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CONTENTS

A LATTICE POINT ASSIGNMENT THEOREM

HARRY D. RUDERMAN

Let L be a rectangular array of lattice points having at least two rows and two columns. Two points of L will be called *adjacent* if they agree in one of their rectangular coordinates and differ by 1 in the other. Thus the points (a, b) and $(a, b\pm1)$ are adjacent if they are in L, and the same can be said of the points (a, b) and $(a\pm1, b)$. A *small square* is one that has each of its vertices adjacent to two other vertices. It will always be oriented in the counterclockwise sense, as is the square ABCD in Figure 1, where B is said to *follow* A, C to follow B, etc. Thus, if two small squares have a common side, such as AB in Figure 1, then B follows A in one square, but A follows B in the other.

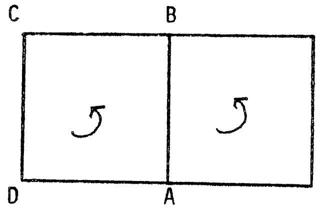

Figure 1

Let $E = \{1, 2, 3, 4\}$. A mapping $f\colon L \to E$ assigns to each point of L one of the numbers in E, and we will say that $f(\mathrm{B})$ *follows* $f(\mathrm{A})$ for a given square if B follows A in that square. The mapping f is arbitrary except for the following two restrictions:

(i) If A and B are on the boundary of the array, B following A, and if $f(\mathrm{A}) = 1$, then $f(\mathrm{B}) \neq 4$. We say that 4 may not follow 1 on the boundary.

(ii) If A and B are adjacent, then $|f(\mathrm{A}) - f(\mathrm{B})| \neq 2$. This means that 4 may follow 1 (except on the boundary, as noted in (i)) or 1 may follow 4, but otherwise adjacent points are assigned numbers differing by at most 1.

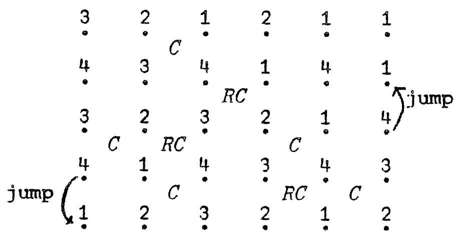

Figure 2

Such a mapping f is illustrated in Figure 2. A *jump* is said to occur if 1 follows 4 *on the boundary*. A small square is said to be *complete (C)* if its vertices' assignments are 1, 2, 3, 4 in counterclockwise order. If the orientation is reversed, we call the square *reversed complete (RC)*. Figure 2 shows 2 jumps, 3 reversed complete squares, and 5 complete squares. It is no accident that here 2 + 3 = 5, for we have the

THEOREM. If the mapping $f: L \to E$ adheres to the restrictions described above, then the number of jumps plus the number of reversed complete squares equals the number of complete squares.

Proof. We assign a number from the set $S = \{-2, -1, 0, 1, 2\}$ of absolutely least residues modulo 5 to each directed edge of each (counterclockwise-oriented) small square as follows: if A and B are adjacent, then to the directed segment $\overline{AB}$ is assigned the number $r \in S$ if $f(B) - f(A) \equiv r \pmod 5$. It follows that the assignments for $\overline{AB}$ and for $\overline{BA}$ have a sum of zero. We now find, in two ways, the sum of all the assignments given in this manner to all the directed segments.

Clearly, all segments having at most one endpoint on the boundary will yield a sum of zero because each such segment will have associated with it two oppositely directed segments with opposite assignments. The segments on the boundary will have but one associated directed segment, oriented counterclockwise. The sum of the boundary segment assignments, which equals the sum of all directed segment assignments, will be 5 times the number of jumps (since the sum must increase by $1+1+1+2 = 5$ from one jump to the next).

Next, we sum the directed segment assignments by adding the sums found separately for each square. If a square has fewer than four distinct numbers assigned to its vertices, the assignments for its directed segments will total zero. See Figure 3 for some examples. That the statement is true in all cases can easily be checked.

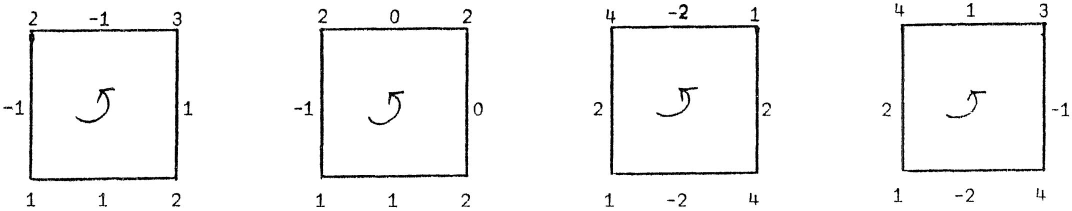

Figure 3

Otherwise the square must be either complete or reversed complete. The sum for each complete square is 5, and it is -5 for each reversed complete square, as can be seen from Figure 4.

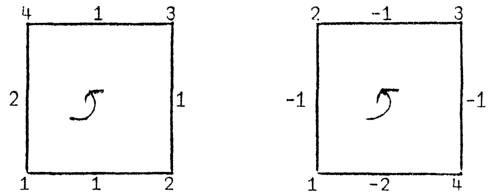

Figure 4

As the sum of the directed segment assignments must be the same for the two ways of summing, we conclude that 5 times the number of jumps must equal 5 times the number of complete squares minus 5 times the number of reversed complete squares. Hence the number of jumps plus the number of reversed complete squares equals the number of complete squares. □

It is an obvious corollary that if the boundary has no jumps, then the number of complete squares equals the number of reversed complete squares.

I feel very sure that this theorem can be generalized to graphs more general than those involving squares. Probably other assignments may be used in the generalization.

In closing, I want to give credit to O.P. Lossers, of Eindhoven University of Technology in The Netherlands. His solution to a problem of mine [1] contains the main ideas of this theorem and its proof.

REFERENCE

1. O.P. Lossers, Solution I to Problem 6192 (proposed by Harry D. Ruderman), *American Mathematical Monthly*, 86 (1979) 598.

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* * *

AN ANGLE TRISECTION METHOD WHICH (USUALLY) DOES NOT WORK

ED BARBEAU and JOHN IM

Problem 298 in Book 3 of *1001 Problems in High School Mathematics* (collected and edited by E. Barbeau, M. Klamkin and W. Moser, and published in 1978 by the Canadian Mathematical Society), asks the reader to determine, for the following proposed construction for a straightedge-and-compasses trisection of the arbitrary acute angle POQ, those angles for which it works, and to show that it fails for other angles:

From any point B *on* OQ, *drop a perpendicular to meet* OP *at* A. *Construct an equilateral triangle* ABC *with* C *and* O *on opposite sides of line* AB. *Then* $\angle POC = \frac{1}{3}\angle POQ$. (See Figure 1.)

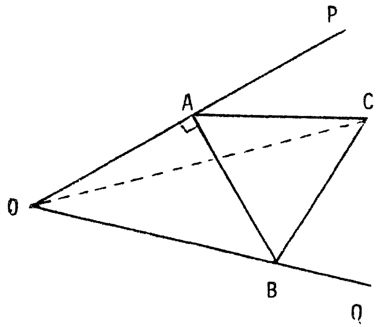

Figure 1

Two relatively complicated trigonometric solutions are given in Book 4 of the collection (published in 1980), one of them requiring the analysis of a cubic equation. At a meeting of the Metro Toronto Mathematics Club, E. Barbeau suggested that it would be more satisfactory to have an argument using only

Euclidean methods and offered a two-dollar prize for the first student to find one. There were two takers: John Im, then at the Toronto French School, and David Atwood, then a Level 6 student at Northern Secondary School. Although duplicate prizes were awarded, we give here only Im's solution, since Atwood's proved rather less than the problem required, merely that the construction fails when $\angle POQ = 60°$.

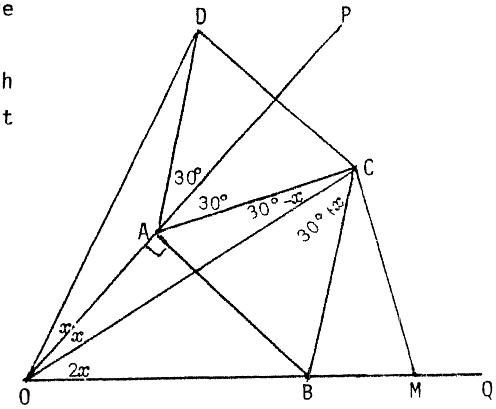

Figure 2

Solution by John Im.

We show that the construction works if and only if $\angle POQ = 45°$. It is quickly apparent from Figure 1 that $\angle POC = 15°$ if $\angle POQ = 45°$, so the condition is sufficient.

To prove necessity, suppose that, in Figure 2, $\angle POC = x$ (degrees) and $\angle COQ = 2x$. If D is the point symmetric to C with respect to line OP, and OM = OC, it is easy to see that triangles ACD and ABC are equilateral and congruent, and that triangles OCD and OMC are isosceles and congruent. It now follows that CB = CM. Now

$$\angle CBM = 2x + (30°+x) = 3x + 30°$$

and

$$\angle CMB = \angle OCM = \tfrac{1}{2}(180°-2x) = 90°-x.$$

Since $\angle CBM = \angle CMB$, we have $3x+30° = 90°-x$, so $x = 15°$ and $\angle POQ = 45°$, as required.

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* * *

NOTES ON NOTATION: II

LEROY F. MEYERS

The rules for omitting parentheses are incompletely specified. How do *you* interpret

$$\sin 2x, \quad \sin \pi/3, \quad \sin a(x-y), \quad \sin x/x\ ?$$

The usual interpretation (as in this journal) seems to be to associate to the right —up to the next "outside" +, -, or / sign or the end of the expression—except possibly for the second example. Compare, for example: $\sin \frac{\alpha}{2}$ and $\sin (\alpha/2)$ [1979: 261],

$\cos\frac{1}{2}A/F$ [1981: 51]. However, I have seen the third expression used to mean $(\sin a)(x-y)$. How is the customary use of association to the right reconciled with the usual (even in *CM*) interpretation of $\sin x \sin y$ as $(\sin x)(\sin y)$ and not as $\sin(x \sin y)$? What if an explicit multiplication symbol is used, as in $\sin a\cdot(x-y)$?

