Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

Volume 43 2010/2011 Number 2

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Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year and consists of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor

Solving problems

Mathematics is about solving problems – discuss! But how do you go about it? And is it an individual or a group activity? Or both? These questions are prompted by a book which has come our way from the *American Regions Math League* (*ARML*) – see reference 1.

The ARML was founded in 1976 as the *Atlantic Region Mathematics League*. Originally an interstate competition covering the eastern seaboard of the US, it soon attracted students and teachers from all over the country. Currently it takes place simultaneously at four sites across the US, bringing together some 1700 students in the US and Canada. More recently teams from Russia, Taiwan, The Philippines, Colombia, and Hong Kong have taken part. Turkey, Bulgaria, and Vietnam competed in the *International Regions Math League* which began in 2008.

ARML is a competition between regions. A region may be as large as a US state, or a county, it may be a city, or even an individual school. Each team consists of 15 students. The contest consists of six parts. Round one is the *Team Round* in which a team works together on ten problems. Next comes the *Power Question* in which teams are given 60 minutes to solve a number of in-depth questions. Following that, teams come together in a large auditorium for the *Individual Round*. Next comes the *Relay Round* in which each team is divided into threes, with each of the three given a problem to solve. The solution to Number 1's problem is needed by Number 2, whose solution is needed to solve Number 3's problem. The fifth round is the *Super Relay* and is just for fun, with the whole team working on 15 problems, with the first team to get the correct answer to the last question earning a thunderous round of applause. The contest ends dramatically with the *Tiebreaker Round* in which those tied for the individual prizes go to the front of the auditorium and compete to solve a problem flashed on a screen. Quite an ordeal!

As one of the organizers writes, the ARML brings together great students. It is one of the few mathematics contests that involves exciting travel, meeting lots of other students, and renewing friendships. It also involves a lot of hard work both by the organizers and those taking part.

You can try out some of the problems set in 2004 to 2008, which are in the present volume, together with full solutions – if needed! So why not get hold of this volume, get together with your fellow-students and have a go? You have nothing to lose.

Reference

1 D. Barry, T. Kilkelly and P. Dreyer, *American Regions Math League & Power & Local Contests: 2004–2008* (ARML, 2009).

Heads or tails

If four coins are tossed, what are the odds against throwing two heads and two tails?

Midsomer Norton, Bath, UK **Bob Bertuello**

Message in a Bottle

PRITHWIJIT DE

If aesthetics and thickness of the raw material are ignored, designing a plastic bottle may be regarded as a pure optimization problem. The goal of the manufacturer is to minimize the manufacturing cost subject to meeting the volume requirements. The purpose of this article is to reach the manufacturer's goal. In what follows, we shall treat manufacturing cost as just the cost incurred to purchase a unit amount of the raw material. As bottles come in various shapes and sizes it is difficult to choose a template for them. But a cone surmounting a right-circular cylinder may serve as an appropriate candidate for a template (see figure 1). Here we are assuming that the cross-sectional radius of the mouth of the bottle is negligible in comparison with the cross-section of the base of the bottle.

If the radius of the base of the cylinder is r, its height h, and the semi-vertical angle of the cone is α, it follows that the total surface area of the bottle is

$$A = r\pi\left(r + \frac{r}{\sin\alpha} + 2h\right), \tag{1}$$

and that the volume of the bottle is

$$V = r^2\pi\left(h + \frac{r}{3}\cot\alpha\right). \tag{2}$$

The semi-vertical angle α lies between 0 and $\pi/2$. If $\alpha = \pi/2$ then $A = 2r\pi(r+h)$ and $V = r^2\pi h$. In this case the shape of the bottle is a closed cylinder. For a closed cylinder

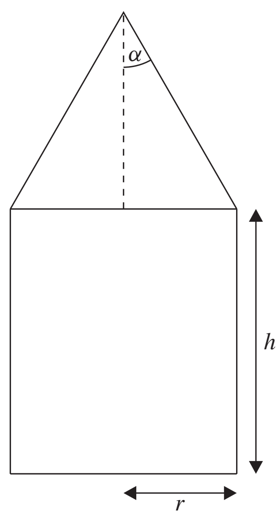

Figure 1 A schematic of the template of the bottle.

with fixed volume V, height h, and base radius r, the total surface area A is minimized when $h = 2r$. To see this write $A = 2\pi r^2 + 2V/r$ and differentiate it with respect to r to obtain

$$\frac{\mathrm{d}A}{\mathrm{d}r} = 4\pi r - \frac{2V}{r^2}.$$

At a stationary point, $\mathrm{d}A/\mathrm{d}r = 0$. Solving this gives us $r^3 = V/2\pi$ and at this value of r the value of $\mathrm{d}^2A/\mathrm{d}r^2$ is positive. This shows that A attains a local minimum when $r^3 = V/2\pi$ and for this value of r the height h is given by

$$h = \frac{V}{\pi r^2} = \frac{2\pi r^3}{\pi r^2} = 2r.$$

Also, as the base radius tends towards zero or infinity the value of A tends to infinity, thus ruling out the possibility of a minimum value at the boundaries of the interval containing the possible values of r. Hence the local minimum is also the global minimum. This validates our claim. The minimum value of A is $\sqrt[3]{54\pi V^2}$.

The reader may note that if the volume V is prespecified then the manufacturer can always make a cylinder with radius $r = \sqrt[3]{V/2\pi}$ and of height h which is twice as much as r to produce the shape requiring the minimum amount of raw material. But in reality this is seldom done. We rarely come across plastic bottles cylindrical in shape. The thickness of the material prevents us from going for this choice of shape. Let us therefore focus on the shape in figure 1 and try to determine the height, radius, and the semi-vertical angle which will guarantee the smallest possible total surface area A for a given volume V.

From (1) and (2) we may write, by eliminating the height h,

$$A = r^2\pi\left(1 + \frac{3 - 2\cos\alpha}{3\sin\alpha}\right) + \frac{2V}{r}. \tag{3}$$

This is a function of two independent variables, r and α. We need to obtain r and α such that A is minimized. To that end, we differentiate (3) with respect to r and α in succession to obtain

$$\frac{\partial A}{\partial \alpha} = r^2\pi\left(\frac{2 - 3\cos\alpha}{3\sin^2\alpha}\right), \tag{4}$$

$$\frac{\partial A}{\partial r} = 2\pi r\left(1 + \frac{3 - 2\cos\alpha}{3\sin\alpha}\right) - \frac{2V}{r^2}. \tag{5}$$

Setting (4) and (5) equal to zero we get

$$\cos\alpha = \frac{2}{3} \quad \text{and} \quad r = \sqrt[3]{\frac{3V}{(3+\sqrt{5})\pi}}.$$

By substituting the values of V and α in (2) we obtain $h = ((\sqrt{5}+1)/\sqrt{5})r$. By using (3) we also get $A = \sqrt[3]{9(3+\sqrt{5})\pi V^2}$.

It remains to be seen whether the solution

$$\left(\sqrt[3]{\frac{3V}{(3+\sqrt{5})\pi}}, \cos^{-1}\left(\frac{2}{3}\right)\right)$$

obtained above is a local minimum point. To ascertain the nature of an extreme point of a multivariate function we need to use the following theorem (see reference 1, p. 379).

Theorem 1 *Let f be a real-valued function of two variables, x and y, with continuous second-order partial derivatives at a stationary point P in R^2, i.e. $\nabla f(P) = 0$. If $\Delta = f_{xx} f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} + f_{yy} > 0$ then P is a local minimum.*

We now explain the notation used in theorem 1:

$$\nabla f(P) = \left(\frac{\partial f(P)}{\partial x}, \frac{\partial f(P)}{\partial y}\right),$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \qquad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}.$$

In the present context,

$$P \equiv \left(\sqrt[3]{\frac{3V}{(3+\sqrt{5})\pi}}, \cos^{-1}\left(\frac{2}{3}\right)\right),$$

$f = A$, $x = r$, and $y = \alpha$. Now,

$$\frac{\partial^2 A}{\partial r^2} = 2\pi\left(1 + \frac{3 - 2\cos\alpha}{3\sin\alpha}\right) + \frac{4V}{r^3},$$

$$\frac{\partial^2 A}{\partial \alpha^2} = \frac{r^2\pi}{\sin\alpha}\left[\left(\frac{3 - 2\cos\alpha}{3\sin\alpha}\right)^2 + \frac{5}{9}\cot^2\alpha\right],$$

$$\frac{\partial^2 A}{\partial r \partial \alpha} = \frac{2r\pi}{\sin\alpha}\left(\frac{2 - 3\cos\alpha}{3\sin\alpha}\right).$$

At P,

$$\cos\alpha = \frac{2}{3} \quad \text{and} \quad \left(\frac{\partial^2 A}{\partial r \partial \alpha}\right)_P = 0.$$

It is easily seen that

$$\left(\frac{\partial^2 A}{\partial r^2}\right)_P > 0 \quad \text{and} \quad \left(\frac{\partial^2 A}{\partial \alpha^2}\right)_P > 0.$$

Hence P is a local minimum.

We find that the ratio of the total surface area of the bottle and that of a cylinder of identical volume is $\sqrt[3]{3+\sqrt{5}} : \sqrt[3]{6} = 0.96 : 1$ (approximately). Thus the manufacturer saves roughly 4% on the material by adopting the design discussed in this article.

Reference

1 T. M. Apostol, *Mathematical Analysis*, 2nd edn. (Addison-Wesley, Reading, MA, 1974).

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Lopsided Numbers

JONNY GRIFFITHS

I was discussing inversion with my students the other day, the transformation that takes (r, θ) to $(1/r, \theta)$ in polar coordinates.

'Think of these points as being best friends', I said whimsically, 'On the understanding that your best friend's best friend will be yourself'.

'Aren't there such things as *friendly numbers*?' asked Daniel, a bright student who reads around the subject. The number 284 came into my mind, and a visit to Wikipedia (see reference 1) revealed why – 284 and 220 are *amicable numbers*.

'What does that mean?' Tilly asked.

'Define $\sigma(n)$ as the sum of all the divisors of n, including 1 and n, and $\sigma'(n)$ as the sum of all the proper divisors of n, that is, not including n', I replied. 'Now $\sigma'(284) = 220$, while $\sigma'(220) = 284$'. Wikipedia obligingly told us further that a *friendly number* was something different. Specifically, $\sigma(n)/n$ takes a range of values as n varies. If $\sigma(m)/m = \sigma(n)/n$, then n and m are *friends*, and are thus both *friendly numbers*. If n takes a unique value for some $\sigma(n)/n$, then n is a *solitary number*. It turns out that all primes are solitary. (Proving this is included in this issue's Problems and Solutions section.) (It is worthwhile mentioning that $\sigma(n)$ is a *multiplicative function*, that is, if n and m have no common factor, then $\sigma(nm) = \sigma(n)\sigma(m)$.)

Alongside this information, Wikipedia helpfully offered a large box of further definitions: *hyper-perfect numbers*, *practical numbers*, *weird numbers*, *sublime numbers*, *frugal numbers*, and so on. This much was immediately clear, some hypotheses that are extremely simple to state remain unresolved in this area. Is there an odd *perfect number*? (A *perfect number* is one where $\sigma'(n) = n$, for example 6, 28, 496, $\ldots$.) Is 10 a solitary number? How many sublime numbers are there? (A *sublime number* is one which has a perfect number of positive divisors (including itself) and whose positive divisors add up to another perfect number. The number 12, for example, is a sublime number. It has a perfect number of positive divisors (six): 1, 2, 3, 4, 6, and 12, and the sum of these is again a perfect number: $1 + 2 + 3 + 4 + 6 + 12 = 28$. There are only two known sublime numbers, 12 and

$$6\,086\,555\,670\,238\,378\,989\,670\,371\,734\,243\,169\,622\,657\\ 830\,773\,351\,885\,970\,528\,324\,860\,512\,791\,691\,264.$$

Readers are invited to check this!)

Tilly had been thinking. '$\sigma(n)$ is always bigger than n, while $\sigma'(n)$ can be less than, or equal to, or greater than n', she said. 'Can you find a pair of numbers n and m (with n less than m) so that

$$\sigma(n) = m \quad \text{and} \quad \sigma'(m) = n?\text{'}$$

We looked at this for a moment, and couldn't see any good reason why such a pair of numbers should not exist. Pete, my Head of Department scratched his head when presented with the problem later: 'They would be *lopsided numbers* then'.

So we seek a pair, $(n, \sigma(n))$ so that $\sigma'(\sigma(n)) = n$. Playing with small numbers, we found that such pairs are certainly possible:

$$\begin{aligned} &\sigma(3) = 4 \quad \text{and} \quad \sigma'(4) = 3, \qquad \text{so } (3, 4) \text{ is a lopsided pair,} \\ &\sigma(7) = 8 \quad \text{and} \quad \sigma'(8) = 7, \qquad \text{so } (7, 8) \text{ is a lopsided pair.} \end{aligned}$$

Is $(2^k - 1, 2^k)$ always a lopsided pair? No, since $\sigma(15) = 24$ and $\sigma'(24) = 36$.

What if $2^k - 1$ is prime? Now $\sigma(2^k - 1) = 2^k$ and $\sigma'(2^k) = 1 + 2 + 4 + \cdots + 2^{k-1} = 2^k - 1$.

So $(2^k - 1, 2^k)$ is a lopsided pair if $2^k - 1$ is prime. (Such primes are called *Mersenne primes*.)

Our question now becomes, are there any lopsided pairs that are not of this form? At this point, a simple EXCEL program provides us with a lot of help. It reveals no lopsided pairs (n, m) that are not of the above form for n up to 50 000. It does, however, supply us with some near misses:

$$\begin{aligned} &\sigma(18) = 39 && \text{and} \quad \sigma'(39) = 17, && \text{so } (18, 39) \text{ is almost a lopsided pair,} \\ &\sigma(242) = 399 && \text{and} \quad \sigma'(399) = 241, && \text{so } (242, 399) \text{ is almost a lopsided pair,} \\ &\sigma'(94) = 50 && \text{and} \quad \sigma(50) = 93, && \text{so } (50, 94) \text{ is almost a lopsided pair,} \\ &\sigma'(2457) = 2023 && \text{and} \quad \sigma(2023) = 2456, && \text{so } (2023, 2457) \text{ is almost a lopsided pair.} \end{aligned}$$

Notice that there is a 'falling one short' each time, it makes sense to call these *almost lopsided numbers*, since *almost perfect numbers* are those where $\sigma(n) = 2n - 1$.

