EDITORIAL

Dear readers,

With this issue *CRUX with MAYHEM* is taking on a new helmsman. It is a very daunting task to try and fill the shoes of the previous Editor-in-Chief, Bruce Shawyer, and the Associate Editor, Clayton Halfyard. We are well aware of the respect they commanded, both inside and outside of the *CRUX with MAYHEM* community. Our goal is to earn some fraction of that same respect from you, the readers.

Under Bruce's capable leadership, and under Bill Sands and Robert Woodrow before him, *CRUX with MAYHEM* gained an excellent reputation as one of the world's top problem-solving journals. This stature has been achieved not only because of their stellar leadership, but also as a result of the quality of the submissions from the readership. We encourage you to continue to provide this.

One of the mandates I have as the new Editor-in-Chief is to try to make the journal more accessible to a younger audience. We have been told by teachers that many students (and some teachers, also) have picked up an issue and started on page 1; not finding any problem they could do in the first few pages, the journal was returned to the shelf and never looked at again. As a result we have decided to present the sections in order of increasing mathematical sophistication.

Each issue now opens with the Skoliad Corner, which we see as the setting for problems most appealing to our younger (potential) readers. Following this comes *Mathematical Mayhem*, then the Olympiad Corner. Next we have book reviews and/or articles, and finally the *CRUX with MAYHEM* Problems and Solutions sections. Please let us know your thoughts on these changes.

Our Editorial Board has remained largely intact. We have added three new Problems Editors, and we have a new Assistant Editor for *Mathematical Mahyem*. Their names all appear on the inside front cover. With a great deal of continuity on the Board and the energy of a number of new Board Members, we should be able to maintain the high-quality journal you have come to expect.

Jim Totten, Editor-in-Chief   *Bruce Crofoot, Associate Editor*

MOT DE LA RÉDACTION

Chers lecteurs,

Ce numéro de *CRUX with MAYHEM* marque l'arrivée à bord d'un nouveau capitaine. Nous sommes tout à fait conscients de la tâche colossale de succéder à Bruce Shawyer et à Clayton Halfyard, qui occupaient respectivement les postes de rédacteur en chef et de rédacteur associé, et nous savons très bien à quel point ces grands navigateurs ont gagné le respect des lecteurs du Crux et du reste de la communauté mathématique. Nous aurons atteint notre objectif si nous réussissons à recueillir une fraction de ce respect de votre part.

Sous la houlette de Bruce, et de Bill Sands et Robert Woodrow avant lui, *CRUX with MAYHEM* s'est taillé une excellente réputation. On le considère d'ailleurs maintenant comme l'une des meilleures revues de résolution de problèmes au monde. Une telle réputation est non seulement redevable à l'excellence de son équipe de rédaction, mais aussi à la qualité des problèmes et solutions envoyés par les lecteurs. Merci, et continuez sur cette belle lancée !

L'un de mes mandats à titre de nouveau rédacteur en chef sera de rendre la revue plus accessible aux jeunes élèves. Des enseignants nous ont déjà dit que de nombreux élèves (et même des enseignants), n'ayant trouvé aucun problème à leur portée dans les premières pages, ont remis la revue sur les tablettes pour ne plus jamais y revenir. De tels commentaires nous ont amenés à présenter les sections en ordre croissant de difficulté mathématique.

Chaque numéro commencera désormais par la section « Skoliad », que nous croyons la plus adaptée à notre (futur) jeune lectorat. Suivront dans l'ordre le *Mathematical Mayhem*, la chronique « Olympiade », les critiques de livres ou articles, et finalement les problèmes et solutions du *CRUX with MAYHEM*. N'hésitez pas à nous faire part de votre opinion sur ces changements.

Notre conseil de rédaction n'a pas beaucoup changé. Nous accueillons toutefois trois nouveaux rédacteurs de problèmes et un nouveau rédacteur adjoint du *Mahyem*. Leurs noms figurent tous en deuxième de couverture. Grâce à cette relative stabilité au conseil et à l'énergie des nouveaux membres, nous avons bon espoir d'arriver à maintenir le niveau de qualité auquel nous êtes maintenant habitués.

Jim Totten, rédacteur en chef   *Bruce Crofoot, rédacteur associé*

SKOLIAD No. 67

Shawn Godin

Solutions may be sent to Shawn Godin, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, ON, CANADA, K1C 2Z5, or emailed to

mayhem-editors@cms.math.ca.

We are especially looking for solutions from high school students. Please include your name, school or other affiliation (if applicable), city, province or state, and country on any correspondence. High school students should also include their grade in school. Please send your solutions to the problems in this edition by *1 April 2003*. A copy of **MATHEMATICAL MAYHEM Vol. 1** will be presented to the pre-university reader(s) who send in the best solutions before the deadline. The decision of the editor is final.

Starting with this issue we have a slight change. Certain items in the problem sets will be marked with an asterisk (∗). We will only print solutions to these problems if we receive them from students in grade **10** or under (or equivalent), or if we receive a unique solution or a generalization.

Our entry this issue comes from the Manitoba Mathematical Contest. My thanks go to Diane Dowling of the University of Manitoba for forwarding the material to me.

MANITOBA MATHEMATICAL CONTEST, 2002

For students in Senior 4

9:00 a.m. – 11:00 a.m.   Wednesday, February 20, 2002

Sponsored by

The Actuaries' Club of Winnipeg, The Manitoba Association of Mathematics Teachers, The Canadian Mathematical Society, and The University of Manitoba

Answer as much as possible. You are not expected to complete the paper. See both sides of this sheet. Hand calculators are not permitted. Numerical answers only, without explanation, will not be given full credit.

1. (a) (∗) Solve the equation $x^4 - 3x^2 + 2 = 0$.

(∗) Résous l'équation $x^4 - 3x^2 + 2 = 0$.

(b) (∗) Solve the equation $\dfrac{4}{(x-3)^2} - \dfrac{4}{(x-3)} + 1 = 0$.

(∗) Résous l'équation $\dfrac{4}{(x-3)^2} - \dfrac{4}{(x-3)} + 1 = 0$.

2. (a) (∗) Solve the equation $9x^3 - 9x^2 - 4x + 4 = 0$.

(∗) Résous l'équation $9x^3 - 9x^2 - 4x + 4 = 0$.

4

(b) (∗) Thirty-six students took a final exam. The average score of those who passed was **60**, the average score of those who failed was **42** and the average of all the scores was **53**. How many students did not pass the exam?

(∗) Trente-six étudiants ont passé un examen. La note moyenne de ceux qui ont réussi était **60**, la note moyenne de ceux qui ont échoué était **42**, et la note moyenne de toutes les notes était **53**. Combien d'étudiants n'ont pas réussi à cet examen ?

3. (a) (∗) The area of a rectangle is **3** and its perimeter is **7**. What is the length of the diagonal of this rectangle?

(∗) L'aire d'un rectangle est égale à **3** et son périmètre est égal à **7**. Quelle est la longueur d'une diagonale de ce rectangle ?

(b) (∗) In this problem O is the origin, A is the point $(3, 4)$ and B is a point in the first quadrant on the line joining O and A. If the length of AB is **6** what are the coordinates of B?

(∗) Soit O l'origine, soit A le point $(3, 4)$, et soit B un point dans le premier quadrant sur la droite qui passe par O et A. Si la longueur de AB est égale à **6**, quelles sont les coordonnées de B ?

4. (a) (∗) Solve the equation $\sqrt{3 - x} + \sqrt{12 - 4x} = \sqrt{x - 1}$.

(∗) Résous l'équation $\sqrt{3 - x} + \sqrt{12 - 4x} = \sqrt{x - 1}$.

(b) (∗) If p, q and r are the three roots of the equation $x^3 - 7x^2 + 3x + 1 = 0$, find the value of $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$.

(∗) Si p, q et r sont les trois racines de l'équation $x^3 - 7x^2 + 3x + 1 = 0$, trouve la valeur de $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$.

5. (a) (∗) If a and b are real numbers such that $\sqrt{a} - \sqrt{b} = \sqrt{2}$ and $a - b = 10$, find a and b.

(∗) Soient a et b des nombres réels tels que $\sqrt{a} - \sqrt{b} = \sqrt{2}$ et $a - b = 10$. Trouve les valeurs de a et b.

(b) (∗) If k is a real number such that $3\left(2^{k+3}\right) - 2^{2k} = 128$, what are the possible values of k?

(∗) Si k est un nombre réel tel que $3\left(2^{k+3}\right) - 2^{2k} = 128$, quelles sont les valeurs possibles de k ?

6. (a) In triangle ABC, $\angle BAC = 60°$, $\angle ACB = 90°$ and D is on BC. If AD bisects $\angle BAC$ prove that $DB = 2CD$.

Soit ABC un triangle tel que $\angle BAC = 60°$ et $\angle ACB = 90°$. Soit D un point sur le côté BC. Si AD bissecte $\angle BAC$, prouve que $DB = 2 \cdot CD$.

(b) In triangle ABC, $AC = BC = 5$ and $AB = 8$. What is the radius of the circle which passes through A, B, and C?

Soit ABC un triangle tel que $AC = BC = 5$ et $AB = 8$. Quel est le rayon du cercle qui passe par les points A, B, et C?

7. A, B and C are points on a circle of radius 3. In triangle ABC, $\angle ACB = 30°$ and $AC = 2$. Find BC.

Les points A, B et C se trouvent sur un cercle dont le rayon est égal à 3. Étant donné que $\angle ACB = 30°$ et $AC = 2$ dans le triangle ABC, trouve BC.

8. If x and y are real numbers, prove that $x^3y + xy^3 \leq x^4 + y^4$.

Soient x et y des nombres réels quelconques. Prouve que $x^3y + xy^3 \leq x^4 + y^4$.

9. A, B, C, and D are points on a circle. AB is the diameter. CD is perpendicular to AB and meets AB at E. If AB and CD are integers and $AE - EB = \sqrt{7}$, find AE.

Les points A, B, C, et D se trouvent sur un cercle. AB est un diamètre de ce cercle. CD est perpendiculaire à AB et croise AB au point E. Étant donné que AB et CD sont des nombres entiers et que $AE - EB = \sqrt{7}$, trouve AE.

10. Nine points, no three of which lie on the same straight line, are located inside an equilateral triangle of side 4. Prove that some three of these points are vertices of a triangle whose area is not greater than $\sqrt{3}$.

Neuf points, dont trois quelconques ne se trouvent pas sur la même droite, sont situés à l'intérieur d'un triangle équilatéral dont la longeur des côtés est égale à 4. Prouve que parmi ces points il y en a trois qui sont les sommets d'un triangle, l'aire duquel ne surpasse pas $\sqrt{3}$.

Next we give the solutions to the 2001 Canadian Open Mathematics Challenge.
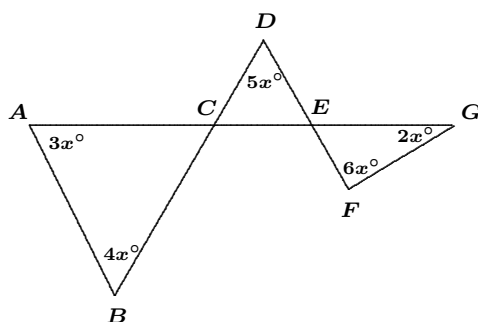
2001 Canadian Open Mathematics Challenge
PART/PARTIE A

1. On définit une opération "$\triangledown$" comme suit : $a\triangledown b = a^2 + 3^b$. Quelle est la valeur de $(2\triangledown 0) \triangledown (0\triangledown 1)$?

Solution par Robert Bilinski, Outremont, QC.
$(2\triangledown 0) \triangledown (0\triangledown 1) = (2^2 + 3^0) \triangledown (0^2 + 3^1) = 5\triangledown 3 = 5^2 + 3^3 = 52$.

2. Dans le diagramme ci-contre, quelle est la valeur de x?

Solution par Robert Bilinski, Outremont, QC.

Dans le triangle ABC, on a $\angle ACB = 180° - 3x° - 4x° = 180° - 7x°$. Ainsi $\angle DCE = \angle ACB = 180° - 7x°$ car ce sont des angles opposés par leurs sommets. De la même manière, en travaillant dans le triangle EFG, on obtient que $\angle DEC = \angle FEG = 180° - 8x°$. Puisque la somme des angles dans un triangle donne $180°$, on a dans le triangle CDE :
$180° - 7x° + 180° - 8x° + 5x° = 180°$ ou $x = 18°$.

3. Un hexagone régulier est un polygone à **6** côtés dont tous les angles sont congrus et tous les côtés sont congrus. Soient P et Q des points sur un hexagone régulier dont les côtés ont une longueur de **1**. Quelle est la longueur maximale possible du segment PQ ?

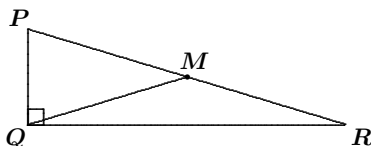*Solution par Robert Bilinski, Outremont, QC.*

Un hexagone régulier peut se diviser en **6** triangles équilatéraux, de même côté que l'hexagone. Ainsi la grande diagonale de l'hexagone est de longueur **2** (deux fois plus long qu'un côté de triangle, donc d'hexagone). Puisque les hexagones réguliers sont inscrits, la plus longue longueur entre deux points de l'hexagone sera le diamètre du cercle, soit une grande diagonale de l'hexagone. La réponse est donc **2**.

4. Résoudre l'équation suivante : $2\left(2^{2x}\right) = 4^x + 64$.

Solution par Robert Bilinski, Outremont, QC.

$$2\left(2^{2x}\right) = 4^x + 64 \quad \Longleftrightarrow \quad 2 \cdot 4^x - 4^x = 64$$
$$\Longleftrightarrow \quad 4^x = 4^3 \quad \Longleftrightarrow \quad x = 3$$

5. Le diagramme illustre un triangle rectangle PQR, dans lequel $PQ = 14$ et $QR = 48$. M est le milieu du côté PR. Déterminer le cosinus de l'angle MQP.

Solution par Robert Bilinski, Outremont, QC.

M étant le milieu de PR, donc l'hypoténuse du triangle rectangle PQR, on a M le centre du cercle circonscrit au triangle PQR. On a donc $MP = MQ = MR$ (des rayons de ce cercle). Donc le triangle MPQ est isocèle en M. Ainsi, on a $\angle MPQ = \angle MQP$. Donc

$$\cos \angle MQP = \cos \angle MPQ = \frac{PQ}{PR} = \frac{7}{25}\,.$$

6. On définit une suite de nombres, t_1, t_2, t_3, $\dots$, comme suit : $t_1 = 2$ et $t_{n+1} = \dfrac{t_n - 1}{t_n + 1}$, pour tout entier strictement positif n. Déterminer la valeur numérique de t_{999}.

Solution par Robert Bilinski, Outremont, QC.
On a

$$
\begin{aligned}
t_{n+1} &= \frac{t_n - 1}{t_n + 1} = 1 - \frac{2}{t_n + 1} = 1 - \frac{2}{\frac{t_{n-1}-1}{t_{n-1}+1} + 1} \\
&= 1 - \frac{2}{\frac{2 t_{n-1}}{t_{n-1}+1}} = \frac{-1}{t_{n-1}} = t_{n-3}
\end{aligned}
$$

Donc la suite $\{t_n\}$ est de période 4. Il suffit donc de connaître t_1, t_2, t_3, et t_4 pour connaître toute la suite.