Recently, on a student's homework paper, I noticed the following confusion, which I have only lightly edited:

$$F(x,y) = \cos \pi y\ \mathrm{i} - \pi x \sin \pi y\ \mathrm{j} = P(x,y)\mathrm{i} + Q(x,y)\mathrm{j}.$$

$$\text{Since } \frac{\partial P(x,y)}{\partial y} = \cos \pi = -1\ \&\ \frac{\partial Q(x,y)}{\partial x} = -\pi \sin \pi y\ ,$$

$$F(x,y) \text{ is not a gradient unless } \sin \pi y = \frac{1}{\pi},\ \text{or } y \approx \frac{1}{10}\ .$$

What is needed is an explicit convention about the omission of parentheses in the neighborhood of function symbols written with several Roman or Gothic symbols (such as sin, log, sgn, max), where the argument of the function is not usually enclosed in parentheses. Can you make up a rule which agrees with common practice, insofar as there is agreement?

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* * *

AREAS OF TRIANGLES INSCRIBED IN A TRIANGLE

M.S. KLAMKIN

This note is a postscript to two notes published earlier in this journal by Dan Pedoe [1979: 191] and Howard Eves [1979: 280]. The well-known problem considered by Pedoe and Eves was the following:

Two triangles whose vertices lie on the sides of a given triangle at equal distances from their midpoints are equal in area.

Thus, in each of the Figures 1 to 4, $[P_1P_2P_3] = [Q_1Q_2Q_3]$, where the brackets denote area. Observe that "mixed orientation" of the P_i and Q_i on each side is allowed, in general giving (except for labeling) four distinct pairs of equal triangles. Eves appropriately calls such pairs *isotomic triangles* because pairs of corresponding vertices are isotomic points on the sides of the given triangle (i.e., each pair is equidistant from the midpoint of the side).

The problem is given and solved in Johnson [1], a reference given by Pedoe. The proofs of Pedoe and Eves are simple and interesting, but Johnson's proof is also quite simple and it has the advantage of leading most naturally to other interesting results. Here, essentially, is Johnson's solution:

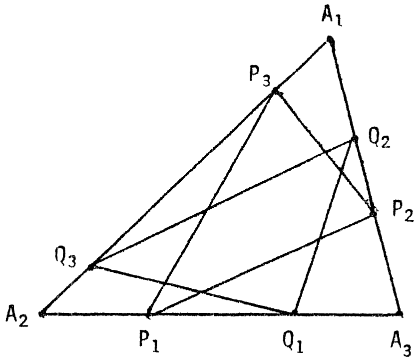

Figure 1

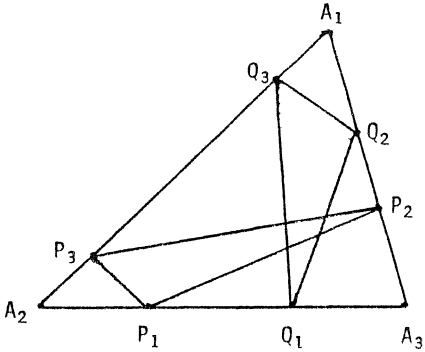

Figure 2

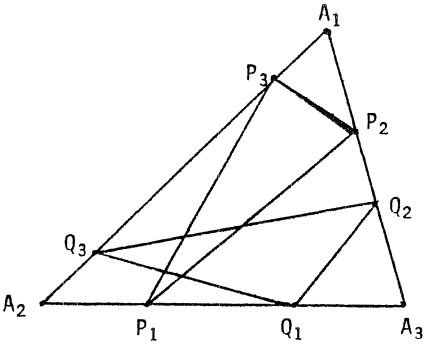

Figure 3

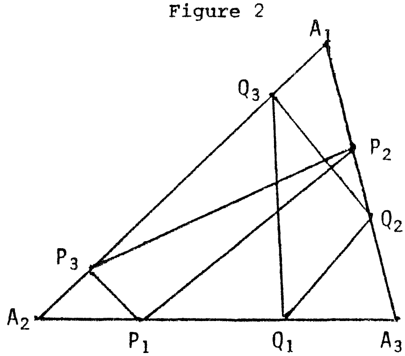

Figure 4

We have given a triangle $A_1A_2A_3$, with sides a_1, a_2, a_3, and two points P_i and Q_i (not necessarily distinct) on each side a_i, and we wish to determine conditions under which

$$[P_1P_2P_3] = [Q_1Q_2Q_3] \ . \tag{1}$$

If we use signed areas for triangles, the points P_i and Q_i need not be restricted to the segment a_i but can be anywhere on its line of support. Regardless of the positions of the P_i and Q_i, (1) is equivalent to

$$[A_1P_3P_2] + [A_2P_1P_3] + [A_3P_2P_1] = [A_1Q_3Q_2] + [A_2Q_1Q_3] + [A_3Q_2Q_1] \ . \tag{2}$$

We assign numerical values to directed segments as follows:

$$\overline{A_2P_1} = x_1, \qquad \overline{A_3P_2} = x_2, \qquad \overline{A_1P_3} = x_3,$$

$$\overline{Q_1A_3} = y_1, \qquad \overline{Q_2A_1} = y_2, \qquad \overline{Q_3A_2} = y_3.$$

Since the areas of two triangles having a common angle are proportional to the products of the including sides, we have

$$[A_1P_3P_2] = \frac{x_3(a_2-x_2)}{a_3a_2} \cdot [A_1A_2A_3], \text{ etc.,}$$

and (2) is equivalent to

$$\sum_{\text{cyclic}} \left\{\frac{x_1}{a_1} - \frac{x_2x_3}{a_2a_3}\right\} = \sum_{\text{cyclic}} \left\{\frac{y_1}{a_1} - \frac{y_2y_3}{a_2a_3}\right\}. \qquad (3)$$

When the points P_i and Q_i are isotomic on a_i for $i = 1, 2, 3$, then we have

$$\frac{x_1}{a_1} = \frac{y_1}{a_1}, \qquad \frac{x_2}{a_2} = \frac{y_2}{a_2}, \qquad \frac{x_3}{a_3} = \frac{y_3}{a_3},$$

and (3) follows immediately. □

But (3) also follows if

$$\frac{x_1}{a_1} = \frac{y_3}{a_3}, \qquad \frac{x_2}{a_2} = \frac{y_1}{a_1}, \qquad \frac{x_3}{a_3} = \frac{y_2}{a_2}.$$

In this case, $Q_2P_3 \parallel A_2A_3$, $Q_3P_1 \parallel A_3A_1$, and $Q_1P_2 \parallel A_1A_2$. This result appeared subsequently as a problem in the Hungarian Kürschák prize competition for high school students and graduates in the following form [2]: *In the convex hexagon* ABCDEF, *suppose that every pair of opposite sides are parallel. Prove that the triangles* ACE *and* BDF *have equal areas.* A solution by orthogonal projection was given by this author in a joint paper on problem solving [3]. But the solution given here is simpler and shows the general pattern of which the problem is a special case.

The remaining "cyclic" equality case,

$$\frac{x_1}{a_1} = \frac{y_2}{a_2}, \qquad \frac{x_2}{a_2} = \frac{y_3}{a_3}, \qquad \frac{x_3}{a_3} = \frac{y_1}{a_1},$$

also gives three pairs of parallel lines.

Among the remaining cases for which (3) holds, there are at least the three "noncyclic" equality cases, one of which is

$$\frac{x_1}{a_1} = \frac{y_1}{a_1}, \qquad \frac{x_2}{a_2} = \frac{y_3}{a_3}, \qquad \frac{x_3}{a_3} = \frac{y_2}{a_2}.$$

Finally, we mention a more special result, also given in Johnson, which makes the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ congruent:

If the three lines ΩA_1, ΩA_2, ΩA_3 *rotate as a rigid system about* Ω, *and intersect the sides* A_1A_2, A_2A_3, A_3A_1 *respectively at* P_1, P_2, P_3; *while* $\Omega'A_1$, $\Omega'A_2$, $\Omega'A_3$ *rotate through the same angle* θ *about* Ω' *in the opposite direction, meeting* A_1A_3, A_3A_2, A_2A_1 *respectively at* Q_1, Q_2, Q_3; *then triangles* $P_1P_2P_3$ *and* $Q_1Q_2Q_3$ *are congruent, and similar to the given triangle* ... .

Here Ω and Ω' are the two Brocard points of triangle $A_1A_2A_3$, and Johnson shows that the ratio of similitude is $\sin\omega/\sin(\omega+\theta)$, where ω is the Brocard angle.

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2. J. Aczél, "A Look at Mathematical Competitions in Hungary," *American Mathematical Monthly,* 67 (1960) 435-437.

3. M.S. Klamkin and D.J. Newman, "The Philosophy and Applications of Transform Theory," *SIAM Review,* 3 (1961) 10-36.

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THE OLYMPIAD CORNER: 24

M.S. KLAMKIN

Problem 3 of the Twelfth Canadian Mathematics Olympiad (1980) reads as follows:

Among all triangles ABC *having* (i) *a fixed angle* A *and* (ii) *an inscribed circle of fixed radius r, determine which triangle has the least perimeter.*

A solution by elementary trigonometry was published earlier in this column [1980: 242]. It showed that both the perimeter and side BC are minimized when $\angle B = \angle C$. Subsequently, in the Canadian Mathematical Society *Notes* 13 (1981) 16-17, J. Aczél, W. Gilbert, and C.T. Ng gave two rather nice elementary geometric solutions. Solution 1 below essentially reproduces one of their arguments. I shall later give an alternate geometric solution.

