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CONTENTS

MAXIMUM AND MINIMUM OF THE SUM OF RECIPROCALS OF THE SIDES OF A TRIANGLE

V.N. MURTY and M. PERISASTRY

Let a, b, c be the side lengths of a variable triangle with fixed area Δ and let

$$S = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

It is well known that

$$\begin{aligned} 16\Delta^2 &= (a+b+c)(b+c-a)(c+a-b)(a+b-c) \\ &= 2(b^2c^2 + c^2a^2 + a^2b^2) - (a^4 + b^4 + c^4). \end{aligned}$$

We will find in this note the following sharp upper and lower bounds for S:

$$\frac{\sqrt{5}\,\sqrt[4]{5+2\sqrt{5}}}{2\sqrt{\Delta}} \le S \le \frac{3\sqrt[4]{3}}{2\sqrt{\Delta}}. \tag{1}$$

The upper bound in (1) is already known in the literature, but the lower bound is, we believe, new.

Using the Lagrange multiplier technique, our task amounts to finding values of a, b, c (or relations among them) such that

$$f(a,b,c,\lambda) \equiv \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \lambda(2\Sigma b^2c^2 - \Sigma a^4 - K) \tag{2}$$

is a maximum or a minimum, where $K = 16\Delta^2$ is a constant.

The values of a, b, c that maximize or minimize (2) must necessarily satisfy

$$\frac{\partial f}{\partial a} = -\frac{1}{a^2} + 4\lambda a(b^2 + c^2 - a^2) = 0, \tag{3}$$

$$\frac{\partial f}{\partial b} = -\frac{1}{b^2} + 4\lambda b(c^2 + a^2 - b^2) = 0, \tag{4}$$

$$\frac{\partial f}{\partial c} = -\frac{1}{c^2} + 4\lambda c(a^2 + b^2 - c^2) = 0, \tag{5}$$

$$\frac{\partial f}{\partial \lambda} = 2\Sigma b^2c^2 - \Sigma a^4 - K = 0. \tag{6}$$

We observe now that, in each of equations (3)-(5), the first term is nonzero and hence the second term also does not vanish. Thus equations (3)-(5) are equivalent to

$$4\lambda = \frac{1}{a^3(b^2+c^2-a^2)} = \frac{1}{b^3(c^2+a^2-b^2)} = \frac{1}{c^3(a^2+b^2-c^2)}, \tag{7}$$

and we may solve, instead of the system (3)-(6), the equivalent system (6)-(7). Now

$$\begin{aligned}
(7) \Longleftrightarrow\ & \frac{b^3c^3}{b^2+c^2-a^2} = \frac{c^3a^3}{c^2+a^2-b^2} = \frac{a^3b^3}{a^2+b^2-c^2} = 4\lambda a^3b^3c^3 \\
\Longleftrightarrow\ & \frac{a^3(b^3+c^3)}{2a^2} = \frac{b^3(c^3+a^3)}{2b^2} = \frac{c^3(a^3+b^3)}{2c^2} = 4\lambda a^3b^3c^3 \\
\Longleftrightarrow\ & a(b^3+c^3) = b(c^3+a^3) = c(a^3+b^3) = 8\lambda a^3b^3c^3 \\
\Longrightarrow\ & b^3(c-a) = ca(c^2-a^2) \text{ and } c^3(a-b) = ab(a^2-b^2)\ .
\end{aligned} \tag{8}$$

The first equation in (8) is equivalent to

$$c = a \quad \text{or} \quad b^3 = ca(c+a)$$

and the second to

$$a = b \quad \text{or} \quad c^3 = ab(a+b).$$

Hence (7) implies the following multiple disjunction:

$$c = a \text{ and } a = b \Longleftrightarrow a = b = c \tag{9}$$

or

$$c = a \text{ and } c^3 = ab(a+b) \Longleftrightarrow c = a \text{ and } a^2 = b(a+b) \tag{10}$$

or

$$a = b \text{ and } b^3 = ca(c+a) \Longleftrightarrow a = b \text{ and } b^2 = c(b+c) \tag{11}$$

or

$$b^3 = ca(c+a) \text{ and } c^3 = ab(a+b) \Longleftrightarrow b = c \text{ and } c^2 = a(c+a)\ . \tag{12}$$

When (9) holds, we have $S = 3/a$ and (6) yields $K = 3a^4$, so $1/a = \sqrt[4]{3/K}$ and $S = 3\sqrt[4]{3/K}$. When (10) holds, we have

$$\frac{a}{1+\sqrt{5}} = \frac{b}{2} = \frac{c}{1+\sqrt{5}} = t, \text{ say,} \tag{13}$$

from which

$$S = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{\sqrt{5}}{2} \cdot \frac{1}{t}. \tag{14}$$

We now substitute from (13) into (6) and find

$$\frac{1}{t} = \frac{2\sqrt[4]{5+2\sqrt{5}}}{\sqrt[4]{K}},$$

so (14) becomes

$$S = \frac{\sqrt{5}\sqrt[4]{5+2\sqrt{5}}}{\sqrt[4]{K}},$$

and the same result is obtained from (11) and (12). Since $\sqrt{5}\sqrt[4]{5+2\sqrt{5}} < 3\sqrt[4]{3}$, we have

$$\frac{\sqrt{5}\sqrt[4]{5+2\sqrt{5}}}{\sqrt[4]{K}} \le S \le \frac{3\sqrt[4]{3}}{\sqrt[4]{K}}. \tag{15}$$

Remembering that $K = 16\Delta^2$, we see that we have in fact obtained (1), as desired.

Equality holds on the right in (15) if and only if $a = b = c$, and it holds on the left if and only if (10), (11), or (12) holds.