On a $t_1 = 2 = t_{4k+1}$, pour $k \in \mathbb{N}$. On a $t_2 = \frac{1}{3} = t_{4k+2}$, pour $k \in \mathbb{N}$. On a $t_3 = -\frac{1}{2} = t_{4k+3}$, pour $k \in \mathbb{N}$. On a $t_4 = -3 = t_{4k+4}$, pour $k \in \mathbb{N}$. Ainsi, puisque $999 = 4 \cdot 249 + 3$, nous avons que $t_{999} = -\frac{1}{2}$.

7. Sachant que a peut prendre la valeur de n'importe quel entier strictement positif et que

$$
\begin{aligned}
2x + a &= y\,, \\
a + y &= x\,, \\
x + y &= z\,,
\end{aligned}
$$

déterminer la valeur maximale possible de l'expression $x + y + z$.

Solution par Robert Bilinski, Outremont, QC.

En prenant les deux premières équations et en réarrangeant les termes, on obtient le système de deux équations à **2** inconnues suivant (en considérant le a fixé)
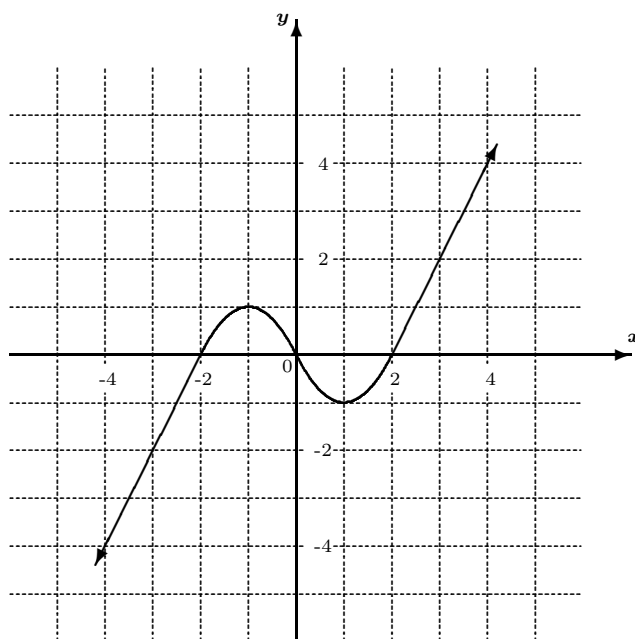
$$
\begin{aligned}
2x - y &= -a \\
-x + y &= -a
\end{aligned}
$$

qui se résout par

$$
\begin{aligned}
x &= -2a \\
y &= -3a
\end{aligned}
$$

8

Par la troisième équation, on obtient facilement que $z = -5a$. Ainsi, la somme $x + y + z = -10a$. Puisque $a > 0$ par hypothèse, on obtient que le maximum de la somme $x + y + z$ est atteint lorsque $a = 1$ et ce maximum est -10.

8. The graph of the function $y = g(x)$ is shown. Determine the number of solutions of the equation $\left||g(x)| - 1\right| = \frac{1}{2}$.

Official Solution.
From the original equation $\left||g(x)| - 1\right| = \frac{1}{2}$, using the definition of absolute value, we obtain,

$$|g(x)| - 1 = \tfrac{1}{2} \quad \text{or} \quad |g(x)| - 1 = -\tfrac{1}{2}$$

$$|g(x)| = \tfrac{3}{2} \quad \text{or} \quad |g(x)| = \tfrac{1}{2}$$

$$g(x) = \pm\tfrac{3}{2} \quad \text{or} \quad g(x) = \pm\tfrac{1}{2}$$

From the original graph $g(x) = \frac{3}{2}$ has **1** solution, $g(x) = -\frac{3}{2}$ has **1** solution, $g(x) = \frac{1}{2}$ has **3** solutions and $g(x) = -\frac{1}{2}$ has **3** solutions. Therefore, $\left||g(x)| - 1\right| = \frac{1}{2}$ has **8** solutions.
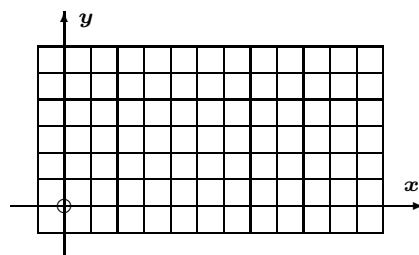
PART /PARTIE B

1. The triangular region T has its vertices determined by the intersections of the three lines: $x + 2y = 12$, $x = 2$, and $y = 1$.

(a) Determine the coordinates of the vertices of T, and show this region on the grid provided.

(b) The line $x + y = 8$ divides the triangular region T into a quadrilateral Q and a triangle R. Determine the coordinates of the vertices of the quadrilateral Q.

(c) Determine the area of the quadrilateral Q.

Solution by Robert Bilinski, Outremont, QC.

(a) To get the coordinates, we need to solve the three following systems of equations (one for each vertex):

$$
\begin{aligned}
x &= 2 & x &= 2 & x + 2y &= 12 \\
y &= 1 & x + 2y &= 12 & y &= 1
\end{aligned}
$$

which give (in order) the vertices $A(2, 1)$, $B(2, 5)$ and $C(10, 1)$. [*Editor's note:* This triangular region can be easily shown on the grid.]

(b) The new line crosses BC and AC, thus defining two new points with systems:

$$
\begin{aligned}
x + 2y &= 12 & y &= 1 \\
x + y &= 8 & x + y &= 8
\end{aligned}
$$

which define, respectively, $D(4, 4)$ on BC and $E(7, 1)$ on AC. Hence A, B, D and E are the vertices of Q.

(c) We can split Q into two triangles, namely ABD and ADE, since their areas are easy to evaluate. We have $[ABD] = \dfrac{(4)(2)}{2} = 4$ and $[ADE] = \dfrac{(5)(3)}{2} = 7.5$. Hence, Q has area 11.5.

2. (a) Alphonse and Beryl are playing a game, starting with a pack of **7** cards. Alphonse begins by discarding at least one but not more than half of the cards in the pack. He then passes the remaining cards in the pack to Beryl. Beryl continues the game by discarding at least one but not more than half of the remaining cards in the pack. The game continues in this way with the pack being passed back and forth between the two players. The loser is the player who, at the beginning of his or her turn, receives only one card. Show, with justification, that there is always a winning strategy for Beryl.

(b) Alphonse and Beryl now play a game with the same rules as in (a), except this time they start with a pack of **52** cards, and Alphonse goes first

again. As in (a), a player on his or her turn must discard at least one and not more than half of the remaining cards from the pack. Is there a strategy that Alphonse can use to be guaranteed that he will win? (Provide justification for your answer.)

Official Solution

(a) Alphonse starts with **7** cards, and so can remove **1**, **2**, or **3** cards, passing **6**, **5**, or **4** cards to Beryl. Beryl should remove **3**, **2**, or **1** cards, respectively, leaving **3** cards only, and pass these **3** cards back to Alphonse. Alphonse now is forced to remove **1** card only, and pass **2** back to Beryl. Beryl removes **1** card (her only option) and passes **1** back to Alphonse, who thus loses. Therefore, Beryl is guaranteed to win.

(b) Alphonse removes **21** cards from original **52**, and passes **31** cards to Beryl. If Beryl removes b_1 cards with $1 \leq b_1 \leq 15$, then Alphonse removes $16 - b_1$ cards to reduce the pack to **15** cards. [This is always a legal move, since $2(16 - b_1) = 32 - 2b_1 \leq 31 - b_1$ so $16 - b_1$ is never more than half of the pack.] If Beryl removes b_2 cards with $1 \leq b_2 \leq 7$, then Alphonse removes $8 - b_2$ to reduce the pack to **7** cards. [This move is always legal, by a similar argument.] Since Beryl now has **7** cards, Alphonse can adopt Beryl's strategy from (a). Thus, Alphonse has a winning strategy.

3. (a) If $f(x) = x^2 + 6x + c$, where c is an integer, prove that $f(0) + f(-1)$ is odd.

(b) Let $g(x) = x^3 + px^2 + qx + r$, where p, q and r are integers. Prove that if $g(0)$ and $g(-1)$ are both odd, then the equation $g(x) = 0$ cannot have three integer roots.

Solution by Robert Bilinski, Outremont, QC.

(a) $f(0) + f(-1) = c + 1 - 6 + c = 2c - 5 = (2c - 6) + 1 = 2(c - 3) + 1$, which is always odd.

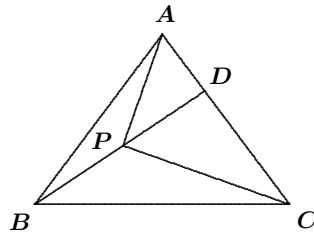(b) $g(0) = r$ is odd. Thus, $r = 2s + 1$ for some s. Therefore,

$$g(-1) = -1 + p - q + r = -1 + p - q + 2s + 1 = p - q + 2s.$$

Since $g(-1)$ is odd, p and q cannot be of the same parity.

Let us suppose that $g(x)$ has three integer roots, say h, i and j. Then we have $g(x) = (x - h)(x - i)(x - j) = x^3 - (h + i + j)x^2 + (hi + hj + ij)x + hij$. We see that $r = -hij$ is odd, Thus, all three of h, i and j must be odd, and $h + i + j$ is odd. But $p = -(h + i + j)$, which implies p is also odd. We also have hi, hj and ij odd. Therefore, $(hi + hj + ij) = q$ is also odd.
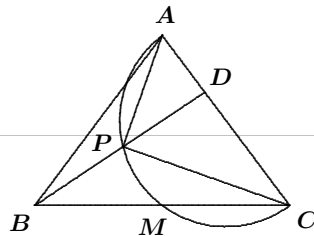
But we saw before that p and q cannot have the same parity. Hence, there is a contradiction, and it is impossible for all three roots of $g(x)$ to be integers.

4. Triangle ABC is isosceles with $AB = AC = 5$ and $BC = 6$. Point D lies on AC, and P is the point on BD so that $\angle APC = 90°$. If $\angle ABP = \angle BCP$, determine the ratio $AD : DC$.

Official Solution

Draw a perpendicular from A to meet BC at M. Then, since $AB = AC$, $BM = MC = 3$ and so $AM = 4$. Let $\alpha = \angle BCP = \angle ABP$ and $\theta = \angle ACP$. Then $\angle PBC = \theta$, since $\triangle ABC$ is isosceles. Draw a circle with AC as diameter. This circle passes through both P and M since $\angle APC = \angle AMC = 90°$. Join P to M.

Then $\angle PAM = \alpha$ since $\angle PAM = \angle PCM$ (subtended by the same chord). Also $\angle AMP = \theta$ for similar reasons. Therefore, $\triangle MPA$ is similar to $\triangle BPC$. Thus, $\frac{PA}{PC} = \frac{MA}{BC} = \frac{4}{6}$, whence $\tan \theta = \frac{PA}{PC} = \frac{2}{3}$.

Now we compute the length of DC. Consider $\triangle BDC$. By the Sine Law,

$$\frac{DC}{\sin\theta} = \frac{BC}{\sin(\angle BDC)}$$

$$DC = \frac{6\sin\theta}{\sin(180° - \theta - \angle DCB)} = \frac{6\sin\theta}{\sin(\theta + \angle DCB)}$$

$$= \frac{6\sin\theta}{\sin\theta\cos(\angle DCB) + \cos\theta\sin(\angle DCB)}$$

$$= \frac{6}{\cos(\angle DCB) + \cot\theta\sin(\angle DCB)} = \frac{6}{\frac{3}{5} + \frac{3}{2}\cdot\frac{4}{5}} = \frac{10}{3}.$$

Then $AD = 5 - \frac{10}{3} = \frac{5}{3}$, and thus $AD : DC = 1 : 2$.

That ends another Skoliad Corner. Send us your problems and solutions. Remember, we have past volumes of MAYHEM for pre-university solvers.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of *Crux Mathematicorum with Mathematical Mayhem*.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z7**. The electronic address is

mayhem-editors@cms.math.ca

The Mayhem Editor is Shawn Godin (Cairine Wilson Secondary School). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Paul Ottaway (Dalhousie University) and Larry Rice (University of Waterloo).

Mayhem Editorial

Shawn Godin

Welcome to another year with MAYHEM! There are a few things that we should mention at the start of the year.

First, it is with sadness that I bid farewell to CHRIS CAPPADOCIA, my assistant editor for the last two years. Chris has been invaluable to me as I took over the helm from NAOKI SATO. Unfortunately, the demands of school (Chris is an undergraduate at the University of Waterloo) are becoming too great for him to keep helping me. I wish Chris all the best. He is an exceptional young man with great potential, and it has been my pleasure to have him as a student, assistant and friend. We will miss you, Chris.

Next, we have two welcomes. First to my new Editor-in-Chief, JIM TOTTEN. Jim has been involved with Crux longer than I have, and checking my back issues, I see that he was involved every year that I have been a subscriber (this will be my 10th year). I first met Jim yesterday (as it would be), and I feel confident that *CRUX with MAYHEM* is in good hands. I look forward to working with Jim over the next x years. By the way, Bruce always allowed me to hand in my material several days late ...

One more welcome, to another name familiar to Crux: JOHN GRANT McLOUGHLIN has agreed to take over from Chris as my assistant editor. John will take over the problem proposals, and we will be searching for some new problem editors to help with selecting solutions. John has been involved

with many activities that will be valuable to us and to you as a reader. He has taught at a number of faculties of education, written a problem column in the *Ontario Mathematics Gazette* a few years ago, is working with the Canadian Mathematics Competitions and other local mathematics competitions as a problem setter, and has numerous other related experiences. I feel John will make a very positive contribution to Mayhem, and I think the two of us will make a good team.

A few minor changes are being made. In SKOLIAD, we will now ask for solutions to only some of the problems. You may send solutions to the other problems, but we will print a solution only if we receive one from a "junior" student (grade **10** and equivalent or earlier), a unique solution, or a nice generalization. We will continue to give out past copies of MAYHEM to some pre-university solvers.

The MAYHEM TAUNT is officially over. A couple of solutions to these problems appear in this issue. Over the rest of the volume the other TAUNT solutions will appear. Also, starting in the next issue, we will publish the names of the winners of the prizes for each section.

We are hoping to continue giving some sort of prizes to our pre-university solvers, but we are still brainstorming as to what form they will take. Details will follow in later issues. Keep sending us your solutions.

I hope you enjoy your **2003** volume of Mayhem. Keep sending us your problems, solutions, comments and articles. We are *your* journal, and its success is a function of the involvement of you, the readers.

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Je vous souhaite la bienvenue en cette nouvelle année du MAYHEM ! Je profite de l'occasion pour vous transmettre quelques bribes d'information.

En premier lieu, c'est avec tristesse que je fais mes adieux à CHRIS CAPPADOCIA, mon rédacteur adjoint des deux dernières années. Chris m'a été d'un grand soutien à partir du moment où j'ai pris la relève de NAOKI SATO. Malheureusement, une charge d'étude devenue trop lourde (Chris est un étudiant de premier cycle à l'Université de Waterloo) l'empêche de continuer à m'aider. Je lui souhaite une bonne continuation. Chris est un jeune homme exceptionnel qui a beaucoup de potentiel, et je suis enchanté de l'avoir eu comme étudiant, comme assistant et comme ami. Tu nous manqueras !

