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## GENERAL INFORMATION

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Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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# THE OLYMPIAD CORNER
## No. 118
R.E. WOODROW

*All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.*

The Olympiad problems we give this issue were sent to me by Professor Francisco Bellot of Valladolid, Spain. They are the problems proposed for the 25th Spanish Mathematics Olympiad, First Round.

### 25TH SPANISH MATHEMATICS OLYMPIAD
First Round
November 11–12, 1988

**1.** Let $n$ be an even number which is divisible by a prime bigger than $\sqrt{n}$. Show that $n$ and $n^3$ cannot be expressed in the form $1 + (2l + 1)(2l + 3)$, i.e., as one more than the product of two consecutive odd numbers, but that $n^2$ and $n^4$ can be so expressed.

**2.** In the square $ABCD$, let $M$ and $H$ be the midpoints of $AB$ and $CD$, respectively. Consider the distance preserving transformation $T$ of points in the plane of the square such that $T(A) = H$, $T(H) = B$, and $T(D) = M$. Is there a straight line $l$ which is preserved by $T$ (i.e. $T(l) = l$)? Is there a point $X$ of the plane with $T(X) = X$?

**3.** The natural numbers $1,2,\ldots,n^2$ are arranged to form an $n \times n$ matrix

$$A = \begin{bmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \cdot & \cdot & & \vdots \\ \cdot & \cdot & & \\ \cdot & \cdot & & n^2 \end{bmatrix}.$$

A sequence $a_1,a_2,\cdots$ of elements of $A$ is chosen as follows. The first element $a_1$ is chosen at random and the row and column containing it are deleted. As long as elements remain, the next element is chosen at random from among the elements that remain, and its row and column are deleted. The process continues until only *one* element is left. Calculate the sum of this last number and all the numbers previously chosen. Show that this sum is independent of the choices made.

4. The angles of triangle $A_1B_1C_1$ are given. For each natural number $n$, let $A_{n+1}$, $B_{n+1}$, $C_{n+1}$ be the points of contact of the incircle of $\Delta A_nB_nC_n$ with the sides $B_nC_n$, $C_nA_n$, $A_nB_n$, respectively. Determine, in terms of $n$, the angles of the triangle $A_nB_nC_n$.

5. Let $ABCD$ be a square and let $E$ be a point inside the square such that $\Delta ECD$ is isosceles with $\angle C = \angle D = 15^\circ$. What can one say about $\Delta ABE$?

6. Several collections of natural numbers are written, such that: (a) in each collection, each one of the digits 0,1,2,...,9 is used once; (b) no number in a collection has 0 as left–most digit; (c) the sum of the numbers in each collection is less than 75. How many such collections can be written in this way? (The order in which the numbers of a collection are written is immaterial.)

7. Find the maximum value of the function

$$f(x) = \prod_{k=0}^{7} |x - k|$$

for $x$ in the closed interval [3,4].

8. Let $m$ be odd. Show that for each integer $n > 2$, the sum of the $m$th powers of the numbers less than $n$ that are relatively prime to $n$ is a multiple of $n$.

* * *

Our first solution this month was sent in by Dr. Peter Taylor, The University of Canberra, who found it in his files.

6. [1982: 134] *1982 Australian Mathematical Olympiad.*

A number is written into each square of an $n \times n$ table. We know that any two rows of the table are different. Prove that the table contains a column such that, on omitting it, the remaining table has no equal rows either.

(*Note.* The rows 1,1,2,7,5 and 1,1,7,2,5 are different, that is, not equal.)

*Solution by U.I. Lydna, Soviet Institute of Medieval Studies.*

For $1 \le i \le n$, let row $R_i$ be represented by vertex $v_i$. Consider the $i$th column. If no two rows differ only in this column, then it can be removed without leaving behind two identical rows. Hence we may assume that there are two such rows. Choose two and join the two vertices representing these two rows by an edge labelled $i$. We now have a graph with $n$ vertices and $n$ edges. Hence it has a cycle

$(v_1,v_2,\cdots,v_k)$ for some $k \leq n$. We may assume that for $1 \leq i \leq k-1$ the edge joining $v_i$ and $v_{i+1}$ is labelled $i$, while the edge joining $v_k$ and $v_1$ is labelled $k$. Now let the row $R_1$, represented by vertex $v_1$, be $R_1 = \langle x_1,x_2,\cdots,x_n\rangle$. Since $v_1$ is joined to $v_2$ by an edge labelled 1 we must have $R_2 = \langle y_1,x_2,\cdots,x_n\rangle$ with $y_1 \neq x_1$. Similarly

$$R_i = \langle y_1,y_2,\cdots,y_{i-1},x_i,x_{i+1},\cdots,x_n\rangle$$

for $3 \leq i \leq k$, with $y_i \neq x_i$ for $2 \leq i \leq k-1$. Now $v_1$ is joined to $v_k$ by an edge labelled $k$. We must have

$$R_1 = \langle y_1,y_2,\cdots,y_k,x_{k+1},\cdots,x_n\rangle \qquad \text{with} \quad y_k \neq x_k .$$

This contradicts the fact that $R_1 = \langle x_1,x_2,\cdots,x_n\rangle$.

*

The next solutions are also from U.I. Lydna, and are to the problems from the *31st Bulgarian Mathematical Olympiad* [1982: 237] for which solutions have not been considered in the Corner. (The solution to problem 2 is in [1984: 110].)

1. Find all pairs of natural numbers $(n,k)$ for which $(n+1)^k - 1 = n!$.

*Solution by U.I. Lydna, Soviet Institute of Medieval Studies.*

Let $p$ be a prime divisor of $n+1$. Then $p \leq n+1$. Since $(n+1)^k = n! + 1$, $p$ divides $n! + 1$. However $n! + 1$ is not divisible by any prime less than $n+1$. It follows that $n+1$ is itself a prime. If $n+1 = 2$, then $2^k = 1 + 1$ and $(n,k) = (1,1)$. If $n+1 = 3$, then $3^k = 2 + 1$ and $(n,k) = (2,1)$. If $n+1 = 5$, then $5^k = 24 + 1$ and $(n,k) = (4,2)$.

We now show that there are no other solutions. Suppose $n+1$ is a prime greater than 5. Then $n = 2m$ for some integer $m > 2$. Since $(n-1)!$ contains both 2 and $m$ among its factors $n$ divides $(n-1)!$ and $n^2$ divides $n!$. If we have

$$n! = (n+1)^k - 1 = n^k + kn^{k-1} + \cdots + kn,$$

then $n^2$ divides $kn$ and so $n$ divides $k$. Hence $k \geq n$. It follows that

$$n! \geq (n+1)^n - 1 > n^n > n!,$$

a contradiction.

3. Given is a regular prism whose bases are the regular $2n$-gons $A_1A_2\cdots A_{2n}$ and $B_1B_2\cdots B_{2n}$, each with circumradius $R$. Prove that if the length of the edges $A_iB_i$ varies, then the angle between the line $A_1B_{n+1}$ and the plane through the points $A_1$, $A_3$, and $B_{n+2}$ is maximal when $A_iB_i = 2R\cos(\pi/2n)$.

*Solution by U.I. Lydna, Soviet Institute of Medieval Studies.*

We may assume that $R = 1$. Set up a coordinate system so that

$$A_1 = (1,0,z)\ ,\quad A_3 = (\cos\frac{2\pi}{n},\sin\frac{2\pi}{n},z) = (1 - 2\sin^2\frac{\pi}{n},2\sin\frac{\pi}{n}\cos\frac{\pi}{n},z)\ ,$$
$$B_{n+1} = (-1,0,0)\ ,\quad B_{n+2} = (-\cos\frac{\pi}{n},-\sin\frac{\pi}{n},0)\ ,$$

where $z$ is the height of the prism. Now

$$\mathbf{u} = \overrightarrow{A_1B_{n+1}} = (2,0,z)\ ,$$
$$\overrightarrow{A_1A_3} = (2\sin^2\frac{\pi}{n},-2\sin\frac{\pi}{n}\cos\frac{\pi}{n},0) = 2\sin\frac{\pi}{n}(\sin\frac{\pi}{n},-\cos\frac{\pi}{n},0)\ ,$$
$$\overrightarrow{A_1B_{n+2}} = (1 + \cos\frac{\pi}{n},\sin\frac{\pi}{n},z)$$

and

$$\mathbf{v} = \frac{\overrightarrow{A_1A_3}\times\overrightarrow{A_1B_{n+2}}}{2\sin\pi/n} = (-z\cos\frac{\pi}{n},-z\sin\frac{\pi}{n},1 + \cos\frac{\pi}{n})\ .$$

To maximize the angle in question, it is necessary and sufficient to minimize the angle $C$ between $\mathbf{u}$ and $\mathbf{v}$. We seek the maximum value of $\cos C$, that is the minimum value of

$$\frac{1}{\cos^2C} = \frac{|\mathbf{u}|^2|\mathbf{v}|^2}{(\mathbf{u}\cdot\mathbf{v})^2} = \frac{(z^2 + 4)(z^2 + (1 + \cos\pi/n)^2)}{z^2(1 - \cos\pi/n)^2}$$
$$= \frac{1}{(1 - \cos\pi/n)^2}\left[z^2 + 4 + (1 + \cos\pi/n)^2 + \frac{4(1 + \cos\pi/n)^2}{z^2}\right]\ ,$$

that is the minimum value of

$$z^2 + \frac{4(1 + \cos\pi/n)^2}{z^2}\ .$$

Since the two terms have constant product, the minimum occurs when they are equal, i.e. when

$$z^2 = 2(1 + \cos\frac{\pi}{n}) = 4\cos^2\frac{\pi}{2n}\ ,$$

yielding $z = 2\cos(\pi/2n)$, as required.

4. Let $x_1,x_2,\cdots,x_n$ be arbitrary numbers in the interval $[0,2]$. Prove that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|^2 \le n^2\ .$$

When is there equality?