Solution 1. Figure 1 shows an isosceles triangle ABC and its incircle tangent to BC at its midpoint P, and a general triangle AB'C' with the same incircle. Note that AB'C' will have a minimum perimeter if and only if B'C' is a minimum. (This follows immediately from the equality of tangents drawn to a circle from an exterior point.) If B"C" is drawn through P parallel to B'C', then, from similar triangles,

$$B'C'/B''C'' = AB'/AB'' \geq 1,$$

so $B'C' \geq B''C''$ and it suffices to show that $B''C'' \geq BC$. Equivalently, if PD is the reflection of PC" in the bisector AP of $\angle A$, we have to show that $B''P + PD \geq 2PB$. Now PB is the bisector of the angle at P in triangle DPB" so, if PM is the corresponding median, the well-known inequality involving adjacent sides and the included median of a triangle yields $B''P + PD \geq 2PM \geq 2PB$, as required. □

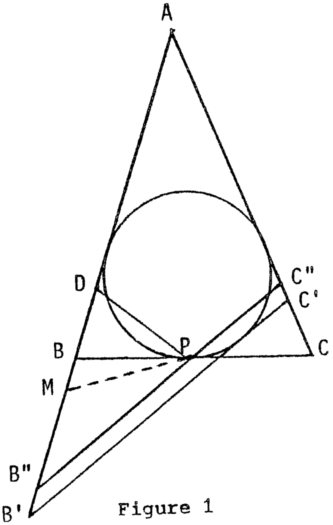

Figure 1

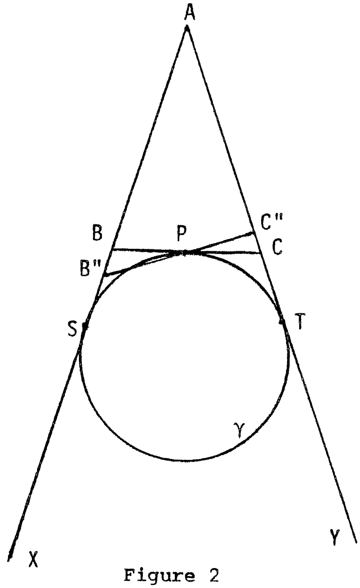

Figure 2

The second geometric solution I shall give is a practically immediate consequence of the following theorem which has long been well-known and which, in addition, has already been proved twice in this journal (Problem 120 [1976: 103-106] where references to the theorem in the literature are given, and Problem 2 in an article by Howard Eves [1980: 232-237]):

Given a point P *inside an angle* XAY, *let* γ *be the unique circle through* P *that is tangent to the sides of the angle and such that* A *and* P *lie on the same side of the chord of contact* (= ST in Figure 2). *Then the segment* BC *through* P *terminating in the sides of the angle such that triangle* ABC *has minimum perimeter is the one tangent to* γ *at* P.

For an elegant method of *constructing* circle γ for an arbitrary point P inside angle XAY, see Sokolowsky's comment in this journal [1976: 139].

Solution 2. Figure 2 shows an isosceles triangle ABC and the midpoint P of its base (determined by the given incircle of the triangle). Since P lies on the bisector of $\angle A$, the circle γ described in the above theorem is an excircle of the triangle. If B"C" is any segment through P, then it suffices to show (as in solution 1 above) that

$$\text{perimeter of } AB''C'' \geq \text{perimeter of } ABC.$$

But this follows immediately from the theorem cited above.

*

We now give solutions received from readers for some problems published earlier in this column. Solutions to the remaining unsolved problems, or improved solutions to those already solved, will be gratefully received.

O-3. [1980: 146] Which is larger, $\sin(\cos x)$ or $\cos(\sin x)$?

Solution by Gali Salvatore, Perkins, Québec.

The result is that $\sin(\cos x) < \cos(\sin x)$ for all real x. For

$$\begin{aligned}\cos(\sin x) - \sin(\cos x) &= \cos(\sin x) + \cos((\pi/2)+\cos x)\\ &= 2\cos\frac{(\pi/2)+\cos x+\sin x}{2}\cos\frac{(\pi/2)+\cos x-\sin x}{2}\\ &> 0,\end{aligned}$$

since

$$|\cos x \pm \sin x| = \sqrt{1 \pm \sin 2x} \le \sqrt{2} < \pi/2. \square$$

This proof is based on that given in [1]. The problem was later proposed in [2], but the solution given in a later issue [3] is not valid for all real x. But the problem is much older, having already appeared (without solution) in Hobson [4]. The problem is also given and solved in [5].

It is natural to wonder whether, for example,

$$\sin(\cos(\sin x)) < \cos(\sin(\cos x))$$

holds for all real x. The answer is no, for this inequality is sometimes true (e.g., for $x = 1$) and sometimes false (e.g., for $x = 0$).

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2. Sidney H.L. Kung (proposer), Problem 835, *Mathematics Magazine*, 45 (1972) 167.

3. Leon Bankoff, Solution of Problem 835, *Mathematics Magazine*, 46 (1973) 109.

4. E.W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Dover, New York, 1957, p. 136, Problem 7 (originally published in 1891).

5. E. Barbeau, M. Klamkin, W. Moser, *1001 Problems in High School Mathematics*, *Book* 3, Canadian Mathematical Society, 1978, pp. 12, 74.

*

J-2. [1980: 145, 312] Solve in rational numbers x, y, z, and t:

$$(x+y\sqrt{2})^2 + (z+t\sqrt{2})^2 = 5 + 4\sqrt{2}.$$

Solution by B.C. Rennie, James Cook University of North Queensland, Australia.

A rational solution (x,y,z,t) of the given equation would also satisfy

$$(x-y\sqrt{2})^2 + (z-t\sqrt{2})^2 = 5 - 4\sqrt{2}.$$

This gives a contradiction since $5 - 4\sqrt{2} < 0$, and hence the given equation has no rational solution.

*

J-8, [1980: 146] Prove that $x\cos x < 0.71$ for all $x \in [0,\pi/2]$.

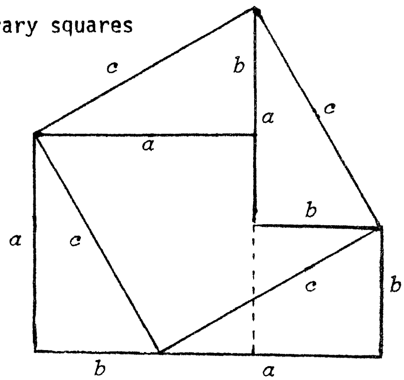
Solution by K.S. Murray, Brooklyn, N.Y.

Let $x = \pi/2 - y$; then $0 \le y \le \pi/2$ and (with $(\sin y)/y = 1$ for $y = 0$, as usual) we have

$$\begin{aligned} x\cos x &= y(\pi/2 - y)(\sin y)/y \\ &\le \{\max y(\pi/2 - y)\}\{\max (\sin y)/y\} \\ &= \pi^2/16 \approx 0.617 < 0.71. \end{aligned}$$

*

J-9, [1980: 146] Is it possible to cut two arbitrary squares into polygons which would form a new square?



Comment.

That the answer is yes follows from any dissection proof of the Pythagorean Theorem. In particular, the dissection proof indicated in the adjoining figure was given by H. Perigal in 1873 [1]. Here the sum of the two squares on the legs a and b of a right triangle is cut into three pieces which, by rearrangement, can be formed into the square on the hypotenuse c.

More generally, we have the Bolyai-Gerwin Theorem [2]: *Two polygons which have equal areas are equidecomposable.* Note that two figures are said to be *equidecomposable* if it is possible to decompose one of them into a finite number of parts which can be rearranged to form the second figure. It is an unsolved problem in general to determine the minimum number of cuts required. See also [3].

REFERENCES

1. Howard Eves, *A Survey of Geometry*, Revised Edition, Allyn and Bacon, Boston, 1972, p. 197.

2. V.G. Boltyanski, *Equivalent and Equidecomposable Figures*, D.C. Heath, Boston, 1963, p. 7.

3. H. Lindgren, *Recreational Problems in Geometrical Dissections & How to Solve Them*, Dover, New York, 1972.

*

J-10. [1980: 146] Solve the system of equations:

$$y(x+y)^2 = 9, \qquad (1)$$

$$y(x^3-y^3) = 7. \qquad (2)$$

Solution by Gali Salvatore, Perkins, Québec.

It is easy to verify that $(x,y) = (2,1)$ is a solution. We show that it is the only real solution using the fact, obvious from (1) and (2), that $x > y > 0$ for any solution (x,y).

Suppose $0 < y \le 1$; then, from (1),

$$(x+1)^2 \ge (x+y)^2 \ge 9,$$

from which we get $x \ge 2$. Also,

$$\frac{9}{7} = \frac{(x+y)^2}{x^3-y^3} \le \frac{(x+1)^2}{x^3-1},$$

an inequality equivalent to $(x-2)(9x^2+11x+8) \le 0$ which yields $x \le 2$. Thus $x = 2$, and then $y = 1$ from (1)

If we assume $y \ge 1$, all the inequalities in the steps above are reversed; so we obtain again $x = 2$, and then $y = 1$.

Comment by M.S.K.

That there are other complex solutions follows by letting $x = ty$, giving

$$y^3(1+t)^2 = 9, \qquad y^4(t^3-1) = 7,$$

and then eliminating y gives

$$7^3(1+t)^8 = 9^4(t^3-1)^3.$$

*

J-11. [1980: 316] Which is larger, $\sqrt[3]{413}$ or $6 + \sqrt[3]{3}$?

Solution by Donald Cross, University of Exeter, England.