En second lieu, c'est avec joie que nous accueillons deux nouvelles recrues. D'abord notre nouveau rédacteur en chef, JIM TOTTEN. Jim participe aux activités du Crux depuis plus longtemps que moi, et il m'a suffi de consulter mes anciens numéros pour constater qu'il a contribué à la revue au moins une fois par année depuis que je suis abonné (et j'entreprends ma 10e année). J'ai rencontré Jim pour la première fois tout récemment, et je sais que le *CRUX with MAYHEM* est entre bonnes mains avec lui. Je serai heureux de travailler avec Jim au cours des prochaines années. À propos,

Bruce me permettait toujours de remettre mes textes avec plusieurs jours de retard...

Nous souhaitons également la bienvenue à une autre personne qui n'est pas étrangère au Crux. JOHN GRANT McLOUGHLIN a accepté de remplacer Chris comme rédacteur adjoint du Mayhem. John reprendra le dossier des propositions de problèmes, et nous chercherons de nouveaux rédacteurs de problèmes qui nous aideront à choisir des solutions. La feuille de route de John sera un atout pour notre équipe, ainsi que pour vous, en tant que lecteurs. John a enseigné dans plusieurs facultés d'éducation, il a signé une chronique de résolution de problèmes dans l'*Ontario Mathematics Gazette* il y a quelques années, il participe à la rédaction des problèmes du Concours canadien de mathématiques et d'autres concours locaux, et il a bien d'autres cordes à son arc. Je crois que John sera un excellent atout pour le Mayhem et que nous formerons une bonne équipe.

Signalons également quelques changements. Dans la chronique « SKOLIAD », nous ne solliciterons plus des solutions à tous les problèmes. Vous pourrez tout de même résoudre les autres problémes, mais nous ne publierons que les solutions venant d'un « jeune » élève (équivalent de la 10e année ou moins), les solutions uniques ou les généralisations élégantes. Nous conserverons notre habitude de remettre d'anciens numéros du MAYHEM aux élèves de niveau préuniversitaire qui nous font parvenir de bonnes solutions.

C'est maintenant officiel, nous ne relancerons plus le DÉFI MAYHEM. Nous publions quelques solutions ce mois-ci, et d'autres paraîtront au cours de l'année. À partir du prochain numéro, nous publierons le nom des gagnants dans chaque section.

Nous espérons être en mesure de continuer à offrir des prix à nos élèves de niveau préuniversitaire, mais nous n'avons pas encore déterminé quelle en sera la nature. Nous vous tiendrons au courant dans les mois à venir. Entre-temps, continuez à nous envoyer vos solutions !

J'espère que vous apprécierez le volume **2003** du Mayhem. Nous attendons vos problèmes, solutions, commentaires et articles avec impatience. Le Mayhem est **votre** revue, et une partie de son succès dépend de **votre** participation.

Mayhem Problems

Proposals and solutions may be sent to **Mathematical Mayhem, 2191 Saturn Crescent, Orleans, Ontario, K4A 3T6** or emailed to
mayhem-editors@cms.math.ca
Please include in all correspondence your name, school, grade, city, province or state, and country. We are especially looking for solutions from high school students.

Please send your solutions to the problems in this edition by *1 August 2003*. Solutions received after this time will be considered only if there is time before publication of the solutions.

M76. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Two buildings A and B are twenty feet apart. A ladder thirty feet long has its lower end at the base of building A and its upper end against building B. Another ladder forty feet long has its lower end at the base of building B and its upper end against building A.

How high above the ground is the point where the ladders intersect?

. . . .

Deux bâtiments A et B sont distants de vingt pieds. Une échelle, longue de trente pieds, a son extrémité inférieure à la base de A et est appuyée contre le bâtiment B. Une autre échelle, de longueur de quarante pieds celle-là, a son extrémité inférieure à la base de B et est appuyée contre le bâtiment A.

À quelle hauteur au-dessus du sol se trouve le point d'intersection des deux échelles ?

M77. *Proposed by Richard Hoshino, Dalhousie University, Halifax, NS.*

Find *all* ordered pairs of integers (a, b) such that the equation $x^2 + |y^2 - 6ay + b| = b - a^2 + 6$ has *exactly* 2001 solutions in positive integers (x, y).

. . . .

Trouver *toutes* les paires ordonnées d'entiers (a, b) telles que l' équation $x^2 + |y^2 - 6ay + b| = b - a^2 + 6$ possède *exactement* 2001 solutions en entiers positifs (x, y).

M78. *Proposed by K. R. S. Sastry, Bangalore, India.*

In a right-angled triangle we consider the two vertices at the two acute angles and draw medians from them to the opposite sides. Determine the maximum (acute) angle between these medians.

. . . .

Dans un triangle rectangle, on considère les deux sommets d'angles aigus, et les médianes abaissées sur le côté opposé. Déterminer l'angle (aigu) maximal entre ces médianes.

M79. *Proposed by the Mayhem Staff.*

Three people play the following game. N marbles are placed in a bowl and the players, in turn, remove 1, 2, or 3 marbles from the bowl. The person who removes the last marble loses. For what values of N can the first and

third player work together to force the second player to lose? (Inspired by a recent problem on the Canadian Open Mathematics Challenge.)

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Trois personnes jouent le jeu suivant. A tour de rôle, chaque joueur retire **1**, **2**, ou **3** billes d'une urne qui en contient N. La personne qui retire la dernière bille a perdu. Pour quelles valeurs de N le premier et le troisième joueur peuvent-ils collaborer pour forcer le second joueur à perdre ? (Inspiré par un récent problème du Défi Ouvert Canadien de Mathématiques.)

M80. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Compute the number of ways that 4 tires can be rotated so that each tire is relocated. (*Editor's note:* "rotating" a car's tires means changing their position on the car so that they can wear more evenly.)

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Trouver le nombre de rotations qu'on peut effectuer sur les 4 pneus d'une voiture pour qu'ils se trouvent dans une autre position. (*Note de l'éditeur :* "rotation" signifie ici : changement de position pour assurer une usure uniforme des pneus.)

M81. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let $a \neq 0$, b, c be integers and $\sin\theta$, $\cos\theta$ be the rational roots of the equation $ax^2 + bx + c = 0$. Show that $a \pm 2c$ are perfect squares.

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Soit $a \neq 0$, b, c des entiers et $\sin\theta$, $\cos\theta$ les racines rationnelles de l'équation $ax^2 + bx + c = 0$. Montrer que $a \pm 2c$ sont des carrés parfaits.
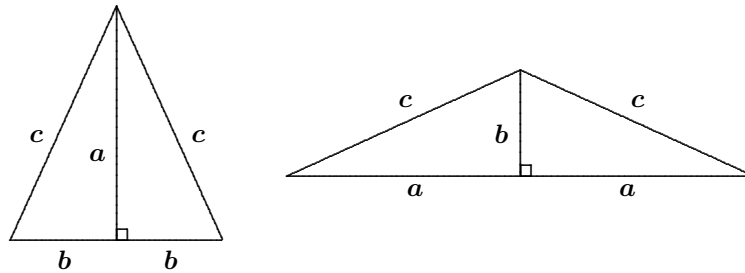
Mayhem Solutions

M26. *Proposed by the Mayhem staff.*

Find two isosceles triangles, with two sides **106** units long and the other side an integer, that have the same area.

Solution by Geneviève Lalonde, Massey, ON.

Two different isosceles triangles can be created that have the same length for their double side and the same area by creating them from two scalene right-angled triangles as in the diagram below, where (a, b, c) is a Pythagorean triple.

For our problem we need a Pythagorean triple $(a, b, 106)$. If we can find two positive integers x and y with $x > y$, such that $x^2 + y^2 = 106$, then we can let $a = x^2 - y^2$ and $b = 2xy$. Since $106 = 9^2 + 5^2$, we have $a = 9^2 - 5^2 = 56$ and $b = 2 \cdot 9 \cdot 5 = 90$. Thus, the triangles with sides $106, 106, 112$ and $106, 106, 180$ both have area $ab = 5040$.

M27. *Proposed by the Mayhem staff.*

Find $\sqrt{ab + 1}$ where $a = \overbrace{111 \cdots 11}^{2002 \text{ 1's}}$ and $b = 1\overbrace{00 \cdots 00}^{2001 \text{ 0's}}5$.

Solution by Mihály Bencze, Brasov, Romania.

In general, if $a = \overbrace{111 \cdots 11}^{n \text{ 1's}} = \dfrac{10^n - 1}{9}$ and $b = 1\overbrace{00 \cdots 00}^{n-1 \text{ 0's}}5 = 10^n + 5$, then

$$\sqrt{ab + 1} = \sqrt{\left(\frac{10^n + 2}{3}\right)^2} = \frac{10^n + 2}{3}.$$

Thus, for this problem the result is $\dfrac{10^{2002} + 2}{3} = \overbrace{333 \cdots 3}^{2001 \text{ 3's}}4$.

Also solved by Gustavo Krimker, Universidad CAECE, Argentina. One incomplete solution was received.

M28. *Proposed by the Mayhem staff.*

Shawn tosses **2001** fair coins and Bruce tosses **2002** fair coins. What is the probability that Bruce gets more heads than Shawn?

Solution by Geneviève Lalonde, Massey, ON.

To start out, let Shawn toss his **2001** coins and Bruce toss **2001** of his. By symmetry $P(\text{Bruce more heads}) = P(\text{Shawn more heads}) = \frac{1-p}{2}$ where p is the probability that they have the same number of heads. If Bruce has more heads, he has already won and the last toss is immaterial. If they have the same number of heads, then Bruce can get more by flipping heads, but if Shawn already has more heads, Bruce cannot get more with his last toss. Thus, the probability that Bruce will have more heads after his **2002**[nd] toss is

$$\frac{1-p}{2} + \frac{1}{2}p = \frac{1}{2}.$$

Also solved by José L. Díaz–Barrero and Juan J. Egozcue, UPC, Barcelona, Spain.

M29. *Proposed by the Mayhem staff.*

Define the "silly product" of two numbers as the sum of the product of all the corresponding digits. So $235 \times_s 718 = 2 \times 7 + 3 \times 1 + 5 \times 8 = 57$. Find two numbers A and B so that $A \times_s B = 2002$ and $A + B$ is a minimum.

Solution by Antonio Lei, year 12, *Colchester Royal Grammar School, Colchester, UK.*

A and B should have the same number of digits. Otherwise, some digits would be multiplied by zero which has no contribution to the "silly product". It contradicts the condition that $A + B$ is minimum.

Let $A = a_n a_{n-1} \cdots a_1$, and $B = b_n b_{n-1} \cdots b_1$ where the a_i and b_i are the digits of A and B. Then

$$A \times_s B = a_n b_n + a_{n-1} b_{n-1} + \cdots + a_1 b_1 \leq 9 \times 9 + 9 \times 9 + \cdots + 9 \times 9,$$

whence $2002 \leq 81n$. Thus, $n \geq \frac{2002}{81}$, but since n is an integer we must have $n \geq 25$.

In order to keep $A + B$ minimum, we want the least number of digits. Therefore, $a_i b_i$ must be as great as possible. But $a_i b_i$ is at most 81 when $a_i = b_i = 9$, and $25 \times 81 = 2025 > 2002$. Hence, only the least significant 24 digits can be 9. Hence, there remains $2002 - 24 \times 81 = 58 = 2 \times 29$. Since 29 is prime, we cannot make 58 with the silly product of two one-digit numbers. Thus, we need two more digits for our number, say a_n, a_{n+1} for A and b_n, b_{n+1} for B. Hence, $a_{n+1} b_{n+1} + a_n b_n = 58$ and a_{n+1} and b_{n+1} should be kept as small as possible. The smallest occurs for $1 \times 2 + 7 \times 8 = 58$. Thus, the two possibilities for A and B that keep the sum $A + B$ a minimum are $1899 \ldots 9$, $2799 \ldots 9$ and $1799 \ldots 9$, $2899 \ldots 9$ (there are twenty-four 9's in each number).

Also solved by Robert Bilinski, Outremont, PQ; Jack Gu, grade 11, *Rachel Li, grade* 12, *Alvin Miao, grade* 10, *Molly Yan, grade* 11, *and Corey Zhou, grade* 12, *Dalian Maple Leaf International School, Dalian, China. Three incorrect solutions were received.*

M30. *Proposed by Haralampy Steryion, Chalkis, Greece.*

Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ with the property

$$f(x + y) = f(x)e^{f(y)-1} \qquad \text{for every } x, y \in \mathbb{R}.$$

Solution by Shien Jin Ong, MIT, USA.
Claim: $f(x) = 1$ or $f(x) = 0$ for all $x \in \mathbb{R}$.

Note that the function $f(x) = 0$ for all $x \in \mathbb{R}$ is a solution. Now assume that $f(x) \neq 0$ for some $x \in \mathbb{R}$, say for $x = x_1$. Substitute $y = 0$ and $x = x_1$ into the equation. We conclude that $f(0) = 1$. Next, substitute $x = 0$ into the same equation. We get $f(y) = e^{f(y)-1}$ for all $y \in \mathbb{R}$. Let $g(t) = t - e^{t-1}$. Note that $g'(t) < 0$ if $t > 1$, $g'(t) > 0$ if $t < 1$, and $g(1) = 0$. Hence, $g(t)$ has only one root, at $t = 1$. This means that the only solution to $f(y) = e^{f(y)-1}$ is $f(y) = 1$ for all $y \in \mathbb{R}$. A simple check verifies that the solution indeed fits into the given equation.

Also solved by Jack Gu, grade 11, *Rachel Li, grade* 12, *and Corey Zhou, grade* 12, *Dalian Maple Leaf International School, Dalian, China. One incorrect solution was received.*

Pólya's Paragon

Paul Ottaway

In this installment of Pólya's Paragon, we will examine the *Extreme Principle*. Here is the basic premise:

If possible, assume that the elements of your problem are "in order". Focus on the "largest" and "smallest" elements, as they may be constrained in interesting ways.

Using this simple idea, we will be able to provide elegant solutions to difficult problems. Let's illustrate this with several examples.

Problem 1: On the plane, we colour a *finite* number of points either black or white. We choose the points and their colours so that every line segment which joins two points of the same colour contains a point of the other colour. Prove that all the points must lie on a single line segment.

Solution: Suppose that the points do not all lie on a single line segment. Then there must exist at least one set of three points that form a triangle (that is, these three points are not all on the same line). Of all such triangles that can be formed, consider the triangle ABC of *smallest* area. We will obtain a contradiction by finding a triangle whose area is smaller than the area of $\triangle ABC$.

Each of the points A, B, and C are coloured black or white. So at least two of the points must be coloured the same. Without loss of generality, suppose that B and C are both coloured white. Then, there must be a black point D somewhere between B and C. Then $\triangle ABD$ is a triangle whose area is strictly smaller than the area of $\triangle ABC$, which contradicts the fact that $\triangle ABC$ was the triangle of smallest area.

Since we have a contradiction, we conclude that all the points must lie on a single line segment.

Problem 2: Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that
$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$
for all x and y in $\mathbb{N}$.

Solution: This problem appeared as the last question of the 2002 Canadian Mathematical Olympiad. The proposer's solution was purely number-theoretic and spanned several pages. It was fully expected that very few people would solve this problem, and that the only correct solutions would be similar to the highly technical approach obtained by the proposer.

To the CMO committee's surprise (and delight!), the following solution was submitted by David Han from Woburn Collegiate Institute, who was the winner of the 2002 CMO Contest.