*Solution by U.I. Lydna, Soviet Institute of Medieval Studies.*

Let

$$f(n) = \max\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2.$$

We may assume that the $x$'s are in non–decreasing order. We claim that $f(2k) = 4k^2$, attained if and only if $x_1 = x_2 = \cdots = x_k = 0$ and $x_{k+1} = x_{k+2} = \cdots$

$= x_n = 2$. Clearly we must have $x_1 = 0$ and $x_{2k} = 2$. We may assume that $x_1 = x_2 = \cdots = x_r = 0$ and $x_{s+1} = x_{s+2} = \cdots = x_{2k} = 2$, where $r \leq s$, $x_{r+1} = u > 0$ and $x_s = v < 2$. Suppose first that $r < k$. If we redefine $x_{r+1}$ to be 0, then $f(2k)$ will increase by

$$\sum_{i=r+2}^{2k} (x_i^2 - (x_i - u)^2) - ru^2 = u\left(2\sum_{i=r+2}^{2k} x_i - (2k-1)u\right).$$

Since $x_i \geq u$ for all $i \geq r + 2$, and $2(2k - r - 1) > 2k - 1$, the increase is positive, contradicting the definition of $f$. Thus $r \geq k$. Next, if $s > k$, we redefine $x_s = 2$. Then $f(2k)$ will increase by

$$\sum_{i=1}^{s-1} ((2 - x_i)^2 - (v - x_i)^2) - (2k - s)(2 - v)^2$$

$$= (2 - v)\left(2(2s - 2k - 1) + (2k - 1)v - 2\sum_{i=1}^{s-1} x_i\right).$$

Since $x_i \leq v$ for $i \leq s - 1$, the increase is at least

$$(2 - v)[2(2s - 2k - 1) + (2k - 1)v - 2v(s - 1)] = (2 - v)^2(2s - 2k - 1) > 0,$$

again a contradiction. Hence $s \leq k$, and we must have $r = s = k$.

Similarly, we can show that $f(2k + 1) = 4k(k + 1)$, attained if and only if $x_1 = x_2 = \cdots = x_k = 0$, $x_{k+1} = 0$ or 2, and $x_{k+2} = x_{k+3} = \cdots = x_{2k+1} = 2$.

5. Determine all values of the parameters $a$ and $b$ for which the polynomial

$$x^4 + (2a + 1)x^3 + (a - 1)^2x^2 + bx + 4$$

can be factored into a product of two polynomials $P(x)$ and $Q(x)$ of degree 2 (with leading coefficients 1) such that the equation $Q(x) = 0$ has two different roots $r$ and $s$ with $P(r) = s$ and $P(s) = r$.

*Solution by U.I. Lydna, Soviet Institute of Medieval Studies.*

We put $Q(x) = (x - r)(x - s)$ and $P(x) = x^2 + cx + d$. Then $r^2 + cr + d = s$ and $s^2 + cs + d = r$. Subtracting the second from the first we have

$$(r + s)(r - s) + c(r - s) = -(r - s)$$

or $c = -r - s - 1$. Substituting back into the first equation, we have

$$d = s - r^2 + r(r + s + 1) = rs + r + s.$$

Let $u = r + s$ and $v = rs$. Then

$$Q(x) = x^2 - ux + v \quad \text{and} \quad P(x) = x^2 - (u + 1)x + (u + v).$$

Hence

$$x^4 + (2a + 1)x^3 + (a - 1)^2x^2 + bx + 4$$
$$= x^4 - (2u + 1)x^3 + (u^2 + 2u + 2v)x^2 - (u^2 + 2uv + v)x + v(u + v) \; .$$

Comparing coefficients, we have $2a + 1 = -2u - 1$, so

$$a = -u - 1 \; .$$

Also $(a - 1)^2 = u^2 + 2u + 2v$, which yields

$$u = v - 2 \; .$$

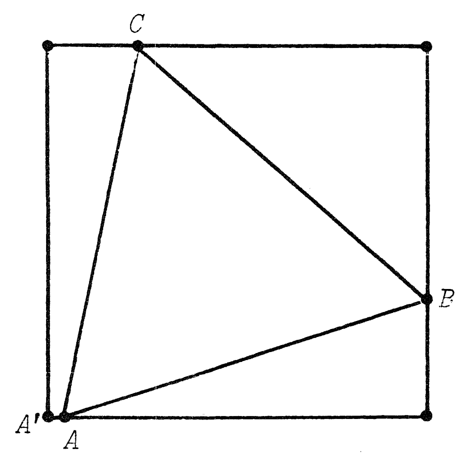
Finally $4 = v(u + v) = 2v(v - 1)$, and

$$0 = v^2 - v - 2 = (v + 1)(v - 2) \; .$$

For $v = 2$, $u = 0$, and so $a = -1$ and $b = -(u^2 + 2uv + v) = -2$. For $v = -1$, $u = -3$, and so $a = 2$ and $b = -14$.

6. Determine the set of centroids of the equilateral triangles whose vertices lie on the sides of a given square.

*Solution by U.I. Lydna, Soviet Institute of Medieval Studies.*




We embed the whole diagram in the complex plane with one corner, $A'$, of the square as the origin and two sides of the square as the axes. Let the triangle be $ABC$ with centroid $O$. We may assume that the square is of side length 1. By symmetry, we may restrict our attention to the case where $A$ lies on the line $y = 0$, $B$ on $x = 1$ and $C$ on $y = 1$, with $A$ moving from $A'$ until $CA$ is perpendicular to $A'A$. Let $B'$, $C'$ and $O'$ be the respective images of $B$, $C$ and $O$ under the translation $\overrightarrow{AA'}$. Let $A = a$ and $B = 1 + bi$ for some real numbers $a$ and $b$. Then $B' = (1 - a) + bi$. Since $C'$ is obtained from $B'$ by a $60^\circ$ rotation about $A'$, $C' = B'(1 + \sqrt{3}i)/2$. Since $C'$ lies on $y = 1$, we have

$$\frac{b}{2} + \frac{\sqrt{3}(1 - a)}{2} = 1$$

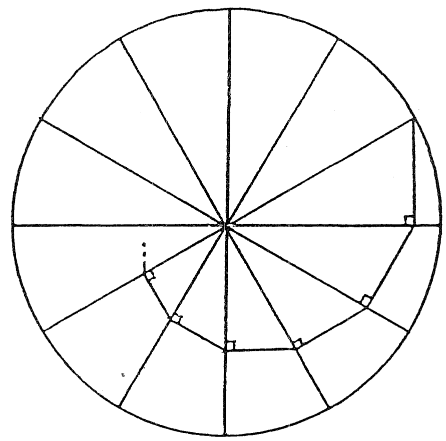
or $b = 2 - \sqrt{3} + \sqrt{3}a$. Since $O'$ is obtained from $B'$ by a $30^\circ$ rotation about $A'$ followed by a contraction towards $A'$ by a factor of $1/\sqrt{3}$, $O' = B'((\sqrt{3} + i)/2)/\sqrt{3}$. The real part of $O'$ is given by $(\sqrt{3}(1 - a) - b)/2\sqrt{3}$. Hence the real part of $O$ is equal to

$$a + \frac{\sqrt{3}(1 - a) - b}{2\sqrt{3}} = \frac{2\sqrt{3}a + \sqrt{3}(1 - a) - (2 - \sqrt{3} + \sqrt{3}a)}{2\sqrt{3}} = 1 - \frac{1}{\sqrt{3}} \; .$$

It follows that the locus of $O$ is the square defined by the lines $x = 1 - 1/\sqrt{3}$, $y = 1/\sqrt{3}$, $x = 1/\sqrt{3}$ and $y = 1 - 1/\sqrt{3}$.

We now turn our attention to the January 1989 issue, and the solutions to the problems from the final round of the *1985–86 Flanders Mathematics Olympiad* [1989: 4].

1. A circle with radius $R$ is divided into twelve equal parts. The twelve dividing points are connected with the centre of the circle, producing twelve rays. Starting from one of the dividing points a segment is drawn perpendicular to the next ray in the clockwise sense; from the foot of this perpendicular another perpendicular segment is drawn to the next ray, and the process is continued *ad infinitum*. What is the limit of the sum of these segments (in terms of $R$)?



*Solution and generalization by J. Chris Fisher, Department of Mathematics and Statistics, The University of Regina, Saskatchewan.*

This can, of course, be solved in one's head (the sum is by definition

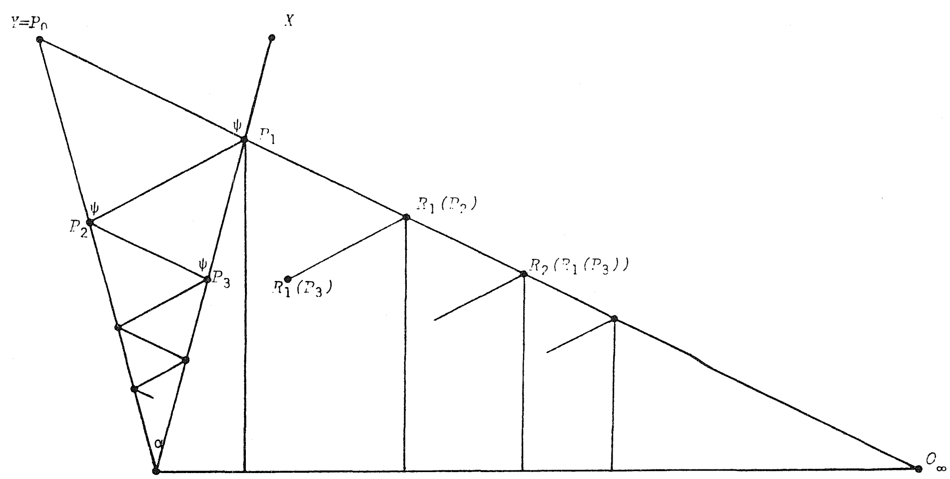
$$R\ \sin(\pi/6) + R\ \sin(\pi/6)\cos(\pi/6) + R\ \sin(\pi/6)\cos^2(\pi/6) + \cdots$$

which equals

$$R\ \sin(\pi/6)/(1 - \cos(\pi/6)) = R/(2 - \sqrt{3})) ,$$

but why do it that way when it is possible to do it by a much harder method?