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MAXIMUM REPRESENTATIONS BY NINE-DIGIT DETERMINANTS

CHARLES W. TRIGG

"The absolute value of a third-order determinant is invariant under interchange of columns, interchange of rows, and interchange of columns and rows. Thus any square array with nine distinct elements is a member of a family of $2\cdot(3!)^2$ or 72 arrays, all with determinants having the same absolute value. ... If the elements are the nine nonzero digits, one member of the family may be designated as the 'parent' if, using the notation

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

$a_1 = 1$, $a_1 < b_1 < c_1$, the smallest of the other six digits is in the second row, and $a_2 > b_1$, $a_3 > b_1$. There are $9!/72$ or 5040 parents." [1]

The largest integer representable by a nine-digit determinant (see [2], [3]) has a single parent; it is

$$412 = \begin{vmatrix} 1 & 4 & 8 \\ 7 & 2 & 6 \\ 5 & 9 & 3 \end{vmatrix}.$$

Other largest representable integers that have only one parent are:

the largest palindrome, $$393 = \begin{vmatrix} 1 & 5 & 8 \\ 7 & 2 & 4 \\ 6 & 9 & 3 \end{vmatrix};$$

the largest prime, $$389 = -\begin{vmatrix} 1 & 4 & 9 \\ 5 & 7 & 2 \\ 8 & 3 & 6 \end{vmatrix};$$

the largest cube, $$343 = -\begin{vmatrix} 1 & 4 & 6 \\ 7 & 9 & 2 \\ 8 & 3 & 5 \end{vmatrix}.$$

The largest representable square has two parents; it is

$$400 = \begin{vmatrix} 1 & 4 & 8 \\ 9 & 2 & 6 \\ 5 & 7 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 8 \\ 5 & 9 & 2 \\ 7 & 3 & 6 \end{vmatrix}.$$

Between 0 (see [4]) and 412 there are twenty-four integers with no nine-digit determinant representation. They are:

324, 329, 355, 357, 358, 362, 364, 365, 367, 373, 375, 378,

381, 383, 386, 387, 394, 397, 399, 401, 403, 406, 409, 411.

There are eight integers with more than forty parents:

9 and 33 with 41 parents each,

11 and 65 with 42 parents each,

55 with 43 parents,

27 with 45 parents,

5 with 47 parents,

and 45 with 48 parents. [1]

We start the enumeration of the 48 parents of that well-disciplined child, 45. The following six collections of two or three parents exemplify determinants invariant under element interchange [5]. In each collection, the first rows are identical and each succeeding parent is obtained from the preceding by the interchange of a single digit pair:

$$45 = -\begin{vmatrix}1&2&3\\4&9&7\\6&5&8\end{vmatrix} = -\begin{vmatrix}1&2&3\\4&9&5\\6&7&8\end{vmatrix} = -\begin{vmatrix}1&2&3\\5&9&4\\6&7&8\end{vmatrix} \qquad 45 = \begin{vmatrix}1&3&6\\8&2&5\\9&4&7\end{vmatrix} = \begin{vmatrix}1&3&6\\9&2&5\\8&4&7\end{vmatrix}$$

$$= -\begin{vmatrix}1&3&4\\8&5&2\\6&7&9\end{vmatrix} = -\begin{vmatrix}1&3&4\\8&9&2\\6&7&5\end{vmatrix} \qquad = \begin{vmatrix}1&3&8\\5&2&7\\6&4&9\end{vmatrix} = \begin{vmatrix}1&3&8\\6&2&7\\5&4&9\end{vmatrix}$$

$$= \begin{vmatrix}1&3&4\\6&2&5\\8&7&9\end{vmatrix} = \begin{vmatrix}1&3&4\\6&2&7\\8&5&9\end{vmatrix}, \qquad = -\begin{vmatrix}1&4&6\\8&2&3\\7&5&9\end{vmatrix} = \begin{vmatrix}1&4&6\\8&2&3\\5&7&9\end{vmatrix}.$$

Interchanges in two digit pairs accomplish the conversions in the following four couples of parents. The first three conversions involve two independent operations done simultaneously, while in the last conversion the two interchanges are successive:

$$45 = \begin{vmatrix}1&2&3\\5&4&8\\6&9&7\end{vmatrix} = \begin{vmatrix}1&2&3\\6&4&7\\5&9&8\end{vmatrix} \qquad 45 = \begin{vmatrix}1&2&9\\5&3&4\\8&6&7\end{vmatrix} = -\begin{vmatrix}1&2&9\\3&5&4\\6&8&7\end{vmatrix}$$

$$= \begin{vmatrix}1&2&4\\6&3&9\\7&5&8\end{vmatrix} = \begin{vmatrix}1&2&4\\9&3&6\\8&5&7\end{vmatrix}, \qquad = \begin{vmatrix}1&2&6\\5&7&3\\4&8&9\end{vmatrix} = -\begin{vmatrix}1&2&6\\3&7&8\\4&5&9\end{vmatrix}.$$

In the following three couples of parents, the first conversion is by a cyclic permutation of the elements in one row; the other two are by cyclic permutation of the elements in two rows:

$$45 = -\begin{vmatrix}1&2&8\\3&6&9\\4&5&7\end{vmatrix} = -\begin{vmatrix}1&2&8\\3&6&9\\5&7&4\end{vmatrix}$$

$$= -\begin{vmatrix}1&2&3\\5&6&4\\9&7&8\end{vmatrix} = \begin{vmatrix}1&2&3\\6&4&5\\8&9&7\end{vmatrix}$$

$$= -\begin{vmatrix}1&2&5\\3&9&6\\4&7&8\end{vmatrix} = \begin{vmatrix}1&2&5\\6&3&9\\8&4&7\end{vmatrix}.$$

The array in the last determinant above is an antimagic square with the eight straight line sums 8, 9, 11, 15, 16, 18, 19, and 21. The only other parent having this value that has an antimagic array is

$$45 = \begin{vmatrix} 1 & 2 & 9 \\ 4 & 3 & 7 \\ 8 & 6 & 5 \end{vmatrix},$$

with straight line sums of 9, 11, 12, 13, 14, 19, 20, and 21. Completing the group of 48 parents are:

$$45 = \begin{vmatrix} 1 & 2 & 3 \\ 8 & 4 & 9 \\ 7 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 5 & 3 & 8 \\ 6 & 7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 9 & 6 & 3 \\ 5 & 7 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 6 & 3 & 4 \\ 8 & 7 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 5 \\ 4 & 3 & 8 \\ 6 & 7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 8 & 3 & 7 \\ 9 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 6 \\ 4 & 3 & 5 \\ 9 & 8 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 7 \\ 3 & 4 & 8 \\ 5 & 9 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 7 \\ 6 & 3 & 9 \\ 5 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 9 \\ 6 & 3 & 4 \\ 7 & 5 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 9 \\ 3 & 5 & 8 \\ 6 & 7 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 4 \\ 5 & 6 & 2 \\ 8 & 9 & 7 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 5 \\ 7 & 6 & 2 \\ 8 & 9 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 5 \\ 9 & 2 & 6 \\ 8 & 4 & 7 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 6 \\ 9 & 4 & 2 \\ 8 & 5 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 5 \\ 7 & 8 & 2 \\ 6 & 9 & 3 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 4 & 5 \\ 9 & 7 & 2 \\ 8 & 6 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 6 \\ 5 & 2 & 3 \\ 9 & 7 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 6 \\ 7 & 3 & 2 \\ 5 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 7 \\ 5 & 3 & 2 \\ 9 & 8 & 6 \end{vmatrix}.$$

Thus there are 48•72 or 3456 (note the consecutive digit sequence) square arrays of the nine nonzero digits with determinants of absolute value 45. Half of these determinants are positive and half negative, so their sum is zero.

