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REMARKS ON A GENERALIZATION OF EULER'S INEQUALITY

Murray S. Klamkin

In a recent note in this journal [1995: 1], F. Ardila extended Euler's classical triangle inequality[1] by replacing the circumcircle with an arbitrary ellipse through the vertices and showing that *if a is the major semi-axis of an ellipse containing the vertices of a triangle whose inradius is r, then $a \geq 2r$.* Here we give a less computational proof and other results.

Starting with a triangle ABC inscribed in an ellipse with major and minor semi-axes a, b, we orthogonally project the ellipse and the triangle into a circle of radius a and inscribed triangle $A'B'C'$. The inscribed circle projects into an ellipse of semi-axes r and ar/b inscribed in $A'B'C'$. If r' is the inradius of $A'B'C'$, then $a \geq 2r'$ by the Euler inequality. It now suffices to show that $r' \geq r$; we can assume $a > b$.

We replace the ellipse of semi-axes r and ar/b by a concentric circle of radius r. This circle, in general, lies inside $A'B'C'$ (it can however, touch one side). Hence $r' \geq r$, with equality only if $a = b$. There is equality in $a \geq 2r$ only if $a = b$ and ABC is equilateral.

In a similar fashion one can show that the semi-major axis of any ellipse inscribed in a triangle is not less than the inradius of the triangle.

Another generalization of $R \geq 2r$ is that $R \geq nr$ where R and r here are the circumradius and inradius of an n-dimensional simplex [2]. One can also extend this result as above for a n-dimensional simplex inscribed in a n-dimensional ellipsoid in a similar way. The inequality here is that the major semi-axis of the ellipsoid is not less than n times the inradius of the simplex.

A more difficult and open problem (even for $n = 2$) is to find the maximum inradius r of n-dimensional simplexes inscribed in an n-dimensional ellipsoid of given semi-axes.

References:

[1] J. S. Mackay, Historical notes on a geometrical theorem and its developments, *Proc. Edinburgh Math. Soc.*, 5(1887), pp. 62–63.

[2] M. S. Klamkin and G. A. Tsintsifas, The circumradius–inradius inequality for a simplex, *Math. Mag.*, 52(1979), pp. 20–22.

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[1] Although this result is usually attributed to Euler (1767), it was already known to Chapple (1746) [1].

THE SKOLIAD CORNER

No. 9

R. E. WOODROW

As a problem set this issue we feature the IBM U.K. Junior Mathematical Olympiad which is organized by the U.K. Mathematics Foundation. The contest was written May 26, 1994, and is restricted to students in school year 8 or below. Two hours are allowed for both parts. Students are also told to leave exact answers, involving π, square roots, etc. Many thanks go to Tony Gardiner of The University of Birmingham for sending the contest to me.

IBM U.K. JUNIOR MATHEMATICAL OLYMPIAD 1994

Section A

A1. What is the angle between the hands of a clock at 9:30?

A2. For how many two digit numbers is the sum of the digits a multiple of 6?

A3. While watching their flocks by night the shepherds managed to lose two thirds of their sheep. They found four fifths of these again in the morning. What fraction of their original flock did they then have left?

A4. Two cubical dice each have faces numbered 0, 1, 2, 3, 4, 5. When both dice are thrown what is the probability that the total score is a prime number?

A5. $ABCD$ is a rectangle with AB twice as long as BC. E is a point such that ABE is an equilateral triangle which overlaps the rectangle $ABCD$. M is the midpoint of the side BE. How big is the angle CMB?

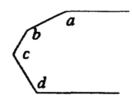
A6. A normal duck has two legs. A lame duck has one leg. A sitting duck has no legs. Ninety-nine ducks have a total of one hundred legs. Given that there are half as many sitting ducks as normal ducks and lame ducks put together, find the number of lame ducks.

A7. How many different solutions are there to this division? (Different letters stand for different digits, and no number begins with a zero.)

$$\begin{array}{r} O\ K \\ U\ K\,\overline{)\,J\ M\ O} \end{array}$$

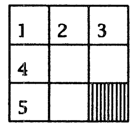
A8. Moses is twice as old as Methuselah was when Methuselah was one third as old as Moses will be when Moses is as old as Methuselah is now. If the difference in their ages is 666, how old is Methuselah?

A9. What is the sum of the four angles a, b, c, d in the diagram?

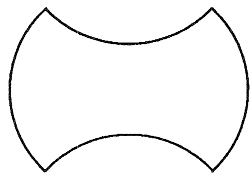


A10. A crossnumber is like a crossword except that the answers are numbers, with one digit in each square. What is the sum of all eight digits in this crossnumber?

Across	Down
1. Square of a prime	1. Square of another prime
4. Prime	2. Square
5. Square	3. Prime



Section B

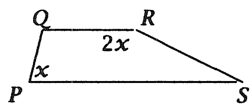


B1. A circle of radius 1 is cut into four equal arcs, which are then arranged to make the shape shown here. What is its area? Explain!

B2. (a) Find three prime numbers such that the sum of all three is also a prime.

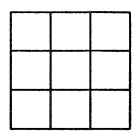
(b) Find three positive integers such that the sum of any two is a perfect square. Can you find other sets of three integers with the same property?

B3. In the trapezium $PQRS$ angle QRS is twice angle QPS, QR has length a and RS has length b. What is the length of PS? Explain!



B4. A sequence of fractions obeys the following rule: given any two successive terms a, b of the sequence, the next term is obtained by dividing their product $a \cdot b$ by their sum $a + b$. If the first two terms are $1/2$ and $1/3$, write down the next three terms. What is the tenth term? Explain clearly what is going on, and how you can be sure.

B5. In this grid, small squares are called *adjacent* if they are next to each other either horizontally or vertically. When you place the digits 1-9 in the nine squares, how many adjacent *pairs* of numbers are there?



You have to arrange the digits 1-9 in the grid so that the total T of all the differences between adjacent pairs is as large as possible. Show how this can be done. Explain clearly why no other arrangement could give a larger total T than yours.

B6. In the figure described in problem A5 what fraction of the rectangle is covered by the equilateral triangle ABE?

*   *   *   *   *

Last issue we gave the problems of the 1992 Saskatchewan Senior Mathematics Contest. Here are the solutions.

1. Let $f(x) = 3x^2 - \frac{1}{x+1}$.

(a) Write expressions for the following:

(i) $f(x-1)$ (ii) $f\left(\frac{1}{x}\right)$ (iii) $f(0)$ (iv) $f(2x)+1$

There is no need to simplify the expressions.

(b) If $g(x)$ is a function and if z is a real number such that $g(z) = 0$, we shall call z a *zero* of $g(x)$.

(i) Find a zero of the function $f(2x) + 1$ in part (a) above.

(ii) How many zeros does this function have?

Solution. $f(x) = 3x^2 - \dfrac{1}{x+1}$.

(a) (i) $f(x-1) = 3(x-1)^2 - \dfrac{1}{(x-1)+1}$ $\left(= 3x^2 - 6x + 3 - \dfrac{1}{x}\right)$.

(ii) $f\left(\dfrac{1}{x}\right) = \dfrac{3}{x^2} - \dfrac{1}{\frac{1}{x}+1}$ $\left(= \dfrac{3}{x^2} - \dfrac{x}{1+x}\right)$.

(iii) $f(0) = -1$.

(iv) $f(2x) + 1 = 3(2x)^2 - \dfrac{1}{2x+1} + 1$ $\left(= 12x^2 - \dfrac{1}{2x+1} + 1\right)$.

(b) (i) $f(2x)+1$ has the obvious zero $x = 0$ since $f(0) + 1 = -1 + 1 = 0$ by (iii).

(ii) Setting $12x^2 - \dfrac{1}{2x+1} + 1 = 0$ gives

$$12x^2(2x+1) - 1 + (2x+1) = 0$$

and

$$2x(6x(2x+1)+1) = 0$$

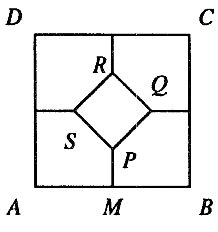
so

$$2x(12x^2 + 6x + 1) = 0.$$

Now the discriminant of $12x^2 + 6x + 1 = 0$ is $36 - 4 \times 12 \times 1 < 0$ so the only solution is $x = 0$.

2. A linoleum company currently produces a product in which the pattern is a repetition of the figure, opposite.

ABCD and *PQRS* are concentric squares. The diagonals of *PQRS* are parallel to the sides of *ABCD*. If the length of *AB* is one unit and if the length of *PQ* is 1/2 unit, compute the length of *PM* where *M* is the midpoint of *AB*.



Solution. Let O be the centre of the two squares. Then $OM = \frac{1}{2}$ and $OP = \frac{1}{2}RP = \frac{1}{2}\sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \frac{\sqrt{2}}{4}$. So

$$PM = OM - OP = \frac{1}{2} - \frac{\sqrt{2}}{4} = \frac{2-\sqrt{2}}{4}.$$

3. Let $\log_{15} 5 = a$. Write $\log_{15} 9$ as a function of a. (No calculators are necessary.)

Solution. $1 = \log_{15} 15 = \log_{15} 5 + \log_{15} 3$. So $\log_{15} 3 = 1 - a$ and

$$\log_{15} 9 = \log_{15} 3^2 = 2\log_{15} 3 = 2(1-a) = 2 - 2a.$$

4. Find all solutions of the equation

$$\cos 2x \sec x + \sec x + 1 = 0$$

which lie in the interval $[-\pi, 2\pi]$ or $[-180°, 360°]$. (Both intervals are the same.)