First, let's complicate matters by choosing an arbitrary (but fixed) angle $\alpha$ between successive rays. Next, instead of dropping a perpendicular, let $\psi$ be the angle between the next ray and the segment from the previous point (as in the figure). Finally, replace the spiral by a zig–zag path of the same length within one angular region. More precisely, let $X$ and $Y$ be two points on the unit circle (note



that letting the unit of measurement be $R$ does nothing but complicate the notation even further) that subtends an angle $\alpha < \pi/2$ at the centre $O$ : $\angle XOY = \alpha$. Define a zig–zag path $P_0P_1P_2\cdots$ by letting $P_0 = Y$, $P_1$ be the unique point on $OX$ for which $\angle XP_1P_0 = \psi \le \pi/2$, $P_2$ be the unique point of $OY$ for which $\angle P_1P_2Y = \psi$, and so on, as indicated in the figure. *Find the length of the path.*

The length is

$$\cos(\alpha/2)/\cos(\psi - \alpha/2).$$

This can be seen by unfolding the zig–zag one segment at a time along the line $P_0P_1$ (the line of the first segment) via a sequence of reflections. Let $R_1$ be the reflection in the bisector of the angle between $P_1P_2$ and the extension of $P_0P_1$. Note that this bisector (call it the first "mirror") makes an angle of $\alpha/2$ with the line $OP_1$ and that the image of $P_2P_3$ under the reflection is parallel to $P_1P_2$. To unfold the image of $P_2P_3$ reflect it in the line through $R_1(P_2)$ that is parallel to the first mirror. Continue in this manner, letting $R_n$ be the reflection in the line through $R_{n-1}(R_{n-2}(\cdots(R_1(P_n))\cdots))$ that is parallel to the first mirror. Since the limit of the sequence of points $P_i$ is $O$, and since $O$ is moved by the successive reflections along a line perpendicular to the mirrors, the limit of the sequence of unfolded segments must be the point, call it $O_\infty$, where $P_0P_1$ meets that perpendicular. Thus the length of the path is $|P_0O_\infty|$. As can be seen from the figure, the triangle $OO_\infty P_0$ has angles $(\pi + \alpha)/2$ and $(\pi + \alpha)/2 - \psi$ at either end of the line $OO_\infty$. The law of sines says

$$|P_0O_\infty| : 1 = \sin((\pi + \alpha)/2) : \sin((\pi + \alpha)/2 - \psi),$$

which reduces to $\cos(\alpha/2)/\cos(\psi - (\alpha/2))$, as claimed.

As a final observation, note that if $\alpha$ is allowed to shrink to zero, the polygonal spiral tends toward the *equiangular spiral* whose polar equation is $r = e^{\theta\cot\psi}$; this is the point of view taken by E.H. Lockwood in *A Book of Curves* (Cambridge U. Press, 1971, Chapter 9, pp. 98–109). It follows that the length of the equiangular spiral from (1,0) to its pole is

$$\lim_{\alpha\to 0} \frac{\cos(\alpha/2)}{\cos(\psi - (\alpha/2))} = \frac{1}{\cos\,\psi}\,.$$

[*Editor's note.* Solutions were also submitted by Douglass L. Grant, Department of Mathematics, University College of Cape Breton; by Botand Köszegi, student, Halifax West High School; by Antonio Leonardo P. Pastor, Universidade de São Paulo, Brasil; as well as by M.A. Selby, Department of Mathematics, The University of Windsor.]

2. Prove that, for every natural number $n$, we have

$$n! \leq \left(\frac{n+1}{2}\right)^n .$$

*Solutions by Botand Köszegi, student, Halifax West High School; Bob Prielipp, University of Wisconsin–Oshkosh; and by Edward T.H. Wang, Department of Mathematics, Wilfrid Laurier University.*

By the arithmetic–geometric–mean inequality, we have

$$(n!)^{1/n} \leq \frac{1+2+\cdots+n}{n} = \frac{n+1}{2} .$$

Thus

$$n! \leq \left(\frac{n+1}{2}\right)^n .$$

Equality holds if and only if $n = 1$.

*Inductive solutions by Mathew Englander, Toronto, Ontario; by Douglass L. Grant, Department of Mathematics, University College of Cape Breton; and by M.A. Selby, Department of Mathematics, The University of Windsor.*

We show $(n!)2^n \leq (n+1)^n$ for each $n > 0$, by mathematical induction. Equality is clear for $n = 1$, so we assume the inequality for some $k$. Then

$$(k+1)!2^{k+1} = 2(k+1)(k!)2^k \leq 2(k+1)(k+1)^k = 2(k+1)^{k+1} .$$

Also

$$\begin{aligned}(k+2)^{k+1} &= [(k+1)+1]^{k+1}\\ &= (k+1)^{k+1} + (k+1)(k+1)^k + \text{positive terms}\\ &> 2(k+1)^{k+1} .\end{aligned}$$

Therefore $(k+1)!2^{k+1} < (k+2)^{k+1}$.

3. A sequence of numbers $\{a_k\}$ is defined as follows:

$$a_0 = 0 ,$$
$$a_{k+1} = 3a_k + 1 , \qquad k \geq 0 .$$

Show that $a_{155}$ is divisible by 11.

*Solutions by Mathew Englander, Toronto, Ontario; Douglass L. Grant, Department of Mathematics, University College of Cape Breton; Botand Köszegi, student, Halifax West High School; Antonio Leonardo P. Pastor, Universidade de São Paulo, Brasil; Bob Prielipp, University of Wisconsin–Oshkosh; Kevin Santosuosso, London, Ontario; M.A. Selby, Mathematics Department, The University of Windsor; and Edward T.H. Wang, Mathematics Department, Wilfrid Laurier University, Waterloo, Ontario.*

$a_0 = 0$ is divisible by 121. By direct calculation $a_{n+5} = 243a_n + 121$. By induction it follows easily that 121 divides $a_{5n}$ for all $n$.

[*Editor's note.* Mr. Santosuosso noticed that $a_{5n}$ is in fact divisible by 121. The solution of A. Pastor found that

$$a_n = \frac{3^n - 1}{2}$$

so

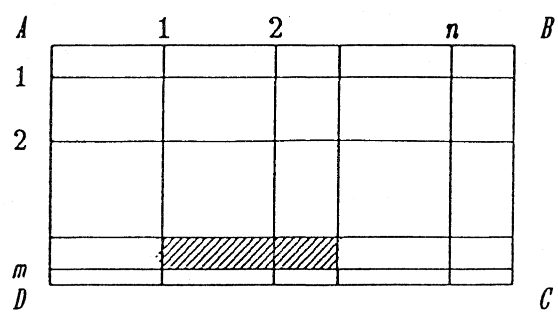
$$a_{155} = \frac{(3^5)^{31} - 1}{2} = \frac{3^5 - 1}{2}\cdot k = 121k ,$$

while that of M. Selby used Fermat's "little" theorem.]

*

The next set of solutions are to the final round of the *1986–87 Flanders Mathematics Olympiad* [1989: 5].

1. A rectangle $ABCD$ is given. On the side $AB$, $n$ different points are chosen strictly between $A$ and $B$. Similarly, $m$ different points are chosen on the side $AD$. Lines are drawn from the points parallel to the sides. How many rectangles are formed in this way? (One possibility is shown in the figure.)




*Solution by Antonio Leonardo P. Pastor, Universidade de São Paulo, Brasil, and also by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Note that a rectangle is uniquely determined by a pair of horizontal lines and a pair of vertical lines. Since there are $m + 2$ horizontal lines and $n + 2$ vertical lines (counting the sides), the number of rectangles is

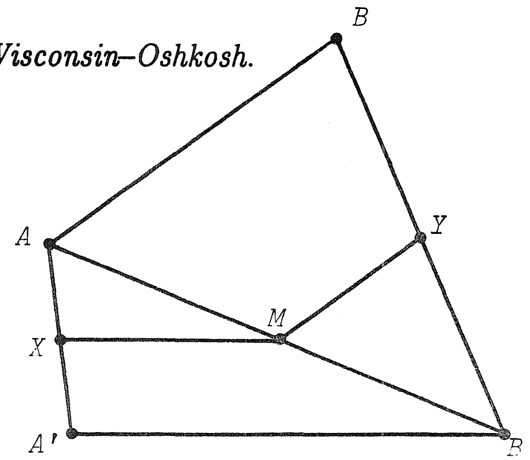
$$\binom{m+2}{2}\cdot\binom{n+2}{2} = \frac{(m+2)(m+1)(n+2)(n+1)}{4} .$$

2. Two parallel lines $a$ and $b$ meet two other lines $c$ and $d$. Let $A$ and $A'$ be the points of intersection of $a$ with $c$ and $d$, respectively. Let $B$ and $B'$ be the points of intersection of $b$ with $c$ and $d$, respectively. If $X$ is the midpoint of the line segment $AA'$ and $Y$ is the midpoint of the segment $BB'$, prove that

$$|XY| \le \frac{|AB| + |A'B'|}{2} .$$

($|XY|$ represents the length of the line segment $XY$.)

*Solution by Bob Prielipp, University of Wisconsin–Oshkosh.*




We shall show that the equivalent result holds for any convex quadrilateral $A'ABB'$ where $X$ is the midpoint of $AA'$ and $Y$ is the midpoint of $BB'$. Let $M$ be the midpoint of $AB'$. Recall that a line segment connecting the midpoints of two sides of a triangle is parallel to the third side of the triangle and the length of this line segment is one–half the length of the third side. Hence from triangle $AA'B'$, $|XM| = \frac{1}{2}|A'B'|$. Similarly from triangle $B'AB$, $|YM| = \frac{1}{2}|AB|$. Thus

$$\frac{|AB| + |A'B'|}{2} = |XM| + |YM| \geq |XY| .$$

3. Determine all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(f(x))^3 = \frac{-x}{12}[x^2 + 7xf(x) + 16(f(x))^2] .$$

*Solutions by Nicos Diamantis, Patras, Greece, by Mathew Englander, Toronto, Ontario, and by M.A. Selby, Department of Mathematics, The University of Windsor, Ontario.*

The given equation is equivalent to

$$12(f(x))^3 + 16x(f(x))^2 + 7x^2f(x) + x^3 = 0$$

and this factors as

$$(2f(x) + x)^2(3f(x) + x) = 0$$

giving that for each $x$ we have $f(x) \in \{-x/2,-x/3\}$. Now consider $f(x)$ on $(-\infty,0)$. Setting $S_1 = \{x < 0 : f(x) \neq -x/2\}$ and $S_2 = \{x < 0 : f(x) \neq -x/3\}$ we see that $S_1 \cap S_2 = \phi$, $S_1 \cup S_2 = (-\infty,0)$ and each set is open since $f(x)$ is continuous. Thus $f(x) = -x/2$ for $x$ in $(-\infty,0)$ or $f(x) = -x/3$ for $x$ in $(-\infty,0)$. Similarly $f(x) = -x/2$ or $f(x) = -x/3$ on $(0,\infty)$. There are thus four possibilities for $f(x)$:

$$f_1(x) = -x/2 , \qquad f_2(x) = -x/3 ,$$

$$f_3(x) = \begin{cases} -x/2 & \text{if } x \leq 0 , \\ -x/3 & \text{if } x > 0 , \end{cases}$$

and

$$f_4(x) = \begin{cases} -x/3 & \text{if } x \leq 0 , \\ -x/2 & \text{if } x > 0 . \end{cases}$$

4. Prove that, for every $r \in \mathbb{R}$ with $r > 1$,

$$\lim_{n\to\infty}\left(\frac{1^r+2^r+\cdots+(n-1)^r+n^r+(n-1)^r+\cdots+2^r+1^r}{n^2}\right) = +\infty .$$

What is the value of the limit when $r = 1$?