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THE OLYMPIAD CORNER: 8

MURRAY S. KLAMKIN

In *The Olympiad Corner*: 3 [1979:62-69], there were brief descriptions of mathematical competitions being held in Canada and the U.S.A. There are many more such contests; these will be listed and described in a subsequent issue. But we start this month giving broad coverage to the field with a bibliography of articles and publications which treat mathematical competitions. A number of these references (marked with *) are good sources of challenging problems. If a journal is not available for xeroxing, you may be able to get reprints, if still available, by writing to the authors. Addresses may be obtained from the NCTM directory and the Combined Membership List of the American Mathematical Society, Mathematical Association of America, and Society for Industrial and Applied Mathematics. The NML publications are obtainable from the Mathematical Association of America (members receive discounts), the NCTM publications are available from the National Council of Teachers of Mathematics (also discounts for members), and the Canadian Mathematical Olympiads are available from the Canadian Mathematical Society. Our coverage of the field will continue, in a coming issue, with an extensive list of problem books, problem-solving books, and other books which are recommended for students and teachers wishing to supplement their regular mathematical texts.

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*

No Olympiad Corner would be complete without a few problems to chew on. We give this month the problems proposed at the 15th British Mathematical Olympiad, held on 4 March 1979. Solutions will appear next month in this column. We are grateful to Willie S.M. Yong of South Wales, one of our readers, for sending us these problems and their official solutions.

15TH BRITISH MATHEMATICAL OLYMPIAD (1979)

Time: 3½ hours

1. Find all triangles ABC for which

$$AB + AC = 2 \text{ cm} \quad \text{and} \quad AD + BC = \sqrt{5} \text{ cm},$$

where AD is the altitude through A, meeting BC at right angles in D.

2. From a point O in 3-dimensional space, three given rays OA, OB, OC emerge, the angles BOC, COA, AOB being α, β, γ, respectively ($0 < \alpha,\beta,\gamma < \pi$).

Prove that, given $2s > 0$, there are unique points X, Y, Z on OA, OB, OC, respectively, such that the triangles YOZ, ZOX, and XOY have the same perimeter $2s$, and express OX in terms of s and $\sin\frac{1}{2}\alpha$, $\sin\frac{1}{2}\beta$, $\sin\frac{1}{2}\gamma$.

3. S is a set of distinct positive odd integers $\{a_i\}$, $i=1, \ldots, n$. No two differences $|a_i - a_j|$ are equal, $1 \le i < j \le n$. Prove that

$$\sum_{i=1}^{n} a_i \ge \frac{1}{3}n(n^2+2).$$

4. The function f is defined on the rational numbers and takes only rational values. Prove that f is constant if, for all rational x and y,

$$f(x+f(y)) = f(x)f(y).$$

5. For n a positive integer, denote by $p(n)$ the number of ways of expressing n as the sum of one or more positive integers. Thus $p(4) = 5$, because there are 5 different sums, namely,

$$1+1+1+1, \quad 1+1+2, \quad 1+3, \quad 2+2, \quad 4.$$

Prove that, for $n > 1$,

$$p(n+1) - 2p(n) + p(n-1) \ge 0.$$

6. Prove that in the infinite sequence of integers

10001, 100010001, 1000100010001, ...

there is no prime number.

Note that each integer after the first (ten thousand and one) is obtained by adjoining 0001 to the digits of the previous integer.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before January 1, 1980, although solutions received after that date will also be considered until the time when a solution is published.

471. *Proposed by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.*

Restore the digits in this multiplication, based on a sign in a milliner's shop window:

$$\begin{array}{r} \mathrm{WE} \\ \underline{\mathrm{DO}} \\ \mathrm{TAM} \\ \underline{\mathrm{HAT}} \\ \mathrm{TRIM} \end{array}$$

472. *Proposé par Jordi Dou, Escola Tecnica Superior Arquitectura de Barcelona, Espagne.*

Construire un triangle connaissant le côté b, le rayon R du cercle circonscrit, et tel que la droite qui joint les centres des cercles inscrit et circonscrit soit parallèle au côté a.

473.* *Proposed by A. Liu, University of Regina.*

The set of all positive integers is partitioned into the (disjoint) subsets $T_1, T_2, T_3, \ldots$ as follows: for each positive integer m, we have $m \in T_k$ if and only if k is the largest integer such that m can be written as the sum of k distinct elements from one of the subsets. Prove that each T_k is finite.

(This is a variant of Crux 342 [1978: 133, 297].)

474. *Proposed by James Propp, Harvard College, Cambridge, Massachusetts.*

Suppose (s_n) is a monotone increasing sequence of natural numbers satisfying $s_{s_n} = 3n$ for all n. Determine all possible values of s_{1979}.

475. *Proposed by Hayo Ahlburg, Benidorm, Spain.*

Consider the products

$$(341+\tfrac{2}{3})(205-\tfrac{2}{5}) = 341\cdot 205, \qquad (43+\tfrac{2}{5})(31-\tfrac{2}{7}) = 43\cdot 31,$$

$$(781+\tfrac{1}{2})(521-\tfrac{1}{3}) = 781\cdot 521, \qquad (57+\tfrac{1}{3})(43-\tfrac{1}{4}) = 57\cdot 43.$$

Find an infinite set of products having the same property.

476. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Construct an isosceles right triangle such that the three vertices lie each on one of three concurrent lines, the vertex of the right angle being on the inside line.