Solution.

$$\cos 2x \sec x + \sec x + 1 = 0.$$

Note that $x \neq -\pi/2, \pi/2, 3\pi/2$. Now multiplying by $\cos x$ gives

$$\cos 2x + 1 + \cos x = 0.$$

Since $\cos 2x = 2\cos^2 x - 1$

$$2\cos^2 x - 1 + 1 + \cos x = 0$$

or

$$\cos x(2\cos x + 1) = 0$$

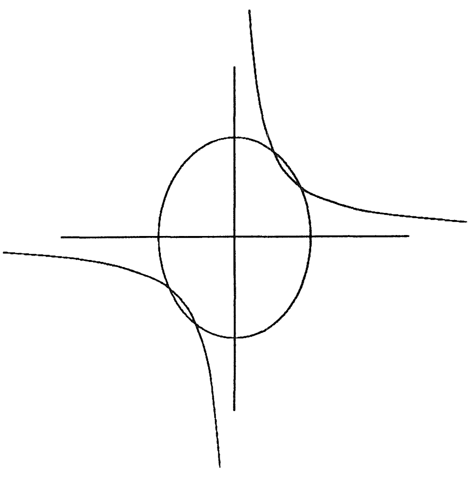
But $\cos x \neq 0$ so $\cos x = -1/2$ and $x = -2\pi/3,\ 2\pi/3,\ 4\pi/3$ in the given domain.

5. Sketch the graphs of

$$2x^2 + y^2 = 3 \quad \text{and} \quad xy = 1$$

using the same axes for both graphs. Determine the coordinates of the points of intersection of the two curves.

Solution.

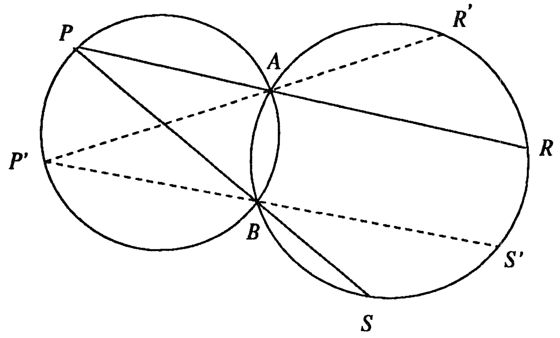


From $y = 1/x$ we get $2x^2 + 1/x^2 = 3$ so $2x^4 - 3x^2 + 1 = 0$ and

$$(2x^2 - 1)(x^2 - 1) = 0.$$

This gives $x^2 - 1 = 0$ or $2x^2 - 1 = 0$ so $x = \pm 1,\ \pm 1/\sqrt{2}$.
The points of intersection are $(1,1)$, $(-1,-1)$, $(1/\sqrt{2}, \sqrt{2})$, $(-1/\sqrt{2}, -\sqrt{2})$.

6. Two circles intersect at A and B. P is any point on an arc AB of one circle. The lines PA, PB intersect the other circle at R and S, as shown below. If P' is any other point on the same arc of the first circle and if R', S' are the points in which the lines $P'A$, $P'B$ intersect the other circle, prove that the arcs RS and $R'S'$ are equal.



Solution. Because opposite angles of a cyclic quadrilateral are supplementary we have that $\angle PBA = \pi - \angle ABS = \angle ARS$. Similarly $\angle PAB = \angle BSR$. Thus ΔPAB and ΔPSR are similar, from which

$$\frac{PA}{PS} = \frac{PB}{PR} = \frac{AB}{RS} .$$

(Notice that this also gives the 'power of the point' result for P, $PA \cdot PR = PB \cdot PS$.) Similarly

$$\frac{P'A}{P'S} = \frac{P'B}{P'R'} = \frac{AB}{R'S'} .$$

Consider now triangles APS and $AP'S'$. We have $\angle APS = \angle APB = \angle AP'B = \angle AP'S'$, because P, P' lie on the same arc of chord AB of the one circle. From the fact that S and S' lie on the same arc of chord AB of the second circle $\angle AS'P' = \angle ASP$. But then ΔAPS and $\Delta AP'S'$ are similar. Thus

$$\frac{PA}{PS} = \frac{P'A}{P'S'}, \quad \text{so} \quad \frac{AB}{RS} = \frac{AB}{R'S'} .$$

Thus $RS = R'S'$ and the arcs are equal.

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That completes this number of the Skoliad Corner. Send me your suggestions, and suitable contest materials.

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THE OLYMPIAD CORNER

No. 169

R. E. WOODROW

All communications about this column should be sent to Professor R. E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this number with some of the problems proposed to the jury but not used at the 35th I.M.O. in Hong Kong in July 1994. My thanks go to Andy Liu, The University of Alberta, member of the Problems Selection Committee and Mathematics Editor of the beautiful booklet of problems and solutions, for forwarding them to me. The booklet does not attribute the country of origin so that I will have to depart from past practice by not listing that information. Because the booklet has been relatively widely circulated, I would solicit your different, nice solutions to the problems.

PROBLEMS PROPOSED BUT NOT USED AT THE 35th I.M.O. IN HONG KONG

Selected Problems

1. M is a subset of $\{1, 2, 3, \ldots, 15\}$ such that the product of any three distinct elements of M is not a square. Determine the maximum number of elements in M.

2. A wobbly number is a positive integer whose digits in base 10 are alternately non-zero and zero, the units digit being non-zero. Determine all positive integers which do not divide any wobbly number.

3. A semicircle Γ is drawn on one side of a straight line ℓ. C and D are points on Γ. The tangents to Γ at C and D meet ℓ at B and A respectively, with the center of the semicircle between them. Let E be the point of intersection of AC and BD, and F be the point on ℓ such that EF is perpendicular to ℓ. Prove that EF bisects $\angle CFD$.

4. A circle ω is tangent to two parallel lines ℓ_1 and ℓ_2. A second circle ω_1 is tangent to ℓ_1 at A and to ω externally at C. A third circle ω_2 is tangent to ℓ_2 at B, to ω externally at D and to ω_1 externally at E. AD intersects BC at Q. Prove that Q is the circumcenter of triangle CDE.

5. A line ℓ does not meet a circle ω with center O. E is the point on ℓ such that OE is perpendicular to ℓ. M is any point on ℓ other than E. The tangents from M to ω touch it at A and B. C is the point on MA such that EC is perpendicular to MA. D is the point on MB such that ED is perpendicular to MB. The line CD cuts OE at F. Prove that the location of F is independent of that of M.

6. On a 5×5 board, two players alternately mark numbers on empty cells. The first player always marks 1's, the second 0's. One number is marked per turn, until the board is filled. For each of the nine 3×3 squares the sum of the nine numbers on its cells is computed. Denote by A the maximum of these sums. How large can the first player make A, regardless of the responses of the second player?

7. Peter has three accounts in a bank, each with an integral number of dollars. He is only allowed to transfer money from one account to another so that the amount of money in the latter is doubled.

(a) Prove that Peter can always transfer his money into two accounts.
(b) Can Peter always transfer his money into one account?

8. At a round table are 1994 girls, playing a game with a deck of n cards. Initially, one girl holds all the cards. In each turn, if at least one girl holds at least two cards, one of these girls must pass a card to each of her two neighbours. The game ends when and only when each girl is holding at most one card.

(a) Prove that if $n \geq 1994$, then the game cannot end.
(b) Prove that if $n < 1994$, then the game must end.

9. Let $a_0 = 1994$ and $a_{n+1} = a_n^2/(a_n + 1)$ for any non-negative integer n. Prove that $1994 - n$ is the greatest integer less than or equal to a_n, $0 \leq n \leq 998$.

10. Let $f(x) = \frac{x^2+1}{2x}$ for $x \neq 0$. Define $f^{(0)}(x) = x$ and $f^{(n)}(x) = f(f^{(n-1)}(x))$ for all positive integers n and $x \neq 0$. Prove that for all non-negative integers n and $x \neq -1$, 0 or 1,

$$\frac{f^{(n)}(x)}{f^{(n+1)}(x)} = 1 + \frac{1}{f((\frac{x+1}{x-1})^{2n})}.$$

* * * * *

We now turn to readers' solutions to problems of the 1994 A.H.S.M.C. Part II [1994: 65–66].

1. Find all polynomials $P(x)$ that satisfy the equation

$$P(x^2) + 2x^2 + 10x = 2xP(x + 1) + 3.$$

Solutions by Seung-Jin Bang, Seoul, Korea; by Tim Cross, Wolverley High School, Kidderminster, U.K.; and by Beatriz Margolis, Paris, France. We give Margolis' solution.

If $P(x)$ is a polynomial of degree n, then $P(x^2)$ has degree $2n$, while $2xP(x+1)$ has degree $n+1$. Thus $n = 1$, i.e. $P(x)$ is linear, say $P(x) = a+bx$.

From $x = 0$, $P(0) = 3 = a$, and from $x = -1$, $P(1) + 2 - 10 = -2P(0) + 3 = -3$.

So $P(1) = 5 = a + b$. Thus $b = 2$ and $p(x) = 2x + 3$.

3. (a) Show that there is a positive integer n so that the interval

$$\left(\left(n + \frac{1}{1994}\right)^2, \left(n + \frac{1}{1993}\right)^2\right)$$

contains an integer N.

(b) Find the smallest integer N which is contained in such an interval for some n.

Solutions by Tim Cross, Wolverley High School, Kidderminster, U.K.; and by Panos E. Tsaoussoglou, Athens, Greece. We give Cross's answer.

Clearly, a solution to (b) automatically establishes (a).

