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PROPRIÉTÉS GÉOMÉTRIQUES ET APPROXIMATION DES FONCTIONS: I

J.G. DHOMBRES

Réaliser l'approximation d'une fonction donnée, c'est remplacer cette fonction par une autre considérée comme connue tout en donnant une estimation de l'erreur commise. Pour fixer les idées, les fonctions dont nous chercherons une approximation seront des fonctions numériques, ou à valeurs complexes, définies sur un intervalle $[a,b]$ de l'axe réel avec $a<b$. Les fonctions considérées comme "connues" constituent une classe notée F. Par exemple, F peut être formée par l'ensemble de toutes les fonctions polynômiales réelles ou complexes sur $[a,b]$, ou par le sous-ensemble de telles fonctions pour un degré inférieur ou égal à n, ou encore par les fonctions de Bessel, etc. En général, les fonctions de la classe F sont tabulées ou figurent dans les mémoires d'ordinateurs. Nous supposerons ici que F est un sous-espace vectoriel de l'ensemble $\mathfrak{F}[a,b]$ des fonctions à valeurs complexes définies sur $[a,b]$. Cette hypothèse permet d'utiliser les méthodes linéaires classiques de l'Analyse Fonctionnelle. Pour mathématiser complètement notre problème d'approximation, il reste encore à préciser ce que nous entendons par erreur commise en remplaçant la fonction f de $\mathfrak{F}[a,b]$ par la fonction g appartenant à F. La bonne notion est celle de norme. Rappelons que si E est un sous-espace vectoriel réel ou complexe de $\mathfrak{F}[a,b]$, une norme sur E est une application définie pour tout h appartenant à E, à valeurs positives ou nulles, et telle qu'en notant $\|h\|$ la norme de h, on dispose des relations:

$$\begin{cases} \|h\| = 0 & \text{si et seulement si } h=0, \\ \|\lambda h\| = |\lambda| \; \|h\| & \text{pour tout } h \text{ de } E \text{ et tout scalaire } \lambda \text{ du corps de base de } E, \\ \|h+h'\| \leq \|h\| + \|h'\| & \text{pour tout couple } (h,h') \text{ d'éléments de } E. \end{cases}$$

Soit une fonction f supposée appartenir à un sous-espace vectoriel E de $\mathfrak{F}[a,b]$ lequel contient F. La précision d'une approximation $g \in F$ de f est alors fournie par la valeur $\|f-g\|$. Quelques remarques simples découlent de notre formulation.

— Ce n'est pas tant l'approximation d'une fonction f déterminée que nous souhaitons obtenir, mais celle de tout élément de l'espace E. Du point de vue algorithmique, nous devons obtenir un moyen de calcul et une caractérisation valables pour tout élément de E.

— Les choix de E et d'une norme sur E sont relativement arbitraires lorsque l'on part d'une f bien déterminée. Un certain doigté, de l'intuition, et certaines facilités mathématiques doivent guider.

Notre propos est de montrer comment les propriétés géométriques de l'espace E, muni d'une norme $\|\cdot\|$, déterminent fondamentalement la recherche d'approximations.

1. *Existence et unicité d'une meilleure approximation.*

La première quantité à considérer est évidemment la borne inférieure

$$\alpha_F(f) = \inf_{g \in F} \|f - g\|$$

obtenue en faisant parcourir à g l'espace vectoriel F. Deux cas se présentent:

(1) Si pour tout f de E on a $\alpha_F(f) = 0$, on dit que F est *dense* dans E. Cela revient à dire que toute fonction f de E s'obtient comme limite d'une suite de vecteurs de F.

(2) Si pour un f de E on a $\alpha_F(f) > 0$, on dira que $g \in F$ est une meilleure approximation de f dans F (ou encore de f par F) lorsque

$$\|f - g\| = \alpha_F(f).$$

Nous n'étudierons ici que le second cas et pour ce faire nous éliminons le premier cas en faisant l'hypothèse que F $(\neq E)$ contient toutes les limites de suites d'éléments de F qui convergent dans E. On abrège en disant que F est un sous-espace *fermé* propre de E.

THÉORÈME 1. *Soit E un espace vectoriel normé. Il existe au plus une meilleure approximation de toute $f \in E$ dans F, et ce pour tout sous-espace vectoriel fermé propre de E, si et seulement si la sphère unité de E ne contient aucun sous-ensemble convexe non réduit à un point.*

La condition est évidemment suffisante puisque si g_1 et g_2 sont deux meilleures approximations de f dans F, on a, avec $0 \leq \lambda \leq 1$,

$$\begin{aligned}\alpha_F(f) \leq \|f - (\lambda g_1 + (1-\lambda) g_2)\| &= \|\lambda(f - g_1) + (1-\lambda)(f - g_2)\| \\ &\leq \lambda \|f - g_1\| + (1-\lambda)\|f - g_2\| = \alpha_F(f).\end{aligned}$$

Par suite $(1/\alpha_F(f))\Big(f - (\lambda g_1 + (1-\lambda) g_2)\Big)$ engendre, lorsque λ varie entre 0 et 1, un sous-ensemble convexe de la sphère unité de E. Pour que ce sous-ensemble soit réduit à un point, on doit avoir $g_1 = g_2$.

Nous laissons la démonstration de la nécessité au lecteur, lequel pourra montrer que l'on peut se contenter de n'utiliser que les sous-espaces vectoriels de dimension finie de E (cf. par exemple [1] ou [2]).

Signalons que les espaces normés satisfaisant la condition du Théorème 1 sont appelés espaces strictement convexes.

THÉORÈME 2. *Soit E un espace vectoriel normé. Il existe une unique meilleure approximation de toute $f \in E$ dans F lorsque F est un sous-espace vectoriel complet de E et lorsque E est uniformément convexe.*

La notion d'uniforme convexité quantifie (et spécialise) la notion de stricte convexité. On dit que $(E, \|\cdot\|)$ est uniformément convexe lorsque pour tout ε, $2 > \varepsilon > 0$, il existe $\delta(\varepsilon)$, $1 > \delta(\varepsilon) > 0$, et pour tous f_1 et f_2 appartenant à la boule unité de E tels que $\|f_1 - f_2\| \geq \varepsilon$, on a

$$\|(f_1 + f_2)/2\| \leq 1 - \delta(\varepsilon).$$

Abordons la démonstration du Théorème 2. Il existe une suite $\{g_n\}_{n\geq 1}$ d'éléments de F pour laquelle

$$\|f - g_n\| \leq \alpha_F(f)(1 + (1/n)).$$

Montrons que cette suite est de Cauchy. On raisonne par l'absurde. Si la suite n'est pas de Cauchy il existe un $\varepsilon > 0$ et, pour tout n, on peut trouver un m tel que $m > n$ et $\|g_m - g_n\| > \varepsilon$. Or considérons

$$f_1 = (f - g_n)/\alpha_F(f)(1 + (1/n)) \quad \text{et} \quad f_2 = (f - g_m)/\alpha_F(f)(1 + (1/n))$$

auxquels nous appliquons l'inégalité d'uniforme convexité puisque

$$\|f_1 - f_2\| = (1/\alpha_F(f))\Big(1/(1 + (1/n))\Big)\|g_m - g_n\| \geq (\varepsilon/2)\alpha_F^{-1}(f) = \varepsilon'.$$

On choisira ε assez petit pour que $\varepsilon' < 2$. Soit

$$(1/\alpha_F(f))\Big(1/(1 + (1/n))\Big)\|f - (g_m + g_n)/2\| \leq 1 - \delta(\varepsilon').$$

Mais en prenant n assez grand, c'est-à-dire tel que $(1 - \delta(\varepsilon'))(1 + (1/n)) < 1$, on déduit

$$\|f - \frac{g_m + g_n}{2}\| < \alpha_F(f),$$

ce qui est impossible par définition de $\alpha_F(f)$, car $(g_m + g_n)/2$ appartient à F.

La suite $\{g_n\}_{n\geq 1}$ étant de Cauchy dans F converge dans ce sous-espace complet vers une fonction g de F et naturellement

$$\alpha_F(f) = \lim_{n\to\infty} \|f - g_n\| = \|f - g\|.$$

Nous laissons de côté l'étude d'une condition nécessaire (cf. [2]).

THÉORÈME 3. *Soit E un espace vectoriel normé. Pour tout sous-espace vectoriel de dimension finie F, il existe une meilleure approximation de $f \in E$ dans F.*

L'application $g \to \|f-g\|$ est continue sur l'espace F. Puisque $\|f-g\|$ est grand lorsque $\|g\|$ devient grand, il suffit de restreindre cette fonction à l'intersection A de F et d'une boule fermée centrée en g. Parce que F est de dimension finie, l'ensemble A est compact d'après le théorème de Bolzano-Weierstrass, c'est-à-dire qu'une fonction continue y atteint son minimum. Ceci termine la démonstration.