If $\sigma(n) = 2n + 1$, then n is called *quasiperfect*, but no such numbers have been discovered thus far. My EXCEL program revealed no 'quasilopsided' pairs for n under 50 000. Why should it be so much easier to fall short by 1 rather than to overreach by 1?

So our unproven conjecture remains as follows:

$$(n, m) \text{ is a lopsided pair} \iff (n, m) \text{ is of the form } (2^k - 1, 2^k) \text{ where } 2^k - 1 \text{ is prime.}$$

This may be no easier to prove than any of the other conjectures in the field, but *Mathematical Spectrum* readers are invited to try.

Reference

1 Wikipedia, http://en.wikipedia.org

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Alternative Continued Nested Radical Fractions for Some Constants

TEIK-CHENG LIM

1. Introduction

A continued nested radical fraction (CNRF),

$$\mathrm{CNRF} = \sqrt[m]{a_0 + \cfrac{b_1}{\sqrt[n]{a_1 + \cfrac{b_2}{\sqrt[n]{a_2 + \cdots}}}}}, \tag{1}$$

is a hybrid of a continued fraction and a nested radical. It can be easily seen that substitution of $m = n = 1$ and $m = -n = 2$ into (1) gives a continued fraction (CF),

$$\mathrm{CF} = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cdots}},$$

and a nested radical (NR),

$$\mathrm{NR} = \sqrt{a_0 + b_1\sqrt{a_1 + b_2\sqrt{a_2 + \cdots}}},$$

respectively. Simple continued fractions and nested radicals are obtained when $b_i = 1$, for $i = 1, 2, 3, \ldots$, to give

$$\mathrm{CF} = [a_0; a_1, a_2, a_3, \ldots]$$

and

$$\mathrm{NR} = \sqrt{a_0 + \sqrt{a_1 + \sqrt{a_2 + \cdots}}}.$$

2. CNRFs of some constants

The plastic constant p, silver ratio δ, and golden section ϕ, can be expressed by CNRFs as follows:

$$p = \sqrt{1 + \cfrac{1}{\sqrt{1 + \cfrac{1}{\sqrt{1 + \cdots}}}}},$$

$$\phi = \sqrt[n]{\mathrm{F}_{n+1} + \cfrac{\mathrm{F}_n}{\sqrt[n]{\mathrm{F}_{n+1} + \cfrac{\mathrm{F}_n}{\sqrt[n]{\mathrm{F}_{n+1} + \cdots}}}}}, \tag{2}$$

$$\delta = \sqrt[n]{\mathrm{P}_{n+1} + \cfrac{\mathrm{P}_n}{\sqrt[n]{\mathrm{P}_{n+1} + \cfrac{\mathrm{P}_n}{\sqrt[n]{\mathrm{P}_{n+1} + \cdots}}}}}$$

respectively, with F_n and P_n referring to the nth Fibonacci and Pell numbers respectively; see references 1 and 2. The plastic constant is defined as the real unique solution of the cubic equation $p^3 - p - 1 = 0$. Two quantities, say, $x < y$, are said to be in golden section if $(x + y) : y = y : x$. It follows that the golden section is the positive solution to the quadratic equation $\phi^2 - \phi - 1 = 0$, in analogy to the plastic constant. The silver ratio, defined as $1 + \sqrt{2}$, is so named via allusion to the golden section, by way of analogy, where the former is the limiting ratio of consecutive Pell numbers

$$\delta = \lim_{n\to\infty} \frac{\mathrm{P}_{n+1}}{\mathrm{P}_n}$$

while the latter is the limiting ratio of consecutive Fibonacci numbers

$$\phi = \lim_{n\to\infty} \frac{\mathrm{F}_{n+1}}{\mathrm{F}_n}.$$

Whilst the CNRF form of plastic constant satisfies $a_0 = b_1 = a_1 = b_2 = a_2 = \cdots$, it is generally not so for the CNRFs of the golden section and the silver ratio. In fact $\mathrm{P}_{n+1} \neq \mathrm{P}_n$, while $\mathrm{F}_{n+1} = \mathrm{F}_n$ if and only if $n = 1$. For this value of n, (2) is no longer a CNRF as it reduces to a continued fraction. To obtain $a_0 = b_1 = a_1 = b_2 = a_2 = \cdots$, we use alternate expressions of $n\phi$ as

$$n\phi = \sqrt{n^2 + n(n\phi)} \tag{3}$$

and

$$n\phi = n + \frac{n^2}{n\phi}. \tag{4}$$

By substituting (4) into the parentheses of (3) and alternating with the substitution of (3) into the parenthesis of (4), we have

$$n\phi = \sqrt{2n^2 + \cfrac{n^3}{\sqrt{2n^2 + \cfrac{n^3}{\sqrt{2n^2 + \cdots}}}}}. \tag{5}$$

When we let $n = 2$, (5) becomes

$$\phi = \frac{1}{2}\sqrt{8 + \cfrac{8}{\sqrt{8 + \cfrac{8}{\sqrt{8 + \cfrac{8}{\sqrt{8 + \cdots}}}}}}}.$$

In the same manner, we write

$$n\delta = \sqrt{n^2 + 2n(n\delta)} \tag{6}$$

and

$$n\delta = 2n + \frac{n^2}{n\delta}. \tag{7}$$

The substitution of (7) and (6) into the parentheses of (6) and (7) respectively gives

$$n\delta = \sqrt{5n^2 + \cfrac{2n^3}{\sqrt{5n^2 + \cfrac{2n^3}{\sqrt{5n^2 + \cdots}}}}}. \tag{8}$$

Let $n = \frac{5}{2}$. Then (8) becomes

$$\delta = \tfrac{2}{5}\sqrt{\tfrac{125}{4} + \cfrac{\frac{125}{4}}{\sqrt{\frac{125}{4} + \cfrac{\frac{125}{4}}{\sqrt{\frac{125}{4} + \cdots}}}}}.$$

Hence we have shown that the plastic constant, silver ratio, and golden section can be generally written in the form

$$C = k\sqrt{m + \cfrac{m}{\sqrt{m + \cfrac{m}{\sqrt{m + \cdots}}}}}.$$

Taking a step further, these CNRFs of slightly different forms can be obtained as follows. Using

$$np = \begin{cases} \sqrt[3]{n^3 + n^2 \times (np)} \\ \sqrt{n^2 + n^3 \div (np)} \end{cases},$$

$$n\phi = \begin{cases} \sqrt[3]{n^3 + 2n^2 \times (n\phi)} \\ \sqrt{2n^2 + n^3 \div (n\phi)} \end{cases},$$

$$n\delta = \begin{cases} \sqrt[3]{2n^3 + 5n^2 \times (n\delta)} \\ \sqrt{5n^2 + 2n^3 \div (n\delta)} \end{cases},$$

and expanding them in similar fashion leads to

$$np = \sqrt[3]{n^3 + n^2\sqrt{n^2 + \cfrac{n^3}{\sqrt[3]{n^3 + n^2\sqrt{n^2 + \cfrac{n^3}{\sqrt[3]{n^3 + \cdots}}}}}}}, \tag{9}$$

$$n\phi = \sqrt[3]{n^3 + 2n^2\sqrt{2n^2 + \cfrac{n^3}{\sqrt[3]{n^3 + 2n^2\sqrt{2n^2 + \cfrac{n^3}{\sqrt[3]{n^3 + \cdots}}}}}}}, \tag{10}$$

$$n\delta = \sqrt[3]{2n^3 + 5n^2\sqrt{5n^2 + \cfrac{2n^3}{\sqrt[3]{2n^3 + 5n^2\sqrt{5n^2 + \cfrac{2n^3}{\sqrt[3]{2n^3 + \cdots}}}}}}} \tag{11}$$

respectively. Substituting $n = 1$, $n = 2$, and $n = \frac{5}{2}$ into (9), (10), and (11) respectively gives

$$p = 1 \times \sqrt[3]{1 + 1 \times \sqrt{1 + 1 \div \sqrt[3]{1 + 1 \times \sqrt{1 + 1 \div \sqrt[3]{1 + \cdots}}}}},$$

$$\phi = \tfrac{1}{2} \times \sqrt[3]{8 + 8 \times \sqrt{8 + 8 \div \sqrt[3]{8 + 8 \times \sqrt{8 + 8 \div \sqrt[3]{8 + \cdots}}}}},$$

$$\delta = \tfrac{2}{5} \times \sqrt[3]{\tfrac{125}{4} + \tfrac{125}{4} \times \sqrt{\tfrac{125}{4} + \tfrac{125}{4} \div \sqrt[3]{\tfrac{125}{4} + \tfrac{125}{4} \times \sqrt{\tfrac{125}{4} + \tfrac{125}{4} \div \sqrt[3]{\tfrac{125}{4} + \cdots}}}}}$$

respectively.

3. Conclusion

The constant C can be written as

$$C = \begin{cases} k \times \sqrt[2]{m + m \div \sqrt[2]{m + m \div \sqrt[2]{m + m \div \sqrt[2]{m + m \div \sqrt[2]{m + \cdots}}}}} \\ k \times \sqrt[3]{m + m \times \sqrt[2]{m + m \div \sqrt[3]{m + m \times \sqrt[2]{m + m \div \sqrt[3]{m + \cdots}}}}} \end{cases},$$

where $C = p$ if $k = m = 1$, $C = \phi$ if $k^{-3} = m = 8$, and $C = \delta$ if $5k^{-2} = m = \frac{125}{4}$.

References

1 T.-C. Lim, Continued nested radical fractions, *Abstract Book of the 25th Journées Arithmétiques*, University of Edinburgh, July 2007, pp. 27–28. Available at http://atlas-conferences.com/c/a/t/q/34.htm.

2 T.-C. Lim, Continued nested radical fractions, *Math. Spectrum* **42** (2009/2010), pp. 59–63.

Teik-Cheng Lim *earned his PhD from the Faculty of Engineering, National University of Singapore, and thereafter pioneered research on nanotechnology, mathematical chemistry, and auxetic materials in Singapore. He is currently a faculty member in the School of Science and Technology, SIM University. Dr Lim has published more than 100 papers in international peer-reviewed journals, three book chapters, and a book. He is the Founder and Editor-in-Chief of the International Journal of Novel Materials and is serving on the editorial board of the International Journal of Chemical Modeling and The Open Industrial & Manufacturing Engineering Journal.*

Mathematical Spectrum **Awards for Volume 42**

Prizes have been awarded to the following student readers for contributions in Volume 42:

George Bignall

for various contributions;

Joshua Lam

for various contributions.

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £50 for letters, solutions to problems, and other items.

Integral Triangles with a 120° Angle

KONSTANTINE ZELATOR

1. Introduction

The aim of this work is to offer a parametric description of the family or set of all integral triangles with a 120° angle. By the term *integral triangles*, we simply mean triangles whose three side lengths are integers. As an historical note, in reference 1 Böttcher is mentioned as having found examples of integral triangles with a 60° or 120° angle (see p. 200). In reference 2, the interested reader can find a detailed and careful analysis of integral triangles with a 60° angle. Also in reference 2 we can find the derivation of parametric formulas describing the entire set of integral triangles with a 60° angle.

Let us say a few words about the organization of this article. In Section 4, we present four parametric families of integral triangles. The union of these families is the set of all integral triangles with a 120° angle.

In Section 5, we present numerical examples. Specifically, we list 24 such triangles. (We give the three side lengths in each case.) In Section 2, we state a result in number theory which in turn is used in Section 3 in order to find all the positive integer solutions of the (three variable) Diophantine equation $x^2 + xy + y^2 = z^2$. We make use of these solutions in Section 4.

2. A result from number theory

Below, we state parametric formulas which describe the entire set of positive integer solutions to the Diophantine equation $x^2+3y^2 = z^2$. Note that this is a special case ($k = 3$) of the more general equation $x^2 + ky^2 = z^2$, where k is a positive integer with $k \geq 2$. A brief discussion of the derivation of the general solution (in positive integers) of this more general Diophantine equation can be found in reference 1, pp. 420–421. For a detailed step-by-step derivation of all positive integer solutions of this Diophantine equation, we refer the reader to reference 3. The case $k = 3$ is also presented in reference 3.

Result 1 *All the positive integer solutions of the Diophantine equation*

$$x^2 + 3y^2 = z^2 \tag{1}$$

can be described by the parametric formulas

$$x = \frac{\delta|k^2 - 3\lambda^2|}{2}, \qquad y = \delta k\lambda, \qquad z = \frac{\delta(k^2 + 3\lambda^2)}{2}, \tag{2}$$

where δ, k, and λ are positive integers such that k and λ are relatively prime; $(k, \lambda) = 1$ (and when k and λ have different parities, δ is an even integer).

When $(x, y) = 1$, the solutions given by (2) are called *primitive*.

As it is easily seen from (1), $(x, y) = 1$ if, and only if, $(x, z) = 1 = (z, y)$ as well. Also, note that for a (positive integer) solution to (1) to be primitive, it is necessary and sufficient that, in addition to the condition $(x, y) = 1$, we must also have either ($\delta = 2$, $k + \lambda \equiv 1 \pmod 2$, and $(k, 3) = 1$) or ($\delta = 1$, $k \equiv \lambda \equiv 1 \pmod 2$, and $(k, 3) = 1$).

3. The Diophantine equation $x^2 + xy + y^2 = z^2$

To be able to achieve our goal of describing the set of all integral triangles with a 120° angle, we must find the positive integer solutions of the Diophantine equation

$$\begin{aligned} x^2 + xy + y^2 &= z^2 \\ \text{or, equivalently,} \qquad x^2 + xy + y^2 - z^2 &= 0. \end{aligned} \tag{3}$$

Note that there are no positive integer solutions to (3) with $x = y$, by virtue of the fact that $\sqrt{3}$ is an irrational number.

