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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

G.W. Sands
Department of Mathematics & Statistics
University of Calgary
Calgary, Alberta
Canada, T2N 1N4

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Graham P. Wright
Canadian Mathematical Society
577 King Edward
Ottawa, Ontario
Canada K1N 6N5

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THE OLYMPIAD CORNER

No. 110

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this month with problems sent to me by Professor Walther Janous of Innsbruck, Austria, whom we thank for his faithful support of this column.

11th AUSTRIAN–POLISH MATHEMATICS COMPETITION

Individual Competition

1st Day: July 6, 1988

Time allowed: 4 1/2 hours each day

1. Let $P(x)$ be a polynomial having integer coefficients. Show that if $Q(x) = P(x) + 12$ has at least six distinct integer roots, then $P(x)$ has no integer roots.

2. Let $a_1,a_2,\cdots,a_n$ $(n \geq 2)$ be natural numbers such that

$$1 \leq a_1 \leq a_2 \leq \cdots \leq a_n.$$

Show that the inequality

$$\sum_{i=1}^{n} a_i x_i^2 + 2\sum_{i=1}^{n-1} x_i x_{i+1} > 0$$

holds for all n–tuples $(x_1,\cdots,x_n) \neq (0,\cdots,0)$ of real numbers if and only if $a_2 \geq 2$.

3. Let $ABCD$ be a convex quadrilateral (i.e. all interior angles are less than 180°) having no pair of sides parallel. We now consider the two angles made by two pairs of opposite sides. Their (interior) angle bisectors intersect the four sides of $ABCD$ in points P, Q, R and S, respectively, so that $PQRS$ is a convex quadrilateral. Show that $ABCD$ can be inscribed in a circle (i.e. it has a circumcircle) if and only if $PQRS$ is a rhombus.

2nd Day: July 8, 1988

4. Determine all strictly monotone increasing functions f: $\mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$f(f(x) + y) = f(x + y) + f(0)$$

for all x, $y \in \mathbb{R}$.

5. The two sequences $\{a_k\}$, $k \geq 0$ and $\{b_k\}$, $k \geq 0$ of integers are given by

$$b_k = a_k + 9 \ , \qquad a_{k+1} = 8b_k + 8 \ , \qquad k \geq 0 \ .$$

Furthermore, the number 1988 appears either in the sequence $\{a_k\}$, $k \geq 0$ or in the sequence $\{b_k\}$, $k \geq 0$. Show that the sequence $\{a_k\}$, $k \geq 0$ does not contain the square of any natural number.

6. Let a point O and three rays h_1, h_2 and h_3 emanating from O be given so that h_1, h_2 and h_3 are not coplanar. Show that if for each triple of points $A_1 \in h_1$, $A_2 \in h_2$, $A_3 \in h_3$ $(A_i \neq O,\ i = 1,2,3)$ the triangle $A_1A_2A_3$ is acute angled, then h_1, h_2, h_3 are pairwise perpendicular.

Team Competition

July 8, 1988

Time allowed: 4 hours

7. In a regular octagon each side is coloured blue or yellow. From such a colouring another colouring will be obtained "in one step" as follows: if the two neighbours of a side have different colours, the "new" colour of the side will be blue, otherwise the colour will be yellow. [*Editor's note*: the colours are modified simultaneously.] Show that after a finite number, say N, of moves all sides will be coloured yellow. What is the least value of N that works for all possible colourings?

8. We are given 1988 congruent cubes (of edge length 1). Using some or all of these cubes, we form three "quadratic boards" A, B, C of side lengths a, b, and c, respectively, with $a \leq b \leq c$ and dimensions $a \times a \times 1$, $b \times b \times 1$, and $c \times c \times 1$, respectively. Now place the board C onto the first quadrant of the x–y plane so that one vertex of C lies at the origin. Place board B on board C so that each cube of B is precisely above a cube of C and B does not overlap C. Similarly, place A on B. This gives us a "three–floor" tower. What choice of a, b and c gives the maximum number of such "three–floor" towers?

9. Consider a rectangle R having positive integer side lengths a and b. Let $D(a,b)$ be the number of ways of covering R by congruent rectangles of integral side lengths formed by a family of cuts parallel to one side of

R. Determine the perimeter P of the rectangle R for which $D(a,b)/P$ is maximized.

* * *

The second contest we give this month is the second round of the 24th Spanish Mathematics Olympiad. Thanks go to Andy Liu, Mathematics Department, The University of Alberta for forwarding a copy.

24th SPANISH MATHEMATICS OLYMPIAD

2nd Round

Madrid, February 1988

1. Let $\{x_n\}$, $n \in \mathbb{N}$ be a sequence of integers, such that $x_1 = 1$, $x_n < x_{n+1}$ for all $n \geq 1$, and $x_{n+1} \leq 2n$ for all $n \geq 1$. Show that for each positive integer k, there exist two terms x_r, x_s of the sequence such that $x_r - x_s = k$.

2. We choose n points $(n > 3)$ on a circle, numbered from 1 to n in any order. We say that two non–adjacent points A and B are related if, in one of the arcs with A and B as endpoints, all the points are marked with numbers smaller than those of A and B. Show that the number of pairs of related points is exactly $n - 3$.

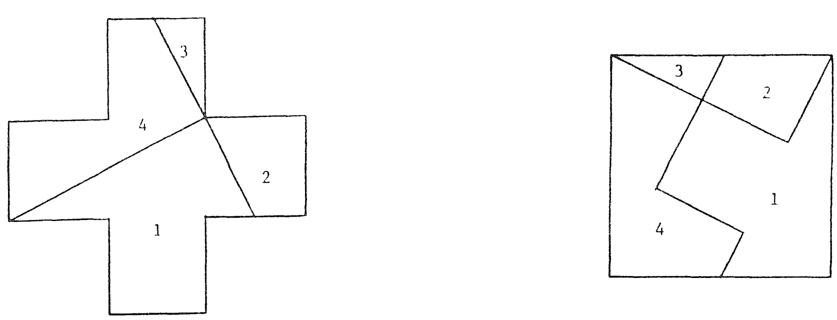
3. Show that $25x + 3y$ and $3x + 7y$ are multiples of 41 for the same integer values of x and y.

4. The celebrated Fibonacci sequence is defined by

$$a_1 = 1\ , \qquad a_2 = 2\ , \qquad a_i = a_{i-2} + a_{i-1} \qquad (i > 2)\ .$$

Express a_{2n} in terms of only a_{n-1}, a_n, and a_{n+1}.

5. A well–known puzzle (see figure) asks for the partition of a cross into 4 parts which are to be reassembled into a square. One solution is exhibited. Show that there are infinitely many distinct solutions. (Some solutions split the cross into 4 equal parts!)



6. For all integer values of the parameter t, find all integer solutions (x,y) of the equation

$$y^2 = x^4 - 22x^3 + 43x^2 - 858x + t^2 + 10452(t + 39).$$

* * *

Next a correction to a solution given earlier this year.

Correction to the solution of France 2 [1987: 277] *appearing in* [1989: 102] *by Murray S. Klamkin, Department of Mathematics, The University of Alberta.*

The last paragraph of the solution, on page 105, is misleading. It should read as follows:

For Levi's theorem to apply, we also assume $f(x) = |x|$ for all real x. Then the Stankovic result extends to the x_i being vectors in E^m.

* * *

Since the files were "closed" on the February 1988 number for preparation of last month's article enough new solutions have come in to warrant continuing this month with more of them. First we would mention that *Nicos Diamantis*, mathematics student, Patras, Greece, submitted solutions to two problems from the 18th Austrian Mathematics Olympiad which were discussed in the November Corner. These were **3** [1988: 34; 1989: 264] from the 2nd Round and **4** [1988: 34; 1989: 265] from the Final Round. While we didn't use these two solutions, we look forward to using his solutions in the future.

3. [1988: 34] *18th Austrian Mathematics Olympiad, Final Round.*

Let $x_1,\cdots,x_n$ be positive real numbers. Prove that

$$\sum_{k=1}^{n} x_k + \sqrt{\sum_{k=1}^{n} x_k^2} \le \frac{n+\sqrt{n}}{n^2}\left(\sum_{k=1}^{n}\frac{1}{x_k}\right)\left(\sum_{k=1}^{n} x_k^2\right).$$

Generalization and solution by Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton.

