Mathematical Spectrum

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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year consisting of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing.

From the Editor

GIMPS comes up trumps again

A former student recently emailed me to ask if I had seen the BBC website (http://www.bbc.co.uk) announcement about prime numbers. (Isn't it strange how former students are more enthusiastic than current ones!) I hadn't, but I soon checked it out and found that a new largest prime number had been discovered, namely

$$2^{13\,466\,917} - 1,$$

found by Michael Cameron, a 20-year-old Canadian participant in a world-wide computer project known as GIMPS (which stands for Great Internet Mersenne Primes Search, rather than a mild term of abuse). We are told that this new largest prime has 4 053 946 digits and would take three weeks to write out in longhand. There is a reward of \$100 000 for the discoverer of a ten-million digit prime number, so you had better get searching. I would advise you to join GIMPS first!

Prime numbers of the form $2^p - 1$ are called *Mersenne primes*, named after Father Marin Mersenne, a French monk who lived between 1588 and 1648. It is not difficult to see that, if 2^p-1 is prime, then so must be p. But what about the converse? Try $2^{11}-1$. In the preface of his *Cogitata Physica-Mathematica* (1644), he asserted that $2^p - 1$ is prime when $p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257$, but composite for all other primes p up to 257. No one knows how he came up with these figures; he could not possibly have tested all these for primality. In 1772, the great Euler verified that $2^{31} - 1$ is prime by testing it for divisibility by all primes up to $46\,339$, but he couldn't handle the three larger ones.

David M. Burton tells a wonderful story in his book *Elementary Number Theory* (3rd edn, William Brown, Dubuque, IA, 1994 — my favourite mathematics book) of the American mathematician Frank Cole who spent the Sunday afternoons of 20 years to find that

$$2^{67} - 1 = 193\,707\,721 \times 761\,838\,257\,287,$$

so disproving Mersenne's claim that $2^{67} - 1$ is prime. It was not until 1947 that the task of checking all the 54 numbers $2^p - 1$, with p prime, up to $2^{257} - 1$ was completed. Mersenne made five mistakes: he should have included $2^{61} - 1$, $2^{89} - 1$ and $2^{107} - 1$ in his list and not included $2^{67} - 1$ and $2^{257} - 1$.

To date, 39 Mersenne primes are known; the previous largest was $2^{6\,972\,593} - 1$. The last five discoveries have been made by GIMPS (you can find a complete list at http://www.mersenne.org/). The big question is: are there infinitely many Mersenne primes?

That there are infinitely many primes was proved in ancient Greek times — see Book IX of Euclid's *Elements*. Or, alternatively, have a go at proving it yourself. Here's a start:

Suppose that there are only finitely many primes, and write them all down: $p_1, p_2, \ldots, p_n$. Now consider the 'magic number'

$$N = (p_1 p_2 \cdots p_n) + 1,$$

obtained by multiplying them all together and adding 1 for luck. Now, N can be uniquely factorized into prime numbers; that's the fundamental theorem of arithmetic, also in Euclid's *Elements*. But do $p_1, \ldots, p_n$ divide N? And, if not, where do its prime factors come from?

The BBC website tells readers that the GIMPS project spent 13 000 years of computer time to find this new largest prime number; Cameron had his PC running for 45 days to find it. Overall, GIMPS has 130 000 volunteer students, schools, universities, businesses and home users worldwide. All the necessary software can be downloaded for free; so what are you waiting for? As George Woltman, the GIMPS founder, says, 'There are more primes out there, and anyone can participate.' Quite so!

David Sharpe

***Mathematical Spectrum* Awards**
for Volume 33

Prizes have been awarded to the following student readers for contributions in Volume 33:

Chun Chung Tang for various contributions;

Daniel Lamy for various contributions.

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £50 for letters, solutions to problems and other items.

To Infinity and Beyond — Further Mathematical Journeys

P. GLAISTER

In my recent article 'A Tale of Two Cities — a Dickens of an Integral' (reference 1), I described a mathematical journey which started at the definite integral $\int_0^1 x^x\,\mathrm{d}x$ and ended, quite unexpectedly, at the series $1/1^1 - 1/2^2 + 1/3^3 - \cdots$. At the end of the article, readers were invited to find the starting point for a journey which ended at the series $1/1^1 + 1/2^2 + 1/3^3 + \cdots$. In this follow-up article we show that the solution to this is part of a whole series of mathematical journeys relating similar series and integrals. Most of the journeys are straightforward, although one is quite tricky, and there are many other completely unrelated but equally interesting journeys which could be made.

Our starting point is the definite integral

$$I = \int_0^1 x^{\alpha x+\beta}\,\mathrm{d}x\,, \tag{1}$$

where α, β are real numbers. This integral is clearly a generalisation of the one considered above with $\alpha = 1$, $\beta = 0$. Following the first few steps taken for this simpler case, we begin by rewriting and then expanding the integrand in (1) as

$$\begin{aligned} x^{\alpha x+\beta} &= x^\beta x^{\alpha x} = x^\beta \exp\left(\ln(x^{\alpha x})\right) = x^\beta \exp(\alpha x \ln x) \\ &= x^\beta \Big(1 + \alpha x \ln x + \frac{(\alpha x \ln x)^2}{2!} \\ &\qquad + \frac{(\alpha x \ln x)^3}{3!} + \cdots\Big) \end{aligned}$$

so that

$$\begin{aligned} I &= \int_0^1 x^\beta \Big(1 + \alpha x \ln x + \frac{(\alpha x \ln x)^2}{2!} \\ &\qquad + \frac{(\alpha x \ln x)^3}{3!} + \cdots\Big)\,\mathrm{d}x \\ &= \int_0^1 x^\beta\,\mathrm{d}x + \alpha \int_0^1 x^{\beta+1} \ln x\,\mathrm{d}x \\ &\quad + \frac{\alpha^2}{2!}\int_0^1 x^{\beta+2}(\ln x)^2\,\mathrm{d}x \\ &\quad + \frac{\alpha^3}{3!}\int_0^1 x^{\beta+3}(\ln x)^3\,\mathrm{d}x + \cdots \end{aligned}$$

by integrating term-by-term. Thus the individual terms require the determination of integrals of the form

$$J_{m,n} = \int_0^1 x^m (\ln x)^n\,\mathrm{d}x\,, \tag{2}$$

where $n \geq 0$ is an integer, so that

$$I = J_{\beta,0} + \alpha J_{\beta+1,1} + \frac{\alpha^2}{2!} J_{\beta+2,2} + \frac{\alpha^3}{3!} J_{\beta+3,3} + \cdots\,. \tag{3}$$

To evaluate $J_{m,n}$ we integrate by parts, as in reference 1, to give

$$\begin{aligned} J_{m,n} &= \int_0^1 x^m (\ln x)^n\,\mathrm{d}x \\ &= \frac{x^{m+1}}{m+1}(\ln x)^n \Big|_0^1 - \frac{n}{m+1}\int_0^1 x^m (\ln x)^{n-1}\,\mathrm{d}x \\ &= -\frac{n}{m+1} J_{m,n-1} \end{aligned}$$

provided $m > -1$ and $n \geq 1$. Thus,

$$\begin{aligned} J_{m,n} &= \frac{-n}{m+1}\,\frac{-(n-1)}{m+1}\,\frac{-(n-2)}{m+1}\cdots\frac{-1}{m+1} J_{m,0} \\ &= \frac{(-1)^n n!}{(m+1)^n} J_{m,0} \end{aligned} \tag{4}$$

and since $J_{m,0} = \int_0^1 x^m\,\mathrm{d}x = 1/(m+1)$ for $m > -1$, we have from (4) that

$$J_{m,n} = \frac{(-1)^n n!}{(m+1)^{n+1}} \tag{5}$$

for $m > -1$ and integers $n \geq 0$. Hence, $J_{\beta,0} = 1/(\beta+1)^1$, $J_{\beta+1,1} = -1/(\beta+2)^2$, $J_{\beta+2,2} = 2!/(\beta+3)^3$, $J_{\beta+3,3} = -3!/(\beta+4)^4$, etc., and from (3) the formula for I becomes

$$\begin{aligned} I &= \frac{1}{(\beta+1)^1} - \alpha\frac{1}{(\beta+2)^2} + \frac{\alpha^2}{2!}\frac{2!}{(\beta+3)^3} \\ &\quad - \frac{\alpha^3}{3!}\frac{3!}{(\beta+4)^4} + \cdots \\ &= \frac{1}{(\beta+1)^1} - \frac{\alpha}{(\beta+2)^2} + \frac{\alpha^2}{(\beta+3)^3} - \frac{\alpha^3}{(\beta+4)^4} + \cdots\,. \end{aligned}$$

Therefore, the value of the definite integral $\int_0^1 x^{\alpha x+\beta}\,\mathrm{d}x$ is the same as the sum of the series S, where

$$\begin{aligned} I &= \int_0^1 x^{\alpha x+\beta}\,\mathrm{d}x \\ &= \frac{1}{(\beta+1)^1} - \frac{\alpha}{(\beta+2)^2} + \frac{\alpha^2}{(\beta+3)^3} - \frac{\alpha^3}{(\beta+4)^4} + \cdots \\ &= S\,. \end{aligned} \tag{6}$$

The previous case with $\alpha = 1$, $\beta = 0$ is readily obtained from (6) as

$$\int_0^1 x^x \, dx = \frac{1}{1^1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots ,$$

as is the corresponding positive series referred to earlier by taking $\beta = 0$ and $\alpha = -1$, to give

$$\int_0^1 x^{-x} \, dx = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots .$$

That is only the beginning, however. There are clearly many more interesting results which can be deduced from (6). We shall concentrate on the cases where $|\alpha| = 1$ and leave readers to consider other values of α, for example noninteger values, as well as some of the other aspects highlighted in reference 1. These might include the numerical determination of the integral I and the sum of the series S, particularly the case of an alternating series where it is known in advance how many terms are required to evaluate the sum to any specified degree of accuracy.

With $\alpha = 1$ we have from (6)

$$\int_0^1 x^{x+\beta} \, dx = \frac{1}{(\beta+1)^1} - \frac{1}{(\beta+2)^2} + \frac{1}{(\beta+3)^3} - \frac{1}{(\beta+4)^4} + \cdots$$

valid for $\beta > -1$. The first obvious new case is with $\beta = 1$, giving

$$\int_0^1 x^{x+1} \, dx = \frac{1}{2^1} - \frac{1}{3^2} + \frac{1}{4^3} - \frac{1}{5^4} + \cdots$$

and then on to $\beta = 2$ and so on, where the base of the first term is increased by unity each time. Similarly for $\alpha = -1$ and $\beta = 1$

$$\int_0^1 x^{-x+1} \, dx = \frac{1}{2^1} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^4} + \cdots .$$

Unlike the previous cases, however, we are also allowed noninteger values of β as well as α, and we again leave readers to investigate this.

Summarising, we have two sets of results, one alternating, the other positive:

$$\int_0^1 x^{\pm x} \, dx = \frac{1}{1^1} \pm \frac{1}{2^2} + \frac{1}{3^3} \pm \frac{1}{4^4} + \cdots ,$$
$$\int_0^1 x^{\pm x+1} \, dx = \frac{1}{2^1} \pm \frac{1}{3^2} + \frac{1}{4^3} \pm \frac{1}{5^4} + \cdots ,$$
$$\int_0^1 x^{\pm x+2} \, dx = \frac{1}{3^1} \pm \frac{1}{4^2} + \frac{1}{5^3} \pm \frac{1}{6^4} + \cdots ,$$

etc., and the alternating series are clearly examples of a more general class of the form

$$S(n,k) = \frac{1}{k^n} - \frac{1}{(k+1)^{n+1}} + \frac{1}{(k+2)^{n+2}} - \cdots ,$$

where k, n are positive integers. Essentially, this is an alternating series of reciprocal powers where the base and power increase by unity, and we are interested in determining a definite integral representation for $S(n,k)$. As before, we leave readers to investigate journeys starting with the corresponding positive series, and then maybe fractional powers or bases.