We use the equivalence

$$a^3-b^3-c^3 > 3abc \iff a > b+c \qquad (1)$$

mentioned and proved in Klamkin's comment on Problem J-6 [1980: 315]. If we set $a = \sqrt[3]{413}$, $b = 6$, and $c = \sqrt[3]{3}$, the left side of (1) becomes $194 > 18\sqrt[3]{1239}$, which is equivalent to $912673 > 903231$. Hence, from the right side of (1), $\sqrt[3]{413} > 6 + \sqrt[3]{3}$.

Comment by M.S.K.

One can also use the power-mean inequality:

$$\left\{\frac{(\frac{6}{n})^3+(\frac{6}{n})^3+\ldots+(\frac{6}{n})^3+3}{n+1}\right\}^{1/3} > \frac{\frac{6}{n}+\frac{6}{n}+\ldots+\frac{6}{n}+\sqrt[3]{3}}{n+1},$$

which is equivalent to

$$\{(n+1)^2(3+6^3/n^2)\}^{1/3} > 6+\sqrt[3]{3}.$$

Setting $n=4$ yields $\sqrt[3]{412.5} > 6+\sqrt[3]{3}$, and the desired result follows.

*

J-12. [1980: 316] Solve the equation $x^y = y^x$ in positive integers.

Comment.

If $x < y$, the only solution is $(x,y) = (2,4)$. Note that this problem is very old and has already appeared in this journal. See Problem 188 [1977: 73-74] for two elegant solutions and many references. The most recent discussion of the equation $x^y = y^x$ is in Knoebel [1].

REFERENCE

1. R. Arthur Knoebel, "Exponentials Reiterated", *American Mathematical Monthly*, 88 (April 1981) 235-252.

*

J-13. [1980: 316] Prove that every convex polygon of area 1 contains a triangle of area $\frac{1}{4}$.

Solution by K.S. Murray, Brooklyn, N.Y.

More generally, we show that every closed convex figure of area 1 contains a triangle of area $\frac{1}{4}$. Let F be such a figure. The areas of all the triangles inscribed in F form a compact set of positive real numbers. Consequently, there exists an inscribed triangle t of maximum area $|t|$. At each vertex of t, there must be a line of support of F which is parallel to the opposite side of t (otherwise we can obtain a larger triangle by moving the vertex). The three lines of support form a triangle T whose area is $|T| = 4|t|$. Since clearly $|T| \geq 1$, it follows that $|t| \geq \frac{1}{4}$ and, by continuity, F contains a triangle of area $\frac{1}{4}$. □

For related results, see the comment and the references at the end of the next problem (J-14).

*

J-14. [1980: 316] Show that the following statement is false: *A convex polyhedron of volume* 1 *contains a tetrahedron of volume* 1/8.

A natural estimate is not $(1/2)^3 = 1/8$ but $(1/3)^3 = 1/27$. Obtain this estimate and try to prove it.

Solution by K.S. Murray, Brooklyn, N.Y.

We prove, more generally, that *every closed* 3*-dimensional convex region of volume* 1 *contains a tetrahedron of volume* 1/27, and that the statement is false if 1/27 is replaced by 1/8.

That 1/8 is too large follows by considering the ratio of the volume V_t of a regular tetrahedron to the volume V_s of its circumscribed sphere. Here we have

$$V_t/V_s = 2\sqrt{3}/9\pi < 1/8.$$

Our proof of the first part parallels that of J-13 above. Let R be a closed 3-dimensional convex region of volume 1. R contains an inscribed tetrahedron τ of maximum volume $|\tau|$. At each vertex of τ, there is a plane of support of R which is parallel to the opposite face of τ. The four planes of support form a tetrahedron T whose volume is $|T| = 27|\tau|$ (we will prove this later). Since clearly $|T| \geq 1$, it follows that $|\tau| \geq 1/27$ and, by continuity, R contains a tetrahedron of volume 1/27.

To prove that $|T| = 27|\tau|$ let T = A-BCD and let $3\vec{a}$, $3\vec{b}$, $3\vec{c}$, $3\vec{d}$ be the position vectors of A, B, C, D, respectively, from some common origin. It is easy to see that the vertices of τ are the centroids of the faces of T, with position vectors

$$\vec{b}+\vec{c}+\vec{d}, \qquad \vec{c}+\vec{d}+\vec{a}, \qquad \vec{d}+\vec{a}+\vec{b}, \qquad \vec{a}+\vec{b}+\vec{c}.$$

With the usual notation $[\vec{u}, \vec{v}, \vec{w}]$ for the scalar triple product $\vec{u}\cdot\vec{v}\times\vec{w}$, we have

$$6|T| = |[3(\vec{b}-\vec{a}),\ 3(\vec{c}-\vec{a}),\ 3(\vec{d}-\vec{a})]|$$

and

$$6|\tau| = |[\vec{a}-\vec{b},\ \vec{a}-\vec{c},\ \vec{a}-\vec{d}]|,$$

so that $|T|/|\tau| = 27$. □

With respect to this and the preceding problem (J-13), a stronger result is that the circle minimizes the maximum-area inscribed triangle over all convex sets of the same area, with similar results in n dimensions. See references [1]-[4], which contain all the information available at this time.

REFERENCES

1. W. Blaschke, *Leipzig Gesellschafte Berichte,* 69 (1917).

2. Gross, *ibid.,* 70 (1918).

3. A.M. Macbeath, "An Extremal Property of the Hypersphere", *Proceedings of the Cambridge Philosophical Society,* 47 (1951) 245-247.

4. H. Groemer, "The Average Size of Polytopes in a Convex Set," (preprint).

J-15. [1980: 316] Does there exist an infinite family of pairwise noncongruent right triangles such that the lengths of the sides are natural numbers and the lengths of the two short sides differ by 1?

Comment.

Bob Prielipp, University of Wisconsin-Oshkosh, refers to the solution in Sierpiński [1]. See also Crux 460 [1980: 159].

REFERENCE

1. W. Sierpiński, *Elementary Theory of Numbers*, Hafner, New York, 1964, pp. 44-46.

*

J-17. [1980: 316] Prove the inequality $(\sin x)^{-2} \geq x^{-2} + 1 - 4/\pi^2$ for $0 < x < \pi/2$.

Comment.

The inequality is incorrect as stated. The inequality sign should be reversed. A proof follows by considering the function

$$F(x) = (\sin x)^{-2} - x^{-2}.$$

We have

$$\lim_{x\to 0} F(x) = 1/3, \qquad F(\pi/2) = 1 - 4/\pi^2,$$

and it follows from

$$\left(\frac{\sin x}{x}\right)^3 \geq \cos x$$

(proved in Problem J-1 [1980: 312]) that

$$F'(x) = \frac{-2\cos x}{\sin^3 x} + \frac{2}{x^3} \geq 0,$$

so that F is nondecreasing. Thus $F(x) < F(\pi/2)$ for $0 < x < \pi/2$.

*

J-18. [1980: 316] Let ABCDE be a convex pentagon with the property that each of the five triangles ABC, BCD, CDE, DEA, and EAB has area 1. Find the area of pentagon ABCDE.

Comment.

The answer is $(5 + \sqrt{5})/2$. This problem was posed in the First U.S.A. Mathematical Olympiad (1972). For a solution see Greitzer [1]. A more general problem was given by Möbius and solved by Gauss; for solutions and references, see our Problem 232 [1977: 238].

REFERENCE

1. S. Greitzer, "The First U.S.A. Mathematical Olympiad," *American Mathematical Monthly*, 80 (1973) 276-281.

*

J-20. [1980: 316] Let $\{x\}$ denote the fractional part of x. Find

$$\lim_{n\to\infty} \{(2+\sqrt{3})^n\}.$$

Solution by Gali Salvatore, Perkins, Québec.

For any positive integer n, we have

$$(2+\sqrt{3})^n = [(2+\sqrt{3})^n] + f_n, \qquad 0 < f_n < 1,$$

where f_n is the function whose limit we are seeking. Since $0 < 2 - \sqrt{3} < 1$, we also have

$$0 < (2-\sqrt{3})^n \equiv g_n < 1.$$

Now $(2+\sqrt{3})^n + (2-\sqrt{3})^n$ is an integer; hence $f_n + g_n$ is an integer strictly between 0 and 2, and thus equals 1. Since clearly $\lim_{n\to\infty} g_n = 0$, it follows that $\lim_{n\to\infty} f_n = 1$.

*

J-21. [1980: 316] Prove that the first thousand digits after the decimal point in the decimal expansion of $(6+\sqrt{35})^{1979}$ are nines.

Solution by I.S. Pressman, Carleton University, Ottawa, Ontario.

It is clear that

$$I_n \equiv (6+\sqrt{35})^n + (6-\sqrt{35})^n \qquad (1)$$

is an integer for $n = 1,2,3,\ldots$. Also, $0 < 6 - \sqrt{35} < 0.1$, so that

$$0 < (6-\sqrt{35})^n < 10^{-n}, \qquad n = 1,2,3,\ldots . \qquad (2)$$

By (1) and (2), $(6+\sqrt{35})^n$ is less than an integer (I_n) by at most 10^{-n}. Taking $n = 1979$ gives the required result.

*

I cannot help wondering how the professors who set the above "Jewish" problems would have fared if they themselves had been asked to solve these or similar problems under the same time and psychological constraints that the students who took these tests had to face.

I have two new problem sets to present this month. As usual, I solicit solutions from all readers, and will publish some of the more elegant ones from time to time. The first set is taken from *Középiskolai Matematikai Lapok* 60 (1979) 140 (for Problems 1-4) and 61 (1980) 99 (for Problem 5). I am grateful to Frank Papp who translated them from Hungarian into English.

H-1. If a, b, c are the sides of a triangle with $a \le b \le c$, determine the best possible upper and lower bounds for the expression $(a+b+c)^2/bc$.

