Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor

Waiting for a bus

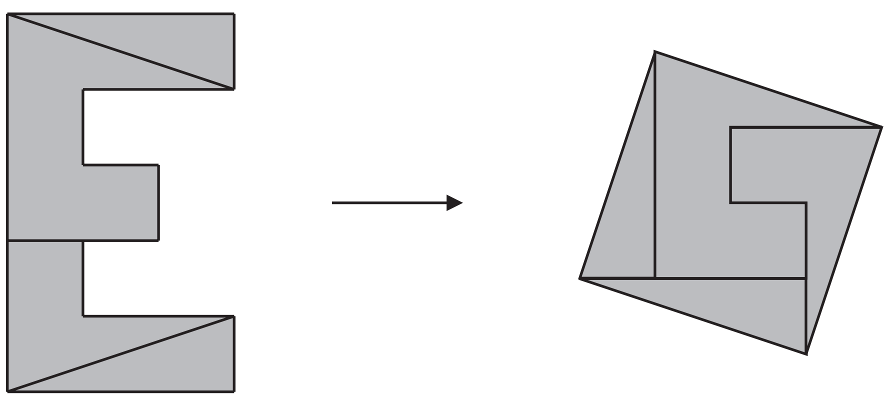
Most of us have been in the situation of waiting ages for a bus only to see two or three or more arrive together. Maybe someone could write us an article to analyse this phenomenon mathematically. A similar thing can happen in mathematics and the sciences. Newton and Leibniz independently introduced Calculus, and got into an acrimonious argument as to who was first. Andrew Wiles proved Fermat's Last Theorem in the 1990s, having hidden himself away for six years working on it, knowing that Gerd Faltings and others were working on it at the same time (see reference 1). It seems as though the time is right for a breakthrough on a problem in that the machinery needed for a solution has been put in place. So similar work is done concurrently and independently by different people. Or it may be that known results are rediscovered or reproved by someone unaware that they are already known. A famous case in point is the self-taught Indian mathematician Srinivasa Ramanujan (1887–1920), who rediscovered many known results in Number Theory.

Which leads me to explain why, in this issue, there are two articles both of which pursue the same idea of introducing trigonometric formulae based on a non-right-angled triangle. They arrived in the office from independent sources about the same time, so it seemed right to include them both. Maybe the same idea had occurred to you!

Reference

1 S. Singh, *Fermat's Last Theorem* (Fourth Estate, London, 1997).

Make a square from E



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Abbas Rouholamini Gugheri

Trigonometry *Without* Right Angles

COLIN FOSTER

This article describes how trigonometric functions can be defined for non-right-angled triangles. In particular, the ratios esin, ecos, and etan, corresponding to the sine, cosine, and tangent ratios, are defined for triangles containing a $\pi/3$ angle. Some of the implications are discussed, including the graphs of these functions and the small-angle approximations.

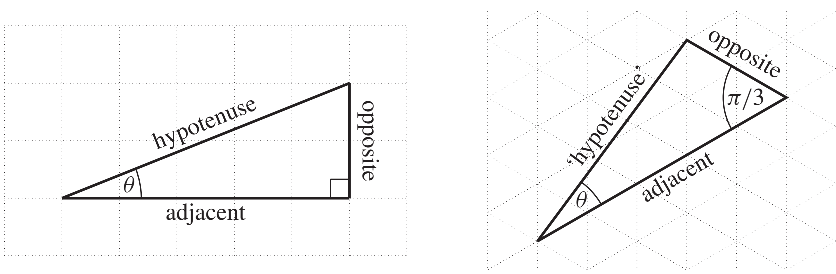
How important are right angles for trigonometry? The standard trigonometric functions are defined as the ratios of sides within right-angled triangles, but is this essential? On a square lattice it makes sense to draw right-angled triangles and label the sides 'opposite', 'adjacent', and 'hypotenuse', but what about on an *isometric* grid?

We will label the sides of a triangle containing a $\frac{\pi}{3}$ angle as 'adjacent' and 'opposite' to a given angle (θ in figure 1) and call the side opposite the $\frac{\pi}{3}$ angle the 'hypotenuse'. Then we can define 'equilateral' trigonometric functions esin, ecos, and etan by analogy with the sin, cos, and tan functions.

We can express the e-trigonometric functions in terms of the ordinary trigonometric functions by using the sine rule and the cosine rule on the $\frac{\pi}{3}$ triangle. Using o for opposite, a for adjacent, and h for 'hypotenuse' in the $\frac{\pi}{3}$ triangle we have, from the sine rule, that

$$\frac{o}{\sin\theta} = \frac{h}{\sin(\frac{\pi}{3})} = \frac{a}{\sin(\pi - \frac{\pi}{3} - \theta)}. \tag{1}$$

(a) (b)



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{esin}\,\theta = \frac{\text{opposite}}{\text{`hypotenuse'}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{ecos}\,\theta = \frac{\text{adjacent}}{\text{`hypotenuse'}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \text{etan}\,\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Figure 1 (a) Trigonometry in a right-angled triangle; (b) e-trigonometry in a $\frac{\pi}{3}$ triangle.

The first equality in (1) reduces to

$$\frac{o}{\sin\theta} = \frac{2\sqrt{3}h}{3},$$

so

$$\operatorname{esin}\theta = \frac{o}{h} = \frac{2\sqrt{3}}{3}\sin\theta. \tag{2}$$

So the esine function is simply proportional to the ordinary sine function (and the constant of proportionality is not very different from 1).

The second inequality in (1) gives

$$\operatorname{ecos}\theta = \frac{a}{h} = \frac{\sin(\frac{2\pi}{3} - \theta)}{\sin(\frac{\pi}{3})} = \frac{\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta}{\frac{\sqrt{3}}{2}} = \cos\theta + \frac{\sqrt{3}}{3}\sin\theta, \tag{3}$$

which we could also write as

$$\operatorname{ecos}\theta = \frac{2\sqrt{3}}{3}\sin\left(\theta + \frac{\pi}{3}\right).$$

This time there is the same factor of $\frac{2\sqrt{3}}{3}$ at the front, so the ecos function has the same amplitude as the esin function, but there is a phase difference of $\frac{\pi}{3}$. (This contrasts with the phase difference of $\frac{\pi}{2}$ between the sin and cos functions.)

Finally, from (1) we obtain

$$\begin{aligned}
\operatorname{etan}\theta &= \frac{o}{a} \\
&= \frac{\sin\theta}{\sin(\frac{2\pi}{3} - \theta)} \\
&= \frac{\sin\theta}{\sin(\frac{2\pi}{3})\cos\theta - \cos(\frac{2\pi}{3})\sin\theta} \\
&= \frac{\sin\theta}{\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta} \\
&= \frac{2}{1 + \sqrt{3}\cot\theta}.
\end{aligned} \tag{4}$$

The graphs of $y = \operatorname{esin}\theta$, $y = \operatorname{ecos}\theta$, and $y = \operatorname{etan}\theta$ are shown in figure 2.

Notice that the three graphs are coincident at $(\frac{\pi}{3} + 2n\pi, 1)$, where n is an integer, where $\operatorname{esin}\theta = \operatorname{ecos}\theta = \operatorname{etan}\theta$, corresponding to an equilateral triangle in which all three ratios are 1. This contrasts with the fact that there are no values of θ for which $\sin\theta = \cos\theta = \tan\theta$.

Now using the cosine rule, we have

$$h^2 = o^2 + a^2 - 2ao\cos\left(\tfrac{\pi}{3}\right),$$

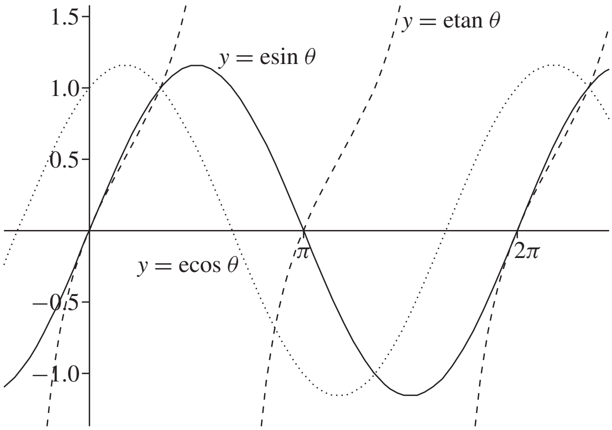


Figure 2 The graphs of $y = \operatorname{esin}\theta$, $y = \operatorname{ecos}\theta$, and $y = \operatorname{etan}\theta$.

which reduces to $h^2 = o^2 + a^2 - ao$. Dividing throughout by h^2 gives

$$\operatorname{esin}^2\theta + \operatorname{ecos}^2\theta - \operatorname{esin}\theta\operatorname{ecos}\theta = 1. \tag{5}$$

Alternatively, we could write

$$h^2 = o^2 + a^2 - ao = (o-a)^2 + ao.$$

Adding these equations, we obtain $2h^2 = o^2 + a^2 + (o-a)^2$ and dividing through by h^2 this time gives

$$\operatorname{esin}^2\theta + \operatorname{ecos}^2\theta + (\operatorname{esin}\theta - \operatorname{ecos}\theta)^2 = 2.$$

From (2), we can see that, for small values of θ in radians,

$$\operatorname{esin}\theta \approx \tfrac{2\sqrt{3}}{3}\theta.$$

From (5), using this approximation for $\operatorname{esin}\theta$, we have that

$$\operatorname{ecos}^2\theta - \tfrac{2\sqrt{3}}{3}\theta\operatorname{ecos}\theta + \left(\tfrac{2\sqrt{3}}{3}\theta\right)^2 - 1 = 0,$$

so

$$\operatorname{ecos}^2\theta - \tfrac{2\sqrt{3}}{3}\theta\operatorname{ecos}\theta + \tfrac{4}{3}\theta^2 - 1 = 0.$$

Completing the square, we have

$$\left(\operatorname{ecos}\theta - \tfrac{\sqrt{3}}{3}\theta\right)^2 - \tfrac{1}{3}\theta^2 + \tfrac{4}{3}\theta^2 - 1 = 0,$$

so

$$\left(\operatorname{ecos}\theta - \tfrac{\sqrt{3}}{3}\theta\right)^2 = 1 - \theta^2,$$

giving $\operatorname{ecos}\theta = \frac{\sqrt{3}}{3}\theta + \sqrt{1-\theta^2}$, taking the positive square root. Expanding the root binomially, we end up with

$$\operatorname{ecos}\theta \approx 1 + \tfrac{\sqrt{3}}{3}\theta - \tfrac{1}{2}\theta^2,$$

ignoring terms in θ^4 and above. As a check on our working, we can obtain the same result more straightforwardly from (3), since

$$\operatorname{ecos}\theta = \cos\theta + \tfrac{\sqrt{3}}{3}\sin\theta \approx \left(1 - \tfrac{1}{2}\theta^2\right) + \tfrac{\sqrt{3}}{3}\theta,$$

using the usual small-angle approximations for $\sin\theta$ and $\cos\theta$. Interestingly, the version with the binomial approximation to the square root gives a much better approximation for $\theta > \frac{\pi}{6}$ than the original approximation does,

$$\operatorname{ecos}\theta \approx \frac{\sqrt{3}}{3}\theta + \sqrt{1-\theta^2},$$

with

$$\operatorname{ecos}\theta \approx 1 + \frac{\sqrt{3}}{3}\theta - \tfrac{1}{2}\theta^2$$

not leading to huge discrepancies until θ is significantly greater than $\frac{2\pi}{3}$.

Finally, from (4) when θ is small we have

$$\operatorname{etan}\theta \approx \frac{2}{\sqrt{3}\cot\theta} = \frac{2\sqrt{3}}{3}\tan\theta,$$

so $\operatorname{etan}\theta \approx \frac{2\sqrt{3}}{3}\theta$. Since $\frac{2\sqrt{3}}{3}$ is fairly close to 1, in fact we are not very far from

$$\operatorname{esin}\theta \approx \operatorname{etan}\theta \approx \sin\theta \approx \tan\theta \approx \theta$$

anyway.

In a world without right angles we could still do e-trigonometry and solve triangles in a similar way to the familiar one. We could use a calculator with esin, ecos, and etan buttons (or e-trigonometry tables) just as easily.

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How good is your algebra?

$$\begin{aligned} p &= a(x^2-1) + 2b(x+1) - 2c(x-1),\\ q &= b(x^2-1) + 2c(x+1) - 2a(x-1),\\ r &= c(x^2-1) + 2a(x+1) - 2b(x-1).\end{aligned}$$

What is $p^2+q^2+r^2$?

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Generalised Trigonometry

STUART SIMONS

A comprehensive treatment is developed of how trigonometry becomes modified when the right angle present in the basic triangle of conventional trigonometry is replaced by some other constant (but arbitrary) angle σ. The relationship between this generalised trigonometry and the conventional version is explored in some detail.

1. Introduction

Trigonometry, with all its manifold ramifications, is based essentially on the definitions of the six standard trigonometrical functions, each being the ratio of the lengths of two sides of a right-angled triangle. But how would the subject be modified if instead of initially considering triangles with one angle equal to $\pi/2$, we worked with triangles which had one angle fixed at a constant, but arbitrary value σ? My thoughts on this topic led to the content of the present article which essentially explores some of the consequences of this replacement of the right angle in the standard approach by a fixed angle σ. As we shall see, the approach is straightforward and leads to results of some interest in their own right. However, of greater importance is that these results may be used to provide a variety of novel exercises for students to hone their skills in trigonometry at different levels and to develop a deeper understanding of the subject. The main goal of our work is thus to investigate how the known results of standard trigonometry are modified when $\sigma \neq \pi/2$, and we base our approach on initially establishing some simple relations between the standard trigonometrical functions on the one hand and their corresponding analogues with $\sigma \neq \pi/2$ on the other. These relations in turn lead to the generalised results we seek and the fact that the latter are more complicated in form and more cumbersome in application presumably accounts for the choice of $\sigma = \pi/2$ in the standard formulation of the subject.