Now, the case $n = k = 1$ was the series we began with, and the cases $n = 1$, $k \geq 1$ have also been considered above with

$$S(1,k) = \frac{1}{k^1} - \frac{1}{(k+1)^2} + \frac{1}{(k+2)^3} - \cdots = \int_0^1 x^{x+k-1} \, dx .$$

Furthermore, all cases where $k \geq n$ are covered by this with a finite number of terms removed from the beginning. For example,

$$\begin{aligned} S(3,5) &= \frac{1}{5^3} - \frac{1}{6^4} + \cdots \\ &= \left(\frac{1}{3^1} - \frac{1}{4^2} + \frac{1}{5^3} - \frac{1}{6^4} + \cdots\right) - \left(\frac{1}{3^1} - \frac{1}{4^2}\right) \\ &= S(1,3) - \left(\frac{1}{3^1} - \frac{1}{4^2}\right) \\ &= \int_0^1 x^{x+2} \, dx - \left(\frac{1}{3^1} - \frac{1}{4^2}\right). \end{aligned}$$

Thus, all that remains is to consider the cases where $n > k \geq 1$. As above, once the cases $k = 1$, $n > 1$ have been determined, all others are covered by removing a finite number of terms from the beginning. Hence, we consider

$$S(n,1) = \frac{1}{1^n} - \frac{1}{2^{n+1}} + \frac{1}{3^{n+2}} - \cdots$$

for $n > 1$, with the initial member

$$S(2,1) = \frac{1}{1^2} - \frac{1}{2^3} + \frac{1}{3^4} - \cdots .$$

Recalling that we already have

$$S(1,1) = \frac{1}{1^1} - \frac{1}{2^2} + \frac{1}{3^3} - \cdots = \int_0^1 x^x \, dx ,$$
$$S(1,2) = \frac{1}{2^1} - \frac{1}{3^2} + \frac{1}{4^3} - \cdots = \int_0^1 x^{x+1} \, dx ,$$

we would expect that $S(2,1)$ can be expressed in terms of $\int_0^1 x^{x-1} \, dx$. However, we already have from (6) with $\alpha = 1$ that

$$\int_0^1 x^{x+\beta} \, dx = \frac{1}{(\beta+1)^1} - \frac{1}{(\beta+2)^2} + \frac{1}{(\beta+3)^3} - \frac{1}{(\beta+4)^4} + \cdots , \tag{7}$$

and clearly the right-hand side of (7) is not defined for $\beta = -1$. Moreover, it can be shown that when $\beta = -1$

the left-hand side of (7) is a divergent improper integral. All is not lost, though, because with the exception of the first term on the right-hand side of (7), the remainder of the series in the case $\beta = -1$ is precisely the series $S(2,1)$, i.e. from (7)

$$\frac{1}{(\beta+2)^2} - \frac{1}{(\beta+3)^3} + \frac{1}{(\beta+4)^4} - \cdots = \frac{1}{(\beta+1)^1} - \int_0^1 x^{x+\beta}\,\mathrm{d}x\,. \qquad (8)$$

Thus by letting $\beta \to -1$ in (8) we have

$$\begin{aligned} S(2,1) &= \frac{1}{1^2} - \frac{1}{2^3} + \frac{1}{3^4} - \cdots \\ &= \lim_{\beta\to -1}\Big(\frac{1}{(\beta+1)^1} - \int_0^1 x^{x+\beta}\,\mathrm{d}x\Big)\,, \end{aligned} \qquad (9)$$

and hence the reciprocal and integral should be combined. Certainly the series in (9) is convergent because it is an alternating series whose terms decrease monotonically to zero. Considering the right-hand side of (9) as a means of determining the sum of the series, we rewrite the reciprocal as an integral as follows

$$\begin{aligned} &\lim_{\beta\to -1}\Big(\frac{1}{(\beta+1)^1} - \int_0^1 x^{x+\beta}\,\mathrm{d}x\Big) \\ &= \lim_{p\to 0}\Big(\frac{1}{p} - \int_0^1 x^{x+p-1}\,\mathrm{d}x\Big) \\ &= \lim_{p\to 0}\Big(\frac{x^p}{p}\Big|_0^1 - \int_0^1 x^{x+p-1}\,\mathrm{d}x\Big) \\ &= \lim_{p\to 0}\Big(\int_0^1 x^{p-1}\,\mathrm{d}x - \int_0^1 x^{x+p-1}\,\mathrm{d}x\Big) \\ &= \lim_{p\to 0}\int_0^1 (x^{p-1} - x^{x+p-1})\,\mathrm{d}x \\ &= \lim_{p\to 0}\int_0^1 x^p\,\frac{1-x^x}{x}\,\mathrm{d}x \\ &= \int_0^1 \frac{1-x^x}{x}\,\mathrm{d}x\,. \end{aligned}$$

(Note that $\lim_{x\to 0}((1-x^x)/x) = \lim_{x\to 0}((-x^x\ln x)/1) = \infty$ by l'Hôpital's rule.) We can establish that the improper integral $\int_0^1 (1-x^x)/x\,\mathrm{d}x$ is convergent by using the well-known inequality $\exp(y) \geq 1+y$, so that the integrand satisfies

$$\begin{aligned} \frac{1-x^x}{x} &= \frac{1-\exp(\ln(x^x))}{x} = \frac{1-\exp(x\ln x)}{x} \\ &\leq \frac{1-(1+x\ln x)}{x} = -\ln x\,, \end{aligned}$$

and thus

$$\begin{aligned} \int_0^1 \frac{1-x^x}{x}\,\mathrm{d}x &\leq \int_0^1 -\ln x\,\mathrm{d}x \\ &= (-x\ln x + x)\big|_0^1 = 1 < \infty\,. \end{aligned} \qquad (10)$$

Hence, the integral is convergent, and direct computation using the results in (2) and (5) yields

$$\begin{aligned} &\int_0^1 \frac{1-x^x}{x}\,\mathrm{d}x \\ &= \int_0^1 \frac{1-\big(1+x\ln x+\frac{(x\ln x)^2}{2!}+\frac{(x\ln x)^3}{3!}+\cdots\big)}{x}\,\mathrm{d}x \\ &= \int_0^1 \Big(-\ln x - \frac{x(\ln x)^2}{2!} - \frac{x^2(\ln x)^3}{3!} - \cdots\Big)\,\mathrm{d}x \\ &= -J_{0,1} - \frac{J_{1,2}}{2!} - \frac{J_{2,3}}{3!} - \cdots \\ &= \frac{1}{1^2} - \frac{2!}{2!2^3} + \frac{3!}{3!3^4} - \cdots \\ &= \frac{1}{1^2} - \frac{1}{2^3} + \frac{1}{3^4} - \cdots = S(2,1)\,, \end{aligned}$$

as required. Note that this integral can be rewritten as $\int_0^1 ((1/x) - x^{x-1})\,\mathrm{d}x$. Similar expressions hold for the more general form $S(n,1)$ for $n > 2$.

In the case of the corresponding positive series we have the first member

$$\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^4} + \cdots\,, \qquad (11)$$

where we already have

$$\frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots = \int_0^1 x^{-x}\,\mathrm{d}x$$

and

$$\frac{1}{2^1} + \frac{1}{3^2} + \frac{1}{4^3} + \cdots = \int_0^1 x^{-x+1}\,\mathrm{d}x\,.$$

Here we can establish convergence of the series (11) by comparing with the series $1/1^1 + 1/2^2 + 1/3^3 + \cdots$ since $1/i^{i+1} \leq 1/i^i$ for $i \geq 1$. The corresponding integral representation for (11) is $\int_0^1 (x^{-x}-1)/x\,\mathrm{d}x$, but note that this time the convergence of this improper integral is more difficult to establish. The simplest inequality to use is via a Taylor series, as $x^{-x} = \exp(\ln(x^{-x})) = \exp(-x\ln x) \leq 1 - x\ln x + \frac{1}{2}(x\ln x)^2 \mathrm{e}^{1/\mathrm{e}}$ for $x \in [0,1]$, and thus

$$\begin{aligned} \int_0^1 \frac{x^{-x}-1}{x}\,\mathrm{d}x &\leq \int_0^1 -\ln x + \tfrac{1}{2}x(\ln x)^2\mathrm{e}^{1/\mathrm{e}}\,\mathrm{d}x \\ &= 1 + \tfrac{1}{8}\mathrm{e}^{1/\mathrm{e}} < \infty\,. \end{aligned}$$

We leave readers to fill in the details, particularly verifying that the series (11) is obtained by expanding the integrand and integrating term-by-term.

Summarising, we have the basic results

$$\begin{aligned} \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots &= \int_0^1 x^{-x}\,\mathrm{d}x\,, \\ \frac{1}{2^1} + \frac{1}{3^2} + \frac{1}{4^3} + \cdots &= \int_0^1 x^{-x+1}\,\mathrm{d}x\,, \\ \frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^4} + \cdots &= \int_0^1 \frac{x^{-x}-1}{x}\,\mathrm{d}x\,, \end{aligned}$$

$$\frac{1}{1^1} - \frac{1}{2^2} + \frac{1}{3^3} + \cdots = \int_0^1 x^x \,\mathrm{d}x\,,$$
$$\frac{1}{2^1} - \frac{1}{3^2} + \frac{1}{4^3} + \cdots = \int_0^1 x^{x+1} \,\mathrm{d}x\,,$$
$$\frac{1}{1^2} - \frac{1}{2^3} + \frac{1}{3^4} + \cdots = \int_0^1 \frac{1-x^x}{x} \,\mathrm{d}x\,,$$

as well as many others, and leave readers to determine numerical approximations for these sums and integrals.

Finally, we observe on our journey that there are many other avenues that could be strolled along, some of which have already been signposted. For example, where do $\int_0^1 (x^2)^x \,\mathrm{d}x$ and $\int_0^1 x^{(x^2)} \,\mathrm{d}x$ lead to, and how different is the scenery along the way?

Bon voyage!

References

1. P. Glaister, A Tale of Two Series — a Dickens of an Integral, *Math. Spectrum* **33** (2000/01), pp. 25–27.

The author lectures in mathematics at Reading University. His research interests include computational fluid dynamics, numerical analysis and perturbation methods as well as mathematics and science education. Sadly, despite his many offers, neither of his children has yet been persuaded to join him on any kind of 'mathematical journey'!

Georg Friedrich Bernhard Riemann: 1826–1866

H. BURKILL

1. Riemann's youth

Bernhard Riemann was born on the 17 September 1826 in a village near Dannenberg, close to the river Elbe. His father had served as a lieutenant on the Allied side during the Napoleonic Wars (in Germany then called the Wars of Liberation), but he was now a Lutheran pastor. Not long afterwards the family moved to the parish of Quickborn, just north of Hamburg; there were four girls and two boys, the elder of whom was Bernhard.

From an early age Bernhard enjoyed hearing his father's tales from history, such as the tragic story of unhappy Poland. Initially the pastor did all the teaching of his children, but eventually he had to engage help. It was then that mathematics replaced history as Bernhard's chief interest. The teacher usually had to exert himself to follow his young pupil's quicker and better solutions of exercises. At this time one of Bernhard's amusements was to invent difficult problems and to set these to his siblings.

At the age of 13 Bernhard was confirmed by his father and thereupon left the parental home. Nevertheless the whole family was always bound by the strongest ties of affection; and Bernhard's letters show that, at every age, he was keenly interested in the smallest detail of life in his parents' household.

In 1840 Bernhard started grammar school in Hanover, where he then lived with his grandmother. In view of his unorthodox earlier schooling he at first encountered some difficulties, though his academic performance was always excellent. His letters home vividly described events at school, while begging to be allowed to spend his vacations in Quickborn. Well aware of his parents' straitened circumstances, he devised ways to minimise the cost of the journey. However, occasionally Bernhard allowed himself a *cri de cœur* regarding his awkwardness when obliged to mix with strangers. His shyness in fact never entirely left him, even when he had achieved international renown.

Figure 1. Carl Friedrich Gauss (1777–1855).

Two years after his arrival in Hanover his grandmother died and, at his own request, Bernhard entered the grammar school in Lüneburg, a town about 30 miles from Quickborn. The relative proximity of the two places was, no doubt, an important factor: it facilitated the journey to and from home, though a good deal of walking was involved, much to his mother's anxiety.

Soon after he arrived in Lüneburg the great fire of Hamburg devastated the city's ancient centre. Bernhard was fascinated and sent exhaustive accounts to his parents. If later he told his daughter stories from history, as his father had done for him, he would have included an account of the painstaking reconstruction of the mediaeval Hamburg churches; but the renewed destruction of the Second World War was still in the future.

During the latter part of his school career Bernhard lived with the family of a teacher whose help and support were much appreciated. The school's headmaster also took a lively interest in him, lending him mathematical books such as Legendre's monumental *Number Theory* and parts of Euler's works, normally well beyond a schoolboy's comprehension, but very much grist to Bernhard's mill. Of course, there was also Bernhard's normal school work at which he always did well, but not as spectacularly as in mathematics.

2. Riemann's student days

At Easter 1846, when 19 years old, Riemann enrolled in the University of Göttingen as a student of philology and theology. He may have chosen these subjects because he knew that his father would be pleased to see him following in his footsteps. Moreover the Church offered the quickest route to his own financial independence, a sure means of helping all the family. However, Riemann also attended some mathematical lectures and these showed him how great a sacrifice he would be making by cutting himself off from mathematics. He therefore sought — and obtained — his father's permission to change his degree course to mathematics, thus decisively influencing the development of the subject. Nevertheless, it was soon clear to Riemann

Figure 2. P. G. Lejeune Dirichlet (1805–1859).

that the curriculum at Göttingen was undistinguished. For instance, the only use that the faculty made of the great Gauss (figure 1) was to allocate to him a course of lectures on the method of least squares. So, after a year in Göttingen, Riemann migrated to Berlin, where he remained from 1847 to 1849. In real analysis he attended lectures by such luminaries as Dirichlet (figure 2) and Jacobi (figure 3), while Jacob Steiner lectured to him on geometry. Riemann also learned about the then highly fashionable subject of elliptic functions from F. G. Eisenstein who was his senior by only three years. These functions subsequently played an important part in Riemann's life, and a brief explanation of their nature will be given in Section 5.