H-2. Let n be a positive integer. As a first step, we have given the sequence $\{a_1, a_2, \ldots, a_k\}$, where $k = 2^n$ and each a_i is 1 or -1. As a second step, we form the new sequence $\{a_1a_2, a_2a_3, \ldots, a_ka_1\}$, and continue to repeat this process to generate new sequences. Show that, by at most the 2^nth iterated step, we arrive at a constant sequence with every term equal to 1.

H-3. Given that $a_i = \pm 1$ for $i = 1,2,\ldots,n$, prove that

$$2\sin\frac{\pi}{4}(a_1 + a_1a_2/2 + \ldots + a_1a_2\ldots a_n/2^{n-1}) = a_1\sqrt{2 + a_2\sqrt{2 + \ldots + a_n\sqrt{2}}}.$$

H-4. Show that, for every natural number $n \ge 4$, there is always at least one integer between $n!$ and $(n+1)!$ which is divisible by n^3.

H-5. Let "$\circ$" denote a binary operation on the integers such that
(i) $0\circ a = a$ for all integers a;
(ii) $(a\circ b)\circ c = c\circ(ab) + (a\circ c) + (b\circ c) - 2c$ for all integers a, b, c.
Determine $8\circ 9$.

*

The second set consists of problems which have appeared in some recent mathematical competitions in Bulgaria. I am grateful to J. Tabov and P. Kenderov for sending them to me in English.

B-1. Two given circles K_1 and K_2, with centers O_1 and O_2 and different radii, are tangent externally at point A. Also given inside one of the circles is a point M which does not lie on the line of centers O_1O_2. Show how to determine a line l through M such that, for some triangle AB_1B_2 with $B_1 \in l\cap K_1$ and $B_2 \in l\cap K_2$, the circumcircle of the triangle is tangent to line O_1O_2.

B-2. Let n be a positive integer, and let x and a be real numbers such that $0 < a < 1$ *and* $a^{n+1} \le x \le 1$. Prove that

$$\prod_{k=1}^{n}\left|\frac{x-a^k}{x+a^k}\right| \le \prod_{k=1}^{n}\frac{1-a^k}{1+a^k}.$$

B-3. Given is a truncated triangular pyramid with lateral area S and parallel bases of areas B_1 and B_2. Prove that, if there is a cross section parallel to the bases such that it divides the given truncated pyramid into two truncated pyramids in which spheres can be inscribed, then

$$S = (\sqrt{B_1} + \sqrt{B_2})(\sqrt[4]{B_1} + \sqrt[4]{B_2})^2.$$

B-4. Let the vertices A and C of a quadrilateral ABCD be fixed points on a circle K, while B moves on one and D moves on the other of the two arcs of K with endpoints A and C, in such a way that always BC = CD. If M is the point of intersection of AC and BD and F is the circumcenter of triangle ABM, prove that the locus of F is an arc of a circle.

B-5. N points are given in general position in space (i.e., no four in a plane). There are $\binom{N}{4}$ possible tetrahedra with vertices only at the given points. Prove that, if a plane does not contain any of the N given points, then it can intersect the $\binom{N}{4}$ tetrahedra in at most $N^2(N-2)^2/64$ plane quadrangular cross sections.

B-6. Let K be one of the arcs into which a given circle is divided by a chord AB and let C be the midpoint of K. Let P be an arbitrary point on K and let M be a point on segment PC such that PM = |PA - PB|/2. Find the set of all possible points M for all points P of K.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before October 1, 1981, although solutions received after that date will also be considered until the time when a solution is published.

622. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

The Diophantine equation

$$8x^3 - 21x^2y + 35xy^2 - 83y^3 = z^3$$

has the obvious solutions $(x,y,z) = (k,0,2k)$, where k is any integer. Find at least two solutions for which $y \neq 0$.

623*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

If PQR is the equilateral triangle of smallest area inscribed in a given triangle ABC, with P on BC, Q on CA, and R on AB, prove or disprove that AP, BQ, and CR are concurrent.

624. *Proposed by Dmitry P. Mavlo, Moscow, U.S.S.R.*

ABC is a given triangle of area K, and PQR is the equilateral triangle of smallest area K_0 inscribed in triangle ABC, with P on BC, Q on CA, and R on AB.

(a) Find the ratio

$$\lambda = K/K_0 \equiv f(A,B,C)$$

as a function of the angles of the given triangle.

(b) Prove that λ attains its minimum value when the given triangle ABC is equilateral.

(c) Give a Euclidean construction of triangle PQR for an arbitrary given triangle ABC.

625. *Proposed by Gali Salvatore, Perkins, Québec.*

(a) Let R denote the real field and let P be a polynomial in $R[x]$. Prove that if there are *positive definite* polynomials $Q_1,Q_2 \in R[x]$ such that $P = Q_1-Q_2$, then there are infinitely many such pairs (Q_1,Q_2).

(b) Exhibit one such pair (Q_1,Q_2) for the polynomial P defined by

$$P(x) = a_0 + a_1x + \ldots + a_nx^n, \qquad a_i \in R.$$

626. *Proposed by A. Liu, University of Alberta.*

A (v,b,r,k,λ)-configuration on a set with v elements is a collection of b k-subsets such that

(i) each element appears in exactly r of the k-subsets;

(ii) each pair of elements appears in exactly λ of the k-subsets.

Prove that $k^r \geq v^\lambda$ and determine the value of b when equality holds.

627. *Proposed by F. David Hammer, Santa Cruz, California.*

Consider the double inequality

$$6 < 3^{\sqrt{3}} < 7.$$

Using *only* the elementary properties of exponents and inequalities (no calculator, computer, table of logarithms, or estimate of $\sqrt{3}$ may be used), prove that the first inequality implies the second.

628. *Proposed by Roland H. Eddy, Memorial University of Newfoundland.*

Given a triangle ABC with sides a,b,c, let T_a,T_b,T_c denote the angle bisectors extended to the circumcircle of the triangle. If R and r are the circum- and in-radii of the triangle, prove that

$$T_a + T_b + T_c \le 5R + 2r,$$

with equality just when the triangle is equilateral.

629*. *Proposed by S.C. Chan, Singapore.*

For which constants λ and m does the infinite series

$$\frac{1^3\lambda}{1!}e^{-m} + \frac{2^3\lambda^2}{2!}e^{-2m} + \frac{3^3\lambda^3}{3!}e^{-3m} + \frac{4^3\lambda^4}{4!}e^{-4m} + \cdots$$

converge, and to what sum?

630. *Proposed by Charles W. Trigg, San Diego, California.*

Partition the palindrome 2662 into two integers, one of which is divisible by 29 and the other by 43.

631. *Proposed by Sidney Kravitz, Dover, New Jersey.*

Richard Bedford BENNETT (1870-1947) and Sir Wilfrid LAURIER (1841-1919) are former Canadian prime ministers; and Pierre-Elliott TRUDEAU is, of course, the current one. They have buried their political differences and find themselves united in the following alphametic, which you are asked to solve:

$$\begin{array}{r} \text{BENNETT} \\ \underline{\text{TRUDEAU}} \\ \text{LAURIER} \end{array}.$$

632. *Proposed by Leroy F. Meyers, The Ohio State University.*

Let ABCD be a plane quadrilateral in Euclidean 3-dimensional space. Find a simple formula for the area of ABCD.

* * *

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

492. [1979: 291; 1980: 291; 1981: 50] Late solution: BIKASH K. GHOSH, Bombay, India.

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525. [1980: 78; 1981: 86] Late solution: F.G.B. MASKELL Algonquin College, Ottawa.

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533. [1980: 113] *Proposed by James Gary Propp, Harvard College, Cambridge, Massachusetts.*

Consider the following products over the complex field:

$$\prod_{k=1}^{n}(x+2k-1) \quad \text{and} \quad \prod_{k=1}^{n}(x-2k+1).$$

(a) For $n=1,2,3,4,5$, find all complex x such that each product is real and rational.

(b) Are there, for any $n>5$, any real irrational x such that each product is rational?

Solution of part (a) by the proposer.

Let A_n and B_n denote the first and second products, respectively.

For $n = 1$. we have $A_1 = x+1$ and $B_1 = x-1$. These are both rational if and only if $\frac{1}{2}(A_1+B_1) = x \in Q$, where Q is the set of all rationals.

For $n = 2$, we have $A_2 = (x+1)(x+3)$ and $B_2 = (x-1)(x-3)$. These are both rational if and only if $\frac{1}{8}(A_2-B_2) = x \in Q$.

For $n = 3$, we find from the expressions for A_3 and B_3 that these are both rational if and only if

$$\tfrac{1}{2}(A_3+B_3) = x(x^2+23)$$

and

$$\frac{1}{18}(A_3-B_3+384) = x^2+23$$

and both rational, and this occurs just when

$$x \in Q \qquad \text{or} \qquad x = \pm i\sqrt{23}.$$

For $n = 4$, it turns out that A_4 and B_4 are both rational if and only if

$$\tfrac{1}{2}(A_4+B_4) = x^4+86x^2+105 \tag{1}$$

and

$$\frac{1}{32}(A_4-B_4) = x(x^2+11) \tag{2}$$

are both rational. For any $r \in Q$, (1) equals r just when $x^2 = -43\pm\sqrt{q}$, where $q = 1744+r$ is rational; and (2) will then be rational only if

$$x^2(x^2+11)^2 = (-43\pm\sqrt{q})(-32\pm\sqrt{q})^2 \tag{3}$$

is rational. Since the coefficient of $\sqrt{q}$ in (3) is $\pm(3776+q)$, either $q = -3776$ or $\sqrt{q}$ is rational. If $q = -3776$, then $x^2(x^2+11)^2 = 600^2$ from (3), and (2) is rational. This yields the solutions $x = \pm\sqrt{-43+i\sqrt{3776}}$. If $\sqrt{q}$ is rational, then $x^2 = -43\pm\sqrt{q}$ is rational and so is x^2+11; and then (2) is rational if and only if $x \in Q$ or

