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CRUX

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THE OLYMPIAD CORNER
No. 105
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this month with the problems of the 1988 Austrian Olympiad. Thanks go to Professor Walther Janous of Innsbruck for forwarding them to us.

19th AUSTRIAN MATHEMATICAL OLYMPIAD, 2nd Round

May 10, 1988

Time: 4 1/2 hours

1. Determine the sum of all the divisors d of $N = 19^{88} - 1$ which are of the form $d = 2^a \cdot 3^b$ with $a, b > 0$.
2. Let $A_1A_2A_3A_4$ be an isosceles trapezoid with the parallel sides being A_1A_2 and A_3A_4 . Let M_1, M_2, M_3, M_4 be the midpoints of sides A_1A_2 , A_2A_3 , A_3A_4 , and A_4A_1 , respectively, and let N_1, N_2 be the midpoints of the diagonals A_1A_3 and A_2A_4 , respectively. Show that the parallelograms $N_1M_3N_2M_1$ and $M_1M_2M_3M_4$ are similar if and only if one of M_1 or M_3 is the circumcentre of the trapezoid.
3. Determine the range of the function

$$f(x,y) = \frac{x+y}{[x][y] + [x] + [y] + 1}$$

over the domain x, y real, $x, y > 0$ with $xy = 1$. (Here $[z]$ is the greatest integer that does not exceed z .)

4. Show that for any $a_0 > 0$ the recurrence relation

$$a_{n+1} = \frac{1}{2} \left(a_n - \frac{1988}{a_n} \right), \quad n \geq 0$$

defines an infinite sequence which must contain infinitely many positive and infinitely many negative numbers.

*

19th AUSTRIAN MATHEMATICAL OLYMPIAD, Final Round

First Day – June 8, 1988

Time: 4 1/2 hours

1. Let a_1, \dots, a_{1988} be positive real numbers whose arithmetic mean equals 1988. Show

$$\sqrt[1988]{\prod_{i=1}^{1988} \prod_{j=1}^{1988} \left(1 + \frac{a_i}{a_j}\right)} \geq 2^{1988}$$

and determine when equality holds.

2. Let A_1, A_2, A_3 be an equilateral triangle. Joining A_4, A_5, A_6 , the midpoints of the sides, we get four smaller equilateral triangles. Let A_7, \dots, A_{15} be the midpoints of the sides of these smaller triangles, enumerated in any order. The 15 points A_1, \dots, A_{15} are each coloured either green or blue. Show that with any such colouring there are always three points A_i, A_j, A_k having the same colour and being equidistant (i.e. $A_i A_j = A_j A_k = A_k A_i$).

3. Show that there is precisely one sequence a_1, a_2, \dots of integers which satisfies $a_1 = 1$, $a_2 > 1$, and

$$a_{n+1}^3 + 1 = a_n a_{n+2}, \quad n \geq 1.$$

Second Day – June 9, 1988

Time: 4 1/2 hours

4. Let a_{ij} be non-negative integers such that $a_{ij} = 0$ if and only if $i > j$. Furthermore, let

$$\sum_{j=1}^{1988} a_{ij} = 1988$$

for all i with $1 \leq i \leq 1988$. Determine all real solutions of the system of equations

$$\sum_{j=1}^{1988} (i + a_{ij})x_j = i + 1, \quad 1 \leq i \leq 1988.$$

5. The angle-bisectors of angles B and C of triangle ABC intersect the sides opposite B and C in points B' and C' , respectively. Show that the line $B'C'$ intersects the incircle of the triangle. [Editor's note: see the problem "Rumania 1" and its solution on p. 135 of this issue.]

6. Determine all polynomials

$$p(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

having real coefficients a_i and the following property: whenever a is a (real or complex) root of $p(x)$, then so are $1/a$ and $1 - a$.

*

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The next problems are from the new regional Olympiad. This differs from many international contests in that a limited number of contestants in each of several widely dispersed participating countries are invited to write it by the appropriate national bodies. The students who are selected then sit the examination, under supervision at centres near their homes. Thanks go to Professor Bruce Shawyer, Memorial University of Newfoundland and Chairman of the I.M.O. committee of the Canadian Mathematics Society, for sending me this new contest which was written the first week of May.

1989 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time: 4 hours

1. Let x_1, x_2, \dots, x_n be positive real numbers, and let $S = x_1 + x_2 + \dots + x_n$. Prove that

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) \leq 1 + S + \frac{S}{2!} + \frac{S}{3!} + \cdots + \frac{S}{n!}.$$

2. Prove that the equation

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solution in integers except $a = b = c = n = 0$.

3. Let A_1, A_2, A_3 be three points in the plane, and for convenience, let $A_4 = A_1, A_5 = A_2$. For $n = 1, 2$ and 3 suppose that B_n is the midpoint of $A_n A_{n+1}$, and suppose that C_n is the midpoint of $A_n B_n$. Suppose that $A_n C_{n+1}$ and $B_n A_{n+2}$ meet at D_n , and that $A_n B_{n+1}$ and $C_n A_{n+2}$ meet at E_n . Calculate the ratio of the area of triangle $D_1 D_2 D_3$ to the area of triangle $E_1 E_2 E_3$.

4. Let S be a set consisting of m pairs (a, b) of positive integers with the property that $1 \leq a < b \leq n$. Show that there are at least

$$4m \cdot \frac{m - n^2/4}{3n}$$

triples (a, b, c) such that $(a, b), (a, c)$ and (b, c) belong to S .

5.

Determine all functions f from the reals to the reals for which

(i) $f(x)$ is strictly increasing,

(ii) $f(x) + g(x) = 2x$ for all real x where $g(x)$ is the composition inverse function to $f(x)$. (Note: f and g are said to be composition inverses if $f(g(x)) = x$ and $g(f(x)) = x$ for all real x .)

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As promised, here are the numerical answers for the 1989 A.I.M.E. These problems and their official solutions are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. Detailed solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, Nebraska, U.S.A., 68588-0322.

1. 869	2. 968	3. 750
4. 675	5. 283	6. 160
7. 925	8. 334	9. 144
10. 994	11. 947	12. 137
13. 905	14. 490	15. 108
*	*	*

We now return to solutions submitted by readers to problems, given in the November 1987 number of *Crux*, that were proposed but not used at the 28th I.M.O. in Havana, Cuba.

Hungary 1. [1987: 278]

(i) Suppose that $\gcd(m,k) = 1$. Prove that there are integers a_1, a_2, \dots, a_m and b_1, b_2, \dots, b_k such that each product $a_i b_j$ ($i = 1, 2, \dots, m$, $j = 1, 2, \dots, k$) gives a different residue when divided by mk .

(ii) Assume $\gcd(m,k) > 1$. Prove that for any integers a_1, a_2, \dots, a_m and b_1, b_2, \dots, b_k there must be two products $a_i b_j$ and $a_s b_t$ ($(i,j) \neq (s,t)$) which give the same residue when divided by mk .

Solution by George Evangelopoulos, Law student, Athens, Greece.

(i) Consider

$$a_i = ki + 1, \quad i = 1, 2, \dots, m,$$

$$b_j = mj + 1, \quad j = 1, 2, \dots, k.$$

Assume that mk divides

$$\begin{aligned} a_i b_j - a_s b_t &= (ki + 1)(mj + 1) - (ks + 1)(mt + 1) \\ &= km(ij - st) + m(j - t) + k(i - s). \end{aligned}$$

Since m divides the last sum, we have that m divides $k(i - s)$. Using that $\gcd(m, k) = 1$ we obtain that m divides $i - s$ from which $i = s$. Similarly $j = t$, and this proves part (i).

(ii) We argue indirectly. Assume all residues are distinct; then the 0 residue must occur, and with renumbering we may take this to be for $a_1 b_1$, i.e., mk divides $a_1 b_1$. Then for some a' and b' we have

$$a' | a_1, \quad b' | b_1 \quad \text{and} \quad mk = a' b'.$$

Suppose that for some $i \neq s$, a' divides $a_i - a_s$. Then $mk = a' b' | a_i b_1 - a_s b_1$, a contradiction. From this we see that no two a_i 's can have the same residue mod a' , so that $a' \geq m$. Similarly $b' \geq k$, and since $mk = a' b'$ we have that $a' = m$, $b' = k$ and thus that no two a_i 's can have the same residue modulo m and no two b_j 's can have the same residue modulo k .