We solve (3) as a quadratic equation in x as follows:

$$x = \frac{-y \pm \sqrt{4z^2 - 3y^2}}{2};$$

and since $x \geq 1$ and $y \geq 1$ we see that the plus sign must hold, i.e.

$$x = \frac{-y + \sqrt{4z^2 - 3y^2}}{2}. \tag{4}$$

With x, y, and z being positive integers, it is clear that the discriminant $4z^2 - 3y^2$ must be a perfect square, i.e.

$$(2z)^2 - 3y^2 = w^2$$

for some positive integer w or, equivalently,

$$(2z)^2 = w^2 + 3y^2. \tag{5}$$

Clearly, the triple $(w, y, 2z)$ is a positive integer solution to (1). Accordingly, we must have

$$w = \frac{d|r^2 - 3t^2|}{2}, \qquad y = drt, \qquad 2z = \frac{d(r^2 + 3t^2)}{2}, \tag{6}$$

for positive integers d, r, and t, such that $(r, t) = 1$.

In view of the third equation in (6), we must actually have

$$\begin{aligned} &\text{either } d \equiv 0 \pmod 4, (r, t) = 1, \text{and } r + t \equiv 1 \pmod 2, \\ &\text{or alternatively (no condition on } d), (r, t) = 1, \text{and } r \equiv t \equiv 1 \pmod 2. \end{aligned} \tag{7}$$

From (4) and (5) we obtain

$$x = \frac{-y + w}{2},$$

and by (6) we arrive at

$$x = \frac{d(-2rt + |r^2 - 3t^2|)}{4}, \qquad y = drt, \qquad z = \frac{d(r^2 + 3t^2)}{4}. \tag{8}$$

4. Integral triangles with a 120° angle

If a, b, c are the three side lengths of an integral triangle containing a $120°$ angle, with c being the largest of the three integers, then the side of length c must lie opposite the angle $120°$, which is the largest of the three interior triangle angles. By the cosine rule, we have

$$c^2 = a^2 + b^2 - 2ab\cos 120°$$

and, since $\cos 120° = -\frac{1}{2}$,

$$c^2 = a^2 + b^2 + ab.$$

Now, the three lengths a, b, c must satisfy the three triangle inequalities $a + b > c$, $b + c > a$, and $a + c > b$. Since c is the largest of the three integers, it follows that two of the three triangle inequalities are automatically satisfied. Therefore, the only one that needs special mention is $a + b > c$.

It is now clear that all $120°$ triples (a, b, c) are precisely those satisfying

$$c^2 = a^2 + b^2 + ab;\ a, b, c \in \mathbb{Z}^+ \text{ and with } a + b > c.$$

Using (8) and the conditions (7) we obtain (up to switching a with b)

$$\left\{\begin{array}{l} a = \dfrac{d(-2rt + |r^2 - 3t^2|)}{4},\ b = drt,\ c = \dfrac{d(r^2 + 3t^2)}{4}, \\ \text{with } d, r, t \in \mathbb{Z}^+,\ (r, t) = 1; \\ \text{and with either } d \equiv 0 \pmod 4 \text{ and } r + t \equiv 1 \pmod 2 \\ \text{or alternatively with } r \equiv t \equiv 1 \pmod 2; \\ \text{and in addition, with } a + b > c. \end{array}\right\} \tag{9}$$

The next thing we must do is work out and analyze the condition $a + b > c$.

We have,

$$a + b > c \quad \Longleftrightarrow \quad \frac{d(-2rt + |r^2 - 3t^2|)}{4} + drt > \frac{d(r^2 + 3t^2)}{4}.$$

By virtue of $d > 0$ and a bit of basic algebra, we see that the last inequality is equivalent to

$$|r^2 - 3t^2| + 2rt > r^2 + 3t^2. \tag{10}$$

To proceed further, we need to drop the absolute value of the first term in (10). First assume that $r^2 - 3t^2 > 0$. Under this assumption,

$$\begin{aligned} (10) \quad &\Longleftrightarrow \quad \left\{\begin{array}{c} r^2 - 3t^2 + 2rt > r^2 + 3t^2 \\ r^2 - 3t^2 > 0 \end{array}\right\} \\ &\Longleftrightarrow \quad \left\{\begin{array}{c} 2rt > 6t^2 \\ r^2 > 3t^2 \end{array}\right\} \\ &\Longleftrightarrow \quad \text{(since } r \text{ and } t \text{ are positive)} \left\{\begin{array}{c} r > 3t \\ r^2 > 3t^2 \end{array}\right\} \\ &\Longleftrightarrow \quad r > 3t. \end{aligned}$$

On the other hand, when $r^2 - 3t^2 < 0$, we have $|r^2 - 3t^2| = 3t^2 - r^2$. So, in this case,

$$\begin{aligned}
(10) \quad &\iff \quad \left\{ \begin{array}{c} 3t^2 - r^2 + 2rt > r^2 + 3t^2 \\ r^2 - 3t^2 < 0 \end{array} \right\} \\
&\iff \quad \left\{ \begin{array}{c} 2rt > 2r^2 \\ r^2 < 3t^2 \end{array} \right\} \\
&\iff \quad \text{(since } r \text{ and } t \text{ are positive)} \left\{ \begin{array}{c} t > r \\ r^2 < 3t^2 \end{array} \right\} \\
&\iff \quad r < t.
\end{aligned}$$

In light of the above analysis and due to the conditions in (9), we conclude the entire set of the $120°$ triples (a, b, c) can be thought of as the nondisjoint or overlapping union of four groups or families of integral triangles with $120°$ angles.

These four groups are:

$$\boldsymbol{F}_1 : \left\{ \begin{array}{l} a = \dfrac{d(-2rt + r^2 - 3t^2)}{4},\ b = drt,\ c = \dfrac{d(r^2 + 3t^2)}{4}, \\ \text{with } d, r, t \in \mathbb{Z}^+,\ (r, t) = 1,\ d \equiv 0 \,(\mathrm{mod}\, 4),\ r + t \equiv 1 \,(\mathrm{mod}\, 2), \\ \text{and } r > 3t \end{array} \right\}, \tag{11}$$

$$\boldsymbol{F}_2 : \left\{ \begin{array}{l} a = \dfrac{d(-2rt + 3t^2 - r^2)}{4},\ b = drt,\ c = \dfrac{d(r^2 + 3t^2)}{4}, \\ \text{with } d, r, t \in \mathbb{Z}^+,\ (r, t) = 1,\ d \equiv 0 \,(\mathrm{mod}\, 4),\ r + t \equiv 1 \,(\mathrm{mod}\, 2), \\ \text{and } r < t \end{array} \right\}, \tag{12}$$

$$\boldsymbol{F}_3 : \left\{ \begin{array}{l} a = \dfrac{d(-2rt + r^2 - 3t^2)}{4},\ b = drt,\ c = \dfrac{d(r^2 + 3t^2)}{4}, \\ \text{with } d, r, t \in \mathbb{Z}^+,\ (r, t) = 1,\ r \equiv t \equiv 1 \,(\mathrm{mod}\, 2) \text{ and } r > 3t, \end{array} \right\} \tag{13}$$

$$\boldsymbol{F}_4 : \left\{ \begin{array}{l} a = \dfrac{d(-2rt + 3t^2 - r^2)}{4},\ b = rt,\ c = \dfrac{d(r^2 + 3t^2)}{4}, \\ \text{with } d, r, t \in \mathbb{Z}^+,\ (r, t) = 1,\ r \equiv t \equiv 1 \,(\mathrm{mod}\, 2) \text{ and } r < t, \end{array} \right\} \tag{14}$$

5. Numerical examples

In table 1 we list all $120°$ integral triples (a, b, c) with the parameters d, r, t satisfying, not only the conditions in (11)–(14), but also the conditions $rt \leq 14$ and $(r, 3) = 1$; $d = 4$ in families $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$; while $d = 1$ in families $\boldsymbol{F}_3$ and $\boldsymbol{F}_4$. In this list there are 24 triples. However, the triple $(a, b, c) = (3, 5, 7)$ in $\boldsymbol{F}_3$, also appears as $(a, b, c) = (5, 3, 7)$ in $\boldsymbol{F}_4$. Likewise, the triple $(9, 56, 61)$ in $\boldsymbol{F}_1$ reappears as $(56, 9, 61)$ in $\boldsymbol{F}_4$.

Note that the condition $(r, 3) = 1$, simply means that r is not divisible by 3. If r is divisible by 3, then all three, a, b, and c are multiples of 3.

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Table 1

	r	t	a	b	c
family $\boldsymbol{F}_1$	4	1	5	16	19
	8	1	45	32	67
	10	1	77	40	103
	14	1	165	56	199
	7	2	9	56	61
family $\boldsymbol{F}_2$	1	2	7	8	13
	1	6	95	24	109
	1	8	175	32	193
	1	10	279	40	301
	1	12	407	48	433
	1	14	559	56	589
	2	3	11	24	31
	2	5	51	40	79
	2	7	115	56	151
family $\boldsymbol{F}_3$	5	1	3	5	7
	7	1	8	7	13
	11	1	24	11	31
	13	1	35	13	43
family $\boldsymbol{F}_4$	1	3	5	3	7
	1	5	16	5	19
	1	7	33	7	37
	1	9	56	9	61
	1	11	85	11	91
	1	13	120	13	127

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Almost an integer

Bob Bertuello, one of our readers, spotted the following intriguing result in *Mathematica Algorithms.pdf*

$$e^{\pi\sqrt{163}} = 262\,537\,412\,640\,768\,743.999\,999\,999\,999\,250\,072\,597\,2\ldots.$$

Pyramids of 3-Power-full Primes

CHRIS K. CALDWELL and ANDREW RUPINSKI

1. Introduction

If we start with $3^0 = 1$ as our top row, we can build a pyramid of primes (other than the first row), by inserting the powers of 3 consecutively (see figure 1).

Our question in this article is simple – can we continue this process of creating new '3-power-full primes' indefinitely? Or is there some limit to how tall such pyramids can be?

2. Does our pyramid have a next row?

Call the rows of our pyramid $a_0 = 1, a_1, a_2, \ldots, a_k$. We are asking if we can always insert 3^{k+1} into a_k to form a new prime row a_{k+1}. Let $\ell(x) = \lfloor 1 + \log x \rfloor$ be the length (in digits) of the integer x where $\log x$ denotes the logarithm of x to the base 10. There are $\ell(a_k) + 1$ places in which we can insert this power of 3: immediately to the right of any digit of a_k, or in front of the first digit. By the prime number theorem (see reference 1), the probability that a random integer $m > 0$ is prime is about $1/\ln m$. So if we think of the potential new rows as being random, we would expect about $r(k+1)$ of them to be prime, where

$$r(k+1) = \frac{\ell(a_k) + 1}{(\ell(a_k) + \ell(3^{k+1})) \ln 10},$$

since for any number m, $\ln m = \log m \ln 10 \approx (\ell(m) - 1) \ln 10$. But the potential rows are not random; they are never divisible by 2, 3, or 5. Usually all but $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = \frac{4}{15}$ of random numbers are divisible by one of these three primes, so we must multiply our estimate by $\frac{15}{4}$ to obtain the better estimate as follows:

$$r(k+1) = \frac{\ell(a_k) + 1}{(\ell(a_k) + \ell(3^{k+1})) \ln 10} \times \frac{15}{4}. \tag{1}$$

(1)
1**3**
13**9**
1**27**39
12**81**739
1281739**243**
128173**729**9243
12817372**187**99243
1281737**6561**2218799243
12**19683**81737656122187992443
121968381737**59049**5612218799243
12196838**177147**17376590495612218799243
12196838**531441**177147173765904956122187992443
12196838531441**1594323**17714717376590495612218799243
1219683853144115943231771471**4782969**17376590495612218799243

Figure 1 An example of a pyramid of 3-power-full primes.

Table 1 The number $\#(3, k)$ of pyramids of height k.

k	$\#(3,k)$	$\frac{\#(3,k)}{\#(3,k-1)}$	$r(k)$	k	$\#(3,k)$	$\frac{\#(3,k)}{\#(3,k-1)}$	$r(k)$
1	2	2.000	1.629	16	1 118	1.505	1.470
2	2	1.000	1.629	17	1 598	1.429	1.468
3	2	1.000	1.221	18	2 422	1.516	1.484
4	6	3.000	1.357	19	3 616	1.493	1.482
5	10	1.667	1.267	20	5 316	1.470	1.495
6	14	1.400	1.357	21	8 061	1.516	1.494
7	30	2.143	1.323	22	12 284	1.524	1.505
8	47	1.567	1.384	23	18 675	1.520	1.515
9	62	1.319	1.368	24	26 946	1.443	1.513
10	91	1.468	1.411	25	39 146	1.453	1.521
11	134	1.473	1.402	26	59 339	1.516	1.520
12	217	1.619	1.435	27	90 790	1.530	1.527
13	316	1.456	1.429	28	137 602	1.516	1.526
14	483	1.528	1.454	29	206 390	1.500	1.533
15	743	1.538	1.450	30	316 184	1.532	1.532

As k increases, this ratio approaches

$$\frac{15}{4 \ln 10} \approx 1.6286,$$

so this heuristic argument suggests that the number of pyramids of height k should eventually increase by about 62.86% each time k is incremented by one. It follows that there should be 3-power-full prime pyramids of every height!

To test these claims, in table 1 we give the number of such pyramids $\#(3, k)$ of height $k \leq 30$, and compare the actual ratios for each height $\#(3, k)/\#(3, k-1)$ to our estimate (1). The 943 930 primes that make up these pyramids were first calculated using the function `isprime` in MAPLE® (see reference 2), which is a probabilistic test for larger numbers. The results were then verified in about twelve CPU-days using the function `isprime` in PARI (see reference 3) which is a deterministic test using the Selfridge–Pocklington–Lehmer $p-1$ test (see reference 4), or the APR-CL test (see reference 5), depending on the number.