Let $S_p = \sum_{k=1}^{n} x_k^p$. The given inequality is then

$$S_1 + \sqrt{S_2} \le \left(\frac{n+\sqrt{n}}{n^2}\right)S_{-1}S_2 .$$

Let α, β, γ be constants with $\alpha > \gamma \ge 0$ and $\beta > 0$. Then by the power mean inequality

$$\{S_\alpha/n\}^{1/\alpha} \geq \{S_\gamma/n\}^{1/\gamma}, \qquad (1)$$

and

$$\{S_\alpha/n\}^{1/\alpha} \geq \{S_{-\beta}/n\}^{-1/\beta} . \qquad (2)$$

It now follows immediately that

$$\{S_\alpha/n\}^{2/\alpha}\{S_{-\beta}/n\}^{1/\beta} \geq \{S_\gamma/n\}^{1/\gamma}, \qquad (3)$$

and

$$\{S_\alpha/n\}^{2/\alpha}\{S_{-\beta}/n\}^{1/\beta} \geq \{S_\alpha/n\}^{1/\alpha} . \qquad (4)$$

We now multiply (3) by $n^{1/\gamma}$ and (4) by $n^{1/\alpha}$ and add to obtain

$$(S_\gamma)^{1/\gamma} + (S_\alpha)^{1/\alpha} \leq \left(\frac{n^{1/\gamma} + n^{1/\alpha}}{n^{2/\alpha+1/\beta}}\right)(S\alpha)^{2/\alpha}(S_{-\beta})^{1/\beta} . \qquad (5)$$

The current problem corresponds to the special case $\alpha = 2$, $\beta = \gamma = 1$.

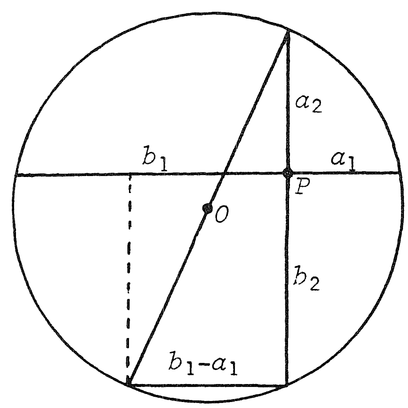
*

1. [1988: 35] *10th Austrian–Polish Mathematics Competition.*

Let P be a point in the interior of a sphere. Through P three pairwise orthogonal lines are drawn. Each line determines a chord on the great circle determined by the points of intersection of the line and the sphere. Show that the sum of the squares of the lengths of the three chords is independent of the directions of these lines.

Solutions by Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton.

Firstly it is simpler to state that "Each line determines a chord of the sphere". Secondly, this problem is very closely related to the known problem in two dimensions of finding the sum of the squares of the 4 segments formed. We solve the problem in two ways. The first and simpler method exploits the symmetry of the sphere. The second method uses notation of a coordinate system. This method can be very useful for more involved problems and will be illustrated.



We start with the analogous problem for a circle. Let the 4 segments formed be denoted by a_1, b_1, and a_2, b_2 as shown in the figure, where we have assumed without loss of generality that $b_1 \geq a_1$ and $b_2 \geq a_2$. It now follows that

$$\begin{aligned} 4R^2 &= (b_1 - a_1)^2 + (b_2 + a_2)^2 \\ &= b_1^2 + a_1^2 + b_2^2 + a_2^2 \end{aligned}$$

since by the power of a point theorem

$$a_1b_1 = a_2b_2 = R^2 - (OP)^2.$$

Here R is the radius and O the center of the circle, and P is the point of concurrency of the chords. It is immediate that

$$(b_1 + a_1)^2 + (b_2 + a_2)^2 = 8R^2 - 4(OP)^2 = \text{constant}.$$

For the spherical case, let the other pair of segments be a_3 and b_3. Then we consider the above figure for each pair of chords. It follows that

$$\begin{aligned} 4R^2 &= (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 + a_3)^2 \\ &= (b_1 - a_1)^2 + (b_2 + a_2)^2 + (b_3 - a_3)^2 \\ &= (b_1 + a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2. \end{aligned}$$

Using the power of a point theorem again, we get

$$(b_1 + a_1)^2 + (b_2 + a_2)^2 + (b_3 + a_3)^2 = 12R^2 - 8(OP)^2.$$

Our second solution is analytic. Let the equation of the sphere be

$$(x' - h)^2 + (y' - k)^2 + (z' - l)^2 = R^2$$

where P is at the origin. To get an arbitrary orientation of the 3 perpendicular axes x', y', z' we rotate the axes so that for the new axes we have

$$\begin{aligned} x' &= \alpha_1x + \alpha_2y + \alpha_3z \\ y' &= \beta_1x + \beta_2y + \beta_3z \\ z' &= \gamma_1x + \gamma_2y + \gamma_3z\ . \end{aligned}$$

The 9 coefficients are the elements of a 3×3 orthonormal matrix. The equation of the sphere with respect to the new axes is now

$$\begin{aligned} (\alpha_1x + \alpha_2y + \alpha_3z - h)^2 &+ (\beta_1x + \beta_2y + \beta_3z - k)^2 \\ &+ (\gamma_1x + \gamma_2y + \gamma_3z - l)^2 = R^2. \end{aligned}$$

The square of the length of the chord through the origin in the x direction is obtained by setting $y = z = 0$ and squaring the difference between the two roots of the resulting quadratic in x. This quadratic reduces to

$$x^2 - 2x(\alpha_1h + \beta_1k + \gamma_1l) + h^2 + k^2 + l^2 - R^2 = 0\ .$$

The square C_x^2 of the chord length is given by

$$C_x^2 = 4(\alpha_1h + \beta_1k + \gamma_1l)^2 + 4R^2 - 4(OP)^2$$

(here $(OP)^2 = h^2 + k^2 + l^2$). Similarly

$$C_y^2 = 4(\alpha_2h + \beta_2k + \gamma_2l)^2 + 4R^2 - 4(OP)^2$$

and

$$C_z^2 = 4(\alpha_3h + \beta_3k + \gamma_3l)^2 + 4R^2 - 4(OP)^2.$$

Then using the properties of the orthonormal matrix, (e.g. $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$, $\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0$, etc.), we get

$$C_x^2 + C_y^2 + C_z^2 = 4(h^2 + k^2 + l^2) + 12R^2 - 12(OP)^2$$
$$= 12R^2 - 8(OP)^2,$$

as before. As another application of the latter method, one can show that the sum of the squares of the reciprocals of the lengths of any three mutually perpendicular chords of an ellipsoid which are concurrent at its center is constant.

Finally, the above properties hold for n–dimensional spheres and ellipsoids. For the n–dimensional sphere, we have

$$4R^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + \cdots + (b_{n-1} - a_{n-1})^2 + (b_n + a_n)^2$$

or

$$\sum_{i=1}^{n} (b_i + a_i)^2 = 4nR^2 - 4(n-1)(OP)^2.$$

*   *   *

Send me your contest problems and nice solutions!

*   *   *

M I N I - R E V I E W S

by

ANDY LIU

BOOKS FROM W.H. FREEMAN AND COMPANY

W.H. Freeman has a relatively small selection of titles for a major publisher, but what a selection! Among the more advanced books are the acclaimed *The Fractal Geometry of Nature* by B.B. Mandelbrot and the long–awaited *Tilings and Patterns* by B. Grünbaum and G.C. Shephard. We will concentrate on books at the high school and beginning college levels. We point out that three of the books in Martin Gardner's *Scientific American* series are published by Freeman.

***Mathematics: Problem Solving Through Recreational Mathematics*, by B. Averbach and O. Chein, 1980. (hardcover, 400 pp.)**

This book presents a minimum of theory and plenty of problems. After a general discussion on problem–solving techniques, problems in logic, algebra, number theory, graph theory, games and puzzles are posed and solved.

***aha! Insight*, by Martin Gardner, 1978. (paperback, 179 pp.)**

This is the book form of six filmstrips titled *Combinatorial aha!*, *Geometry aha!*, *Number aha!*, *Logic aha!*, *Procedural aha!*, and *Word aha!*. Selected sequences of frames featuring problems and paradoxes are shown with accompanying text.

***aha! Gotcha*, by Martin Gardner, 1982. (paperback, 164 pp.)**

This is the book form of six filmstrips titled *aha Logic!*, *aha Number!*, *aha Geometry!*, *aha Probability!*, *aha Statistics!*, and *aha Time!*. Selected sequences of frames featuring problems and paradoxes are shown with accompanying text.

***Geometry*, by H.R. Jacobs, 1987. (hardcover, 701 pp.)**

The second edition of this outstanding textbook covers standard Euclidean plane and solid geometry in a way which students will find enjoyable. The serious mathematics is interlaced with cartoons, anecdotes and practical problems. The Teacher's Guide provides specific lesson plans.

