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The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

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Pascal's Pyramid

LIU ZHIQING, *Beijing, People's Republic of China*

The author wrote this article when he was 17 years old and a student at No. 4 Middle School in Beijing. His main interest now is in computer science.

Readers will probably be familiar with Pascal's triangle to display the binomial coefficients, which occur when we expand $(a+b)^n$ for successive values of n by the binomial theorem (see figure 1).

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Figure 1. Pascal's triangle

An important property of Pascal's triangle which enables it to be built up easily is that each number in the triangle is the sum of the two immediately above it. Thus, for example, $10 = 4+6$. (The 1's which occur at the ends of each row obey an obvious degenerate form of this rule.)

But what happens if we look at the expansions of $(a+b+c)^n$ for successive powers of n? Is there a similar way of arranging these which enables us easily to work out the coefficients for a given value of n from those for the previous value? Indeed there is. The fact that *Mathematical Spectrum* is not printed in three dimensions makes it more difficult to visualise, but here goes! We have

$$\begin{aligned}
(a+b+c)^0 &= \quad 1, \\
(a+b+c)^1 &= \quad a+b+c, \\
(a+b+c)^2 &= \quad a^2 \\
&\quad\; 2ab+2ac \\
&\quad\; +b^2+2bc+c^2, \\
(a+b+c)^3 &= \quad a^3 \\
&\quad\; 3a^2b+3a^2c \\
&\quad\; +3ab^2+6abc+3ac^2 \\
&\quad\; b^3+3b^2c+3bc^2+c^3,
\end{aligned}$$

$$
\begin{aligned}
(a+b+c)^4 = \qquad & a^4 \\
& 4a^3b+4a^3c \\
& 6a^2b^2+12a^2bc+6a^2c^2 \\
& 4ab^3+12ab^2c+12abc^2+4ac^3 \\
& b^4+4b^3c+6b^2c^2+4bc^3+c^4,
\end{aligned}
$$

and so on. We now arrange these coefficients in a pyramid with a triangular cross-section, as shown in figure 2. The coefficients for a given exponent all occur on the same level, as shown. On each face of this 'infinite pyramid' we find Pascal's triangle. But the crucial property is that each number in the pyramid is the sum of the *three* immediately above it. (Degenerate versions of this rule apply to those numbers on the faces of the pyramid.) We can illustrate this easily if we superimpose the numbers at level $n = 5$ on the triangle at level $n = 4$ (see figure 3). Thus $20 = 4+4+12$ and $30 = 6+12+12$ (with $10 = 4+6$, etc. on the edges of the triangle).

To prove this relation between the coefficients at the two levels, we write

$$(a+b+c)^n = \sum_{r+s+t=n} \theta(r,s,t)a^r b^s c^t,$$

so that, in fact,

$$\theta(r,s,t) = \frac{(r+s+t)!}{r!s!t!}.$$

Then

$$(a+b+c)^n = a(a+b+c)^{n-1}+b(a+b+c)^{n-1}+c(a+b+c)^{n-1}$$

and it is easy to see from this that

$$\theta(r,s,t) = \theta(r-1,s,t)+\theta(r,s-1,t)+\theta(r,s,t-1)$$

(where we better define $\theta(r,s,t) = 0$ if any of r, s and t is -1, to cover the degenerate cases). It is this simple observation that provides the relation between the numbers at successive levels of the pyramid, and enables us to build up the pyramid level by level. Readers may like to use it to construct level $n = 6$ from level $n = 5$.

The next thing to attempt is to display the coefficients of the expansions of $(a+b+c+d)^n$ for successive values of n in a four-dimensional Pascal pyramid. But this would probably be beyond the ingenuity even of the *Spectrum* printers to reproduce!

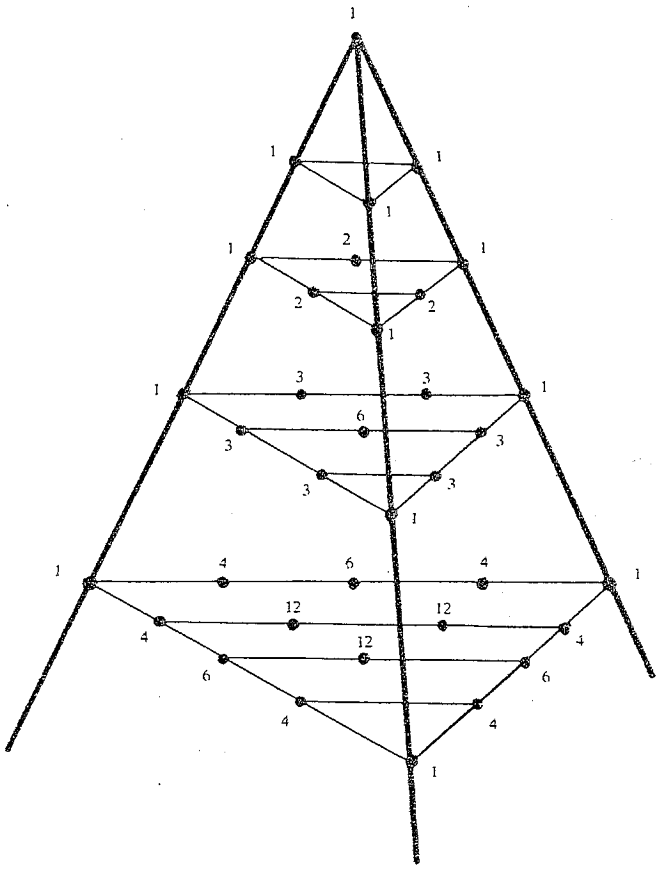


Figure 2. Pascal's pyramid

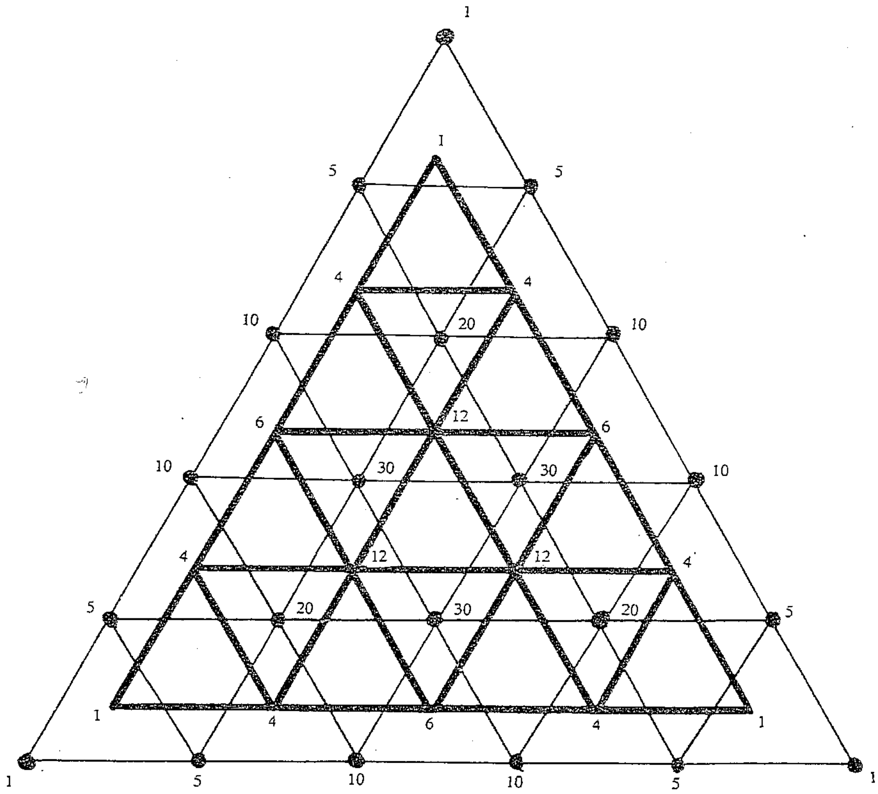


Figure 3

Errors in Navigation

DAVID SAUNDERS, *British Broadcasting Corporation*

The author graduated from Cambridge University in 1970 and now works at the BBC, producing mathematics television programmes for the Open University. Although he is principally interested in pure mathematics, his work has led him into several other mathematics topics. He has recently been involved with the use of computer graphics to teach dynamic concepts in probability and statistics, and has shown examples of this work at meetings of the Royal Statistical Society.

Navigation—the art of finding one's position at sea—is one of the most vital arts of sailing, not only for ocean-going vessels but also for small boats which always remain in coastal waters. In fact, the navigator of any boat near to a coastline will need to be particularly vigilant about his position to avoid running into shallow water. Several electronic navigation aids using radio beacons or satellites have come on the market in recent years, but although these are excellent devices they tend to be very expensive. Thus, many small boats still rely on the traditional method of using an accurately calibrated 'hand-bearing' compass and plotting the bearings so obtained on an Admiralty chart.

This traditional method uses prominent landmarks with fixed positions. A typical landmark might be a lighthouse, a tall building, or even a buoy—any clearly visible object which is marked on the chart will do. By sighting the compass at several such landmarks, the (clockwise) angle between magnetic north and each of them can be found. Then, by drawing a line on a chart from each object at the corresponding inclination to magnetic north, one finds the location of the boat at the point where the lines cross. Difficulties arise, however, because all measurements are subject to error. Although modern hand-bearing compasses are accurate to within a degree, such accuracy can only be obtained when standing still. Anyone who has sailed in a boat in even moderately rough seas will appreciate that the unpredictable rocking motion is liable to introduce much larger errors into each reading, of the order of five degrees or even more. When accurate positioning is neeeded, some technique must be used to reduce the effect of this error.

