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Managing Editor: F.G.B. Maskell, Algonquin College, 200 Lees Ave., Ottawa, Ontario, Canada K1S 0C5.

Typist-compositor: Kuen Huynh.

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CONTENTS

A Dice-Tossing Problem . . . . Curtis Cooper and Robert E. Kennedy 134

Laying to Rest that Mersenne Number . . . . Clayton W. Dodge 138

On a Triangle Inequality . . . . M.S. Klamkin 139

The Olympiad Corner: 55 . . . . M.S. Klamkin 141

The Puzzle Corner . . . . Alan Wayne 154

Problems - Problèmes 155

Solutions 157

A DICE-TOSSING PROBLEM

CURTIS COOPER and ROBERT E. KENNEDY

1. *Introduction.*

Coin-tossing is an event often studied in courses on probability theory. One way to generalize coin-tossing problems is to consider tossing polyhedral dice with m faces labeled $0, 1, \ldots, m-1$. The face that "scores" is the "down" face, not the "up" face which may not exist (e.g., in a tetrahedral die). Problems involving the tossing of n m-faced dice have been studied in [5] and [6]. In the first part of this paper, we will introduce a notation to represent the number of ways that a person tossing n m-faced dice can obtain a sum of k, and then develop some of the properties of this notation. The remainder of the paper will use this notation to solve a generalization to polyhedral dice of a well-known coin-tossing problem.

2. *The notation and its properties.*

Let n, m, and k be integers with $n,m \geq 1$. The symbol $\binom{n}{k}_m$ is defined as follows:

$$\binom{1}{k}_m = \begin{cases} 1, & \text{if } 0 \leq k \leq m-1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\binom{n}{k}_m = \sum_{i=0}^{m-1} \binom{n-1}{k-i}_m, \qquad n > 1.$$

For $m = 2$, the quantity $\binom{n}{k}_m$ is the binomial coefficient $\binom{n}{k}$. In general, $\binom{n}{k}_m$ represents the number of ways that a person tossing n m-faced dice can obtain a sum of k. To show this, we use induction on n. The statement is clearly true for $n = 1$. Suppose it holds for some $n \geq 1$. Then the number of ways of obtaining a sum of k by tossing $n+1$ m-faced dice is the sum, for $i = 0,1,\ldots,m-1$, of the number of ways of obtaining a sum of $k-i$ by tossing n m-faced dice times the number of ways $(= 1)$ of obtaining a score of i by tossing one m-faced die. By the induction hypothesis, this sum is equal to

$$\sum_{i=0}^{m-1} \binom{n}{k-i}_m = \binom{n+1}{k}_m,$$

and the induction is complete.

The first theorem we present will give some identities involving $\binom{n}{k}_m$.

THEOREM 1.

(i) $\binom{n}{0}_m = 1.$

(ii) $\binom{n}{1}_m = n$, if $m > 1$.

(iii) $\binom{n}{s}_m = \binom{n}{t}_m$, if $s+t = n(m-1)$.

(iv) $\sum_{k=0}^{n(m-1)} \binom{n}{k}_m = m^n$.

Proof of (i). A person tossing n m-faced dice can obtain a sum of 0 in exactly one way: all the n dice score 0.

Proof of (ii). A person tossing n m-faced dice can obtain a sum of 1 in exactly n ways: for each $i = 1,2,\ldots,n$, the ith die scores 1 and all the other dice score 0.

Proof of (iii). Let $(x_1, x_2, \ldots, x_n)$ be an ordered n-tuple where each x_i represents the score of the ith die, and suppose $x_1+x_2+\ldots+x_n = s$. Associate with this ordered n-tuple the ordered n-tuple

$$(m-1-x_1,\ m-1-x_2,\ \ldots,\ m-1-x_n).$$

Then we have a bijection from the set of all dice-tossings with a sum of s onto the set of all dice-tossings with a sum of t, where $t = n(m-1) - s$, or $s+t = n(m-1)$.

Proof of (iv). The sum on the left is the total number of outcomes of tossing n m-faced dice. Since we have n dice with m faces each, this number is m^n. $\square$

The generating function which produces the quantity $\binom{n}{k}_m$ is

$$f(x) = \sum_{k=0}^{\infty} \binom{n}{k}_m x^k = (1 + x + \ldots + x^{m-1})^n.$$

The following theorem is proved by using this generating function.

THEOREM 2.

(i) $\sum_{k=0}^{n(m-1)} (-1)^k \binom{n}{k}_m = \begin{cases} 1, & \text{if } m \text{ is odd,} \\ 0, & \text{if } m \text{ is even.} \end{cases}$

(ii) $\sum_{k=1}^{n(m-1)} k\binom{n}{k}_m = \frac{nm^n(m-1)}{2}$.

(iii) $\sum_{k=1}^{n(m-1)} (-1)^{k-1} k\binom{n}{k}_m = \begin{cases} \frac{n(1-m)}{2}, & \text{if } m \text{ is odd,} \\ 0, & \text{if } m \text{ is even.} \end{cases}$

(iv) $\sum_{i=0}^{k} \binom{n_1}{i}_m \cdot \binom{n_2}{k-i}_m = \binom{n_1+n_2}{k}_m$.

Proof of (i). The sum on the left equals $f(-1)$.

Proof of (ii). The sum on the left equals $f'(1)$.

Proof of (iii). The sum on the left equals $f'(-1)$.

Proof of (iv). The result follows by equating the coefficients of x^k on both sides of the identity

$$(1+x+\ldots+x^{m-1})^{n_1}\cdot(1+x+\ldots+x^{m-1})^{n_2} = (1+x+\ldots+x^{m-1})^{n_1+n_2}. \quad \square$$

The following theorem [1] gives a method of calculating $\binom{n}{k}_m$ in terms of binomial coefficients. However, unlike [1], our proof is strictly combinatorial.

THEOREM 3. $\binom{n}{k}_m = \sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{n-1+k-mi}{n-1}$.

Proof. The quantity $\binom{n}{k}_m$ is the number of integral solutions of

$$x_1 + x_2 + \ldots + x_n = k, \tag{1}$$

where each x_i satisfies $0 \le x_i \le m-1$. Let U be the set of *all* integral solutions of (1) where each $x_i \ge 0$; and, for each $j = 1,2,\ldots,n$, let A_j be the set of integral solutions of (1) for which $x_i \ge 0$ for $1 \le i \le j-1$, $x_j \ge m$, and $x_i \ge 0$ for $j+1 \le i \le n$. With the horizontal bar denoting complementation relative to U and the vertical bars set cardinality, the principle of inclusion-exclusion [4] gives

$$\begin{aligned}\binom{n}{k}_m &= |\bar{A}_1\cap\bar{A}_2\cap\ldots\cap\bar{A}_n| \\ &= |U| - \sum_i|A_i| + \sum_{i,j}|A_i\cap A_j| - \sum_{i,j,k}|A_i\cap A_j\cap A_k| + \ldots + (-1)^n|A_1\cap A_2\cap\ldots\cap A_n| \\ &= |U| - \binom{n}{1}|A_1| + \binom{n}{2}|A_1\cap A_2| - \binom{n}{3}|A_1\cap A_2\cap A_3| + \ldots + (-1)^n|A_1\cap A_2\cap\ldots\cap A_n|.\end{aligned}$$

But the number of integral solutions of (1) for which $x_i \ge r_i$ for $i = 1,2,\ldots,n$ is

$$\binom{n-1+k-r_1-r_2-\ldots-r_n}{n-1}.$$

Therefore the value last obtained for $\binom{n}{k}_m$ equals

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{n-1+k-mi}{n-1},$$

as required.

3. *A dice-tossing problem with polyhedral dice.*

The principal aim of this paper is to propose and solve the following problem, which is motivated by [2], [3], and [7]:

If A tosses n $(n+m)$-faced dice and B tosses $n+m$ n-faced dice, what is the probability $P(n,m)$ that A obtains a larger sum than B?

Solution. For $0 \le r \le (n-1)(n+m)$, B can toss a sum of r in

$$\binom{n+m}{r}_n$$

ways. For each such r, A can toss a larger sum than B by tossing a sum of $r+s$ for some s with $1 \le s \le (n+m-1)n - r$, and this A can do in

$$\binom{n}{r+s}_{n+r}$$

ways. Hence the required probability is

$$P(n,m) = \sum_{r=0}^{(n-1)(n+m)} \frac{\binom{r+m}{r}_n}{n^{n+m}} \sum_{s=1}^{(n+m-1)n-r} \frac{\binom{n}{r+s}_{n+m}}{(n+m)^n}$$

$$= \frac{1}{n^{n+m}(n+m)^n} \sum_{r=0}^{(n-1)(n+m)} \sum_{s=1}^{(n+m-1)n-r} \binom{n+m}{r}_n \binom{n}{r+s}_{n+m}. \quad \square \qquad (2)$$

As an example, we calculate $P(3,2)$.

$$P(3,2) = \frac{1}{3^5 \cdot 5^3} \sum_{r=0}^{10} \sum_{s=1}^{12-r} \binom{5}{r}_3 \binom{3}{r+s}_5$$

$$= \frac{1}{3^5 \cdot 5^3} (1\cdot 124 + 5\cdot 121 + 15\cdot 115 + 30\cdot 105 + 45\cdot 90 + 51\cdot 72$$

$$+ 45\cdot 53 + 30\cdot 35 + 15\cdot 20 + 5\cdot 10 + 1\cdot 4)$$

$$= \frac{17115}{30375} \approx 0.563.$$

As Klamkin notes in [3], it is doubtful that the double sum in (2) can be reduced to a "simple" single one. But for one special case the calculation is particularly easy, and the result surprising: *if $m=1$, then $P(n,1) = \frac{1}{2}$ for all n.* For

$$P(n,1) = \sum_{j>i} \frac{\binom{n}{j}_{n+1}}{(n+1)^n} \cdot \frac{\binom{n+1}{i}_n}{n^{n+1}}.$$

But the function $F(i,j) = (n^2-1-i,\ n^2-j)$ maps the set of lattice points

$$\{(i,j) \mid j > i,\ 0 \le j \le n^2,\ 0 \le i \le n^2-1\}$$

bijectively onto the set of lattice points

$$\{(i,j) \mid j \le i,\ 0 \le j \le n^2,\ 0 \le i \le n^2-1\}$$

and

$$\binom{n}{j}_{n+1}\binom{n+1}{i}_n = \binom{n}{n^2-j}_{n+1}\binom{n+1}{n^2-1-i}_n.$$

Therefore

$$P(n,1) = \frac{1}{2}\sum_{i,j} \frac{\binom{n}{j}_{n+1}}{(n+1)^n} \cdot \frac{\binom{n+1}{i}_n}{n^{n+1}} = \frac{1}{2} \cdot \frac{1}{(n+1)^n n^{n+1}} \cdot (n+1)^n n^{n+1} = \frac{1}{2}.$$

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Department of Mathematics, Central Missouri State University, Warrensburg, Missouri 64093.