***Elementary Algebra*, by H.R. Jacobs, 1979. (hardcover, 876 pp.)**

This textbook covers functions and graphs, number systems, equations, polynomials, exponents and radicals, inequalities and number sequences. It is done in a lucid and refreshing manner, much in the style of its companion volume *Geometry.* The Teacher's Guide provides specific lesson plans.

***Mathematics: A Human Endeavor*, by H.R. Jacobs, 1982. (hardcover, 649 pp.)**

The subtitle of this book is "A Book for Those Who Think They Don't Like the Subject". It is a stimulating survey of number theory, algebra, geometry, combinatorics, probability, statistics and topology.

***Mathematics: An Introduction to Its Spirit and Use*, edited by M. Kline, 1979. (paperback, 249 pp.)**

This book contains forty articles reprinted from *Scientific American*, with fourteen from Martin Gardner's column *Mathematical Games.* They are classified into six categories: history, number and algebra, geometry, statistics and probability, symbolic logic and computers, and applications. Note that W.H. Freeman also handles offprints of other *Scientific American* articles.

***Mathematics: A Man-Made Universe*, by S.K. Stein, 1976. (hardcover, 573 pp.)**

The third edition of this classic contains nineteen chapters, covering number theory, geometry, graph theory, modern algebra, number systems, constructibility problems and infinite sets. Its outstanding feature is a large collection of exercises that urge the reader to explore and discover. Despite the ease in which various topics are handled, the book has tremendous depth.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor , whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **July 1, 1990,** *although solutions received after that date will also be considered until the time when a solution is published.*

1491. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

In triangle ABC, the internal bisector of $\angle A$ meets BC at D, and the external bisectors of $\angle B$ and $\angle C$ meet AC and AB (produced) at E and F respectively. Suppose that the normals to BC, AC, AB at D, E, F, respectively, meet. Prove that $\overline{AB} = \overline{AC}$.

1492. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be a triangle inscribed in a triangle ABC, so that $A' \in BC$, $B' \in CA$, $C' \in AB$. Suppose also that $BA' = CB' = AC'$.

(a) If either the centroids G, G' or the circumcenters O, O' of the triangles coincide, prove that ΔABC is equilateral.

(b)* If either the incenters I, I' or the orthocenters H, H' of the triangles coincide, characterize ΔABC.

1493. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Two squares $ABDE$ and $ACFG$ are described on AB and AC outside the triangle ABC. P and Q are on line EG such that BP and CQ are perpendicular to BC. Prove that

$$BP + CQ \geq BC + EG.$$

When does equality hold?

1494. *Proposed by Murray S. Klamkin, University of Alberta.*

Three numbers x, y, z are chosen independently at random and uniformly in [0,1]. What is the probability that x, y, z can be the lengths of the sides of a triangle whose altitudes are also the sides of some triangle?

1495. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Show that there exist infinitely many positive integer solutions to

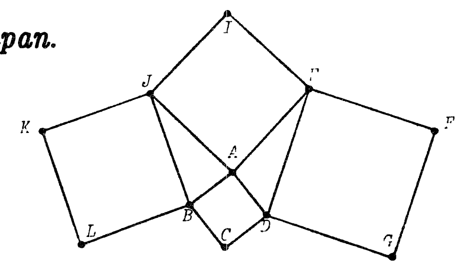
$$c^2 = \binom{a}{2} - \binom{b}{2} .$$

(b)* If $k > 2$ is an integer, are there infinitely many solutions in positive integers to

$$c^k = \binom{a}{k} - \binom{b}{k} ?$$

1496. *Proposed by H. Fukagawa, Aichi, Japan.*

There are four squares $ABCD$, $DEFG$, $AEIJ$, $BJKL$ as shown in the figure. Show that L, C, G are collinear if and only if $2AD = AE$.



1497. *Proposed by Ray Killgrove and Robert Sternfeld, Indiana State University, Terre Haute.*

A *translate* g of a function f is a function $g(x) = f(x + a)$ for some constant a. Suppose that one translate of a function f: $\mathbb{R} \to \mathbb{R}$ is odd and another translate is even. Show that f is periodic. Is the converse true?

1498*. *Proposed by D.M. Milošević, Pranjani, Yugoslavia.*

Show that

$$\prod_{i=1}^{3} h_i^{a_i} \leq (3r)^{2s},$$

where a_1, a_2, a_3 are the sides of a triangle, h_1, h_2, h_3 its altitudes, r its inradius, and s its semiperimeter.

1499. *Proposed by Herta T. Freitag, Roanoke, Virginia.*

A second-order linear recursive sequence $\{A_n\}_1^\infty$ is defined by $A_{n+2} = A_{n+1} + A_n$ for all $n \geq 1$, with A_1 and A_2 any integers. Select a set S of any $2m$ consecutive elements from this sequence, where m is an odd integer. Prove that the sum of the numbers in S is always divisible by the $(m + 2)$nd element of S, and the multiplying factor is L_m, the mth Lucas number.

1500. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

A parallelogram is called *self–diagonal* if its sides are proportional to its diagonals. Suppose that $ABCD$ is a self–diagonal parallelogram in which the bisector of angle ADB meets AB at E. Prove that $AE = AC - AB$.

* * *

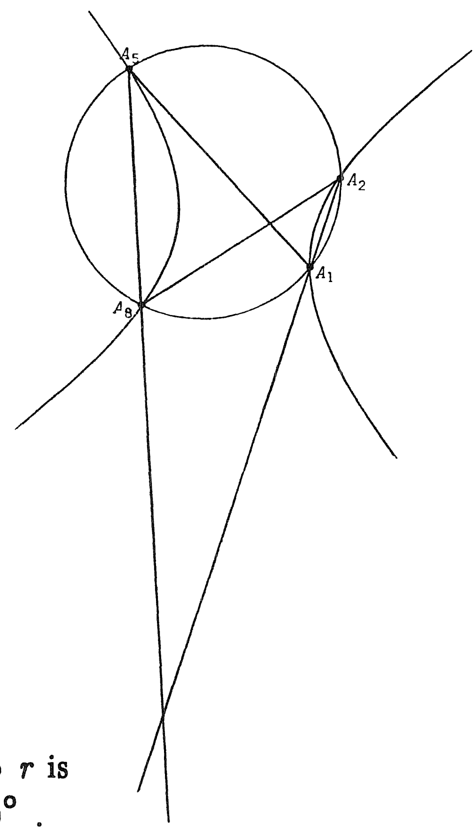
SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1374. [1988: 235] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $A_1A_2\cdots A_9$ be a regular 9–gon. Show that A_1, A_2 lie on one branch of a hyperbola and A_5, A_8 lie on the other, and determine the ratio of the axes of this hyperbola.

Solution by Jordi Dou, Barcelona, Spain.



There are infinitely many hyperbolas having the conditions of the problem. I interpret "determine the ratio" in the sense of "determine the range of the ratio". The ratio of the axes, given the angle α of the asymptotes containing a branch of the hyperbola, is

$$r = \tan\frac{\alpha}{2}\ .$$

Intuitively, it is clear that the infimum and the supremum of α are the angles formed by the pairs of lines (degenerate conics) $\{A_5A_1,A_8A_2\}$ and $\{A_5A_8,A_1A_2\}$, equal to $\alpha_1 = 80^\circ$ and $\alpha_2 = 160^\circ$ respectively. Thus the range of the ratio r is

$$\tan 40^\circ < r < \tan 80^\circ\ .$$

[*Editor's note*: Dou then gives a more detailed justification by describing the pencil of conics passing through the points A_1, A_2, A_5, A_8. He shows, for instance, that the axes of all such conics are parallel and perpendicular to the line A_1A_7.]

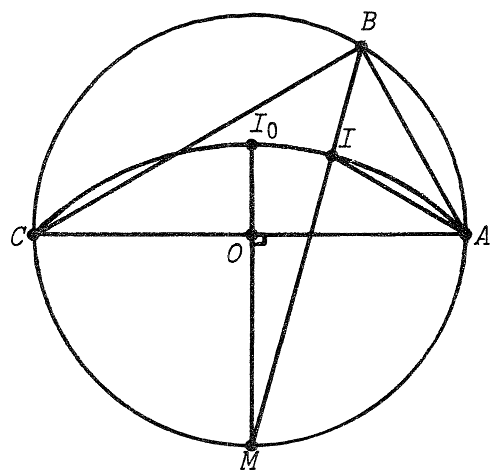
P. PENNING, Delft, The Netherlands, also gave the range of the required ratio. The proposer only gave one value inside this range.