A standard technique for taking account of measurement errors has been developed over the years. It is to take bearings not on two fixed objects but on three. If all three bearings were accurate, then the three lines drawn on the chart would all meet at a single point—the position of the boat. Typically, however, the inevitable measurement errors mean that the three lines will not meet at a point but instead will form a triangle, as shown in figure 1. The conventional belief is that this triangle gives an idea of the uncertainty of

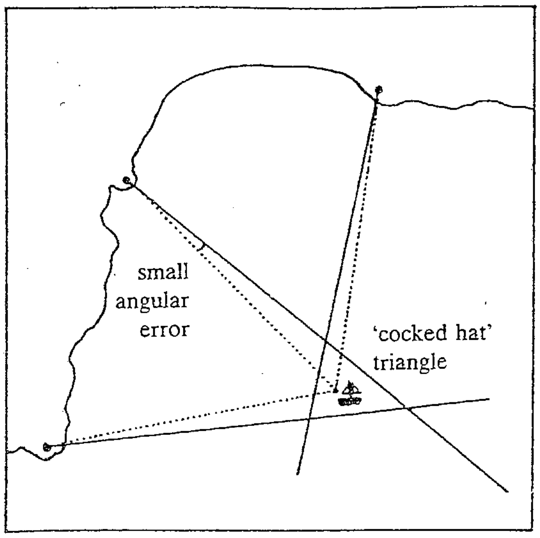


Figure 1

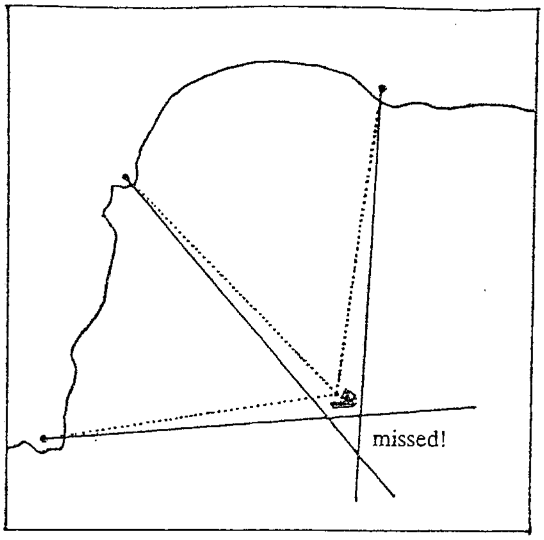


Figure 2

the measurements and that we cannot be more precise than to say that the true position of the boat will be somewhere within it.

Incidentally, this triangle is known in nautical jargon as the 'cocked hat'. The question is, however, whether this conventional belief is correct.

Now, given the situation shown in figure 1, the cocked hat certainly does contain the true position of the boat. But is this always so? Figure 2 shows why the conventional belief is too optimistic. The true position of the boat can quite easily be outside the cocked hat. Whereas this possibility is now well understood in nautical circles (see reference 1), it is less well known that the accuracy of the cocked hat can actually be quantified using statistical ideas.

Making an inference about a measurement subject to a random error is a common problem in statistics. The simplest case involves a one-dimensional measurement, where the quantity being measured can be represented by a real number. Although the true value of the quantity is unknown, we can make a set of measurements, where each measurement is independent and subject to a random error. We can then try to infer the true value of the quantity. There is much statistical literature explaining how to construct a 'confidence interval', a range of possible values likely to contain the true value, from a set of measurements. A numerical confidence level, such as 95%, is attached to the interval, and this is interpreted in the following sense: if we perform the set of measurements not once but a large number of times, and thus obtain a large number of intervals, then in the long run 95% of those intervals will contain the true value. Since 'relative frequency over a long run' is a way of approximating *probability*, we can say that the probability that intervals constructed by this procedure contain the unknown value is 0.95. In general, for a given confidence level, the procedure for constructing the confidence interval from the set of measurements will depend on the way the errors are distributed, that is, on the probabilities that the error takes particular sets of values.

A similar idea can be applied to measurements in two dimensions. Again, if the true value is unknown, then a set of measurements can be taken—but this time a confidence *region* would be constructed instead of a confidence interval. For (say) a 95% confidence region, the probability that regions generated by the chosen procedure actually contain the true value must be 0.95. Just as in the one-dimensional case, for a given confidence level, the way the region is constructed from the set of measurements will generally depend on the way the errors are distributed. However, the calculations involved will obviously be more complicated in two dimensions.

How do these ideas apply to the 'cocked hat' triangle? The triangle is generated by a predetermined procedure from measurements with random errors, and so can be considered a confidence region. We can therefore ask: what is the confidence level of the region? In other words, what is the probability that triangles generated by this procedure contain the true, unknown position of the boat? From the remarks above, it might be imagined that the answer would depend on the distribution of the errors, and in any event would be difficult to calculate. However, if we assume a very simple model for the errors, it turns out that the calculations are very straightforward for this particular method of constructing a confidence region.

The argument runs like this. The question of whether the triangle contains the true position depends on the relationship of that true position to the three lines forming the triangle. The distance of the point from each line is irrelevant; the only relevant factor is whether the point is on one side of the line or the other. We can express this in terms of the sign of the error in the corresponding angular measurement—that is, whether the angular error is clockwise or anticlockwise. We ignore the chance of absolutely no error at all, assuming that the probability that this event happens, is 0.

The example shown in figure 3 will make this clear. Since each of the three angular errors can be either clockwise or anticlockwise, there are eight possible cases. We can see, if we inspect the diagram, that the triangle will contain the point only when all three errors are clockwise or all three errors are anticlockwise. If, and this is the simple model for the errors, the compass is *unbiased* in the sense that clockwise and anticlockwise errors are equally likely, we can then calculate the probability that the triangle contains the point. The probability of a clockwise error on any one measurement is a half, and since the three measurements are independent we can find the probability of three clockwise errors by multiplying the three individual probabilities, giving one eighth. In the same way, the probability of three anticlockwise errors is also one eighth, and since these two cases are mutually exclusive we can add their probabilities to give a total probability of one

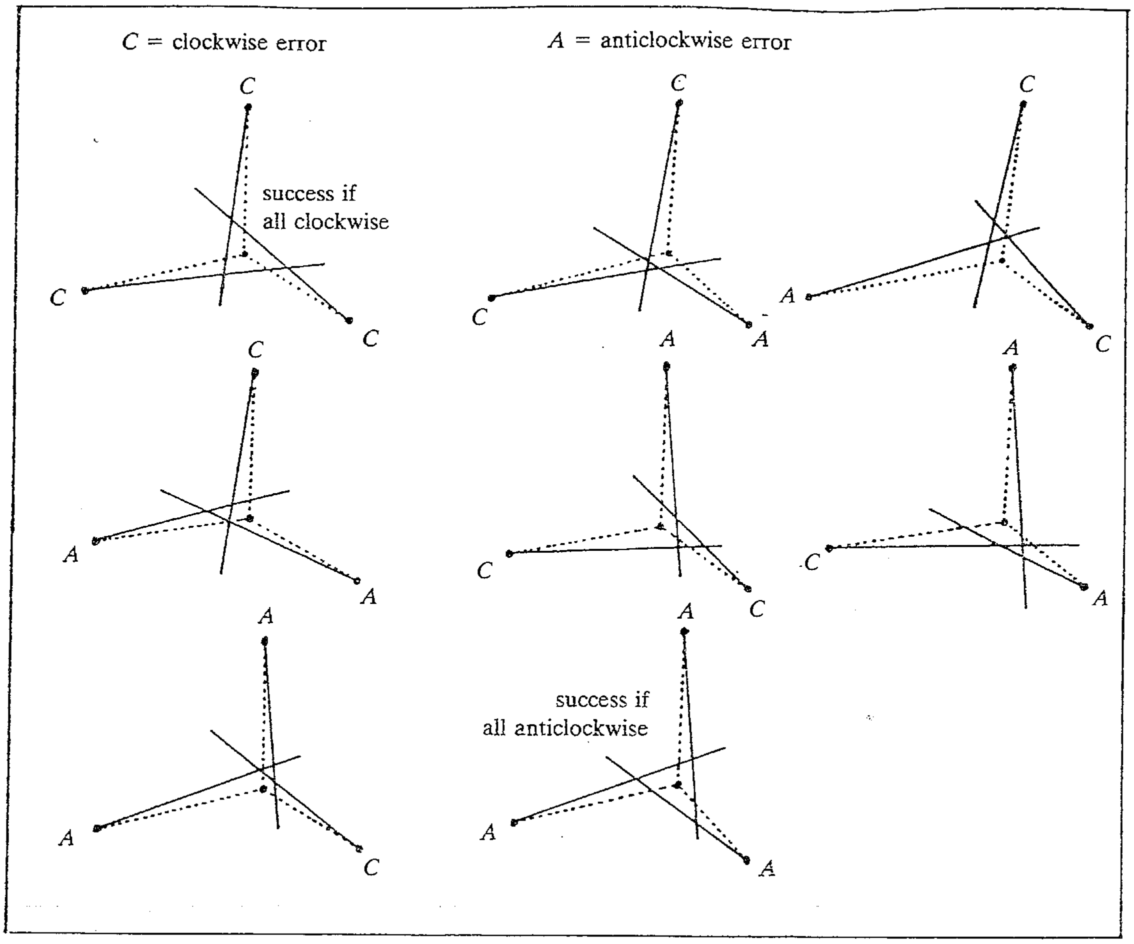


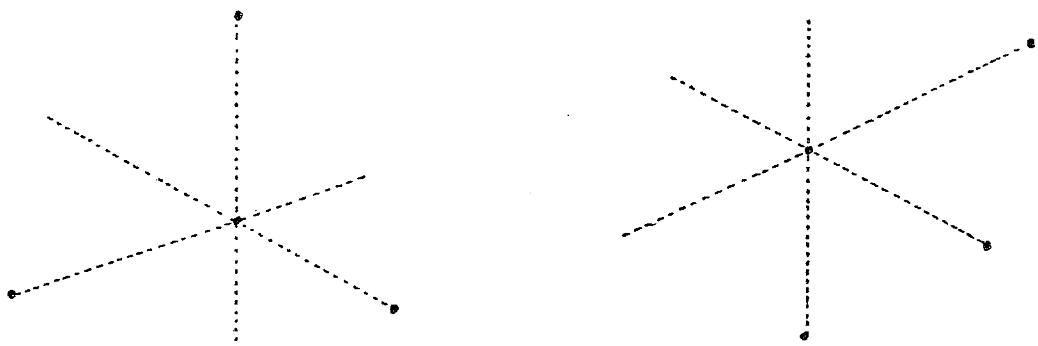
Figure 3

quarter that the triangle will contain the point. In other words, the triangle is only a 25 per cent confidence region for the position. Note, however, that the triangle may, in this model, be large or small depending on the size of the errors, which we have not taken into account.