* * *

LAYING TO REST THAT MERSENNE NUMBER

Following the two recent items in this journal [1984: 66, 69] about the Mersenne number $M_{251} = 2^{251} - 1$, it might be useful to have in one place the complete information about this number: its complete decimal expansion and its prime factorization. My only contribution to this is the decimal expansion of M_{251}, which I found with the aid of a Timex Sinclair 1000 computer.

$$\begin{aligned} 2^{251} - 1 &= 3\ 618\ 502\ 788\ 666\ 131\ 106\ 986\ 593\ 281\ 521\ 497\ 120\ 414 \\ &\quad\ 687\ 020\ 801\ 267\ 626\ 233\ 049\ 500\ 247\ 285\ 301\ 247 \\ &= 503 \\ &\times\ 54\ 217 \\ &\times 178\ 230\ 287\ 214\ 063\ 289\ 511 \\ &\times\ 61\ 676\ 882\ 198\ 695\ 257\ 501\ 367 \\ &\times\ 12\ 070\ 396\ 178\ 249\ 893\ 039\ 969\ 681. \end{aligned}$$

CLAYTON W. DODGE
University of Maine at Orono

* * *

ON A TRIANGLE INEQUALITY

M.S. KLAMKIN

It is well known [4, p. 18] that, if A,B,C are the angles of a triangle, then

$$\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}, \tag{1}$$

with equality if and only if A = B = C. (A proof is immediate from the concavity of the sine function on the interval $[0,\pi]$.) Vasić [1] generalized (1) to

$$x \sin A + y \sin B + z \sin C \le \frac{\sqrt{3}}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right), \tag{2}$$

where $x,y,z > 0$. In [2], the author showed that (2) was a special case of the two-triangle inequality

$$4(xx'\sin A \sin A' + yy'\sin B \sin B' + zz'\sin C \sin C')$$
$$\le \left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right)\left(\frac{y'z'}{x'} + \frac{z'x'}{y'} + \frac{x'y'}{z'}\right). \tag{3}$$

Here we strengthen (2) to

$$x \sin A + y \sin B + z \sin C \le \frac{1}{2}(yz+zx+xy)\sqrt{\frac{x+y+z}{xyz}}. \tag{4}$$

We start with the polar moment of inertia inequality [3],

$$(w_1+w_2+w_3)(w_1R_1^2+w_2R_2^2+w_3R_3^2) \ge w_2w_3a_1^2 + w_3w_1a_2^2 + w_1w_2a_3^2,$$

in which w_1,w_2,w_3 are arbitrary nonnegative numbers; a_1,a_2,a_3 are the sides of a triangle $A_1A_2A_3$; and R_1,R_2,R_3 are the distances from an arbitrary point to the vertices of the triangle. Taking $R_1 = R_2 = R_3 = R$, the circumradius of the triangle, and using the power mean inequality,

$$\frac{w_2w_3a_1^2 + w_3w_1a_2^2 + w_1w_2a_3^2}{w_2w_3 + w_3w_1 + w_1w_2} \ge \left\{\frac{w_2w_3a_1 + w_3w_1a_2 + w_1w_2a_3}{w_2w_3 + w_3w_1 + w_1w_2}\right\}^2,$$

we obtain

$$R(w_1+w_2+w_3)\sqrt{w_2w_3+w_3w_1+w_1w_2} \ge w_2w_3a_1 + w_3w_1a_2 + w_1w_2a_3. \tag{5}$$

Now letting

$$w_1^2 = \frac{yz}{x}, \qquad w_2^2 = \frac{zx}{y}, \qquad w_3^2 = \frac{xy}{z},$$

and using $a_i = 2R\sin A_i$ in (5), we obtain (4).

There is equality if and only if

$$a_1 = a_2 = a_3$$

and the centroid of the weights w_1, w_2, w_3 at the respective vertices of the triangle coincides with the circumcenter. This entails that

$$w_1 = w_2 = w_3,$$

or, equivalently, that

$$x = y = z.$$

We have therefore shown that equality holds in (4) if and only if the triangle is equilateral.

Finally, to show that (4) is stronger than (2), we must establish that

$$\sqrt{3}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right) \geq (yz + zx + xy)\sqrt{\frac{x + y + z}{xyz}}$$

or, equivalently, that

$$3(y^2z^2 + z^2x^2 + x^2y^2)^2 \geq xyz(x + y + z)(yz + zx + xy)^2.$$

Letting $x = 1/u$, $y = 1/v$, and $z = 1/w$ shows that this is equivalent to

$$3(u^2 + v^2 + w^2)^2 \geq (u + v + w)^2(vw + wu + uv).$$

Since

$$\left(\frac{u^2 + v^2 + w^2}{3}\right)^2 \geq \left(\frac{u + v + w}{3}\right)^4$$

by the power mean inequality, it suffices finally to show that

$$\left(\frac{u + v + w}{3}\right)^2 \geq \frac{vw + wu + uv}{3},$$

and this is equivalent to

$$(v-w)^2 + (w-u)^2 + (u-v)^2 \geq 0.$$

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Department of Mathematics. University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

THE OLYMPIAD CORNER: 55

M.S. KLAMKIN

I present two new problem sets this month. The first consists of the problems set at the Second Round of the 1981 Leningrad High School Olympiad. I am grateful to Alex Merkurjev and to Larry Glasser for the transmittal and translation, respectively, of these problems. Since the problems are intended for students of Grades 8, 9, or 10, they should be accessible to all readers, from whom I solicit, for possible later publication in this column, elegant solutions and, if possible, nontrivial generalizations. I particularly welcome solutions from high school students, who should include the name of their school. All solutions should include the volume, page, and number of the problem.

Next I give the problems of the 2nd Annual American Invitational Mathematics Examination (AIME), which took place on March 20, 1984. See [1983: 170] for more information about AIME. Students and teachers with questions or comments about this AIME may write to the AIME Chairman, Professor George Berzsenyi, Department of Mathematics, Lamar University, Beaumont, Texas 77710. The answers (only) of the 1984 AIME will appear in my next column.

*

1981 LENINGRAD HIGH SCHOOL OLYMPIAD

Second Round

1. Find all integers x for which $|4x^2 - 12x - 27|$ is a prime number. (Grade 8)

2. Find $(a+b)/(a-b)$ if $a > b > 0$ and $a^2+ b^2= 6ab$. (Grades 8, 9)

3. From a point M outside an angle with vertex A two straight line segments are drawn, one of which cuts off on the sides of the angle two congruent segments AB and AC, and the other intersects these sides at the points D and E, respectively. Prove that $|BD|/|CE| = |MD|/|ME|$. (Grade 8)

4. Given that $x^x + y^y = x^y + y^x$, where x and y are natural numbers, prove that $x = y$. (Grades 8, 9)

5. Show that in order for the diagonals of a quadrilateral to be perpendicular, it is necessary and sufficient that the midlines of the quadrilateral be congruent. (A *midline* is a line segment connecting the midpoints of opposite sides.) (Grade 8)

6. Show that the two common tangents to the circle $x^2+y^2 = 2$ and the parabola $y = x^2/8$ are perpendicular. (Grade 9)

7. The angle bisectors AD, BE, CF of a triangle ABC concur in the incenter I. If

$$\frac{AI}{ID} = \frac{BI}{IE} = \frac{CI}{IF},$$

prove that triangle ABC is equilateral. (Grade 9)

8. Which number is larger, 48^{25} or 344^{17}? (Grade 9)

9. Prove that the positive root of the equation

$$x(x+1)(x+2)\ldots(x+1980)(x+1981) = 1$$

is less than $1/1981!$. (Grade 10)

10. Find all primes a and b such that $a^{a+1} + b^{b+1}$ is also a prime. (Grade 10)

11. Let $0 < x < \pi/6$. Show that for all natural numbers n

$$\sin x + \tan^2 x + \sin^3 x + \ldots + \tan^{2n} x < 1.4. \qquad \text{(Grade 10)}$$

12. Find the lengths of the edges of a rectangular parallelepiped with a square base if they are natural numbers and the total surface area is numerically equal to the sum of all the edge lengths. (Grade 9)

13. In the convex quadrilateral ABCD, $\angle BAC = \angle CBD$ and $\angle ACD = \angle BDA$. Show that $|AC|^2 = |BC|^2 + |AD|^2$. (Grade 9)

14. Show that $2^{b+c} + 2^{c+a} + 2^{a+b} < 2^{a+b+c+1} + 1$ for all $a, b, c > 0$. (Grades 9, 10)

15. Show that, for all postitive x and y,

$$x^3 + y^3 \le x^3 \cdot \sqrt[3]{\frac{x}{y}} + y^3 \cdot \sqrt[3]{\frac{y}{x}}. \qquad \text{(Grade 10)}$$

*

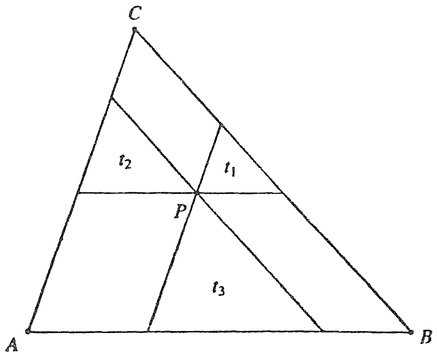
2nd ANNUAL AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

March 20, 1984 - Time: 2½ hours

1. Find the value of $a_2+a_4+a_6+\ldots+a_{98}$ if $a_1, a_2, a_3, \ldots$ is an arithmetic progression with common difference 1, and $a_1+a_2+a_3+\ldots+a_{98} = 137$.