2. Definitions and basic relations

Consider a triangle ABC as shown in figure 1 with angle C retaining throughout a fixed value σ. We let σ play a role analogous to the right angle in standard trigonometry and by analogy with the logarithmic function, we describe our generalised 'trigonometrical' functions as being taken to base σ. If $f_\sigma(\theta)$ denotes the generalised 'trigonometrical' function $f(\theta)$ to base σ, we define these functions by

$$\sin_\sigma \theta = \frac{b}{c} = \operatorname{cosec}\sigma \sin\theta, \tag{1}$$

$$\cos_\sigma \theta = \frac{a}{c} = \operatorname{cosec}\sigma \sin(\theta+\sigma) = \cos\theta + \cot\sigma \sin\theta, \tag{2}$$

$$\tan_\sigma \theta = \frac{b}{a} = \operatorname{cosec}(\theta+\sigma)\sin\theta = \frac{\sec\sigma \tan\theta}{\tan\theta + \tan\sigma}, \tag{3}$$

with $\operatorname{cosec}_\sigma \theta$, $\sec_\sigma \theta$, and $\cot_\sigma \theta$, respectively, being the reciprocals of these functions. We refer to $f_\sigma(\theta)$ with σ arbitrary as a *generalised* trigonometrical function and $f_{\pi/2}(\theta) \equiv f(\theta)$ as

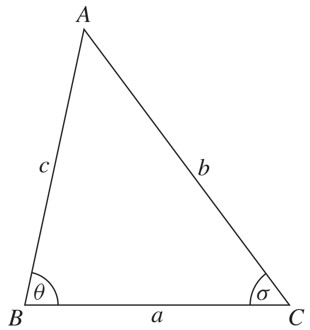

Figure 1

a *standard* function. The special values $\sin_\sigma \sigma = 1$, $\cos_\sigma \sigma = 2\cos\sigma$, and $\tan_\sigma \sigma = \frac{1}{2}\sec\sigma$ follow immediately from the definitions (1)–(3). Now, the above relationships (1)–(3) define $f_\sigma(\theta)$ in terms of $f(\theta)$, and later work requires the inversion of these results, that is, the expression of $f(\theta)$ in terms of $f_\sigma(\theta)$. This is a straightforward procedure and readily yields

$$\sin\theta = \sin\sigma \sin_\sigma \theta, \tag{4}$$

$$\cos\theta = \cos_\sigma \theta - \cos\sigma \sin_\sigma \theta, \tag{5}$$

$$\tan\theta = \frac{\tan\sigma \tan_\sigma \theta}{\sec\sigma - \tan_\sigma \theta}. \tag{6}$$

At this stage we should emphasise the following point. There are many expressions in this article, for example, the right-hand sides of (4)–(6), which purport to involve only generalised trigonometrical functions of θ and not standard functions. Nevertheless the involvement of standard trigonometrical functions of σ (such as $\cos\sigma$ or $\sin\sigma$) in such expressions is legitimate since these functions are just alternative measures of the value of the constant angle σ in the triangle.

It is clear that the above definitions (1)–(3) of $f_\sigma(\theta)$ as the ratio of the lengths of two sides of a triangle are satisfactory when σ, θ, and $\sigma + \theta$ all lie in the interval $[0, \pi]$. If this is not so, we simply use the standard trigonometric expressions on the right-hand sides of (1)–(3) in order to *define* the relevant generalised trigonometric function on the corresponding left-hand side. A consequence of this is that, just as the standard trigonometric functions are periodic with period 2π, so also the generalised functions will be periodic with the same period. We note that, for a specified value of σ, the general behaviour of $f_\sigma(\theta)$ as a function of θ can be readily obtained from the known behaviour of the right-hand sides of (1)–(3). Details are left to the reader.

If $f_\sigma(\theta) = x$, we define the inverse function $f_\sigma^{-1}(x)$ as θ. It then follows from (1)–(3) that

$$\sin_\sigma^{-1} x = \sin^{-1}(x \sin\sigma), \tag{7}$$

$$\cos_\sigma^{-1} x = \cos^{-1}(x \sin\sigma) + \frac{\pi}{2} - \sigma, \tag{8}$$

$$\tan_\sigma^{-1} x = \tan^{-1}\left(\frac{x \tan\sigma}{\sec\sigma - x}\right). \tag{9}$$

Corresponding to the fact that $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$ are all many-valued functions, it follows from (7)–(9) that $\sin_\sigma^{-1} x$, $\cos_\sigma^{-1} x$, and $\tan_\sigma^{-1} x$ will also be many-valued. We therefore define the principal values of the latter as corresponding to the situation when the right-hand sides of (7), (8), and (9) take their principal values. Inversion of (7), (8), and (9) readily yields

$$\sin^{-1} y = \sin_\sigma^{-1}(y \operatorname{cosec} \sigma), \tag{10}$$

$$\cos^{-1} y = \cos_\sigma^{-1}(y \operatorname{cosec} \sigma) + \sigma - \frac{\pi}{2}, \tag{11}$$

$$\tan^{-1} y = \tan_\sigma^{-1}\left(\frac{y \sec \sigma}{y + \tan \sigma}\right); \tag{12}$$

these results will be required in Section 6.

Finally we make the point that our definitions of generalised trigonometrical functions (1)–(3) are somewhat reminiscent of logarithms which are generally defined initially with respect to an arbitrary base. However, for computational work the base 10 clearly plays a special role in making life easier, as does the base e for theoretical work, since in general the latter yields simpler expressions, for example, for the differential coefficient or for power series expansions. In much the same way it transpires that for generalised trigonometrical functions the base $\pi/2$ plays a special role in simplifying matters, both for computational work (see Section 3) and also for theoretical work (see Sections 4–7). As mentioned at the end of Section 1, this presumably accounts for the fact that in the standard development of trigonometry the base $\pi/2$ is used throughout.

3. Solution of triangles

From a historical perspective the main motivation for the initial development of trigonometry was presumably its application to the solution of triangles; that is, solving the problem of calculating all sides and angles of a triangle when given the minimum information on these which defines the triangle uniquely. Can the generalised trigonometrical functions we have introduced be used to tackle this problem? The short answer is yes, but the relevant formulae can be significantly more complicated in structure than those using standard functions and hence more cumbersome in application. Now, generally triangles can be solved by application of the sine and cosine rules, depending on what information about the triangle is provided. Thus, if the length of one side, together with two angles is given, the sine rule alone is sufficient to solve the triangle and it follows from (4) that this rule takes exactly the same form in terms of generalised sines as it does in terms of standard ones; that is

$$\frac{p}{\sin_\sigma P} = \frac{q}{\sin_\sigma Q} = \frac{r}{\sin_\sigma R}$$

for triangle PQR and all values of σ. In other cases the cosine rule,

$$p^2 = q^2 + r^2 - 2qr \cos P,$$

is required and when generalised functions are used this takes the form (see (5)),

$$p^2 = q^2 + r^2 - 2qr(\cos_\sigma P - \cos \sigma \sin_\sigma P),$$

so that even if values of $\cos_\sigma \theta$ and $\sin_\sigma \theta$ are readily available for some specified σ (for example, they are computer generated), the computation will be longer than when standard functions are used. Other formulae employed in the solution of triangles are listed in the standard trigonometrical textbooks, and these again can be readily formulated in terms of generalised functions, using (4)–(6). The structure of these generalised formulae, however, may be considerably more complicated than those using standard functions.

4. Identities and equations

As with standard functions, very many identities exist between the different generalised functions. Here we shall consider just three of these, involving somewhat different approaches.

(i) Prove that $\cos_\sigma \theta \equiv \sin_\sigma(\theta + \sigma)$. The simplest way to tackle this is to used the basic definitions of the generalised functions as ratios of two sides of a triangle. Thus, with reference to figure 1,

$$\begin{aligned}
\cos_\sigma \theta &= \frac{a}{c} \\
&= \sin_\sigma A \\
&= \sin_\sigma(\pi - \theta - \sigma) \\
&= \operatorname{cosec} \sigma \sin(\pi - \theta - \sigma) \\
&= \operatorname{cosec} \sigma \sin(\theta + \sigma) \\
&= \sin_\sigma(\theta + \sigma),
\end{aligned}$$

on making use of (1) and the standard result $\sin \varphi = \sin(\pi - \varphi)$.

(ii) What identity involving generalised functions is equivalent to the standard identity $\cos^2 \theta + \sin^2 \theta \equiv 1$? We make use of (4) and (5) to transform $\cos^2 \theta + \sin^2 \theta$ into

$$\cos^2_\sigma \theta - 2\cos\sigma \sin_\sigma \theta \cos_\sigma \theta + \cos^2 \sigma \sin^2_\sigma \theta + \sin^2 \sigma \sin^2_\sigma \theta,$$

and hence obtain the modified identity in the form

$$\cos^2_\sigma \theta + \sin^2_\sigma \theta - 2\cos\sigma \cos_\sigma \theta \sin_\sigma \theta \equiv 1.$$

Note that other identities follow immediately from this, such as

$$\tan^2_\sigma \theta - 2\cos\sigma \tan_\sigma \theta + 1 \equiv \sec^2_\sigma \theta$$

which is, of course, the generalised analogue of the standard result $\tan^2 \theta + 1 \equiv \sec^2 \theta$ to which it reduces when $\sigma = \pi/2$.

(iii) Express $\sin_\sigma(\theta + \varphi)$ in terms of $f_\sigma(\theta)$ and $f_\sigma(\varphi)$. The procedure here is to use (1) to transform $\sin_\sigma(\theta + \varphi)$ into standard functions, to expand the latter in terms of $f(\theta)$, $f(\varphi)$, and finally to use (4) and (5) to transform back to generalised functions. Thus,

$$\begin{aligned}
\sin_\sigma(\theta + \varphi) &= \operatorname{cosec} \sigma \sin(\theta + \varphi) \\
&= \operatorname{cosec} \sigma (\sin\theta \cos\varphi + \cos\theta \sin\varphi) \\
&= \operatorname{cosec} \sigma [\sin\sigma \sin_\sigma \theta(\cos_\sigma \varphi - \cos\sigma \sin_\sigma \varphi) \\
&\quad + \sin\sigma \sin_\sigma \varphi(\cos_\sigma \theta - \cos\sigma \sin_\sigma \theta)] \\
&= \sin_\sigma \theta \cos_\sigma \varphi + \cos_\sigma \theta \sin_\sigma \varphi - 2\cos\sigma \sin_\sigma \theta \sin_\sigma \varphi.
\end{aligned}$$

As a corollary of this,

$$\sin_\sigma 2\theta = 2\sin_\sigma\theta\cos_\sigma\theta - 2\cos\sigma\sin_\sigma^2\theta.$$

We note that a similar procedure, albeit of greater algebraic complexity, can be used to obtain $\sin_\sigma n\theta$, $\cos_\sigma n\theta$, or $\tan_\sigma n\theta$ in terms of generalised functions of θ for specified integer values of n. However, a quicker approach to this will be developed in Section 7.

A final comment concerns the solution of trigonometrical equations involving generalised trigonometrical functions. Usually the best way to tackle these is to transform the equation into one involving standard functions (using (1)–(3)) and then to solve this. Occasionally the reverse procedure may be effective if values of the generalised functions and their inverses for arbitrary σ are readily available. Thus, one way of tackling the equation $l\cos\theta + m\sin\theta = n$ (for specified values of l, m, and n) would be to express it in the form

$$\cos\theta + \beta\sin\theta = \gamma$$

(where $\beta = m/l$ and $\gamma = n/l$) which in turn can be expressed as $\cos_\sigma\theta = \gamma$ where $\sigma = \cot^{-1}\beta$ (see (2)). Thus $\theta = \cos_\sigma^{-1}\gamma$.

5. Differentiation and integration

In common with the approach of Section 4 we tackle the differentiation of generalised trigonometrical functions by first transforming to standard functions, then performing the differentiation and finally transforming back to generalised functions. Thus we readily obtain from (1)–(6)

$$\frac{\mathrm{d}\sin_\sigma\theta}{\mathrm{d}\theta} = \operatorname{cosec}\sigma\cos_\sigma\theta - \cot\sigma\sin_\sigma\theta, \tag{13}$$

$$\frac{\mathrm{d}\cos_\sigma\theta}{\mathrm{d}\theta} = \cot\sigma\cos_\sigma\theta - \operatorname{cosec}\sigma\sin_\sigma\theta, \tag{14}$$

$$\frac{\mathrm{d}\tan_\sigma\theta}{\mathrm{d}\theta} = \operatorname{cosec}\sigma\sec_\sigma^2\theta,$$

and multiplying these three results by $-\operatorname{cosec}_\sigma^2\theta$, $-\sec_\sigma^2\theta$, and $-\cot_\sigma^2\theta$ respectively yields the differential coefficients of $\operatorname{cosec}_\sigma\theta$, $\sec_\sigma\theta$, and $\cot_\sigma\theta$. Note that $\sin_\sigma\theta$ and $\cos_\sigma\theta$ each satisfy the relation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\theta^2} + f = 0$$

since each is a linear combination of $\sin\theta$ and $\cos\theta$. The above approach to deriving (13) and (14) can similarly be used for obtaining the nth differential coefficients of $\sin_\sigma\theta$ and $\cos_\sigma\theta$ and these in turn may be applied to deriving the power series expressions for the two functions. The latter, however, are more simply found from (1) and (2), making use of the known power series expansions for $\sin\theta$ and $\cos\theta$. It should be noted that for all values of σ the function $\sin_\sigma\theta$ is an odd function with power series expansion involving only odd powers of θ, while for $\sigma \neq \pi/2$ the function $\cos_\sigma\theta$ is neither odd nor even and its power series involves both even and odd powers of θ.

A similar procedure to that used above for differentiating the generalised trigonometrical functions can be followed for integrating $\sin_\sigma \theta$ and $\cos_\sigma \theta$. Alternatively, (13) and (14) allow $\cos_\sigma \theta$ and $\sin_\sigma \theta$ each to be expressed as a linear combination of $\mathrm{d}\sin_\sigma \theta/\mathrm{d}\theta$ and $\mathrm{d}\cos_\sigma \theta/\mathrm{d}\theta$. Hence $\int \cos_\sigma \theta \, \mathrm{d}\theta$ and $\int \sin_\sigma \theta \, \mathrm{d}\theta$ can be found, each as a linear combination of $\sin_\sigma \theta$ and $\cos_\sigma \theta$.