*Generalization by M.A. Selby, Department of Mathematics and Statistics, The University of Windsor, Ontario.*

Let $r > 0$. We show that

$$\lim_{n\to\infty}\left(\frac{1^r+2^r+\cdots+(n-1)^r+n^r+(n-1)^r+\cdots+2^r+1^r}{n^k}\right) = \begin{cases} 0 & \text{if } k > r+1 , \\ 2/k & \text{if } k = r+1 , \\ \infty & \text{if } k < r+1 . \end{cases}$$

Consider

$$S_n = \sum_{j=0}^{n-1} f\left(\frac{j}{n}\right)\frac{1}{n} ,$$

for $f(x) = x^r$. This sum represents a Riemann sum for $x^r$ on $[0,1]$ with equal subintervals. Hence as $n \to \infty$,

$$S_n \longrightarrow \int_0^1 x^r dx = \frac{1}{r+1} . \tag{1}$$

Now note that

$$\frac{1^r + 2^r + \cdots + (n-1)^r}{n^{r+1}} = S_n ,$$

and therefore

$$\frac{1^r+2^r+\cdots+(n-1)^r+n^r+(n-1)^r+\cdots+2^r+1^r}{n^k} = \frac{2n^{r+1}S_n}{n^k} + \frac{n^r}{n^k} .$$

If $k > r+1$, $n^{r+1}/n^k \to 0$ and $n^r/n^k \to 0$ as $n \to \infty$, and thus from (1) the given expression goes to 0. When $k = r+1$ we have $2S_n + 1/n \to 2/(r+1) = 2/k$ as $n \to \infty$. If $k < r+1$ then the expression exceeds $2n^{r+1}S_n/n^k$ and $n^{r+1}/n^k \to \infty$. Thus from (1) the expression tends to $+\infty$.

[*Editor's note*: A different solution using integration was given by Douglass L. Grant, University College of Cape Breton, Sydney, Nova Scotia.]

* * *

This is all the space we have this issue. Don't forget to send me your nice solutions and good problem sets.

* * *

# MINI-REVIEWS

by

ANDY LIU

## THE DOLCIANI MATHEMATICAL EXPOSITIONS SERIES OF THE MATHEMATICAL ASSOCIATION OF AMERICA

The books in this hardcover series are selected both for clear, informal style and for stimulating mathematical content. Some are collections of articles and problems while others deal with specific topics. Each has an ample supply of exercises, many with accompanying solutions. This series would appeal to high school and university undergraduate students.

**Volume 1. *Mathematical Gems I*, by R. Honsberger, 1973. (176 pp.)**

This book presents thirteen articles from elementary combinatorics, number theory and geometry. Topics include combinatorial geometry, recurrence relations, Hamiltonian circuits, perfect numbers, primality testing, Morley's Theorem and a story about a Hungarian prodigy, Louis Pósa.

**Volume 2. *Mathematical Gems II*, by R. Honsberger, 1976. (182 pp.)**

This book presents fourteen articles from elementary combinatorics, number theory and geometry. Topics include combinatorial geometry, box–packing problems, Fibonacci sequence, Hamiltonian circuits, Tutte's Theorem, linear Diophantine equations, the generation of prime numbers, the harmonic series, the isosceles tetrahedron and inversion.

**Volume 3. *Mathematical Morsels*, by R. Honsberger, 1978. (249 pp.)**

This book contains 91 elegant problems and 25 exercises. Most are taken from the problem sections of various journals, in particular the *American Mathematical Monthly* and *Mathematics Magazine.*

**Volume 4. *Mathematical Plums*, edited by R. Honsberger, 1979. (191 pp.)**

This book contains two articles by Honsberger and eight by other writers. The respective titles are "Some surprises in probability", "Kepler's conics", "Chromatic graphs", "How to get (at least) a fair share of the cake", "Some remarkable sequences of integers", "Existence out of chaos", "Anomalous cancellation", "A distorted view of geometry", "Convergence, divergence and the computer" and "The Skewes number".

**Volume 5.** ***Great Moments in Mathematics*** **(*before 1650*), by H. Eves, 1980. (also in paperback, 270 pp.)**

This is a history of mathematics before 1650 presented in twenty lectures, each highlighted by one of what the author considers a great moment. One such great moment is Euclid's *Elements*. Although the text is a condensed version of the author's presentation, it retains much of the vitality and smoothness of a gifted lecturer.

**Volume 6.** ***Maxima and Minima Without Calculus*, by I. Niven, 1981. (323 pp.)**

The central result considered in this book is the Isoperimetric Theorem, which states that among all figures of fixed perimeter, the circle has the greatest area. It is a problem which is not easy to handle using standard techniques in calculus. After an introduction to inequalities, in particular the Arithmetic–Mean–Geometric–Mean Inequality and Jensen's Inequality, these elementary tools are applied to various maxima and minima problems. Several related topics are also discussed.

**Volume 7.** ***Great Moments in Mathematics*** **(*after 1650*), by H. Eves, 1981. (also in paperback, 270 pp.)**

This is a history of mathematics after 1650 presented in twenty lectures. As in its companion volume, each lecture is centred around a great moment in mathematics. Here the choice of topics, by the author's own admission, is more difficult. One of the great moments chosen was the invention of the differential calculus. Another is the impact of computers and the resolution of the four–colour conjecture.

**Volume 8.** ***Map Coloring, Polyhedra and the Four–Colour Problem*, by D. Barnette, 1983. (184 pp.)**

The central result considered in this book is the Four–Colour Theorem, but only one of eight chapters is devoted to the computer–assisted proof by Appel and Haken. After giving an early history of the problem, the book discusses many related concepts and results, including Euler's Formula, Hamiltonian circuits and convex polyhedra. Map colouring on other surfaces is also considered.

**Volume 9.** ***Mathematical Gems III*, by R. Honsberger, 1985. (260 pp.)**

This book presents eighteen articles from elementary combinatorics, number theory and geometry. Topics include combinatorial geometry, generating functions, Fibonacci and Lucas numbers, probability, Ramsay's Theorem, cryptography, Helly's Theorem and a selection of problems from various Olympiads.

# PROBLEMS

*Problem proposals and solutions should be sent to the editor, B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.*

*Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.*

*To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before* **May 1, 1991,** *although solutions received after that date will also be considered until the time when a solution is published.*

**1571**. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $ABC$ be a triangle with circumradius $R$ and area $F$, and let $P$ be a point in the same plane. Put $AP = R_1$, $BP = R_2$, $CP = R_3$, $R'$ the circumradius of the pedal triangle of $P$, and $p$ the power of $P$ relative to the circumcircle of $\Delta ABC$. Prove that

$$18R^2R' \geq a^2R_1 + b^2R_2 + c^2R_3 \geq 4F\sqrt{3|p|} .$$

**1572**. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Consider a "signed harmonic series"

$$\sum_{n=1}^{\infty} \frac{\epsilon_n}{n} , \quad \epsilon_n = \pm 1 \quad \text{for each } n .$$

Assuming that plus and minus signs occur with equal frequency, i.e.

$$\lim_{n\to\infty} \frac{\epsilon_1 + \cdots + \epsilon_n}{n} = 0,$$

prove or disprove that the series necessarily converges.

**1573**. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let $M$ be the midpoint of $BC$ of a triangle $ABC$. Suppose that $\angle BAM = \angle C$ and $\angle MAC = 15^\circ$. Calculate angle $C$.

**1574**. *Proposed by Murray S. Klamkin, University of Alberta.*

Determine sharp upper and lower bounds for the sum of the squares of the sides of a quadrilateral with given diagonals $e$ and $f$. For the upper bound, it is assumed that the quadrilateral is convex.

**1575***. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

The rational number 5/2 has the property that when written in decimal expansion, i.e. 2,5 (2.5 in North America), there appear exactly the (base 10) digits of numerator and denominator in permuted form. Do there exist infinitely many $m, n \in \mathbb{N}$, neither ending in 0, so that $m/n$ has the same property?

**1576**. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Circles $\Gamma_1$ and $\Gamma_2$ have a common chord $PQ$. $A$ is a variable point of $\Gamma_1$. $AP$ and $AQ$ intersect $\Gamma_2$ for the second time in $B$ and $C$ respectively. Show that the circumcentre of $\Delta ABC$ lies on a fixed circle. (This problem is not new. A reference will be given when the solution is published.)

**1577***. *Proposed by Isao Ashiba, Tokyo, Japan.*

Suppose $\alpha$, $\beta$, $\gamma$ are arbitrary angles such that $\cos\alpha \neq \cos\beta$, and $x$ is a real number such that

$$x^2\cos\beta\cos\gamma + x(\sin\beta + \sin\gamma) + 1 = 0$$

and

$$x^2\cos\gamma\cos\alpha + x(\sin\gamma + \sin\alpha) + 1 = 0\ .$$

Prove that

$$x^2\cos\alpha\cos\beta + x(\sin\alpha + \sin\beta) + 1 = 0\ .$$

**1578**. *Proposed by O. Johnson and C.S. Goodlad, students, King Edward's School, Birmingham, England.*

For each fixed positive real number $a_n$, maximise

$$\frac{a_1a_2\cdots a_{n-1}}{(1+a_1)(a_1+a_2)(a_2+a_3)\cdots(a_{n-1}+a_n)}$$

over all positive real numbers $a_1,a_2,\cdots,a_{n-1}$.

**1579**. *Proposed by S. Kotani, Akita, Japan and H. Fukagawa, Aichi, Japan.*

Given a triangle $ABC$, we erect three similar rectangles $ABDE$, $CAFG$, $BCHI$ outside $ABC$. Let $M_a$ and $N_a$ be the midpoints of $BC$ and $EF$ respectively. Define $M_b$, $N_b$, $M_c$, $N_c$ analogously. Prove that the lines $M_aN_a$, $M_bN_b$, $M_cN_c$ are concurrent.