2. The integer n is the smallest positive multiple of 15 such that every digit of n is either 0 or 8. Compute $n/15$.

3. A point P is chosen in the interior of ΔABC so that when lines are drawn through P parallel to the sides of ΔABC, the resulting smaller triangles t_1, t_2, and t_3 (see figure next page) have areas 4, 9, and 49, respectivley. Find the area of ΔABC.



4. Let S be a list of positive integers (not necessarily distinct) in which the number 68 appears. The average (arithmetic mean) of the numbers in S is 56. However, if 68 is removed, the average of the remaining numbers drops to 55. What is the largest number that can appear in S?

5. Determine the value of ab if $\log_8 a + \log_4 b^2 = 5$ and $\log_8 b + \log_4 a^2 = 7$.

6. Three circles, each of radius 3, are drawn with centers at (14,92), (17,76), and (19,84). A line passing through (17,76) is such that the total area of the parts of the three circles to one side of the line is equal to the total area of the parts of the three circles to the other side of it. What is the absolute value of the slope of this line?

7. The function f is defined on the set of integers and satisfies

$$f(n) = \begin{cases} n-3, & \text{if } n \geq 1000, \\ f(f(n+5)), & \text{if } n < 1000. \end{cases}$$

Find $f(84)$.

8. The equation $z^6 + z^3 + 1 = 0$ has one complex root with argument (angle) θ between 90° and 180° in the complex plane. Determine the degree measure of θ.

9. In tetrahedron ABCD, edge AB has length 3 cm. The area of face ABC is 15 cm^2 and the area of face ABD is 12 cm^2. These two faces meet each other at a 30° angle. Find the volume of the tetrahedron in cm^3.

10. Mary told John her score on the American High School Mathematics Examination (AHSME), which was over 80. From this, John was able to determine the number of problems Mary solved correctly. If Mary's score had been any lower, but still over 80, John could not have determined this. What was Mary's score? (Recall that the AHSME consists of 30 multiple-choice problems and that one's score, s, is

computed by the formula $s = 30+4c-w$, where c is the number of correct and w is the number of wrong answers; students are not penalized for problems left unanswered.)

11. A gardener plants three maple trees, four oak trees, and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Let m/n in lowest terms be the probability that no two birch trees are next to each other. Find $m+n$.

12. A function f is defined for all real numbers and satisfies

$$f(2+x) = f(2-x) \quad \text{and} \quad f(7+x) = f(7-x)$$

for all real x. If $x = 0$ is a root of $f(x) = 0$, what is the least number of roots $f(x) = 0$ must have in the interval $-1000 \le x \le 1000$?

13. Find the value of $10\cot(\cot^{-1}3 + \cot^{-1}7 + \cot^{-1}13 + \cot^{-1}21)$.

14. What is the largest even integer which cannot be written as the sum of two odd composite numbers? (Recall that a positive integer is said to be *composite* if it is divisible by at least one positive integer other than 1 and itself.)

15. Determine $x^2 + y^2 + z^2 + w^2$ if

$$\frac{x^2}{2^2-1^2} + \frac{y^2}{2^2-3^2} + \frac{z^2}{2^2-5^2} + \frac{w^2}{2^2-7^2} = 1,$$

$$\frac{x^2}{4^2-1^2} + \frac{y^2}{4^2-3^2} + \frac{z^2}{4^2-5^2} + \frac{w^2}{4^2-7^2} = 1,$$

$$\frac{x^2}{6^2-1^2} + \frac{y^2}{6^2-3^2} + \frac{z^2}{6^2-5^2} + \frac{w^2}{6^2-7^2} = 1,$$

$$\frac{x^2}{8^2-1^2} + \frac{y^2}{8^2-3^2} + \frac{z^2}{8^2-5^2} + \frac{w^2}{8^2-7^2} = 1.$$

*

I now present solutions to several problems published earlier in this column.

J-7. [1980: 146] *From a list of Russian "Jewish" problems.*

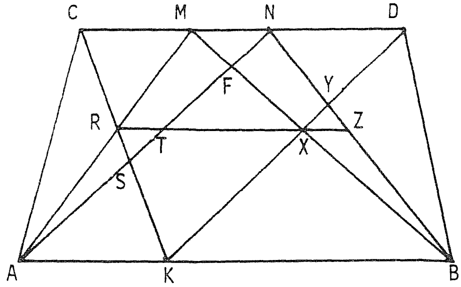
Let ABDC be a trapezoid with the bases AB and CD, and let K be a point in AB. Find a point M in CD such that the area of the quadrangle which is the intersection of the triangles AMB and CDK is maximal.

Solution by Daniel Sokolowsky, California State University at Los Angeles.

We show that if M is chosen so that

$$\frac{\text{CM}}{\text{MD}} = \frac{\text{AK}}{\text{KB}},$$

then the intersection quadrangle KXMR (see figure) will have the maximum area. M being so chosen, Let N be any point other than M on CD. To be specific, we assume that N lies between M and D (the proof is similar if N lies between C and M). With square brackets denoting area, it is clear from the figure that, if RX || CD, then



$$RT = XZ \implies [RTNM] = [XZNM] \implies [SFMR] > [XYNF] \implies [KXMR] > [KYNS],$$

so it suffices to show that RX || CD and RT = XZ. By similar triangles and the choice of M, we have

$$\frac{KR}{RC} = \frac{AK}{CM} = \frac{KB}{MD} = \frac{KX}{XD}.$$

Therefore RX || CD and $\frac{KR}{KC} = \frac{KX}{KD}$. Now

$$\frac{RT}{MN} = \frac{AR}{AM} = \frac{KR}{KC} = \frac{KX}{KD} = \frac{BX}{BM} = \frac{XZ}{MN},$$

and RT = XZ follows.

*

17. [1981: 17] *From a 1973 Moscow Olympiad.*

Twelve painters live in 12 houses which are built along a circular street and are painted some white, some blue. Each month one of the painters, taking with him enough white and blue paint, leaves his house and walks along the road in the clockwise sense. On the way, he repaints every house (starting with his own) the opposite colour. He stops work as soon as he repaints some white house blue. In a year, each painter undertakes such a journey exactly once. Show that at the end of a year each house will be painted its original colour, provided that at the beginning of the year at least one house was painted blue.

Solution by R.B. Killgrove, University of South Carolina at Aiken.

We show more generally that the same conclusion holds for any finite number of houses. Let there be $n+1$ houses, where $n \geq 0$. We first number the houses consecutively $0,1,2,\ldots,n$ in the clockwise sense, starting at an arbitrarily chosen house. Independently of the house number, we assign to each house at any given time a *chromatic digit*: the digit 0 or 1 according as the house is then colored white or blue, respectively. Thus, at any given time there is a *chromatic street number*, an $(n+1)$-digit binary number

$$\alpha = d_n d_{n-1} \ldots d_1 d_0,$$

where $d_k = 0$ or 1 according as house k is then colored white or blue, respectively.

For $k = 0,1,\ldots,n$, let α_k and α_{k+1} be the street numbers when the painter from house k starts and finishes his round, respectively. It is not hard to show that

$$\alpha_{k+1} \equiv \alpha_k + 2^k \pmod{2^{n+1}-1},$$

and summing these $n+1$ relations gives

$$\alpha_{n+1} \equiv \alpha_0 + 2^0 + 2^1 + \ldots + 2^n \pmod{2^{n+1}-1}.$$

Since $2^0+2^1+\ldots+2^n = 2^{n+1}-1$, we therefore have

$$\alpha_{n+1} \equiv \alpha_0 \pmod{2^{n+1}-1}.$$

Finally, since $\alpha_0 \geq 1$ (at least one house was originally painted blue), we conclude that $\alpha_{n+1} = \alpha_0$, that is, at the end each house is painted its original color.

*

1. [1981: 42] *From the 1980 Austrian-Polish Competition.*

Given three infinite arithmetic progressions of natural numbers such that each of the numbers 1, 2, 3, 4, 5, 6, 7, and 8 belongs to at least one of them, prove that the number 1980 also belongs to at least one of them.

Solution by Andy Liu, University of Alberta.

If one of the progressions has common difference 1, it contains 1980. So we assume there is no common difference 1. If one of the progressions has common difference 2, it contains 1980 unless all its terms are odd. But then one of the remaining progressions has common difference 2, or else both remaining progressions have common difference 4. Either way, we obtain 1980. Finally, suppose all three progressions A,B,C have common differences at least 3. We can let $1 \in A$, $2 \in B$, and $3 \in C$. Then we must have $4 \in A$, $5 \in B$, $6 \in C$, and C contains 1980.

*

2. [1981: 42] *From the 1980 Austrian-Polish Competition.*

Let $\{x_n\}$ be a sequence of natural numbers such that

(a) $1 = x_1 < x_2 < x_3 < \ldots$; (b) $x_{n+1} \leq 2n$ for all n.

Prove that, for every natural number k, there exist terms x_r and x_s such that $x_r - x_s = k$.

Solution by Andy Liu, University of Alberta.

Let k be a natural number. Since $x_{k+1} \leq 2k$ and the terms of the sequence are strictly increasing, at least $k+1$ of the $2k$ numbers $1,2,\ldots,2k$ belong to the sequence.

Hence, by the pigeonhole principle, one of the pairs

$$(1,k+1),\quad (2,k+2),\quad \ldots,\quad (k,2k)$$

will be in the sequence, and for this pair the difference of the terms is k.

*

3. [1981: 42] *From the 1980 Austrian-Polish Competition.*

Prove that the sum of the six angles subtended at an interior point of a tetrahedron by its six edges is greater than 540°.

Solution by M.S.K.