The inverse generalised trigonometrical functions are readily differentiated by transforming them to inverse standard functions whose differential coefficients are known, making use of (7)–(9). We restrict ourselves to principal values and thus obtain

$$\begin{aligned}
\frac{\mathrm{d}\sin_\sigma^{-1} x}{\mathrm{d}x} &= \frac{\sin\sigma}{(1 - x^2\sin^2\sigma)^{1/2}}, \\
\frac{\mathrm{d}\cos_\sigma^{-1} x}{\mathrm{d}x} &= \frac{-\sin\sigma}{(1 - x^2\sin^2\sigma)^{1/2}}, \\
\frac{\mathrm{d}\tan_\sigma^{-1} x}{\mathrm{d}x} &= \frac{\sin\sigma}{1 - 2x\cos\sigma + x^2}.
\end{aligned}$$

6. Changing the base

So far we have supposed that the chosen base retains a constant value σ throughout. We now proceed to consider some situations where the base can vary. The first of these is to tackle the problem of changing the base from a given value σ to some other value ρ. That is, we require to obtain an expression for $f_\rho(\theta)$ in terms of $f_\sigma(\theta)$. The key to this is (4)–(6) which express a given standard trigonometrical function in terms of generalised functions with a specified base. Equating such expressions with two different bases, σ and ρ, enables the required result to be found. Thus from (4),

$$\sin\theta = \sin\sigma \sin_\sigma \theta = \sin\rho \sin_\rho \theta;$$

whence,

$$\sin_\sigma \theta = \frac{\sin\rho}{\sin\sigma} \sin_\rho \theta. \tag{15}$$

From (5), $\cos\theta = \cos_\sigma \theta - \cos\sigma \sin_\sigma \theta = \cos_\rho \theta - \cos\rho \sin_\rho \theta$ which, with the help of (15) yields

$$\cos_\sigma \theta = \cos_\rho \theta + \frac{\sin(\rho - \sigma)}{\sin\sigma} \sin_\rho \theta,$$

while from (6) we finally obtain

$$\tan_\sigma \theta = \frac{\sec\sigma \tan\rho \tan_\rho \theta}{(\tan\rho - \tan\sigma)\tan_\rho \theta + \tan\sigma \sec\rho}.$$

A similar approach may be employed to change the base in the case of inverse generalised functions, making use of (10)–(12). Details are left to the interested reader.

When the base σ is allowed to vary it becomes meaningful to consider differentiation with respect to σ. This is easily done by reference to (1)–(6). Thus, from (1),

$$\frac{\partial}{\partial\sigma}\sin_\sigma \theta = \frac{\partial}{\partial\sigma}\operatorname{cosec}\sigma \sin\theta = -\cot\sigma \operatorname{cosec}\sigma \sin\theta = -\cot\sigma \sin_\sigma \theta.$$

Similarly,

$$\frac{\partial}{\partial\sigma}\cos_\sigma \theta = -\operatorname{cosec}\sigma \sin_\sigma \theta$$

and

$$\frac{\partial}{\partial\sigma}\tan_\sigma\theta = \operatorname{cosec}\sigma\tan_\sigma^2\theta - \cot\sigma\tan_\sigma\theta,$$

after a little manipulation. A similar approach allows the inverse functions to be differentiated with respect to the base.

In Section 2 we defined the six trigonometric functions, taking as base σ the angle C of triangle ABC and as current angle θ the angle B. However, for a triangle ABC the base σ could be chosen as any of the three angles A, B, C with θ taken as either of the two remaining angles. There are thus six possible choices for the pair (σ, θ) and for each such pair six generalised functions can be defined, leading to a total of $6 \times 6 = 36$ possible generalised trigonometrical functions for a specified triangle ABC. Since each such generalised function is the ratio of the lengths of two specified sides of the triangle and since there are just six such possible ratios for a given triangle, it follows that there must be $36 - 6 = 30$ equalities existing between all the possible different generalised functions. These are readily obtained by considering

$$\frac{b}{c} = \sin_C B = \cos_C A = \tan_A B = \cot_A C = \operatorname{cosec}_B C = \sec_B A$$

together with five other sets, each of five equalities between generalised functions arising from the side ratios c/b, c/a, a/c, b/a, a/b. Among other equalities that readily follow from a similar approach is that, if $f_\beta(\alpha)$ represents any of the generalised trigonometrical functions, then

$$f_A(B)f_B(C)f_C(A) = 1,$$

since the left-hand side is the product of three side ratios of the triangle with a, b, and c each appearing once in the numerator and once in the denominator.

7. Introducing complex numbers

The introduction of complex numbers into the study of standard trigonometrical functions leads to some very fruitful developments, for example showing a connection between trigonometrical functions on the one hand and exponential and hyperbolic functions on the other. In this section we shall begin to explore some simple consequences of introducing complex numbers into the study of generalised trigonometrical functions. Our work is based on considering how the expression $e^{i\theta} = \cos\theta + i\sin\theta$ transforms when generalised functions are introduced. Making use of (4) and (5) we readily obtain

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ &= \cos_\sigma\theta - \cos\sigma\sin_\sigma\theta + i\sin\sigma\sin_\sigma\theta \\ &= \cos_\sigma\theta - e^{-i\sigma}\sin_\sigma\theta \end{aligned} \tag{16}$$

and

$$e^{-i\theta} = \cos\theta - i\sin\theta = \cos_\sigma\theta - e^{i\sigma}\sin_\sigma\theta. \tag{17}$$

We note that the structure of these expressions in terms of generalised functions is similar to that in terms of standard functions (i.e. $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$) with the former being obtained from the latter by replacing $\pm i$ by $-e^{+i\sigma}$. However, when standard functions are used the two terms on the right-hand side are the real and imaginary parts of $e^{\pm i\theta}$, whereas this is not so in the case of generalised functions ($\sigma \neq \pi/2$).

The first application we make of (16) is to the polar representation of a general complex number z which allows it to be expressed in the form $z = r(\cos\theta + i\sin\theta)$ where r and θ are the modulus and argument of z respectively. Identity (16) shows that an equivalent representation is through

$$z = r(\cos_\sigma \theta - e^{-i\sigma} \sin_\sigma \theta).$$

Our next application is to the generalised function analogue of De Moivre's theorem which for standard functions takes the form

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

valid for n a positive or negative integer or a rational fraction in its lowest terms. (Note that in the last case, $\cos n\theta + i\sin n\theta$ is only one of the several values that $(\cos\theta + i\sin\theta)^n$ can take.) On making use of (16) we immediately obtain De Moivre's theorem for generalised functions in the form

$$(\cos_\sigma \theta - e^{-i\sigma} \sin_\sigma \theta)^n = \cos_\sigma n\theta - e^{-i\sigma} \sin_\sigma n\theta. \tag{18}$$

In the context of generalised functions this result has several uses, modelled on the corresponding uses for standard functions. The first is to obtain the expansions of $\cos_\sigma n\theta$ and $\sin_\sigma n\theta$ as a sum of powers of $\cos_\sigma \theta$ and $\sin_\sigma \theta$, for a specified integer value of n. This is simply done by expanding the left-hand side of (18) by the binomial theorem and then equating in turn the real and imaginary parts of both sides of this relation. Thus, for example, when $n = 3$,

$$\begin{aligned}(\cos_\sigma \theta - e^{-i\sigma} \sin_\sigma \theta)^3 &= \cos_\sigma^3 \theta - 3e^{-i\sigma} \cos_\sigma^2 \theta \sin_\sigma \theta \\ &\quad + 3e^{-2i\sigma} \cos_\sigma \theta \sin_\sigma^2 \theta - e^{-3i\sigma} \sin_\sigma^3 \theta \\ &= \cos_\sigma^3 \theta - 3(\cos\sigma - i\sin\sigma) \cos_\sigma^2 \theta \sin_\sigma \theta \\ &\quad + 3(\cos 2\sigma - i\sin 2\sigma) \cos_\sigma \theta \sin_\sigma^2 \theta - (\cos 3\sigma - i\sin 3\sigma) \sin_\sigma^3 \theta,\end{aligned}$$

and on equating real and imaginary parts of the last expression to those of $\cos_\sigma 3\theta - e^{-i\sigma} \sin_\sigma 3\theta$, we obtain

$$\begin{aligned}&\cos_\sigma 3\theta - \cos\sigma \sin_\sigma 3\theta \\ &\quad = \cos_\sigma^3 \theta - 3\cos\sigma \cos_\sigma^2 \theta \sin_\sigma \theta + 3\cos 2\sigma \cos_\sigma \theta \sin_\sigma^2 \theta - \cos 3\sigma \sin_\sigma^3 \theta\end{aligned}$$

and

$$\sin\sigma \sin_\sigma 3\theta = 3\sin\sigma \cos_\sigma^2 \theta \sin_\sigma \theta - 3\sin 2\sigma \cos_\sigma \theta \sin_\sigma^2 \theta + \sin 3\sigma \sin_\sigma^3 \theta,$$

from which $\sin_\sigma 3\theta$ and $\cos_\sigma 3\theta$ may each be readily obtained as a sum of powers of $\cos_\sigma \theta$ and $\sin_\sigma \theta$. The reverse problem of expressing $\sin_\sigma^n \theta$ or $\cos_\sigma^n \theta$ as a sum of terms each being a generalised sine or cosine of a multiple of θ can also be readily tackled – such a procedure is of use in obtaining the integrals of $\sin_\sigma^n \theta$ or $\cos_\sigma^n \theta$. In the case of $\sin_\sigma^n \theta$, we begin by expressing $\sin_\sigma \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, either by eliminating $\cos_\sigma \theta$ between (16) and (17) or by using (1) together with

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

Either way, we obtain

$$\sin_\sigma \theta = \frac{\operatorname{cosec}\sigma}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}).$$

Thus,

$$\sin_\sigma^n \theta = \left(\frac{\operatorname{cosec}\sigma}{2\mathrm{i}}\right)^n (\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta})^n$$

and the term $(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta})^n$ can now be expanded by the binomial theorem to yield a polynomial in $\mathrm{e}^{\mathrm{i}\theta}$ together with a polynomial in $\mathrm{e}^{-\mathrm{i}\theta}$, in each case the powers lying in the interval $[0, n]$. Finally, a further application of (16) and (17) allows each term of the form $\mathrm{e}^{\pm\mathrm{i}m\theta}$ $(0 \le m \le n)$ to be expressed in terms of $\cos_\sigma m\theta$ and $\sin_\sigma m\theta$. In the case of $\cos_\sigma^n \theta$ we begin with the result

$$\cos_\sigma \theta = \operatorname{cosec}\sigma \sin(\theta + \sigma)$$

(see (2)). The procedure is then identical to that for $\sin_\sigma^n \theta$, but with θ replaced by $\theta + \sigma$, and hence finally yields $\cos_\sigma^n \theta$ as a sum of terms of the form $\cos_\sigma m(\theta + \sigma)$ and $\sin_\sigma m(\theta + \sigma)$. If the purpose of the transformation was to integrate $\cos_\sigma^n \theta$, this can then be done immediately; alternatively $\cos_\sigma m(\theta + \sigma)$ and $\sin_\sigma m(\theta + \sigma)$ can each be expressed in terms of $\cos_\sigma m\theta$ and $\sin_\sigma m\theta$ as discussed in Section 4. Finally, we apply our generalised De Moivre's theorem (18) to the summation of certain series and illustrate the technique by showing in outline our approach to

$$S_1 = \sum_{m=0}^{N-1} \cos_\sigma m\theta \quad \text{and} \quad S_2 = \sum_{m=0}^{N-1} \sin_\sigma m\theta.$$

We consider

$$S = S_1 - \mathrm{e}^{-\mathrm{i}\sigma} S_2 = \sum_{m=0}^{N-1} (\cos_\sigma m\theta - \mathrm{e}^{-\mathrm{i}\sigma} \sin_\sigma m\theta) = \sum_{m=0}^{N-1} (\cos_\sigma \theta - \mathrm{e}^{-\mathrm{i}\sigma} \sin_\sigma \theta)^m \tag{19}$$

using (18). The latter sum is a geometric progression yielding

$$S = \frac{(\cos_\sigma \theta - \mathrm{e}^{-\mathrm{i}\sigma} \sin_\sigma \theta)^N - 1}{\cos_\sigma \theta - \mathrm{e}^{-\mathrm{i}\sigma} \sin_\sigma \theta - 1} = \frac{\cos_\sigma N\theta - \mathrm{e}^{-\mathrm{i}\sigma} \sin_\sigma N\theta - 1}{\cos_\sigma \theta - \mathrm{e}^{-\mathrm{i}\sigma} \sin_\sigma \theta - 1}, \tag{20}$$

making use of (18) again. Now, from (19) we see that the imaginary part of S is $\sin\sigma S_2$ and equating this to the imaginary part of the right-hand side of (20) thus yields S_2. The real part of S is clearly $S_1 - \cos\sigma S_2$ and equating this to the real part of the right-hand side of (20), together with knowledge of S_2 (obtained above), thus enables us to calculate S_1.

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Does this Ring a Bell?

MARTIN GRIFFITHS

Bell numbers arise when enumerating partitions of sets, and there are a number of well-known results associated with them. In this article we look at some of the less well-known, nonetheless accessible and interesting, properties of these numbers.

1. Introduction

Whilst doing some mathematical exploration recently, using the powerful piece of software MATHEMATICA®, I came across the first few terms of the following rather odd-looking sequence:

$$1,\ 3,\ 13,\ 12,\ 781,\ 39,\ 137\,257,\ 24,\ 39,\ 2\,343,\ 28\,531\,167\,061,\ 156, \ldots .$$

Do you recognise it? I certainly had never seen this sequence anywhere before, and in fact thought that I may have been the first person to 'discover' it. However, its appearance as sequence A054767 in *The On-Line Encyclopedia of Integer Sequences* (see reference 1) soon put me in my place. I will keep you guessing a little while longer as to exactly how this sequence arises, although the title of this article does at least hint at where we are headed.

Some readers may already be familiar with the *Bell numbers*. For the sake of completeness, however, we describe them here. The nth Bell number B_n enumerates all possible partitions of a set consisting of n distinct elements. For example, the partitions of $\{1, 2, 3, 4\}$ are given by

$$\begin{array}{lll}
\{\{1,2,3,4\}\}, & \{\{1,2\},\{3,4\}\}, & \{\{1,4\},\{2\},\{3\}\}, \\
\{\{1,2,3\},\{4\}\}, & \{\{1,3\},\{2,4\}\}, & \{\{2,3\},\{1\},\{4\}\}, \\
\{\{1,2,4\},\{3\}\}, & \{\{1,4\},\{2,3\}\}, & \{\{2,4\},\{1\},\{3\}\}, \\
\{\{1,3,4\},\{2\}\}, & \{\{1,2\},\{3\},\{4\}\}, & \{\{3,4\},\{1\},\{2\}\}, \\
\{\{2,3,4\},\{1\}\}, & \{\{1,3\},\{2\},\{4\}\}, & \{\{1\},\{2\},\{3\},\{4\}\},
\end{array}$$

showing that $B_4 = 15$. Several other combinatorial interpretations of these numbers are given in reference 2. In order to place the Bell numbers in some sort of context, I find it helpful to keep in mind the interpretation of B_n as the number of ways n people can be assigned to n indistinguishable tables (not all of which are necessarily occupied). Table 1 gives the first few Bell numbers, noting that $B_0 = 1$ by definition.