We claim that f is a constant function. Suppose, for a contradiction, that there exist x and y with $f(x) < f(y)$. Choose x and y such that $f(y) - f(x) = d > 0$ is minimal. Then,

$$f(x) \; = \; \frac{xf(x) + yf(x)}{x + y} \; < \; \frac{xf(y) + yf(x)}{x + y} \; < \; \frac{xf(y) + yf(y)}{x + y} \; = \; f(y) \, .$$

Letting $z = x^2 + y^2$, we have shown that $f(x) < f(z) < f(y)$. Hence, the integers x and z satisfy $0 < f(z) - f(x) < d$, which contradicts the minimality of d. Therefore, no such x and y exist, and so we conclude that f must be a constant function. [*Editor's Note:* The above argument requires $x \neq 0$ and $y \neq 0$. To complete the proof, we need only show that $f(0) = f(x)$ for some non-zero $x \in \mathbb{N}$. Now, select $y = 0$ and $x = 1$ in the given functional equation, and we are done.]

We quickly see that for all $c \in \mathbb{N}$, the function $f(x) = c$ satisfies the given functional equation. Therefore, we have solved the problem.

Problem 3: Prove that the equation $x^4 + y^4 = z^4$ has no solutions in positive integers x, y, z.

Solution: This is the most famous application of the Extreme Principle. This is the $n = 4$ case of Fermat's Last Theorem.
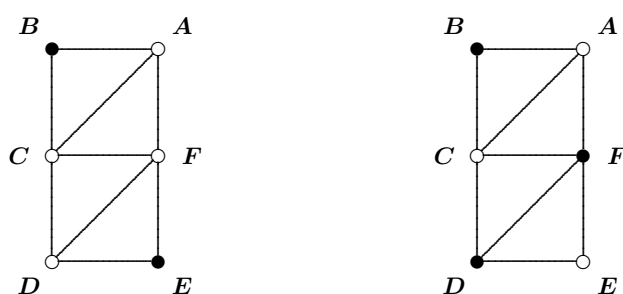
We outline the proof here, and we invite you to fill in the details!

1. Define (a, b, c) to be a *special triple* if a, b, c are positive integers for which $a^4 + b^4 = c^2$.

2. Suppose that a special triple (a, b, c) exists. Consider the *smallest* one; that is, one where c is minimized.

3. We have $a^4 + b^4 = c^2$, so (a^2, b^2, c) is a Pythagorean Triple. Explain why there must exist positive integers p and q for which $a^2 = p^2 - q^2$, $b^2 = 2pq$, and $c = p^2 + q^2$.

4. Show that there exist positive integers d and e for which $a = d^2 - e^2$, $q = 2de$, and $p = d^2 + e^2$.

5. Explain why d, e, and $d^2 + e^2$ are all perfect squares. Conclude that there must exist a special triple (r, s, t) with $t < c$, which gives you the desired contradiction.

6. Explain why no special triples (r, s, t) exist, and thus conclude that the equation $x^4 + y^4 = z^4$ has no solutions in positive integers x, y, z.

We conclude this article by providing some more questions where the Extreme Principle may be used.

1. Imagine an infinite chessboard that contains a positive integer in each square. If the value in each square is equal to the average of its four neighbours to the north, south, west, and east, prove that the values in all the squares are equal.

2. Consider a graph with finitely many points, some of which are joined to one another by lines. We shall colour each point either black or white, and call the graph "integrated" if each white point has at least as many black as white neighbours, and vice versa. The example below shows two different colourings of the same graph. The one on the left is not integrated, because point A has two white neighbours (C and F), and only one black neighbour (B). The graph on the right is integrated.

Given *any* graph, can we colour the points so that the graph is integrated?

3. Prove that the equation $x^2 + y^2 = 3z^2$ has no solutions in positive integers (x, y, z).

4. On a large flat field, n people are positioned so that for each person the distances to all the other people are different. Each person holds a water pistol and, at a given signal, fires and hits the person who is closest. When n is odd, show that there is at least one person left dry. Is this always true when n is even?
   (1987 CMO, Question 4.)

5. Consider finitely many points in the plane such that, if we choose any three points A, B, C among them, the area of triangle ABC is always less than 1. Prove that all of these points lie within the interior or on the boundary of a triangle with area less than 4.
   (1995 Korean Mathematical Olympiad.)

THE OLYMPIAD CORNER
No. 227

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, AB, Canada. T2N 1N4.

Thank you to our contributors and to Joanne Longworth, who so capably changes my scrawl into readable type.

Mohammed Aassila	Murray S. Klamkin
Miguel Amengual Covas	Raul A. Simon Lamb
Jean Claude Andrieux	Elena Lee
Ed Barbeau	Hojoo Lee
Michel Bataille	Robert Barrington Leigh
Robert Bilinski	Brett Link
Pierre Bornsztein	Andy Liu
Christopher J. Bradley	David Loeffler
Bob Cao	Pavlos Maragoudakis
Babis Stergiou Chalkida	Greg Martin
Adrian Chan	Stewart Metchette
Peter de Caux	Vedula N. Murty
Peter Du	Moubinool Omarjee
George Evagelopoulos	Toshio Seimiya
Dora Ho	Achilleas Sinefakopoulos
Oleg Ivrii	D.J. Smeenk
Geoffrey A. Kandall	Panos E. Tsaoussoglou
April Keller	Edward T.H. Wang

To start off the new year we give the problems of the 2000 Korean Mathematical Olympiad. Thanks go to Andy Liu, Canadian Team Leader to the IMO in Korea, for collecting them.

2000 KOREAN MATHEMATICAL OLYMPIAD

1. Prove that for any prime p, there exist integers x, y, z, and w such that $x^2 + y^2 + z^2 - wp = 0$ and $0 < w < p$.

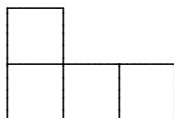**2**. Determine all functions f from the set of real numbers to itself such that for every x and y,

$$f(x^2 - y^2) = (x - y)\big(f(x) + f(y)\big).$$

3. A rectangle $ABCD$ is inscribed in a circle with centre O. The exterior bisectors of $\angle ABD$ and $\angle ADB$ intersect at P; those of $\angle DAB$ and $\angle DBA$ intersect at Q; those of $\angle ACD$ and $\angle ADC$ intersect at R; and those of $\angle DAC$ and $\angle DCA$ intersect at S. Prove that P, Q, R, and S are concyclic.

4. Let p be a prime such that $p \equiv 1 \pmod 4$. Evaluate

$$\sum_{k=1}^{p-1} \left(\left\lfloor \frac{2k^2}{p} \right\rfloor \right) - 2 \left(\left\lfloor \frac{k^2}{p} \right\rfloor \right).$$

5. Prove that an $m \times n$ rectangle can be constructed using copies of the following shape if and only if mn is a multiple of 8.

6. The real numbers a, b, c, x, y, and z are such that $a > b > c > 0$ and $x > y > z > 0$. Prove that

$$\frac{a^2 x^2}{(by + cz)(bz + cy)} + \frac{b^2 y^2}{(cz + ax)(cx + az)} + \frac{c^2 z^2}{(ax + by)(ay + bx)} \geq \frac{3}{4}.$$

As a second set we give the problems of the 2000 Bulgarian Mathematical Olympiad. Thanks again go to Andy Liu, Canadian Team Leader to the IMO in Korea, for collecting them.

2000 BULGARIAN MATHEMATICAL OLYMPIAD

1. In the Cartesian plane, a set of 2000 points $M_i(x_i, y_i)$ is said to be *good* if $0 \leq x_i \leq 83$ and $0 \leq y_i \leq 1$ for $i = 1, 2, \ldots, 2000$ and $x_i \neq x_j$ for $i \neq j$. Find all positive integers n such that for any good set, some n of its points lie in a square of side length 1, but there exists a good set such that no $n + 1$ of its points lie in a square of side length 1. (A point on a side of a square lies in the square).

2. Let ABC be an acute triangle.

(a) Prove that there exist unique points A', B', and C', on BC, CA, and AB, respectively, such that A' is the mid-point of the orthogonal projection of $B'C'$ onto BC, B' is the mid-point of the orthogonal projection of $C'A'$ onto CA, and C' is the mid-point of the orthogonal projection of $A'B'$ onto AB.

(b) Prove that $A'B'C'$ is similar to the triangle formed by the medians of ABC.

3. Let p be an odd prime and a_1, a_2, ..., a_{p-2} be a sequence of positive integers such that for all $k = 2, 3, \ldots, p - 2$, the prime p does not divide both a_k and a_{k-1}. Prove that the product of some elements of this sequence is congruent to 2 modulo p.

4. Find all polynomials $P(x)$ with real coefficients such that we have $P(x)P(x + 1) = P(x^2)$ for all real x.

5. In triangle ABC, we have $CA = CB$. Let D be the mid-point of AB and E an arbitrary point on AB. Let O be the circumcentre of $\triangle ACE$. Prove that the line through D perpendicular to DO, the line through E perpendicular to BC, and the line through B parallel to AC are concurrent.

6. Let $\mathcal{A}$ be the set of all binary sequences of length n, and let $0 \in \mathcal{A}$ be the sequence all terms of which are zeroes. The sequence $c = \langle c_1, c_2, \ldots, c_n \rangle$ is called the sum of $a = \langle a_1, a_2, \ldots, a_n \rangle$ and $b = \langle b_1, b_2, \ldots, b_n \rangle$ if $c_i = 0$ when $a_i = b_i$ and $c_i = 1$ when $a_i \neq b_i$. Let $f : \mathcal{A} \to \mathcal{A}$ be a function such that $f(0) = 0$ and if the sequences a and b differ in exactly k terms then the sequences $f(a)$ and $f(b)$ differ also exactly in k terms. Prove that if a, b, and c are sequences from $\mathcal{A}$ such that $a+b+c = 0$, then $f(a)+f(b)+f(c) = 0$.

As a final set of puzzles for this number we give the problems of the 2000 Vietnamese Mathematical Olympiad. Again, thanks go to Andy Liu, Canadian Team Leader to the IMO in Korea, for collecting them.

2000 VIETNAMESE MATHEMATICAL OLYMPIAD

1. Given a real number $c > 0$ and an initial value x_0 where $0 < x_0 < c$, a sequence $\{x_n\}$ of real numbers is defined by $x_{n+1} = \sqrt{c - \sqrt{c + x_n}}$ for $n \geq 0$. Find all positive real numbers c such that for each initial value x_0 in $(0, c)$, the sequence $\{x_n\}$ is defined for all n and has a finite limit.

2. Two circles Ω_1 and Ω_2 with respective centres O_1 and O_2 are given on the plane. Let M_1 and M_2 be two points on Ω_1 and Ω_2, respectively, such that the lines O_1M_1 and O_2M_2 intersect at Q. Starting simultaneously from these positions, the points M_1 and M_2 move clockwise, each on its own circle, with the same angular velocity.

(a) Determine the locus of the mid-point of M_1M_2.

(b) Prove that the circumcircle of the triangle M_1QM_2 passes through a fixed point.

3. Consider the polynomial $P(x) = x^3 + 153x^2 - 111x + 38$.

(a) Prove that the closed interval $[1, 3^{2000}]$ contains at least 9 integers a for which $P(a)$ is divisible by 3^{2000}.

(b) Determine the number of integers a in the closed interval $[1, 3^{2000}]$ for which $P(a)$ is divisible by 3^{2000}.

4. For every integer $n \geq 3$ and any given angle α in $(0, \pi)$, let $P_n(x) = x^n \sin \alpha - x \sin n\alpha + \sin(n-1)\alpha$.

(a) Prove that there is only one polynomial of the form $f(x) = x^2 + ax + b$ such that for every $n \geq 3$, $P_n(x)$ is divisible by $f(x)$.

(b) Prove that there does not exist a polynomial $g(x)$ of the form $g(x) = x+c$ such that for every $n \geq 3$, $P_n(x)$ is divisible by $g(x)$.

5. Determine all integers $n \geq 3$ such that there exists n points A_1, A_2, ..., A_n in space, with no three on a line and no four on a circle, such that all the circumcircles of the triangles $A_i A_j A_k$ are congruent.

6. Let $P(x)$ be a non-zero polynomial such that, for all real numbers x, we have $P(x^2 - 1) = P(x)P(-x)$. Determine the maximal number of roots of $P(x)$.

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Next we have some tidying up and apologies to make.

First, my apologies to Robert Bilinski, Outremont, QC, who sent in a batch of problems, only one of which was cited (and the other two misfiled with that month's "solved problems"). He should have been listed as a solver of problem #2 of the 47th Latvian Mathematical Olympiad [2000 : 324; 2002 : 425], as well as one of the solvers of Problem #1 of the XXIII All Russian Olympiad [2000 : 388–389; 2002 : 493–494].

Continuing in this vein, I have also received two solutions from Pavlos Maragoudakis, Pireas, Greece which were not cited last year. He sent in a solution to problem #3 of the 47th Latvian Mathematical Olympiad [2000 : 324; 2002 : 425–426], as well as a different solution to problem #1 of the XXIII All Russian Olympiad [2000 : 388–389; 2002 : 493–494]. We give his alternate solution here.

1. Solve, in integers, the equation

$$(x^2 - y^2)^2 = 1 + 16y.$$

Alternate solution by Pavlos Maragoudakis, Pireas, Greece.

If (x, y) is a solution then $1 + 16y \geq 0$, so $y \geq 0$. Also, if (x, y) is solution, $(-x, y)$ is also a solution. We find the solutions (x, y) with $x, y \geq 0$.

- If $x = 0$, then $y^4 - 16y - 1 = 0$, which has no integer solutions.

- If $y = 0$, then $x = 1$ or $x = -1$.
- If $x, y \geq 1$, then

$$2(x - y)^2 \leq (x + y)(x - y)^2 = \frac{1 + 16y}{x + y} < \frac{16x + 16y}{x + y} = 16\,,$$

so $(x - y)^2 < 8 \implies |x - y| < 3$.

Since $1 + 16y$ is odd and $x - y$ divides $1 + 16y$, we get that $x - y$ is odd. So $x - y = 1$ or -1.

If $x - y = 1$, then $(2y + 1)^2 = 1 + 16y \iff y = 0$ or $y = 3$.

If $x - y = -1$, then $(2y - 1)^2 = 1 + 16y \iff y = 0$ or $y = 5$.

All solutions are: $(1, 0)$, $(-1, 0)$, $(4, 3)$, $(-4, 3)$, $(4, 5)$, $(-4, 5)$.

While going through files, I noticed that somehow we managed to skip over the solutions we had received to some of the problems from the 20th Austrian-Polish Mathematical Competition and Selected Problems from the Israel Mathematical Olympiads that normally should have appeared in the October 2002 number. Interested readers must have wondered where their solutions had gone astray. We start to set this straight by giving readers' solutions to problems 6 through 9 of the 20th Austrian-Polish Mathematical Competition [2000 : 197–199].

6. Prove that there does not exist a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x + f(y)) = f(x) - y$ for all integers x and y.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Pontoise, France. We give Bataille's write-up.

Suppose that such a function exists. We denote by (R) the identity $f(x + f(y)) = f(x) - y$. Let $a = f(0)$. Suitable choices for x and y in (R) yield successively:

$$\begin{aligned}
f(a) &= a & [x = y = 0] \\
f(2a) &= 0 & [x = y = a] \\
a &= 0 & [x = 0,\, y = 2a]\,; \quad \text{that is, } f(0) = 0\,.
\end{aligned}$$

Taking $x = 0$ in (R), we obtain the identity

$$f(f(y)) = -y\,, \qquad\qquad (R')$$

valid for all integers y. From this, $-f(y) = f(f(f(y))) = f(-y)$, so that f is an odd function.