Let ABCD be the tetrahedron, P the interior point, and A' the intersection of line AP with face BCD. Then, if

$$S = \angle APB + \angle APC + \angle APD \quad\text{and}\quad S' = \angle BPC + \angle CPD + \angle DPB,$$

then $S = 3\pi - (\angle A'PB + \angle A'PC + \angle A'PD)$, and $S > 3\pi - S'$ follows from Problem J-36 [1984: 82]. Therefore $S + S' > 3\pi = 540^\circ$.

To see that $S + S'$ can be made arbitrarily close to 3π, we consider a tetrahedron whose edges BC,CD,DB are very small, whose altitude from vertex A is very large, and with interior point P very close to A.

*

5. [1981: 42] *From the 1980 Austrian-Polish Competition.*

Let $A_1A_2A_3$ be a triangle and, for $1 \le i \le 3$, let B_i be an interior point of the edge opposite A_i. Prove that the perpendicular bisectors of A_iB_i for $1 \le i \le 3$ are not concurrent.

Solution by M.S.K.

Our proof is indirect. Suppose the three perpendicular bisectors are concurrent at point P. If the vertices are labeled so that $PA_1 \ge PA_2 \ge PA_3$, then the circle with center P and radius PA_1 cannot intersect segment A_2A_3 in its interior. This gives the required contradiction.

*

H-1. [1981: 114] *From Középiskolai Matematikai Lapok 60 (1979) 140.*

If a, b, c are the sides of a triangle with $a \le b \le c$, determine the best possible upper and lower bounds for the expression $(a+b+c)^2/bc$.

Solution by M.S.K.

In triangle inequalities, the extreme cases usually hold for triangles with sides (a,b,c) proportional to $(1,1,1)$, $(0,1,1)$, $(1,1,2)$, or $(1,1,\sqrt{2})$. These lead to the tentative bounds 9 and 4, corresponding to the cases $(1,1,1)$ and $(0,1,1)$, respectively. To establish that these bounds are indeed valid, we use the representation

$$a = y+z, \qquad b = z+x, \qquad c = x+y,$$

with which, when $a \le b \le c$, the often bothersome constraints $b+c \ge a$, etc., are replaced by the simpler $x \ge y \ge z \ge 0$ (see Third proof of Problem 2 [1984: 47]).

We find that

$$I \equiv \frac{(a+b+c)^2}{bc} = \frac{4(x+y+z)^2}{(x+y)(x+z)}$$

and

$$I - 4 = \frac{4(y^2+z^2+yz+zx+xy)}{(x+y)(x+z)} \ge 0,$$

with equality if and only if $y = z = 0$, corresponding to the degenerate triangle with sides (a,b,c) proportional to $(0,1,1)$. Also,

$$9 - I = \frac{5x^2-4y^2-4z^2+yz+zx+xy}{(x+y)(x+z)} \ge \frac{5x^2-3y^2-2z^2}{(x+y)(x+z)} \ge 0,$$

with equality if and only if $x = y = z$, corresponding to the equilateral triangle.

*

H-2. [1981: 114] *From Középiskolai Matematikai Lapok 60 (1979) 140.*

Let n be a positive integer. As a first step, we have given the sequence $\{a_1, a_2, \ldots, a_k\}$, where $k = 2^n$ and each a_i is 1 or -1. As a second step, we form the new sequence $\{a_1a_2, a_2a_3, \ldots, a_ka_1\}$, and continue to repeat this process to generate new sequences. Show that, by at most the 2^nth iterated step, we arrive at a constant sequence with every term equal to 1.

Solution by M.S.K.

To get an idea for the proof, we first consider the special case $n = 2$. Here the sequences for the first 4 steps are:

Step 0. $\{a_1^1, a_2^1, a_3^1, a_4^1\}$.

Step 1. $\{a_1^1a_2^1, a_2^1a_3^1, a_3^1a_4^1, a_4^1a_1^1\}$.

Step 2. $\{a_1^1a_2^2a_3^1, a_2^1a_3^2a_4^1, a_3^1a_4^2a_1^1, a_4^1a_1^2a_2^1\}$.

Step 3. $\{a_1^1a_2^3a_3^3a_4^1, a_2^1a_3^3a_4^3a_1^1, a_3^1a_4^3a_1^3a_2^1, a_4^1a_1^3a_2^3a_3^1\}$.

Since $a_i^3 = a_i$, each term in Step 3 equals $a_1a_2a_3a_4$, and so each term in Step 4 equals $(a_1a_2a_3a_4)^2 = 1$.

For arbitrary n, one can show by induction that, after Step m, where $m \le 2^n-1$, for each term of the sequence the exponents are the coefficients of $(x+y)^m$. In particular, when $m = 2^n-1$, the exponents are all odd and each term of the sequence equals $a_1a_2\ldots a_{m+1}$. Therefore, after Step $m+1 = 2^n$, each term of the sequence equals $(a_1a_2\ldots a_{m+1})^2 = 1$.

*

B-3. [1981: 114] This problem appeared again later [1981: 267] with a slightly different wording and notation. For a proof, see [1983: 15].

*

4. [1983: 108] *From the 1983 British Mathematical Olympiad.*

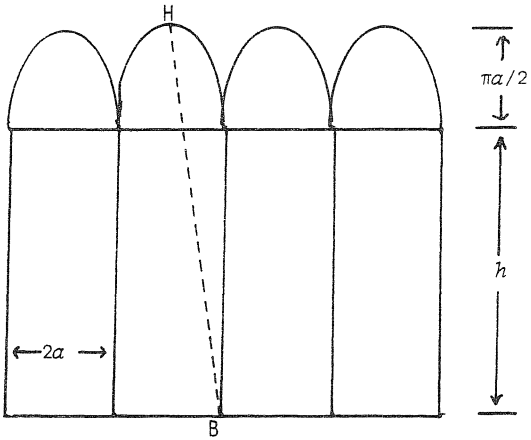
The two cylindrical surfaces

$$x^2 + z^2 = a^2, \qquad z > 0, \qquad |y| \le a$$

and

$$y^2 + z^2 = a^2, \qquad z > 0, \qquad |x| \le a$$

intersect, and with the plane $z = 0$ enclose a dome-like shape which is here called a *cupola*. The cupola is placed on top of a vertical tower of height h whose horizontal cross-section is a square of side $2a$. Find the shortest distance from the highest point of the cupola to a corner of the base of the tower, over the surface of the cupola and tower.



Solution by Donald Aitken, Edmonton, Alberta.

The surfaces of the tower and cupola are developable, as shown in the figure. The shortest distance from H to B along the surface is

$$HB = \sqrt{a^2 + (h+\pi a/2)^2}.$$

*

2. [1983: 137] *From the 1982 Netherlands Olympiad.*

M is the midpoint of AB and P is an arbitrary point on AC. Using only a pencil and a straightedge, construct a point Q on BC such that P and Q are at equal distance from CM. Justify your construction.

Solution by Noam Elkies, student, Columbia University.

Let O = CM ∩ PB. Then the required point Q = AO ∩ BC.

This is the classical straightedge construction of a line PQ || AB, given P on AC and the midpoint M of AB. (One proof of this classical construction follows from an orthogonal projection of ΔABC into an isosceles ΔA'B'C' with A'C' = B'C'.) [One can also use cross ratios; see almost any text on projective geometry. (M.S.K.)]

To complete the proof, let M' = PQ $\cap$ CM, and let the orthogonal projections of P and Q upon CM be P' and Q', respectively. Then PM' = M'Q, so triangles PP'M' and QQ'M' are congruent (*ASA*) and PP' = QQ'.

*

F. 2414. [1983: 237] *From Középiskolai Matematikai Lapok (March 1983).*

Solve the system of equations

$$x + y + z = 5 \tag{1}$$
$$x^2 + y^2 + z^2 = 9 \tag{2}$$
$$xy + u + vx + vy = 0$$
$$yz + u + vy + vz = 0$$
$$zx + u + vz + vx = 0.$$

Solution by M.S.K.

Subtracting the last three equations in pairs gives

$$(y-z)(x+v) = 0, \qquad (z-x)(y+v) = 0, \qquad (x-y)(z+v) = 0.$$

Since $x = y = z$ does not satisfy (1) and (2), we may, on account of the symmetry of the system in x,y,z, assume that $y = z \neq x$; then also $y = z = -v$ and $u = y^2$. Eliminating z from (1) and (2) yields the system

$$x + 2y = 5, \qquad x^2 + 2y^2 = 9,$$

from which we obtain the satisfactory solutions

$$\{x, y, z\} = \{1, 2, 2\}, \qquad (u, v) = (4, -2)$$

and

$$\{x, y, z\} = \{\tfrac{7}{3}, \tfrac{4}{3}, \tfrac{4}{3}\}, \qquad (u, v) = (\tfrac{16}{9}, -\tfrac{4}{3}).$$

*

F. 2415. [1983: 237] *From Középiskolai Matematikai Lapok (March 1983).*

Choose 400 different points inside a unit cube. Show that 4 of these points lie inside some sphere of radius 4/23.

Solution by Lones Smith, student, Carleton University, Ottawa.

Divide the unit cube into 125 congruent cubes of side 1/5. By the pigeonhole principle, one of these smaller cubes must contain at least 4 points. The sphere circumscribing this cube has radius $\sqrt{3}/10 < 4/23$, and the desired result follows.

*

1. [1983: 268] *From the 1983 Austrian Mathematical Olympiad.*

For natural numbers x, let $Q(x)$ be the sum and $P(x)$ the product of the digits of x (in base ten). Show that, for each natural number n, there exist infinitely many natural numbers x such that

$$Q(Q(x)) + P(Q(x)) + Q(P(x)) + P(P(x)) = n.$$

Solution by Ravi Ramakrishna, student, Essex Junction High School, Vermont.