Let p be a common prime divisor of m and k . Now since the a_i 's have all the possible residues modulo m , there are $m - m/p$ among them which are not divisible by p . Similarly, $k - k/p$ of the b_j 's are not divisible by p . However, we assumed above that the products $a_i b_j$ must give all possible residues mod mk , and there are $mk - mk/p$ among them which are not divisible by p . As

$$mk - \frac{mk}{p} \neq (m - \frac{m}{p})(k - \frac{k}{p})$$

we obtain a contradiction, establishing (ii).

Iceland 1. [1987: 278]

Five distinct numbers are drawn successively and at random from the set $\{1, \dots, n\}$. Show that the probability that the first three numbers drawn, as well as all five numbers, can be arranged to form an arithmetic progression, is greater than $6/(n - 2)^3$.

Solution by George Evangelopoulos, Law student, Athens, Greece.

Denote the set $\{1, \dots, n\}$ by A . The total number of five-element subsets of A is

$$\binom{n}{5} = \frac{n!}{(n - 5)!5!}.$$

The number of five-element subsets whose elements form an arithmetic progression is equal to the number of two-element subsets $\{x, y\}$ such that $x \equiv y \pmod{4}$; the elements x, y are the endpoints of the progression. Denote this number by $a(n)$.

Given a subset $\{y_1, \dots, y_5\}$ such that $y_1 < y_2 < y_3 < y_4 < y_5$ form an arithmetic progression, there are $\binom{5}{3} = 10$ different three-element subsets of this that could correspond to the first three elements drawn. Of these 10, exactly four produce a three-element arithmetic progression, namely

$$\{y_1, y_2, y_3\}, \{y_2, y_3, y_4\}, \{y_3, y_4, y_5\} \text{ and } \{y_1, y_3, y_5\}.$$

Denoting the required probability by p we conclude that

$$p = \frac{a(n) \cdot 4/10}{\binom{n}{5}} = \frac{48a(n)}{n(n-1)(n-2)(n-3)(n-4)}.$$

It remains to determine $a(n)$. Let $A_i = \{x \in A : x \equiv i \pmod{4}\}$, $i = 0, 1, 2, 3$, and let $m = n - 2$. We now consider four cases.

Case (i): $n = 4k$, $k \geq 2$. Then each of the sets A_0, A_1, A_2, A_3 have k elements and

$$a(n) = 4 \binom{k}{2} = 2k(k-1) = \frac{n(n-4)}{8}.$$

Thus

$$p = \frac{6}{(n-1)(n-2)(n-3)} = \frac{6}{m(m^2-1)} > \frac{6}{m^3}.$$

Case (ii): $n = 4k-1$, $k \geq 2$. Then A_0 has $k-1$ elements and each of the sets A_1, A_2, A_3 has k elements. Thus

$$a(n) = \binom{k-1}{2} + 3 \binom{k}{2} = (k-1)(2k-1) = \frac{(n-1)(n-3)}{8},$$

and so

$$p = \frac{6}{n(n-2)(n-4)} = \frac{6}{m(m^2-4)} > \frac{6}{m^3}.$$

Case (iii): $n = 4k-2$, $k \geq 2$. Then each of the sets A_0 and A_3 has $k-1$ elements, while A_1 and A_2 each have k elements. Then

$$a(n) = 2 \binom{k-1}{2} + 2 \binom{k}{2} = 2(k-1)^2 = \frac{(n-2)^2}{8}.$$

Thus

$$p = \frac{6(n-2)}{n(n-1)(n-3)(n-4)} = \frac{6m}{(m^2-1)(m^2-4)} > \frac{6}{m^3}.$$

Case (iv): $n = 4k-3$, $k \geq 2$. Then each of the sets A_0, A_2, A_3 has $k-1$ elements, while A_1 has k elements. Thus

$$a(n) = 3 \binom{k-1}{2} + \binom{k}{2} = (k-1)(2k-3) = \frac{(n-1)(n-3)}{8}.$$

Hence

$$p = \frac{6}{n(n-2)(n-4)} = \frac{6}{m(m^2-4)} > \frac{6}{m^3}.$$

Now as each $n \geq 5$ can be represented in the form $4k - i$ for some $k \geq 2$ and $0 \leq i \leq 3$, we are done.

Rumania 1. [1987: 278]

The bisectors of the angles B and C of a triangle ABC intersect the opposite sides in B' and C' , respectively. Prove that the straight line $B'C'$ cuts the inscribed circle.

Solution by George Evangelopoulos, Law student, Athens, Greece.

Let a, b, c be the lengths of the sides opposite A, B, C respectively. From the law of sines

$$\frac{AB'}{B'C} = \frac{c}{a} \quad \text{and} \quad \frac{AC'}{C'B} = \frac{b}{a}.$$

Let the tangent to the inscribed circle parallel to BC intersect AC and AB in M and N respectively. It suffices to show that B' lies on the segment MC and C' lies on NB , that is

$$\frac{MA}{MC} \leq \frac{AB'}{B'C} = \frac{c}{a}$$

and

$$\frac{NA}{NB} \leq \frac{AC'}{C'B} = \frac{b}{a}.$$

With h_a the altitude from A , r the inradius, Δ the area, and s the semiperimeter of $\triangle ABC$,

$$\frac{MA}{MC} = \frac{NA}{NB} = \frac{h_a - 2r}{2r} = \frac{2\Delta/a - 2\Delta/s}{2\Delta/s} = \frac{s-a}{a}.$$

It remains to remark that

$$s-a < c \quad \text{and} \quad s-a < b.$$

U.S.A. 1. [1987: 279]

The *runs* of a decimal number $d_1d_2\dots d_n$ are its maximal increasing or decreasing blocks of digits (here each d_i is a digit and we allow $d_1 = 0$). Thus 024379 has three runs: 024, 43 and 379. Determine the average number of runs for a decimal number in the set $\{d_1d_2\dots d_n \mid d_k \neq d_{k+1}, k = 1, 2, \dots, n-1\}$ for any $n \geq 2$.

Solution by George Evangelopoulos, Law student, Athens, Greece.

The digit d_1 never ends a run, the digit d_n always ends a run, and for $k = 2, 3, \dots, n-1$ the probability that d_k equals i and ends a run is

$$\begin{aligned} P(d_k = i)P[(d_{k-1} < i \text{ and } d_{k+1} < i) \text{ or } (d_{k-1} > i \text{ and } d_{k+1} > i)] \\ = \frac{1}{10} \left[\left(\frac{i}{9} \right)^2 + \left(1 - \frac{i}{9} \right)^2 \right]. \end{aligned}$$

Therefore the average number of runs for $d_1d_2\dots d_n$ is

$$\begin{aligned} 1 + (n-2) \sum_{i=0}^9 \left(\frac{1}{10}\right) \left[\left(\frac{i}{9}\right)^2 + \left(1 - \frac{i}{9}\right)^2 \right] &= 1 + \frac{n-2}{10} \cdot \frac{2}{81} \sum_{i=0}^9 i^2 \\ &= 1 + \frac{19}{27}(n-2). \end{aligned}$$

U.S.A. 2. [1987: 279]

At a party attended by n couples, each person is at any time in some conversational grouping (called a clique). A person and his or her spouse are never in the same clique, but every other pair of persons takes part in the same clique exactly once. Prove: If the total number of cliques formed during the party is k and if $n \geq 4$ then $k \geq 2n$.

Solution by George Evangelopoulos, Law student, Athens, Greece.

Let d_i denote the number of cliques of which person i is a member. Clearly $d_i \geq 2$. We now distinguish two cases.

(a) For some i , $d_i = 2$. Suppose that i is a member of only two cliques C_p and C_q . Then $|C_p| = |C_q| = n-1$, since for each couple other than i and his or her spouse, one member must be in C_p while the other is in C_q . There are thus $(n-1)(n-2)$ pairs $\{r,s\}$, where $r \in C_p$, $s \in C_q$, each pair contained in a distinct clique. Otherwise, we find two members of C_p which belong to a clique other than C_p , or two members of C_q which belong to a clique other than C_q . It follows that $k \geq 2 + (n-1)(n-2) \geq 2n$. [This last inequality uses that $n \geq 4$.]