It is well-known that the Bell numbers satisfy the recurrence relation

$$B_n = \sum_{k=1}^{n} \binom{n-1}{k-1} B_{n-k},$$

Table 1 Bell numbers.

n	B_n
0	1
1	1
2	2
3	5
4	15
5	52
6	203
7	877
8	4 140
9	21 147
10	115 975

as is shown in references 2 and 3. This result is actually quite easy to prove by way of a combinatorial argument, as follows. By definition, B_n enumerates the partitions of

$$S_n = \{1, 2, \ldots, n\}.$$

The number of partitions of S_n containing the singleton part $\{n\}$ is simply B_{n-1}. Next, the number of partitions of S_n containing the part $\{n, m\}$ for some $m \in S_{n-1}$ is B_{n-2} multiplied by $\binom{n-1}{1}$, the number of ways of choosing the element m from S_{n-1}. More generally, the number of partitions of S_n for which the element n is in a part with exactly $k-1$ other elements from S_n is

$$\binom{n-1}{k-1} B_{n-k},$$

as required.

Bell numbers are named after Eric Temple Bell (1883–1960) who did a considerable amount of work on them, culminating in a paper in 1934 (see reference 4). However, the amazing Indian mathematician Srinivasa Ramanujan (1887–1920) was known to have been studying these numbers at least 25 years earlier. Bell is actually better known for his writing on the history of mathematics and his science-fiction novels, the latter of which were written under the pseudonym John Taine, than for his mathematics.

Some of the less well-known properties of these numbers are interesting, and certainly deserve a wider audience. Our intention here is both to bring some of these properties to the attention of readers who may not be familiar with them and to provide relatively simple proofs of these results (which, from my own experience at least, do not appear to be commonly available).

2. Another way of calculating B_n

We now demonstrate a method of generating the Bell numbers that is possibly not so well known. Table 2 shows the first seven rows of what is known as the *Bell triangle*. We use $b(i, j)$ to denote the element in row i and column j. So, for example, $b(5, 3) = 27$. The Bell triangle

Table 2 The Bell triangle.

i	$b(i,1)$	$b(i,2)$	$b(i,3)$	$b(i,4)$	$b(i,5)$	$b(i,6)$	$b(i,7)$
1	1						
2	1	2					
3	2	3	5				
4	5	7	10	15			
5	15	20	27	37	52		
6	52	67	87	114	151	203	
7	203	255	322	409	523	674	877

gives another way to obtain the Bell numbers recursively. Indeed, B_n appears in the table as entry $b(n,n)$; alternatively as entry $b(n+1,1)$.

Table 2 can be constructed in the following way. First, set $b(1,1) = b(2,1) = 1$. Now calculate rows recursively using the relations

$$b(i, j+1) = b(i, j) + b(i-1, j), \qquad j = 1, 2, \ldots, i-1,$$

and

$$b(i+1, 1) = b(i, i).$$

It is not at all obvious, however, that this process does actually give rise to the first n Bell numbers, as claimed above. In order to show that this is in fact the case, we will prove that $b(i, j)$ enumerates the partitions of S_i for which the element i is not in a part with any of the elements from $S(j, i-1)$, where

$$S(a, b) = \{a, a+1, \ldots, b\}.$$

(The set $S(a,b)$ is deemed to be empty when $a > b$.) Note that the definitions of $b(i, j)$ and B_n do indeed coincide when $i = j = n$. The following argument is rather intricate, and can easily be omitted on a first reading of this article.

Consider the entry $b(i, j)$, $i \geq j$, in the triangle. To this we assign the number $n = i(i-1)/2 + j$. It is very straightforward to prove first that any $n \in \mathbb{N}$ may be written as $n = i(i-1)/2 + j$ for some $i, j \in \mathbb{N}$ with $i \geq j$ and second that this representation is unique.

We now proceed by induction on $n = i(i-1)/2 + j$. The claim is clearly true for the basis case $n = 1$. Now assume that $b(i, j)$ enumerates the partitions of S_i for which the element i is not in a part with any of the elements from $S(j, i-1)$ for all n such that $1 \leq n \leq m$ for some $m \in \mathbb{N}$. There are two cases to consider:

(i) $m = i(i-1)/2 + j$ such that $j = i$,

(ii) $m = i(i-1)/2 + j$ such that $1 \leq j < i$.

If $j = i$ then, by assumption, $b(i, j)$ gives the number of ways of partitioning S_i, which is B_i. It follows, from the way the Bell triangle is constructed, that $b(i+1, 1) = B_i$. However, the number of ways of partitioning S_i is the same as the number of ways of partitioning S_{i+1} such that the element $i+1$ is not in a part with any of the elements from $S(1, i)$.

Now suppose that $1 \leq j < i$. Let X and Y denote the partitions of S_i for which i is not in a part with any of the elements from $S(j+1, i-1)$ and $S(j, i-1)$ respectively. Any partition of S_i for which i is not in a part with any element from $S(j, i-1)$ must certainly be one for which i is not in a part with any elements from $S(j+1, i-1)$. From this it follows that $Y \subseteq X$.

Let us next characterise the elements in $X \setminus Y$ (the elements of X that are not in Y). These are in fact all the partitions of S_i for which i is in a part with j but not in a part with any of $S(j+1, i-1)$. As i is always with j in $X \setminus Y$ then, for the purposes of the enumeration of the partitions of this set, we can ignore j and just count the partitions of

$$\{1, 2, \ldots, j-1, j+1, \ldots, i\}$$

for which i is not in a part with any of $S(j+1, i-1)$. This, however, is equivalent to the enumeration of the partitions of S_{i-1} for which $i-1$ is not in a part with any of $S(j, i-2)$. The number of such partitions is, by the inductive hypothesis, equal to $b(i-1, j)$. Furthermore, the inductive hypothesis tells us, via the definition of Y, that $b(i, j)$ is the number of partitions in Y. From the algorithm we know that

$$b(i, j+1) = b(i-1, j) + b(i, j).$$

From the above, however, it follows that $b(i-1, j) + b(i, j)$ gives the number of elements in $Y \cup (X \setminus Y) = X$, which is the set consisting of all partitions of S_i for which i is not in a part with any of the elements from $S(j+1, i-1)$, thereby proving the claim by the principle of mathematical induction.

3. Row sums of the Bell triangle

Other than the appearance of the Bell numbers, the most obvious feature of the Bell triangle, as far as table 2 goes at least, is probably that the sum of all the elements in the kth row is equal to $b(k+1, k)$. For example,

$$\begin{aligned}\sum_{k=1}^{4} b(4, k) &= 5 + 7 + 10 + 15 \\ &= 37 \\ &= b(5, 4).\end{aligned}$$

That this is in fact true follows very easily from the way that the triangle is constructed. As can be seen in table 2, the diagonal consisting of entries $b(k+1, k)$, $k \geq 1$, gives the row sums.

A far more interesting observation concerning the row sums is in connection with *Stirling numbers of the second kind*, denoted by $S(n, k)$. They enumerate the ways of partitioning S_n into exactly k nonempty parts; see reference 3, for example. To illustrate this idea, here are all the possible ways in which $\{1, 2, 3, 4\}$ can be partitioned into three parts:

$$\begin{array}{lll} \{\{1\}, \{2\}, \{3, 4\}\}, & \{\{1\}, \{3\}, \{2, 4\}\}, & \{\{1\}, \{4\}, \{2, 3\}\}, \\ \{\{2\}, \{3\}, \{1, 4\}\}, & \{\{2\}, \{4\}, \{1, 3\}\}, & \{\{3\}, \{4\}, \{1, 2\}\}. \end{array}$$

This tells us that $S(4, 3) = 6$. Table 3 gives values of $S(n, k)$ for $1 \leq n$ and $k \leq 7$. You might notice that

$$\sum_{k=1}^{n} kS(n, k)$$

Table 3 Stirling numbers of the second kind.

n	$S(n,1)$	$S(n,2)$	$S(n,3)$	$S(n,4)$	$S(n,5)$	$S(n,6)$	$S(n,7)$
1	1						
2	1	1					
3	1	3	1				
4	1	7	6	1			
5	1	15	25	10	1		
6	1	31	90	65	15	1	
7	1	63	301	350	140	21	1

appears to be giving the nth row sum of the Bell triangle. For example,

$$\begin{aligned}\sum_{k=1}^{5} kS(5,k) &= S(5,1) + 2S(5,2) + 3S(5,3) + 4S(5,4) + 5S(5,5)\\ &= 1 + 30 + 75 + 40 + 5\\ &= 151\\ &= \sum_{j=1}^{5} b(5,i).\end{aligned}$$

We now prove that this conjecture is in fact true.

These numbers obey the well-known recurrence relation

$$S(n+1,k) = S(n,k-1) + kS(n,k), \tag{1}$$

with boundary values given by $S(n,n) = 1$ for $n \geq 0$ and with $S(n,0) = 0$ for $n \geq 1$. There is a very simple combinatorial explanation of the above recurrence relation. Suppose that we are in a canteen with k indistinguishable tables. Let us consider the number of ways of seating $n+1$ (with $n+1 \geq k$) distinguishable people at these tables such that all of them are occupied by at least one person; this would give us $S(n+1,k)$. We could seat one particular person X on his/her own and then spread the remaining n people amongst the remaining $k-1$ tables. There would be $S(n,k-1)$ ways of doing this. Alternatively, suppose that X arrives late and that the remaining n people are already spread out amongst the k tables. Then X could join any one of these tables. There are $kS(n,k)$ ways of doing this. This is enough to show that the recurrence relation is true, bearing in mind the fact that all possibilities for sitting $n+1$ distinguishable people at k indistinguishable tables such that all tables are occupied have now been taken into account.

From the definition of the Bell numbers and Stirling numbers of the second kind, it is clear that

$$\sum_{k=1}^{n} S(n,k) = B_n.$$

Then, on utilising (1), it follows that

$$\begin{aligned}\sum_{k=1}^{5} kS(n,k) &= \sum_{k=1}^{n}(S(n+1,k) - S(n,k-1)) \\ &= (B_{n+1} - S(n+1,n+1)) - (S(n,0) + B_n - S(n,n)) \\ &= B_{n+1} - B_n .\end{aligned}$$

We have already mentioned that $b(n+1,n)$ gives the sum of all the entries in the nth row of the Bell triangle. Thus, since $b(n,n) = B_n$, $b(n+1,n+1) = B_{n+1}$, and $b(n+1,n) = b(n+1,n+1) - b(n,n)$, we have shown what was required.

4. Remainders of Bell numbers

Note where the even numbers are situated in the Bell triangle. We might conjecture that $b(i,j)$ is even (i.e. $b(i,j) \equiv 0 \pmod 2$) if, and only if, $i + j \equiv 1 \pmod 3$, and this does indeed turn out to be the case. This divisibility result can be proved by induction on $n = i(i-1)/2 + j$ once more. It is true when $n = 1$ and $n = 2$, so assume that it is true for all n such that $1 \le n \le m$ for some $m \in \mathbb{N}$.

We may, as before, write $m = i(i-1)/2 + j$ for some $i, j \in \mathbb{N}$ such that $1 \le j \le i$. Suppose first that $i = j$. Since

$$\begin{aligned} i + i \equiv 1 \pmod 3 &\iff i \equiv 2 \pmod 3 \\ &\iff (i+1) + 1 \equiv 1 \pmod 3, \end{aligned}$$

it follows, from both the inductive hypothesis and the way the triangle is constructed, that

$$b(i+1,1) \equiv 0 \pmod 2 \iff (i+1) + 1 \equiv 1 \pmod 3.$$

Now suppose that $1 \le j < i$. We know that $b(i,j+1) = b(i,j) + b(i-1,j)$, and consider the three separate cases $i + (j+1) \equiv k \pmod 3$, $k = 0, 1, 2$. If $k = 0$ then $i + j \equiv 2 \pmod 3$ and $(i-1) + j \equiv 1 \pmod 3$. This tells us, via the inductive hypothesis, that $b(i,j)$ is odd and $b(i-1,j)$ is even, and hence that $b(i,j+1)$ is odd. When $k = 1$ and $k = 2$ we find, using similar arguments, that $b(i,j+1)$ is even and odd respectively, thereby obtaining our sought-after divisibility result. This proves in particular that B_i is even if, and only if, $i \equiv 2 \pmod 3$.

From the above result we may deduce that two out of three Bell numbers are odd. It is interesting next to consider the remainders when the Bell numbers are divided by numbers bigger than 2. Here, for example, is a list of remainders when the first 30 Bell numbers (starting from B_1) are divided by 3:

$$1,\ 2,\ 2,\ 0,\ 1,\ 2,\ 1,\ 0,\ 0,\ 1,\ 0,\ 1,\ 1,\ 1,\ 2,\ 2,\ 0,\ 1,\ 2,\ 1,\ 0,\ 0,\ 1,\ 0,\ 1,\ 1,\ 1,\ 2,\ 2,\ 0,\ \ldots .$$

You have probably noticed that there appears to be a cyclic pattern of period 13. It can be proved that this pattern does indeed continue indefinitely. From this we may glean that four out of thirteen Bell numbers are multiples of three, and so on.

The nth term of the sequence given at the beginning of this article does indeed give the period of the remainders modulo n of B_k, $k = 1, 2, 3, \ldots$. Note that long periods result even when dividing by small positive integers. For example, the period when dividing by 5 is 781, while that for 11 is a staggering 28 531 167 061. Here is a piece of MATHEMATICA code to calculate the fractions of Bell numbers possessing each of the possible remainders when they are divided by 5.

```
For[n=0,n<=4,c[n]=0;n++];
For[n=0,n<=4,For[k=1,k<=781,If[Mod[BellB[k],5]==n,c[n]++];k++];
n++];
For[n=0,n<=4,Print[c[n]];n++]
```

The output from this programme tells us that the fractions of Bell numbers leaving remainders 0, 1, 2, 3, and 4 are

$$\frac{156}{781},\quad \frac{160}{781},\quad \frac{155}{781},\quad \frac{170}{781},\quad \text{and}\quad \frac{140}{781}$$

respectively. For readers interested in taking this further, see reference 5.