Of course it is always dangerous to argue from diagrams, and we could ask whether the particular configuration of points shown in figure 3 affects the result. Discerning readers will have already noticed that, in figure 1, the triangle contained the true point but that two errors were clockwise and one anticlockwise. In fact the configuration of the points does not affect the end result, and we can see this by observing first that *small* changes in the relative positions of the three landmarks and the boat can have no effect on the argument. A 'topological' change in the configuration only occurs when, if we change those relative positions, two of the lines in the diagram coincide, and therefore the triangle disappears from one side of a line and reappears on the other. It turns out that there are only two possible configurations which are 'topologically distinct', and they are shown in figure 4. We have already examined the first of these, where the three landmarks span an angle of more than 180 degrees at the true position. In an obvious sense, the three landmarks are opposite one another. The other possible configuration is

At the true position, the three landmarks span more than 180°

At the true position, the three landmarks span less than 180°



All three errors must be in the same direction

The errors in the outer two bearings must be in the same direction, and the error in the middle bearing must be in the opposite direction

Figure 4

when the three landmarks span less than 180 degrees, and in this case we can clearly refer to one of the landmarks as the middle landmark. By examining the eight different possibilities for the error direction in this second case, we find that the triangle contains the true position only when both outer bearing errors are in the same direction and the error in the middle bearing is in the opposite direction, again two cases out of eight. So in both the possible configurations we have found two cases out of eight when the triangle contains the true position, and we can confirm our claim that the 'cocked hat' triangle is only a 25 per cent confidence region.

We can now compare this mathematical result with the conventional nautical belief. It certainly isn't true that the cocked hat always contains the true position of the boat, and we have now found that over a long run this only happens 25 per cent of the time—not a very comforting thought. Furthermore, if the size of the errors is large, the cocked hat will itself be too large to give an accurate navigational position for the boat. So, if we had to offer the navigator some advice, perhaps the most appropriate would be: 'If your cocked hat ever gets close to a danger area then watch out, for the technique is less accurate than you think.'

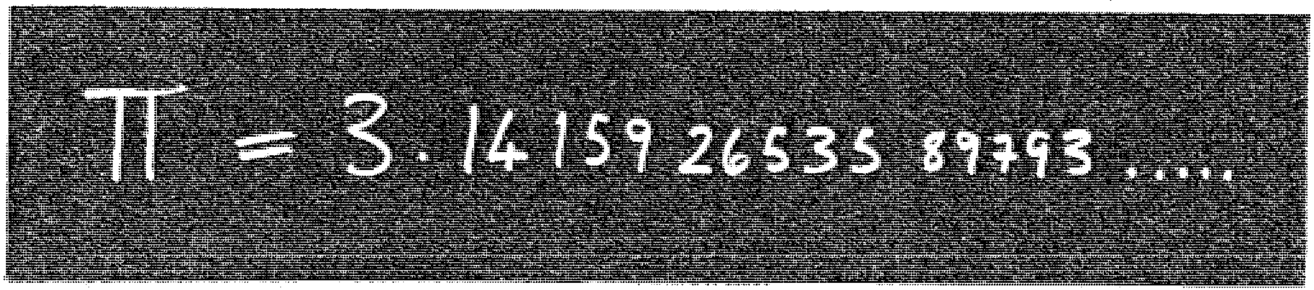
Reference

A. Sharples, Errors in fix, *Yachting Monthly*, July 1982.

Pi and Chips

KEITH DEVLIN, *University of Lancaster*

In November 1983 Keith Devlin wrote in the *Guardian* about the calculation of π to a record eight million decimal places. He subsequently received a letter from one of the two Japanese record breakers, Yasumasa Kanada of the University of Tokyo Computer Centre, informing him that they had shattered this record by going up to sixteen million decimal places. This article appeared in the *Guardian* on 2 February 1984.



It is a feature of the methods used to calculate π to many places of decimals that there is a possibility of errors occurring 'well out' along the decimal expression. To guard against this, the standard procedure is to make a second calculation of π using a different method and compare the two results. Using an S-810 model 20 supercomputer made by Hitachi, Kanada made a second calculation of π to just over ten million places of decimals. By comparing the two results, he concluded that he (or rather his computer) now knows the decimal representation of π accurately to 10 013 395 places. For those who are interested, the ten millionth digit in the expansion is 7.

The computer used for this calculation, mentioned above, is one of a small group of very advanced machines which are so much more powerful than 'ordinary' computers that they are referred to as 'supercomputers'. The CRAY-1 and the CYBER-205 (both American) are examples of such machines. Comparison of the speed and power of supercomputers is very difficult to make, since any machine will have various idiosyncracies that make it more suited to one type of calculation than another. The CRAY-1 has hitherto been regarded as the 'world's most powerful computer' because of its very short 'clock time' (the time taken to change from one discrete state to another), but for many types of calculation the CYBER-205 will perform much better.

In fact, neither the CRAY-1 nor the CYBER-205 can be indisputably described as faster than the ILLIAC IV computer, a 'one-off' computer built by the Burroughs Corporation in the early 1970s for use by NASA. (ILLIAC IV—now no longer used—was the first large-scale computer to make use of silicon-chip technology for its central memory.)

One measure of the calculating speed of any computer is to count the number of single arithmetic operations that can be performed in a second. (The figure must include the time taken to collect the numbers from memory and store the result in memory.) Home micros are capable of handling a few hundred arithmetic operations a second. (If any home-micro manufacturer is reading this and feels I am being unfair to micros, please let me know. I should be happy to receive some precise figures.) For supercomputers you have to talk in terms of 'megaflops'. One megaflop is one million arithmetic operations per second. The CRAY-1 and CYBER-205 have, according to their manufacturers, peak (!) computing speeds of around 200 and 400 megaflops, respectively.

According to Kanada, his Hitachi computer averaged over 450 megaflops for the entire 24-hour period that the calculation took. Whilst I am not for a moment pretending to be an authority on supercomputers, that would look to me as though the world computer speed record has just passed from America to Japan. (Again, counter-facts would be gratefully received.)

Why calculate π to so many decimal places? One reason, of course, is that, as with the search for record prime numbers, such feats provide some sort of measure of the power of a computer in a form that everyone can understand. But in the case of π there are other, mathematical, reasons. As you probably know, π is defined to be the ratio of the circumference of any circle to its diameter.

Expressed as a decimal, π requires infinitely many places of decimals, and its decimal expansion does not recur because it is irrational. The first few terms are given by

$$\pi = 3.14159\,26535\,89793\ldots.$$

However, no pattern is known in the digits in the π decimal and there is a slight chance that by examining long expressions for π it may be possible to discern one. Kanada has used his computer to perform various tests on the number. He has shown that, although the digits cannot constitute a 'random' sequence of numbers (because they are obtained from a formula), they do behave like one, satisfying various standard tests for 'randomness'. In other words, there does indeed seem to be no pattern.

Puzzles from the Greek Anthology

DAVID SINGMASTER, *Polytechnic of the South Bank*

David Singmaster is an American who has been at the Polytechnic of the South Bank since 1970. His basic interests are in combinatorics and number theory, particularly the elementary aspects of them. Recently he has been interested in various aspects of recreational mathematics, particularly Rubik's cube and similar mathematical puzzles. He has recently begun editing a series of books on recreational mathematics and is compiling a book on sources of recreational topics, which led him to the material discussed in this article.

1. Introduction

I have recently been studying the history of recreational mathematics and have found mathematical problems in suprising places. One of the most unlikely places is a collection of Greek poetry known as *The Greek Anthology*. This is a collection of short epigrams, both in verse and in prose, including hymns to the gods, epitaphs, dedications, eulogies, examples of exotic poetic metres, riddles, charades, oracles, poems forming shapes and even mathematical problems!

The *Anthology* comes down to us via a unique manuscript in the Palatine Library, Heidelberg, compiled by one Constantine Cephalas in the 10th century. The epigrams range from the 7th century BC to the 5th century AD. Some are attributed to definite authors, but many are anonymous. Some of the epigrams are known via other sources.

The *Anthology* has been translated into English, both completely and as excerpts. W. R. Paton produced a dual-language version in five volumes for the Loeb Classical Library (reference 1), and I shall quote from this version. Paton has put all the arithmetical epigrams in Book XIV of Volume 5. There are 44 arithmetical problems in Book XIV. Many of these (Paton's 116–146) are specifically attributed to Metrodorus, a grammarian of the 4th or 5th century AD. Paton asserts that the other problems (Paton's 1–4, 6, 7, 11–13, 48–51) are in the same style and can be also attributed to Metrodorus.

The problems fall into three main types, each with a few minor variants, and a few miscellaneous problems. These are presented in the next sections, with Paton's numbers given in parentheses. All the problems are easily solved by a little algebra and many have been used in school books ever since. I shall give the answers at the end.

2. Type A — Diophantos' age

The first problem is famous as it is one of the few references to Diophantos.