Figure 3. Carl Gustav J. Jacobi (1804–1851).

During his time in Berlin, Riemann not only profited mathematically; he also experienced at first hand the Prussian revolution of 1848 which was briefly and improbably led by the king, Frederick William IV. Nevertheless the resulting constitution lasted until the end of the monarchy in 1918. Riemann's sympathies were with the established order: he joined the loyalist Student Corps and even did a spell of guard duty in the Royal Palace.

3. The beginning of complex analysis

Augustin Louis Cauchy (1789–1857) proved in 1825 the first major theorem of complex analysis, now called Cauchy's theorem: *If f is differentiable inside and on a closed contour γ, then* $\int_\gamma f(z)\,\mathrm{d}z = 0$. As the subject developed it became clear that complex differentiability was a crucial and frequently occurring condition with a natural link to real analysis. If $f(z) = f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y)$, then by consideration of increments of z parallel to the real and imaginary axes it is seen that $u_x + \mathrm{i}v_x = f' = (1/\mathrm{i})(u_y + \mathrm{i}v_y)$. Hence,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x\,. \tag{1}$$

Riemann, having naturally proceeded to research in mathematics, outlined a rather different approach to complex analysis. Now partial differentiation of (1) with respect to x and y leads to

$$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0\,,$$

since $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$. Thus, Riemann based his theory on so-called *harmonic functions* $h = h(x, y)$ which satisfy identities of the form

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0. \tag{2}$$

These functions frequently occur in mathematical physics, particularly in the theory of gravitation.

The equations (1) are called the Cauchy–Riemann equations in acknowledgement of the parts played by both men in the early days of complex analysis.

Gauss was greatly impressed by Riemann's thesis. In his report on the work he spoke of 'Riemann's creative, active, truly mathematical mind and of his gloriously fertile originality'. One of the novel ideas in the thesis is the notion of a *Riemann surface* (not so designated by the shy young student). This is a device, introduced by Riemann, for circumventing the difficulties associated with many-valued functions. Instead of these he allows generalised regions in which distinct points may correspond to the same complex number. Points which occupy the same place must now be distinguishable by other means, for instance by different colours or labels. Points with the same label are considered to lie in the same *sheet*. It is possible to treat Riemann surfaces rigorously, but here they are confined to an illustrative role.

A simple Riemann surface arises from the mapping $w = z^n$, where n is an integer greater than 1. It is easy to see from figures 4 and 5 that there is a one-to-one correspondence between any two of the sectors S, W and, in particular, between each sector

$$S_k = \left\{ z \in \mathbb{C}\colon \frac{2(k-1)\pi}{n} < \arg z < \frac{2k\pi}{n} \right\} \quad (k = 1, \ldots, n)$$

and the whole w-plane with the positive real axis removed. The image of each S_k is obtained by making a cut along the positive real axis in the w-plane; and this cut is considered to have an upper and a lower edge. To the n sectors S_k $(k = 1, \ldots, n)$ in the z-plane there correspond n identical copies of the w-plane cut along the positive real axis. These copies are regarded as the n sheets of the Riemann surface, and they are distinguished by the label k which identifies the corresponding sector S_k. When z moves in its plane, the point $w = z^n$ should be free to move on the Riemann surface. This is built up by glueing the lower edge of the first sheet to the upper edge of the second sheet, then the lower edge of the second sheet to the upper edge of the third sheet, and so on. Finally, the lower edge of the nth sheet is glued to the upper edge of the first sheet. The result of the construction is a Riemann surface whose points are in one-to-one correspondence with the points of the z-plane. Of course the construction of the Riemann surface is possible only in the mind's eye, not in a physical sense.

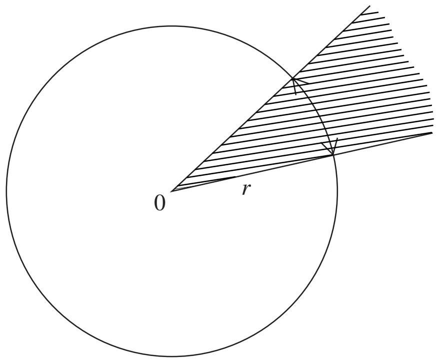

Figure 4. $S = \{z \in \mathbb{C} : 0 < \alpha < \arg z < \beta < \alpha + 2\pi\}$.

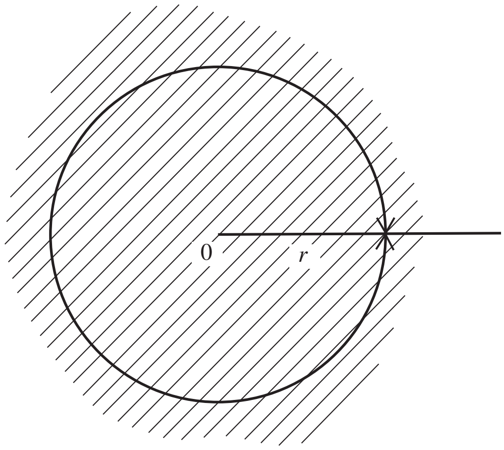

Figure 5. $W = \{w \in \mathbb{C} : 0 < \arg w < 2\pi\}$.

4. Riemann's habilitation

Upon completion of his doctoral thesis in 1851, Riemann was appointed assistant to Wilhelm Weber (1804–1891), a professor of physics at Göttingen who had recently returned to his former position after having been expelled from the university for political reasons, together with seven other professors. When Riemann applied for the assistantship Gauss was one of his referees. However, Gauss later regretted the appointment, believing that too much of Riemann's energy would thereby be diverted from his 'habilitation', the qualification which would entitle Riemann to lecture in the University of Göttingen. In fact, although Riemann found his work very demanding, he remained in his post for 18 months since through it he not only gained a strong background in mathematical physics, but also acquired a feel for experimental physics. Actually, Riemann's interests were unusually broad: he was happy to devote as much time to physics as to mathematics.

In 1854 Riemann submitted his habilitation thesis which consisted of two distinct parts. The first part was a description of previous work on the representation of functions by trigonometric series, while the second part was new material on the same topic.

Fourier had noted that, if

$$f(x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{3}$$

so that also

$$f(x)\cos kx = \tfrac{1}{2}\,a_0\cos kx + \sum_{n=1}^{\infty}(a_n\cos nx\cos kx + b_n\sin nx\cos kx)\,,$$

and if this series may be integrated term by term, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\,kx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} a_k\cos^2 kx\,\mathrm{d}x = a_k \qquad (k = 0, 1, 2, \ldots)\,, \quad (4)$$

while similarly

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\,kx\,\mathrm{d}x = b_k \qquad (k = 1, 2, \ldots)\,. \quad (5)$$

Fourier had asserted that, if a_n, b_n are the *Fourier coefficients* of f given by (4) and (5), then (3) holds, i.e. the *Fourier series* on the right of (3) converges to the function f. Whilst this is certainly not generally true, it is true under suitable conditions which need to be specified.

The first requirement was that integration, which had been taken for granted, should be assigned a meaning, and Riemann filled this gap. He began with several definitions and some simple deductions from them.

(i) Riemann called a set $\mathcal{D}$ of numbers

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

a *dissection* of $[a, b]$, and he called $\max(x_r - x_{r-1})$ the *norm* of $\mathcal{D}$.

(ii) When ξ_r belonged to $[x_{r-1}, x_r]$ and M_r, m_r were the supremum and infimum of f on $[x_{r-1}, x_r]$, then evidently $m_r \le f(\xi_r) \le M_r$. Hence, by summation over $r = 1, 2, \ldots, n$,

$$s(\mathcal{D}) \le \sigma(\mathcal{D}) \le S(\mathcal{D})\,,$$

where

$$s(\mathcal{D}) = \sum_{r=1}^{n} m_r(x_r - x_{r-1})\,,$$

$$\sigma(\mathcal{D}) = \sum_{r=-1}^{n} f(\xi_r)(x_r - x_{r-1})\,,$$

and

$$S(\mathcal{D}) = \sum_{r=1}^{n} M_r(x_r - x_{r-1})\,.$$

(iii) If $\sigma(\mathcal{D})$ tended to a limit I as the norm of $\mathcal{D}$ tended to 0, then Riemann called f *integrable* over $[a, b]$ and he called I the value of the integral.

Figure 6 illustrates Riemann's geometrical thinking behind his analytic definition of an integral.

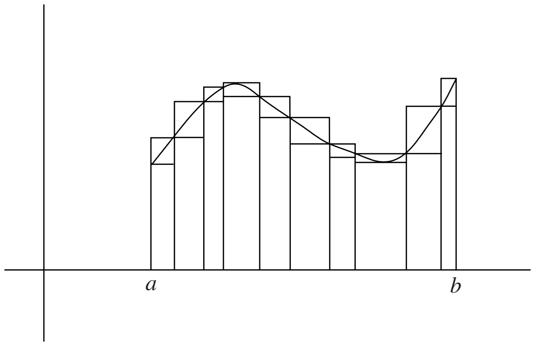

Figure 6. Riemann's approximative sums.

A very useful criterion for the integrability of a function is that $S - s \to 0$ as the norm of the dissection $\mathcal{D}$ tends to 0. The proof is not difficult, but is lengthy and is therefore omitted. It follows easily from this theorem that continuous functions and monotonic functions (i.e. increasing or decreasing functions) are integrable. The next theorem, also deduced from the $S - s$ criterion, provides a ready approach to a convergence theory for Fourier series.

Riemann's theorem. *If the function f is integrable over the interval $[a, b]$, then*

$$\int_a^b f(x)\cos\lambda x\,\mathrm{d}x \quad \textit{and} \quad \int_a^b f(x)\sin\lambda x\,\mathrm{d}x$$

tend to 0 *as* $\lambda \to \infty$.

Proof. The two integrals are treated in very similar ways. It is therefore sufficient to consider in detail only one of them, say the cosine integral. In addition, any desired elementary property of the integral may reasonably be assumed.

Given $\varepsilon > 0$, there is a dissection $\mathcal{D}$ of $[a, b]$ such that

$$S - s = \sum_{r=1}^{n}(M_r - m_r)(x_r - x_{r-1}) < \varepsilon\,.$$

Next, the function g on $(x_{r-1}, x_r]$ is defined by $g(x) = m_r$ for $x_{r-1} < x \le x_r$. Then

$$\int_a^b f(x)\cos\lambda x\,\mathrm{d}x = \int_a^b \{f(x) - g(x)\}\cos\lambda x\,\mathrm{d}x + \int_a^b g(x)\cos\lambda x\,\mathrm{d}x = A + B\,, \quad \text{say.}$$

Now

$$|A| \le \int_a^b |f(x) - g(x)|\,\mathrm{d}x \le \sum_{r=1}^{n}(M_r - m_r)(x_r - x_{r-1}) < \varepsilon\,,$$

while

$$B = \sum_{r=1}^{n} m_r \int_{x_{r-1}}^{x_r} \cos\lambda x\,\mathrm{d}x = \sum_{r=1}^{n} \frac{m_r}{\lambda}(\sin\lambda x_r - \sin\lambda x_{r-1})$$

so that

$$|B| \le \frac{1}{\lambda}\sum_{r=1}^{n} 2m_r = \frac{K}{\lambda} < \varepsilon$$

if λ is sufficiently large.

Corollary. *If a_n, b_n are the Fourier coefficients of the integrable function f, then $a_n, b_n \to 0$ as $n \to \infty$.*

Figure 7. G. F. Bernhard Riemann (1826–1866).

Riemann's theorem often occurs in the cognate Lebesgue form. Henri-Léon Lebesgue (1875–1941) was a great French mathematician whose fame is based upon his work on integration and trigonometric series. The Lebesgue integral is 'more powerful than the Riemann integral' in the sense that every Riemann integrable function is also Lebesgue integrable. The analogue of Riemann's theorem now simply states that, if f on $[a, b]$ is Lebesgue integrable, then $\int_a^b f(x)\cos\lambda x\,\mathrm{d}x$ and $\int_a^b f(x)\sin\lambda x\,\mathrm{d}x$ tend to 0 as $\lambda \to \infty$. There are numerous ways in which the Lebesgue integral scores over the Riemann integral, but this does not detract from Riemann's achievement of seeing the need for and creating the first ever theory of integration.

Although Riemann had submitted his habilitation thesis (on trigonometric series) in 1854, his test was not complete until he had also given a public lecture. He was expected to prepare three lectures and Gauss then decided which of these was actually to be delivered. Riemann had prepared two lectures on electricity and one on geometry, but nevertheless Gauss opted for the geometry lecture entitled (in English translation) 'On the hypotheses which are basic to geometry'. Gauss must have been greatly intrigued; other members of the audience may well have been puzzled by Riemann's revolutionary ideas.