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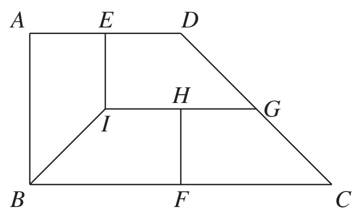
Cutting a trapezium into four equal trapeziums



$$AD = AB = \frac{BC}{2},$$
$$\angle DAB = \angle ABC = 90^\circ.$$

Students' Investigation House,
Shariati Avenue, Sirjan, Iran

Abbas Rouholamini Gugheri

Making 100

Digits in ascending order using addition and subtraction only:

$$\begin{aligned}
100 &= 12 + 3 - 4 + 5 + 67 + 8 + 9 \\
&= 123 + 4 - 5 + 67 - 89 \\
&= 12 + 3 - 4 + 5 + 67 + 8 + 9 \\
&= 123 - 4 - 5 - 6 - 7 + 8 - 9 \\
&= 123 + 4 - 5 + 67 - 89 \\
&= 123 + 45 - 67 + 8 - 9 \\
&= 123 - 45 - 67 + 89 \\
&= 12 - 3 - 4 + 5 - 6 + 7 + 89 \\
&= 12 + 3 + 4 + 5 - 6 - 7 + 89 \\
&= 1 + 23 - 4 + 5 + 6 + 78 - 9 \\
&= 1 + 23 - 4 + 56 + 7 + 8 + 9 \\
&= 1 + 2 + 3 - 4 + 5 + 6 + 78 + 9.
\end{aligned}$$

Digits in descending order using addition and subtraction only:

$$\begin{aligned}
100 &= 98 - 76 + 54 + 3 + 21 \\
&= 9 - 8 + 76 + 54 - 32 + 1 \\
&= 98 + 7 + 6 - 5 - 4 - 3 + 2 - 1 \\
&= 98 - 7 - 6 - 5 - 4 + 3 + 21 \\
&= 9 - 8 + 76 - 5 + 4 + 3 + 21 \\
&= 98 - 7 + 6 + 5 + 4 - 3 - 2 - 1 \\
&= 98 + 7 - 6 + 5 - 4 + 3 - 2 - 1 \\
&= 98 + 7 - 6 + 5 - 4 - 3 + 2 + 1 \\
&= 98 - 7 + 6 + 5 - 4 + 3 - 2 + 1 \\
&= 98 - 7 + 6 - 5 + 4 + 3 + 2 - 1 \\
&= 98 - 7 - 6 + 5 + 4 + 3 + 2 + 1 \\
&= 9 + 8 + 76 + 5 + 4 - 3 + 2 - 1 \\
&= 9 + 8 + 76 + 5 - 4 + 3 + 2 + 1 \\
&= 9 - 8 + 7 + 65 - 4 + 32 - 1.
\end{aligned}$$

Digits in descending order and any mathematical signs:

$$\begin{aligned}
100 &= (9 + 8) \times 7 - 6 - 5 - 4 - 3 - 2 + 1 \\
&= (98 \div 7 + 6) \times 5 + 4 - 3 - 2 + 1 \\
&= 9 - 8 - 7 - 6 + 5! - 4 - 3 - 2 + 1 \\
&= (9 \times 8 \times 7 + 6) \div 5 + 4 - 3 - 2 - 1.
\end{aligned}$$

Digits in any order and making use of the decimal point:

$$100 = 9 + 87.6 + 5.4 - 3 + 2 - 1 = 98.3 + 6.4 - 5.7 + 2 - 1 = (8 \times 9.125) + 37 - 6 - 4.$$

Folkestone Academy **David Benjamin**

The Nine-Point Circle as a Locus

AYOUB B. AYOUB

A different look at the nine-point circle.

In 1763, the Swiss mathematician Leonhard Euler proved that, in the nonequilateral triangle ABC, the circumcentre O, the centroid G, and the orthocentre H are collinear and $HG = 2GO$ (see reference 1). In 1821, the French mathematicians C. B. Brianchon and J. V. Poncelet proved that the circle that passes through the midpoints A', B', C' of the sides BC, CA, AB also passes through the feet D, E, F of the altitudes AD, BE, CF and through the midpoints A'', B'', C'' of the segments AH, BH, CH. Later, this circle became known as the *nine-point circle*. The centre O' of this circle is midway between H and O (see reference 2). From these two results, we have $HO' : O'G : GO = 3 : 1 : 2$ (see figure 1). This implies that the centre of the nine-point circle divides the segment OG externally in the ratio $3 : 1$.

In this article we will show that the nine-point circle of a triangle is the locus of a point moving, under a certain condition, in the plane of the triangle.

Let K be a point on the smaller arc AB of the circumcircle. Join KG and extend it to N such that $GN = \frac{1}{2}KG$. Now we prove that N is a point of the nine-point circle (see figure 2). Join ON and erect on it at N, a perpendicular that meets the circumcircle at L and M. Then N is the midpoint of LM. Consequently, G is the centroid of $\triangle KLM$. Thus, the triangles KLM and ABC have the same centroid, G. Since they also have the same circumcentre O, then the two triangles have the same nine-point circle, whose centre O' divides OG externally

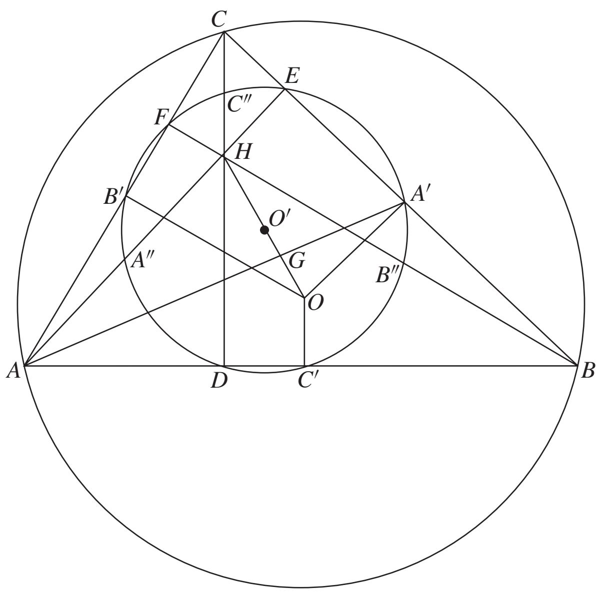


Figure 1

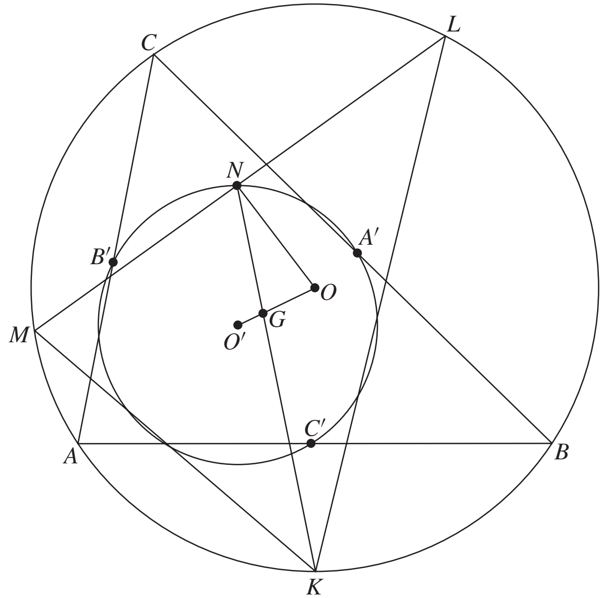


Figure 2

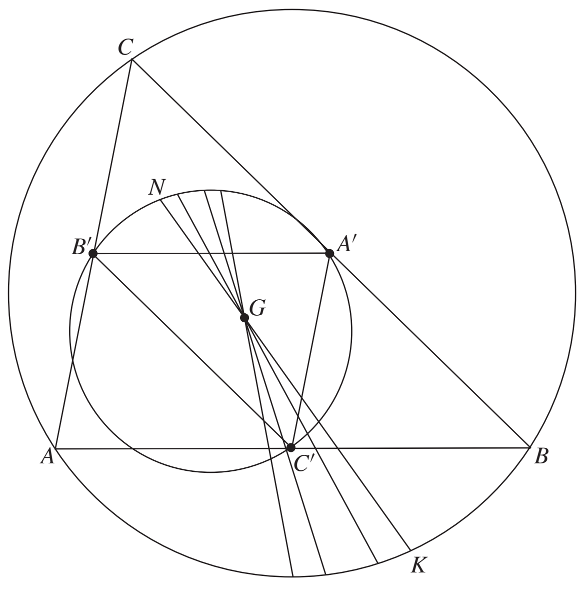


Figure 3

in the ratio 3 : 1 (see the opening paragraph). Hence, N lies on the nine-point circle of $\triangle ABC$. So if K moves along the arc AB, N traces part of the nine-point circle. More precisely N will trace the arc $A'B'$ of the nine-point circle. The reason is, when K coincides with A,

N coincides with A', and when K coincides with B, N coincides with B'. Similarly, if the same procedure is followed when L moves along the arc BC and when M moves along the arc CA, N traces the arcs $B'C'$ and $C'A'$ respectively. Thus, the nine-point circle is the locus of N. A byproduct of this proof is that there are infinitely many triangles that share the same nine-point circle and the same Euler line.

We could treat the problem of the locus by making use of the similarity of the two triangles ABC and $A'B'C'$. Since the corresponding sides are parallel, then the lines AA', BB', and CC' meet in the center of similarity. This point happens to be the common centroid G of the two triangles. Then the ratio of similarity is $2:1$. Now if K moves along the circumcircle of triangle ABC such that $KG : GN = 2:1$, then N traces a circle that passes through A', B', and C' which is the nine-point circle (see figure 3).

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Picture-perfect squares

$$\begin{aligned} 12^2 &= 144,\\ 21^2 &= 441,\\ 12012^2 &= 144288144,\\ 21021^2 &= 441882441. \end{aligned}$$

The pattern continues with the squares of

$$\begin{aligned} &12012000012,\\ &12(10^{18}+10^{15}+10^6+1),\\ &12(10^{33}+10^{24}+10^{21}+10^6+1),\\ &12(10^{57}+10^{54}+10^{45}+10^{21}+10^6+1), \end{aligned}$$

and their ‘reversals’.

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A Note on 'Oblique-Angled Diameters'

SANDRO CAPARRINI

In some recent articles by Thomas Osler and other authors, attention has been given to a paper by Euler on the *oblique-angled diameters* of conics. As a matter of fact, the so-called oblique-angled diameters have a history that goes back to Apollonius of Perga (late third century BC). For centuries they were an important part of the curriculum of students of mathematics under the name of *conjugate diameters*. Today, most of the classic theorems on conjugate diameters can be easily demonstrated by analytic geometry.

In recent issues of *Mathematical Spectrum*, Thomas Osler, in collaboration with other authors, has published three articles on the *oblique-angled diameters* of conics (see references 1–3). The subject was brought to his attention when he and Edward Greve were translating a paper by Euler (see reference 4). Euler, in effect, merely hints at conics, a case considered too commonly known to be worth mentioning, and is interested only in the generalization of the *diamètres obliquangles* to algebraic curves of higher order. 'Apparently,' wrote Osler and Greve, 'these ideas were familiar to mathematicians in [Euler's] day[s], but have been ignored in the education of modern mathematicians'. The recovery of this now-lost knowledge was the motivation behind the research of Osler and his collaborators. Osler and Robert Buonpastore (see reference 2), for example, wrote that their article 'is about one such ignored property of the ellipse which we rediscovered in order to understand Euler's work'.

Osler and Greve were right in suspecting that these problems were better appreciated in the past, for their so-called 'oblique-angled diameters' have a history that goes back to Greek times. Until a few decades ago they were well known even to undergraduates under the name of *conjugate diameters*. Before the time of Euler, conjugate diameters had been studied by, among others, Apollonius of Perga (*Conics*, late third century BC), Kepler (*Harmonices Mundi*, 1619), Desargues (*Brouillon Project d'une Atteinte aux Événemens des Rencontres du Cone avec un Plan*, 1639), de la Hire (*Nouvelle Méthode en Géométrie pour les Sections des Superficies Coniques et Cylindriques*, 1673), Newton (*Principia*, 1687; *Enumeratio Linearum Tertii Ordinis*, 1711), and de l'Hôpital (*Traité Analytique des Sections Coniques*, 1707). As for Euler, he gave a systematic treatment of conic sections in general and of conjugate diameters in particular in the second volume of his classic treatise *Introductio in Analysin Infinitorum*, published in 1748 (see reference 4). In fact, the theorems demonstrated by Osler, Buonpastore, and Adam Romasko on triangles of equal area inside an ellipse and a hyperbola appear in the *Introductio* (Volume II, Chapter V, Section 116). For details, the reader is referred to general histories of mathematics (see references 5 and 6) or to more specialized texts on the history of geometry (see references 7 and 8). The theory of conjugate diameters was then presented in every textbook on conics until, say, the 1960s. Things changed with the advent of Bourbakist mathematics, when classic geometry was banned from our universities. This is not the place to go into detail about the pros and cons of that decision; suffice to say that, for these historical reasons, most mathematicians of today are much less familiar with geometry than they ought to be. However, history has its cycles and revivals: thanks to the resources provided by the

Internet, elementary synthetic geometry currently enjoys something of a new life. A Google search for 'conjugate diameters' will uncover more information on the subject than is really needed.

The venerable properties of conjugate diameters afford modern mathematicians the chance to gauge their own strengths in analytic geometry. The purpose of the present note is to give simple proofs of two classic theorems of Apollonius. For convenience, we will refer to the notation and the results of Osler and his collaborators.

A *diameter* of a conic is a straight line that bisects all chords that are parallel to a given direction. In a central conic any chord through the centre is a diameter and every diameter must pass through the centre, while in the parabola a diameter is simply any straight line parallel to the axis. If, in the case of a central conic, we have two diameters, each one bisecting the chords parallel to the other, we say that they are *conjugate* to each other. Let us now restrict our attention to the ellipse, represented in orthogonal Cartesian coordinates by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Then, if the extremities M and M' of a diameter are given respectively by $(a\cos\theta, b\sin\theta)$ and $(-a\cos\theta, -b\sin\theta)$, where θ is the *eccentric angle*, the coordinates of the end points of the diameter conjugate to MM' are $m(-a\sin\theta, b\cos\theta)$ and $m'(a\sin\theta, -b\cos\theta)$.