* * *

1377. [1988: 235] *Proposed by Colin Springer, student, Waterloo, Ontario.*

In right triangle ABC, hypotenuse AC has length 2. Let O be the midpoint of AC and let I be the incentre of the triangle. Show that $\overline{OI} \geq \sqrt{2} - 1$.

I. *Solution by Hans Engelhaupt, Gundelsheim, Federal Republic of Germany.*



In the figure, B lies arbitrarily on the upper half of the circle of Thales (center O, radius 1), and BM is the bisector of the right angle CBA. Then

$$\angle IAM = \angle OAM + \angle OAI = 45^\circ + \alpha/2$$

and

$$\angle AIM = \angle IBA + \angle IAB = 45^\circ + \alpha/2 ,$$

so ΔAIM is isosceles. Thus $\overline{MI} = \overline{MA} = \sqrt{2}$, so I lies on the circle with center M and passing through A. Therefore

$$\overline{OI} \geq \overline{OI_0} = \sqrt{2} - 1 .$$

II. *Solution and extensions by Murray S. Klamkin, University of Alberta.*

Since O is the circumcenter of ΔABC and since (as is known for any triangle) $\overline{OI}^2 = R^2 - 2Rr$, where R is the circumradius and r the inradius, $\overline{OI}$ will be a minimum when r is a maximum. Intuitively, this will occur for an isosceles right triangle. More generally, we consider the same problem for a triangle with a given angle A and inscribed in a given circle of radius R. We will show that $\overline{OI}$ will be a minimum, or, equivalently, that r will be a maximum, when the triangle is isosceles.

Since it is known that

$$r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} ,$$

we maximize

$$\sin\frac{B}{2} \sin\frac{C}{2}$$

subject to $B + C = \pi - A$. Since also

$$2 \sin \frac{B}{2} \sin \frac{C}{2} = \cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{B+C}{2}\right) ,$$

r increases monotonically from $B = 0$ to $B = (\pi - A)/2$ and then decreases monotonically from $B = (\pi - A)/2$ to $B = \pi - A$. Hence r is maximal when $B = C = (\pi - A)/2$, i.e.,

$$r_{\max} = 4R \sin \frac{A}{2} \sin^2\left(\frac{\pi - A}{4}\right) ,$$

so

$$(\overline{OI}^{\,2})_{\min} = R^2 - 8R^2 \sin \frac{A}{2} \sin^2\left(\frac{\pi - A}{4}\right) .$$

Also

$$(\overline{OI}^{\,2})_{\max} = R^2 ,$$

and is taken on for the degenerate triangle of angles A, $\pi - A$, 0. For the given problem, $R = 1$ and $A = \pi/2$ so that

$$(\overline{OI}^{\,2})_{\min} = 3 - 2\sqrt{2} , \qquad \overline{OI}_{\min} = \sqrt{2} - 1.$$

As further extensions, we now determine the extreme values of (i) $\overline{OH}$, (ii) $\overline{OG}$, (iii) $\overline{GH}$, (iv) $\overline{IH}$, and (v) $\overline{IG}$, for the more general problem above (where R and angle A are given). Here G is the centroid and H the orthocenter of the triangle.

(i) Since

$$\overline{OH}^{\,2} = R^2 - 8R^2 \cos A \cos B \cos C , \tag{1}$$

we want the extreme values of $\cos B \cos C$ where $B + C = \pi - A$. We also assume here and below that $B \le C$. Since

$$2 \cos B \cos C = \cos(B - C) + \cos(B + C) ,$$

$\overline{OH}$ increases or decreases monotonically for $B = 0$ to $B = (\pi - A)/2$ according as $\cos A < 0$ or $\cos A > 0$. Thus, for $A > \pi/2$,

$$(\overline{OH}^{\,2})_{\max} = R^2 - 8R^2 \cos A \sin^2\frac{A}{2} ,$$

$$(\overline{OH}^{\,2})_{\min} = R^2 + 8R^2 \cos^2 A .$$

For $A < \pi/2$, the above max and min are interchanged.

(ii) and (iii) Since $\overline{OG} = \overline{OH}/3$ and $\overline{GH} = 2\overline{OH}/3$, these cases are equivalent to (i).

(iv) Since

$$\begin{aligned}
\frac{\overline{IH}^{\,2}}{4R^2} &= \frac{r^2}{2R^2} - \cos A \cos B \cos C \\
&= (1 - \cos A)(1 - \cos B)(1 - \cos C) - \cos A \cos B \cos C \\
&= 1 - \cos A + \cos B \cos C(1 - 2 \cos A) - (\cos B + \cos C)(1 - \cos A) \\
&= 1 - \cos A + \frac{(\cos(C+B) + \cos(C-B))(1 - 2 \cos A)}{2} - 2\cos\left(\frac{C+B}{2}\right)\cos\left(\frac{C-B}{2}\right)(1 - \cos A)
\end{aligned}$$

$$= 1 - \cos A - \frac{\cos A(1 - 2\cos A)}{2} + \frac{\cos(C - B)(1 - 2\cos A)}{2}$$
$$- 2\sin\frac{A}{2}(1 - \cos A)\cos\left(\frac{C - B}{2}\right),$$

we want the extreme values of

$$(1 - 2\cos A)x^2 - 2\sin\frac{A}{2}(1 - \cos A)x\,, \tag{2}$$

where

$$x = \cos\left(\frac{C - B}{2}\right)$$

and $B + C = \pi - A$, $0 \le B \le C$, so $\sin(A/2) \le x \le 1$. By completing the square, (2) becomes

$$(1 - 2\cos A)(x - f)^2 - (1 - 2\cos A)f^2\,, \tag{3}$$

where

$$f = \frac{\sin(A/2)(1 - \cos A)}{1 - 2\cos A}\,.$$

If $A \le \pi/3$, then both f and its denominator are negative, so the minimum for (3) occurs when $x = 1$, i.e., $B = C$, and the maximum when $x = \sin(A/2)$, i.e., $B = 0$ and $C = \pi - A$.

If $A > \pi/3$, then by considering

$$\frac{df}{dA} = \frac{\cos(A/2)(2\cos A + 1)(\cos A - 1)}{2(1 - 2\cos A)^2}$$

it follows that as A increases from $\pi/3$ to $2\pi/3$, f decreases from ∞ to $3\sqrt{3}/8$, and as A increases from $2\pi/3$ to π, f increases from $3\sqrt{3}/8$ to $2/3$. Also, $f = 1$ for

$$A = \alpha \approx 73.3^\circ.$$

So for $\pi/3 < A \le \alpha$, $f > 1$ and the minimum occurs for $B = C$ while the maximum occurs for $B = 0$, $C = \pi - A$. For $f \le 1$, the minimum occurs for $x = f$ provided that any solutions for B and C are feasible angles of a triangle. Here we must have

$$\pi - A = C + B \ge C - B = 2\cos^{-1}x = 2\cos^{-1}f$$

or

$$\sin\frac{A}{2} = \cos\left(\frac{\pi - A}{2}\right) \le f = \frac{\sin(A/2)(1 - \cos A)}{1 - 2\cos A}\,,$$

so $A \le \pi/2$. Then the maximum must occur either for $B = C$ or for $B = 0$. Since here $\alpha \le A \le \pi/2$ and $f \ge \sin(A/2)$, from (3) we have to compare $1 - f$ (case $B = C$, or $x = 1$) with $f - \sin(A/2)$ (case $B = 0$, or $x = \sin(A/2)$). Now

$$1 - f \gtrless f - \sin(A/2)$$

according as

$$1 - 2\cos A \gtrless \sin(A/2).$$

The critical value is

$$\sin\frac{A}{2} = \frac{1 + \sqrt{17}}{8}\,,$$

or

$$A = \beta \approx 79.64^\circ.$$

Hence for $\alpha \le A \le \beta$ the maximum occurs for $B = 0$, while for $\beta \le A \le \pi/2$ the maximum occurs for $B = C$.

For the last case, $\pi/2 < A \le \pi$ and

$$f = \frac{\sin(A/2)(1 - \cos A)}{1 - 2\cos A} < \sin\frac{A}{2}.$$

Hence the maximum of (3) occurs for $B = C$ ($x = 1$) and the minimum for $B = 0$ ($x = \sin(A/2)$).

