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CONTENTS

ON EQUILATERAL AND EQUIANGULAR POLYGONS

MURRAY S. KLAMKIN

I. *Introduction.*

It is well known and elementary that if a triangle is equilateral (i.e., having congruent sides), then the triangle is also equiangular (i.e., having congruent interior angles), and conversely. This result does not immediately generalize to higher-order planar polygons. As counterexamples, the interior angles of a rhombus need not be congruent, and the sides of a rectangle need not be congruent. Consequently, to obtain generalizations, we restrict the polygons considered to the following two classes:

(I) polygons inscribed in a given circle

and

(C) polygons circumscribed about a given circle.

After this paper was completed, the author came across the following problem of G. Pólya in the 1946 Stanford University Competitive Examination in Mathematics for high school students [1], which considered some of the same generalizations:

> Consider the following four propositions, which are not necessarily true.
> I. If a polygon inscribed in a circle is equilateral, it is also equiangular.
> II. If a polygon inscribed in a circle is equiangular, it is also equilateral.
> III. If a polygon circumscribed about a circle is equilateral, it is also equiangular.
> IV. If a polygon circumscribed about a circle is equiangular, it is also equilateral.
> (A) State which of the four propositions are true and which are false, giving a proof of your statement in each case.
> (B) If, instead of general polygons, we should consider only quadrilaterals, which of the four propositions are true and which are false? And if we consider only pentagons?
> In answering (B) you may state conjectures, but prove as much as you can and separate clearly what is proved and what is not.

We will solve the questions raised by Pólya and also give further extensions.

II. *Inscribed and Circumscribed Polygons.*

It follows easily that, for Class (I) polygons, equilateral $\Longrightarrow$ equiangular but not conversely. The counterexample for the converse result is again the rectangle. For Class (C) polygons, equiangular $\Longrightarrow$ equilateral but also not conversely. The counterexample here is the circumscribed rhombus. However, it was noted by M. Riesz (cited by Coxeter in [2]) that, for Class (I) polygons, equiangular $\Longrightarrow$ equilateral provided we further restrict the polygons to have an odd number of sides. A simple algebraic proof follows by referring to Figure 1.

Here we have

$$\begin{aligned} \theta_1 + \theta_2 &= k, \\ \theta_2 + \theta_3 &= k, \\ \vdots \quad \vdots \quad &\vdots \\ \theta_n + \theta_1 &= k, \end{aligned}$$

from which

$$\theta_1 = \theta_3 = \theta_5 = \ldots = \theta_{n-1},$$
$$\theta_2 = \theta_4 = \theta_6 = \ldots = \theta_n.$$

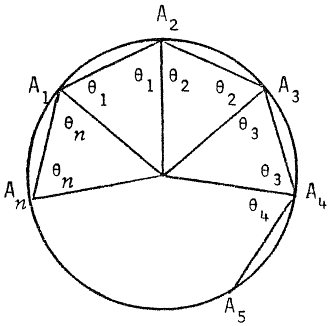

Figure 1.

If n is odd, θ_i=constant and the polygon is regular. If n is even, there are an infinite number of possible polygons.

To extend the result of Riesz, we first define a *d-diagonal* of an n-gon (with $2d \le n$) as a diagonal which "skips" $d-1$ vertices (e.g., in Figure 1, A_1A_2 is a 1-diagonal and A_1A_4 is a 3-diagonal). We now show that

THEOREM. If all the d-diagonals (d fixed) of an inscribed n-gon are congruent and $(n,d)=1$, *then the polygon is regular.*

Proof. Referring again to Figure 1, the angles θ_i satisfy the system

$$\begin{aligned} \theta_1 + \theta_2 + \ldots + \theta_d &= k, \\ \theta_2 + \theta_3 + \ldots + \theta_{d+1} &= k, \\ \vdots \quad \vdots \qquad \vdots \quad &\vdots \\ \theta_n + \theta_1 + \ldots + \theta_{d-1} &= k. \end{aligned}$$

Consequently,

$$\theta_1 = \theta_{d+1} = \theta_{2d+1} = \ldots = \theta_{id+1},$$

where $i = 0,1,\ldots, n-1$ and the indices are taken modulo n. Since, for the indicated values of i, $id+1$ generates a complete residue set modulo n if and only if $(n,d)=1$, the desired result follows. □

This proof also implies that the circulant determinant (see [3])

$$C(1,1,\ldots,1,0,0,\ldots,0) \equiv \begin{vmatrix} 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 1 & 1 & 0 & \ldots & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 1 & 1 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{vmatrix} \neq 0$$

if and only if $(n,d)=1$. Here each row contains d ones and $n-d$ zeros. More generally, it is known [3] that

$$C(a,a,\ldots,a,b,b,\ldots,b) = \begin{cases} (pa+qb)(a-b)^{n-1}, & \text{if } (p,q)=1, \\ 0, & \text{if } (p,q)>1, \end{cases}$$

where there are p a's and q b's. A related unsolved problem is to evaluate simply (as above)

$$C(a,a,\ldots,a,b,b,\ldots,b,c,c,\ldots,c),$$

where there are p a's, q b's, and r c's.

A dual of the Riesz result is that all equilateral circumscribed $(2n+1)$-gons must be regular. For $2n$-gons, we need only have the alternating interior angles congruent. A proof follows simply from the theorem that tangents to a circle from the same point are congruent. One extension of these results is that, given a circumscribed n-gon (Figure 2) such that

$$\sum_{i=1}^{d} \tan\theta_{i+j} = k, \quad j=0,1,\ldots,n-1,$$

then the polygon is regular if and only if $(n,d)=1$.

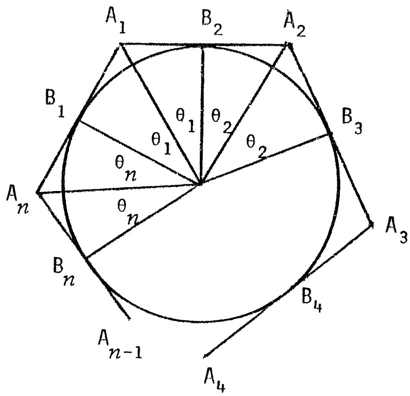

Figure 2

III. *Open Problems.*

In a recent paper, Grünbaum (in Kelly [4]) shows that there are still quite a number of interesting and "elementary" conjectures and results about polygons which are not widely known. One particularly nice result, of interest to organic chemists, is that an equilateral and equiangular pentagon is either 2-dimensional or else 4-dimensional. We end by listing some open problems which, for the most part, were suggested by Grünbaum's paper.

First we give some definitions (the first one is given by Grünbaum [4, p. 169]). An n-gon P with vertices $A_1, A_2, \ldots, A_n$ in E^m is *k-equilateral* (with *parameters* c_j) if $A_iA_{i+j} = c_j$ for all $i \pmod n$ and for $j=1,2,\ldots,k$. The n-gon P is *k-equiangular* (with *parameters* θ_j) if $\angle A_iA_{i+j}A_{i+2j} = \theta_j$ for all $i \pmod n$ and for $j=1,2,\ldots,k$.

It follows that 2-equilateral is equivalent to equilateral and equiangular. Also, in the plane, 2-equilateral $\Longrightarrow$ 2-equiangular. It is not immediate whether or not this implication is also valid in higher-dimensional spaces. The converse implication, 2-equiangular $\Rightarrow$ 2-equilateral, is in general not valid, even in the plane. Just consider a $2n$-gon whose $2n$ vertices are the vertices of two regular n-gons with the same centroid.

Now for the open problems.

1. For which n-gons in E^m does k-equiangular $\Rightarrow$ 2-equilateral? It appears that for n odd, $m = 2$, it suffices to take $k = 2$. [This was proved subsequently by A. Liu and the author. The proof will appear soon in this journal. (Editor)]

2. Grünbaum had conjectured that k-equilateral $(2k+1)$-gons span only even-dimensional spaces. This was subsequently proved by Lawrence [4, p. 185]. Analogously, what dimensions are spanned by k-equiangular n-gons?

The next set of problems are related ones but with one or two extra constraint conditions of having the vertices of the n-gon P lie on a sphere of dimension μ (indicated by $P(\mathrm{I}_\mu)$) and/or having its edges tangent to a sphere of dimension ν (indicated by $P(\mathrm{C}_\nu)$). If P is subject to both constraints, we denote it by $P(\mathrm{I}_\mu, \mathrm{C}_\nu)$. What are the possible dimensions spanned by the n-gon P if

3. $P(\mathrm{I}_\mu)$ is k-equiangular and l-equilateral?

