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Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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THE OLYMPIAD CORNER

No. 120

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The set of problems we give this number are the problems of the *20th Austrian Mathematical Olympiad.* My thanks to Professor Walther Janous, Innsbruck, Austria for sending them along.

20TH AUSTRIAN MATHEMATICAL OLYMPIAD

Advanced Level

1st Day

1. Let S_n be the set of all the 2^n numbers of the type

$$2 \pm \sqrt{2 \pm \sqrt{2 \pm \ldots}}\ .$$

Here the number 2 appears $n + 1$ times and there are n square roots preceded by $+1$ or -1.

(a) Show that all members of S_n are real.

(b) Compute the product P_n of all members of S_n.

2. Determine all triples (a,b,c) of whole numbers such that $a \cdot b \cdot c = 1989$ and $a + b - c = 89$.

3. Show that it is possible to situate eight parallel planes at equal distances such that every plane determined contains precisely one vertex of a (fixed) cube. Also find with proof the number of different solutions with the required property (each solution consists of eight planes).

2nd Day

4. We are given a circle κ and two non–parallel tangents t_1, t_2 at points $P_1, P_2 \in \kappa$. Let A_0 be the point of intersection of t_1 and t_2. How must the point P_3 be chosen on the smaller arc P_1P_2 so that $\Delta A_0A_1A_2$ has maximum area? (Here A_1 and A_2 are the points of intersection of tangent t_3 (at P_3) with the intervals A_0P_1 and A_0P_2, respectively.)

5. Determine all real solutions of the system

$$x^2 + 2yz = x$$
$$y^2 + 2xz = y$$
$$z^2 + 2xy = z\ .$$

6. Determine all functions $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that for $n \in \mathbb{N}_0$

$$f(f(n)) + f(n) = 2n + 6$$

($\mathbb{N}_0 = \{0,1,2,\cdots\}$).

20TH AUSTRIAN MATHEMATICAL OLYMPIAD
Final Round, Beginner's Level

1. Let a, b, c, d, where $a \le b \le c \le d$, be natural numbers such that $a + b + c + d = 30$. Find the maximum value of the product $P = a\cdot b\cdot c\cdot d$.

2. Let a and b be non–negative real numbers such that $a^2 + b^2 = 4$. Show that

$$\frac{ab}{a + b + 2} \le \sqrt{2} - 1\ ,$$

and determine when equality occurs.

3. Let a be a real number. Show that if the equation $x^2 - ax + a = 0$ has two real solutions x_1 and x_2, then

$$x_1^2 + x_2^2 \ge 2(x_1 + x_2)\ .$$

4. Show that for any triangle each exradius is less than four times the circumradius.

* * *

The first solution in response to the appeal for solutions to "old" problems is one I missed using last month. Andy also submitted a solution to F2414. My apologies.

F2412. [1983: 237] *From Középiskolai Matematikai Lapok.*

We have d cartons and one box. The cartons are numbered from 1 to d and some of them (maybe all) contain balls. We want to collect the contents of all the cartons in the box. Carton i may be emptied if it contains exactly i balls, and this is done in such a way that one ball is placed into the box and the remaining $i - 1$ balls are placed one by one into the cartons $1,2,\cdots,i - 1$, respectively. For which n is it possible to place n balls into an appropriate number of cartons so that they all could be collected?

Solution by Andy Liu, The University of Alberta, Edmonton, Alberta.

The task is possible for all n. We use induction on n. The case $n = 1$ is trivial. Suppose n balls are distributed in such a way that the task is possible. We add the $(n + 1)$st ball and redistribute some of the others as follows. Let the kth carton be the first one that is empty. We add the $(n + 1)$st ball to this carton and transfer to it one ball from each of the first $k - 1$ cartons. In our first move we empty the kth carton by the prescribed procedure. This is allowed as it contains exactly k balls. After this first move, the distribution of the remaining balls is exactly the same as before and the task can be completed by the induction hypothesis.

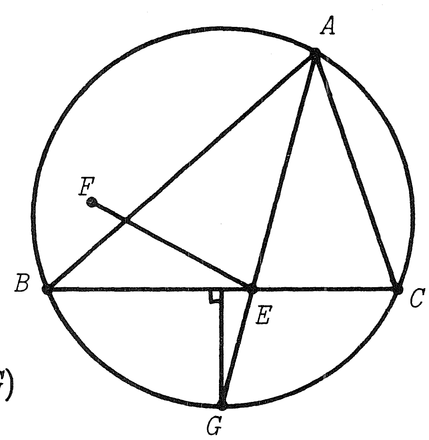
*

The next three solutions are to problems from *Középiskolai Matematikai Lapok* [1984: 75].

GY2147. Construct a triangle ABC (with sides a, b, c), given the side a, the ratio $b{:}c$, and the difference of the angles B and C.

Solution by Andy Liu, The University of Alberta, Edmonton, Alberta.

Construct $BC = a$, and let E be the point of BC such that $BE/CE = c/b > 1$ (without loss of generality). Draw EF so that $\angle BEF = \angle C - \angle B$. Let the bisector of $\angle CEF$ intersect at G the perpendicular bisector of BC, and at A the circumcircle of BCG. We have $\angle BAG = \angle CAG$. Hence

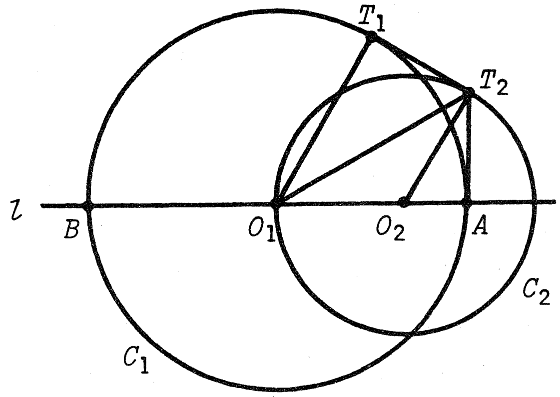


$$\begin{aligned}\angle BEF &= \angle BEA - \angle CEA\\ &= \angle C + \angle CAG - (\angle B + \angle BAG)\\ &= \angle C - \angle B\end{aligned}$$

as desired. Finally $\dfrac{AB}{AC} = \dfrac{BE}{CE} = \dfrac{c}{b}$ since AE bisects $\angle BAC$.

GY2148. Given are a circle C_1 with centre O_1 and a line l passing through the centre. Consider all circles C_2 which pass through O_1, have their centres on l, and intersect C_1 in two points. Draw the common tangents of these circles with C_1. Find the locus of the points of contact with C_2 of these tangents for all positions of C_2.

Solutions by R.K. Guy, The University of Calgary, and by Andy Liu, The University of Alberta, Edmonton.

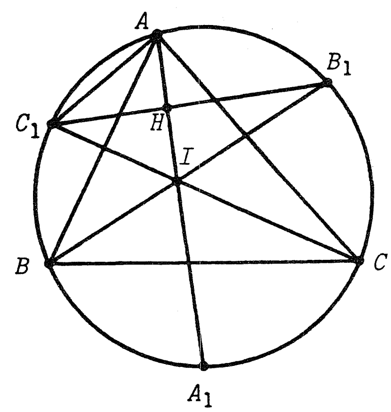


Let AB be the diameter of C_1 on l. We claim that the desired locus consists of the two tangents of C_1 at A and B. Draw an arbitrary C_2 (on the side of A, as shown) and let it cut the tangent of C_1 at A at the point T_2. Let T_1 be the point of contact of the other tangent from T_2 to C_1. Let the line through T_2 parallel to T_1O_1 cut l at O_2. It is sufficient to prove that O_2 is the centre of C_2. Now $\angle O_2O_1T_2 = \angle T_1O_1T_2 = \angle O_1T_2O_2$. Hence $O_2O_1 = O_2T_2$. Since the centre of C_2 lies on l, O_2 is indeed the centre of C_2.

[*Editor's note*: R.K. Guy's solution included a second approach, first inverting with respect to C_1 so that C_1 and l invert into themselves.]

GY2149. Let I be the incentre of a triangle ABC. The lines AI, BI, CI meet the circumcircle of the triangle in A_1, B_1, C_1, respectively. Prove that $AA_1 \perp B_1C_1$.

Solutions by R.K. Guy, The University of Calgary, and by Andy Liu, The University of Alberta, Edmonton.



Let AA_1 cut B_1C_1 at H. Note that

$$\angle BAC_1 = \angle BCC_1 = \frac{1}{2}\angle BCA\ ,$$

$$\angle AC_1B_1 = \angle ABB_1 = \frac{1}{2}\angle ABC\ ,$$

and $\angle BAA_1 = \frac{1}{2}\angle BAC$. It follows that $\angle C_1AH + \angle AC_1H = 90^\circ$ and $\angle AHC_1 = 90^\circ$ as desired.