Summarizing: for $0 \le A \le \alpha$,

$$\overline{IH}_{\min} = 2R\sin\frac{A}{2}\left|2\sin\frac{A}{2} - 1\right|,$$

$$\overline{IH}_{\max} = 2R\cos A;$$

for $\alpha \le A \le \pi/2$,

$$\overline{IH}_{\min} = R\cos A\sqrt{\frac{2(1 - 3\cos A)}{1 - 2\cos A}},$$

$$\overline{IH}_{\max} = \begin{cases} 2R\cos A & \text{for } \alpha \le A \le \beta, \\ 2R\sin\frac{A}{2}\left(2\sin\frac{A}{2} - 1\right) & \text{for } \beta \le A \le \frac{\pi}{2}; \end{cases}$$

and for $A \ge \pi/2$,

$$\overline{IH}_{\min} = -2R\cos A,$$

$$\overline{IH}_{\max} = 2R\sin\frac{A}{2}\left(2\sin\frac{A}{2} - 1\right).$$

(v) Since it is known that

$$9\overline{IG}^2 = 4R^2 - 12Rr + 6r^2 + 4R^2\cos A\cos B\cos C$$

and

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

we have

$$9\overline{IG}^2 = 4R^2\left(1 - 12\prod\sin\frac{A}{2} + 24\prod\sin^2\frac{A}{2} + \prod\cos A\right),$$

the sums and products here and subsequently being cyclic over A, B, C. It then follows from

$$1 + 4\prod\sin\frac{A}{2} = 1 + \frac{r}{R} = \sum\cos A$$

that

$$\frac{9\overline{IG}^2}{4R^2} - 1 = 3\prod(1 - \cos A) - 3\left(\sum\cos A - 1\right) + \prod\cos A$$

$$= 6 - 6\sum\cos A + 3\sum\cos B\cos C - 2\prod\cos A. \quad (4)$$

Since R and A (and thus $B + C$) are known, we want the extreme values of

$$-(\cos B + \cos C)(6 - 3 \cos A) + \cos B \cos C(3 - 2 \cos A)$$

or equivalently the extreme values of

$$-2 \sin \frac{A}{2} \cos\left(\frac{C - B}{2}\right)(6 - 3 \cos A) + \frac{\cos(C - B)(3 - 2 \cos A)}{2}\ . \tag{5}$$

We now let

$$x = \cos\left(\frac{C - B}{2}\right)$$

so that $\sin(A/2) \le x \le 1$ and $\cos(C - B) = 2x^2 - 1$. Then, by completing the square, (5) becomes

$$(3 - 2 \cos A)(x - g)^2 - (3 - 2 \cos A)(g^2 + 1/2) \tag{6}$$

where

$$g = \frac{\sin(A/2)(6 - 3 \cos A)}{3 - 2 \cos A}\ .$$

It can be shown that $dg/dA > 0$ so that g increases from 0 to 9/5 as A increases from 0 to π. Also $g = 1$ for

$$A = \alpha \approx 49.7^\circ.$$

Hence for $A \ge \alpha$, (6) (and thus $\overline{IG}$) is a maximum for $B = 0$ ($x = \sin(A/2)$) and a minimum for $B = C$ ($x = 1$).

For $0 \le A \le \alpha$, the minimum of (6) occurs for $x = g$ provided that any solutions B, C are feasible angles of a triangle. Here we must have

$$\pi - A = C + B \ge C - B = 2 \cos^{-1} g$$

or $g \ge \sin(A/2)$. The latter is valid since

$$\frac{6 - 3 \cos A}{3 - 2 \cos A} > 1$$

for all A. The maximum of (6) occurs for either $B = C$ or $B = 0$ depending on whether or not

$$1 - g \ge g - \sin \frac{A}{2}\ .$$

Now

$$2g - \sin \frac{A}{2} = \frac{\sin(A/2)(5 + 8 \sin^2(A/2))}{1 + 4 \sin^2(A/2)}$$

is increasing in A and equals 1 for

$$g = \beta \approx 25.64^\circ.$$

Thus, for $0 \le A \le \beta$ the maximum of (6) occurs for $B = C$, while for $\beta \le A \le \alpha$ the maximum of (6) occurs for $B = 0$.

Summarizing:

$$\overline{IG}_{\min} = \begin{cases} \dfrac{R(1-\cos A)}{3}\sqrt{\dfrac{2(5\cos A - 3)}{3-2\cos A}} & \text{for } 0 \le A \le \alpha\,, \\[2ex] \dfrac{2R}{3}\left(1-\sin\dfrac{A}{2}\right)\left|1-2\sin\dfrac{A}{2}\right| & \text{for } \alpha \le A \le \pi\,; \end{cases}$$

$$\overline{IG}_{\max} = \begin{cases} \dfrac{2R}{3}\left(1-\sin\dfrac{A}{2}\right)\left(1-2\sin\dfrac{A}{2}\right) & \text{for } 0 \le A \le \beta\,, \\[2ex] \dfrac{2R}{3}\sin A & \text{for } \beta \le A \le \pi\,. \end{cases}$$

Three other related problems are to determine the maximum values of $\overline{OH}$, $\overline{IH}$, and $\overline{IG}$ just given R. This follows from the above results by maximizing over A. We obtain:

$\overline{OH}_{\max} = 3R$, only occurring for the degenerate triangle of angles 0, 0, π;

$\overline{IH}_{\max} = 2R$, only occurring for the degenerate triangle of angles 0, 0, π;

$\overline{IG}_{\max} = 2R/3$, occurring for the degenerate triangle of angles 0, $\pi/2$, $\pi/2$.

Reference:

[1] D.S. Mitrinović, J.E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, 1989.

Also solved by SEUNG–JIN BANG, AJOU University, Suwon, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSION LOPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; ANDREW CHOW, student, Albert Campbell C.I., Scarborough, Ontario; JORDI DOU, Barcelona, Spain; R.H. EDDY, Memorial University of Newfoundland; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE–WAI LAU, Hong Kong; DAVE MCDONALD, Crimson Elk, Alberta; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; M. PARMENTER, Memorial University of Newfoundland; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; D.J. SMEENK, Zaltbommel, The Netherlands; J. SUCK, Essen, Federal Republic of Germany (two solutions); GEORGE TSINTSIFAS, Thessaloniki, Greece; C. WILDHAGEN, Breda, The Netherlands; and the proposer. The solutions of Dou, McDonald, and the proposer were the same as Engelhaupt's solution.

Janous mentions the "complementary" inequality

$$\frac{r}{h} \ge \sqrt{2} - 1$$

where r is the inradius and h is the altitude to the hypotenuse AC (item 11.22 *of Bottema et al,* Geometric Inequalities*).*

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1378*. [1988: 235] *Proposed by J. Walter Lynch, Georgia Southern College, Statesboro, Georgia.*

Suppose $a_0, a_1, a_2, \ldots$ is a sequence of positive real numbers such that $a_0 = 1$ and $a_n = a_{n+1} + a_{n+2}$, $n \geq 0$. Find a_n.

Solution par Roger Cuculière, Rabat, Morocco.

Les raisons des suites géométriques qui vérifient la même relation de récurrence $a_n = a_{n+1} + a_{n+2}$ sont les racines de l'équation caractéristique: $1 = x + x^2$. Ce sont les réels

$$\alpha = \frac{\sqrt{5} - 1}{2} \qquad \text{et} \qquad \beta = \frac{-\sqrt{5} - 1}{2},$$

et l'on a donc:

$$a_n = \lambda\alpha^n + \mu\beta^n,$$

avec $\lambda + \mu = 1$ puisque $a_0 = 1$. Si l'on suppose $\mu \neq 0$, il existe n tel que

$$\left|\frac{\lambda}{\mu}\cdot\frac{\alpha^n}{\beta^n}\right| < 1 .$$

(Pour le prouver, il suffit de prendre n tel que $|\beta/\alpha|^n > |\lambda/\mu|$, ce qui est toujours possible puisque

$$\left|\frac{\beta}{\alpha}\right| = \frac{\sqrt{5} + 1}{\sqrt{5} - 1} > 1 .)$$

Mais si n est ainsi choisi, on a

$$a_n = \mu\beta^n\left(1 + \frac{\lambda}{\mu}\cdot\frac{\alpha^n}{\beta^n}\right),$$

et

$$1 + \frac{\lambda}{\mu}\cdot\frac{\alpha^n}{\beta^n} > 0 .$$

Donc le réel a_n est du signe de $\mu\beta^n$ pour n suffisamment grand. Ce qui prouve que l'on a $a_n < 0$, pour $n = 2m$ si $\mu < 0$, ou pour $n = 2m + 1$ si $\mu > 0$. Ceci étant impossible par hypothèse, il faut en conclure que $\mu = 0$, donc $\lambda = 1$, et enfin

$$a_n = \alpha^n = \left(\frac{\sqrt{5} - 1}{2}\right)^n .$$

Also solved by SEUNG-JIN BANG, AJOU University, Suwon, Republic of Korea; NICOS D. DIAMANTIS, student, University of Patras, Greece; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; GUO-GANG GAO, *Université de Montréal; JÖRG HÄRTERICH, Winnenden, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; BRUCE MCLEAN and DAVID R. STONE, Georgia Southern College, Statesboro; LEROY F. MEYERS, The Ohio State University; ROBERT E. SHAFER, Berkeley, California; DANIEL B. SHAPIRO, Ohio State University; HUME SMITH, Acadia University; COLIN SPRINGER, student, University of Waterloo; EDWARD T.H. WANG, Wilfrid Laurier University; and C. WILDHAGEN, Breda, The Netherlands. Four other readers obtained the correct answer but failed to show it was unique.*