The prime number theorem is an asymptotic estimate, so we do not expect our estimate $r(k)$ to fit too well for small values of k, but it fits quite well with larger values of k.

Another way to test our claims would be to show that a very tall pyramid can be calculated. In figure 2 we present one with height 113. Since the final prime in this pyramid has 3130 digits, we can't list them all. Instead we will show where to insert the next power of 3. Starting from the 0th level which is the number 1, the first number in the list above is 0. This means we insert 3^1 in the 0th position from the right to obtain 13. The second number is 1, so we insert $3^2 = 9$ in the 1st position from the right, giving 193 as our second level, and so on. We used Marcel Martin's ECPP implementation `Primo` (see, for example, reference 6, Section 5.10), to prove all 113 rows were prime. This took two CPU-months.

[0, 1, 1, 4, 3, 5, 0, 14, 14, 16, 15, 9, 14, 16, 35, 50, 1, 48, 50, 99, 87, 78, 25, 137, 142, 6, 102, 89, 21, 24, 214, 10, 26, 159, 248, 256, 317, 289, 89, 187, 69, 126, 365, 135, 261, 388, 243, 156, 158, 129, 414, 565, 19, 248, 699, 282, 99, 271, 364, 135, 459, 631, 909, 718, 697, 966, 200, 949, 491, 408, 904, 260, 93, 802, 1294, 69, 115, 94, 655, 639, 293, 754, 887, 875, 1109, 975, 732, 1584, 1280, 1907, 1236, 1827, 571, 1333, 637, 1526, 14, 2100, 1705, 399, 2297, 775, 1523, 1775, 2542, 1815, 2028, 2080, 2163, 2344, 890, 1595, 1348]

Figure 2

3. Powers of bases other than 3

The key difficulty in obtaining pyramids is the start. For example, using powers of the base $b = 6$, there is only one odd possibility for the next row, 61. It does happen to be prime, but then for the next row there are just two odd possibilities, 3661 and 6361; neither is prime. (This same difficulty can also be seen in the slow starting growth of $\#(3, k)$ in table 1.) Once the number at any particular level gets large, however, perhaps the numbers will grow (on the average) similar to (1). A depth-first search for the pyramids based on powers $b = 3c \leq 300$ up to height 30 also supports these conclusions (see table 2). We will leave it to the reader to explore these bases further.

4. Keeping the powers together

Above we allowed ourselves to insert 3^{k+1} anywhere into a_k to create new 3-power-full primes, but what if we only allow ourselves to insert the powers of 3 so that they never break up a previous power of 3? We call these *contiguous* 3*-power-full primes*. For example,

Table 2 Probable-prime b-power-full prime pyramid of maximal height.

l	bases b for which the tallest pyramid has height l
0	12, 30, 36, 45, 48, 72, 78, 84, 87, 90, 96, 108, 111, 114, 126, 135, 141, 144, 150, 156, 159, 165, 168, 177, 189, 192, 198, 204, 207, 210, 219, 240, 243, 246, 258, 261, 264, 270, 276, 282, 288, 294
1	9, 15, 18, 24, 51, 54, 75, 93, 102, 105, 153, 180, 186, 195, 222, 225, 234
2	6, 57, 99, 120, 171, 183, 255
3	60, 63, 81, 138, 147, 279
4	267
5	69, 162
6	237, 291
7	249, 300
9	39
12	297
≥ 30	3, 21, 27, 33, 42, 66, 117, 123, 129, 132, 174, 201, 213, 216, 228, 231, 252, 273, 285

Table 3 All contiguous 3-power-full primes.

h	contiguous 3-power-full primes of height h
1	13, 31
2	139, 193
3	12 739, 19 273
4	1 273 981, 1 278 139, 1 819 273, 1 927 813, 8 119 273
5	1 243 278 139, 1 243 819 273, 1 273 981 243, 1 819 243 273, 8 119 273 243
6	1 243 278 139 729, 1 243 819 273 729, 1 243 819 277 293, 8 119 273 729 243
7	12 187 243 278 139 729, 12 432 781 321 879 729, 81 192 737 292 432 187, 81 218 719 273 729 243
8	121 872 432 781 396 561 729, 121 872 432 781 397 296 561, 121 872 432 781 656 139 729, 811 927 372 965 612 432 187, 811 965 612 737 292 432 187, 812 187 192 765 613 729 243
9	11 968 321 872 432 781 397 296 561, 12 187 243 278 139 196 837 296 561, 19 683 121 872 432 781 397 296 561, 19 683 811 927 372 965 612 432 187, 81 196 561 271 968 337 292 432 187, 81 196 831 965 612 737 292 432 187
10	1 218 724 359 049 278 139 196 837 296 561, 1 968 312 187 243 278 139 590 497 296 561, 1 968 312 187 590 492 432 781 397 296 561, 8 119 656 127 590 491 968 337 292 432 187

1 218 724 359 049 278 139 196 837 296 561 is such a 3-power-full prime as its digits can be partitioned as follows:

$$[1]\,[2187]\,[243]\,[59049]\,[27]\,[81]\,[3]\,[9]\,[19683]\,[729]\,[6561].$$

This changes (1) by greatly reducing the numerator. The corresponding limit then becomes zero, so the number of contiguous b-power-full primes should be finite for any base b. Indeed, for $b = 3$ we find the *complete list* contains only the 40 primes in table 3.

There are other possibilities that the reader might want to explore. What, for example, if we start our towers with a digit other than 1? Suppose that we do not restrict our constructions to pyramids of primes, and instead just ask how large a prime can be formed in each of these manners? Good luck!

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Anagrams

$$\begin{aligned} 21 \times 32 &= 672, & 221 \times 312 &= 68\,952,\\ 12 \times 23 &= 276, & 122 \times 213 &= 25\,986,\end{aligned}$$

$$\begin{aligned} 12 \times 12 &= 144, & 13 \times 13 &= 169,\\ 21 \times 21 &= 441, & 31 \times 31 &= 961,\end{aligned}$$

$$\begin{aligned} 102 \times 102 &= 10\,404, & 103 \times 103 &= 10\,609,\\ 201 \times 201 &= 40\,401, & 301 \times 301 &= 90\,601,\end{aligned}$$

$$\begin{aligned} 112 \times 112 &= 12\,544, & 122 \times 122 &= 14\,884,\\ 211 \times 211 &= 44\,521, & 221 \times 221 &= 48\,841.\end{aligned}$$

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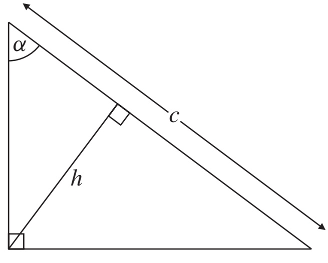
Express h in terms of c and α.



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Markov Processes in Management Science

ALAN OXLEY

Introduction

Markov process models are used to study how systems behave over time periods. As an example, assume that we have a group of consumers. Each consumer purchases a product every week. There are only two brands of the product – A and B. We can set up a model to find out the share of the market that brand A has and the share that brand B has, in the long run. We would need to know the relevant probabilities – the probability that a consumer purchasing brand A stays loyal to the product from one week to the next, and the probability that a consumer purchasing brand B stays loyal to it. Whilst these probabilities easily describe what happens in the first week, the amounts purchased of brands A and B in subsequent weeks will change week by week. However, as the weeks roll by, the size of the change reduces. Ultimately we reach a steady state solution. As far as the mathematics is concerned, we can go straight to calculating this steady state solution. For example,

$$\begin{aligned}
\text{P(consumer stays loyal to A)} &= \tfrac{9}{10},\\
\text{P(consumer stays loyal to B)} &= \tfrac{4}{5},\\
\text{(number of A customers this week)} &= \tfrac{9}{10} \times \text{(number of A customers last week)}\\
&\quad + \tfrac{1}{5} \times \text{(number of B customers last week).}
\end{aligned}$$

The difference between weeks gets smaller and smaller as the weeks go by so we end up with

$$\begin{aligned}
\text{(number of A customers in the long run)} &= \tfrac{9}{10} \times \text{(number of A customers in the long run)}\\
&\quad + \tfrac{1}{5} \times \text{(number of B customers in the long run).}
\end{aligned}$$

Simplifying gives

$$\text{(number of A customers in the long run)} = 2 \times \text{(number of B customers in the long run).}$$

That is, A will end up with $\frac{2}{3}$ of the market share; B will be left with $\frac{1}{3}$ of the share.

A similar analysis can be undertaken for a situation where three brands are competing; this is left to the reader.

There are a wide range of applications in which the above analysis can be applied. Let us now consider an example concerned with estimating bad debts.

Estimating bad debts

Any company needs to keep track of the money that it is owed from its customers as well as from other companies. We could classify individual amounts as follows:

$$\begin{gathered}\leq 30 \text{ days overdue,}\\ \text{31–90 days overdue,}\\ \text{91 or more days overdue.}\end{gathered}$$

The company must wonder whether it will ever receive those amounts in the latter category. They are so long overdue we could call them 'bad debts'.

From time to time a company must estimate what it will lose in the way of bad debts over a period of time in the future. In time, some of the amounts currently between 31 and 90 days old will become bad debts. When more time elapses, similarly, some of those in the first category will become bad debts.

Problem Given the amount owed today in the '≤ 30 days overdue' category and the amount owed in the '31–90 days overdue' category, how much of this will end up as bad debt?

Solution We can look at the historical records of the company to work out what proportion of the total amount ever to have been in the '≤ 30 days overdue' category eventually ended up as a bad debt. Similarly, we can work out what proportion of the total amount ever to have been in the '31–90 days overdue' category eventually ended up as a bad debt.

Example We have

$$\begin{aligned}\text{P('}\leq 30\text{ days overdue' amount ends up as bad debt)} &= \tfrac{1}{7},\\ \text{P('31–90 days overdue' amount ends up as bad debt)} &= \tfrac{1}{3},\\ \text{actual amount, today, in '}\leq 30\text{ days overdue' category} &= £1\,000,\\ \text{actual amount, today, in '31–90 days overdue' category} &= £2\,000.\end{aligned}$$

How much of this £3 000 do we predict will end up as bad debt? The answer is

$$1\,000 \times \tfrac{1}{7} + 2\,000 \times \tfrac{1}{3} = £809.52.$$

Can the company's management use a strategy to reduce the level of bad debt? Yes they could. For example, they could give a discount for prompt payment in the hope that it would increase the likelihood that amounts in the '≤ 30 days overdue' category would be paid.

Problem How can we estimate the reduction in debt due to this policy?

Solution 1 – Guess how the overall probabilities change

Here we want to guess new values for

$$\begin{gathered}\text{P('}\leq 30\text{ days overdue' amount ends up as bad debt),}\\ \text{P('31–90 days overdue' amount ends up as bad debt).}\end{gathered}$$

The problem with this is that it is difficult to get a feel for these probabilities.

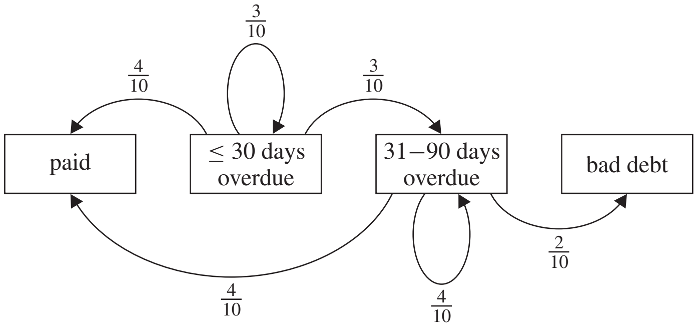

Figure 1

Solution 2 – Guess how the constituent probabilities change

In order to understand this solution, we must get an idea of what these constituent probabilities are and how the overall probabilities are derived from them.

At any point in time an account is in one of the following four states: paid, $\leq$ 30 days overdue, 31–90 days overdue, or bad debt. We can look again at the historical records of the company, this time in more detail. We need to work out the probability that an account in one state will have moved to another state, or stayed in the same state, in, say, 1 week's time.

Let us consider an example. We will use a state diagram to show the probabilities of transferring from one state to another (see figure 1).

Alternatively, we can tabulate this information; see table 1.

Let us now try to find the probabilities in the long run. We will use a similar analysis to that which we used in solving the market share problem given earlier, i.e.

$$\begin{aligned}(\text{amount in `paid' this week}) = {} & (\text{amount in `paid' last week}) \\ & + \tfrac{4}{10} \times (\text{amount in `$\leq$ 30 days overdue' last week}) \\ & + \tfrac{4}{10} \times (\text{amount in `31–90 days overdue' last week}).\end{aligned}$$

Ignoring new accounts, we end up with the following degenerate solution:

$$(\text{amount in `paid' in the long run}) = (\text{amount in `paid' in the long run}) + 0 + 0.$$

We therefore have to use another approach.

Table 1

		next period			
		paid	$\leq$ 30 days overdue	31–90 days overdue	bad debt
current period	paid	1	0	0	0
	$\leq$ 30 days overdue	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	0
	31–90 days overdue	$\frac{4}{10}$	0	$\frac{4}{10}$	$\frac{2}{10}$
	bad debt	0	0	0	1

Table 2

N		next state	
		≤ 30 days overdue	31–90 days overdue
current state	≤ 30 days overdue	$\frac{10}{7}$	$\frac{5}{7}$
	31–90 days overdue	0	$\frac{5}{3}$

We need to calculate the proportion of an account currently in the '$\leq$ 30 days overdue' category that will eventually end up as bad debt, and the proportion of an account currently in the '31–90 days overdue' category that will eventually end up as bad debt. This requires the use of matrix algebra.