*

We now turn to solutions to problems given in the February 1989 number of the Corner.

4. [1989: 33] *1987 Bulgarian Olympiad.*

The sequence $x_1, x_2, \cdots$ is defined by the equalities $x_1 = x_2 = 1$ and

$$x_{n+2} = 14x_{n+1} - x_n - 4,\ n \geq 1.$$

Prove that each member of the given sequence is a perfect square.

Solutions by Mathew Englander, Toronto, Ontario; Robert E. Shafer, Berkeley, California; Bob Prielipp, The University of Wisconsin–Oshkosh; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Consider the sequence defined by $y_1 = y_2 = 1$ and $y_{n+2} = 4y_{n+1} - y_n$. We shall prove by induction that

$$x_n = y_n^2 \ , \quad x_{n+1} = y_{n+1}^2 \ , \quad \text{and} \quad 4y_{n+1}y_n = x_{n+1} + x_n + 2 \ .$$

The statements clearly hold when $n = 1$. So assume that for some $k \geq 1$

$$x_k = y_k^2 \ , \quad x_{k+1} = y_{k+1}^2 \ , \quad \text{and} \quad 4y_{k+1}y_k = x_{k+1} + x_k + 2 \ .$$

Now

$$\begin{aligned} 4y_{k+2}y_{k+1} &= 4(4y_{k+1} - y_k)y_{k+1} = 16y_{k+1}^2 - 4y_ky_{k+1} \\ &= 16x_{k+1} - x_{k+1} - x_k - 2 = (14x_{k+1} - x_k - 4) + x_{k+1} + 2 \\ &= x_{k+2} + x_{k+1} + 2 \ , \end{aligned}$$

and

$$\begin{aligned} y_{k+2}^2 &= (4y_{k+1} - y_k)^2 = 16y_{k+1}^2 - 8y_{k+1}y_k + y_k^2 \\ &= 16x_{k+1} - 2(x_{k+1} + x_k + 2) + x_k = 14x_{k+1} - x_k - 4 = x_{k+2} \ . \end{aligned}$$

This completes the inductive step, and the result follows.

[*Editor's note*: Robert Shafer gave the following generalization. Let

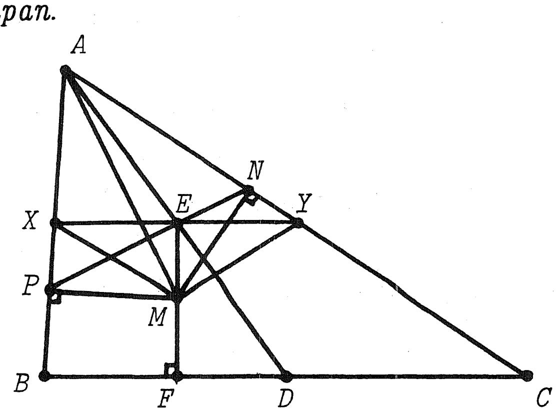
$$y_0 = u^2, \ y_1 = (ux + t)^2 \text{ and } y_{n+2} = (4x^2 - 2)y_{n+1} - y_n - 2(u^2x^2 - t^2 - u^2).$$

Then $y_n(x,t)$ is the square of a polynomial with integer coefficients. The example has $x = 2$, $t = -1$, $u = 1$.]

5. [1989: 33] *1987 Bulgarian Olympiad.*

From a point E on the median AD of the triangle ABC, the perpendicular EF is dropped to the side BC. From a point M, lying on EF, two perpendiculars MN and MP are dropped to the sides AC and AB respectively. If the points N, E, and P are collinear show that the point M lies on the internal bisector of the angle BAC.

Solution by Toshio Seimiya, Kawasaki, Japan.



Draw a line through E parallel to BC and intersecting AB, AC at X, Y respectively. As $XY \perp EF$ and $MP \perp AB$ we get that the four points E, X, P and M are concyclic. Similarly $ENYM$ are concyclic. Therefore $\angle PXM = \angle PEM = \angle NYM = \angle AYM$. Hence the four points A, X, M, Y are concyclic. Therefore

$$\angle XAM = \angle XYM\ , \qquad \angle MAY = \angle MXY\ . \tag{1}$$

As $XY \| BC$, we have $XE{:}BD = AE{:}AD = EY{:}DC$. Since $BD = DC$, we get $XE = EY$. It follows that $\Delta EMX \equiv \Delta EMY$ so we have $\angle EXM = \angle EYM$. Hence from (1) we obtain $\angle XAM = \angle MAY$, i.e. M lies on the internal bisector of $\angle ABC$.

*

The remaining solutions this month are to problems of the 1986 *Swedish Mathematical Competition (final round)* [1989: 34].

1. Prove that the polynomial

$$x^6 - x^5 + x^4 - x^3 + x^2 - x + \frac{3}{4}$$

has no real root.

Solutions by Indy Lagu, The University of Calgary; H.M. Lee, John Abbott College, Ste. Anne de Bellevue, Quebec; Bob Prielipp, University of Wisconsin–Oshkosh; Michael Rubenstein, student, Princeton University; M. Selby, University of Windsor, Ontario; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $f(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 3/4$. We will show that $f(x) > 0$ for all x. Now

$$f(x) + f(-x) = 2(x^6 + x^4 + x^2 + 3/4) \tag{1}$$

and

$$f(x) - f(-x) = -2(x^5 + x^3 + x)\ . \tag{2}$$

We see that if $x \le 0$, (1) is positive and (2) is nonnegative so the sum $2f(x)$ is positive. Now writing

$$f(x) = (x^6 - x^5) + (x^4 - x^3) + (x^2 - x) + \frac{3}{4}$$

it is easy to see that for $x \ge 1$, each bracketed term is nonnegative so $f(x) > 0$ for $x \ge 1$. The case $x \in (0,1)$ remains. But if $0 < x < 1$ then

$$f(x) = -x(1 - x)(x^4 + x^2 + 1) + \frac{3}{4} > 0$$

since $0 < x(1 - x) \le 1/4$ and $x^4 + x^2 + 1 < 3$. Thus $f(x) > 0$ for all x.

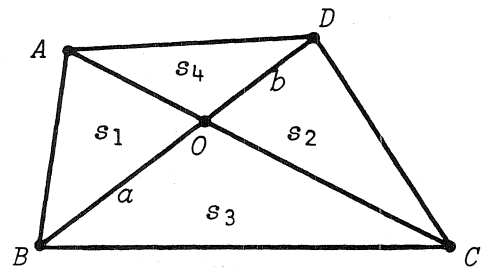
2. The diagonals AC and BD of the quadrilateral $ABCD$ intersect at the interior point O. The areas of the triangles AOB and COD are s_1 and s_2, respectively, and the area of the quadrilateral is s. Prove that

$$\sqrt{s_1} + \sqrt{s_2} \le \sqrt{s}\ .$$

Also prove that equality holds if and only if the lines AB and CD are parallel.

Solutions by George Evagelopoulos, Law student, Athens, Greece, and by Michael Rubinstein, student, Princeton University, Princeton, N.J.

Let the area of ΔAOD be s_4 and the area of ΔBOC be s_3. Let $BO = a$ and $OD = b$. Now $\sqrt{s_1} + \sqrt{s_2} \le \sqrt{s}$ is equivalent to



$$s_1 + 2\sqrt{s_1 s_2} + s_2 \le s = s_1 + s_2 + s_3 + s_4 ,$$

that is,

$$2\sqrt{s_1 s_2} \le s_3 + s_4 .$$

But

$$\frac{s_4}{s_1} = \frac{b}{a} \quad \text{and} \quad \frac{s_3}{s_2} = \frac{a}{b} .$$

Thus, applying the geometric–arithmetic mean inequality,

$$s_3 + s_4 = \frac{as_2}{b} + \frac{bs_1}{a} \ge 2\sqrt{\frac{as_2}{b}.\frac{bs_1}{a}} = 2\sqrt{s_1 s_2}$$

and the result falls out. Equality holds if and only if $as_2/b = bs_1/a$, or

$$\frac{a}{b} = \frac{s_1/a}{s_2/b} = \frac{AO}{CO} ,$$

i.e., $AB \| CD$.

3. Let n be a positive integer greater than or equal to 3, and let S be the set of all pairs (a,b) of positive integers such that $1 \le a < b \le n$. Prove that the sets

$$\{(a,b) \in S : b < 2a\}$$

and

$$\{(a,b) \in S : b > 2a\}$$

have the same number of elements.