(b) For every i , $d_i \geq 3$. For $i = 1, 2, \dots, 2n$ assign to person i an indeterminant x_i and for $j = 1, 2, \dots, k$

$$y_j = \sum_{i \in C_j} x_i$$

where C_1, \dots, C_k are the cliques.

From linear algebra we know that if $k < 2n$, then there exist x_1, x_2, \dots, x_{2n} , not all zero, such that

$$y_1 = y_2 = \dots = y_k = 0.$$

Thus, the proof will be complete if we show that $y_1 = y_2 = \dots = y_k = 0$ implies that $x_1 = x_2 = \dots = x_{2n} = 0$. Let $M = \{\{i,j\} : i \text{ and } j \text{ are married}\}$ and let $M' = \{\{i,j\} : i \neq j, i \text{ and } j \text{ are not married}\}$. In view of the assumed properties of C_1, \dots, C_k , it follows that

$$\begin{aligned} \sum_{j=1}^k y_j^2 &= \sum_{i=1}^{2n} d_i x_i^2 + 2 \sum_{\{i,j\} \in M'} x_i x_j \\ &= \left(\sum_{i=1}^{2n} x_i \right)^2 + \sum_{i=1}^{2n} (d_i - 1)x_i^2 - 2 \sum_{\{i,j\} \in M} x_i x_j. \end{aligned}$$

Since $2ab \leq a^2 + b^2$ for all real a, b we have

$$2 \sum_{\{i,j\} \in M} x_i x_j \leq \sum_{i=1}^{2n} x_i^2.$$

Thus

$$\sum_{j=1}^k y_j^2 \geq \left(\sum_{i=1}^{2n} x_i \right)^2 + \sum_{i=1}^{2n} (d_i - 2)x_i^2 \geq \sum_{i=1}^{2n} x_i^2$$

and the desired conclusion follows immediately.

West Germany 1. [1987: 280]

Let S be a set of n elements. We denote the number of all permutations of S which have exactly k fixed points by $P_n(k)$. Prove

$$\sum_{k=0}^n (k-1)^2 P_n(k) = n!.$$

(Compare with I.M.O. problem #1 [1987: 210]!)

Solution by M.A. Selby, Department of Mathematics and Statistics, The University of Windsor, Ontario.

Let D_k denote the number of derangements on k distinct elements, i.e. the number of fixed point free permutations on a k element set. If we define $D_0 = 1$, then the number of permutations of n distinct elements with exactly k fixed points is

$$P_n(k) = \binom{n}{k} D_{n-k}$$

where of course

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

Also, it is known that

$$D_j - jD_{j-1} = (-1)^j. \quad (1)$$

This follows easily by induction on $j \geq 1$. For $j = 1, 2$ the identity is obvious, while for $j > 2$ notice that derangements τ of $\{1, 2, \dots, j\}$ can be partitioned into those for which j lies in a transposition ($\tau(j) = i, \tau(i) = j$ for some $i \neq j$) and those for which

the cycle containing j is of length at least three. There are clearly $(j-1)D_{j-2}$ of the first kind, and there are $(j-1)D_{j-1}$ of the second. (For each σ a derangement of $\{1, 2, \dots, j-1\}$ and each $i < j$ let $\tau_{i\sigma}$ be obtained by setting

$$\tau_{i\sigma}(k) = \begin{cases} \sigma(k) & k \neq i \\ j & k = i \\ \sigma(i) & k = j \end{cases};$$

then $\tau_{i\sigma}$ is a derangement of the second kind. Moreover, this correspondence is reversible.) Thus

$$\begin{aligned} D_j &= (j-1)D_{j-2} + (j-1)D_{j-1} \\ &= D_{j-1} - (-1)^{j-1} + (j-1)D_{j-1} \\ &= jD_{j-1} + (-1)^j, \end{aligned}$$

which is (1). (Notice that (1) holds with $j = 0$ as well.)

We also require the following three identities which are easily derived from the binomial formula:

$$\sum_{k=0}^n \binom{n}{k}(-1)^k = 0, \quad n \geq 1 \quad (2)$$

$$\sum_{k=0}^n k \binom{n}{k}(-1)^k = 0, \quad n \geq 2 \quad (3)$$

$$\sum_{k=0}^n k^2 \binom{n}{k}(-1)^k = 0, \quad n \geq 3. \quad (4)$$

(2) follows from

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

and substitution of $x = -1$. For (3), differentiation and then multiplication by x gives

$$n(1+x)^{n-1}x = \sum_{k=0}^n k \binom{n}{k} x^k$$

and substitution of $x = -1$ yields the result. For (4) a second differentiation yields

$$n(n-1)(1+x)^{n-2}x^2 = \sum_{k=0}^n k(k-1) \binom{n}{k} x^k$$

and thus

$$n(n-1)(1+x)^{n-2}x^2 + n(1+x)^{n-1}x = \sum_{k=0}^n k^2 \binom{n}{k} x^k.$$

Substituting $x = -1$ we obtain (4) for $n \geq 3$.

Let

$$F(n) = \sum_{k=0}^n (k-1)^2 P_n(k) = \sum_{k=0}^n (k-1)^2 \binom{n}{k} D_{n-k}.$$

By direct calculation $F(2) = 2$. For $n \geq 3$,

$$\begin{aligned} F(n) &= \sum_{k=0}^n (k-1)^2 \binom{n}{k} [(-1)^{n-k} + (n-k)D_{n-k-1}] \\ &= \sum_{k=0}^n (k-1)^2 \binom{n}{k} (-1)^{n-k} + n \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-k-1)!k!} (k-1)^2 D_{n-k-1}. \end{aligned}$$

Using (2), (3) and (4) we obtain

$$\begin{aligned} \sum_{k=0}^n (k-1)^2 \binom{n}{k} (-1)^{n-k} &= \sum_{k=0}^n k^2 \binom{n}{k} (-1)^{n-k} - 2 \sum_{k=0}^n k \binom{n}{k} (-1)^{n-k} \\ &\quad + \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} = 0. \end{aligned}$$

Therefore

$$F(n) = n \sum_{k=0}^{n-1} \binom{n-1}{k} (k-1)^2 D_{n-1-k} = nF(n-1), \quad n \geq 3.$$

Since $F(2) = 2$ it follows that $F(n) = n!$ for $n \geq 2$.

A combinatorial solution by Denis Hanson, The University of Regina, Saskatchewan.

We first show that for each r , $0 \leq r \leq n$, we have that

$$S_r = \sum_{k=r}^n k(k-1)\dots(k-r+1)P_n(k) = n! .$$

[The case $r = 1$ is problem 1 of the 1987 I.M.O. [1987: 210].]

Now

$$P_n(k) = \binom{n}{k} P_{n-k}(0)$$

(fix k elements and derange the remaining $n-k$), thus

$$\begin{aligned}
 S_r &= \sum_{k=r}^n k(k-1)\dots(k-r+1) \binom{n}{k} P_{n-k}(0) \\
 &= \sum_{k=r}^n \frac{k!}{(k-r)!} \cdot \frac{n!}{k!(n-k)!} \cdot P_{n-k}(0) \\
 &= \sum_{k=r}^n n(n-1)\dots(n-r+1) \binom{n-r}{n-k} P_{n-k}(0) \\
 &= \frac{n!}{(n-r)!} \sum_{k=r}^n \binom{n-r}{n-k} P_{n-k}(0) \\
 &= \frac{n!}{(n-r)!} \sum_{l=0}^{n-r} \binom{n-r}{l} P_l(0).
 \end{aligned}$$

But since this last sum just counts the number of permutations of $n-r$ things we have

$$S_r = \frac{n!}{(n-r)!} \cdot (n-r)! = n!$$

as claimed.

In particular

$$\begin{aligned}
 \sum_{k=0}^n (k-1)^2 P_n(k) &= \sum_{k=0}^n (k(k-1) - k + 1) P_n(k) \\
 &= S_2 - S_1 + S_0 \\
 &= n! - n! + n! = n!.
 \end{aligned}$$

A problem: what functions $f_n(k)$ share with $P_n(k)$ the property that

$$\sum_{k=r}^n k(k-1)\dots(k-r+1) f_n(k) = n!$$

for all $0 \leq r \leq n$?

