Mathematical Spectrum

25TH YEAR OF PUBLICATION

1992/3
Volume 25
Number 4

A magazine for students and teachers of mathematics in schools, colleges and universities

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

Volume 25 of *Mathematical Spectrum* will consist of four issues, of which this is the fourth. The first was published in September 1992, the second in November 1992 and the third in February 1993.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

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Women in Mathematics

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Women mathematicians, like women in other walks of life, are often invisible. But there are more of us than you might think—certainly more than I had realised, and going back further into the past than I had expected. I have read about women learning and contributing to mathematics as far back as the eighteenth century, sometimes in the face of enormous difficulties imposed by the conventions of their societies. At first I thought there would not be much human interest, that I was going to find worthy industrious women living lives unremarkable except for their involvement in mathematics. But some of those I read about were involved in war or revolution. Others contributed to less dramatic but ultimately more far-reaching social and technical revolutions. Many have been written about with enormous gratitude and affection by friends, protégés and students. Pioneers who encountered many difficulties and obstacles themselves were generous in helping others, especially other women, to succeed in less hostile climates. Altogether it makes an encouraging story for all who have hope in the ultimate goodness of the human spirit, so I hope these episodes from it will be enjoyable and cheering.

Sophie Germain was born in Paris in 1776, at a time when strife and revolution swept France at frequent intervals. Her family was quite affluent, and their fortunes were not much affected by the political turmoil. Nevertheless Sophie found it deeply disturbing, and when she read an account of the murder of Archimedes who remained absorbed in a mathematical problem while his town was invaded and indeed until a soldier struck him down, she decided that mathematics would provide her with exactly the distraction she needed from the distressing events around her. She was 13, and began at once to read and study mathematics and science. At first her parents tried to turn her back to studies which they thought more suitable for a girl, and she was reduced to sitting wrapped in blankets while she read by candlelight, having been deprived of both fire and clothing by her parents who thought thus to to force her to give up and go to bed. There was a tradition in France of women engaging in intellectual pursuits, and eventually her parents gave in and allowed her to pursue her interests unhindered. She never had to worry about earning a living, so in that way at least she was lucky. Her progress in mathematics was rapid—soon she taught herself Latin, so that she could read the works of Newton and Euler. When she was 18 the Ecole Polytechnique was founded in Paris and among the first teachers was Lagrange. French liberalism did not extend to allowing women to register as students or attend lectures, but copies of the lecture notes could be obtained by anyone. The students could submit notes or observations on the courses to their lecturers for comments, and Germain sent in notes to Lagrange under the pseudonym LeBlanc (the name of a young student known to her). In this way she entered into a fruitful correspondence with Lagrange, and when he found out who she was he continued his support, becoming a lifelong friend and colleague. She also corresponded with Gauss after she read his book on number theory. She made important contributions to this field, including an extension of the then very few cases for which Fermat's theorem ($x^n + y^n = z^n$ has no positive integer solutions for $n > 2$) could be proved. In this correspondence she again signed herself LeBlanc.

When France invaded Germany in 1806, Germain became concerned that her teacher and friend would meet the same fate as Archimedes, so she wrote to General Pernety, who was a family friend, asking him to be sure that Gauss was not harmed. Accordingly the astonished Gauss was visited by the general, who told him that his friend Sophie Germain was concerned for his safety. Gauss said that he knew no such person, but a subsequent letter from his friend LeBlanc explained. Gauss's friendly and grateful reply ended with a note about a problem in number theory they were discussing by post. The long correspondence was mutually enriching—Gauss said it was her early letters which turned his attention back to number theory—but the two never met. By this time Germain was accepted by the French mathematicians of the day as one of themselves, but she was never able to occupy any mathematical job and always worked as an 'amateur'. However, in 1808 Cladni proposed a problem in the elasticity of surfaces which Lagrange said could not be solved with the mathematical tools then available. A prize was offered by the Institut de France for its successful solution, and in 1811 Germain submitted a partial solution. It was the only one submitted, as was her second attempt in 1813. Both these attempts hinted at a successful approach, but were marred by errors in the mathematics—as a self-taught mathematician she had gaps and weaknesses in her toolkit. But in 1815 she submitted a third attempt and won the prize, though a complete and rigorous solution of the most general case did not come for many more years. It is surprising that the winning of this prize and the fundamental contribution she made to elasticity did not ensure her lasting fame, but she is almost unknown even among mathematicians.

A commemorative plaque on the Eiffel Tower lists those whose work made its construction possible, but her name is not there although those who extended her fundamental results are so remembered. It seems clear that among these Poisson used, without acknowledgement, the results included in her second prize submission, the manuscript of which was never returned. Germain died in 1831 at the age of 55. Only the previous year she wrote one of her last papers while the guns boomed in the streets below—mathematics had certainly served for her the purpose for which she had first pursued it.

Sonia Kovalevskaya was born in Russia in 1850 and grew up at a time when many groups were meeting in Moscow and St Petersburg to discuss reform and revolution of Russia's antiquated social and political structures. Both Sonia and her elder sister were active in these groups. Once she married, Sonia moved to Heidelberg (her husband went elsewhere to study geology). Thereafter she immersed herself in her mathematical work, later moving to Berlin where she worked with Weierstrass, producing in two years three doctoral dissertations, two in pure mathematics and one on the form of Saturn's rings. As a woman she was not allowed to present these at the University of Berlin but instead submitted them to Göttingen, from which she received the Ph.D. *summa cum laude*. In 1874 she returned to Russia. She lived there for the next eight years, during which she did no new mathematics but was involved with her husband in a disastrous commercial venture. She began writing as a theatre critic during this time, and she also wrote a novel and an autobiography which is still read today. But in 1882 she went back to Europe and to mathematics, working in Paris on the refraction of light in crystals. The successful completion of this work brought her some notice and in 1883 she was invited to lecture at the University of Stockholm. This was a new university and much more liberal than the old centres of learning in Europe—there were women students and staff, and an innovative approach to the curriculum. For the first year she lectured in German (all her papers were written in French or German), but thereafter she used Swedish which she had been studying since her arrival.

Early in her days in Sweden she worked on a classical problem in the rotation of solid bodies, and her solution won her a prize awarded by the Paris Academy of Sciences. The problem was still not completely solved, but she had made a significant advance on the previous efforts, including those of Euler, Lagrange and Poisson. This work, together with that on partial differential equations with Weierstrass which resulted in the Cauchy–Kovalevsky theorem, is her main claim to fame. She is far more widely known than Sophie Germain, perhaps in part because she had the opportunity to hold an academic post. She was a creative mathematician of the first rank, in spite of being distracted for some of her best years by events at home as Russia rumbled slowly towards revolution. She was fortunate to have been provided with a teacher in her teenage years—her parents finally gave in to her determination to learn mathematics when a family friend pointed out her ability after she had reinvented the trigonometric functions when she had no book from which to learn. Her mathematical interest was first aroused by old lecture notes on calculus with which the nursery wall was temporarily papered. From such eccentric beginnings, and through domestic and political turmoil, she emerged as one of the first paid women mathematicians, and the first to be widely known and remembered for her work. She died in Stockholm when she was only 41.

Emmy Noether was a German Jew and had the misfortune to be at the height of her powers at the time that Hitler came into his in 1933. She fled to America but died less than 2 years later, at the age of only 53. She was a pure mathematician with enormous influence on the development of algebra and topology for about 20 years, but she never had a proper academic job in her native Germany in spite of the efforts of her great contemporaries Hilbert and Weyl at Göttingen. Hilbert in exasperation pointed out that he was proposing her for membership of the academic staff and the senate was 'not a bathhouse, that a woman cannot be allowed in'. But he was unsuccessful, and during all her years at Göttingen she received only a small honorarium for the courses she taught, and this had to be approved annually by the Ministry. Of necessity she lived simply, but she was always welcoming and hospitable and many students and colleagues remembered happy evenings in her garret. Her politics were radical, and she looked with hope towards Russia. She had close ties with Russian mathematicians from the time she worked with Alexandroff in Holland in the winter of 1925–26. She spent the winter of 1928–29 in Moscow and taught at the university. Her friends and colleagues tried to arrange a job for her in 1933 when she had to flee Germany, but the Russian bureaucracy was as slow then as now and it was not done in time—she had to accept the offer from Bryn Mawr in the USA, and there she spent the last short stage of her life, learning another language in the process. Her brother was at the University of Tomsk. It seems strange now, looking back across the years of the Cold War and Stalin's purges, to read that the Noethers travelled freely to and from Russia and enjoyed the warmest personal and professional relationships with the mathematicians there.

Heida Geiringer was another Jew who fled to America from Hitler's Europe. She came from Vienna and was teaching at the University of Berlin in 1933. She left for Belgium and then for Istanbul, where she learned Turkish and taught at the university till 1939. From there she reached America and taught first at Bryn Mawr and then at Wheaton.

Top left:	Mary Somerville. From a painting by Thomas Phillips, c. 1834, reproduced by permission of the National Galleries of Scotland.
Top right:	Ada Byron, Countess of Lovelace. Drawn by A. E. Chalon and engraved by W. H. Mote. Reproduced by permission of the National Portrait Gallery.
Left:	Emmy Noether. Reproduced by permission of Birkhäuser Verlag AG from *Emmy Noether* by A. Dick (1970).