Riemann's definition of a geometry involved two ingredients. The 'points' of the geometry were ordered n-tuples $(x^1, x^2, \dots, x^n)$, but there was no question of specifying the exact nature of $(x^1, x^2, \dots, x^n)$. His other requirement was a formula for an 'element of arc' or for the distance between two points infinitesimally close to one another. Thus, in the case of triples,

$$\begin{aligned}(\mathrm{d}s)^2 &= g_{11}(\mathrm{d}x)^2 + g_{12}(\mathrm{d}x)(\mathrm{d}y) + g_{13}(\mathrm{d}x)(\mathrm{d}z)\\ &\quad + g_{21}(\mathrm{d}y)(\mathrm{d}x) + g_{22}(\mathrm{d}y)^2 + g_{23}(\mathrm{d}y)(\mathrm{d}z)\\ &\quad + g_{31}(\mathrm{d}z)(\mathrm{d}x) + g_{32}(\mathrm{d}z)(\mathrm{d}y) + g_{33}(\mathrm{d}z)^2,\end{aligned}$$

where each g_{ik} is independent of $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$. In general, an n-dimensional Riemannian space consists of points $\boldsymbol{x} = (x^1, x^2, \dots, x^n)$ which can be combined according to certain rules to yield the nondegenerate quadratic form

$$(\mathrm{d}s)^2 = \sum_{1\le i\le k\le n} g_{ik}(\mathrm{d}x^i)(\mathrm{d}x^k)$$

in which each g_{ik} is independent of $\mathrm{d}x^1, \mathrm{d}x^2, \dots, \mathrm{d}x^n$.

Evidently, if the element of arc is given by

$$(\mathrm{d}s)^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + \cdots + (\mathrm{d}x^n)^2,$$

then the geometry reduces to n-dimensional Euclidean space. In pure mathematics, the differential form $(\mathrm{d}s)^2$ is usually taken to be positive definite, i.e. $(\mathrm{d}s)^2 > 0$ whenever $(\mathrm{d}x^1, \mathrm{d}x^2, \dots, \mathrm{d}x^n) \ne (0, 0, \dots, 0)$. This condition ensures the preservation of certain desirable consequences. On the other hand, in general relativity $(\mathrm{d}s)^2$ is assumed to be hyperbolic, i.e. reducible to a sum of squares minus the square of a linear differential form.

Figure 8. The Frederick William IV University, Berlin, in about 1840.

Riemannian spaces have shortest lines, now called geodesics. Moreover, curvature can be defined at all their points and zero curvature characterises locally Euclidean spaces. Finally, a Riemannian space with constant curvature is a sphere if the constant is positive and is non-Euclidean if the constant is negative.

5. Elliptic functions and their generalisations

The definitions of all these functions begin with a rational function $F(x, y)$ and a polynomial $g(x)$. It was well known in the eighteenth century that, if g was of degree 1 or 2, then

$$u(x) = \int_0^x F\big(s, \sqrt{g(s)}\big)\,\mathrm{d}s \tag{6}$$

was an elementary function, while if g was of degree 3 or 4 and had no repeated factor then (6) was a nonelementary integral. The latter is called an *elliptic integral* because the arc length of an ellipse (which is not a circle) is an integral of this kind.

Up to the time of Niels Henrik Abel (1802–1829) these integrals were the objects of study. However, Abel divined that even more interesting were the *inverses* of the functions u of x (in the sense that, for instance, the function $y = x^2$ $(x \geq 0)$ has the inverse function $x = y^{1/2}$ $(y \geq 0)$). It is convenient to denote these functions by $J(u)$, and to write them as $J(z)$ $(z \in \mathbb{C})$ when they can be extended to all of $\mathbb{C}$; Abel then called these *elliptic functions*. The initial difficulty, overcome by Abel, was proving the existence of $J(z)$ for all $z \in \mathbb{C}$. The functions J have remarkable properties. Perhaps the most striking is *double periodicity:* to each J there correspond complex numbers w_1, w_2 with w_1/w_2 nonreal, such that, for all z,

$$J(z + w_1) = J(z) = J(z + w_2).$$

Consequently $J(z)$ needs only to be explored in the fundamental parallelogram with vertices 0, $w_1, w_2, w_1 + w_2$ (see figure 9).

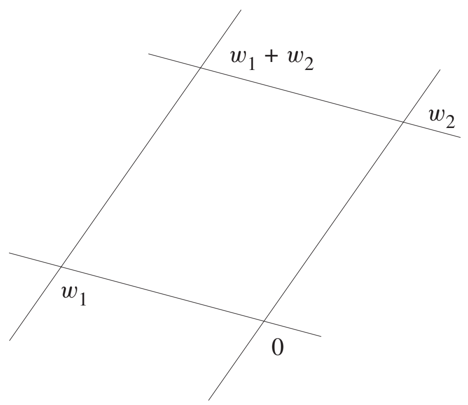

Figure 9. The fundamental parallelogram arising from w_1, w_2.

After disposing of the inversion problem for elliptic integrals, Abel introduced *hyperelliptic integrals* which are of the form (6) with $F(x, y)$ again a rational function, but g now of degree greater than 4. Sadly, time was now running out for the sick Abel, and it was left to Jacobi to prove the existence of *hyperelliptic functions*.

Finally, there is the very large class of *Abelian integrals*

$$\int_0^x F(s, t)\,\mathrm{d}s\,,$$

where F is, as before, a rational function, while t is now an algebraic function of s. Riemann then spent several years establishing the existence of *Abelian functions*. This particularly difficult work was embodied in a course of lectures that he gave in 1855/6 to an audience of three. One of the three was R. Dedekind who, years later, coedited Riemann's collected papers and wrote a brief but discerning biography for the collection. A favourite tool of Riemann's, used by him also in this investigation, was the following principle.

Dirichlet's principle. *Let G be a region in $\mathbb{R}^2$ and let $\mathcal{G}$ be the set of functions differentiable on G and taking specified values on the boundary ∂G. Then, subject to certain conditions, there is a unique function $z \in \mathcal{G}$ which minimises the integral*

$$\frac{1}{2}\int\!\!\int_G \left(z_x^2 + z_y^2\right)\mathrm{d}x\,\mathrm{d}y$$

among all functions in $\mathcal{G}$. Moreover, the minimising function is harmonic in G (see (2)) and takes the required values on ∂G.

Although Karl Weierstrass (1815–1897) now noticed that Dirichlet's principle, as stated, did not guarantee the existence of a minimising function, David Hilbert (1862–1943) in 1901 obtained a version that validated Riemann's argument. Riemann himself had never doubted the essential correctness of his proof.

It is of interest to note the great range of topics on which Riemann lectured after gaining his habilitation. At one end of the scale there was the previously mentioned course on Abelian functions which Riemann was happy to provide for a tiny audience. However, there was also a much admired course on partial differential equations with applications to physics, which was reprinted in 1938, eighty years after its original publication.

6. The last decade

Gauss died in 1855 and Dirichlet was his natural successor. At that time Riemann's friends urged the authorities to promote him by the conferment of a personal chair, but the attempt was unsuccessful owing, it was averred, to the university's lack of funds. However, in 1857 Riemann was promoted to Assistant Professor, and when Dirichlet died in 1859 Riemann became a full Professor.

In 1859 Riemann published the famous paper in which his hypothesis regarding the zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ first appeared.

If p_k $(k = 1, 2, \ldots)$ is the sequence of primes 2, 3, 5, ..., then, when $\mathrm{Re}\,s > 1$, the Euler product

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{k=1}^{\infty}(1 - p_k^{-s})^{-1}$$

holds. This identity is the first indication of an association between ζ and the sequence (p_k).

A function which continually occurs in analysis is the so-called gamma function $\Gamma(z)$ defined as

$$\int_0^{\infty} \mathrm{e}^{-t} t^{z-1}\,\mathrm{d}t\,.$$

From the formula $\Gamma(z + 1) = z\Gamma(z)$, proved by integration by parts, it follows that, when n is a positive integer,

$$\Gamma(n) = (n - 1)!\,.$$

Thus the complex Γ function is a generalisation of the elementary factorial function.

Riemann now proved that the function $\eta(s)$ defined as $\zeta(s)\Gamma(\frac{1}{2}s)\pi^{-(1/2)s}$ has the useful property of invariance under the substitution of $1-s$ for s. He then asserted without proof that the number of zeros of ζ with $\operatorname{Im} z$ between 0 and T (> 0) is

$$\frac{1}{2\pi}T\log T - \frac{1+\log 2\pi}{2\pi}T + O(\log T);$$

and, still without proof, he suggested that all nonreal zeros of ζ have real part $\frac{1}{2}$. This is the famous (or notorious) Riemann hypothesis. It is not even known how Riemann came to make this suggestion.

Since Riemann's time there has been much numerical work which supports (but does not prove) the Riemann hypothesis. Two arguments, pulling in opposite directions, may also be quoted.

(i) G. H. Hardy (1877–1947) showed in 1914 that infinitely many zeros of ζ have real part $\frac{1}{2}$.

(ii) J. E. Littlewood (1885–1977), probably the foremost classical analyst of his time, eventually came to the conclusion that the Riemann hypothesis may well be false.

A proof of the hypothesis or a counterexample seems to be as distant as ever.

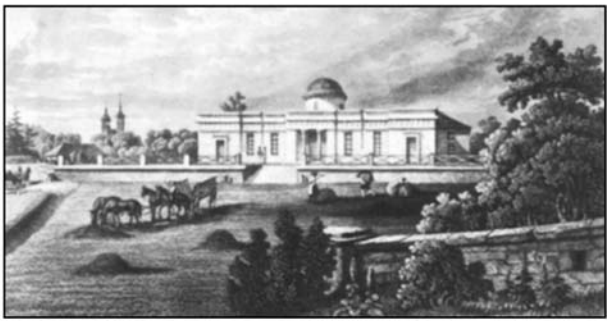

Figure 10. The New Observatory at Göttingen.

Riemann's father and eldest sister Clara died in 1855, and the three remaining sisters went to live with the other, younger, brother who was a postal clerk in Bremen, consequently rather more affluent than a university teacher. Then, in 1857, the brother died and the three sisters joined Riemann's establishment in Göttingen, soon to be diminished by one with the death of the youngest sister Marie. Because of the pressure on space in his household the university authorities allowed Riemann to occupy the Observatory as living quarters, as Gauss had done during his lifetime (see figure 10).

At last, in 1862, Riemann felt sufficiently secure financially to be able to support a wife, and he married Elise Koch, a friend of his sisters. Unfortunately only a month later he contracted pleurisy, and from then on he suffered from tuberculosis. The Government awarded Riemann the funds to convalesce in the balmier air of Italy. However, he never made a genuine recovery. One more sister died in the summer of 1864, so by now his parents and four of their six children had died. Riemann tried several times to return to Göttingen, but each time he did so his health rapidly deteriorated.

Throughout his life Riemann had remained modest, gentle and kind, unspoilt by the honours that recognition finally brought with it: thus, in March 1866 he was made a Corresponding Member of the French Academy, and in June 1866 a Foreign Member of the Royal Society of London. Yet another visit to Italy in search of better health took him to Selasca on Lake Maggiore; and there he died in July 1866, not yet forty years old, sustained by his wife and by his faith. Dedekind, in his account of Riemann's life, says 'He served his God faithfully, as his father had done, but in a different way'.

Acknowledgments

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Before retiring, ***Harry Burkill*** *was a reader in pure mathematics at the University of Sheffield; he edited* Mathematical Spectrum *from 1974 to 1979.*

Series Expansions of $1/\pi$ in Various Bases and How to Discover Your Own

MILTON CHOWDHURY

Introduction

Put your calculator into radians mode and calculate, successively, $\sin(2^1) = 0.9092\ldots$, $\sin(2^2) = -0.7568\ldots$, $\sin(2^3) = 0.9893\ldots$, writing 0 if the answer is positive and 1 if it is negative. Your answers give the binary digits after the binary point for $1/\pi$. The same thing is true for the sequence $\tan(2^0), \tan(2^1), \tan(2^2), \ldots$. These are known results, although the second is quite recent (reference 1).

In this article, we show how to prove these results together with a new result involving cos, before describing a systematic method for deriving analogous series expansions for $1/\pi$ in other bases, including base 10. It is worth stressing that, intriguing though these series are, they are not useful in the practical calculation of the digits of $1/\pi$, since they presuppose accurate values of the trigonometric functions involved. Different types of series do lead to efficient digit extraction algorithms for π as in references 2 (pp. 663–676) and 3 (pp. 479–483).

Throughout this article, we shall write $\lfloor x \rfloor$ for the *integer part* of x and $\{x\}$ for the *fractional part*, $x - \lfloor x \rfloor$, and make extensive use of the jump function $u(x)$ defined by

$$u(x) = \begin{cases} 1 & \text{if } x \le 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Base 2 results

Our initial observation may be recast as: $u(\sin 2^n)$ is a_n, the nth binary digit after the binary point of $1/\pi$, or

$$\sum_{n=1}^{\infty} \frac{1}{2^n} u(\sin 2^n).$$

To see this, notice that a_n is the leading digit after the binary point of $\{2^{n-1}/\pi\}$. (Since π is irrational, $2^{n-1}/\pi$ cannot take the values 0 or $\frac{1}{2}$; this explains the exclusive use of open intervals in the tables that follow. Table 1 shows that $a_n = u(\sin(2\pi)(2^{n-1})/\pi) = u(\sin 2^n)$, as claimed. The proof of the tan result is very similar.