4. $P(\mathrm{C}_\nu)$ is k-equiangular and l-equilateral?

5. $P(\mathrm{I}_\mu, \mathrm{C}_\nu)$ is k-equiangular and l-equilateral?

It is to be noted that if k or l is zero, then there is no implicit condition concerning equiangular or equilateral, respectively.

6. It is known [5] that, for any convex n-gon in E^2, there are at least $n-2$ sides which are shorter than the longest diagonal and that, furthermore, $n-2$ is best possible. Given a convex polytope of a given type, what is the least number of edges which are shorter than the longest diagonal? In particular, consider a polyhedron which is topologically equivalent to a cube.

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4. L.M. Kelly, Ed., *The Geometry of Metric and Linear Spaces*, Springer-Verlag, Heidelberg, 1975, pp. 147-184, plus references therein.

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Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

A 1981 GALLIMAUFRY

CHARLES W. TRIGG

The editor of this journal has graciously conferred upon my unworthy self the prestigious title of Prince of Digit Delvers [1980: 195]. I must now make a frantic effort, before the King comes along and makes me pale into insignificance, to show what a prince can do. So gather around, peasants. I shall delve into the numeral of the new year, 1981. The editor may then realize that the more modest title of Count would have been more appropriate.

A

The year is not a prime one since $1981 = 7 \cdot 283$ (and note that $7 = 2 + 8 - 3$), nor is its reverse $1891 = 31 \cdot 61$. But if we concatenate the two numbers with a bang (to make the middle digits meld), we get 1981891, a palindromic prime [1980: 267]. Right at the outset, we must face the disappointing fact that 1981 is greatly deficient, since $1 + 7 + 283 = 291 < 1981$.

B

Of the twelve four-digit integers that can be formed from the digits 1, 9, 8, 1, three are prime, six are the products of two primes, and three have three prime factors.

8191 is prime	$1189 = 29 \cdot 41$
9181 is prime	$1198 = 2 \cdot 599$
9811 is prime	$1819 = 17 \cdot 107$
$1918 = 2 \cdot 7 \cdot 137$	$1891 = 31 \cdot 61$
$8911 = 7 \cdot 19 \cdot 67$	$1981 = 7 \cdot 283$
$9118 = 2 \cdot 47 \cdot 97$	$8119 = 23 \cdot 353$

C

Of the twenty-two distinct one-, two-, and three-digit integers that can be formed from the digits 1, 9, 8, 1, three are squares: 1, 9, 81; and seven are primes: 11, 19, 89, 181, 191, 811, 911.

D

$1 \cdot 9 = \sqrt{81}$ and $1 \cdot 9 - 8 = 1$.

E

$1 - 9 + 8 + 1 = 1$, $1 \cdot \sqrt{9\sqrt{81}} = 9$, $1^9 \cdot 8 \cdot 1 = 8$, $1 \cdot 9 - 8 \cdot 1 = 1$.

F

$19 \cdot 81 = 1539$, in which all four digits are odd. The missing odd digit, 7, is a factor of 1981. Furthermore, $19 + 81 = 10^2$, the square of the base in the decimal system.

G

The Collatz algorithm (if it is odd, triple it and add 1; if it is even, divide it by 2) applied to 1980, 1981, and 1982 takes

$$1 + 98 + 0 = 1 \cdot 98 + 1 = -1 + 98 + 2 = 99$$

operations for each sequence to reach the inevitable 1. The sequences for 1980 and 1981 join forces as a result of the 3rd operation, and the sequence for 1982 joins the other sequences as a result of the 12th operation (see adjoining table). The common sequence, as it continues on to a final 4-2-1, can be found earlier in this journal [1980: 202].

1980	1981	1982
990	5944	991
495	2972	2974
1486	1486	1487
	743	4462
	2230	2231
	1115	6694
	3346	3347
	1673	10042
	5020	5021
	2510	15064
	1255	7532
	3766	3766

H

A modified Collatz algorithm, wherein 1 is subtracted rather than added, takes 28 operations to reach the 18-member regenerative loop.

1981				
5942	2504	1054	74	182
2971	1252	527	37	91
8912	626	1580	110	272
4456	313	790	55	136
2228	938	395	164	68
1114	469	1184	82	34
557	1406	592	41	17
1670	703	296	122	50
835	2108	148	61	25
				74

I

The topsy-turvy transformation, the upsetting rotation through 180°, converts 1981 into the prime 1861, an upsetting year for the USA, marking the start of its Civil War. Note that 1861 is a prime prime, since $18 + 61 = 79$, and its digital root is 7, which is also the arithmetic mean of its internal digits.

J

$$19811981 = 2^{24} + 2^{21} + 2^{20} - 2^{16} - 2^{15} - 2^{13} - 2^{12} - 2^{9} + 2^{7} + 2^{4} - 2^{1} - 2^{0}$$

$$= 1981 \cdot 137 \cdot 73 \cdot 1,$$ with a palindromic tail of factors.

K

$$1981 = 2^{11} - 2^7 + 2^6 - 2^1 - 2^0$$
$$= 3^7 - 3^5 + 3^3 + 3^2 + 3^0$$
$$= 0^9 + 1^8 + 2^7 + 3^6 + 4^5 + 5^1 + 6^2 + 7^2 + 8^1 + 9^0.$$

L

Fermat (according to L.E. Dickson, *History of the Theory of Numbers*, Vol. II, p. 6) made the famous comment: "I was the first to discover the very beautiful and entirely general theorem [that, in effect, every positive integer can be expressed as the sum of m or fewer m-gonal numbers]." The nth m-gonal number is

$$M_n = n\{(m-2)n - (m-4)\}/2.$$

The "fewer" category embraces 1981, as evidenced by:

$m = 3 \quad 1981 = 28 + 1953 = T_7 + T_{62}$

$m = 4 \quad = 9 + 36 + 1936 = S_3 + S_6 + S_{44}$

$m = 5 \quad = 5 + 12 + 247 + 1717 = P_2 + P_3 + P_{13} + P_{34}$

$m = 6 \quad = 91 + 120 + 1770 = X_7 + X_8 + X_{30}$

$m = 7 \quad = 1 + 7 + 55 + 1918 = H_1 + H_2 + H_5 + H_{28}$

$m = 8 \quad = 21 + 280 + 1680 = E_3 + E_{10} + E_{24}$

$m = 9 \quad = 1 + 24 + 1956 = N_1 + N_3 + N_{24}$

Three of the permutations of 1981 are polygonal numbers, namely, $1891 = X_{31}$, $1918 = H_{28}$, and $8911 = T_{133} = X_{67}$.

M

This triangular array contains seven 1981's written in sequence from the left. Four of the row sums are primes and two (including 1) are cubes. The remaining one, undaunted by a mere prince, is just there to spoil the effect.

1	1
9 8	17
1 1 9	11
8 1 1 9	19
8 1 1 9 8	27
1 1 9 8 1 1	21
9 8 1 1 9 8 1	37

N

The reverse of 1981 is the well-primed $1891 = 31 \cdot 61$, with $3 - 1 = 2$ and $6 - 1 = 5$. Also, the internal duo is 89 and the extremities form 11, with $8 + 9 = 17$, $1 + 1 = 2$, and $17 + 2 = 19$.

O

The reiterative routine in which an integer and its reverse are added, when applied to 1981, produces a palindrome in three operations, and immediately two more palindromes:

$$
\begin{array}{rrrrr}
1981 & 3872 & 6655 & 12221 & 24442 \\
\underline{1891} & \underline{2783} & \underline{5566} & \underline{12221} & \underline{24442} \\
3872 & 6655 & 12221 & 24442 & 48884
\end{array}.
$$

Extension to over 400 operations produces no more palindromes.

P

1981 is part of a 1560-digit additive bracelet wherein each element is the units' digit of the sum of the four preceding digits:

... 1 9 8 1 9 7 5 2 3 7 7 9 6 9 1 5 1 6 3 5 5 9 2 1 7 ...

The complete bracelet is included in my "A Digital Bracelet for 1971," *The Mathematics Teacher*, 64 (October 1971) 567-571.