Klamkin showed more generally (proof as above) that for $r < 0$ and $s \geq 0$, the unique sequence $a_0, a_1, a_2, \ldots$ of positive numbers satisfying $a_0 = 1$ and $a_{n+2} = ra_{n+1} + sa_n$ is

$$a_n = \left(r + \sqrt{r^2 + 4s}\right)^n ;$$

for $r \geq 0$ and $r^2 + 4s \geq 0$, one can choose

$$a_n = A\left(r + \sqrt{r^2 + 4s}\right)^n + (1 - A)\left(r - \sqrt{r^2 + 4s}\right)^n$$

for any $A \geq 1/2$; and for any other values of r and s, a_n cannot always be positive.

Janous remarks that the problem was posed in the first round of the 1972 *Olympiad in Austria. Smith observes that the problem is identical in nature to Problem* A–5 *of the* 1988 *Putnam.*

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1379. [1988: 235] *Proposed by P. Penning, Delft, The Netherlands.*

Given are an arbitrary triangle ABC and an arbitrary interior point P. The pedal–points of P on BC, CA, and AB are D, E, and F respectively. Show that the normals from A to EF, from B to FD, and from C to DE are concurrent.

I. *Solution by R.H. Eddy, Memorial University of Newfoundland.*

Because AP, BP, CP are concurrent, the theorem of Ceva implies that

$$\frac{\sin \angle PAB}{\sin \angle PAC} \cdot \frac{\sin \angle PBC}{\sin \angle PBA} \cdot \frac{\sin \angle PCA}{\sin \angle PCB} = -1 .$$

Since $AFPE$ is a cyclic quadrilateral, $\angle FPA = \angle FEA$ and so $\angle PAF = \angle EAX$, where X is the intersection of the normal from A to FE. Thus AX is the reflection of AP in the internal angle bisector of $\angle CAB$, and similarly for the other two normals. The product of the sines for the angles at the vertices formed by the normals is just the reciprocal of those formed by AP, BP, CP above, hence the normals are concurrent at a point P', called the isogonal conjugate of P. See Roger A. Johnson, *Advanced Euclidean Geometry (Modern Geometry)*, Dover, NY, 1960.

II. *Solution by Dan Pedoe, Minneapolis, Minnesota.*

In most textbooks on analytical geometry the following theorem appears as an exercise:

If ABC and $A'B'C'$ are triangles such that the perpendiculars from A onto $B'C'$, from B onto $C'A'$, and from C onto $A'B'$ are concurrent, then the perpendiculars from A' onto BC, from B' onto CA, and from C' onto AB are also concurrent.

This theorem dates back at least to Steiner, and a discussion can be found in [1] below.

In the given problem, the perpendiculars from the vertices D, E and F onto the sides of ABC are DP, EP and FP, which pass through P.

Reference:

[1] Daniel Pedoe, On (what should be) a well-known theorem in geometry, *American Math. Monthly* **74** (1967) 839–841.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; JÖRG HÄRTERICH, Winnenden, Federal Republic of Germany; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student, University of Waterloo; and the proposer.

Hut and Smeenk gave similar proofs to Eddy's. Klamkin also pointed out the theorem of Steiner given in solution II.

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<u>1380</u>. [1988: 235] *Proposed by Kee-Wai Lau, Hong Kong.*

Prove the inequality

$$\sin(\tan x) < \tan(\sin x)$$

for $0 < x < \pi$, $x \neq \pi/2$.

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.

If $0 < x < \pi/2$, the proposed inequality is proved in *Mathematics Magazine* Vol **53**, no. 5 (November 1980) pp. 302–303, by C.S. Gardner, in his solution of his problem 1082. In order to extend the result to the interval $(\pi/2,\pi)$, it suffices to observe that

$$\sin(\pi - x) = \sin x \qquad \text{and} \qquad \tan(\pi - x) = -\tan x ,$$

and therefore, if $\pi/2 < \beta < \pi$, that

$$\sin(\tan \beta) \le |\sin(\tan \beta)| = |\sin(\tan(\pi - \beta))| < \tan(\sin(\pi - \beta)) = \tan(\sin \beta) ,$$

because $0 < \pi - \beta < \pi/2$. [*Editor's note:* it should also be mentioned that for those values of $\pi - \beta$ for which $\sin(\tan(\pi - \beta)) < 0$, it follows

$$\frac{\pi}{2} > \pi - \beta > \tan^{-1}\pi \approx 1.2626 > 0.9033 \approx \sin^{-1}\frac{\pi}{4}$$

and hence $\tan(\sin(\pi - \beta)) > 1 > |\sin(\tan(\pi - \beta))|$.]

Two references related to this problem are *Monthly* problem E2720, solution in October 1979, and R.P. Boas, Inequalities for a collection, *Math. Magazine* **52** (1979) 28–31, a nice note, with generalizations and several examples of similar inequalities.

Also solved by HAYO AHLBURG, Benidorm, Spain; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT E. SHAFER, Berkeley, California; and the proposer.

Ahlburg mentions the similar problem

$$\sin(\cos x) < \cos(\sin x)$$

for all x, which appeared in Crux [1981: 107], *having been around for years before (Léo Sauvé gives a short history), and was also proposed as problem* 1020 *in the* Journal of Recreational Mathematics, *with solution in Vol.* 14 (1982) *p.* 315.

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1381. [1988: 268] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

ABC is a triangle with circumcircle Ω. AA_1, BB_1, CC_1 are three parallel chords of Ω, and A_2, B_2, C_2 are the feet of the perpendiculars from A_1 to BC, B_1 to AC, C_1 to AB, respectively. Prove that

(a) A_1A_2, B_1B_2, C_1C_2 intersect in a point on Ω;

(b) A_2, B_2, C_2 all lie on a line parallel to AA_1.

(This problem is not new. References will be given when a solution is published.)

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.

(a) It is well known that there exists *only one* point M such that the Simson line of M with respect to ΔABC has a given direction; the construction of that point is precisely that given in the problem (see [1], [2], [3]). Namely, let AA_1 be the given direction, with $A_1 \in \Omega$, and let the perpendicular from A_1 to BC meet BC at A_2 and Ω at M.

The points N and P (similarly constructed from BB_1 and CC_1 respectively) must coincide with M: the angle formed by the Simson lines of M and N equals half the arc length MN, and the three Simson lines are parallel ($AA_1 \| BB_1 \| CC_1$); therefore

$$\frac{\widehat{MN}}{2} = \frac{\widehat{MP}}{2} = \frac{\widehat{NP}}{2} = 0 \ ,$$

and so $M = N = P$.

(b) A_2, B_2, C_2 are the feet of the perpendiculars from M to the sides of ABC. They determine the Simson line of M, which is parallel to AA_1.

References:

[1] Lalesco, La Géométrie du Triangle, p. 9, note 2.9.

[2] H.S.M. Coxeter and S. Greitzer, *Geometry Revisited*, Vol. 19 NML, M.A.A., 1967, pp. 43–45.

[3] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960, p. 207.

Also solved by JORDI DOU, Barcelona, Spain; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

The proposer gives the reference Dubois, Journal de Mathématiques Elementaires, *Vol.* 42 *no.* 1 (1918). *Janous refers to the result as the theorem of Aubert, with reference G. Titeica,* Problems in Geometry (*Romanian*), *Bucharest,* 1982, *problems* 271 *and* 272.

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1382. [1988: 268] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let A be a nonconstant arithmetic sequence of n terms. Characterize, in terms of n only, those positive integers k such that A can be partitioned into k subsets of equal size having equal sums of their elements.

Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.

