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CONICS AND CONVERSES

LEROY F. MEYERS

In many textbooks on plane analytic geometry, "ellipse" and "hyperbola" are defined essentially as follows:

The ellipse with constant $2a$ and foci F and F' consists of just those points P in the plane for which

$$FP + F'P = 2a.$$

The hyperbola with constant $2a$ and foci F and F' consists of just those points P in the plane for which

$$|FP - F'P| = 2a.$$

In the above definitions, it is assumed that $a \geq 0$. (Some authors also assume that $F \neq F'$, but we do not make this assumption now.)

Derivations of the standard equations of the ellipse and hyperbola are then given. Suppose $F = (c, 0)$ and $F' = (-c, 0)$, where $c \geq 0$. For the ellipse we have:

(0) the point (x, y) is on the ellipse with constant $2a$ and foci at $(c, 0)$ and $(-c, 0)$,

$$(1) \quad \sqrt{(x+c)^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a,$$

$$(2) \quad \sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2},$$

$$(3) \quad (x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2,$$

$$(4) \quad a\sqrt{(x-c)^2 + y^2} = a^2 - cx,$$

$$(5) \quad a^2((x-c)^2 + y^2) = (a^2 - cx)^2,$$

$$(6) \quad (a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

Now what does this derivation mean? Equation (1) is a direct translation of the definition as applied to line (0), and each line following (1) is a consequence of preceding lines. Hence we have proved:

If the point (x, y) is on the ellipse with constant $2a$ and foci $(c, 0)$ and $(-c, 0)$, then $(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$.

However, we have *not* proved that (6) is an equation of the ellipse, for we have not shown the converse:

If $(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$, then (x, y) is a point on the ellipse with constant $2a$ and foci $(c, 0)$ and $(-c, 0)$.

This note is, in a certain sense, an amplification of the editor's remarks about the solvers of Problem 177 [1977: 52], only two of whom indicated that in order to *solve* a locus problem one must find a new condition which is satisfied by *all those* and *only those* points which satisfy the given condition. In the present case, we have found a new condition (equation (6)) which is satisfied by *all* points satisfying the given condition (line (0): being on the ellipse); but we have *not* shown that the equation is satisfied by *only those* points which are on the ellipse.

In many cases, it is easy to supply the converse argument by merely reversing the order of the steps. It is not so easy in this case. (Try it!) I made a random survey of analytic geometry and calculus textbooks and found that, for those books using the definitions given above:

- 9 did not mention that a converse argument was needed;
- 7 mentioned that the converse argument could be obtained by merely reversing the steps (one of these said it was "not obvious");
- 2 left the converse as an exercise (one with no hint, and one with a confusing hint); and
- 7 gave a correct converse argument ([1] - [7]).

It is interesting to note that if we want to find an equation for the hyperbola with constant $2a$ and foci at $(a,0)$ and $(-a,0)$, that is, if we replace equation (1) by

$$(1') \quad \sqrt{(x+a)^2 + (y-0)^2} - \sqrt{(x-a)^2 + (y-0)^2} = \pm 2a$$

and carry through calculations as before, then we find that we obtain precisely the same equation (6) as for the ellipse. "But," you exclaim, "a hyperbola isn't an ellipse!" Of course it isn't! However for an ellipse we have, by the triangle inequality,

$$2a = FF' \leq FP + F'P = 2a, \quad \text{so } a \geq c;$$

whereas for the hyperbola

$$2a = FF' \geq |FP - F'P| = 2a, \quad \text{so } a \leq c.$$

If $a = c > 0$, then the "ellipse" reduces to the closed segment FF' , whereas the "hyperbola" reduces to what is left of the line through F and F' when the open segment FF' is removed. If $a = c = 0$, then the "ellipse" reduces to just one point, and the "hyperbola" is the entire plane. If $c > a = 0$, then the "hyperbola" is the perpendicular bisector of the segment FF' . These degenerate cases are customarily excluded. The nondegenerate cases thus require that the assumption $0 < 2a \neq FF'$ be part of the definition. Thus $0 < a \neq c$, and equation (6) now implies

$$(1') \quad \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1,$$

where

(8e) $a > c \geq 0$ for an ellipse and

(8h) $c > a > 0$ for a hyperbola.

We now go through the converse argument for the ellipse. (The argument for the hyperbola is left as an exercise for the reader. In other words, the author is too lazy to write it down, or he can't do it himself. It is somewhat trickier.)

In reversing the steps from (7) to (1), we note that the only troublesome stages occur when we try to derive steps (4) and (2) from steps below them. In each of these cases, a square root is taken, and we must be sure that we are getting the same square root on both sides. Although the left members of (4) and (2) are obviously nonnegative, the signs of the right members must be determined from the values of a , c , x , and y .

Suppose that (7) and (8e) hold. Then $\frac{y^2}{x^2 - c^2} \geq 0$, and so

(9) $x^2 \leq a^2$, that is, $-a \leq x \leq a$.

Hence

$$a^2 - cx \geq a^2 - ca = a(a - c) > 0,$$

and so (4) is correct. Then, from (4) and (9),

$$2a - \sqrt{(x - c)^2 + y^2} = 2a - \frac{a^2 - cx}{a} = a + \frac{cx}{a} \geq a - \frac{ca}{a} = a - c > 0,$$

and so (2) is correct. Hence the point (x, y) is on the ellipse *if and only if* (7) holds, provided that $a > c$. (But of course you knew this already!)

An elegant derivation of (7) from (1), which can be easily modified to give a derivation of (7) from (1'), is found in [8]. But, according to [9], this derivation is due to the Marquis de L'Hospital [10] and is described in [11]. Unfortunately, the converse derivation from (7) to (1) or (1') is not indicated, but it is not difficult to supply.

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3. Richard E. Johnson, Neal H. McCoy and A.F. O'Neill, *Introduction to mathematical analysis*; Holt, Rinehart and Winston, New York, 1962, pp. 153-156. (Awkwardly stated proof.)

4. Charles B. Morrey, Jr., *University calculus with analytic geometry*; Addison-Wesley, Reading (Mass.), 1962, pp. 244-245. (Proof same as given here.)

5. J.M.H. Olmsted, *Calculus with analytic geometry*, vol. 1; Appleton-Century-Crofts, New York, 1966, pp. 486-488. (Direct proof squares (1) directly without going to (2); then only one step in the converse requires special treatment.)

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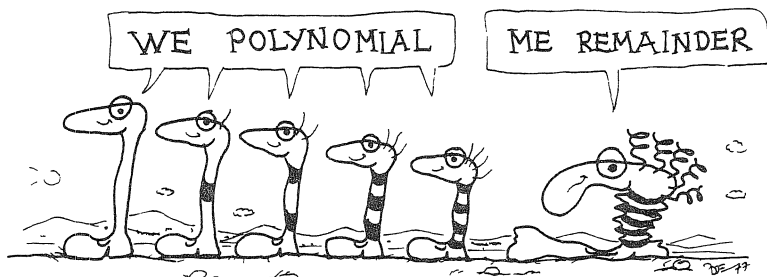
Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210.

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FOOTIES AT THE TAYLOR'S



Andres Dunkels,
University of Luleå, Sweden

Reminiscences

HIGH SCHOOL: LETTERS, LOGIC AND SIN

R. ROBINSON ROWE

In 1908, farm children learned all there was (or all they needed) to know in the eight primary grades. So I thought, too: I could add, subtract, multiply, divide, extract square and cube roots, and write numbers as big as I pleased. No one from our country school had ever gone on to high school. The nearest one was 6 miles away in Grand Rapids. There were no school buses: one could walk $2\frac{1}{2}$ miles to the streetcar line and ride on for a nickel, but nickels didn't grow on our apple trees. Besides, one had to pass a written examination and pay tuition.¹ But my sister and I passed the examination, our college-bred parents enrolled us, and a generous grandfather paid the tuition. We went. The first from Hurd School.

