the arithmetic-geometric mean inequality, we have

$$\begin{aligned} &\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)\left(\frac{x_2+\cdots+x_n}{n-1}\right)\cdots\left(\frac{x_n+x_1+\cdots+x_{n-2}}{n-1}\right)\\ &\quad\ge (x_1\cdots x_{n-1})^{1/(n-1)}(x_2\cdots x_n)^{1/(n-1)}\cdots(x_n x_1\cdots x_{n-2})^{1/(n-1)}\\ &\quad= x_1x_2\cdots x_n. \end{aligned}$$

37.3 Let k, n be integers such that $2 \le k \le n$. Determine the number of sequences of k integers $a_1, \dots, a_k$ with $1 \le a_1 < a_2 < \cdots < a_k = n$, and such that $a_2-a_1, a_3-a_2, \dots, a_{k-1}-a_{k-2}$ are odd and $a_k - a_{k-1}$ is even.

Solution by Farshid Arjomandi, who proposed the problem

Denote the required number by $f(n,k)$. We prove by induction on n that

$$f(n,k) = \binom{\left[\frac{n+k-3}{2}\right]}{k-1},$$

(the binomial coefficient), where $[x]$ denotes the greatest integer $\le x$. For $n = 2$, we have $k = 2$, $a_1 = 1$, $a_2 = 2$, $f(2,2) = 0$, and the binomial coefficient is taken to be zero. Now let $n > 2$ and assume the result for $n-1$. We have

$$f(n,k) = f_1(n,k) + f_2(n,k),$$

where $f_1(n,k)$ is the number of such sequences with $a_1 = 1$, and $f_2(n,k)$ is the number with $a_1 \ge 2$. For a sequence with $a_1 \ge 2$, we can subtract 1 from each term, so that

$$f_2(n,k) = f(n-1,k).$$

Now suppose that $a_1 = 1$. If k is odd, this means that $a_3, a_5, \ldots, a_{k-2}$ are odd, a_{k-1} is even, and $a_k = n$ is even so that $n+k$ is odd. If k is even, then, by the same token, n is odd and again $n+k$ is odd. Thus, when $n+k$ is even, $f_1(n,k) = 0$ and, by the inductive hypothesis,

$$\begin{aligned} f(n,k) &= f_2(n,k) \\ &= f(n-1,k) \\ &= \binom{\left[\frac{n+k-4}{2}\right]}{k-1} \\ &= \binom{\left[\frac{n+k-3}{2}\right]}{k-1}, \end{aligned}$$

and the inductive step is proved when $n+k$ is even.

Now suppose that $n+k$ is odd, and consider a sequence with $a_1 = 1$. Then

$$1 = a_1 < a_2 < \cdots < a_{k-1} < a_k = n,$$

so that

$$1 \le a_2 - 1 < a_3 - 1 < \cdots < a_{k-1} - 1 < a_k - 1 = n - 1,$$

with $a_2 - 1$ odd. But this is a typical sequence for $n-1$ and $k-1$. (If $1 \le b_1 < b_2 < \cdots < b_{k-2} < b_{k-1} = n-1$ were such a sequence with b_1 even, then with k even b_{k-2} is odd, and $b_{k-1} = n-1$ is odd so n is even, yet $n+k$ is odd; use a similar argument if k is odd.) Hence

$$f_1(n,k) = f(n-1,k-1)$$

and, by the inductive hypothesis,

$$\begin{aligned}
f(n,k) &= f_1(n,k) + f_2(n,k) \\
&= f(n-1,k-1) + f(n-1,k) \\
&= \left(\begin{matrix} \left[\frac{n+k-5}{2}\right] \\ k-2 \end{matrix}\right) + \left(\begin{matrix} \left[\frac{n+k-4}{2}\right] \\ k-1 \end{matrix}\right) \\
&= \left(\begin{matrix} \frac{n+k-5}{2} \\ k-2 \end{matrix}\right) + \left(\begin{matrix} \frac{n+k-5}{2} \\ k-1 \end{matrix}\right) \\
&= \left(\begin{matrix} \frac{n+k-3}{2} \\ k-1 \end{matrix}\right) \\
&= \left(\begin{matrix} \left[\frac{n+k-3}{2}\right] \\ k-1 \end{matrix}\right),
\end{aligned}$$

which proves the inductive step when $n+k$ is odd.