With the help of these few basic notions, we can immediately demonstrate our first result.

Theorem 1 *The sum of the squares of two conjugate diameters of an ellipse is constant.*

Proof Since the lengths of the two conjugate axes MM' and mm' are respectively

$$2\sqrt{a^2\cos^2\theta+b^2\sin^2\theta} \quad \text{and} \quad 2\sqrt{a^2\sin^2\theta+b^2\cos^2\theta},$$

the sum of their squares is $4(a^2+b^2)$, which is a constant.

In Apollonius's *Conics* this is Proposition XII of Book VII (see reference 9). An analytic proof is given by Euler in the *Introductio* (Volume II, Chapter V, Section 119). There is an equivalent result for the hyperbola: *the difference of the squares of two conjugate diameters of a hyperbola is constant* (Apollonius's *Conics*, Book VII, Proposition XIII). The proof is left to the reader. The main difference from the ellipse is that the length of the diameter having the same direction of a set of chords of the hyperbola $x^2/a^2-y^2/b^2=1$ is determined by the intersections with the *conjugate hyperbola* $x^2/a^2-y^2/b^2=-1$.

Let us now turn to our second theorem.

Theorem 2 *The area of the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant.*

Proof The tangents at points M, m, and M' are

$$x\frac{\cos\theta}{a}+y\frac{\sin\theta}{b}=1, \qquad x\frac{\sin\theta}{a}-y\frac{\cos\theta}{b}=-1, \qquad x\frac{\cos\theta}{a}+y\frac{\sin\theta}{b}=-1$$

respectively. The tangents at M and m intersect at the point $A(a(\cos\theta-\sin\theta),\ b(\cos\theta+\sin\theta))$, and those at m and M' intersect at $B(-a(\cos\theta+\sin\theta),\ b(\cos\theta-\sin\theta))$.

The intersections of the four tangents are the vertices of a parallelogram whose centre is at the origin O of the axes. The area of this parallelogram is four times the area of the triangle OAB, that is,

$$2\begin{vmatrix} 0 & 0 & 1 \\ a(\cos\theta - \sin\theta) & b(\cos\theta + \sin\theta) & 1 \\ -a(\cos\theta + \sin\theta) & b(\cos\theta - \sin\theta) & 1 \end{vmatrix},$$

whose value is easily found to be $4ab$, a quantity independent of the two diameters.

This is Proposition XXXI of Book VII in Apollonius's *Conics*. It also appears in Newton's *Principia* (see reference 10, Book I, Lemma XII).

Theorem 2 remains valid word-for-word in the case of the hyperbola; this result is demonstrated by reasoning similar to that given above. We leave to the reader the task of proving it, as well as of showing that the asymptotes of the hyperbola are the diagonals of the parallelogram formed by the tangents (Apollonius's *Conics*, Book II, Proposition XXI).

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The Ends of Square–Triangular Numbers

THOMAS KOSHY

Triangular numbers have been studied for centuries. Finding square–triangular numbers is closely related to solving the Pell's equation $x^2 - 8y^2 = 1$, and hence to Pell and Pell–Lucas numbers. Since there are infinitely many such numbers, there are that many square–triangular numbers. Using recursion, we determine the possible unit digits.

Introduction

Figurate numbers are a delightful bridge between number theory and geometry; they are positive integers that can be represented by geometric patterns. The Pythagoreans have been credited with their discovery, although the Chinese seem to have originated such representations about 500 years before Pythagoras (see reference 1). In 1665, the French mathematician Blaise Pascal (1623–1662) wrote a book on them, *Treatise on Figurate Numbers*.

Triangular numbers

Triangular numbers are a special class of planar figurate numbers; they can be represented by triangular patterns. The first four triangular numbers, 1, 3, 6, and 10, are represented in figure 1. The number of pins (10) in bowling and the number of balls (15) in the game of pool are triangular numbers. Triangular numbers can also be found in floral designs on tablecloths, and arrangements of apples and oranges in triangular bins in grocery stores. In fact, they pop up in numerous unexpected places (see references 2 and 3).

Triangular numbers t_n are often defined by the recurrence $t_n = t_{n-1} + n$, where $t_1 = 1$ and $n \geq 2$. They can be computed explicitly also using the formula

$$t_n = \frac{n(n+1)}{2} = \binom{n+1}{2}.$$

Consequently, they can be read directly from the northeast diagonal 2 in Pascal's triangle.

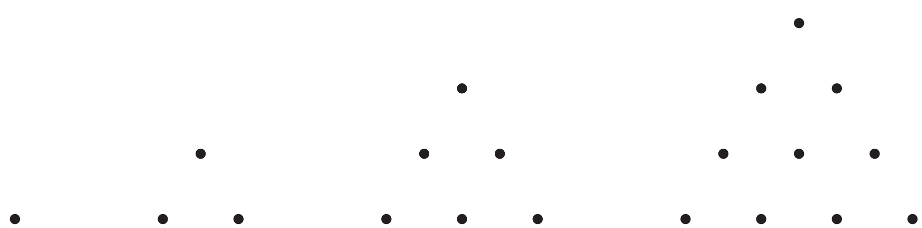

Figure 1 The first four triangular numbers.

They satisfy a number of interesting properties (see references 2, 4, 5, and 6). One of them is the well-known *Diophantus' theorem*: $8t_n + 1 = (2n+1)^2$. For example, $8t_{36} = 8 \cdot 666 + 1 = (2 \cdot 36 + 1)^2$, where 666 is the infamous *beastly number* mentioned in the *Book of Revelation* in the *Bible*.

As a byproduct, Diophantus' theorem can be used to determine whether or not a given integer N is a triangular number. To see this, N is triangular if and only if $8N + 1 = (2n+1)^2$; that is, if and only if

$$n = \frac{\sqrt{8N+1} - 1}{2}$$

is a positive integer. For example, since $(\sqrt{8 \cdot 78 + 1} - 1)/2 = 12$ is a positive integer, 78 is a triangular number. But $(\sqrt{8 \cdot 90 + 1} - 1)/2$ is not an integer, so 90 is not a triangular number.

Square–triangular numbers

There are triangular numbers that are also squares (see references 4 and 6). Three such numbers are $t_1 = 1 = 1^2$, $t_8 = 36 = 6^2$, and $t_{49} = 1225 = 35^2$. Are there others? If there are, how do we find them in a systematic way?

To answer these two questions, let $t_n = y^2$. Then, by Diophantus' theorem, $8y^2 + 1$ must be a square, say, x^2. So (x, y) must be a solution of the Pell's equation $x^2 - 8y^2 = 1$.

To solve this equation, we invoke the following result from the theory of Pell's equations (see references 2 and 7).

Theorem 1 *Let (α, β) denote the least positive solution of $x^2 - Ny^2 = 1$, where N is a positive nonsquare integer. Then the equation has infinitely many solutions (x_k, y_k), given by*

$$x_k = \frac{(\alpha + \beta\sqrt{N})^k + (\alpha - \beta\sqrt{N})^k}{2},$$

$$y_k = \frac{(\alpha + \beta\sqrt{N})^k - (\alpha - \beta\sqrt{N})^k}{2\sqrt{N}},$$

where $k \geq 1$.

Since $(\alpha, \beta) = (3, 1)$ is the least positive solution of the Pell's equation $x^2 - 8y^2 = 1$, it follows by theorem 1 that it has infinitely many solutions (x_k, y_k):

$$x_k = \frac{(3 + 2\sqrt{2})^k + (3 - 2\sqrt{2})^k}{2},$$

$$y_k = \frac{(3 + 2\sqrt{2})^k - (3 - 2\sqrt{2})^k}{4\sqrt{2}}.$$

Since the Pell's equation has an infinite number of solutions, it follows that there are infinitely many square–triangular numbers y_k^2, as the great Swiss mathematician Leonhard Euler (1707–1783) discovered in 1730.

Pell and Pell–Lucas numbers

The general solution (x_k, y_k) can be rewritten in terms of *Pell numbers* P_m and *Pell–Lucas numbers* Q_m. To this end, let $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ be solutions of the quadratic

Table 1

k	x_k	y_k	Square–triangular number y_k^2	Corresponding subscript n_k in t_{n_k}
1	3	1	1	1
2	17	6	36	8
3	99	35	1 225	49
4	577	204	41 616	288
5	3 363	1 189	1 413 721	1 681
6	19 601	6 930	48 024 900	9 800
7	114 243	40 391	1 631 432 881	57 121
8	665 857	235 416	55 420 693 056	332 928

equation $x^2 = 2x + 1$. Then P_m and Q_m are given by the *Binet-like* formulas

$$P_m = \frac{\gamma^m - \delta^m}{\gamma - \delta} \quad \text{and} \quad Q_m = \frac{\gamma^m + \delta^m}{2}$$

(see reference 7). The first five Pell numbers are 1, 2, 5, 12, and 29; and the first Pell–Lucas numbers are 1, 3, 7, 17, and 41. They both satisfy the same recurrence $u_n = 2u_{n-1} + u_{n-2}$, where $n \geq 3$.

So the solutions (x_k, y_k) can be rewritten as follows:

$$x_k = \frac{\gamma^{2k} + \delta^{2k}}{2} = Q_{2k},$$
$$y_k = \frac{\gamma^{2k} - \delta^{2k}}{2(\gamma - \delta)} = \frac{1}{2}P_{2k}.$$

Since $P_{2k} = 2P_k Q_k$, $y_k = P_k Q_k$.

Suppose that y_k^2 is the n_kth triangular number. Then, again by Diophantus' theorem,

$$n_k = \frac{\sqrt{8y_k^2 + 1} - 1}{2} = \frac{\sqrt{2P_{2k}^2 + 1} - 1}{2}.$$

Since $Q_n^2 - 2P_n^2 = (-1)^n$, this implies that $n_k = (Q_{2k} - 1)/2$. Thus $y_k^2 = t_{(Q_{2k}-1)/2}$.

For example, $y_4^2 = 41\,616 = 204^2 = (12 \cdot 17)^2$ is a square–triangular number. Then $n_4 = (Q_8 - 1)/2 = (577 - 1)/2 = 288$. So $41\,616 = t_{288}$, the 288th triangular number.

Table 1 shows the first eight solutions (x_k, y_k) of the Pell's equation $x^2 - 8y^2 = 1$, and the corresponding square–triangular numbers and their subscripts.

Hidden treasures

Table 1 contains several hidden treasures.

(i) The factors P_k and Q_k in $y_k = P_k Q_k$ are relatively prime. This follows from the facts that: $P_k + P_{k-1} = Q_k$; any common divisor d of P_k and Q_k also divides P_{k-1}; and hence, by the Pell recurrence, d divides $P_{k-2}, \ldots, P_2$, and P_1.

(ii) The sequence $\{x_k\}$ can be defined recursively:

$$\begin{aligned} x_1 &= 3, \\ x_2 &= 17, \\ x_k &= 6x_{k-1} - x_{k-2}, \qquad k \geq 3. \end{aligned}$$

Since $x_k = Q_{2k}$, this recurrence follows by the Pell recurrence.

(iii) The sequence $\{y_k\}$ also satisfies the same recurrence, where $y_1 = 1$ and $y_2 = 6$.

(iv) Let $c_k = y_k^2$. The sequence $\{c_k\}$ satisfies the nonhomogeneous recurrence

$$c_k = 34c_{k-1} - c_{k-2} + 2,$$

where $c_1 = 1$, $c_2 = 36$, and $k \geq 3$. This follows from the identity

$$P_n^2 = 6P_{n-1}^2 - P_{n-2}^2 - 2(-1)^n.$$

(v) The sequence $\{n_k\}$ satisfies a similar nonhomogeneous recurrence,

$$n_k = 6n_{k-1} - n_{k-2} + 2,$$

where $n_1 = 1$, $n_2 = 8$, and $k \geq 3$. Since $n_k = \frac{1}{2}(Q_{2k} - 1)$, this follows by the recurrence $Q_{2k} = 6Q_{2k-2} - Q_{2k-4}$.

These recurrences, coupled with the corresponding initial conditions, can be used to find explicit formulas for x_k, y_k, y_k^2, and n_k, and the generating functions for the corresponding sequences.

The ends of x_k, y_k, y_k^2, and n_k

Table 1 manifests four additional treasures.

(vi) The sequence $\{x_k \pmod{10}\}$ shows an interesting periodic pattern with period 6:

$$\underbrace{3\ 7\ 9\ 7\ 3\ 1}\ \underbrace{3\ 7\ 9\ 7\ 3\ 1}\ \cdots.$$

That is,

$$x_k \equiv \begin{cases} 3 \pmod{10} & \text{if } k \equiv 1 \pmod 6 \\ 7 \pmod{10} & \text{if } k \equiv 2 \pmod 6 \\ 9 \pmod{10} & \text{if } k \equiv 3 \pmod 6 \\ 7 \pmod{10} & \text{if } k \equiv 4 \pmod 6 \\ 3 \pmod{10} & \text{if } k \equiv 5 \pmod 6 \\ 1 \pmod{10} & \text{if } k \equiv 0 \pmod 6. \end{cases}$$

(vii) We have

$$y_k \equiv \begin{cases} 1 \pmod{10} & \text{if } k \equiv 1 \pmod 6 \\ 6 \pmod{10} & \text{if } k \equiv 2 \pmod 6 \\ 5 \pmod{10} & \text{if } k \equiv 3 \pmod 6 \\ 4 \pmod{10} & \text{if } k \equiv 4 \pmod 6 \\ 9 \pmod{10} & \text{if } k \equiv 5 \pmod 6 \\ 0 \pmod{10} & \text{if } k \equiv 0 \pmod 6. \end{cases}$$

(viii) We have

$$y_k^2 \equiv \begin{cases} 1 \pmod{10} & \text{if } k \equiv 1 \pmod 6 \\ 6 \pmod{10} & \text{if } k \equiv 2 \pmod 6 \\ 5 \pmod{10} & \text{if } k \equiv 3 \pmod 6 \\ 6 \pmod{10} & \text{if } k \equiv 4 \pmod 6 \\ 1 \pmod{10} & \text{if } k \equiv 5 \pmod 6 \\ 0 \pmod{10} & \text{if } k \equiv 0 \pmod 6. \end{cases}$$

(ix) We have

$$n_k \equiv \begin{cases} 1 \pmod{10} & \text{if } k \equiv 1 \pmod 6 \\ 8 \pmod{10} & \text{if } k \equiv 2 \pmod 6 \\ 9 \pmod{10} & \text{if } k \equiv 3 \pmod 6 \\ 8 \pmod{10} & \text{if } k \equiv 4 \pmod 6 \\ 1 \pmod{10} & \text{if } k \equiv 5 \pmod 6 \\ 0 \pmod{10} & \text{if } k \equiv 0 \pmod 6. \end{cases}$$

Property (vi) follows by working modulo 10 from the recursive definition of $\{x_k\}$ in (ii); property (vii) from the recursive definition of $\{y_k\}$ in (iii); property (viii) by squaring (vii) modulo 10; and property (ix) from the recursive definition of $\{n_k\}$ in (v).