Solutions by Michael Rubinstein, student, Princeton University, Princeton, N.J., and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

For each fixed $b \ge 3$, let

$$G_b = \{(a,b) : b > 2a\} , \quad L_b = \{(a,b) : b < 2a\} , \quad G = \bigcup_{b\ge 3} G_b , \quad L = \bigcup_{b \ge 3} L_b .$$

Now if $b = 2q$ is even, then the map $(i,b) \longmapsto (q + i,b)$, $i = 1,2,\cdots,q - 1$, is clearly a bijection from G_b to L_b. Similarly if $b = 2q + 1$ is odd then $(i,b) \longmapsto (q + i,b)$, for $i = 1,2,\cdots,q$, is still a bijection. Hence

$$|G| = \sum_{b=3}^{n} |G_b| = \sum_{b=3}^{n} |L_b| = |L| .$$

Furthermore since

$$|S| = 1 + 2 + \cdots + (n - 1) = \tfrac{1}{2}n(n - 1) ,$$

we easily find that

$$|G| = |L| = \begin{cases} \frac{1}{2}\left(\frac{n(n-1)}{2} - \frac{n}{2}\right) = \frac{1}{4}n(n-2) & \text{if } n \text{ is even },\\ \frac{1}{2}\left(\frac{n(n-1)}{2} - \frac{n-1}{2}\right) = \frac{1}{4}(n-1)^2 & \text{if } n \text{ is odd }. \end{cases}$$

4. Show that the only positive solution of the system

$$x + y^2 + z^3 = 3$$
$$y + z^2 + x^3 = 3$$
$$z + x^2 + y^3 = 3$$

is $x = y = z = 1$.

Solutions by Michael Rubinstein, student, Princeton University, and by M. Selby, The University of Windsor, Ontario.

We argue by contradiction. If not all of x, y, z are equal to 1, at least one of them is strictly greater than 1. By the symmetry there is no loss in assuming $x > 1$. We consider three cases.

Case (i): $y > 1$. Then $y^2 < y^3$, so that $x + z^3 > z + x^2$. But $x, y > 1$ imply $z < 1$ so that $x + z^3 < z + x^2$, a contradiction.

Case (ii): $y < 1$. Then $z^2 + x^3 < x + z^3$. Now $z \leq 1$ gives $z^2 + x^3 > x + z^3$, a contradiction. Also $z > 1$ cannot arise by the argument of case (*i*).

Case (iii): $y = 1$. Then $x + z^3 = 2 = x^3 + z^2 = x^2 + z$. But then $z^2 < z < z^3$, a contradiction. The only possibility is when $x = y = z = 1$.

[*Editor's note*: Selby's solution was less elementary as it turned on a clever use of the Cauchy–Schwarz inequality.]

6. The union of a finite number of intervals cover the interval [0,1]. Show that one can choose among these intervals pairwise disjoint intervals of total length at least 1/2.

Solution by Michael Rubinstein, student, Princeton University, Princeton, New Jersey.

We prove the result by induction on the number n of intervals. The result is trivial if $n = 1$. Assume that the result is true for any covering by n intervals. Suppose [0,1] is covered by intervals $I_1, \cdots, I_{n+1}$. There are three possibilities to be analyzed.

(a) At least one of the intervals is completely covered by the other intervals.

(b) There exist disjoint adjacent intervals.

(c) Neither (a) or (b): each interval partially overlaps 2 intervals, except possibly $[0,a]$ and $[b,1]$ which may only partially overlap one interval.

Case (a). Since one interval is completely covered by other intervals, we can choose n intervals which cover $[0,1]$, and we are done by induction.

Case (b). Without loss I_n and I_{n+1} are disjoint but touching. For the moment consider them to be one interval I_n'. Then $I_1,\cdots,I_{n-1}$, I_n' cover $[0,1]$, and the induction hypothesis yields a subcollection of disjoint intervals with total length at least $1/2$. Of course if I_n' is used we are free to use both I_n and I_{n+1} giving the desired collection.

Case (c). In this case the intervals may be assumed to be listed in left to right order (by the "left end") of the intervals. Now the intervals $I_1,I_3,I_5,\cdots$ are pairwise disjoint as are $I_2,I_4,I_6,\cdots$. As the total length of all the intervals is at least 1 we must have that either the sum of the lengths of the intervals in the first group or the sum for the second is at least $1/2$.

This completes the inductive step and completes the proof.

* * *

This is all the room we have this month. Send me your problems and nice solutions.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **July 1, 1991,** *although solutions received after that date will also be considered until the time when a solution is published.*

1591. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let D be a point on the side BC of a triangle ABC. Suppose that $\overline{AC} = \overline{BD}$, $\angle ADC = 30^\circ$, and $\angle ACB = 48^\circ$. Calculate angle B.

1592. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

If P is a monic polynomial of degree $n > 1$, having n negative roots (counting multiplicities), show that

$$P'(0)P(1) \geq 2n^2P(0) ,$$

and find conditions for equality.

1593. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Let the rigid triangle ABC move in the Cartesian plane such that B moves along the y axis and C moves along the x axis. Then it is well known that A will describe an ellipse $\mathcal{E}$. Find all points A' of the plane such that the ellipse described by A' when $\Delta A'BC$ moves as above is congruent to $\mathcal{E}$.

1594. *Proposed by Murray S. Klamkin, University of Alberta.*

Express

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 - 4x_1x_2x_3x_4$$

as a sum of squares of rational functions with real coefficients. (By the A.M.–G.M. inequality, this polynomial is nonnegative for all real values of its variables, and so by a theorem of Hilbert it can be so expressed.)

1595. *Proposed by Isao Ashiba, Tokyo, Japan.*

P is a variable point on the circumcircle of a triangle ABC. Show that

$$PA^2\sin 2A + PB^2\sin 2B + PC^2\sin 2C$$

is constant.

1596. *Proposed by Mark Kisin, student, Monash University, Clayton, Australia.*

Given an integer $n \geq 1$, what is the least integer $f(n)$ such that, from any collection of $f(n)$ integers, one can always choose n of them so that their sum is divisible by n?

1597. *Proposed by Jordi Dou, Barcelona, Spain.*

Given three lines a_1, a_2, a_3 and a point S, find the line s through S so that the lines s_1, s_2, s_3 symmetric to s with respect to a_1, a_2, a_3, respectively, are concurrent.

1598*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $\lambda > 0$. Determine the maximum constant $C = C(\lambda)$ such that for all non-negative real numbers x_1, x_2 there holds

$$x_1^2 + x_2^2 + \lambda x_1 x_2 \geq C(x_1 + x_2)^2 .$$

1599. *Proposed by Milen N. Naydenov, Varna, Bulgaria.*

A convex quadrilateral with sides a, b, c, d has both an incircle and a circumcircle. Its circumradius is R and its area F. Prove that

$$abc + abd + acd + bcd \leq 2\sqrt{F}(F + 2R^2).$$

1600. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, and Wan–Di Wei, Sichuan University, Chengdu, China.*

Find the number of unordered triples $\{a,b,c\}$ of positive integers such that $\mathrm{lcm}(a,b,c) = 1600$.

* * *

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1394. [1988: 302; 1990: 46] *Proposed by Murray S. Klamkin, University of Alberta.*

If x, y, $z > 0$, prove that

$$\sqrt{y^2 + yz + z^2} + \sqrt{z^2 + zx + x^2} + \sqrt{x^2 + xy + y^2} \geq 3\sqrt{yz + zx + xy} .$$

IV. *Comments by the proposer.*

More generally, let $\mathbf{A}_1,\mathbf{A}_2,\cdots,\mathbf{A}_n$ be n given coplanar vectors concurrent at point O and with respective lengths $x_1,x_2,\cdots,x_n$. Consider the polygon $\mathcal{P} = A_1A_2\cdots A_n$ whose vertices are the endpoints of the given vectors. Letting $\angle A_iOA_{i+1} = \alpha_i$, $i = 1,2\cdots,n$, with $A_{n+1} = A_1$, the perimeter and area of $\mathcal{P}$ are given respectively by

$$p = \sum_{i=1}^{n} \sqrt{x_i^2 - 2x_ix_{i+1}\cos\alpha_i + x_{i+1}^2} ,$$

$$F = \frac{1}{2}\left|\sum_{i=1}^{n} x_ix_{i+1}\sin\alpha_i\right| .$$

By the isoperimetric inequality the ratio of F to p^2 is a maximum when $\mathcal{P}$ is regular. Hence

$$\frac{p^2}{F} \geq 4n \tan\frac{\pi}{n}\,.$$

Letting $n = 3$, we obtain the inequality given by Janous on [1990: 47].