The requisite algebra did not appeal to me, since I thought I knew all about mathematics. I was soon jolted. Here was a new game. You didn't know a number, but you knew something about it. You couldn't express it with numerals, so used a letter, like x . Then whatever you knew about the number was stated as a modification of x , making an equation. With that, you found x . Miraculous! The game was competitive—to beat classmates to the right answer. The game grew to more letters until x , y , z were like 1, 2, 3, and in three semesters we went through factoring and simultaneous quadratics. I could now find out how old was Ann. Most amazing, x could be the littlest or biggest number, positive or negative, integer or surd.²

Having finished algebra, I expected geometry would teach me even more exciting things about numbers, but I was soon disillusioned. Figures, yes, but of lines instead of numerals. Day after day, we were asked to prove what had been proved 2000 years before—many things so obvious as to be taken for granted.³ But, thanks be to a wonderful teacher⁴, whom I appreciated more and more with the years. To her, syllogistic logic was the backbone of geometry, and geometry a means of teaching logic.

I recall one day when a reciting pupil had copied his proof of a theorem on the blackboard and tried to explain it orally. Teacher looked at it with a grimace, then

¹Only \$40, but a fortune to a farmer.

²Real ones; imaginaries would come much later.

³As Will Rogers said, when asked for his birth certificate, "Down in Oklahoma, when you see a man alive and walking around, you assume he has been born."

⁴Alice James, also Vice-Principal, Central High School, Grand Rapids.

said, "Major premise, All rooms have ceilings. Minor premise, There are 28 people in this room. Conclusion, Someone is an idiot." This was really eloquent, for it was something we 27 pupils would always remember—that the three parts of a syllogism must cohere. It came back to me often, especially years later when reading Dodgson's *Symbolic Logic*.

Each class was assigned to a "Session Room," where we studied when not attending classes. In my Junior year, the Master of our session room was Mr. Snell, and his class in Trigonometry recited in the front of the room while I was supposedly studying at my desk halfway back. As I was slated to take this subject in my Senior year, I watched and listened with curiosity.

I had supposed that trigonometry was an advanced geometry (just lines and circles and logic), so I was pleased to see the blackboard regularly filled with a lot of numerals, meaningless though they were. But they were intermixed with puzzling nonwords, like COS, CSC, SEC, and other irrelevant words, like TAN, LOG and SIN. That last intrigued me: as a clergyman's son, I knew sin was something you didn't do, but which must be very pleasurable because so many others did, even though they knew they would go to Hell. At last, I was going to learn about SIN.

I was, of course, disappointed in that expectation, but thrilled nevertheless. At last, arithmetic, algebra and geometry were all put together to solve real practical problems: how high is a tree you can't climb? how wide is a river you can't wade? SIN, COS, TAN, LOG and all the others were just numbers, a new kind of numbers, wonderful numbers. I wonder now why so many high schools have dropped trigonometry from their curricula. Allegedly, too few students would elect it. Perhaps, as it did to me, the word "trigonometry" implies to many "advanced geometry" instead of "practical application of algebra and geometry."

In epilogue⁵, that following summer (1912), my father took me along with him and a friend to inspect the progress of a miner being grubstaked by this friend and two others in Grand Rapids. The site was in the Bitterroot Mountains near Adair, Idaho. The prospect was a quartz outcrop on the mountainside and the miner had been tunneling in from a lower level to intercept it. He had tunneled over 200 feet; the question was, how much farther?

He had taped the distance from tunnel adit to outcrop and with an Abney level

⁵Needing some prologue. As the only farm boy in my Freshman class, I was dubbed "Hayseed." After two years of walking in from the farm, the family moved into the city. During my other two years at high school, I studied piano, violin, went to dancing school, wore a necktie and polished my shoes, but the boys still called me "Hey, Hayseed."

had measured the slope of the mountainside and the dip of the quartz lode. With my new-found friend, trigonometry, I solved⁶ the triangle and found he had to go another 25 feet. He was elated; he had feared he had missed the lode or it had pinched out.

2701 Third Avenue, Sacramento, CA 95818.

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LETTERS TO THE EDITOR

Dear Editor:

As a result of the announcement you were good enough to publish in [1977: 88], I received from Canadian and other EUREKA readers several unsolved problems for inclusion in the forthcoming new edition of my book *Open Questions in Mathematics*.

Unfortunately, the bag containing all my Canadian mail was stolen. So I hope you will allow me to ask, through the pages of EUREKA, all Canadians who responded to my original appeal to resubmit their contributions to me. At the same time, I renew my invitation to all EUREKA readers to send me their favorite unsolved problems. Manuscripts, not exceeding two or three pages, should be submitted in duplicate and in final form for reproduction.

DAGMAR HENNEY,
Associate Professor of Mathematics,
George Washington University,
Washington, D.C. 20052.

Dear Editor:

It has come to my attention that some of the personal exchanges in the columns of EUREKA have at times become excessively vitriolic and intemperate, thereby abandoning one of our most cherished scientific principles—reliance on proven fact rather than on speculative conjecture. I refer specifically to the controversy that has been raging regarding the positive or negative aspects of Edith Orr's pulchritude—whether it is to be measured in millihelens or in milliclocks [1977: 129, 224].

I am pleased to report that I have acquired a portrait that will put this question to rest for all time—a rare drawing of Edith Orr (see figure), executed by her third husband, a talented painter and sculptor. It must be said that his portraits were not as bad as they were painted. As a sculptor, he was known chiefly for his absent-mindedness, a trait that finally spelled the end of his marriage to Edith. It was his habit to kiss his statues good-bye and chisel on his wife.

This rare portrait of Edith came into my possession



⁶Without sine-cosine tables! How? Easy: the Abney level has a slope-tangent scale.

at one of the weekly meetings of the Benevolent Order of Solitudinous Hermits. After Edith's husband finished his keynote address, "Sociability—its Cause and Cure," I prevailed upon him to part with his masterpiece for the benefit of the readers of EUREKA.

With my customary altruism, I now offer this treasure to you with the hope that it will soon be recognized as one of the most outstanding items in your archives. If you choose to conduct a poll on the evaluation of Edith Orr's beauty, may I be the first to cast my vote: 427 milliclocks.

LEON BANKOFF,
6360 Wilshire Blvd.,
Los Angeles, CA 90048.

Editor's comment.

Dr. Bankoff is greatly mistaken. The formidable dowager in full sail pictured on the opposite page may well be named Edith Orr, but she is not *our* Edith Orr.

In fact, I believe Dr. Bankoff is not being completely candid with us. He clearly knows the Mrs. Orr of the portrait (than whom it is impossible to look more crushingly uxorial). He shows us only a bust of the lady, but it is easy to guess that her costume ends in a bustle¹, which certainly dates her—and him. His preposterous story of obtaining the portrait from her third husband at a weekly meeting of the Benevolent Order of Solitudinous Hermits is clearly all BOSH—which, fittingly enough, is the acronym of the Order. This third husband, claims Dr. Bankoff, is a talented painter and sculptor. Now it happens that Dr. Bankoff is himself a talented painter and sculptor. His painterly ability is vouched for by his EUREKA Valentine drawing [1977: 60], and a comparison of this with Mrs. Orr's portrait shows that the latter was probably drawn by the same hand. As a sculptor, Dr. Bankoff is best known for his statue² of Galatea. (I am not making this up: the statue exists.) Mrs. Orr, who was then between husbands, was an old flame of his, but she left him in a huff when she discovered that a young neighbour, whose name was Cynisca³, had posed for the statue.

What may have happened is this. Dr. Bankoff saw the name of Edith Orr in EUREKA, thought she was *his* Edith Orr, and one night, when in his cups, with the embers of remembrance glowing more brightly in his heart(h)—for absinthe makes the heart grow

¹For those who have forgotten or never knew, a bustle can best be described as a deceitful seatful.