For the purpose of this problem we may, without loss of generality, confine our attentions to the sequence $\{1,2,...,n\}$, for if a satisfactory partition of this sequence is found we may convert it to a partition for a sequence with first member a and common difference d by applying the transformation $A_i = a + d(i - 1)$ to each $i \in \{1,2,...,n\}$.

Let the sum of the members of the sequence be $S = n(n + 1)/2$. Clearly a successful partitioning cannot be performed if $k = n$, nor if k is not a factor of both n and S. Thus a necessary condition is that

if n is even, k is a divisor of $n/2$, if n is odd, k is an aliquot divisor of n.

To prove that these conditions are sufficient, we will describe a partitioning that will work in all cases (there are many others). There are two possibilities.

(a) *n/k is even.* Assign the integers to the subsets as shown in the following table:

Subset							
1	1	$2k$	$2k+1$	$4k$	$\cdots$	$n-2k+1$	n
2	2	$2k-1$	$2k+2$	$4k-1$	$\cdots$	$n-2k+2$	$n-1$
3	3	$2k-2$	$2k+3$	$4k-2$	$\cdots$	$n-2k+3$	$n-2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
k	k	$k+1$	$3k$	$3k+1$	$\cdots$	$n-k$	$n-k+1$

The number of terms in each subset is n/k and the sum of each subset is $n(n+1)/(2k)$.

(b) *n/k is odd (so that n and k are also odd).* Proceed as follows for the first three assignments to each subset, and then assign the remaining $n - 3k$ integers according to the scheme in part (a):

Subset			
1	1	$k+(k+1)/2$	$3k$
2	3	$k+(k-1)/2$	$3k-1$
3	5	$k+(k-3)/2$	$3k-2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(k+1)/2$	k	$k+1$	$2k+(k+1)/2$
$(k+3)/2$	2	$2k$	$2k+(k-1)/2$
$(k+5)/2$	4	$2k-1$	$2k+(k-3)/2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$k-1$	$k+(k+3)/2$	$2k+1$

Note that the first two columns are discontinuous across the horizontal line, while the third column is not. The sum of the three members of each subset is $(9k+3)/2$, but when the other members of each subset are assigned by the method of part (a), the sum of each subset will be $n(n+1)/(2k)$.

For example, let $n = 81$ and $k = 9$. Then $S = 81 \cdot 82/2 = 3321$, $S/k = 369$, and the partitioning is as follows:

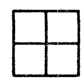
1	14	27	28	45	46	63	64	81
3	13	26	29	44	47	62	65	80
5	12	25	30	43	48	61	66	79
7	11	24	31	42	49	60	67	78
9	10	23	32	41	50	59	68	77
2	18	22	33	40	51	58	69	76
4	17	21	34	39	52	57	70	75
6	16	20	35	38	53	56	71	74
8	15	19	36	37	54	55	72	73

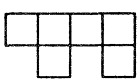
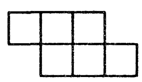
Also solved by HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; MARCIN E. KUCZMA, Warszawa, Poland; and the proposer. A further reader gave the correct conditions without proof.

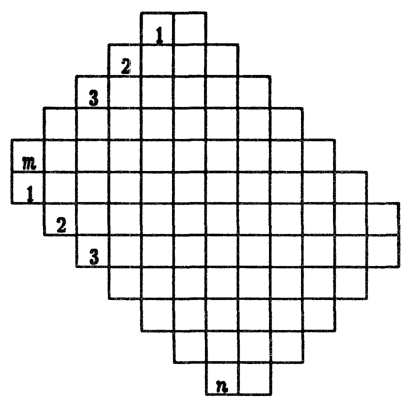
Problem 1 *of the* 1989 *I.M.O.* [1989: 196] *follows as a special case.*

* * *

1383. [1988: 268] *Proposed by Carl Friedrich Sutter, Viking, Alberta.*

A polyomino (rook–wise connected set of squares from a chessboard) is called *square–free* if it does not contain a 2×2 square  . For example,

 is square–free while  isn't.



Consider the jagged $m \times n$ "rectangle" R illustrated at the right, where m and n are each at least 2. Obviously R can be tiled by $m + n$ pairwise non–overlapping square–free polyominoes; just use the rows (or the columns). Prove that R cannot be tiled by less than $m + n$ pairwise nonoverlapping square–free polyominoes.

Solution by Marcin E. Kuczma, Warszawa, Poland.

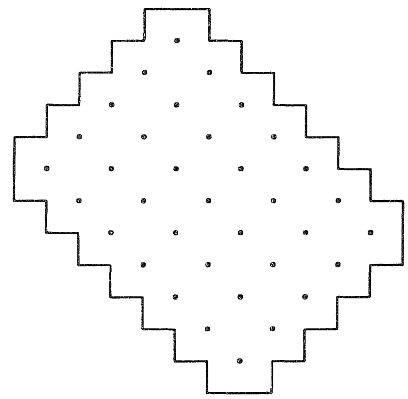
Lemma. For every polyomino P,

$$\text{perimeter}(P) \leq 2(1 + \text{Area}(P)). \tag{1}$$

Proof. In any polyomino P of area > 1 we can find a "field" (= unit square) whose removal does not disconnect P. To see this, choose any field in P as the "origin" and assign to any other field in P its distance (= minimum length of a rook's route within P) from the origin; any field whose distance from the origin is a maximum has the asserted property.

Thus, in other words, P is obtained by adjoining a field to a (connected) polyomino P'. By this operation, the area increases by 1 and the perimeter increases at most by 2 (it increases by 2, remains unchanged, or decreases by 2 or 4, according as the field in question is adjacent to 1, 2, 3 or 4 fields in P). Since (1) holds for P being a single square, induction proves (1) for any polyomino P. $\square$

Let R be the $m \times n$ jagged "rectangle" under consideration. Suppose it is partitioned into r square-free polyominoes $P_1,\ldots,P_r$. The partition lines consist of a certain number of unit segments, say k, in the interior of R. Each one of the mn lattice points marked in the figure at right is an endpoint of at least two partition segments, because the P_i's are square-free; moreover distinct points determine distinct segments. Hence



$$k \geq 2mn.$$

By the lemma,

$$\sum_{i=1}^{r} \text{perimeter}(P_i) \leq 2r + 2\sum_{i=1}^{r} \text{Area}(P_i)$$
$$= 2r + 2\ \text{Area}(R)$$
$$= 2r + 2(2mn + m + n). \qquad (2)$$

On the other hand,

$$\sum_{i=1}^{r} \text{perimeter}(P_i) = \text{perimeter}(R) + 2k$$
$$= 4(m + n) + 2k$$
$$\geq 4(m + n) + 4mn. \qquad (3)$$

Inequalities (2) and (3) yield $r \geq m + n$.

Also solved (in much the same way) by JÖRG HÄRTERICH, Winnenden, Federal Republic of Germany; and the proposer.

* * *

1384. [1988: 269] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

If the center of curvature of every point on an ellipse lies inside the ellipse, prove that the eccentricity of the ellipse is at most $1/\sqrt{2}$.

I. *Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.*

From the equation of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

we easily obtain

$$y' = -\frac{b^2x}{a^2y}, \qquad y'' = -\frac{b^4}{a^2y^3},$$

and the radius of curvature at (x,y) is

$$r = \frac{(1 + y'^2)^{3/2}}{y''} = -\frac{(a^4y^2 + b^4x^2)^{3/2}}{a^4b^4}.$$

The coordinates (x_1,y_1) of the center of curvature are

$$x_1 = x + r\cos\theta = \frac{(a^2 - b^2)x^3}{a^4},$$

$$y_1 = y + r\sin\theta = -\frac{(a^2 - b^2)y^3}{b^4},$$

where

$$\theta = \tan^{-1}\left(\frac{-1}{y'}\right) = \tan^{-1}\left(\frac{a^2y}{b^2x}\right).$$

The square of the distance from the center of curvature to the center of the ellipse is

$$x_1^2 + y_1^2 = \frac{(a^2 - b^2)^2}{a^2}\left[\frac{a^2}{b^2}\left(\frac{y^2}{b^2}\right)^3 + \left(\frac{x^2}{a^2}\right)^3\right]$$

$$= \frac{(a^2 - b^2)^2}{a^2}\left[\frac{a^2 - b^2}{b^2}\left(\frac{y^2}{b^2}\right)^3 + \left(\frac{x^2}{a^2}\right)^3 + \left(\frac{y^2}{b^2}\right)^3\right].$$