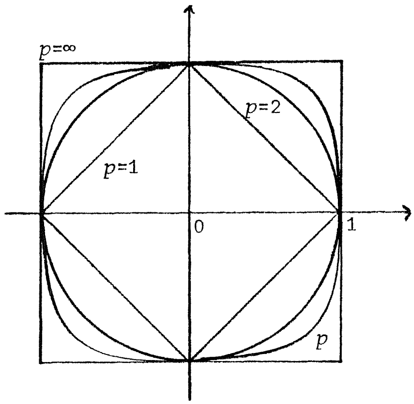
Exemple 1. Prenons $E = R^3$, l'espace vectoriel réel ordinaire à trois dimensions rapporté à la base canonique. Pour tout $x = (x_1, x_2, x_3) \in E$, on pose

$$\|x\|_p = \sqrt[p]{|x_1|^p + |x_2|^p + |x_3|^p}, \quad p \geq 1$$

et

$$\|x\|_\infty = \sup(|x_1|, |x_2|, |x_3|).$$

Nous avons dessiné les boules unités de E pour $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_p$ $(p > 2)$ et $\|\cdot\|_\infty$ et nous laissons l'image parler d'elle-même pour interpréter les Théorèmes 3 et 1.



$\begin{cases} p = 1,\ p = \infty\text{: il n y a pas unicité;} \\ 1 < p < \infty\text{: il } y \text{ a unicité} \end{cases}$

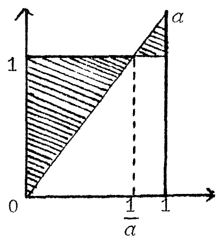
Exemple 2. Cherchons à approximer la fonction $f(x) = 1$ pour $x \in [0,1]$, par l'espace F des fonctions numériques linéaires nulles à l'origine et pour une norme déterminée par p où $p \geq 1$:

$$\|f\|_p = \left(\int_0^1 |f(x)|^p dx\right)^{1/p}$$

ou

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|.$$

Pour $p = \infty$, on note sur un graphique que toute fonction g de la forme $g(x) = ax$ avec $0 \leq a \leq 2$ est une meilleure approximation de f.



Pour $\infty > p \geq 1$, on calcule (avec $a \geq 1$ pour d'évidentes raisons)

$$\int_0^1 |1-ax|^p dx = \frac{(a-1)^{p+1}+1}{a(p+1)} .$$

La dérivée de cette expression en a s'annule pour une unique valeur de $a>1$, celle qui réalise $(ap+1)(a-1)^p=1$. En cette valeur, la puissance p-ième de la norme vaut $1/(ap+1)$. Or, au point $a=1$, l'expression vaut $1/(p+1)$, quantité toujours supérieure à la précédente. Par suite, le minimum de la norme est atteint.

Pour $p=1$, on calcule que le minimum vaut en $a=\sqrt{2}$ la valeur $\sqrt{2}-1$, ce que l'on peut vérifier géométriquement. Pour $p=2$, on calcule que le minimum est en $a=3/2$ pour une norme égale à $\frac{1}{2}$. Plus généralement, la norme minimale est $a-1$ pour le a réalisant le minimum.

Il y a donc une unique meilleure approximation pour $\infty>p\geq 1$. Pour $\infty>p>1$, ce résultat est conséquence aussi du Théorème 2, car on peut montrer que $\|\cdot\|_p$ est une norme strictement convexe, contrairement à $\|\cdot\|_\infty$ ou $\|\cdot\|_1$. Le résultat obtenu avec $p=1$ ne contredit rien car ici F est fixé.

2. *Un critère de meilleure approximation.*

Un exemple. Commençons par un exemple bien classique en reprenant l'espace $E=R^3$ de la géométrie ordinaire muni de la norme euclidienne

$$\|x\| = \sqrt{x_1^2+x_2^2+x_3^2}.$$

Soit F un plan passant par l'origine et x un point n'appartenant pas à ce plan. La meilleure approximation Px de x par F existe et est unique. Elle possède deux propriétés bien remarquables:

La première est que Px se trouve à l'intersection de F et de la droite passant par x et orthogonale au plan F.

La deuxième (Théorème de Thalès) est que la meilleure approximation de $x+y$ dans F est obtenue comme la somme des meilleures approximations dans F de x et de y. On écrit

$$P(x+y) = Px+Py.$$

De même

$$P(\lambda x) = \lambda Px \quad \text{pour tout } \lambda \text{ réel}.$$

Chacun sait l'importance de la linéarité d'une projection orthogonale dans les raisonnements géométriques. Il est très remarquable de noter (mais nous ne le démontrerons pas) que, si la dimension de F dépasse deux, la linéarité de la meilleure approximation pour tout sous-espace de dimension finie (voire fixée) caractérise les espaces normés dont la norme dérive d'un produit scalaire (cf. [1] ou [2]). Nous ne pouvons donc pas partir de la seconde propriété pour caractériser une meilleure approximation dans le cas général d'un espace vectoriel normé quelconque. Essayons de modifier la première propriété sous une forme ne faisant plus intervenir l'orthogonalité, notion qui fait nécessité de l'existence d'un produit scalaire.

Soit $ay_1 + by_2 + cy_3 = 0$ $(a^2 + b^2 + c^2 \neq 0)$ une équation analytique du plan F. La droite passant par x et orthogonale à F est parallèle au vecteur (a,b,c). Par suite, le pied y de la perpendiculaire abaissée sur F a pour coordonnées des nombres notés y_1, y_2, y_3. Cherchons à calculer la distance de x à F, c'est-à-dire l'expression

$$\|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} = \alpha_F(x).$$

En supposant $a \neq 0$ pour fixer les idées, ce qui revient à supposer que F ne contient pas l'axe des x, on a

$$y_2 - x_2 = \frac{b}{a}(y_1 - x_1), \quad y_3 - x_3 = \frac{c}{a}(y_1 - x_1), \quad \text{et} \quad ay_1 + by_2 + cy_3 = 0.$$

Par suite, en écrivant $a(x_1 - y_1) + b(x_2 - y_2) + c(x_3 - y_3) = ax_1 + bx_2 + cx_3$ il vient

$$ax_1 + bx_2 + cx_3 = -\frac{1}{a}(y_1 - x_1)(a^2 + b^2 + c^2)$$

tandis que

$$\|x - y\|^2 = \frac{1}{a^2}(y_1 - x_1)^2(a^2 + b^2 + c^2).$$

D'où le résultat, valable en fait pour toutes valeurs de a,

$$|ax_1 + bx_2 + cx_3| = \|x - y\|\sqrt{a^2 + b^2 + c^2} = \alpha_F(x)\sqrt{a^2 + b^2 + c^2}\ . \tag{A}$$

En résumé, si y est la meilleure approximation de x par F, il existe des constantes réelles a', b', c', satisfaisant les trois conditions suivantes:

$$a'^2 + b'^2 + c'^2 = 1, \tag{1}$$

$$a'x_1 + b'x_2 + c'x_3 = \|x - y\| = \inf_{z \in F} \|x - z\|, \tag{2}$$

et

$$a'y_1 + b'y_2 + c'y_3 = 0 \quad \text{pour tout } y = (y_1, y_2, y_3) \text{ de } F. \tag{3}$$

L'application de E dans R définie par $y = (y_1, y_2, y_3) \to a'y_1 + b'y_2 + c'y_3$ est une forme linéaire $L : E \to R$. On transcrit les propriétés (2) et (3) en termes de la forme L:

$$L(x) = \|x-y\|, \tag{2) bis}$$

$$L(y) = 0 \quad \text{pour tout } y \text{ de } F. \tag{3) bis}$$

Il reste donc à interpréter (1) en termes de la seule forme linéaire L. Considérons la borne supérieure des quantités réelles $|a'z_1+b'z_2+c'z_3|$ lorsque $z=(z_1,z_2,z_3)$ parcourt la boule unité de $E=R^3$. On note

$$\|L\| = \sup_{\substack{\|z\|\le 1\\ z\in E}} |a'z_1+b'z_2+c'z_3| = \sup_{\substack{\sqrt{z_1^2+z_2^2+z_3^2}\le 1\\ z\in E}} (|a'z_1+b'z_2+c'z_3|).$$

L'égalité (A) va nous permettre de calculer effectivement $\|L\|$.

D'une part, en effet, puisque 0 appartient à F et donc que $\alpha_F(z)\le\|z-0\|=\|z\|$, on dispose grâce à (A), utilisée avec les a' et les z_i au lieu des a et des x_i, de l'inégalité

$$(a'z_1+b'z_2+c'z_3)^2 \le (z_1^2+z_2^2+z_3^2)(a'^2+b'^2+c'^2),$$

ce qui fournit

$$\|L\| \le \sqrt{a'^2+b'^2+c'^2}.$$

Cependant, tenant compte de (3), l'égalité (A) peut encore s'écrire sous la forme

$$\{a'(x_1-y_1)+b'(x_2-y_2)+c'(x_3-y_3)\}^2 = \{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\}(a'^2+b'^2+c'^2).$$

Par suite en posant $z=(z_1,z_2,z_3)$, où

$$z_1 = \frac{x_1-y_1}{\|x-y\|},\quad z_2 = \frac{x_2-y_2}{\|x-y\|},\quad \text{et}\quad z_3 = \frac{x_3-y_3}{\|x-y\|},$$

on dispose de $z\in E$; $\|z\|=1$ et $|a'z_1+b'z_2+c'z_3| = a'^2+b'^2+c'^2$.

