# Crux

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**Mathematicorum**

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## CONTENTS

This issue is dedicated

to the founding Managing Editor of this journal

F.G.B. MASKELL

# THE OLYMPIAD CORNER: 60

## M.S. KLAMKIN

The following letter addresses itself to a statement I had made in my October column [1984: 260] which was based on information presented at the 1984 International Mathematical Olympiad Jury:

Dear Murray,

I would like the opportunity to set the record straight on what happened with respect to the problems submitted by Canada to the International Mathematical Olympiad in Prague. In your article in the 1984 October *Crux*, you make the following statement:

"Two other reasonably nice Canadian problems were discarded since, due to a misunderstanding, the Canadian leader had given them to his team for practice."

This is not accurate. Professor Bowden, the leader, sent me some problems for consideration. I put a couple of them with some of my own, and sent them off to Prague to be considered for the competition, sending off a copy to Bowden for his records. At no time were any of these problems revealed to the team, either by Bowden or myself. Apparently, what happened at the jury meeting for the making up of the competition, was that the Canadian problems were brought up for consideration. Bowden recognized having seen them before, but unfortunately had forgotten the context, since they had been sent up about three months previously and he had neglected to review what we had sent in. Consequently, he came to the conclusion that they must have been used in one of the practice sets.

This was most unfortunate, since I felt that the problems were possibilities. However, the fact remains that, had they been used, the Canadian team would have had no special advantage.

This issue is of some importance and I hope that you will be able to find room in your column to deal with my response.

> Yours sincerely,
>
> (signed) ED BARBEAU, Deputy
> 1984 Canadian IMO team

*

I present two new problem sets this month. They consist of the problems of the First Mathematical Balkaniad (through the courtesy of Jordan B. Tabov, Bulgaria) and those of the ¯th Austrian-Polish Mathematical Competition (through the courtesy of Walther Janous, Austria). I solicit from all readers elegant solutions to these problems.

### FIRST MATHEMATICAL BALKANIAD
### (for secondary school students)
#### Athens, Greece, May 6-10, 1984. Time: 4 hrs.

1. Let $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ be positive numbers whose sum is 1. Prove that

$$\frac{x_1}{1+x_2+x_3+\ldots+x_n} + \frac{x_2}{1+x_1+x_3+\ldots+x_n} + \ldots + \frac{x_n}{1+x_1+x_2+\ldots+x_{n-1}} \geq \frac{n}{2n-1}.$$

2. Let $A_1A_2A_3A_4$ be a cyclic quadrilateral. If $H_1$, $H_2$, $H_3$, $H_4$ denote the orthocenters of triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$, respectively, prove that quadrilaterals $A_1A_2A_3A_4$ and $H_1H_2H_3H_4$ are congruent.

3. Prove that for every positive integer $m$ there exists an integer $n > m$ such that the decimal representation of $5^n$ can be obtained from the decimal representation of $5^m$ by including some digits on the left.

4. Find all real solutions $(x,y,z)$ of the system

$$\begin{cases} by + cz = (y - z)^2 \\ cz + ax = (z - x)^2 \\ ax + by = (x - y)^2 \end{cases}$$

where $a,b,c$ are given positive numbers.

7th AUSTRIAN-POLISH MATHEMATICAL COMPETITION (Poznán, Poland)

First day: July 4, 1984. Time: $4\frac{1}{2}$ hrs.

1. In a given tetrahedron, the foot of the altitude issued from each vertex coincides with the incenter of the opposite face. Prove that this tetrahedron is regular.

2. Let $A$ be the set of all natural numbers between 1000 and 9999 which contain precisely two different digits, both different from zero. Interchanging the two digits yields for every $n \in A$ another number $f(n) \in A$ (e.g., $f(3111) = 1333$). Determine $n \in A$, with $n > f(n)$, such that gcd $(n,f(n))$ is a maximum. (No calculators are to be used.)

3. If $a,x_1,x_2,\ldots,x_n$ $(n \geq 2)$ are positive real numbers, prove that

$$\frac{a^{x_1-x_2}}{x_1+x_2} + \frac{a^{x_2-x_3}}{x_2+x_3} + \ldots + \frac{a^{x_n-x_1}}{x_n+x_1} \geq \frac{n^2}{2(x_1+x_2+\ldots+x_n)}$$

and determine when there is equality.

Second day: July 5, 1984. Time: $4\frac{1}{2}$ hrs.

4. If $A_1A_2\ldots A_7$ is a regular heptagon with circumcircle $C$ and P is a point on minor arc $A_7A_1$, prove that

$$\overline{PA_1} + \overline{PA_3} + \overline{PA_5} + \overline{PA_7} = \overline{PA_2} + \overline{PA_4} + \overline{PA_6}.$$

5. In a folk-dance, the dancers form two lines facing each other. One line is formed by $n$ ladies, the other line by $n$ gentlemen. Each person joins his or her left hand to either the person opposite him or her, or to the left neighbor, or to the person standing opposite the left neighbor. A similar rule holds for the right hand (change "left" to "right" in previous rule). No person is allowed to join both hands to the same person. How many ways are there to meet these rules?

6. Let $a_1, a_2, \ldots, a_n$ be $n$ distinct nonnegative integers, with $n \geq 2$. Determine all $(n+1)$-tuples $(y, x_1, x_2, \ldots, x_n)$ of nonnegative integers with gcd $(x_1, x_2, \ldots, x_n) = 1$ such that

$$
\begin{aligned}
a_1 x_1 + a_2 x_2 + \ldots + a_n x_n &= y x_1, \\
a_2 x_1 + a_3 x_2 + \ldots + a_1 x_n &= y x_2, \\
\vdots \qquad \vdots \qquad\qquad \vdots \qquad &\quad \vdots \\
a_n x_1 + a_1 x_2 + \ldots + a_{n-1} x_n &= y x_n.
\end{aligned}
$$

Third day (team competition), July 6, 1984. Time: 4 hrs.

7. Let $A = (a_{ij})$, where $1 \leq i \leq m$ and $1 \leq j \leq n$, be an $m \times n$ matrix of real numbers in which $|a_{ij}| \leq 1$ for all $i, j$ and all column sums are zero. Show that it is possible to permute the entries of each column to obtain a matrix $B = (b_{ij})$ such that

$$
|\sum_{j=1}^{n} b_{ij}| < 2, \qquad \text{for } i = 1, 2, \ldots, n.
$$

8. The functions $F_0$ and $F_1$ are defined for $x > 1$ by

$$
F_0(x) = 2x \qquad \text{and} \qquad F_1(x) = \frac{x}{x-1}.
$$

Prove that for arbitrary real numbers $a, b$ such that $1 \leq a < b$ there exists a natural number $k$ and a sequence of indices $i_1, i_2, \ldots, i_k$ with $i_j \in \{0, 1\}$ such that

$$
a < F_{i_k}(F_{i_{k-1}}(\ldots(F_{i_1}(2))\ldots)) < b.
$$

9. Determine all real-valued functions $f$ defined on the set $Q$ of rational numbers and satisfying, for all $x, y \in Q$,

$$
f(x+y) = f(x)f(y) - f(xy) + 1.
$$

∗

I now present the official solutions to the problems, given earlier in this column [1984: 181], of the 1984 Canadian Mathematics Olympiad. The committee responsible for organizing this Olympiad consisted of:

C.M. Reis (Chairman), University of Western Ontario
D. Borwein, University of Western Ontario
S.Z. Ditor, University of Western Ontario
B.L.R. Shawyer (Secretary-Treasurer), University of Western Ontario
T.J. Griffiths, A.B. Lucas Secondary School, London, Ontario
G.J. Butler, University of Alberta
P. Arminjon, Université de Montréal
N.S. Mendelsohn, University of Manitoba
E. Mendelsohn, University of Toronto
M.A. McKiernan, University of Waterloo

### 16th CANADIAN MATHEMATICS OLYMPIAD

May 2, 1984 - Time: 3 hours

1. Prove that the sum of the squares of 1984 consecutive positive integers cannot be the square of an integer.

*Official solution.*

Let $n$ be a nonnegative integer. The sum of the squares of the $k$ consecutive integers $n+1$, $n+2$, ..., $n+k$ is

$$S(n,k) = \sum_{i=1}^{k} (n+i)^2 = \sum_{i=1}^{k} (n^2 + 2ni + i^2)$$

$$= kn^2 + 2n\sum_{i=1}^{k} i + \sum_{i=1}^{k} i^2$$

$$= kn^2 + 2n \cdot \frac{k(k+1)}{2} + \frac{k(k+1)(2k+1)}{6}$$

$$= \frac{k}{6}\{6n^2 + 6n(k+1) + (k+1)(2k+1)\}.$$

Setting $k = 1984$, we get

$$S(n,1984) = 992 \cdot \frac{(2s+1)}{3}$$

for some positive integer $s$. Now $(2s+1)/3$ is an odd integer and $992 = 2^5 \cdot 31$. It follows that $2^5$ is the highest power of 2 dividing $S(n,1984)$, proving that $S(n,1984)$ cannot be a perfect square for any $n$.

2. Alice and Bob are in a hardware store. The store sells coloured sleeves that fit over keys to distinguish them. The following conversation takes place:

Alice:   Are you going to cover your keys?