Q

As a last princely gesture, I obtain 1981 by concatenating four distinct nine-nonzero-digit determinants:

$$
\begin{vmatrix} 1 & 2 & 6 \\ 4 & 7 & 3 \\ 5 & 9 & 8 \end{vmatrix}
\begin{vmatrix} 1 & 2 & 6 \\ 4 & 3 & 7 \\ 9 & 5 & 8 \end{vmatrix}
\begin{vmatrix} 1 & 2 & 6 \\ 4 & 3 & 7 \\ 8 & 5 & 9 \end{vmatrix}
\begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 9 \\ 8 & 5 & 7 \end{vmatrix}
= 1981.
$$

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* * *

LES MATHÉMATIQUES AU SERVICE DE CASANOVA

HIPPOLYTE CHARLES

"Ses axiomes sont des paradoxes faits pour faire éternuer l'esprit."

Cette phrase (oũ il est question de J.-J. Rousseau) est de Jacques Casanova de Seingalt (1725-1798). Elle est tirée de l'Avant-Propos de son *Histoire de ma fuite des Prisons de la République de Venise qu'on appelle les Plombs, écrite à Dux en Bohême en l'année 1787*, publiée à Leipzig en 1788. Quiconque est capable d'accoucher d'une telle phrase doit avoir un esprit porté vers les mathématiques. En est-il ainsi de Casanova? Oui, il s'intéresse aux mathématiques, et s'en sert à sa façon.

Charlatan à ses heures, il se sert à l'occasion de nombres pour embobiner ses victimes. Par exemple, il raconte dans ses *Mémoires* comment un hasard lui permit de "guérir" M. de Bragadin, un riche sénateur qui deviendra son bienfaiteur. Celui-ci se persuade que Casanova n'a pu le guérir que par la cabale. Notre héros n'a garde de le détromper. "Ne voulant pas choquer sa vanité en lui disant qu'il se trompait, je pris la folle résolution de lui faire en présence de ses deux amis la fausse et extravagante confidence que je possédais un calcul numérique par lequel, moyennant une question que j'écrivais et que je changeais en nombres, j'obtenais, également en nombres, une réponse qui m'instruisait de tout ce que je voulais savoir."

Écoutons-le raconter, dans les trois paragraphes qui suivent et qui sont tirés de son *Histoire de ma fuite,* comment il décide quel jour serait le plus propice à son évasion des Plombs.

"Ne sachant pas de quelle méthode me servir pour me faire révéler le moment de ma liberté par la Bible, je me suis déterminé à consulter le divin poème du *Roland furieux* de Messire Lodovico Ariosto, que j'avais lu cent fois et qui faisait encore là-haut mes délices. J'idolâtrais son génie et je le croyais beaucoup plus propre que Virgile à me prédire mon bonheur.

Dans cette idée, j'ai couché une courte question, dans laquelle je demandais, à une intelligence que je supposais, dans quel chant de l'Arioste se trouvait la prédiction du jour de ma délivrance. Après cela j'ai formé une pyramide à rebours, composée des nombres résultant des paroles de mon interrogation et, avec la soustraction du nombre 9 de chaque couple de chiffres, j'ai trouvé pour le dernier nombre le 9 et j'ai cru que dans le neuvième chant il y avait ce que je cherchais. J'ai suivi la même méthode pour savoir dans quelle stance de ce chant se trouvait cette prédiction et j'ai trouvé le nombre 7 et, curieux enfin de savoir dans quel vers de la stance se trouvait l'oracle, j'ai reçu l'1. J'ai d'abord pris entre mes mains l'Arioste avec le coeur palpitant et j'ai trouvé que le premier vers de la septième strophe du neuvième chant était: *Tra il fin d'Ottobre e il capo di Novembre.*

La précision de ce vers et l'à-propos me parurent si admirables que je ne dirai pas d'y avoir ajouté foi, mais le lecteur me pardonnera si je me suis disposé de mon côté à faire tout ce qui dépendait de moi pour aider à la vérification de l'oracle. Le singulier de ce fait est que *Tra il fin d'Ottobre e il capo di Novembre* il n'y a que minuit et que ce fut positivement au son de la cloche de minuit du trente-un d'octobre que je suis sorti de là, comme le lecteur va voir."

Mais son intérêt, sinon ses capacités, pour les mathématiques est véritable, au point qu'il croit avoir réussi à résoudre le problème de la duplication du cube! En effet, il publie en 1790, juste avant de commencer la rédaction de ses *Mémoires* en 1791, deux opuscules sur les mathématiques: *Solution du Problème Deliaque démontrée* et *Deux corollaires à la duplication de l'hexaèdre.*

Il s'intéresse aussi, mais toujours à sa façon, aux applications des mathématiques: il écrit en 1793 des *Rêveries sur la mesure moyenne de notre année selon la calendrier grégorien.*

Et la suite des entiers, qui est assez longue, est toujours à sa disposition pour dénombrer ses conquêtes amoureuses.

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THE OLYMPIAD CORNER: 21

MURRAY S. KLAMKIN

In a previous issue [1979: 102-105], I gave subscription information about a number of publications containing problem sections. Since then, I have received a number of requests for information about other journals, so I give the following supplementary list:

1. *Fun With Mathematics.*

This is an informal publication of the Ontario Institute for Studies in Education. It is published 8 times a year and is sold in sets of 10 copies. A single issue (10 copies) costs \$1.50; a year's subscription (10 copies of all 8 issues) costs \$10.00. It is especially designed to provide continuous material for children's individual reading in mathematics and to supplement material studied at school by problems, games, and investigations in which the child can be involved on his own. (For students in the grades 5-8 range and also for bright children in grade 4 or for older children taking a general mathematics or remedial arithmetic course.) The editors are Shmuel Avital and Mary Stager.

Sample problem (1980): *You are told that the Highest Common Factor (H.C.F.) of a pair of natural numbers is* 6 *and that the Lowest Common Multiple (L.C.M.) of the same pair is* 60. *Find such a pair of numbers, if you can! Now can you find more than one pair of numbers which have* 6 *as their H.C.F. and* 60 *as their L.C.M.?*

Write to Fun With Mathematics,
c/o Mary Stager,
Ontario Institute for Studies in Education,
252 Bloor Street West,
Toronto, Ontario,
M5S 1V6.

2. *Mathematical Spectrum.*

This is a magazine for the instruction and entertainment of student mathematicians in schools, colleges, and universities, as well as the general reader interested in mathematics. It is published 3 times a year at a cost of \$9.00/yr. The editor is D.W. Sharpe.

Sample problem (1979/80): (i) *In a sequence of real numbers, the sum of every N consecutive terms is negative, whereas the sum of every M consecutive terms is positive. Show that the sequence must have fewer than M+N-D terms, where D is the highest common factor of M and N.*

(ii) *In a sequence of positive real numbers, the product of every N consecutive terms is less than* 1, *whereas the product of every M consecutive terms is greater than* 1. *Show that, again, the sequence must have fewer than M+N-D terms.*

Write to The Editor, Mathematical Spectrum,
Hicks Building,
The University,
Sheffield S3 7RH England.

3. *The Fibonacci Quarterly.*

This is a magazine devoted to the study of integers with special properties. It is the official journal of the Fibonacci Association. Dues are $18.00/yr. The problem editors are A.P. Hillman (Elementary Section) and R.E. Whitney (Advanced Section).

Sample problem (1980): *Let k and n be positive integers with $k<n$ and let S consist of all k-tuples $X=(x_1,x_2,\ldots,x_k)$ with each x_j an integer and*

$$1 \le x_1 < x_2 < \ldots < x_k \le n.$$

For $j=1,2,\ldots,k$, find the average value $\bar{x}_j$ of x_j over all X in S.

Write to Professor L. Klosinski,
Mathematics Department,
University of Santa Clara,
Santa Clara, CA 95053.

4. *Elemente der Mathematik.*

This journal is devoted to "Elementary Mathematics" and most of the material is in German (not really suitable for secondary schools). It appears 6 times a year at a cost of 39 Swiss Fr./yr. The problem editor is H. Kappus.

Sample problem (1980): *Für natürliche n beweise man*

$$\exp\frac{n(n-1)}{2} \le 1^1\cdot 2^2\cdot\ldots n^n \le \exp\frac{n(n-1)(2n+5)}{12}.$$

Write to Birkhaüser Verlag,
4010 Basel,
Switzerland.

5. *Nieuw Archief voor Wiskunde.*

This journal is published 3 times a year at a cost of Dfl. 72/yr. for nonmembers. The problem editor is M.L.J. Hautus and the problems are generally of an advanced type.