Yugoslavia 1. [1987: 280]

Prove that for each $k = 2, 3, 4, \dots$ there exists an irrational number r such that

$$[r^m] \equiv -1 \pmod{k}$$

for every natural number m , where $[x]$ is the greatest integer $\leq x$.

Solution by George Evangelopoulos, Law student, Athens, Greece.

It is enough to prove the existence of natural numbers p and q such that the roots of

$$x^2 - kpx + kq = 0$$

are irrational real numbers r, s with $0 < s < 1$ and $0 < r$. If r, s are such roots then $r + s = kp$ and $rs = kq$; thus we have $r + s \equiv 0 \pmod{k}$ from which it easily follows that $r^m + s^m \equiv 0 \pmod{k}$ for each natural number m . But since $0 < s^m < 1$ we must have $[r^m] \equiv -1 \pmod{k}$, and this yields the assertion.

To prove the existence of such p and q , we want $(kp)^2 - 4kq > 0$ and

$$0 < \frac{kp - \sqrt{(kp)^2 - 4kq}}{2} < 1.$$

Since $k > 0$, these are equivalent to

$$p^2 > \frac{4q}{k}$$

and

$$q > 0, \quad p > q + \frac{1}{k},$$

respectively. There are infinitely many such choices of p for any given $q > 0$. To obtain irrationality of the roots we may take $q = k$ because

$$(kp)^2 - 4kq = k^2(p^2 - 4)$$

is not a perfect square, for if $p^2 - 4 = t^2$ for some natural number t , then we should have $(p - t)(p + t) = 4$, which is impossible. [Alternatively, s is irrational since $0 < s < 1$ is a root of a monic polynomial with integer coefficients, and thus, from $r + s = kp$, r is irrational.]

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With this we complete the solutions we have received for the problems from the November 1987 *Crux*. Next month we will discuss problems from December 1987. Send in your nice solutions to problems not discussed. Also don't forget to send me the national and regional Olympiads you have collected!

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M I N I - R E V I E W S
by
ANDY LIU

[Introducing a new feature, to occur only finitely often and probably irregularly: short reviews, written by Andy Liu of the University of Alberta, of series of books on popular mathematics. (The word "series" is to be taken loosely, sometimes meaning only that the books have the same author or publisher, for example.) These reviews were first printed, slightly altered, in a publication of the Alberta Teachers' Association, and are limited to books which are in English (perhaps in translation) and are still in print as far as the reviewer knows. Many of these books have been used by Andy in his mathematics enrichment programs for junior and senior high school students in Edmonton; some of them have also been given several times as prizes in the Alberta Mathematics Contests. They range from the familiar to the less well known; for *Crux's* North American readers at least, perhaps the first series below falls in the latter category. — Ed.]

MIR PUBLISHERS LITTLE MATHEMATICS LIBRARY SERIES

This superb series on popular mathematics is translated from Russian by Mir Publishers of Moscow. The books are paperbacks and may be obtained from Progress Books of Toronto at \$2.50 a copy! The series is written for high school students. Each volume is a stimulating study of a particular topic, and the mathematics is of the highest quality.

***Dividing a Segment in a Given Ratio*, by N.M. Beskin, 1975. (71 pp.)**

This book begins with an analysis of the very elementary problem of how to divide a line segment in a given ratio. From this, the reader is led to concepts such as parallel projections, ideal points, separation, cross ratio and complete quadrilaterals. It is an excellent introduction to projective geometry.

***Inequalities*, by P.P. Korovkin, 1975. (72 pp.)**

This book begins with the basic Arithmetic-Mean-Geometric-Mean Inequality and uses the Bernoulli Inequality to generalize it to the Power-Mean Inequality. Various applications are then given.

***The Method of Mathematical Induction*, by I.S. Sominsky, 1975. (62 pp.)**

This is an excellent introduction to the important method of mathematical

induction. After a detailed discussion of the basic idea, numerous examples from arithmetic, algebra and trigonometry are provided, including many proofs of identities and inequalities.

***The Monte Carlo Method*, by I.M. Sobol, 1975. (72 pp.)**

This is an introduction to the theory of statistical sampling. The Monte Carlo method calculates a certain parameter by running a sequence of simulations, taking the average value and estimating the error. It is based on the probabilistic concept of a random variable. In the case of a continuous variable, calculus is used.

***Recursion Sequences*, by A.I. Markushevich, 1975. (48 pp.)**

This book deals with the counting technique based on recurrence relations. After a review of geometric progressions, the method of characteristic equations is introduced to solve recurrence relations.

***Lobachevskian Geometry*, by A.S. Smogorzhevsky, 1976. (71 pp.)**

This is an introduction to hyperbolic geometry. The principal tool is the inversive transformation in Euclidean geometry. This is used to construct a model of the hyperbolic plane, and various theorems in hyperbolic geometry are proved. The book ends with a discussion of the hyperbolic functions and their uses for computation in the hyperbolic plane.

***Pascal's Triangle/Certain Applications of Mechanics to Mathematics*, by V.A. Uspensky, 1976. (87 pp.)**

This volume consists of two independent booklets by the same author. The first consists of an introduction to the binomial coefficients and Pascal's Formula which lead to the construction of Pascal's triangle. The second uses the mechanical principle that in a position of equilibrium, the potential energy of a weight attains its lowest value, to solve a number of problems in geometry and number theory.

***The Shortest Line*, by L.A. Lyusternik, 1976. (104 pp.)**

This may be considered as a non-technical introduction to differential geometry. The concepts of space curve, curvature and geodesics are discussed. However, the emphasis is on the shortest-line problem on special surfaces which can be solved by elementary methods. Plenty of applications are considered.

***Algebraic Equations of Arbitrary Degrees*, by A.G. Kurosh, 1977. (31 pp.)**

High school students are familiar with polynomial equations of degrees one and two, the latter usually solved by the Quadratic Formula. This book gives a brief

description of the Cubic Formula and goes on to discuss the general theory of polynomial equations.

***Differentiation Explained*, by V.G. Boltyansky, 1977. (63 pp.)**

This book introduces the reader to differential calculus by considering problems in physics, such as the problem of a free-falling body. This is followed by an informal discussion of differential equations, which are applied to tackle the problem of harmonic oscillations. Other applications of differential calculus are also given.

***Stereographic Projection*, by B.A. Rosenfeld and N.D. Sergeeva, 1977. (51 pp.)**

The stereographic projection is a projection of a sphere from one of its points onto the plane tangent to the sphere at the diametrically opposite point. Applications to astronomy, geography and hyperbolic geometry are included.

***Proof in Geometry*, by A.I. Fetisov, 1978. (62 pp.)**

This book raises and answers the following questions: What is proof? Why is proof a necessity? What conditions should a proof satisfy for us to call it a correct one? What propositions may be accepted without proof? The discussion is illustrated with numerous examples from geometry.

***An Unusual Algebra*, by I.M. Yaglom, 1978. (128 pp.)**

This is an introduction to Boolean algebra, or the algebra of sets. Comparison with the algebra of numbers is made. Applications to propositional logic and switching circuits are discussed.

***The Fundamental Theorem of Arithmetic*, by L.A. Kaluzhnin, 1979. (36 pp.)**

The Fundamental Theorem of Arithmetic states that prime factorization is unique. This theorem is usually taken for granted, but the book argues that its importance calls for a rigorous proof. Examples of number systems in which prime factorization is not unique are given. The proof of the theorem also leads to a method of solving linear Diophantine equations.

***Induction in Geometry*, by L.I. Golovina and I.M. Yaglom, 1979. (132 pp.)**

This book gives an excellent account of the power of mathematical induction, applied to problems in geometry. There are problems on computation, proof, construction and locus. Induction is also used to define concepts and to generalize results to higher dimensions.

***Method of Successive Approximation*, by N.Ya. Vilenkin, 1979. (110 pp.)**

This is an introduction to numerical analysis. Starting with the simplest form of successive approximation, the reader is led to the method of iteration, the method

of chords and Newton's method which uses differential calculus.