Let v be any natural number the sum of whose digits is n. Clearly, such numbers exist. Then let $r = 11...1$, a number consisting of $10v$ ones, and let x be any element of the infinite set

$$\{10r,\ 10^2r,\ 10^3r,\ ...\}.$$

Since x is a multiple of 10, $P(x) = 0$. Also, $Q(x) = 10v$, since the nonzero digits of x consist of $10v$ ones. Finally,

$$\begin{aligned} Q(Q(x)) + P(Q(x)) + Q(P(x)) + P(P(x)) &= Q(10v) + P(10v) + Q(0) + P(0) \\ &= Q(v) + 0 + 0 + 0 \\ &= n. \end{aligned}$$

*

2. [1983: 269] *From the 1983 Austrian Mathematical Olympiad.*

Let x_1,x_2,x_3 be the roots of

$$x^3 - 6x^2 + ax + a = 0.$$

Determine all real numbers a such that

$$(x_1 - 1)^3 + (x_2 - 2)^3 + (x_3 - 3)^3 = 0.$$

Also, for each such a, determine the corresponding values of x_1,x_2,x_3.

Solution by Daniel Ropp, student, Stillman Valley High School, Illinois.

Setting $m = x_1-1$, $n = x_2-2$, $p = x_3-3$ in the identity

$$m^3 + n^3 + p^3 - 3mnp = \frac{1}{2}(m+n+p)\{(n-p)^2+(p-m)^2+(m-n)^2\},$$

and noting that $m+n+p = x_1+x_2+x_3-6 = 0$, we find that

$$0 = (x_1-1)^3 + (x_2-2)^3 + (x_3-3)^3 = 3(x_1-1)(x_2-2)(x_3-3).$$

The desired results follow upon setting, $x = 1,2,3$ successively in the given cubic:

$$\begin{aligned} &x_1 = 1, \qquad a = \frac{5}{2}, \qquad x_2,x_3 = \frac{5\pm\sqrt{35}}{2}; \\ &x_2 = 2, \qquad a = \frac{16}{3}, \qquad x_3,x_1 = \frac{6\pm2\sqrt{15}}{3}; \\ &x_3 = 3, \qquad a = \frac{27}{4}, \qquad x_1,x_2 = \frac{3\pm3\sqrt{2}}{2}. \end{aligned}$$

*

4. [1983: 269] *From the 1983 Austrian Mathematical Olympiad.*

The sequence $\{x_n\}$ is defined as follows: $x_1 = 2$, $x_2 = 3$, and

$$x_{2m+1} = x_{2m} + x_{2m-1}, \qquad m \ge 1; \qquad (1)$$

$$x_{2m} = x_{2m-1} + 2x_{2m-2}, \qquad m \ge 2. \qquad (2)$$

Determine x_n (as a function of n).

Solution by Ravi Ramakrishna, student, Essex Junction High School, Vermont.

We have from (1)

$$x_{2m} = x_{2m+1} - x_{2m-1} \quad \text{and} \quad x_{2m-2} = x_{2m-1} - x_{2m-3}, \qquad m \ge 2,$$

and substituting these values in (2) yields

$$x_{2m+1} - 4x_{2m-1} + 2x_{2m-3} = 0, \qquad m \ge 2.$$

With $v_m = x_{2m-1}$, this equation becomes

$$v_{m+1} - 4v_m + 2v_{m-1} = 0, \qquad m \ge 2. \qquad (3)$$

With $v_1 = x_1 = 2$ and $v_2 = x_3 = 5$, the solution of difference equation (3) is found to be

$$v_m = \left(\frac{3-\sqrt{2}}{4}\right)(2+\sqrt{2})^m + \left(\frac{3+\sqrt{2}}{4}\right)(2-\sqrt{2})^m = x_{2m-1}, \qquad m \ge 1. \qquad (4)$$

Proceeding likewise, we have from (2)

$$x_{2m-1} = x_{2m} - 2x_{2m-2} \quad \text{and} \quad x_{2m+1} = x_{2m+2} - 2x_{2m}, \qquad m \ge 2,$$

and substituting these values in (1) yields

$$x_{2m+2} - 4x_{2m} + 2x_{2m-2} = 0, \qquad m \ge 2.$$

With $v_m = x_{2m}$, we obtain again difference equation (3). But here $v_1 = x_2 = 3$ and $v_2 = x_4 = 11$, and the solution is

$$v_m = \left(\frac{1+2\sqrt{2}}{4}\right)(2+\sqrt{2})^m + \left(\frac{1-2\sqrt{2}}{4}\right)(2-\sqrt{2})^m = x_{2m}, \qquad m \ge 1. \qquad (5)$$

Finally, to express x_n explicitly in terms of n, as the problem requires, we need only replace m by $(n+1)/2$ and $n/2$, respectively, in (4) and (5).

*

5. [1983: 269] *From the 1983 Austrian Mathematical Olympiad.*

Let N be the set of natural numbers. For all $(a,b) \in N\times N$, find all the solutions $(x,y) \in N\times N$ of the equation

$$x^{a+b} + y = x^a y^b.$$

Solution by Daniel Ropp, student, Stillman Valley High School, Illinois.

The given equation is equivalent to $y = kx^a$, where $k = y^b - x^b$. In terms of k and x, the equation can be written

$$x^{a+b} + kx^a = k^b x^{a(b+1)},$$

which is equivalent to both

$$x^b = k(k^{b-1}x^{ab} - 1) \qquad (1)$$

and

$$k = x^b(k^b x^{b(a-1)} - 1). \qquad (2)$$

Since all exponents are nonnegative, it is evident from (1) and (2) that $k = x^b$, and then $y = kx^a = x^{a+b}$. The initial equation now reduces to

$$x^{b(a+b-1)} = 2,$$

which is satisfied only for $x = 2$, $b = 1$, $a+b-1 = 1$. Thus there is only the solution $(x,y) = (2,4)$, which occurs for $(a,b) = (1,1)$.

*

6. [1983: 269] *From the 1983 Austrian Mathematical Olympiad.*

Let π_1 and π_2 be two planes in Euclidean space R^3. These planes effect a partition of the *reduced space* $S \equiv R^3 - (\pi_1 \cup \pi_2)$ into several components. Show that, for any cube in R^3, at least one of the components of S has a nonempty intersection with at least three faces of the cube.

Solution by Ravi Ramakrishna, student, Essex Junction High School, Vermont.

If one vertex of the cube lies in one component of the partition, then the three faces concurring in that vertex are all partially within this component. Now we need only concern ourselves with cubes all of whose vertices lie in $\pi_1 \cup \pi_2$. One plane, π_1 say, contains exactly four vertices of the cube, and these are the vertices of a square or a rectangle in π_1. In the first case, $\pi_1 \parallel \pi_2$ and the component between them has a nonempty intersection with four faces of the cube. In the second case, $\pi_1 \perp \pi_2$ and each of the four components has a nonempty intersection with three faces of the cube.

*

M796. [1983: 269] *Proposed by L.D. Kurliandchik (in Kvant, April 1983).*

Find $\angle APB$ if P is a point inside a square ABCD such that

$$PA : PB : PC = 1 : 2 : 3.$$

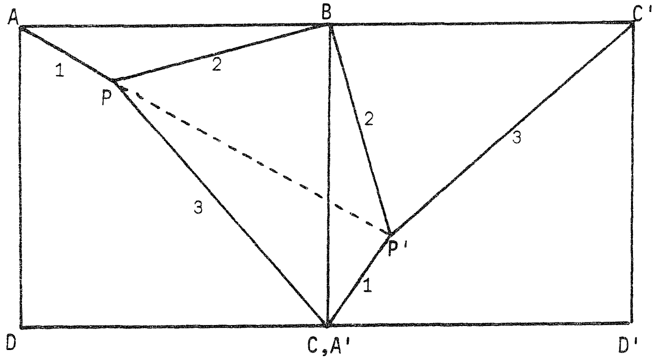
Solution by M.S.K.

We rotate the square about B through 90° in the counterclockwise sense (see figure). Then PBP' is an isosceles right triangle and $PP' = 2\sqrt{2}$. Now

$$3^2 = 1^2 + (2\sqrt{2})^2,$$

so $\angle PP'C = 90^\circ$. Therefore

$$\angle APB = \angle A'P'B = 90^\circ + 45^\circ = 135^\circ.$$



Editor's Note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

THE PUZZLE CORNER

Puzzle No. 53: Enigmatic rebus (11)

#

He failed to give the SAID.
The sentry shot him dead.

Puzzle No. 54: Rebus (8)

M + S

To add, please keep this rule in sight:
Be sure you copy TOTAL right.

ALAN WAYNE
Holiday, Florida

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before December 1, 1984, although solutions received after that date will also be considered until the time when a solution is published.

941. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Independently solve each of the following alphametics in base ten:

$$6 \cdot \text{GEESE} = \text{FLOCK},$$
$$7 \cdot \text{GEESE} = \text{FLOCK},$$
$$8 \cdot \text{GEESE} = \text{FLOCK}.$$

942. *Proposed by Freeman Dyson, University of California, Davis (permanent address, Institute for Advanced Study, Princeton).*

The infinite tree T is defined as the unique connected graph having three edges at every vertex and no closed cycles. We consider functions $f(\mathrm{P})$ defined on the vertices P of T. The neighbor-averaging operator A is defined by

$$Af(\mathrm{P}) = \tfrac{1}{3}\{f(\mathrm{Q}) + f(\mathrm{R}) + f(\mathrm{S})\},$$

where Q,R,S are the neighbors of P on T. A is a linear bounded operator on the function f. The eigen-set E of A is the set of complex numbers λ for which a non-zero function f_λ exists with

$$Af_\lambda(\mathrm{P}) = \lambda f_\lambda(\mathrm{P}).$$

The problem: What is E?

943. *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

For positive integer n, the numbers $u_k(n)$, $k = 0,1,2,\ldots$, are defined by

$$\frac{x^n}{(1-x)(1-x^n)} = \sum_{k=0}^{\infty} u_k(n)x^k, \qquad |x| < 1.$$

Find a simple expression for $u_k(n)$.

944. *Proposed by the Cops of Ottawa.*

Find all primes p such that $2^p + p^2$ is also a prime.

945. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Solve the system

$$\begin{aligned} x + y + z + t &= 2, \\ x^2 + y^2 + z^2 + t^2 &= 118, \\ x^3 + y^3 + z^3 + t^3 &= 176, \\ x^4 + y^4 + z^4 + t^4 &= 6514. \end{aligned}$$

946*. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

The nth differences of a function f at r are defined as usual by $\Delta^0 f(r) = f(r)$ and

$$\Delta^1 f(r) = \Delta f(r) = f(r+1) - f(r), \qquad \Delta^n f(r) = \Delta(\Delta^{n-1} f(r)), \qquad n = 1,2,3,\ldots .$$

Prove or disprove that, if $\Delta^n f(1) = n$ for $n = 0,1,2,\ldots$, then

$$f(n) = (n-1)\cdot 2^{n-2}.$$

947. *Proposed by Jordi Dou, Barcelona, Spain.*

Let ABCD be a quadrilateral (not necessarily convex) with AB = BC, CD = DA, and AB $\perp$ BC. The midpoint of CD being M, points K and L are found on line BC such that AK = AL = AM. If P,Q,R are the midpoints of BD,MK,ML, respectively, prove that PQ $\perp$ PR.

948. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

If a,b,c are the side lengths of a triangle of area K, prove that

$$27K^4 \le a^3b^3c^2,$$

and determine when equality occurs.

949. *Proposed by Charles W. Trigg, San Diego, California.*

In all bases, find all two-digit integers that are four times their reverses.

950. *Proposed by F.G.B. Maskell, Algonquin College, Ottawa.*

Let I be the incentre and Ω,Ω' the Brocard points of a nonequilateral triangle ABC, so that

$$\angle\Omega BC = \angle\Omega CA = \angle\Omega AB \quad \text{and} \quad \angle\Omega' CB = \angle\Omega' AC = \angle\Omega' BA.$$

Show that Ω,I,Ω' can never be collinear.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

816. [1983: 46] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let a,b,c be the sides of a triangle with semiperimeter s, inradius r, and circumradius R. Prove that, with sums and product cyclic over a,b,c,

(a) $$\Pi(b+c) \le 8sR(R+2r),$$

(b) $$\Sigma bc(b+c) \le 8sR(R+r),$$

(c) $$\Sigma a^3 \le 8s(R^2-r^2).$$

Solution by M.S. Klamkin, University of Alberta.

We prove the following sharper results:

$$\Pi(b+c) \le 4s(2R^2 + 3Rr + 2r^2) \le 8sR(R + 2r),$$

$$\Sigma bc(b+c) \le 4s(2R^2 + Rr + 2r^2) \le 8sR(R + r),$$

$$\Sigma a^3 \le 4sR(2R - r) \le 8s(R^2 - r^2).$$

Each of the three right inequalities is equivalent to the well-known $R \ge 2r$. With the known relations

$$\Pi(b+c) = 2s(s^2+2Rr+r^2),\ \Sigma bc(b+c) = 2s(s^2-2Rr+r^2),\ \Sigma a^3 = 2s(s^2-6Rr-3r^2),$$

each of the three left inequalities is equivalent to

$$s^2 \le 4R^2 + 4Rr + 3r^2.$$

The last is a known inequality, which results from

$$\mathrm{IH}^2 = 4R^2 + 4Rr + 3r^2 - s^2 \ge 0,$$

where I and H are the incenter and orthocenter, respectively, of a triangle. Equality holds throughout just when $a = b = c$.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India (two solutions); and the proposer.

* * *

817. [1983: 46] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

(a) Suppose that to each point on the circumference of a circle we arbitrarily assign the color red or green. Three distinct points of the same color will be said to form a *monochromatic triangle*. Prove that there are monochromatic isosceles triangles.

(b) Prove or disprove that there are monochromatic isosceles triangles if to every point on the circumference of a circle we arbitrarily assign one of k colors, where $k \geq 2$.

Joint solution by M.S. Klamkin and A. Meir, University of Alberta.

Instead of a circle, we consider more generally any rectifiable curve γ in which each point is colored with one of k colors, where $k \geq 2$. For any points $X, Y \in \gamma$, we use $\overset{\frown}{XY}$ and $\overline{XY}$ to denote the arc length and chord length, respectively, from X to Y. Note that, if γ is an arc of a circle, then $\overset{\frown}{XY} = \overset{\frown}{YZ}$ if and only if $\overline{XY} = \overline{YZ}$. We will show that, for any integer $r \geq 3$, there are distinct points $A_1, A_2, \ldots, A_r$ on γ, all of the same color, such that

$$\overset{\frown}{A_1A_2} = \overset{\frown}{A_2A_3} = \ldots = \overset{\frown}{A_{r-1}A_r}. \tag{1}$$

When γ is an arc of a circle, part (b) of our problem corresponds to the case $r = 3$, and part (a) when also $k = 2$.

A theorem of van der Waerden [1] states that, given r and k, there is an integer $n = n(r,k)$ such that, if the set $\{1,2,\ldots,n\}$ is partitioned into k classes, then at least one class contains an r-term arithmetic progression. We first choose distinct points $P_1, P_2, \ldots, P_n$ on γ such that

$$\overset{\frown}{P_1P_2} = \overset{\frown}{P_2P_3} = \ldots = \overset{\frown}{P_{n-1}P_n}.$$

This can always be done regardless of the value of n, for the points P_1 and P_2 can be taken arbitrarily close (by arc length). We then partition the set of subscripts $\{1,2,\ldots,n\}$ into k classes, each class containing the subscripts of points of the same color. By the theorem, at least one class contains an r-term arithmetic progression

$$a,\ a+d,\ a+2d,\ \ldots,\ a+(r-1)d.$$

If we now let $A_i = P_{a+(i-1)d}$, then the points $A_1, A_2, \ldots, A_r$ satisfy (1).

How large n has to be as a function of r and k is a Ramsey Problem. But when $r = 3$ and $k = 2$, there is a particularly simple solution. Let R_1 and R_2 be two distinct points of the same color (say red) on γ, and choose points A and B, one on each side of $\overset{\frown}{R_1R_2}$, such that

$$\overset{\frown}{AR_1} = \overset{\frown}{R_1R_2} = \overset{\frown}{R_2B}.$$

Here it is assumed that A, R_1, R_2, B are all distinct. This can always be ensured by making $\overset{\frown}{R_1R_2}$ arbitrarily small (of three points arbitrarily close (by arc length), at least two must be of the same color). If either A or B is red, we are done. If both A and B are green, and if C is the midpoint of $\overset{\frown}{R_1R_2}$, then either A,C,B are all green or else R_1,C,R_2 are all red. □

Finally we give, jointly with the late Ernst Straus, a related conjecture: *Let γ be a continuous curve which is k-colored, where $k \geq 2$. Then for any $r \geq 3$ there exist distinct points $A_1, A_2, \ldots, A_r$ of the same color on γ such that*

$$\overline{A_1A_2} = \overline{A_2A_3} = \ldots = \overline{A_{r-1}A_r}.$$

Also solved by FRED GALVIN, University of Kansas, Lawrence, Kansas; JAN VAN DE CRAATS, Leiden University, The Netherlands; and JOSEPH ZAKS, University of Haifa, Israel. Partial solutions were received from W.J. BLUNDON, Memorial University of Newfoundland; HOWARD EVES, University of Maine; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; BASIL C. RENNIE, James Cook University of North Queensland, Australia; and the proposer.

REFERENCE

1. R.L. Graham, *Rudiments of Ramsey Theory*, American Mathematical Society, Providence, 1981, p. 9.

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818. [1983: 46] *Proposed by A.P. Guinand, Trent University, Peterborough, Ontario.*

Let ABC be a scalene triangle with circumcentre O and orthocentre H, and let P be the point where the internal bisector of angle A intersects the Euler line OH. If O, H, P only are given, construct an angle equal to angle A, using only ruler and compass.

Solution by the proposer.

We assume that $P \neq H$ (otherwise A is a right angle). If R is the circumradius, then the metric trilinear coordinates of O with respect to ΔABC (i.e., the directed distances from O to the sides of ΔABC) are

$$(R\cos A,\ R\cos B,\ R\cos C);$$

and for H they are

$$(2R\cos B\cos C,\ 2R\cos C\cos A,\ 2R\cos A\cos B).$$

If P divides OH in the ratio $1:2\lambda$, then the metric trilinear coordinates of P are

$$\left(\frac{2R(\lambda\cos A + \cos B\cos C)}{2\lambda+1},\ \frac{2R(\lambda\cos B + \cos C\cos A)}{2\lambda+1},\ \frac{2R(\lambda\cos C + \cos A\cos B)}{2\lambda+1}\right).$$

Since P is on the internal bisector of angle A, the last two coordinates must be equal, whence

$$\lambda\cos B + \cos C\cos A = \lambda\cos C + \cos A\cos B.$$

But $\cos B \neq \cos C$ since the triangle is scalene, so $\lambda = \cos A$. Consequently the size of angle A is uniquely determined by the ratio in which P divides OH. If G is the centroid of ΔABC and H' is the reflection of H in O, then G and H' divide OH in the ratios 1:2 and 1:-2, respectively. Since $2\lambda = 2\cos A$ lies in the interval $(-2,2)$, it follows that P cannot lie in the closed segment GH'.

To construct the required angle, draw a circle on diameter OP and a circle with centre P and radius $\frac{1}{2}$PH. Let Q be one of the points of intersection of these two circles. Then $\angle OQP = 90^\circ$, so

$$\cos OPQ = \frac{PQ}{OP} = \frac{1}{2}\frac{PH}{OP} = \frac{1}{2}|2\lambda| = |\lambda| = \pm\cos A.$$

Hence if P lies in the open segment GH, then the required angle is the acute angle OPQ. For P in the other permissible ranges, the required angle is the obtuse angle between PQ and the Euler line OH.

Also solved by LEON BANKOFF, Los Angeles, California; W.J. BLUNDON, Memorial University of Newfoundland; JORDI DOU, Barcelona, Spain; HOWARD EVES, University of Maine; J.T. GROENMAN, Arnhem, The Netherlands; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

Editor's comment.

The proposal did specify that P was on the Euler *line* OH. Yet nearly all other solvers tacitly assumed that P was on the *segment* OH, resulting, inevitably, always in an acute angle A. The obtuseness was there, but transferred from the triangle to the solvers.