²What are you doing down here? Statue² is not an invitation to read a footnote. It means, literally, *statue squared*, for a statue of Galatea is a statue of a statue.

³It is probably no coincidence that Cynisca is also the name of Pygmalion's wife in W.S. Gilbert's comedy *Pygmalion and Galatea* (1871). But it is not clear whether Dr. Bankoff's Cynisca was named after Gilbert's or the other way around.

fonder—decided to try to reestablish contact with his long-lost inamorata by writing the above letter to the editor. The depth of his resurgent feeling for her can best be gauged by his generously rating her at *only* 427 milliclocks.

To help Dr. Bankoff, I have had discreet inquiries made. Aided by the fact that Clayton W. Dodge, who lives in Orono, Maine, claimed to have seen Mrs. Orr [1977: 224], my investigators soon located her in the nearby town of Kenduskeag, Maine, where she heads the local chapter of WAVE-EM (Women Against Violence to Everything—Except Men). Dr. Bankoff can take it from there—or (my advice) leave it.

Our own Edith Orr is a sweet young thing—not a frowzy old battle-axe. A gifted Canadian poetess, she has ably assisted this French-speaking editor by serving as the arbiter of linguistic propriety in English for EUREKA. As any thoroughly modern young woman, she has enthusiastically espoused the cause of woman's lib and is assiduously preparing herself to assault the bastions of male power. In particular, she stands ready and willing to take over the editorship of this journal when, as must soon happen, the present editor falters in his arduous task of turning out an issue every month. She argues persuasively that the transition could be effected very smoothly: all she would have to do would be to drop the last letter of each of her names; and, she adds, the fact that she knows no mathematics whatsoever would ensure the strict impartiality of all her editorial decisions.

She would not thank me for saying that her pulchritude rating is in the very high millihelens, for she abhors being considered as a sex object. Indeed she constantly strives, and fails spectacularly, to attenuate those charms with which a generous nature has endowed her. Recently R. Robinson Rowe wrote to give her some good grandfatherly advice. He said:

$$\int_0^{\infty} LA \, dy \implies SE \frac{dx}{dt} = 0,$$

meaning, of course, that a completely integrated lady implies that the differentiation of sex in time becomes zero.

Have no fear, Mr. Rowe. No red-blooded male will ever be able to look at *our* Edith Orr without wanting to shout lustily: *Vive la différence!*

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A riddle

What is purple and commutes?

An Abelian grape, of course.

LEIGH JANES,
Rocky Hill, Connecticut.

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PROBLEMS - - PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before April 1, 1978, although solutions received after that date will also be considered until the time when a solution is published.

301. *Proposed by Herman Nyon, Paramaribo, Surinam.*

The following cryptarithmic decimal subtraction is dedicated to two of the outstanding digit-delvers of our day, J.A.H. Hunter and Charles W. Trigg:

$$\text{HUNTER} - \text{TRIGG} = \text{DIGITS}.$$

There are, as one would expect, exactly two solutions. That our two protagonists tower equally in the field of digital recreations is shown by the fact that the sums of the digits of HUNTER and TRIGG in one solution are equal, respectively, to the sums of the digits of TRIGG and HUNTER in the other solution.

302. *Proposed by Leroy F. Meyers, The Ohio State University.*

Show that if p is a prime, then $p^2 + 5$ is not a prime. (I first heard of this problem from H.J. Ryser.)

303. *Proposed by Viktors Linis, University of Ottawa.*

Huygens' inequality $2 \sin \alpha + \tan \alpha \geq 3\alpha$ was proved in Problem 115.

Prove the following hyperbolic analogue:

$$2 \sinh x + \tanh x \geq 3x, \quad x \geq 0.$$

304. *Proposed by Viktors Linis, University of Ottawa.*

Prove the following inequality:

$$\frac{\ln x}{x-1} \leq \frac{1 + \sqrt[3]{x}}{x + \sqrt[3]{x}}, \quad x > 0, x \neq 1.$$

305*: *Proposed by Bruce McColl, St. Lawrence College, Kingston, Ontario.*

How many distinct values does $\cos(\frac{1}{3} \sin^{-1} x)$ have? What is the product of these values?

306. *Proposed by Irwin Kaufman, South Shore H.S., Brooklyn, N.Y.*

Solve the following inequality, which was given to me by a student:

$$\sin x \sin 3x > \frac{1}{4}.$$

307. *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

It was shown in Problem 153 that the equation $ab = a + b$ has only one solution in positive integers, namely $(a, b) = (2, 2)$. Find the least and greatest values of x (or y) such that

$$xy = nx + ny,$$

if n, x, y are all positive integers.

308. *Proposed by W.A. McWorter, Jr., The Ohio State University.*

Some restaurants give only one pat of butter (of negligible thickness!) with two rolls. To get equal shares of butter on each roll, one can cut the butter square along a diagonal with a knife.

(a) What regular n -gons can be cut in half with only a straightedge?

(b) What convex n -gons can be cut in half with a straightedge and compass (saucer?)?

309. *Proposed by Peter Shor, student, California Institute of Technology.*

Let ABC be a triangle with $a \geq b \geq c$ or $a \leq b \leq c$. Let D and E be the midpoints of AB and BC, and let the bisectors of angles BAE and BCD meet at R. Prove that

(a) $AR \perp CR$ if and only if $2b^2 = c^2 + a^2$;

(b) if $2b^2 = c^2 + a^2$, then R lies on the median from B.

Is the converse of (b) true? (See Problem 210 [1977: 197; 1978: 13]).

310. *Proposed by Jack Garfunkel, Forest Hills H.S., Flushing, N.Y.*

Prove that

$$\frac{a}{\sqrt{a^2 + b^2}} + \frac{b}{\sqrt{9a^2 + b^2}} + \frac{2ab}{\sqrt{a^2 + b^2} \cdot \sqrt{9a^2 + b^2}} \leq \frac{3}{2}.$$

When is equality attained?

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Mathematics is not as difficult as ordinarily supposed and even division may be mastered by diligence.

MELANCHTHON (Luther's associate)

Quoted in *Mathematics*, edited by Samuel Rapoport and Helen Wright, Washington Square Press, New York, 1964, p. 13.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

210. [1977: 10, 160, 196] *Proposed by Murray S. Klamkin, University of Alberta.*

P, Q, R denote points on the sides BC, CA and AB, respectively, of a given triangle ABC. Determine all triangles ABC such that if

$$\frac{BP}{BC} = \frac{CQ}{CA} = \frac{AR}{AB} = k \quad (\neq 0, 1/2, 1),$$

then PQR (in some order) is similar to ABC.

IV. *Comment by W.J. Blundon, Memorial University of Newfoundland.*

In a comment on this problem [1977: 197], the editor asked for properties of triangles ABC with $a > b > c$ and $2b^2 = c^2 + a^2$. Because this formula is symmetrical in a and c , it is sufficient to require b to be the "middle" side. We will, in a sense, find all triangles ABC such that

$$2b^2 = c^2 + a^2, \quad \text{with } a \geq b \geq c \text{ or } a \leq b \leq c, \quad (1)$$

by assuming A and C to be fixed and finding the locus of B. Note that no strictly isosceles (nonequilateral) triangle satisfies (1), for $b=c$ or $c=a$ or $a=b$ all imply $a=b=c$.