Table 1.

Interval for $\{x\}$	$\left(0, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 1\right)$
$u(\sin(2\pi x))$	0	1

Replacing sin by cos gives the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} u(\cos 2^{n-1}) = 0.4756260767\ldots.$$

This is almost certainly another irrational number (although this is not known), but we can give a new and unusual closed form for the sum of this series; it is the 'bitwise *xor* product' of $1/\pi$ and $1/2\pi$, meaning that a binary digit in the binary expansion of the series is 1 if and only if the corresponding binary digit in the expansion of $1/\pi$ and $1/2\pi$ are different. To see this consider table 2. Inspecting the entries, we see that $u(\cos 2^{n-1})$ is 1 if and only if $u(\sin 2^{n-1})\, u(\sin 2^n)$ are different, i.e. a_{n-1} and a_n are different. Thus, as claimed,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} u(\cos 2^{n-1}) \frac{1}{\pi} \text{ Bitxor } \left(\frac{1}{2\pi}\right).$$

Table 2.

Interval for $\{x\}$	$\left(0, \frac{1}{4}\right)$	$\left(\frac{1}{4}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, \frac{3}{4}\right)$	$\left(\frac{3}{4}, 1\right)$
$u(\sin 2\pi x)$	0	0	1	1
$u(\sin 4\pi x)$	0	1	0	1
$u(\cos 2\pi x)$	0	1	1	0

Base 4 expansions

We can derive a base 4 expansion immediately by noting that the top two rows of table 2 yield the leading base 4 digit for $\{x\}$ as $2u(\sin 2\pi x) + u(\sin 4\pi x)$, so that

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \left(2u(\sin \tfrac{1}{2} 4^n) + u(\sin 4^n)\right) = \frac{1}{\pi}$$

because the nth digit after the point in $1/\pi$ is the leading digit of $\{4^{n-1}/\pi\}$. A similar approach produces expansions in base 8, base 16, ... Here we prefer to use an alternative method to dig out the base 4 digits; this will set the scene for other bases.

Table 3.

Interval for $\{x\}$	$\left(0, \frac{1}{4}\right)$	$\left(\frac{1}{4}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, \frac{3}{4}\right)$	$\left(\frac{3}{4}, 1\right)$
$A = u(\sin 2\pi x)$	0	0	1	1
$B = u(\cos 2\pi x)$	0	1	1	0
Leading base 4 digit	0	1	2	3

Consider table 3. We seek a combination of A and B, $\Psi(A, B)$ say, which reproduces the bottom row of the base 4 digits. Then

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \Psi(A, B) = \frac{1}{\pi},$$

Table 5.

Interval for $\{x\}$	$(0, 0.1)$	$(0.1, 0.2)$	$(0.2, 0.3)$	$(0.3, 0.4)$	$(0.4, 0.5)$	$(0.5, 0.6)$	$(0.6, 0.7)$	$(0.7, 0.8)$	$(0.8, 0.9)$	$(0.9, 1)$
$A = u(\sin 2\pi x)$	0	0	0	0	0	1	1	1	1	1
$B = u(\sin 4\pi x)$	0	0	?	1	1	0	0	?	1	1
$C = u(\sin 10\pi x)$	0	1	0	1	0	1	0	1	0	1
$D = u(\cos 2\pi x)$	0	0	?	1	1	1	1	?	0	0
$E = u(\cos 4\pi x)$	0	?	1	?	0	0	?	1	?	0

it being understood that in this series $\Psi(A, B)$ is evaluated at $x = 4^{n-1}/\pi$ for each term. One way is to use a *switch function*, so called because $u(x)$ mimics the behaviour of a switch. We write

$$\begin{aligned}\Psi(A, B) &= (1 - A)B + 2AB + 3A(1 - B)\\ &= 3A + B - 2AB\,.\end{aligned}$$

Other bases

In base b, a table such as table 3 will have b open subintervals for $\{x\}$ to consider, so we shall need at least r functions $A, B, C, \ldots$ of the type above where $b \le 2^r$. There is scope for choice in the functions involved. Even so, we may encounter problems. When b is not a power of 2, our sin and cos functions may change sign in one of the subintervals rather than just at the endpoints. We may then have to check carefully that our function $\Psi(A, B, C, \ldots)$ still correctly reproduces all base b digits for all such ambiguities. We illustrate this by deriving a base 6 expansion.

Table 4.

Interval for $\{x\}$	$(0, \frac{1}{6})$	$(\frac{1}{6}, \frac{2}{6})$	$(\frac{2}{6}, \frac{3}{6})$	$(\frac{3}{6}, \frac{4}{6})$	$(\frac{4}{6}, \frac{5}{6})$	$(\frac{5}{6}, 1)$
$A = u(\sin 2\pi x)$	0	0	0	1	1	1
$B = u(\sin 6\pi x)$	0	1	0	1	0	1
$C = u(\cos 2\pi x)$	0	?	1	1	?	0
Leading base 6 digit	0	1	2	3	4	5

The unambiguous entries of table 4 determine the switch function:

$$\begin{aligned}\Psi(A, B, C) &= (1 - A)B + 2(1 - A)(1 - B)C + 3ABC\\ &\quad + 4A(1 - B) + 5AB(1 - C)\\ &= 4A + B + 2C - 2AC - 2BC\,,\end{aligned}$$

which reconstructs each digit in all cases. The upshot is the base 6 formula

$$\sum_{n=1}^{\infty} \frac{1}{6^n} \Psi(A, B, C) = \frac{1}{\pi}\,,$$

where $\Psi(A, B, C)$ is evaluated as $x = 6^{n-1}/\pi$ for each term. Modifying the coefficients in the switch function easily gives a base 3 formula

$$\begin{aligned}\Psi(A, B, C) &= (1 - A)(1 - B)C + ABC\\ &\quad + 2A(1 - B) + 2AB(1 - C)\\ &= 2A + C - AC - BC\,.\end{aligned}$$

The same method will generate a base c formula from a base b formula whenever c is a factor of b. Finally, as promised, we give a base 10 series expansion for $1/\pi$. At first sight, because $2^3 < 10 \le 2^4$, it looks as though we will need four functions here.

The switch function derived from A, B, C, D in table 5 does not suffice to resolve the ambiguities in B and D, but that obtained from A, B, C, D and E is successful. The result is

$$\sum_{n=1}^{\infty} \frac{1}{10^n} \Psi(A, B, C, D, E) = \frac{1}{\pi}\,,$$

where $\Psi(A, B, C, D, E)$, evaluated at $x = 10^{n-1}/\pi$ for each term, is given by

$$\begin{aligned}&\Psi(A, B, C, D, E)\\ &\quad = C - BC + 4BD - CD\\ &\qquad + 2(-1 + C)(-1 + 2BD)E\\ &\qquad + A\big(-C + 2B(4 + C - 9D) + 6D - 2E\\ &\qquad\quad + \big(4BD + C(9 - 9B - 5D + 10BD)\big)E\big)\,.\end{aligned}$$

MATHEMATICA® and `FullSimplify` were used to simplify the switch function into the form above. This justifies the assertions in my letter in Volume 33, Number 3, p. 65. We leave it to the reader to check the details and to investigate whether a four-function formula can be constructed.

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Pythagorean Quadruples

GUIDO LASTERS and DAVID SHARPE

We are all familiar with triples of positive integers such as $(3, 4, 5)$ and $(5, 12, 13)$ which satisfy the equation

$$x^2 + y^2 = z^2 .$$

But what about the equation

$$x^2 + y^2 + z^2 = t^2 ?$$

An example of a 'Pythagorean quadruple' would be $(1, 2, 2, 3)$, since $1^2 + 2^2 + 2^2 = 3^2$.

There are well-known formulae which describe all Pythagorean triples. To be precise, a triple (x, y, z) of positive integers is said to be a *Pythagorean triple* if $x^2 + y^2 = z^2$. A Pythagorean triple in which x, y, z have highest common factor 1 is called a *primitive* Pythagorean triple. It is easy to see that, if (x, y, z) is a primitive Pythagorean triple, then one of x, y is even and the other is odd, and z is odd. If we take y to be even (after all, x, y can always be interchanged), then the following formulae describe all primitive Pythagorean triples:

$$x = a^2 - b^2 , \qquad y = 2ab , \qquad z = a^2 + b^2 ,$$

where a, b are coprime positive integers with $a > b$, one even and the other odd. To obtain formulae for a general Pythagorean triple, we simply multiply through by a positive integer k.

In Volume 34, Number 1, pp. 10–12, we gave a geometrical argument which obtained these formulae. A similar argument will produce formulae which give all Pythagorean quadruples, i.e. all quadruples (x, y, z, t) of integers (x, y, z, t) such that

$$x^2 + y^2 + z^2 = t^2 .$$

We have not initially restricted ourselves to only positive integer solutions.

The equation

$$x^2 + y^2 + z^2 = 1 \tag{1}$$

is the equation of the unit sphere, centre the origin, relative to rectangular axes $0xyz$. One point of this sphere is A, with coordinates $(-1, 0, 0)$. Consider a straight line through A. This has equations of the form

$$\frac{x+1}{a} = \frac{y}{b} = \frac{z}{c} , \tag{2}$$

where a, b, c are real numbers not all zero. These equations need interpretation when one or two of a, b, c are zero. For example, if $a = 0$ but $b, c \neq 0$, they are read as $x = -1$, $y/b = z/c$; if $a = b = 0$ and $c \neq 0$, they are interpreted as $x = -1$, $y = 0$. To find where this straight line intersects the unit sphere, we substitute (2) into (1) to give

$$x^2 + \frac{b^2}{a^2}(x+1)^2 + \frac{c^2}{a^2}(x+1)^2 = 1 ,$$

i.e.

$$\Big(\frac{b^2}{a^2} + \frac{c^2}{a^2} + 1\Big)x^2 + 2\Big(\frac{b^2}{a^2} + \frac{c^2}{a^2}\Big)x + \Big(\frac{b^2}{a^2} + \frac{c^2}{a^2} - 1\Big) = 0 .$$

Of course, one solution of this equation must be $x = -1$, and this helps us to factorize to give

$$(x+1)\Big(\Big(\frac{b^2}{a^2} + \frac{c^2}{a^2} + 1\Big)x + \Big(\frac{b^2}{a^2} + \frac{c^2}{a^2} - 1\Big)\Big) = 0 .$$

Thus, this straight line through A cuts the sphere again when

$$x = \frac{a^2 - b^2 - c^2}{a^2 + b^2 + c^2} ,$$

when also, from (2),

$$y = \frac{b}{a}\Big(\frac{2a^2}{a^2 + b^2 + c^2}\Big) = \frac{2ab}{a^2 + b^2 + c^2} ,$$
$$z = \frac{c}{a}\Big(\frac{2a^2}{a^2 + b^2 + c^2}\Big) = \frac{2ac}{a^2 + b^2 + c^2} ,$$

i.e. the line cuts the sphere again at the point P with coordinates

$$\Big(\frac{a^2 - b^2 - c^2}{a^2 + b^2 + c^2}, \frac{2ab}{a^2 + b^2 + c^2}, \frac{2ac}{a^2 + b^2 + c^2}\Big) . \tag{3}$$

Although this argument required $a \neq 0$, when $a = 0$ the point (3) is $(-1, 0, 0)$. In this case the straight line has equations $x = -1$, $y/b = z/c$ and is tangential to the sphere, and the two intersection points coincide.

If a, b, c are integers not all zero, then the coordinates (3) of P are rational numbers. Conversely, starting with a point $\mathrm{P} \neq \mathrm{A}$ of the sphere with rational coordinates (r, s, t) (say), the straight line AP is given by

$$\frac{x+1}{r+1} = \frac{y}{s} = \frac{z}{t}$$

and $r + 1, s, t$ are rational numbers not all zero. We can multiply $r + 1, s, t$ by the least common multiple of their denominators to make these equations in the form of (2), where a, b, c are integers not all zero. It follows that the formulae (3) give *all* points of the sphere with rational coordinates, $(-1, 0, 0)$ included (when $a = 0$). But now

$$(a^2 - b^2 - c^2)^2 + (2ab)^2 + (2ac)^2 = (a^2 + b^2 + c^2)^2 ,$$

so this gives a solution of the equation

$$x^2 + y^2 + z^2 = t^2$$

in integers with $t \neq 0$. We can divide through by the highest common factor h of the four numbers and multiply them by any nonzero integer k, so that

$$\Big(\frac{k(a^2-b^2-c^2)}{h}, \frac{2kab}{h}, \frac{2kac}{h}, \frac{k(a^2+b^2+c^2)}{h}\Big) \quad (4)$$

is a solution of the equation $x^2 + y^2 + z^2 = t^2$ in integers with $t \neq 0$. It is a simple matter to verify this directly. But what about the converse?