Sample problem (1980): *Prove or disprove the following statement: Let n be an integer ≥ 4. Then n points can be chosen in the plane such that their $n(n-1)/2$ perpendicular bisectors dissect the plane into convex pieces among which an $(n-1)$-gon occurs.*

Write to Adm. of Mathematisch Centrum,
Tweede Boerhaavestraat 49,
1091 A1 Amsterdam,
The Netherlands.

6. *The MATYC Journal.*

This is the journal of two-year-college mathematics and computer education. It is published 3 times a year at a cost of \$8.50/yr. or \$16.00/2 yrs. The problem editor is M.J. Brown.

Sample problem (1980): *If a, b, c, d are in arithmetic progression, then the roots of*

$$\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} + \frac{1}{x-d} = 0$$

are also in arithmetic progression.

Write to The MATYC Journal,
Department of Math./Stat./Comp.,
Nassau Community College,
Garden City, N.Y. 11530.

7. *James Cook Mathematical Notes.*

This informal journal appears 3 times a year and the editor is B.C. Rennie. The first 17 issues have been reproduced in a single bound volume available at a cost of \$5.75 US (includes postage).

Sample problem (1980): *Prove that*

$$\sum_{r=0}^{2n} (-1)^r \Big/ \binom{2n}{r} = \frac{2n+1}{n+1}.$$

Write to Professor B.C. Rennie,
Mathematics Department,
James Cook University of North Queensland,
Townsville 4811,
Australia.

8. *The Mathematical Gazette.*

This is the journal of the Mathematical Association of Great Britain, an association of teachers and students of elementary mathematics. It is published 4 times a year and is included in the membership fee. Although it does not have a formal problem section, it does have a Problem Bureau, and many problems can be extracted from the papers and notes it publishes.

Sample problem (1979): *Given sn numbers, s each of the numbers* $1,2,\ldots,n$, *to find whether they can be arranged in a row in such a way that successive occurrences of each number k are separated by exactly k other elements of the row. For example,* 4, 1,3,1,2,4,3,2 *for* $(s,n)=(2,4)$, *and* 1,9,1,6,1,8,2,5,7,2,6,9,2,5,8,4,7,6,3,5,4,9,3,8, 7,4,3 *for* $(s,n)=(3,9)$.

Write to Honorary Treasurer, Math. Assoc.,
259 London Road,
Leicester LE2 3BE,
Great Britain.

9. *Scientific American.*

Available monthly at newsstands at \$2.00/issue or by subscription at \$21.00/yr. The editor of the Mathematical Games section is Martin Gardner (who is due to retire at the end of 1981, so hurry up and read him while you can).

Sample problem (1980): *In any row of* $mn+1$ *distinct real numbers there is either an increasing subsequence of length* $m+1$ *or a decreasing subsequence of length* $n+1$.

Write to Scientific American,
415 Madison Avenue,
New York, N.Y. 10017.

10. *Canadian Mathematical Bulletin.*

This is a journal of the Canadian Mathematical Society which is published 4 times a year. The editor of the Problem Section is E.J. Barbeau. The cost is \$15.00/yr. for members of the Society and \$30.00/yr. for nonmembers. The membership fee varies from \$5.00/yr. to \$30.00/yr. for different categories of members.

Sample problem (1980): *Find the radius of the smallest circle inside which disks of radii* $1/n$ $(n=1,2,\ldots)$ *can all be packed.*

Write to G.P. Wright, Executive Secretary,
Canadian Mathematical Society,
577 King Edward Avenue,
Ottawa, Ontario, Canada K1N 6N5.

*

I am now able to give, through the courtesy of Willie Yong, the preparation problem set for the thirty-sixth Moscow Olympiad (1973). This set was translated by Mark E. Saul, and other sets of problems translated by him are slated to appear. I will publish from time to time selected elegant solutions to these problems that I receive from readers (particularly from high school students, who should include their grade, school, and location).

Readers of this journal in the U.S.S.R. (the editor tells me there are a few) could render a service by sending me, for publication in this column, preparation and final problem sets for various recent Russian Olympiads. They should send the original Russian text, and an English translation if possible.

THIRTY-SIXTH MOSCOW OLYMPIAD (1973)

(Preparation Problem Set)

1. A square island consists of several estates. Can one divide these estates into smaller estates in such a way that no new intersection points of boundaries are introduced, and so that the entire map of the island can be coloured with only two colours (estates with a common boundary being coloured differently)?

2. Can an integer consisting of six hundred digits 6 and any number of digits 0 be the square of another integer?

3. For any natural number p, consider the equation $1/x + 1/y = 1/p$. We are looking for solutions (x,y) of this equation in natural numbers (solutions (x,y) and (y,x) being considered distinct unless $x = y$). Show that the equation has exactly 3 solutions if p is prime, and more than 3 solutions if p is composite.

4. Three grasshoppers lie on a square ABCD, one at each of the vertices A, B, C. They start to play "symmetric leapfrog": if the grasshopper at A, for example, jumps over that at C, it lands at a point A' symmetric to A with respect to C. Can it happen that, after a number of jumps, one of the grasshoppers lands on vertex D?

5. A point is chosen on each side of a parallelogram in such a way that the area of the quadrilateral whose vertices are these four points is one-half the area of the parallelogram. Show that at least one of the diagonals of the quadrilateral is parallel to a side of the parallelogram.

6. A square is divided into convex polygons. Show that one can further subdivide these polygons into smaller convex polygons so that, in the new division of the square, each polygon has an odd number of neighbours (*neighbours* are polygons with a common side).

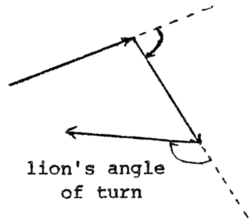
7. Given is a polynomial with integral coefficients. For three integral values of the variable, it takes on the value 2. Show that for no integral value of the variable can it take on the value 3.

8. The faces of a cube are numbered 1,2,...,6 in such a way that the sum of the numbers on opposite faces is always 7. We have a chessboard of 50×50 squares, each square congruent to a face of the cube. The cube "rolls" from the lower left-hand corner of the chessboard to the upper right-hand corner. The "rolling" of the cube consists of a rotation about one of its edges so that one face rests on a square of the chessboard. The cube may roll only upward and to the right (never downward or to the left). On each square of the chessboard that was occupied during the

trip is written the number of the face of the cube that rested there. Find the largest and the smallest sum that these numbers may have.

9. On a piece of paper is an inkblot. For each point of the inkblot, we find the greatest and smallest distances from that point to the boundary of the inkblot. Of all the smallest distances we choose the maximum and of all the greatest distances we choose the minimum. If these two chosen numbers are equal, what shape can the inkblot have?

10. A lion runs about the circular arena (radius 10 metres) of a circus tent. Moving along a broken line, he runs a total of 30 km. Show that the sum of the angles through which he turns (see figure) is not less than 2998 radians.



11. Show that in any convex equilateral (but not necessarily regular) pentagon one may place an equilateral triangle so that one of its sides coincides with a side of the pentagon and the entire triangle lies within the pentagon.

12. On an infinite chessboard, a closed simple (i.e., non-self-intersecting) path is drawn, consisting of sides of squares of the chessboard. Inside the path are k black squares. What is the largest area that can be enclosed by the path?

13. The following operation is performed on a 100-digit number: a block of 10 consecutively-placed digits is chosen and the first five are interchanged with the last five (the 1st with the 6th, the 2nd with the 7th, ..., the 5th with the 10th). Two 100-digit numbers which are obtained from each other by repeatedly performing this operation will be called *similar*. What is the largest number of 100-digit integers, each consisting of the digits 1 and 2, which can be chosen so that no two of the integers will be similar?

14. If $n \geq 2$ is a given natural number, show that there exists a natural number k such that

$$\frac{k+\sqrt{k^2-4}}{2} = \left(\frac{n+\sqrt{n^2-4}}{2}\right)^5.$$

15. On each endpoint of a line segment the number 1 is placed. As a first step, between these two numbers is placed their sum, 2. For each successive step, between every two adjacent numbers we place their sum. (After the second step we have 1,3,2,3,1; after the third we have 1,4,3,5,2,5,3,4,1; and so on.)

How many times will the number 1973 appear after a million steps?

16. The bisectors of the face angles of a trihedral angle are drawn. Show that the angles between these bisectors (taken in pairs) are either all acute, all right, or all obtuse.