With the help of (R) and (R'), it follows that

$$f(u + v) = f(u + f(-f(v))) = f(u) + f(v)$$

for all integers u and v and, using induction, that $f(m) = mf(1)$ for all integers m. Taking $m = f(1)$, we get $f(f(1)) = (f(1))^2$ while (R') yields $f(f(1)) = -1$, a clear contradiction. The result follows.

7. (a) Prove that for all real numbers p and q the inequality $p^2 + q^2 + 1 > p(q + 1)$ holds.

(b) Determine the greatest real number b such that for all real numbers p and q the inequality $p^2 + q^2 + 1 > bp(q + 1)$ holds.

(c) Determine the greatest real number c such that for all integers p and q the inequality $p^2 + q^2 + 1 > cp(q + 1)$ holds.

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Pontoise, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Aassila's write-up.

[*Editor's Note*: Unless we replace the strict inequality $>$ by $\geq$, we cannot find a *greatest* real number for (b) or a *greatest* integer for (c).]

(a)–(b) It is easy to see that we can suppose $p, q \geq 0$. We have

$$p^2 + q^2 + 1 \; \geq \; p^2 + 2\left(\frac{q+1}{2}\right)^2 \; \geq \; 2\sqrt{\frac{p^2(q+1)^2}{2}} \; = \; \sqrt{2}\,p(q+1)\,;$$

equality holds for $p = \sqrt{2}$, $q = 1$. Hence $b = \sqrt{2}$.

(c) If $p = q = 1$, we have $1 + 1 + 1 = \frac{3}{2}1(1 + 1)$, whence $\frac{3}{2}$ is an upper bound for the maximum. In fact, it is the maximum, since:

- if $q = 0$, then $p^2 + 1 > \frac{3}{2}p$, true.
- if $q = 1$, then $p^2 + 2 \geq 3p$; that is, $(p - 1)(p - 2) \geq 0$, true.
- if $q = 2$, then $p^2 + 5 \geq \frac{9}{2}p$; that is, $(p - 2)(p - \frac{5}{2}) \geq 0$, true.
- if $q \geq 3$, then $p(3 - q) - 2 < 0 \leq 2(p - q)^2$; hence

$$p^2 + q^2 + 1 > \frac{3}{2}p(q + 1)\,.$$

8. Let n be a natural number and let M be a set with n elements. Find the biggest integer k with the property: there exists a k-element family K of three-element subsets of M such that any two sets from K are non-disjoint.

Comment by Pierre Bornsztein, Pontoise, France.

This is a special case ($p = 3$) of a well-known theorem of Erdős, Ko and Rado: "The largest size of an intersecting p-family in an n-set is $\binom{n-1}{p-1}$."

Reference:

[1] M. Aigner, G.M. Ziegler, *Proofs from The Book*, Springer, p. 136–137.

9. Let P be a parallelepiped, let V be its volume, S its surface area, and L the sum of the lengths of the edges of P. For $t \geq 0$ let P_t be the solid

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consisting of points having distance from P not greater than t. Prove that the volume of P_t is equal to

$$V \;+\; St \;+\; \frac{\pi}{4}Lt^2 \;+\; \frac{4}{3}\pi t^3 \,.$$

Comment by Murray S. Klamkin, University of Alberta, Edmonton, AB.
It is a known result [1] that the volume $V(P_t)$ of a parallel body of a convex polyhedron P at a distance t from P is the sum of

(a) $V(P)$,

(b) the volumes of right prisms of height t whose bases are the faces of P, altogether a volume St, where S is the total surface area of P,

(c) the volumes of the cylindrical segments whose heights are the lengths e_i of the edges and whose bases are circular sectors of radius t and centre angles α_i equal to the angles between the normals to the faces intersecting in e_i, and

(d) the volumes of the spherical sectors at the vertices of P, which altogether equal the volume of one spherical ball of radius t.

Thus,

$$V(P_t) \;=\; V(P) + St + \left(\sum \frac{e_i \alpha_i}{2}\right) t^2 + \frac{4\pi t^3}{3} \,.$$

For the special case where P is a rectangular parallelepiped, $\alpha_i = \frac{\pi}{2}$, and the given result follows. The result for general parallelepipeds is not valid.

Reference:

[1] H. Guggenheimer, *Applicable Geometry*, R. Krieger Publishing Co., N.Y., 1977, p. 116.

Next we give solutions from the readers to Selected Problems from Israel Mathematical Olympiads, given [2000 : 199].

1. Prove that there are at most 3 primes between 10 and 10^{10} all of whose digits in base ten are 1 (for example, 11).

Solved by Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's write-up and comment.
In number theory, a positive integer each digit of which in base 10 is 1 is called a *repunit*. We show that, in fact, 11 is the *only* repunit between 10 and 10^{10} which is also a prime.

Let $n = 111\cdots 11$ be a repunit with k digits where $2 \le k \le 10$.

- If $k = 3$, 6, or 9, then clearly $n > 3$ and is divisible by 3. Hence it is not a prime.

- If $k = 4$, 8, or 10, then by a well-known test for divisibility, n is divisible by 11. Since $n > 11$, it is not a prime.

- If $k = 5$, then $n = 11111 = 41 \times 271$, which is composite.

- If $k = 7$, then $n = 1111111 = 239 \times 4649$, which is composite.

Finally, since $n = 11$ is a prime, our proof is complete.

Comment. This problem can be found as Exercise Number 14 on page 164 of the book *Elementary Number Theory and its Applications* by Kenneth H. Rosen, 3rd edition.

2. Is there a planar polygon whose vertices have integer coordinates, whose area is $\frac{1}{2}$, such that this polygon is

(a) a triangle with at least two sides longer than 1000?

(b) a triangle whose sides are all longer than 1000?

(c) a quadrangle?

Solution by Murray S. Klamkin, University of Alberta, Edmonton, AB.
(a) Yes. Just consider the one with vertices $(0, 0)$, $(1, 0)$, and $(1001, 1)$.

(b) Yes. Consider a triangle with integer coordinates $(0, 0)$, (a, b), and (c, d) where the three sides are > 1000. The area is given by $|ad - bc|/2$. Thus, we need $ad - bc = \pm 1$. For instance, let $(a, b) = (1000, 999)$ and $(c, d) = (10001, 9991)$. We can even make all the sides arbitrarily large. We first would choose a and b to be arbitrarily large and relatively prime. Then we can find arbitarily large c and d satisfying $|ad - bc| = 1$. For example, for the choice of (a, b) above, $(c, d) = (1000m + 1, 999m + 1)$ for large m.

(c) No. By Pick's Theorem, the area of such a quadrangle is $I + \frac{B}{2} - 1$ where I is the number of interior lattice points and B is the number of boundary lattice points. Since B is at least 4, the area of any quadrangle is at least 1.

3. Find all real solutions of

$$\sqrt[4]{13 + x} + \sqrt[4]{4 - x} = 3.$$

Solved by Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's write-up.
Let $u = \sqrt[4]{13 + x}$ and $v = \sqrt[4]{4 - x}$. Then $u + v = 3$ and $u^4 + v^4 = 17$. Thus,

$$17 + 2u^2v^2 = (u^2 + v^2)^2 = ((u + v)^2 - 2uv)^2 = (9 - 2uv)^2.$$

Simplifying, we get $u^2v^2 - 18uv + 32 = 0$, whence $(uv - 2)(uv - 16) = 0$.

If $uv = 2$, then $(13 + x)(4 - x) = 16$ yields $x^2 + 9x - 36 = 0$ or $(x - 3)(x + 12) = 0$. Hence $x = 3$ or -12.

If $uv = 16$, then $(13 + x)(4 - x) = 16^4$ yields $x^2 + 9x + 65484 = 0$ which clearly has no real solutions.

Since it is easy to see that both $x = 3$ and $x = -12$ satisfy the given equation, we conclude that the only real solutions are $x = 3$ and $x = -12$.

4. Prove that if two altitudes of a tetrahedron intersect, then so do the other two altitudes.

Solved by Michel Bataille, Rouen, France; and Murray S. Klamkin, University of Alberta, Edmonton, Alberta. First we give Bataille's write-up and comment.

Suppose that the altitudes AH and BK of tetrahedron $ABCD$ concur, say at U (with H and K in the planes (BCD) and (ACD), respectively). Since CD is orthogonal to AH and BK, CD is orthogonal to the plane (ABU), and hence to AB.

Conversely, suppose $AB \perp CD$ and let B' be the foot of the altitude from B in $\triangle BCD$. Then CD is orthogonal to the plane (ABB'), hence to AB', so that B' is also the foot of the altitude from A in $\triangle ACD$. Thus, AH and BK are both contained in the plane (ABB') and, as such, are concurrent (obviously they cannot be parallel).

It follows that the concurrency of AH and BK is *equivalent* to the condition $AB \perp CD$, and therefore also to the concurrency of the altitudes issued from C and D. This completes the proof.

Comment. With the hypotheses above, UB' is the third altitude in $\triangle ABB'$ (the first two being AH and BK) so that $UB' \perp AB$ and $UB' \perp CD$. Reasoning similarly with the intersection V of the altitudes from C and D, we may conclude that, when $V \neq U$, the line UV is the common perpendicular to the orthogonal lines AB and CD.

Next we give Klamkin's solution.
This is a known result and is included in the following results from [1].

203 Theorem. If a pair of opposite edges of a tetrahedron are orthogonal, the two altitudes of the tetrahedron issued from the ends of each of these two edges are coplanar.

Proof. If the two opposite edges BC, AD of the tetrahedron $ABCD$ are orthogonal, it is possible to draw through BC a plane BMC orthogonal to AD at M. This plane is orthogonal to the two planes ADB, ADC passing through AD. Hence, the altitude from C to ABD and the altitude from B to ADC both lie in the plane BCM. Similarly, for the altitudes issued from the ends A, D of AD.

204 Converse Theorem. If two altitudes of a tetrahedron are coplanar, the edge joining the two vertices from which these altitudes issue is orthogonal

to the opposite edge of the tetrahedron.

Proof. If the altitudes issued from B and C meet in a point H, the plane BHC is orthogonal to the planes ADC, ADB and, therefore, also to their intersection AD.

205 Corollary I. If two altitudes of a tetrahedron intersect, the remaining two intersect also.

Comment. We leave it as a related exercise to show that if one altitude intersects two other altitudes, then the four altitudes are concurrent.

Reference:

[1] N. Altshiller-Court, *Modern Pure Solid Geometry*, Macmillan, N.Y., 1935, pp. 61–62.

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Now we move to readers' solutions to problems from the Estonian Mathematical Olympiads 1996–1997, Final Round of the National Olympiad [2000 : 452–453].

1. On a square table of size $3n \times 3n$ each unit square is coloured either red or blue. Each red square not lying on the edge of the table has exactly five blue squares among its eight neighbours. Each blue square not lying on the edge of the table has exactly four red squares among its eight neighbours. Find the number of red and blue squares on the table.

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Pontoise, France; and Tobias Reuter, student, Wilnsdorf, Germany. We give Bornsztein's solution.

We will show that there are exactly $5n^2$ blue squares and $4n^2$ red squares. Divide the square table of size $3n \times 3n$ into n^2 pairwise disjoint square tables of size 3×3. For each of these squares:
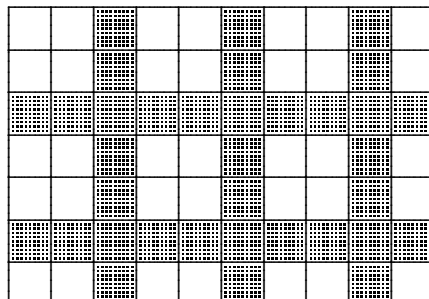
● If the unit square at the centre is red, then there are exactly **5** blue unit squares on the edge and, thus, exactly **3** red unit squares on the edge.

● If the unit square at the centre is blue, then there are exactly **4** red unit squares on the edge and, thus, exactly **4** blue unit squares on the edge.

Then, in each case, there are exactly **5** blue unit squares and **4** red unit squares in a square table of size 3×3. Thus, there are $5n^2$ blue squares and $4n^2$ red squares in the whole $3n \times 3n$ square table, as claimed.

It remains to show that such a colouring is possible. It suffices to use n^2 square tables of size 3×3 of the type:

to obtain

2. Find

(a) all quadruples of positive integers (a, k, l, m) for which the equality $a^k = a^l + a^m$ holds;

(b) all 5-tuples of positive integers (a, k, l, m, n) for which the equality $a^k = a^l + a^m + a^n$ holds.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; and Tobias Reuter, student, Wilnsdorf, Germany. We give the solution of Maragoudakis.

(a) Obviously $a \geq 2$ and $k \geq l, m$. If $l \neq m$, for example, $l > m$, then $a^{k-m} = a^{l-m} + 1$ with $k - m, l - m \geq 1$. Therefore, a divides $a^{k-m} - a^{l-m} = 1$, a contradiction.

Thus, $l = m$ and $a^k = 2 \cdot a^m \iff a^{k-m} = 2 \iff a = 2, k - m = 1$. All quadruples are $(2, m + 1, m, m)$, $m = 1, 2, 3, \ldots$.

(b) Obviously $a \geq 2$, $k \geq l, m, n$. Without loss of generality, we suppose that $l \geq m \geq n$. As above, we see that $m = n$, so the equation becomes $a^{k-m} = a^{l-m} + 2$.

• If $l = m$, then $a^{k-m} = 3$, then $a = 3, k - m = 1$.

• If $l > m$, then $a | 2$. Thus, $a = 2$ and $2^{k-m} = 2^{l-m} + 2$. By (a), $k - m = 2$, $l - m = 1$.

All 5-tuples are $(3, m + 1, m, m, m)$, $(2, m + 2, m + 1, m, m)$, where $m = 1, 2, 3, \ldots$, with appropriate permutations.

3. Prove that, for any real numbers x and y, the following inequality holds:
$$x^2 + y^2 + 1 > x\sqrt{y^2 + 1} + y\sqrt{x^2 + 1}.$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; Tobias Reuter, student, Wilnsdorf, Germany; D.J. Smeenk, Zaltbommel, the

Netherlands; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. Many found similarly nice solutions. We give Aassila's write-up as an example.

First Solution. We have $\left(x - \sqrt{y^2 + 1}\right)^2 \geq 0$, $\left(y - \sqrt{x^2 + 1}\right)^2 \geq 0$. Hence,

$$\left(x - \sqrt{y^2 + 1}\right)^2 + \left(y - \sqrt{x^2 + 1}\right)^2 \geq 0.$$

That is,

$$x^2 + y^2 + 1 \geq x\sqrt{y^2 + 1} + y\sqrt{x^2 + 1}.$$

The equality holds if and only if $x^2 = y^2 + 1$ and $y^2 = x^2 + 1$, a system which has no solutions. Consequently,

$$x^2 + y^2 + 1 > x\sqrt{y^2 + 1} + y\sqrt{x^2 + 1}.$$

Second Solution. By the AM–GM inequality we have

$$\frac{x^2 + (y^2 + 1)}{2} \geq \sqrt{x^2(y^2 + 1)} \geq x\sqrt{y^2 + 1}$$

$$\frac{(x^2 + 1) + y^2}{2} \geq \sqrt{y^2(x^2 + 1)} \geq y\sqrt{x^2 + 1}.$$

Hence

$$x^2 + y^2 + 1 \geq x\sqrt{y^2 + 1} + y\sqrt{x^2 + 1}.$$

The rest of the proof is the same as the first solution.