37.4 Let n be a natural number and α a non-zero real number. Evaluate

$$\lim_{x\to 0} \frac{1-(\cos x \cos 2x \cdots \cos nx)}{\sin^2 \alpha x}.$$

Solution by Bor-Yann Chen, University of California, Irvine

By L'Hôpital's rule, the limit is

$$\begin{aligned}
&\lim_{x\to 0} \frac{\sin x \cos 2x \cdots \cos nx + 2\cos x \sin 2x \cos 3x \cdots \cos nx + \cdots}{2\alpha \sin \alpha x \cos \alpha x} \\
&\quad = \frac{1}{2\alpha} \lim_{x\to 0} \left(\frac{\sin x}{\sin \alpha x}\right)\left(\frac{\cos 2x \cdots \cos nx}{\cos \alpha x}\right) + \cdots .
\end{aligned}$$

Now

$$\lim_{x\to 0} \frac{\sin nx}{\sin \alpha x} = \frac{n}{\alpha} \lim_{x\to 0} \left(\frac{\sin nx}{nx}\right)\left(\frac{\alpha x}{\sin \alpha x}\right) = \frac{n}{\alpha}.$$

Hence, the given limit is equal to

$$\begin{aligned}
\frac{1}{2\alpha}\left(\frac{1}{\alpha} + 2\frac{2}{\alpha} + 3\frac{3}{\alpha} + \cdots + n\frac{n}{\alpha}\right) &= \frac{1}{2\alpha^2}(1^2 + 2^2 + \cdots + n^2) \\
&= \frac{1}{12\alpha^2} n(n+1)(2n+1).
\end{aligned}$$

Also solved by James West and Ovidiu Furdui, Western Michigan University, who considered the more general limit where the cosine terms are raised to power p; the limit is then multiplied by p.

Reviews

Strange Curves, Counting Rabbits, and Other Mathematical Explorations. By Keith Ball. Princeton University Press, 2003. Paperback, 296 pages, £19.95 (ISBN 0-691-11321-1).

Much of the material in this book arose from 'popular' maths lectures given in schools, or to school pupils visiting University College London, where the author works. From reading the book it is very clear that I would have enjoyed the lectures and so would my sixth-form maths students. The book illustrates very clearly that investigating maths is fun, challenging and rewarding. The author's love of the subject shines through, as does his desire to pass on his mathematical enjoyment to others. The author also believes that maths is a subject that you do and investigate for yourself, rather than just read. He presents problems in each chapter to encourage the reader to do this. The book has a very clear style and is straightforward to follow. The mathematics that is used to explore the various topics is within the grasp of a sixth-form student taking A-level maths or higher-level maths in the International Baccalaureate. In a relaxed, exploring and lucid style the author is able to use this level of maths to delve into the topics much further than a school syllabus would go. I would recommend this book for every school library that has sixth formers doing maths. It is also a good source for teachers who have students requiring ideas for them to investigate, for example, for the extended essay in maths in the International Baccalaureate. If I were to criticize the book, it would be for its dust jacket and title. It could be dismissed as another recreational maths book on the usual topics, for example, Fibonacci, which would be a pity as it is much more than this; even with the more popular topics there are nice twists and extensions and there are neat proofs and explanations that make you smile. Although each chapter is basically self-contained, some ideas and techniques are used to unite topics, for example Fibonacci points and Pick's theorem. There are ten chapters with the following topics:

Codes: error detecting and error correcting, plus transmission rates;

Pick's theorem: an interesting connection to the coprimality theorem is shown;

Fermat's little theorem: with connections to the decimal expansions of the reciprocals of primes;

Space-filling curves: including programs to generate the curves;

Probability and the normal curve: shared birthdays; explanation of the formula for the normal distribution;

Stirling's formula: shows how to make approximations more accurate;

Testing procedures: efficiency of binary tests;

Fibonacci numbers: golden ratio, continued fractions, Lucas numbers;

Approximating curves: Padé approximations using rational functions (I found this particularly interesting);

Rational and irrational numbers: $\sqrt{d}$, e, π (even here, only reasoning that can be followed using school level maths is employed).

This book shows the young student who is interested in maths that it is not a cut and dried subject. It gives good insights into how a mathematician thinks and how proofs are constructed. I hope that the young students who work through this book gain as much enjoyment from it as I did.