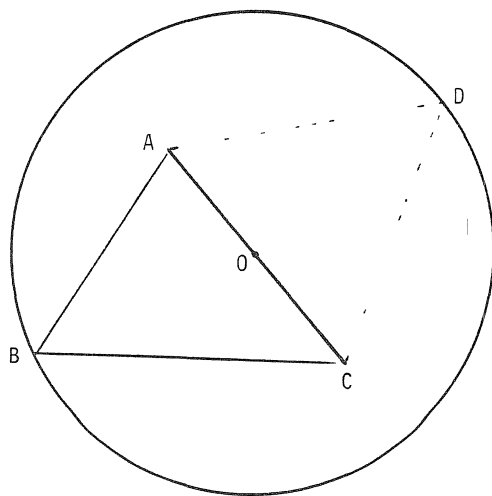
Vertices A and C being given, let O be the midpoint of AC (see figure). By a well-known formula for the length of a median (see [1], for example), ABC satisfies (1) if and only if

$$OB = \frac{1}{2} \sqrt{2c^2 + 2a^2 - b^2} = \frac{\sqrt{3}}{2} b.$$

If we draw equilateral triangle ACD, then $OD = \frac{\sqrt{3}}{2} b$, and so the locus of B is the circle with centre O and radius OD.

V. *Comment by Leon Bankoff, Los Angeles, California.*

Triangles which satisfy (1) have been very extensively investigated in the past, and it is possible to mention here only a very small part of what is known about them.



For example, in a series of articles in *Mathesis* [2] are listed, mostly without proof and with an abundance of earlier references, 69 properties of such triangles. I state and prove below 10 of the more obvious properties from [2].

These triangles are consistently referred to in [2] as *automedian triangles*, a name suggested by merely one of their many properties (see Property 1 below). But it seems to me inappropriate that any single geometrical property should be used to name these triangles and overshadow all their other beautiful properties. Accordingly, I propose to rename them *Root-Mean-Square (R-M-S) triangles*, after their neutral algebraic characterization (1), which implies $b = \sqrt{(c^2 + a^2)/2}$. But I won't be surprised if nobody else salutes when I run this name up the flagpole.

Let m_a, m_b, m_c denote the medians and k_a, k_b, k_c the symmedians to sides a, b, c , respectively, of triangle ABC; let H, O, G, and K denote the orthocenter, circumcenter, centroid, and symmedian point, respectively; and as usual let $2s = a + b + c$, h_b = altitude from B, and R = circumradius.

$$\text{PROPERTY 1.} \quad m_a = \frac{\sqrt{3}}{2}c, \quad m_b = \frac{\sqrt{3}}{2}b, \quad m_c = \frac{\sqrt{3}}{2}a.$$

Proof. This follows immediately upon substituting $2b^2 = c^2 + a^2$ in the median formulas

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}, \quad m_b = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}, \quad m_c = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}.$$

Note that m_a, m_b , and m_c can be rearranged by translation to form a triangle inversely similar to triangle ABC (hence the qualifier *automedian* given to triangle ABC). The ratio of similitude of corresponding sides implies that the area of the median triangle is $3/4$ that of ABC. It is also seen that $\Sigma m_a = s\sqrt{3}$, a well-known property of the equilateral triangle, which is a trivial special case of an R-M-S triangle.

$$\text{PROPERTY 2.} \quad b^2 = 2ca \cos B.$$

Proof. This follows from (1) and $b^2 = c^2 + a^2 - 2ca \cos B$.

$$\text{PROPERTY 3.} \quad 2 \cot B = \cot C + \cot A.$$

Proof. Since $a^2 - b^2 = b^2 - c^2$, we have

$$\sin^2 A - \sin^2 B = \sin^2 B - \sin^2 C,$$

from which

$$\sin(A+B) \sin(A-B) = \sin(B+C) \sin(B-C)$$

or

$$\frac{\sin(A-B)}{\sin(B+C)} = \frac{\sin(B-C)}{\sin(A+B)}.$$

Now, dividing each side by $\sin B$ and expanding the numerators, we get

$$\cot B - \cot A = \cot C - \cot B,$$

and the desired result follows.

$$\text{PROPERTY 4. } 2 \cos 2B = \cos 2C + \cos 2A.$$

Proof. As in the proof of Property 3, we have

$$\sin^2 A - \sin^2 B = \sin^2 B - \sin^2 C,$$

and this implies

$$\cos^2 A - \cos^2 B = \cos^2 B - \cos^2 C.$$

Subtracting these two relations gives

$$\cos 2A - \cos 2B = \cos 2B - \cos 2C,$$

from which the desired result follows.

$$\text{PROPERTY 5. } AG^2 + BG^2 + CG^2 = b^2.$$

Proof.

$$AG^2 + BG^2 + CG^2 = \frac{4}{9}(m_a^2 + m_b^2 + m_c^2) = \frac{1}{3}(a^2 + b^2 + c^2) = b^2.$$

$$\text{PROPERTY 6. } 4(\Delta ABC) = b^2 \tan B.$$

Proof. By Property 2,

$$\cos B = \frac{b^2}{2ca} = \frac{\sin^2 B}{2 \sin C \sin A};$$

then we easily get

$$\frac{2 \sin C \sin A}{\sin B} = \frac{\sin B}{\cos B}, \quad 2h_b = b \tan B,$$

and finally

$$4(\Delta ABC) = 2bh_b = b^2 \tan B.$$

$$\text{PROPERTY 7. } 2m_b^2 = m_c^2 + m_a^2.$$

Proof. This follows immediately from (1) and Property 1.

$$\text{PROPERTY 8. } 2BH^2 = CH^2 + AH^2.$$

Proof. This is an immediate consequence of the well-known relations $BH = 2R \cos B$, etc., and $2 \cos^2 B = \cos^2 C + \cos^2 A$ from the proof of Property 4.

PROPERTY 9. If B' and B'' are the vertices of equilateral triangles constructed externally and internally on side AC , then the angle $B'BB''$ is a right angle.

Proof. It is known (see [3]) that $B'B^2 + B''B^2 = a^2 + b^2 + c^2$. Since $B'B'' = b\sqrt{3}$, it follows that $B'B''^2 = 3b^2 = a^2 + b^2 + c^2$ and that triangle $B'BB''$ is a right triangle.

PROPERTY 10. $\frac{m_c}{k_b} = 2 \cos B.$

Proof. Since

$$m_b = \frac{1}{2} \sqrt{2c^2 + 2a^2 - b^2} \quad \text{and} \quad k_b = \frac{ca}{c^2 + a^2} \sqrt{2c^2 + 2a^2 - b^2},$$

it follows from Property 2 that

$$\frac{m_b}{k_b} = \frac{c^2 + a^2}{2ca} = \frac{b^2}{ca} = 2 \cos B.$$

Editor's comment.

Further properties of R-M-S triangles will be found in Problem 309 in this issue.

Blundon has shown, in Comment IV, how to construct all R-M-S triangles geometrically. But it may be helpful, for purposes of illustration or neatness of calculation, to be able to generate specific examples of R-M-S triangles with, say, integral sides.

A fairly complete discussion of the Diophantine equation $2b^2 = c^2 + a^2$ (with $a > b > c$) can be found in Dickson [4], but a particularly simple derivation of a parametric solution for this equation occurs in Bankoff [5]. The result there is as follows:

All primitive solutions are given by

$$a = u^2 + 2uv - v^2, \quad b = u^2 + v^2, \quad c = \pm(u^2 - 2uv - v^2), \quad (2)$$

where $u > v$, with u, v relatively prime positive integers of different parity, and the sign is selected that renders c positive.

But not all solutions (2) yield R-M-S triangles: they must satisfy the triangle inequality as well. For example, $u = 5$ and $v = 2$ yield $a = 41$, $b = 29$, $c = 1$, and there is no triangle with these sides.

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3. R.A. Johnson, *Modern Geometry*, Houghton Mifflin, 1929; reissued as *Advanced Euclidean Geometry*, Dover, 1960, p. 220.
4. Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, 1952, Vol. II, pp. 435 ff.
5. Leon Bankoff, Solution to Problem 91 (A Long Probate), *Mathematics Magazine* (Nov-Dec 1968) 291-293.