We also give Klamkin's generalization.
We show more generally that for all real x, y, z,

$$x^2 + y^2 + z^2 > x\sqrt{y^2 + z^2} + y\sqrt{x^2 + z^2}.$$

By Cauchy's Inequality,

$$(x^2 + y^2)\left[(y^2 + z^2) + (x^2 + z^2)\right] \geq \left[x\sqrt{y^2 + z^2} + y\sqrt{x^2 + z^2}\right]^2.$$

Hence, it suffices to prove the stronger inequality

$$\left(x^2 + y^2 + z^2\right)^2 \geq \left(x^2 + y^2\right)\left[(y^2 + z^2) + (x^2 + z^2)\right],$$

which, on expanding out, reduces to $z^4 \geq 0$.

Comment. It also follows in a similar fashion that

$$\sqrt{2}\left(x^2 + y^2 + z^2 + w^2\right)$$
$$\geq x\sqrt{y^2 + z^2 + w^2} + y\sqrt{z^2 + x^2 + w^2} + z\sqrt{x^2 + y^2 + w^2}.$$

Here, using Cauchy's Inequality as before, we have the stronger inequality

$$2\left(x^2 + y^2 + z^2 + w^2\right)^2 \geq \left(x^2 + y^2 + z^2\right)\left[\left(y^2 + z^2 + w^2\right) \right.$$
$$\left. + \left(z^2 + x^2 + w^2\right) + \left(x^2 + y^2 + w^2\right)\right] ,$$

with equality if and only if $w = 0$.

4. In a triangle ABC the values of $\tan \angle A$, $\tan \angle B$ and $\tan \angle C$ relate to each other as $1 : 2 : 3$. Find the ratio of the lengths of the sides AC and AB.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; and Panos E. Tsaoussoglou, Athens, Greece. We give Bataille's solution.

From $\tan B = 2 \tan A$, $\tan C = 3 \tan A$ and the well-known relation $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, we obtain $6 \tan A = 6(\tan A)^3$.

Certainly $\tan A \neq 0$; hence, $\tan A = 1$ or $\tan A = -1$. But the latter cannot hold since otherwise all the angles of $\triangle ABC$ would be obtuse. Thus, $\tan A = 1$ and $A = 45°$.

Now, let H be the foot of the altitude from C. We have

$$\frac{CH}{AH} = \tan A = 1, \quad \frac{CH}{BH} = \tan B = 2,$$

and $H \in [AB]$ (since A, B are acute). It follows that

$$AB = AH + HB = \frac{3}{2}CH .$$

Since we clearly have $AC = CH\sqrt{2}$, we finally get $\dfrac{AC}{AB} = \dfrac{2\sqrt{2}}{3}$.
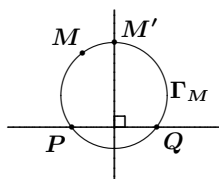
5. There are n points ($n \geq 3$) in the plane, no three of which are collinear. Is it always possible to draw a circle through three of these points so that it has no other given points
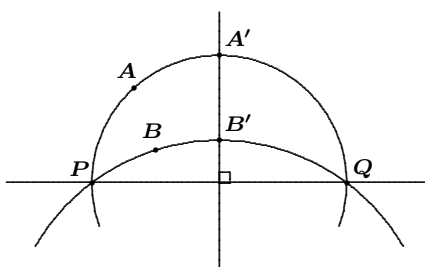
(a) in its interior?   (b) in its interior nor on the circle?

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Tobias Reuter, student, Wilnsdorf, Germany. We give Bornsztein's solution.

(a) Yes. Let E be a set of $n \geq 3$ points in the plane, no three of which are collinear. Let P, Q be two adjacent vertices of the convex hull of E. We then have P, Q in E. For each point $M \in E - \{P, Q\}$, let Γ_M denote the circle through M, P, Q; and let M' denote the intersection of Γ_M and the perpendicular bisector of $[PQ]$ which is on the same side of (PQ) as M.

Since three points in E are never collinear, and from the choice of P and Q, all the points in $E - \{P, Q\}$ are on the same side of (PQ). Since E is finite, we may consider a point $A \in E$ such that the distance from A' to (PQ) is minimal.

Suppose that $B \in E$ is interior to Γ_A. Then the arc of Γ_B whose endpoints are P and Q, and which contains B (and B'), is in the interior of Γ_A (P and Q excepted). Thus,

$$0 \; < \; d(B', (PQ)) \; < \; d(A', (PQ)),$$

which contradicts the minimality of A'. Then Γ_A has no other point of E in its interior, and we are done.

(b) For $n = 3$, the answer is obviously yes. For $n \geq 4$, the answer is no. It suffices to choose the n points on a given circle.

6. For positive integers m, n denote $T(m, n) = \gcd\left(m, \frac{n}{\gcd(m,n)}\right)$.

(a) Prove that there exist infinitely many pairs of integers (m, n) such that $T(m, n) > 1$ and $T(n, m) > 1$.

(b) Does there exist a pair of integers (m, n) such that
$T(m, n) = T(n, m) > 1$?

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Pavlos Maragoudakis, Pireas, Greece; Tobias Reuter, student, Wilnsdorf, Germany; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution by Reuter.
In what follows we simply denote $\gcd(a, b)$ by (a, b).

(a) Let $(m, n) = d$, $m = dm_1$, $n = dn_1$, $(m_1, n_1) = 1$. Now we have

$$T(m, n) \; = \; (dm_1, n_1) \; > \; 1 \quad \text{if and only if} \quad (d, n_1) > 1\,.$$

Similarly, we obtain

$$T(n, m) = (dn_1, m_1) > 1 \quad \text{if and only if } (d, m_1) > 1 .$$

It is clear that there are infinitely many pairs (m, n) which satisfy these two conditions: for example, if we set $d = 2^p 3^q$, $m_1 = 2$ and $n_1 = 3$, we obtain infinitely many such pairs, since there are infinitely many pairs (p, q) with natural numbers p, q.

(b) If we have the required relation, then using the notation from (a) we must have

$$(dn_1, m_1) = (dm_1, n_1) > 1 \quad \Longleftrightarrow \quad (d, m_1) = (d, n_1) = d_1 .$$

It follows that $d_1 = 1$, since otherwise we would have $(n_1, m_1) > 1$. Hence, there are no integers which satisfy the conditions.

7. A function f satisfies the condition

$$f(1) + f(2) + \cdots + f(n) = n^2 f(n)$$

for any positive integer n. Given that $f(1) = 999$, find $f(1997)$.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; Tobias Reuter, student, Wilnsdorf, Germany; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use Bilinski's solution.

We have:

$$f(1) + \cdots + f(n) = n^2 f(n) ,$$
$$\overline{f(1) + \cdots + f(n-1) = (n-1)^2 f(n-1) .}$$

Subtracting and isolating $f(n)$, we get (for $n \geq 2$)

$$f(n) = \frac{(n-1)^2}{n^2 - 1} f(n-1) = \frac{n-1}{n+1} f(n-1) .$$

By telescoping this formula, we obtain

$$f(n) = 2 \frac{(n-1)!}{(n+1)!} f(1) = \frac{2f(1)}{(n+1)n} .$$

Since $f(1) = 999$, we have

$$f(1997) = \frac{2 \cdot 999}{(1998)(1997)} = \frac{1}{1997} .$$

That completes this number of the Corner. As we enter Olympiad season, remember to send me contests and of course, your nice solutions and generalizations.

BOOK REVIEWS

John Grant McLoughlin

The Golden Section
by Hans Walser, as translated from the original German by Peter Hilton, with the assistance of Jean Pedersen, published by the Mathematical Association of America (Spectrum Series), 2001,
ISBN 0-88385-534-8, softcover, 136 + pages, US$26.95.
Reviewed by **Walter M. Reid**, *University of Wisconsin-Eau Claire, Eau Claire, Wisconsin.*

This book presents many perspectives on the golden section (golden ratio), keeping a clear focus on its topic. In independent chapters, the book presents the golden section first by examples and definition, then through fractals, golden (constructive) geometry, paper folds (origami) and cuts, number (Fibonacci) sequences, regular and semi-regular solids, and finishes with a chapter on graph intersections, extremal values, and probabilities of winning games.

I have often wondered about the golden ratio and its origin, and yet had never happened upon a concise and satisfactory presentation. Therefore, to begin with, the book's topic caught my attention and interest. It is also an attractive book. Its cover features Leonardo da Vinci's *Mona Lisa* with facial and other golden rectangles superimposed, one of which serves to box in the defining equation for the golden ratio. The book proper then presents a very clean typographical style with a wealth of fascinating graphics and illustrations throughout. The clean look of the text is due in part to a somewhat terse narrative. Few words are wasted. Each section begins with a brief argument or explanation of a relationship, graph or diagram, followed by questions inviting the reader to continue developing the material in the same vein. The book is inviting to pick up and thumb through.

The book begins quickly with several examples of graphs demonstrating the ratio. As it turns out, the book returns later to develop and explore each of those introductory examples. The definition of the golden section appears (boxed in) immediately on page 2:

"We say that a line-segment is divided in the **ratio of the Golden Section**, or the **Golden Ratio**, if the larger segment is related to the smaller exactly as the larger segment is related to the whole segment."

This definition leads directly to the defining (cover) equation:

$$\frac{x}{1-x} = \frac{1}{x},$$

which, in turn, leads to the golden ratio $\tau = \frac{\sqrt{5}+1}{2} \approx 1.61803$ and its reciprocal $\rho = \frac{\sqrt{5}-1}{2} \approx 0.61803$.

My favourite chapters were Chapter 1, the introductory chapter; Chapter 2 on fractals, where the development of fractal dimension intrigued me; Chapter 5 on number (Fibonacci and generalized Fibonacci) sequences with an intriguing development of the family tree of a drone (bee); and Chapter 7, the last chapter, on intersections of graphs, extremal values, and probabilities of winning games.

Chapters 3 and 5 are the longest. Chapter 3 on golden geometry begins with a challenging argument. It establishes the construction of numbers whose decimal representations differ from those of their reciprocals by a whole number n, thus sharing with them the same fractional parts. This generalizes the case of the Golden Ratio, where $n = 1$. Later in this chapter, the topics of equal areas of ellipses and circles and the geometry of a musical cassette are also fascinating.

The questions, 80 of them in all, begin in the second chapter, on fractals, and then continue (not uniformly) throughout the rest of the book. Chapter 7 finishes with a burst of them, the last 20 questions appearing in five of the last pages of the book. The questions are, for the most part, presented in a casual and inviting way. Most of them are interesting but many are non-routine. All the answers, but not the solutions, are offered at the end of the book. While the questions are usually posed in a casual fashion, it is often unclear what the essence of a question is. I often found myself reading the answer for guidance and then returning to the question. Some of the answers contained surprises, for example, topological equivalence, which was not found in the index.

Though the topics are interesting and the variety of applications is enticing, their development seems somewhat uneven. The more algebraic arguments seem smoother than many of the geometric arguments. While this perception most likely reflects my own preferences, it struck me that the general audience for which the Spectrum Series reportedly is written, might share similar reactions. Furthermore, some arguments included geometric series, limits, elementary matrix theory and eigenvalues, and mathematical induction. While the book lists no prerequisites, having a rudimentary grasp of elementary calculus, linear algebra, number theory, topology, plane and solid geometry would serve the reader well. The high school graduate may find this book a challenge to read, the college graduate in mathematics with a strong geometry background should find it a pleasure to read, and even the geometer might find some fascinating nuggets awaiting. I personally enjoyed the book, and yes, it did satisfy my curiosity about the golden ratio.

On a final note, I must say that the book seemed free of errors in the parts I read carefully.

Brahmagupta Quadrilaterals: A Description

K.R.S. Sastry

Introduction

Heron of Alexandria (Egypt) gave the formula $\sqrt{s(s-a)(s-b)(s-c)}$ for the area of a triangle in terms of its sides a, b, c and $s = (a+b+c)/2$. Right-angled triangles having sides and area that are integers were determined long before Heron. But to his credit he found such a triangle that is **not** a right-angled one: 13, 14, 15; 84. Because of this we honour Heron by naming triangles with integer sides and area *Heron triangles*.

The Indian mathematician Brahmagupta determined Heron triangles by adjoining two right-angled triangles along a common side. He took his principle further and gave us a construction to obtain a cyclic (inscribable in a circle) quadrilateral with integer sides, diagonals, and area. Later mathematicians were intrigued by the Brahmagupta process. But it took Kummer to demystify it. We call an inscribable quadrilateral a *Brahmagupta quadrilateral* if it has integer sides, diagonals, and area. Our present aim is to provide a description of Brahmagupta quadrilaterals via *Heron angles* (see [1], pp. 191–224, and [4]).

Background Material

An angle θ is called a Heron angle if both $\sin\theta$ and $\cos\theta$ are rational. Hence a parametrization of Heron angles is given by

$$\sin\theta = \frac{m^2 - n^2}{m^2 + n^2}, \quad m > n, \quad \gcd(m,n) = 1. \tag{1}$$

In (1) the integers m and n may both be odd. This enables us to obtain the Heron angle $\pi/2 - \theta$ also. The reader will see the advantage of this in the proof of Theorem 2 later on.

Furthermore, we need the following well-known results from circle geometry. We refer to Figure 1. Let AB be a chord of a circle. If two angles are inscribed on the same side of AB, then they will be equal. If they are inscribed on opposite sides, then they will be supplementary. The extended Sine Rule says that
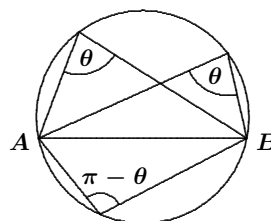

Figure 1

$$AB = (\text{diameter})(\sin\theta).$$

Here is an important observation: Suppose an inscribable qudrilateral has rational sides, diagonals and area. If all these rationals are integers, then it is a Brahmagupta quadrilateral. Otherwise we multiply all these rationals by the least common multiple of their denominators. This process yields a quadrilateral similar to the original one but with integer sides, diagonals, and area; that is, a Brahmagupta quadrilateral.

Let θ_1, θ_2, θ_3 be Heron angles. Invoking (1), we have, for appropriate pairs of natural numbers m_i, n_i ($i = 1, 2, 3$),

$$\sin\theta_i \; = \; \frac{m_i^2 - n_2^2}{m_i^2 + n_i^2} \quad \text{and} \quad \cos\theta_i \; = \; \frac{2m_i n_i}{m_i^2 + n_i^2} \, .$$

We will need expressions for $\sin(\theta_1 + \theta_2)$, $\cos(\theta_1 + \theta_2)$, $\sin(\theta_2 + \theta_3)$ and $\sin(\theta_1 + \theta_2 + \theta_3)$ in terms of m_i and n_i. The required expressions can be obtained very easily from the above formulas by using standard trigonometric identities. For example, $\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)$ and

$$\sin(\theta_1 + \theta_2 + \theta_3) = \sin(\theta_1 + \theta_2)\cos(\theta_3) + \cos(\theta_1 + \theta_2)\sin(\theta_3) \, . \qquad (2)$$

Description of Brahmagupta Quadrilaterals.