**1580***. *Proposed by Ji Chen, Ningbo University, China.*

For every convex $n$-gon, if one circle with centre $O$ and radius $R$ contains it and another circle with centre $I$ and radius $r$ is contained in it, prove or disprove that

$$R^2 \geq r^2\sec^2\frac{\pi}{n} + IO^2\ .$$

*　　　　*　　　　*

# SOLUTIONS

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

**1449**. [1989: 149] *Proposed by David C. Vaughan, Wilfrid Laurier University.*

Prove that for all $x \ge y \ge 1$,

$$\frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{y+1}} + \frac{1}{\sqrt{x+1}} \ge \frac{y}{\sqrt{x+y}} + \frac{x}{\sqrt{x+1}} + \frac{1}{\sqrt{y+1}} .$$

Determine where equality holds.

I. *Solution by Seung–Jin Bang, Seoul, Republic of Korea.*

Let

$$f(x,y) = \frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{y+1}} + \frac{1}{\sqrt{x+1}} - \left(\frac{y}{\sqrt{x+y}} + \frac{x}{\sqrt{x+1}} + \frac{1}{\sqrt{y+1}}\right)$$

$$= \frac{x-y}{\sqrt{x+y}} - \frac{x-1}{\sqrt{x+1}} + \frac{y-1}{\sqrt{y+1}} .$$

Note that if $x = y$ or $y = 1$ then $f = 0$. Assume $1 < y < x$. Some calculation shows that

$$\frac{\partial f}{\partial x} = \frac{1}{2}(g(y) - g(1))$$

where

$$g(y) = \frac{x+3y}{(x+y)^{3/2}} .$$

Since

$$g'(y) = \frac{3(x-y)}{2(x+y)^{5/2}} > 0 ,$$

$g(y)$ is strictly increasing as a function of $y$, that is, $g(y) - g(1) > 0$. This implies that $\partial f/\partial x > 0$, so $f(x,y)$ is strictly increasing as a function of $x$. Since $f(y,y) = 0$ for all $y$, we conclude that $f(x,y) > 0$ for all $x > y > 1$. q.e.d.

II. *Solution by Murray S. Klamkin, University of Alberta.*

Let $x + 1 = 2p^2$ and $y + 1 = 2q^2$ (so that $p \ge q \ge 1$) to give successively

$$\frac{2p^2 - 1}{\sqrt{p^2 + q^2 - 1}} + \frac{2q^2 - 1}{q} + \frac{1}{p} \ge \frac{2q^2 - 1}{\sqrt{p^2 + q^2 - 1}} + \frac{2p^2 - 1}{p} + \frac{1}{q} ,$$

$$\frac{p^2 - q^2}{\sqrt{p^2 + q^2 - 1}} \ge (p - q) + \frac{p - q}{pq} ,$$

and (if $p \ne q$)

$$pq(p + q) \ge (1 + pq)\sqrt{p^2 + q^2 - 1} .$$

Now

$$p + q - 1 \geq \sqrt{p^2 + q^2 - 1}\ ,$$

since by squaring out we obtain $2(p - 1)(q - 1) \geq 0$. It now suffices to show that

$$pq(p + q) \geq (1 + pq)(p + q - 1)\ .$$

On expanding out, the latter reduces to $(p - 1)(q - 1) \geq 0$. Thus the given inequality holds. There is equality if and only if $p = 1$ or $q = 1$ or $p = q$; equivalently, if $x = y$ or $y = 1$.

*Remark.* By the rearrangement inequality [1], among all three–term sums with numerators $x \geq y \geq 1$ and denominators some permutation of $\sqrt{x+1}$, $\sqrt{y+1}$, $\sqrt{x+y}$, the largest and smallest are respectively

$$\frac{x}{\sqrt{y+1}} + \frac{y}{\sqrt{x+1}} + \frac{1}{\sqrt{x+y}} \quad\text{and}\quad \frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{x+1}} + \frac{1}{\sqrt{y+1}}\ .$$

*Reference*:


[1] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge Univ. Press, London, 1964, p. 261, Th. 368.


III. *Solution by David Poole, Trent University, Peterborough, Ontario.*

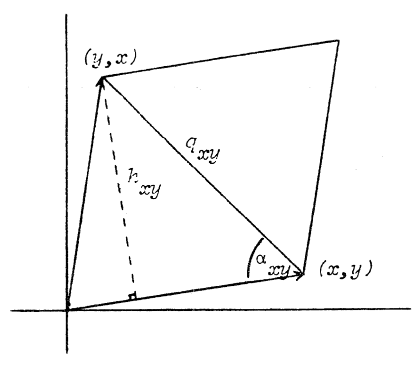
We prove, more generally [*See below. –Ed.*], that for all positive real numbers $x$, $y$, $z$ such that $x \geq y \geq z$,

$$\frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{y+z}} + \frac{z}{\sqrt{x+z}} \geq \frac{y}{\sqrt{x+y}} + \frac{x}{\sqrt{x+z}} + \frac{z}{\sqrt{y+z}}\ . \tag{1}$$

Since $x \geq y \geq z > 0$ if and only if $x^2 \geq y^2 \geq z^2 > 0$, (1) is equivalent to the assertion that for all $x \geq y \geq z > 0$,

$$\frac{x^2 - z^2}{\sqrt{x^2 + z^2}} \leq \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} + \frac{y^2 - z^2}{\sqrt{y^2 + z^2}}\ . \tag{2}$$

Let $A_{xy}$ and $h_{xy}$ denote, respectively, the area and height of the rhombus with sides determined by the vectors $(x,y)$ and $(y,x)$, where $x \geq y$. Then




$$A_{xy} = \det\begin{bmatrix} x & y \\ y & x \end{bmatrix} = x^2 - y^2$$

and

$$h_{xy} = \frac{A_{xy}}{\|(x,y)\|}\ .$$

Thus (2) becomes

$$h_{xz} \leq h_{xy} + h_{yz} \qquad \text{for all} \quad x \geq y \geq z > 0\ . \tag{3}$$

If for $x \geq y$ we set

$$\mathbf{Q}_{xy} = (x,y) - (y,x) = (x - y, y - x)\ , \qquad q_{xy} = \|\mathbf{Q}_{xy}\|\ ,$$

then $\mathbf{Q}_{xz} = \mathbf{Q}_{xy} + \mathbf{Q}_{yz}$ for $x \geq y \geq z$, and since $\mathbf{Q}_{xy}$ is parallel to $(1,-1)$ for any

$x > y$ we have

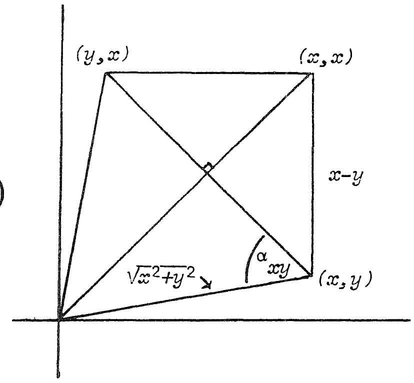
$$q_{xz} = q_{xy} + q_{yz}$$

for all $x \geq y \geq z > 0$. Also,

$$h_{xy} = q_{xy}\sin\,\alpha_{xy}$$

where $\alpha_{xy}$ is the acute angle between the vectors $(x,y)$ and $(1,-1)$. Note that

$$\cos\,\alpha_{xy} = \frac{(x-y)/\sqrt{2}}{\sqrt{x^2+y^2}}\ ;$$




thus, since $y \geq z$,

$$\cos\,\alpha_{xy} = \frac{x-y}{\sqrt{2}\sqrt{x^2+y^2}} \leq \frac{x-z}{\sqrt{2}\sqrt{x^2+z^2}} = \cos\,\alpha_{xz}\ .$$

Hence $\alpha_{xz} \leq \alpha_{xy}$. Similarly [*See below once more! –Ed.*], since $x \geq y$, $\alpha_{xz} \leq \alpha_{yz}$. We therefore have

$$\begin{aligned} h_{xz} = q_{xz}\sin\,\alpha_{xz} &= q_{xy}\sin\,\alpha_{xz} + q_{yz}\sin\,\alpha_{xz} \\ &\leq q_{xy}\sin\,\alpha_{xy} + q_{yz}\sin\,\alpha_{yz} = h_{xy} + h_{yz}\ , \end{aligned}$$

thus establishing (3). Equality holds if and only if

$$q_{xy}\sin\,\alpha_{xy} = q_{xy}\sin\,\alpha_{xz} \qquad \text{and} \qquad q_{yz}\sin\,\alpha_{xz} = q_{yz}\sin\,\alpha_{yz}$$

which is easily seen to be equivalent to $x = y$ or $y = z$.

[*Editor's quibbles.* Two points in this very nice proof need comment. First, the "generalization" (1) isn't really, since it follows immediately by replacing $x$ in the original proposal by $x/z$ and $y$ by $y/z$. Second, the proof that $\alpha_{xz} \leq \alpha_{yz}$, rather than being "similar" to the proof that $\alpha_{xz} \leq \alpha_{xy}$, seems to the editor to require knowing that the function

$$f(x) = \frac{x-z}{\sqrt{x^2+z^2}}$$

is *increasing* for $x \geq z$. This, however, is easy to show by differentiation. Alternatively, let $\theta_{xy}$ be the acute angle between $(x,y)$ and the $x$–axis; then

$$\theta_{xy} = \frac{\pi}{4} - \left(\frac{\pi}{2} - \alpha_{xy}\right) = \alpha_{xy} - \frac{\pi}{4}\ ,$$

and it is clear that $\theta_{xz} \leq \theta_{xy}$ and $\theta_{xz} \leq \theta_{yz}$, so $\alpha_{xz} \leq \alpha_{xy}$ and $\alpha_{xz} \leq \alpha_{yz}$ both follow.]

*Also solved by C. FESTRAETS–HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE–WAI LAU, Hong Kong; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; and the proposer. One partial solution was sent in.*

*Lau's solution was the same as solution* I *by Bang. Janous also noticed the more general "triangle inequality"* (2) *given in Poole's proof, but without the neat* *geometric interpretation* (3). *In fact Janous went on to show that the function*

$$d(x,y) = \frac{|x - y|}{\sqrt{x + y}}$$

*satisfies the triangle inequality* $d(x,z) \le d(x,y) + d(y,z)$ *for all* $x,y,z > 0$, *and thus induces a metric on* $\mathbb{R}^+$.

*Is there a generalization to* $n$ *variables?*

* * *

**1451**. [1989: 177] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find positive integers $a$ and $b$ such that $a \le 2000 \le b$ and 2, $a(a + 1)$, $b(b + 1)$ are in arithmetic progression.