17. Twelve painters live in 12 houses which are built along a circular street and are painted some white, some blue. Each month one of the painters, taking with him enough white and blue paint, leaves his house and walks along the road in the clockwise sense. On the way, he repaints every house (starting with his own) the opposite colour. He stops work as soon as he repaints some white house blue. In a year, each painter undertakes such a journey exactly once. Show that at the end of a year each house will be painted its original colour, provided that at the beginning of the year at least one house was painted blue.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

FIG. NEWTON

MRS BASHAM [Isaac Newton's housekeeper]... However, since you have one of your calculating fits on I wonder would you mind doing a little sum for me to check the washing bill. How much is three times seven?

NEWTON. Three times seven? Oh, that is quite easy.

MRS BASHAM. I suppose it is to you, sir; but it beats me. At school I got as far as addition and subtraction; but never could do multiplication or division.

NEWTON. Why, neither could I: I was too lazy. But they are quite unnecessary: addition and subtraction are quite sufficient. You add the logarithms of the numbers; and the antilogarithm of the sum of the two is the answer. Let me see: three times seven? The logarithm of three must be decimal four seven seven or thereabouts. The logarithm of seven is, say, decimal eight four five. That makes one decimal three two two, doesnt it? What's the antilogarithm of one decimal three two two? Well, it must be less than twentytwo and more than twenty. You will be safe if you put it down as—

Sally returns.

SALLY. Please, maam, Jack says it's twentyone.

NEWTON. Extraordinary! Here was I blundering over this simple problem for a whole minute; and this uneducated fish hawker solves it in a flash! He is a better mathematician than I.

GEORGE BERNARD SHAW, in Act I of *In Good King Charles's Golden Days*.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before April 1, 1981, although solutions received after that date will also be considered until the time when a solution is published.

596. *Proposed by Leroy F. Meyers, The Ohio State University.*

Automorphic numbers were discussed in my comment II to Crux 321 [1978: 252]. An *automorphic number* (in base ten) is a positive integer k whose square ends in k. (Initial zeros are permitted.) Not counting the trivial solutions 1, 01, 001, ..., there are exactly two n-digit automorphic numbers for each positive integer n. Examples are

5, 6, 25, 76, 376, 625, 0625, 9376.

For an arbitrary positive integer n, find explicit formulas for the two nontrivial n-digit automorphic numbers.

597. *Proposed by Roland H. Eddy, Memorial University of Newfoundland.*

Consider the equalities

$$\sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}} \quad \text{and} \quad \sqrt{a\frac{b}{c}} = a\sqrt{\frac{b}{c}}\ .$$

The first occurs in W. Knight's item "... But Don't Tell Your Students" [1980: 240], which inspired this problem. Find all positive integer triples (a,b,c), with b and c square-free and $(b,c) = 1$, that satisfy the second.

598. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Given a triangle ABC and a segment PQ on side BC, find, by Euclidean construction, segments RS on side CA and TU on side AB such that, if equilateral triangles PQJ, RSK, and TUL are drawn outside the given triangle, then JKL is an equilateral triangle.

599. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Prove that 36 divides the sum of the 36 integers composing a sixth-order magic square that is *pandiagonal* (magic also along the broken diagonals) or *symmetrical* (pairs symmetrical with respect to the center have a constant sum).

600. *Proposed by Jordi Dou, Escola Tecnica Superior Arquitectura de Barcelona, Spain.*

(Propuesta para CRUX dedicada al Prof. Léo Sauvé.)

En una urna hay 4 bolas señaladas [marked] con las letras C, R, U, X. Se extraen sucesivamente n bolas con devolución [with replacement]. Sea P_n la probabilidad de que aparezca CRUX en 4 extracciones consecutivas.

(a) Calcular el valor mínimo de n para que $P_n > 0.99$.

(b) Hallar una fórmula explícita de P_n en función de n.

601. *Proposed by J.A.H. Hunter, Toronto, Ontario.*

Even amateur gardeners know the value of the maxim *weed'em and reap*. But first you must find the WEEDS. Find them in

$$\begin{array}{r} \text{GEE} \\ \text{TAM} \\ \text{SEE} \\ \underline{\text{THEM}} \\ \text{WEEDS} \end{array}.$$

602. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Given are twenty natural numbers a_i such that

$$0 < a_1 < a_2 < \ldots < a_{20} < 70.$$

Show that at least one of the differences $a_i - a_j$, $i > j$, occurs at least four times.

(A student proposed this problem to me. I don't know the source.)

603. *Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.*

If k is a positive integer, show that $n^5 + 1$ is a factor of

$$(n^4 - 1)(n^3 - n^2 + n - 1)^k + (n+1)n^{4k-1}.$$

604. *Proposed by Stanley Wagon and Joan P. Hutchinson, Smith College, Northampton, Massachusetts.*

Show that one can determine with only $n(n-1)$ additions whether a real $n \times n$ matrix is such that the sum of the elements of D is constant whenever D is an n-element set having exactly one element from each row and one from each column. (The most naïve approach requires $n!(n-1)$ additions.)

*　　　　*　　　　*

MAMA-THEMATICS

Mrs. Haken to Mrs. Appel: "No forthright person can say that our boys Wolfgang and Kenneth are colourless!"

CHARLES W. TRIGG

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

491. [1979: 291; 1980: 290] *Proposé par Alan Wayne, Pasco-Hernando Community College, New Port Richey, Floride.*

(Dédié au souvenir de Victor Thébault, jadis inspecteur d'assurances à Le Mans, France.)

Résoudre la cryptarithmie décimale suivante:

$$\text{UN} + \text{DEUX} + \text{DEUX} + \text{DEUX} + \text{DEUX} + \text{DEUX} = \text{ONZE}.$$

III. *Comment by Herman Nyon, Paramaribo, Surinam.*

In his Comment II [1980: 291], Simpson gives only one solution to his Spanish variant of this problem, although he implies there may be more solutions that he did not bother to find. There are indeed: I found 30 solutions, which makes it a very, very bad alphametic. But it can easily be redeemed by imposing additional conditions to reduce the number of solutions. For example, if we require that only the digits 1 to 7 be used (or else that ONCE be minimal), then

$$\text{UNO} + \text{DOS} + \text{DOS} + \text{DOS} + \text{DOS} + \text{DOS} = \text{ONCE}$$

has the unique solution

$$451 + 217 + 217 + 217 + 217 + 217 = 1536.$$

* * *

495. [1979: 292; 1980: 322] *Proposed by J.L. Brenner, Palo Alto, California; and Carl Hurd, Pennsylvania State University, Altoona Campus.*

Let S be the set of lattice points (points having integral coordinates) contained in a bounded convex set in the plane. Denote by N the minimum of two measurements of S: the greatest number of points of S on any line of slope 1, -1. Two lattice points are *adjoining* if they are exactly one unit apart. Let the n points of S be numbered by the integers from 1 to n in such a way that the largest difference of the assigned integers of adjoining points is minimal. This minimal largest difference we call the *discrepancy* of S.

(a) Show that the discrepancy of S is no greater than $N+1$.

(b) Give such a set S whose discrepancy is $N+1$.

(c) Show that the discrepancy of S is no less than N.

Solution by the proposers.

(a) (The points under discussion are all in S.) We must show that we may number the points of S in such a way that the difference of the associated integers of any two adjoining points is no greater than $N+1$. Consider the set of parallel lines none of which contains more than N points. We may assume that these lines all have slope 1 and, for geometric clarity, we rotate them 45°, thus visualizing them as a set of vertical lines.

Figure 1 illustrates the meaning of some terms we will use in our proof. The terms refer to the relative positions of points on a line immediately to the right of a given line.

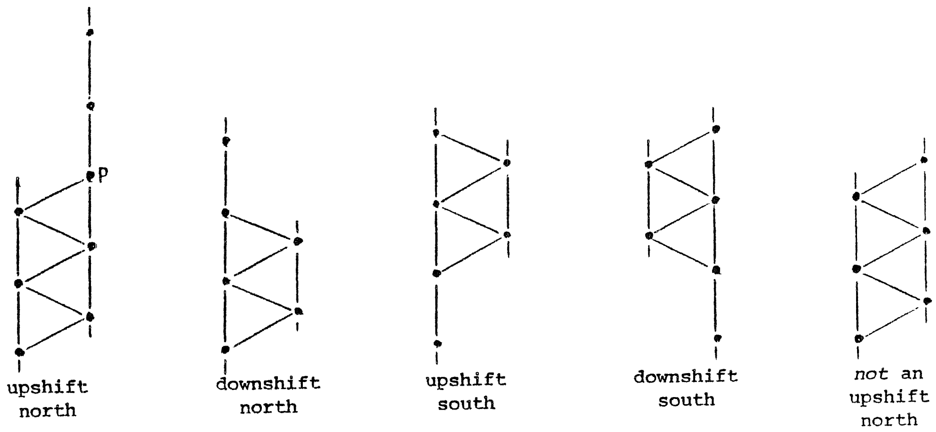


Figure 1

In all of S, due to convexity,

(1) we cannot have a downshift north to the left of an upshift north, and

(2) we cannot have an upshift south to the left of a downshift south.