Since

$$\left(n + \frac{1}{1994}\right)^2 = n^2 + \frac{2n}{1994} + \frac{1}{(1994)^2}$$

and

$$\left(n + \frac{1}{1993}\right)^2 = n^2 + \frac{2n}{1993} + \frac{1}{(1993)^2},$$

for minimality we want

$$\frac{2n}{1994} + \frac{1}{1994^2} < k < \frac{2n}{1993} + \frac{1}{1993^2}$$

for some positive integer k, with k minimal. When $k = 1$ we have

$$\left(1 - \frac{1}{1993^2}\right) 996\frac{1}{2} < n < \left(1 - \frac{1}{1994^2}\right) 997$$

i.e. $996\frac{1}{2} - \delta < n < 997 - \varepsilon$ for some small positive reals δ and ε. This gives $n < 997$ and $n > 996$, for which no such integer exists.

However, for $k = 2$, we get

$$996\frac{1}{2} - \delta + 996\frac{1}{2} < n < 997 - \varepsilon + 997.$$

So

$$1993 - \delta < n < 1994 - \varepsilon$$

so that $n = 1993$ and $N_{\text{min}} = (1993)^2 + 2$.

4. *ABCDE* is a convex pentagon in the plane. Through each vertex draw a straight line which cuts the pentagon into two parts of the same area. Prove that for some vertex, the line through it must intersect the "opposite side" of the pentagon. (Here the opposite side to vertex A is the side CD, the opposite side to B is DE, and so on.)

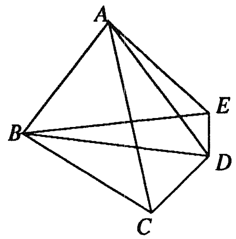
Solution by Toshio Seimiya, Kawasaki, Japan.

We denote the area of the polygon $A_1A_2\ldots A_n$ by $[A_1A_2\ldots A_n]$, and we put $[ABCDE] = S$. We assume that the straight line through A which bisects the area of the pentagon does not intersect the opposite side CD.

Then either $[ABC] = \frac{1}{2}S$ or $[ADE] > \frac{1}{2}S$.

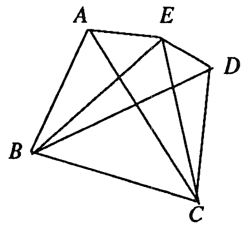
We may assume without loss of generality that $[ABC] > \frac{1}{2}S$. (1)

Now we assume that the straight line through B which bisects the area of the pentagon does not intersect the opposite side DE. Then either $[BAE] > \frac{1}{2}S$ or $[BCD] > \frac{1}{2}S$.



Case 1. $[BAE] > \frac{1}{2}S$. (2)

From (1) we get that $[ADE] < [ACDE] < \frac{1}{2}S$ and from (2) we get $[BCD] < [BCDE] < \frac{1}{2}S$. Therefore the line through D which bisects the area of the pentagon must intersect the opposite side AB.



Case 2. $[BCD] > \frac{1}{2}S$. (3)

From (1) we get $[ECD] < [ACDE] < \frac{1}{2}S$ and $[EAB] < [EABD] < \frac{1}{2}S$. Therefore the line through E which bisects the area of the pentagon must intersect the opposite side BC.

5. Let a, b, c be real numbers. Their pairwise sums $a+b$, $b+c$ and $c+a$ are written on three round cards and their pairwise products ab, bc and ca are written on three square cards. We call (a,b,c) a *tadpole* if we can form three pairs of cards, each consisting of one round card and one square card with the same number on both. An example of a tadpole is $(0,0,0)$.

(a) Find all possible tadpoles of the form (a,a,a).

(b) Prove that there is a tadpole that is not of the form (a,a,a). (You do not have to find the actual values of a, b, and c.)

Solution by Tim Cross, Wolverley High School, Kidderminster, U.K.

(a) If $b = c = a$, there are 3 pairs of cards $2a$ and $\boxed{a^2}$. Now, $a^2 = 2a$ iff $a = 0$ or $a = 2$. Thus there are exactly two tadpoles of the form (a,a,a), namely $(0,0,0)$ and $(2,2,2)$.

(b) *Case I.* Two of a, b, c are equal: say a, a, b $(b \neq a)$. This leads to pairings

(i) $2a$ $\boxed{a^2}$, $a+b$ $\boxed{ab}$, and $a+b$ $\boxed{ab}$.

The first pair of which implies $a = 0$ or 2 $a = 0 \to ab = 0 \Rightarrow a+b = 0 \Rightarrow b = 0 = a, \#$ $a = 2 \Rightarrow b+2 = 2b \Rightarrow b = 2 = a, \#$ or

(ii) $2a$ $\boxed{ab}$, $a+b$ $\boxed{a^2}$, and $a+b$ $\boxed{ab}$. Here, the last two pairs give $a^2 = ab$ implying $a = 0$ or $a = b$.

Now $a = 0 \Rightarrow b = 0 = a, \#$.

Case II. a, b, c are all distinct. [Throughout, the assumption is made implicitly that a, b, c do not take values which would render any of the others unbounded].

(i) $a+b\ \boxed{ab}$, $\quad b+c\ \boxed{bc}$, $\quad$ and $c+a\ \boxed{ca}$.

$$ab = a + b \Rightarrow (a-1)(b-1) = 1 \quad \text{and} \quad b = 1 + \frac{1}{a-1}.$$

Similarly

$$c = \frac{1}{1+\frac{1}{b-1}} \quad \text{and} \quad a = 1 + \frac{1}{c-1} = 1 + \frac{1}{1+\frac{1}{b-1}} = b\#.$$

(ii) $a+b\ \boxed{bc}$, $b+c\ \boxed{ab}$, and $c+a\ \boxed{ac}$.

Now $a+b = bc \Rightarrow c = 1 + 1/b$. Then $b + 1 + a/b = ab \Rightarrow b + 1 = a(b - 1/b) \Rightarrow a = b/(b-1)$ or $b = -1$.

$$a = \frac{b}{b-1} \Rightarrow c = 1 + \frac{1}{b-1} = \frac{b}{b-1} = a\#.$$

$$b = -1 \Rightarrow a^2 - a + 1 = 0 \text{ and } a \text{ is not real } \#.$$

(iii) $a+b\ \boxed{ca}$, $\quad c+a\ \boxed{ab}$, $\quad$ and $c+a\ \boxed{bc}$.

Now

$$\left.\begin{array}{l} a+b = ca \Rightarrow b = a(c-1) \\ b+c = ab \Rightarrow b = \frac{c}{a-1} \end{array}\right\} \Rightarrow a(a-1) = \frac{c}{c-1}.$$

Also

$$c + a = ac(c-1) \Rightarrow a = \frac{c}{c^2-c-1}.$$

so that

$$\left(\frac{c}{c^2-c-1}\right)\left(\frac{2c-c^2+1}{c^2-c-1}\right) = \frac{c}{c-1}.$$

This gives

$$c^4 - c^3 - 4c^2 + 3c + 2 = 0 \Rightarrow (c-2)(c^3 + c^2 - 2c - 1) = 0.$$

Now $c = 2$ gives $a^2 - a - 2 = 0 \Rightarrow a = 2$, a contradiction, or $a = -1 \Rightarrow b = -1$, a contradiction.

Thus $c^3 + c^2 - 2c + 1 = 0$ is required to have a real solution, which it must do since complex roots occur in conjugate pairs. (Note only (iii) is necessary in order to establish the existence of solutions, provided it is now checked that $c - 1 \neq 0$ and $c^2 - c - 1 \neq 0$. The other cases are included for completeness.)

In fact, using Cardan's method and complex numbers, the three real roots of the cubic $c^3 + c^2 - 2c - 1 = 0$ can be found. They are

$$c_1 = \frac{2\sqrt{7}}{3}\cos\frac{\theta}{3} - \frac{1}{3},$$

$$c_2 = \frac{2\sqrt{7}}{3}\cos\left(\frac{2\pi-\theta}{3}\right) - \frac{1}{3}$$

and $$c_3 = \frac{2\sqrt{7}}{3}\cos\left(\frac{2\pi+\theta}{3}\right) - \frac{1}{3},$$

where $\theta = \cos^{-1}(\frac{1}{\sqrt{28}})$.

Unless I have overlooked something, there are then 6 tadpoles not of the form (a, a, a) being the six permutations of (c_1, c_2, c_3). (N.B. $c_1 \approx 1.2469796$, $c_2 \approx -0.4450419$, $c_3 \approx -1.8019377$).

* * *

That completes our file of solutions for the Alberta contest and the space this month. Send me your nice solutions and your Olympiad sets.

* * * * *

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

Lion Hunting and Other Mathematical Pursuits: A Collection of Mathematics, Verse and Stories by Ralph P. Boas Jr., edited by G. L. Alexanderson and D. H. Mugler. Published by the Mathematical Association of America, Washington, 1994. ISBN 0-88385-323-X, paperbound, 320+ pages, US$35.00. *Reviewed by Andy Liu.*

This book offers a panoramic view of the many facets and diverse talents of the distinguished mathematician, the late Ralph P. Boas Jr. It begins with his own self-profile of a "quasi-mathematician", with many photographs from his own album. This is followed by three reminiscences containing more photographs, including one by his son Harold. Then comes an obituary by P. J. Davis.

The main body of the book consists of Boas' writings, some serious and others whimsical. These are organized into sixteen sections. Section 1 recounts a famous episode in the folklore of mathematics. Under the pseudonym of H. Pétard, Boas and F. Smithies wrote an article on various mathematical methods for catching lions. Responding to this inspirational piece, many others contributed other ways of doing so, and a sample of their papers are also included here.

Section 12 features another piece of mathematical folklore, a "feud" between Boas and Bourbaki in which various parties accused one another of either non-existence or non-uniqueness. Section 15 unearths various references to mathematics and mathematicians in literature. Section 16 contains three samples of Boas' mathematical reviews and an article titled "Are mathematicians unnecessary?"