232. [1977: 104, 238] Proposed by Viktors Lints, University of Ottawa.

Given are five points A, B, C, D, E in the plane, together with the segments joining all pairs of distinct points. The areas of the five triangles BCD, EAB, ABC, CDE, DEA being known, find the area of the pentagon ABCDE.

The above problem with a solution by Gauss was reported by Schumacher [*Astronomische Nachrichten*, Nr. 42, November 1823]. The problem was given by Möbius in his book (p. 61) on the Observatory of Leipzig, and Gauss wrote his solution in the margins of the book.

IV. *Solution by O. Bottema, Delft, The Netherlands.*

Given are five points A, B, C, D, E in the plane. We denote the areas of triangles

ABC, BCD, CDE, DEA, EAB

by

$a, b, c, d, e,$

respectively. If these five numbers are given, one asks for the area of the pentagon ABCDE. To the solutions in [1977: 238-240], we add here a construction for all pentagons satisfying the given data. Furthermore, we suppose that all areas concerned are given a sign, depending on the order of the vertices (the areas (ABC) and (ACB), for example, have opposite signs); hence any of the five numbers can be positive, zero, or negative.

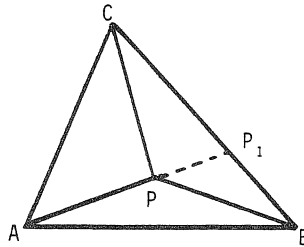
We make use of the barycentric coordinates of a point P with respect to a triangle ABC (see figure):

$$x = (PBC), \quad y = (PCA), \quad z = (PAB);$$

obviously,

$$x + y + z = (ABC).$$

It is easy to construct P if its coordinates are given; for example, if AP intersects BC at P_1 , we have $CP_1:P_1B = y:z$, etc.



As Klamkin remarked [1977: 240], the pentagon is not uniquely determined by the data: if we transform a solution by an equiaffine transformation, we obtain again a solution. This implies that we may take for ABC any triangle with area a , and the pentagon will be found if we calculate the barycentric coordinates of D and E.

Let $D = (x_1, y_1, z_1)$ and $E = (x_2, y_2, z_2)$. We have obviously

$$x_1 = b, \quad z_1 = a - b - y_1, \quad z_2 = e, \quad x_2 = a - e - y_2.$$

If F is the area of the triangle with vertices (x_i, y_i, z_i) , $i = 1, 2, 3$, we have

$$a^2 F = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Hence, from $(CDE) = c$ we get

$$a^2 c = \begin{vmatrix} b & y_1 & a - b - y_1 \\ a - e - y_2 & y_2 & e \\ 0 & 0 & a \end{vmatrix},$$

which reduces to

$$y_1 y_2 - (a - e) y_1 + b y_2 - ca = 0; \quad (1)$$

and similarly $(DEA) = d$ implies

$$y_1 y_2 + e y_1 - (a - b) y_2 - da = 0. \quad (2)$$

Subtracting (1) and (2) gives

$$y_1 - d = y_2 - c, \quad (3)$$

and it is easy to show that both expressions in (3) are equal to $a - \omega$, where ω is the area of pentagon ABCDE. Eliminating y_2 from (1) and (3), we obtain the following quadratic for y_1 :

$$y_1^2 - (a - b - c + d - e) y_1 - (ac + bd - bc) = 0.$$

The final results are

$$2y_1 = (a - b - c + d - e) \pm \sqrt{\Delta},$$

$$2y_2 = (a - b + c - d - e) \pm \sqrt{\Delta},$$

$$2\omega = (a + b + c + d + e) \mp \sqrt{\Delta},$$

with

$$\Delta = (a + b + c + d + e)^2 - 4(ab + bc + cd + de + ea).$$

There are two real solutions if $\Delta \geq 0$.

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240. [1977: 105, 264, 299] A solution by HERMAN NYON, Paramaribo, Surinam, confirms anew that the three solutions given in [1977: 264] are the only ones. This problem can now be considered as closed, and may CARL × F = GAUSS rest in peace along with CARL F. GAUSS.

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244. [1977: 130] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

Solve the following problem, which can be found in *Integrated Algebra and Trigonometry*, by Fisher and Ziebur, Prentice-Hall (1957) p. 259:

A rectangular strip of carpet 3 ft. wide is laid diagonally across the floor of a room 9 ft. by 12 ft. so that each of the four corners of the strip touches a wall. How long is the strip?

I. *Solution by Leroy F. Meyers, The Ohio State University.*

Let x and α be as shown in the figure;

then

$$x \cos \alpha + 3 \sin \alpha = 12,$$

$$3 \cos \alpha + x \sin \alpha = 9.$$

Putting $x = 3y$ and then eliminating successively $\sin \alpha$ and $\cos \alpha$ give

$$(y^2 - 1) \cos \alpha = 4y - 3,$$

$$(y^2 - 1) \sin \alpha = 3y - 4,$$

whence

$$(y^2 - 1)^2 = (4y - 3)^2 + (3y - 4)^2,$$

which reduces to

$$f(y) \equiv y^4 - 27y^2 + 48y - 24 = 0. \quad (1)$$

Note that

$$3 + x = PS + PQ \geq SQ \geq AB = 12 \quad \text{and} \quad x \leq AC = 15;$$

so we must have $9 \leq x \leq 15$ and hence $3 \leq y \leq 5$.

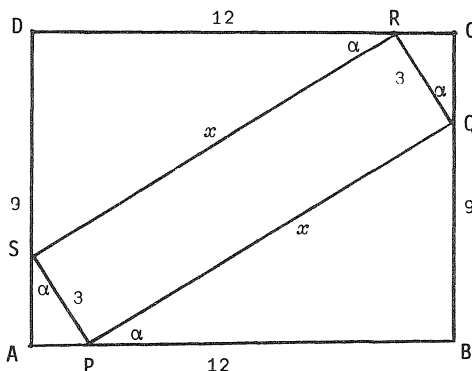
It will be shown below that $f(y)$ is irreducible over the rationals. Equation (1) has a satisfactory positive root $y \approx 4.085$, a negative root $y \approx -5.975$, and two imaginary roots. Thus the unique solution is

$$x = 3y \approx 3 \times 4.085 = 12.255.$$

By Gauss's theorem, if $f(y)$ is reducible over the rationals, then it can be factored into a product of polynomials with integral coefficients. None of the divisors of 24 is a root of (1); so $f(y)$ has no linear factor with integral coefficients. We have only left to try a factorization of the form

$$f(y) = (y^2 + ay + b)(y^2 + cy + d), \quad (2)$$

where a, b, c, d are integers. A comparison of the coefficients of y^n for



$n = 3, 1, 0$ in (1) and (2) yields the equations

$$a + c = 0, \quad ad + bc = 48, \quad bd = -24.$$

Then $c = -a$ and $a(d - b) = 48$, that is, $d - b$ divides 48. However, a direct calculation shows that, if b and d are appropriately paired divisors of -24 , then $d - b$ never divides 48. We conclude that $f(y)$ is irreducible over the rationals.

II. *Solution by Bruce McColl, St. Lawrence College, Kingston, Ontario.*

[Having found equation (1) as in solution I or otherwise], the classical method of solving a quartic (is it still being taught?) gives as the appropriate root

$$y = -\sqrt{z} + \sqrt{\frac{27}{2} - z + \frac{12}{\sqrt{z}}},$$

where

$$z = \frac{9}{2} + \frac{\sqrt[3]{49}}{4} \left(\sqrt{-9 + 4\sqrt{2}} + \sqrt{-9 - 4\sqrt{2}} \right).$$

Those who don't insist on exactness will find that

$$y \approx 4.08511 \ 58663 \ 45067$$

and so

$$x = 3y = 12.25534 \ 75990 \ 35201 \text{—approximately.}$$

Also solved by LEON BANKOFF, Los Angeles, California; DOUG DILLON and HUGH GRANT, Brockville, Ontario (jointly); J.D. DIXON, Haliburton Highlands H.S., Haliburton, Ontario; CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montréal, Québec; HERMAN NYON, Paramaribo, Surinam; DANIEL ROKHSAR, Susan Wagner H.S., Staten Island, N.Y.; R. ROBINSON ROWE, Sacramento, California; DAVID R. STONE, University of Kentucky, Lexington; CHARLES W. TRIGG, San Diego, California (two solutions); and KENNETH M. WILKE, Washburn University, Topeka, Kansas.