Based on the result related to Heron angles given in (2) we state

Theorem 1 An inscribable quadrilateral is a Brahmagupta quadrilateral if and only if the sides a, b, c, d and the diagonals e, f are proportional to

$$
\begin{aligned}
a \;&=\; (m_1^2 - n_1^2)(m_2^2 + n_2^2)(m_3^2 + n_3^2) \, , \\
b \;&=\; (m_1^2 + n_1^2)(m_2^2 - n_2^2)(m_3^2 + n_3^2) \, , \\
c \;&=\; (m_1^2 + n_1^2)(m_2^2 + n_2^2)(m_3^2 - n_3^2) \, , \\
d \;&=\; 4m_3 n_3 [m_2 n_2(m_1^2 - n_1^2) + m_1 n_1(m_2^2 - n_2^2)] + \\
&\qquad (m_3^2 - n_3^2)[4m_1 n_1 m_2 n_2 - (m_1^2 - n_1^2)(m_2^2 - n_2^2)] \, , \\
e \;&=\; 2(m_3^2 + n_3^2)[m_2 n_2(m_1^2 - n_1^2) + m_1 n_1(m_2^2 - n_2^2)] \, , \\
f \;&=\; 2(m_1^2 + n_1^2)[m_3 n_3(m_2^2 - n_2^2) + m_2 n_2(m_3^2 - n_3^2)] \, ,
\end{aligned}
$$

where m_i, n_i are relatively prime natural numbers such that $m_i > n_i$ for $i = 1, 2, 3$ and $a, b, c, d, e, f > 0$.

Proof: Let us consider a cyclic quadrilateral in a circle of diameter 1 (see Figure 2). Let $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = e$, $BD = f$, $\angle ADB = \theta_1$, $\angle BDC = \theta_2$, $\angle DAC = \theta_3$. Then $\angle ACD = \pi - (\theta_1 + \theta_2 + \theta_3)$. From our earlier observations we have $\angle ACB = \theta_1$, $\angle BAC = \theta_2$, $\angle DBC = \theta_3$, $a = \sin\theta_1$, $b = \sin\theta_2$, $c = \sin\theta_3$, $d = \sin(\theta_1 + \theta_2 + \theta_3)$, $e = \sin(\theta_1 + \theta_2)$, and $f = \sin(\theta_2 + \theta_3)$. We also have

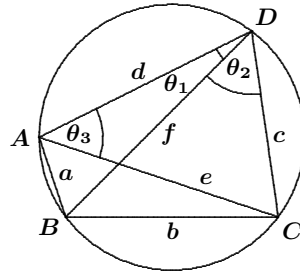$$\text{area } (ABCD) \;=\; \tfrac{1}{2}(ab + cd)\sin(\theta_1 + \theta_2) \, .$$

Figure 2

The lengths a, b, c, d, e, f, and the area of the quadrilateral $ABCD$ are rational if and only if the angles θ_i are Heron angles. Furthermore, as observed earlier, these rationals can be converted to integers, leading to a Brahmagupta quadrilateral. Substituting for the angles in terms of integers m_i and n_i (using our expressions from the last section), and multiplying by the least common multiple of the denominators, we obtain the expressions in the statement of Theorem 1.

We give a numerical illustration before we deduce two important theorems from Theorem 1. Suppose we set $m_1 = 2$, $n_1 = 1$, $m_2 = 3$, $n_2 = 2$, $m_3 = 4$, $n_3 = 1$. Then we have a Brahmagupta quadrilateral $ABCD$ given by

$$
\begin{aligned}
a &= 663, & b &= 425, & c &= 975, \\
d &= 943, & e &= 952, & f &= 1100.
\end{aligned}
$$

We first find $\sin(\theta_1 + \theta_2)$ from (2) and then $\frac{1}{2}(ab + cd)\sin(\theta_1 + \theta_2)$ in order to compute the area of $ABCD$. This area may also be computed by using Brahmagupta's formula (see the concluding section below).

Originally, Brahmagupta's quadrilaterals had perpendicular diagonals. Kummer derived general expressions to obtain such quadrilaterals.

Theorem 2 (Kummer) A Brahmagupta quadrilateral has perpendicular diagonals if and only if it has sides and diagonals proportional to

$$
\begin{aligned}
a &= (m_1^2 - n_1^2)(m_2^2 + n_2^2), & b &= (m_1^2 + n_1^2)(m_2^2 - n_2^2), \\
c &= 2m_1n_1(m_2^2 + n_2^2), & d &= 2m_2n_2(m_1^2 + n_1^2), \\
e &= 2[m_1n_1(m_2^2 - n_2^2) + m_2n_2(m_1^2 - n_1^2)], \\
f &= 4m_1n_1m_2n_2 + (m_1^2 - n_1^2)(m_2^2 - n_2^2).
\end{aligned}
$$

Proof: We refer to Figure 2. The diagonals AC and BD will be perpendicular to each other if and only if $\theta_3 = \pi/2 - \theta_1$. Hence, $\sin\theta_3 = \cos\theta_1$. Therefore, we put $m_3 = m_1 + n_1$ and $n_3 = m_1 - n_1$ in Theorem 1 and then divide by the greatest common divisor to get the expressions listed.

If a trapezium (trapezoid) is cyclic, then it is easy to see that it must be isosceles. Hence, Brahmagupta trapeziums are isosceles trapeziums with integer sides, diagonals, and area.

Theorem 3 An inscribable trapezium is a Brahmagupta trapezium if and only if the sides and diagonals are proportional to

$$
\begin{aligned}
a = c &= (m_1^2 - n_1^2)(m_1^2 + n_1^2)(m_2^2 + n_2^2), \\
b &= (m_1^2 + n_1^2)^2(m_2^2 - n_2^2), \\
d &= 8m_1n_1m_2n_2(m_1^2 - n_1^2) \\
&\quad + (m_2^2 - n_2^2)(2m_1n_1 + m_1^2 - n_1^2)(2m_1n_1 - m_1^2 + n_1^2), \\
e = f &= 2(m_1^2 + n_1^2)[m_2n_2(m_1^2 - n_1^2) + m_1n_1(m_2^2 - n_2^2)].
\end{aligned}
$$

Proof: The sides AD and BC are parallel if and only if $\theta_3 = \theta_1$ (see Figure 2). Hence, we set $m_3 = m_1$ and $n_3 = n_1$ in Theorem 1 to obtain the above expressions.

Conclusions:

Brahmagupta found several important results involving cyclic quadrilaterals. The reader may enjoy rediscovering some of them independently.

1. Brahmagupta's remarkable formula for the area of a cyclic quadrilateral is similar to Heron's formula for a triangle. Actually the converse holds too. Let the sides of a quadrilateral be denoted by a, b, c, d and let $s = (a + b + c + d)/2$. Prove that the quadrilateral is cyclic if and only if its area is $\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)}$. Note that when $d = 0$, the quadrilateral degenerates into a triangle. In fact, this beautiful formula raises the general open question: Given an n–gon with sides $a_1, a_2, \ldots, a_n$, let $s = \left(\sum_{i=1}^{n} a_i \right) /2$. Determine those n–gons whose area is $\Delta_n = \sqrt{(s-a_1)(s-a_2)\cdots(s-a_n)}$. A partial solution is given in [5].

2. Express the lengths e, f of the diagonals of a cyclic quadrilateral in terms of the sides a, b, c, d.

3. Kummer has given a complex method to determine Heron quadrilaterals, the ones with integral sides, diagonals, and area. This is outlined in [1], pp. 191–224. The reader may attempt to give simpler descriptions of Heron quadrilaterals, at least of special Heron quadrilaterals such as Heron parallelograms, and Heron trapeziums.

One can find more on the life of Brahmagupta in [2], pp. 49–51. A partial solution, in terms of a special family of Heron triangles, to the general problem discussed in the present paper can be found in [3], pp. 49–52.

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[1] L.E. Dickson, *History of the Theory of Numbers*, Vol. II, Chelsea, N.Y. (1971).

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[3] K.R.S. Sastry, *A Family of Heron Triangles*, Mathematical Spectrum, 33 (2000/1).

[4] K.R.S. Sastry, *Heron Angles*, Mathematics and Computer Education, 35 (2001), 51–60.

[5] K.R.S. Sastry, *Polygonal Area in the manner of Brahmagupta*, Mathematics and Computer Education, 35 (2001), 147–151.

[6] K.R.S. Sastry, *Heron Triangles: A New Perspective*, Australian Mathematical Society Gazette, 26 (1999), 160–168.

K.R.S. Sastry
Jeevan Sandhya
Doddakalsandra Post
Raghuvana Halli
Bangalore 560062, India

PROBLEMS

Problem proposals and solutions should be sent to Jim Totten, Department of Mathematics and Statistics, University College of the Cariboo, Kamloops, BC, Canada, V2C 5N3. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 September 2003**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in LaTeX.*) *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5 and 7, English will precede French, and in issues 2, 4, 6 and 8, French will precede English.

In the solutions section, the problem will be given in the language of the primary featured solution.

2801. *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany.*

Suppose that $\triangle ABC$ is not obtuse. Denote (as usual) the sides by a, b, and c and the circumradius by R. Prove that

$$\left(\frac{2A}{\pi}\right)^{\frac{1}{a}}\left(\frac{2B}{\pi}\right)^{\frac{1}{b}}\left(\frac{2C}{\pi}\right)^{\frac{1}{c}} \leq \left(\frac{2}{3}\right)^{\frac{\sqrt{3}}{R}}.$$

When does equality hold?

$$\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot$$

Supposons que le triangle ABC n'aie pas d'angle obtus et soit a, b, et c ses côtés et R le rayon du cercle circonscrit. Montrer que

$$\left(\frac{2A}{\pi}\right)^{\frac{1}{a}}\left(\frac{2B}{\pi}\right)^{\frac{1}{b}}\left(\frac{2C}{\pi}\right)^{\frac{1}{c}} \leq \left(\frac{2}{3}\right)^{\frac{\sqrt{3}}{R}}.$$

Quand l'égalité a-t-elle lieu ?

44

2802. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Four positive integers, a, b, c, d, are said to have property $\mathcal{PS}$ if all of $bc + cd + db$, $ac + cd + da$, $ab + bd + da$, and $ab + bc + ca$ are $\mathcal{P}$erfect $\mathcal{S}$quares.

Suppose that the positive integers m, p, q, and r satisfy $p \leq q \leq r$ and $pq + qr + rp = m^2$. Let $s = p + q + r + 2m$.

Prove that p, q, r, and s have property $\mathcal{PS}$.

.·.

On dit que quatre entiers positifs, a, b, c, d, possèdent la propriété $\mathcal{CP}$ si tous les nombres $bc + cd + db$, $ac + cd + da$, $ab + bd + da$, et $ab + bc + ca$ sont des $\mathcal{C}$arrés $\mathcal{P}$arfaits.

Supposons que les entiers positifs m, p, q, et r satisfont $p \leq q \leq r$ et $pq + qr + rp = m^2$. Soit $s = p + q + r + 2m$.

Montrer que p, q, r, et s possèdent la propriété $\mathcal{CP}$.

2803. *Proposed by I.C. Draghicescu, Bucharest, Romania.*

Suppose that $x_1, x_2, \ldots, x_n$ $(n > 2)$ are real numbers such that the sum of any $n - 1$ of them is greater than the remaining number. Let $s = \sum_{k=1}^{n} x_k$. Prove that

$$\sum_{k=1}^{n} \frac{x_k^2}{s - 2x_k} \geq \frac{s}{n - 2}\,.$$

.·.

Soit x_1, x_2, $\ldots$, x_n $(n > 2)$ des nombres réels tels que la somme de $n - 1$ d'entre eux est plus grande que le nombre restant. On pose $s = \sum_{k=1}^{n} x_k$. Montrer que

$$\sum_{k=1}^{n} \frac{x_k^2}{s - 2x_k} \geq \frac{s}{n - 2}\,.$$

2804. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Given three non-concentric circles Γ_j (M_j, R_j), let μ_j denote the power of a point P with respect to Γ_j.

Determine the locus of P if $2\mu_2 = \mu_1 + \mu_3$.

.·.

On donne trois cercles non concentriques Γ_j (M_j, R_j) et soit μ_j la puissance d'un point P par rapport à Γ_j.

Déterminer le lieu des points P tels que $2\mu_2 = \mu_1 + \mu_3$.

2805. *Proposed by Mihály Bencze, Brasov, Romania.*

Let k be a fixed positive integer. For all positive integers n, prove that there exist positive integers $a_1, a_2, \ldots, a_n$, such that $(n, a_n) = 1$ and

$$\sum_{j=1}^{n} \frac{j^k}{a_j} = 1.$$

. . . .

Soit k un entier positif donné. Montrer que, pour tous les entiers positifs n, il existe des entiers positifs $a_1, a_2, \ldots, a_n$, tels que $(n, a_n) = 1$ et

$$\sum_{j=1}^{n} \frac{j^k}{a_j} = 1.$$

2806. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $x, y, z > 0$, $\alpha \in \mathbb{R}$ and $x^\alpha + y^\alpha + z^\alpha = 1$. Prove that

(a) $x^2 + y^2 + z^2 \geq x^{\alpha+2} + y^{\alpha+2} + z^{\alpha+2} + 2x^2 y^2 z^2 \left(x^{\alpha-2} + y^{\alpha-2} + z^{\alpha-2}\right)$,

(b) $\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{z^2} \geq x^{\alpha-2} + y^{\alpha-2} + z^{\alpha-2} + \dfrac{2\left(x^{\alpha+1} + y^{\alpha+1} + z^{\alpha+1}\right)}{xyz}$.

. . . .

Soit $x, y, z > 0$, $\alpha \in \mathbb{R}$ et $x^\alpha + y^\alpha + z^\alpha = 1$. Montrer que

(a) $x^2 + y^2 + z^2 \geq x^{\alpha+2} + y^{\alpha+2} + z^{\alpha+2} + 2x^2 y^2 z^2 \left(x^{\alpha-2} + y^{\alpha-2} + z^{\alpha-2}\right)$,

(b) $\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{z^2} \geq x^{\alpha-2} + y^{\alpha-2} + z^{\alpha-2} + \dfrac{2\left(x^{\alpha+1} + y^{\alpha+1} + z^{\alpha+1}\right)}{xyz}$.

2807. *Proposed by Aram Tangboondouangjit, student, University of Maryland, College Park, Maryland, USA.*

In $\triangle ABC$, denote its area by $[ABC]$ (and its semi-perimeter by s). Show that

$$\min\left\{ \frac{2s^4 - (a^4 + b^4 + c^4)}{[ABC]^2} \right\} = 38.$$

. . . .

Soit $[ABC]$ l'aire d'un triangle ABC, et s son demi-périmètre. Montrer que

$$\min\left\{ \frac{2s^4 - (a^4 + b^4 + c^4)}{[ABC]^2} \right\} = 38.$$

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2808. *Proposed by Aram Tangboondouangjit, student, University of Maryland, College Park, Maryland, USA.*

In $\triangle ABC$, we have $b < c$ and $a\left(3b^2 + c^2 - a^2\right) = 2b\left(c^2 - b^2\right)$. Determine the ratio $a : b : c$.

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Dans le triangle ABC, on a $b < c$ et $a\left(3b^2 + c^2 - a^2\right) = 2b\left(c^2 - b^2\right)$. Déterminer les rapports $a : b : c$.