En définitive $\|L\|=\sqrt{a'^2+b'^2+c'^2}$. Dès lors on interprète (1) en écrivant

$$\|L\| = 1. \tag{1) bis}$$

Note. La démonstration de (1) bis a été faite en partant d'une propriété géométrique. Une démonstration directe (inégalité de Schwarz et cas d'égalité) consiste à écrire la positivité de $\|x+\lambda y\|^2$ pour tout λ. Enfin l'égalité de Lagrange, mémorisable en se souvenant qu'en diminution du second membre doit venir le produit vectoriel, permet encore une autre démonstration:

$$(az_1+bz_2+cz_3)^2 = (a^2+b^2+c^2)(z_1^2+z_2^2+z_3^2) - \{(az_2-bz_1)^2+(az_3-cz_1)^2+(bz_3-cz_2)^2\}.$$

THÉORÈME 4. *Soient E un espace vectoriel normé réel et F un sous-espace vectoriel fermé propre de E. Soit x un point de E n'appartenant pas à F. Un* *vecteur y de F est une meilleure approximation de x par F si et seulement s'il existe une application linéaire L de E dans R telle que*

(1) $$\sup_{\substack{\|x\| \leq 1 \\ x \in E}} |L(x)| = \|L\| = 1,$$

(2) $$L(x) = \alpha_F(x) = \inf_{z \in F} \|x - z\|,$$

et

(3) $L(z) = 0$ *pour tout* z *de* F.

Démonstration.

(i) *La condition est suffisante.* Supposons donc $L : E \to R$ telle que décrite par le théorême. Soit z un point de F et majorons $\|x - y\|$.

$$\|x - y\| = L(x) = L(x - z)$$

puisque z appartient à F et que L est linéaire. Or (1) fournit par homogénéité

$$|L(x - z)| \leq \|x - z\|,$$

ce qui fournit bien l'inégalité caractéristique d'une meilleure approximation y de x par F:

$$\|x - y\| \leq \|x - z\| \text{ pour tout } z \text{ de } F.$$

(ii) *La condition est nécessaire.* Soit y une meilleure approximation de x par F et soit $\alpha_F(x) = \|x - y\|$. Sur l'espace vectoriel engendré par F et le vecteur x, c'est-à-dire l'espace vectoriel G des vecteurs de la forme

$$t = z + \lambda x,$$

où λ parcourt l'axe réel, on pose

$$L' : G \to R \quad \text{selon} \quad L'(t) = L'(z + \lambda x) = \lambda \alpha_F(x).$$

Il est clair que L' est une forme linéaire sur G. En outre, montrons que l'on dispose de l'inégalité

$$|L'(t)| \leq \|t\| \quad \text{pour tout } t \text{ de } F.$$

En effet, cela signifie

$$|\lambda| \alpha_F(x) \leq \|z + \lambda x\|.$$

Si $\lambda = 0$, l'inégalité est évidente. Si $\lambda \neq 0$, puisque $-z/\lambda$ est un vecteur de F, on dispose de

$$\|t\| = |\lambda| \; \|x - (-z/\lambda)\| \geq |\lambda| \alpha_F(x).$$

Il n'est d'ailleurs pas difficile d'en déduire que

$$\|L'\| = \sup_{\substack{\|t\| \leq 1 \\ t \in G}} |L'(t)| = 1.$$

A ce niveau, nous devons faire appel à un théorème d'extension des formes linéaires, théorème dit de Hahn-Banach[1], qui assure l'existence d'une forme linéaire $L : E \to R$, laquelle satisfait

(a) $L(t) = L'(t)$ pour tout t de E

et

(b) $\sup_{\substack{\|z\| \leq 1 \\ z \in E}} |L(z)| = 1.$

Dès lors L remplit les conditions du Théorème 4, à savoir (1) par (b), (2) par construction puisque

$$L(x) = L'(x) = \alpha_F(x),$$

et enfin (3) par construction puisque z est dans F, donc dans G, et

$$L(z) = L'(z) = 0.$$

On remarquera comment une propriété géométrique euclidienne a pu être transcrite dans le cadre des espaces vectoriels normés de dimension quelconque. Plusieurs conséquences peuvent être déduites de ce remarquable théorème. Nous fournirons dans un prochain numéro une seule utilisation, récemment élaborée, et qui nous mènera directement à des problèmes non encore résolus.

Note. Si l'on suppose E complexe, et F sous-espace vectoriel complexe de E, le Théorème 4 reste vrai à condition de prendre pour L une forme linéaire complexe $L : E \to C$ et de remplacer la valeur absolue par le module.

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[1]Pour la démonstration de ce théorème très classique, consulter par exemple [3], [4], ou tout manuel d'Analyse Fonctionnelle.

LINEARITY IN DIFFERENTIAL OPERATORS

ALFRED AEPPLI

In this journal [1979: 152-155] Leroy F. Meyers proved that every linear, differential operator (of order n, on some interval I) is a linear differential operator. We would like to give here a few examples as applications of this theorem and its proof.

(a) *If*

$$(Tf)(x) = F(x, f(x), f'(x), \ldots, f^{(n)}(x)) \tag{1}$$

is a differential operator of order n on the real interval I, with $f \in D^n(I)$, *where* $D^n(I)$ *is the space of n-times differentiable functions on I, and if*

$$fg\,T(f+g) = (f+g)(g\,Tf + f\,Tg) \tag{2}$$

and

$$T(\alpha f) = \alpha^2\,Tf \qquad \text{for } \alpha \in R, \tag{2_α}$$

then

$$Tf = f\,Lf \tag{3}$$

for

$$Lf = a_0 f + a_1 f' + a_2 f'' + \ldots + a_n f^{(n)}, \tag{4}$$

where $a_j = a_j(x)$, $x \in I$.

Proof. For $g \equiv 1$, (2) implies

$$\frac{Tf}{f} = T(f+1) - T(f) - (f+1)T(1), \tag{5}$$

that is, $\frac{Tf}{f}$ is well defined for all $f \in D^n(I)$. Now (1) and (2),(2_α) express the condition that $\frac{Tf}{f}$ is a differential operator which is linear; hence $\frac{Tf}{f} = Lf$ for Lf of the form (4) by the quoted theorem.

By (1) and (5),

$$\begin{aligned}\frac{Tf}{f}(x) &= F(x, f(x)+1, f'(x), f''(x), \ldots, f^{(n)}(x)) \\ &\quad - F(x, f(x), f'(x), f''(x), \ldots, f^{(n)}(x)) \\ &\quad - (f(x)+1)F(x, 1, 0, 0, \ldots, 0) \\ &= G(x, f(x), f'(x), f''(x), \ldots, f^{(n)}(x)),\end{aligned}$$

and the proof in Meyers' article gives

$$a_0(x) = G(x,1,0,0,\dots,0,0),$$

$$a_1(x) = G(x,0,1,0,\dots,0,0),$$

$$a_2(x) = G(x,0,0,1,\dots,0,0),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_n(x) = G(x,0,0,0,\dots,0,1).$$

(b) *Let* Φ *be a given linear differential operator of nth order on* I *and assume* $(\Phi h)(x) \neq 0$ *for all* $x \in I$ *for some* $h \in D^n(I)$. *If* Tf *as in* (1) *fulfills*

$$\Phi(f)\,\Phi(g)\,T(f+g) = \Phi(f+g)\,\big(\Phi(g)\,T(f) + \Phi(f)\,T(g)\big) \tag{2_Φ}$$

and

$$T(\alpha f) = \alpha^2\, Tf, \tag{2_α}$$

then

$$Tf = \Phi(f)\,L(f) \tag{3_Φ}$$

for Lf *of the form* (4).

Proof. For $g = h$, (2_Φ) implies

$$\frac{Tf}{\Phi f} = \frac{1}{\Phi h}\left\{T(f+h) - Tf - \frac{\Phi(f+h)}{\Phi h}\,Th\right\}, \tag{5_Φ}$$

so $\frac{Tf}{\Phi f}$ is well defined, even if $(\Phi f)(x) = 0$ for some $x \in I$. (2_Φ) and (2_α) are the linearity conditions for $\frac{Tf}{\Phi f}$, a differential operator by (1) and by the assumptions on Φ, hence $\frac{Tf}{\Phi f} = Lf$ as above for Lf of the form (4). In analogy to (a), the coefficients $a_j = a_j(x)$, $0 \le j \le n$, are given by F and Φ via (5_Φ).

(c) In what way does the statement in (b) remain true for more general operators Φ,

$$(\Phi f)(x) = H\big(x, f(x), f'(x), \dots, f^{(n)}(x)\big), \tag{1_Φ}$$

with $x \in I$ and $f \in D^n(I)$? The main problem is to prove that the partially defined operator $\frac{Tf}{\Phi f}$ can be extended to an everywhere defined linear operator. The following version of (b) has been communicated to me by Leroy F. Meyers.

For fixed $x \in I$, $(Tf)(x) = T_x$ and $(\Phi f)(x) = \Phi_x$ depend on the $(n+1)$-tuple

$$\big(f(x), f'(x), f''(x), \dots, f^{(n)}(x)\big) \in R^{n+1},$$

hence $T_x, \Phi_x : R^{n+1} \to R$.