The latter was too small an arena for her energy and talents, but by then she had married Richard von Mises, whose flight from Berlin via Turkey had ended with a job at Harvard. She applied for several jobs at universities in the Boston area without success. In some cases the reason for not considering her was openly stated as prejudice against women faculty members. But she did not let bitterness deflect her from her work, which she described as the deepest need of her life. She had a life-long interest in probability and worked on applications in genetics as well as on theoretical problems. After the death of von Mises she extended his work on the fundamentals of probability and published new editions of his *Probability, Statistics and Truth* and *The Mathematical Theory of Probability*. She was a dedicated teacher, and some of her Wheaton students went on to mathematical careers. In 1956 she received belated recognition from the University of Berlin from which she had been driven out more than 20 years before. She was made emeritus professor with full salary. She loved to climb mountains—along with chess and music, this is a traditional relaxation for mathematicians. In spite of the traumas of her middle years she lived to be 79. At the age of 61 she received a grant from the Office of Naval Research to work as a research fellow at Harvard, and continued as a productive mathematician into old age.

Mary Fairfax Somerville was not a great creative mathematician or scientist, but she had a great gift for communication and was fluent in French. These talents combined with her fortunate social position to give her considerable influence at a time when science was just beginning to be recognised as more than a harmless amusement for rich intellectuals. She was born in Jedburgh in 1780 and spent her early life in Burntisland and Edinburgh. She had almost no formal education, but she was helped in her studies by several Edinburgh mathematicians, notably John Playfair. Her brief first marriage put an end to her studies, but as a young widow she returned to work. Her second husband, William Somerville, encouraged her, and their house in London became a meeting place for the mathematicians and scientists of the early nineteenth century. Almost all of them were still amateurs. There was no course or training for young recruits to this select company, but the older men helped the younger ones with their studies in an informal way. Mary Somerville was helped and encouraged in just this way, and later helped others in her turn, including the young Ada Lovelace. Somerville did some original work, but her most influential writings were a translation with extensive additional notes of Laplace's *Celestial Mechanics*, and a book on physical sciences. The former made available to English-speaking mathematicians some of the most important new French work in analysis. At this time mathematics in France was more advanced than in England, and with typical insularity the English had ignored developments across the channel.

Somerville's contribution brought the French work to a wide audience. Her later work *On the Connexion of the Physical Sciences* appeared at a time when the boundaries between scientific disciplines were still extremely fluid but the content was very rapidly expanding. Some systematisation was greatly needed, and her book was translated into Italian, French and Swedish. Her definition of the physical sciences as those concerning matter and energy permanently influenced our view of what constitutes physical science. James Clerk Maxwell described this book as a seminal work because of its insistence on viewing physical science as whole. J. C. Adams, who discovered Neptune, said that his first ideas about the existence of the new planet came from a passage in this book.

During Somerville's long life (she lived to be 92) the Royal Society went through an acrimonious period of reform, and new scientific societies were formed to deal with new specialities. The professionalisation of science had begun: so had moves to free women from their long domestic slavery. Somerville's was the first signature on John Stuart Mill's petition to give women the vote. Whewell, Airy, Sedgewick and Peacock tried to get her elected to the Cambridge Philosophical Society. Women were admitted but later excluded from lectures at King's College London. Overall the gains for women were few and temporary, but Somerville and a few of her contemporaries raised the profile of women in science at a time of great change, and we should not let them fall back into obscurity.

One of Somerville's young protégés was **Ada Lovelace**, the daughter of Lord Byron, whom she introduced to Charles Babbage. Lovelace worked with him on problems with the difference and analytical engines. She was responsible for seminal ideas about computer programming, and is remembered today in the language ADA named for her. She described the use of cards to determine arithmetic operations and variable cards to indicate the operands. She produced detailed examples, listing the cards needed to solve algebraic and trigonometric equations. The two basic techniques of looping and recursion are part of her work—she showed how looping could be used to solve a set of nine simultaneous equations, and wrote a program to calculate Bernoulli numbers recursively. This work began as a translation of a paper by Menabrea (she too was fluent in French), but at Babbage's suggestion she added her own work, which went far beyond that of the original. She published it under her initials A. A. L. Women could still not be expected to be taken seriously by what was becoming a scientific establishment. Ada Lovelace died when she was only 32, in 1852, amid speculation that her mounting financial problems were the result of trying to use the analytical engine to develop a system for betting on horse races. It appears that many letters which would throw light on her life and work were destroyed by Babbage, possibly to protect her reputation if the gabling rumours were true, but perhaps just because he was not a great one for keeping everything.

Among women mathematicians who participated in social revolution we perhaps owe most to the black Americans. The first black women to gain Ph.D.s in mathematics from American universities were **Marjorie Lee Browne** at Michigan and **Evelyn Boyd Granville** at Yale, both in 1949. (The first white woman was Winifred Edgerton Merrill, Columbia 1886, and the first black man was Elbert Cox, Cornell 1925.) It is perhaps significant that both these women came from the south and attended segregated schools and colleges up to first-degree level. There were some superb teachers, and in this environment they were encouraged to achieve the best of which they were capable. By the time they moved into the white male-dominated world, they were already confident professionals. Though they did encounter discrimination on the grounds of both race and sex, they did not succumb to discouragement and bitterness. Their superb academic credentials ensured that, though some of their applications were never even considered, they did get good jobs, and both had very productive and enjoyable careers. Boyd Granville was heavily involved in the space race for 16 years and Lee Browne in the beginnings of the academic use of computers. Both invested their enormous energies in mathematics teaching at undergraduate and postgraduate level, Lee Browne also working at teaching secondary school teachers. Their students found them inspiring, and their influence especially on other women and other blacks can hardly be overestimated. By being highly visible and successful, they helped to erode the stereotypes of women and blacks as being unable to deal with mathematics and science. Since 1949 many more black women have become professional mathematicians in America, but it is perhaps worth noting that some of those who grew up in the north and never attended segregated schools report having to struggle with personal and professional isolation in their early years. These people encountered the racism of many of their compatriots at an age when those trapped in more obviously racist structures were encouraged by a warm personal environment and by teachers dedicated to their advancement.

Many of these women pioneers spoke and wrote warmly of the help and encouragement they received from men in mathematics, and of course from older women breaking the trail ahead. It is an indication of the enormous power of patriarchy that even a man like Hilbert could not persuade his university to accept a woman like Noether on to the staff, and that the successes of many women have not seriously damaged the stereotypes even now. All of us in mathematics, as in life, should do what we can to end oppression and prejudice.

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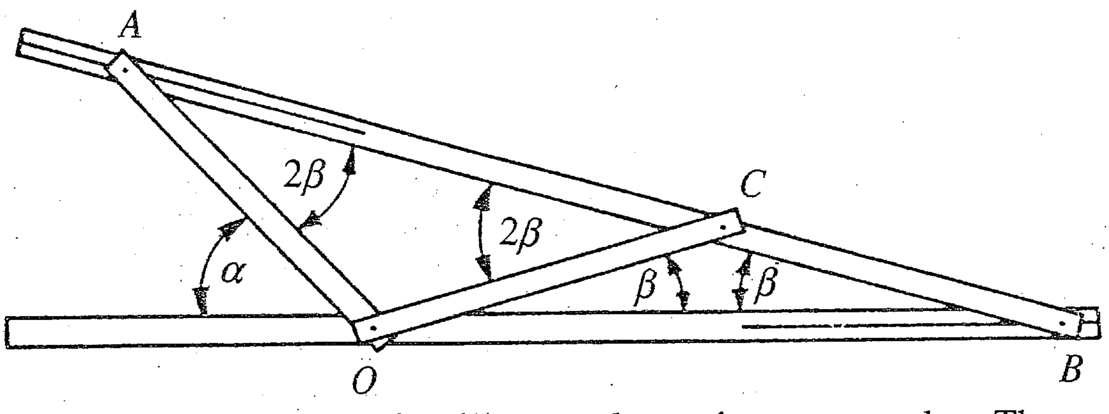
A device for angle trisection



We have designed the device illustrated to trisect an angle. There are four rods with hinges at O and C and with A and B free to slide in runners. Also $OA = OC = CB$. The angle α to be trisected is set as shown; its trisection is β.

Yours sincerely,

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Knights' Tours without Counts

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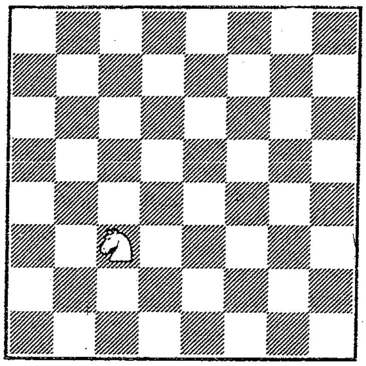


1. Introduction

The problem of finding a Knight's tour—a sequence of Knight's moves which visits every square of a chessboard exactly once—dates back certainly to Euler (see reference 1) and probably to 12th century Arabia. It is just one of the more general class of problems which involve the determination of closed cycles and tours in which every node is visited exactly once; such paths are now named after the mathematician W. R. Hamilton (1805–1865), who studied them extensively in his algebraic work.