*Solution by Hayo Ahlburg, Benidorm, Spain.*

We want

$$a(a + 1) - 2 = b(b + 1) - a(a + 1) .$$

Multiplied by 4 and rearranged, this gives

$$2(2a + 1)^2 - (2b + 1)^2 = 9 .$$

$A = 2a + 1$ and $B = 2b + 1$ must both be multiples of 3, since otherwise $2A^2 - B^2 \not\equiv 0 \bmod 3$. With $A = 3x$ and $B = 3y$, we need the solutions of the Fermat–Pell equation $2x^2 - y^2 = 1$. They are

$$x = \frac{(1 + \sqrt{2})^{2n+1} - (1 - \sqrt{2})^{2n+1}}{2\sqrt{2}} = \left[\frac{(1 + \sqrt{2})^{2n+1}}{2\sqrt{2}}\right] + 1 ,$$

$$y = \frac{(1 + \sqrt{2})^{2n+1} + (1 - \sqrt{2})^{2n+1}}{2} = \left[\frac{(1 + \sqrt{2})^{2n+1}}{2}\right] ,$$

where $[K]$ is the largest integer $\le K$. The first few values are:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|---|---|
| $x$ | 1 | 5 | 29 | 169 | 985 | 5741 | 33461 | |
| $y$ | 1 | 7 | 41 | 239 | 1393 | 8119 | 47321 | |
| $a$ | 1 | 7 | 43 | 253 | **1477** | 8611 | 50191 | |
| $b$ | 1 | 10 | 61 | 358 | **2089** | 12178 | 70981 | |

$n = 4$ gives the solution to our problem.

[Dr. Ahlburg ended with kind words about the late Dr. Groenman, saying in part "he proposed several problems leading to Fermat–Pell equations which I have liked for half a century". – Ed.]

*Also solved by SEUNG–JIN BANG, Seoul, Republic of Korea; GINGER BOLTON, Swainsboro, Georgia; DUANE M. BROLINE, Eastern Illinois University,* *Charleston; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; O. JOHNSON, student, King Edward's School, Birmingham, England; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE–WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; P. PENNING, Delft, The Netherlands; STANLEY RABINOWITZ, Westford, Massachusetts; ROBERT E. SHAFER, Berkeley, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.*

*One other reader (dividing each term by 2) noted that the problem asks for three triangular numbers in arithmetic progression, and generalized to show that, ignoring the constraint on a and b, there are solutions for any initial triangular number. But he never answered the question!*

*The proposer mentioned that the numbers $n(n+1)$ for $n \in \mathbb{N}$ are called "pronikgetallen" in Dutch.*

* * *

**1452**. [1989: 177] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $x_1, x_2, x_3$ be positive reals satisfying $x_1 + x_2 + x_3 = 1$, and consider the inequality

$$(1 - x_1)(1 - x_2)(1 - x_3) \geq c_r(x_1x_2x_3)^r. \qquad (1)$$

For each real $r$, find the greatest constant $c_r$ such that (1) holds for all choices of the $x_i$, or prove that no such constant $c_r$ exists.

*Solution by Murray S. Klamkin, University of Alberta.*

More generally, we consider the homogeneous inequality

$$(s - x_1)(s - x_2)\cdots(s - x_n) \geq c_r(n)(x_1x_2\cdots x_n)^r s^{n-nr} \qquad (2)$$

where $n > 2$, $s = x_1 + x_2 + \cdots + x_n$, and all $x_i > 0$.

*Case* (i): $r < 1/(n-1)$. Let $x_1 = N$ ($N$ large) and all the remaining $x_i = 1$. Then we must have

$$(n-1)(N + n - 2)^{n-1} \geq c_r(n)N^r(N + n - 1)^{n-nr},$$

so for large $N$

$$c_r(n) \leq \frac{(n-1)N^{n-1}}{N^rN^{n-nr}} = \frac{n-1}{N^{1-r(n-1)}} .$$

Hence $c_r(n) = 0$.

*Case* (ii): $r \geq 1/(n-1)$. It is known that

$$(s - x_1)(s - x_2)\cdots(s - x_n) \geq (n - 1)^n(x_1x_2\cdots x_n)^{1/(n-1)}\left(\frac{s}{n}\right)^{n(n-2)/(n-1)}$$

(Popoviciu's inequality [1] with $f(x) = -\log x$ and $k = n - 1$). Hence it suffices to satisfy

$$(n - 1)^n(x_1x_2\cdots x_n)^{1/(n-1)}\left(\frac{s}{n}\right)^{n(n-2)/(n-1)} \geq c_r(n)(x_1x_2\cdots x_n)^r s^{n-nr},$$

or

$$c_r(n)(x_1x_2\cdots x_n)^{r-1/(n-1)} \leq (n - 1)^n n^{n(r-1)}\left(\frac{s}{n}\right)^{n(n-2)/(n-1) + nr-n}$$

$$= (n - 1)^n n^{n(r-1)}\left(\frac{s}{n}\right)^{n(r-1/(n-1))}. \qquad (3)$$

Now by the A.M.–G.M. inequality, $(s/n)^n \geq x_1x_2\cdots x_n$, so (3) is satisfied if $c_r(n) = (n - 1)^n n^{n(r-1)}$.

In summary,

$$c_r(n) = \begin{cases} (n - 1)^n n^{n(r-1)} & \text{if } r \geq 1/(n - 1)\ , \\ 0 & \text{if } r < 1/(n - 1)\ . \end{cases}$$

The proposed inequality corresponds to the case $n = 3$.

Incidentally, the above value of $c_r(n)$ for $r \geq 1$ can be obtained more easily by using the A.M.–G.M. in (2) for $s$ and for each of the terms $s - x_i$. Then

$$s - x_1 = x_2 + \cdots + x_n \geq (n - 1)(x_2x_3\cdots x_n)^{1/(n-1)} = \frac{(n-1)}{x_1^{1/(n-1)}}\left(\prod_{i=1}^{n} x_i\right)^{1/(n-1)}, \quad \text{etc.,}$$

and

$$s = x_1 + x_2 + \cdots + x_n \geq n\left(\prod_{i=1}^{n} x_i\right)^{1/n},$$

so for r $\geq$ 1,

$$s^{n(r-1)}(s - x_1)\cdots(s - x_n)$$

$$\geq n^{n(r-1)}\left(\prod_{i=1}^{n} x_i\right)^{r-1}(n - 1)^n\left(\prod_{i=1}^{n} x_i\right)^{n/(n-1)} \Bigg/ \left(\prod_{i=1}^{n} x_i\right)^{1/(n-1)}$$

$$= (n - 1)^n n^{n(r-1)}\left(\prod_{i=1}^{n} x_i\right)^{r}.$$

Thus (2) is satisfied if $c_r(n) = (n - 1)^n n^{n(r-1)}$.

Inequality (2) for the case $n = 3$ can be converted into a triangle inequality by letting

$$a = x_2 + x_3\ , \quad b = x_3 + x_1\ , \quad c = x_1 + x_2\ ,$$

and rewriting $r$ as $\rho$. Then $s$ is in fact the semiperimeter, and (2) becomes

$$abc \ge c_\rho(3)(s(s-a)(s-b)(s-c))^\rho s^{3-4\rho} = c_\rho(3)F^{2\rho}s^{3-4\rho}\ ,$$

where $F$ is the area of the triangle. Using $abc = 4RF$ and $F = rs$, where $R$ is the circumradius and $r$ the inradius, we get

$$4RF \ge c_\rho(3)F(rs)^{2\rho-1}s^{3-4\rho}$$

or

$$4R \ge c_\rho(3)r^{2\rho-1}s^{3-4\rho}. \tag{4}$$

In this form, it is immediately apparent that for $\rho < 1/2$, $c_\rho(3)$ must be 0 (just choose triangles so that $r$ but not $s$ is arbitrarily small). For $1/2 \le \rho \le 1$, we use the known result that for given $R$, $r$ and $s$ are maximized when the triangle is equilateral. Thus (4) gives

$$c_\rho(3) = \frac{4R}{(R/2)^{2\rho-1}(3\sqrt{3}R/2)^{2-2\rho}} = 8(27)^{\rho-1}.$$

For $\rho > 1$, we obtain the same value for $c_\rho(3)$ from the result that for given $s$ or for given $R$, $r$ is maximized when the triangle is equilateral.

*Reference*:

[1] D.S. Mitrinović, *Analytic Inequalities*, Springer–Verlag, 1970, p. 174.

*Also solved by ROBERT E. SHAFER, Berkeley, California; and the proposer. Two incorrect solutions were sent in.*

* * *

**1453**. [1989: 177] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Triangle $ABC$ moves in such a way that $AB$ passes through a fixed point $P$ and $AC$ through a fixed point $Q$. Prove that throughout the motion $BC$ is tangent to a fixed circle.

*Comment by Dan Pedoe, Minneapolis, Minnesota.*

This problem occurs as a lemma in my article "The Bobillier envelope theorem", *Crux Mathematicorum* 5 (1979) 41–43. It is a special case of the theorem, due to Bobillier: *if a triangular lamina $ABC$ moves so that $AB$ touches a fixed circle and $AC$ touches another fixed circle, then the envelope of the third side $BC$ is a third fixed circle.*

*Also solved by C. FESTRAETS–HAMOIR, Brussels, Belgium; JILL HOUGHTON, Cammeray, Australia; SHIKO IWATA and HIDETOSI FUKAGAWA, Gifu and Aichi, Japan; P. PENNING, Delft, The Netherlands; STANLEY RABINOWITZ, Westford, Massachusetts; TOSHIO SEIMIYA, Kawasaki, Japan; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.*

*Iwata and Fukagawa, and Seimiya, also recalled the result as being due to Bobillier. Sokolowsky and the proposer also gave the stronger result mentioned by Pedoe.*

*The editor apologizes for the duplication, but the result is a nice one, and maybe deserves to be mentioned in* Crux *every ten years. (Stay tuned in 1999!)*

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<u>1454</u>. [1989: 177] *Proposed by Marcin E. Kuczma, Warsaw, Poland.*

Given a convex pentagon of area $S$, let $S_1,\cdots,S_5$ denote the areas of the five triangles cut off by the diagonals (each triangle is spanned by three consecutive vertices of the pentagon). Prove that the sum of some four of the $S_i$'s exceeds $S$.