Systems of Linear Inequalities, by A.S. Solodovnikov, 1979. (123 pp.)

This is an introduction to linear programming. Systems of linear inequalities in two or three unknowns are visualized geometrically followed by a brief discussion of convexity. The simplex method is then presented, with a proof of the Duality Theorem and an application to a transportation problem.

Elements of Game Theory, by Ye.S. Venttsel, 1980. (69 pp.)

Game theory deals with conflict scenarios which are resolved according to definite rules. Each party in the conflict has a finite number of options known to all others. After presenting the basic concepts, the book discusses pure and mixed strategies as well as general and approximate methods for solving games.

The Kinematic Method in Geometrical Problems, by Yu.I. Lyubich and L.A. Shor, 1980. (55 pp.)

When solving a geometrical problem, it is helpful to imagine what would happen to the elements of the figure under consideration if some of its points start moving. After a review of vector algebra, this book shows how kinematics, or the theory of velocities, can be applied to tackle geometric problems.

Method of Coordinates, by A.S. Smogorzhevsky, 1980. (47 pp.)

This is an introduction to analytic geometry, covering the most basic concepts. There is also a brief discussion of polar coordinates.

Remarkable Curves, by A.I. Markushevich, 1980. (47 pp.)

This book presents a collection of very attractive curves and their interesting properties. Starting with the basic conic sections, the reader is led on to the lemniscate of Bernoulli, the cycloid, the spiral of Archimedes, the catenary and the logarithmic spiral.

Area and Logarithm, by A.I. Markushevich, 1981. (69 pp.)

This book presents a geometric theory of logarithm, in which logarithms are introduced as various areas. Properties of logarithms are then derived from those of areas. The reader is introduced to rudimentary integral calculus without first going through differential calculus.

Solving Equations in Integers, by A.O. Gelfond, 1981. (56 pp.)

The subject is Diophantine equations. Detailed study of the linear Diophantine equation, Pythagoras' equations and Pell's equation are presented, with continued fractions playing a central role. There is a brief discussion of other

Diophantine equations.

Calculus of Rational Functions, by G.E. Shilov, 1982. (51 pp.)

This book is an informal introduction to differential and integral calculus with sufficient rigor when attention is restricted to the rational functions, that is, functions expressible as a quotient of two polynomials. There is an illuminating preamble on graph sketching.

Complex Numbers and Conformal Mappings, by A.I. Markushevich, 1982. (61 pp.)

Not assuming prior acquaintance with complex numbers, the book introduces them to the reader in geometric form as directed line segments. Functions of a complex variable are considered as geometric transformations. Of particular interest is the class of conformal mappings, or angle-preserving transformations.

Post's Machine, by V.A. Uspensky, 1983. (88 pp.)

The book deals with a certain abstract computing machine. Though this machine does not exist physically, calculations on it involve many important features inherent in the computations on computers. This machine is also known as Turing's machine.

Images of Geometric Solids, by N.M. Beskin, 1985. (78 pp.)

The subject of this book is descriptive geometry, that is, the two-dimensional representation of three-dimensional objects. The concept of projection plays a central role. Numerous practical exercises are included.

Fascinating Fractions, by N.M. Beskin, 1986. (87 pp.)

The fascinating fractions are the continued fractions. After two introductory problems, the book gives an algorithm for converting a real number into a continued fraction. Further applications follow, including the solution to Diophantine equations and the approximation of real numbers by rational numbers.

Geometrical Constructions with Compass Only, by A. Kostovskii, 1986. (77 pp.)

This book presents a proof of the Mascheroni-Mohr Theorem that the ruler is a redundant tool in Euclidean constructions. Of course, the compass alone cannot draw straight lines, but we may consider a straight line constructed if at least two points on it are obtained. However, if further use of this line is made to intersect other lines and circles, those points of intersection have to be constructed explicitly. The main idea behind the proof is inversion. Later, further constraints are placed on

the compass.

Godel's Incompleteness Theorem, by V.A. Uspensky, 1987. (103 pp.)

Godel's Incompleteness Theorem says roughly that under certain very reasonable conditions in a mathematical system, there exist true but unprovable statements. This book provides the necessary background in mathematical logic and gives a formal proof of this important result.

The address of Progress Books is

71 Bathurst Street, 3/F
Toronto, Ontario M2V 2P6
Canada

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P R O B L E M S

Problem proposals and solutions should be sent to the editor , whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before December 1, 1989, although solutions received after that date will also be considered until the time when a solution is published.

1441*. Proposed by Stanley Rabinowitz, Westford, Massachusetts.

Let

$$S_n = (x - y)^n + (y - z)^n + (z - x)^n.$$

It is easy to see that if p is a prime, S_p/p is a polynomial with integer coefficients. Prove that

$$\begin{array}{c|c} \frac{S_2}{2} & | S_{2+6k} , \\ \hline \end{array} \quad \begin{array}{c|c} \frac{S_3}{3} & | S_{3+6k} , \\ \hline \end{array}$$
$$\begin{array}{c|c} \frac{S_5}{5} & | S_{5+6k} , \\ \hline \end{array} \quad \begin{array}{c|c} \frac{S_7}{7} & | S_{7+6k} , \\ \hline \end{array}$$

for all $k = 1,2,3,\dots$, where $|$ denotes polynomial divisibility.

1442. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle. If P is a point on the circumcircle, and D, E, F are the feet of the perpendiculars from P to BC, AC, AB respectively, then it is well known that D, E, F are collinear (Wallace line or Simson line). Find P such that E is the midpoint of the segment DF .

1443. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Given an integer $n \geq 2$, determine the minimum value of

$$\sum_{\substack{1 \leq i, j \leq n \\ i \neq j}} \left(\frac{x_i^2}{x_j} \right)$$

over all positive real numbers x_1, \dots, x_n such that $x_1^2 + \dots + x_n^2 = 1$.

1444. *Proposed by Jordi Dou, Barcelona, Spain.*

Given the centre O of a conic γ and three points A, B, C lying on γ , construct those points X on γ such that XB is the bisector (interior or exterior) of $\angle AXC$.

1445. *Proposed by M.S. Klamkin and Andy Liu, University of Alberta.*

Determine the minimum value of

$$\frac{x^3}{1-x^8} + \frac{y^3}{1-y^8} + \frac{z^3}{1-z^8}$$

where $x, y, z \geq 0$ and $x^4 + y^4 + z^4 = 1$.

1446. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be an equilateral triangle inscribed in a triangle ABC , so that $A' \in BC, B' \in CA, C' \in AB$. We denote by G', G the centroids, by O', O the circumcenters, by I', I the incenters and by H', H the orthocenters of triangles $A'B'C'$ and ABC respectively. Prove that in each of the four cases

- (a) $G = G'$, (b) $O = O'$, (c) $I = I'$, (d) $H = H'$,
 ABC must be equilateral.

1447. *Proposed by Henjin Chi and Raymond Killgrove, Indiana State University, Terre Haute.*

For each natural number n , how many integer-sided right triangles are there such that the area is n times the perimeter? How many of these are primitive (the sides have no common factor)?

1448. *Proposed by Jack Garfunkel, Flushing, N.Y.*

If A, B, C are the angles of a triangle, prove that

$$\frac{2}{3} \left(\sum \sin \frac{A}{2} \right)^2 \geq \sum \cos A,$$

with equality when $A = B = C$.

1449. *Proposed by David C. Vaughan, Wilfrid Laurier University.*

Prove that for all $x \geq y \geq 1$,

$$\frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{y+1}} + \frac{1}{\sqrt{x+1}} \geq \frac{y}{\sqrt{x+y}} + \frac{x}{\sqrt{x+1}} + \frac{1}{\sqrt{y+1}}.$$

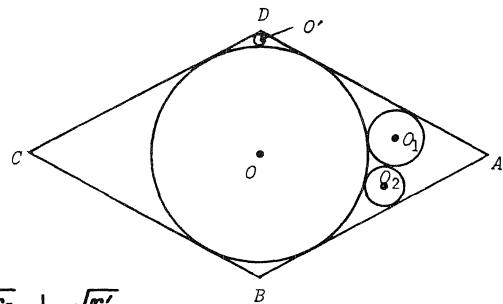
Determine where equality holds.