The problems of the existence and generation of such tours, which are interlinked, have not been solved for general regions, and for this reason the discovery of algorithms which succeed, even for very small regions as in the Knight's tour, may be useful in shedding light on more difficult problems. Note that the aim is to produce just one of the many tours from a given starting position, bearing in mind that a random 'Knight's walk' almost certainly results in an impasse and hence an incomplete tour. The simple successful methods so far expounded have usually been variants of the method proposed by Warnsdorff (reference 3) and have all relied on numerical calculation to determine the next move.

Warnsdorff's rule involves counting, for every possible move, the number of possible subsequent moves, and choosing the one for which this number is least; he did not provide a means of choosing between more than one position with equal values for this number, possibly because this very rarely matters. This is computationally demanding, especially when a further level of search is introduced, as was suggested by Stonebridge (reference 2) to overcome the occasional failures of the rule. For larger problems, this complexity could be a hindrance, and a set of rules which does not rely on computations but upon simple decisions has an advantage. Such a textual (i.e. non-numerical) algorithm might be a reformulation of Warnsdorff's rule, so producing identical results, or it might be a completely different algorithm. We propose a different, simple rule and prove its validity by using it to construct tours from all the symmetrically distinct starting positions. By comparison with Warnsdorff's algorithm, or modifications of it, the following algorithm is non-numerical, in that it does not involve counting or other computations. We shall use the concept of 'nearness' and take it to be an intuitive notion, essentially non-numerical, and 'nearer' to be a property which can be decided by a simple comparison.

2. Simplification of the positions of the board

If we regard the board as an 8×8 square, by considerations of its symmetry about a vertical axis through its centre, the number of starting positions can be reduced from 64 to 16, a 4×4 square, as shown in figure 1.

$\downarrow i$ \ $\rightarrow j$				
	D	U	U	U
	L	D	U	U
	L	L	D	U^*
	L	L	L^*	D^*

Figure 1. A quarter of the chessboard

Of these 16, the 12 off-diagonal elements L and U in figure 1 are related by a twofold axis of symmetry along the diagonal $DDDD$, so reducing the number of distinct positions to 10. Suitable choices for unique starting positions are thus those marked D and U. Any position may be reduced to one of these, and we shall therefore use the indices i and j, written as a pair, (i, j), with $1 \leqslant i \leqslant j \leqslant 4$, to denote the 10 positions D and U in the obvious way.

3. The algorithm

Input: The starting position as an ordered pair of integers (i, j) with $1 \leqslant i \leqslant j \leqslant 4$.

Output: A Knight's tour from this position.

If the starting position is either $(1,2)$ or $(3,3)$ then, in operating the rule, use the following.

Exception Always move to a position $(4,4)$ in preference to $(3,4)$, that is, D^* rather than L^* or U^* in figure 1.

Rule: Select the next position by applying the following steps in order until a step succeeds uniquely.

1. Choose the position(s) nearest to an edge of the board.
2. (Amongst these) choose the position(s) nearest to a different edge of the board.
3. (For each of these positions) apply this complete rule again, to generate subsequent positions until there is
   either a *difference* (one path leads to a position which satisfies steps 1 and 2 better than any other equal-length path from that position),
   or a *coincidence* (two or more paths lead to the same position on the board), in which case either (or any) of the paths may be used,
   or a tour is completed.

It may appear to be a disadvantage that the rule refers to itself (recursively), but for this problem the depth of call never exceeds 4 and it is thus easy to implement. In fact, most of the moves are determined by either step 1 or step 2.

The diagram which is inset into figure 2 displays the path, ..., 41, 42*, A, B, C, ..., obtained by using the *unmodified* rule from the starting position (1, 2), which reaches an impasse at T, and shows how the 'exception' provides a means of avoiding an impasse to provide a tour, ..., 41, 42*, 43, ...

4. Proof

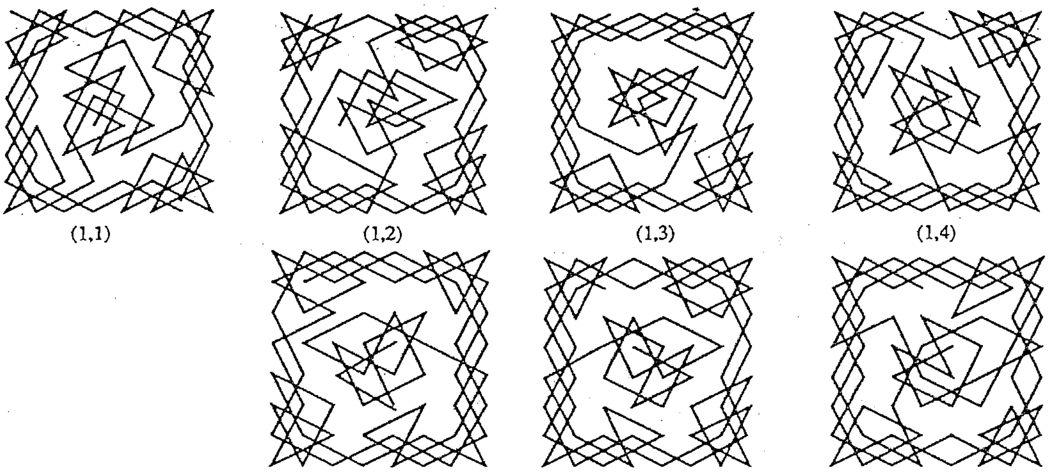
The algorithm is proved by demonstrating that for all starting positions it results in a complete tour, as opposed to reaching some premature state in which no move is possible.

In figure 2 the squares of the board are represented by points and the moves are shown as arcs between them.

5. Comment

The rule alone, without the special 'exception', provides a tour for eight out of the ten possible starting positions. With the addition of the 'exception', it succeeds for the two previous failures and, incidentally, still succeeds for four of the eight previously successful starting positions. However, it fails for the remaining four of the eight previous successes, and a modification incorporating the 'exception' as a general rule, rather than as one for special starting positions, is thus unsatisfactory.

It is interesting that five of the ten tours, namely (1, 4), (2, 3), (2, 4), (3, 3) and (4, 4) are cyclic, in that the last position is a Knight's move from the starting position; this is a high proportion amongst tours generally. Particularly remarkable are (2, 3) and (4, 4), as completing their cycles leads to identical paths, as can be seen from figure 2.



4	1	6	21	38	41	16	19
7	22	3	42*	17	20	37	40
2	5	L	Q	N	39	18	15
23	8	A	T	43		O	36
B	K	R	M	P	I	14	29
9	24	E	J	S	30	35	32
F	C	26	11	H	33	28	13
25	10	G	D	27	12	31	34

(1,2) with unmodified Rule:- ...,41,42*,A,B,C,....

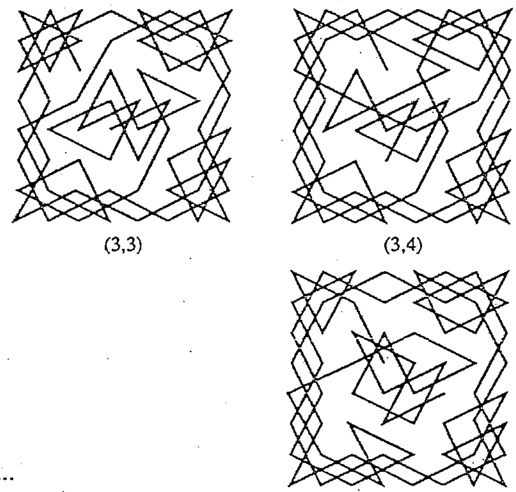


Figure 2. Tours generated from all distinct starting positions

An exhaustive method of proof, such as we have used, can only be contemplated for a small problem for which the consideration of all distinct possibilities is feasible. It would be interesting to investigate whether this algorithm could be extended to the generation of Hamiltonian tours in other regions.

The rule is sufficiently simple that it can be implemented without computational assistance—something which could hardly be said of the extended Warnsdorff rule (reference 3). Nevertheless, it seems likely that there is an even simpler algorithm with a *uniform* set of a very few rules and without special exceptions for certain starting positions. The subtleties of the problem and the difficulty of finding a simple set of rules to solve it can only be appreciated by experimenting for oneself with some trial sets of rules.

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From Stamps to Diophantine Equations

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Consider the following problem:

> The Post Office sells stamps of only two values, 5p and 7p. What amounts could be put on an envelope (up to 40p) ?

One solution strategy might be to produce an array of the numbers 1 to 40 and to highlight the amounts that can be made up from 5p and 7p:

1	2	3	4	**5**	6	**7**	8	9	**10**
11	**12**	13	**14**	**15**	16	**17**	18	**19**	**20**
21	**22**	23	**24**	**25**	**26**	**27**	**28**	**29**	**30**
31	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**

From this array one then makes the observation that all amounts from the *threshold* value 24p are possible (at least up to 40p).