Let (x, y, z, t) be any such solution with $t \neq 0$. We can divide through by t^2 to give

$$\Big(\frac{x}{t}\Big)^2 + \Big(\frac{y}{t}\Big)^2 + \Big(\frac{z}{t}\Big)^2 = 1\,,$$

so that $(x/t, y/t, z/t)$ is a rational point of the unit sphere, and so, from (3), there exist integers a, b, c not all zero such that

$$\frac{x}{t} = \frac{a^2-b^2-c^2}{a^2+b^2+c^2}\,, \qquad \frac{y}{t} = \frac{2ab}{a^2+b^2+c^2}\,,$$
$$\frac{z}{t} = \frac{2ac}{a^2+b^2+c^2}\,.$$

It follows that x, y, z, t are as in (4), so that (4) gives all integer-valued solutions of the equation $x^2 + y^2 + z^2 = t^2$ with $t \neq 0$. (We could even include $t = 0$ by allowing $k = 0$.) When $k = 1$, this gives the 'primitive solutions', with highest common factor 1. We can restrict ourselves to solutions in positive integers by making a, b, c positive, putting a modulus sign round $a^2 - b^2 - c^2$ in x and not allowing (b, c, a) to be a Pythagorean triple.

We can analyse when the Pythagorean quadruple is primitive. First, for (x, y, z, t) to be a primitive Pythagorean quadruple, we must have two of x, y, z even and one odd, and z is odd. This follows from the fact that $a^2 \equiv 0 \pmod 4$ when a is even and $a^2 \equiv 1 \pmod 4$ when a is odd. Thus, if one of x, y, z is even and two are odd, then $x^2+y^2+z^2 \equiv 2 \pmod 4$, if all are odd, $x^2 + y^2 + z^2 \equiv 3 \pmod 4$, and if all are even, then t would also be even. Since x, y, z are interchangeable, we could take x odd and y, z even.

Now consider the Pythagorean quadruple

$$(|a^2-b^2-c^2|,\ 2ab,\ 2ac, a^2+b^2+c^2).$$

If $a^2 = b^2 + c^2$, the first term is zero (and all terms are even). If an even number of a, b, c are odd (none or two), then again all numbers are even. Any common divisor of a and $b^2 + c^2$ will divide all terms. Thus, for this quadruple to be primitive, we must have

(i) $a^2 \neq b^2 + c^2$;
(ii) an odd number (i.e. one or three) of a, b, c are odd;
(iii) $a, b^2 + c^2$ are coprime.

Conversely, if a, b, c satisfy these conditions and if p is a prime divisor of $a^2-b^2-c^2$, $2ab$, $2ac$, $a^2+b^2+c^2$, then p divides

$$2a^2, \quad 2ab, \quad 2ac, \quad 2(b^2+c^2)$$

and p is odd (since $a^2 + b^2 + c^2$ is odd), so $p|a$ and $p|(b^2+c^2)$, which is impossible. It follows that the formulae

$$\begin{aligned} x &= |a^2-b^2-c^2|\,, & y &= 2ab\,, \\ z &= 2ac\,, & t &= a^2+b^2+c^2\,, \end{aligned} \quad (5)$$

where a, b, c are positive integers satisfying (i)–(iii) above, describe all primitive Pythagorean quadruples with y, z even.

We began with the Pythagorean quadruple $(1, 2, 2, 3)$. In (4), this comes by putting $a = 2$, $b = c = k = 1$, since $h = 2$. The more obvious $a = b = c = k = 1$ gives $(-1, 2, 2, 3)$. In (5), we put $a = b = c = 1$.

Presumably one can keep going. At the next stage, the complete solution of the equation

$$x^2 + y^2 + z^2 + t^2 = u^2$$

in integers would be

$$(x, y, z, t, u) = \Big(\frac{k(a^2-b^2-c^2-d^2)}{h}, \frac{2kab}{h}, \frac{2kac}{h}, \frac{2kad}{h}, \frac{k(a^2+b^2+c^2+d^2)}{h}\Big),$$

where a, b, c, d are integers not all zero, h is the highest common factor of $a^2 - b^2 - c^2 - d^2$, $2ab$, $2ac$, $2ad$, $a^2 + b^2 + c^2 + d^2$ and k is any integer. Thus $a = b = c = d = k = 1$ gives the solution $(-1, 1, 1, 1, 2)$; $a = 2$, $b = c = d = k = 1$ gives $(1, 4, 4, 4, 7)$. Which values of the parameters give $(1, 1, 1, 1, 2)$? Analysis of when we have a primitive quintuple is complicated by the fact that x, y, z, t can all be odd; they are either all odd or exactly one of them is odd.

Returning to quadruples, as an alternative we could start with the hyperboloid with equation

$$x^2 + y^2 - z^2 = 1\,,$$

in place of the unit sphere (1), and the straight line (2) as before. A similar argument will show that this meets the hyperboloid again in the point

$$\Big(\frac{a^2-b^2+c^2}{a^2+b^2-c^2}, \frac{2ab}{a^2+b^2-c^2}, \frac{2ac}{a^2+b^2-c^2}\Big),$$

which is the analogue of (3). But now we require the condition $a^2 + b^2 \neq c^2$. This will give the solutions

$$\Big(\frac{k(a^2-b^2+c^2)}{h}, \frac{2abk}{h}, \frac{2ack}{h}, \frac{k(a^2+b^2-c^2)}{h}\Big) \quad (6)$$

of the equation $x^2 + y^2 = z^2 + t^2$ in integers, where a, b, c, k are integers and h is the highest common factor of $a^2 - b^2 + c^2$, $2ab$, $2ac$, $a^2 + b^2 - c^2$. These are solutions whether or not $a^2+b^2=c^2$, and can easily be checked directly.

But in fact (6) gives all integer solutions of the equation. When $t \neq 0$, so that $a^2+b^2 \neq c^2$, this is checked as previously. When $a^2+b^2=c^2$, (6) becomes

$$\left(\frac{k(a^2-b^2+c^2)}{h}\,,\ \frac{2abk}{h}\,,\ \frac{2ack}{h}\,,\ 0\right) = \left(\frac{2a^2k}{h}\,,\ \frac{2abk}{h}\,,\ \frac{2ack}{h}\,,\ 0\right),$$

and h is the highest common factor of $2a^2, 2ab, 2ac$, so we can write this as

$$\left(\frac{ka}{h'}\,,\ \frac{kb}{h'}\,,\ \frac{kc}{h'}\,,\ 0\right),$$

where $h' = \mathrm{hcf}(a, b, c)$. Since $a^2+b^2=c^2$, this trivially gives all solutions of $x^2+y^2=z^2+0$.

Thus we have found all integer solutions of the equation $x^2+y^2=z^2+t^2$. This incidentally finds lots of different pairs of integers the sums of whose squares are the same. For example, $a=1$, $b=5$, $c=3$, $k=1$ gives $(-15)^2+10^2=6^2+17^2$, $a=4$, $b=9$, $c=15$, $k=1$ gives $20^2+9^2=15^2+(-16)^2$ (since $h=8$).

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Mathematics in the Classroom

Why Mathematics Matters: a Conference for Schools 'Down Under'[1]

JOE GANI

A response to the decline in numbers of mathematics students in Australian universities.

For the past several years, the numbers of students enrolling for mathematics degrees in Australian universities has been declining; this has also been true of subjects such as physics and chemistry. Students appear to show a preference for computer science, economics and management science as topics of study, possibly because these are thought to offer better chances of a job after graduation. Yet there are many practical areas of employment in which a university training in mathematics is essential, engineering, operations research and finance among them.

The problem of encouraging students to pursue their studies in mathematics is exacerbated by the shortage of secondary school teachers with degrees in the subject. Only a minor proportion of mathematics teachers in Australian schools have graduated in mathematics; most of them are teachers of science or geography with some mathematical background, whose services in teaching mathematics are enlisted because schools have no other choice. This often means that students may not be taught with the understanding and enthusiasm which make mathematics the fascinating subject that it can be.

Concerned by the fall in numbers of its mathematics students, the School of Mathematical Sciences of the Australian National University, Canberra, organized a conference on 24–25 November 2000, to celebrate the World Mathematical Year 2000. This conference, sponsored by the Australian Mathematical Society and the Australian Academy of Science, was entitled 'Why Mathematics Matters'; its purpose was to explain to secondary schoolchildren, their parents and their teachers the importance of mathematics in modern life, and to indicate the various jobs open to mathematical graduates.

The organizing committee of the conference consisted of Professor Cheryl Praeger of the University of Western Australia, Professor Ian Sloan of the University of New South Wales and the author. We contacted colleagues in all parts of Australia, asking them to nominate speakers who were known for their skills as communicators. The response was very gratifying, and the final selection consisted of 14 speakers, two from Canberra, two from the state of Victoria, three from New South Wales, one from Tasmania, two from Queensland, two from South Australia and two from Western Australia. The intention was to provide a broad range of mathematical talks of a non-technical nature, accessible to secondary school pupils, their parents and their teachers, and given by representative men and women mathematicians from all over Australia.

The conference was widely advertised by letters and posters mailed to more than 30 schools in Canberra, as well as schools in the surrounding districts of New South Wales. The *Canberra Times* newspaper and two Canberra radio stations also informed the public of the event. The attendance proved respectable: 72 participants on Friday 24 November and 48 on Saturday 25 November, but only a minor proportion turned out to be school students. Other participants included a medical practitioner, a public servant, a Visiting Professor and colleagues from the Australian National University and other institutions. A schoolgirl reported that her school had neglected to post the advertisements mailed to it; it was her grandmother, after reading the *Canberra Times*, who told her about the conference and suggested she attend it. Communication with mathematics teachers in the schools had clearly broken down; links with them need to be much improved in future.

The speakers proved to be uniformly excellent; they chose relevant topics, presented them clearly, and sparkled with infectious enthusiasm. Their names and the titles of their talks were:

> Professors Neil Trudinger FAA, FRS[2] and Bernhard Neumann FAA, FRS (Australian National University, Canberra) who introduced the conference speakers.
>
> Professor Peter Taylor (University of Canberra). Competitions: direct and indirect influences on mathematics learning.
>
> Dr Christine O'Keefe (CSIRO, the Australian Government Research Organization, Adelaide). Geometry and the sharing of secrets.

[1]This is an expanded version of a report that appeared in *Austral. Math. Soc. Gaz.* **28** (2000), p. 12.

[2]An FAA is a Fellow of the Australian Academy of Science, while an FRS is a Fellow of the Royal Society of London.

Dr Clio Cresswell (University of New South Wales, Sydney). The role of mathematics in scientific discovery.

Dr Rodney Wolff (Queensland University of Technology, Brisbane). Be a mathematician and see the world.

Professor Alf van der Poorten (Macquarie University, Sydney). On the equation $a + b = c$.

Professor Larry Forbes (University of Tasmania, Hobart). How mathematics changed the twentieth century.

Dr Alice Niemeyer (University of Western Australia, Perth). A quick tour through computer algebra.

Professor Rodney Baxter FAA, FRS (Australian National University, Canberra). Mathematics *vis-à-vis* physics.

Dr Peter Taylor (University of Adelaide). Mathematics and mathematicians in the telecommunications industry.

Dr Mark Berman (CSIRO, Sydney). Digital image processing and analysis — A mathematical chocolate box.

Dr Mark Reynolds (Murdoch University, Murdoch, Western Australia). Dominoes, tiles and the logic of time.

Professor Kathy Horadam (Royal Melbourne Institute of Technology, Melbourne). Information transmission in the Information Age: reliability, privacy, efficiency.

Dr Darryn Bryant (University of Queensland, Brisbane). Combinatorial mathematics and DNA sequencing.

Dr Antoinette Tordesillas (University of Melbourne). COLORBOND, moths, HMMWVs and sands: mathematical adventures round the world.

I personally enjoyed every one of these talks. Some were expositions of several areas of mathematics: analysis, geometry and the tiling of surfaces, algebra, computer algebra, and combinatorics. Others covered various applications: in scientific discovery, physics, biology, telecommunications, digital image processing, information transmission, and the interaction of tyres with solid surfaces. Last, but not least, the social effects of mathematics were outlined, among them the effect of competitions on the learning of mathematics, the peregrinations of mathematicians, and the enormous effect which mathematics has had on the shaping of the twentieth century.

The conference gave every sign of modest success: many questions were asked after each talk, and conversation was animated during the tea breaks. I, for one, also learned some valuable lessons:

(a) next time, I shall try to collect the talks into a volume of proceedings, as the interest of the talks was more than ephemeral;
(b) the best way of advertising such a conference is to visit the local schools and talk to the mathematics teachers and students; one school which had received a recent visit from one of our staff members, sent 15 students to the conference;
(c) universities need to advertise the value of mathematics, or they will continue to lose potential students. Mathematics now has many other disciplines competing for talented students; mathematicians must fight hard to maintain their fair share of talent.