There are also generalizations to n-dimensional space. Here we consider concurrent vectors $\mathbf{A}_0,\mathbf{A}_1,\cdots,\mathbf{A}_n$ equally inclined to each other and with respective lengths $x_0,x_1,\cdots,x_n$. As is known, the cosine of the angle between each pair of vectors is $-1/n$. The endpoints of the vectors will form a simplex $\mathcal{S}$ of total edge length and volume given respectively by

$$E = \sum_{i<j} \sqrt{x_i^2 + \frac{2x_ix_j}{n} + x_j^2}\ ,$$

$$V = V_0x_0x_1\cdots x_n \sum_{i=0}^{n} \frac{1}{x_i}\ ,$$

where V_0 is the volume of an n-dimensional regular pyramid formed by n unit vectors equally inclined at angles $\arccos(-1/n)$ ($n + 1$ of these pyramids form a regular n-dimensional simplex of side $\sqrt{2 + 2/n}$). The isoperimetric inequality here is

$$\sum_{i<j} \sqrt{x_i^2 + \frac{2x_ix_j}{n} + x_j^2} \;\geq\; k\left(x_0x_1\cdots x_n \sum_{i=0}^{n} \frac{1}{x_i}\right)^{1/n}. \tag{1}$$

Since there is equality here when the simplex is regular ($x_i = 1$ for all i), the constant k is given by

$$k = \frac{\binom{n+1}{2}\sqrt{2 + 2/n}}{(n + 1)^{1/n}}\,.$$

For $n = 2$, (1) reduces to the original proposed inequality.

One need not have the angles between pairs of vectors to be constant. However in this case one must include the not too simple compatibility conditions for these $\binom{n+1}{2}$ angles. These conditions can be obtained by ensuring that there exists an n-dimensional simplex of oriented sides of lengths $\sin\theta_{ij}/2$, where the θ_{ij}'s are the angles between pairs of the vectors. The necessary and sufficient conditions for this are given in terms of the Cayley–Menger determinant [1]. For $n = 2$, the condition is just that the sum of the three angles is 2π.

For generalizations in a different direction, we consider

$$\sum_{i<j} \sqrt{x_i^2 + x_i x_j + x_j^2} \geq \alpha \left(\sum_{i<j} x_i x_j \right)^{1/2}, \tag{2}$$

$$\sum_{i<j} \sqrt{x_i^2 + x_i x_j + x_j^2} \geq \beta \left(\sum_{i=1}^{n} x_i x_{i+1} \right)^{1/2}, \tag{3}$$

and

$$\sum_{i=1}^{n} \sqrt{x_i^2 + x_i x_{i+1} + x_{i+1}^2} \geq \gamma \left(\sum_{i=1}^{n} x_i x_{i+1} \right)^{1/2}, \tag{4}$$

where $x_1, x_2, \cdots, x_n$ $(x_{n+1} = x_1)$ are arbitrary positive real numbers and α, β, γ are constants depending only on n.

We prove that (2) is valid for

$$\alpha = \sqrt{3\binom{n}{2}} ,$$

with equality if all the x_i's are equal. Since

$$2\sqrt{x_i^2 + x_i x_j + x_j^2} \geq \sqrt{3}(x_i + x_j) ,$$

it suffices to show that

$$\sqrt{3}(n-1) \sum_{i=1}^{n} x_i \geq 2\alpha \left(\sum_{i<j} x_i x_j \right)^{1/2} .$$

The latter is the known Maclaurin inequality [2]

$$\frac{1}{n} \sum_{i=1}^{n} x_i \geq \left(\sum_{i<j} x_i x_j \bigg/ \binom{n}{2} \right)^{1/2} .$$

Proceeding in the same way, (3) will be valid if

$$\sqrt{3}(n-1) \sum_{i=1}^{n} x_i \geq 2\beta \left(\sum_{i=1}^{n} x_i x_{i+1} \right)^{1/2} .$$

If there is to be equality for the x_i's equal, the latter becomes

$$\sum_{i=1}^{n} x_i \geq \sqrt{n} \left(\sum_{i=1}^{n} x_i x_{i+1} \right)^{1/2} ,$$

where $\beta = (n-1)\sqrt{3n}/2$. This inequality is valid for $n = 3$ and 4, being equivalent to

$$x_1^2 + x_2^2 + x_3^2 \geq x_1 x_2 + x_2 x_3 + x_3 x_1 \quad \text{and} \quad (x_1 + x_3 - x_2 - x_4)^2 \geq 0 ,$$

respectively, but is not valid for $n > 4$. For $n > 3$, we do have

$$\sum_{i=1}^{n} x_i \ \geq\ 2\left(\sum_{i=1}^{n} x_i x_{i+1}\right)^{1/2},$$

with a proof by mathematical induction (see [1985: 284]). There is equality if two consecutive x_i's are equal and the rest are zero. In view of this result, it is conjectured that (3) is valid for $n > 4$ with $\beta = 2(n-2) + \sqrt{3}$.

Again proceeding in the same way, (4) will be valid if

$$\sqrt{3}\sum_{i=1}^{n} x_i \ \geq\ \gamma\left(\sum_{i=1}^{n} x_i x_{i+1}\right)^{1/2}.$$

As before, this inequality (and hence (4)) will be valid for $\gamma = \sqrt{3n}$, $n = 3$ or 4, with equality if all the x_i's are equal. For $n \geq 5$, it is conjectured that (4) will be valid for $\gamma = 2 + \sqrt{3}$, this critical value occurring for two consecutive x_i's being equal and the rest zero.

References:

[1] L.M. Blumenthal, *Theory and Applications of Distance Geometry*, Chelsea, N.Y., 1970, pp. 97–99.

[2] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities* (2nd ed.), Cambridge Univ. Press, 1967, Theorem 52.

* * *

<u>1469</u>. [1989: 208] *Proposed by Andy Liu, University of Alberta.*

The following is an excerpt from U.I. Lydna's *Medieval Justice.*

"Justice on Pagan Island was administered by a Tribunal of seven Chiefs, at most one from each tribe. In a trial, each Chief would deliver a preliminary verdict of "innocent" or "guilty". There was no problem when unanimity was achieved, but it was felt that the final verdict of any case in between should be left to the Big Chief in the Sky.

Accordingly, the following procedure was established. A pouch and a token were made for each Chief on the Tribunal, both bearing the tribal insignia of that Chief. The tokens were put into the pouches at random, not necessarily one token in each pouch. (The pouches were made in such a way that, without opening them, the defendant could not tell how many tokens, if any, were inside.) After each Chief had delivered a verdict, the pouches were brought before the defendant, who then chose a number of them equal to the number of "innocent" verdicts received. The defendant opened the chosen pouches and kept the tokens inside. Any pouch bearing the same insignia as one of the defendant's tokens was also opened and the token(s) inside added to the defendant's collection. This continued until no further pouches could be opened. Finally, the defendant would be deemed innocent if all seven tokens had been collected, and guilty otherwise."

Suppose a defendant received k "innocent" verdicts, $0 \leq k \leq 7$. What is his probability of acquittal?

Solution by the proposer.

His probability of acquittal is k/n.

Each distribution of the tokens induces a partition of n, namely into the sum of the numbers of tokens in the pouches. For a fixed partition π of n, let $S(\pi,n)$ denote the set of distributions according to π. We claim that exactly k/n of the distributions in $S(\pi,n)$ lead to acquittal. (The result then follows.) We fix k and use an inductive argument on n. The case $n = k$ being trivial, we assume $n > k$ hereafter.

We may assume that the defendant chooses pouches 1 to k. By symmetry, $1/n$ of the distributions in $S(\pi,n)$ put the nth token in the nth pouch, and the defendant is doomed. Each of the remaining distributions induces a distribution in $S(\pi_1,n-1)$ for some partition π_1 of $n-1$ as follows. Let the nth token be in the ith pouch for some i, $1 \leq i \leq n-1$. Then the opening of the ith pouch leads immediately to the opening of the nth pouch. Hence we may discard the nth token and pouch and transfer the contents of the latter into the ith pouch.

By symmetry, there is a constant many-to-one correspondence between $S_1(\pi,n)$ and $S(\pi_1,n-1)$, where $S_1(\pi,n)$ is that subset of $S(\pi,n)$ consisting of all distributions which induce a distribution in $S(\pi_1,n-1)$. By the induction hypothesis, exactly $k/(n-1)$ of the distributions in $S(\pi_1,n-1)$, and hence in $S_1(\pi,n)$, lead to acquittal. The same holds for other partitions $\pi_2,\pi_3,\cdots$ of $n-1$ that may be induced by some distribution in $S(\pi,n)$. Thus the probability of acquittal is

$$\frac{n-1}{n}\cdot\frac{k}{n-1} = \frac{k}{n},$$

as desired.

For a nice non-inductive proof of the special case when there is exactly one token in each pouch, see Problem 3-6, p. 25 and pp. 198-199 of L. Lovász, *Combinatorial Problems and Exercises*, North-Holland, New York, 1979.

Also solved by MARCIN E. KUCZMA, Warszawa, Poland.