Now consider the generalisation of this problem:

> Given stamp values of a pence and b pence ($1 < a < b$), is it possible to make all amounts from beyond a certain threshold value v? If so:
> (a) what property must a and b possess;
> (b) what is the relationship between the stamp values a and b and the threshold value v?

Obviously a given amount of n pence must be a linear combination of a and b, that is, $n = xa+yb$ for some non-negative integers x and y. If we temporarily drop the reference to stamps, it is clear that, given positive integers a and b, we seek solutions of the Diophantine equation $n = ax+by$. The term *Diophantine equation*, named after the Greek mathematician Diophantus who lived in Alexandria in the third century AD, applies to any equation in one or more unknowns which is to be solved in the integers.

In particular, we need to find the smallest positive integer v such that, for all integers $k \geqslant v$, there exist non-negative integers x_k and y_k such that

$$k = ax_k + by_k. \qquad (*)$$

The case where a and b have a common divisor $d > 1$ can be discounted. For, if $d = \gcd(a,b)$, then $a = t_a d$ and $b = t_b d$ for some positive integers t_a and t_b and consequently d divides k. Thus possible values of (*) must differ by at least d (> 1), rendering it impossible for consecutive (stamp) values to be obtained. Therefore we need only consider the case where the pairs a, b are coprime, i.e. $\gcd(a, b) = 1$, and to that effect a selection of coprime pairs a, b together with their corresponding threshold values v are given in the table. From this table it appears that $v = (a-1)(b-1)$, and indeed this relation is established for the general case below in the main theorem. But first a few preliminary results.

a	5	5	5	6	6	6
b	7	8	9	7	11	13
v	24	28	32	30	50	60

Lemma 1 (see p. 40 of Burton's book, reference 1). Given $\gcd(a, b) = 1$, then the linear Diophantine equation $ax+by = n$ possesses a solution. If x_0, y_0 is any particular solution of this equation, then all the other solutions are given by $x = x_0+bt$, $y = y_0-at$, where t is any integer.

Corollary 2. Given $\gcd(a, b) = 1$, a particular solution x_0, y_0 of the Diophantine equation $ax+by = n$, where $0 < n \leqslant a < b$, can be chosen such that $0 < x_0 < b$ and $-a < y_0 \leqslant 0$.

Proof of Corollary. To show that x_0 can be chosen to satisfy $0 < x_0 < b$, recall that, from Lemma 1, $x = x_0+bt$, which implies that x_0 can be chosen such that $0 \leqslant x_0 < b$. Now observe for $x_0 = 0$ that $n = ax_0+by_0$ implies $y_0 = nb^{-1}$, which contradicts the fact that y_0 is an integer (since $0 < n \leqslant a < b$), and thus only the strict inequality holds.

Now for y_0 observe that $ax_0+by_0 = n$ and the established inequality $0 < x_0 < b$ imply that

$$1 > \frac{n}{b} > y_0 > \frac{n}{b} - a > -a,$$

which in turn implies $0 \geqslant y_0 > -a$.

Theorem 3. For coprime pairs a, b, the threshold value v is equal to $(a-1)(b-1)$.

Proof. Let $v = (a-1)(b-1)$. Then we are required to show:

(a) for all $k \geqslant v$ there exist non-negative integers x_k and y_k such that $k = ax_k+by_k$;

(b) for $v-1$ there do not exist non-negative integers x_0 and y_0 such that $v-1 = ax_0+by_0$.

Fix a and b and suppose without loss of generality that $1 < a < b$.

(a) First observe that, if the first a consecutive integers from a given value u, inclusive, each possess non-negative solutions of (*), then all subsequent integers will possess non-negative solutions of (*). For, if $u+l$ for $l \in \{0, 1, \ldots, a-1\}$ satisfies (*) with non-negative solutions, i.e. $u+l = ax+by$ for some non-negative integers x and y, then $(u+l)+a$ satisfies (*) with non-negative solutions since

$$(u+l)+a = (ax+by)+a = (x+1)a+by.$$

But then the next a consecutive integers from $u+a$ each possess non-negative solutions of (*) and so the previous argument can be repeated from $u+a$, etc.

Now

$$v+l = (a-1)(b-1)+l = ab-a-b+(l+1).$$

Hence we require to show that $v, v+1, \ldots, v+(a-1)$, that is, $ab-a-b+1$, $ab-a-b+2$, ..., $ab-a-b+a$ all possess non-negative solutions to (*). Thus for $1 \leqslant n \leqslant a$ consider

$$\begin{aligned} k &= ab-a-b+n \\ &= ab-a-b+(ax_0+by_0) \\ &= a(x_0-1)+b(a+y_0-1), \end{aligned}$$

where $0 < x_0 < b$ and $-a < y_0 \leqslant 0$ by Corollary 2. Now observe that $x_0-1 \geqslant 0$ and $a+y_0-1 \geqslant 0$. Hence $k = ax_k+by_k$, where $x_k = x_0-1 \geqslant 0$ and $y_k = a+y_0-1 \geqslant 0$, for $k \in \{v, v+1, \ldots, v+a-1\}$.

(b) Consider

$$v-1 = (a-1)(b-1)-1 = ab-(a+b) = a(b-1)-b.$$

Suppose that $a(b-1)-b = k_1a+k_2b$ for some integers k_1 and k_2. Since $b-1, -1$ is a particular solution of $v-1 = k_1a+k_2b$, it follows from Lemma 1 that

$$k_1 = b-1+tb \tag{1}$$

$$k_2 = -1-ta = -(1+ta) \tag{2}$$

for an integer t. But $k_2 \geqslant 0$ implies that $t < 0$ (from (2)), which in turn implies that $k_1 < 0$ (from (1)). Hence k_1 and k_2 cannot simultaneously be non-negative integers.

Reference

1. D. M. Burton *Elementary Number Theory* (Allyn and Bacon, London, 1980).

Golden Pentagons

K. R. S. SASTRY, *Box 21862, Addis Ababa, Ethiopia*

Consideration of self-median triangles in which the sides are proportional to the medians (*Mathematical Spectrum* Volume 22 Number 2) and self-altitude triangles in which the sides are proportional to the altitudes (Volume 22 Number 3), coupled with a different perception of Morley's theorem (Volume 23 Number 1), led the author to study convex pentagons in which the diagonals are parallel to the sides. As you will see, he thereby rediscovered a very beautiful theorem of geometry that deserves to be well known.

We exhibit a pentagon which shares several important properties with the regular pentagon but has several curiosities of its own: if it has one pair of equal sides then there will be another pair: this is also the case with the angles. It is unique, up to similarity, if it contains a pair of right angles; this is also the case if an equilateral triangle or a square is embedded in it. If you erect regular pentagons on each side of it, inwardly or outwardly, then their centres themselves form a regular pentagon! These are some of the properties of a golden pentagon.

What is a golden pentagon?

A golden pentagon is a convex pentagon in which the diagonals divide each other in the golden section. At first it might appear that such a pentagon ought to be regular. Our first theorem shows that this definition is equivalent to having the diagonals parallel to the sides of the convex pentagon.

Theorem 1. Each diagonal of a convex pentagon is parallel to a side if and only if it is a golden pentagon.

Proof. First we suppose that the diagonals are parallel to the sides of the pentagon $ABCDE$. See figure 1. So we have to prove that P, Q, R, S and T are the golden section points of various diagonals. Now $ATDE$ and $CDES$ are parallelograms. This yields $AT = DE = CS$, $AS = TC$, etc. From the similarity of the triangles ASR and ACD we have

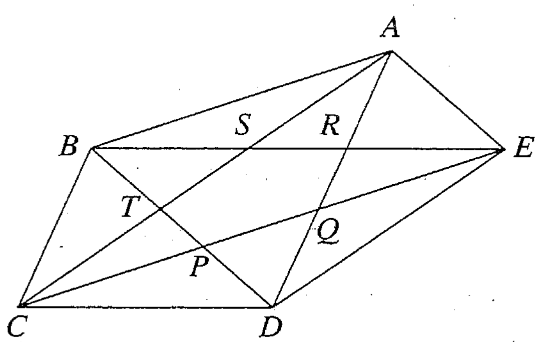

Figure 1

$$\frac{CS}{AC} = \frac{DR}{AD},$$

and so

$$\frac{DE}{AC} = \frac{BC}{AD}.$$

From the similarity of the triangles BTC and AED we have

$$\frac{BC}{AD} = \frac{BT}{AE} = \frac{CT}{DE}.$$

These show that $AC.CT = DE^2 = AT^2$, implying that T is a golden section point of AC. So are the points P, Q, R and S of the various diagonals.

To prove the converse, assume that P, Q, R, S and T are the golden section points of the diagonals. Then, in particular, $AC.CT = AT^2$ and $CE.CP = PE^2$. Therefore

$$\frac{AT}{CT} = \frac{AC}{AT} = \tau = \frac{PE}{CP} = \frac{CE}{PE}, \tag{1}$$

where $\tau = \frac{1}{2}(\sqrt{5}+1)$ is the golden ratio, and so the diagonal BD is parallel to the side AE. The parallelism of the other diagonals to the other sides follows similarly.