Let $\boldsymbol{Q}$ be a matrix showing the probabilities for the '$\leq$ 30 days overdue' and '31–90 days overdue' states only, i.e.

$$\boldsymbol{Q} = \begin{bmatrix} \frac{3}{10} & \frac{3}{10} \\ 0 & \frac{4}{10} \end{bmatrix}.$$

We then evaluate $\boldsymbol{N} = (\boldsymbol{I} - \boldsymbol{Q})^{-1}$, which is referred to as the *fundamental matrix*,

$$\begin{aligned} \boldsymbol{N} &= (\boldsymbol{I} - \boldsymbol{Q})^{-1} \\ &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{10} & \frac{3}{10} \\ 0 & \frac{4}{10} \end{bmatrix} \right\}^{-1} \\ &= \begin{bmatrix} \frac{7}{10} & -\frac{3}{10} \\ 0 & \frac{3}{5} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{10}{7} & \frac{5}{7} \\ 0 & \frac{5}{3} \end{bmatrix}. \end{aligned}$$

The n_{ij} entry gives the average time the process is in state j given that it began in state i. Consider table 2. The $\frac{10}{7}$ entry is the average number of weeks that a debt spends in the state '$\leq$ 30 days overdue' given that it started in the state '$\leq$ 30 days overdue'. The $\frac{5}{7}$ entry is the average number of weeks that a debt spends in the state '31–90 days overdue' given that it started in the state '$\leq$ 30 days overdue', and so on.

We now need to use

$$\boldsymbol{R} = \begin{bmatrix} \frac{4}{10} & 0 \\ \frac{4}{10} & \frac{2}{10} \end{bmatrix},$$

where $\boldsymbol{R}$ has the transition probabilities shown in table 3. Now,

$$\begin{aligned} \boldsymbol{N}\boldsymbol{R} &= \begin{bmatrix} \frac{10}{7} & \frac{5}{7} \\ 0 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} \frac{4}{10} & 0 \\ \frac{4}{10} & \frac{2}{10} \end{bmatrix} \\ &= \begin{bmatrix} \frac{6}{7} & \frac{1}{7} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}. \end{aligned}$$

Table 3

	in 1 week is in the state 'paid'	in 1 week is in the state 'bad debt'
currently in the state '$\leq$ 30 days overdue'	$\frac{4}{10}$	0
currently in the state '31–90 days overdue'	$\frac{4}{10}$	$\frac{2}{10}$

Looking at the second column we have:

$$\text{P('}\leq 30\text{ days overdue' amount ends up as bad debt)} = \tfrac{1}{7},$$
$$\text{P('31–90 days overdue' amount ends up as bad debt)} = \tfrac{1}{3}.$$

These are the same figures we arrived at when we initially looked at the historical records, before we started to consider different states. If we use a discount policy to reduce the level of debt then we need to guess how the constituent probabilities change.

Adopting a new policy

Let us now return to the original problem. Management's task is to attempt to reduce one or both of the following probabilities:

$$\text{P('}\leq 30\text{ days overdue' amount ends up as bad debt)},$$
$$\text{P('31–90 days overdue' amount ends up as bad debt)}.$$

They have proceeded by looking at historical records and producing the state diagram (see figure 1). They are considering offering a discount for prompt payment. They must estimate the new probabilities for the transitions between states. Let us assume that the company believes that offering some sort of early payment incentive will change the original state diagram to the one in figure 2. Here, two of the probabilities have changed. The proportion of accounts that are settled within 30 days has risen to $\frac{6}{10}$ whilst the proportion moving into the '31–90 days overdue' category has fallen to $\frac{1}{10}$.

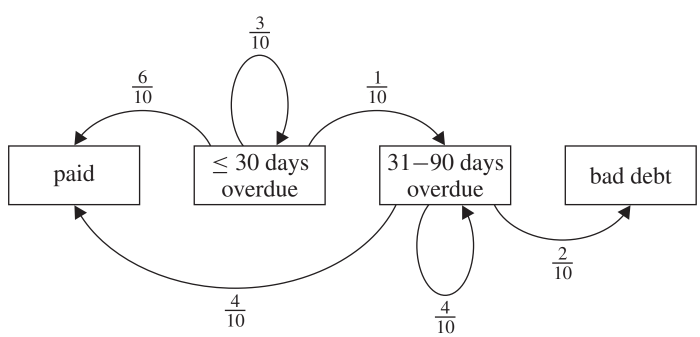

Figure 2

Next the company perform the matrix calculations, as described above. This will give them new overall probabilities from which they can calculate an estimate of the total amount that is likely to become bad debt as follows:

$$\boldsymbol{Q} = \begin{bmatrix} \frac{3}{10} & \frac{1}{10} \\ 0 & \frac{4}{10} \end{bmatrix}, \qquad \boldsymbol{N} = \begin{bmatrix} \frac{10}{7} & \frac{10}{42} \\ 0 & \frac{10}{6} \end{bmatrix}, \qquad \boldsymbol{R} = \begin{bmatrix} \frac{6}{10} & 0 \\ \frac{4}{10} & \frac{2}{10} \end{bmatrix}, \qquad \boldsymbol{NR} = \begin{bmatrix} \frac{20}{21} & \frac{1}{21} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

This gives

$$\text{P('}\leq 30\text{ days overdue' amount ends up as bad debt)} = \tfrac{1}{21},$$
$$\text{P('31–90 days overdue' amount ends up as bad debt)} = \tfrac{1}{3}.$$

In the original problem we had £3 000 owing to the company. The new estimate for the amount that will end up as bad debt is

$$1\,000 \times \tfrac{1}{21} + 2\,000 \times \tfrac{1}{3} = £714.29.$$

The company can compare the two bad debt amounts – one without the early settlement policy in place and one with it in place. There is a £95.23 difference. Finally, the company can decide whether the savings justify the usage of the policy, bearing in mind that some of the £95.23 will have to be spent on incentives.

There are a large number of texts that describe Markov processes. An example in the Management Science application area is given by reference 1.

Reference

1 D. R. Anderson, D. Sweeney, T. A. Williams and K. Martin, *An Introduction to Management Science*, 12th edn. (Thomson South-Western, Mason, OH, 2007).

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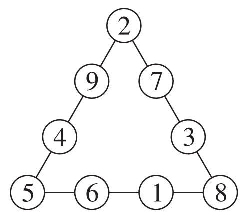
A magic triangle

$$2 + 9 + 4 + 5 = 20 \qquad 2^2 + 7^2 + 3^2 + 8^2 = 126$$
$$2 + 7 + 3 + 8 = 20 \qquad 2^2 + 9^2 + 4^2 + 5^2 = 126$$
$$8 + 1 + 6 + 5 = 20 \qquad 5^2 + 6^2 + 1^2 + 8^2 = 126$$



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Strung Out on the M25

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Introduction

Every school-child and adult involved in the practice of map reading will know that a working estimate for distances travelled on foot or otherwise can be gleaned from a scaled, relevant map using a piece of string. The purpose here is to provide a mathematical model for such an exercise with particular emphasis on an application to distances travelled around the M25 orbital ring road that encircles Greater London and the area enclosed therein. There has been much interest of late in just such endeavours (see reference 1), where it was deemed appropriate to view the M25 as an ellipse in order to model matters of interest. This is a worthy mathematical exercise but it is one that raises also the question of a comparison with reality. A look at any road map of the Greater London area will indicate that the ring road in question is anything but the shape of an ellipse and the results of the exercise proposed here will indicate as much although it will be seen that it is possible to approximate the shape of the M25 in some sense by an ellipse.

Most of the sums associated with this exercise involve only algebra, simple trigonometric geometry, and possibly numerical integration. Moreover, the example affords an instance of applied mathematics at a fundamental and practical level.

To keep matters simple, it will be assumed, as is usual, that the terrain is flat, even though to the motorists on the north side of the M25 adjacent to the Bell Common tunnel and on the south side straddling the M23 valley the gradients appear to be anything but benign. Nonetheless, this is a reasonable assumption given the large distances involved, and the exercise can be repeated for any orbital ring road surrounding any large city in any country provided that a suitable map to work from is available.

The flat string model

If, from a $1 : 200\,000$ scaled map of the M25 (see references 2 and 3), a sufficient number of map pins – N say – are inserted at various points, or nodes, around the perimeter to support an encircling string, then it is possible to estimate the circumference simply by measuring the string and scaling the result. Alternatively, with a suitably chosen origin, the node coordinates can be scaled from the map in tabular form and the circumference can be viewed as the sum of a series of concatenated line segment lengths (to be calculated) between adjacent nodes. To keep matters simple, the nodes are assumed to lie, say, along the central reservation and the width of the motorway is assumed to be negligible, so that traffic going one way round will travel the same distance as traffic going round the other way. Other second-order effects due to radiusing that would provide a smooth transition from one line segment to the next are also ignored (a more detailed map would be required to accommodate these effects).

For the present purpose, the origin of coordinates is chosen to lie more or less at the middle of Westminster Bridge. This particular choice enables us to treat the area enclosed by the M25 as a straightforward superposition of areas of independent contiguous triangles.

For the calculations regarding the number of line segments and nodes a value of $N = 100$ was chosen. A higher value could have been chosen but it is likely that not much more

Table 1 M25 node coordinates in mm extracted from a $1:200\,000$ scaled map.

N	X	Y	N	X	Y	N	X	Y
1	−25	−134	35	109.5	86	69	−142	57
2	−15.5	−129	36	103	89.5	70	−145	36.5
3	13	−135	37	84.5	101.5	71	−141.5	30
4	22.5	−134.5	38	76	105.5	72	−140	24.5
5	30	−134.5	39	69	106.5	73	−135	21
6	39.5	−129	40	59	104	74	−131.5	16.5
7	53	−128	41	51	100.5	75	−131	3.5
8	60.5	−126	42	44.5	101	76	−129	−8.5
9	67	−125	43	37.5	101	77	−136	−24
10	77.5	−121	44	17	103	78	−138	−30
11	84	−118	45	8.5	106.5	79	−144	−40
12	94	−114.5	46	1.5	106	80	−144	−48
13	97.5	−109.5	47	−8	103	81	−144	−55.5
14	101	−105	48	−15	104	82	−140.5	−64.5
15	100	−94	49	−21	101.5	83	−134	−71
16	99	−84.5	50	−28.5	102.5	84	−130.5	−79.5
17	99.5	−75	51	−32	102	85	−125	−88
18	108	−66.5	52	−36	101.5	86	−124	−95
19	112.5	−57	53	−40.5	103	87	−119	−99.5
20	118.5	−48.5	54	−48	114	88	−110	−102.5
21	126	−39.5	55	−55	117.5	89	−102.5	−107.5
22	126.5	−32.5	56	−64	114.5	90	−95	−110
23	126	−26.5	57	−72	115.5	91	−84.5	−107
24	132	−20	58	−81	114	92	−75	−105
25	135	−13	59	−89.5	116.5	93	−68.5	−107
26	136	−6.5	60	−99.5	116	94	−63.5	−115
27	135.5	7.5	61	−108.5	114	95	−59	−117
28	140	20	62	−114	108	96	−52	−118.5
29	141.5	29	63	−115	100.5	97	−47.5	−121
30	143	35	64	−118.5	95	98	−43.5	−128.5
31	139.5	46.5	65	−130	90	99	−38	−133.5
32	136	51.5	66	−134	75.5	100	−32.5	−136
33	133.5	63.5	67	−138	70.5			
34	121	73.5	68	−141	65			

useful information could be gleaned using the available scale. Specifically, adopting the usual rectangular coordinate axis system, (X, Y) where the positive Y-axis points northwards, coordinates can be read from the scaled map, more or less as per table 1.

In order to relate the coordinates in table 1 to measurements on the ground it will suffice to multiply the coordinate numbers in table 1 by 0.2 and call the result km. Data modified in this way can be plotted on a spreadsheet to reproduce a recognisable M25 map (see figure 1).

It may be appreciated from figure 1 that the map of the M25 does not look like an ellipse. However, it is appropriate to raise the issue as to just how well it can be approximated by one. To this end, it will suffice to recall that the area A enclosed within an ellipse of eccentricity e

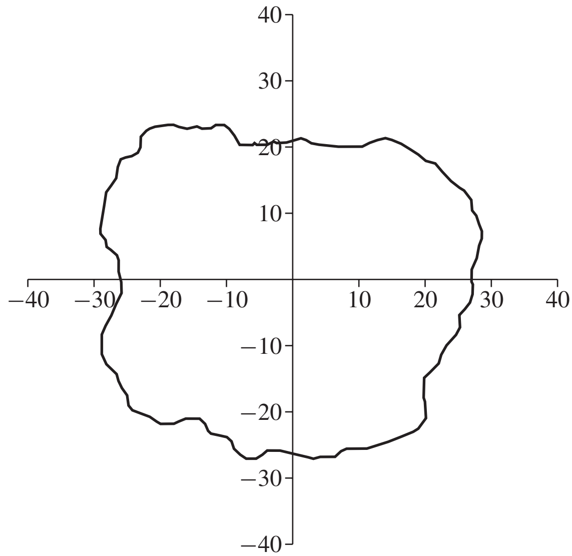

Figure 1 M25 map (scale in km, north vertical).

is given by

$$A = \pi a b, \tag{1}$$

where a and b denote the lengths of the semi-major and semi-minor axes respectively, and $b = a\sqrt{1-e^2}$. The circumference C of the ellipse is given by

$$C = 4aE(e), \tag{2}$$

where $E(e)$ denotes the complete elliptic integral of the second kind, i.e.

$$E(e) = \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2(t)}\,\mathrm{d}t,$$

which, strictly, is a matter for numerical integration. However, there are useful approximations to the elliptic integral (see reference 4) that can be used here to good effect. Specifically, they can be used to relate the sum of the semi-major and semi-minor axes of the ellipse to its circumference C and its enclosed area A, in a simple quadratic form. One of the more accurate of these gives $C \approx \pi(3(a+b) - \sqrt{(3a+b)(3b+a)}$. It is *Ramanujan's first approximation* and it can be manipulated using (1) to produce the quadratic equation

$$\alpha(a+b)^2 - \beta(a+b) + \gamma = 0, \tag{3}$$

where

$$\begin{aligned} \alpha &= 1, \\ \beta &= \frac{C}{\pi}, \\ \gamma &= \frac{1}{6}\biggl(\Bigl(\frac{C}{\pi}\Bigr)^2 - 4\Bigl(\frac{A}{\pi}\Bigr)\biggr). \end{aligned} \tag{4}$$

Another useful but not as accurate quadratic arises if, in the above, the parameters β and γ are replaced by $\beta = 0$ and $\gamma = -\frac{1}{3}(4(A/\pi) + 2(C/\pi)^2)$; these parameter values arise from a less accurate approximation (see reference 4). Attention hereafter will focus on the use of the first quadratic but the arguments apply just as well to the simpler, less accurate, alternative. The manner in which these expressions can be used is quite straightforward. First, an area A and a circumference C of a closed region of interest is determined – by measurement or otherwise. These values are then substituted into (4) to determine the coefficients of the quadratic equation (3), which is solved in the usual way to determine the roots $a + b$, denoted hereafter by μ. Moreover, since

$$b = \frac{1}{a}\frac{A}{\pi},$$

it follows that

$$a + \frac{1}{a}\frac{A}{\pi} = \mu. \tag{5}$$

This is another quadratic that is dependent on the solution to the first one. It can be solved to produce a value for a, and hence b. It is now fruitful to examine the scenario with reference to the M25.