*Solution by Murray S. Klamkin, University of Alberta.*

From L.F. Meyers' solution of *Crux* 232 [1977: 238] we have

$$2S = A + \sqrt{A^2 - 4B}\ ,$$

where

$$A = \sum_{i=1}^{5} S_i\ , \qquad B = S_1S_2 + S_2S_3 + S_3S_4 + S_4S_5 + S_5S_1\ . \tag{1}$$

Letting $C = \min S_i$, we want to show that $A - C > S$, or that

$$A - 2C > \sqrt{A^2 - 4B}\ .$$

Squaring both sides, we get

$$B + C^2 > CA\ ,$$

which immediately follows by replacing $A$ and $B$ by their values in (1). By allowing the pentagon to degenerate into a triangle, we can have equality: if $V_1, V_2,\cdots$ are the vertices of the pentagon, let $V_5$ coincide with $V_1$ and let $V_3$ lie on the side $V_2V_4$.

In [1], p. 412, the following problem, M193 from *Kvant* 11 (1973) 43–44, is referenced: *Let ABCDE be a convex pentagon of area $F_{ABCDE}$. Then*

$$F_{ABE} + F_{AED} + F_{EDC} + F_{BCD} \geq F_{ABCDE}\ ,$$

*where $F_{ABE}$ is the area of triangle ABE, etc.* As stated this is incorrect (take

$AEDC$ close to collinear with $A$ close to $E$ and $D$ close to $C$). It would be of interest to obtain the *Kvant* solution to see if the statement was corrected.

*Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer. Janous just gave the above references.*

* * *

**1455**. [1989: 178] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be a triangle inscribed in a triangle $ABC$, so that $A' \in BC$, $B' \in CA$, $C' \in AB$. Suppose that

$$\frac{BA'}{A'C} = \frac{CB'}{B'A} = \frac{AC'}{C'B} \neq 1$$

and that $\Delta A'B'C'$ is similar to $\Delta ABC$. Prove that the triangles are equilateral.

I. *Comment by the editor.*

Both Dan Sokolowsky, Williamsburg, Virginia, and Dan Pedoe, Minneapolis, Minnesota, write to point out (gently) that this problem is essentially the same as *Crux* 210 [1977: 160, 196]. As the published solutions to *Crux* 210 show, the present problem is *false* unless the similarity of triangles $ABC$ and $A'B'C'$ is direct (which the proposer and most solvers did in fact assume).

Pedoe also mentions that the problem appeared as problem 47.5 in his 1970 book *A Course of Geometry for Colleges and Universities* (it's on page 184 of the 1988 Dover reprint *Geometry: A Comprehensive Course*). And Sokolowsky notes that it follows from his embarrassingly recent problem *Crux* 1208 [1988: 111].

This makes the third *Crux* problem in a row which turned out to be already known, two of them having previously appeared in *Crux*, the last only 2 1/2 years ago. The editor rejects premature senility as the cause of these lapses on his part, and instead (observing that the offending problems all appeared in a June issue) suspects sunstroke.

II. *Generalization by Murray S. Klamkin, University of Alberta.*

[Klamkin first gave a proof of the original problem. –Ed.]

We can prove a restricted extension to $n$–gons. Let $z_1, z_2, \cdots, z_n$ be the complex numbers corresponding to the respective vertices of an $n$–gon $P$: $A_1A_2\cdots A_n$ in the complex plane with origin at its centroid. Let $z_1, z_2, \cdots, z_n$ be the complex numbers corresponding to the respective vertices of an associated $n$–gon $P'$: $A_1A_2\cdots A_n$, where

$$z_i = az_i + bz_{i+1}\ , \quad i = 1,2,\cdots,n\ , \qquad (1)$$

$a$ and $b$ being fixed complex numbers, $b \neq 0$, and $z_{n+1} = z_1$. Then *$P'$ is directly similar to $P$ if and only if $P$ is regular.*

$P'$ and $P$ have the same centroid (add up the $z_i$'s and use that the sum of the $z_i$'s is 0). If $P$ is regular then $z_i = \omega^i z_n$ where $\omega$ is a primitive $n$th root of unity. It then follows immediately that $z_i = \omega^i z_n$ so that $P'$ is also regular and thus similar to $P$. Conversely, if $P'$ is directly similar to $P$ then $z_i/z_i = \lambda$ for all $i$. It then follows that $(\lambda - a)z_i = bz_{i+1}$ for all $i$, whence $(\lambda - a)^n = b^n$. Since the vertices $z_i$ are distinct, $\lambda - a = \omega b$ where $\omega$ is again a complex $n$th root of unity. Thus $\omega z_i = z_{i+1}$ so that both $P$ and $P'$ are regular. It is to be noted that $P$ need not be convex, it can be a regular star polygon.

It is an open problem whether similarity still implies regularity if we replace conditions (1) by the more general conditions

$$z_i = u_1 z_i + u_2 z_{i+1} + \cdots + u_n z_{i+n-1} ,$$

where $z_{i+n} = z_i$, $i = 1,2,\cdots,n$, and the $u_i$ satisfy some mild restrictions.

*Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; STANLEY RABINOWITZ, Westford, Massachusetts; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.*

*Seimiya found a counterexample by considering the case that the similarity of the given triangles is indirect. The other solvers took the similarity to be direct.*

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<u>1456</u>. [1989: 178] *Proposed by Murray S. Klamkin, University of Alberta.*

(a) Find a pair of integers $(a,b)$ such that

$$x^{13} - 233x - 144 \qquad \text{and} \qquad x^{15} + ax + b$$

have a common (nonconstant) polynomial factor.

(b)* Is the solution unique?

I. *Solution to* (a) *by Mathew Englander, Kitchener, Ontario.*

Consider the Fibonacci sequence $f_0 = f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$. For $n \geq 0$ let $p_n$ be the polynomial

$$x^n + x^{n-1} + 2x^{n-2} + 3x^{n-3} + 5x^{n-4} + \cdots + f_{n-1}x + f_n .$$

Then

$$\begin{aligned}(x^2 - x - 1)p_n &= x^{n+2} + x^{n+1} + 2x^n + \cdots + f_{n-1}x^3 + f_n x^2 \\ &\qquad - (x^{n+1} + x^n + 2x^{n-1} + \cdots + f_{n-1}x^2 + f_n x) \\ &\qquad - (x^n + x^{n-1} + 2x^{n-2} + \cdots + f_{n-1}x + f_n) \\ &= x^{n+2} - f_{n+1}x - f_n .\end{aligned}$$

Observe that $f_{11} = 144$, $f_{12} = 233$, $f_{13} = 377$, $f_{14} = 610$. So

$$(x^2 - x - 1)p_{11} = x^{13} - 233x - 144$$

and

$$(x^2 - x - 1)p_{13} = x^{15} - 610x - 377 .$$

Thus $(a,b) = (-610,-377)$ is one solution to the problem.

II. *Solution to* (b) *by Stanley Rabinowitz, Westford, Massachusetts.*

[Of course, Rabinowitz first answered part (a). –Ed.]

My computer tells me that $x^{13} - 233x - 144$ factors as

$$(x^2-x-1)(x^{11}+x^{10}+2x^9+3x^8+5x^7+8x^6+13x^5+21x^4+34x^3+55x^2+89x+144) , \qquad (1)$$

where both factors are irreducible. (This factorization, involving Fibonacci numbers, can be found in [1] and [2].) If $x^{15} + ax + b$ has a (nonconstant) polynomial factor in common with $x^{13} - 233x - 144$, then this common factor must be one of the two factors in (1) (or their product).

*Case* (i). Suppose that $x^{15} + ax + b$ is divisible by $x^2 - x - 1$. Then since $x^{15} - 610x - 377$ is also divisible by $x^2 - x - 1$, the difference between these two fifteenth degree polynomials must also be divisible by $x^2 - x - 1$. In other words, $(a + 610)x + (b + 377)$ would be divisible by $x^2 - x - 1$. The only way this could happen is if $a = -610$ and $b = -377$.

*Case* (ii). Suppose that $x^{15} + ax + b$ is divisible by

$$x^{11} + x^{10} + 2x^9 + 3x^8 + 5x^7 + 8x^6 + 13x^5 + 21x^4 + 34x^3 + 55x^2 + 89x + 144.$$

In this case we have

$$x^{15} + ax + b = (x^4 + px^3 + qx^2 + rx + s)\cdot\frac{x^{13} - 233x - 144}{x^2 - x - 1} ,$$

or

$$(x^{15} + ax + b)(x^2 - x - 1) = (x^4 + px^3 + qx^2 + rx + s)(x^{13} - 233x - 144) .$$

But this equation cannot hold, since the coefficient of $x^5$ is 0 on the left and $-233$ on the right.

Thus the solution in part (a) is unique.

*References*:


[1] Charles R. Wall, Problem B–55, *Fibonacci Quarterly* 3 (1965) 158.

[2] Russell Euler, Problem 8, *Missouri Journal of Math. Sciences 1* (1989) 44–45.


*Part* (a) *also solved by HAYO AHLBURG, Benidorm, Spain; HARRY ALEXIEV, Zlatograd, Bulgaria; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The*

*Netherlands; ROBERT E. SHAFER, Berkeley, California; and the proposer.*

*In addition Hess and Shafer gave almost–complete proofs of part* (b). *Hess's dealt with the stronger problem of showing that no integers* $(a,b) \neq (-610,-377)$ *exist such that* $x^{15} + ax + b$ *has a root in common with* $x^{13} - 233x - 144$. *Shafer's solution was like Rabinowitz's, but without his computer, and so was much longer. Shafer also wonders about the irreducibility of the polynomial*

$$\frac{x^n - F_{n+1}x - F_n}{x^2 - x - 1} = x^{n-2} + x^{n-3} + 2x^{n-4} + \cdots + F_{n-1}x + F_n ,$$

*where* $F_n$ *is the nth Fibonacci number. A computer verifies that these are all irreducible for* $n \leq 13$. *Can anyone come up with further information?*

* * *

<u>1457</u>. [1989: 178] *Proposed by Colin Springer, student, University of Waterloo.*

In $\Delta ABC$, the sides are $a$, $b$, $c$, the perimeter is $p$ and the circumradius is $R$. Show that

$$R^2p \geq \frac{a^2b^2}{a + b - c} .$$

Under what conditions does equality hold?

I. *Solution by Harry Alexiev, Zlatograd, Bulgaria.*

Evidently the required inequality is equivalent to

$$\frac{a^2b^2}{R^2} \leq (a + b + c)(a + b - c) = (a + b)^2 - c^2 = 2ab + 2ab \cos C ,$$

or

$$\frac{ab}{2R^2} \leq 1 + \cos C = 2 \cos^2(C/2) ,$$

or

$$\cos^2(C/2) \geq \frac{a}{2R}\cdot\frac{b}{2R} = \sin A \sin B .$$

But

$$\sin A \sin B = \frac{\cos(A - B) - \cos(A + B)}{2}$$
$$\leq \frac{1 - \cos(A + B)}{2} = \sin^2\left(\frac{A + B}{2}\right) = \cos^2\left(\frac{C}{2}\right) .$$

We have equality for $A = B$.