$x = \pm i\sqrt{11}$. All the solutions for the case $n = 4$ are therefore given by

$$x \in Q \qquad \text{or} \qquad x = \pm i\sqrt{11} \qquad \text{or} \qquad x = \pm\sqrt{-43\pm i\sqrt{3776}}.$$

For $n = 5$, it turns out that A_5 and B_5 are both rational if and only if

$$\tfrac{1}{2}(A_5+B_5) = x(x^4+230x^2+1689) \tag{4}$$

and

$$\frac{1}{10}(A_5-B_5) = 5x^4+190x^2+189 \tag{5}$$

are both rational. For any $r \in Q$, (5) equals r just when $x^2 = -19\pm\sqrt{q}$, where $q = (1616+r)/5$ is rational; and (4) will then be rational only if

$$x^2(x^4+230x^2+1689)^2 = (-19\pm\sqrt{q})(-2320+192\sqrt{q}+q)^2 \tag{6}$$

is rational. Since the coefficient of $\sqrt{q}$ in (6) is $\pm(q^2+24928q+22309120)$, either

$$q^2 + 24928q + 22309120 = 0 \tag{7}$$

or $\sqrt{q}$ is rational. Since (7) has no rational solution, for (4) to be rational it is necessary that $\sqrt{q}$ be rational and that $x = \pm\sqrt{s}$, where $s = -19\pm\sqrt{q}$ is rational. The condition is also sufficient if $x \in Q$. But if $x \notin Q$ sufficiency requires the additional condition

$$x^4 + 230x^2 + 1689 = s^2 + 230s + 1689 = 0, \tag{8}$$

a condition which is never realized since (8) has no rational solution s. So for $n = 5$ only $x \in Q$ makes A_5 and B_5 both rational.

Part (a) was partially solved by J.T. GROENMAN, Arnhem, The Netherlands.

Editor's comment.

It is clear that A_n and B_n are both rational for all n if $x \in Q$; and part (a) shows that, if $1 \le n \le 5$, A_n and B_n are *never* both rational for any (real) irrational x. Part (b), which remains open, asks if the last statement remains true for all $n > 5$.

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534. [1980: 113] *Proposed by Leroy F. Meyers, The Ohio State University.*

Some time ago I noticed that the exponent of 2 in the prime factorization of $n!$ seems to be approximately twice the exponent of 3 in the same factorization, at least for small values of n, say up to 100. Is this true in general? What about the exponents of other primes? More precisely, if n is any positive integer and p is prime, let $e_p(n)$ be the exponent of p in the prime factorization of $n!$. Is it true that $\lim_{n\uparrow} e_2(n)/e_3(n) = 2$? What about $\lim_{n\uparrow} e_p(n)/e_q(n)$ for primes p and q?

Solution de Marc Sackur, Collège Stanislas, Montréal, Québec.

D'après un résultat bien connu de Legendre (voir la solution du Problème 90 [1976:34] pour plusieurs références), si

$$n = a_0 + a_1p + \ldots + a_rp^r, \qquad 0 \le a_i \le p-1,\ a_r \ne 0,$$

alors

$$e_p(n) = \frac{n - (a_0+a_1+\ldots+a_r)}{p-1}.$$

Il résulte l'encadrement

$$\frac{n-(r+1)(p-1)}{p-1} \le e_p(n) \le \frac{n}{p-1},$$

d'où

$$\frac{1}{p-1} - \frac{r}{n} - \frac{1}{n} \le \frac{e_p(n)}{n} \le \frac{1}{p-1}. \qquad (1)$$

Or il est clair que $p^r \le n$, donc $r \log p \le \log n$ et

$$0 \le \lim_{n\uparrow} \frac{r}{n} \le \frac{1}{\log p} \cdot \lim_{n\uparrow} \frac{\log n}{n} = 0.$$

On obtient donc de (1) $\lim_{n\uparrow} e_p(n)/n = 1/(p-1)$, d'où le résultat recherché:

$$\lim_{n\uparrow} e_p(n)/e_q(n) = (q-1)/(p-1).$$

En particulier, $\lim_{n\uparrow} e_2(n)/e_3(n) = 2$.

Also solved by JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; ERNEST W. FOX and MARIO D'ANGELO, Marianopolis College, Montréal, Québec (jointly); J.T. GROENMAN, Arnhem, The Netherlands; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; DAVID SINGMASTER, Polytechnic of the South Bank, London, England; ROBERT A. STUMP, Hopewell, Virginia; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

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535. [1980: 113] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Given a triangle ABC with sides a,b,c, let T_a, T_b, T_c denote the angle bisectors extended to the circumcircle of the triangle. Prove that

$$T_aT_bT_c \ge \tfrac{8}{9}\sqrt{3}abc,$$

with equality attained in the equilateral triangle.

I. *Solution by Howard Eves, University of Maine.*

Denote the bisector of $\angle A$ by t_a and let D and E denote the other ends of t_a and T_a, respectively. From similar triangles ABD and AEC it follows that $t_a = bc/T_a$, with similar expressions for t_b and t_c. Hence

$$t_a t_b t_c = a^2b^2c^2/T_aT_bT_c$$

and it suffices to show that

$$t_a t_b t_c < \frac{3}{8}\sqrt{3}\,abc\,. \tag{1}$$

This is easily accomplished by means of inequalities 8.8 and 1.12 in the standard reference [1]:

$$t_a \le \sqrt{s(s-a)} \quad \text{and} \quad 64s^3(s-a)(s-b)(s-c) \le 27a^2b^2c^2, \tag{2}$$

where s is the semiperimeter of triangle ABC. For we have, using those inequalities in turn,

$$t_a^2t_b^2t_c^2 \le s^3(s-a)(s-b)(s-c) \le \frac{27}{64}a^2b^2c^2$$

and (1) follows. As in (2), equality holds if and only if the triangle is equilateral. Inequality (1) deserves to be better known.

II. *Solution by Roland H. Eddy, Memorial University of Newfoundland.*

Let $\angle A = 2\alpha$, $\angle B = 2\beta$, $\angle C = 2\gamma$. [With E as in solution I],

$$\mathrm{BE} = \mathrm{EC} = c\sin\alpha/\sin 2\gamma.$$

Then, by Ptolemy's Theorem, we have

$$\begin{aligned} aT_a &= b\cdot\mathrm{BE} + c\cdot\mathrm{EC} \\ &= c(b+c)\sin\alpha/\sin 2\gamma \end{aligned}$$

and two similar results. These yield

$$\begin{aligned} T_aT_bT_c &= (b+c)(c+a)(a+b)\,\sin\alpha\sin\beta\sin\gamma/\sin 2\alpha\sin 2\beta\sin 2\gamma \\ &= \frac{1}{8}(b+c)(c+a)(a+b)/\cos\alpha\cos\beta\cos\gamma\,. \end{aligned}$$

The required result is obtained immediately by applying inequalities 1.4 and 2.28 of [1]:

$$(b+c)(c+a)(a+b) \ge 8abc \quad \text{and} \quad \cos\alpha\cos\beta\cos\gamma \le \frac{3}{8}\sqrt{3}. \tag{3}$$

As in (3), equality holds if and only if the triangle is equilateral.

Also solved by LEON BANKOFF, Los Angeles, California; W.J. BLUNDON, Memorial University of Newfoundland; ERNEST W. FOX, Marianopolis College, Montréal, Québec; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; V.N. MURTY, Pennsylvania State University, Capitol Campus; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

Editor's comment.

Nearly all solvers used inequalities from the "Bottema Bible" [1] in their solutions. See Crux 628 in this issue for another inequality involving T_a, T_b, T_c.

REFERENCE

1. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen, 1969.

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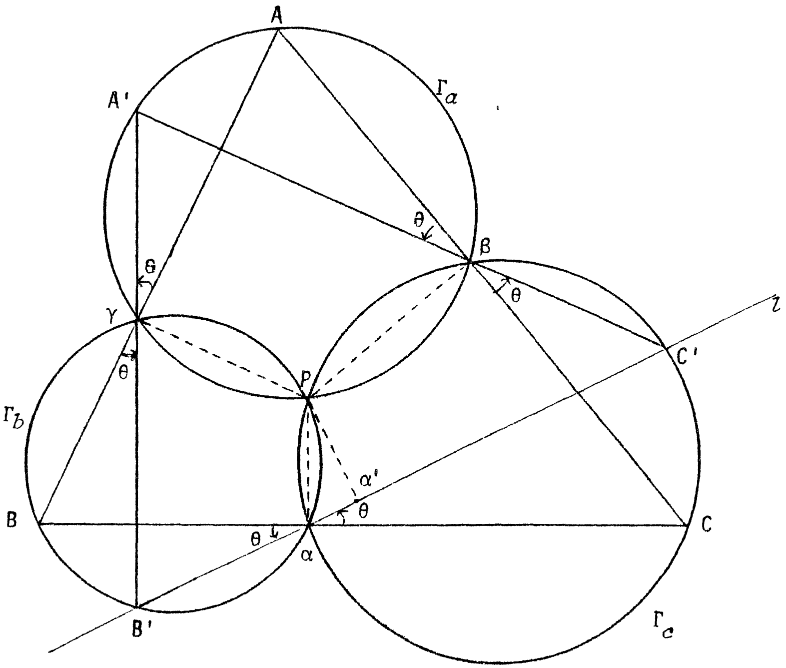
536. [1980: 113] *Proposed by B. Leeds, Forest Hills, New York.*

Through each of the midpoints of the sides of a triangle ABC, lines are drawn making an acute angle θ with the sides. These lines intersect to form a triangle A'B'C'. Prove that A'B'C' is similar to ABC and find the ratio of similitude.

Solution by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.

More generally, let P be any point in the plane of triangle ABC (it will become apparent that the proposed problem corresponds to the special case when P is the circumcenter of the triangle) and let α, β, γ be the vertices of the pedal triangle of the point P with respect to triangle ABC, as shown in the figure. It is clear that $A\gamma P\beta$, $B\alpha P\gamma$, $C\beta P\alpha$ are all cyclic quadrilaterals, being inscribed in (say) circles Γ_a, Γ_b, Γ_c, respectively.