Acknowledgment

The author would like to thank the Editor for his valuable suggestions.

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Thomas Koshy *received his PhD in Algebraic Coding Theory from Boston University in 1971. He continues his passion for the beauty, power, and ubiquity of Fibonacci and Lucas numbers, Catalan numbers, and Pell and Pell–Lucas numbers. Two of his most recent books are 'Catalan Numbers with Applications' (2009) and 'Triangular Numbers with Applications' (2011), both published by Oxford University Press.*

In Search of Square Centred Square Numbers

M. A. NYBLOM

We prove the existence of infinitely many square centred numbers which are simultaneously perfect squares via a completely elementary approach which circumvents the need to use the theory of Pell's equation. The advantage of this approach is that we uncover an interesting one-to-one correspondence between the square centred square number and the square triangular numbers.

1. Introduction

Frequently the simplest questions require the most complex solutions. This maxim is nowhere more appropriate in Mathematics than in the field of Number Theory. The study of numbers abounds in problems which are easily understood by the non-expert, but whose solution can remain frustratingly elusive to the most knowledgeable of experts in the field. However, as we shall see, if one is fortunate enough in the choice of one's problem, then an elementary solution may well suffice. In a recent paper (see reference 1) Euler *et al.* returned to a variation on the theme of figurate numbers known as the *centred figurate numbers*. A centred figurate number is the number of dots formed from a central dot surrounded by successive polygonal layers, each side containing one dot more than a side in the previous layer, so starting from the second layer each layer, for example, in a triangular and square centred number would contain three and four more dots respectively than the previous layer as pictured in figure 1.

Counting the accumulation of dots in each successive layer in the geometrical arrangements of figure 1 results in the two sequences of numbers, 1, 4, 10, 19, . . . and 1, 5, 13, 25, . . . , known as the *centred triangular number sequence* and *centred square number sequence* respectively.

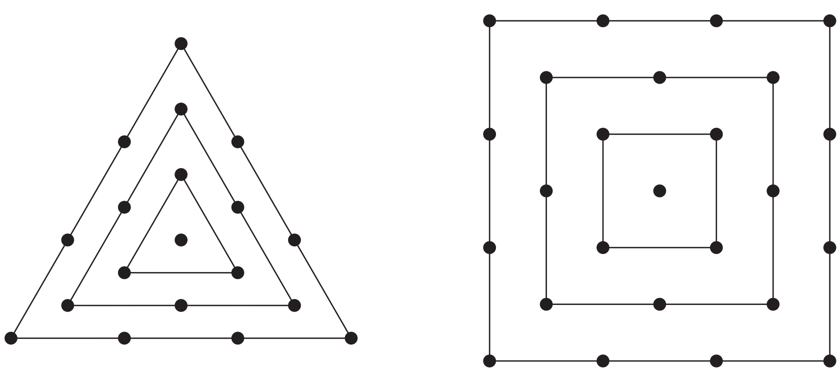

Figure 1

From the above description, both the *centered triangular numbers*, t_n, and the *centred square numbers*, s_n, must satisfy a recurrence relation of the form $u_n = u_{n-1} + d(n-1)$ for $n \geq 2$, with $u_1 = 1$, where $d = 3$ and 4 respectively. Thus

$$\begin{aligned} t_n &= 1 + 3((n-1) + (n-2) + \cdots + 1) \\ &= 1 + 3\frac{(n-1)n}{2} \\ &= \frac{3n^2 - 3n + 2}{2}, \end{aligned}$$

for $n \geq 1$, while similarly $s_n = 2n^2 - 2n + 1$, for $n \geq 1$. In reference 1 Euler *et al.* posed the following question. Given there are infinitely many square triangular numbers (see reference 2), that is, numbers of the form $m(m+1)/2$ which are simultaneously square, can there similarly be infinitely many centred triangular numbers that are triangular? This question was answered in the affirmative in reference 1 by first setting $t_n = m(m+1)/2$, then using a series of algebraic manipulations the solvability of this equation in integers was reduced to that of Pell's equation $X^2 - 3Y^2 = 1$, whose solutions (X_n, Y_n) are known (see reference 3) to be generated from the smallest positive solution $(X_1, Y_1) = (2, 1)$ via the equality $X_n + Y_n\sqrt{3} = (2 + \sqrt{3})^n$. By solving for X_n and Y_n and back-substituting it was finally shown that all of the centred triangular numbers that are triangular are given by

$$\tfrac{3}{16}\big((2+\sqrt{3})^{2n+1} + (2-\sqrt{3})^{2n+1} + \tfrac{4}{3}\big), \tag{1}$$

for $n \geq 1$. In this article we shall investigate the related problem of determining whether there are infinitely many centred square numbers which are perfect squares. In place of a Pell's equation, our approach will instead concentrate on reducing the solvability of the equation $s_n = m^2$ in integers to that of $x(x-1) = 2y(y-1)$, which after applying a series of elementary arguments can then be shown to have infinitely many integer solutions. One advantage of this elementary approach is that we can uncover a surprising connection between the square centred square numbers and the square triangular numbers. In particular, if T_n denotes the square root of the nth square triangular number, then the nth square centred square number will be shown to be equal to $12T_n^2 + 4T_n\sqrt{8T_n^2 + 1} + 1$, which in turn yields a formula similar to (1) and a recursive procedure for the calculation of the nth square centred square number.

2. An elementary approach

First, to determine the existence of centred square numbers which are also perfect squares we examine the solvability in positive integers of the equation $s_l = 2l^2 - 2l + 1 = m^2$. Clearly as m must be an odd integer, we see that the equation above transforms to

$$x(x-1) = 2y(y-1), \tag{2}$$

after making the substitutions $m = 2y - 1$ and $l = x$, for some positive integers x and y. Our task is thus reduced to demonstrating the existence of positive integer solutions (x, y) to (2). To this end, first observe that if $0 < x \leq y$, then $x(x-1) < 2y(y-1)$, while if $x \geq 2y$ then $x(x-1) > 2y(y-1)$. So, for an arbitrary positive integer y, the only integer values x may assume in order that (2) may possibly be satisfied are those for which $y < x < 2y$. Thus, if (x, y) is a positive integer solution of (2), there must exist a positive integer r such that $x = y + r$, with $(y + r)(y + r - 1) = 2y(y - 1)$, which upon expanding and simplifying yields the quadratic equation $0 = y^2 - y(2r + 1) + (r - r^2)$, whose only positive (not necessarily integer) solution is given by

$$y = \frac{(2r + 1) + \sqrt{8r^2 + 1}}{2}. \tag{3}$$

Now the left-hand side of (3) can be a positive integer if and only if the term $8r^2 + 1$ is an odd perfect square. Consequently, by setting $8r^2 + 1 = (2s + 1)^2$, for some positive integer s, we find after rearrangement that

$$r^2 = \frac{s(s + 1)}{2},$$

and so r^2 must be a square triangular number, for which it is well known there are infinitely many solutions. As the solvability in integers of $s_l = m^2$ is equivalent to that of $x(x - 1) = 2y(y - 1)$, we can conclude there are infinitely many square centred square numbers.

Next, to make explicit the connection between the square triangular numbers and the square centred square numbers, let T_n denote the square root of the nth square triangular number. As $r = T_n$ and recalling $s_l = m^2$, we find from (3) and the substitution $m = 2y - 1$ that the nth square centred square number is given by

$$\begin{aligned} s_l &= \left(2T_n + \sqrt{8T_n^2 + 1}\,\right)^2 \\ &= 12T_n^2 + 4T_n\sqrt{8T_n^2 + 1} + 1. \end{aligned} \tag{4}$$

We now examine two consequences of (4), the first of which is a closed-form expression similar to (1) for the nth square centred square numbers, denoted here by S_n. To this end consider the following formula attributable to Leonhard Euler (see reference 4) for the nth square triangular number, namely

$$T_n^2 = \left(\frac{(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n}{4\sqrt{2}}\right)^2. \tag{5}$$

Upon expanding and simplifying (5) we obtain

$$8T_n^2 + 1 = \left(\frac{(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n}{2}\right)^2,$$

from which it is also easily deduced that

$$4T_n\sqrt{8T_n^2 + 1} = \frac{(3 + 2\sqrt{2})^{2n} - (3 - 2\sqrt{2})^{2n}}{2\sqrt{2}}. \tag{6}$$

Substituting (5) and (6) into the right-hand side of (4) finally yields

$$\begin{aligned} S_n &= 3\left(\frac{(3 + 2\sqrt{2})^{2n} - 2 + (3 - 2\sqrt{2})^{2n}}{8}\right) + \left(\frac{(3 + 2\sqrt{2})^{2n} - (3 - 2\sqrt{2})^{2n}}{2\sqrt{2}}\right) + 1 \\ &= (3 + 2\sqrt{2})^{2n}\left(\frac{3 + 2\sqrt{2}}{8}\right) + (3 - 2\sqrt{2})^{2n}\left(\frac{3 - 2\sqrt{2}}{8}\right) + \frac{1}{4} \\ &= \tfrac{1}{8}((3 + 2\sqrt{2})^{2n+1} + (3 - 2\sqrt{2})^{2n+1} + 2). \end{aligned} \tag{7}$$

Table 1 The first five square centred square numbers.

V_{2n+1}	S_n
198	25
6 726	841
228 486	28 561
7 761 798	970 225
263 672 646	32 959 081

Although of interest in its present form, (7) is not practicable for the purposes of calculating the nth square centred square numbers. However, the presence of the term

$$(3+2\sqrt{2})^{2n+1}+(3-2\sqrt{2})^{2n+1}$$

in (7) gives a clue to constructing a recursive procedure for calculating S_n. To conclude, we shall derive this recursive procedure using some elementary properties of *generalized Lucas sequences* as follows. A sequence of the form $V_n=\alpha^n+\beta^n$, where α and β are real conjugate surds, is called a *generalized Lucas sequence*. It is well known (see reference 5) that the solution of the recurrence relation

$$V_n-6V_n+V_{n-1}=0, \quad \text{for } n\geq 2, \tag{8}$$

is of the form

$$V_n=A(3+2\sqrt{2})^n+B(3-2\sqrt{2})^n,$$

where $A+B=V_0$ and

$$A(3+2\sqrt{2})+B(3-2\sqrt{2})=V_1.$$

If we take $V_0=2$ and $V_1=6$, this gives $A=B=1$. Thus from (7), we find that S_n can be calculated recursively using (8) and $S_n=\frac{1}{8}(V_{2n+1}+2)$, the result of which can be found displayed in table 1.

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***Michael Nyblom** is a lecturer at the Department of Mathematics at RMIT University. His general research interests include Number Theory, Combinatorics, and Analysis.*

Letters to the Editor

Dear Editor,

Closure under multiplication

There are examples of sets A of numbers such that, if $a, b \in A$, then $ab \in A$. An example is

$$A = \{x^2 + y^2 \mid x, y \in \mathbb{Z}\},$$

since

$$(x^2 + y^2)(x'^2 + y'^2) = (xx' + yy')^2 + (xy' - x'y)^2.$$

In Volume 42, Number 1, p. 46, Bob Bertuello gave the identity

$$(a^2 + nb^2)(c^2 + nd^2) = (ac + nbd)^2 + n(bc - ad)^2,$$

so that, for a fixed integer n,

$$A = \{x^2 + ny^2 \mid x, y \in \mathbb{Z}\}$$

is another example. A third example is

$$A = \{x^2 + y^2 + z^2 + t^2 \colon x, y, z, t \in \mathbb{Z}\},$$

since

$$\begin{aligned}(a^2 + b^2 + c^2 + d^2)&(x^2 + y^2 + z^2 + t^2)\\ &= (ax + by + cz + dt)^2 + (bx - ay + dz - ct)^2\\ &\quad + (cx - dy - az + bt)^2 + (dx + cy - bz - at)^2.\end{aligned}$$

Readers may be interested in this example:

$$A = \{x^3 + y^3 + z^3 - 3xyz \mid x, y, z, \in \mathbb{Z}\},$$

which follows from the identity

$$(x^3 + y^3 + z^3 - 3xyz)(a^3 + b^3 + c^3 - 3abc) = p^3 + q^3 + r^3 - 3pqr,$$

where

$$p = ax + bz + cy, \qquad q = by + cx + az, \qquad r = cz + ay + bx.$$

Yours sincerely,

Abbas Rouholamini Gugheri
(Students' Investigation House
Shariati Avenue
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Dear Editor,

Calculation of $s_k = 1^k + 2^k + \cdots + n^k$ for odd k

In Volume 40, Number 1, p. 40, M. A. Khan used a beautiful method to calculate s_3 which I used in Volume 42, Number 1, p. 46 to calculate s_5. Here we use this method to obtain a recurrence formula for s_n for odd n. We write $T_r = r^k(r+1)^k$, so that

$$\begin{aligned} T_r - T_{r-1} &= r^k(r+1)^k - (r-1)^k r^k \\ &= 2\left[\binom{k}{1} r^{2k-1} + \binom{k}{3} r^{2k-3} + \binom{k}{5} r^{2k-5} + \cdots\right]. \end{aligned}$$

If we sum these expressions for $r = 1, 2, \ldots, n$, we obtain

$$\binom{k}{1} s_{2k-1} + \binom{k}{3} s_{2k-3} + \binom{k}{5} s_{2k-5} + \cdots = \tfrac{1}{2} n^k (n+1)^k.$$