Bob:     I would like to, but there are only seven colours and I have eight keys.

Alice:   Yes, but you could always distinguish a key by noticing that the red key next to the green key was different from the red key next to the blue key.

Bob:     You must be careful what you mean by "next to" or "three keys over from" since you can turn the key ring over and the keys are arranged in a circle.

Alice:   Even so, you don't need eight colours.

*The problem*:  What is the smallest number of colours needed to distinguish $n$ keys if all the keys are to be covered?

*Official solution.*

For 1, 2 and 3 keys, 1, 2 and 3 colours, respectively, are needed.

For 4 or 5 keys, clearly 1 colour won't work. Consider 2 colours, say $B$ and $W$. We have the following possibilities (up to cyclic permutations and colour interchanges):

$$BWWW(W), \qquad BBWW(W), \qquad BWBW(W).$$

None of these provides an orientation from which to count. Thus two colours do not suffice. Three colours, say $B$, $W$, $R$, will work. For example, $(B)BBWR$ works.

For 6 or more keys, 2 colours suffice since $BWBBWW...W$ provides a suitable orientation from which to count.

3.  An integer is *digitally divisible* if (a) none of its digits is zero; (b) it is divisible by the sum of its digits (e.g., 322 is digitally divisible). Show that there are infinitely many digitally divisible integers.

*Official solution.*

For any natural number $n$ let $\sigma(n)$ denote the sum of its digits, and let $m$ be a digitally divisible number. Let $l$ be the number of digits in the decimal representation of $m$ and let

$$p = m(10^{2l} + 10^{l} + 1).$$

Now

$$10^{2l} + 10^{l} + 1 = (10^{l}-1)^2 + 3(10^{l}-1) + 3$$

is divisible by 3, proving that $p$ is divisible by $3m$. But $\sigma(p) = 3\sigma(m)$ and, since $\sigma(m)|m$, it follows that $\sigma(p)|p$. Thus, starting with the digitally divisible number 1, we form the infinite sequence

$$1, \qquad 111, \qquad 111111111, \qquad ...,$$

in which the $i$th term has $3^{i-1}$ ones, and it follows by the above analysis that each term of the sequence is digitally divisible.

4. An acute-angled triangle has area 1. Show that there is a point inside the triangle whose distances from each of the vertices is at least $2/\sqrt[4]{27}$.

I. *Official solution (edited to save space).*

The following theorem is well known (a proof can be found, for example, on page 104 of *Geometric Inequalities*, by Nicholas D. Kazarinoff, New Mathematical Library, No. 4, published by the M.A.A):

*Of all triangles inscribed in a given circle, the equilateral triangle has the greatest area.*

By inverting this result, we conclude that among all triangles of a given fixed area the equilateral triangle has the smallest circumradius. Now the triangle in question is acute-angled, so its circumcentre falls inside the triangle. The desired result will therefore follow if we can prove that *the circumradius of an equilateral triangle of unit area is $2/\sqrt[4]{27}$ units in length*, and a simple calculation shows that this statement is true.

II. *Alternative solution by M.S.K.*

Let ABC be the given triangle, $a, b, c$ its sides, and $R$ its circumradius. As noted in solution I, the circumcentre falls inside the triangle. Since

$$\text{Area} = 1 = \frac{abc}{4R} = 2R^2 \sin A \sin B \sin C,$$

the problem reduces to finding the maximum value of $\sin A \sin B \sin C$. Now $\ln \sin x$ is concave in $(0,\pi)$, so $\sin A \sin B \sin C \le \sin^3 60°$, with equality if and only if $A = B = C$. Thus

$$R^2 = \frac{1}{2 \sin A \sin B \sin C} \ge \frac{4}{\sqrt{27}},$$

which implies the desired result.

(The more general problem of maximizing $\sin^\alpha A \sin^\beta B \sin^\gamma C$ is given in Problem 908 [1984: 19].)

5. Given any seven real numbers, prove that there are two of them, say $x$ and $y$, such that

$$0 \le \frac{x - y}{1 + xy} \le \frac{1}{\sqrt{3}}.$$

*Official solution.*

Let the given numbers be $x_1, x_2, \ldots, x_7$. Since the range of the function $y = \tan x$ $(-\pi/2 < x < \pi/2)$ is the set of all real numbers, for each $x_i$ there exists $\theta_i$ in the interval $(-\pi/2 \ \pi/2)$ such that $x_i = \tan \theta_i$. Divide the interval $(-\pi/2, \pi/2)$ into six equal subintervals each of length $\pi/6$. By the pigeonhole principle, two of the $\theta_i$,

say $\theta_s$ and $\theta_t$, lie in the same subinterval. Thus, assuming $\theta_s \geq \theta_t$, we have $0 \leq \theta_s - \theta_t < \pi/6$. Since $y = \tan x$ is an increasing function of $x$ on $[0, \pi/6)$, it follows that

$$\tan 0 \leq \tan(\theta_s - \theta_t) < \tan \frac{\pi}{6}.$$

Therefore

$$0 \leq \frac{\tan \theta_s - \tan \theta_t}{1 + \tan \theta_s \tan \theta_t} < \frac{1}{\sqrt{3}}, \qquad \text{that is,} \qquad 0 \leq \frac{x_s - x_t}{1 + x_s x_t} < \frac{1}{\sqrt{3}}.$$

\*

Finally, I present solutions to the problems of the Second Round of the 1983 Dutch Olympiad, which I published in a recent column [1984: 213]. Two of these are official solutions, which were translated into English by Andy Liu.

### DUTCH MATHEMATICAL OLYMPIAD
Second Round, September 16, 1983. Time: 3 hours

1. A triangle ABC can be divided into two isosceles triangles by a line through A. Given that one of the angles of the triangle is $30°$, determine all possible values of the other two angles.

*Solution by Gali Salvatore, Perkins, Québec.*

It is known (see Problem 200 [1977: 134, 228]) that the following triangles, and only those, can be divided into two isosceles triangles by a line through a vertex:

(i) all right triangles;

(ii) all triangles in which one *acute* angle is twice another;

(iii) all triangles in which one angle is three times another.

If one angle is $30°$, we therefore have the following solution triangles:

*Case (i).* Triangle $(30°, 60°, 90°)$.

*Case (ii).* Triangle $(15°, 30°, 135°)$ and the right triangle of Case (i).

*Case (iii).* Triangles $(10°, 30°, 140°)$, $(30°, 37\frac{1}{2}°, 112\frac{1}{2}°)$, and the right triangle of Case (i).

So there are exactly four distinct solution triangles.

2. Prove that if $n$ is an odd positive integer, then the last two digits in base ten of $2^{2n}(2^{2n+1} - 1)$ are 28.

*Solution by Andy Liu, University of Alberta.*

Let $n = 2k+1$. It is easy to verify (by induction on $k$ or otherwise) that

$$2^{2n} = 4^{2k+1} \equiv 4 \pmod{10}.$$

Since also $4 | 2^{2n}$, we therefore have (all congruences from now on being modulo 100)

$$2^{2n} \equiv 4,\ 24,\ 44,\ 64,\ \text{or } 84.$$

Now

$$2^{2n} \equiv 4 \implies 2^{2n+1} - 1 \equiv 7, \text{ and } 4 \cdot 7 = 28 \equiv 28 \, ;$$

$$2^{2n} \equiv 24 \implies 2^{2n+1} - 1 \equiv 47, \text{ and } 24 \cdot 47 = 1128 \equiv 28 \, ;$$

$$2^{2n} \equiv 44 \implies 2^{2n+1} - 1 \equiv 87, \text{ and } 44 \cdot 87 = 3828 \equiv 28 \, ;$$

$$2^{2n} \equiv 64 \implies 2^{2n+1} - 1 \equiv 27, \text{ and } 64 \cdot 27 = 1728 \equiv 28 \, ;$$

$$2^{2n} \equiv 84 \implies 2^{2n+1} - 1 \equiv 67, \text{ and } 84 \cdot 67 = 5628 \equiv 28.$$

3. Let $a, b, c, p$ be real numbers, with $a, b, c$ not all equal, such that

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a} = p.$$

Determine all possible values of $p$ and prove that $abc + p = 0$.

*Official solution.*

From $bc + 1 = cp$ and $ca + 1 = ap$, we get

$$ap^2 = cap + p = a(bc+1) + p = abc + a + p.$$

Hence $a(p^2 - 1) = abc + p$. With two similar results obtained by cyclic permutations, we therefore have

$$a(p^2 - 1) = b(p^2 - 1) = c(p^2 - 1) = abc + p.$$

Since $a, b, c$ are not all equal, we must have $p = \pm 1$, and so $abc + p = 0$.

Both values of $p$ are seen to be possible by considering the triples

$$(2, -1, \tfrac{1}{2}) \quad \text{and} \quad (-2, 1, -\tfrac{1}{2}).$$

4. Within an equilateral triangle of side 15 are 111 points. Prove that it is always possible to cover at least 3 of these points by a round coin of diameter $\sqrt{3}$, part of which may lie outside the triangle.