In theorem 2 we shall derive an algebraic relation between four consecutive side lengths of the golden pentagon. It will follow that a golden pentagon is completely determined if three consecutive side lengths are known. In what follows, let a, b, c, d and e denote the respective side lengths of AB, BC, CD, DE and EA of the golden pentagon $ABCDE$. Then from (1)

$$CE = a\tau, \qquad AD = b\tau, \qquad BE = c\tau, \qquad AC = d\tau, \qquad BD = e\tau. \tag{2}$$

Note also that τ satisfies

$$\tau^2 - \tau - 1 = 0. \tag{3}$$

Theorem 2.

$$d^2 = a^2 + \frac{1}{\tau}(b^2 - c^2); \qquad e^2 = c^2 - \frac{1}{\tau}(a^2 - b^2). \tag{4}$$

Proof. The parallel property of the golden pentagon yields

$$\angle BAC = \angle ACE = \angle CED, \qquad AC = d\tau, \qquad CE = a\tau.$$

Equating the expressions for $\cos BAC$ and $\cos CED$ from the triangles ABC and CED, respectively, and using (3), we deduce the first relation. Also, by cyclic interchange of letters,

$$c^2 = e^2 + \frac{1}{\tau}(a^2 - b^2),$$

i.e.

$$e^2 = c^2 - \frac{1}{\tau}(a^2 - b^2).$$

Construction of the golden pentagon

Earlier, prior to theorem 2, it was mentioned that a golden pentagon is completely determined with the knowledge of three consecutive side lengths—say a, b and c. To see this we use (4) and determine d and e. Thus $\triangle ABC$ is constructible, $AB = a$, $BC = b$ and $AC = \tau d$; so is $\triangle BCD$ because $CD = c$ and $BD = e\tau$. Finally, E is obtainable either from $AE = e$ and $DE = d$ or as the intersection of lines through A and D parallel to BD and CA, respectively.

Curious properties of a golden pentagon

Theorems 3 and 4 deal with curious properties of a golden pentagon. To establish them we make use of the following characterizing properties of an isosceles trapezium:

(a) its base angles are equal in size;
(b) its oblique sides are equal in length;
(c) its diagonals are equal in length.

Theorem 3. A golden pentagon has a pair of equal sides if and only if it has a pair of equal angles. Then it has yet another pair of equal sides and equal angles.

Proof. First suppose that the golden pentagon $ABCDE$ (figure 2) has a pair of equal sides. These sides may or may not be consecutive. Hence we consider the two essentially different cases.

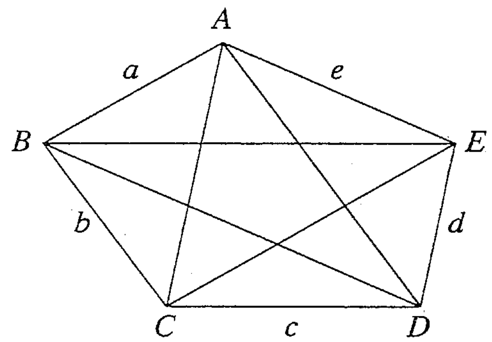

Figure 2

(a) $a = b$. Then (4) yields $c = e$. So $ACDE$ is an isosceles trapezium and the equality of the angles D and E follows. Also from $a = b$ and $\angle EAC = \angle DCA$, we have equal angles A and C.

(b) $c = e$. Then $CDEA$ is an isosceles trapezium. Hence $\hat{D} = \hat{E}$ and $CE = AD$. From the latter follows $a = b$, see (2). Furthermore it yields $\hat{A} = \hat{C}$.

To establish the reverse implications, assume

(a′) $\hat{D} = \hat{E}$. Then $CDEA$ is an isosceles trapezium. So $e = c$, $a = b$ and $\hat{A} = \hat{C}$ follow.

(b′) $\hat{A} = \hat{C}$. A similar argument now yields $a = b$, $c = e$ and $\hat{D} = \hat{E}$.

Theorem 4 gives the maximum number of right angles that a golden pentagon can have. When that maximum is realized the pentagon is determined uniquely, up to similarity (see construction of the golden pentagon).

Theorem 4. A golden pentagon can have at most two right angles. If it does have two right angles then its sides are proportional to 1, $\sqrt{\tau}$, $\sqrt{\tau}$, 1 and $\sqrt{2}$.

Proof. Without loss of generality, let $\angle ABC$ be a right angle. See figure 3. Then $ABCQ$ is a rectangle. Hence neither angle BAE nor angle BCD can be a right angle. This leaves the angles CDE and AED, of which only one can be a right angle.

Again, without loss of generality, let $\angle CDE$ be the second right angle. Then from theorem 3 (b′), with $\hat{B} = \hat{D}$ in place of $\hat{A} = \hat{C}$, it follows that

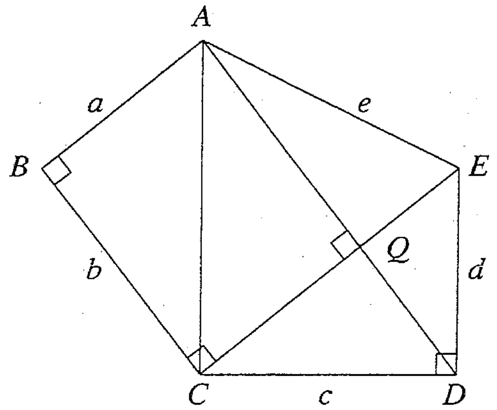

Figure 3

$$a = d, \qquad b = c, \qquad \angle BAE = \angle AED.$$

The Pythagorean theorem applied to triangle ABC then gives

$$a^2+b^2 = d^2\tau^2 = a^2\tau^2.$$

Using (3) and (4) we see that a, b, c, d, and e are, respectively, proportional to 1, $\sqrt{\tau}$, $\sqrt{\tau}$, 1 and $\sqrt{2}$ as claimed.

Further discussion

To encourage further discovery of properties of a golden pentagon, here is a (non-exhaustive) list of problems.

1. If the squares of the side lengths a, b and c are in arithmetic progression then show that the squares of the side lengths d , b and e are also in arithmetic progression.

2. Show that a golden pentagon is uniquely determined (up to similarity) if an equilateral triangle or a square is embedded in its configuration.

Let M_1, M_2, M_3, M_4 and M_5 denote the midpoints of the sides CD, DE, EA, AB and BC, respectively. Then AM_1, BM_2, …, EM_5 are defined to be the medians of the golden pentagon $ABCDE$.

3. Prove that the medians of a golden pentagon are concurrent and that each median bisects the area of the pentagon.

4. Let G denote the centroid, the concurrence point of the medians, of the golden pentagon $ABCDE$. Prove that

$$GA^2+GB^2+GC^2+GD^2+GE^2 = a^2+\tau b^2+c^2.$$

Conclusion

A golden pentagon is an *affinely regular pentagon*, that is, the image of a regular pentagon (as we know in the Euclidean sense) under an *affine* transformation. This is the definition in *affine geometry*. For the benefit of those unfamiliar with affine geometry, the distinguished geometer Professor H. S. M. Coxeter (reference 2) provided the following equivalent definition of an affinely regular n-gon:

> $A_0A_1 \ldots A_{n-1}$ is called an affinely regular n-gon if all the lines $A_\mu A_\nu$ with $\mu+\nu \equiv c \pmod n$ are parallel for each residue c.

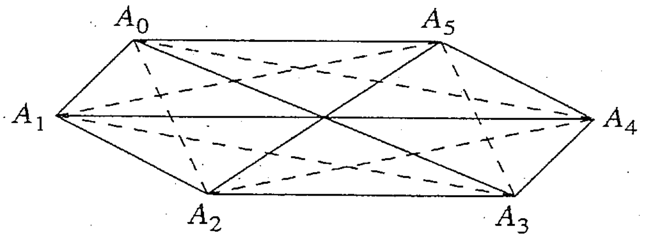

Figure 4

As an illustration, we exhibit an *affinely* regular hexagon $A_0A_1A_2A_3A_4A_5$ in figure 4. For $c = 0$ we have A_1A_5 and A_2A_4 parallel. For $c = 1$, A_0A_1, A_2A_5 and A_3A_4 are parallel. For $c = 2$, A_0A_2 and A_3A_5 are parallel. For $c = 3$, A_0A_3, A_1A_2 and A_5A_4 are parallel to each other, and so on for $c = 4$ and $c = 5$.

A beautiful theorem on affinely regular n-gons is this:

> Erect a Euclidean regular n-gon (inwardly or outwardly) on each side of an affinely regular n-gon. Then the centres of these n Euclidean regular n-gons themselves form another Euclidean n-gon.

This means the following.

(a) For $n = 3$, take any triangle and erect equilateral triangles on each side of the original triangle. Then the centres of these equilateral triangles form an equilateral triangle. This theorem is attributed to Napoleon.

(b) For $n = 4$, take any parallelogram and erect squares on each side of the original parallelogram. Then the centres of these squares form a square. This is known as Thébault's theorem.

(c) For $n = 5$, take any golden pentagon and erect regular pentagons on the sides of the golden pentagon. Then the centres of these regular pentagons form a regular pentagon, see figure 5.