1450. *Proposed by H. Fukagawa, Yokosuka High School, Aichi, Japan.*

There is a rhombus $ABCD$ and inscribed circles $O(r)$ and $O'(r')$ as shown in the figure. We draw any two circles $O_1(r_1)$ and $O_2(r_2)$ touching the sides AB and AD respectively, and also touching the circle $O(r)$ and each other. Show the simple relation

$$\sqrt{r_1 + r_2 + r} = \sqrt{r_1} + \sqrt{r_2} + \sqrt{r'}.$$

*



SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1332. [1988: 109] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

It is known that if A, B, C are the angles of a triangle,

$$\sin A/2 + \sin B/2 + \sin C/2 \geq 1,$$

with equality if and only if the triangle is degenerate with angles $\pi, 0, 0$. Establish the related non-comparable inequality

$$\sin A/2 + \sin B/2 + \sin C/2 \geq 5r/R - 1,$$

where r and R are the inradius and circumradius respectively.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We show the following inequality, stronger than both given inequalities:

$$\sum \sin(A/2) \geq 1 + r/R \quad (1)$$

where the sum is cyclic. [Note $1 + r/R \geq 5r/R - 1$ follows easily from $2r \leq R$.]

Indeed, we first prove

$$\sum \cos A \geq 1 + 4 \prod \cos A. \quad (2)$$

Noting that

$$\sum \cos A = 1 + r/R \quad (3)$$

and

$$\prod \cos A = \frac{s^2 - (2R + r)^2}{4R^2}$$

(s being the semiperimeter), we have to show for (2) that

$$Rr \geq s^2 - (2R + r)^2,$$

i.e.

$$s^2 \leq 4R^2 + 5Rr + r^2. \quad (4)$$

From

$$s^2 \leq 4R^2 + 4Rr + 3r^2$$

([1], item 5.9) we get immediately (4). (In fact

$$4R^2 + 4Rr + 3r^2 \leq 4R^2 + 5Rr + r^2$$

is equivalent to $2r \leq R$.)

Finally, replacing in (2) A by $(\pi - A)/2$, etc., the latter also being angles of a triangle, we obtain

$$\sum \sin A/2 \geq 1 + 4 \prod \sin A/2 = 1 + r/R.$$

Remark. From (3), inequality (1) can also be written

$$\sum \sin A/2 \geq \sum \cos A.$$

Applying to this inequality the transformation

$$A \rightarrow \frac{\pi - A}{2}, \text{ etc.}$$

we get the nice inequality

$$\sum \sin\left(\frac{\pi - A}{4}\right) \geq \sum \sin \frac{A}{2},$$

i.e.

$$\sum \cos \frac{A}{4} - \sum \sin \frac{A}{4} \geq 2\sqrt{2} \sum \sin \frac{A}{4} \cos \frac{A}{4}.$$

Let's now similarly improve the inequality

$$\sum \cos \frac{A}{2} > 2$$

([1], item 2.27). We claim

$$\sum \cos \frac{A}{2} \geq 2 + (3\sqrt{3} - 4) \frac{r}{R}. \quad (5)$$

We put $\lambda = 3\sqrt{3} - 4$ and will first show

$$\sum \sin A \geq 2 + 4\lambda \prod \cos A, \quad (6)$$

or equivalently

$$\frac{s}{R} \geq 2 + \frac{\lambda(s^2 - (2R + r)^2)}{R^2},$$

i.e.

$$\lambda s^2 - Rs + 2R^2 - \lambda(2R + r)^2 \leq 0.$$

Thus we have to show that always $s_1 \leq s \leq s_2$ where

$$s_1 = \frac{1}{2\lambda} \left(R - \sqrt{R^2 - 8\lambda R^2 + 4\lambda^2(2R + r)^2} \right),$$

$$s_2 = \frac{1}{2\lambda} \left(R + \sqrt{R^2 - 8\lambda R^2 + 4\lambda^2(2R + r)^2} \right).$$

It is easily checked that $s_1 < 0$ (indeed, the root is $> R$). Now for $s \leq s_2$. From [1], item 5.4 we take

$$s \leq 2R + \lambda r.$$

Thus it is enough to prove

$$2R + \lambda r \leq s_2,$$

i.e.

$$(4\lambda R + 2\lambda^2 r - R)^2 \leq R^2 - 8\lambda R^2 + 4\lambda^2(2R + r)^2.$$

Collecting terms yields

$$4\lambda^4 r^2 + 16\lambda^3 R r \leq 20\lambda^2 R r + 4\lambda^2 r^2,$$

or

$$R(5 - 4\lambda) \geq (\lambda^2 - 1)r,$$

or, since $\lambda = 3\sqrt{3} - 4 < 5/4$,

$$R \geq \frac{\lambda^2 - 1}{5 - 4\lambda} r. \quad (7)$$

Since

$$\frac{\lambda^2 - 1}{5 - 4\lambda} = 2,$$

(7) and thus (6) follows. Applying to (6) the transformation

$$A \rightarrow \frac{\pi - A}{2}, \text{ etc.}$$

we get

$$\sum \cos \frac{A}{2} \geq 2 + \lambda \cdot \frac{r}{R},$$

i.e. (5).

Reference:

- [1] Bottema et al, *Geometric Inequalities*, Groningen, 1968.

The stronger inequality (1) above was also proved by SVETOSLAV J. BILCHEV and EMILIA A. VELIKOVA, Technical University, Russe, Bulgaria. The original problem was also solved by KEE-WAI LAU, Hong Kong; BOB PRIELIPP, University of Wisconsin-Oshkosh; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student, University of Waterloo; and the proposer. Two incorrect solutions were sent in.

The proposer's proof reduced the given inequality to showing

$$1 + \sum \cos A \geq 20 \prod \cos A,$$

which has been proved in Crux [1983: 23]. Janous's inequality (2) is similar, and is stronger, because of

$$\prod \cos A \leq \frac{1}{8}.$$

* * *

1333. [1988: 109] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

If a, b, c and a', b', c' are the sides of two triangles and F, F' are their areas, show that

$$\sum a[a' - (\sqrt{b'} - \sqrt{c'})^2] \geq 4\sqrt{3}FF',$$

where the sum is cyclic. (This improves Crux 1114 [1987: 185].)

I. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We prove a stronger result: for $0 < p, q \leq 1$,

$$\sum a'^{2q}[a^{2p} - (b^p - c^p)^2] \geq 3\left(\frac{4}{\sqrt{3}}\right)^{p+q} F^p F'^q.$$

(We have changed the roles of a, b, c and a', b', c' for notational convenience.) When $p = q = 1/2$ we obtain the stated inequality.

We first show

$$\sum a'^2[a^2 - (b - c)^2] \geq 16FF'. \quad (1)$$

Referring to Crux 1181 [1988: 25–26] we know that for $x, y, z > 0$

$$\sum yza'^2 \geq 4F'\sqrt{xyz(x + y + z)}.$$

Putting

$$x = -a + b + c, \quad y = a - b + c, \quad z = a + b - c$$

and noting

$$xyz(x + y + z) = 16s(s - a)(s - b)(s - c) = 16F^2,$$

where s is the semiperimeter, we get (1).

Next we employ the following result of Oppenheim (cf. [1]): for $0 < p \leq 1$, a^p , b^p , c^p are the sides of a triangle of area

$$F_p \geq \left(\frac{\sqrt{3}}{4}\right)^{1-p} F^p.$$

This and (1) yield for $0 < p, q \leq 1$

$$\begin{aligned} \sum a'^{2q} [a^{2p} - (b^p - c^p)^2] &\geq 16F_p F_q' \\ &\geq 16 \left(\frac{\sqrt{3}}{4}\right)^{2-p-q} F^p F'^q \\ &= 3 \left(\frac{4}{\sqrt{3}}\right)^{p+q} F^p F'^q, \end{aligned}$$

as claimed.