*Official solution.*

Cover the triangle with $1 + 2 + \ldots + 10 = 55$ congruent regular hexagons, as shown in the figure. At least 3 of the points lie within one of the hexagons, because $2 \cdot 55 < 111$. Since the diameter of the hexagon is $\sqrt{3}$, it and the 3 or more points inside it can be covered by a coin of diameter $\sqrt{3}$.
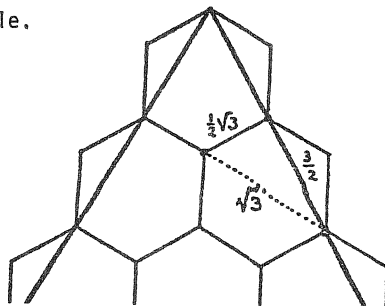


*Editor's note.* Address all communications about this column directly to Professor Klamkin.

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# P R O B L E M S - - P R O B L È M E S

*Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (\*) after a number indicates a problem submitted without a solution.*

*Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.*

*To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before May 1, 1985, although solutions received after that date will also be considered until the time when a solution is published.*

**991.** *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Prove that the synonymical addition

$$\begin{array}{c} \text{STAIN} \\ \underline{\text{SPOT}} \\ \text{TAINT} \end{array}$$

has a unique solution in base seven and in base eight, and at least one solution in every base $b \geq$ nine.

**992.** *Proposed by Harry D. Ruderman, Bronx, N.Y.*

Let $\alpha = (a_1, a_2, \ldots, a_{mn})$ be a sequence of positive real numbers such that $a_i \leq a_j$ whenever $i < j$, and let $\beta = (b_1, b_2, \ldots, b_{mn})$ be a permutation of $\alpha$. Prove that

(a) $\displaystyle\sum_{j=1}^{n} \prod_{i=1}^{m} a_{m(j-1)+i} \geq \sum_{j=1}^{n} \prod_{i=1}^{m} b_{m(j-1)+i}$ ;

(b) $\displaystyle\prod_{j=1}^{n} \sum_{i=1}^{m} a_{m(j-1)+i} \leq \prod_{j=1}^{n} \sum_{i=1}^{m} b_{m(j-1)+i}$.

**993.** *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $P$ be the product of the $n+1$ positive real numbers $x_1, x_2, \ldots, x_{n+1}$. Find a lower bound (as good as possible) for $P$ if the $x_i$ satisfy

(a) $\displaystyle\sum_{i=1}^{n+1} \frac{1}{1+x_i} = 1$ ;

(b)\* $\displaystyle\sum_{i=1}^{n+1} \frac{a_i}{b_i+x_i} = 1$, where the $a_i$ and $b_i$ are given positive real numbers.

**994.\*** *Proposed by Ernest J. Eckert, University of Wisconsin-Green Bay.*

Can two different primitive Pythagorean triangles with sides $(a,b,c)$ and $(r,s,t)$ be such that $abc = rst$?

995. *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokai-City, Aichi, Japan.*

A square sheet of paper ABCD is folded as shown in the figure, with D falling on D' along BC, A falling on A', and A'D' meeting AB in E. A circle is inscribed in triangle EBD'. Prove that the radius of this circle equals A'E.
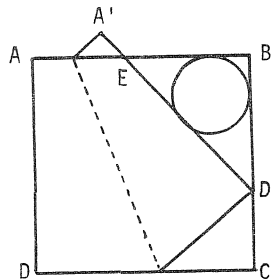
996. *Proposed by Herta T. Freitag, Roanoke, Virginia.*

If $G = (\sqrt{5}+1)/2$ is the golden ratio, prove that, for every positive integer $n$,

$$\sum_{k=1}^{n} \sum_{i=0}^{2(k-1)} (-1)^{i} G^{2(k-i-1)}$$

is a Fibonacci number of even subindex.

997. *Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.*

Suppose that $f(x)$ is bounded in a deleted neighborhood of zero and suppose that $a$ and $b$ are real numbers less than 1 in magnitude such that $\lim_{x \to 0} \{f(x) + af(bx)\}$ exists (and is finite). Prove that $\lim_{x \to 0} f(x)$ exists.

998. *Proposed by Andrew P. Guinand, Trent University, Peterborough, Ontario.*

If just one angle of a triangle is $60°$, show that the inverse of the orthocentre with respect to the circumcircle lies on the side (or side produced) opposite that angle.

999.[*] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Let $R,r,s$ be the circumradius, inradius, and semiperimeter, respectively, of an acute-angled triangle. Prove or disprove that

$$s^2 \geq 2R^2 + 8Rr + 3r^2.$$

When does equality occur?

1000. *Proposed by H.S.M. Coxeter, University of Toronto.*

In a tetrahelix (see R. Buckminster Fuller, *Synergetics*, Macmillan, New York, 1975, pp. 518-524), points $A_0, A_1, A_2, \ldots$ are arranged so that every consecutive four are the vertices of a regular tetrahedron; in other words, there is an infinite sequence of regular tetrahedra $A_0A_1A_2A_3$, $A_1A_2A_3A_4$, $A_2A_3A_4A_5$, $\ldots$ . In terms of the edge length $A_0A_1$ as a unit of measurement, find the distance $d$ and angle $\delta$ between the two skew lines $A_0A_1$ and $A_nA_{n+1}$.

*            *            *

# SOLUTIONS

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

836. ⌜1983: 113 ; 1984: 228⌝ *Proposed by Vedula N. Murty, Pennsylvania State University, The Capitol Campus.*

(a)  If A,B,C are the angles of a triangle, prove that

$$(1 - \cos A)(1 - \cos B)(1 - \cos C) \geq \cos A \cos B \cos C,$$

with equality if and only if the triangle is equilateral.

(b)  Deduce from (a) Bottema's triangle inequality [1982: 296]:

$$(1 + \cos 2A)(1 + \cos 2B)(1 + \cos 2C) + \cos 2A \cos 2B \cos 2C \geq 0.$$

II.  *Comment by N.S. Mendelsohn, University of Manitoba.*

In a comment ⌜1984: 229⌝ the editor asked for a proof or disproof of the following generalization of Bottema's inequality

$$(1 + \cos 2^n A)(1 + \cos 2^n B)(1 + \cos 2^n C) + \cos 2^n A \cos 2^n B \cos 2^n C \geq 0 \qquad (1)$$

for $n = 1,2,3,\ldots$, which had been conjectured by Curtis Cooper.  The inequality is true, and surely the published solution for $n = 1$ already contains the general proof.

The idea is simply this.  Proceed inductively.  If $A \geq B \geq C$ are the angles of a triangle then either

$$\pi - 2A, \ \pi - 2B, \ \pi - 2C \qquad \text{or} \qquad 2A - \pi, \ 2B, \ 2C$$

are angles of a triangle.  Replacing A,B,C by either of these sets in the inequality (1) increases the value of $n$ by 1.  The only thing to note is that

$$\cos 2^n(\pi - 2X) = \cos 2^n(2X - \pi) = \cos 2^{n+1}X, \qquad n = 1,2,3,\ldots \ .$$

This brings up the point that any inequality involving only angles of a triangle can be "generalized" by noting that if A,B,C are angles of a triangle, then so are

$$\lambda\pi + (1-3\lambda)A, \qquad \lambda\pi + (1-3\lambda)B, \qquad \lambda\pi + (1-3\lambda)C,$$

provided these angles are positive.  A sufficient condition is $0 < \lambda < 1/3$.  Naturally, the resulting formulas in most cases are quite uninspiring.  Also, some formulas regarding angles of a triangle follow simply from $A+B+C = \pi$ and are valid even when negative values for A, B, or C are used.  In this case, the restriction on $\lambda$ can be removed.

*Editor's comment.*

Cooper's proof of his generalization (1) was essentially the same as that given above, and the editor erred in calling it "an incomplete proof".

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861. [1983: 208] *Proposed by J.A.H. Hunter, Toronto, Ontario.*

```
    S A M'S
    I D E A
    M A D E
   ─────────
   S E N S E
```

Of course, it would not be an odd IDEA, but what makes SENSE?

I. *Solution by Edwin M. Klein, University of Wisconsin-Whitewater.*

Since S = 1 or 2, A is even, and A+S = 10, it follows that A = 8 and S = 2.
With the 8 already in place, the leading 2 in the sum can occur only if {I,M} = {7,9}.
Now M+D = 11 in the tens' column. Since M = 9 implies D = 2, a duplication, we must
have M = 7, I = 9, and D = 4. Finally, N = 1 and the unique answer is

```
    2872
    9408
    7840.
   ──────
   20120
```

II. *Comment by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New
Hampshire.*

Apostrophes usually have no arithmetical significance in problems of this kind,
but to cover all the bases (baseball idiom—no arithmetical significance) we decided
to find out what happens if M' ≠ M. There is then one solution in base ten with I
and M interchangeable:

```
    2852                    2852
    7608                    9608
    9860        and         7860.
   ──────                  ──────
   20320                   20320
```

III. *Comment by Glen E. Mills, Pensacola Junior College, Florida.*

Since there are only seven different letters in this alphametic, it may be
interesting to list the solutions for bases seven, eight, and nine. There is no
solution in base seven, and the solutions in bases eight and nine are, respectively,

```
    2652                    1821
    7406                    4708
    5640        and         2870.
   ──────                  ──────
   20120                   10610
```

Note that the sums make the same SENSE in bases eight and ten.