Atlantic College **Paul Belcher**

Writing Projects For Mathematics Courses: Crushed Clowns, Cars, and Coffee to Go. By Annalisa Crannell, Gavin LaRose, Thomas Ratliff and Elyn Rykken. MAA, Washington, DC, 2004. Paperback, 128 pages, \$28.50 (ISBN 0-88385-735-9).

This book is comprised of 35 writing projects in mathematics that have been set by the four authors for their undergraduate mathematics students. The projects are written in idiomatic and colloquial American English, rather than precise mathematical language or standard English. Although the projects were designed for American undergraduates, almost all the mathematics that is required to answer the questions could be handled by a good sixth-form British mathematics student. I found it frustrating, as would the students, that the instructors' solutions contained assumptions that had not been stated in the problem, for example, 'this assumes that the cannon sits precisely at the edge of the canyon', 'we assume that this velocity has reached terminal velocity when it is within 1% of its final value'. I am always looking for good projects that I can use as assignments for students. Sadly this book did not contain any that I would use, or even adapt to use, as I would be much better off starting from scratch myself. In many cases the solution was shorter than the original rambling letter that raised the question. The book concludes with some sample solutions and marking checklists. I did not appreciate this book; its attempted humour was irritating rather than funny. I will not ever refer back to it.

Atlantic College **Paul Belcher**

Essentials of Mathematics. By Margie Hale. MAA, Washington, DC, 2003. Hardback, 180 pages, \$47.50 (ISBN 0-88385-729-4).

This book is aimed at students making the transition from school to university. It consists of a curious mixture of observations on mathematical culture interspersed with an introduction to university mathematics, moving from logic and set theory to number systems (natural, rational, real and complex). As the picture on the front cover demonstrates, mathematics is a tree built on these foundations, leading to branches of analysis, algebra and geometry, and subdividing further thereafter.

But I didn't really find the treatment of logic and set theory particularly compelling. To start on symbolic logic with little motivation seems an odd decision, and many students may be put off by the formal nature of this initial chapter, especially when the following chapter delves into set theory (ending with the Zermelo–Fraenkel axioms). Although the later chapters are an improvement, the book in itself is a little unmotivated, and would work better as a companion to a lecture course. All of the chapters have a large number of exercises, many of which are valuable.

Throughout the text, mostly at the beginning of the chapters, the author includes sections discussing an aspect of mathematical culture, for example how to write proofs, well-known paradoxes, philosophy of mathematics, famous mathematical objects, famous mathematicians (a rather idiosyncratic selection), famous theorems, professional organisations, and much more. These sections work better, although the treatment is rather subjective, and there is an understandable bias towards American institutions which will not be relevant to most UK readers. I wasn't sure that integrating the culture and the mathematics in this way succeeded completely.

I know of no other book on mathematical culture, so this attempt to fill the gap should be welcomed. However, it is clearly a difficult task to discuss mathematical culture within a text book, and unfortunately this book is not as successful as it might have been.

University of Sheffield **Frazer Jarvis**

Mathematics in Nature: Modelling Patterns in the Natural World. By John A. Adam. Princeton University Press, 2003. Paperback, 448 pages, £26.95 (ISBN 0-691-11429-3).

This is a delightful book that uses applied mathematics to explain natural phenomena. It is well written with nice touches of unassuming humour. It is a book that you would keep referring back to, when you had seen a natural formation and wondered what caused it. It is very diverse in scope, looking at rainbows, rivers, clouds, animal markings, mud cracks, bird flight, and much more. Much of the mathematical analysis could be followed and well-understood by a British sixth-form student taking mathematics at A-level, or an equivalent award. It shows how mathematical models are constructed and simplifications made in order to be able to apply mathematics. There is a nice chapter on how to apply estimation theory, dealing with problems such as 'estimate the number of blades of grass on the Earth'. Dimensional analysis and the effect of scale are concepts that are used throughout the book. There are some beautiful photographs illustrating the natural patterns that are investigated. The bibliography is particularly extensive and allows you to follow up on any particular phenomenon that catches your interest. I would strongly recommend this book for all school libraries, as it can be considered as a classic.

Atlantic College **Paul Belcher**

When Least is Best: How Mathematicians Discovered Many Clever Ways to Make Things as Small (or as Large) as Possible. By Paul J. Nahin. Princeton University Press, 2004. Paperback, 328 pages, £19.95 (ISBN 0-691-07078-4).