Problem 1 (126). This tomb holds Diophantos. Ah, how great a marvel! the tomb tells scientifically the measure of his life. God granted him to be a boy for the sixth part of his life, and adding a twelfth part to this, he clothed his cheeks with down; He lit him the light of wedlock after a seventh part, and five years after his marriage He granted him a son. Alas! late-born wretched child; after attaining the measure of half his father's life, chill Fate took him. After consoling his grief by this science of numbers for four years he ended his life.

The next problem is of the same type and has survived because it refers to Pythagoras. In the *Anthology* it is attributed to Socrates, but Paton presumably includes it in his attribution to Metrodorus.

Problem 2 (1—Socrates).

POLYCRATES SPEAKS

Blessed Pythagoras, Heliconian scion of the Muses, answer my question: How many in thy house are engaged in the contest for wisdom performing excellently?

PYTHAGORAS ANSWERS

I will tell thee then, Polycrates. Half of them are occupied with belles lettres; a quarter apply themselves to studying immortal nature; a seventh are all intent on silence and the eternal discourse of their hearts. There are also three women, and above the rest is Theano. That is the number of interpreters of the Muses I gather round me. (Theano is not counted—DS.)

There are 19 problems of this form, though some give several numerical values (e.g. the 5 years and the 3 women above). There is another problem of this general type which is a slight variation of the above.

Problem 3 (143). The father perished in the shoals of the Syrtis, and this, the eldest of the brothers, came back from that voyage with five talents. To me he gave twice two-thirds of his share, on our mother he bestowed two-eighths of my share, nor did he sin against divine justice.

3. Type B—Several equations

In the next problem, a day is 12 hours.

Problem 4 (6). 'Best of clocks, how much of the day is past?'. There remain twice two-thirds of what is gone.

There are 8 problems of this form. The next three problems extend the form, the first only slightly.

Problem 5 (13). We both of us together weigh twenty minae, I, Zethus, and my brother; and if you take the third part of me and the fourth part of Amphion here, you will find it makes six, and you will have found the weight of our mother.

Problem 6 (51).
A. I have what the second has and the third of what the third has.
B. I have what the third has and the third of what the first has.
C. And I have ten minae and the third of what the second has.

Problem 7 (49). Make me a crown weighing sixty minae, mixing gold and brass, and with them tin and much-wrought iron. Let the gold and bronze (i.e. brass—DS) together form two-thirds, the gold and tin together three-fourths, and the gold and iron three-fifths. Tell me how much gold you must put in, how much brass, how much tin, and how much iron, so as to make the whole crown weigh sixty minae.

There are two further problems of this basic form but which form a special subtype. This subtype is often attributed to Euclid, but not in the *Anthology*.

Problem 8 (145).
A. Give me ten minae and I become three times as much as you.
B. And if I get the same from you I am five times as much as you.

4. Type C — Cistern problems

These problems and their complications survived as favourites to torture school students well into this century. Recall that a day is 12 hours.

Problem 9 (135). We three Loves stand here pouring out water for the bath, sending streams into the fair-flowing tank. I on the right, from my long-winged feet, fill it full in the sixth part of a day; I on the left, from my jar, fill it in four hours; and I in the middle, from my bow, in just half a day. Tell me in what a short time we should fill it, pouring water from wings, bow, and jar all at once.

Problem 10 (7). I am a brazen lion; my spouts are my two eyes, my mouth, and the flat of my right foot. My right eye fills a jar in two days, my left eye in three, and my foot in four. My mouth is capable of filling it in six hours; tell me how long all four together will take to fill it.

There are four problems with three spouts and two with four spouts. There are two similar problems where the individual rates are given differently as in the following.

Problem 11 (136). Brick-makers, I am in a great hurry to erect this house. Today is cloudless, and I do not require many more bricks, but I have all I want but three hundred. Thou alone in one day couldst make as many, but thy son left off working when he had finished two hundred, and thy son-in-law when he had made two hundred and fifty. Working all together, in how many hours can you make these?

5. Miscellaneous

There are three problems remaining which I cannot squeeze into the above types. The latter two have an extra twist. For the first you need to know that one mina = 100 drachms. For the second, you must know there are three Graces.

Problem 12 (12). Croesus the king dedicated six bowls weighing six minae, each one drachm heavier than the other (i.e. than the previous one—DS).

Problem 13 (48). The Graces were carrying a basket of apples, and in each was the same number. The nine Muses met them and asked them for apples, and they gave the same number to each Muse, and the nine and the three had each of them the same number. Tell me how many they gave and how they all had the same number.

I have found Problem 13 in a recent book.

Problem 14 (144).
A. How heavy is the base I stand on together with myself.
B. And my base together with myself weighs the same number of talents.
A. But I alone weight twice as much as your base.
B. And I alone weight three times the weight of yours.

In addition there is an epigram which could be classified as arithmetical but is not really a problem.

Epigram 147. Answer of Homer to Hesiod when he asked the number of the Greeks who took part in the War against Troy.
There are seven hearths of fierce fire, and in each were fifty spits and fifty joints on them. About each joint were nine hundred Achaeans.

Solutions and comments

1. Diophantos lived 84 years.
2. 28.
3. Paton asserts that the elder has $1\frac{5}{7}$ talents, the younger $2\frac{2}{7}$ and the mother has 1. I get $\frac{15}{8}$, $\frac{5}{2}$ and $\frac{5}{8}$ respectively.
4. $5\frac{1}{7}$ hours are gone.
5. Zethus weighs 12, Amphion 8.
6. A has 45, B has $37\frac{1}{2}$, C has $22\frac{1}{2}$.
7. Gold $30\frac{1}{2}$, brass $9\frac{1}{2}$, tin $14\frac{1}{2}$, iron $5\frac{1}{2}$.
8. $A = 15\frac{5}{7}$, $B = 18\frac{4}{7}$.
9. $\frac{1}{11}$ of a day.
10. $\frac{12}{37}$ of a day $= 3\frac{33}{37}$ hours. Paton remarks that some commmentators tried to avoid fractions.
11. $\frac{2}{5}$ of a day.

12. The weights are $97\frac{1}{2}$, $98\frac{1}{2}$, $99\frac{1}{2}$, $100\frac{1}{2}$, $101\frac{1}{2}$, and $102\frac{1}{2}$.
13. The twist is that you have not enough data to solve the problem. Any multiple of 12 apples works. If we have $12a$ apples, then each Grace has $4a$ at first and each Grace and Muse has a after sharing.
14. Again there is insufficient information. Let A be the weight of statue A and a be the weight of its base and let B and b be similarly defined for B. Then we have $A+a = B+b$, $A = 2b$, $B = 3a$, and the most we can deduce is $A = 4a$, $B = 3a$, $b = 2a$.

Epigram 147. $315\,000 = 7 \times 50 \times 900$.

It is clear from the problems that most of them were well known before Metrodorus. D. E. Smith traces the cistern problems back to Heron (c. 50BC) and traces its many descendants (reference 2).

References

1. W. R. Paton, *The Greek Anthology*, Vol. 5, Book XIV, pp. 25–105 (Loeb Classical Library, Heinemann, London, 1916-1918).
2. D. E. Smith, *History of Mathematics*, Vol. 2, pp. 532–541 (Dover, New York, 1958).

Powers of 3

Mr. L. B. Dutta of Keshabpur, Jessore, Bangladesh, has sent us the following sums.

$$
\begin{aligned}
1 &= 3^0 \\
1+2 &= 3^1 \\
2+3+4 &= 3^2 \\
2+3+4+5+6+7 &= 3^3 \\
5+6+7+\ldots+13 &= 3^4 \\
5+6+7+\ldots+22 &= 3^5 \\
14+15+16+\ldots+40 &= 3^6 \\
14+15+16+\ldots+67 &= 3^7 \\
41+42+43+\ldots+121 &= 3^8
\end{aligned}
$$

Can you continue the pattern for 3^9, 3^{10} etc. and can you find a general formula?

Graph Embeddings

P. A. FIRBY AND C. F. GARDINER, *University of Exeter*

Dr Firby is a lecturer in pure mathematics at Exeter. He bowls. Mr Gardiner is also a lecturer in pure mathematics at Exeter. He does not bowl, but has previously contributed to *Mathematical Spectrum* (Beauty in mathematics, Vol. 16 (1983/84), pages 78–84.)

1. Surfaces

We all have our own intuitive notions of what a surface is. However, our vague intuitions are no adequate foundation for a precise theory able to solve significant problems. Let us begin, therefore, by giving a simple but precise definition of what we shall mean by a surface in this article. Later we shall relate it to more familiar notions.

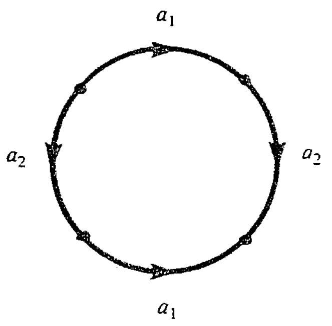
Definition 1.1. A surface is a regular $2n$-gon (i.e. a regular plane polygon with $2n$ sides) marked in the following way. Each member of the set $\{a_1, a_2, \ldots, a_n\}$ is used to label two distinct edges of the polygon and then each edge is marked with an arrow, so that the arrows on edges marked with the same label point in opposite senses relative to movement round the perimeter of the polygon.

For simplicity, we draw our $2n$-gons as circular discs with $2n$ vertices equally spaced around the circumference. Figure 1.1 (a) gives some idea of what we have in mind.

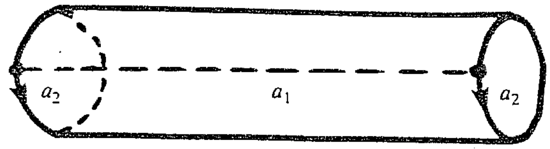
But how is the 'surface' of this definition related to our everyday notion of surface? To see this, let us interpret the $2n$-gon with labelled edges in the following way.