Editor's comment.

This problem is taken from a 1957 freshman text. How many freshpersons today would be able to handle this problem, which 1957 freshmen were supposed to take in their stride?

All the references were sent in by solvers, to whom the editor is grateful (especially to Trigg who sent in four).

REFERENCES

1. Problem E 1225, *American Mathematical Monthly*, 64 (1957) 114-116. (More references given here on p. 116.)
2. Problem E 1584, *American Mathematical Monthly*, 71 (1964) 209-210.

3. Robert Blasch, Solution to Problem 30, *The Pentagon*, 10 (Fall 1950) 42-43.
4. James F. Hurley, *Litton's Problematical Recreations*, Van Nostrand Reinhold, 1971, p. 205. (Problem 41.)
5. Samuel I. Jones, *Mathematical nuts for lovers of mathematics*, Samuel I. Jones (sic), Publisher, Nashville, Revised Edition, 1936, p. 178. (Problem 14. The problem is the same as our own, except that the room is 30×40 ft. They had rooms in those days.)
6. William R. Ransom, *One hundred mathematical curiosities*, Wallach, Portland, Maine, 1955, pp. 60-63.
7. Edmund H. Umberger, Solution to Problem 1275, *School Science and Mathematics*, 33 (June 1933) 678.

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245. [1977: 130] *Proposed by Charles W. Trigg, San Diego, California.*

Find the volume of a regular tetrahedron in terms of its bimedian b .
(A bimedian is a segment joining the midpoints of opposite edges.)

Solution by Sahib Ram Mandan, Indian Institute of Technology, Kharagpur, India.

By drawing pairs of parallel planes through the three pairs of opposite edges of any tetrahedron T , we obtain (uniquely) a parallelepiped P whose edges are equal and parallel to the bimedians of T ([1], p. 67), and the volume of T is one-third that of P ([1], p. 95). If T is isosceles (when opposite edges are equal in pairs), then P becomes rectangular ([1], p. 103). Consequently, when T is regular (all edges equal), P becomes a cube whose volume is b^3 , where b is any one of the three equal bimedians of T . Hence the volume V of a regular tetrahedron in terms of its bimedian b is $V = \frac{1}{3}b^3$.

Also solved by LEON BANKOFF, Los Angeles, California; LOUIS H. CAIROLI, graduate student, Kansas State University, Manhattan, Kansas; DOUG DILLON, Brockville, Ontario; CLAYTON W. DODGE, University of Maine at Orono; HERMAN NYON, Paramaribo, Surinam; DANIEL ROKHSAR, Susan Wagner H.S., Staten Island, N.Y.; R. ROBINSON ROWE, Sacramento, California; and the proposer (two solutions). Two incorrect solutions were received.

REFERENCE

1. Nathan Altshiller-Court, *Modern Pure Solid Geometry*, Chelsea, 1964.

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246. [1977: 131] *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

Let p_i denote the i th prime and let P_n denote the product of the first n primes. Prove that the number N defined by

$$N = \frac{P_n}{p_i p_j \dots p_r} \pm p_i p_j \dots p_r,$$

where p_i, p_j, \dots, p_r are any of the first n primes, all different, or unity, is a prime whenever $N < p_{n+1}^2$. (This is known as Tallman's Formula.)

Example:

$$N = \frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11}{2 \cdot 3 \cdot 5} \pm 2 \cdot 3 \cdot 5 = 107 \text{ or } 47, \text{ both primes.}$$

I. *Solution by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

The last sentence in the proposal should read: "Prove that if the number N is defined by..., then $|N|$ is a prime whenever $1 < |N| < p_{n+1}^2$."

As N is defined, p_i does not divide N for $i = 1, 2, \dots, n$. So if $|N|$ is composite it has at least two prime factors each greater than p_n , that is, $|N| \geq p_{n+1}^2$.

II. *Comment by Leroy F. Meyers, The Ohio State University.*

The restriction $1 < |N|$ is necessary, as can be seen from the examples $\pm(2-1)$, $\pm(2 \cdot 3 - 5)$, $\pm(3 \cdot 5 - 2 \cdot 7)$. Are there any others?

III. *Comment by the proposer.*

T.J. Stieltjes essentially used this formula in 1890 to prove the infinitude of primes [1], and his proof is closely related to the classical one of Euclid. Stieltjes' variation of Euclid's proof occasionally appears as an exercise in texts on number theory (see [2], for example).

But the use of this method to *generate* primes appears to be due to Malcolm H. Tallman. The method is mentioned in Hunter and Madachy [3]¹, and it is there stated that the proof of Tallman's formula was published in [6]. However, what appears in [6] is simply a letter to the editor by Malcolm H. Tallman, which it is worth quoting briefly: "The claim by Mr. Kravitz in his article [our reference [4] below] that 'No polynomial can represent primes exclusively' is punctured by the following which was written by me in 1935 and published in the March 1952² SCRIPTA MATHEMATICA." Tallman then states his formula¹ but offers no proof, merely a *verification* that it holds for $n=3$. In [5] Tallman states his formula¹, verifies it for $n=3$, and

¹But incompletely, as in our proposal (Editor).

²Wrong. It should be 1950 [as in our reference [5] below].

suggests that more primes can be generated by replacing any prime in the formula by one of its powers, giving as examples the replacement of 2 by 2^5 , 3 by 3^3 , etc. (and making several mistakes in the process).

Also solved by DOUG DILLON, Brockville, Ontario; CLAYTON W. DODGE, University of Maine at Orono; LEROY F. MEYERS, The Ohio State University (solution as well); BOB PRIELIPP, The University of Wisconsin-Cshkosh; R. ROBINSON ROWE, Sacramento, California; and the proposer (solution as well).

Editor's comment.

The proposer's comment III appears to encompass the complete history of Tallman's Formula. If so, then not only is our solution I, easy though it is, the first published proof, but also it contains the first correct and complete formulation of the problem.

REFERENCES

1. Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, Bronx, N.Y., 1952, Vol. I, p. 414.
2. Underwood Dudley, *Elementary Number Theory*, W.H. Freeman and Co., San Francisco, 1969, p. 18 (Problem 18).
3. J.A.H. Hunter and Joseph S. Madachy, *Mathematical Diversions*, Dover, New York, 1975, p. 6.
4. Sidney Kravitz, Prime Generating Polynomials, *Recreational Mathematics Magazine* (June 1961), p. 50.
5. Malcolm H. Tallman, A Sequel to Eratosthenes, *Scripta Mathematica*, Vol. XVI (March-June 1950), p. 106.
6. ———, letter to the Editor, *Recreational Mathematics Magazine* (August 1961), pp. 63-64.

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247. [1977: 131] Proposed by Kenneth S. Williams, Carleton University, Ottawa.

On page 215 of *Analytic Inequalities* by D.S. Mitrinović, the following inequality is given: if $0 < b \leq a$ then

$$\frac{1}{8} \frac{(a-b)^2}{a} \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{1}{8} \frac{(a-b)^2}{b}.$$

Can this be generalized to the following form: if $0 < a_1 \leq a_2 \leq \dots \leq a_n$ then

$$k \frac{\sum_{1 \leq i < j \leq n} (a_i - a_j)^2}{a_n} \leq \frac{a_1 + \dots + a_n}{n} - \sqrt[n]{a_1 \dots a_n} \leq k \frac{\sum_{1 \leq i < j \leq n} (a_i - a_j)^2}{a_1},$$

where k is a constant?