If T *and* Φ *as in* (1) *and* (1_Φ) *fulfill, for every* $x \in I$,

$$\Phi_x(u)\,\Phi_x(v)\,T_x(u+v) = \Phi_x(u+v)\big(\Phi_x(v)\,T_x(u) + \Phi_x(u)\,T_x(v)\big) \tag{2_Φ}$$

and

$$\Phi_x(u)\,T_x(\alpha u) = \alpha\,\Phi_x(\alpha u)\,T_x(u) \tag{2_α^Φ}$$

for $u, v \in R^{n+1}$ *and if, for every* $x \in I$,

$$W_x = \ker \Phi_x = \left\{ u \in R^{n+1} : \Phi_x(u) = 0 \right\}$$

is a proper linear subspace of R^{n+1}, *then*

$$Tf = \Phi(f)\,L(f) \tag{3_Φ}$$

for a uniquely determined $L(f)$ *of the form* (4).

Proof. $\frac{Tf}{\Phi f}$ is a linear, differential operator of order n on I since for every $x \in I$ there is a well defined linear function $S_x : R^{n+1} \to R$ such that $T_x(u) = \Phi_x(u)\,S_x(u)$. S_x is defined by

$$S_x(u) = \begin{cases} \dfrac{T_x(u)}{\Phi_x(u)}, & \text{for } u \notin W_x \\[2ex] S_x(u+w) - S_x(w), & \text{for } u \in W_x \end{cases}$$

for some fixed $w \in R^{n+1} - W_x$.

(d) If (1) and (2),(2_α) hold for Tf, that is, if Tf is of the form (3) and if, in addition,

$$T(fg) = (Tf)\,g^2 + f^2\,(Tg), \tag{6}$$

then

$$Tf = a\,f\,f' \tag{7}$$

for $a = a(x)$ and $x \in I$.

(e) The condition (6) leads to the following consideration. If $\mathcal{L}$ is a differential operator, like T in (1), with

$$\mathcal{L}(fg) = \mathcal{L}(f) + \mathcal{L}(g), \qquad \mathcal{L}(f^{\alpha}) = \alpha\mathcal{L}(f), \tag{8}$$

then $\mathcal{L}f = L\log f$ for L of the form (4) since $\mathcal{L}\exp$ is a linear, differential operator. Observe that (6) is a special case of (8) by setting $\mathcal{L}f = \frac{Tf}{f^2}$.

(f) Similarly to (e), if $\mathcal{E}$ is a differential operator, like T in (1), with $(\mathcal{E}f)(x) > 0$ for all $f \in D^n(I)$ and $x \in I$, and with

$$\mathcal{E}(f+g) = \mathcal{E}(f)\mathcal{E}(g), \qquad \mathcal{E}(\alpha f) = (\mathcal{E}(f))^{\alpha},$$

then $\mathcal{E}f = \exp Lf$ for L of the form (4) since $\log\mathcal{E}$ is a linear, differential operator.

(g) As a consequence of (c) and (e), if T is an operator for which (1) and (2_Φ),(2_α^Φ) hold, with the conditions on Φ in (c) fulfilled, that is, T is of the form (3_Φ), and if in addition (6) is assumed, then

$$\frac{Tf}{f^2} = \frac{1}{f^2}\Phi f\, Lf = M\log f\,, \tag{9}$$

that is,

$$\Phi f = f^2\frac{M\log f}{Lf} \tag{9_Φ}$$

for some linear differential operator M (of the form (4)).

(7) is an illustration of (9) for $Lf = Mf = af'$ and $\Phi f = f$.

The author thanks Leroy F. Meyers for his stimulating suggestions.

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* * *

CRUX MATHEMATICORUM

Quotation from *The Thirteen Books of Euclid's Elements*, Dover, New York, 1956, Vol. 3, pp. 8-9:

> Simon Stevin (1548-1620) gave an *Appendice des incommensurables grandeurs en laquelle est sommairement déclaré le contenu du Dixiesme Livre d'Euclide (Oeuvres mathématiques,* Leyde, 1634, pp. 218-22); he speaks thus of the book: "La difficulté du dixiesme Livre d'Euclide est à plusieurs devenue en horreur, voire jusque à l'appeler la croix des mathématiciens, matière trop dure à digérer, et en la quelle n'aperçoivent aucune utilité," ... .

* * *

THE OLYMPIAD CORNER: 15

MURRAY S. KLAMKIN

In the *Notices of the American Mathematical Society*, 27 (April 1980) 279-280, appeared three letters, one by Raymond Ayoub, Pennsylvania State University, one by Andrei Sakharov, Academician, and one by Boris M. Schein, a Russian Jewish refugee, all three letters dealing with the discrimination against Jews and other minority groups in entrance examinations to Russian universities. Jews and other candidates considered to be undesirable are given to solve more difficult problems, which have come to be known as "Jewish" problems, while easier problems are given to "others". The letters contained several examples of both types of problems. These problems are given at oral examinations, with a time limit of 20 minutes per problem. "If one problem was solved, a second, third, ..., nth problem was given," writes Schein. Some of the "Jewish" problems were given previously in written Mathematical Olympiads where much more time per problem was allowed, with a consequent lessening of tension.

The problems appearing in the above-mentioned letters are given below. I expect to publish more "Jewish" problems in subsequent columns, through the courtesy of Boris M. Schein.

For more information concerning Soviet mathematics and mathematicians, see the following issues of the *Notices of the A.M.S.*: 25 (November 1978) 495-497, 26 (February 1979) 115-117, 26 (June 1979) 230-231, 26 (August 1979) 305-308, 26 (October 1979) 398-401, 27 (January 1980) 68-72, and 27 (February 1980) 181-182. Also there are the two recent books:

G. Freiman, *It Seems I Am A JEW*: Samizdat Essay on Soviet Mathematics;

M.B. Nathanson, *Soviet Mathematics*.

Both are from the Southern Illinois University Press.

"JEWISH" PROBLEMS

J-1. Prove that $\left(\dfrac{\sin x}{x}\right)^3 \geq \cos x;\quad 0 < x \leq \dfrac{\pi}{2}$.

J-2. Solve in rational numbers x, y, z, and t:

$$(x + y\sqrt{2})^2 + (z + t\sqrt{2})^2 = 5 + 4\sqrt{2}.$$

J-3. A function $f(x,y)$ of two variables takes on at least 3 values. For some fixed numbers a and b, we have $f(a,y) \neq$ constant, $f(x,b) \neq$ constant.

Prove that there exist numbers p,q,r,s such that $f(p,q)$, $f(r,q)$, $f(p,s)$ are three pairwise-distinct values of $f(x,y)$.

J-4. Let $ab = 4$, $c^2 + 4d^2 = 4$. Prove the inequality

$$(a-c)^2 + (b-d)^2 \geq 1.6.$$

J-5. Let ABCD be a tetrahedron with $DB \perp DC$ such that the perpendicular to the plane ABC coming through the vertex D intersects the plane of the triangle ABC at the orthocenter of this triangle. Prove that

$$(|AB| + |BC| + |AC|)^2 \leq 6(|AD|^2 + |BD|^2 + |CD|^2).$$

For which tetrahedra does the equality take place?

J-6. What is more: $\sqrt[3]{60}$ or $2 + \sqrt[3]{7}$?

J-7. Let ABCD be a trapezoid with the bases AB and CD, and let K be a point in AB. Find a point M in CD such that the area of the quadrangle which is the intersection of the triangles AMB and CDK is maximal.

J-8. Prove that $x \cos x < 0.71$ for all $x \in [0, \pi/2]$.

J-9. Is it possible to cut two arbitrary squares into polygons which would form a new square?

J-10. Solve the system of equations:

$$y(x+y)^2 = 9,$$
$$y(x^3 - y^3) = 7.$$

"OTHER" PROBLEMS

O-1. Write the cosine law.

O-2. Does the function

$$y = \begin{cases} x \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

have a derivative at 0?

O-3. Which is larger, $\sin(\cos x)$ or $\cos(\sin x)$?

O-4. Construct the graph of $2^{|x-1|}$.

I would be happy to receive (especially from secondary school students) and consider for publication elegant solutions to *some* of these problems (but not to problems 0-1, 0-2, or 0-4, which are quite straightforward). I urge solvers to indicate on each solution the time they spent on the problem.

*

SOLUTIONS TO PRACTICE SET 12

12-1. (a) Show that one root of the equation

$$x^4 + 5x^2 + 5 = 0$$

is $x = \omega - \omega^4$, where ω is a complex fifth root of unity (i.e., $\omega^5 = 1$).

(b) Determine the other three roots as polynomials in ω.

Solution.

We can assume that ω is a primitive fifth root of unity, that is, $\omega \neq 1$. For any such root, we have

$$\omega^4 + \omega^3 + \omega^2 + \omega + 1 = \frac{\omega^5 - 1}{\omega - 1} = 0.$$

(a) The proof is straightforward. For $x = \omega - \omega^4$, we get, using $\omega^5 = 1$ whenever possible,

$$x^2 = (\omega - \omega^4)^2 = \omega^3 + \omega^2 - 2$$

and

$$x^4 = (\omega^3 + \omega^2 - 2)^2 = \omega^4 - 4\omega^3 - 4\omega^2 + \omega + 6.$$

Hence,

$$\begin{aligned} x^4 + 5x^2 + 5 &= (\omega^4 - 4\omega^3 - 4\omega^2 + \omega + 6) + 5(\omega^3 + \omega^2 - 2) + 5 \\ &= \omega^4 + \omega^3 + \omega^2 + \omega + 1 \\ &= 0. \end{aligned}$$

(b) Since the quartic polynomial in our equation is an even function, its zeros are of the form $\pm a$ and $\pm b$, where $a = \omega - \omega^4$. The sum of the products of the roots taken two at a time is

$$-a^2 - b^2 = 5,$$

from which we get

$$\begin{aligned} b^2 &= -5 - a^2 = -5 - (\omega^3 + \omega^2 - 2) = -3 - \omega^2 - \omega^3 \\ &= -3 - \omega^2 - \omega^3 + (1 + \omega + \omega^2 + \omega^3 + \omega^4) \\ &= \omega^4 - 2 + \omega = \omega^4 - 2\omega^5 + \omega^6 = (\omega^2 - \omega^3)^2. \end{aligned}$$

Finally, the roots are $\pm(\omega - \omega^4)$ and $\pm(\omega^2 - \omega^3)$.