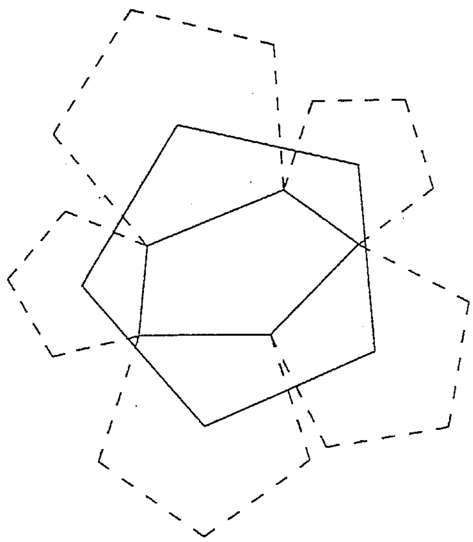

Figure 5. A cluster of regular pentagons around a golden pentagon

Euclidean proofs for the cases $n = 3$ and $n=4$ are not too hard. For $n > 4$ it is generally believed that the tools of affine geometry are essential, see references 1 and 3. This author proved the cases $n = 5$ and $n = 6$ using Euclidean tools and lengthy computations. Is there a reader who can defend the honour of Euclid by providing an elegant Euclidean proof of the general case?

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1. Adriano Barlotti, *Boll. Un. Math. Ital* (3) **10** (1955), 96–98.
2. H. S. M. Coxeter, *Introduction to Geometry*, 2nd edn. (Wiley, New York, 1969).
3. Leon Gerber, Napoleon's theorem and the parallelogram inequality for affine-regular polygons, *American Mathematical Monthly* **87** (1980), 644–648.

Superbrain

In 1983, at University College Cork, Ireland, over coffee after a student mathematics society meeting, a discussion arose as to who were the best mathematical students in college. Students of electrical engineering claimed that, because of the high points requirement for the course, they were obviously the best, but honours students of mathematics in sciences and arts hotly disputed this. A challenge went out which led to the organization of an annual competitive examination open to all full-time registered students of the college, regardless of subjects or faculty, including undergraduates and postgraduates. It was dubbed the 'Superbrain Competition'. So as not to give an advantage to students who had taken advanced courses, the topics were those of the Irish schools' Honours Leaving Certificate, though, of course, the standard was a good deal more difficult.

The Superbrain examination is contested at the beginning of the second term each year, and has been dominated to a large extent by science and electrical engineering students. In recent years it has served as the selection examination for the UCC team for the Mathematical Intervarsity, initiated and hosted by UCC in 1990. The perpetual trophy for this annual competition was won in both 1990 and 1991 by Trinity College Dublin, with UCC close runners-up on both occasions. In 1992 it was won by UCC with University College Dublin in second place. The 1993 competition was held at University College Galway, with all major Irish colleges participating.

Copies of the Superbrain papers 1984–1992 may obtained be at a cost of £2 including postage by writing to me:

Professor D. MacHale
Department of Mathematics
University College
Cork
Ireland.

Solutions to these problems are not normally available because we have found that solutions deprive people of the greatest thrill to be derived from mathematics—the pleasure of solving the problem oneself, perhaps after many years of trying.

DES MACHALE
University College Cork

Bracketing

In the expression $a_1 - a_2 - \cdots - a_n$, in how many ways can pairs of terms be bracketed to give different answers? For example, when $n = 4$,

$$((a_1-a_2)-a_3)-a_4 = a_1-a_2-a_3-a_4,$$

$$(a_1-a_2)-(a_3-a_4) = a_1-a_2-a_3+a_4,$$

$$(a_1-(a_2-a_3))-a_4 = a_1-a_2+a_3-a_4,$$

$$a_1-((a_2-a_3)-a_4) = a_1-a_2+a_3+a_4,$$

$$a_1-(a_2-(a_3-a_4)) = a_1-a_2+a_3-a_4,$$

but two of these are the same.

D. BOFFEY
(12 Wood Road, Halewood,
Liverpool L26 1UZ)

Letters to the Editor

Dear Editor,

APR

I write to make a few comments on the article about APR by Alan Fearnehough in Volume 25 Number 1. If one keeps to the equation in F and M, then, setting $x = 1+M$, one obtains the polynomial equation

$$f(x) = 12nx^{12n+1}-(1+nF+12n)x^{12n}+1+nF = 0.$$

Differentiation shows that $f(x)$ has a maximum at $P(0, 1+nF)$ and a minimum at $x = 1+nF/(12n+1)$. So there are exactly three real roots, one negative, one at $x = 1$, and the required root $x_1 > 1$. The root $x = 1$ ($M = 0$) has been introduced by summing the geometric series and multiplying by M; it is not a solution to the original problem. The tangent at P, $y = 1+nF$, meets the curve $y = f(x)$ again at $x = 1+(1+nF)/12n$. So we have the inequalities

$$1+\frac{nF}{12n+1} < x_1 < 1+\frac{1+nF}{12n}.$$

From this it follows that $x_1 \to 1+\frac{1}{12}F$ as $n \to \infty$. This case corresponds to the situation where the borrower repays the interest each month but repays no capital, and the result $M = \frac{1}{12}F$ merely reflects this. Clearly, if capital is being repaid we have $M > \frac{1}{12}F$, and this also follows mathematically from the observation that $f(1+\frac{1}{12}F) = 1+nF-(1+\frac{1}{12}F)^{12n}$ is negative. This now gives the bounds $\frac{1}{12}F < M < \frac{1}{12}F+\frac{1}{12}n^{-1}$, with corresponding bounds for $A = (1+M)^{12}-1$. The lower bound $(1+\frac{1}{12}F)^{12}-1$ for A is not bounded above by $2F$, but only extortionists charge 150% interest! It is clear from the graph of $y = f(x)$ that Newton's method will converge for initial values greater than $1+nF/(12n+1)$; $1+\frac{1}{12}F$ would seem a natural starting point. However, this need not improve on the numerical results already given in Alan Fearnehough's article.

Yours sincerely,

A. O. Sands
(Department of Mathematics,
University of Dundee)

Dear Editor,

Solution of financial equations by fixed-point iteration

Fearnehough's method (reference 3) of finding the annual percentage rate (APR) from a flat annual percentage for a loan is an interesting application of fixed-point iteration for solving an equation of the form $x = f(x)$. (See section 2.2 of reference 2 for an account of this method.) It requires a fair number of iterations to get an accurate result, and so it is desirable to have a reasonably good starting value.

One suggested starting value is given by Daykin's approximation (p. 258 of reference 4):

$$i = \frac{2F}{1+\dfrac{1}{n}+\dfrac{(n-3m+2)F}{3m}},$$

where $100F$ is the flat rate of interest per annum, n is the number of equal instalments and m is the number of payments per year, in arrears. In Fearnehough's numerical example, $100F = \frac{445}{42}$, $n = 36$, $m = 12$ and Daykin's formula gives a starting value of $i = 0.2050$ against Fearnehough's $i \doteqdot 2F = 0.2119$. This is a slight improvement, because the required value is $i = 0.2063$.

Another financial problem to which the fixed-point algorithm turns out to be applicable concerns calculations involving the probability of ultimate ruin of an insurance company. For this problem, suppose, taking burglary insurance as an example, that

- claims against a company occur randomly (so that the number of claims in a year follows a Poisson distribution);
- the distribution of claim size is exponential with mean α;
- the premium charged for a policy is $(1+\theta)\alpha$, where θ is a 'relative security loading' which also allows, in the long run, for the company to cover running expenses and make a profit;
- the company's initial reserves amount to u.

It is known that the probability, $\psi(u)$, of the company's ultimate ruin is given by

$$\psi(u) = \frac{1}{1+\theta}\exp\left(-\frac{\theta u}{(1+\theta)\alpha}\right).$$

(See reference 1 for details.) If u is fixed, α is estimated and $\psi(u)$ is decided upon by the company directors, then this equation has to be solved for the security loading θ which is used in determining premiums. Writing

$$k = \frac{\alpha}{u\exp\left(\dfrac{u}{\alpha}\right)\psi(u)} \quad \text{and} \quad X = \frac{u}{\alpha(1+\theta)},$$

it becomes $\exp(-X) = kX$ or $X = -\log kX$, which is to be solved for X. This is done iteratively by choosing an X_0 and taking

$$X_{n+1} = -\log kX_n \quad (n = 0, 1, 2, \ldots).$$

For example, if $\alpha = 5000$, $u = 500\,000$ and $\psi(u) = 0.002$ then $k = 5\exp(-100) = 1.86\times 10^{-43}$, and an arbitrary starting value $X_0 = 1$ leads to

$$X_1 = 98.390\,58, \qquad X_2 = 93.8016, \qquad \ldots, \qquad X_6 = 93.8489$$

and $\theta = u/\alpha X - 1 = 0.065\,54$ or 6.554%. The convergence is rapid even for a very inaccurate starting value, and for realistic values of α, u and $\psi(u)$ appears to be far superior to the Newton–Raphson method despite the latter being a second-order iterative method.