* * *

819. [1983: 46] *Proposed by H. Kestelman, University College, London, England.*

Let A and B be $n\times n$ Hermitian matrices. Prove that $AB - BA$ is singular if A and B have a common eigenvector. Prove that the converse is true if $n = 2$ but not if $n > 2$.

Solution by the proposer.

If $Av = kv$ and $Bv = jv$, $v \neq 0$, then $ABv = jkv = BAv$, and so $AB - BA$ is singular.

$AB - BA$ has zero trace for all n, and so if, conversely, $AB - BA$ is singular (i.e., 0 is one of its eigenvalues) and $n = 2$, it follows that the skew-Hermitian matrix $AB - BA$ has 0 for its only eigenvalue; this implies $AB - BA = 0$. To prove that A and B have a common eigenvector (when $n = 2$), we can ignore the trivial case where A is a scalar multiple of I and assume that $Av = kv$ for some $v \neq 0$ and that every solution of $(A-kI)x = 0$ is a scalar multiple of v. Then, since $A(Bv) = B(Av) = k(Bv)$, it follows that Bv is a scalar multiple of v, i.e., v is an eigenvector of A.

If $n \geq 3$, let A and B be the matrices given on the following page. The (r,s) element of AB and of BA is 0 if $r+s \geq n+2$, and the $(n,1)$ element of AB and of BA

$$A = \begin{pmatrix} 1 & & & & & \\ & 2 & & & & \\ & & 3 & & & \\ & & & \ddots & & \\ & & & & n-1 & \\ & & & & & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is 1; it follows that det $(AB-BA) = 0$. The eigenvectors of A are the scalar multiples of $e_2, e_3, \ldots, e_{n-1}$ and all vectors of the form je_1+ke_n. Since

$$B(je_1+ke_n) = (j+k)e_1 + j(e_2+\ldots+e_n),$$

it is easy to see that no eigenvector of A is one of B.

* * *

820. [1983: 47] *Proposed by W.R. Utz, University of Missouri-Columbia.*

Let P be a polynomial with real coefficients. Devise an algorithm for summing the series

$$\sum_{n=q}^{\infty} \frac{P(n)}{n!}.$$

Solution by Gali Salvatore, Perkins, Québec.

Let the polynomial P be of degree k with complex coefficients a_i (why real?), so that

$$P(n) = a_0n^k + a_1n^{k-1} + \ldots + a_{k-1}n + a_k.$$

By the method of undetermined coefficients, we can, in a finite number of steps, express $P(n)$ in the form

$$P(n) = b_0 + b_1n + b_2n(n-1) + \ldots + b_kn(n-1)\ldots(n-k+1).$$

(It turns out, for example, that

$$b_0 = a_k, \quad b_1 = P(1)-a_k, \quad b_2 = \tfrac{1}{2}(P(2)-2P(1)+a_k), \quad \text{and} \quad b_k = a_0.)$$

We then have

$$\sum_{n=0}^{\infty} \frac{P(n)}{n!} = \sum_{n=0}^{\infty} \frac{b_0}{n!} + \sum_{n=1}^{\infty} \frac{b_1}{(n-1)!} + \ldots + \sum_{n=k}^{\infty} \frac{b_k}{(n-k)!}.$$

Since, for $r = 0,1,\ldots,k$,

$$\sum_{n=r}^{\infty} \frac{1}{(n-r)!} = \sum_{n=0}^{\infty} \frac{1}{n!} = e,$$

it follows that

$$S_1 \equiv \sum_{n=0}^{\infty} \frac{P(n)}{n!} = Be,$$

where $B = b_0+b_1+\ldots+b_k$. The finite sum

$$S_2 = \sum_{n=0}^{q-1} \frac{P(n)}{n!}$$

can now be calculated in a finite number of steps, and then the required sum is

$$S = S_1 - S_2.$$

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

* * *

821. [1983: 78] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Solve the alphametic

CRUX = [MATHEMAT/CORUM],

where the brackets indicate that the remainder of the division, which is less than 500, is to be discarded.

Solution by J.A. McCallum, Medicine Hat, Alberta.

The unique solution is

1670 = [32854328/19673],

with discarded remainder 418.

If, instead of the greatest integer function $[x]$, the proposer had used $[x+\frac{1}{2}]$, the nearest integer function, he could have obtained a closer approximation:

$$2759 = [(68301683/24756) + \tfrac{1}{2}],$$

with discarded remainder -121.

This problem constituted the baptism of fire of a programmable calculator which I recently acquired. Unfortunately, the calculator will only handle up to 4 levels of nested loops, and this problem needs 5, so that I spent hours transcribing hundreds of partial results, each of which had then to be punched in to a second program.

Also solved by MEIR FEDER, Haifa, Israel; ALLAN WM. JOHNSON JR., Washington, D.C.; GLEN E. MILLS, Pensacola Junior College, Florida; and the proposer.

* * *

822. [1983: 78] *Proposed by Charles W. Trigg, San Diego, California.*

Arrange nine consecutive digits in a 3×3 array so that each of the six three-digit integers in the columns (reading downward) and rows is divisible by 7.

Solution by the proposer.

Since the three-digit integers in the middle row and middle column of the square array have a digit in common, we group the distinct-digit multiples of 7 by their middle digits. Thus

105	210	126	231	140	154	168	175	182	196
203	217	329	238	147	259	364	273	189	294
301	315	420	532	245	350	462	371	280	392
308	413	427	539	546	357	469	378	287	490
406	518	623	630	742	651	560	476	385	497
504	714	721	637	749	658	567	574	483	693
602	812	728	735	840	756	763	672	581	791
609	819	826	931	847	854	861	679	784	798
805	910	924	938	945	952		875	980	896
903	917						973	987	

Using the zero-free triads, we form the seventeen distinct-digit central crosses of the square arrays that have a central 1 (crosses resulting from row-column interchange being considered equivalent).

$$\begin{array}{ccc} &3& \\ 2&1&7 \\ &5& \end{array}\quad \begin{array}{ccc} &4& \\ 2&1&7 \\ &3& \end{array}\quad \begin{array}{ccc} &\underline{5}& \\ 2&1&7 \\ &8& \end{array}\quad \begin{array}{ccc} &8& \\ 2&1&7 \\ &9& \end{array}\quad \begin{array}{ccc} &7& \\ 3&1&5 \\ &\underline{4}& \end{array}\quad \begin{array}{ccc} &8& \\ 3&1&5 \\ &2& \end{array}\quad \begin{array}{ccc} &8& \\ 3&1&5 \\ &9& \end{array}\quad \begin{array}{ccc} &9& \\ 3&1&5 \\ &7& \end{array}\quad \begin{array}{ccc} &5& \\ 4&1&3 \\ &8& \end{array}$$

$$\begin{array}{ccc} &8& \\ 4&1&3 \\ &\underline{2}& \end{array}\quad \begin{array}{ccc} &8& \\ 4&1&3 \\ &9& \end{array}\quad \begin{array}{ccc} &9& \\ 4&1&3 \\ &7& \end{array}\quad \begin{array}{ccc} &7& \\ 5&1&\underline{8} \\ &4& \end{array}\quad \begin{array}{ccc} &9& \\ \underline{5}&1&8 \\ &7& \end{array}\quad \begin{array}{ccc} &8& \\ 7&1&4 \\ &2& \end{array}\quad \begin{array}{ccc} &8& \\ 7&1&4 \\ &9& \end{array}\quad \begin{array}{ccc} &9& \\ 8&1&2 \\ &7& \end{array}$$

In five of these arrays one digit is underscored. Every eligible triad with this central digit has a digit duplicating a digit in the cross. We now add another eligible triad to each of the other twelve crosses. The triads chosen were the only ones available that do not duplicate digits of the cross and having a particular middle digit.

$$\begin{array}{ccc} \underline{8}&\underline{3}& \\ 2&1&7 \\ 6&5& \end{array}\quad \begin{array}{ccc} 9&3& \\ 2&1&7 \\ \underline{4}&\underline{5}& \end{array}\quad \begin{array}{ccc} 8&4& \\ 2&1&7 \\ 6&3& \end{array}\quad \begin{array}{ccc} \underline{6}&\underline{8}& \\ 2&1&7 \\ 3&9& \end{array}\quad \begin{array}{ccc} \underline{6}&\underline{8}& \\ 3&1&5 \\ 7&2& \end{array}\quad \begin{array}{ccc} \underline{6}&\underline{8}& \\ 3&1&5 \\ 7&9& \end{array}\quad \begin{array}{ccc} 2&9& \\ 3&1&5 \\ 8&7& \end{array}$$

$$\begin{array}{ccc} &5&6 \\ 4&1&3 \\ &8&7 \end{array}\quad \begin{array}{ccc} 2&8&7 \\ 4&1&3 \\ &9& \end{array}\quad \begin{array}{ccc} 8&9&6 \\ 4&1&3 \\ &7& \end{array}\quad \begin{array}{ccc} 3&8&5 \\ 7&1&4 \\ &2& \end{array}\quad \begin{array}{ccc} 3&8&5 \\ 7&1&4 \\ &9& \end{array}\quad \begin{array}{ccc} \underline{6}&9&3 \\ \underline{8}&1&2 \\ &7& \end{array}$$

There are no triads that start with 83, 45, or 68, so the six squares with underlined pairs cannot be completed. Again, using the only triads available with a particular digit pair, triads are added to the other seven arrays.

```
8 4 7  2 9    7 5 6  2 8 7  8 9 6  3 8 5  3 8 5  3 8 5
2 1 7  3 1 5  4 1 3  4 1 3  4 1 3  7 1 4  7 1 4  7 1 4
6 3    8 7 5    8 7  5 9 5  7 7    1 2    8 2    8 9
```

In every array there is a duplicated digit. Thus no 3-by-3 array of the nine nonzero digits with rows and columns divisible by 7 exists with a central 1. In like manner, all arrays with the other central nonzero digits are eliminated.