477. *Proposé par Hippolyte Charles, Waterloo, Québec.*

Pour $n = 0, 1, 2, \ldots$, évaluer l'intégrale

$$I_n = \int_0^{\pi} \frac{\cos nx}{5 - 4\cos x}\, dx.$$

478. *Proposed by Murray S. Klamkin, University of Alberta.*

Consider the following theorem:

If the circumcircles of the four faces of a tetrahedron are mutually congruent, then the circumcentre O *of the tetrahedron and its incentre* I *coincide.*

An editor's comment following Crux 330 [1978: 264] claims that the proof of this theorem is "easy". Prove it.

479. *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

A car, of wheelbase L, makes a left turn in such a way that the locus of

A, the point of contact of the left front wheel, is a circle of radius $R > L$. B is the point of contact of the left rear wheel. Before the turn, the car was travelling in a straight line with A moving toward the circle along a tangent. Find the locus of B.

480. *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

In a Cartesian plane let l_1 and l_2 be two nonparallel lines intersecting in a point P and $Q(x_1,y_1)$ a point distinct from P. Let l be a line which does not pass through either P or Q, is not parallel to PQ, and intersects PQ at the point $R(x_2,y_2)$.

If $ax+by = c$, $a_1x+b_1y = c_1$, and $a_2x+b_2y = c_2$ are equations for l, l_1, and l_2, respectively, find, as simply as possible, the coordinates of R in terms of

$$a,\ b,\ c;\quad a_1,\ b_1,\ c_1;\quad a_2,\ b_2,\ c_2;\quad \text{and}\quad x_1,\ y_1.$$

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

394. [1978: 283] *Proposed by Harry D. Ruderman, Hunter College Campus School, New York.*

A wine glass has the shape of an isosceles trapezoid rotated about its axis of symmetry. If R, r, and h are the measures of the larger radius, smaller radius, and altitude of the trapezoid, find $r:R:h$ for the most economical dimensions.

Solution by G.P. Henderson, Campbellcroft, Ontario.

Let θ be the upper base angle of the trapezoid. We assume for now that $r > 0$ and $0 < \theta < \pi/2$; we will investigate later the possibilities $r = 0$ and $\theta = \pi/2$. We interpret "most economical" as meaning "yielding minimal surface area A for a given volume V."

We first find

$$h = (R-r)\tan\theta\ , \tag{1}$$

$$V = \frac{\pi}{3}(R^3 - r^3)\tan\theta\ , \tag{2}$$

and

$$A = \pi[r^2 + (R^2 - r^2)\sec\theta]\ . \tag{3}$$

For fixed θ, (2) defines R as a function of r for which $dR/dr = r^2/R^2$ and then, from (3),

$$\frac{dA}{dr} = 2\pi \frac{r}{R} \sec\theta \, [r - R(1 - \cos\theta)] \,. \tag{4}$$

This derivative vanishes if and only if

$$\frac{r}{R} = 1 - \cos\theta \,, \tag{5}$$

and then (2) and (5) determine unique values of r and R. To prove that A is actually a minimum for these values, it is sufficient to verify that the last factor in (4) changes sign from negative to positive as r increases through its critical value; but this follows from

$$\frac{d}{dr}[r - R(1 - \cos\theta)] = \left(1 - \frac{r^2}{R^2}\right) + \frac{r^2}{R^2}\cos\theta > 0.$$

Using (5), we now find from (2) and (3) that

$$\frac{A^3}{9\pi V^2} = \frac{3 - 3\cos\theta + \cos^2\theta}{\sin^2\theta} \,, \tag{6}$$

where A now stands for the minimum area for a given θ. Differentiating (6) with respect to θ, we find

$$\operatorname{sgn}(dA/d\theta) = \operatorname{sgn}(3\cos^2\theta - 8\cos\theta + 3), \tag{7}$$

which vanishes if and only if

$$\cos\theta = \frac{4 - \sqrt{7}}{3},$$

that is, for $\theta \approx 63.2^\circ$. Since it is clear from (7) that $dA/d\theta$ changes sign from negative to positive as θ increases through its critical value, the area is a minimum at this point. From (1) and (5), we now find

$$r : R : h = 1 : \frac{1}{1 - \cos\theta} : \frac{\sin\theta}{1 - \cos\theta} = 1 : \frac{1 + \sqrt{7}}{2} : \sqrt[4]{7}, \tag{8}$$

that is, $r : R : h \approx 1 : 1.82 : 1.63$. When $r > 0$ and $0 < \theta < \pi/2$, these dimensions will yield minimum area A, which can then be calculated, with the aid of (6), from

$$\frac{A^3}{9\pi V^2} = 1 + \frac{\sqrt{7}}{2} \,.$$

When $r = 0$ (and, necessarily, $0 < \theta < \pi/2$), we have $A^3/9\pi V^2 = 1/\sin^2\theta\cos\theta$, which takes its minimum value $3\sqrt{3}/2 > 1 + \sqrt{7}/2$ when $\cos\theta = 1/\sqrt{3}$. Finally, when $\theta = \pi/2$

(and, necessarily, $r > 0$), we have $A = \pi r^2 + 2V/r$; this attains its minimum when $r = \sqrt[3]{V/\pi}$, and then $A^3/9\pi V^2 = 3 > 1 + \sqrt{7}/2$.

We conclude that the most economical dimensions are those given by (8).

Also solved by FRIEND H. KIERSTEAD, Cuyahoga Falls, Ohio; and the proposer.

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395. [1978: 283] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

In Crux 247 [1977: 131; 1978: 23, 37] the following inequality is proved: if $0 < a_1 \le a_2 \le \ldots \le a_n$, then

$$\frac{1}{2n^2}\,\frac{\sum\limits_{1\le i<j\le n}(a_i - a_j)^2}{a_n} \le A - G \le \frac{1}{2n^2}\,\frac{\sum\limits_{1\le i<j\le n}(a_i - a_j)^2}{a_1},$$

where A (resp. G) is the arithmetic (resp. geometric) mean of $a_1, \ldots, a_n$. This is a refinement of the familiar inequality $A \ge G$. If H denotes the harmonic mean of $a_1, \ldots, a_n$, that is,

$$\frac{1}{H} = \frac{1}{n}\left\{\frac{1}{a_1} + \ldots + \frac{1}{a_n}\right\},$$

find the corresponding refinement of the familiar inequality $G \ge H$.

Solution by the proposer.