Take the points on one edge labelled a_i to represent the same points in space as the corresponding points on the second edge labelled a_i. More suggestively, imagine the pair of edges labelled a_i $(i = 1, 2, \ldots, n)$ to be identified, or glued together, with the corresponding arrows pointing in the same direction. When we do this, we get a set of points in three-dimensional space which looks like a surface in the everyday sense. We can realise this in a concrete way by 'cutting out' the $2n$-gon in paper, say, and then glueing together as suggested above. Such a realisation is called a *paper model* of the surface. In doing this, it should be borne in mind that, since we are interested in the surfaces from the topological point of view, the rigid Euclidean shape is not important. We are allowed to stretch and distort our shapes providing that no tearing or cutting takes place.

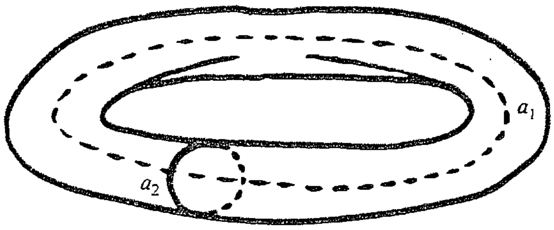
In identifying pairs of edges of the $2n$-gon, we are allowed some latitude. For example, we get the same set in three-dimensional space no matter in what order we identify pairs of edges. This is illustrated in figure 1.1. First



(a) Dots indicate vertices of the 4-gon

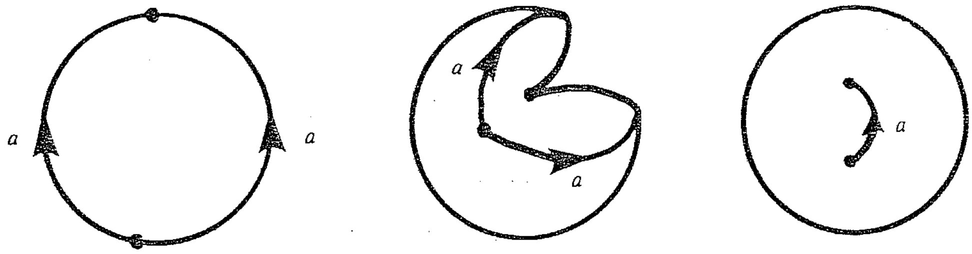


(b)

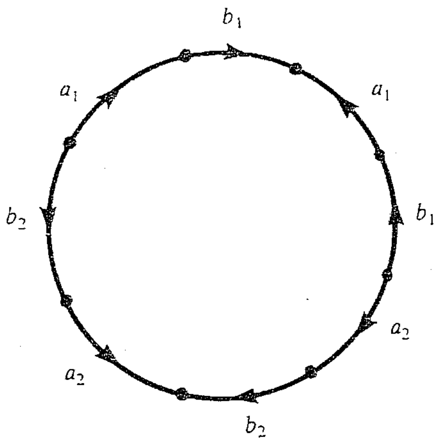


(c)

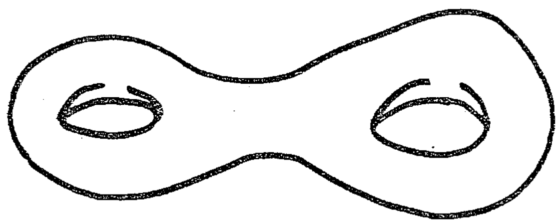
Figure 1.1. Torus T



(a) Sphere S



(b) Double torus $2T$



(c) Space model of $2T$

Figure 1.2

identifying the edges a_1 of the surface shown in figure 1.1 (a), we get the hollow cylinder in figure 1.1 (b). Then, identifying the edges a_2 by joining the circular boundaries of this cylinder, we get the torus T shown in figure 1.1 (c). Reversing the order of these operations gives exactly the same surface as far as topology is concerned. This means that by some stretching and squeezing without cutting or tearing, we can distort one shape into the other. Figure 1.2 shows two more examples of surfaces and their realisations as sets in three-dimensional space. In figure 1.2 (a), we have a sphere denoted by S, and figure 1.2 (b) shows a two-holed torus, or double torus, which we denote by $2T$. In general, an m-holed torus is denoted by mT.

A fundamental theorem in surface topology implies that any surface satisfying definition 1.1, when realised as above in three-dimensional space, looks like either a sphere S or an m-holed torus mT. This means that any surface given by our definition, when realised in three-dimensional space, looks like the surface of a solid object in everyday life. The converse is also true i.e. the surface of any solid object in everyday life is the realisation in three-dimensional space of some labelled $2n$-gon, as given in definition 1.1.

For ease of future reference, it is convenient to call such a realisation a *space model* of the surface.

It is usually rather awkward to work with the space models of surfaces. For this reason, we use the $2n$-gons of definition 1.1. In doing so, it is useful to be able to find the number of holes a surface has without having to go through the construction process which realises it in three-dimensional space. This can be done as follows.

First we count the number of distinct points on the space model of the surface produced by the vertices of the $2n$-gon. Rather than try to describe the process in the general case, we illustrate it for the surface shown in figure 1.3. To facilitate our description, we give the following definition.

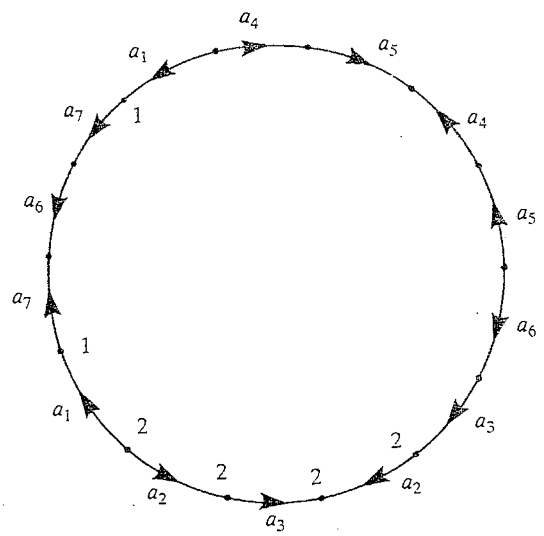


Figure 1.3

Definition 1.2. The *final vertex* of an edge is the vertex towards which the arrow on that edge points. The other vertex of the edge is called the *initial vertex*.

We begin by attaching the label 1 to the final vertex of an edge a_1. When the edges a_1 are identified, this vertex is attached to the final vertex of the second edge a_1. Hence we attach the label 1 to this vertex also. Now this latter vertex is also the initial vertex of an edge a_7. However, when we look at the initial vertex of the second edge a_7, we find that we have returned to our starting point. This means that these two vertices labelled 1 represent a single point on the space model of the surface. The other vertices are distinct from this point, so we start our labelling process afresh by attaching the label 2 to the final vertex of an edge a_2, and the same label to the final vertex of the second edge a_2. This latter vertex is also a final vertex for an edge a_3, so we also attach the label 2 to the final vertex of the second edge a_3. Continuing in this way, using figure 1.3, we find that all other vertices are labelled 2, and therefore represent just one point on the space model of the surface.

Let v denote the total number of points on the space model of the surface represented by the vertices of the $2n$-gon. (Here $v = 2$.) Is there a connection between the number of holes in the surface and this number v?

Using the above method on figures 1.1 and 1.2 gives the data shown in table 1.1.

Table 1.1

Surface ($2n$-gon)	Number of holes	v	n
Sphere S	0	2	1
Torus T	1	1	2
Double torus $2T$	2	1	4

Definition 1.3. The number of holes in the surface M is called the *genus* of the surface and is denoted by $g(M)$.

From table 1.1 we conjecture the following formula:

$$g(M) = \tfrac{1}{2}(1+n-v).$$

Although we shall not do so here, we can prove that this formula does hold.

Exercise 1.1. With figure 1.2(b) as a guide, by identifying the edges of the surface in figure 1.3 find the number of holes in the surface and hence check the above formula.

Notice that the same surface from a topological point of view may be described by many different $2n$-gons, i.e. the *space models* of the surfaces defined by the $2n$-gons may be distorted into one another by suitable stretching and squeezing without cutting or tearing. However, the above formula enables us to determine quickly and easily which surface mT is defined by any given $2n$-gon.

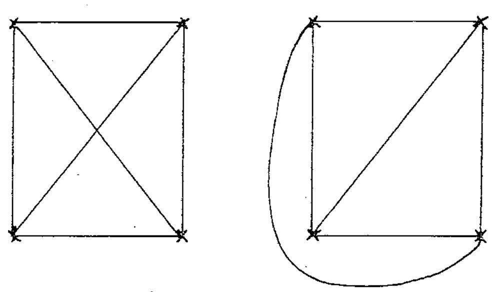
2. Graphs

There are several different definitions of the term graph. As far as this article is concerned, we shall work with a restricted type of graph defined as follows.

Definition 2.1. A graph G is a finite set of points in the plane, called the *vertices* of G, and a finite set of lines (not necessarily straight) in the plane, called the *edges* of G, which join certain pairs of distinct vertices in such a way that any pair of distinct vertices determines at most one edge. A graph is said to be *planar* if it can be drawn so that its edges meet only at vertices.

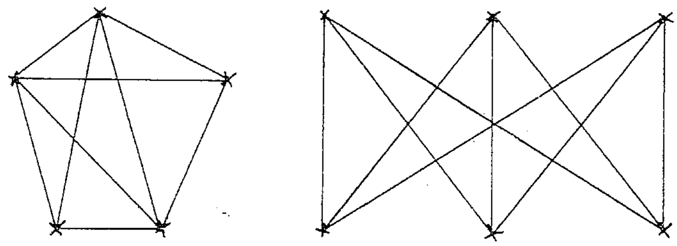
For example, figure 2.1 shows two drawings of the same graph. Drawing (b) shows that the graph is planar. Vertices are indicated by crosses.