Remark. The Neuberg–Pedoe inequality (cf. [2], item 10.8) reads

$$\sum a'^2(-a^2 + b^2 + c^2) \geq 16FF',$$

i.e.

$$M := \sum a^2 \sum a'^2 - 2 \sum a^2 a'^2 \geq 16FF'.$$

As (1) can also be written

$$-\sum a^2 \sum a'^2 + 2 \sum a^2 a'^2 + 2 \sum a'^2 bc \geq 16FF',$$

we obtain

$$\sum a'^2 bc \geq 8FF' + \frac{M}{2} \geq 16FF'.$$

References:

- [1] A. Oppenheim, Inequalities involving elements of triangles, quadrilaterals or tetrahedra, *Univ. Belgrade Publ. Fak. Ser. Mat. Fiz.* No. 461–497 (1974) 257–263.
- [2] O. Bottema et al, *Geometric Inequalities*, Groningen, 1968.
- II. *Solution by Svetoslav J. Bilchev and Emilia A. Velikova, Technical University, Russe, Bulgaria.*

We shall use the well-known polar moment of inertia inequality (see [1] – [4], [6])

$$4F \sqrt{\sum yz} \leq \left| \sum a^2 x \right|, \quad (2)$$

where x, y, z are arbitrary real numbers with $\sum yz \geq 0$.

[*Editor's note:* As most of the given references are likely inaccessible to most *Crux* readers, here is an alternate derivation of (2), kindly supplied by Murray Klamkin. Start with the general polar moment of inertia inequality as given in [1989: 28], i.e.

$$(x + y + z)(xR_1^2 + yR_2^2 + zR_3^2) \geq a^2yz + b^2zx + c^2xy,$$

where x, y, z are arbitrary real numbers and R_1, R_2, R_3 are the distances of a point P to the vertices of the triangle. To get (2), let P be the circumcenter, so that

$$R_1 = R_2 = R_3 = R = \frac{abc}{4F},$$

and replace x, y, z by a^2x, b^2y, c^2z respectively.]

Apply (2) to a triangle with sides $a_1 = \sqrt{a}, b_1 = \sqrt{b}, c_1 = \sqrt{c}$ and area

$$F_1 = \frac{1}{2}\sqrt{r(4R + r)} \quad (3)$$

where r, R are the inradius and circumradius of the a, b, c triangle (see [4]).

[*Editor's note:* Formula (3) can also be derived from

$$16F_1^2 = 2 \sum b_1^2c_1^2 - \sum a_1^4 = 2 \sum bc - \sum a^2$$

using

$$\sum bc = s^2 + r^2 + 4Rr$$

and

$$\sum a^2 = 2(s^2 - r^2 - 4Rr),$$

where s is the semiperimeter. For these see equations (3) and (4) of J. Steinig, Inequalities concerning the inradius and circumradius of a triangle, *Elemente der Mathematik* 18 (1963) 127–131. Thanks again to Murray Klamkin!]

We get

$$2\sqrt{r(4R + r)} \sum yz \leq \left| \sum ax \right|. \quad (4)$$

Further, let

$$x = a'^2 - (b' - c')^2, \quad y = b'^2 - (c' - a')^2, \quad z = c'^2 - (a' - b')^2,$$

i.e.

$$\begin{aligned} \sum yz &= \sum [b'^2 - (c' - a')^2][c'^2 - (a' - b')^2] \\ &= 16 \sum (s' - c')(s' - a')(s' - a')(s' - b') \\ &= 16(s' - a')(s' - b')(s' - c')s' = 16F'^2. \end{aligned}$$

From (4) follows

$$8F'\sqrt{r(4R+r)} \leq \sum a[a'^2 - (b' - c')^2]. \quad (5)$$

Now we apply inequality (5) to ΔABC and to the triangle with sides $\sqrt{a'}$, $\sqrt{b'}$, $\sqrt{c'}$ and area

$$\frac{1}{2}\sqrt{r'(4R'+r')}.$$

We obtain

$$\sum a[a' - (\sqrt{b'} - \sqrt{c'})^2] \geq 4\sqrt{rr'(4R+r)(4R'+r')}. \quad (6)$$

Inequality (6) is sharper than the given inequality because of the well-known inequalities

$$4R + r \geq s\sqrt{3}, \quad 4R' + r' \geq s'\sqrt{3}$$

(see [5], item 5.5) and $rs = F$, $r's' = F'$.

References:

- [1] A. Oppenheim, Some inequalities for a triangle, *Publ. El. Fak. Univ. u Beogradu, Ser. Math. i Fiz.*, No. 357–380 (1971) 21–28.
- [2] M.S. Klamkin, Geometric inequalities via the polar moment of inertia, *Math. Magazine* 48 (1975) 44–46.
- [3] Sv. Bilchev and Hr. Lesov, Varlin edno ravenstvo i svarzanite s nego geometricni neravenstva, *Obuch. po Matematika*, No. 4 (1986) 33–42.
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- [5] O. Bottema, R.Ž. Djordjević, R.R. Janić, D.S. Mitrinović, and P.M. Vasić, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.
- [6] D.S. Mitrinović, J.E. Pečarić, S.J. Bilchev, and E.A. Velikova, On an inequality of O. Kooi, *Matematika i Matematičesko Obrazovanie*, 1988, BAN, Sofia, pp. 566–568.

Also solved by the proposer, who also obtained inequality (1) in the Janous solution.

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1334. [1988: 109] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

(a) Suppose Fibonacci had wanted to set up an annuity that would pay F_n lira on the n th year after the plan was established, for $n = 1, 2, 3, \dots$ ($F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n > 2$). To fund such an annuity, Fibonacci had planned to invest a sum of money with the Bank of Pisa, which paid 70%

interest per year, and instruct them to administer the trust. How much money did he have to invest so that the annuity could last in perpetuity?

(b) When he got to the bank, Fibonacci found that their interest rate was only 7% (he had misread their ads), not enough for his purposes. Despondently, he went looking for another bank with a higher interest rate. What rate must he seek in order to allow for a perpetual annuity?

Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.

(b) Let P_n be the principal amount required to provide the payment at the end of the n th year, and let I be the interest rate. Then

$$F_n = P_n(1 + I)^n. \quad (1)$$

A well-known formula for F_n is

$$F_n = \frac{a^n - b^n}{a - b}, \quad (2)$$

where

$$a = \frac{1 + \sqrt{5}}{2}, \quad b = \frac{1 - \sqrt{5}}{2}.$$

Combining (1) and (2) gives

$$P_n = \frac{a^n - b^n}{\sqrt{5}(1 + I)^n}.$$

Fibonacci's total investment is therefore

$$P = \sum_{i=1}^{\infty} P_i = \frac{1}{\sqrt{5}} \left[\sum_{i=1}^{\infty} \left(\frac{a}{1+I} \right)^i - \sum_{i=1}^{\infty} \left(\frac{b}{1+I} \right)^i \right]. \quad (3)$$

Since the absolute value of b is less than 1, the second series on the right always converges. However, the first series converges only when

$$I > a - 1 = .618033989\dots.$$

Thus Fibonacci must find an interest rate greater than 61.8033989%.

(a) With an interest rate of 70%, both series in (3) converge, and use of the well-known formula for the sum of a geometric series gives

$$\begin{aligned} P &= \frac{1}{\sqrt{5}} \left(\frac{a}{1+I-a} - \frac{b}{1+I-b} \right) \\ &\approx 8.94736842 \quad \left[= \frac{170}{19} \right]. \end{aligned}$$

Also solved by M.A. SELBY, University of Windsor; COLIN SPRINGER, student, University of Waterloo; and the proposer. Solved (with a minor error) by RICHARD I. HESS, Rancho Palos Verdes, California.

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1335. [1988: 109] *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Let A and B be points on the positive x and y axes, respectively, such that AB is tangent to the curve $xy = 1$. Let Q be on AB so that $OQ \perp AB$. Find the area in the first quadrant enclosed by the locus of Q .

Solution by Jordi Dou, Barcelona, Spain.