IV. *Comment by Allan Wm. Johnson Jr., Washington, D.C.*
Variations on a theme by Hunter.
Were Sam's IDEA ODD as well as even, in other words if

```
S A M'S
  O D D
I D E A
M A D E
S E N S E
```
(1)

is to be solved for even IDEA, then (1) also has a unique solution:

```
 1071
  299
 6950
 7095.
15415
```
(2)

In this solution, appropriately, ODD is odd.

If we replace ODD by EVEN in (1) and use the commutativity of addition, we obtain

```
E V E N
S A M'S
I D E A
M A D E.
S E N S E
```

Here, as befits EVEN, there are just two solutions with even IDEA:

```
 5459              4540
 1071              2862
 2350              9748
 7035     and      6874.
15915             24024
```

Observe that the SAMS wear the same disguise in the first of these and in (2).

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay; CLAYTON W. DODGE, University of Maine at Orono; RICHARD I. HESS, Rancho Palos Verdes, California; ALLAN WM. JOHNSON JR., Washington, D.C.; ROBERT S. JOHNSON, Montréal, Québec; MARK KANTROWITZ, student, Maimonides School, Brookline, Massachusetts; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Pensacola Junior College, Florida; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; RAVI RAMAKRISHNA, Essex Junction High School, Vermont; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer (two solutions).

✻       ✻       ✻

862. [1983: 208] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

P is an interior point of a triangle ABC. Lines through P parallel to the sides of the triangle meet those sides in the points $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$, as shown in the figure. Prove that

(a) $[A_1B_1C_1] \leq \frac{1}{3}[ABC]$; (b) $[A_1C_2B_1A_2C_1B_2] \geq \frac{2}{3}[ABC]$,

where the brackets denote area.
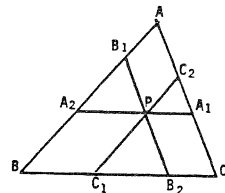
*Solution by Stan Wagon, Smith College, Northampton, Massachusetts (revised by the editor).*

More generally, we use *signed* areas and show that, for *any* point P in the plane of the triangle, the following signed areas are all equal:

$$[A_1B_1C_1], \quad [A_2B_2C_2], \quad [AB_1C_2] + [BC_1A_2] + [CA_1B_2], \quad [ABC] - [A_1C_2B_1A_2C_1B_2]. \quad (1)$$

(The signed area of the hexagon can be defined, for example, by

$$[A_1C_2B_1A_2C_1B_2] = [A_1C_2B_1] + [A_1B_1A_2] + [A_1A_2C_1] + [A_1C_1B_2].)$$

Our two desired results (and more!) will then follow immediately when we have shown that the common value, $\alpha$, of the areas (1) satisfies $\alpha \leq \frac{1}{3}[ABC]$ when ABC is oriented positively.

The problem is one of affine geometry, since affine transformations preserve collinearity, parallelism, and area ratios, so it suffices to prove it for one special case, say when ABC is an isosceles right triangle, as shown in the adjoining figure.
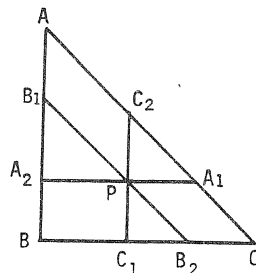


We introduce a coordinate system in which the vertices have coordinates

$$A(0, 1), \quad B(0, 0), \quad C(1, 0).$$

Then $[ABC] = 1/2$. If $P(x, y)$ is any point in the plane, then the remaining special points of the figure have the following coordinates:

$$A_1(1-y, y), \quad B_1(0, x+y), \quad C_1(x, 0),$$
$$A_2(0, y), \quad B_2(x+y, 0), \quad C_2(x, 1-x).$$

Using these coordinates, we easily find that each of the signed areas (1) has the value

$$\alpha = \frac{1}{2}(x + y - x^2 - xy - y^2)$$
$$= [ABC] \cdot \{\frac{1}{3} - (x + \frac{y-1}{2})^2 - \frac{3}{4}(y - \frac{1}{3})^2\}$$
$$\leq \frac{1}{3}[ABC].$$

Equality occurs just when $x = y = 1/3$, that is, just when P is the centroid of ABC.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; O. BOTTEMA, Delft, The Netherlands; MICHAEL BROZINSKY, Queensborough Community College, Bayside, N.Y.; GENG-ZHE CHANG, University of Science and Technology, Hefei, Anhui, People's Republic of China; JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; DAN PEDOE, Minneapolis, Minnesota; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

     *              *              *

863. ⌐1983: 208] *Proposed by (the late) H. Kestelman, University College, London, England.*

If $A$ is an $n \times j$ matrix with rank $j$, and $B$ is an $n \times k$ matrix with rank $k$, then the equation $Ax = By$ has a solution other than $x = y = 0$ if and only if

$$B^+B - B^+A(A^+A)^{-1}A^+B$$

is singular.

(The $(r,s)$ element of $M^+$ is the complex conjugate of the $(s,r)$ element of $M$.)

*Solution by Kenneth S. Williams, Carleton University, Ottawa.*

Suppose the equation

$$Ax = By \qquad (1)$$

has a solution $(x,y) \neq (0,0)$. Then $y \neq 0$, for otherwise $Ax = 0$ and so $x = 0$ as $A$ has rank $j$. Now, from (1), $A^+Ax = A^+By$ and, as $A^+A$ is invertible (being a $j \times j$ matrix of rank $j$), we obtain

$$x = (A^+A)^{-1}A^+By. \qquad (2)$$

Hence

$$\begin{aligned}(B^+B - B^+A(A^+A)^{-1}A^+B)y &= B^+By - B^+A(A^+A)^{-1}A^+By \\ &= B^+Ax - B^+Ax, \qquad \text{from (1) and (2),} \\ &= 0,\end{aligned}$$

and $y \neq 0$ implies that

$$B^+B - B^+A(A^+A)^{-1}A^+B \qquad (3)$$

is singular, as required.

Conversely, suppose (3) is singular. Then there exists $y \neq 0$ such that

$$(B^+B - B^+A(A^+A)^{-1}A^+B)y = 0.$$

If we now take for $x$ the value given by (2), this equation becomes

$$B^+(By - Ax) = 0,$$

and (1) follows since $B^+$ has rank $k$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; and the proposer.

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864. [1983: 208] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*
Find all $x$ between $0$ and $2\pi$ such that

$2\cos^2 3x - 14\cos^2 2x - 2\cos 5x + 24\cos 3x - 89\cos 2x + 50\cos x > 43.$

*Solution by D.J. Smeenk, Zaltbommel, The Netherlands.*

We seek the values of $x \in [0, 2\pi]$ such that $f(x) > 0$, where

$$f(x) = 2\cos^3 3x - 14\cos^2 2x - 2\cos 5x + 24\cos 3x - 89\cos 2x + 50\cos x - 43.$$

With $c = \cos x$ and the known results

$$\cos 2x = 2c^2 - 1, \qquad \cos 3x = 4c^3 - 3c, \qquad \cos 5x = 16c^5 - 20c^3 + 5c,$$

we obtain

$$f(x) \equiv g(c) = 32c^6 - 32c^5 - 104c^4 + 136c^3 - 104c^2 - 32c + 32.$$

One can always try to factor $g(c)$ by trial and error, but in this case it is possible to proceed more methodically. For $c \neq 0$,

$$g(c) = c^3\{32(c^3 + \tfrac{1}{c^3}) - 32(c^2 + \tfrac{1}{c^2}) - 104(c + \tfrac{1}{c}) + 136\}.$$

We now set $t = c + 1/c$, so that $c^2 + 1/c^2 = t^2 - 2$ and $c^3 + 1/c^3 = t(t^2 - 3)$, and we easily find

$$
\begin{aligned}
g(c) &= 8c^3(t-1)(2t+5)(2t-5) \\
&= 8(c^2 - c + 1)(2c^2 + 5c + 2)(2c^2 - 5c + 2) \\
&= 8(c^2 - c + 1)(2c + 1)(c + 2)(2c - 1)(c - 2) \\
&= 32(c^2 - c + 1)(c + \tfrac{1}{2})(c + 2)(c - \tfrac{1}{2})(c - 2),
\end{aligned}
$$

where now $c = 0$ is allowed. Now $|c| \leq 1$, so $g(c) > 0$ if and only if $-\tfrac{1}{2} < c < \tfrac{1}{2}$, and the solutions to the original inequality are

$$\frac{\pi}{3} < x < \frac{2\pi}{3} \qquad \text{and} \qquad \frac{4\pi}{3} < x < \frac{5\pi}{3}.$$

Also solved by CURTIS COOPER, Central Missouri State University at Warrensburg; CLAYTON W. DODGE, University of Maine at Orono; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; GLEN E. MILLS, Pensacola Junior College, Florida; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

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865. [1983: 209] *Proposed by Clark Kimberling, University of Evansville, Indiana.*

Let $x$, $y$, $z$ be the distances from the sides $a = BC$, $b = CA$, $c = AB$, respectively, to a variable point inside a triangle ABC. Prove that, for $0 \neq t \neq 1$, the critical point of $x^t + y^t + z^t$ satisfies

$$x : y : z = a^p : b^p : c^p, \tag{1}$$

where $p = 1/(t-1)$. Discuss limiting cases.