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Yours sincerely,

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(Department of Statistics
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University of the Witwatersrand,
Johannesburg, South Africa)

Editor's note: In Alan Fearnehough's example the value $i = 0.2062$ would be given by modifying Daykin's approximation to

$$i = \frac{2F}{1+\dfrac{1}{n+\dfrac{(n-3m+2)F}{3m}}}.$$

Can any reader provide a mathematical justification for this?

Dear Editor,

Happy numbers

Problem 24.1 can be generalized to bases other than 10. I have observed the results given below, which may be proved in ways very similar to the solution to 24.1 given by Amites Sarkar (Volume 24, Number 3, pages 90–91). Essentially all we have to do is to observe that, in base g,

$$\sum x_i^2 \leqslant g(1+\log_g n) < n \quad \text{for } n \geqslant n_0(g)$$

and verify the results for $n < n_0(g)$.

Bases 2 and 4. All integers end up in the fixed point 1.

Base 3. Fixed points 1, 12, 22; cycle 2 → 11 → 2.

Base 5. Fixed points 1, 23, 33; cycle 4 → 31 → 20 → 4.

Base 6. Fixed point 1; cycle 5 → 41 → 25 → 45 → 105 → 42 → 32 → 21 → 5.

Base 7. Fixed points 1, 13, 34, 44, 63; cycles 2 → 4 → 22 → 11 → 2 and 16 → 52 → 41 → 23 → 16.

Base 8. Fixed points 1, 24, 64; cycles 4 → 20 → 4, 5 → 31 → 12 → 5 and 15 → 32 → 15.

Base 9. Fixed points 1, 45, 55; cycles 58 → 108 → 72 → 58 and 75 → 82 → 75.

I conjecture that, for bases $g = 3, 5, 6, 7, 9$ and 10, the numbers that end up in 1 are not divisible by $g-1$.

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2. W. Sierpinski, *Elementary Theory of Numbers*, p. 289 (PWN Polish Scientific Publishers, Warsaw/North-Holland, Amsterdam, 1987).

Yours sincerely,

ANAND KUMAR
(B N College,
Patna, India)

Dear Editor,

Problem 24.7

In Volume 25 Number 1 there appeared a solution to Problem 24.7:

> What is the maximum number of regions into which a circle can be divided by n straight lines?

Am I alone in feeling somewhat uncomfortable with that solution? It asserts:

> In order to maximise the number of regions created (by n lines), each line must cross every other line ...

and goes on to deduce that $P(n) = 1+\frac{1}{2}n(n+1)$. Whilst the assertion is indeed true, it seems to me to be a consequence of the solution to the problem and not a wise assumption from which to derive a solution.

Instead, why not argue as follows? Adding an $(r+1)$th line creates at most $r+1$ new regions; it creates exactly $r+1$ if and only if the new line crosses each existing line in a different point inside the circle. Hence n lines can divide the circle into *at most*

$$P(0)+1+2+\cdots+n = 1+\tfrac{1}{2}n(n+1)$$

regions. This bound is attained if and only if there is equality as each line is added, i.e. if every line crosses every other line, with all points of intersection distinct and inside the circle. The bound can be attained because it is possible (a) to add n lines to the Euclidean plane, one by one, such that each added line is parallel to none of the existing lines and passes through none of the existing points of intersection, and then (b) to scale the whole design down so that all of the (finitely many) points of intersection fit inside the given circle.

Yours sincerely,

JEREMY BYGOTT
(23 Capel Close,
Oxford OX2 7LA)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues, and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

25.10 (Submitted by David Brackin, St Catherine's College, Cambridge)
Does there exist a plane through the origin which intersects the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

with $a \geqslant b \geqslant c > 0$, in a circle?

25.11 (Submitted by Gregory Economides, a medical student at the University of Newcastle upon Tyne)
Archimedes, a mathematics student, has a cart with square wheels, each with a side of length 2 units. The cart can remain stationary on the rough road whatever the position of the wheels. Find the equation of the curve describing the road, and show that the cart can move along the road without bumping.

25.12 (Submitted by Matthew Christophe, Wilmington, Delaware, USA)
Evaluate

$$\lim_{\lambda\to\infty}\left(\frac{\lambda-\alpha}{\lambda-\beta}\right)^{\lambda\gamma}.$$

Solutions to Problems in Volume 25 Number 2

25.4 Evaluate

$$\int_0^1 \ln(1+x)\ln(1-x)\,\mathrm{d}x.$$

Solution by Gregory Economides
Denote the integral by I. If we substitute $y=-x$ we obtain

$$I=\int_0^{-1}\ln(1-y)\ln(1+y)\,(-\mathrm{d}y)$$

$$=\int_{-1}^{0}\ln(1+x)\ln(1-x)\,\mathrm{d}x,$$

so

$$I=\tfrac{1}{2}\int_{-1}^{1}\ln(1+x)\ln(1-x)\,\mathrm{d}x.$$

Put $x = 2y-1$. Then

$$
\begin{aligned}
I &= \tfrac{1}{2}\int_0^1 \ln(2y)\ln(2-2y)\ (2\,\mathrm{d}y) \\
&= \int_0^1 (\ln 2+\ln y)[\ln 2+\ln(1-y)]\ \mathrm{d}y \\
&= \int_0^1 (\ln 2)^2\ \mathrm{d}y + \ln 2\int_0^1 \ln y\ \mathrm{d}y + \ln 2\int_0^1 \ln(1-y)\ \mathrm{d}y + \int_0^1 \ln y \ln(1-y)\ \mathrm{d}y \\
&= (\ln 2)^2 + \ln 2\int_0^1 \ln y\ \mathrm{d}y + \ln 2\int_1^0 \ln x\ (-\mathrm{d}x) - \int_0^1 \ln y \sum_{n=1}^{\infty} \frac{y^n}{n}\ \mathrm{d}y \\
&= (\ln 2)^2 + 2\ln 2\{y\ln y - y]_0^1 - \sum_{n=1}^{\infty} \frac{1}{n}\int_0^1 y^n \ln y\ \mathrm{d}y \\
&= (\ln 2)^2 - 2\ln 2 - \sum_{n=1}^{\infty} \frac{1}{n}\int_0^1 y^n \ln y\ \mathrm{d}y.
\end{aligned}
$$

Now

$$
\begin{aligned}
\sum_{n=1}^{k} \frac{1}{n}\int_0^1 y^n \ln y\ \mathrm{d}y &= \sum_{n=1}^{k} \frac{1}{n}\left(\left[\frac{y^{n+1}}{n+1}\ln y\right]_0^1 - \int_0^1 \frac{y^{n+1}}{n+1}\,\frac{1}{y}\ \mathrm{d}y\right) \\
&= -\sum_{n=1}^{k} \frac{1}{n(n+1)}\int_0^1 y^n\ \mathrm{d}y \\
&= -\sum_{n=1}^{k} \frac{1}{n(n+1)}\left[\frac{y^{n+1}}{n+1}\right]_0^1 \\
&= -\sum_{n=1}^{k} \frac{1}{n(n+1)^2} \\
&= -\sum_{n=1}^{k} \left\{\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2}\right\} \\
&= \left\{\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{k}-\frac{1}{k+1}\right)\right\} + \sum_{n=1}^{k} \frac{1}{(n+1)^2} \\
&= -1 + \frac{1}{k+1} + \sum_{n=2}^{k+1} \frac{1}{n^2} \\
&= -1 + \frac{1}{k+1} + \sum_{n=1}^{k+1} \frac{1}{n^2} - 1 \\
&\to -2 + \tfrac{1}{6}\pi^2 \quad (\text{as } k \to \infty).
\end{aligned}
$$

Hence

$$I = (\ln 2)^2 - 2\ln 2 + 2 - \tfrac{1}{6}\pi^2.$$

25.5 For a natural number n, denote by $T(n)$ the sum of its digits when written in decimal form, and define

$$S(n) = \begin{cases} T(n) & (\text{if } T(n) \leqslant 9), \\ S(T(n)) & (\text{if } T(n) > 9). \end{cases}$$

Determine (i) $S(2^p)$ when p is prime and $p > 3$, and (ii) $\displaystyle\sum_{k=1}^{\infty} \frac{S(2^{k-1})}{2^k}$.

Solution by Gregory Economides

(i) $S(2^k) = 1, 2, 4, 8, 7, 5, 1$ for $k = 0, 1, 2, 3, 4, 5, 6$, respectively. Now $S(n)$ is the number between 1 and 9 for which n is congruent modulo 9, so, for $m \geqslant 0$, $S(2^{m+6}) = S(2^m)$. Hence, when p is prime and $p > 3$, $S(2^p) = 2$ if $p \equiv 1 \pmod 6$ and $S(2^p) = 5$ if $p \equiv 5 \pmod 6$.