We will employ only two numbering schemes for the points of S. On each vertical line we number adjacent points by adjacent integers, and number adjacent lines by adjacent intervals of integers; these intervals increase from left to right. In scheme A the integers increase for higher points, and in scheme B the integers decrease for higher points (on each line). If scheme B fails (i.e., produces a discrepancy greater than $N+1$) for an upshift north, then that upshift is called *severe* (the total number of points to the left of or above point P in the first diagram of Figure 1 is greater than N). A *severe* downshift south causes scheme A to fail.

(3) A severe upshift north [downshift south] implies an upshift south [downshift north] for the same two lines.

Now for the actual numbering. If there are no severe upshifts morth, scheme B succeeds.

Suppose there is a severe upshift north. We use scheme A. By (3), there is an upshift south for the same two lines. By (2), there are no downshifts south to the right of this upshift south. If there were a severe downshift south to the left of this upshift south, there would be (by (3)) a downshift north to the left of an upshift north, which (1) prohibits. Thus scheme A succeeds.

(b) Consider the set S pictured in Figure 2 (usual orientation again). (These points can be surrounded by a closed convex curve that contains no other lattice points, as the dotted line suggests.) As the slanted lines show, $N = 2$. The discrepancy, however, is 3. We can show this most easily by exhausting the possible integers for point P.

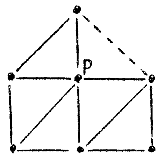


Figure 2

(c) This assertion was proved for a rectangular lattice by Chvátalova [1] and by a different method in *A.M.M.* Problem E 2732 [2]. Still required is a proof (or disproof) for an arbitrary convex region.

Editor's comment.

This argument and conclusion generalize the assertion of Problem E 2732 [2]. The result had previously been obtained by Jarmila Chvátalova [1]. The argument given here seems to be more direct. In another article, to appear in the *Two-Year College Mathematics Journal*, J.L. Brenner has generalized the result (and argument) to several dimensions, to Möbius strips, etc. (The fact that a simpler argument was eventually found to prove her theorem must not detract from the significance of the original discovery of Chvátalova.)

REFERENCES

1. Jarmila Chvátalova, "Optimal Labelling of a Product of Two Paths," *Discrete Mathematics*, 11 (1975) 249-253.

2. Peter Sjögren (proposer), Problem E 2732, *American Mathematical Monthly*, 86 (1979) 867-868.

* * *

502. [1980: 15] *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

Given $n > 3$ points in the plane, no three collinear, we are interested in "triangulating" their convex hull, that is, in covering it with nonoverlapping triangles, each having three of the given points as vertices.

(a) For a fixed set of points, there are several ways of triangulating. Do they all give the same number of triangles?

(b) For fixed n, different sets of n points may be triangulated with different numbers of triangles. What bounds can be given for the number of triangles?

I. *Solution by Ferrell Wheeler, student, Forest Park H.S., Beaumont, Texas.*

Let $T(n)$ be the number of triangles required to triangulate the convex hull of a set of n points, no three of which are collinear. The convex hull is a q-gon, where $3 \le q \le n$, with $n-q$ of the points in its interior. Let $A(n,q)$ be the sum of the radian measures of all the angles in the triangulation. Each interior point contributes 2π to A, and the q-gon itself contributes $(q-2)\pi$. Thus we have

$$A(n,q) = 2\pi(n-q) + \pi(q-2) = \pi(2n-q-2).$$

Since each triangle contributes π to A, we have

$$T(n) = A(n,q)/\pi = 2n-q-2. \tag{1}$$

If both n and q are fixed, so is T, and this answers part (a) in the affirmative. For part (b), where only n is fixed, the bounds are found by letting q take on its two extreme values, n and 3, resulting in the sharp bounds

$$n-2 \le T(n) \le 2n-5. \tag{2}$$

II. *Adapted from a comment by George Tsintsifas, Thessaloniki, Greece.*

In the notation of solution I, relation (1) is easily established by induction by noting that, for fixed $q \ge 3$, it holds for $n=q$, and that increasing n by 1 (i.e., adding an interior point) increases T by 2.

Induction is also useful to establish corresponding results in 3-dimensional space. Suppose n points are given in general position (no three in a line, no four in a plane) whose convex hull has q vertices, where $4 \le q \le n$. Then one easily shows by induction that a triangulation of the surface of the convex hull necessarily contains $2q-4$ triangles, and , with the help of this result, that the "tetrahedration" of the interior of the convex hull contains

$$\tau(n) = 3n - q - 7$$

tetrahedra, from which, when only n is fixed, we get the sharp bounds

$$2n-7 \le \tau(n) \le 3n-11.$$

Also solved by L.F. MEYERS, The Ohio State University; GEORGE TSINTSIFAS, Thessaloniki, Greece (3 solutions); and the proposer.

Editor's comment.

Implicit in solution I and comment II (also in the proposer's solution) is the assumption that all $n-q$ interior points of the convex hull are vertices of the triangulation. But the wording of the proposal does allow a less restrictive interpretation. Meyers showed that, if only m of the $n-q$ interior points are used (the remaining points being "covered" but not used as vertices), then (1) becomes

$$T(n) = 2m+q-2, \qquad 0 \le m \le n-q,$$

and (2) becomes

$$1 \le T(n) \le 2n-5,$$

where the bounds both occur for $q=3$, when $m=0$ and $m=n-3$, respectively.

* * *

503. [1980: 15] *Proposed by Meir Katchalski, Technion, Israel and Andy Liu, University of Regina.*

Let the *north-east corner* of a compact set S in the plane be the point (a,b) such that

$$a = \max\{x \mid (x,y) \in S\} \quad \text{and} \quad b = \max\{y \mid (a,y) \in S\}.$$

Let F be a family of at least two compact convex sets in the plane with nonempty intersection. Prove that there exist two sets in F such that the north-east corner of their intersection coincides with the north-east corner of the intersection of the entire family.

Solution by the proposers.

Let (a,b) be the north-east corner of $\cap\mathrm{F}$ and define

$$D = \{(x,y) \mid x > a\} \cup \{(a,y) \mid y > b\}.$$

D is clearly a convex set. If we let $\mathrm{F}^* = \mathrm{F} \cup \{D\}$, it is easy to see that $\cap\mathrm{F}^*$ is empty. By Helly's Theorem, F^* contains a three-member subfamily G^* with empty intersection. Since $\cap\mathrm{F}$ is nonempty it follows that $D \in \mathrm{G}^*$. Let

$$\mathrm{G} = \mathrm{G}^* - \{D\} = \{A,B\}, \text{ say,}$$

and let (m,n) be the north-east corner of $A \cap B$. Since $\cap\mathrm{F}$ is a subset of $A \cap B$, we have either

$$a < m \quad \text{or} \quad a = m \text{ and } b < n \quad \text{or} \quad a = m \text{ and } b = n.$$

However, if the first or second cases hold, then $A \cap B \cap D$ is nonempty by the definition of D, a contradiction. Hence $(a,b) = (m,n)$, as required. □

The generalized form of this problem is a critical step in the argument of the paper "A Problem of Geometry in R^n" by the proposers. The paper appeared in the *Proceedings of the American Mathematical Society*, 75 (1979) 284-288.

Also solved by MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; and GEORGE TSINTSIFAS, Thessaloniki, Greece.

* * *

504. [1980: 15] *Proposed by Leon Bankoff, Los Angeles, California and Jack Garfunkel, Flushing, N.Y.*

Given is a triangle ABC and its circumcircle. Find a Euclidean construction for a point J inside the triangle such that, when the chords AD, BE, CF are all drawn through J, then triangle DEF is equilateral.

I. *Solution by Jordi Dou, Escola Tecnica Superior Arquitectura de Barcelona, Spain.*

Take points P, Q on the circumcircle such that arc $PQ = 120^\circ$, and let $R = AP \cap BQ$ and $S = BP \cap CQ$. The intersection ($\neq B$) of the circumcircles of triangles ABR and BCS is the required point J (which is not always inside the given triangle).