*Solution by the proposer.*

Let $\triangle$ denote the area of triangle ABC. Then

$$ax + by + cz = 2\triangle. \qquad (2)$$

If $f(x,y) = x^t + y^t + z^t$, where $z$ satisfies (2), then

$$\frac{\partial f}{\partial x} = tx^{t-1} - tz^{t-1}(a/c),$$

which vanishes when $x/z = (a/c)^p$. Similarly, $\partial f/\partial y = 0$ when $y/z = (b/c)^p$. Thus the one and only point where both derivatives vanish satisfies (1). Now if we set $\alpha = \partial^2 f/\partial x^2$, $\beta = \partial^2 f/\partial x\partial y$, and $\gamma = \partial^2 f/\partial y^2$, we find that

$$\alpha = \frac{t(t-1)}{c^2}(c^2 x^{t-2} + a^2 z^{t-2}), \qquad \gamma = \frac{t(t-1)}{c^2}(c^2 y^{t-2} + b^2 z^{t-2}),$$

and

$$\beta^2 - \alpha\gamma = -\frac{t^2(t-1)^2}{c^2}\{a^2(yz)^{t-2} + b^2(zx)^{t-2} + c^2(xy)^{t-2}\}.$$

Thus $\beta^2 - \alpha\gamma < 0$ for all $t$ except 0 and 1; $\alpha < 0$ and $\gamma < 0$ for $0 < t < 1$; and $\alpha > 0$ and $\gamma > 0$ for $t < 0$ and $t > 1$. Therefore the critical point is a maximum or minimum according as $0 < t < 1$ or not.

As $|t| \to \infty$, the critical point approaches the incenter of the triangle, since $x:y:z \to 1:1:1$.

If a point P (inside triangle ABC) is the critical point of $x^t + y^t + z^t$, then its isogonal conjugate is the critical point of $x^{2-t} + y^{2-t} + z^{2-t}$. This follows from the fact that the isogonal conjugate $(x',y',z')$ of a point with trilinear coordinates $(x,y,z)$ satisfies $x':y':z' = x^{-1}:y^{-1}:z^{-1}$. As special cases, P is the symmedian point of ABC for $t = 2$, and P is the centroid (a limiting case) for $t = 0$.

The trilinear coordinates of the critical point for $0 \neq t \neq 1$ are

$$\left(\frac{2\triangle}{a + b(\frac{b}{a})^p + c(\frac{c}{a})^p}, \quad \frac{2\triangle}{a(\frac{a}{b})^p + b + c(\frac{c}{b})^p}, \quad \frac{2\triangle}{a(\frac{a}{c})^p + b(\frac{b}{c})^p + c}\right).$$

From these it is easily seen that as $t \to 1$ the critical point approaches vertex C if $a < b < c$, and it approaches the midpoint of side BC if $a < b = c$.

Also solved by JORDI DOU, Barcelona, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; and GEORGE TSINTSIFAS, Thessaloniki, Greece (incomplete solution).

*Editor's comment.*

Klamkin extended the problem a bit by allowing the variable point to lie also on the boundary of triangle ABC. Since, for $t > 0$, $x^t + y^t + z^t$ is then a continuous function over a closed set, it has both a maximum and a minimum value (but only a

minimum value for $t < 0$). The following results of Klamkin complete the proposer's solution for the extended problem:

  (a)  If $t > 1$, the maximum occurs at the vertex of the smallest angle.

  (b)  If $0 < t < 1$, the minimum occurs at the vertex of the largest angle.

  ✻                              ✻                              ✻

866.  [1983: 209] *Proposed by Jordi Dou, Barcelona, Spain.*

  Given a triangle ABC with sides $a, b, c$, find the minimum value of

$$a \cdot \overline{XA} + b \cdot \overline{XB} + c \cdot \overline{XC},$$

where X ranges over all the points of the plane of the triangle.

  I. *Solution by Jordan B. Tabov, Sofia, Bulgaria.*

  In this solution square brackets denote the signed area of a triangle, and we assume that ABC is oriented positively. It is known that the area of any quadrilateral does not exceed half the product of its diagonals. We will use this fact in the following more general form: *If P,Q,R,S are any four points in the plane, then*

$$PR \cdot QS \geq 2([PRS] - [PRQ]). \tag{1}$$

  Let $f(X) = a \cdot XA + b \cdot XB + c \cdot XC$, where X ranges over the entire plane. Then, from (1),

$$\begin{aligned} f(X) &= BC \cdot XA + CA \cdot XB + AB \cdot XC \\ &\geq 2([BCA] - [BCX] + [CAB] - [CAX] + [ABC] - [ABX]) \tag{2} \\ &= 4[ABC]. \end{aligned}$$

  Let A be the largest angle of the triangle. If angle $A \leq 90°$, then the ortho-centre H does not lie outside the triangle, and equality holds in (2) when X = H. In this case, therefore, min $f(X) = 4[ABC]$, attained when X = H. The situation is different, however, if angle $A > 90°$. We will show that then min $f(X) = 2bc > 4[ABC]$, attained when X = A.

  The function $f(X)$ is continuous, strictly positive, and $f(X) \to \infty$ when $X \to \infty$. Therefore $f(X)$ has a minimum value $f(X_0)$. We first show that $X_0$ cannot lie outside triangle ABC. Suppose, for example, that $X_0$ and C lie on opposite sides of line AB. Then, if $X_1$ is symmetric to $X_0$ with respect to AB, we have $X_0A = X_1A$, $X_0B = X_1B$, but $X_0C > X_1C$. Therefore $f(X_0) > f(X_1)$, a contradiction. Next, the function $f(X)$ is differentiable at all points except at the vertices A,B,C. Hence $X_0$ must be one of the points A,B,C or else a point Y such that grad $f(Y) = \vec{0}$. Such a point Y, if one exists, must satisfy

$$\text{grad } f(Y) = \frac{a}{YA} \cdot \vec{YA} + \frac{b}{YB} \cdot \vec{YB} + \frac{c}{YC} \cdot \vec{YC} = \vec{0}. \tag{3}$$

Let $B_1\vec{C}_1$, $C_1\vec{A}_1$, $A_1\vec{B}_1$ (whose sum is $\vec{0}$) denote, respectively, the three vector terms in (3). Then the segments $B_1C_1$, $C_1A_1$, $A_1B_1$ have lengths $a$, $b$, $c$, respectively. Consequently, in triangle $A_1B_1C_1$, angle $A_1$ = angle A and the angle between $C_1\vec{A}_1$ and $A_1\vec{B}_1$ equals $180°$ - A. Hence angle BYC = $180°$ - A < A, which means that Y lies outside triangle ABC. Since outside points have already been eliminated, we conclude that $X_0$ = A, B, or C. Now $f(A) = 2bc$, $f(B) = 2ca$, and $f(C) = 2ab$; and since $a$ is the longest side, we conclude that $X_0$ = A. Thus min $f(X) = 2bc$, attained when X = A.

  II. *Comment by O. Bottema, Delft, The Netherlands.*

  This is a particular case of a well-known problem. Let ABC and PQR be two triangles in a plane, with sides $a,b,c$ and $p,q,r$, respectively. The point S in the plane such that

$$p \cdot SA + q \cdot SB + r \cdot SC \tag{4}$$

is a minimum is called the *metapole* of ABC with respect to PQP. One of its properties is

$$\angle BSC = \pi-P, \quad \angle CSA = \pi-Q, \quad \angle ASB = \pi-R.$$

If $p = a$, $q = b$, $r = c$, the metapole coincides with the *orthocentre* of ABC. The minimum value of (4) is then $4\ulcorner ABC \urcorner$.

  Also solved by J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; M. PARMENTER, Memorial University of Newfoundland; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer. A comment was received from ROLAND H. EDDY, Memorial University of Newfoundland.

  *Editor's comment.*

  Not all solutions were entirely satisfactory. Some solvers considered only points X inside the triangle, and some tacitly assumed that angle A $\leq 90°$, thereby achieving only a partial result. Two solvers based their proofs on an interesting inequality of Steensholt [1], but this result is applicable to our problem only if the points X are restricted to the interior of the triangle *and* if angle A $\leq 90°$. Satyanarayana apparently misunderstood the problem and proved instead the following interesting related result: *For all points X in the plane,*

$$a \cdot \vec{XA} + b \cdot \vec{XB} + c \cdot \vec{XC} = \vec{0}$$

*if and only if* X = I, *the incentre of the triangle.*

  Bottema gave no reference for his "well-known problem", but Groenman wrote that it can be found in Bottema's own book [2]. We have not seen this reference, but it

seems clear from Bottema's comment that his problem generalizes our own only for the case A ≤ 90°. That Bottema was right in claiming that his problem was "well-known" is confirmed by the following quotation, which Eddy found in [3]:

"... Simpson treats the more general problem,

Three points A B C being given, to find the position of a fourth point P, so that if lines be drawn from thence to the three former, the sum

$$a \cdot AP + b \cdot BP + c \cdot CP$$

where $a$ $b$ $c$ denote given numbers, shall be a minimum.