We apply the generalized A.M.-G.M. inequality mentioned in the proposal to the increasing sequence

$$\left(\frac{1}{a_n}, \frac{1}{a_{n-1}}, \ldots, \frac{1}{a_1}\right).$$

We obtain (summations throughout are for $1 \le i < j \le n$)

$$\frac{a_1}{2n^2}\sum\left(\frac{1}{a_i} - \frac{1}{a_j}\right)^2 \le \frac{1}{H} - \frac{1}{G} \le \frac{a_n}{2n^2}\sum\left(\frac{1}{a_i} - \frac{1}{a_j}\right)^2$$

and multiplying throughout by GH gives

$$\frac{a_1 GH}{2n^2}\sum\frac{(a_i - a_j)^2}{a_i^2 a_j^2} \le G - H \le \frac{a_n GH}{2n^2}\sum\frac{(a_i - a_j)^2}{a_i^2 a_j^2}.$$

Now, as $a_1 \le H \le G \le a_n$, we have

$$\frac{a_1 GH}{2n^2}\sum\frac{(a_i - a_j)^2}{a_i^2 a_j^2} \ge \frac{a_1 H^2}{2n^2 a_n^4}\sum(a_i - a_j)^2 \ge \frac{1}{2n^2}\cdot\frac{a_1^3}{a_n^4}\sum(a_i - a_j)^2$$

and

$$\frac{a_n GH}{2n^2}\sum\frac{(a_i-a_j)^2}{a_i^2a_j^2} \le \frac{a_n G^2}{2n^2a_1^4}\sum(a_i-a_j)^2 \le \frac{1}{2n^2}\cdot\frac{a_n^3}{a_1^4}\sum(a_i-a_j)^2 .$$

Thus the required inequality is

$$\frac{1}{2n^2}\cdot\frac{a_1^3}{a_n^4}\sum(a_i-a_j)^2 \le G - H \le \frac{1}{2n^2}\cdot\frac{a_n^3}{a_1^4}\sum(a_i-a_j)^2 .$$

Also solved by MURRAY S. KLAMKIN, University of Alberta.

* * *

396. [1978: 283] *Proposed by Viktors Linis, University of Ottawa.*

Given is the following polynomial with some undetermined coefficients denoted by stars:

$$x^{10} + {*}x^9 + {*}x^8 + \ldots + {*}x^2 + {*}x + 1.$$

Two players, in turn, replace one star by a real number until all stars are replaced. The first player wins if all zeros of the polynomial are imaginary, the second if at least one zero is real. Is there a winning strategy for the second player?

Editor's comment.

Our proposer found this problem in the Russian journal *Kvant* (Problem 458), where it was proposed by D. Bernshtein. The proposer's solution which I give below was edited from a translation supplied by V. Linis. It appeared in *Kvant*, No. 6, 1978, pp.44-45.

Solution by D. Bernshtein (proposer).

There is indeed a winning strategy for the second player.

If the first player, A, chooses the coefficient of an even-degree term, the second player, B, selects that of an odd-degree term, and vice versa. After seven moves, at least one of the remaining two terms is of odd degree, since there are five odd-degree terms.

It is now B's turn to play. Let the given polynomial, as it now stands, be

$$f(x) = P(x) + \alpha x^k + \beta x^l,$$

where α and β are the remaining two stars and l is odd.

(a) If k is even, then

$$f(1) + f(-1) = P(1) + P(-1) + 2\alpha$$

and B selects $\alpha = -\frac{1}{2}[P(1)+P(-1)]$, which results in $f(1)+f(-1) = 0$. Thus, either $f(1) = f(-1) = 0$ and there are two real roots, or else $f(1)$ and $f(-1)$ have opposite signs and there is (by continuity) a real root in the interval $(-1, 1)$.

(b) If k is odd, then

$$2^l f(-1) + f(2) = 2^l P(-1) + P(2) + (2^k - 2^l)\alpha$$

and B selects

$$\alpha = -\frac{2^l P(-1) + P(2)}{2^k - 2^l},$$

which results in $2^l f(-1) + f(2) = 0$. Thus, as before, either $f(-1) = f(2) = 0$ or there is a real root in the interval $(-1, 2)$.

A should now realize that he is "checkmated" and resign without playing his last move.

Also solved by MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; MURRAY S. KLAMKIN, University of Alberta; F.G.B. MASKELL, Algonquin College, Ottawa; and LEROY F. MEYERS, The Ohio State University. One incorrect solution was received.

Editor's comment.

It is obvious, as Meyers pointed out, that the same strategy applies if the given polynomial is of any even degree ≥ 4. This is one of the rare (to the editor's knowledge) games where one player plays both first *and* last, and still cannot be assured of a win.

Our incorrect solver apparently misread the problem and gave what purported to be a winning strategy for the *first* player. That player is in for a big surprise if the second player adopts the strategy described above. He had better not bet too heavily on the game.

* * *

397. [1978: 283] *Proposed by Jack Garfunkel, Forest Hills H.S., Flushing, N.Y.*

Given is ΔABC with incenter I. Lines AI, BI, CI are drawn to meet the incircle (I) for the first time in D, E, F respectively. Prove that

$$(AD + BE + CF)\sqrt{3}$$

is not less than the perimeter of the triangle of maximum perimeter that can be inscribed in circle (I).

I. *Solution by Leon Bankoff, Los Angeles, California.*

Since the triangle of maximum perimeter that can be inscribed in circle (I) is equilateral (see, e.g., Kazarinoff [1, p.100]), with perimeter $3\sqrt{3}r$, where r is the radius of the circle, the problem reduces to showing that

$$AD + BE + CF \geq 3r \qquad (1)$$

or, equivalently, that

$$AI + BI + CI \geq 6r. \qquad (2)$$

But (2) follows at once from the Erdős-Mordell inequality [1, p.78].

II. *Solution by Roland H. Eddy, Memorial University of Newfoundland.*

[As in solution I], the problem is equivalent to (1). Since $AD = r\csc(A/2) - r$, etc., we have

$$AD + BE + CF = r\left(\csc\frac{A}{2} + \csc\frac{B}{2} + \csc\frac{C}{2} - 3\right),$$

and the inequality

$$\csc\frac{A}{2} + \csc\frac{B}{2} + \csc\frac{C}{2} \geq 6,$$

which can be found in Bottema [2], immediately yields (1).