I look forward to the next conference of the same type, and hope that the attendance to it will double!

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Computer Column

A mathematical card trick

With Christmas still in the air, my thoughts have been turning increasingly to parties and party tricks. Rather than regale you with another of my (cough) insightful analyses, therefore, I thought I'd teach you a trick instead. The following is one of my favourites, and, yes, you can try this at home. (No animals were harmed in the production of this column, either, though I did test it out on a couple of small, furry creatures from Alpha Centauri.)

The trick consists of a small deck of cards with columns of numbers printed on them, which you give to your poor, unsuspecting target. They then have to think of a number, and give back to you the cards that have that number printed on them somewhere. You then, with a flourish and possibly a fanfare if you have one handy, *tell them what the number is*.

So, what is the secret? Well, the cards are made up so that, if you add up the numbers in the top left-hand corners of all the cards you get back, you get the only number which is common to all of them, the number your target must have thought of.

The way this works is that the corner numbers are all the powers of 2; 1 on the first card, then 2, 4, 8, and so on. Now, we all know that any number can be represented as a sum of powers of 10, for example $123 = 3$ (3×10^0) plus 20 (2×10^1) plus 100 (1×10^2). This is the basis of our decimal number system, but there is nothing special about the number 10: we could equally well have used 2 (binary), 8 (octal), 16 (hexadecimal) or any other number. Binary is particularly useful, especially in computing, because it only needs the digits 0 and 1 (since two 2s, say, can be represented as one 4 instead). Computers use this to represent numbers using a series of on or off states, and the trick uses it to mark each number only on the cards for powers of 2 which are in the sum.

By this point, you may well have put two and two together (ahem) and guessed that a program listing which will create these cards for you is coming next, and you would be absolutely right. The BASIC program in the listing below asks for a range of numbers to use, and how you would like the cards to be formatted (how many entries you want on each line, and how widely to separate the columns), then generates a suitable deck and saves it as `CARDS.TXT`. You can print this and cut out the cards, ready to surprise your unsuspecting public. Having worked out how many cards you need, it actually includes as many numbers as will work with that number of cards, which may be more than you asked for. It is also best to specify a number of columns that is a power of two, as this will fill the last row on all the cards. (For example, asking for 100 numbers and 8 entries per row will give you seven cards, each with 8 rows of 8 numbers, enough to work the trick with numbers up to 127.)

The way the program works is fairly straightforward. Having asked its initial questions, the program then calculates the logarithm to base 2 of the input number and rounds it up to find the nearest whole power of 2 above the input number. (Since BASIC cannot handle base 2, it takes advantage of the fact that

$$\log_2 x = \frac{\log_n x}{\log_n 2}$$

for any base n.) The largest power that we will need to put on the cards is 1 less than this, and the rest of the program loops through the powers needed to create the cards. The tricky thing is working out which numbers go on each card.

This part got me scratching my head as well, until inspiration struck. Take, for example, 100 (4 in binary). The following numbers are 101 (5), 110 (6), 111 (7) and 1000 (8). Thus the third digit is 1 for four numbers in a row, until you reach 1000. After that, there are four more numbers (1000, 1001, 1010 and 1011) before the third digit comes back again. With a bit of thought, it became clear that the third digit would keep being 'on' for four numbers, 'off' for four, then 'on' for four again, while the fourth digit would do the same at intervals of eight numbers and so on. The program therefore starts at the power of 2 for the current card (say x), includes the next x numbers, misses out x, then includes the following x and so on up to the largest number to be included. The remainder of the program is just concerned with making sure that the printout has the right number of columns and spacing.

Well, that's all folks! I hope that you had a merry Christmas and have a happy New Year, and if you come across anything to beat my mathemagical cards, you know where to send it . . .

Program listing

```
REM Program to generate cards for trick

REM Ask what range of numbers and card
REM format the user wants
INPUT "How many numbers would you like to
                             use? ", usernum
INPUT "Entries per line? ", epl
INPUT "Entry spacing? ", espace

REM Create format string for later printing
FOR n = 1 TO espace
   formstr$ = formstr$ + "#"
NEXT n

REM Find next power of 2 greater than input,
REM as this determines the number of numbers
REM to consider, and the number of cards
power = LOG(usernum) / LOG(2)
IF (power - INT(power) > 0) THEN
   power = INT(power) + 1
END IF

nonumbers = 2 ^ power

REM Open file and create cards
OPEN "cards.txt" FOR OUTPUT AS #1
   entry = 0

REM Loop over powers, then over blocks of
REM numbers with that power in their sum
   FOR lhpower = 0 TO power - 1
      lhnumber = 2 ^ lhpower
      FOR n = lhnumber TO nonumbers
                         STEP (2 * lhnumber)
         FOR m = n TO n + lhnumber - 1

REM Print nicely formatted entries, then
REM check whether we're at the end of a line
            PRINT #1, USING formstr$; m;
            entry = entry + 1
            IF (entry = epl) THEN
               PRINT #1,
               entry = 0
            END IF
         NEXT m
      NEXT n

REM Print space between cards,
REM then loop back
      PRINT #1, : PRINT #1,
   NEXT lhpower
CLOSE
PRINT "All done, and saved as CARDS.TXT."
END
```

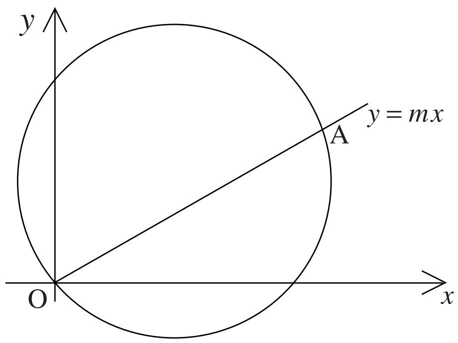
Peter Mattsson

Letters to the Editor

Dear Editor,

A problem about rational numbers

In his letter in Volume 33, Number 3, Bob Bertuello finds rational solutions of the equation $x^2 + y^2 = x + y$, and asks whether there are formulae which give all rational solutions.



The equation can be rewritten as

$$(x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 = \tfrac{1}{2}\,,$$

which is the equation of a circle, centre $(\frac{1}{2}, \frac{1}{2})$, radius $1/\sqrt{2}$. The straight line $y = mx$ through the origin, with slope m, meets the circle again at the point

$$\left(\frac{1+m}{1+m^2}, \ \frac{m(1+m)}{1+m^2}\right).$$

If m is a rational number, this point has rational coordinates. Conversely, if we take a point A on the circle with rational coordinates other than $(0, 1)$, the line OA will have a rational slope. (When $\mathrm{A} = (0, 0)$, OA is taken to be the tangent to the circle at O, having slope -1.) Thus,

$$(x, y) = \left(\frac{1+m}{1+m^2}, \ \frac{m(1+m)}{1+m^2}\right)$$

give all rational solutions of the equation except for $(0, 1)$. We can even include $(0, 1)$ if we are prepared to allow the limiting values as $m \to \infty$. For example, $m = 2$ gives the rational solution $(\frac{3}{5}, \frac{6}{5})$.

This can be extended to three variables. The equation

$$x^2 + y^2 + z^2 - x - y - z = 0$$

is the equation of a sphere passing through the origin. The straight line

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$$

through the origin with a, b, c rational numbers not all zero (interpreted, for example, to mean $x = 0$ and $y/b = z/c$ when $a = 0$ and $b, c \neq 0$), meets the sphere again at the point

$$(x, y, z) = \left(\frac{a(a+b+c)}{a^2+b^2+c^2}, \frac{b(a+b+c)}{a^2+b^2+c^2}, \frac{c(a+b+c)}{a^2+b^2+c^2}\right),$$

and a similar argument shows that this gives all rational solutions of the equation. Here, two sets of values of a, b, c which have the same ratios $a : b : c$ will give the same solution.

Yours sincerely,
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(Ganzendries 245,
3300 Tienen/Oplinter,
Belgium.)

[Prof. Roy Davies of Leicester points out that Bob Bertuello's other equation $x^2 + y^2 = x - y$ can now be solved by replacing y by $-y$ — Ed.]

Dear Editor,

A problem about rational numbers

There is a simple answer to Bob Bertuello's question in Volume 33, Number 3. He asks for all rational solutions of $x^2+y^2 = x+y$. If k is any rational number, then, following the line of Mr Bertuello's method, letting $4x = k(a+1)$ and $1 - x = (a-1)/k$ will yield a general solution.

However, a simpler solution is more easily obtained directly from the original equation. Simply substitute $y = kx$ into $x^2 + y^2 = x + y$ and we obtain $x = (1+k)/(1+k^2)$ and $y = k(1+k)/(1+k^2)$. This yields all solutions (except $x = 0$ and $y = 1$, which requires infinite k) without repetition as k ranges over the complete set $\mathbb{Q}$ of rational numbers. For example $k = 2$ and $k = -\frac{1}{3}$ give the first two of Mr Bertuello's solutions $x = \frac{3}{5}$ and $y = \frac{6}{5}$ or $-\frac{1}{5}$.

Such a simple method, without recourse to quadratic equations (or worse), will work whenever the degrees of all the terms in the initial equation differ by at most one. It is in fact a standard method for plotting algebraic curves parametrically which I recall discovering when I was in the sixth form at school (in about 1960) and then going on to fill a graph book with drawings of curves like the folium of Descartes: $x^3+y^3 = xy$. In this case, letting $y = kx$ yields $x = k/(1+k^3)$, $y = k^2/(1+k^3)$.

Yours sincerely,
ALASTAIR SUMMERS
(57 Conduit Road,
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Dear Editor,

Riffle shuffle

I read a brief article in the *Daily Telegraph* (10 October 2000), stating that it had now been proved that five 'riffle' shuffles are sufficient to randomise a pack of cards. I found that I couldn't agree.

It is necessary to define the terms used.

1. A pack of cards: 52 uniquely identifiable cards (ignoring jokers and/or Bridge score cards), normally from 1 (Ace) to King in four 'suits' (Hearts, Clubs, Diamonds and Spades).
2. Random: in relation to a pack of cards, any prediction made about the 'pack', or about a single card (or position), would have the same likelihood as that required by chance.
3. A 'riffle' shuffle: this is where the 'pack' is 'cut' roughly in half (ideally in two piles of 26 cards, but possibly anything up to a split of 30 : 22). The edges of the two 'halves' are then brought together, and fanned downwards to the table, allowing cards from either 'half' to interleave themselves to make one pack. The 'pack' is then straightened up to provide a new pack, ready for dealing, or for a further 'riffle' shuffle.

Using the above definitions, it is necessary to consider the effects of five 'riffle' shuffles. There are several possibilities for the method used to determine the outcome of 'riffle' shuffles, and so some clarification is required. The variables for a 'riffle' shuffle are as follows:

(i) the number of cards taken into each hand (presumably between 20 and 32, otherwise the piles will be noticeably uneven);
(ii) in which hand the top of the original pack is taken;
(iii) which hand releases the first card (the bottom card of the 'new' pack);
(iv) from which hand the last card is released to form the 'top' of the new pack;
(v) how many cards are interleaved on each 'flick' of the (half) packs (normally 1 or 2, generally not more than 5), possibly varying with each flick.

In real life, any of the above criteria may be affected by the person handling the cards. *But* if all these are treated as being completely random (i.e. 50 : 50 for (i)–(iv), and 1 in 5 for (v), then, after five 'riffle' shuffles, we do *not* have a randomised pack.

Consider items (ii) and (iv), whatever the original top card (ace of clubs for a brand new pack of Waddingtons). For each 'riffle' shuffle, the top card of the pack is unchanged *if* one hand takes the top of the pile *and* releases the last card. Even with purely 'random' actions, the top card will change only 50% of the time. Thus, after five 'riffle' shuffles, there is a 1 in 32 chance that the top card is unchanged. Thus, if one knows the starting arrangement (as with a new pack), one has a 1 in 32 chance of making a '1 in 52' prediction (i.e. name the top card).

A similar argument can be made for the bottom card of the pack. Any further arguments become more difficult to quantify, but must be equally valid as, if there is an 'unfair' chance of the top and bottom cards being unchanged, then any other predictions must also be affected.

By this reasoning, it is thus impossible that five 'riffle' shuffles will randomise a pack of cards.

The same article commented on the ineffectiveness of the normal 'overhand' shuffle. But, alternating the two styles of shuffle is (probably) the most effective system.

Yours sincerely,

NIGEL PARSONS
(56 Fairfield Avenue,
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[See Problem 34.5 in this issue — Ed.]

