Mathematical Excalibur

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Olympiad Corner

Below are the problems of the *2015 International Mathematical Olympiad* held in July 10-11, 2015.

Problem 1. We say that a finite set $\boldsymbol{S}$ of points in the plane is *balanced* if, for any two different points A and B in $\boldsymbol{S}$, there is a point C in $\boldsymbol{S}$ such that $AC=BC$. We say that $\boldsymbol{S}$ is *center-free* if for any three different points A, B and C in $\boldsymbol{S}$, there is no point P in $\boldsymbol{S}$ such that $PA=PB=PC$.

(a) Show that for all integers $n\geq 3$, there exists a balanced set consisting of n points.

(b) Determine all integers $n\geq 3$ for which there exists a balanced center-free set consisting of n points.

Problem 2. Determine all triples (a,b,c) of positive integers such that each of the numbers

$$ab-c,\ \ bc-a,\ \ ca-b$$

is a power of 2.

(A power of 2 is an integer of the form 2^n, where n is a non-negative integer.)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 27, 2015***.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO 2015 – Problem Report

Law Ka Ho

IMO 2015 was held in Chiang Mai, Thailand from July 4 to 16. The examinations were held in the mornings of July 10 and 11 (contestants unable to adhere to this schedule with religious reasons were allowed to be quarantined in the day and sit the Day 2 paper after sunset). The Hong Kong team was consisted of the following students:

CHEUNG Wai Lam (Queen Elizabeth School, Form 5)
KWOK Man Yi (Baptist Lui Ming Choi College, Form 4)
LEE Shun Ming Samuel (CNEC Christian College, Form 4)
TUNG Kam Chuen (La Salle College, Form 6)
WU John Michael (Hong Kong International School, Form 4)
YU Hoi Wai (La Salle College, Form 4)

Cheung and Yu were in the IMO team last year, while the rest are first-timers.

Since Hong Kong will host IMO 2016, we sent a total of 14 observers in addition to the contestants, the leader and the deputy leader.

The following consists mainly of the discussions of the problems, marking schemes, performance etc., rather than of the solutions. The problems can be found from the Olympiad Corner in this issue. (Some readers may want to try the problems before reading this section.)

Problem 1. This is quite a standard question in combinatorial geometry. Clearly odd polygons would work for both (a) and (b). The construction for even n in (a) would take some effort, although there were a number of ways to get it done. In (b), the proof that even n does not work involves a standard double counting technique. The Hong Kong team did very well in this question, with five perfect scores plus a 6 out of 7. This question allows partial progress to various degrees. One may complete the whole question. Those who didn't may just figure out the odd polygons, or in addition they could complete the rest of either part (a) or (b). This is better than an all-or-nothing problem. (The marking scheme does not require students to give any proof that their constructions are balanced and/or centre-free.)

Students raised quite a lot of queries on this question during the contest. The most popular question was whether the point C has to be unique. There were also questions like whether the points must be lattice points, and whether the points A, B, C could be collinear.

Problem 2. This looks like a typical number theory problem. The problem is easy to understand. However, all known solutions involve a heavy amount of considerations of different cases, and very limited number theory techniques were involved. It ended up more like an algebra problem, where one deals with the different algebraic expressions by inequality bounds and so on.

Although the known solutions were not particularly elegant, the answers turned out to be surprisingly nice. While most contestants would get (2,2,2) and (2,2,3) (and its permutations) by trial-and-error or whatever methods, there are two other sets of solutions (3,5,7) and (2,6,11) (and their permutations).

The problem was much more difficult than imagined. Very few students managed to get a complete solution, even among the strongest teams. Most of our team members obtained partial results on this one. The question also killed a lot of the contestants' time, leaving them with little time for the last problem of Day 1.

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During the problem selection, there were discussions of whether the note defining what a power of 2 is should be included. Some leaders felt that this destroyed the beauty and elegance of the paper. Some others insisted that it should be there because it would otherwise lead to heaps of questions as for whether 1 is a power of 2. Some even said that in their countries a power of 2 would mean 2 to the power 2 or above!