In contrast, figure 2.2 shows two graphs, denoted by K_5 and $K_{3,3}$ respectively, which are non-planar. It is not possible to draw these graphs on a plane so that the edges do not cross.



(a) (b)

Figure 2.1



(a) K_5 (b) $K_{3,3}$

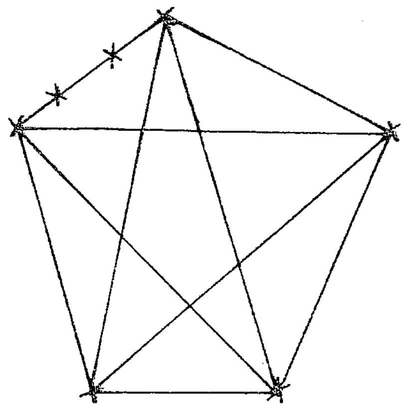
Figure 2.2

Naturally, if any part of a graph G looks like K_5, $K_{3,3}$, or any other non-planar graph, then G too must be non-planar.

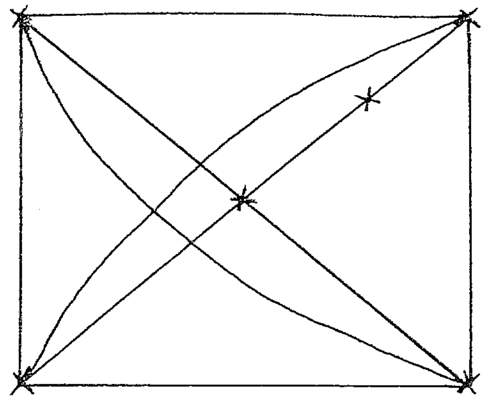
It is convenient to give the following definitions.

Definition 2.2. The graphs G and H are *homeomorphic* if they are isomorphic (i.e. essentially the same graph, as, for example (a) and (b) in figure 2.1), or if each graph can be obtained from a graph J by the introduction of new vertices on the edges of J.

The latter is illustrated in figure 2.3, where the homeomorphic graphs G and H can each be obtained from K_5 in this way.



G

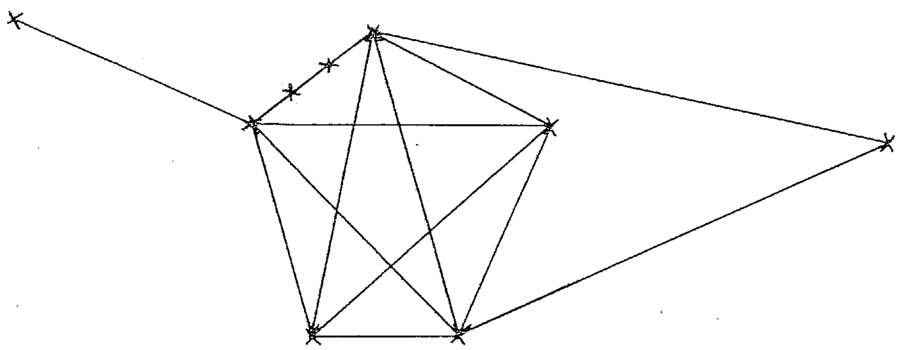


H

Figure 2.3

Definition 2.3. We call the graph A a *subgraph* of the graph B if each vertex of A is a vertex of B and each edge of A is an edge of B.

For example, the graph G of figure 2.3 is a subgraph of the graph L shown in figure 2.4.



L

Figure 2.4

We are now in a position to state a famous theorem of Kuratowski which tells us exactly when a graph is planar.

Kuratowski's theorem. A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.

This situation is described by saying that K_5 and $K_{3,3}$ form a set of *forbidden graphs* for the plane. From Kuratowski's theorem we deduce that the graph L in figure 2.4 is non-planar.

Exercise 2.1. Use Kuratowski's theorem to show that the graph in figure 2.5 is non-planar.

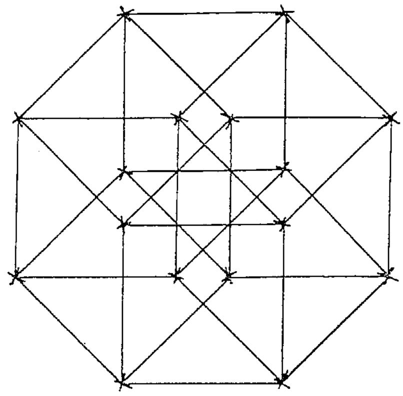


Figure 2.5

3. Embedding graphs in surfaces

By drawing the 2-gon representing the sphere S around the graph G, as shown in figure 3.1, we can see that G is planar if and only if it can be drawn on a sphere so that the edges of the graph meet only at vertices. This leads us to the following definition.

Definition 3.1. A graph G is said to be *embedded* in the surface M if G is drawn on M so that its edges meet only at vertices of G.

Notice that when we draw an edge joining two vertices of the graph, we can take the edge of the graph up to an edge of the surface. It then 'reappears' at the corresponding point on the equivalent edge of the surface. This can be seen in figure 3.2.

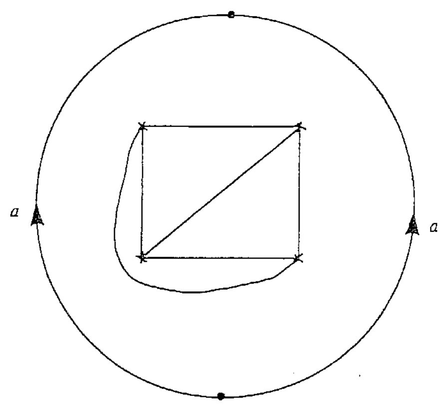


Figure 3.1 Planar graph on a sphere

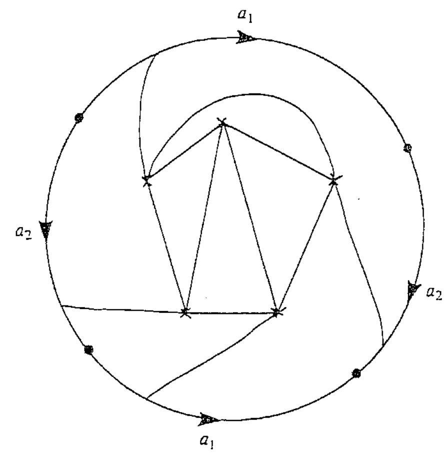


Figure 3.2 K_5 embedded in a torus

The more holes a surface M has, i.e. the higher the value of $g(M)$, the more edges of M we have available to use in this way, thereby increasing the likelihood that we can embed a given complicated graph in the surface. For example, K_5 cannot be embedded in S, but it can be embedded in T, as shown in figure 3.2.

Exercise 3.1. Find an embedding of $K_{3,3}$ in T.

Since a graph is planar if and only if it can be embedded in a sphere, Kuratowski's theorem tells us that $\{K_5, K_{3,3}\}$ forms a set of forbidden graphs for S. On the other hand, as we have just seen, K_5 and $K_{3,3}$ are *not* forbidden graphs for T.

It would be useful to have a theorem in the style of Kuratowski's theorem for T. In fact, it was proved in 1968 by Vollmerhaus (reference 3) that there is a finite set of forbidden graphs for *any* given surface M. Unfortunately no one has yet been able to display a set of forbidden graphs for any surface other than S.

Since this form of embedding theorem appeared to have reached an impasse, a different approach was tried. Instead of looking for the types of graph which can be embedded in a given surface, mathematicians turned their attention to particular classes of graphs, and asked the question: 'How many holes does a surface need in order to embed this type of graph?' In other words, they fixed the type of graph and looked for the appropriate surface.

This approach turned out to be much more productive. A result of König plays a large part in the solutions. For our purpose let us state it as follows.

König's theorem. Let G be any 'reasonable' graph (this includes any graph considered here) embedded in the surface M, and suppose that M has the smallest possible number of holes for such an embedding. Let G have V vertices and E edges, and suppose that the embedding produces F faces on M. Then

$$V-E+F = 2-2g(M).$$

Under the conditions of König's theorem, since each face has at least three edges, we have $3F \leqslant 2E$. Hence

$$2-2g(M) \leqslant V-E+\tfrac{2}{3}E = V-\tfrac{1}{3}E.$$

Thus

$$g(M) \geqslant \tfrac{1}{2}(2-V+\tfrac{1}{3}E).$$

Notice that the right-hand side of this inequality involves quantities associated only with the graph G.

It is convenient to give the following definition.

Definition 3.2. The smallest value taken by the genus of the surfaces in which the graph G can be embedded is called the *characteristic* of G and is denoted by $\Gamma(G)$.

From the above, for any graph G,

$$\Gamma(G) \geqslant \tfrac{1}{2}(2-V+\tfrac{1}{3}E).$$

Since $\Gamma(G)$ must be a positive integer, this leads to

$$\Gamma(G) \geqslant \{\tfrac{1}{2}(2-V+\tfrac{1}{3}E)\}, \tag{1}$$

where $\{x\}$ denotes the least integer greater than or equal to the number x. This inequality can be applied to various classes of graphs where we can give values V and E, so obtaining lower bounds for $\Gamma(G)$.

We consider three such classes below.