Now, in view of (1), and since $s^3 + t^3 \leq (s + t)^3$ for nonnegative s and t,

$$x_1^2 + y_1^2 \leq \frac{(a^2 - b^2)^2}{a^2}\left[\frac{a^2 - b^2}{b^2}\left(\frac{y^2}{b^2}\right)^3 + 1\right]$$

$$\leq \frac{(a^2 - b^2)^2}{a^2}\left[\frac{a^2 - b^2}{b^2} + 1\right] = \frac{(a^2 - b^2)^2}{b^2},$$

with equality occurring when $y = \pm b$. Now if

$$\frac{(a^2 - b^2)^2}{b^2} \leq b^2, \tag{2}$$

the center of curvature of every point of the ellipse is not only within the ellipse, but also within a circle of radius b whose center coincides with that of the ellipse. But (2) is equivalent to

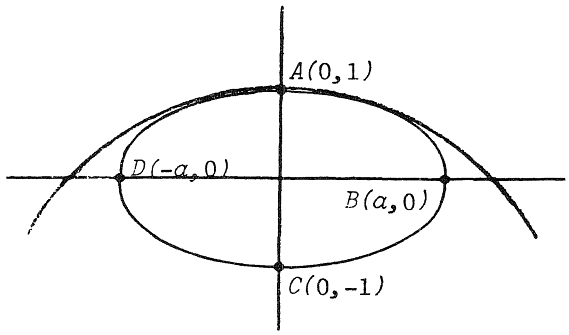
$$a^2 - b^2 \leq b^2 ,$$

i.e. the eccentricity

$$e = \sqrt{1 - \frac{b^2}{a^2}} \leq \frac{1}{\sqrt{2}} .$$

II. *Solution by Colin Springer, student, University of Waterloo.*

Let an ellipse be as shown, on the Cartesian plane, with equation



$$\frac{x^2}{a^2} + y^2 = 1 . \qquad (1)$$

In the extreme case, the centre of curvature at point A is at C, and the corresponding circle will be

$$x^2 + (y + 1)^2 = 4 . \qquad (2)$$

Eliminating x from (1) and (2), we get

$$4 - (y + 1)^2 + a^2y^2 = a^2 ,$$

i.e.

$$(a^2 - 1)y^2 - 2y + (3 - a^2) = 0 .$$

If this circle represents the curvature at A, this equation has a double root:

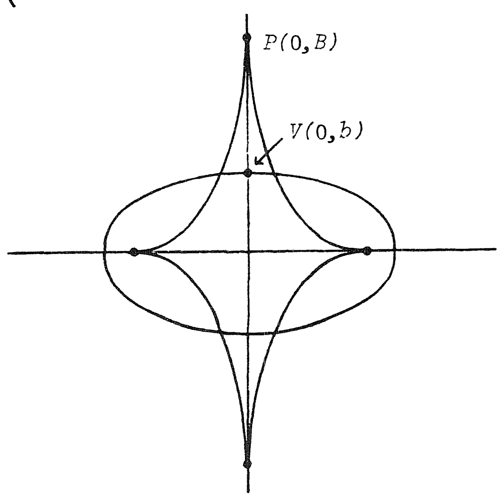
$$4 = 4(a^2 - 1)(3 - a^2) ,$$

$$a^4 - 4a^2 + 4 = 0 ,$$

so $a = \sqrt{2}$, since $a > 0$. In this case the eccentricity is $1/\sqrt{2}$, and for any greater eccentricity the centre of curvature at A is outside the ellipse. Hence for ellipses as given in the problem, the eccentricity may not exceed $1/\sqrt{2}$.

III. *Solution by Murray S. Klamkin, University of Alberta.*

It is known [1] that the evolute (locus of the centers of curvature) of an ellipse is the hypocycloid



$$\left(\frac{x}{A}\right)^{2/3} + \left(\frac{y}{B}\right)^{2/3} = 1 ,$$

where $Aa = Bb = a^2 - b^2$. See figure for a typical evolute (this one does not lie in the interior of the ellipse). The necessary and sufficient condition for the evolute to lie inside the ellipse is that the top point $P(0,B)$ of the evolute must lie below the top vertex

$V(0,b)$ of the ellipse, i.e.,

$$\frac{a^2 - b^2}{b} < b$$

or $2b^2 > a^2$. Since the eccentricity e is given by $e^2 = 1 - b^2/a^2$, $e < 1/\sqrt{2}$.

Reference:

[1] R.C. Yates, *A Handbook on Curves and their Properties*, J.W. Edwards, Ann Arbor, 1947, p.89.

Also solved by SEUNG–JIN BANG, Seoul, Korea; JORDI DOU, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta (a second solution); MARCIN E. KUCZMA, Warszawa, Poland; KEE–WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

* * *

FUNDS FOR A CANADIAN IMO

The Canadian Mathematical Society has issued a provisional invitation to host the International Mathematical Olympiad in Canada in 1995. For this, the CMS must raise over 1 million dollars in private and public support and must raise at least $100,000 by the middle of 1990 so that the Society can confirm its invitation. Corporate and government support is currently being sought, but individual donations are very much encouraged and will be necessary. Any *Crux* readers who would like to contribute should make their cheques or money orders payable to **"CMS c/o IMO 1995"**, and send them to:

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* * *

YEAR-END WRAPUP

Once again, this being the last issue of the year, here is some information on past *Crux* problems and articles which has been received by the editor during 1989.

1179. [1986: 206; 1988: 22, 85, 317]

JOHN RAUSEN, New York, gives two references:

Jesse Douglas, Geometry of polygons in the complex plane, *Journal of Math. and Physics* 19 (1940) 93–130;

Leon Gerber, Napoleon's Theorem and the parallelogram inequality for affine–regular polygons, *Amer. Math. Monthly* 87 (1980) 644–648;

both of which deal with erecting regular n–gons on the sides of an n–gon. He also points out that part (a) of the problem is Exercise 10, §1.8 of H.S.M. Coxeter, *Introduction to Geometry* (Wiley, 1961) with a reference to p. 40 of H.G. Forder, *The Calculus of Extension* (Cambridge Univ. Press, 1941/Chelsea, 1960).

1263. [1987: 215; 1988: 247]

HAYO AHLBURG, Benidorm, Spain, gives the references R. Hoppe, *Archiv Math. Phys.* 64 (1879) 441, and R. Müller, *Archiv Math. Phys.* (2) 5 (1887) 111–112, as examples of past solutions of the problem.

1313. [1988: 45; 1989: 77]

In confirmation of a claim of Bisztriczky [1989: 86], J. CHRIS FISHER, University of Regina, gives an example of a region of the plane of area 1, which however folded (once) will always cover an area of at least $1 - \epsilon$. Namely, he takes N long thin rectangles $R_1, R_2, \cdots, R_N$ and arranges them end to end in staircase fashion, alternately vertically and horizontally. The rectangles also become rapidly longer and thinner: precisely, R_k is a_k by $1/(Na_k)$, where $a_1 = 1$ and $a_k = \sum_{i=1}^{k-1} a_i$ for $k \geq 2$. By the way, Bisztriczky's original idea (which readers may like to work

on) was that any region which is the union of a large number of small identical discs, randomly placed on the plane, should have the same folding property.

*

Late solutions have been received to 1109 **(J. Chris Fisher, University of Regina);** 1287, 1288, 1289, 1293, 1298, 1300, 1304, 1308, 1311, 1314 **(J. Suck, Essen); and** 1331 **(Nicos D. Diamantis, University of Patras, Greece).**

*

Regarding Andy Liu's Mini–Reviews of Martin Gardner's *Scientific American* books (September), HANS HAVERMANN, Weston, Ontario, offers the information that the first three books listed are now being published by the University of Chicago Press (the first, under the title *Hexaflexagons and Other Mathematical Diversions*), while *Mathematical Carnival* has just been put out by the Mathematical Association of America.

*   *   *

Some good ideas eventually occur to the editor, and approaching the end of his fourth year on the job he realizes he has never made a point of thanking publicly those people whose advice and assistance have been so valuable to him. The following contributed to the quality of *Crux* in 1989 by their opinions of articles, problems, and solutions: *L. BOS*, *J.C. FISHER*, *R.K. GUY*, *W. JANOUS*, *J. KONHAUSER*, *D. PEDOE*, *J. SCHAER*, and *D. SOKOLOWSKY*. Apologies to those people the editor has forgotten to include! One person, *M.S. KLAMKIN*, has *not* been forgotten, but has been reserved for special mention because of the enormity of help he gives the editor. Finally, special thanks also to *LAURIE LORO*, who personally types every syllable of *Crux*, and whose skills with the word processor and laser printer have meant a steady improvement in the appearance of *Crux* over the last four years.

* * * * * * * *

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