The M25 approximating ellipse

To address this it will be necessary to determine (from the data in table 1) the circumference of the M25, seen as the sum of the lengths of concatenated line segments, and the enclosed area seen as the sum of areas of contiguous triangles formed by two adjacent nodes and the origin of the coordinate system. This is a straightforward exercise involving coordinate geometry. For example, if two adjacent nodes have coordinates (X_N, Y_N) and (X_{N+1}, Y_{N+1}), then the line segment length L_N, say, is given by

$$L_N = \sqrt{(X_{N+1} - X_N)^2 + (Y_{N+1} - Y_N)^2},$$

and the area of the triangle formed with the origin may be calculated using Heron's formula, i.e.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where

$$\begin{aligned} s &= \tfrac{1}{2}(a + b + c), \\ a &= \sqrt{X_N^2 + Y_N^2}, \\ b &= \sqrt{X_{N+1}^2 + Y_{N+1}^2}, \\ c &= L_N. \end{aligned}$$

In this manner, the perimeter C_m of the M25 and the enclosed area A_m were found to be

$$C_m = 188.382\,\text{km}, \qquad A_m = 2316.185\,\text{km}^2.$$

These values will be referred to as measurement-based data. It is now appropriate to examine in detail just how the formulae in the previous section can be driven to reproduce this data. In the process, it will be assumed that there is an allowable error of $p\,\%$ in respect of the area, so that

Table 2 M25 approximating ellipse parameters b and a for various p and q values.

error input parameters		M25 ellipse approximating parameters	
p	q	b	a
0	0	18.715	39.394
−5.50	5.50	24.111	32.260

$A = A_m(1 - p/100)$, and an allowable error of $q\,\%$ in respect of the circumference, so that $C = C_m(1 - q/100)$. Moreover, there is no restriction on the sign of the parameters p and q. These values for A and C (now dependent on A_m, C_m, p, and q) are then substituted into (4) to facilitate root-finding for both quadratic equations (3) and (5). This procedure was carried out for various values of p and q to produce results as in table 2.

Using equations (1) and (2), it is also possible to verify the data produced in table 2 via a numerical integration routine for the elliptic integral.

Discussion of results

From the results in table 2, it would appear that there is one ellipse that will match the circumference and enclosed area of the M25 without error (p and q both zero) and it is instructive to view such an ellipse against the backdrop of the M25 profile of figure 1. This can be seen in figure 2.

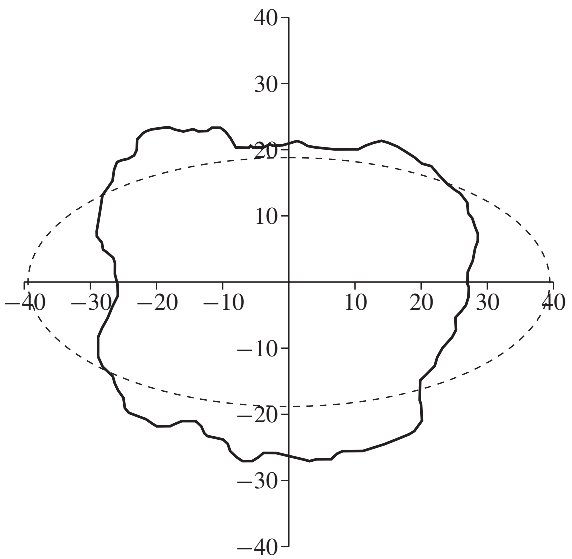

Figure 2 A comparison of the M25 map with a matching ellipse (scale in km, north vertical, $b = 18.715$, $a = 39.394$ km).

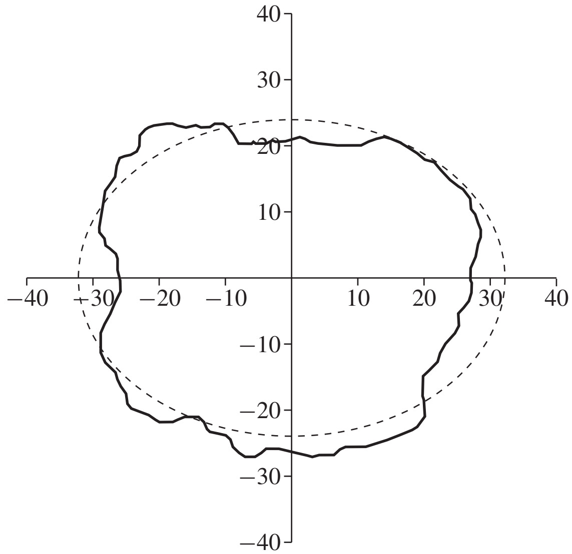

Figure 3 An alternative penalty driven M25 approximating ellipse (scale in km, north vertical, $e = 0.6644$, $a = 32.260$ km).

Not surprisingly, it can be seen from figure 2 that the matching ellipse does not look much like the M25. However, it is possible to achieve a more appealing result, but at some cost. Accordingly, the approximating ellipse for the other values of p and q displayed in table 2 is shown against the same backdrop in figure 3.

The picture shown in figure 3 is more appealing but the approximating ellipse is still not properly centred. The fit can be improved by relocating the origin of the ellipse to a point about 2.5 km directly southwards. This would put the origin at a point somewhere on the Albert embankment between Lambeth and Vauxhall bridges. It is left as an exercise to the interested reader to explore other equally appealing approximations for different sets of the input error parameters p and q.

Conclusion

When our attention is drawn to the business of the M25 and its resemblance to an ellipse, we must be quite clear in our mind as to what is meant otherwise there is the risk of being strung out and left to ponder. With a keener eye and a larger scale map it would be possible to refine further the data obtained above, but it is unlikely that the overall picture would improve significantly to produce a penalty-free ellipse looking like the M25.

Acknowledgement

The author is pleased to acknowledge suggestions from an anonymous referee concerning approximations to the elliptic integral.

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4 http://local.wasp.uwa.edu.au/∼pbourke/geometry/ellipsecirc/.

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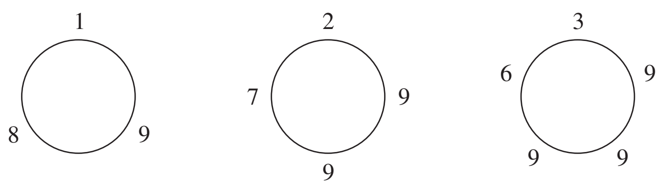
Circular numbers



$$\begin{aligned} 198 &= 19 + 98 + 81, \\ 2997 &= 299 + 997 + 972 + 729, \\ 39996 &= 3999 + 9996 + 9963 + 9639 + 6399. \end{aligned}$$

The pattern continues with

$$499995, \quad 5999994, \quad 69999993, \quad 799999992, \quad 8999999991.$$

Examples in bases other than 10 are

$$(121)_3, \quad (132)_4, \quad (2331)_4, \quad (143)_5, \quad (2442)_5, \quad (34441)_5,$$
$$(154)_6, \quad (2553)_6, \quad (35552)_6, \quad (455551)_6,$$

with similar examples in bases 7, 8, and 9. In base n, the 3-digit circular number is

$$(1(n-1)(n-2))_n.$$

Similar formulae can be found for four or more digits.

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Intimations of Immortality – Will the Game go on for Ever?

JOHN FERDINANDS

Introduction

The fat schoolboy Billy Bunter was at one time one of the most famous characters in English fiction. An incident in the book *Billy Bunter's Beanfeast* (see reference 1) suggests an interesting mathematical problem. A schoolboy named Vernon-Smith works out a system for winning the French gambling game *la boule*. Here is how Vernon-Smith describes the game to his friend Redwing:

> There's a sort of shallow bowl on a green baize table, with numbers marked round it, one to nine. The man spins the ball in the bowl, and it falls into a numbered slot. That's the winning number.

One possible gamble is to bet on the ball ending up in an even (or odd) slot, with 5 being considered neither even nor odd. A bet of A units has a potential payoff of A. Now listen to Vernon-Smith describe his system.

> You wait till odd, say, has come up four times. Then you back even. The chances are that you will win. But if there's a slip, and odd comes up for a fifth time, you double your stake on even next. And if odd still comes up, you re-double. That makes it a cert. You win, and get back all you've lost, with an amount over and above. I tell you, Reddy, that a man playing on this system couldn't lose.

Redwing was sceptical; 'he doubted very much whether professional gamesters could ever be beaten at their own game'. And his scepticism was well founded – it is not hard to see why Vernon-Smith's conclusion is incorrect. The strategy would work perfectly for a gambler with infinite resources, but is likely to be less effective for a real-life gambler with a finite number of chips, who could be wiped out by a sufficiently long string of losses.

The strategy of doubling your bet after each loss is known as the *martingale system*, and has appealed to many gamblers over the years. Grimmett and Stirzaker's book (see reference 2, pp. 302–303) mentions two rather famous people who were aware of the system. Apparently 'Thackeray's advice was to avoid its use at all costs', while Casanova's *Memoirs* record two occasions when he used it. All went well the first time, but speaking of the second time he said: 'I still played the martingale, but with such bad luck that I was soon left without a sequin'.

The mathematics of the martingale system has some interesting similarities to the mathematics of another popular gambling system: that of life insurance. The theory of probability originated in an exchange of letters between Pascal and Fermat about a gambling question. From such an apparently frivolous beginning, the theory has grown to embrace such weighty matters as life insurance. This article reverses the historical sequence: we begin with a fact in the mathematics of life insurance, and show that an analogous result holds for the martingale system of gambling.

For a person whose age is x years, let $A(n)$ be the probability of surviving to age $x+n$, and let $D(n)$ be the probability of dying at age $x+n$. It is clear that the sum of the infinite series $\sum_{n=0}^{\infty} D(n)$ gives the probability that a person aged x years will eventually die. As Benjamin Franklin pointed out long ago, this implies that $\sum_{n=0}^{\infty} D(n) = 1$. ('In this world, nothing can be said to be certain except death and taxes' (see reference 3).)

We shall assume that the gambler uses the martingale sytem in the following way: he repeatedly makes the same bet in a game such as *la boule* or roulette, so that there is a fixed probability of winning on each play. His initial bet is 1 chip and he doubles after each loss, returning to a bet of 1 chip after each win. Each time he wins, his payoff is equal to the amount that he bet. He will quit the game when the bet required by the martingale system is greater than his current fortune. We define $A(n)$ and $D(n)$ in ways appropriate to this game, and such that the infinite sum $\sum_{n=0}^{\infty} D(n)$ is the probability that the game will eventually end. Our main result is that under certain conditions $\sum_{n=0}^{\infty} D(n)$ is less than 1; in other words, there is a chance that the game will be immortal.

The function *A*(*n*)

Let us suppose that our gambler has a fixed probability p of winning on each bet, that he begins the game with 50 chips, and that he initially bets 1 chip. He will double his stake after each loss, and will quit the game, or 'die', if he does not have enough chips to cover his next bet. If he loses five straight times, his total loss is $1+2+2^2+2^3+2^4 = 31$ chips. His next bet should be $2^5 = 32$ chips, but his fortune is only $50-31 = 19$ chips; therefore his game is over. We generalize this result with the following lemma.

Lemma 1 *Suppose that the gambler's fortune is $F = 2^n + i$, for $n \geq 1$ and $-1 \leq i < 2^n - 1$. Then n straight losses would eliminate him.*

Proof If $n = 1$, then the only possibilities for F are 1 and 2, and it is easy to see that one loss would end the game. So suppose that $n \geq 2$. After n straight losses, his fortune is $(2^n+i) - (1+2+2^2+2^3+\cdots+2^{n-1}) = (2^n+i) - (2^n-1) = i+1$. Now if $i < 2^n - 1$, then $i+1 < 2^n$, and he does not have enough to cover a bet of 2^n chips; therefore his game is over after n straight losses.

It is convenient to think of the outcomes of the game in terms of clusters rather than individual plays. The first cluster begins when the gambler makes the first play, and ends either with the first win or with his elimination. The nth cluster begins after his $(n-1)$th win, and ends either with a win or with 'death'. Suppose that during a cluster the gambler loses k straight times and then wins. His net gain is $2^k - (1+2+2^2+\cdots+2^{k-1}) = 1$. It follows that when a cluster ends, the gambler has either increased his fortune by 1 chip or has been forced to quit the game.

We have assumed that the gambler's initial capital is 50 chips. Define $A(n)$ to be the probability that he will survive the first n clusters (analagous to the probability that a person of a given age will survive for n more years). We have seen that the gambler will 'die' on the first sequence if and only if he loses five straight times. The probability of winning a single play is p, so the probability of losing a single play is $q = 1-p$; hence the probability of dying on the first cluster is q^5. It follows that the probability of surviving the first cluster is $1-q^5$, i.e. $A(1) = 1-q^5$.