12-2. Prove that the midpoints of the six edges of a tetrahedron are cospherical (i.e., lie on a common sphere) if and only if the four altitudes of the tetrahedron are concurrent.

Solution.

We will use the known fact that, in any tetrahedron ABCD, the three bimedians (segments joining the midpoints of pairs of opposite edges) are concurrent and bisect one another in a point G, called the *centroid* of the tetrahedron.

The condition is sufficient. Suppose the four altitudes concur in a point H. (When such a point H exists, it is called the *orthocenter* of the tetrahedron.) Let

$$\overrightarrow{HA} = \vec{a}, \qquad \overrightarrow{HB} = \vec{b}, \qquad \overrightarrow{HC} = \vec{c}, \qquad \overrightarrow{HD} = \vec{d}.$$

From the orthogonality conditions $HA \perp BC$, $HA \perp CD$, etc., we get

$$\vec{a}\cdot(\vec{c}-\vec{b}) = \vec{a}\cdot(\vec{d}-\vec{c}) = \vec{b}\cdot(\vec{d}-\vec{c}) = \vec{b}\cdot(\vec{a}-\vec{d}) = \vec{c}\cdot(\vec{a}-\vec{d}) = 0,$$

from which

$$\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c} = \vec{a}\cdot\vec{d} = \vec{b}\cdot\vec{c} = \vec{b}\cdot\vec{d} = \vec{c}\cdot\vec{d} = k,$$

where k is some constant. We now show that the three bimedians are equal, that is,

$$\left|\frac{\vec{a}+\vec{b}-\vec{c}-\vec{d}}{2}\right| = \left|\frac{\vec{a}+\vec{c}-\vec{b}-\vec{d}}{2}\right| = \left|\frac{\vec{a}+\vec{d}-\vec{b}-\vec{c}}{2}\right| . \tag{1}$$

Since the bimedians bisect one another at G, it will then follow that the midpoints of the six edges lie on a sphere with center G. The square of the first quantity in (1) is easily found to be

$$\frac{|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + |\vec{d}|^2 - 4k}{4},$$

and the symmetry of this expression shows that the same result would be obtained from the other two quantities in (1).

The condition is necessary. Suppose the midpoints of the six edges lie on a common sphere. Since the three bimedians concur and bisect one another at G, for any bimedian MN we have

$$GM \cdot GN = GM^2 = GN^2.$$

Since, furthermore, for any number of concurrent chords of a sphere, the product of the two segments of a chord is the same for all chords, we conclude that the six midpoints of the edges are all equidistant from G and that the three bimedians are equal.

We now take the circumcenter O of the tetrahedron as the origin of vectors and write $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, etc., so that

$$|\vec{a}|^2 = |\vec{b}|^2 = |\vec{c}|^2 = |\vec{d}|^2 = R^2,$$

where R is the circumradius. The equality of the bimedians now gives

$$\left|\frac{\vec{a}+\vec{b}-\vec{c}-\vec{d}}{2}\right|^2 = \left|\frac{\vec{a}+\vec{c}-\vec{b}-\vec{d}}{2}\right|^2 = \left|\frac{\vec{a}+\vec{d}-\vec{b}-\vec{c}}{2}\right|^2$$

or, upon expansion,

$$R^2 - T + (\vec{a}\cdot\vec{b}+\vec{c}\cdot\vec{d}) = R^2 - T + (\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{d}) = R^2 - T + (\vec{a}\cdot\vec{d}+\vec{b}\cdot\vec{c}),$$

where

$$T = \tfrac{1}{2}(\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{a}\cdot\vec{d}+\vec{b}\cdot\vec{c}+\vec{b}\cdot\vec{d}+\vec{c}\cdot\vec{d}).$$

Thus

$$\vec{a}\cdot\vec{b}+\vec{c}\cdot\vec{d} = \vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{d} = \vec{a}\cdot\vec{d}+\vec{b}\cdot\vec{c},$$

from which

$$(\vec{a}-\vec{d})\cdot(\vec{b}-\vec{c}) = (\vec{a}-\vec{c})\cdot(\vec{d}-\vec{b}) = (\vec{a}-\vec{b})\cdot(\vec{c}-\vec{d}) = 0. \tag{2}$$

Observe that (2) shows that each edge is orthogonal to the opposite edge. A tetrahedron with this property is called *rectangular* or *orthocentric* because, as we will now show, it has an orthocenter.

Let H be the point symmetric to the circumcenter O with respect to the centroid G. This point is given by

$$\overrightarrow{OH} = \vec{h} = \tfrac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d}).$$

We show that H is the orthocenter of the tetrahedron by proving that

$$\text{HA} \perp \text{face BCD}, \quad \text{HB} \perp \text{face CDA}, \quad \text{etc.} \tag{3}$$

From

$$2(\vec{h}-\vec{a})\cdot(\vec{b}-\vec{c}) = (\vec{b}+\vec{c}+\vec{d}-\vec{a})\cdot(\vec{b}-\vec{c}) = |\vec{b}|^2 - |\vec{c}|^2 + (\vec{d}-\vec{a})\cdot(\vec{b}-\vec{c}) = 0,$$

we conclude that $HA \perp BC$, and similarly $HA \perp BD$. This establishes the first relation in (3), and the others are proved in the same way.

12-3. If Alice and Bob toss 11 and 9 fair coins, respectively, show that the probability that Alice gets more heads than Bob is

$$\frac{1}{2} + \frac{1}{2^{21}}\binom{20}{10} \approx 0.588.$$

Solution.

More generally, let Alice and Bob toss $m+n$ and n coins, respectively. For

$0 \le r \le n$, Bob can toss exactly r heads in

$$\binom{n}{r} \text{ of the } 2^n \text{ ways}$$

he can toss his n coins. For each such r, Alice can toss more heads than Bob by tossing $r+s$ heads for some s, $1 \le s \le m+n-r$; and this she can do in

$$\binom{m+n}{r+s} \text{ of the } 2^{m+n} \text{ ways}$$

she can toss her $m+n$ coins. Hence the required probability is

$$P(m,n) \equiv \sum_{r=0}^{n} \frac{\binom{n}{r}}{2^n} \sum_{s=1}^{m+n-r} \frac{\binom{m+n}{r+s}}{2^{m+n}} .$$

If we interchange the order of the summations and note that

$$\binom{m+n}{r+s} = 0 \quad \text{if } r+s > m+n,$$

we can write

$$P(m,n) = \frac{1}{2^{m+2n}} \sum_{s=1}^{m+n} \sum_{r=0}^{n} \binom{n}{r}\binom{m+n}{r+s} . \tag{1}$$

To evaluate the summation over r in (1), we first determine the coefficient of t^s on both sides of the identity

$$(1+t)^{m+n}\left(1+\frac{1}{t}\right)^n = \frac{(1+t)^{m+2n}}{t^n} ,$$

obtaining

$$\sum_{r=0}^{n} \binom{m+n}{r+s}\binom{n}{r} = \binom{m+2n}{n+s} ,$$

so that (1) becomes

$$P(m,n) = \frac{1}{2^{m+2n}} \sum_{s=1}^{m+n} \binom{m+2n}{n+s} . \tag{2}$$

We now employ the substitution $t = m+n-s$ to obtain

$$\sum_{s=1}^{m+n} \binom{m+2n}{n+s} = \sum_{s=1}^{m+n} \binom{m+2n}{m+n-s} = \sum_{t=0}^{m+n-1} \binom{m+2n}{t}$$

and (2) becomes

$$P(m,n) = \frac{1}{2^{m+2n}} \sum_{t=0}^{m+n-1} \binom{m+2n}{t} . \tag{3}$$

For particular small values of m, it is easy to evaluate (3) by using

$$\sum_{t=0}^{m+2n} \binom{m+2n}{t} = 2^{m+2n} \quad \text{and} \quad \binom{m+2n}{t} = \binom{m+2n}{m+2n-t} .$$

Thus we obtain

$$P(0,n) = \frac{1}{2} - \frac{1}{2^{2n+1}} \binom{2n}{n} ,$$

$$P(1,n) = \frac{1}{2},$$

$$P(2,n) = \frac{1}{2} + \frac{1}{2^{2n+3}} \binom{2n+2}{n+1} .$$

In particular, in our problem where we have $m+n=11$ and $n=9$, we get

$$P(2,9) = \frac{1}{2} + \frac{1}{2^{21}} \binom{20}{10} = \frac{1}{2} + \frac{19 \cdot 17 \cdot 13 \cdot 11}{2^{19}} \approx 0.588. \ \square$$

The surprising result that $P(1,n) = \frac{1}{2}$ is independent of n can be obtained much more simply by using a symmetry argument, and we leave this for the reader (or see Uspensky [1]). The problem is fully discussed in [2], where also appears a generalization to unfair coins for each of which the probability of getting a head and a tail is p and q, respectively.