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1470*. [1989: 208] *Proposed by Michael Somos, Cleveland, Ohio.*

Consider the sequence (a_n) where $a_0 = a_1 = \cdots = a_5 = 1$ and

$$a_n = \frac{a_{n-1}a_{n-5} + a_{n-2}a_{n-4} + a_{n-3}^2}{a_{n-6}}$$

for $n \geq 6$. Computer calculations show that $a_6, a_7, \cdots, a_{100}$ are all integers. Consequently it is conjectured that all the a_n are integers. Prove or disprove.

Comment by the editor.

There were no solutions submitted for this problem. However the editor hears that the problem *has been solved* (the conjecture is true), and generalized, and related problems proposed, some of which are still open. All this and more is revealed in a fascinating article in the column Mathematical Entertainments (by new column editor David Gale), supposed to appear in the January 1991 issue of the *Mathematical Intelligencer.*

Graham Denham, a student at the University of Alberta, sent in a proof of two variations, where the above recursion is replaced by the simpler one

$$a_n a_{n-4} = a_{n-1}a_{n-3} + a_{n-2}^2$$

or by the analogous fifth–order recursion. These cases are mentioned in the Gale article as having been done by several people.

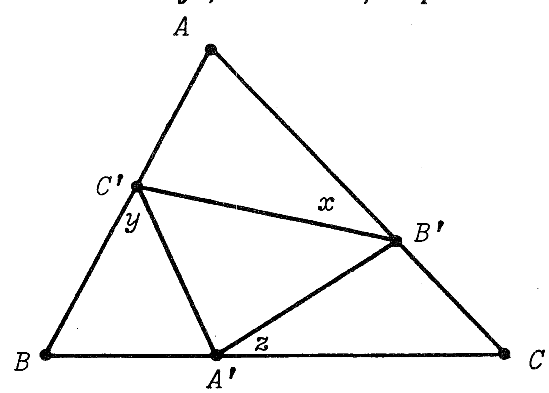
* * *

1471. [1989: 232] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be a triangle inscribed in a triangle ABC, so that $A' \in BC$, $B' \in CA$, $C' \in AB$, and so that $A'B'C'$ and ABC are directly similar. If $BA' = CB' = AC'$, prove that the triangles are equilateral.

Counterexample and corrected solution by Toshio Seimiya, Kawasaki, Japan.

We assume that $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$. As $\angle A = \angle A'$ and $B'C'$ is the common side of $\Delta AB'C'$ and $\Delta A'B'C'$, the circumradii of $\Delta AB'C'$ and $\Delta A'B'C'$ are equal. Thus [by symmetry] the circumradii of $\Delta AB'C'$, $\Delta BC'A'$, $\Delta CA'B'$ and $\Delta A'B'C'$ are equal.

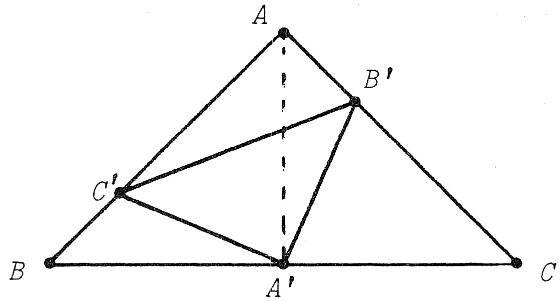


We put

$$\angle AB'C' = x\ , \quad \angle BC'A' = y\ , \quad \angle CA'B' = z\ .$$

Since the circumradii of $\Delta AB'C'$ and $\Delta BC'A'$ are equal, and $AC' = BA'$, we get either $x = y$ or $x + y = \pi$. Similarly we have either $x = z$ or $x + z = \pi$. Thus there may occur four cases: (i) $x = y$, $x = z$; (ii) $x = y$, $x + z = \pi$; (iii) $x + y = \pi$, $x = z$; (iv) $x + y = \pi$, $x + z = \pi$.

When $x = y$, we get from $\angle A + x = \angle BC'B'$ that $\angle A = \angle A'C'B' = \angle C$. Similarly when $x = z$ we get $\angle C = \angle B$. Therefore if (i) occurs ΔABC is equilateral. In the other cases we can deduce that ΔABC is isosceles, but we cannot necessarily deduce that ΔABC is equilateral. We shall give a counterexample. Let ABC be an isosceles right triangle such that $AB = AC$. Let A' be the midpoint of BC, and take B' on CA and C' on AB so that $CB' = AC' = BA'$ $(= A'C = A'A)$. Since $\Delta A'CB' \equiv \Delta A'AC'$,



we get $A'B' = A'C'$ and $\angle CA'B' = \angle AA'C'$, so that $\angle B'A'C' = \angle CA'A = 90^\circ$. Therefore $\Delta A'B'C' \sim \Delta ABC$. ΔABC is not equilateral, but there exist points A', B', C' which satisfy the conditions.

Also solved correctly by JORDI DOU, Barcelona, Spain; M.S. KLAMKIN and ANDY LIU, University of Alberta; and D.J. SMEENK, Zaltbommel, The Netherlands. The proposer's false solution was similar to Seimiya's but did not consider cases (ii)–(iv). One other reader sent in a false solution of the proposer's claim.

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<u>1472</u>. [1989: 232] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For each integer $n \geq 2$, find the largest constant c_n such that

$$c_n \sum_{i=1}^{n} |a_i| \leq \sum_{i<j} |a_i - a_j|$$

for all real numbers $a_1, \cdots, a_n$ satisfying $\sum_{i=1}^{n} a_i = 0$.

Solution by the proposer.

We show $c_n = n/2$. Indeed, for fixed i, $1 \leq i \leq n$, we have

$$n|a_i| = |(n-1)a_i - (-a_i)| = \left| \sum_{j\neq i} (a_i - a_j) \right| \leq \sum_{j \neq i} |a_i - a_j| .$$

Summing up these inequalities we get

$$n \sum_{i=1}^{n} |a_i| \leq 2 \sum_{i<j} |a_i - a_j| ,$$

which implies $c_n \geq n/2$. But $a_1 = \cdots = a_{n-1} = a$, $a_n = -(n-1)a$ then shows

$c_n = n/2$.

Also solved (but not as neatly!) by MARCIN E. KUCZMA, Warszawa, Poland; and C. WILDHAGEN, Breda, The Netherlands.

The proposer actually showed that the result holds when the a_i's come from an arbitrary normed space (just replace $|\ |$ by $\|\ \|$ throughout).

* * *

1473*. [1989: 232] *Proposed by Murray S. Klamkin, University of Alberta.*

Given is a unit circle and an interior point P. Find the convex n–gon of largest area and/or perimeter which is inscribed in the circle and passes through P.

Solution by Leroy F. Meyers, The Ohio State University.

Let $A_1A_2\cdots A_n$ be a convex n–gon inscribed in the unit circle, with the point P on side A_nA_1. Let θ_i be the central angle subtended by side A_iA_{i+1}, with $\theta = \theta_n$ for A_nA_1, and all angles measured counter–clockwise. If the angle subtended by any side (there can be at most one) is greater than π, then we can obtain an n–gon of larger area and perimeter by replacing each of the vertices not an endpoint of this side by the "other" point in which the circle intersects the line which is perpendicular to the offending side and passes through the vertex. Hence we may assume that $0 \le \theta_i \le \pi$ for $i = 1,\cdots,n$ and $\theta_1 + \cdots + \theta_n = 2\pi$.

Twice the area of the n–gon is $\sin\theta_1 + \cdots + \sin\theta_n$. But, as proved in [1], this is largest when $\theta_1 = \cdots = \theta_n = 2\pi/n$, provided that there is a chord through P short enough to subtend a central angle of $2\pi/n$, which occurs just when the distance d of P from the center of the circle is at least $\cos(\pi/n)$. Otherwise, for fixed θ, the area is largest, again by [1], when

$$\theta_1 = \cdots = \theta_{n-1} = \frac{2\pi - \theta}{n-1}.$$

If we set

$$f(\theta) = \sin\theta + (n-1)\sin\left(\frac{2\pi-\theta}{n-1}\right), \qquad 0 \le \theta \le \pi,$$

then

$$f'(\theta) = \cos\theta - \cos\left(\frac{2\pi-\theta}{n-1}\right),$$

which is 0 precisely when

$$\theta = \frac{2\pi-\theta}{n-1},$$

i.e. $\theta = 2\pi/n$, as expected, but

$$f''(\theta) = -\sin\theta - \frac{1}{n-1}\sin\left(\frac{2\pi-\theta}{n-1}\right) < 0,$$

so that the area decreases strictly as θ recedes from $2\pi/n$ in either direction. Now the shortest chord through P is the one in which P is the midpoint of the chord, and its central angle θ satisfies $\cos(\theta/2) = d$ or $\theta = 2$ Arccos d. Thus, if 2 Arccos $d \geq 2\pi/n$, then the n-gon through P of largest area is the one in which the side through P subtends a central angle of $\theta = 2$ Arccos d and each of the other $n - 1$ sides subtends a central angle of $(2\pi - \theta)/(n - 1)$.