(ii)
$$\sum_{k=1}^{\infty} \frac{S(2^{k-1})}{2^k} = \left(\frac{1}{2} + \frac{2}{2^2} + \frac{4}{2^3} + \frac{8}{2^4} + \frac{7}{2^5} + \frac{5}{2^6}\right)\left(1 + \frac{1}{2^6} + \frac{1}{2^{12}} + \cdots\right)$$
$$= \left(2 + \frac{7}{2^5} + \frac{5}{2^6}\right)\left(1 - \frac{1}{2^6}\right)^{-1}$$
$$= \tfrac{7}{3}.$$

25.6 Find the angles subtended at the centre by non-adjacent vertices in a dodecahedron and in an icosahedron.

Solution by Gregory Economides

See Dermot Roaf's article 'Angles in Platonic solids' in Volume 25 Number 2, pages 47–51. The angle subtended by vertices separated by two edges of a dodecahedron (e.g. AH) is that subtended by one edge of a cube, i.e. $\arccos \frac{1}{3}$. Vertices separated by three edges (e.g. AB) subtend $\arccos(-\frac{1}{3})$. Vertices separated by four edges subtend $\arccos(-\frac{1}{3}\sqrt{5})$. Note that five edges are needed to travel from a vertex to its opposite vertex, when the angle is π radians.

The angle subtended by two vertices separated by two edges of an icosahedron (e.g. QS) is $\arccos(-rp) = \arccos(-1/\sqrt{5})$. Note that only three edges are needed to travel from a vertex to its opposite vertex.

Prizes for Student Contributors

The Editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £25 for letters, solutions to problems, and other items.

Reviews

New Applications of Mathematics. Edited by CHRISTINE BONDI. Penguin, London, 1991. Pp. x+289. Paperback £12.99 (ISBN 0-14-012491-8).

Any book that encourages students to read mathematics is welcome. This is one such book. Produced by the Institute of Mathematics and its Applications, it aims to answer the familiar classroom plea: 'What is the use of this topic?' It comprises a set of essays on areas which emphasise both old and new applications of mathematics, written by a variety of people who are engineers as well as mathematicians, and in industry as well as academia.

The varied and welcome background of the authors, which reflect the needs of the A-level mathematician, many of whom use mathematics as a service subject, give us chapters on diverse topics such as cryptography, biological modelling, supercomputers, scaling methods, vibrations in musical instruments and mathematical programming.

In a few places the book struggles by trying to do too much. For example, in the first chapter on functions and graphs, we have some excellent examples (I particularly liked the study of a plume of pollutant in a river—it is a nice result with a good interplay of physics, probability and geometry, and yet not difficult to follow), but the chapter is bogged down by seven pages of explanation of basic calculus. While there is no objection to A-level mathematics being developed in the book, this seems excessive.

There are one or two disturbing points which have crept into this book. I don't think many A-level teachers will be too happy with the expression $dy = dx/x$ which occurs in Chapter 7! I also found the 'conversations' between characters in 'real life' situations rather immature.

Despite these faults, overall we have an impressive book. Its level is such that an average A-level student will benefit from reading it, after having done a little calculus. I would recommend it both to students and to teachers, who will find it a useful source of ideas and valuable background reading.

Undergraduate at St John's College, Oxford — MARK FRENCH

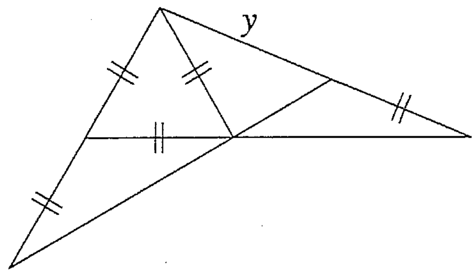
The Penguin Dictionary of Curious and Interesting Geometry. By DAVID WELLS. Penguin, London, 1991. Pp. xiv+285. Paperback £10.99 (ISBN 0-14-011813-6).

It is unusual for a dictionary to be read through from start to finish, but that's what will often happen to this one. Other readers will browse; for wherever the book is opened there are diagrams to attract the attention. The author's selection of what is interesting or curious is of course personal and no doubt governed by considerations of space; I find it hard to suggest any geometry which would be disqualified from appearing in a book with this title.

So while this book has the form of a dictionary, with alphabetically arranged entries, it is not a true dictionary; there is even an index. Rather, the author tries to interest the general mathematical reader from bright school pupil to ageing recreational mathematician. Typically, an entry is about a page long with a couple of diagrams, results are quoted without proof and diagrams are allowed to speak for themselves.

Dead mathematicians who are mentioned in the dictionary are listed chronologically with their dates and nationalities. France won with 24, followed by Germany and Greece with 16 each; Belgium beat Scotland by four dead mathematicians to three.

I much enjoyed this book and was continually tempted to try to take an idea further or prove the more elementary results: for instance, if the line-segments marked as equal in the diagram are of unit length, prove that the length y is the cube root of 2.



The book would benefit from having a glossary of those mathematical terms which are used but not given entries in the main dictionary. Nonetheless it is an excellent purchase for a school library or as a gift for anyone who enjoys mathematics.

Census Computer Services Ltd, Wolverhampton JOHN MACNEILL

Other books received

Mathematical Methods and Theory in Games, Programming, and Economics. By SAMUEL KARLIN. Volume I: **Matrix Games, Programming and Mathematical Economics.** Pp. x+433. Volume II: **The Theory of Infinite Games.** Pp. ix+386. (Two volumes bound as one.) Dover, New York, 1992. Paperback £18.95 (ISBN 0-486-67020-1).

This Dover edition is a republication of the work first published by Addison-Wesley, Reading, MA, 1959.

On Formally Undecidable Propositions of Principia Mathematica and Related Systems. By KURT GÖDEL. Dover, New York, 1992. Pp. viii+72. Paperback £4.95 (ISBN 0-486-66980-7).

This Dover edition is a republication of the work first published by Basic Books, New York, 1962.

Elements of the Topology of Plane Sets of Points. By M. H. A. NEWMAN. Dover, New York, 1992. Pp. vii+214. Paperback £6.95 (ISBN 0-486-67037-6).

This Dover edition is a republication of the second edition (1951) of the work first published by Cambridge University Press, 1939.

Methods of Applied Mathematics. By FRANCIS B. HILDEBRAND. Dover, New York, 1992. Pp. ix+362. Paperback £9.95 (ISBN 0-486-67002-3).

This Dover edition is a slightly corrected republication of the second edition (1965) of the work first published by Prentice-Hall, Englewood Cliffs, NJ, 1952.

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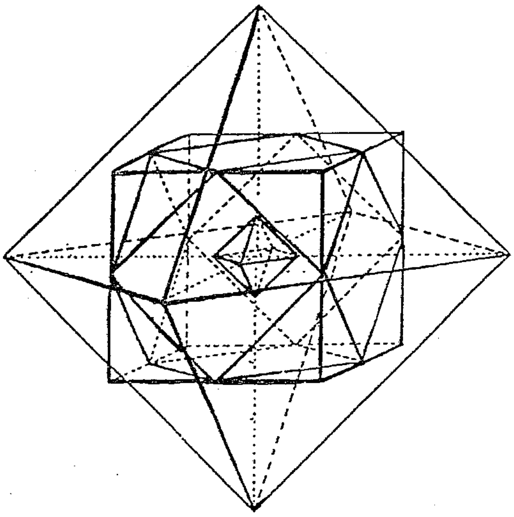

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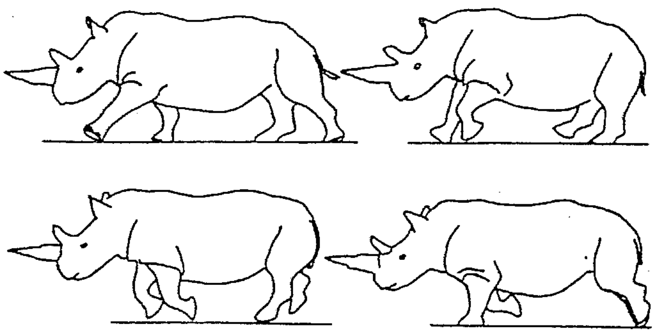

Optimization of Running and Jumping

EDINBURGH Commences at 2.00 pm, 3.00 pm refreshments, ends 4.30. Lecture Theatre 4, Appleton Tower, George Square, Edinburgh EH9 3JZ. Admission is free. Enquires to Dr P. Heywood, Department of Mathematics & Statistics, Edinburgh University, James Clerk Maxwell Building, The King's Buildings, Edinburgh EH9 3JZ.

LEEDS Commences at 7.00 pm, 8.00 pm refreshments, ends at 9.30. Rupert Beckett Lecture Theatre, Arts Building, University of Leeds. Admission free, with ticket in advance. Apply by Monday 21 June to Mr L. Smith, Department of Mathematics, University of Leeds, Leeds LS2 9JT. A stamped addressed envelope would be appreciated.

LONDON Commences at 7.30 pm, 8.30 pm refreshments, ends at 10.00 pm. The Great Hall, Sherfield Building, Imperial College, South Kensington, London SW7. Admission free, with ticket in advance. Apply by Monday 28 June to Miss S.M. Oakes, London Mathematical Society, Burlington House, Piccadilly, London W1V 0NL. A stamped addressed envelope would be appreciated.

CONTENTS

ISSN 0025-5653

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Editor—Mathematical Spectrum,
Hicks Building,
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Published by the Applied Probability Trust
Typeset by The Pi-squared Press, Nottingham, UK
Printed by Galliard (Printers) Ltd, Great Yarmouth, UK