Since the contact point of AB (with the curve $xy = 1$) is its midpoint, the area of ΔOAB is $2xy = 2$. Put $OQ = \rho$ and $\angle AOQ = \alpha$. Then

$$\begin{aligned} 2 &= \frac{1}{2}OQ(AQ + QB) \\ &= \frac{1}{2}\rho(\rho \tan \alpha + \rho \cot \alpha) \\ &= \frac{\rho^2}{2 \sin \alpha \cos \alpha}, \end{aligned}$$

so

$$\rho^2 = 4 \sin \alpha \cos \alpha = 2 \sin 2\alpha.$$

The area bounded by the locus of Q is therefore

$$\frac{1}{2} \int_0^{\pi/2} \rho^2 d\alpha = \int_0^{\pi/2} \sin 2\alpha d\alpha = -\frac{\cos 2\alpha}{2} \Big|_0^{\pi/2} = 1.$$

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student, University of Waterloo; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

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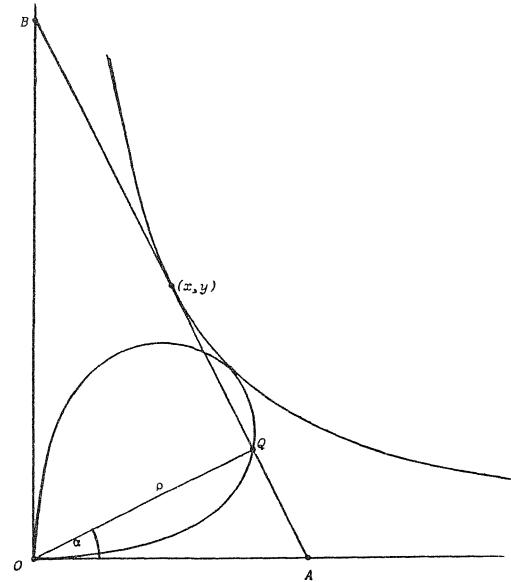
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1336. [1988: 109] *Proposed by Jordan Stoyanov, Bulgarian Academy of Sciences, Sofia.*

For each natural number $n \geq 2$, express the number

$$\underbrace{5599\cdots9}_{n-2} \underbrace{8933\cdots39}_{n-1}$$



as a sum of squares of three natural numbers.

Solution by Leroy F. Meyers, The Ohio State University.

We have

$$\begin{aligned} \underbrace{5599\cdots98933\cdots39}_{n-2} &= 55 \cdot 10^{2n} + (10^{2n} - 10^{n+2}) + 89 \cdot 10^n + \frac{10^n - 10}{3} + 9 \\ &= \frac{1}{9}(504 \cdot 10^{2n} - 96 \cdot 10^n + 51) \end{aligned} \quad (1)$$

for $n \geq 2$. We look for integers u, v, w, x, y, z such that

$$(10^n u + x)^2 + (10^n v + y)^2 + (10^n w + z)^2 = 504 \cdot 10^{2n} - 96 \cdot 10^n + 51. \quad (2)$$

Comparing coefficients of 10^{2n} , $2 \cdot 10^n$ and 1 yields

$$\begin{aligned} u^2 + v^2 + w^2 &= 504 \\ &= 22^2 + 4^2 + 2^2 = 20^2 + 10^2 + 2^2 = 18^2 + 12^2 + 6^2, \\ ux + vy + wz &= -48, \end{aligned} \quad (3)$$

and

$$\begin{aligned} x^2 + y^2 + z^2 &= 51 \\ &= 7^2 + 1^2 + 1^2 = 5^2 + 5^2 + 1^2. \end{aligned}$$

If we assume that u, v , and w are positive, then the only ways that (u, v, w) , set equal to $(22, 4, 2)$, $(20, 10, 2)$, or $(18, 12, 6)$, can combine with a permutation of $(\pm 7, \pm 1, \pm 1)$ or $(\pm 5, \pm 5, \pm 1)$ so that (3) holds are given by the following table:

u	v	w	x	y	z
22	4	2	-1	-7	1
20	10	2	1	-7	1
20	10	2	-5	5	1
18	12	6	-1	1	-7
18	12	6	-5	1	5
18	12	6	-1	-5	5

By (1) and (2), we also want

$$\frac{10^n u + x}{3}, \quad \frac{10^n v + y}{3}, \quad \frac{10^n w + z}{3}$$

to be integers, and so we require that $u + x$, $v + y$, and $w + z$ all be divisible by 3, as occurs only in the first three lines of the table. The decompositions of the given number into the sum of three squares are then

$$\underbrace{733\cdots3^2}_n + \underbrace{133\cdots31^2}_{n-1} + \underbrace{66\cdots67^2}_{n-1},$$

$$\underbrace{66\cdots 67^2}_n + \underbrace{33\cdots 31^2}_n + \underbrace{66\cdots 67^2}_{n-1},$$

and

$$\underbrace{66\cdots 65^2}_n + \underbrace{33\cdots 35^2}_n + \underbrace{66\cdots 67^2}_{n-1}.$$

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; P. PENNING, Delft, The Netherlands; COLIN SPRINGER, student, University of Waterloo, KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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1337. [1988: 110] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Consider integers a, b, c such that (*) the equation $ax^2 + by^2 + cz^2 = 0$ has a solution in integers x, y, z not all zero, and every solution $(x,y,z) \neq (0,0,0)$ satisfies

$$\gcd(x,y) > 1, \quad \gcd(y,z) > 1, \quad \gcd(z,x) > 1.$$

- (i) Show that $a = 9, b = 25, c = -98$ satisfies (*).
- (ii) Find infinitely many triples (a,b,c) satisfying (*).

Solution by Chris Wildhagen, Breda, The Netherlands.

Let p, q and r be distinct primes such that

$$p \equiv 3 \text{ or } 5 \pmod{8}, \quad q \equiv 3 \text{ or } 5 \pmod{8}, \quad r \equiv 3 \pmod{4}. \quad (1)$$

Then put

$$a = p^2, \quad b = q^2, \quad c = -2r^2.$$

The equation $ax^2 + by^2 + cz^2 = 0$ can then be rewritten as

$$u^2 + v^2 = 2w^2,$$

where $u = px, v = qy, w = rz$. This diophantine equation has the known general solution

$$\begin{aligned} u &= \pm d(l^2 - m^2 - 2lm), \\ v &= \pm d(m^2 - l^2 - 2lm), \\ w &= \pm d(l^2 + m^2), \end{aligned}$$

where $d, l, m \in \mathbb{Z}$, l, m not both zero, and $\gcd(l,m) = 1$. [In particular there are solutions not all zero.] Therefore

$$px = \pm d[(l - m)^2 - 2m^2],$$

$$qy = \pm d[(l-m)^2 - 2l^2], \\ rz = \pm d(l^2 + m^2).$$

Since by (1) we have the Legendre symbols

$$\left(\frac{2}{p}\right) = \left(\frac{2}{q}\right) = \left(\frac{-1}{r}\right) = -1,$$

it follows (from $\gcd(l,m) = 1$) that

$$p \nmid (l-m)^2 - 2m^2, \quad q \nmid (l-m)^2 - 2l^2, \quad r \nmid l^2 + m^2.$$

Thus d is divisible by p , q and r , which implies

$$p \mid \gcd(y,z), \quad q \mid \gcd(x,z), \quad r \mid \gcd(x,y).$$

There are infinitely many triples (p,q,r) of distinct primes satisfying (1), from which (ii) follows. Part (i) is a particular case ($p = 3$, $q = 5$, $r = 7$).

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; COLIN SPRINGER, student, University of Waterloo; and the proposer. Two other readers sent in incorrect solutions, probably as a result of misreading the problem.

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T W O N E W B O O K S

of particular relevance for readers of *Crux Mathematicorum* are on the market, or soon will be. The editor has seen neither book as yet, but hopes to print reviews of both books eventually. The following information is passed on now for the convenience of *Crux* readers.

Already in print is *Recent Advances in Geometric Inequalities*, by D.S. Mitrinović, J.E. Pecarić, and V. Volenec, published by Kluwer Academic Publishers, Dordrecht, The Netherlands. Over 700 pages of material, with many references to *Crux*. It is *not* intended to replace the earlier *Geometric Inequalities* of Bottema et al, which is so often cited in *Crux*.

To appear in the early fall is *Japanese Temple Geometry Problems*, by H. Fukagawa and D. Pedoe, published by the Charles Babbage Research Centre, University of Manitoba, Winnipeg, Manitoba, Canada. This book contains nearly 300 problems, some of which have appeared in *Crux* over the last four years as proposals of the first author.

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