For the perimeter, there is a similar solution. Half the perimeter is

$$\sin(\theta_1/2) + \cdots + \sin(\theta_n/2) \ ,$$

and a similar analysis goes through.

Reference:

[1] Ivan Niven, *Maxima and Minima without Calculus*, Dolciani Mathematical Expositions no. 6, M.A.A., 1981, pp. 93–95.

Also solved by JORDI DOU, Barcelona, Spain; G.P. HENDERSON, Campbellcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; and the proposer.

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<u>1474</u>. [1989: 232] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with sides a, b, c which we regard as being made of thin homogeneous material. The center of gravity of the perimeter of the triangle is denoted G. If G lies on the incircle of ΔABC, prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{10}{a + b + c} \ .$$

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.

For convenience we call I' the center of gravity of the perimeter of ABC, and as usual let G denote the centroid of ABC. Then I' is the incenter of $\Delta A'B'C'$, the medial triangle of ABC. (See [1], p. 134 or [2], Theorem 412. Incidentally, I' is also the radical center of the excircles of ABC; see [3], 3.16 and 3.17, p. 21.) In the homothety with center G and ratio –1/2, the triangle ABC transforms into the triangle $A'B'C'$; therefore the incenters I of ABC and I' of $A'B'C'$ are collinear with G, and so we have

$$|II'| = \frac{3}{2}|GI| \ .$$

If I' is on the incircle of ABC, then $|II'| = r$ (the inradius), and by means of the identity

$$|GI|^2 = \frac{s^2 + 5r^2 - 16Rr}{9}$$

(see [4], p. 280), where s is the semiperimeter and R the circumradius, we obtain $4r^2 = s^2 + 5r^2 - 16Rr$ or

$$s^2 + r^2 - 16Rr = 0 \ . \tag{1}$$

From the well known expressions for r and R in terms of the sides of ABC, namely

$$r = \sqrt{(s-a)(s-b)(s-c)/s} \ , \qquad R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \ ,$$

we can write (1) as

$$s^3 + (s-a)(s-b)(s-c) - 4abc = 0 \ ,$$
$$(ab + bc + ca)s = 5abc \ ,$$
$$(ab + bc + ca)(a + b + c) = 10abc \ ,$$

and finally

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab + bc + ca}{abc} = \frac{10}{a + b + c} \ ,$$

as required.

References:

[1] Belda, *Mecanica Clasica*, Madrid.

[2] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960.

[3] Lalesco, *La Geometrie du Triangle*, Paris, 1987.

[4] D.S. Mitrinović et al, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, 1989.

Also solved by C. FESTRAETS–HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE–WAI LAU, Hong Kong; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

The problem was taken from the 1987 Dutch book Hoofdstukken uit de Elementaire Meetkunde *of O. Bottema. In this book there appears a solution due to the proposer, which Dr. Hut was kind enough to send.*

* * *

1475. [1989: 233] *Proposed by Richard Katz, California State University, Los Angeles; Raymond Killgrove, Indiana State University, Terre Haute; and Reginald Koo, University of South Carolina, Aiken.*

Is there a function f from the reals onto the reals such that $f(f(x)) = f(x)f(x)$ for all x?

Solution by Mathew Englander, Kitchener, Ontario.

Suppose there were such a function f. Since f is surjective, there is some real number y (not necessarily unique) such that $f(y) = -1$. There is now some x (not necessarily unique) such that $f(x) = y$. Now

$$y^2 = f(x)f(x) = f(f(x)) = f(y) = -1 ,$$

which is impossible for real y. Thus there is no function f as described.

Note. If you replace the word "reals" in the question with "complex numbers" then $f(z) = z^2$ fulfills the conditions.

Also solved by SEUNG–JIN BANG, Seoul, Republic of Korea; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; RODNEY HUTCHINGS, student, Memorial University of Newfoundland, St. John's; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; DAVID E. MANES, SUNY at Oneonta; LEROY F. MEYERS, The Ohio State University, Columbus; SHEILA OATES–WILLIAMS, University of Queensland, St. Lucia, Australia; STANLEY RABINOWITZ, Westford, Massachusetts; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; ST. OLAF PROBLEM SOLVING GROUP, St. Olaf College, Northfield, Minnesota; D.C. VAUGHAN, Wilfrid Laurier University, Waterloo, Ontario; C. WILDHAGEN, Breda, The Netherlands; and the proposers. Three incorrect solutions were received.

The St. Olaf Problem Solving Group proved that there exists a function f from the reals onto the reals such that $f_n(x) = (f(x))^n$, where $f_n(x)$ denotes the composition of $f(x)$ with itself n times (n a natural number), if and only if n is odd.

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1476. [1989: 233] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

A triangle is called *self–altitude* if it is similar to the triangle formed from its altitudes. Suppose ΔABC is self–altitude, with sides $a \ge b \ge c$ and angle bisectors AP, BQ, CR. Prove that the lengths of CP, PB, BR, RA form a geometric progression.

Solution by Jill Houghton, Sydney, Australia.

In any triangle ABC with angle bisectors AP and CR,

$$\frac{PC}{PB} = \frac{b}{c} \quad \text{and} \quad \frac{RB}{RA} = \frac{a}{b} . \tag{1}$$

For a self–altitude triangle ABC with altitudes AD, BE, CF of lengths h_a, h_b, h_c respectively,

$$\sin A = \frac{h_c}{b} = \frac{h_b}{c} , \qquad \text{giving} \quad \frac{h_c}{h_b} = \frac{b}{c} .$$

If $a \ge b \ge c$ then $h_c \ge h_b \ge h_a$ and $a{:}b{:}c = h_c{:}h_b{:}h_a$. Thus

$$\frac{a}{b} = \frac{h_c}{h_b} = \frac{b}{c}$$

for all self–altitude triangles. Using this with (1) gives

$$\frac{PC}{PB} = \frac{RB}{RA} = k \quad \left(= \frac{a}{b} = \frac{b}{c}\right) .$$

Now $PC + PB = a$ and $RB + RA = c$, i.e.

$$(k + 1)PB = a \qquad \text{and} \qquad \left(1 + \frac{1}{k}\right)RB = c ,$$

giving

$$\frac{PB}{RB} = \frac{a}{kc} = \frac{a}{ac/b} = \frac{b}{c} = k .$$

Therefore

$$\frac{PC}{PB} = \frac{PB}{RB} = \frac{RB}{RA} ,$$

i.e. PC, PB, RB, RA are in geometric progression.

Also solved by WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; LEROY F. MEYERS, The Ohio State University; P. PENNING, Delft, The Netherlands; STANLEY RABINOWITZ, Westford, Massachusetts; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

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1477. [1989: 233] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

1024 tennis players, 128 professionals among them, are participating in a cup championship. All participants are numbered 1 through 1024. In the cup system, in the first round no. 1 plays against no. 2, no. 3 against no. 4, etc.; the winner in match 1 vs. 2 plays in the second round against the winner of 3 vs. 4, etc.; the final match constitutes the tenth round. Professionals bear numbers divisible by 8. In a match between a professional and an amateur, the former wins with probability 0.6. Is the cup winner more likely to be a professional or an amateur?

Solution by J.A. McCallum, Medicine Hat, Alberta.

The winner is more likely to be a professional, and the proposer seems to have carefully chosen the power of 2 to use as total entry to just ensure that result. The probability of a professional winning is about .517647663... according to my calculations.

In the first three levels of play no professional plays against another professional. The probability of any professional surviving his first three games is just $(0.6)^3 = 0.216$.

In any round after the first three, the probability of a professional meeting a professional, and so of a professional winning the game by default, so to speak, is P^2, where P is the probability of a professional being in a particular slot. Similarly, let A be the probability of an amateur being in a particular slot. The probability of a professional meeting an amateur in any match is then $2PA$, or $2P(1 - P)$. And the total probability of a professional winning any game, after the first three rounds, is then

$$P^2 + 2P(1 - P)(0.6) .$$

This expression gives the value of P to use for the next level of play.

Doing this recursively until the field is reduced to the last surviving player we eventually get a probability of a professional win to be 0.518 approximately.

Also solved (the same way) by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; P. PENNING, Delft, The Netherlands; and the proposer.