Sections 3, 5, 7, 9 and 11 are all titled "Recollections and Verse" and contain, naturally, amusing anecdotes and mathematical poems. Here is one example of each.

*A friend of mine wanted to find out what was in a paper in Danish. He knew no Danish but tried to decipher the paper on the principle that short words can be neglected. It was hard work, because he neglected the word "***ikke***", which means "not".* (Page 152)

Envoy

Instructor, ponder this codicil,
An awkward truth that you can't gainsay:
What you're teaching now, with so much good will,
Is tomorrow's math of yesterday. (Page 203)

The remaining sections all have something to do with the teaching of mathematics in general, and of calculus in particular. Section 13 is titled "The Teaching of Mathematics", Section 14 "Polynomials", Section 10 "Inverse Functions", Section 6 "Indeterminate Forms", Section 4 "The Mean Value Theorem", Section 2 "Infinite Series" and Section 8 "Complex Variables". The reviewer, who has been teaching mathematics for fifteen years but calculus for only one, finds these sections very helpful.

The book concludes with three more reminiscences by Boas' former students, including one of the editors of this book, and a comprehensive bibliography of Boas' 264 papers and 7 translations. All in all, one comes away with a deep fascination of this wonderful mathematician. This book offers excellent reading.

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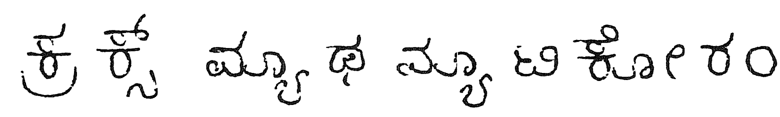
$\frac{1}{2}$(KANNADA + CANADA) = ?

Regular *Crux* contributor K. R. S. Sastry writes that while in his previous home of Ethiopia, he noticed that some Ethiopians had trouble pronouncing the word CANADA. Also, they would ask him the name of his language, and when he said KANNADA, they would try to pronounce it, but would reduce the stress on the N. Finally they found a sort of KANNADA-CANADA "mean" and used it to pronounce both!

Sastry also writes CRUX MATHEMATICORUM in Kannada characters:



The first two characters look to the editor like faces, the first smiling, the second grimacing — perhaps two readers, one having just solved a *Crux* problem, the other not yet!

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **June 1, 1996**, *although solutions received after that date will also be considered until the time when a solution is published.*

2081. *Proposed by K. R. S. Sastry, Dodballapur, India.*

In base ten, if we write down the first double-digit integer (10) followed by the last two single-digit integers (8 and 9) we form a four-digit number (1089) which is a *perfect square* (33^2). What other bases exhibit this same property?

2082. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with $\angle A > 90°$, and AD, BE and CF are its altitudes (with D on BC, etc.). Let E' and F' be the feet of the perpendiculars from E and F to BC. Suppose that $2E'F' = 2AD + BC$. Find $\angle A$.

2083. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

The numerical identity $\cos^2 14° - \cos 7° \cos 21° = \sin^2 7°$ is a special case of the more general identity $\cos^2 2x - \cos x \cos 3x = \sin^2 x$. In a similar manner, find a generalization for each of the following numerical identities:

(a) $\tan 55° - \tan 35° = 2 \tan 20°$;

(b) $\tan 70° = \tan 20° + 2\tan 40° + 4 \tan 10°$;

(c)* $\csc 10° - 4 \sin 70° = 2$.

2084. *Proposed by Murray S. Klamkin, University of Alberta.*

Prove that

$$\cos\frac{B}{2}\cos\frac{C}{2} + \cos\frac{C}{2}\cos\frac{A}{2} + \cos\frac{A}{2}\cos\frac{B}{2} \geq 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

where A, B, C are the angles of a triangle.

2085. *Proposed by Iliya Bluskov, student, Simon Fraser University, Burnaby, B. C., and Gary MacGillivray, University of Victoria, B. C.*

Find a closed-form expression for the n by n determinant

$$\begin{vmatrix} n & -1 & -1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 \\ -1 & 0 & -1 & 3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 3 & -1 & 0 \\ -1 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & -1 & 3 \end{vmatrix} .$$

2086. *Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

If the side AC of the spherical triangle ABC has length $120°$ (i.e., it subtends an angle of $120°$ at the centre), prove that the median from B (i.e. the arc of the great circle from B to the midpoint of AC) is bisected by the other two medians.

2087. *Proposed by Toby Gee, student, The John of Gaunt School, Trowbridge, England.*

Find all polynomials f such that $f(p)$ is a power of two for every prime p.

2088. *Proposed by Šefket Arslanagić, Berlin, Germany.*

Determine all real numbers x satisfying the equation

$$\left\lfloor \frac{2x+1}{3} \right\rfloor + \left\lfloor \frac{4x+5}{6} \right\rfloor = \frac{3x-1}{2},$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.

2089. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

Let $ABCD$ be a trapezoid with $AB \parallel CD$, and let X be a point on segment AB. Put $P = CB \cap AD$, $Y = CD \cap PX$, $R = AY \cap BD$ and $T = PR \cap AB$. Prove that

$$\frac{1}{AT} = \frac{1}{AX} + \frac{1}{AB}.$$

2090. *Proposed by Peter Ivády, Budapest, Hungary.*

For $0 < x < \pi/2$ prove that

$$\left(\frac{\sin x}{x}\right)^2 < \frac{\pi^2 - x^2}{\pi^2 + x^2}.$$

*   *   *   *   *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1894. [1993: 294; 1994: 261] *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.* (Dedicated in memoriam to K. V. R. Sastry.)

Evaluate
$$\sum_{k=0}^{n} \frac{\binom{n+1}{k}\binom{n}{k}}{\binom{2n}{2k}}$$
where n is a nonnegative integer.

III. *Solution by Andy Liu, University of Alberta.*

As pointed out in Solution I [1994: 261], the problem is equivalent to proving
$$\sum_{k=0}^{n} \frac{1}{n-k+1}\binom{2(n-k)}{n-k}\binom{2k}{k} = \binom{2n+1}{n}.$$
Consider the grid of lattice points (x, y), $-1 \le x, y \le n$, and all paths from $(-1,-1)$ to (n,n) consisting of a combination of $n+1$ (unit) steps east and $n+1$ steps north, the first one being eastward. Clearly, the total number of such paths is $\binom{2n+1}{n}$. On the other hand, every path must return to the diagonal $y = x$, if only at the destination (n,n). Let $(n-k, n-k)$, $0 \le k \le n$, be the first point on this diagonal which lies on the path. The number of ways of getting from $(-1,-1)$ to $(n-k, n-k)$ without landing on $y = x$ before is given by the Catalan number
$$\frac{1}{n-k+1}\binom{2(n-k)}{n-k}.$$
[For example, see Chapter 20 of Martin Gardner's book *Time Travel and Other Mathematical Bewilderments.—Ed.*] Since no further restrictions apply, the number of ways of getting from $(n-k, n-k)$ to (n,n) is $\binom{2k}{k}$. The desired result follows.

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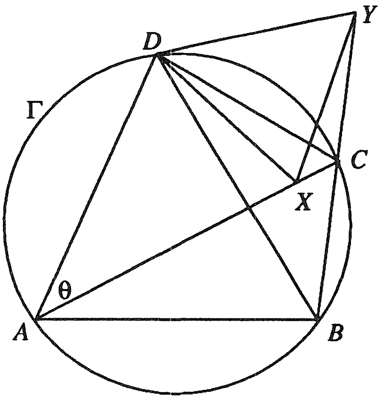
1993. [1994: 285] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

$ABCD$ is a convex quadrilateral inscribed in a circle Γ. Assume that A, B and Γ are fixed and C, D are variable, so that the length of the segment CD is constant. X, Y are the points on the rays AC and BC respectively, such that $AX = AD$ and $BY = BD$. Prove that the distance between X and Y remains constant. [This problem was inspired by *Crux* 1902 [1994: 287].]

I. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Because A, B, C and D are concyclic, $\angle DAC = \angle DBC$, so $\angle DAX = \angle DBY$. As $AD = AX$ and $BD = BY$, $\triangle ADX$ and $\triangle BDY$ are similar. Consequently, $\triangle DAB$ and $\triangle DXY$ are similar [since $AD/DX = BD/DY$ and $\angle ADB = \angle ADX - \angle BDX = \angle BDY - \angle BDX = \angle XDY$]. Therefore,



$$\frac{DA}{DX} = \frac{AB}{XY}. \qquad (1)$$

Because CD is constant, $\angle DAC$ is constant. We put $\angle DAC = \theta$. Since $AD = AX$ and $\angle DAX = \theta$, we get

$$\frac{DX}{2AD} = \sin\frac{\theta}{2}. \qquad (2)$$

From (1) and (2), we have $XY = 2AB\sin(\theta/2)$ which is constant.

Editor's note by Cathy Baker. The term "ray" is significant here. X and C are on the same side of A; Y and C, the same side of B. An analogous result holds if X and Y are selected from the rays CA and CB instead. Namely, (1) still holds, but $\angle DAX = \pi - \theta$ and so

$$XY = 2AB\sin\frac{\pi - \theta}{2} = 2AB\cos\frac{\theta}{2},$$

and the result still holds. One of the solvers (Cyrus Hsia) did the problem this way.