Also solved by HAYO AHLBURG, Benidorm, Spain (partial solution); ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, New Jersey; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; and the proposer.

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* * *

398. [1978: 284] *Proposed by Murray S. Klamkin, University of Alberta.*

Find the roots of the $n \times n$ determinantal equation

$$\left| \frac{1}{x\delta_{rs} + k_r} \right| = 0$$

where δ_{rs} is the Kronecker delta (= 1 or 0 according as $r = s$ or $r \neq s$).

Solution by G.P. Henderson, Campbellcroft, Ontario (revised by the editor).

There is no solution if $n = 1$, so we assume $n > 1$. The equation is then meaningful only if $k_r \neq 0$ for $r = 1, \ldots, n$, and a number x is a root only if

$$x \neq -k_r, \qquad r = 1, \ldots, n. \qquad (1)$$

Let Δ be the determinant in the proposal. If, for $r = 1, \ldots, n$, we multiply the rth row of Δ by the nonzero number $k_r(x+k_r)$, we obtain

$$k_1 \ldots k_n(x+k_1)\ldots(x+k_n)\Delta = |x(1-\delta_{rs}) + k_r|. \tag{2}$$

To evaluate the determinant on the right of (2), we subtract the first column from every other column and then we add all the other rows to the first row. Equation (2) then becomes

$$k_1 \ldots k_n(x+k_1)\ldots(x+k_n)\Delta = (-x)^{n-1}[(n-1)x + \Sigma k_j]. \tag{3}$$

(Summations throughout are for $j = 1, \ldots, n$.) We now consider two cases:

(a) Suppose

$$\Sigma k_j \neq (n-1)k_r, \qquad r = 1, \ldots, n. \tag{4}$$

The equation $\Delta = 0$ is equivalent to

$$x^{n-1}[(n-1)x + \Sigma k_j] = 0.$$

If $\Sigma k_j = 0$, then $x = 0$ is a root of multiplicity n which satisfies (1); otherwise $x = 0$ is a root of multiplicity $n-1$ and the remaining root is $x = -(\Sigma k_j)/(n-1)$ which, in view of (4), also satisfies (1).

(b) Suppose, on the contrary, that there exists an integer ρ, $1 \le \rho \le n$, such that

$$\Sigma k_j = (n-1)k_\rho.$$

Then $(n-1)x + \Sigma k_j = (n-1)(x+k_\rho)$ and the factor $x + k_\rho$ can be cancelled from both sides of (3), yielding

$$k_1 \ldots k_n(x+k_1)\ldots(\qquad)\ldots(x+k_n)\Delta = (-x)^{n-1}(n-1),$$

where the empty parentheses indicate the absence of the cancelled factor. The equation $\Delta = 0$ is now equivalent to the equation $x^{n-1} = 0$, which has only the root $x = 0$, of multiplicity $n-1$.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; G.C. GIRI, Research Scholar, Indian Institute of Technology, Kharagpur, India; ALLAN WM. JOHNSON JR., Washington, D.C.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

Editor's comment.

The equation in this problem contained the n parameters $k_1, \ldots, k_n$. The editor would like to remind readers again (see [1979: 178]) that a solution, to be considered complete, must include an exhaustive discussion of the various solution sets which can arise, depending on the values of the parameters. This is inherent in the meaning of "solving an equation", not a mere caprice of the editor. Of the solutions received, none contained a parametric discussion that was really complete.

* * *

399. [1978: 284] *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

A *prime magic square of order* 3 is a square array of 9 distinct primes in which the three rows, three columns, and two main diagonals all add up to the same *magic constant*. What prime magic square of order 3 has the smallest magic constant

(a) when the 9 primes are in arithmetic progression?

(b) when they are not?

I. *Solution by Charles W. Trigg, San Diego, California.*

The basic theory of third-order magic squares is simple and complete, and there is an abundant literature on the subject (see, e.g., Kraitchik [5]). The fundamental result is the following: *Nine distinct numbers are capable of forming a magic square if and only if they form three 3-term arithmetic progressions having the same common difference and whose first terms are in A.P. If these conditions are satisfied, there is essentially only one (i.e., to within rotations and reflections) magic square derivable from these numbers.*

If the three progressions are first displayed as the rows of a 3×3 array, with the A.P. of first terms forming the first column, the magic square can then be obtained by

i) interchanging the elements at the extremities of each diagonal, or those at the extremities of each bimedian, followed by

ii) rotating (clockwise or counterclockwise) the perimeter one space.

This is the *trade 'n' twist* routine [11]. Thus

$$\boxed{\begin{array}{lll} a & a+d & a+2d \\ \\ a+k & a+k+d & a+k+2d \\ \\ a+2k & a+2k+d & a+2k+2d \end{array}} \longrightarrow \boxed{\begin{array}{lll} a+k & a+2k+2d & a+d \\ \\ a+2d & a+k+d & a+2k \\ \\ a+2k+d & a & a+k+2d \end{array}} \tag{1}$$

and the magic constant is three times the middle term of the second progression: $3(a+k+d)$.

(a) For this part of the problem, we need a 9-term prime A.P. with the smallest possible fifth term. It is known (see, e.g., Agnew [1] or Sierpiński [9]) that, for $n > 2$, in any n-term prime A.P. the common difference is divisible by every prime less than n. So, in any 9-term prime A.P., the common difference must be a multiple of $2 \cdot 3 \cdot 5 \cdot 7 = 210$. The one with smallest fifth term is now easily found (see [2], [12]). It is

$$199, \quad 409, \quad 619, \quad 829, \quad 1039, \quad 1249, \quad 1459, \quad 1669, \quad 1879$$

with common difference 210. Now, using the trade 'n' twist technique as in (1), we obtain the answer to our problem, a square with magic constant $3 \cdot 1039 = 3117$:

$$\begin{array}{|ccc|}\hline 199 & 409 & 619 \\ 829 & 1039 & 1249 \\ 1459 & 1669 & 1879 \\ \hline\end{array} \longrightarrow \begin{array}{|ccc|}\hline 829 & 1879 & 409 \\ 619 & 1039 & 1459 \\ 1669 & 199 & 1249 \\ \hline\end{array}.$$

This square has been attributed to Dudeney by Madachy [6], but without reference.