While discussing the marking scheme, it was decided that no penalty would be levied on students who forgot to list the permutations. In other words, one would not be penalized for saying that there are in total four solutions, namely, (2,2,2), (2,2,3), (3,5,7) and (2,6,11). I also asked for clarification whether points would be deducted for not checking the solutions satisfied the conditions of the problem. The answer was negative.

Problem 3. This is again a difficult question, even for members of some strong teams. Of course, as previously mentioned, most students spent a lot of time dealing with the different cases in Problem 2. So they simply did not have much time left for this one. This could be one of the important reasons for the general poor performance. However, our deputy leader pointed out there is a very simple solution using inversion. Interested readers may wish to try it out.

During problem selection, there had been discussions of whether there should be a note (as in Problem 2) explaining what orthocenter means. It was eventually decided that such note should not appear in the question paper. During the contest, when a question on the meaning of the orthocenter arrived, the leader of UK shouted *"Finally!"*.

Another issue is the possibility of having two different configurations. To avoid making students spend extra time working on the two cases, it was decided to fix one configuration, and so the phrase '*A, B, C, K and Q are all different, and lie on Γ in this order*' was added.

Our team obtained little in this question. Only two students managed to show that Q, M, H are collinear. According to the marking scheme, it is worth 1 point. One of the students, however, did not include much detail of the proof (after all, the question was not to prove that Q, M, H are collinear!), and the coordinator refused to award the point. This went into a long fight. The coordinators referred the case to the problem captain, then the chief coordinator. It turned out that there were many similar cases in which students mentioned the collinearity of the three points but were not accepted by the coordinators as a *proof*.

To prove that Q, M, H are collinear, one simple way is to show that Q, H, A' are collinear (where A' is the point on Γ that is diametrically opposite A), and that H, M, A' are collinear. The coordinators decided that the latter is well-known, but the former requires an explicit mention that $\angle AQH = \angle AQA' = 90^\circ$. To me, it is clear that proving the former is more trivial than the latter. If a student mentioned that A' is the antipodal point of A, then clearly (s)he knew that $\angle AQA'=90^\circ$ (it's the IMO!). Furthermore, $\angle AQH=90^\circ$ is given in the problem. What is the point of penalizing students who failed to copy this again? I didn't really see the consistency in accepting the latter as well-known but requiring such a detailed proof for the former. An urgent Jury Meeting was called to discuss this issue. The motion of sticking to the original marking scheme (i.e. to accept H, M, A' being collinear as well-known but to award 1 point only if $\angle AQH = \angle AQA' = 90^\circ$ is explicitly mentioned) was passed by a narrow margin.

The next day when we went on excursion, the Deputy Leader of Paraguay talked to me saying that many people thought that my speech was really to-the-point (by that time the deputy leaders had moved to the leaders' site and were allowed to sit in the Jury Meetings). But obviously more thought the opposite, as shown by the result of the vote!

Problem 4. This is the first problem of Day 2. It is a geometry problem, phrased carefully to make it as easy as possible. The order of the points was clearly given to ensure that only one configuration is possible. The statement to be proved was also rephrased from the original version so that the word *collinear* could be avoided.

Our team did not do well in this question. Only three students solved it. Another student showed that it suffices to prove $\angle AFK=\angle AGL$, which according to the marking scheme is worth 2 points. This sounds pretty much trivial, and the other two students would probably know it as well (only that they did not write it down because they did not find that useful).

In fact, there had been quite a lot of discussions on this point. Suppose a student showed $\angle AFK = \angle AGL$. How many points should that be worth? According to the original marking scheme, this would be worth 4 points; if a student added that *hence we are done*, that would make it 5; by writing *by symmetry we are done*, that would make it 6. (A perfect score would require some explanation on how symmetry leads to the result.) This led to strong opinion from the leaders. Eventually the (4,5,6) above was revised to (5,6,6).

Problem 5. This is the only question for which no student asked questions. This is interesting because in Problem 1 set notations were deliberately avoided, but in this question notation like $f\colon \mathbb{R} \to \mathbb{R}$ did not lead to any question, which to me is a bit of surprise.