II. *Solution by Jordi Dou, Barcelona, Spain.*

Let G_A^{ω} be the rotation with centre A through $\angle DAC = \angle DBC = \omega$, so $G_A^{\omega}(D) = X$. Define G_B^{ω} similarly and $G_B^{\omega}(D) = Y$. Then $(G_A^{-\omega} * G_B^{\omega})(X) = Y$. Since $\omega - \omega = 0$, $G_A^{-\omega} * G_B^{\omega}$ is a translation [see, for example, Roger C. Lyndon, *Groups and Geometry*, Cambridge University Press (1989), p. 26]. Let $A' = G_B^{\omega}(A)$ [hence $(G_A^{-\omega} * G_B^{\omega})(A) = A'$].

Now $XY = AA' = 2AB\sin(\omega/2)$ [since a translation moves points a fixed distance], and so XY is constant [since ω, and hence the translation, depends only on the length CD].

Editor's note by Cathy Baker. Since a translation moves points in a fixed direction, the direction of XY is also fixed, and thus the **vector** $\overrightarrow{XY}$ is constant. This stronger result was also noted by solvers Bradley and Smeenk, and by the proposer.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.;

CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; P. PENNING, Delft, The Netherlands; CRISTÓBAL SÁNCHEZ-RUBIO, I. B. Penyagolosa, Castellón, Spain; D. J. SMEENK, Zaltbommel, The Netherlands (2 solutions); and the proposer.

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1994. [1994: 285] *Proposed by N. Kildonan, Winnipeg, Manitoba.*

This problem marks the one and only time that the number of a *Crux* problem is equal to the year in which it is published. In particular this is the *first* time that

a problem number is an integer multiple of its publication year. (1)

Assuming that *Crux* continues indefinitely to publish 10 problems per issue and 10 issues per year, will there be a *last* time (1) happens? If so, when will this occur?

Solution by Christopher J. Bradley, Clifton College, Bristol, U. K.

Problem $p + 100k$ in year $1995 + k$, where

$$p + 100k = m(1995 + k), \qquad 2001 \le p \le 2100$$

(m being the multiple required), provides the solution for all possibilities. There is a solution with $m = 99$ [and clearly no larger m is possible — *Ed.*], and $p = 2001$ gives the largest possible k, namely

$$k = 99 \cdot 1995 - p = 99 \cdot 1995 - 2001 = 195504.$$

Hence in the remote future, problem $2001 + 100(195504) =$ **19552401** in year $1995 + 195504 =$ **197499** will be the last time it will occur, the multiple being 99.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; TIM CROSS, Wolverley High School, Kidderminster, U. K.; KEITH EKBLAW, Walla Walla, Washington; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; KATHLEEN E. LEWIS, SUNY at Oswego; J. A. MCCALLUM, Medicine Hat, Alberta; MIKE PARMENTER, Memorial University of Newfoundland, St. John's; P. PENNING, Delft, The Netherlands; PAUL YIU, Florida Atlantic University, Boca Raton; and the proposer. Four incorrect solutions were received.

Many solvers also pointed out that the next time a problem number will be a multiple of its publication year is problem 4030 *in the March* 2015 *issue.*

Yiu finds exactly **eight** *occasions in which a* Crux *problem number* **divided into** *its publication year, the last of which was of course the current problem. Readers may enjoy finding them too!*

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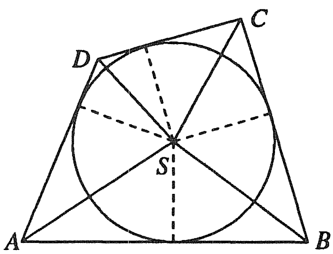
1995. [1994: 285] *Proposed by Jerzy Bednarczuk, Warszawa, Poland.*

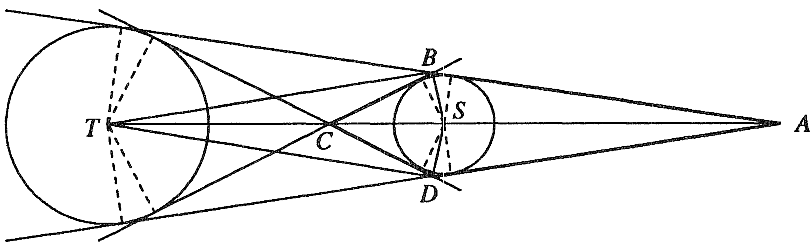
Given two pyramids $SABCD$ and $TABCD$ ($S \neq T$) with a common base $ABCD$. The altitudes of their eight triangular faces, taken from the vertices S and T, are all equal to 1. Prove or disprove that the line ST is perpendicular to the plane containing points A, B, C, D.

Solution by Richard I. Hess, Rancho Palos Verdes, California.



A view of the pyramid looking down from S perpendicular to the plane $ABCD$ will look like the figure at right. S may be raised above the plane to give unit altitudes if the inscribed circle has radius < 1.

Now consider two such circles of different sizes as shown below. Each has a radius < 1. They define the four lines drawn to produce $ABCD$ as shown. Raise S and T out of the plane appropriately so that all altitudes are unit. Clearly the line ST is **not** perpendicular to the plane defined by $ABCD$.



Also solved (the same way) by TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer. Seven incorrect solutions were sent in!

The proposer ("Proof" [1995: 239]*) was inspired to create this problem (and fool several unsuspecting* Crux *readers!) by seeing the following problem in some Polish school books: Given a pyramid* $SA_1 \ldots A_n$ *(*$n \geq 4$*). The altitudes of all its triangular faces from S are equal. Prove that the base* $A_1 \ldots A_n$ *has an incircle. (Also wrong, as readers may like to prove for themselves!)*

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1996. [1994: 285] *Proposed by Murray S. Klamkin, University of Alberta.*

(a) Find positive integers a_1, a_2, a_3, a_4 so that

$$(1 + a_1\omega)(1 + a_2\omega)(1 + a_3\omega)(1 + a_4\omega)$$

is an integer, where ω is a complex cube root of unity.

(b)* Are there positive integers $a_1, a_2, a_3, a_4, a_5, a_6$ so that

$$(1 + a_1\omega)(1 + a_2\omega)(1 + a_3\omega)(1 + a_4\omega)(1 + a_5\omega)(1 + a_6\omega)$$

is an integer, where ω is a complex *fifth* root of unity?

I. *Solution by Kee-Wai Lau, Hong Kong.*

(a) We have $(1 + 2\omega)^4 = 9$. [*Editor's note.* Since $\omega = -1/2 \pm (\sqrt{3}/2)i$, we have $1 + 2\omega = \pm\sqrt{3}\,i$.]

(b) We shall answer in the negative.

We first show that if A, B, C, D are integers such that

$$A + B\omega + C\omega^2 + D\omega^3 = 0 \tag{1}$$

then $A = B = C = D = 0$. We may assume that $\omega = e^{2\pi i/5} = \cos 72^\circ + i \sin 72^\circ$, as the proofs for other choices of ω are similar. The real part and imaginary part of the left hand side of (1) both vanish and so

$$A + B\cos 72^\circ - (C + D)\cos 36^\circ = 0 \tag{2}$$

and

$$B\sin 72^\circ + (C - D)\sin 36^\circ = 0. \tag{3}$$

From (3) we obtain $2B\cos 36^\circ + (C - D) = 0$. Since $\cos 36^\circ$ is irrational [in fact, $\cos 36^\circ = (\sqrt{5} + 1)/4$ — *Ed.*], $B = 0$ and $C = D$. Thus (2) becomes $A - 2C\cos 36^\circ = 0$ and so $A = C = 0$. [*Editor's note.* Alternatively, this result is true because the polynomial $1 + z + z^2 + z^3 + z^4$ is *irreducible* and thus must be the polynomial of smallest degree (with integer coefficients) that has ω as a root. See the (first) Editor's Note in Solution III below.]

Now suppose, on the contrary, that the answer is in the affirmative. Using the fact that $\omega^4 = -(1 + \omega + \omega^2 + \omega^3)$ and $\omega^5 = 1$ we see that the given product equals $P + Q\omega + R\omega^2 + T\omega^3$, where P, Q, R, T are integers and $R = S_2 - S_4$, $T = S_3 - S_4$ where S_k stands for the kth elementary symmetric function of $a_1, a_2, \ldots, a_6$, $k = 1, 2, 3, 4$. Using the above result we see that $R = T = 0$ and so $S_2 - S_3 = 0$. However, since $a_k \geq 1$ we have

$$\begin{array}{lll}
a_1a_2 \leq a_1a_2a_3, & a_1a_3 \leq a_1a_3a_4, & a_1a_4 \leq a_1a_4a_5, \\
a_1a_5 \leq a_1a_5a_6, & a_1a_6 \leq a_1a_2a_6, & a_2a_3 \leq a_2a_3a_4, \\
a_2a_4 \leq a_2a_4a_5, & a_2a_5 \leq a_2a_5a_6, & a_2a_6 \leq a_2a_3a_6, \\
a_3a_4 \leq a_3a_4a_5, & a_3a_5 \leq a_2a_3a_5, & a_3a_6 \leq a_3a_4a_6, \\
a_4a_5 \leq a_4a_5a_6, & a_4a_6 \leq a_2a_4a_6, & a_5a_6 \leq a_3a_5a_6.
\end{array}$$

[*Editor's note.* If the editor may put on his combinatorial hat for a minute, what's going on here is just a demonstration of the known fact that there is a "complete matching" from the 2-element subsets to the 3-element subsets of the set $\{1, 2, 3, 4, 5, 6\}$, that is, a one-to-one function from the 2-element subsets to the 3-element subsets such that each 2-element subset is mapped to a 3-element subset containing it. In fact there is a complete matching from the k-element subsets of an n-element set to the $(k+1)$-element subsets whenever $k < n/2$. See, for example, Corollary 13.3 on page 688 of [2], or (for the whole story and more) Chapters 1 to 3, especially Exercise 2.4 on page 23, of [1].] Therefore

$$S_2 - S_3 \le -(a_1a_2a_4 + a_1a_2a_5 + a_1a_3a_5 + a_1a_3a_6 + a_1a_4a_6) < 0,$$

a contradiction.