The proof (which can vary slightly depending on the shape of triangle ABC and the location of the points P and Q) goes essentially like this: $\angle AJB = \angle ARB$ implies arc DE = arc PQ, and similarly arc EF = arc PQ.

II. *Solution by O. Bottema, Delft, The Netherlands.*

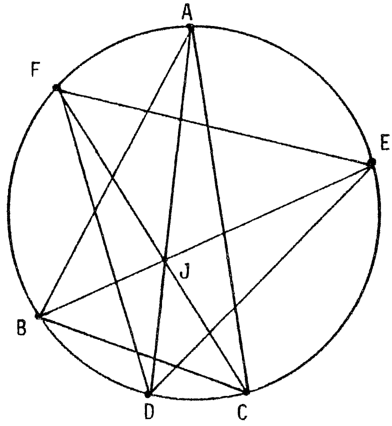
For any point J in the plane, the corresponding triangle DEF is directly similar to the pedal triangle of J with respect to triangle ABC (Johnson [1]). It is known that the pedal triangle of J is equilateral if and only if J coincides with one of the two isodynamic points J_1, J_2 of triangle ABC (see explanation below). So J_1 and J_2 each constitute a solution to our problem (at most one of which, however, lies inside the given triangle).

The construction is now obvious. Describe any two of the three Apollonian circles of triangle ABC. Their intersections J_1 and J_2 are the required points. □

A few words of clarification may be in order. Let the interior and exterior bisectors of angles A, B, C of triangle ABC meet the opposite sides BC, CA, AB in the points U, U'; V, V'; W, W',respectively. The circles on UU', VV', WW' as diameters are called the *Apollonian circles*, or the *circles of Apollonius*, of triangle ABC. The three Apollonian circles of a triangle have two points J_1, J_2 in common. These are called the *isodynamic points* of the triangle. See Altshiller Court [2] for further details.

III. *Solution by Howard Eves, University of Maine.*

It is just as easy to solve the more general problem in which we seek J so that triangle DEF will be of any given shape; we may call triangles ABC and DEF a pair of *cocyclic perspective triangles*. Later we will easily further extend the problem to the situation of finding a point J so that any given cyclic n-gon will yield a second cyclic n-gon of given shape. Since there exist triangles ABC for which there is no point J inside ABC leading to a triangle DEF of given shape, we replace the restriction that J lie inside the given triangle by the really significant restriction that J lie inside the given circumcircle.



We see from the figure that

$$\angle BJC = \tfrac{1}{2}(\text{arc } BDC + \text{arc } FAE) = \angle A + \angle D.$$

Similarly,

$$\angle CJA = \angle B + \angle E \quad \text{and} \quad \angle AJB = \angle C + \angle F.$$

Point J is then easily constructed by drawing the appropriate circular arcs on any two of the chords BC, CA, AB. For triangle DEF to be equilateral, we merely take $\angle D = \angle E = \angle F = 60^\circ$.

The extension to the situation where triangle ABC is replaced by a given cyclic n-gon $A_1A_2 \ldots A_n$ and triangle DEF by a cyclic n-gon $B_1B_2 \ldots B_n$ of given shape is straightforward. If O is the center of the circumcircle of $A_1A_2 \ldots A_n$, then

$$\angle A_1JA_2 = \tfrac{1}{2}(\angle A_1OA_2 + \angle B_1OB_2), \text{ etc.}$$

Finally, we point out that another solution can be found where J lies *outside* the given circumcircle; here we have $\angle A_1JA_2$ given by the difference of angles B_1OB_2 and A_1OA_2 instead of their sum, etc.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; J.T. GROENMAN, Arnhem, The Netherlands; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal et LUC ST-LOUIS, Montréal, Québec (conjointement); GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposers.

Editor's comment.

Tranquille and St-Louis observed that no solution point J lies inside the given triangle if the latter has an angle greater than 120°.

REFERENCES

1. Roger A. Johnson, *Advanced Euclidean Geometry*, Dover, New York, 1960, p. 141, Theorem 199.

2. Nathan Altshiller Court, *College Geometry*, Barnes & Noble, New York, 1952, 260-267.

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505. [1980: 15] *Proposed by Bruce King, Western Connecticut State College and Sidney Penner, Bronx Community College.*

Let

$$F_1 = F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n > 2$$

and

$$G_1 = 1, \qquad G_n = 2^{n-1} - G_{n-1} \quad \text{for} \quad n > 1.$$

Show that (a) $F_n \le G_n$ for each n and (b) $\lim_{n\to\infty} F_n/G_n = 0$.

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

We will use the following explicit formulas for F_n and G_n:

$$F_n = \frac{a^n - b^n}{a-b}, \qquad n = 1,2,3,\ldots, \tag{1}$$

where $a = \frac{1}{2}(1+\sqrt{5})$ and $b = \frac{1}{2}(1-\sqrt{5})$, and

$$G_n = \frac{2^n - (-1)^n}{3}, \quad n = 1,2,3,\ldots\ . \tag{2}$$

The first is the well-known formula for Fibonacci numbers and the second is easily verified by induction.

(a) We show that

$$d_n \equiv G_n - F_n \ge 0, \qquad n = 1,2,3,\ldots\ . \tag{3}$$

Since, for $n > 2$,

$$G_n = 2^{n-1} - (2^{n-2} - G_{n-2}) = 2^{n-2} + G_{n-2} = G_{n-1} + 2G_{n-2},$$

we have

$$d_n = (G_{n-1} + 2G_{n-2}) - (F_{n-1} + F_{n-2}) = d_{n-1} + d_{n-2} + G_{n-2}. \tag{4}$$

Now $d_1 = d_2 = 0$, and (3) follows from (4) by induction.

(b) Observing that $|b| < 1$, so that $\lim b^n = 0$, we have from (1) and (2)

$$\lim_{n\to\infty} F_n/G_n = \lim_{n\to\infty} \frac{3}{\sqrt{5}}\left(\frac{a}{2}\right)^n = 0,$$

since $a/2 \approx 0.809 < 1$.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; UNDERWOOD DUDLEY, DePauw University, Greencastle, Indiana; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; BENGT MÅNSSON, Lund, Sweden; L.F. MEYERS, The Ohio State University; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; BOB PRIELIPP, The University of Wisconsin-Oshkosh; HYMAN ROSEN, student, The Cooper Union, New York; MARK TERRIBILE, Courant Institute, New York University; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; FERRELL WHEELER, Beaumont, Texas; KENNETH M. WILKE, Topeka, Kansas; and the proposers.

Editor's comment.

We regret to announce that our second proposer, Sidney Penner, died unexpectedly on 31 May 1980. He had been since 1977 Problem Editor for the *New York State Mathematics Teachers' Journal.*

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506. [1980: 16] *Proposed by M.S. Klamkin, University of Alberta.*

It is known from an earlier problem in this journal [1975: 28] that if a, b, c are the sides of a triangle, then so are $1/(b+c)$, $1/(c+a)$, $1/(a+b)$. Show more generally that if $a_1, a_2, \ldots, a_n$ are the sides of a polygon then, for $k=1,2,\ldots,n$,

$$\frac{n+1}{S-a_k} \geq \sum_{\substack{i=1 \\ i\neq k}}^{n} \frac{1}{S-a_i} \geq \frac{(n-1)^2}{(2n-3)(S-a_k)},$$

where $S=a_1+a_2+\ldots+a_n$.

Solution by George Tsintsifas, Thessaloniki, Greece.

Since any side may be labeled a_n, there is no loss of generality in proving the two inequalities only for $k=n$.

The first is equivalent to

$$\frac{n+2}{S-a_n} \geq \sum_{i=1}^{n} \frac{1}{S-a_i}.$$

Since $(n+2)/(S-a_n) \geq (n+2)/S$, it suffices to show that

$$\frac{n+2}{S} \geq \sum_{i=1}^{n} \frac{1}{S-a_i}$$

or, equivalently, that

$$2 \geq \sum_{i=1}^{n} \frac{a_i}{S-a_i}. \qquad (1)$$

Now $S-a_i \geq a_i$ holds for any i, and this is equivalent to

$$\frac{2a_i}{S} \geq \frac{a_i}{S-a_i}, \qquad (2)$$

If we sum both sides of (2) for $i=1$ to n, we obtain (1).