* * *

1478*. [1989: 233] *Proposed by D.M. Milošević, Pranjani, Yugoslavia.*

A circle of radius R is circumscribed about a regular n-gon. A point on the circle is at distances $a_1, a_2, \cdots, a_n$ from the vertices of the n-gon. Prove that

$$\sum_{i=1}^{n} a_i^3 \geq 2R^3 n\sqrt{2} .$$

I. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

Let the n-gon be $A_0A_1\cdots A_{n-1}$ with center O, and the point on the circumcircle be P, so that $a_i = PA_i$ (I prefer to enumerate the vertices from 0 to $n - 1$ rather than 1 to n). Then the statement follows immediately from

$$\sum_{i=0}^{n-1} a_i^2 = \sum_{i=0}^{n-1} PA_i^2 = \sum_{i=0}^{n-1}(\overrightarrow{OA_i} - \overrightarrow{OP})^2$$

$$= nR^2 - 2\overrightarrow{OP}\cdot\sum_{i=0}^{n-1}\overrightarrow{OA_i} + nR^2$$

$$= 2nR^2 - 2\overrightarrow{OP}\cdot 0 = 2nR^2 , \qquad (1)$$

by Jensen's inequality applied to the convex function $g(t) = t^{3/2}$:

$$\sum_{i=0}^{n-1} a_i^3 = \sum_{i=0}^{n-1} g(a_i^2)$$

$$\geq ng\left(\frac{1}{n}\sum_{i=0}^{n-1} a_i^2\right) = ng(2R^2) = 2^{3/2}nR^3 . \tag{2}$$

Remarks. The inequality is always strict (equality in (2) would require $a_0 = \cdots = a_{n-1}$, which is absurd). In fact, it is not hard to raise the given lower estimate for the sum of cubes as follows.

For each n, let c_n be the greatest constant such that

$$\sum_{k=0}^{n-1} a_k^3 \geq c_n nR^3 \tag{3}$$

holds for any position of the varying point P on the circle. Let M be the midpoint of arc $A_{n-1}A_0$, and suppose without loss of generality that P lies on arc MA_0, at angular distance 2φ from A_0, $0 \leq \varphi \leq \pi/2n$. Then

$$a_k = 2R \sin\left(\frac{k\pi}{n} + \varphi\right) , \qquad k = 0,1,\cdots,n-1 .$$

Using $4 \sin^3\theta = 3 \sin \theta - \sin 3\theta$ and other familiar trigonometric identities we obtain after small manipulation

$$\sum_{k=0}^{n-1} a_k^3 = 2R^3\left(\frac{3 \cos(\pi/2n - \varphi)}{\sin(\pi/2n)} - \frac{\cos 3(\pi/2n - \varphi)}{\sin(3\pi/2n)}\right) . \tag{4}$$

Routine examination shows that (4) is a decreasing function of φ in $[0,\pi/2n]$. Hence $\sum a_k^3$ attains its minimum when $P = M$ (regardless of whether n is odd or even (!)). Consequently

$$c_n = \frac{2}{n}\left(\frac{3}{\sin(\pi/2n)} - \frac{1}{\sin(3\pi/2n)}\right) , \tag{5}$$

and so

$$c_3 = 10/3 = 3.333\cdots , \quad c_4 = 3.378\cdots , \quad c_5 = 3.388\cdots ,$$

an increasing sequence (calculus; unpleasant!). Thus (3) holds for all $n \geq 3$ with $c_n = 10/3 > 2\sqrt{2}$. (Note that the claimed constant $2\sqrt{2}$ actually appears as c_2, formula (5) making sense for $n = 2$, too; but for some reason 2–gons are not liked by geometers.)

And what if other exponents $r \geq 1$ are considered instead of cubes? For the point P located (as before) at angle 2φ from a vertex, the sum $\sum a_k^r$ equals

$$F(n,r,\varphi) = (2R)^r \sum_{k=0}^{n-1} \sin^r\left(\frac{k\pi}{n} + \varphi\right), \qquad 0 \le \varphi \le \frac{\pi}{2n}.$$

Restricting attention to integer exponents r, we have explicit formulas which represent $\sin^r$ as a trigonometric polynomial. They lead to expressions like (4), with a growing (as r grows) number of summands. In this way we can learn that $F(n,1,\varphi)$ is an increasing function of φ (for every n), and so is $F(n,5,\varphi)$; $F(n,3,\varphi)$ is decreasing in φ, and $F(n,2,\varphi)$ and $F(n,4,\varphi)$ are constant. One is tempted to conjecture that $F(n,r,\varphi)$ is constant in φ for even r and strictly monotonic (alternately: increasing, decreasing) for odd r, independently of n.

But a moment's thought is enough to realize that this can't be the case. For fix n. When r grows to infinity, the rth power norm approaches the sup norm; the greatest summand outweighs the others. Hence, for large r, the sum $\sum a_k^r$ attains its maximum when the greatest a_k is as large as possible; namely, when P lies antipodally to a vertex. This corresponds to $\varphi = 0$ for even n and to $\varphi = \pi/2n$ for odd n.

That's curious! Whether $F(n,r,\varphi)$ increases or decreases in φ seems to depend, for small r, on the arithmetic properties of r (its remainder mod 4), and for large r on the parity of n.

Can anyone tell what happens "in between"? How does $F(n,r,\varphi)$ behave for non-integer r? Is this function monotone in φ, for every fixed value of n and r? If yes, in which sense? For what values of r is the answer to the last question independent of n?

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We shall show the following more general result.

Theorem. A circle C of radius R is circumscribed about a regular n-gon. A point P lies in the plane of the n-gon and is at distances $a_1,\cdots,a_n$ from the vertices of the n-gon and at distance d from the center O of circle C. Then

$$\sum_{i=1}^{n} a_i^t \begin{cases} \ge n(R^2 + d^2)^{t/2} & \text{if } t > 2 \text{ or } t < 0\,, \\ = n(R^2 + d^2) & \text{if } t = 2\,, \\ \le n(R^2 + d^2)^{t/2} & \text{if } 0 < t < 2\,. \end{cases}$$

($t = 3$ and $d = R$ yield the stated inequality).

[*Editor's note.* Janous then proved the equality in the case $t = 2$ (this can be done as in (1) above), and observed that the rest follows by the general means-inequality. For example, if $0 < t < 2$ then

$$\left(\sum_{i=1}^{n} \frac{a_i^t}{n} \right)^{1/t} \leq \left(\sum_{i=1}^{n} \frac{a_i^2}{n} \right)^{1/2},$$

so

$$\sum_{i=1}^{n} a_i^t \leq n\left(\sum_{i=1}^{n} \frac{a_i^2}{n} \right)^{t/2} = n(R^2 + d^2)^{t/2} .]$$

Also solved by G.P. HENDERSON, Campbellcroft, Ontario; and RICHARD I. HESS, Rancho Palos Verdes, California.

Both Henderson and Hess obtained formula (5) *and then showed that* $c_n > 2\sqrt{2}$ *for* $n \geq 3$. *Hess further observed that* $\lim_{n\to\infty} c_n = 32/3\pi \approx 3.3953$.

Kuczma's intriguing solution leaves a wealth of unanswered questions. Further information from readers would be most welcome!

* * *

1479. [1989: 233] *Proposed by Vedula N. Murty, Pennsylvania State University at Harrisburg.*

Given $x > 0$, $y > 0$ satisfying $x^2 + y^2 = 1$, show without calculus that

$$x^3 + y^3 \geq \sqrt{2}xy .$$

I. *Solution by Hugh Thomas, student, Kelvin High School, Winnipeg, Manitoba.*

From

$$0 \leq \left(\frac{x - y}{2}\right)(x^2 - y^2) = \frac{x^3 - x^2y - xy^2 + y^3}{2},$$

we have

$$x^3 + y^3 \geq x^3 + y^3 - \frac{x^3 - x^2y - xy^2 + y^3}{2} = \frac{x^3 + x^2y + xy^2 + y^3}{2}$$

$$= \frac{(x + y)(x^2 + y^2)}{2} = \frac{x + y}{2},$$

and by the AM–GM inequality, we know that

$$\frac{x + y}{2} \geq \sqrt{xy} .$$

To determine the maximum value of xy, find α such that $x = \sin\alpha$ and $y = \cos\alpha$, and note that

$$xy = \sin\alpha\cos\alpha = \frac{\sin 2\alpha}{2} \leq \frac{1}{2} .$$

Thus $\sqrt{2xy} \leq 1$ and so

$$x^3 + y^3 \geq \frac{x + y}{2} \geq \sqrt{xy} \geq \sqrt{xy}\sqrt{2xy} = \sqrt{2}xy ,$$

as required.