... the more general problem [is] discussed (See *Nova Acta Academiae...Petropolitanae* XI. 235-8 (1798)) by Nicolas Fuss in his memoir 'De Minimis quibusdam geometricis, ope principii statici inventis' read to the Petersburg Academy of Sciences on 25th February 1796."

Mathematical archeologists can take it from there.

REFERENCES

1. Gunnar Steensholt, "Note on an elementary property of triangles", *American Mathematical Monthly*, 63 (1956) 571-572.

2. O. Bottema, *Hoofdstukken uit de elementaire meetkunde*, Servire, The Hague, 1944, p. 98.

3. *Proceedings of the Edinburgh Mathematical Society*, XV (1897) 102.

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**867.** [1983: 209] *Proposed by Charles W. Trigg, San Diego, California.*

A strip of four equilateral triangles can be folded along the common edges to form a regular tetrahedron with 3 open edges. (An open edge is one through which there is direct access to the centroid of the polyhedron.) How many triangles must be in the strip to form a tetrahedron with no open edges?

I. *Solution by the proposer.*

Since a regular tetrahedron with two opposite open edges is a deformed cylinder, continuous folding of a strip of $n$ triangles will merely reinforce the faces, but not close the opposite edges. The direction of folding must be changed in order to accomplish closure.

A strip of nine triangles will suffice. Number the triangles consecutively from one end, as shown in Figure 1. Designate an edge by the numbers, in parentheses, of
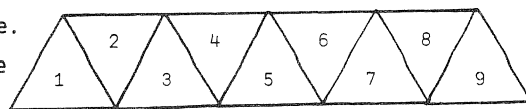


Figure 1

the faces meeting at that edge. Bring one edge of triangle 1 into contact with an edge of 3, and another edge of triangle 1 into contact with edge (4,5). Then bring

1 into coincidence with 5. This will close edge (1,4). Bring 6 into coincidence with 5, then 7 onto 3 thus closing (1,3), then 8 onto 4, and 9 onto 2, thus closing (2,4). A stable collapsible model is obtained by tucking 9 under 2. To make the model, thin stiff paper is desirable.

II. *Comment by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.*

If the term "strip" is interpreted more generally to include any connected set of triangles, it is clear that there must be at least seven triangles, since there are six edges to be covered with folds. There are many seven-triangle solutions, of which the symmetrical solution in Figure 2 is probably the most aesthetically satisfying.
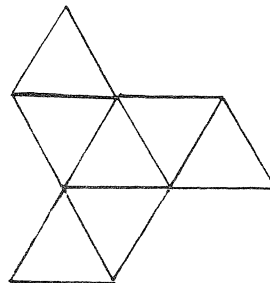


Figure 2

Also solved by CLAYTON W. DODGE, University of Maine at Orono; and JORDI DOU, Barcelona, Spain.

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868. [1983: 209] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

The graph of $x^3 + y^3 = 3axy$ is known as the *folium of Descartes*. Prove that the area of the loop of the folium is equal to the area of the region bounded by the folium and its asymptote $x + y + a = 0$.

*Solution by M.S. Klamkin, University of Alberta.*

A $45°$ rotation of the axes transforms the equations of the curve and its asymptote into

$$y^2 = \frac{x^2(3b-x)}{3(b+x)} \quad \text{and} \quad x = -b,$$

where $b = a/\sqrt{2}$. We show that for the curve

$$y^2 = kx^2(\frac{c-x}{b+x}), \qquad b,c,k > 0,$$

the area $A_1$ bounded by the loop of the curve equals the area $A_2$ bounded by the curve and its asymptote if and only if $c = 3b$ (for arbitrary $k > 0$), corresponding to the folium of Descartes (for $k = 1/3$).

We set $b+x = (b+c)\sin^2\theta$ and obtain

$$A_1 = 2\sqrt{k} \int_0^c x\sqrt{\frac{c-x}{b+x}}\,dx = 4\sqrt{k}(b+c) \int_\lambda^{\pi/2} f(\theta)\,d\theta$$

and

$$A_2 = 2\sqrt{k}\int_{-b}^{0} -x\sqrt{\frac{c-x}{b+x}}\,dx = -4\sqrt{k}(b+c)\int_{0}^{\lambda} f(\theta)\,d\theta,$$

where $\lambda = \arcsin\sqrt{b/(b+c)}$ and $f(\theta) = \{c - (b+c)\cos^2\theta\}\cos^2\theta$. It follows that

$$A_1 = A_2 \iff \int_{0}^{\pi/2} f(\theta)\,d\theta = 0.$$

From standard integrals we easily find that, to within an additive constant,

$$F(\theta) \equiv \int f(\theta)\,d\theta = c\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) - (b+c)\left(\frac{3\theta}{8} + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}\right).$$

Since $F(\pi/2) = (c-3b)\pi/16$ and $F(0) = 0$, we conclude that

$$A_1 = A_2 \iff F(\pi/2) - F(0) = 0 \iff c = 3b. \quad \square$$

Note that in this solution the actual values of $A_1$ and $A_2$ for the folium of Descartes, not asked for in the proposal, did not have to be calculated. Their common value is $3a^2/2$, and their separate calculation is given as exercises in many calculus textbooks, for example in [1]. More generally, we have the following problem (from St. John's College, 1882) as stated in [2]:

Trace the curve

$$x^{2n+1} + y^{2n+1} = (2n+1)ax^n y^n,$$

when $n$ is even, and when $n$ is odd, $n$ being a positive integer; and prove that the area of the loop is $(2n+1)a^2/2$. Prove that this is also the area between the infinite branches of the curve and the asymptote.

Also solved by G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta (second solution); BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANA-RAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer. A comment was received from O. BOTTEMA, Delft, The Netherlands.

*Editor's comment.*

The proposer had carefully worded his problem in the hope of eliciting from readers a solution like the one we have featured, in which the actual values of $A_1$ and $A_2$ did not have to be calculated. But all other solvers except Satyanarayana calculated $A_1$ and $A_2$ separately, thus transforming their solutions into laborious but straightforward calculus exercises.

REFERENCES

1. ?. Williamson, *An Elementary Treatise on the Integral Calculus*, Longmans, Green & Co., London, 1906, pp. 192, 218.

2. Joseph Edwards, *A Treatise on the Integral Calculus*, Chelsea, New York, 1955, Vol. I, p. 432.

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869. [1983: 209] *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

(a) Prove that every integer $n > 6$ can be expressed as the sum of two relatively prime integers both of which exceed 1.

(b) Prove that every integer $n > 11$ can be expressed as the sum of two composite positive integers.

I. *Solution to part (a) by Edwin M. Klein, University of Wisconsin-Whitewater.*

If $n = 2k+1 > 6$, then $n = 2 + (2k-1)$, and the two summands both exceed 1 and are relatively prime.

If $n = 4k > 6$ or $n = 4k+2 > 6$, then we have, respectively,

$$n = (2k-1) + (2k+1) \quad \text{or} \quad n = (2k-1) + (2k+3).$$

In each case, the two summands both exceed 1, and they are relatively prime since they are both odd and differ by a power of 2.

II. *Solution to part (b) by F.D. Hammer, Palo Alto, California.*

If $n > 11$, then either $n-8$ or $n-9$ is composite, and 8 and 9 are.

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay; CUPTIS COOPER, Central Missouri State University at Warrensburg; CLAYTON W. DODGE, University of Maine at Orono; F.D. HAMMER, Palo Alto, California (also part (a)); RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARK KANTROWITZ, student, Maimonides School, Brookline, Massachusetts; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; EDWIN M. KLEIN, University of Wisconsin-Whitewater (also part (b)); SIDNEY KRAVITZ, Dover, New Jersey; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, University of Wisconsin-Oshkosh; LAWRENCE SOMER, Washington, D.C.; JORDAN B. TABOV, Sofia, Bulgaria; KENNETH M. WILKE, Topeka, Kansas (two solutions); KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

*Editor's comment.*

Wilke found part (a), with solution, in Sierpiński [1]; and Klein found part (b), with solution, in Sierpiński [2]. In [1] Sierpiński also states and proves the following extension (Ex. 48*, pp. 4, 38-39):

Prove that every integer > 17 can be represented as a sum of three integers > 1 which are pairwise relatively prime, and show that 17 does not have this property.

REFERENCES

    1.  W. Sierpiński, *250 Problems in Elementary Number Theory*, American Elsevier, New York, 1970, pp. 4, 37-38, Ex. 47.

    2.  _____, *Elementary Theory of Numbers*, Warszawa, 1964, p. 120, Ex. 1.

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871. [1983: 241] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*
        In each of the following four (independent) cryptarithms, assign each X a different decimal digit to obtain decimal integers that make the cryptarithm arithmetically true.

    (a)  X·X·X·X = XX = X·X,
    (b)  X·X·X·X = XX simultaneously with X·X = X,
    (c)  X·X·X·X = XX simultaneously with X·X = X + X,
    (d)  X·X·X·X = XX = (X + X)·(X + X).