If the consecutive digits are 0,1,...,8, the zero must fall in the lower right 2-by-2 square of the array. Working with the 9-free triads that are multiples of 7 and the routine employed with the nonzero digits, we find six basic squares with the desired divisibility property, namely:

```
6 3 7  6 3 7  1 6 8  8 6 1  7 4 2  7 4 2
5 0 4  5 0 4  4 2 0  4 2 0  5 8 1  5 1 8
8 1 2  1 8 2  7 3 5  7 3 5  6 3 0  6 3 0
```

The arrays occur in pairs. In each pair, one array goes into the other by interchanging 1 and 8. Finally, we note that six more solutions are found by reflection in the principal diagonal:

```
6 5 8  6 5 1  1 4 7  8 4 7  7 5 6  7 5 6
3 0 1  3 0 8  6 2 3  6 2 3  4 8 3  4 1 3
7 4 2  7 4 2  8 0 5  1 0 5  2 1 0  2 8 0
```

Also solved by S.C. CHAN, Singapore; MEIR FEDER, Haifa, Israel; STEWART METCHETTE, Culver City, California; and GLEN E. MILLS, Pensacola Junior College, Florida. One incorrect solution was received.

Editor's comment.

The proposer's solution shows how this problem can be solved by brute force. Because of its length, we would have preferred to publish another solution, but the fact is that it was the only complete one. Only Metchette found all six basic solutions (presumably computer-generated), but even he forgot to mention that six more could be obtained by reflection in the principal diagonal.

Metchette wrote: "Your readers may enjoy the same problem for divisibility by 11 or 13. For 11, there is an interesting relationship between the integers in the solutions. For 13, there is a single solution using consecutive digits."

* * *

823. [1983: 79] *Proposé par Olivier Lafitte, élève de Mathématiques Supérieures au Lycée Montaigne à Bordeaux, France.*

(a) Soit $\{a_1, a_2, a_3, \ldots\}$ une suite de nombres réels strictement positifs.

Si

$$v_n = \left(\frac{a_1 + a_{n+1}}{a_n}\right)^n, \qquad n = 1,2,3,\ldots,$$

montrer que $\limsup_{n\to\infty} v_n \geq e$.

(b) Trouver une suite $\{a_n\}$ pour laquelle intervient l'égalité dans (a).

Comment by M.S. Klamkin, University of Alberta.

This problem appears, with a very well motivated solution, in G. Pólya, *Mathematics and Plausible Reasoning*, Princeton University Press, Princeton, N.J., 1954, Volume I, pages 80-84. And on page 86 an extension is given, an exercise asking for a proof that

$$\limsup_{n\to\infty} \left(\frac{a_1 + a_{n+p}}{a_n}\right)^n \geq e^p,$$

where the $a_i > 0$ and p is a positive integer, and asking also for a sequence for which equality is attained. The sequence given is

$$a_1 = 1, \qquad a_n = n \log n, \qquad n = 2,3,4,\ldots .$$

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LEROY F. MEYERS, The Ohio State University; et par le proposeur.

* * *

824. [1983: 79] *Proposed by J.C. Fisher and E.L. Koh, University of Regina.*

(a) *A True Story*: One problem on a recent calculus exam was to find the length of a particular curve $y = f(x)$ from $x = a$ to $x = b$. Some of our more primitive students found instead the area under the curve and, to our surprise and annoyance, came up with the same answer. What must $f(x)$ have been?

(b) If we don't want the same thing to happen on the next exam dealing with surface areas, what functions must we avoid? That is, what functions $z = f(x,y)$ will yield the same numerical answer for the volume under the surface as for the surface area, both over the same region?

I. *Solution to part (a) by M.S. Klamkin, University of Alberta.*

This part of the problem has appeared in [1], where the Problem Editor noted that the problem was also located as Ex. 9, p. 45 of *Ordinary Differential Equations*, by R.E. Langer; as Ex. 8, p. 25 of *Elementary Differential Equations*, by G.E.F. Sherwood and A.E. Taylor; and on pp. 149-150 of *Through the Mathescope*, by C.S. Ogilvy.

For each interval $[a,b]$, and for every function f that is smooth over $[a,b]$, the equation

$$\int_a^b f(x)\,dx = \int_a^b \sqrt{1 + (f'(x))^2}\,dx \qquad (1)$$

establishes a relation between a and b, and then f is a solution to our problem for the interval $[a,b]$ if and only if a and b satisfy that relation. For example, $f(x) = x$ is a solution for the interval $[a,b]$ provided

$$\int_a^b x\,dx = \int_a^b \sqrt{2}\,dx, \qquad \text{or} \qquad \frac{b^2-a^2}{2} = \sqrt{2}(b-a),$$

that is, provided $a = b$ or $a+b = 2\sqrt{2}$.

We now find solution functions f which have the desired property for *all* intervals $[a,b]$. It follows from (1) that $y = f(x)$ is a solution if y is a smooth function satisfying

$$y = \sqrt{1 + (y')^2}.$$

This is a first-order nonlinear ordinary differential equation for which, by standard elementary techniques, we easily find the solutions

$$y \equiv 1 \qquad \text{and} \qquad y = \cosh(x-k),$$

where k is an arbitrary constant. These two functions have the same value at $x = k$, and the same is true of their first derivatives, so we have in addition the two-parameter family of smooth solutions

$$y = \begin{cases} \cosh(x-l), & x \le l, \\ 1, & l < x < m, \\ \cosh(x-m), & x \ge m, \end{cases}$$

each of which has the desired property for all intervals $[a,b]$.

II. *Solution to part (b) by the Cops of Ottawa (revised by the editor).*

We will call a function $z = f(x,y)$ a *solution* if it is a function to be avoided "on the next exam".

For each "good" region R, and for every function $z = f(x,y)$ that is smooth over R, the equation

$$\iint_R z\,dx\,dy = \iint_R \sqrt{1 + z_x^2 + z_y^2}\,dx\,dy \qquad (2)$$

establishes a condition on the boundary of R, and then z is a solution to our problem for the region R if and only if the boundary of R satisfies that condition. For example, $z = x+y$ is a solution for the square $0 \le x,y \le a$ provided

$$\int_0^a\int_0^a (x+y)\,dx\,dy = \int_0^a\int_0^a \sqrt{3}\,dx\,dy, \qquad \text{or} \qquad a^3 = \sqrt{3}a^2,$$

that is, provided $a = 0$ or $a = \sqrt{3}$.

We now find solution functions which have the desired property for *all* "good" regions R. It follows from (2) that $z = f(x,y)$ must be an integral of the nonlinear first-order partial differential equation

$$z = \sqrt{1 + z_x^2 + z_y^2}. \tag{3}$$

To solve this equation, we can set $z = g(t)$, where $t = x\cos\alpha + y\sin\alpha$ and α is an arbitrary constant. The equation then becomes

$$g = \sqrt{1 + (g')^2},$$

and it follows from part (a) that

$$z \equiv 1 \qquad \text{or} \qquad z = \cosh(x\cos\alpha + y\sin\alpha - k), \tag{4}$$

where k is an arbitrary constant. [As in solution I], we have in addition the three-parameter family of solutions

$$z = \begin{cases} \cosh(x\cos\alpha + y\sin\alpha - l), & x\cos\alpha + y\sin\alpha \le l, \\ 1, & l < x\cos\alpha + y\sin\alpha < m, \\ \cosh(x\cos\alpha + y\sin\alpha - m), & x\cos\alpha + y\sin\alpha \ge m, \end{cases}$$

each of which has the desired property for every "good" region R.

Finally, we show how to obtain complete integrals different from that given in (4). We can specify $k = \phi(\alpha)$ for various ϕ, and then determine the envelope of the resulting one-parameter family. For example, let $k = a\cos\alpha + b\sin\alpha$, where a and b are arbitrary but fixed. We then have

$$z = \cosh\{(x-a)\cos\alpha + (y-b)\sin\alpha\},$$

and eliminating α from this and $\partial z/\partial\alpha = 0$ yields

$$z \equiv 1 \qquad \text{or} \qquad z = \cosh\sqrt{(x-a)^2 + (y-b)^2}.$$

If we now let a and b vary, we have a second two-parameter family of solutions. Many more are possible. In fact, equation (3) becomes an identity for any function $z = \cosh h(x,y)$ for which $h_x^2 + h_y^2 = 1$.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain (part (a) only); the COPS of Ottawa (also part (a)); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. Klamkin, University of Alberta (also part (b)); KENNETH M. WILKE, Topeka, Kansas; and the proposers.

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1. Problem E 1549 (proposed by C.R. MacCluer, solution by D.A. Moran), *American Mathematical Monthly*, 70 (1963) 893.

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825. [1983: 79] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Of the two triangle inequalities (with sum and product cyclic over A,B,C)

$$\Sigma \tan^2\frac{A}{2} \geq 1 \quad \text{and} \quad 2 - 8\Pi \sin\frac{A}{2} \geq 1,$$

the first is well known and the second is equivalent to the well-known inequality $\Pi \sin(A/2) \leq 1/8$. Prove or disprove the sharper inequality

$$\Sigma \tan^2\frac{A}{2} \geq 2 - 8\Pi \sin\frac{A}{2}.$$

Solution by Leon Bankoff, Los Angeles, California.

The inequality is true. I gave in [1] a short proof that the inequality

$$2\Sigma \cos A + \Sigma \tan^2\frac{A}{2} \geq 4 \tag{1}$$

holds, with equality if and only if the triangle is equilateral. The proof hinged on showing that (1) was equivalent to the inequality of O. Kooi [2],

$$R(4R + r)^2 \geq 2s^2(2R - r),$$

where R,r,s have their usual meanings. Since

$$\Sigma \cos A = 1 + 4\Pi \sin\frac{A}{2},$$

inequality (1) becomes

$$2 + 8\Pi \sin\frac{A}{2} + \Sigma \tan^2\frac{A}{2} \geq 4,$$

and this is equivalent to the proposed inequality.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain (partial solution); W.J. BLUNDON, Memorial University of Newfoundland; LÁSZLÓ CSIRMAZ, Mathematical Institute of the Hungarian Academy of Sciences, Budapest; J.T. GROENMAN, Arnhem, The Netherlands; G.P. HENDERSON, Campbellcroft, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India (two solutions); and GEORGE TSINTSIFAS, Thessaloniki, Greece.

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