REFERENCES

1. J.V. Uspensky, *Introduction to Mathematical Probability*, McGraw-Hill, New York, 1937, pp. 38, 59.

2. Murray S. Klamkin, "A Probability of More Heads," *Mathematics Magazine*, 44 (May 1971) 146-149.

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PRACTICE SET 13

13-1. In n-dimensional Euclidean space E^n, determine the least and greatest distances between the point $A = (a_1, a_2, \ldots, a_n)$ and the n-dimensional rectangular parallelepiped whose vertices are $(\pm v_1, \pm v_2, \ldots, \pm v_n)$ with $v_i > 0$. (Some may find it helpful first to do the problem in E^3 or even in E^2.)

13-2. If A,B,C are positive angles whose sum does not exceed π, and such that the sum of any two of the angles is greater than the third, show that

(a) there exists a triangle with sides sin A, sin B, sin C;

(b) $4+\sum \sin^2 A\csc^2 B\csc^2 C \le 2\sum \csc^2 A$ (cyclic sums).

13-3. (a) If $0 \le x_i \le a$, $i=1,2,\ldots,n$, determine the maximum value of

$$A \equiv \sum_{i=1}^{n} x_i - \sum_{1\le i<j\le n} x_i x_j.$$

(b) If $0 \le x_i \le 1$, $i=1,2,\ldots,n$ and $x_{n+1}=x_1$, determine the maximum value of

$$B_n \equiv \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i x_{i+1}.$$

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

PROBLEMS -- PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before October 1, 1980, although solutions received after that date will also be considered until the time when a solution is published.

541. *Proposed by Herman Nyon, Paramaribo, Surinam.*

Solve the following colourful alphametic, in which BLUE is a perfect square:

$$\begin{array}{r} \text{RED} \\ \text{BLUE} \\ \underline{\text{GREEN}} \\ \text{BROWN} \end{array} .$$

542. *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

Let p be a permutation of the digits 0 through 9. This induces, in a natural way, a function p^* defined on the interval (0,1) by writing

$$x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$$

(using 0's instead of 9's in the finite decimal case) and putting

$$p^*(x) = \sum_{n=1}^{\infty} \frac{p(a_n)}{10^n} .$$

These functions were defined earlier in this journal [1979: 184], where it was stated that some of them were bijective. Find a necessary and sufficient condition for a function p^* to be bijective.

543. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $F_1, F_2, \ldots, F_n$ be ovals (compact convex bodies) in the real Euclidean plane R^2. We suppose that, for all i and j,

$$\Phi_{ij} \equiv F_i \cap F_j \neq \emptyset.$$

Prove that, if each three of the Φ's are intersected by a straight line, then the F_i's have a common point, that is,

$$\bigcap_{i=1}^{n} F_i \neq \emptyset.$$

544. *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.*

Prove that, in any triangle ABC,

$$2\left(\sin\frac{B}{2}\sin\frac{C}{2} + \sin\frac{C}{2}\sin\frac{A}{2} + \sin\frac{A}{2}\sin\frac{B}{2}\right) \leq \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} ,$$

with equality if and only if the triangle is equilateral.

545. *Proposed by Jack Garfunkel, Flushing, New York.*

Given three concentric circles, construct an equilateral triangle having one vertex on each circle.

546. *Proposed by John A. Winterink, Albuquerque Technical Vocational Institute, Albuquerque, New Mexico.*

Prove the validity of the following simple method for finding the center of a conic, which is not given in most current texts:

For the central conic

$$\phi(x,y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \qquad ab - h^2 \neq 0,$$

the center is the intersection of the lines

$$\frac{\partial\phi}{\partial x} = 0 \quad \text{and} \quad \frac{\partial\phi}{\partial y} = 0.$$

547. *Proposed by Charles W. Trigg, San Diego, California.*

There are powers 2^k whose decimal digits sum to k; for example, $2^5 = 32$ and $3 + 2 = 5$. Find another.

548. *Proposed by M.S. Klamkin, University of Alberta.*

If three equal cevians of a triangle divide the sides in the same ratio and same sense, must the triangle be equilateral?

549.* *Proposed by Ferrell Wheeler, student, Forest Park H.S., Beaumont, Texas.*

The centers of eight congruent spheres of radius r are the vertices of a cube of edge length r.

(a) Find the volume of the intersection of the eight spheres.

(b) Into how many parts do these spheres divide the cube?

550. *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

Prove that

$$\int_0^{\frac{\pi}{2}} \cos(\cos\theta)\cdot\cosh(\sin\theta)\,d\theta = \frac{\pi}{2}.$$

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

457. [1979: 167] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Here are examples of two n-digit squares whose juxtaposition forms a

$2n$-digit square:

$$\begin{array}{rcrcrcl} 4 & \text{and} & 9 & \text{form} & 49 & = & 7^2, \\ 16 & \text{and} & 81 & \text{form} & 1681 & = & 41^2, \\ 225 & \text{and} & 625 & \text{form} & 225625 & = & 475^2. \end{array}$$

Is there at least one such juxtaposition for each $n = 4, 5, 6, \ldots$?

Solution by F.G.B. Maskell, Algonquin College, Ottawa.

The answer is YES. We will show that, for $n = 1,2,3,\ldots$, there are positive integer triples (x,y,z) such that

$$x^2 \cdot 10^n + y^2 = z^2, \tag{1}$$

where x^2 and y^2 are n-digit squares, that is,

$$10^{n-1} \le x^2, y^2 < 10^n. \tag{2}$$

Equation (1) is clearly satisfied for any λ and p if

$$x^2 \cdot 10^n = 4\lambda^2 p^2, \qquad y^2 = (\lambda^2 - p^2)^2, \qquad z^2 = (\lambda^2 + p^2)^2. \tag{3}$$

We will select λ and p so that x and y are positive integers that satisfy (2). We consider two cases:

Case 1: $n = 2k$, $k = 1,2,3,\ldots$. We obtain from (3) a solution of (1) for every even n by setting $\lambda = \frac{1}{2} \cdot 10^k$ and $p = \lambda - 1$:

$$x = \tfrac{1}{2} \cdot 10^k - 1, \qquad y = 10^k - 1, \qquad z = \tfrac{1}{2} \cdot 10^{2k} - 10^k + 1; \tag{4}$$

and it is easy to verify that here x and y are integers that satisfy (2). Thus, for

$$\begin{array}{llll} n = 2: & 4^2 = 16, & 9^2 = 81, & 1681 = 41^2; \\ n = 4: & 49^2 = 2401, & 99^2 = 9801, & 24019801 = 4901^2; \\ n = 6: & 499^2 = 249001, & 999^2 = 998001, & 249001998001 = 499001^2; \end{array}$$

etc. However, for some even n at least, there are solutions other than (4). For example, for $n = 8$ there is the following solution different from (4):

$$3829^2 = 14661241, \qquad 8751^2 = 76580001, \qquad 1466124176580001 = 38290001^2.$$

Case 2: $n = 2k+1$, $k = 0,1,2,\ldots$. The first example in the proposal gives a solution for $n = 1$, so we consider only the cases $k \ge 1$. We will obtain from (3) a solution of (1) for every odd $n \ge 3$ by setting $\lambda = \frac{1}{2} \cdot 10^k \sqrt{10}$ and taking for p the unique positive integer such that $\frac{1}{2} < |\lambda - p| < 1$. Thus, if we denote by $[\lambda]$ and $\{\lambda\}$

the integral and fractional parts of the irrational number λ, then we have $p = [\lambda]$ or $[\lambda]+1$ according as $\{\lambda\}$ is greater or less than $\frac{1}{2}$. The values thus obtained from (3),

$$x = p, \qquad y = |\lambda^2 - p^2|, \qquad z = \lambda^2 + p^2, \tag{5}$$

are clearly integers, and we have only to make sure that here x and y satisfy (2), that is,

$$10^k \le x,y < 10^k\sqrt{10}. \tag{6}$$

Observe that $[\lambda] \le p \le [\lambda]+1$ holds for all k, with equality on one side for some k and equality on the other side for all the other k. Since

$$10^k < \left[\frac{\sqrt{10}}{2}\cdot 10^k\right] = [\lambda] \le p \le [\lambda]+1 < \lambda+1 < 2\lambda = 10^k\sqrt{10},$$

it follows that (6) holds for x for all k. Since $\frac{1}{2} < |\lambda - p|$, we have

$$10^k < \left[\frac{\sqrt{10}}{2}\cdot 10^k\right] = [\lambda] < \tfrac{1}{2}(\lambda + [\lambda]) \le \tfrac{1}{2}(\lambda + p) < |\lambda^2 - p^2|$$

and the left inequality in (6) holds for y for all k. We prove the right inequality in (6) for y in two steps. If $p = [\lambda]$, we use $|\lambda - p| < 1$ to get

$$|\lambda^2 - p^2| = |\lambda - p|(\lambda + p) < \lambda + p = \lambda + [\lambda] < 2\lambda = 10^k\sqrt{10}.$$

Finally, if $p = [\lambda]+1$, then $p-1 = [\lambda] < \lambda$ and $(p-1)^2 < \lambda^2$. Since $(p-1)^2$ and λ^2 are integers, we must have $(p-1)^2 \le \lambda^2 - 1$, from which

$$|\lambda^2 - p^2| = p^2 - \lambda^2 \le 2(p-1) = 2[\lambda] < 2\lambda = 10^k\sqrt{10},$$

which completes the proof.