For example, if $k = 4$,

$$4s_7 + 4s_5 = \tfrac{1}{2} n^4 (n+1)^4.$$

Yours sincerely,

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Dear Editor,

The indefinite integral for $|x - c|^p$

We propose to find the indefinite integral

$$I = \int |x - c|^p \,\mathrm{d}x, \qquad p \geq 0.$$

This can be achieved by first finding the corresponding definite integral

$$I_0 = \int_a^b |x - c|^p \,\mathrm{d}x, \qquad p \geq 0,\ a < b.$$

The integral I_0 may be evaluated in the normal fashion by utilizing the definition of $|x - c|$ given by

$$|x - c| = \begin{cases} x - c, & x \geq c, \\ c - x, & x \leq c. \end{cases}$$

By using this definition we can cater for three different cases arising out of the values a and b relative to c, that is $a < b \leq c$, $a < c < b$, and $c \leq a < b$. This leads to the following possible values of I_0:

$$\begin{aligned}
I_0 &= \int_a^b (c-x)^p \,\mathrm{d}x \\
&= -\left[\frac{(c-x)^{p+1}}{p+1}\right]_a^b \\
&= \frac{(c-a)^{p+1}-(c-b)^{p+1}}{p+1}, \quad \text{when } a<b\le c, && (1)\\
I_0 &= \int_a^c (c-x)^p \,\mathrm{d}x + \int_c^b (x-c)^p \,\mathrm{d}x \\
&= -\left[\frac{(c-x)^{p+1}}{p+1}\right]_a^c + \left[\frac{(x-c)^{p+1}}{p+1}\right]_c^b \\
&= \frac{(c-a)^{p+1}+(b-c)^{p+1}}{p+1}, \quad \text{when } a<c\le b, && (2)\\
I_0 &= \int_a^b (x-c)^p \,\mathrm{d}x \\
&= \left[\frac{(x-c)^{p+1}}{p+1}\right]_a^b \\
&= \frac{(b-c)^{p+1}-(a-c)^{p+1}}{p+1}, \quad \text{when } c\le a<b. && (3)
\end{aligned}$$

The three values of I for the different cases as given in (1)–(3) may be combined to yield

$$I_0 = \left[\frac{|x-c|^p(x-c)}{p+1}\right]_a^b, \quad \text{for all } a, b.$$

Consequently, we can write the corresponding indefinite integral as

$$I = \frac{|x-c|^p(x-c)}{p+1}. \tag{4}$$

The formula given in (4) may be generalized to give

$$\int |lx+m|^p \,\mathrm{d}x = \frac{|lx+m|^p(lx+m)}{l(p+1)}, \qquad l \ne 0.$$

Yours sincerely,

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Indian Overseas Bank
Ashok Marg
Lucknow
India)

Dear Editor,

Sums of squares and cubes

In Volume 43, Number 3, p. 132, I gave a rational solution $x = \frac{5}{3}$, $y = \frac{4}{3}$ of the equation $x^3 + y^3 = 7$. The equation $x^3 + y^3 = 6$ has a rational solution $x = \frac{17}{21}$, $y = \frac{37}{21}$. More generally, if a, b are positive integers, the equation

$$x^3 + y^3 = ab(a + b)$$

has a rational solution

$$x = \frac{a^3 - b^3 + 3ab(2a + b)}{3(a^2 + ab + b^2)}, \qquad y = \frac{b^3 - a^3 + 3ab(2b + a)}{3(a^2 + ab + b^2)}.$$

The case $a = 1$, $b = 2$ gives the solution $x = \frac{17}{21}$, $y = \frac{37}{21}$ of $x^3 + y^3 = 6$.

I have also found two families of solutions of the equation $x^2 + y^2 + z^2 = t^2$ in integers. One is

$$(a^2)^2 + (2b^2)^2 + (2ab)^2 + (a^2 + 2b^2)^2;$$

the other is the less obvious

$$\begin{aligned} x &= 2a(2a + b - 3c) - 2(a^2 + b^2 - c^2), \\ y &= 2b(2a + b - 3c) - (a^2 + b^2 - c^2), \\ z &= 2(a^2 + b^2 - c^2), \\ t &= 3(a^2 + b^2 + c^2) - 2c(2a + b), \end{aligned}$$

where a, b, c are integers.

Yours sincerely,

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Dear Editor,

The sum of the first n squares

In Volume 41, Number 3, p. 137 and Volume 42, Number 3, p. 142 I gave two methods for finding the formula for the sum of the first n squares, the second one geometrical in nature. A third method uses the formula

$$(m + 1)^3 = m^3 + 3m^2 + 3m + 1,$$

so that

$$3m^2 = (m + 1)^3 - m^3 - 3m - 1.$$

If we sum these expressions for $m = 1, 2, \ldots, n$, we obtain

$$\begin{aligned} 3(1^2 + 2^2 + \cdots + n^2) &= (n + 1)^3 - 1^3 - 3(1 + 2 + \cdots + n) - n \\ &= n^3 + 3n^2 + 3n + 3 \times \tfrac{1}{2}n(n + 1) - n \\ &= n^3 + \tfrac{3}{2}n^2 + \tfrac{1}{2}n, \end{aligned}$$

which gives the familiar formula

$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

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Dear Editor,

The equation $x^2 + y^2 = xy$

We ask what are the rational solutions of this equation. It can be rewritten as

$$(2x-1)^2 + (2y-1)^2 = 2.$$

From my letter in Volume 43, Number 2, pp. 90–91, we can write the rational solutions of this as

$$2x - 1 = \pm\frac{s^2 - t^2 + 2st}{s^2 + t^2}, \qquad 2y - 1 = \pm\frac{s^2 - t^2 - 2st}{s^2 + t^2},$$

where s, t are coprime positive integers with $s > t$ and one of them even. This gives

$$x = \frac{s(s+t)}{s^2+t^2} \quad \text{or} \quad x = \frac{t(t-s)}{s^2+t^2}$$

and

$$y = \frac{s(s-t)}{s^2+t^2} \quad \text{or} \quad y = \frac{t(s+t)}{s^2+t^2}.$$

By replacing s by t and t by $-s$, we can simplify these to

$$x, y = \frac{s(s+t)}{s^2+t^2}, \frac{t(s+t)}{s^2+t^2}$$

(in either order), where s, t are integers not both zero. For example, $s = 2$ and $t = 3$ gives the solution $x, y = \frac{10}{13}, \frac{15}{13}$.

We can generalize this to the equation

$$x_1^2 + \cdots + x_n^2 = x_1 + \cdots + x_n.$$

To obtain a family of rational solutions, let $s_1, \ldots, s_n$ be any integers, not all zero. Then

$$x_i = \frac{s_i(s_1 + \cdots + s_n)}{s_1^2 + \cdots + s_n^2}, \quad \text{for } i = 1, \ldots, n,$$

is a family of rational solutions.

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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st November will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

44.9 In figure 1, $BE = DF$. Obtain a relationship involving the sides of the quadrilaterals $ABCD$ and $AECF$, and deduce that

$$AE^2 + AF^2 = CE^2 + CF^2$$

when angles BAD and BCD are right angles.

(Submitted by Indika Shameera Amarasinghe, University of Kelaniya, Sri Lanka)

44.10 The tetrahedron $ABCD$ is such that angles ADB, BDC, CDA are right angles. Prove that

$$(\text{area } \triangle ABC)^2 = (\text{area } \triangle ABD)^2 + (\text{area } \triangle BCD)^2 + (\text{area } \triangle CAD)^2.$$

(Submitted by Zhang Yun, Sunshine High School of Xi An Jiaotong University, Shan Xi Province, China)

44.11 A number of the form $2^n - 1$, where n is a positive integer, is called a *Mersenne number* after Père Marin Mersenne (1588–1648). Find all positive integer cubes which can be written as the sum of two Mersenne numbers.

(Submitted by Tom Moore, Bridgewater State University, USA)

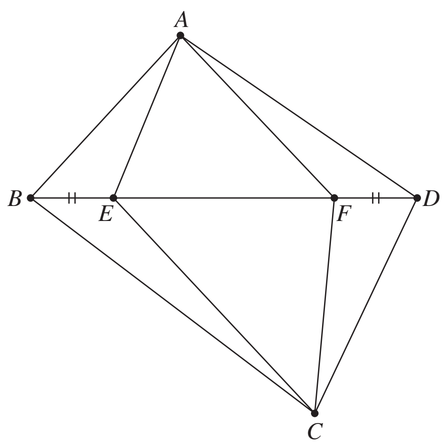


Figure 1

44.12 Tatyana loves track and trig. She runs one lap around a circular track with centre O and radius r. Let T be her position on the track at any time and S her starting point. Let $\theta = \angle SOT$, $0 \le \theta \le 2\pi$, and d the distance she has run along the track from S to T. For fixed r, how many values of θ are there such that $d = \sec^2\theta$?

(Submitted by William Gosnell, Amherst, USA and Herb Bailey, Rose-Hulman Institute of Technology, USA)

Solutions to Problems in Volume 44 Number 1

44.1 Find all positive integers n such that

$$\log_{2009} n = \log_{2010} n + \log_{2011} n.$$

Solution by Henry Ricardo, Tappan, New York

Clearly $n = 1$ is a solution. For any integers $k > 1$, $n > 1$,

$$\begin{aligned}
&\log_{k+2} n + \log_{k+1} n - \log_k n \\
&\quad = \frac{1}{\log_n(k+2)} + \frac{1}{\log_n(k+1)} - \frac{1}{\log_n k} \\
&\quad = \frac{\log_n k \log_n(k+1) + \log_n k \log_n(k+2) - \log_n(k+2)\log_n(k+1)}{\log_n(k+2)\log_n(k+1)\log_n k} \\
&\quad = \frac{\log_n(k+1)[\log_n k^2 - \log_n(k+2)] + \log_n(k+2)[\log_n k^2 - \log_n(k+1)]}{2\log_n(k+2)\log_n(k+1)\log_n k} \\
&\quad > 0
\end{aligned}$$

since

$$k^2 > k+2 > k+1.$$

Taking $k = 2009$, we see that

$$\log_{2011} n + \log_{2010} n > \log_{2009} n,$$

so $n = 1$ is the only solution.

44.2 What is

$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7}?$$

Solution by Abbas Rouholamini Gugheri, Sirjan, Iran

We have

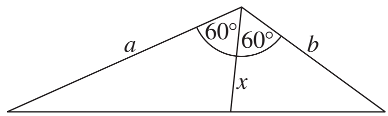
$$\begin{aligned}
&2\sin\frac{\pi}{7}\left(\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7}\right) \\
&\quad = 2\sin\frac{\pi}{7}\left(\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7}\right) \\
&\quad = \sin\frac{2\pi}{7} + \left(\sin\frac{4\pi}{7} - \sin\frac{2\pi}{7}\right) + \left(\sin\frac{6\pi}{7} - \sin\frac{4\pi}{7}\right) \\
&\quad = \sin\frac{6\pi}{7} \\
&\quad = \sin\frac{\pi}{7},
\end{aligned}$$

so that

$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}.$$

Also solved by Henry Ricardo.

44.3 Find a relation connecting x, a, and b.



Solution by Abbas Rouholamini Gugheri

Considering areas of triangles, we have

$$\tfrac{1}{2}ab\sin 120^\circ = \tfrac{1}{2}ax\sin 60^\circ + \tfrac{1}{2}bx\sin 60^\circ$$

which gives

$$\frac{1}{x} = \frac{1}{a} + \frac{1}{b}.$$

44.4 Let A, B, C be the angles of a triangle with $A \le B \le C$. Prove that

$$\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} \le \sqrt{3}$$

if and only if $B \le \pi/3$. (See the article ‘A Trip from Trig to Triangle’ in Volume 44, Number 1, pp. 19–23.)

Solution by Michel Bataille, Rouen, France, who proposed the problem

Since

$$\begin{aligned}\cos A + \cos B + \cos C &= \cos A + \cos B - \cos(A+B)\\ &= \cos A(1-\cos B) + \cos B + \sin A\sin B\end{aligned}$$

and

$$A \le B < \frac{\pi}{2},$$

the denominator is positive, so the inequality is equivalent to

$$\left(\frac{\sqrt{3}}{2}\cos A - \frac{1}{2}\sin A\right) + \left(\frac{\sqrt{3}}{2}\cos B - \frac{1}{2}\sin B\right) + \left(\frac{\sqrt{3}}{2}\cos C - \frac{1}{2}\sin C\right) \ge 0,$$

i.e.

$$\sin\left(\frac{\pi}{3} - A\right) + \sin\left(\frac{\pi}{3} - B\right) + \sin\left(\frac{\pi}{3} - C\right) \ge 0. \tag{1}$$

The following identity was obtained at the beginning of the article:

$$\sin(x+y-z)+\sin(y+z-x)+\sin(z+x-y)$$
$$=\sin(x+y+z)+4\sin x\sin y\sin z.$$

When $x+y+z=0$, this gives

$$\sin 2x+\sin 2y+\sin 2z=-4\sin x\sin y\sin z.$$

Since

$$\left(\frac{\pi}{6}-\frac{A}{2}\right)+\left(\frac{\pi}{6}-\frac{B}{2}\right)+\left(\frac{\pi}{6}-\frac{C}{2}\right)=0,$$

this gives

$$\sin\left(\frac{\pi}{3}-A\right)+\sin\left(\frac{\pi}{3}-B\right)+\sin\left(\frac{\pi}{3}-C\right)$$
$$=-4\sin\left(\frac{\pi}{6}-\frac{A}{2}\right)\sin\left(\frac{\pi}{6}-\frac{B}{2}\right)\sin\left(\frac{\pi}{6}-\frac{C}{2}\right).$$

Now

$$0<A\le\frac{\pi}{3}\quad\text{and}\quad\frac{\pi}{3}\le C<\pi,$$

so that

$$\sin\left(\frac{\pi}{6}-\frac{A}{2}\right)\ge 0$$

and

$$\sin\left(\frac{\pi}{6}-\frac{C}{2}\right)\le 0.$$

Hence (1) holds if and only if

$$\sin\left(\frac{\pi}{6}-\frac{B}{2}\right)\ge 0,$$

i.e. if and only if $B\le\pi/3$.

$$1+5+8+12=2+3+10+11,$$
$$1^2+5^2+8^2+12^2=2^2+3^2+10^2+11^2,$$
$$1^3+5^3+8^3+12^3=2^3+3^3+10^3+11^3.$$

$$13!=112296^2-79896^2,$$
$$240^4+340^4+430^4+599^4=651^4.$$

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