(a) *The complete graphs* K_p. (Here K denotes Kuratowski not Komplete!) We denote by K_p the complete graph on p vertices. It has p vertices, each vertex being joined to every other vertex by an edge. The graph K_5 described earlier is one such example. For K_p, $V = p$ and $E = {}^pC_2$. Hence, from (1) above, we have

$$\Gamma(K_p) \geqslant \{\tfrac{1}{2}[2-p+\tfrac{1}{6}p(p-1)]\}.$$

Thus

$$\Gamma(K_p) \geqslant \{\tfrac{1}{12}(12-7p+p^2)\}. \tag{2}$$

Notice that, if $p = 4$, the right-hand side of (2) is zero, while figure 3.1 shows that K_4 can be embedded in a sphere S. We conclude that equality holds in (2) and $\Gamma(K_4) = 0$. Similarly, if $p = 5$, (2) and figure 3.2 show that equality holds in (2) and $\Gamma(K_5) = 1$. To prove that we always have equality in (2), so that

$$\Gamma(K_p) = \{\tfrac{1}{12}(12-7p+p^2)\},$$

it is necessary to display, for each $p \geqslant 3$, an embedding of K_p in a surface M with genus

$$g(M) = \{\tfrac{1}{2}[2-p+\tfrac{1}{6}p(p-1)]\}.$$

In fact this has been done for every value of p. In 1891, Heffter did this for $p = 8$ to 12. Ringel found the embedding for $p = 13$ in 1952. Continued work up to 1968 by Ringel, Youngs, Gustin, Terry and Welch settled all cases except for $p = 18$, 20 and 23. Finally, the embeddings for these remaining cases were found at the end of the 1960s by Jean Mayer, a professor of French literature.

Thus for all values of p we have

$$\Gamma(K_p) = \{\tfrac{1}{12}(12-7p+p^2)\}. \tag{3}$$

Exercise 3.2. If you evaluate (3) above for $p = 6$ and 7, you will find that not only can K_5 be embedded in T but so can K_6 and K_7. Can you find such embeddings?

(b) *The complete bipartite graphs $K_{m,n}$.* The complete bipartite graph $K_{m,n}$ consists of two disjoint sets of m and n vertices respectively, with each vertex in one set being joined by an edge to each vertex in the other set. Thus the graph $K_{3,3}$ discussed above is a special case. An embedding of the complete bipartite graph $K_{m,n}$ has no triangles. This means that $4F \leqslant 2E$. Hence

$$2-2\Gamma(K_{m,n}) = V-E+F \leqslant V-\tfrac{1}{2}E.$$

Bearing in mind that $\Gamma(K_{m,n})$ is a positive integer, we conclude that

$$\Gamma(K_{m,n}) \geqslant \{\tfrac{1}{2}(2-V+\tfrac{1}{2}E)\}.$$

Since $K_{m,n}$ has $m+n$ vertices and mn edges, we deduce the inequality

$$\Gamma(K_{m,n}) \geqslant \{\tfrac{1}{2}(2-m-n+\tfrac{1}{2}mn)\}.$$

Once again, to prove that we have equality here, so that

$$\Gamma(K_{m,n}) = \{\tfrac{1}{2}(2-m-n+\tfrac{1}{2}mn)\}, \tag{4}$$

we must display suitable embeddings.

In exercise 3.1 you have done this for $m = n = 3$. In fact this has now been done for all values of $m, n \geqslant 3$. The final embeddings needed to prove formula (4) were found in 1965.

(c) *The graphs C_n of the cube.* Many other simple classes of graphs can be described and you may like to try out the above procedure for such classes. For example, take the graph C_n formed from the vertices and edges of an n-dimensional cube. To help in forming a picture of this, we remark that a square is a two-dimensional cube, a cube is a three-dimensional cube (!) and figure 2.5 shows a four-dimensional cube.

Obviously C_2 and C_3 can be embedded in S. On the other hand, you have shown in exercise 2.1 that C_4 cannot. These considerations lead naturally to the following exercise.

Exercise 3.3. Find an inequality for the characteristic $\Gamma(C_n)$ of the graph C_n and determine the exact value of $\Gamma(C_4)$.

References

1. P. A. Firby and C. F. Gardiner, *Surface Topology* (Ellis Horwood, Chichester, 1982).
2. F. Harary, *Graph Theory* (Addison-Wesley, Reading, U.S.A, 1969).
3. H. Vollmerhaus, Über die Einbettung von Graphen in zweidimensionale orientierbare Mannigfaltigkeiten kleinsten Geschlechts. *Beiträge zur Graphentheorie* (Teubner, Leipzig, 1968).

Computer Column

MIKE PIFF, *University of Sheffield*

You have a garden path. It starts at your back door, and leads in a straight line (well, fairly straight, anyway— it was a pity about that plum tree getting in the way, but that's beside the point) to the bottom of your long, thin garden. The trouble is, you find it rather monotonous, in that endless miscellany of all shades of grey, stretching off, seemingly, to infinity.

One day, whilst visiting the local builders' suppliers, you discover that square slabs are available in colours other than grey! Yes, you can get blue ones, red ones and even green ones! You decide to brighten up your garden! Rushing home, with your estate car rearing on its hind wheels, you set to work, to dispel that drab image once and for all. You tear up those old slabs, and donate them to the local council tip. Then you set to work... blue, red, green, blue, red, green, blue,... .

But hang on a minute, isn't this as monotonous as light grey, medium grey, dark grey, light grey,... ? OK, start again... blue, red, green, blue, green, red,... .

You quickly realise that as there are only finite number of sequences of length n using b, r and g, you must content yourself with *some* repetition, but decide that, starting anywhere, and walking n slabs down the garden the next n slabs should never *exactly* repeat the same pattern, for *any* n. Thus, *rbrgbgbrgb*... is definitely out, as, whichever way it is continued, it leads to such a repitition. You have fortunately bought large numbers of all three colours, as you will definitely need all three.

Next day, with a path so far only ten slabs long, you decide to use your micro to take the backache out of this problem. You feed in the total length of your garden in units of one slab, and, a short while later, lay your path.

So, what was *your* program to solve the problem?

Dividing by 7 or 13

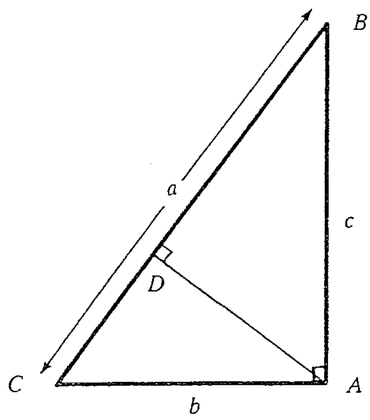
The following curious fact was sent in by Anthony Higgins of Harlow. Suppose we want to test whether a number is divisible by 7. Take 27 720 as an example. Divide by 50 to give 554 with remainder 20. Now divide 554 by 50 to give 11 with remainder 4. Now divide 11 by 50 to give 0 with remainder 11. The sum of the remainder is $20+4+11 = 35$, which is divisible by 7. It follows that 27 720 is divisible by 7. This works for every number. Can you explain why, and can you devise a similar test for divisibility by 13?

Letter to the Editor

Dear Editor,

A proof of Pythagoras' theorem

I had set a group of pupils the task of amassing proofs of Pythagoras' theorem. So my mind was on these matters and I thought of the following proof in bed this morning ('being indolent', to quote the biography of C. F. Gardiner in *Mathematical Spectrum* Volume 16, Number 3).



Since the areas of similar triangles are proportional to the squares of corresponding sides,

$$\triangle BCA : \triangle ACD : \triangle BAD = a^2 : b^2 : c^2.$$

But

$$\triangle BCA = \triangle ACD + \triangle BAD.$$

So

$$a^2 = b^2 + c^2.$$

According to A. W. Siddons and K. S. Snell (*A New Geometry*, Cambridge University Press, 1938), 'it is probable that Pythagoras himself used the proof ... which we give ...', referring to a proof essentially that quoted by C. F. Gardiner. How do they know?

Yours sincerely,
JOHN MACNEILL
(The Royal Wolverhampton School)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

17.1. (Submitted by L. B. Dutta, Maguradanga, Keshabpur, Jessore, Bangladesh)
Consider, for example, the numbers 12 493 526 and 92 493 576. Now

$$(6+5+9+2)-(2+3+4+1) = 12, \qquad (6+5+9+2)-(7+3+4+9) = -1,$$

which have remainders 1 and 10, respectively, on division by 11, and these are also the remainders when the original numbers are divided by 11. Prove a general result on divisibility by 11 suggested by these examples.

17.2. (Submitted by J. N. MacNeill, The Royal Wolverhampton School)
Given a quadrilateral with sides of fixed lengths, show that the maximum area is when the quadrilateral is cyclic.

17.3. A destroyer sights the periscope of an enemy submarine and immediately heads towards it. The submarine retracts its periscope and the captain of the destroyer decides that the submarine will proceed in a straight line (of unknown bearing) with speed 10 knots. If the destroyer's speed is 30 knots, how should the captain steer in order to ensure that, at some subsequent time, the destroyer will pass over the submarine?

Solutions to Problems in Volume 16, Number 2

16.4. How many squares are crossed by the diagonal of a 1984×1066 chessboard?

Solution by Richard Dobbs (Winchester College)

We consider an $\alpha \times \beta$ chessboard. The number of squares crossed by the diagonal equals

$$\begin{aligned}&1+\text{the number of horizontal lines crossed}\\ &\quad+\text{the number of vertical lines crossed}\\ &\quad-\text{the number of corners which the diagonal passes through.}\end{aligned}$$

For a diagonal to go through a corner, there must exist positive integers a and b such that $a < \alpha$, $b < \beta$ and $a/b = \alpha/\beta$. Now $1066 = 2\times13\times41$ and $1984 = 2^6\times31$, so $a = 13\times41$, $b = 2^5\times31$ and the diagonal passes through one corner. Hence the number of squares crossed by the diagonal is $1+1065+1983-1 = 3048$.

Also solved by T. B. H. Hall (Rugby School), Matthew Richards (Millfield School), James McKee (King Edward VI School, Louth) and Ruth Lawrence (St. Hugh's College, Oxford).