To establish the second inequality, we first observe that the arithmetic-harmonic mean inequality yields

$$\frac{1}{n-1}\sum_{i=1}^{n-1} \frac{1}{S-a_i} \geq \frac{n-1}{(n-2)S+a_n},$$

so it suffices to show that

$$\frac{(n-1)^2}{(n-2)S+a_n} \geq \frac{(n-1)^2}{(2n-3)(S-a_n)}.$$

But this is equivalent to $S - a_n \geq a_n$, which is obviously true.

The proposed inequalities are both strict for nondegenerate polygons.

Also solved by the proposer.

Editor's comment.

The proposer made two observations about the special case $n = 3$: he noted that the earlier result mentioned in the proposal is now improved to

$$\frac{1}{b+c} + \frac{1}{c+a} \geq \frac{4}{3} \cdot \frac{1}{a+b}\text{ , etc.,}$$

and that the special case of inequality (1) occurs in Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969, p. 15.

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507. [1980: 16] *Proposed by Rufus Isaacs, Baltimore, Maryland.*

The sequence of numbers $(t_1, t_2, \ldots, t_n)$ is symmetric in that $t_k = t_{n+1-k}$ for $k = 1,2,\ldots,n$. The sequences (x_k) and (y_k) are defined by

$$\begin{aligned} x_0 &= 0, \quad x_1 = 1; \quad x_{k+1} = t_k x_k + x_{k-1}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad k = 1,2,\ldots,n. \\ y_0 &= 1, \quad y_1 = 0; \quad y_{k+1} = t_k y_k + y_{k-1}. \end{aligned}$$

Show that $x_n = y_{n+1}$.

Solution by Underwood Dudley, DePauw University, Greencastle, Indiana.

Before trying thought, look in Chrystal's *Algebra*. The following comes, essentially, from pages 407 and 408 of the second volume of the 1889 edition [pages 431-433 in the more accessible 1900 edition reprinted by Chelsea in 1952].

The required relation holds by definition for $n = 0$, so we assume $n \geq 1$. If we put $p_k = x_{k+1}$, $q_k = y_{k+1}$ and $a_k = t_k$, then, in standard symbols, p_k/q_k is the kth convergent in the continued fraction expansion of the rational number whose partial quotients are a_1, a_2, $\ldots$, a_n. As Chrystal points out,

$$p_n/p_{n-1} = [a_n, a_{n-1}, \ldots, a_1],$$

and, because of symmetry, this is the same as

$$[a_1, a_2, \ldots, a_n] = p_n/q_n.$$

Since convergents are in lowest terms, $p_{n-1} = q_n$, as was to be shown.

Also solved by L.F. MEYERS, The Ohio State University (two solutions, one of which was similar to the above).

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508. [1980: 16] *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

Problem 24 in W. Sierpiński's *250 Problems in Number Theory* (American Elsevier, 1970) asks for an infinite set of pairs (x,y) of positive integers such that $x^x | y^y$ but $x \nmid y$. The answer given is $x = 2^k$, $y = 2p$, where $k > 1$ is an integer and $p > k \cdot 2^{k-1}$ is a prime. Show that there is an infinite set of pairs (x,y) with the same property in which x contains an odd factor.

Solution by Leroy F. Meyers, The Ohio State University.

The solution given in Sierpiński's book is easily generalized. Let $m \geq 2$ and $k \geq 2$ be integers, and let p be any integer greater than km^{k-1} and relatively prime to m. Then, for $x = m^k$ and $y = mp$, we have

$$x^x = m^{km^k} = (m^m)^{km^{k-1}} \mid (m^m)^p \mid (mp)^{mp} = y^y,$$

and so $x^x \mid y^y$. But $x \nmid y$, for otherwise $m^k \mid mp$ and $m^{k-1} \mid p$, which is impossible.

Also solved by UNDERWOOD DUDLEY, DePauw University, Greencastle, Indiana; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; VIKTORS LINIS, University of Ottawa; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; BOB PRIELIPP, The University of Wisconsin-Oshkosh; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; FERRELL WHEELER, Beaumont, Texas; and the proposer.

Editor's comment.

Ecker and Stone both showed that, for every positive integer $x \geq 4$ that is *not* square-free, infinitely many suitably defined y exist such that (x,y) is a solution. The proof is notationally messy but otherwise not essentially different from the one given above.

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509. [1980: 16] *Proposed by Charles W. Trigg, San Diego, California.*

Is there any system of notation in which there is a repdigit aa such that

$$(aa)^2 = ccdd, \quad \text{with} \quad c = a - 1 \quad \text{and} \quad d = c - 1?$$

Solution by Underwood Dudley, DePauw University, Greencastle, Indiana.

Suppose there is a solution in base r, a positive integer. Then

$$(a + ar)^2 = (a - 2) + (a - 2)r + (a - 1)r^2 + (a - 1)r^3$$

$$= a(r + 1)(r^2 + 1) - (r + 1)(r^2 + 2).$$

Cancelling an $r+1$ gives, upon rearrangement,

$$(r+1)a^2 = a(r+1)(r-1)+2a-(r+1)(r-1)-3$$

or

$$(r+1)\{a^2-(r-1)a+r-1\} = 2a-3. \tag{1}$$

Thus $r+1$ divides $2a-3$. But a is a digit in base r, so

$$2a-3 < 2r-3 < 2(r+1),$$

and it follows that $r+1 = 2a-3$. Now substitution in (1) gives

$$a^2-(2a-5)a + 2a-5 = 1,$$

which is true for $a=1$ and $a=6$. The first is impossible, so $a=6$, $r=8$, and the unique solution is $66^2=5544$ in base eight.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; NGO TAN, student, J.F.Kennedy H.S., Bronx, N.Y.; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Ahlburg noted that 5544 in base eight becomes 2211 in base eleven.

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510. [1980: 16] *Proposed by Gali Salvatore, Perkins, Québec.*

There is only one integer n for which the expression

$$E(n) \equiv \frac{12n^3-5n^2-251n+389}{6n^2-37n+45}$$

is an integer. Find this value of n and show there are no others.

Solution by W.J. Blundon, Memorial University of Newfoundland.

We have

$$E(n) = 2n+9+\frac{(3n-4)(5n+4)}{(3n-5)(2n-9)}.$$

Since the consecutive integers $3n-4$ and $3n-5$ are relatively prime, $E(n)$ is integral only if $3n-5$ divides both

$$5n+4 \quad \text{and} \quad 3(5n+4) - 5(3n-5) = 37.$$

Thus $3n-5=\pm 1$ or ± 37. Now n is not an integer if $3n-5=-1$ or -37, and $E(n)$ is not an integer if $3n-5=1$ (and $n=2$). The last possibility, $3n-5=37$, yields the unique solution $n=14$, since $E(14)=41$ is an integer.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; UNDERWOOD DUDLEY, DePauw University, Greencastle, Indiana; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; VIKTORS LINIS, University of Ottawa; BENGT MÅNSSON, Lund, Sweden; J.A. McCALLUM, Medicine Hat, Alberta; HARRY L. NELSON, Livermore, California; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; SANJIB KUMAR ROY, Indian Institute of Technology, Kharagpur, India; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; CHARLES W. TRIGG, San Diego, California; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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501. [1980: 15, 330] *Proposed by J.A.H. Hunter, Toronto, Ontario.*

For the second time in less than a year, Canadians are undergoing the throes of a national election. In the alphametic

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you won't be sure about the FOOLS (until after the election), but what's the unique value of OTTAWA?

Editor's comment.

In last month's issue [1980: 330], we reported that, with the assumption $L' \neq L$, Kenneth M. Wilke had found two pairs of solutions to this problem, yielding two distinct values of OTTAWA. The published version did an injustice to Wilke, since it was edited from an incomplete solution: it has just now come to light that page 2 of Wilke's solution had been accidentally misfiled with a later problem. On that page, Wilke gave a third pair of solutions,

$$\begin{array}{r} 81 \\ 51127 \\ 49186 \\ \underline{40} \\ 100434 \end{array} \quad \text{and} \quad \begin{array}{r} 81 \\ 51126 \\ 49187 \\ \underline{40} \\ 100434 \end{array},$$

which are identical except for the interchange of 6 and 7, and provide a third value for OTTAWA. (It must be for Ottawa, Ill.)

Wilke had also given, on the same misfiled page, the unique value of OTTAWA, 100373, which he realized very well was the one intended by the proposer and which results from the assumption $L' = L$.

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