To compute $A(n)$ for general n, we need two observations. First, after surviving a cluster the gambler's fortune increases by 1 chip, so when he has survived n clusters his fortune will be $50+n$. Second, since surviving the $(n+1)$th cluster is independent of surviving the first n clusters,

$$A(n+1) = (\text{the probability of surviving the } (n+1)\text{th cluster})A(n).$$

After surviving the first cluster the gambler's fortune is 51 chips, and it will take five straight losses to end the game on the second cluster; hence $A(2) = (1-q^5)A(1) = (1-q^5)^2$. By the same reasoning, $A(n) = (1-q^5)^n$ for $1 \le n \le 13$.

After 13 successful clusters, the gambler's fortune has grown to $63 = 2^6 - 1$ chips, and it follows from lemma 1 that it will now take six straight losses to 'kill' him on the next cluster. Therefore, $A(14) = (1-q^6)(1-q^5)^{13}$. Moreover, $A(n) = (1-q^5)^{13}(1-q^6)^{n-13}$ for $14 \le n \le 77$, while for $78 \le n \le 205$, $A(n) = (1-q^5)^{13}(1-q^6)^{64}(1-q^7)^{n-77}$.

We now state the general formula for $A(n)$. A proof by induction is straightforward, and the details are left to the reader.

Formula for *A*(*n*)

Suppose that the gambler's initial fortune is $F = 2^a + i$, where $a \ge 1$ and $0 \le i \le 2^a - 1$.

(a) If $F + n \le 2^{a+1} - 1$, then $A(n) = (1-q^a)^n$.

(b) If $F + n = 2^b + j$, where $b \ge a+1$ and $0 \le j \le 2^b - 1$, then

$$A(n) = (1-q^a)^{2^a-i-1}(1-q^{a+1})^{2^{a+1}}(1-q^{a+2})^{2^{a+2}}\cdots(1-q^{b-1})^{2^{b-1}}(1-q^b)^{j+1}.$$

We will illustrate this result with a numerical example. Suppose that our gambler plays roulette and bets on the ball ending up in a black slot. An American roulette wheel has 38 slots, of which 18 are black; so the probability of winning is $p = \frac{18}{38}$ and $q = \frac{20}{38}$. Let the initial fortune be 50 chips. Then $A(13) = 0.5851$ and $A(50) = 0.2642$. The latter number is the surprisingly high probability that the gambler will live to double his initial capital of 50 chips.

Will the game ever end?

We define $D(n)$ to be the probability that the gambler survives the first n clusters and 'dies' on the $(n+1)$th cluster. Then the probability that the game will eventually end is equal to the sum of the series $\sum_{n=0}^{\infty} D(n)$.

We derive two equations involving $A(n)$ and $D(n)$. Since the gambler either survives or does not survive the first cluster, $D(0) = 1 - A(1)$. For $n \ge 1$, it follows by independence that

$$D(n) = (\text{probability of 'dying' on the } (n+1)\text{th sequence})A(n).$$

Now recall that

$$A(n+1) = (\text{the probability of surviving the } (n+1)\text{th sequence})A(n).$$

Adding the last two equations, we see that $D(n) + A(n+1) = A(n)$. So we have the following relationship:

$$\begin{aligned} D(0) &= 1 - A(1), \\ D(n) &= A(n) - A(n+1), \qquad \text{for } n \ge 1. \end{aligned}$$

This implies that the series $\sum_{n=0}^{\infty} D(n)$ telescopes, and its partial sum is $\sum_{i=0}^{n} D(i) = 1 - A(n+1)$. By letting $n \to \infty$, we see that

$$\sum_{n=0}^{\infty} D(n) = \lim_{n\to\infty} (1 - A(n+1)). \tag{1}$$

Since $\sum_{n=0}^{\infty} D(n)$ is the probability that the gambler will eventually 'die', it follows that the probability of 'dying' is $\lim_{n\to\infty}(1 - A(n+1))$, and the probability of immortality is $1 - \sum_{n=0}^{\infty} D(n) = \lim_{n\to\infty} A(n+1)$.

Conditions for death and for immortality

Whether or not the game will end depends on the probability p of winning a single bet. Since the cases $p = 0$ or $p = 1$ are trivial, we assume that $0 < p < 1$.

Theorem 1 *If $0 < p \le \frac{1}{2}$, there is a probability of 1 that 'death' will occur, or that the game will end. If $\frac{1}{2} < p < 1$, 'immortality' is possible; there is a nonzero probability that the game will never end.*

The following claim, together with (1), implies the truth of theorem 1.

(a) If $0 < p \le \frac{1}{2}$, then $\sum_{n=1}^{\infty} A(n)$ converges, and hence the sequence $A(n)$ has limit 0.

(b) If $\frac{1}{2} < p < 1$, the sequence $A(n)$ has a limit that is strictly between 0 and 1.

Proof of part (a) of the claim By elementary calculus it can be shown that $1 - x \le \mathrm{e}^{-x}$ when $x > 0$, which implies that $(1-x)^{1/x} \le \mathrm{e}^{-1}$ when $0 < x < 1$. Since $0 < q < 1$, it follows that, for every positive integer b,

$$(1 - q^b)^{2^b} = ((1 - q^b)^{1/q^b})^{(2q)^b} \le (\mathrm{e}^{-1})^{(2q)^b} = \mathrm{e}^{-(2q)^b}.$$

If $0 < p \le \frac{1}{2}$, then $2q \ge 1$, so $(2q)^b \ge 2q$, and hence $\mathrm{e}^{-(2q)^b} \le \mathrm{e}^{-2q}$. Therefore

$$(1 - q^b)^{2^b} \le \mathrm{e}^{-2q}$$

for every positive integer b.

From the formula for $A(n)$, we see that

$$\begin{aligned}\sum_{n=1}^{\infty} A(n) = &\sum_{n=1}^{2^a-i-1} (1-q^a)^n + (1-q^a)^{2^a-i-1} \sum_{n=1}^{2^{a+1}} (1-q^{a+1})^n \\ &+ (1-q^a)^{2^a-i-1}(1-q^{a+1})^{2^{a+1}} \sum_{n=1}^{2^{a+2}} (1-q^{a+2})^n \\ &+ (1-q^a)^{2^a-i-1}(1-q^{a+1})^{2^{a+1}}(1-q^{a+2})^{2^{a+2}} \sum_{n=1}^{2^{a+3}} (1-q^{a+3})^n + \cdots .\end{aligned}$$

The inequality $(1-q^b)^{2^b} \le \mathrm{e}^{-2q}$ and the fact that $(1-q^b)^n < 1$ for all positive integers b and n imply that

$$\begin{aligned}
\sum_{n=1}^{\infty} A(n) \le & \sum_{n=1}^{2^a-i-1} (1-q^a)^n + (1-q^a)^{2^a-i-1} \sum_{n=1}^{2^{a+1}} 1 \\
& + (1-q^a)^{2^a-i-1} \mathrm{e}^{-2q} \sum_{n=1}^{2^{a+2}} 1 + (1-q^a)^{2^a-i-1} (\mathrm{e}^{-2q})^2 \sum_{n=1}^{2^{a+3}} 1 + \cdots \\
= & \sum_{n=1}^{2^a-i-1} (1-q^a)^n + (1-q^a)^{2^a-i-1} \{2^{a+1} + \mathrm{e}^{-2q} 2^{a+2} + (\mathrm{e}^{-2q})^2 2^{a+3} + \cdots\} \\
= & \sum_{n=1}^{2^a-i-1} (1-q^a)^n + (1-q^a)^{2^a-i-1} 2^{a+1} \{1 + 2\mathrm{e}^{-2q} + (2\mathrm{e}^{-2q})^2 + \cdots\}.
\end{aligned}$$

The series $\{1 + 2\mathrm{e}^{-2q} + (2\mathrm{e}^{-2q})^2 + \cdots\}$ is a geometric series with common ratio $2\mathrm{e}^{-2q} \le 2\mathrm{e}^{-1} < 1$, and hence it converges with sum $1/(1-2\mathrm{e}^{-2q})$. Therefore the partial sums of $\sum_{n=1}^{\infty} A(n)$ are bounded above. Since this is a series of positive terms, it must converge. Therefore when $0 < p \le \frac{1}{2}$, the sequence $A(n)$ has limit 0.

Proof of part (b) of the claim Recall that

$$A(n+1) = (\text{the probability of surviving the } (n+1)\text{th sequence}) A(n).$$

Clearly the sequence $A(n)$ is strictly decreasing, and is bounded below by 0. Hence it converges, and its limit is nonnegative and strictly less than 1. Furthermore, every subsequence of $A(n)$ must converge to this same limit. Consider the subsequence consisting of terms for which the last factor has a power of 2 in the exponent; that is, the subsequence f_k, $k \ge a+1$, where

$$f_k = (1-q^a)^{2^a-i-1} (1-q^{a+1})^{2^{a+1}} \cdots (1-q^{k-1})^{2^{k-1}} (1-q^k)^{2^k}.$$

The ingenious argument that follows is due to my colleague Tom Jager. Let $S = (1-q^a)^{2^a-i-1}$; then

$$f_k = S(1-q^{a+1})^{2^{a+1}} \cdots (1-q^{k-1})^{2^{k-1}} (1-q^k)^{2^k}.$$

Now observe that $1 - x > \mathrm{e}^{-2x}$ when $0 < x < \frac{1}{2}$. Since $\frac{1}{2} < p < 1$, $0 < q < \frac{1}{2}$, and hence $1 - q^i > \mathrm{e}^{-2q^i}$ for $a+1 \le i \le k$. Therefore

$$\begin{aligned}
f_k &= S(1-q^{a+1})^{2^{a+1}} \cdots (1-q^{k-1})^{2^{k-1}} (1-q^k)^{2^k} \\
&> S(\mathrm{e}^{-2q^{a+1}})^{2^{a+1}} \cdots (\mathrm{e}^{-2q^k})^{2^k} \\
&= S\mathrm{e}^{-2((2q)^{a+1} + (2q)^{a+2} + \cdots + (2q)^k)}.
\end{aligned}$$

The geometric series $\sum_{k=a+1}^{\infty} (2q)^k$ converges, since $0 < 2q < 1$, and its sum is

$$\frac{(2q)^{a+1}}{1-2q}.$$

Hence

$$(2q)^{a+1} + (2q)^{a+2} + \cdots + (2q)^k < \frac{(2q)^{a+1}}{1-2q}.$$

Therefore

$$e^{-2((2q)^{a+1}+(2q)^{a+2}+\cdots+(2q)^k)} > e^{-2((2q)^{a+1}/(1-2q))},$$

which implies that, for all $k \geq a+1$,

$$f_k > Se^{-2((2q)^{a+1}/(1-2q))}.$$

It follows that

$$\lim_{k\to\infty} f_k \geq Se^{-2((2q)^{a+1}/(1-2q))}.$$

Since the subsequence f_k has the same limit as the sequence $A(n)$,

$$\lim_{n\to\infty} A(n) \geq (1-q^a)^{2^a-i-1} e^{-2((2q)^{a+1}/(1-2q))}. \tag{2}$$

It follows that $\lim_{n\to\infty} A(n)$ is strictly positive. We have already seen that it is strictly less than 1, so the proof is complete.

To recap the conditions for death and immortality stated in theorem 1, we recall (1). If $0 < p \leq \frac{1}{2}$, then $\lim_{n\to\infty} A(n) = 0$, and hence $\sum_{n=0}^{\infty} D(n) = 1$, so the probability of death is 1. Since $\sum_{n=0}^{\infty} D(n)$ is the probability of death, $1 - \sum_{n=0}^{\infty} D(n) = \lim_{n\to\infty} A(n+1)$ is the probability of immortality. If $\frac{1}{2} < p \leq 1$, then $\lim_{n\to\infty} A(n) > 0$, implying that there is a nonzero probability that the game will never end.

Let us consider two examples of possibly immortal games. Suppose that $p = \frac{20}{38}$, so that $q = \frac{18}{38}$, and the initial fortune $F = 50 = 2^5 + 18$. When we apply (2) with $a = 5$ and $i = 18$, we see that the probability that the game will never end is at least 8.56×10^{-13}. If p increases to 0.6 and F remains at 50, the lower bound for this probability jumps to 0.0636.

Acknowledgements

I am grateful to an anonymous referee and the Editor for several helpful suggestions. I am also indebted to my colleagues, Tom Jager and Paul Zwier, for their help and encouragement.

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1 Frank Richards, *Billy Bunter's Beanfeast* (Cassell, London 1953).

2 G. R. Grimmett and D. R. Stirzaker, *Probability and Random Processes*, 2nd edn. (Oxford University Press, 1992).

3 Benjamin Franklin, *Quotations*, available at http://www.quotedb.com/quotes/1007.

John Ferdinands *has taught mathematics at Calvin College since September 1988. A few years ago he became reacquainted with some old friends of his youth, the Billy Bunter books. An incident in one of these books gave him a mathematical idea, which eventually resulted in this article.*

Letters to the Editor

Dear Editor,

Identities using imaginary numbers

In Volume 42, Number 1, Page 46, Bob Bertuello used imaginary numbers to prove identities. Consider the problem of writing $(a^2+1)(b^2+1)(c^2+1)$ as a sum of two squares, where a, b, c are real numbers. We can write

$$\begin{aligned}(a^2+1)(b^2+1)(c^2+1) &= (a+\mathrm{i})(b+\mathrm{i})(c+\mathrm{i})(a-\mathrm{i})(b-\mathrm{i})(c-\mathrm{i}) \\ &= ((abc-a-b-c)+\mathrm{i}(bc+ca+ab-1)) \\ &\quad \times ((abc-a-b-c)-\mathrm{i}(bc+ca+ab-1)) \\ &= (abc-a-b-c)^2+(bc+ca+ab-1)^2.\end{aligned}$$

If we put $a=b=c$, we obtain

$$(a^2+1)^3 = (a^3-3a)^2+(3a^2-1)^2,$$

so that $x=a^3-3a$, $y=3a^2-1$, $z=a^2+1$ satisfy the equation $x^2+y^2=z^3$. When a is an integer, this will give a family of lattice points on the curve $x^2+y^2=z^3$, i.e. points whose coordinates are integers.