By nature this problem is quite similar to Problem 2. Most students managed to make some partial progress, as one naturally starts by plugging in certain values of x and y into the functional equation, leading to some preliminary discoveries. However not many students obtained full solutions. We are glad everyone in our team got partial marks.

The solution to this problem depends heavily on fixed points, which in hindsight is reasonable considering that the expression $x+f(x+y)$ occurs on both sides. This also justifies starting the problem with setting $y=1$ as it would equate the terms $f(xy)$ and $yf(x)$ on the two sides of the equation. Completing the solution, on the other hand, is much more difficult, as there are too many equations and sometimes it is not clear what to put into which equation.

There were heated debates when discussing the marking scheme to this problem. As there were two functions satisfying the equation, most solutions could be divided into two parts (e.g. according to whether $f(0)=0$ or not). Each part would lead to one solution, and then one needs to check that the two solutions obtained, namely, $f(x)=x$ and $f(x)=2-x$, indeed satisfy the equation in the question. In the original proposal of the marking scheme, the coordinators said that they would accept students directly claiming that the former is a solution, while for the latter, it must be explicitly checked (expanding brackets and showing that the two sides are equal).

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***November 27, 2015.***

Problem 471. For $n \ge 2$, let $A_1, A_2, \ldots, A_n$ be positive integers such that $A_k \le k$ for $1 \le k \le n$. Prove that $A_1+A_2+\cdots+A_n$ is even if and only if there exists a way of selecting + or – signs such that

$$A_1 \pm A_2 \pm \cdots \pm A_n = 0.$$

Problem 472. There are $2n$ distinct points marked on a line, n of them are colored red and the other n points are colored blue. Prove that the sum of the distances of all pairs of points with same color is less than or equal to the sum of the distances of all pairs of points with different color.

Problem 473. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x,y \in \mathbb{R}$,

$$f(x)\, f(y\, f(x) - 1) = x^2 f(y) - f(x).$$

Problem 474. Quadrilateral $ABCD$ is convex and lines AB, CD are not parallel. Circle Γ passes through A, B and side CD is tangent to Γ at P. Circle L passes through C, D and side AB is tangent to L at Q. Circles Γ and L intersect at E and F. Prove that line EF bisects line segment PQ if and only if lines AD, BC are parallel.

Problem 475. Let a, b, n be integers greater than 1. If b^n-1 is a divisor of a, then prove that in base b, a has at least n digits not equal to zero.

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Solutions

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Problem 466. Let k be an integer greater than 1. If $k+2$ integers are chosen among $1,2,3,\ldots,3k$, then there exist two of these integers m,n such that $k<|m-n|<2k$.

Solution. **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania).

Let S be the set of the $k+2$ chosen integers and a be the smallest number in S. Subtracting $a-1$ from each element in S do not change the differences between the elements of S. So, without loss of generality, we can suppose $1 \in S$.

If S contains an element b such that $k+2 \le b \le 2k$, then take $m=b$ and $n=1$ to get $k<|m-n|=b-1<2k$. Otherwise, none of the numbers $k+2, k+3, \ldots, 2k$ belong to S. The $k+1$ numbers from $S\backslash\{1\}$ are then among the components of the k pairs $(2,2k+1)$, $(3,2k+2)$, … , $(k+1,3k)$. By the pigeonhole principle, there is a pair containing two numbers m, n from $S\backslash\{1\}$. Then we have $k<|m-n|=2k-1<2k$.

Other commended solvers: **Prithwijit DE** (HBCSE, Mumbai, India), **Ángel PLAZA** (Universidad de Las Palmas de Gran Canaria, Spain), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Simon YAU**.

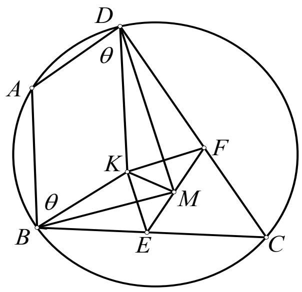
Problem 467. Let p be a prime number and q be a positive integer. Take any pq consecutive integers. Among these integers, remove all multiples of p. Let M be the product of the remaining integers. Determine the remainder when M is divided by p in terms of q.