Since $\frac{1}{2}\sqrt{10} = 1.58113883008\ldots$, we have

$$p = [\lambda] \qquad \text{for} \qquad k = 1,5,6,\ldots$$

and

$$p = [\lambda]+1 \qquad \text{for} \qquad k = 2,3,4,7,8,9,\ldots\ .$$

We will use this to find a few examples. For

$n = 3$: $\quad 15^2 = 225, \quad 25^2 = 625, \quad 225625 = 475^2;$

$n = 5$: $\quad 159^2 = 25281, \quad 281^2 = 78961, \quad 2528178961 = 50281^2;$

$n = 7$: $\quad 1582^2 = 2502724, \quad 2724^2 = 7420176, \quad 25027247420176 = 5002724^2;$

etc. Here also, for some odd n at least, (5) is not the only solution. For example,

$$n = 3: \quad 12^2 = 144, \quad 20^2 = 400, \quad 144400 = 380^2;$$

$$n = 5: \quad 126^2 = 15876, \quad 155^2 = 24025, \quad 1587624025 = 39845^2.$$

Also solved by CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montréal, Québec; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University; HARRY L. NELSON, Livermore, California; CHRIS NYBERG, East York, Ontario; CHARLES W. TRIGG, San Diego, California; ALAN WAYNE, Pasco-Hernando Community College, New Port Richey, Florida; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Several of the other solutions were incomplete in one way or another, some admittedly so. Some solvers, for example, found infinite families of solutions of (1), by solving a Pell equation or otherwise, but failed to show that, for every $n = 1,2,3,\ldots$, at least one solution satisfied the crucial inequality (2). One solver proved the theorem for even n but "disproved" it for odd n by sketching an argument (he said he was "omitting the boring details") showing there is no solution for $n = 11$. Wilke, on the other hand, exhibited *seven* solutions for $n = 11$.

The proposer extended his theorem. He showed that, for every $n = 1,2,3,\ldots$, there are three n-digit squares whose concatenation forms a $3n$-digit square; and also five n-digit squares whose concatenation forms a $5n$-digit square. He exhibited a few examples of each kind, of which I mention only one,

$$44944 = 212^2,$$

a palindrome which is also the square of a palindrome. Although the examples

$$1444 = 38^2, \quad 36493681 = 6041^2, \quad 121121400625 = 348025^2$$

show that there are four n-digit squares whose concatenation forms a $4n$-digit square, the proposer failed in his attempt to show that such exist for all $n = 1,2,3,\ldots$.

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458. [1979: 167] *Proposed by Harold N. Shapiro, Courant Institute of Mathematical Sciences, New York University.*

Let $\phi(n)$ denote the Euler function. It is well known that, for each fixed integer $c > 1$, the equation $\phi(n) = n - c$ has at most a finite number of solutions for the integer n. Improve this by showing that any such solution, n, must satisfy the inequalities $c < n \le c^2$.

Solution by Kenneth M. Wilke, Topeka, Kansas.

It is known that $\phi(n) = n-1$ if n is prime and that

$$\phi(n) = n\prod_{p_i|n}\left(1 - \frac{1}{p_i}\right), \tag{1}$$

where the p_i are the distinct prime divisors of n if $n>1$ and $\phi(1)=1$. It is clear from (1) that $\phi(n)$ is even for $n>2$. Since here $c>1$, we have $n-c<n-1$ and n must be composite, which implies that $2\le\phi(n)=n-c$. This establishes the sharper left inequality in

$$c+2 \le n \le c^2. \tag{2}$$

To prove the right inequality, we note that

$$n-c = \phi(n) \le n\left(1 - \frac{1}{p^*}\right) \le n\left(1 - \frac{1}{\sqrt{n}}\right) = n-\sqrt{n},$$

where p^* is the smallest prime divisor of n. Thus $n-c\le n-\sqrt{n}$ and $n\le c^2$.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; and the proposer.

Editor's comment.

The bounds in (2) are best possible. The lower bound is attained when $c=2$ (and $n=4$) and, as the proposer pointed out, the upper bound is attained whenever c is prime, for then the greatest solution of the equation $\phi(n)=n-c$ is $n=c^2$.

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459. [1979: 167] *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.*

If n is a positive integer, prove that

$$\sum_{k=1}^{\infty}\frac{1}{k^{2n}} \le \frac{\pi^2}{8}\cdot\frac{1}{1-2^{-2n}}.$$

Solution and comment by M.S. Klamkin, University of Alberta.

All the information we use in our solution and comment can be found in [1]. We will need the functions

$$\zeta(m) = \sum_{k=1}^{\infty} k^{-m}, \qquad m=2,3,\ldots$$

for which it is known that

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}\,|B_{2n}|, \qquad n = 1,2,\ldots$$

where the B_{2n} are Bernoulli numbers, and

$$\lambda(m) = \sum_{k=0}^{\infty} (2k+1)^{-m} = (1-2^{-m})\zeta(m), \quad m = 2,3,\ldots .$$

Our problem itself is easily settled. It is equivalent to $\lambda(2n) \le \pi^2/8$, which is obvious since

$$\lambda(2n) \le \lambda(2) = \pi^2/8,$$

with equality if and only if $n = 1$. □

The upper bound given in the problem is a rather trivial one. We can improve it greatly by using the inequalities

$$1 < \frac{(2\pi)^{2n}}{2(2n)!}\,|B_{2n}| < \frac{1}{1-2^{1-2n}} < \frac{\pi^2}{8}\cdot\frac{1}{1-2^{-2n}},$$

the first two of which hold for all $n \ge 1$ according to [1], and the last of which holds for all $n \ge 2$, as is easily verified. These yield

$$1 < \zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} < \frac{1}{1-2^{1-2n}}, \qquad n = 1,2,\ldots;$$

here the upper bound is sharp for $n \to \infty$, and it is better than the proposed upper bound for all $n \ge 2$.

Also solved by PAUL BRACKEN, student, University of Toronto; CLAYTON W. DODGE, University of Maine at Orono; ALLAN WM. JOHNSON JR., Washington, D.C.; LEROY F. MEYERS, The Ohio State University; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

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1. Milton Abramowitz and Irene A. Stegun (Editors), *Handbook of Mathematical Functions*, Dover, New York, 1965, pp. 805, 807-808.

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460. [1979: 167] *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Problem 124 in the *MATYC Journal* (12 (Fall 1978) 254), proposed by C.W. Trigg, asks: "Can two consecutive odd integers be the sides of a Pythagorean triangle?" It is easy to show that the answer is "Yes" provided the two consecutive odd integers are a leg and the hypotenuse and their mean is an even square integer. Answer the more difficult question: Can two consecutive even integers be the sides of a Pythagorean triangle? Show how to find all such Pythagorean triangles.

Solution by M.S. Klamkin, University of Alberta.

If two sides of a Pythagorean triangle are even, so is the third side. There are two cases to consider, the first of which is trivial.

Case 1. The hypotenuse and a leg are consecutive even integers. If the sides are $2m$, $2n$, $2n+2$, then

$$m^2 + n^2 = (n+1)^2 \qquad \text{or} \qquad n = \frac{m^2-1}{2},$$

which holds if and only if m is odd. Thus all solution triangles can be found from

$$m = 2k+1, \qquad n = 2k(k+1), \qquad k = 1,2,3,\ldots .$$

Case 2. The two legs are consecutive even integers.

Since all sides are even, it suffices to find all Pythagorean triangles with legs that are consecutive integers and then multiply each side by 2. The reduced problem is an old and well-known one whose solutions depend on the Pell equations $x^2 - 2y^2 = \pm 1$. Dickson [1] devotes two full pages to it. According to [1], Fermat showed that the required triangles form an infinite sequence in which the first triangle is (3,4,5) and the successor of

$$(x, x+1, z) \quad \text{is} \quad (3x+2z+1, 3x+2z+2, 4x+3z+2). \tag{1}$$

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; RICHARD BURNS, East Longmeadow H.S., East Longmeadow, Massachusetts; MILTON P. EISNER, J. Sargeant Reynolds Community College, Richmond, Virginia; HOWARD EVES, University of Maine; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; ALLAN WM. JOHNSON JR., Washington, D.C.; ROBERT S. JOHNSON, Montréal, Québec; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; F.G.B. MASKELL, Algonquin College, Ottawa; BOB PRIELIPP, The University of Wisconsin-Oshkosh; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; CHARLES W. TRIGG, San Diego, California. ALAN WAYNE, Pasco-Hernando Community College, New Port Richey, Florida; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Our problem (particularly Case 2 for triangles with consecutive legs) is fully discussed and proved in several recent books: see references [2] - [5], which were submitted by various solvers. Reference [3] is particularly useful, since it contains on pages 328-329 a list of the first *one hundred* Pythagorean triangles with consecutive legs, the last of which has a 77-digit hypotenuse. Starting with (3,4,5), these triangles can be quickly generated by (1) or by the following method also due to Fermat [1]: if P is the perimeter of any triangle (a,b,c) in the sequence, where $a < b < c$, then the successor of (a, b, c) is $(2P-b, 2P-a, 2P+c)$.