References:

[1] Ian Anderson, *Combinatorics of Finite Sets*, Oxford University Press, 1987.

[2] Ralph P. Grimaldi, *Discrete and Combinatorial Mathematics* (3rd Edition), Addison-Wesley, 1994.

II. *Solution to part* (a) *by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

We show that the only solutions (up to permutations) are

$$(a_1, a_2, a_3, a_4) = (1,2,3,5) \quad \text{and} \quad (2,2,2,2).$$

The given product equals $A + B$, where

$$\begin{aligned} A &= 1 + (a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4)\omega^3 \\ &= 1 + (a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4), \end{aligned}$$

which is an integer, and

$$\begin{aligned} B &= (a_1 + a_2 + a_3 + a_4)\omega \\ &\qquad + (a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4)\omega^2 + a_1a_2a_3a_4\omega^4 \\ &= (a_1 + a_2 + a_3 + a_4)\omega \\ &\qquad + (a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4)\overline{\omega} + a_1a_2a_3a_4\omega. \end{aligned}$$

Now to get rid of any imaginary parts the coefficients of ω and $\overline{\omega}$ must be the same. (And note that since the real parts of ω and $\overline{\omega}$ are $-1/2$, if these coefficients are equal, B, and thus the given product, will be an integer.) So we need

$$\begin{aligned} &a_1 + a_2 + a_3 + a_4 + a_1a_2a_3a_4 \\ &\qquad = a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4. \end{aligned} \tag{4}$$

Without loss of generality assume $1 \le a_1 \le a_2 \le a_3 \le a_4$. If $a_1a_2 \ge 6$, then

$$\begin{aligned} &a_1 + a_2 + a_3 + a_4 + a_1a_2a_3a_4 \\ &\qquad > 6a_3a_4 \ge a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4; \end{aligned}$$

so $a_1a_2 < 6$, and we have two cases: $a_1 = 1$ and $a_1 = 2$.

Case 1: $a_1 = 1$. Then (4) reduces to

$$1 + a_2a_3a_4 = a_2a_3 + a_2a_4 + a_3a_4. \tag{5}$$

If $a_2 \geq 3$ then

$$1 + a_2a_3a_4 > 3a_3a_4 \geq a_2a_3 + a_2a_4 + a_3a_4;$$

so $a_2 = 1$ or 2. If $a_2 = 1$, (5) reduces to $1 = a_3 + a_4$ which has no positive solution. If $a_2 = 2$, (5) reduces to $1 + a_3a_4 = 2(a_3 + a_4)$ which has solution $a_3 = 3$, $a_4 = 5$. [*Editor's note.* We borrow from Janous's solution and write this equation as

$$(a_3 - 2)(a_4 - 2) = 3,$$

which makes it clear that $a_3 = 3$, $a_4 = 5$ is the only possibility.]

Case 2: $a_1 = 2$. Since $a_1a_2 < 6$, $a_2 = 2$ also, and (4) reduces to $a_3a_4 = a_3 + a_4$, which has $a_3 = 2$, $a_4 = 2$ as its only positive solution.

III. *Generalization of part* (b) *by Gerd Baron, Technische Universität Wien, Austria.*

We will consider the following more general situation: *for n, m positive integers and $\omega^m = 1$, determine all sets $\{a_1, \ldots, a_n\}$ of positive integers such that $\prod_{i=1}^n (1 + a_i\omega)$ is an integer.* We will prove:

THEOREM. If $m > 3$ is prime and ω is a complex mth root of unity, then for $m \leq n \leq 2m - 3$ and $a_1, a_2, \ldots, a_n$ nonnegative integers, the product $\prod_{i=1}^n (1 + a_i\omega)$ is an integer only if some $a_i = 0$.

Note that for $m = 5$ and $5 \leq n \leq 7$, we get no solutions and (b) is done.

Proof of the theorem. Assume all $a_i > 0$. As a polynomial in ω,

$$\prod_{i=1}^{n}(1 + a_i\omega) = \sum_{k=0}^{n} p_k\omega^k =: P(\omega)$$

where the p_k are the elementary symmetric functions in a_i, with

$$p_0 = 1, \quad p_1 = \sum a_i, \quad \text{and} \quad p_n = \prod a_i.$$

Let $n \leq 2m - 3$. Reducing $P(\omega)$ modulo the relation $\omega^m = 1$ we get

$$Q(\omega) = \sum_{k=n-m+1}^{m-1} p_k\omega^k + \sum_{k=0}^{n-m} (p_k + p_{k+m})\omega^k.$$

If m is prime, then the polynomial $R(\omega) = \omega^{m-1} + \omega^{m-2} + \cdots + \omega + 1$ is irreducible, and $Q(\omega) = u$ is an integer exactly if the polynomial $Q(\omega) - u$ is an integer multiple of $R(\omega)$; i.e., all coefficients of $Q(\omega)$ but the constant term are equal.

[*Editor's note.* It is known that if m is prime, the polynomial $x^{m-1} + x^{m-2} + \cdots + x + 1$ is irreducible, that is, it cannot be factored into polynomials of smaller degree with integer coefficients. For example, see exercise 19, page 84 of Ed Barbeau's *Polynomials* (Springer-Verlag, 1989), or most any text on abstract algebra. Now since ω is a root of the polynomials $R(\omega)$ and $Q(\omega) - u$, it will be a root of their greatest common divisor, which must be $R(\omega)$ since $R(\omega)$ is irreducible; thus $Q(\omega) - u$ must be a multiple of $R(\omega)$, and a constant multiple since they both have degree $m - 1$.]

If we can show that $p_k \neq p_{k+1}$ for some k with $n - m < k < m - 1$, we are done and there is no solution with all $a_i > 0$. We claim that if m is a prime greater than 3, and $m \leq n \leq 2m - 3$, then there is such a k with $p_{k+1} - p_k > 0$. To prove this we take $a_i = 1 + b_i$ and let $q_0, q_1, \ldots, q_n$ be the elementary symmetric functions in the b_i's. To calculate the p_k's in terms of the q_j's, note that

$$\prod_{i=1}^{n}(x + a_i) = \prod_{i=1}^{n}(x + 1 + b_i)$$

gives

$$\begin{aligned}\sum_{k=0}^{n} p_k x^{n-k} &= \sum_{j=0}^{n} q_j (x+1)^{n-j} \\ &= \sum_{j=0}^{n} q_j \sum_{k=0}^{n-j} \binom{n-j}{k} x^k = \sum_{j=0}^{n} q_j \sum_{k=j}^{n} \binom{n-j}{n-k} x^{n-k},\end{aligned}$$

therefore [exchanging the order of summation and equating coefficients]

$$p_k = \sum_{j=0}^{k} \binom{n-j}{n-k} q_j = \sum_{j=0}^{k} \binom{n-j}{k-j} q_j.$$

Thus the difference $p_{k+1} - p_k$ equals $\sum_{j=0}^{k+1} r_j q_j$, where

$$\begin{aligned} r_j &= \binom{n-j}{k+1-j} - \binom{n-j}{k-j} = \binom{n-j}{k-j}\left[\frac{n-k}{k+1-j} - 1\right] \\ &= \binom{n-j}{k-j}\frac{n-2k+j-1}{k-j+1} \end{aligned} \tag{6}$$

for $j < k + 1$, and $r_{k+1} = 1$. Since a_i is a positive integer it follows that $b_i \geq 0$ and therefore $q_j \geq 0$ and $q_0 = 1 > 0$.

If $m \leq n < 2m - 3$, then $0 < n - m + 1 < m - 2 < m - 1$; therefore we can set $k = n - m + 1$, and $n - m < k < m - 1$ will hold. Moreover $2k + 1 = 2(n - m + 1) + 1 < n$, so from (6) all r_j are positive [in particular, $r_0 > 0$], and therefore $p_{k+1} - p_k \geq r_0 q_0 > 0$.

For $n = 2m - 3$, putting $k = n - m + 1 = m - 2$ again satisfies $n - m < k < m - 1$. Also $2k + 1 = 2m - 3 = n$, so by (6) $r_0 = 0$ but $r_j > 0$ for $j \geq 1$.

Hence $p_{k+1} - p_k > 0$ unless $q_j = 0$ for all $j \geq 1$, i.e. $b_i = 0$ for all i. But then all $a_i = 1$, so that $p_k = \binom{n}{k}$ for all k.

Editor's note. Baron now computes the difference of the coefficients of ω^{m-3} and ω and shows it is nonzero, which finishes the proof. But it is easier to compare the coefficients of ω^{m-2} and ω^{m-3}: we get (since $m - 3 = n - m$) that these coefficients are respectively p_{m-2} and $p_{m-3} + p_{2m-3}$, that is,

$$\binom{2m-3}{m-2} \quad \text{and} \quad \binom{2m-3}{m-3} + \binom{2m-3}{2m-3} = \binom{2m-3}{m-3} + 1;$$

however it is known (and easy to see) that consecutive binomial coefficients $\binom{n}{k}$ and $\binom{n}{k-1}$ for $n > 2$ never differ by only 1, so we're done. □

Baron then proves part (a) separately (finding both solutions), since his theorem doesn't apply to $m = 3$. He also states that if we allow some of the a_i's to be zero, there are two more solutions to (a), namely $(0, 1, 1, 1)$ and $(0, 0, 2, 2)$.