Solution. **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India), **Mark LAU Tin Wai**, **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania), **Alex Kin-Chit O** (G.T. (Ellen Yeung) College) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

For $r = 0, 1, 2, \ldots, p-1$, among the pq consecutive integers, there are q integers having remainders r when divided by p. Then $M \equiv 1^q 2^q \cdots (p-1)^q = (p-1)!^q \pmod p$. By Wilson's theorem, $(p-1)! \equiv -1 \pmod p$. So $M \equiv (-1)^q \pmod p$. Then the remainder when M is divided by p is 1 if q is even and is $p-1$ if q is odd.

Problem 468. Let $ABCD$ be a cyclic quadrilateral satisfying $BC>AD$ and $CD>AB$. E, F are points on chords BC, CD respectively and M is the midpoint of EF. If $BE=AD$ and $DF=AB$, then prove that $BM \perp DM$.

Solution. **George APOSTOLOPOULOS** (2 High School, Messolonghi, Greece), **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India) and **MANOLOUDIS Apostolis** (4 High School of Korydallos, Piraeus, Greece).



Let K be the point such that $ABKD$ is a parallelogram. Let $\theta = \angle ABK = \angle ADK$. Now $BE=AD=BK$, $DF=AB=DK$ and

$$\angle BKE = 90^\circ - \tfrac{1}{2}\angle KBE = 90^\circ - \tfrac{1}{2}(\angle ABC - \theta),$$
$$\angle DKF = 90^\circ - \tfrac{1}{2}\angle KDF = 90^\circ - \tfrac{1}{2}(\angle ADC - \theta),$$
$$\angle BKD = 180^\circ - \theta.$$

Adding these and using $\angle ABC + \angle ADC = 180^\circ$, we get $\angle BKE + \angle BKD + \angle DKF = 270^\circ$. Then $\angle EKF = 90^\circ$, i.e. $KF \perp KE$. So $ME = MK = MF$. Also $BE = BK$ and $DF = DK$. Then $BM \perp KE$ and $DM \perp KF$. So $BM \| KF$ and $DM \| KE$. So $BM \perp DM$.

Other commended solvers: **Prithwijit DE** (HBCSE, Mumbai, India), **Toshihiro SHIMIZU** (Kawasaki, Japan), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 469. Let m be an integer greater than 4. On the plane, if m points satisfy no three of them are collinear and every four of them are the vertices of a convex quadrilateral, then prove that all m of the points are the vertices of a m-sided convex polygon.

Solution. **William FUNG**, **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Let S be the set of the m points and C be the set of the vertices of the convex hull H of S. Then S contains C and C has at least 3 elements. Assume there is a point P in S and not in C. Let n be the number of elements in C. Since H is a convex polygon, H can be decomposed into $n-2$ triangles by selecting a vertex and connecting all other vertices to this vertex. Since no three points of S are collinear, P is in the interior of one of these triangles. This contradicts every four of them are the vertices of a convex quadrilateral. So $S=C$, $m=n$ and S is the set of the vertices of a m-sided convex polygon.

Problem 470. If a, b, $c > 0$, then prove that

$$\frac{a}{b(a^2+2b^2)}+\frac{b}{c(b^2+2c^2)}+\frac{c}{a(c^2+2a^2)}$$
$$\geq \frac{3}{ab+bc+ca}.$$

Solution. **Jon GLIMMS** and **Henry RICARDO** (New York Math Circle, New York, USA).