Solution by Basil C. Rennie, James Cook University of North Queensland, Australia.

Consider the variable t in a fixed interval $0 < m \leq t \leq M$. For any mass-distribution on the interval the arithmetic mean A , geometric mean G , and mean square S are defined in the usual way; in fact for n points t_r with weights w_r (where $w_r \geq 0$ and $\sum w_r = 1$), we put

$$A = \sum t_r w_r, \quad G = \Pi t_r^{w_r}, \quad S = \sum w_r t_r^2.$$

In three dimensions consider the closed arc where $m \leq t \leq M$ of the curve

$$x = t, \quad y = \log t, \quad z = t^2.$$

For any mass-distribution on this arc E the centroid $(A, \log G, S)$ is by definition in the convex hull C of E . Caratheodory's theorem is that any point of C is the centroid of some four points of E . A corollary improves this number from four to three because E is connected.

LEMMA 1. Every point of C is a weighted mean of some three points of E .

Proof. Let P be some fixed point of C . For $0 \leq \theta \leq 1$ consider the subset $E(\theta)$ of E defined by $m \leq t \leq m + \theta(M - m)$, and let $C(\theta)$ be its convex hull. As θ decreases from 1 to 0 the set $C(\theta)$ shrinks continuously to a single point, and so there is some θ for which P is a boundary point of $C(\theta)$. Through any boundary point of a convex set there is a supporting plane which has the set on one side only. Any four points of $E(\theta)$ that have P in their convex hull must be in this plane, and by the two-dimensional Caratheodory theorem one of the four can be left out.

LEMMA 2. Every boundary point of C is in the convex hull of some two points of E .

Proof. Take any linear form $lx + my + nz$ in R^3 and consider its values on the curve E . Since its derivative along the curve is $l + m/t + 2nt$ which cannot have more than two zeros, a plane $lx + my + nz = \text{constant}$ cannot cut the curve in more than three points. Now take any point T in the convex hull C . It is a weighted mean of some three points P, Q and R (in that order) of the curve E . If this is so in a trivial sense, that is, either P, Q and R are collinear or one of the weights is zero, then the result of the lemma is true. Therefore suppose that T is non-trivially a weighted mean of P, Q and R , that is, T is strictly inside the proper triangle PQR . Since the plane of the triangle cannot cut the curve E at any other point, the arc PQ is on one side of the curve (call it above) and the arc QR is below. Take S on the arc PQ sufficiently close to P so that the projection of SQR

on to the plane contains T ; then, since S is above the plane there is a point T^* of the convex hull C perpendicularly up from the point T . Similarly there is another point perpendicularly down from T . Therefore T is an interior point of the convex hull C . If T is a boundary point of the convex hull, then the supposition that T is nontrivially a weighted mean of three points of the curve must be false. This proves Lemma 2.

THEOREM 1. Using the notation above, if an inequality $G \leq f(A, S)$ can be proved for a mass-distribution consisting of two points, then it holds for any other mass-distribution.

Proof. The set of all G that are compatible with any given A and S is given by the intersection of the line $A = \text{constant}$ and $S = \text{constant}$ with the closed convex set C , and therefore it is a closed interval on that line, bounded by two points, G_{\min} and G_{\max} . Both of these points are on the boundary of C and so are the centroids of two-point mass-distributions. Therefore both G_{\min} and G_{\max} are $\leq f(A, S)$, and all other G in the interval also satisfy this inequality.

Corollary. A similar result holds for an inequality the other way.

THEOREM 2. For a two-point mass-distribution there is an inequality

$$A - G \leq B(S - A^2)$$

holding for $B = \frac{1}{2m}$.

This is one part of Williams' inequality.

Proof. Take points $x \geq y$ with weights μ and $1 - \mu$. The means are

$$A = \mu x + (1 - \mu)y, \quad G = x^\mu y^{1-\mu}, \quad S = \mu x^2 + (1 - \mu)y^2.$$

Expand $B(S - A^2) - A + G$ in a Taylor series in x about the value y , treating A , G and S as functions of x , with y , μ and B constant. Since $A' = \mu$, $G' = \mu G/x$ and $S' = 2\mu x$, the first derivative of the function is

$$2B\mu(x - A) - \mu + \frac{\mu G}{x},$$

which vanishes when x takes the value y . The second derivative is

$$\mu(1 - \mu) \left(2B - \frac{G}{x^2} \right) \geq 0.$$

THEOREM 3. For a two-point mass-distribution there is an inequality

$$b(S - A^2) \leq A - G,$$

where $b = \frac{1}{2M}$.

Proof. The calculation is just as for Theorem 2 except that we take $x \leq y$.

The function $A - G - b(S - A^2)$ has a first derivative which is zero at $x = y$ and second derivative

$$\mu(1 - \mu) \left\{ \frac{G}{x^2} - 2b \right\} \geq 0$$

because $x^2 \leq GM$. The second-order Taylor theorem establishes the result.

We therefore have

THEOREM 4. For any mass-distribution on the interval $m \leq t \leq M$,

$$\frac{S - A^2}{2M} \leq A - G \leq \frac{S - A^2}{2m}.$$

Corollary. To prove the inequality suggested by Williams, take equal weights for n points and note that $\sum (\alpha_i - \alpha_j)^2 = n^2(S - A^2)$. The given inequality therefore holds with the constant $k = 1/2n^2$.

No other solutions were received, but CLAYTON W. DODGE, University of Maine at Orono, correctly conjectured that $k = 1/2n^2$.

Editor's comment.

This problem was reprinted with permission in *James Cook Mathematical Notes* No. 10 (July 1977), whose editor is none other than our solver, Basil C. Rennie. *JCMN* readers sent in two solutions to their editor. One, by G. Szekeres, somewhat shorter than our own, was published by Rennie in *JCMN* No. 12 (October 1977); the other, by D.I. Cartwright and M.J. Field, was somewhat longer than our own and was submitted by its authors to the *Duke Mathematical Journal*.

Readers interested in seeing Szekeres' solution can write to Prof. B.C. Rennie, Mathematics Department, James Cook University of North Queensland, Post Office James Cook University, Q.4811, Townsville, Australia.

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248. [1977: 131, 154] (Corrected)

Proposed by Dan Sokolowsky, Yellow Springs, Ohio.

Circle (Q) is tangent to circles (O), (M), (N), as shown in Figure 1, and FG is the diameter of (Q) parallel to diameter AB of (O). W is the radical center of circles (M), (N), (Q). Prove that WQ is equal to the circumradius of $\triangle PFG$.

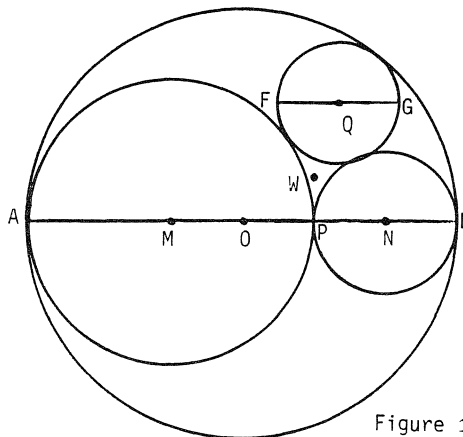


Figure 1

I. *Solution by Clayton W. Dodge, University of Maine at Orono.*

We prove that the conclusion holds for any three circles (M), (N), (Q), externally tangent in pairs, whether or not there exists, as in Figure 1, a circle (O) to which all three are internally tangent. So consider Figure 2 instead. Draw PW and WQ, let T be the point of tangency on MQ, and draw WT. Since W is the radical center, we have

$$WP = WT = t, \text{ say;}$$

and if the radius of circle (Q) is r , then, from right triangle QTW we have

$$WQ^2 = t^2 + r^2.$$

Now if the points P, W, Q are not collinear let C be the fourth vertex of parallelogram PWQC. Since $WP \perp MN$, it follows that $QC \perp FG$. Thus $QC = WP = t$ and

$$CF^2 = CG^2 = t^2 + r^2$$

from right triangles CFQ and CGQ. Also, of course,

$$CP^2 = WQ^2 = t^2 + r^2.$$

Hence C is the circumcenter of $\triangle PFG$ and its circumradius is $CP = WQ$.