Let line l through α make a directed angle θ with $\vec{BC}$, with $0 \le |\theta| \le 90°$, meeting Γ_b in B' and Γ_c in C'; let C'β meet Γ_a again in A', and join γ to A' and to B'. The angles marked θ in the figure are clearly all equal, and it follows that A',γ,B' are collinear. It is now obvious that $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, and hence that $\Delta A'B'C' \sim \Delta ABC$.

Now $\Delta BPC \sim \Delta B'PC'$; so if $P\alpha' \perp B'C'$, which makes $\angle \alpha P\alpha' = \theta$, then the required ratio of similitude is

$$\frac{B'C'}{BC} = \frac{P\alpha'}{P\alpha} = \cos\theta .$$

If $\theta = 0$, then A'B'C' coincides with ABC; and if $|\theta| = 90°$, then A'B'C' collapses into the single point P.

Also solved by JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; W.J. BLUNDON, Memorial University of Newfoundland; HOWARD EVES, University of Maine; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; LAI LANE LUEY, Willowdale, Ontario; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

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537. [1980: 113] *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*

Find all pairs of integers (k,p) such that

$$(k-p)(2^{p+1}-2) = (p+1)(2^p-2).$$

Solution by Kenneth M. Wilke, Topeka, Kansas (revised by the editor).

The given equation, which we write in the form

$$2(k-p)(2^p-1) = (p+1)(2^p-2), \qquad (1)$$

has no solution for $p = 0$, and the two sides vanish if and only if $k = p = \pm 1$, so we have the two solutions (1,1) and (-1,-1). Suppose $p > 1$. The two consecutive positive integers $2^p - 2$ and $2^p - 1$ are relatively prime, so $2^p - 1$ divides $p + 1$ and $2^p - 1 \le p + 1$, an inequality which does not hold for any $p > 2$. And $p = 2$ yields the solution (3,2). Suppose $p < -1$. Setting $p = -r$ in (1) and multiplying both sides by -2^r yield the equivalent equation

$$(k+r)(2^{r+1}-2) = (1-r)(2^{r+1}-1),$$

where $r > 1$. The consecutive positive integers $2^{r+1} - 2$ and $2^{r+1} - 1$ are relatively prime, so $2^{r+1} - 2$ divides $1 - r$ and $2^{r+1} - 2 \le |1-r|$, an inequality which does not hold for any $r > 1$.

The only integral solutions (k,p) are (-1,-1), (1,1), and (3,2).

Also solved by JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; ERNEST W. FOX, Marianopolis College, Montréal, Québec; J.T. GROENMAN, Arnhem, The

Netherlands; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, University of Wisconsin-Oshkosh; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer. Incomplete solutions were received from G.C. GIRI, Midnapore College, West Bengal, India; and MARC SACKUR, Collège Stanislas, Montréal, Québec.

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538. [1980: 113] *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

Find

$$\lim_{n\to\infty} \surd(5 + 1\,\surd(6 + 2\,\surd(7 + 3\,\surd\ldots\surd((n+3) + (n-1)\surd(n+4+ne^n))\ldots))).$$

Solution by the proposer.

For $n \geq 2$ and suitable x, let

$$R_m(x) = \sqrt{m+4+mx}, \qquad m = 1,2,\ldots,n; \quad (1)$$

then the sequence whose limit is sought is $\{f_n\}$, where

$$f_n = R_1 \circ R_2 \circ \ldots \circ R_n(e^n), \qquad n = 2,3,4,\ldots .$$

It follows from (1) that

$$R_m(m+3) = m+2 \quad (2)$$

and it is easily verified that

$$1 < x/y \leq k \quad \Longrightarrow \quad 1 < R_m(x)/R_m(y) < \sqrt{k}. \quad (3)$$

Since $n \geq 2$, we have

$$1 < e^n/(n+3) < e^n, \quad (4)$$

and repeated use of (3) and (2) gives the successive implications

$$\begin{aligned}
(4) \quad &\Longrightarrow \quad 1 < R_n(e^n)/(n+2) < \exp(n/2)\\
&\Longrightarrow \quad 1 < R_{n-1}\circ R_n(e^n)/(n+1) < \exp(n/4)\\
&\Longrightarrow \quad \ldots\\
&\Longrightarrow \quad 1 < R_1\circ R_2\circ\ldots\circ R_n(e^n)/3 < \exp(n/2^n). \quad (5)
\end{aligned}$$

We now have from (5)

$$3 < f_n < 3\exp(n/2^n), \quad (6)$$

and so $\lim_{n\to\infty} f_n = 3$.

Also solved by LEROY F. MEYERS, The Ohio State University.

Editor's comment.

Meyers showed more generally that the limit exists and is again 3 if

$$f_n = R_1 \circ R_2 \circ \ldots \circ R_n(q^n)$$

for any $q > 1$. In fact, if $q = e^a$, where $a > 0$, we get in place of (6), for all sufficiently large n,

$$3 < f_n < 3 \exp(an/2^n),$$

which shows that the limit 3 is independent of the value of q.

In a subsequent letter, Meyers considered the still more general

$$\lim_{n\to\infty} \surd(1+p+1\surd(2+p+2\surd\ldots\surd((n-1)+p+(n-1)\surd(n+p+na_n))\ldots)),$$

where the sequence $\{a_n\}$ is nonnegative and bounded by a geometric sequence. (Our problem has $p = 4$ and $a_n = e^n$.) He conjectured that this limit, call it $l(p)$, depends upon p alone and exists for all $p \ge -1$. The following table, computed by Meyers, gives some values of $l(p)$.

$p=$	$l(p)\approx$	$p=$	$l(p)\approx$	$p=$	$l(p)\approx$
-1	1.7559485	4.5	3.0949200	10^2	10.593024
-0.5	1.9207299	5	3.1867400	10^3	32.151009
0	2.0714907	6	3.3621498	10^4	100.50881
0.5	2.2111897	7	3.5280247	10^5	316.73054
1	2.3418786	8	3.6857352	10^6	1000.5009
1.5	2.4650490	9	3.8363555	10^7	3162.7779
2	2.5818250	10	3.9807453	10^8	10000.500
2.5	2.6930786	20	5.1924418	10^9	31623.277
3	2.7995020	30	6.1542767	10^{10}	100000.50
3.5	2.9016555	40	6.9761944	10^{11}	316228.27
4	3	50	7.7055891	10^{12}	1000000.5

It appears from the table that $l(p) \sim \sqrt{p}$ for large p. It would be interesting to have confirmation of Meyers' conjecture and a closed form expression for $l(p)$.

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539. [1980: 114] *Proposed by Charles W. Trigg, San Diego, California.*

From among the three-digit primes less than 500, form four four-term arithmetic progressions in which the first and last terms contain the same decimal digits.

Solution by the proposer.

Among the three-digit primes less than 500 can be found exactly 18 pairs (a,b) with $a < b$ in which a and b are permutations of the same digit set. These are:

(113, 131)	(127, 271)	(173, 317)	(149, 491)	(239, 293)	(337, 373)
(113, 311)	(137, 173)	(139, 193)	(419, 491)	(241, 421)	(349, 439)
(131, 311)	(137, 317)	(149, 419)	(179, 197)	(313, 331)	(379, 397)

We wish to find all the pairs (a,b) for which

$$a + (b-a)/3 \quad \text{and} \quad a + 2(b-a)/3 \tag{1}$$

are both primes. Now $b-a$ is divisible by 3 in every case (divisible, in fact, by 9: see Crux 576 [1980: 251]), so the numbers (1) are always both integers; but only for the pairs

(131, 311), (137, 317), (349, 439), (419, 491)

are they both primes. The four progressions are:

(131, 191, 251, 311),
(137, 197, 257, 317),
(349, 379, 409, 439),
(419, 443, 467, 491).

Also solved by JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; ERNEST W. FOX, Marianopolis College, Montréal, Québec; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; BOB PRIELIPP, University of Wisconsin-Oshkosh; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; and KENNETH M. WILKE, Topeka, Kansas.

Editor's comment.

It is a matter of editorial policy never to publish a proposer's solution unless it is, in the editor's opinion, clearly superior to (or at least as good as and essentially different from) all the other solutions received. This policy is even more strictly adhered to when, as in this case, the problem is very easy and there is an adequate supply of other solutions to choose from. But here the editor's hand was forced. Three of the other solvers used the same approach as the proposer but carelessly omitted some of the 18 pairs (a,b)—one had 17, one 16, and one 12—and it was sheer luck that the omitted pairs did not lead to solutions. The remaining six solvers submitted correct answers but with little or no indication of how the answers were arrived at. This may be a good place to remind readers that when a problem has an answer, numerical or otherwise, the answer itself does not constitute a solution to the problem. A *solution* is a description (preferably in complete, grammatical, properly punctuated sentences in the language of discourse, which is English or French, not "mathematics") of the process by which the answer is arrived at. Answers alone belong at the back of textbooks.

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540. [1980: 114] *Proposed by Leon Bankoff, Los Angeles, California.*

Professor Euclide Paracelso Bombasto Umbugio has once again retired to his *tour d'ivoire* where he is now delving into the supersophisticated intricacies of the works of Grassmann, as elucidated by Forder's *Calculus of Extension.* His goal is to prove Neuberg's Theorem:

If D, E, F *are the centers of squares described externally on the sides of a triangle* ABC, *then the midpoints of these sides are the centers of squares described internally on the sides of triangle* DEF.

Help the dedicated professor emerge from his self-imposed confinement and enjoy the thrill of hyperventilation by showing how to solve his problem using only high-school, synthetic, Euclidean, "plain" geometry.

Solution by J.T. Groenman, Arnhem, The Netherlands.