(b) Observe on the left side of (1) that the two diagonals as well as the middle row are arithmetic progressions. Hence, for any third-order magic square, it is necessary to have three 3-term arithmetic progressions with a common middle term and first terms in A.P. We will find three such *prime* progressions, with smallest possible common middle term, that can be used to form a prime magic square. This square will be the answer to our problem. Our task will be facilitated if we know that (3,5,7) is the *only* prime triplet in A.P. with a common difference that is *not* divisible by 6 [10] and it cannot be one of our three progressions.

From a table of primes, we find that there are exactly 33 prime triplets in A.P. having a common difference divisible by 6 and middle term ≤ 59 (we won't need to go beyond 59). From these, we find that there are only 6 trios of progressions with common middle term and first terms in A.P. We list them below, each disposed as the diagonals and middle row of a square array.

$$\begin{array}{|ccc|}\hline 5 & & 41 \\ 11 & 29 & 47 \\ 17 & & 53 \\ \hline\end{array} \qquad \begin{array}{|ccc|}\hline 7 & & 67 \\ 13 & 43 & 73 \\ 19 & & 79 \\ \hline\end{array} \qquad \begin{array}{|ccc|}\hline 5 & & 53 \\ 23 & 47 & 71 \\ 41 & & 89 \\ \hline\end{array}$$

$$\begin{array}{|ccc|}\hline 5 & & 101 \\ 11 & 59 & 107 \\ 17 & & 113 \\ \hline\end{array} \qquad \begin{array}{|ccc|}\hline 11 & & 71 \\ 29 & 59 & 89 \\ 47 & & 107 \\ \hline\end{array} \qquad \begin{array}{|ccc|}\hline 5 & & 89 \\ 17 & 59 & 101 \\ 29 & & 113 \\ \hline\end{array}$$

We can discard the first five, because in each case the arithmetic mean of the corner elements in the bottom row is composite. The last one, however, is satisfactory, since the arithmetic means of the corner elements in the top and bottom rows are primes. We can now trade 'n' twist to obtain the required answer,

$$\begin{array}{|ccc|}\hline 5 & 47 & 89 \\ 17 & 59 & 101 \\ 29 & 71 & 113 \\ \hline\end{array} \longrightarrow \begin{array}{|ccc|}\hline 17 & 113 & 47 \\ 89 & 59 & 29 \\ 71 & 5 & 101 \\ \hline\end{array},$$

a prime magic square with smallest possible magic constant $3 \cdot 59 = 177$.

This square has been previously reported by Rudolf Ondrejka, according to Madachy [7].

II. *Comment by Allan Wm. Johnson Jr., Washington, D.C.*

A friendly computer has provided me with much information about various extensions of this problem. [Some of it is given below. The rest will repose in the editor's archives.]

Comment on part (a).

In any prime magic square of order 3, the nine primes must be all of the form $6k+1$ or else all of the form $6k-1$ (otherwise the magic constant would not be a multiple of 3). The square with magic constant 3117 [given in solution I] has primes in A.P. of the form $6k+1$. The corresponding square with nine primes in A.P. of the form $6k-1$ and smallest possible magic constant is

$$\begin{array}{|ccc|}\hline 10883 & 2063 & 8363 \\ 4583 & 7103 & 9623 \\ 5843 & 12143 & 3323 \\ \hline\end{array}$$

with magic constant 21309. My computer was all set to search for 16 primes in A.P. to form a fourth-order magic square, but it breathed a sigh of relief when I discovered the following information in Madachy [7]: "The longest arithmetic progression of primes known, containing but 16 terms, begins with 2236133941, ends with 5582526991, and has a common difference between terms of 223092870." [Trigg reports that Madachy has actually constructed a magic square with these 16 primes, with magic constant 15637321864. It can be found in [8].]

Comment on part (b).

In the prime magic square with magic constant 177 [given in solution I], the primes are all of the form $6k-1$. The corresponding square with primes of the form $6k+1$ and smallest magic constant is

109	7	103
67	73	79
43	139	37

with magic constant 219. There are two essentially distinct prime magic squares of third order each with magic constant 381, which is the smallest constant for which such equality is possible. They are

211	13	157
73	127	181
97	241	43

and

181	43	157
103	127	151
97	211	73

.

Consider the four squares

$$\begin{array}{|cccc|}\hline A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \\ \hline\end{array} \longrightarrow \begin{array}{|cccc|}\hline A & C & B & D \\ I & K & J & L \\ E & G & F & H \\ M & O & N & P \\ \hline\end{array}, \quad \begin{array}{|cccc|}\hline K & I & L & J \\ C & A & D & B \\ O & M & P & N \\ G & E & H & F \\ \hline\end{array}, \quad \begin{array}{|cccc|}\hline K & L & I & J \\ O & P & M & N \\ C & D & A & B \\ G & H & E & F \\ \hline\end{array}. \qquad (2)$$

They are all essentially distinct and, if the first is a magic square, so are the other three and they all have the same magic constant. There are 16 fourth-order prime magic squares with smallest possible magic constant (equal to 120). These are the four squares given on the next page and the 12 additional squares that result when the transformations (2) are applied to each of these four squares, the first three of which, the reader will note, are different arrangements of the same 16 primes!

3	61	19	37
43	31	5	41
7	11	73	29
67	17	23	13

3	67	7	43
37	31	11	41
19	5	73	23
61	17	29	13

3	73	31	13
43	19	17	41
7	23	61	29
67	5	11	37

5	59	43	13
41	37	19	23
3	17	47	53
71	7	11	31

Also solved by ALAN EDELMAN, student, Canarsie H.S., Brooklyn, N.Y. (part (b) only); ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta (part (a) only); HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; and KENNETH M. WILKE, Topeka, Kansas.

Editor's comment.

As our first solver noted, the basic theory of third-order magic squares is simple and complete. But it is not so simple for squares of higher order, even if one considers only normal magic squares. (A magic square is *normal* if it is made up of the first n^2 natural numbers.) According to the fundamental result enunciated in solution I, there is essentially only one normal magic square of order 3; but for order 4 there are 880 essentially distinct normal magic squares (and every last one of them is listed in [4, pp. 188-198]). The corresponding number for higher orders is not known.

There are cogent reasons for not including 1 among the primes, but some people persist in doing so (especially among the older devotees of magic squares). If 1 is considered a prime, the answers to part (b) of our problem for orders 3 and 4 would be

67	1	43
13	37	61
31	73	7

Magic sum 111

and

3	71	5	23
53	11	37	1
17	13	41	31
29	7	19	47

Magic sum 102

.