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4. W. Sierpiński, *Pythagorean Triangles*, Yeshiva University, New York, 1962, pp. 13-22.

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461. [1979: 199] *Proposed by the late R. Robinson Rowe, Rowe Manse, Naubinway, Michigan.*

Restore the digits in the decimal addition

$$\begin{array}{r} \text{C} \\ \text{DODGE} \\ \underline{\text{MAINE}} \\ \text{ORONO} \end{array},$$

with everyone agreeing that DODGE is the greatest.

Solution by Clayton W. Dodge, University of Maine at Orono.

We count columns from the right. From columns 1 and 5, O is even and greater than 2; so $O = 4$, 6, or 8. Thus we must have $E = 2$, 3, or 4 and $G = 0$; or else $E = 7$ or 8 and $G = 9$. From columns 3 and 5, we get $|M - I| < 2$, so $|M - I| = 1$ since $M \neq I$. It is clear that C and A are interchangeable in any solution unless $A = 0$. Finally, if s, t, u, and v denote the carries into columns 2 to 5, respectively, we must have $t \neq v$. This is about all we can tell from a superficial analysis of the problem. Now we must dig deeper.

We are asked to believe that DODGE is the greatest (personally, I prefer General Motors or even Ford products). Observing that $D < O$ from column 5, we will first find the solution(s) with maximal D when $O = 8$. We find [the details are omitted (Editor)] no solution for $D = 7$ or 6, and exactly two solutions for $D = 5$, which are identical except for the interchange of C and A:

$$\begin{array}{r} 7 \\ 58504 \\ \underline{21394} \\ 86898 \end{array} \quad \text{and} \quad \begin{array}{r} 1 \\ 58504 \\ \underline{27394} \\ 86898 \end{array}. \tag{1}$$

When $O = 6$, there is no other solution with $D = 5$; hence (1) represents the complete answer to our problem with the imposed restriction.

Of the two solutions (1), all right-thinking persons will naturally prefer the second, where MAINE is the greatest.

Also solved by ADAM CAROMICOLI, Parkside H.S., Dundas, Ontario; MILTON P. EISNER, J. Sargeant Reynolds Community College, Richmond, Virginia; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; J.A. McCALLUM, Medicine Hat, Alberta; DONVAL R. SIMPSON, Fairbanks, Alaska; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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462. [1979: 199] *Proposé par Hippolyte Charles, Waterloo, Québec.*

Soient A, B, C les angles d'un triangle. Montrer que

$$\begin{vmatrix} \tan\frac{A}{2} & \cos A & 1 \\ \tan\frac{B}{2} & \cos B & 1 \\ \tan\frac{C}{2} & \cos C & 1 \end{vmatrix} = 0.$$

I. *Solution de F.G.B. Maskell, Collège Algonquin, Ottawa.*

Soit D le déterminant donné. On a, toutes les sommes étant cycliques,

$$\begin{aligned} D &= \sum \tan\frac{A}{2}(\cos B - \cos C) = -2\sum \cot\frac{B+C}{2}\sin\frac{B+C}{2}\sin\frac{B-C}{2} \\ &= -2\sum \cos\frac{B+C}{2}\sin\frac{B-C}{2} = -\sum(\sin B - \sin C) \\ &= 0. \end{aligned}$$

II. *Adapted from a comment by W.J. Blundon, Memorial University of Newfoundland.*

Let R, r, s represent respectively the circumradius, inradius, and semiperimeter of a triangle and consider the following two theorems:

THEOREM 1. *At least one angle of the triangle has measure* θ *if and only if*

$$s = 2R\sin\theta + r\cot\frac{\theta}{2}. \qquad (1)$$

THEOREM 2. *At least one angle of the triangle has measure* θ *if and only if*

$$s\tan\frac{\theta}{2} + 2R\cos\theta = 2R + r. \qquad (2)$$

It is an easy exercise to show that equations (1) and (2) are equivalent; hence the theorems also are equivalent. The first is a theorem I had proved in 1967 [1].

Now let $\vec{t}$, $\vec{c}$, and $\vec{u}$ be the column vectors of the matrix whose determinant is given in the proposal. Since (2) holds when θ is replaced by A,B,C we have

$$s\vec{t} + 2R\vec{c} = (2R+r)\vec{u}.$$

Thus the vectors $\vec{t},\vec{c},\vec{u}$ are linearly dependent and consequently the given determinant vanishes. I had in [1] given a similar corollary to Theorem 1:

$$\begin{vmatrix} 1 & \sin A & \cot \frac{A}{2} \\ 1 & \sin B & \cot \frac{B}{2} \\ 1 & \sin C & \cot \frac{C}{2} \end{vmatrix} = 0.$$

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; KAREN L. COLLINS, student, Smith College, Northampton, Massachusetts; CLAYTON W. DODGE, University of Maine at Orono; HOWARD EVES, University of Maine; JACK GARFUNKEL, Flushing, N.Y.; G.C. GIRI, Midnapore College, West Bengal, India; ALLAN WM. JOHNSON JR., Washington, D.C.; M.S. KLAMKIN, University of Alberta; J.A. McCALLUM, Medicine Hat, Alberta; L.F. MEYERS, The Ohio State University; GREGG PATRUNO, student, Stuyvesant H.S., New York; BOB PRIELIPP, The University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; CHARLES W. TRIGG, San Diego, California; et le proposeur.

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463. [1979: 199] *Proposed by Jack Garfunkel, Flushing, N.Y.*

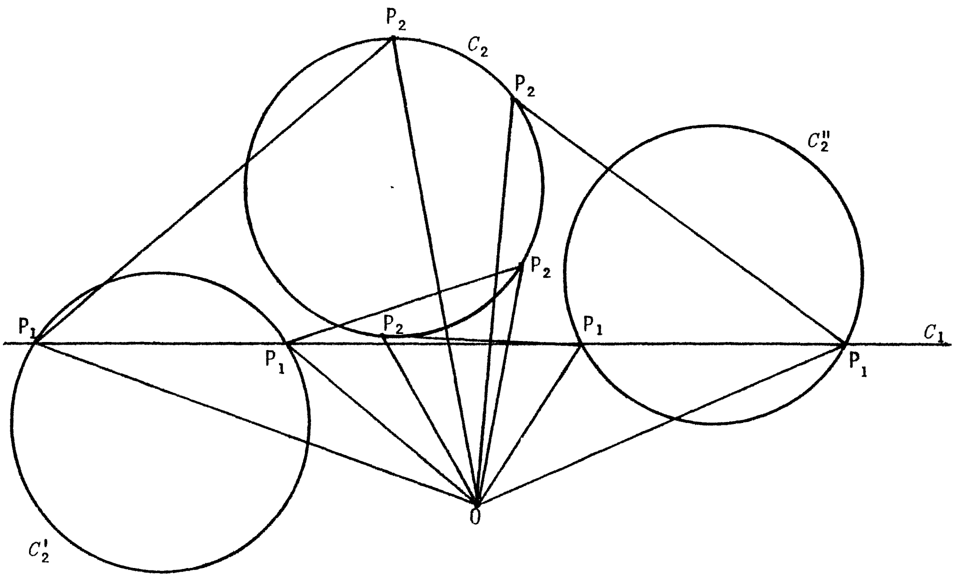
Construct an equilateral triangle so that one vertex is at a given point, a second vertex is on a given line, and the third vertex is on a given circle.

Solution by Howard Eves, University of Maine.

This is a special case of the general problem: *Given a point* O *and two curves* C_1 *and* C_2*, to locate a triangle* OP_1P_2*, where* P_1 *is on* C_1 *and* P_2 *is on* C_2*, directly similar to a given triangle* $O'P_1'P_2'$.

The possible positions of P_1 (see [1]) are the intersections of C_1 and C_2', where C_2' is the map of C_2 under the homology (or spiral rotation) $H(O,\theta,k)$, where θ is the sensed angle $P_2'O'P_1'$ and $k = O'P_1'/O'P_2'$. (In particular, if the given triangle $O'P_1'P_2'$ is equilateral, we take $\theta = \pm 60^\circ$ and $k = 1$.). If each of C_1 and C_2 is a circle or a straight line, the problem is readily solved with Euclidean tools.

Thus, for the given problem, C_1 is a straight line and C_2 is a circle. Rotate



circle C_2 about 0 through 60° and -60° to obtain two new circles C_2' and C_2'' (see figure). Any intersection of these circles with the straight line C_1 yields a position for vertex P_1 of the sought equilateral triangle OP_1P_2. There may be as many as four distinct solutions to the problem.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain (two solutions); SHMUEL AVITAL, I.I.T. Technion, Haifa, Israel; W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; MILTON P. EISNER, J. Sargeant Reynolds Community College, Richmond, Virginia; M.S. KLAMKIN, University of Alberta and A. LIU, University of Regina (jointly); VIKTORS LINIS, University of Ottawa; F.G.B. MASKELL, Algonquin College, Ottawa; L.F. MEYERS, The Ohio State University; DAN PEDOE, University of Minnesota; MARK E. SAUL, Bronx H.S. of Science; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; and the proposer.

Editor's comment. Klamkin gave references [2],[3] for related problems.

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