If the points P, W, Q are collinear, one merely finds the point C on QP such that $QC = PW$, and the proof proceeds as before.

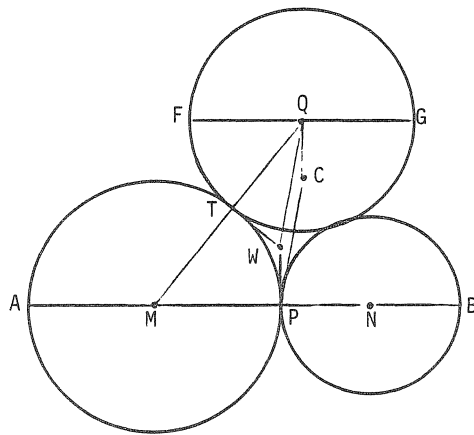


Figure 2

II. *Comment by Leon Bankoff, Los Angeles, California.*

I give, without proof, a few items of interest relating to Figure 1:

- W is the incenter of triangle MNQ, with P, J, K the incircle contacts.
- W is the circumcenter of triangle JPK.
- WP is equal to half the harmonic mean of MP and PN.
- The area of triangle FPG is equal to $\frac{1}{2}FG^2$.
- JK and FG are antiparallel in triangle FPG.
- Triangles PJK and PFG are inversely similar.
- Q lies on the intersection of two ellipses, one with major axis AN and foci M and O, the other with major axis MB and foci O and N.

Also solved by LEON BANKOFF, Los Angeles, California (solution as well); J.D. DIXON, Haliburton Highlands Secondary School, Haliburton, Ontario; SAHIB RAM MANDAN, Indian Institute of Technology, Kharagpur, India; and the proposer.

249. [1977: 131] Proposed by Clayton W. Dodge, University of Maine at Orono.

The positive integers 1, 4, and 6 are not primes and cannot be written as sums of distinct primes. Prove or disprove that all other positive integers are either prime or can be written as sums of distinct primes.

I. Solution by Kenneth M. Wilke, Washburn University, Topeka, Kansas (revised by the editor).

We shall use the following two forms of Bertrand's Postulate. The first was proved by Tchebycheff and the second, stronger form was stated and proved by Dressler in [3].

B₁. For every real $x \geq 2$, there is a prime p such that $x < p < 2x$.

B₂. If p_n denotes the n th prime and $n > 7$, then $p_n < 2p_{n-1} - 10$.

We first verify that our theorem holds for all positive integers $N (\neq 1, 4, 6) \leq 2^5$ as well as, for later use, for $N = 33, 34$.

$$\begin{array}{llll} 8 = 3 + 5 & 15 = 2 + 13 & 22 = 3 + 19 & 28 = 5 + 23 \\ 9 = 2 + 7 & 16 = 3 + 13 & 24 = 5 + 19 & 30 = 7 + 23 \\ 10 = 3 + 7 & 18 = 5 + 13 & 25 = 2 + 23 & 32 = 3 + 29 \\ 12 = 5 + 7 & 20 = 3 + 17 & 26 = 3 + 23 & 33 = 2 + 31 \\ 14 = 3 + 11 & 21 = 2 + 19 & 27 = 3 + 5 + 19 & 34 = 3 + 31 \end{array}$$

Suppose now that our theorem holds for all positive integers $N (\neq 1, 4, 6) \leq 2^k$, where $k \geq 5$. We will show that it holds as well for any integer N such that

$$2^k < N \leq 2^{k+1}, \quad (1)$$

and the desired result will then follow by induction.

We note from (1) that

$$16 \leq 2^{k-1} < \frac{N}{2} \leq 2^k < N. \quad (2)$$

We have seen above that the theorem holds for $N = 33, 34$. For $N > 34$, let p_i be the smallest prime p such that $N/2 < p < N$ (its existence is guaranteed by B₁). Then $p_i > 17 = p_7$ and $N \geq 2p_{i-1}$. Now by B₂ and (2) we have

$$10 < 2p_{i-1} - p_i \leq N - p_i < N - \frac{N}{2} = \frac{N}{2} \leq 2^k.$$

Since $N - p_i (\neq 1, 4, 6) \leq 2^k$, it follows from the induction assumption that it can be represented in the form

$$N - p_i = p_{i_1} + \dots + p_{i_r},$$

where the distinct primes p_{i_1}, \dots, p_{i_r} are all less than $N/2$ (since $N - p_i < N/2$),

while $p_i > \sqrt{N}$. Thus the primes p_i, p_j, \dots, p_n are all distinct and

$$N = p_i + p_j + \dots + p_n,$$

which completes the proof.

Since

$$\begin{array}{lll} 13 = 2 + 11 & 19 = 2 + 17 & 29 = 5 + 11 + 13 \\ 17 = 2 + 3 + 5 + 7 & 23 = 3 + 7 + 13 & 31 = 2 + 29 \end{array}$$

we have in fact established the following stronger result: *every integer greater than 11 is the sum of two or more distinct primes.*

II. *Historical comment by the editor.*

I give below a list of eleven theorems which have been proved or mentioned in the literature in recent years. The progenitor of all this activity seems to have been Richert [7]. The list was compiled with the help of the references given below, a majority of which were sent in by readers.

1. *Considering unity as a prime, show that every positive integer is either a prime or a sum of distinct primes.* [1], [6].

2. *Every integer greater than 6 is either a prime or a sum of distinct primes.* [5], [7], [8], [9], [10].

3. *Every integer greater than 9 is either an odd prime or a sum of distinct odd primes.* [2], [3], [7], [9], [10].

4. *Every integer greater than 11 is the sum of two or more distinct primes.* [9], [10], [11].

5. *Every integer greater than 45 is the sum of distinct primes each of which is at least 11.* [4].

6. *Every integer greater than 55 is a prime of the form $4k-1$ or a sum of distinct primes of the form $4k-1$.* [10].

7. *Every integer greater than 121 is a prime of the form $4k+1$ or a sum of distinct primes of the form $4k+1$.* [10].

8. *Every integer greater than 161 is a prime of the form $6k-1$ or a sum of distinct primes of the form $6k-1$.* [10].

9. *Every integer greater than 205 is a prime of the form $6k+1$ or a sum of distinct primes of the form $6k+1$.* [10].

10. *Every integer greater than 11 is the sum of two composite integers.* [10].

11. *Every integer $n > 6$ is the sum of two relatively prime integers (excluding the trivial representation $n = 1 + (n-1)$).* [12].

Solutions and/or comments were also received from DOUG DILLON, Brockville, Ontario; ROBERT S. JOHNSON, Montréal, Québec; LEROY F. MEYERS, The Ohio State University; HARRY L. NELSON, Livermore, California; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; R. ROBINSON ROWE, Sacramento, California; KENNETH M. WILKE, Washburn University, Topeka, Kansas (second solution); and the proposer.

Editor's comment.

The proposer's solution was actually a proof of the stronger result: *every integer $n > 29$ is a sum of distinct primes, the largest of which does not exceed $n - 7$.* But, after a quick check to see that it holds for $30 \leq n \leq 45$, this follows from the stronger Theorem 5.

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Good luck to EUREKA in 1978 = $2^{11} - 2^6 - 2^2 - 2$.

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