Let x=1/a, y=1/b and z=1/c. Below all sums are cyclic in the order x,y,z. The desired inequality is the same as

$$\sum\frac{y^2}{z(2x^2+y^2)}\geq\frac{3}{x+y+z}.$$

By Cauchy's inequality, we have

$$\sum\frac{y^2}{z(2x^2+y^2)}\geq\frac{(x^2+y^2+z^2)^2}{\sum y^2z(2x^2+y^2)}.$$

It suffices to show

$$\frac{(x^2+y^2+z^2)^2}{\sum y^2z(2x^2+y^2)}\geq\frac{3}{x+y+z}.$$

Cross-multiplying and expanding, this is the same as

$$\sum(x^5+2x^3y^2+x^2y^3+xy^4)$$
$$\geq\sum(2x^4y+4x^2y^2z). \qquad (*)$$

By the AM-GM inequality, we have

(1) $\sum(x^5+x^3y^2)\geq\sum 2x^4y,$

(2) $\sum(x^2y^3+xy^4)=\sum(x^2y^3+yz^4)$
$\geq\sum 2y^2z^2x=\sum 2x^2y^2z.$

Next, (3) $\sum(x^3y^2+x^2y^3)\geq\sum 2x^2y^2z$ is the same as $\sum x\sum x^2y^2\geq 3xyz\sum xy$ after expansion. To get it, we have

$$\sum x\sum xy\geq\sum x\frac{\left(\sum xy\right)^2}{3}\geq 3xyz\sum xy$$

by Cauchy's inequality and the AM-GM inequality. Finally adding up (1), (2), (3), we get (*).

Other commended solvers: **Alex Kin-Chit O** (G.T. (Ellen Yeung) College), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Ángel PLAZA** (Universidad de Las Palmas de Gran Canaria, Spain), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

Olympiad Corner

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Problem 3. Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA$ = 90°, and let K be the point on Γ such that $\angle HKQ$ = 90°. Assume that the points A, B, C, K and Q are all different, and lie on Γ in this order.

Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

Problem 4. Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects segment BC at points D and E, such that B, D, E and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.

Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

Problem 5. Let ℝ be the set of real numbers. Determine all functions f: ℝ → ℝ satisfying the equation

$$f(x+f(x+y))+f(xy)=x+f(x+y)+y\,f(x)$$

for all real numbers x and y.

Problem 6. The sequence a_1, a_2, … of integers satisfies the following conditions:

(i) $1\leq a_j\leq 2015$ for all $j\geq 1$;

(ii) $k+a_k\neq l+a_l$ for all $1\leq k\leq l$.

Prove that there exist two positive integers b and N such that

$$\left|\sum_{j=m+1}^{n}(a_j-b)\right|\leq 1007^2$$

for all integers m and n satisfying $n > m \geq N$.

IMO2015–Problem Report

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This led to strong reactions from almost all the leaders, as the process of checking is indeed trivial, so an indication that the student is aware of the need of checking should be sufficient. This was eventually accepted by the coordinators.

Then the Canadian leader suggested that no mark should be deducted at all for omitting the checking. The UK leader said that he was surprised to hear such a suggestion as omitting the checking constitutes a logical error, but he would be happy to let this suggestion go to a vote. The Jury eventually voted against the suggestion. So in the end a student must somehow mention the checking (but need not actually show it) to get full mark for this question.

Interestingly, not checking that the solutions work would also constitute a logical error in Problem 2, but nobody made a suggestion to deduct points in that case. Also, while the coordinators first expected the checking to be explicitly carried out, in Problem 1 the coordinators did not even expect students to do anything to show that their constructed sets are balanced and center-free. It seems that such inconsistency between different problems is a common phenomenon.

Problem 6. Traditionally, Problem 6 is the most difficult problem of the IMO. This year's Problem 6 turned out to be not as difficult. Although only 11 out of the 577 contestants obtained perfect scores, the mean 0.355 for this question was one of the highest in recent years.

One of our team members solved this question. He mentioned that he got the idea by working on small cases first. So after all, this simple rule sometimes helps us solve not-so-simple problems!

At first sight the problem looks like one in mathematical analysis concerning the convergence of a sequence. One may even be tempted to try to prove that the sequence eventually becomes constant, which is not true.

There is an interesting interpretation of this problem (which is probably how this problem came up in the first place). At each second a ball is thrown upward, and the ball thrown at the i-th second will return to the ground after a_i seconds. So the condition $k+a_k\neq l+a_l$ for all $1\leq k\leq l$ means that no two balls shall return to the ground at the same time. The interested reader may follow this line to see whether a solution could be obtained more easily.