II. *Solution by O. Johnson, student, King Edward's School, Birmingham, England.*

By the formula

$$R = \frac{abc}{4\ \text{Area}(\Delta ABC)} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

($s$ the semiperimeter), we have

$$R^2p = \frac{a^2b^2c^2\cdot 2s}{16s(s-a)(s-b)(s-c)} = \frac{a^2b^2c^2}{(a+b-c)(a-b+c)(-a+b+c)}\,. \tag{1}$$

By the A.M.–G.M. inequality,

$$\sqrt{(a-b+c)(-a+b+c)} \le \frac{a-b+c-a+b+c}{2} = c\,,$$

so by (1)

$$R^2p \ge \frac{a^2b^2}{a+b-c}\,.$$

*Also solved by DUANE M. BROLINE, Eastern Illinois University, Charleston; C. FESTRAETS–HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Flushing, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; KEE–WAI LAU, Hong Kong; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; BOB PRIELIPP, University of Wisconsin–Oshkosh; STANLEY RABINOWITZ, Westford, Massachusetts; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer. Solved for acute triangles only by GRAHAM DENHAM, student, University of Alberta.*

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**1460**. [1989: 178] *Proposed by Mihaly Bencze, Brasov, Romania.*

$P$ is an interior point of a convex $n$–gon $A_1A_2\cdots A_n$. For each $i = 1,\cdots,n$ let $R_i = \overline{PA_i}$ and $w_i$ be the length of the bisector of $\angle P$ in $\Delta A_iPA_{i+1}$ $(A_{n+1} \equiv A_1)$. Also let $c_1,\cdots,c_n$ be positive real numbers. Prove that

$$2\cos\frac{\pi}{n}\sum_{i=1}^{n} c_i^2 \ \ge\ \sum_{i=1}^{n} c_ic_{i+1}w_i\left(\frac{1}{R_i} + \frac{1}{R_{i+1}}\right)$$

$(R_{n+1} \equiv R_1)$.

*Comment by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

This inequality is not new. It can be found as (9) on p. 423 of D.S. Mitrinović, J.E. Pečarić and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, 1989.

*Also solved by MURRAY S. KLAMKIN, University of Alberta; and the proposer. Like Janous, Klamkin just gave the above reference.*

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**1461**. [1989: 206] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $a$, $b$, $c$, $r$, $R$, $s$ be the sides, inradius, circumradius, and semiperimeter of a triangle and let $a'$, $b'$, $c'$, $r'$, $R'$, $s'$ be similarly defined for a second triangle. Show that

$$\left(4ss' - \sum aa'\right)^2 \geq 4(s^2 + r^2 + 4Rr)(s'^2 + r'^2 + 4R'r') ,$$

where the sum is cyclic.

I. *Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

By using the well-known expression

$$s^2 + r^2 + 4Rr = bc + ca + ab ,$$

the proposed inequality can be written as

$$[(a+b+c)(a'+b'+c') - (aa'+bb'+cc')]^2 \geq 4(bc+ca+ab)(b'c'+c'a'+a'b') ,$$

or

$$(ab'+ac'+ba'+bc'+ca'+cb')^2 \geq 4(bc+ca+ab)(b'c'+c'a'+a'b') ,$$

and this inequality is a particular case (in which $a$, $b$, $c$, $a'$, $b'$, $c'$ are the sides of two triangles) of the inequality proposed by Australia at the 28th IMO in Havana (solution in [1988: 297]). Equality holds when $a/a' = b/b' = c/c'$, that is, when the triangles are similar.

II. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

The formulas

$$r = \frac{abc}{4Rs} = \sqrt{(s-a)(s-b)(s-c)/s}$$

yield

$$s^2 + r^2 + 4Rr = s^2 + \frac{(s-a)(s-b)(s-c)}{s} + \frac{abc}{s} = bc + ca + ab = \frac{4s^2 - q^2}{2} ,$$

where $q^2 = a^2 + b^2 + c^2$. The inequality we are to prove is square–homogeneous in $a$, $b$, $c$ as well as in $a'$, $b'$, $c'$. So we may assume without loss of generality that $s = s' = 1/2$, and the inequality becomes

$$[1 - (aa' + bb' + cc')]^2 \geq (1 - q^2)(1 - q'^2) , \qquad (1)$$

with $q'$ defined analogously to $q$. But this last inequality is obvious: the number $(1 - qq')^2$ can be put in between the two sides. [*Editor's note.* The resulting left–hand inequality becomes

$$aa' + bb' + cc' \leq \sqrt{(a^2 + b^2 + c^2)(a'^2 + b'^2 + c'^2)}$$

which follows from the Cauchy–Schwarz inequality, and the right–hand inequality becomes

$$2\sqrt{(a^2 + b^2 + c^2)(a'^2 + b'^2 + c'^2)} \le (a^2 + b^2 + c^2) + (a'^2 + b'^2 + c'^2) ,$$

which follows from the A.M.–G.M. inequality.]

(1) turns into an equality if and only if the triples $(a,b,c)$ and $(a',b',c')$ coincide. On account of the previous homogeneity argument, this means that the original inequality becomes an equality if and only if the two triangles are similar, $a'$ corresponding to $a$, etc.

*Also solved by L.J. HUT, Groningen, The Netherlands; and the proposer.*

* * *

**1462***. [1989: 206] *Proposed by Jack Garfunkel, Flushing, N.Y.*

If $A$, $B$, $C$ are the angles of a triangle, prove or disprove that

$$\sqrt{2}\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) \ge \sqrt{\sin(A/2)} + \sqrt{\sin(B/2)} + \sqrt{\sin(C/2)} ,$$

with equality when $A = B = C$.

*Solution by Marcin E. Kuczma, Warszawa, Poland.*

I choose to prove, but the proof is anything but nice. We wish to minimize

$$F(x,y,z) = \sqrt{2}(\sin x + \sin y + \sin z) - (\sqrt{\sin x} + \sqrt{\sin y} + \sqrt{\sin z})$$

subject to conditions $x,\ y,\ z \ge 0,\ x + y + z = \pi/2$.

Suppose a minimum is attained at an "interior" point $(x,\ y,\ z > 0)$. By the Lagrange multiplier theorem, the partial derivatives of $F$ must be equal at this point. This yields $f(x) = f(y) = f(z)$, where

$$f(x) = \frac{\partial F}{\partial x} = \sqrt{2}\cos x - \frac{1}{2}(\sin x)^{-1/2}\cos x .$$

We examine $f$ on $(0,\pi/2)$ by calculus:

$$\frac{df}{dx} = -\sqrt{2}\sin x + \frac{1}{4}(\sin x)^{-3/2} + \frac{1}{4}(\sin x)^{1/2}.$$

Using the substitution $2\sin x = t^2$ we rewrite this expression as

$$\frac{\sqrt{2}}{8}t^{-3}g(t) ,$$

where

$$g(t) = -4t^5 + t^4 + 4 , \qquad 0 < t < \sqrt{2} .$$

Since

$$\frac{dg}{dt} = 4t^3(1 - 5t) ,$$

$g$ grows in $[0,1/5]$ and falls in $[1/5,\sqrt{2}]$, and since $g(0) = 4$ and $g(\sqrt{2}) < 0$, there exists $t_0$ such that $g > 0$ in $(0,t_0)$ and $g < 0$ in $(t_0,\sqrt{2})$. For $x \in (0,\pi/2)$ we have

$$\operatorname{sign}\frac{df}{dx} = \operatorname{sign} g(\sqrt{2\sin x}) ,$$

showing that $f$ is monotonic in the respective two subintervals of $(0,\pi/2)$. Thus $f$

takes each value at most twice.

Hence, if $(x,y,z)$ is an extreme point of $F$ with $x$, $y$, $z > 0$, then two of the three numbers $x$, $y$, $z$ must be equal. So it remains to examine

$$F\left(x,x,\frac{\pi}{2} - 2x\right) , \qquad 0 < x < \frac{\pi}{4} \tag{1}$$

and the "boundary values"

$$F\left(x,\frac{\pi}{2} - x,0\right) , \qquad 0 \le x \le \frac{\pi}{4} . \tag{2}$$

We begin with (2). Writing $s$ and $c$ for $\sin x$ and $\cos x$ respectively, we claim that

$$F\left(x,\frac{\pi}{2} - x,0\right) = \sqrt{2}(s + c) - (\sqrt{s} + \sqrt{c}) > 0 .$$

This is equivalent to

$$2(s + c)^2 > (\sqrt{s} + \sqrt{c})^2 ,$$

and the latter inequality follows from

$$2(s + c)^2 - (\sqrt{s} + \sqrt{c})^2 = 2(1 - s)(1 - c) + (\sqrt{s} - \sqrt{c})^2 + 2sc > 0 .$$

To be able to do something with (1), we again set $2 \sin x = t^2$. Then $\cos 2x = 1 - t^4/2$ and we get

$$F\left(x,x,\frac{\pi}{2} - 2x\right) = \sqrt{2}(2 \sin x + \cos 2x) - (2\sqrt{\sin x} + \sqrt{\cos 2x})$$

$$= \frac{\sqrt{2}}{2}(-t^4 + 2t^2 - 2t + 2 - \sqrt{2 - t^4}) , \qquad 0 < t < \sqrt[4]{2} .$$

This reduces the problem to showing that

$$-t^4 + 2t^2 - 2t + 2 \ge \sqrt{2 - t^4} , \qquad 0 < t < \sqrt[4]{2} . \tag{3}$$

The polynomial on the left is positive (its derivative is negative, as is easy to verify; so the polynomial attains its minimum at $t = \sqrt[4]{2}$). Thus, squaring transforms (3) into the equivalent inequality

$$P(t) := t^8 - 4t^6 + 4t^5 + t^4 - 8t^3 + 12t^2 - 8t + 2 \ge 0 .$$

Now the identity

$$P(t) = (t - 1)^2[t^2(t - 1)^2(t^2 + 4t + 2) + (2t^2 - 1)^2 + (2t - 1)^2]$$

shows that $P(t) \ge 0$ for $t > 0$, with strict inequality except for $t = 1$. This value corresponds to $x = \pi/6$.

Conclusion: $F \ge 0$, with equality only for $A = B = C = \pi/3$.

*Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and KEE–WAI LAU, Hong Kong. Two other readers sent in incorrect solutions.*

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