Also solved (both parts) by F. J. FLANIGAN, San Jose State University, San Jose, California; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; and VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan. Part (a) *only was also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; TIM CROSS, Wolverley High School, Kidderminster, U. K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany: WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ANDY LIU, University of Alberta; and the proposer.*

Several of these also found both solutions to (a), *most showing that they are the only ones.*

Baron's theorem leaves his general problem still unsolved for many values of m and n; can readers settle the situation for some of these, either finding the required positive integers a_i or proving they don't exist? In particular the problem is wide open for nonprime m. The editor used something like Godin's technique to find a solution when $m = 4$ and $n = 5$; in this case the "complex mth root of unity" is just i of course, and it turns out that

$$(1+i)(1+2i)(1+4i)(1+23i)(1+30i) \qquad \textit{is an integer.}$$

By switching the complex numbers over to polar form, this yields the nice fact:

$$\arctan 1 + \arctan 2 + \arctan 4 + \arctan 23 + \arctan 30 \quad \textit{is a multiple of } \pi$$

(it's 2π, to be exact). No doubt some reader has seen this written down somewhere before.

* * * * *

1997. [1994: 285] *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*

ABC is a triangle which is not equilateral, with circumcentre O and orthocentre H. Point K lies on OH so that O is the midpoint of HK. AK meets BC in X, and Y, Z are the feet of the perpendiculars from X onto the sides AC, AB respectively. Prove that AX, BY, CZ are concurrent or parallel.

Solution by D. J. Smeenk, Zaltbommel, The Netherlands.

Let $D \in BC$ be the foot of the altitude from A, and let M be the midpoint of BC, N the midpoint of AH. Since NH is equal and parallel to OM (see, for example, R. A. Johnson, *Advanced Euclidean Geometry*, §252), $NO \| HM$. Since O and N are midpoints of two sides of $\Delta HKA, NO \| AK$ and so $NO \| AX$. Thus $AX \| NO \| HM$ so that $\Delta ADX \sim \Delta HDM$ and, therefore,

$$AD : DX = HD : DM. \tag{1}$$

We now require four facts about triangles with circumradius R, sides a, b, c and angles α, β, γ :

$$HD = 2R\cos\beta\cos\gamma, \qquad DM = R\sin(\beta-\gamma), \qquad c = 2R\sin\gamma,$$

$$AD = c\sin\beta = 2R\sin\gamma\sin\beta.$$

These can be found or deduced from standard references such as the Johnson book §252. Our ratio (1) is therefore

$$2R\sin\gamma\sin\beta : DX = 2\cos\beta\cos\gamma : \sin(\beta-\gamma),$$

so that

$$DX = \frac{R\sin\beta\sin\gamma\sin(\beta-\gamma)}{\cos\beta\cos\gamma}.$$

[*Editor's comment by Chris Fisher.* You can work out for yourself the case where $\cos\beta\cos\gamma = 0$.] Therefore,

$$MX = DX - R\sin(\beta-\gamma) = R\sin(\beta-\gamma)\left(\frac{\sin\beta\sin\gamma}{\cos\beta\cos\gamma} - 1\right).$$

Finally, after some trigonometry,

$$MX = R\,\frac{\sin\gamma\cos\gamma - \sin\beta\cos\beta}{\cos\beta\cos\gamma}. \tag{2}$$

Ceva's theorem says that if

$$AZ \cdot BX \cdot CY = ZB \cdot XC \cdot YA, \tag{3}$$

then the cevians AX, BY, CZ are concurrent or parallel, which is our goal. Setting $d = MX$, we find

$$AZ = c - (\frac{1}{2}a + d)\cos\beta, \qquad ZB = (\frac{1}{2}a + d)\cos\beta,$$

$$BX = \frac{1}{2}a + d, \qquad XC = \frac{1}{2}a - d,$$

$$CY = (\frac{1}{2}a - d)\cos\gamma, \qquad YA = b - (\frac{1}{2}a - d)\cos\gamma.$$

Substituting into (3) we find that Ceva's theorem applies if and only if

$$d = \frac{c\cos\gamma - b\cos\beta}{2\cos\beta\cos\gamma} = R\,\frac{\sin\gamma\cos\gamma - \sin\beta\cos\beta}{\cos\beta\cos\gamma},$$

which agrees with (2) as desired.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I. B. Leopoldo Cano, Valladolid, Spain; P. PENNING, Delft, The Netherlands; and the proposer.

Ardila managed a very concise argument by invoking a portion of the featured solution (also by Smeenk) to an earlier Bradley problem, Crux 1908 [1994: 293], *which has much in common with the problem here.*

* * * * *

1998. [1994: 286] *Proposed by John Clyde, student, New Plymouth High School, New Plymouth, Idaho.*

Let $a = \sin 10°$, $b = \sin 50°$, $c = \sin 70°$. Prove that

(i) $a + b = c$, (ii) $a^{-1} + b^{-1} = c^{-1} + 6$, (iii) $8abc = 1$.

Solution by Beatriz Margolis, Paris, France.

Since

$$\sin 30° = \sin 150° = -\sin 210° = \frac{1}{2},$$

we have

$$\sin(3 \cdot 10) = \sin(3 \cdot 50) = \sin(3 \cdot -70) = \frac{1}{2}. \qquad (1)$$

But

$$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x,$$

so that by (1)

$$\sin^3 10 = \frac{3}{4}\sin 10 - \frac{1}{8}, \quad \sin^3 50 = \frac{3}{4}\sin 50 - \frac{1}{8}$$

and

$$\sin^3(-70) = \frac{3}{4}\sin(-70) - \frac{1}{8},$$

i.e. a, b, $-c$ are the three roots of

$$f(t) = t^3 - \frac{3}{4}t + \frac{1}{8}.$$

Therefore

$$a + b + (-c) = 0, \qquad \text{i.e. (i)}$$

$$ab(-c) = -\frac{1}{8}, \qquad \text{i.e. (iii)}$$

and

$$ab + b(-c) + (-c)a = -\frac{3}{4}.$$

Dividing by abc and using (iii) this leads to

$$c^{-1} - a^{-1} - b^{-1} = \left(-\frac{3}{4}\right) \cdot 8, \qquad \text{i.e. (ii).}$$

Also solved by HAYO AHLBURG, Benidorm, Spain; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; ŠEFKET ARSLANAGIĆ, Berlin, Germany; SAM BAETHGE, Science Academy, Austin, Texas; LEON BANKOFF, Beverly Hills, California; GERD BARON, Technische Universität Wien, Austria (two solutions); FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, Wolverley High School, Kidderminster, U. K.; COLIN DIXON, Royal Grammar School, Newcastle upon Tyne, England; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RUSSELL EULER, Northwest Missouri State University, Maryville; ANNA FONOLLOSA (student) and JORDI GARCIA (teacher), I. B. Ramon Cid, Benicarló, Spain; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; SOLOMON W. GOLOMB, University of Southern California, Los Angeles; DOUGLASS L. GRANT, University College of Cape Breton, Sydney, Nova Scotia; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; JOE HOWARD, New Mexico Highlands University, Las Vegas, New Mexico; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; R. DANIEL HURWITZ, Skidmore College, Saratoga Springs, New York; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; DAVID E. MANES, State University of New York, Oneonta; J. A. MCCALLUM, Medicine Hat, Alberta; RICHARD MCINTOSH, University of Regina; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; BOB PRIELIPP, University of Wisconsin-Oshkosh; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Spain; ST. OLAF PROBLEM GROUP, St. Olaf College, Northfield, Minnesota; CRISTÓBAL SÁNCHEZ-RUBIO, I. B. Penyagolosa, Castellón, Spain; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; TOSHIO SEIMIYA, Kawasaki, *Japan; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; D. J. SMEENK, Zaltbommel, The Netherlands; DIGBY SMITH, Mount Royal College, Calgary; L. J. UPTON, Mississauga, Ontario; EDWARD T. H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.*

Baron, Chronis, McIntosh, Penning and the proposer gave solutions similar to Margolis's. Most other solvers used trigonometric identities.

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1999. [1994: 286] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Spain.*

Let ABC be a (variable) isosceles triangle with constant sides $a = b$ and variable side c. Denote the median, angle bisector and altitude, measured from A to the opposite side, by m, w and h respectively. Find

$$\lim_{c\to a}\frac{m-h}{w-h}\,.$$

Solution by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.

Note first that

$$\lim_{c\to a}\frac{m-h}{w-h}=\lim_{c\to a}\left(\frac{m^2-h^2}{w^2-h^2}\cdot\frac{w+h}{m+h}\right)=\lim_{c\to a}\frac{m^2-h^2}{w^2-h^2}\,,$$

since

$$\lim_{c\to a}\frac{m+h}{w+h}=1$$

(if $a = b = c$ then $m = w = h$). Now

$$m^2-h^2=\frac{2(b^2+c^2)-a^2}{4}-\frac{4}{a^2}s(s-a)(s-b)(s-c)$$

(where s is the semiperimeter of the triangle), and since $a = b$ we have

$$\begin{aligned} m^2-h^2 &= \frac{a^2+2c^2}{4}-\frac{4}{a^2}\left(a+\frac{c}{2}\right)\cdot\frac{c}{2}\cdot\frac{c}{2}\left(a-\frac{c}{2}\right)\\ &= \frac{a^2+2c^2}{4}-\frac{c^2}{4a^2}(4a^2-c^2)=\frac{(a^2-c^2)^2}{4a^2}\,. \end{aligned}$$

Also

$$w^2-h^2=\frac{4bc}{(b+c)^2}s(s-a)-\frac{4}{a^2}s(s-a)(s-b)(s-c)$$

and putting $a = b$ we easily find that

$$\begin{aligned} w^2-h^2 &= \frac{4ac}{(a+c)^2}\left(a+\frac{c}{2}\right)\frac{c}{2}-\frac{c^2}{4a^2}(4a^2-c^2)\\ &= \frac{c^2(2a+c)}{4a^2(a+c)^2}[4a^3-(2a-c)(a+c)^2]=\frac{c^2(2a+c)^2(a-c)^2}{4a^2(a+c)^2}\,. \end{aligned}$$

Thus

$$\frac{m^2 - h^2}{w^2 - h^2} = \frac{(a + c)^4}{c^2(2a + c)^2},$$

so

$$\lim_{c \to a} \frac{m - h}{w - h} = \lim_{c \to a} \frac{m^2 - h^2}{w^2 - h^2} = \frac{(2a)^4}{a^2(3a)^2} = \frac{16}{9}.$$

[*Editor's note.* Chronis uses the known formulas

$$m = \frac{1}{2}\sqrt{2(b^2 + c^2) - a^2}, \quad w = \frac{2}{b + c}\sqrt{bcs(s - a)}$$

and

$$h = \frac{2}{a}\sqrt{s(s - a)(s - b)(s - c)},$$

for arbitrary triangles ABC. For the convenience of readers who have not seen them, here is how to get these formulas.