Yours sincerely,

Abbas Rooholamini Gugheri
(10 Shahid Azam Lane
Makki Abad Avenue
Sirjan
Iran)

Dear Editor,

What is the probability that the final person on the aircraft sits in his own seat?

After reading Paul Belcher's article in Volume 42, Number 3, pp. 107–110, I realised that there is an easy extension to this problem which gives the last seat as a particular example.

Let $\mathrm{P}(m)$ be the probability that passenger m finds seat m occupied ($m=2,\dots,n$) and $\mathrm{P}(m,j)$ be the probability that passenger m finds seat m occupied by passenger j ($j<m$).

Then $\mathrm{P}(m,1)=1/n$ as the first passenger chooses one of n seats at random and $\mathrm{P}(2)=\mathrm{P}(2,1)=1/n$.

If passenger j ($1<j<n$) finds seat j occupied then seats 2 to ($j-1$) are also occupied (none of them can be empty because its owner is already on board), so passenger j has a choice of $n+1-j$ seats, so chooses seat m ($m>j$) with probability $1/(n+1-j)$.

So $\mathrm{P}(m,j)=\mathrm{P}(j)/(n+1-j)$ and

$$\begin{aligned}\mathrm{P}(m) &= \mathrm{P}(m,1)+\mathrm{P}(m,2)+\mathrm{P}(m,3)+\cdots+\mathrm{P}(m,m-2)+\mathrm{P}(m,m-1) \\ &= \frac{1}{n}+\frac{\mathrm{P}(2)}{n-1}+\frac{\mathrm{P}(3)}{n-2}+\cdots+\frac{\mathrm{P}(m-2)}{n-m+3}+\frac{\mathrm{P}(m-1)}{n-m+2} \\ &= \mathrm{P}(m-1)+\frac{\mathrm{P}(m-1)}{n-m+2}.\end{aligned}$$

So

$$\begin{aligned}(n-m+2)\,\mathrm{P}(m) &= (n-m+3)\,\mathrm{P}(m-1)\\ &= (n-m+4)\,\mathrm{P}(m-2)\\ &\ \ \vdots\\ &= (n-1)\,\mathrm{P}(3)\\ &= n\,\mathrm{P}(2)\\ &= 1\end{aligned}$$

and $\mathrm{P}(m) = 1/(n-m+2)$ and $\mathrm{P}(n) = \frac{1}{2}$.

Yours sincerely,
Dermot Roaf
(Exeter College
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UK)

Dear Editor,

What is the probability that the final person on the aircraft sits in his own seat?

I enjoyed Paul Belcher's article in Volume 42, Number 3, pp. 107–110, but offer the following direct proof of the fundamental identity (1) (in the form (2)) in addition to the probabilistic and inductive proofs given there.

The identity (2) is clearly true if $n = 2$ and, for $n \geq 3$, consider the polynomial $f(x) = (1+2x)(1+3x)\cdots(1+(n-1)x)$. In expanded form,

$$f(x) = 1 + \Bigg(\sum_{2\leq i\leq n-1} i\Bigg)x + \Bigg(\sum_{2\leq i<j\leq n-1} ij\Bigg)x^2 + \cdots + (n-1)(n-2)\cdots 2x^{n-2},$$

and the required sum (2) is therefore

$$f(1) = 3\cdot 4\cdots n = \frac{n!}{2}.$$

Yours sincerely,
Nick Lord
(Tonbridge School
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UK)

Dear Editor,

The equation $x^2 + y^2 = 2z^2$

Suppose that (x, y, z) is a solution of this equation in positive integers with highest common factor 1. We can take $x > y$. Then

$$2x^2 + 2y^2 = 4z^2,$$

so that

$$(x+y)^2 + (x-y)^2 = (2z)^2.$$

Moreover x, y must both be odd, so $x+y$, $x-y$ are even and

$$\left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right) = z^2,$$

so that $((x+y)/2, (x-y)/2, z)$ is a solution of $X^2 + Y^2 = Z^2$ in positive integers, also with highest common factor 1. But the solutions of this equation are known. We can write

$$\frac{x+y}{2} = s^2 - t^2, \qquad \frac{x-y}{2} = 2st, \qquad z = s^2 + t^2,$$

for some coprime positive integers s, t with $s > t$ and one of s, t even; or else $(x+y)/2$ and $(x-y)/2$ can be reversed. Hence

$$x = s^2 - t^2 + 2st, \qquad y = |s^2 - t^2 - 2st|, \qquad z = s^2 + t^2,$$

with $s > t > 0$, $(s,t) = 1$, and one of s, t even. It is easily verified that these are solutions of the equation, so this gives all primitive solutions of the equation.

Yours sincerely,

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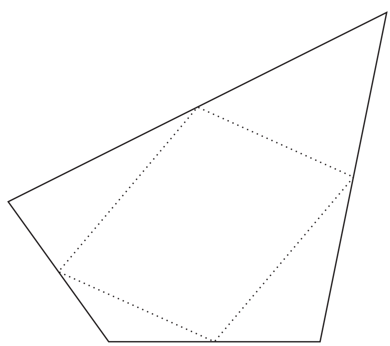
Midpoints

The midpoints of the sides of any quadrilateral form the vertices of a parallelogram. Why?



(Gleaned from Terence Tao, *Solving Mathematical Problems: A Personal Perspective* (Oxford University Press, 2006).)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st March will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

43.5 Prove that all prime numbers are solitary – see the article 'Lopsided numbers' on pp. 53–54 of this issue.

(Submitted by Jonny Griffiths, Paston College, Norfolk, UK)

43.6 For positive integers a and n, sum the finite series

$$aa! + (a+1)(a+1)! + (a+2)(a+2)! + \cdots + (a+n-1)(a+n-1)!$$

(Submitted by Zheng Zhusheng, Chengdu, Sichuan, P. R. China)

43.7 The positive real numbers a, b, c are such that $a^2+b^2=c^2$, $c=b^2/a$, and $b-a=1$. Determine a, b, c.

(Submitted by Will Gosnell, Amherst, MA, USA)

43.8 Determine all nondegerate triangles ABC in which

$$\frac{\sin A}{A} = \frac{\sin B}{B} = \frac{\sin C}{C}.$$

(Submitted by Prithwijit De, Mumbai, India)

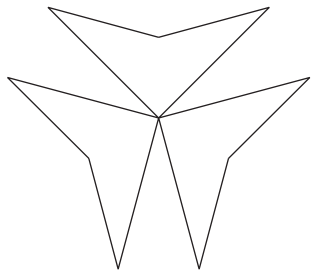
Solutions to Problems in Volume 42 Number 3

42.9 Two 24-hour clocks are set going simultaneously at midnight, both showing the correct time. One gains 2 minutes per hour, the other loses 1 minute per hour. How long is it before they again show the same time simultaneously? How long is it before they again show midnight simultaneously?

Solution by M. A. Khan, Lucknow, India

The faster clock gains 3 minutes per hour on the slower clock, or 1 minute per $\frac{1}{3}$ hour, so 24 hours in $(60 \times 24)/3$ hours, i.e. in 20 days, so after 20 days both clocks will again show the same time. In this time, the slower clock has lost 20×24 minutes, i.e. 8 hours, so after 60 days it will have lost 24 hours and both clocks will again show midnight.

42.10 Banana cross – not quite a Maltese cross! Do a transverse cut on a banana and the regular pattern shown is obtained (idealised). If the long sides are 2 cm in length and the internal angles are $90°$ and $30°$, what is the total internal area?



Solution by Bob Bertuello, who proposed the problem

The area of one leaf of the Maltese cross (see figure 1) is

$$\begin{aligned} \text{area } \triangle OAB - \text{area } \triangle ABC &= \tfrac{1}{2} \times 2 \times 2 - \tfrac{1}{2} \times AB \times CD \\ &= 2 - \tfrac{1}{2} \times 2BD \times BD \tan 15° \\ &= 2 - 2 \tan 15°. \end{aligned}$$

Put $\tan 15° = x$. Then

$$\frac{1}{\sqrt{3}} = \tan 30° = \frac{2x}{1 - x^2}$$

so $x^2 + 2\sqrt{3}x - 1 = 0$ and $x = -\sqrt{3} + 2$ and the area of the Maltese cross is

$$6(1 + \sqrt{3} - 2) = 6(\sqrt{3} - 1) \text{ square cm.}$$

Also solved by M. A. Khan (Lucknow, India).

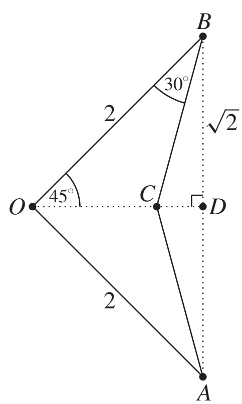


Figure 1

42.11 In an acute-angled triangle ABC, prove that

$$\frac{\tan^3 A}{\tan^2 B}+\frac{\tan^3 B}{\tan^2 C}+\frac{\tan^3 C}{\tan^2 A}\geq \tan A\tan B\tan C.$$

Solution by M. A. Khan, Lucknow, India

Since $A+B+C=\pi$,

$$\tan C=\tan(\pi-A-B)=-\tan(A+B)=-\frac{\tan A+\tan B}{1-\tan A\tan B},$$

so

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

Put $x=\tan A$, $y=\tan B$, $z=\tan C$. Now

$$\begin{aligned}\left(\frac{x^3}{y^2}+\frac{y^3}{z^2}+\frac{z^3}{x^2}\right)-(x+y+z)&=\frac{x(x^2-y^2)}{y^2}+\frac{y(y^2-z^2)}{z^2}+\frac{z(z^2-x^2)}{x^2}\\&=\frac{x(x^2-y^2)}{y^2}+\frac{y(y^2-z^2)}{z^2}-\frac{z(x^2-y^2+y^2-z^2)}{x^2}\\&=\frac{(x^2-y^2)(x^3-y^2z)}{x^2y^2}+\frac{(y^2-z^2)(x^2y-z^3)}{x^2z^2}.\end{aligned}$$

We may suppose that $A\geq B\geq C$, so $x\geq y\geq z$, from which it follows that this expression is greater than or equal to zero (and is equal to zero if and only if the triangle is equilateral).

Also solved by Subramanyam Durbha (Rowan University, Glassboro, NJ, USA) and Jonny Griffiths (Paston College, Norfolk, UK).

42.12 Find all natural numbers n and k such that

$$n^2+(n+1)^2+\cdots+(n+k)^2=(n+k+1)^2+(n+k+2)^2+\cdots+(n+2k)^2.$$

Solution by Bor-Yann Chen, University of California, Irvine, USA

We have

$$\begin{aligned}\text{RHS}-\text{LHS}&=((n+k+1)^2-n^2)+((n+k+2)^2-(n+1)^2)\\&\quad+\cdots+((n+2k)^2-(n+k-1)^2)-(n+k)^2\\&=(k+1)[(2n+k+1)+(2n+k+3)+\cdots+(2n+3k-1)]-(n-k)^2\\&=(k+1)[(2n+k)k+(1+3+\cdots+2k-1)]-(n+k)^2\\&=(k+1)[(2n+k)k+k^2]-(n+k)^2\\&=2k(k+1)(n+k)-(n+k)^2\\&=(n+k)[k(2k+1)-n],\end{aligned}$$

which is zero if and only if $n=k(2k+1)$. (For example, $k=3$ gives $n=21$ and $21^2+22^2+23^2+24^2=25^2+26^2+27^2$.)

Also solved by M. A. Khan (Lucknow, India).

Reviews

Catalan Numbers with Applications. By Thomas Koshy. Oxford University Press, 2008. Hardback, 304 pages, £80.00 (ISBN 978-0-19-533454-8).

Thomas Koshy of Framingham State University in Massachusetts is a prolific author, one whose name is likely to sit alongside 'with Applications' in the titles of his books. He has a stable of at least seven such volumes, of which this is the latest. A *Catalan number*, for the uninitiated, is a number of the form

$$C_n = \frac{1}{n+1}\binom{2n}{n};$$

it is not immediately obvious that C_n is always an integer, but it is, and it transpires that these numbers crop up time and time again in mathematics, pure and applied.

Koshy's text is a homage to these numbers, and by extension the binomial coefficients too – I cannot believe that there is a single known fact about $\binom{n}{r}$ that does not appear in this book. The writing is beautifully organised and extremely clear, and is punctuated by historical mini-biographies that add much to the book (the number of mathematicians who have contributed in this area is vast). Koshy is swift to point to elegance in their work, and has contributed significantly himself; the theorem concerning which C_n are prime is shared by him. The book is encyclopaedic yet an accessible and rewarding read, and would be a valuable addition to any mathematics library.

Paston College, Norfolk **Jonny Griffiths**

A Primer for Mathematics Competitions. By Alexander Zawaira and Gavin Hitchcock. Oxford University Press, 2008. Hardback, 368 pages, £52.00 (ISBN 978-0-19-953987-1).

The aim of the book is to equip students for the British Mathematical Olympiad. It takes the reader from basic material in normal school syllabuses to useful advanced material. On the way it includes a good deal of interesting history. There are nine chapters: Geometry, Algebraic inequalities and mathematical induction, Diophantine equations, Number theory, Trigonometry, Sequences and series, Binomial theorem, Combinatorics, and Miscellaneous problems.

Each chapter contains: a summary, one or more 'appetizers' (which the student may attempt immediately and possibly unsuccessfully or leave until later), basic results, more advanced results, solutions to the appetizers, and problems followed by their solutions later on. There are a total of 230 problems with sometimes more than one solution given.

The text contains proofs and a few worked examples. The mathematics is rigorous, without being complicated. The problems are testing, varying from quite easy to hard.

I would strongly recommend this book to bright students who might expect to be taking the British Mathematical Olympiad 1 and 2 or beyond.

Stamford, Lincolnshire **Alastair Summers**

Mathematical Spectrum

Volume 43 2010/2011 Number 2

ISSN 0025-5653

http://ms.appliedprobability.org
Published by the Applied Probability Trust
Printed by Berforts South East Ltd, Stevenage, Hertfordshire, UK