*Solution by Charles W. Trigg, San Diego, California.*
    Since 2·3·4·5 = 120 > 98, one of the digits in X·X·X·X must be 1; and since 1·4·5·6 = 120 > 98, another digit in X·X·X·X must be 2 or 3. With these restrictions, we easily find that there are exactly seven solutions with six distinct digits to the cryptarithm X·X·X·X = XX, which is common to the four parts of the problem. They, with the corresponding sets of four unused digits isolated, are:

$$1·2·3·9 = 54 \qquad 6, 7, 8, 0$$
$$1·2·4·7 = 56 \qquad 3, 8, 9, 0$$
$$1·2·6·7 = 84 \qquad 3, 5, 9, 0$$
$$1·3·4·5 = 60 \qquad 2, 7, 8, 9$$
$$1·3·4·6 = 72 \qquad 5, 8, 9, 0$$
$$1·3·4·8 = 96 \qquad 2, 5, 7, 0$$
$$1·3·5·6 = 90 \qquad 2, 4, 7, 8$$

Then, upon examining the sets of unused digits, we have the following unique solutions:

    (a)  1·3·4·6 = 72 = 8·9,
    (b)  1·3·5·6 = 90 with 2·4 = 8,
    (c)  1·3·4·5 = 60 with 2·8 = 7+9,
    (d)  1·3·5·6 = 90 = (2+4)·(7+8).

    Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; FAREED ALI, Grade 11 student, St. John's-Ravenscourt School, Winnipeg, Manitoba; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay; RICHARD I. HESS, Rancho Palos Verdes, California; MARK KANTROWITZ, student, Maimonides School, Brookline, Massachusetts; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; EDWIN M. KLEIN, University of Wisconsin-Whitewater; J.A. McCALLUM, Medicine Hat, Alberta; STEWART METCHETTE, Culver City,

California; GLEN E. MILLS, Pensacola Junior College, Florida; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer.

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872. ⌐1983: 241 : 1984: 18⌐ (Corrected) *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $T$ be a given triangle ABC with sides $a, b, c$ and circumradius $R$, and let P be any point in the plane of $T$. It is known [1] that there exists a triangle $T_0 = T_0(P)$, possibly degenerate, with sides $a \cdot PA$, $b \cdot PB$, and $c \cdot PC$. Find the locus of all the points P for which

$$PA \cdot PB \cdot PC \leq R \cdot R_0,$$

where $R_0 = R_0(P)$ is the circumradius of $T_0$. When does equality occur?

*Solution by the proposer.*

For any point P in the plane, let the sides of triangle $T_0(P)$ be

$$a_0 = a \cdot PA, \qquad b_0 = b \cdot PB, \qquad c_0 = c \cdot PC$$

and let $F_0$ be its area; then

$$16F_0^2 = -a_0^4 - b_0^4 - c_0^4 + 2b_0^2 c_0^2 + 2c_0^2 a_0^2 + 2a_0^2 b_0^2. \tag{1}$$

Now let $T_1 = A_1 B_1 C_1$ be the pedal triangle of point P with respect to the given triangle $T$, $a_1, b_1, c_1$ its sides, and $F_1$ its area. It is known ⌐2, p. 136⌐ that

$$a_1 = B_1 C_1 = \frac{a \cdot PA}{2R}, \text{ etc.,}$$

and so

$$a_0 = 2Ra_1, \qquad b_0 = 2Rb_1, \qquad c_0 = 2Rc_1. \tag{2}$$

From (1) and (2) now follows

$$F_0 = 4R^2 F_1. \tag{3}$$

It is also known [2, p. 139] that

$$F_1 = \frac{|R^2 - OP^2|}{4R^2} \cdot F,$$

where O is the circumcenter and $F$ the area of triangle $T$. Hence, from (3),

$$F_0 = |R^2 - OP^2| \cdot F. \tag{4}$$

Since $F_0 = a_0 b_0 c_0 / 4R_0$ and $F = abc/4R$, the inequality $PA \cdot PB \cdot PC \leq R \cdot R_0$ is equivalent to

$$F_0 \leq R^2 F, \tag{5}$$

and it follows from (4) that (5) holds just when $|R^2 - OP^2| \leq R^2$, that is, just when $OP \leq R\sqrt{2}$. The required locus therefore consists of the closed circular disk $\gamma$ with center 0 and radius $R\sqrt{2}$. Equality holds in (5) when P = 0 and when $OP = R\sqrt{2}$, and the inequality is reversed when P lies outside $\gamma$.

Also solved by M.S. KLAMKIN, University of Alberta; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India. A comment was received from J.T. GROENMAN, Arnhem, The Netherlands.

REFERENCES

1. M.S. Klamkin, "Triangle Inequalities from the Triangle Inequality", *Elemente der Mathematik*, Vol. 34 (1979), No. 3.

2. Roger A. Johnson, *Advanced Euclidean Geometry (Modern Geometry)*, Dover, New York, 1960.

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873. [1983: 241] *Proposed by W.R. Utz, University of Missouri-Columbia.*

    Show that the sequence $\{3n^2 + 3n + 1\}$, $n$ an integer, contains an infinite number of squares but only one cube.

*Solution by Edwin M. Klein, University of Wisconsin-Whitewater.*

It is straightforward to show that the sequence $(n_k, m_k)$ defined by

$$(n_0, m_0) = (0, 1), \quad (n_k, m_k) = (7n_{k-1} + 4m_{k-1} + 3, \ 12n_{k-1} + 7m_{k-1} + 6), \quad k = 1,2,3,\ldots,$$

provides an infinite number of integer solutions to the equation

$$(n+1)^3 - n^3 \equiv 3n^2 + 3n + 1 = m^2. \tag{1}$$

This sequence is given in Sierpiński [1, p. 99], where it is further stated (the proof being left to the reader) that these are the only solutions in natural numbers. Sierpiński solves (1) as a special case of the Pell equation $x^2 - 3y^2 = 1$, namely,

$$(2m)^2 - 3(2n+1)^2 = 1.$$

Furthermore, he adds, it has been proved that if the natural numbers $n$ and $m$ satisfy (1), then $m$ is the sum of the squares of two consecutive natural numbers. The first five nontrivial solutions are

$$
\begin{aligned}
&n_1 = 7, &&m_1 = 13 = 2^2 + 3^2, \\
&n_2 = 104, &&m_2 = 181 = 9^2 + 10^2, \\
&n_3 = 1455, &&m_3 = 2521 = 35^2 + 36^2, \\
&n_4 = 20272, &&m_4 = 35113 = 132^2 + 133^2, \\
&n_5 = 282359, &&m_5 = 489061 = 494^2 + 495^2.
\end{aligned}
$$

For the second part of the problem, if $(n+1)^3 - n^3 = m^3$, then $n = 0$ or $-1$ and $m = 1$, since it is well known that there are no integral solutions to the equation $x^3 + y^3 = z^3$ with $xyz \neq 0$ [1, p. 384].

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; W.J. BLUNDON, Memorial University of Newfoundland; CURTIS COOPER, Central Missouri State University at Warrensburg; the COPS of Ottawa (three solutions); J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; LAWRENCE SOMER, Washington, D.C.; STAN WAGON, Smith College, Northampton, Massachusetts; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

*Editor's comment.*

Several solvers used Pellian techniques to give $n_k$ and $m_k$ explicitly in terms of $k$. The most useful formulas, from which $n_k$ and $m_k$ can be quickly evaluated for given $k$ with a pocket calculator, were derived by Ahlburg:

$$n_k = \left[ \frac{(1+\sqrt{3})^{4k+2}}{2^{2k+3}\sqrt{3}} \right], \qquad m_k = \left[ \frac{(1+\sqrt{3})^{4k+2}}{2^{2k+3}} \right] + 1, \qquad k = 0,1,2,\ldots,$$

where the square brackets denote the greatest integer function.

R.C. Lyness has the pleasant and useful habit of including a problem each year with the Christmas cards he sends to his mathematical friends. It will be recalled that his Christmas 1982 problem led to the interesting article "The Mystery of the Double Sevens" in this journal [1983: 194-199]. His Christmas 1984 problem was the following:

*If the difference between two consecutive cubes is a square then it is the square of the sum of two [consecutive] squares.*

The editor takes it upon himself to suggest that readers send solutions to this problem (along with a request to be on his 1985 Christmas list) directly to

> R.C. Lyness
> 2 Godyll Road
> Southwold, Suffolk
> England IP18 6AJ.

REFERENCE

1. W. Sierpiński, *Elementary Theory of Numbers*, Warszawa, 1964.

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**874.** [1983: 241] *Proposed by the COPS of Ottawa.*

Let $S_n$ be the sum of the first $n$ primes. Show that for every $n$ there is at least one square between $S_n$ and $S_{n+1}$.

*Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

More generally, let $(x_n)$ be a sequence of real numbers such that

$$2 \leq x_1 < 4, \quad x_2 \geq x_1+1, \quad \text{and} \quad x_{n+1} \geq x_n+2, \quad n = 2,3,4,\ldots .$$

(In particular, the primes form such a sequence.) An easy induction shows that $x_n \geq 2n-1$ for $n \geq 1$, the inequality being strict for (at least) $n = 1$. We show that, for each $n \geq 1$, there is a natural number $m$ such that

$$S_n < m^2 < S_{n+1}, \tag{1}$$

where $S_k = x_1+x_2+\ldots+x_k$.

