Mediterranean Mathematics Olympiad 2010

$\boxed{1}$ Real numbers a, b, c, d are given. Solve the system of equations (unknowns x, y, z, u)

$$x^2 - yz - zu - yu = a$$

$$y^2 - zu - ux - xz = b$$

$$z^2 - ux - xy - yu = c$$

$$u^2 - xy - yz - zx = d$$

$\boxed{2}$ Given the positive real numbers $a_1, a_2, \ldots, a_n$, such that $n > 2$ and $a_1 + a_2 + \cdots + a_n = 1$, prove that the inequality

$$\frac{a_2 \cdot a_3 \cdot \cdots \cdot a_n}{a_1 + n - 2} + \frac{a_1 \cdot a_3 \cdot \cdots \cdot a_n}{a_2 + n - 2} + \cdots + \frac{a_1 \cdot a_2 \cdot \cdots \cdot a_{n-1}}{a_n + n - 2} \leq \frac{1}{(n-1)^2}$$

does holds.

$\boxed{3}$ Let $A' \in (BC)$, $B' \in (CA), C' \in (AB)$ be the points of tangency of the excribed circles of triangle $\triangle ABC$ with the sides of $\triangle ABC$. Let R' be the circumradius of triangle $\triangle A'B'C'$. Show that

$$R' = \frac{1}{2r}\sqrt{2R\left(2R - h_a\right)\left(2R - h_b\right)\left(2R - h_c\right)}$$

where as usual, R is the circumradius of $\triangle ABC$, r is the inradius of $\triangle ABC$, and h_a, h_b, h_c are the lengths of altitudes of $\triangle ABC$.

$\boxed{4}$ Let p be a positive integer, $p > 1$. Find the number of $m \times n$ matrices with entries in the set $\{1, 2, \ldots, p\}$ and such that the sum of elements on each row and each column is not divisible by p.