16.5. Show that the product of n consecutive integers is divisible by $n!$

Solution 1 by Chris Aubrey (The Royal Grammar School, High Wycombe)

We may as well suppose that the consecutive integers are all positive, say $m+1, m+2, \ldots, m+n$ with $m \geqslant 1$. (If $m = 0$, the result is obvious, if one of the integers is 0 then again it is obvious, and if the integers are all negative, then just change the sign of each one.) But now

$$\frac{(m+1)(m+2)\ldots(m+n)}{n!} = \frac{(m+n)!}{m!n!} = {}^{m+n}C_n,$$

the binomial coefficient, and this is well known to be an integer (it is the number of ways of choosing n objects from $m+n$ when the order choice is immaterial). Hence $n!$ divides $(m+1)(m+2)\ldots(m+n)$.

In case you object that Chris Aubrey's solution begs the question by using the fact that the binomial coefficient is an integer, here is a second solution.

Solution 2 by Matthew Richards (Millfield School)

We pick up from the end of solution 1. We have to show that $(m+n)!$ is divisible by $m!n!$. Consider a prime number p, and denote by N_p and M_p the numbers of times that p divides $(m+n)!$ and $m!n!$, respectively. Then

$$N_p = \sum_{k=1}^{\infty} \left[\frac{m+n}{p^k}\right],$$

where $[x]$ denotes the integer part of the fraction x. But, for any x and y,

$$x+y \geqslant [x]+[y].$$

So

$$[x+y] \geqslant [x]+[y].$$

Hence

$$N_p \geqslant \sum_{k=1}^{\infty} \left[\frac{m}{p^k}\right] + \sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right] = M_p.$$

This is true for all p, so $(m+n)!$ is divisible by $m!n!$

Also solved by T. B. H. Hall (Rugby School) and Ruth Lawrence (St. Hugh's College, Oxford).

16.6. A snowplough clears snow at a constant rate during a snowstorm in which the snow is falling at a constant rate. Between 10 a.m. and 11 a.m. it travels 1 mile, and between 11 a.m. and noon it travels $\frac{1}{2}$ mile. At what time did the snowstorm begin?

Solution by T. B. H. Hall (Rugby School)

If the storm begins at time $t = 0$, then at time t hours the depth of snow is proportional to t. The speed of the plough is proportional to $(\text{depth})^{-1}$, so that we can write the speed of the plough as kt^{-1} for some constant k. If 10 a.m. is time T, s denotes the distance travelled at time t and S the distance travelled at time T, then

$$\frac{ds}{dt} = kt^{-1}$$

and

$$\int_S^{S+1} ds = k\int_T^{T+1} \frac{dt}{t}, \qquad \int_{S+1}^{S+\frac{3}{2}} ds = k\int_{T+1}^{T+2} \frac{dt}{t},$$

from which

$$1 = k\log\frac{T+1}{T}, \qquad \tfrac{1}{2} = k\log\frac{T+2}{T+1},$$

or

$$\frac{T+1}{T} = e^{1/k}, \qquad \frac{T+2}{T+1} = e^{1/2k}.$$

Hence

$$\left(\frac{T+2}{T+1}\right)^2 = \frac{T+1}{T},$$
$$T(T+2)^2 = (T+1)^3,$$
$$T^2+T-1 = 0,$$
$$T = \tfrac{1}{2}(\sqrt{5}-1).$$

It follows that the storm started at 9.23 a.m.

Also solved by Richard Dobbs (Winchester College), Matthew Richards (Millfield School) and Ruth Lawrence (St. Hugh's College Oxford).

The '1984' problem This was to represent the integers 1 to 100 in terms of the digits of 1984 in their correct order using only the operations $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, !. We pointed out that we had been successful for all but two integers. The offending integers were 63 and 83. We received some ingenious suggestions for these integers, all involving the breaking or at least bending of the rules. Here is a selection:

$$63 = -1^9+8^{\sqrt{4}} = -1^9+\sqrt{(8^4)}, \qquad 83 = -1^9+84 = 19+8^{\sqrt{4}}$$

(from Bruce Andrews, Archie Sprott, Paul Tully, N. D. North, R. S. Scowen, A. J. Mansfield, Jerry Callum, Mark Dixon, Nigel Clark, David King and Michael Pegler),

$$63 = (-1+9)\times 8-\cos 4\pi$$

(from Mark Dixon, Nigel Clark, David King and Michael Pegler),

$$83 = -(1\times .\dot{9})+84$$

(from Archie Sprott). But the prize for ingenuity goes to Alvin Kam (Repton School), whose suggestions come nearest to obeying the rules:

$$63 = -1+[(/\sqrt{9})\times 8\times 4!] = -1+[\sqrt{9}\div\div(8\times 4!)]$$

($x\div\div y = y/x$ on a calculator),

$$83 = -1\times(\ldots\sqrt{\sqrt{9}})+84,$$

where $\ldots\sqrt{\sqrt{\ }}$ means take infinitely many square roots.

Book Reviews

Wheels, Life and other Mathematical Amusements. By MARTIN GARDNER. W. H. Freeman and Company, New York, 1983. Pp. ix + 261. Hardback £13·95.

This is the latest collection of Martin Gardner's 'Mathematical Games' columns from *Scientific American*, to which is added further material bringing up to date the situation of the various mathematical recreations. Readers who have some of the author's earlier books will only need telling that the standard is as high as ever, with an abundance of information and entertainment. Others, who have still to discover the delights of Martin Gardner's works, should find this volume an excellent introduction to the pleasure and stimulation of mathematical games and problems.

The word 'Life' in the book's title refers to a solitaire pastime and the final three chapters of the book are devoted to this one topic. Incidentally, this is not typical, as the other nineteen chapters are more or less independent. 'Life' can be played on a large, theoretically infinite chessboard or Go-board with pieces which are placed in the squares of the board. However, the most convenient way to play is on a computer. So if you want a change from the familiar chase/reach/score computer games, then 'Life' could be the answer. Other recreational uses of the computer include finding the number of ways of folding up a 3×3 map, finding the shortest ruler with 12 marks so that the 66 distances between pairs of marks are all integers and all different, and deciding whether all 32 chess men can be placed legally on a chess board so no move is possible.

At one point the author writes of 'a type of two-person game that so far has no agreed-on name.' This highlights a problem facing anyone who wishes to find out what others have done on a particular topic in recreational mathematics. Others may have described the topic in different terms, so that contact is difficult, there is no exchange of ideas, and there is duplication of effort. Fortunately, Martin Gardner's column has helped in some way to overcome this by acting as a central exchange point for different people's ideas and providing an opportunity for agreed names to develop.

On page 167, in figure 106, the 6 should be 0 and the 3 and 5 should be interchanged.

As a final attraction, the book contains '...problems that are ridiculously simple to state but so difficult and deep that long-lasting fame awaits the first person to solve them.'

University of Sheffield KEITH AUSTIN

Mathematical Snapshots. By H. STEINHAUS. A reprint of the third American edition. Oxford University Press, 1983. Pp. 311. £5·95 (paperback).

'What does one do all day when one is a mathematician?' This is the question which prompted Professor Steinhaus to write his book some forty-five years ago. Over the past decades it has become something of a mathematical classic and it is now reissued as a paperback. One of the main features is the use made of diagrams and photographs throughout; but, in spite of these aids, the arguments and hints are sometimes tantalizingly brief. However, there are treats in store on almost every page for the reader who is prepared to exercise his mind (and often his pencil too). Here are a few samples: the beautiful geometrical argument to show that $\sqrt{2}$ cannot be a rational number; the striking use of symmetry applied to the problem of locating fragments of shrapnel within the human body; the fair division of a cake (How does one ensure that everyone receives what *he* considers to be his fair share?); the optical illusion of the 'following eyes' of a portrait. This is a book to dip into rather than to read from beginning to end; wherever you choose to open the book you will find something to set you thinking.

The text has not been updated since the 1969 edition, and in particular, no mention is made that the famous 4-colour problem, described in Chapter 12, was solved in 1976.

University of Sheffield HAZEL PERFECT

Learning Mathematics with Micros. By ADRIAN OLDKNOW and DEREK SMITH. Ellis Horwood Limited, Chichester, 1983. Pp. 268. £15·50 (paperback £8·95).

The microcomputer will no doubt change many things, and the teaching—and learning—of mathematics is no exception. Until textbooks are written with the microcomputer in mind, we shall need books like this one to supply the computing ideas, and also to write programs for those of us who have not the time—or the ability—to write our own. There are plenty of good ideas in this book, and the programs are simple enough for those with minimal competence to use them and even adapt them to their individual needs.

The authors include over a hundred short programs which illustrate various branches of mathematics. The topics covered include: functions and graphs, solving equations, matrices and geometry, statistics and probability, limits and calculus, approximation techniques and numerical analysis.

The style is easy and relaxed, and the descriptions of the programs are clear and helpful. The book is primarily aimed at teachers of mathematics, but it is very suitable for the student as part of a self-teaching programme. Many mathematical ideas can be clarified and reinforced by 'playing' on the computer.

There is also some 'fun' mathematics, such as magic squares and number chains. A program is given to generate the number chain 'divide by two if even, multiply by three and add one if odd', but, of course, the tantalizing conjecture that you will eventually reach the number 1 wherever you start cannot be settled by the computer!

University College of Swansea GEOFFREY V. WOOD

Colson News Volume 1 Number 1 (1984). Eds ELLIS HILLMAN, ALVIN PAUL and CEDRIC SMITH. Pp. 12. £1·00 for four issues. (Enquiries to C. A. B. Smith, 141 Portland Crescent, Stanmore, Middlesex HA7 1LR, England.)

This little quarterly magazine aims to persuade us to turn our arithmetic upside down! 'Two-way' numbers are introduced, and examples given of how their use can simplify the basic operations of arithmetic. More advanced numerical topics, as well as some recreational mathematics, will be pursued from this unusual angle in forthcoming issues, and the history of mathematical notation will be discussed. It is worth having a look at this first issue.

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