It suffices to show that, say, the midpoint M of BC is the center of the square on EF or, equivalently, that ME = MF and ME $\perp$ MF (see Figure 1).

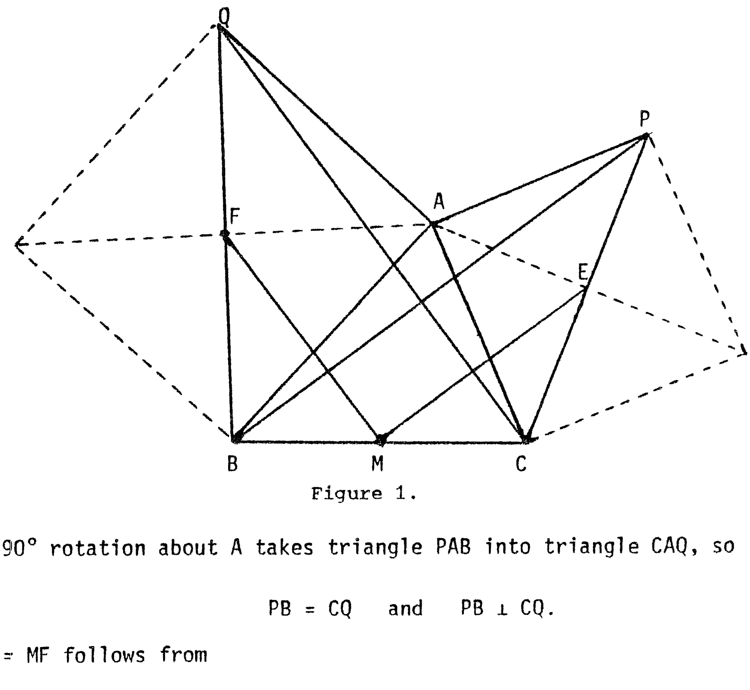

Figure 1.

A 90° rotation about A takes triangle PAB into triangle CAQ, so

$$PB = CQ \quad \text{and} \quad PB \perp CQ.$$

Now ME = MF follows from

$$ME = \tfrac{1}{2}BP = \tfrac{1}{2}CQ = MF$$

and ME $\perp$ MF follows from

$$ME \parallel BP \perp CQ \parallel MF.$$

Also solved by JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; HOWARD EVES, University of Maine; JACK GARFUNKEL, Flushing, N.Y.; ANDY LIU, University of Alberta; LAI LANE LUEY, Willowdale, Ontario; LEROY F. MEYERS, The Ohio State University; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; JAN VAN DE CRAATS, Leiden University, The Netherlands; and the proposer.

Editor's comment.

This theorem is credited to Neuberg by Forder [1], who gives a two-line solution, using the method and notation of Grassmann, which so far has thoroughly defeated (and deflated) Professor Umbugio, the premier mathematician at the University of Guayazuela, despite frequent invocations to his patron saint and namesake Euclide.

In an effort to help the good professor, most of our solvers submitted solutions (some quite lengthy) in "plain" geometry. Dodge and van de Craats gave proofs by transformation geometry and expressed the hope that such proofs would soon be considered "plain" geometry even by Professor Umbugio. In view of the parlous state of geometry in Guayazuela (and North America—we don't know about the rest of the world), that time is not yet. Garfunkel gave a proof by complex numbers, which could only be considered "plain" geometry in an imaginary high school.

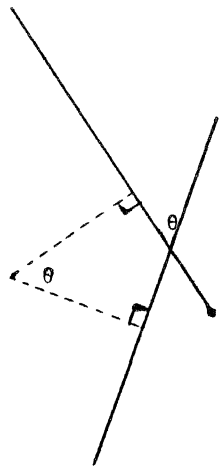

Figure 2.

There is only one statement in our featured solution which is likely to cause Professor Umbugio some concern: that $PB \perp CQ$ results from a $90°$ rotation about A. Figure 2 is all that is needed to bring that statement into the realm of "plain" geometry.

REFERENCE

1. Henry George Forder, *The Calculus of Extension,* Chelsea, New York, 1960, p. 40.

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541. [1980: 152] *Proposed by Herman Nyon, Paramaribo, Surinam.*

Solve the following colourful alphametic, in which BLUE is a perfect square:

$$\begin{array}{r} \text{RED} \\ \text{BLUE} \\ \underline{\text{GREEN}} \\ \text{BROWN} \end{array}.$$

Solution by Lai Lane Luey, Willowdale, Ontario.

It is clear that we must have B = 8 or 9, and examination of a table of squares reduces the possible values of BLUE to 5 four-digit squares having no repeated digits:

8649, 9025, 9216, 9604, 9801.

The requirement D + E = 10 immediately eliminates 9025, 9604, and 9801. Since BLUE = 9216 implies W = 4 = D, we must have BLUE = 8649, and D = 1 and W = 3 follow. Now the hundreds' column requires R ≥ 3, and of the available digits only R = 5 does not result in a duplication. Finally, G = 7, O = 2, and N = 0 fall into place. The unique solution is

$$\begin{array}{r} 591 \\ 8649 \\ \underline{75990} \\ 85230 \end{array} \begin{array}{l} \\ (= 93^2) \\ . \\ \\ \end{array}$$

Also solved by JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; JAMES BOWE, Erskine College, Due West, South Carolina; CLAYTON W. DODGE, University of Maine at Orono; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; J.A. McCALLUM, Medicine Hat, Alberta; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; DONVAL R. SIMPSON, Fairbanks, Alaska; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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542. [1980: 153] *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

Let p be a permutation of the digits 0 through 9. This induces, in a natural way, a function p^* defined on the interval (0,1) by writing (the sums are for $n = 1,2,3,\ldots$) $x = \Sigma a_n/10^n$ (using 0's instead of 9's in the finite decimal case) and putting $p^*(x) = \Sigma\, p(a_n)/10^n$. These functions were defined earlier in this journal [1979: 184] where it was stated that some of them were bijective. Find a necessary and sufficient condition for a function p^* to be bijective.

Solution de Robert Tranquille, Collège de Maisonneuve, Montréal, Québec.

Nous allons montrer que:

I. Une condition nécessaire et suffisante pour qu'une de ces fonctions p^* soit une application de l'intervalle (0,1) dans lui-même est que la permutation p de l'ensemble $\{0,1,\ldots,9\}$ qui l'engendre fasse correspondre $\{0,9\}$ à $\{0,9\}$.

II. Cette condition étant remplie, la fonction p^* est nécessairement une permutation de l'intervalle (0,1), c'est-à-dire une bijection de l'intervalle (0,1) sur lui-même.

Démonstration de I. La condition est nécessaire. Car supposons, au contraire, que $p(\{0,9\}) \neq \{0,9\}$. Alors il existe un chiffre $a \notin \{0,9\}$ tel que $p(a) \in \{0,9\}$, et par suite $x = \Sigma a/10^n \in (0,1)$ mais $p^*(x) = \Sigma p(a)/10^n = 0$ ou $.999\ldots = 1 \notin (0,1)$. La condition est suffisante. Car supposons que $p(\{0,9\}) = \{0,9\}$ et soit $x \in (0,1)$. Si $x = .aaa\ldots$, alors $a \notin \{0,9\}$ et $p^*(x) = \Sigma p(a)/10^n = .bbb\ldots \in (0,1)$, car $b = p(a) \notin \{0,9\}$. Si, par contre, $x = .a_1a_2a_3\ldots$ où, disons, $a_i \neq a_j$, alors $p(a_i) \neq p(a_j)$ et $p^*(x) = \Sigma p(a_n)/10^n \in (0,1)$.

Démonstration de II. Supposons que $p(\{0,9\}) = \{0,9\}$. En vertu de I, p^* est une application de l'intervalle $(0,1)$ dans lui-même. Montrons qu'elle est injective. Soit $x,y \in (0,1)$. Leurs représentations décimales $x = .a_1a_2a_3\ldots$ et $y = .b_1b_2b_3\ldots$ ne contiennent pas de suite infinie de 9. Supposons que $x \neq y$, donc $a_i \neq b_i$ pour un certain i et, par suite, $p(a_i) \neq p(b_i)$. Si l'on a en même temps

$$p^*(x) = \Sigma p(a_n)/10^n = \Sigma p(b_n)/10^n = p^*(y),$$

alors $p^*(x)$ et $p^*(y)$ contiennent, l'un une suite infinie de 0, l'autre une suite infinie de 9. Donc x et y contiennent, l'un une suite infinie de j, l'autre une suite infinie de k, et $p(\{j,k\}) = \{0,9\}$. Puisque $\{j,k\} = p^{-1}(\{0,9\}) = \{0,9\}$, l'un de x et y contient une suite infinie de 9, ce qui est faux. Donc $p^*(x) \neq p^*(y)$ et p^* est injective.

Pour démontrer la surjectivité de p^*, soit $y \in (0,1)$. Si $y = .bbb\ldots$, alors $b \notin \{0,9\}$ et il existe un chiffre $a \notin \{0,9\}$ tel que $p(a) = b$; et pour $x = \Sigma a/10^n \in (0,1)$ on a

$$p^*(x) = \Sigma p(a)/10^n = \Sigma b/10^n = y.$$

Par contre, si $y = .b_1b_2b_3\ldots$ avec, disons, $b_i \neq b_j$, alors $p^{-1}(b_i) \neq p^{-1}(b_j)$ et l'on a $p^*(x) = y$ pour

$$x = \Sigma p^{-1}(b_n)/10^n \in (0,1).$$

Also solved by JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; and the proposer.

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A PANDIAGONAL SIXTH-ORDER

PRIME MAGIC SQUARE

This pandiagonal sixth-order magic square is composed of distinct primes and has the magic sum 630.

ALLAN WM. JOHNSON JR.

23	137	31	127	131	181
173	83	163	103	89	19
263	7	41	67	193	59
47	139	179	223	13	29
107	97	5	37	151	233
17	167	211	73	53	109