For the first, apply the law of cosines to the two triangles formed from ABC by the median from A; we get

$$c^2 = m^2 + \frac{a^2}{4} - ma\cos\theta \quad \text{and} \quad b^2 = m^2 + \frac{a^2}{4} + ma\cos\theta,$$

where θ is the angle between the median and BC; now add. For the second, apply the law of cosines to the analogous two triangles formed by the angle bisector from A; since the angle bisector divides BC into the proportion $c : b$, we get

$$\left(\frac{ac}{b + c}\right)^2 = c^2 + w^2 - 2cw\cos\frac{A}{2} \quad \text{and} \quad \left(\frac{ab}{b + c}\right)^2 = b^2 + w^2 - 2bw\cos\frac{A}{2};$$

now multiply the first by b, the second by c, and subtract. The equation for h follows from Heron's formula for the area of the triangle.]

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; JORDI DOU, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; ST. OLAF PROBLEM GROUP, St. Olaf College, Northfield, Minnesota; CRISTÓBAL SÁNCHEZ-RUBIO, I. B. Penyagolosa, Castellón, Spain; D. J. SMEENK, Zaltbommel, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. Two incorrect solutions were received.

Cabezón's solution was very similar to Chronis's.

Bellot found two related problems in the Romanian journal Gazeta Matematica; *number* 22746 *on p.* 320 *of the* 1993 *volume, and another in issue* 5 *of the* 1994 *volume.*

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2000. [1994: 286] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

A 1000-element set is randomly chosen from $\{1, 2, \ldots, 2000\}$. Let p be the probability that the sum of the chosen numbers is divisible by 5. Is p greater than, smaller than, or equal to $1/5$?

Solution by Robert B. Israel, University of British Columbia.

p is greater than $1/5$, but not by much: it is exactly

$$\frac{1}{5} + \frac{4}{5}\frac{\binom{400}{200}}{\binom{2000}{1000}}$$

which is approximately $1/5 + 4 \times 10^{-482}$.

Divide up the set $\{1, \ldots, 2000\}$ into the 400 5-tuples

$$\{1, \ldots, 5\}, \quad \{6, \ldots, 10\}, \quad \ldots, \quad \{1996, \ldots, 2000\}.$$

Consider the probabilities conditioned on the number of elements chosen in each 5-tuple. Now given that there are k elements chosen in a particular 5-tuple, if k is not 0 or 5 the sum of those k elements is equally likely to be in each of the congruence classes mod 5. An easy way to see this is to consider the symmetry that increases each element of the 5-tuple by 1 except the last, which it takes to the first. This adds k mod 5 to the sum of the k elements in the 5-tuple. So the probability of this sum being congruent to j is equal to its probability of being congruent to $j + k$. Since k and 5 are relatively prime, all five congruence classes have equal probabilities. Note that this is independent of the sum of the numbers chosen outside of this 5-tuple. Thus we find that if any of the 5-tuples contain a number of chosen elements other than 0 or 5, the conditional probability of the sum being divisible by 5 is 1/5. On the other hand, if every 5-tuple contains 0 or 5 chosen elements, the sum is certainly divisible by 5. So $p = 1/5 + (4/5)q$, where q is the probability that every 5-tuple contains 0 or 5 chosen elements. Now each possible choice of the set satisfying this requirement corresponds to a choice of 200 of the 400 5-tuples (which will be the 5-tuples contained in the set). Therefore

$$q = \binom{400}{200} \Big/ \binom{2000}{1000}.$$

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; GERD BARON, Technische Universität Wien, Austria; CURTIS COOPER, Central Missouri State University, Warrensburg; HOE TECK WEE, student, Hwa Chong Junior College, Singapore (two solutions); and the proposer. There was one incorrect solution submitted.

Several of the solutions, including the proposer's, are somewhat similar to Israel's. Ardila and Baron each establish the following more general result (which can also be proved using Israel's approach):

Let P be a prime number and N a positive integer. A K-element set is randomly chosen from $\{1, 2, \ldots, PN\}$. Let p_i be the probability that the sum of the chosen elements is congruent to i modulo P. Then $p_1 = p_2 = \cdots = p_{P-1}$. If P does not divide K then $p_0 = p_1 = \cdots = 1/P$. If P divides K then $p_0 > p_1 = p_2 = \cdots$; in this case we have

$$p_0 = \frac{1}{P} + \frac{P-1}{P}\frac{\binom{N}{K/P}}{\binom{PN}{K}} \qquad \text{and} \qquad p_i = \frac{1}{P} - \frac{1}{P}\frac{\binom{N}{K/P}}{\binom{PN}{K}} \quad \text{for } 1 \le i \le P-1.$$

Some solvers mention that this problem is related to Problem 6 of the 1995 IMO [1995: 269], *which was also proposed by Kuczma.*

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2002. [1995: 19] *Proposed by K. R. S. Sastry, Dodballapur, India.*

Find a positive integer N such that both N and the sum of the digits of N are divisible by both 7 and 13.

I. *Solution by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas.*

Since 7 and 13 are divisors of 1001, any number formed by adding together multiples of numbers of the form $ABCABC$ is divisible by 7 and 13. The sum of the digits of such numbers $ABCABC$ has to be even, so [by stringing together such numbers] we can form numbers still divisible by 7 and 13, and whose digit sums are divisible by $2 \cdot 7 \cdot 13 = 182$, such as

$$N = (0)19019\ 999999\ 999999\ 999999.$$

There exist much smaller N. Start with the number 999999 999999, which is divisible by 7 and 13 and has a digit sum of 108. If we can find a number divisible by 7 and 13 with a digit sum of 17, we will have another N [by subtraction]. 940121 000000 is one such number, so

$$N = 59878\ 999999$$

is a solution.

[*Editor's note.* Remarkably, this last solution is the *smallest possible* one, as several readers noted, the proposer included. Some of these used a computer to prove minimality. Solvers Golomb and Israel, however, gave written proofs.]

II. *Solution by Christopher J. Bradley, Clifton College, Bristol, U. K.*

How about

$$2002\ 2002\ 2002 \ldots 2002$$

with digit sum 364?

III. *Comments by the editor.*

There were many other solutions sent in for this problem, among which were several more of the "repeated pattern" variety, like:

the "repunit" $R_{546} = 1\ 1\ 1 \ldots 1$ (546 ones)
91 91 91 ... (91 copies)

364 364 ... (7 copies)
7 7 7 ... (78 digits)

Each of these was contributed by at least two readers. Other solutions included

$$7^{25} \cdot 13 = 17433892055631543710491, \qquad \text{(Konečný)}$$

$$999999\,999999 \cdot 1001 + 91 = 1000999999999090, \qquad \text{(Cross)}$$

and

$$1729\ 819\ 819\ 819\ 819$$

(Wildstrom — note that each of the five groups is divisible by 7 and 13).

Also solved by HAYO AHLBURG, Benidorm, Spain; C. ARCONCHER, Jundiaí, Brazil; FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; CHARLES ASHBACHER, Cedar Rapids, Iowa; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; PAUL COLUCCI, student, University of Illinois; TIM CROSS, Wolverley High School, Kidderminster, U. K.; PETER DUKES, student, University of Victoria, B.C.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; J. K. FLOYD, Newnan, Georgia; TOBY GEE, student, The John of Gaunt School, Trowbridge, England; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; SOLOMON W. GOLOMB, University of Southern California, Los Angeles; DAVID HANKIN, John Dewey High School, Brooklyn, New York; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; SHOICHI HIROSE, Tokyo, Japan; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; PETER HURTHIG, Columbia College, Burnaby, B. C.; ROBERT B. ISRAEL, University of British Columbia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; INES KLIMANN, student, Université de Paris 6, France; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; DAVID E. MANES, State University of New York, Oneonta; J. A. MCCALLUM, Medicine Hat, Alberta; STEWART METCHETTE, Culver City, California; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ASHISH KR. SINGH, Kanpur, India; D. J. SMEENK, Zaltbommel, The Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; JOHN S. VLACHAKIS, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, The Netherlands; SUSAN SCHWARTZ WILDSTROM, Walt Whitman High School, Bethesda, Maryland; ANA WITT, Austin Peay State University, Clarksville, Tennessee; and the proposer.

Alhburg and Janous both gave generalizations where the numbers 7 *and* 13 *(i.e.* 91*) are replaced by any positive integer not divisible by* 2 *or* 5. *Janous then went on to eliminate this last condition, and prove: for* every *positive integer m there is a positive integer divisible by m and whose digit sum is also divisible by m.*

*   *   *   *   *