As an Engineer, Paul Nahin has spent most of his life calculating maxima and minima. Although he does deal with calculus, much of his effort is towards other methods of optimisation, which are often more elegant. Indeed, nothing pleases Nahin more than slipping in a subtle arithmetic/geometric mean inequality or using linear programming to produce a solution.

Nahin goes back as far as the ancient Greeks and their optics (light takes the shortest path). He then crosses the Mediterranean to Carthage, the site of Dido's classic maximum area problem. We then meet Fermat, Descartes and Steiner, and follow the various disagreements they had. Then we have Galileo's hanging chain forming a catenary or, if loaded, a parabola. Kepler's wine barrel also puts in an appearance.

Nahin adds many of his own examples, such as 'the pipe and corner problem' and 'where to build the bridge' (you can tell he is an engineer!). He complements these examples with explaining how to get the best value from the US postal service, why Delta Airlines have hubs in the middle of nowhere and how to make the perfect basketball shot. A thorough but fun read, Nahin has clearly enjoyed writing this book.

Student, Gresham's School **Jonathan Smith**

Proofs That Really Count: The Art of Combinatorial Proof. By Arthur Benjamin and Jennifer Quinn. MAA, Washington, DC, 2003. Hardback, 208 pages, $41.95 (ISBN 0-88385-333-7).

There cannot be a more appropriate subtitle to this book than *The Art of Combinatorial Proof* because, like the best art, the mathematics contained within is beautiful.

The beauty of the book is in the simplicity of the idea behind many of the proofs given. The idea is that a 'how many' question is asked, for example 'how many ways can we choose k scoops of ice cream from n different flavours?'. The question is then answered in two different ways, yielding two different identities. Since both identities answer the same question they must be equal. Using this procedure, many complicated identities simply fall out when the right 'how many' question is posed. There is no complicated massaging of algebraic expressions or reliance on the often-mystical concept of proof by induction.

In the first chapter, it is shown how Fibonacci numbers can be interpreted as the number of ways of covering a board with tiles of a certain size. By then asking questions related to the number of possible tilings, many unexpected identities involving Fibonacci numbers can easily be proved. Subsequent chapters introduce other interesting numbers that appear in mathematics, such as Lucas numbers, binomial coefficients and Stirling numbers. Each one is given an interpretation where the number 'counts' a particular set, then the 'how many' questions again reveal many fascinating identities. Throughout the book diagrams and cartoons help illustrate the concepts introduced and help keep the proofs short and clear. Each chapter ends with a set of exercises that encourage the reader to prove identities using the methods and interpretations of that chapter. There are hints and solutions provided for some of the exercises often suggesting the right counting question to be posed.

While the book does mention topics that are often covered in the first year of a university course (for example, group theory, continued fractions and some number theory), the topics are developed assuming only a very basic background in mathematics, making the book highly accessible. In fact, the only major prerequisite is the ability to be able to count. For students who are interested in studying mathematics at university, it provides a number of counting argument proofs to theorems often proved by different means in a lecture course. The style is also similar to the presentation of an undergraduate textbook, making it a useful introduction on how to read mathematics. Along with the unfamiliar results, highlights include proofs of more established identities, for example a charming counting argument proof for the formula for the sum of the first n natural numbers, something that is usually proved using a trick of Gauss or by induction.

This book really demonstrates the power of combinatorics and how, with the right interpretation, many results both familiar and new can be proved without resorting to pages of horrendous algebra. It is as enchanting as the *Mona Lisa* but only a fraction of the cost – art I can really get behind!

University of Warwick **Stuart Hall**

USA and International Mathematical Olympiads 2003. Edited by Titu Andreescu and Zuming Feng. MAA, Washington, DC, 2004. Paperback, 104 pages, $26.95 (ISBN 0-88385-817-7).

This volume, as its title implies, contains the problems set in the USA and International Mathematical Olympiads in 2003. The problems are presented in Chapter 1, hints are given in Chapter 2 to get you going, and solutions, often multiple solutions, are given in Chapter 3.

LONDON MATHEMATICAL SOCIETY

POPULAR LECTURES 2005

The dates for this year's lectures are yet to be confirmed but will take place, as usual, in Manchester and London in June/July.

The Speakers will be:

Dr Joan Lasenby
(Cambridge)
'The Maths of Shrek'

Dr Alan Slomson
(Leeds)
'What Computers Can't Do'

Once the details are finalised, the information will be posted on the LMS website (www.lms.ac.uk).

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