The property certainly holds for $n = 1$, since $S_1 < 2^2 < 5 \leq S_2$. We show that it also holds for all $n > 1$. Suppose, on the contrary, that there exist natural numbers $n > 1$ and $t$ such that

$$t^2 \leq S_n < S_{n+1} \leq (t+1)^2. \tag{2}$$

Then

$$S_{n+1} - S_n = x_{n+1} \leq (t+1)^2 - t^2 = 2t + 1.$$

From $2t+1 \geq x_{n+1} \geq 2n+1$ now follows $t \geq n$.

If $t = n$, then

$$S_{n+1} = x_1+x_2+\ldots+x_{n+1} > 1+3+\ldots+(2n+1) = (n+1)^2 = (t+1)^2,$$

contradicting (2).

Suppose now that $t > n$. From $x_{n+1} \leq 2t+1$ we obtain successively

$$x_n \leq x_{n+1} - 2 \leq 2t - 1,$$
$$x_{n-1} \leq x_n - 2 \leq 2t - 3,$$

etc., and finally

$$x_2 \leq 2t - 2n + 3.$$

Moreover, $x_1 \leq x_2-1 \leq 2t-2n+2$, and since $2t-2n+2 \geq 4$, it follows that

$$x_1 < 2t-2n+2 \leq (t-n+1)^2 = 1+3+5+\ldots+(2t-2n+1).$$

Therefore

$$S_n = x_1 + x_2 + \ldots + x_n$$
$$< 1 + 3 + \ldots + (2t-2n+1) + (2t-2n+3) + \ldots + (2t-3) + (2t-1)$$
$$= t^2,$$

again contradicting (2).

Hence for each $n \geq 1$ there is a natural number $m$ satisfying (1).

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; RICHARD I. HESS, Rancho Palos Verdes, California; EDWIN M. KLEIN, University of Wisconsin-Whitewater; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, University of Wisconsin-Oshkosh; LAWRENCE SOMER, Washington, D.C.; STAN WAGON, Smith College, Northampton, Massachusetts; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposers (two solutions).

*Editor's comment.*

The problem is not new, but our featured solution is, as far as we know, the most general one published so far. The following references were all supplied by readers. The problem was originally proposed by R.S. Luthar in the *Monthly*, and a solution by E.W. Trost was published in [1]. It appeared subsequently (without solution) in Apostol [2], in Roberts [3] (with solution), and in Honsberger [4] (with a solution by Ivan Niven).

Wagon asked the following question about the sequence $(p_n)$: Is it true that, for $n$ sufficiently large, there is a cube between $S_n$ and $S_{n+1}$? Readers should try to answer this question for the more general sequence $(x_n)$ used in our featured solution.

REFERENCES

1. E.W. Trost, Solution to Problem E 2164 (proposed by R.S. Luthar), *American Mathematical Monthly*, 76 (1969) 1151.

2. Tom M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976, p. 102.

3. Joe Roberts, *Elementary Number Theory*, The MIT Press, Cambridge, 1977, pp. 23, 27s-28s.

4. Ross Honsberger, *Mathematical Morsels*, Mathematical Association of America, 1978, pp. 221-222.

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875. [1983: 241] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Can a square be dissected into three congruent nonrectangular pieces?

*Comment by Stan Wagon, Smith College, Northampton, Massachusetts.*

I first heard about this problem from Ron Graham in the following form: Suppose $p$ is an odd prime. Prove that a square cannot be divided into $p$ congruent polygons except by using $p$ horizontal strips or $p$ vertical strips.

He also said that he had seen a proof of the case $p = 3$ on a table napkin. Perhaps one of your readers will find the table napkin proof. Or perhaps it will gain

the status of Fermat's Last Theorem!

*Editor's comment.*

Lunch, anyone? And try not to get any ketchup on your table napkin.

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876, ⌜1983: 242⌝ *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with sides $a, b, c$, and let $K_a, K_b, K_c$ be the circles with centers A, B, C, respectively, and radii $\lambda\sqrt{bc}$, $\lambda\sqrt{ca}$, $\lambda\sqrt{ab}$, respectively, where $\lambda \geq 0$. Find the locus of the radical center of $K_a$, $K_b$, $K_c$ as $\lambda$ ranges over the nonnegative real numbers.

*Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India (revised by the editor).*

Let $O$ and $P$ be the circumcentre and circumradius, respectively, of triangle ABC. We introduce a coordinate system with origin $O$ and $x$-axis parallel to BC, as shown in the figure. If L is the projection of A upon BC, then the directed angle $(AO, AL) = C-B$. The coordinates of the vertices are then
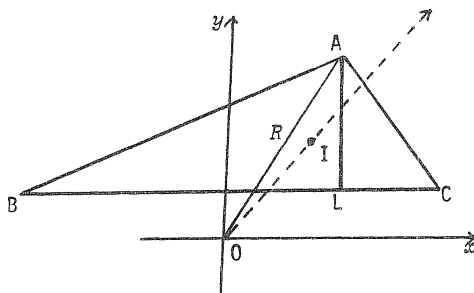


$A = (R\sin(C-B), R\cos(C-B))$,   $B = (-R\sin A, -R\cos A)$,   $C = (R\sin A, -R\cos A)$.

The desired locus consists of all the points $P = P(\lambda)$ whose powers with respect to $K_a, K_b, K_c$ are equal, that is, all points P such that

$$PA^2 - \lambda^2 bc = PB^2 - \lambda^2 ca = PC^2 - \lambda^2 ab. \tag{1}$$

Thus P lies on the locus if and only if

$$PB^2 - PC^2 = \lambda^2 a(c-b) \quad\text{and}\quad PC^2 - PA^2 = \lambda^2 b(a-c).$$

For $P = (x, y)$, this system becomes

$$\begin{cases} ux &= \lambda^2 a(c-b) \\ vx + wy &= \lambda^2 b(a-c), \end{cases} \tag{2}$$

where $u, v, w$ are the constants

$$u = 4R\sin A, \quad v = 2R\{\sin(C-B) - \sin A\}, \quad w = 2R\{\cos(C-B) + \cos A\}.$$

Clearly $u \neq 0$, and

$$w = 4R\cos\frac{A-B+C}{2}\cos\frac{A+B-C}{2} = 4R\cos(\tfrac{\pi}{2}-B)\cos(\tfrac{\pi}{2}-C) = 4R\sin B\sin C \neq 0;$$

hence $uw \neq 0$ and the system $(2)$ has a unique solution $(x,y)$ for every $\lambda \geq 0$. In particular, for $\lambda = 0$ we get $(x,y) = (0,0)$, so $P(0) = 0$. We now consider two cases.

*Case 1.* ABC is equilateral. Then, from $(2)$, $(x,y) = (0,0)$ for every $\lambda \geq 0$, so in this case the desired locus consists of the single point 0.

*Case 2.* ABC is not equilateral. If $b \neq c$, we see from (2) that $|x| \to \infty$ as $\lambda \to \infty$ and $x$ has the same sign for all $\lambda > 0$; and if $b = c$, then $x = 0$, $a \neq c$, $|y| \to \infty$ as $\lambda \to \infty$, and $y$ has the same sign for all $\lambda > 0$. Moreover, eliminating $\lambda$ from $(2)$ yields a linear equation in $x$ and $y$. Our conclusion from all this is that in this case the desired locus is a ray with initial point 0.

If I is the incentre of the triangle, it is not hard to show that

$$IA^2 - bc = IB^2 - ca = IC^2 - ab \; (= -16R^2 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}).$$

It now follows from $(1)$ that $P(1) = I$, and so the required locus in this case is the ray $OI$ (the dotted ray in the figure).

Also solved by JORDI DOU, Barcelona, Spain; and the proposer.

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## FAREWELL

This is the 100th issue of our journal and, for the first time, my name does not appear on the masthead as its Managing Editor. A serious illness, which developed rapidly, has compelled me to relinquish this duty, a duty which for ten years has proved a most rewarding experience, and one in which I have been sustained by many friends and colleagues who became part of the world-wide community of *Crux* problem solvers. Without their steady support, *Crux* could not have continued in existence.

Professor Kenneth S. Williams, of Carleton University, has generously volunteered to step into the breach at short notice and take over my responsibilities as Managing Editor. Professor Williams has undertaken this task not as a favour to me but as a service to the mathematical community. I am happy, grateful, and relieved that, thanks to his generosity, the continued publication of *Crux* is assured. As Managing Editor, Professor Williams will operate from the *Crux* office at Algonquin College. All correspondence intended for him should therefore be addressed as follows:

> Kenneth S. Williams
> Managing Editor, *Crux Mathematicorum*
> Algonquin College
> 200 Lees Ave.
> Ottawa, Ontario
> Canada K1S 0C5

Above all, it has been a great privilege to collaborate with my colleague Léo Sauvé, under whose editorship *Crux* has become a journal of international reputation. May it long continue to thrive!

And now I bid you all a fond farewell.

F.G.B. MASKELL

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## TITLES OF ARTICLES AND NOTES

## PROBLEMS AND SOLUTIONS

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