II. *Generalization by M.S. Klamkin, University of Alberta.*

If $x_1, \cdots, x_n \geq 0$, and $r, s, t, u, v > 0$ satisfy $r \geq s$, $r \geq t$ and $r = su + tv$, then it follows by the power mean inequality that

$$\left(\sum_{i=1}^{n} \frac{x_i^r}{n}\right)^{1/r} \geq \left(\sum_{i=1}^{n} \frac{x_i^s}{n}\right)^{1/s}, \left(\sum_{i=1}^{n} \frac{x_i^t}{n}\right)^{1/t}$$

and thus

$$\sum_{i=1}^{n} \frac{x_i^r}{n} = \left(\sum_{i=1}^{n} \frac{x_i^r}{n}\right)^{su/r + tv/r}$$

$$\geq \left(\sum_{i=1}^{n} \frac{x_i^s}{n}\right)^{u} \left(\sum_{i=1}^{n} \frac{x_i^t}{n}\right)^{v}$$

$$\geq \left(\sum_{i=1}^{n} \frac{x_i^s}{n}\right)^{u} (x_1 x_2 \cdots x_n)^{vt/n},$$

with equality if and only if $x_1 = \cdots = x_n$. If additionally $x_1^s + \cdots + x_n^s = 1$, then

$$\sum_{i=1}^{n} x_i^r \geq n^{1-u-v} \left(\sum_{i=1}^{n} x_i^t\right)^{v} \geq n^{1-u} (x_1 x_2 \cdots x_n)^{vt/n}.$$

The proposed problem corresponds to the very special case $n = 2$, $r = 3$, $s = 2$, $t = 2$, $u = 1/2$, $v = 1$; i.e.

$$x_1^3 + x_2^3 \geq \frac{1}{\sqrt{2}} \geq \sqrt{2} x_1 x_2 .$$

Also solved by SEUNG–JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSION LOPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; JACK GARFUNKEL, Flushing, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; STEPHEN HNIDEI, student, University of British Columbia; JILL HOUGHTON, Sydney, Australia; PETER HURTHIG, Columbia College, Burnaby, B.C.; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE–WAI LAU, Hong Kong; DAVID E. MANES, SUNY at Oneonta; LEROY F. MEYERS, The Ohio State University, Columbus; M. PARMENTER, Memorial University of Newfoundland; TOSHIO SEIMIYA, Kawasaki, Japan; ROBERT E. SHAFER, Berkeley, California; D.C. VAUGHAN, Wilfrid Laurier

University; EDWARD T.H. WANG, Wilfrid Laurier University; C. WILDHAGEN, Breda, The Netherlands; and the proposer. There was one incorrect solution sent in.

Janous gave the generalization:

$$x^{p+1} + y^{p+1} \geq \sqrt[p]{2}xy$$

whenever $x,y > 0$ *and* $p > 1$ *are real numbers such that* $x^p + y^p = 1$, *which follows from Klamkin's result.*

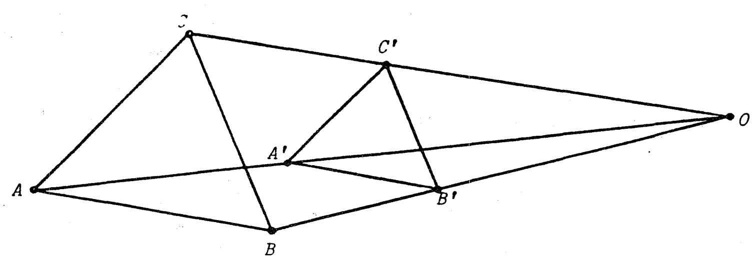
* * *

1480. [1989: 233] *Proposed by Juan Bosco Romero Márquez, Valladolid, Spain.*

ABC and $A'B'C'$ are triangles connected by a dilatation ($BC \| B'C'$, $CA \| C'A'$, $AB \| A'B'$), and $A'' = BC' \cap B'C$, $B'' = AC' \cap A'C$, $C'' = AB' \cap A'B$. Show that $\Delta A''B''C''$ is connected to either of the two given triangles by a dilatation, and that the centroids of the three triangles are collinear.

Solution by Marcin E. Kuczma, Warszawa, Poland.

Lines AA', BB', CC' are concurrent or parallel. When you draw these lines and mark the perimeters of triangles ABC, $A'B'C'$ with bold rule, it is almost impossible not to see two identical vertical triangles in space, illustrated in concurrent or parallel perspective. Quadrilaterals $BCC'B'$ etc. are projective images of the respective rectangles; points A'', B'', C'' represent the centers of these rectangles. Hence the assertion. [*Editor's note*: at least if ABC and $A'B'C'$ are on the same side of O, the centre of homothety.]



A rigorous proof can be written, e.g., in terms of vectors $\mathbf{A} = \overrightarrow{OA}$, $\mathbf{A}'$, etc. (the "parallel" case in which dilatation becomes translation is obvious). Vector $\mathbf{C}''$ is a convex combination of $\mathbf{A}$ and $\mathbf{B}' = \lambda\mathbf{B}$, and simultaneously a convex combination of $\mathbf{B}$ and $\mathbf{A}' = \lambda\mathbf{A}$ (same λ). Clearly, a vector is represented *uniquely* as a linear combination of $\mathbf{A}$ and $\mathbf{B}$. Upon equating the coefficients we soon arrive at

$$\mathbf{C}'' = \frac{\lambda}{1 + \lambda}(\mathbf{A} + \mathbf{B}) \qquad \text{(and cyclically)} ,$$

and the claims result.

Also solved by JILL HOUGHTON, Sydney, Australia; L.J. HUT, Groningen, The Netherlands; P. PENNING, Delft, The Netherlands; JOHN RAUSEN, New York; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

YEAR-END WRAPUP

Another year of *Crux* has passed and so here are a few comments on old problems which have been received from readers during the last twelve months.

522. [1980: 77; 1981: 81]

One of the proposers of this problem, LEROY F. MEYERS, Ohio State University, has come across a better reference to the result than he gave on [1981: 83]. It appears in Edouard Lucas, "Sur les congruences des nombres eulériens et des coefficients différentiels des fonctions trigonométriques, suivant un module premier", *Bulletin de la Société Mathématique de France* 6 (1878) 49–54, especially pp. 51–52.

1370. [1988: 203; 1989: 281]

EDWARD T.H. WANG, Wilfrid Laurier University, writes with what was news to the editor, and it seems many readers, namely that the "$3x + 1$" problem had earlier occurred in this journal as *Crux* 133! Not only that, but in the seven pages of comments (no solution, of course) which appeared three issues later [1976: 144–150], many sets of more than 17 consecutive integers with the same L-value are given, the longest being the 40 integers from 596310 to 596349, all of which have L-value equal to 97. These results are apparently due to an anonymous writer to *Popular Computing* in 1974.

1396. [1988: 302; 1990: 51]

KENNETH S. WILLIAMS, Carleton University, notes that the equation given at the bottom of [1990: 53] is essentially problem E1044 of the *American Math. Monthly* (solution in Vol. 60 (1953) 421–422).

1447. [1989: 148; 1990: 219]

JOHN F. GOEHL JR., Barry University, Miami Shores, Florida, writes that the first part of this problem was solved by him in a letter to the editor in *Mathematics Teacher*, May 1985, pp. 330 and 332. Also included in the letter is a similar result for rectangles.

1454. [1989: 177; 1990: 248]

MURRAY KLAMKIN, University of Alberta, points out a reference that the editor forgot to include in his published solution. Reference [1] should be: D.S. Mitrinović, J.E. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer, 1989.

*

Late solutions were received to 1387 (Kenneth M. Wilke, Topeka, Kansas); and to 1403 (a second solution by Marcin E. Kuczma, Warszawa, Poland).

*

The editor has learned that JACK GARFUNKEL, long–time *Crux* problem proposer and solver, and fan of geometric inequalities, has recently retired from teaching at the impressive age of 80. As a nice and fitting tribute to his contributions to *Crux* over the years, readers are encouraged to submit appropriate problems dedicated to him, which could be included in *Crux* over the next few months. This idea comes thanks to the thoughtfulness of his son (and well–known mathematician) Sol Garfunkel.

*

The following people were of great help to the editor during 1990, by contributing their opinions on articles, problems, and solutions; *LEN BOS*, *HANAFI FARAHAT*, *CHRIS FISHER*, *HIDETOSI FUKAGAWA*, *RICHARD GUY*, *WALTHER JANOUS*, *CLARK KIMBERLING*, *MURRAY KLAMKIN*, *W.O.J. MOSER*, *GEORGE PURDY*, *JONATHAN SCHAER*, *and D.J. SMEENK*. *Crux* benefits greatly by their efforts; a "thank you" to all of them. The editor would also like to thank *Crux* typist *LAURIE LORO* who, after five years of deciphering his handwriting, is moving on to concentrate on the (easier?) task of passing university courses. Next issue, and next year, will see some changes (including a new look!); however, your overworked but gritty editor will be back at his post.

CRUX MATHEMATICORUM

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HAPPY NEW YEAR

with many good ideas for 1991.

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