The first is due to H.E. Dudeney and the second to E. Bergholt and C.D. Shuldham [3]. Note, however, that the second is not the only one with magic sum 102: three more can be found by using transformations (2).

No magic square of any order is known that consists entirely of *consecutive* primes. But if 1 is considered a prime, J.N. Muncey showed in 1913 that the smallest magic square that can be constructed of consecutive *odd* primes is one of the twelfth order, consisting of the primes 1, 3, 5, 7, 11, ..., 827. This information, and the extraordinary square itself, can be found in [4, pp. 36-37].

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12. ________ and Bob Prielipp, Solution of Problem 3544, *School Science and Mathematics*, 75 (February 1975) 199-200.

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400. [1978: 284] *Proposed by Andrejs Dunkels, University of Luleå, Sweden.*

In the false bottom of a chest which had belonged to the notorious pirate Capt. Kidd was found a piece of parchment with instructions for finding a treasure buried on a certain island. The essence of the directions was as follows.

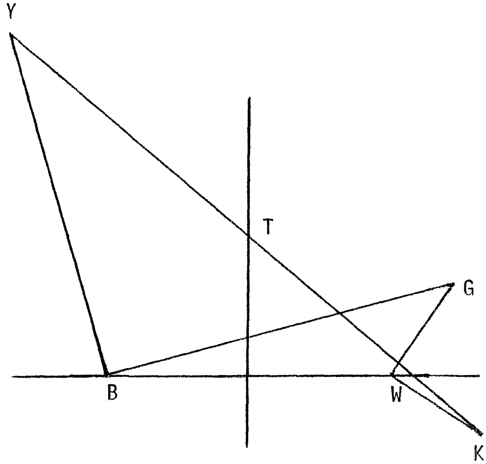
"Start from the gallows and walk to the white rock, counting your paces. At the rock turn left through a right angle and walk the same number of paces. Mark the spot with your knife. Return to the gallows. Count your paces to the black rock, turn right through a right angle and walk the same distance. The treasure is midway between you and the knife."

However, when the searchers got to the island they found the rocks but no trace of the gallows remained. After some thinking they managed to find the treasure anyway. How?

(This problem must be very old. I heard about it in my first term of studies at Uppsala.)

I. *Solution by The Pentagon Problems Group, Washington, D.C.*

Imbed Treasure Island in a complex plane so that the affixes of G (gallows), W (white rock), and B (black rock) are z, 1, and -1, respectively (see figure). Then the affix of K (knife) is



$$z_1 = 1 - i(z - 1)$$

and that of Y (you) is

$$z_2 = -1 + i(z + 1).$$

Hence the affix of T (treasure) is $\frac{1}{2}(z_1 + z_2) = i$.

So, to find the treasure without the aid of the gallows, go to the black rock, walk half the distance to the white rock, turn left through a right angle, walk the same half-distance again, and start digging.

II. *Solution by M.S. Klamkin, University of Alberta.*

More generally, suppose the turns are through an angle θ instead of a right angle and, [in the notation of solution I], WK = mGW and BY = mGB. Then

$$z_1 = 1 - me^{i\theta}(z - 1), \qquad z_2 = -1 + me^{i\theta}(z + 1)$$

and the affix of T is $\frac{1}{2}(z_1 + z_2) = me^{i\theta}$, so the treasure can be found without the aid of the gallows.

Also solved by HAYO AHLBURG, Benidorm, Spain (two solutions); W.J. BLUNDON, Memorial University of Newfoundland; O. BOTTEMA, Delft, The Netherlands; LOU CAIROLI, student-pirate, Kansas State University, Manhattan, Kansas; STEVE CURRAN for the Beloit College Solvers, Beloit, Wisconsin; J.D. DIXON, Haliburton Highlands Secondary School, Haliburton, Ontario; CLAYTON W. DODGE, University of Maine at Orono; G.C. GIRI, Research Scholar, Indian Institute of Technology, Kharagpur, India; T.J. GRIFFITHS, A.B. Lucas Secondary School, London, Ontario; G.P. HENDERSON, Campbellcroft, Ontario; ERIC INCH, Algonquin College, Ottawa; FRIEND H. KIERSTEAD, Cuyahoga Falls, Ohio; ANDRÉ LADOUCEUR, École secondaire De La Salle, Ottawa;

ANDY LIU, University of Regina; HERMAN NYON, Paramaribo, Surinam; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, New Jersey; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; KENNETH M. WILKE, Topeka, Kansas (two solutions); and the proposer (three solutions). Comments were received from F.J. PAPP, University of Lethbridge, Alberta; and L.J. UPTON, Mississauga, Ontario.

Editor's comment.

When you leave Treasure Island with the recovered sea-chest full of pieces of eight, run the Jolly Roger up the mast, and then head for the Admiral Benbow Inn, where you and your friends can celebrate by quaffing and singing:

Fifteen men on the dead man's chest —
Yo-ho-ho, and a bottle of rum!

This problem has been island-hopping for more than 30 years; and yet its popularity continues undiminished, judging from the number of responses it elicited from our readers. Various readers reported its appearance in more than a dozen different places, but I'll only mention two of these. The problem apparently first appeared in George Gamow's book *One Two Three ... INFINITY*, first published in 1947 by Viking Press (and subsequently reprinted many times by the New American Library as a Mentor book). Its latest appearance was in Martin Gardner's "Mathematical Games" column in *Scientific American* (August 1979 issue, solution in September 1979 issue).

* * *

A NOTE ON PRIMES IN A.P.

This note contains information which came to light too late to be included with the solution of Problem 399 (pages 237-242 in this issue). All of it was garnered from Wagstaff [1], where additional references can be found for the statements made below.

The statement made on our page 239 that the longest prime A.P. known contains only 16 terms is no longer true. S. Weintraub found one of 17 terms in 1977. The first term is 3430751869 and the common difference is 87297210. It is still not known if there are arbitrarily long arithmetic progressions of primes. An affirmative answer to this question is implied by an open problem proposed by P. Erdős, for the solution of which he has offered a prize of $3000.

The editor

REFERENCE

1. Samuel S. Wagstaff, Jr., "Some Questions About Arithmetic Progressions", *American Mathematical Monthly*, 86 (August-September 1979) 579-582.