Dear Editor,

A Tale of Two Series — A Dickens of an Integral

In his letter in Volume 33, Number 3, following from P. Glaister's article in Volume 33, Number 2, pp. 25–27, A. G. Summers points out that

$$\int_0^1 x^{-x}\,\mathrm{d}x = \sum_{x=1}^{\infty} x^{-x}$$

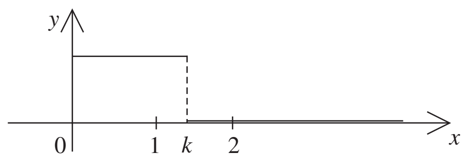
and asks whether there are any other functions f such that

$$\int_0^1 f(x)\,\mathrm{d}x = \sum_{x=1}^{\infty} f(x)\,.$$

Disregarding examples where both sides are divergent, such as $f(x) = 1/x$, we let k be any real number such that $1 \leq k < 2$ and define the function R_k on the nonnegative real line by

$$R_k(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq k\,, \\ 0 & \text{if } x > k\,, \end{cases}$$

as shown.



Then

$$\int_0^1 R_k(x)\,\mathrm{d}x = \int_0^1 1\,\mathrm{d}x = 1$$

and

$$\sum_{x=1}^{\infty} R_k(x) = 1 + 0 + 0 + \cdots = 1\,.$$

This gives a whole family of such functions. It can be generalized by defining, for any function p which is defined and integrable on $[0, 1]$ and any integer $\ell \geq 2$,

$$R_{k,p,\ell}(x) = \begin{cases} p(x) & \text{if } 0 \leq x \leq k, \\ -p(1) + \int_0^1 p(x)\,\mathrm{d}x & \text{if } x = \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, any continuous function f which is of period 1, is odd about $x = \frac{1}{2}$ (so that $f(\frac{1}{2}-x) = -f(\frac{1}{2}+x)$ for $0 \le x \le \frac{1}{2}$) and $f(0) = f(1) = 0$ has this property. An example would be $f(x) = \sin(2n\pi x)$ for any positive integer n.

Another way of constructing such a function would be to start with an infinite series with a known sum, such as the geometric series

$$1 + r + r^2 + \cdots = \frac{1}{1-r}$$

for $0 \le r < 1$. For $0 \le r < 1$, define f_r by

$$f_r(x) = \begin{cases} \dfrac{1}{1-r} & \text{for } 0 \le x < 1, \\ r^{x-1} & \text{for } x = 1, 2, 3, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

If it is required that f_r be continuous, it could be defined linearly between consecutive integer values, with $f_r(0) = (1+r)/(1-r)$.

Any linear combination of such functions is again such a function.

These examples, although they may be continuous for $x > 0$, are not differentiable. It may be asked whether there are examples other than x^{-x} of such functions which are differentiable for $x > 0$.

Yours sincerely,
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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

34.5 A pack of 52 cards is repeatedly shuffled using perfect riffle shuffles. After how many shuffles will every card return to its initial position?

(See the letter by Nigel Parsons in this issue)

34.6 Sets $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$ are such that each of the sets $X_1, \ldots, X_k$ has a nonempty intersection with exactly m of the sets $Y_1, \ldots, Y_k$ and each of the sets $Y_1, \ldots, Y_k$ has a nonempty intersection with exactly n of the sets $X_1, \ldots, X_k$. What is the connection between m and n?

(Submitted by H. A. Shah Ali, Tehran)

34.7 What is the sum of the series

$$1 + \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{15} + \cdots ?$$

(Submitted by J. A. Scott, Chippenham)

34.8 Given five points A, B, C, D, E in the plane, X_1 is the midpoint of AB, X_2 the midpoint of CD, X_3 the midpoint of X_1X_2, X_4 the midpoint of DE and X_5 the midpoint of X_1X_4. In which point do the lines EX_3 and CX_5 intersect?

(Submitted by Guido Lasters, Tienen, Belgium)

Solutions to Problems in Volume 33 Number 3

33.9 Prove that

$$\frac{1}{\pi} = \sum_{n=1}^{\infty} \frac{u(\sin 2^n)}{2^n},$$

where

$$u(x) = \begin{cases} 1 & \text{if } x \le 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Solution This was solved by Daniel Lamy (Nottingham High School) using Fourier Series. A more elementary solution is given in the article on pp.36–37 by Milton Chowdhury, who posed the problem.

33.10 Colin is playing a computer game and his latest win has just brought his life-time success rate, as displayed by the computer, up to 95%, his highest so far. How many further consecutive wins would be needed to bring his displayed success rate up to 96%, given that the success rate is displayed rounded to the nearest 1% and Colin has lost 5 times?

Solution by Daniel Lamy

Denote by n the number of games played to bring his success rate to 95%. His success rate after $n-1$ games is

94% (to the nearest 1%). Since he has had 5 defeats,

$$\frac{n-6}{n-1} < 0.945\,,$$

so

$$0.055n < 5.055\,,$$

and so

$$n < 91\tfrac{10}{11}\,.$$

Since the success rate changes to 95% after n games, $n = 91$. Suppose that, after m games $(m > n)$, with no more defeats, the success rate reaches 96%. Then

$$\frac{m-5}{m} \geq 0.955\,,$$

so

$$0.045m \geq 5\,,$$

and so

$$m \geq 111\tfrac{1}{9}\,.$$

Hence the success rate will reach 96% after 112 games, i.e. after 21 further consecutive wins.

33.11 Solve the simultaneous equations

$$\sum_{i=1}^{n} x_i = 1, \qquad \sum_{i=1}^{n} y_i = 1, \qquad \sum_{i=1}^{n} (x_i - y_i)^2 = 2\,,$$

where $n \geq 2$ and the x_i and y_i are nonnegative real unknowns.

Solution by Daniel Lamy

Since $0 \leq x_i,\ y_i \leq 1$ for each i,

$$\begin{aligned} 2 = \sum_{i=1}^{n} (x_i - y_i)^2 &= \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - 2\sum_{i=1}^{n} x_i y_i \\ &\leq \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i - 2\sum_{i=1}^{n} x_i y_i = 2 - 2\sum_{i=1}^{n} x_i y_i\,. \end{aligned}$$

It follows that, for each i, $x_i y_i = 0$, $x_i^2 = x_i$ and $y_i^2 = y_i$. Thus the solutions are

$$\begin{aligned} x_i &= 1, \qquad x_j = 0 \qquad \text{for } j \neq i\,, \\ y_k &= 1, \qquad y_\ell = 0 \qquad \text{for } \ell \neq k\,, \end{aligned}$$

and $i \neq k\ \ (i, k = 1, \ldots, n)$.

33.12 In the diagram, ABCD is a rectangle with $\mathrm{AB} = a$, $\mathrm{BC} = b\ (a < b)$ and K, L, M, N lie on sides AB, BC, CD, DA (possibly produced) as shown, such that $\mathrm{AK} = b$, $\mathrm{BL} = a$, $\mathrm{CM} = b$, $\mathrm{DN} = a$. When are K, L, M, N collinear?

Solution by Daniel Lamy

By consideration of similar triangles, K, L, M, N are collinear if and only if

$$\frac{\mathrm{KB}}{\mathrm{BL}} = \frac{\mathrm{LP}}{\mathrm{PN}} = \frac{\mathrm{DM}}{\mathrm{ND}}\,,$$

i.e. if and only if

$$\frac{b-a}{a} = \frac{a}{b-2a} \left(= \frac{b-a}{a}\right),$$

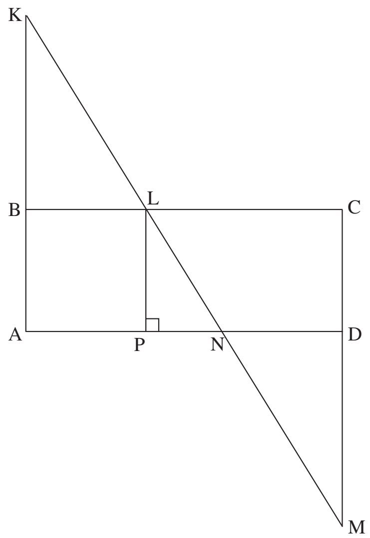
that is

$$b^2 - 3ab + a^2 = 0$$

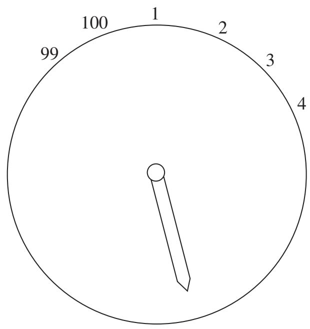
or

$$b = \frac{3+\sqrt{5}}{2}\,a$$

(so that $b = ((1+\sqrt{5})/2)^2 a$ and $b - a = ((1+\sqrt{5})/2)a$, another appearance of the golden ratio!).



Unlucky number



Numbers 1 to 100 are placed round the circumference of a roulette wheel. With each spin of the wheel, a number is chosen at random. In 100 tries, what is the probability that a given number is never chosen? What is the limiting value of this probability if 100 is increased without limit?

Guido Lasters
Tienen, Belgium

Reviews

Hungarian Problem Book III. By ANDY LIU. MAA, Washington, 2001. Pp. 163. Paperback $27.95 (ISBN 0-88385-644-1).

This book serves the very useful purpose of training young problem solvers. It has been translated for American readers, but this hardly affects its value to British readers. Any sixth formers entering the British Maths Olympiads would benefit from studying it. Although the problems come from a different era and another country, they deal with types of material still found in modern competitions. Furthermore, the solutions given have been written recently and are presented in a way that is appropriate today.

The problems are first presented as originally set, i.e. in groups of three problems to be solved in four hours, so the challenge is there to try and do just that. However, they are also regrouped into five different chapters under four headings: Combinatorics, Number Theory, Algebra and Geometry. This covers pretty comprehensively what is found in the BMO. There are several excellent features in the chapters which cause me to recommend the book highly. More than one solution is given to many of the problems; this should encourage students not to be content with the first solution they find to a problem but to search for alternatives; an excellent way of building up a good armoury of different problem solving techniques. Secondly, the solutions, whilst not pedantically spelling out unnecessary detail, always give clear enunciation of key steps, even elementary ones when they are essential to a full proof. Thirdly, and perhaps most useful of all, are the important theorems which are expounded, usually proved and then used in the solutions. Lastly there is a useful topic index.

The geometry is most interesting. In fact 75 pages out of 135 are devoted to geometry. This is surely the Achilles heel of modern students. I was quite humbled reading it because, although brought up on Euclidean geometry in the fifties, I learned subtleties in the development which I never recall having seen before. Anyone anxious to score well in the geometry questions on BMO, or master a branch of mathematics which, sadly, is grossly under represented in modern syllabuses would do well to read this book for the two geometry chapters alone.

I have a few not very significant grumbles. Firstly, I found some typographical errors. Secondly, I would have liked clearer indication of how full a proof should be, but this could perhaps be dealt with better in different type of book. Occasionally I thought a few more details might be required in a competition. On the other hand there are some sections of exposition which might have been omitted. Finally an age-old criticism of mathematics texts — there is not very much guidance on how one might arrive at a solution!

Apart from sixth formers looking to move through the various stages of the Maths Olympiad, this book would serve university students well who are seeking to improve their problem solving skills, or teachers seeking to help their very ablest sixth form students do the same.

Stamford School — ALASTAIR SUMMERS

Angles of Reflection. By JOAN RICHARDS. Freeman, Basingstoke, 2000. Pp. 282. Paperback $14.95 (ISBN 0-7167-9461-6).

British readers may well be familiar with the idea of politician becoming novelist — from the Victorian Lord Beaconsfield (Benjamin Disraeli) to the infamous Lord Archer, with Edwina Currie and Ann Widdecombe thrown in for good measure. But a mathematician writing a novel! Not that this is strictly a novel. The description on the cover, ‘a memoir of logic and a mother’s love’, would be a more accurate description.

Joan Richards writes in an arresting way of her exploration, in particular, of the life and ideas of the Victorian mathematician Augustus De Morgan and his wife Sophia and their family, and of her own family, seeking to make a connection between the two. The link between mathematical ideas, in particular of probability, and Joan Richards’ family fortunes and misfortunes, is not convincingly made. To quote: ‘... in real examples the attempt to mathematize [*sic*] rational thinking was hopelessly inadequate’ (p. 118). It is the vagaries of the medical profession that come over most clearly. Perhaps this book should be required reading for all medics!

University of Sheffield — DAVID SHARPE

Mathematical Puzzle Tales. By MARTIN GARDNER. MAA, Washington, 2000. Pp. 168. Paperback $22.50 (ISBN 0-88385-533-x).

This is a re-issue of thirty-six stories taken from Isaac Asimov’s *Science Fiction Magazine*. Each puzzle in the book is presented in the form of a story, adding much interest in the telling. The puzzles can be solved using logic, and knowledge gained from A-level Mathematics, including logarithms, probability and number sequences.

This book will appeal to enthusiastic mathematics students who enjoy doing more maths at home for fun. Happily, the answers can be found at the back of the book. However, in most cases a new problem, relating to the original, is posed at the end of the answer. These are also solved in a Second Answers section. The iteration continues in some puzzles where there is yet a third question at the end of the second answer that leads to another solution in the Third Answer section. Overall I found this to be an amusing yet challenging collection of brainteasers well worth investigating if you have not seen them before.

Student, Solihull Sixth Form College — SAMANTHA CLARKE

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