2809. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $k \geq 2$ is a fixed integer. For each non-negative integer n, let x_n denote the leftmost digit of n^k.

Prove that the number $0.x_0x_1x_2\ldots x_n\ldots$ is irrational.

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Soit $k \geq 2$ un entier donné. Pour tout entier non négatif n, désignons par x_n le premier chiffre du nombre n^k.

Montrer que le nombre $0,x_0x_1x_2\ldots x_n\ldots$ est irrationnel.

2810. *Proposed by I.C. Draghicescu, Bucharest, Romania.*

Suppose that a, b and $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ are positive real numbers. Let $s = \sum_{k=1}^{n} x_k$. Prove that

$$\prod_{k=1}^{n}\left(a + \frac{b}{x_k}\right) \geq \left(a + \frac{nb}{s}\right)^n.$$

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Supposons que a, b et $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ soient des nombres réels positifs et posons $s = \sum_{k=1}^{n} x_k$. Montrer que

$$\prod_{k=1}^{n}\left(a + \frac{b}{x_k}\right) \geq \left(a + \frac{nb}{s}\right)^n.$$

2811. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy, for all real x,

$$f\left(x^3 + x\right) \leq x \leq f^3(x) + f(x).$$

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Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ satisfaisant, pour tous les x réels,

$$f\left(x^3 + x\right) \leq x \leq f^3(x) + f(x).$$

2812. *Proposed by Mihály Bencze, Brasov, Romania.*
Determine all injective functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$(2a + b)f(ax + b) \geq a\, f^2\left(\frac{1}{x}\right) + b\, f\left(\frac{1}{x}\right) + a$$

for all positive real x, where $a, b \in \mathbb{R}$, $a > 0$, $a^2 + 4b > 0$ and $2a + b > 0$.

. . . .

Déterminer toutes les fonctions injectives $f : \mathbb{R} \to \mathbb{R}$ satisfaisant

$$(2a + b)f(ax + b) \geq a\, f^2\left(\frac{1}{x}\right) + b\, f\left(\frac{1}{x}\right) + a$$

pour tous les x réels positifs, où $a, b \in \mathbb{R}$, $a > 0$, $a^2 + 4b > 0$ et $2a + b > 0$.

2813. *Proposed by Barry R. Monson, University of New Brunswick, Fredericton, NB and J. Chris Fisher, University of Regina, Regina, SK.*
Suppose that M is the mid-point of side AB of the square $ABCD$. Let P and Q be the points of intersection of the line MD with the circle, centre M, radius MA $(= MB)$, where P is inside the square $ABCD$ and Q is outside.

Prove that rectangle $APBQ$ is a golden rectangle; that is,

$$PB : PA = (\sqrt{5} + 1) : 2\,.$$

. . . .

Soit M le point milieu du côté AB du carré $ABCD$. Soit P et Q les points d'intersection de la droite MD et du cercle de centre M, de rayon MA $(= MB)$, où P est à l'intérieur et Q à l'extérieur du carré $ABCD$.

Montrer que $APBQ$ est un rectangle d'or, c'est-à-dire

$$PB : PA = (\sqrt{5} + 1) : 2\,.$$

CORRECTION : The summations in problems **2792** and **2799** require $i \neq j$.

. . . .

LA CORRECTION : Les additions dans les problèmes **2792** et **2799** nécessitent $i \neq j$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologize for omitting the name of NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina from the list of solvers of 2634 and 2683, and the name of DAVID LOEFFLER, student, Trinity College, Cambridge, UK from the list of solvers of 2690.

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2701★. [2002 : 52] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Do there exists infinitely many triplets $(n, n+1, n+2)$ of adjacent natural numbers such that all of them are sums of two positive perfect squares?

(Examples are $(232, 233, 234)$, $(520, 521, 522)$ and $(808, 809, 810)$.)

Compare the 2000 Putnam problem A2 [2001 : 3]

Amalgamated solution by Paul Jefferys, student, Berkhamsted Collegiate School, UK; Gottfried Perz, Pestalozzigymnasium, Graz, Austria; and the proposer.

For all integers $a \geq 3$ consider the triples

$$
\begin{aligned}
(a^2 - a)^2 + (a^2 - a)^2 &= 2a^4 - 4a^3 + 2a^2 \\
(a^2 - 2a)^2 + (a^2 - 1)^2 &= 2a^4 - 4a^3 + 2a^2 + 1 \\
(a^2 - a - 1)^2 + (a^2 - a + 1)^2 &= 2a^4 - 4a^3 + 2a^2 + 2
\end{aligned}
$$

Since these expressions are increasing with respect to a, there are infinitely many triples.

Several solutions began with the observation that $8n^2 = (2n)^2 + (2n)^2$ and $8n^2 + 2 = (2n-1)^2 + (2n+1)^2$. Thus, the problem reduces to expressing $8n^2 + 1$ as a sum of two squares.

$$(2n - a)^2 + (2n + a - 1)^2 = 8n^2 + 1 - (4n - (2a^2 - 2a)).$$

Hence, we require n and a such that $4n - (2a^2 - 2a) = 0$, or $2n = a^2 - a$.

It is possible to require that all members of the triple are the sum of two *different* positive perfect squares. Note that $4n^4 + 4n^2 = (2n^2)^2 + (2n)^2$ and $4n^4 + 4n^2 + 2 = (2n^2 + 1)^2 + 1$. Finally, if $2n^2 + 1 = r^2 + s^2$, then

$$4n^4 + 4n^2 + 1 = (2n^2 + 1)^2 = (r^2 - s^2)^2 + (2rs)^2.$$

By above $2n^2 + 1 = r^2 + s^2$ admits infinitely many solutions.

Also solved by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; PETER HURTHIG, Columbia College, Vancouver, BC; DAVID LOEFFLER, student, Trinity College,

Cambridge, UK; JOEL SCHLOSBERG, student, New York University, NY, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. There was one incorrect solution.

Two solutions constructed triples from solutions to Pell's equation: $u^2 - 2v^2 = 1$. Li Zhou asks the question: Are there infinitely many different similarity classes of Pythagorean triples the hypotenuses of which appear in the set $\{2n^2 + 1\}_{n \in \mathbb{N}}$?

2702. $[2002 : 53]$ *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let λ be an arbitrary real number. Show that

$$\left(\frac{s}{r}\right)^{2\lambda} s^2 \geq 3^{3\lambda+1}\left(s^2 - 8Rr - 2r^2\right),$$

where R, r and s are the circumradius, the inradius and the semi-perimeter of a triangle, respectively.

Determine the cases of equality.

[*Editor's Remark.* The condition $\lambda > 0$ was added in a footnote in $[2002 : 248]$.]

I. *Solution by David Loeffler, student, Trinity College, Cambridge, UK.*

The statement is incorrect, even with the correction $\lambda > 0$. The given statement is equivalent to

$$\left(\frac{s^2}{27r^2}\right)^{\lambda} \geq \frac{3}{s^2}\left(s^2 - 8Rr - 2r^2\right). \qquad (1)$$

Now, recall Guerretsen's Inequalities ($[1996 : 130]$):

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2.$$

We also have

$$R \geq 2r \qquad \Longleftrightarrow \qquad 4Rr \geq 8r^2$$
$$\Longleftrightarrow \qquad 16Rr - 5r^2 \geq 12Rr + 3r^2,$$

which implies $s^2 \geq 12Rr + 3r^2$. Hence,

$$3s^2 - 24Rr - 6r^2 \geq s^2 \qquad \Longleftrightarrow \qquad \frac{3}{s^2}\left(s^2 - 8Rr - 2r^2\right) \geq 1.$$

We note that, crucially, this is strictly greater than 1 if the triangle is not equilateral.

Thus, in the adjusted inequality (1), let $\lambda \to 0$ from above. The left-hand side tends to 1, while for any non-equilateral triangle the right-hand side is strictly greater than 1. Therefore, for some positive λ the inequality does not hold.

II. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

We show that the inequality is valid for $\lambda \geq 2$. The case $0 < \lambda < 2$ is left open.

Dividing the given inequality by r^2 and setting $x = (s/r)^2$, the inequality becomes

$$x^{\lambda+1} + 3^{3\lambda+1}\left(\frac{8R}{r} + 2\right) \geq 3^{3\lambda+1}x\,.$$

Since $R/r \geq 2$, it suffices to prove the stronger inequality

$$t(t^\lambda - 1) \geq 2(t - 1)\,,$$

where now $t = x/27$ and it is known that $t \geq 1$. Since $t^\lambda - 1 \geq t^2 - 1$ and since $t(t+1) \geq 2$, we have

$$t(t^\lambda - 1) \geq t(t^2 - 1) \geq 2(t - 1)\,.$$

There is equality only if the triangle is equilateral.

Also shown incorrect by PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

In an addendum to his solution, Loeffler makes the following comment:
We have
$$R \geq 2r \quad \Longleftrightarrow \quad 16Rr - 5r^2 \geq 27r^2\,.$$

Thus, from Gerretsen, we see that the left-hand side of (1) is also strictly greater than **1** for non-equilateral triangles. It follows that for each non-equilateral triangle ABC, there is some λ_0 such that the inequality holds if and only if $\lambda \geq \lambda_0$, with equality only for $\lambda = \lambda_0$. (Clearly, equality always holds for equilateral trinagles.)

The limit of $\lambda_0(a, b, c)$ as $a, b, c \to 1, 1, 1$ is $\frac{2}{3}$. Over all triangles the largest value of λ_0, and hence the minimum value of λ for which the inequality holds for all triangles, is (to 30 digits):
$$0.702543072697378209700856413\,.$$

This is achieved for a triangle with sides 1, 1, and α where α is a constant equal to about
$$0.773737141433407603891152539846671\,.$$

(Loeffler has run these numbers through the Inverse Symbolic Calculator website, and states that they do not appear to be obviously related to any known constants.)

The constant α may be identified as the value of z maximizing the expression

$$\frac{\log\left[\dfrac{3(3z^2 - 4z + 4)}{(z + 2)^2}\right]}{\log\left[\dfrac{(z + 2)^3}{27z^2(2 - z)}\right]}$$

and the maximum value of this expression is the sought-after maximum of λ_0.

2703. [2002 : 53]
Proposed by Mihály Bencze, Brasov, Romania.
Suppose that a, b, c, d, u, $v \in \mathbb{R}$ and $a + c \neq 0$. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which $f(ax + b) + f(cx + d) = ux + v$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA (modified slightly by the editor).
Suppose f satisfies the required condition. Let

$$g(x) \; = \; f(x) - \frac{u}{a+c}x + \frac{1}{2}\left(\frac{b+d}{a+c}u - v\right).$$

Then g is continuous and

$$\begin{aligned}
g(ax + b) + g(cx + d) \; &= \; ux + v - \frac{u}{a+c}\Big((a+c)x + (b+d)\Big) \\
&\quad + \frac{b+d}{a+c}u - v \\
&= \; 0.
\end{aligned}$$

Depending on the values of a, b, c, and d, we consider three cases separately:

Case (i). If $a = c \neq 0$ and $b = d$, then $2g(ax + b) = 0$ for all $x \in \mathbb{R}$. Hence, $g \equiv 0$ and $f(x) = \frac{u}{2a}x + \frac{1}{2}\left(\frac{b}{a}u - v\right)$.

Case (ii). If $a = c \neq 0$ and $b \neq d$, then $g(ax + b) + g(ax + d) = 0$. Letting $y = ax + d$, we then obtain $g(y + b - d) = -g(y)$. Hence,

$$f(x) \; = \; g(x) + \frac{u}{a+c}x - \frac{1}{2}\left(\frac{b+d}{2a}u - v\right),$$

where g is a continuous function satisfying $g(x + b - d) = -g(x)$. (Thus, g is periodic with period $2(b - d)$.) There are clearly infinitely many such functions; for example, $g(x) = \sin\left(\frac{\pi x}{b - d}\right)$.

Case (iii). If $a \neq c$, then $|a| \neq |c|$ since $a \neq -c$ by assumption. Without loss of generality, we may assume that $|a| > |c|$. Let $p = \frac{d - b}{a - c}$ and $q = \frac{ad - bc}{a - c}$. Then $ap + b = cp + d = q$. Now, for all $x \in \mathbb{R}$, we have

$$g(ax + q) \; = \; g\Big(a(x + p) + b\Big) \; = \; -g\Big(c(x + p) + d\Big) \; = \; -g(cx + q) \quad (1)$$

If $c = 0$, then $q = d$ and $g(ax + q) = -g(q)$ for all $x \in \mathbb{R}$, implying that $g \equiv 0$. If $c \neq 0$, then letting $x = \frac{ay}{c}$, we get from (1) that

$$g\left(a\Big(\frac{a}{c}\Big)y + q\right) \; = \; -g(ay + q) = g(cy + q).$$

Iterating this substitution, we obtain inductively that

$$g\left(a\left(\frac{a}{c}\right)^n z + q\right) = (-1)^{n+1}g(cz+q)$$

for all $z \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Let $z_n = z_n(x) = \dfrac{x-q}{a(a/c)^n}$. Then

$$g(x) = g\left(\left(\frac{a}{c}\right)^n z_n + q\right) = (-1)^{n+1}g(cz_n+q). \qquad (2)$$

Since $\left|\dfrac{a}{c}\right| > 1$ we have $z_n(x) \to 0$ as $n \to \infty$. Let $n \to \infty$ through the odd and even integers, respectively. Then from (2) and the continuity of g, we conclude that $g(x) = g(q)$ and $g(x) = -g(q)$ for all $x \in \mathbb{R}$. Therefore, $g(x) = g(q) = 0$. Thus, $g \equiv 0$ again, from which it follows that

$$f(x) = \frac{u}{a+c}x - \frac{1}{2}\left(\frac{b+d}{a+c}u - v\right).$$

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; JOEL SCHLOSBERG, student, New York University, NY, USA; and the proposer.

2704. [2002 : 53] (Corrected [2002 : 174]) *Proposed by Mihály Bencze, Brasov, Romania.*
  Prove that

$$R - 2r \geq \frac{1}{12}\left(\sum_{\text{cyclic}} \sqrt{2(b^2+c^2) - a^2} - \frac{s^2+r^2+4Rr}{R}\right) \geq 0,$$

 where a, b and c are the sides of a triangle, and R, r and s are the circumradius, the inradius and the semi-perimeter of the triangle, respectively.

Solution by G. Tsintsifas, Thessaloniki, Greece.
  We prove the stronger inequality:

$$R - 2r \geq \frac{1}{8}\left(\sum_{\text{cyclic}} \sqrt{2(b^2+c^2) - a^2} - \frac{s^2+r^2+4Rr}{R}\right) \geq 0,$$

From elementary geometry we use the following formulas:

$$b^2 + c^2 = 2m_a^2 + \frac{a^2}{2} \quad \text{and} \quad s^2 + r^2 + 4Rr = ab + bc + ca,$$

where m_a is the length of the median through A. We set

$$A = \sum_{\text{cyclic}} \sqrt{2(b^2+c^2) - a^2} - \frac{s^2+r^2+4Rr}{R}.$$

This simplifies to:

$$A = 2 \sum_{\text{cyclic}} m_a - \frac{ab + bc + ca}{R} = 2 \sum_{\text{cyclic}} m_a - \frac{2R \sum_{\text{cyclic}} h_a}{R}$$

$$= 2 \left[\sum_{\text{cyclic}} m_a - \sum_{\text{cyclic}} h_a \right].$$

From [1] 8.2 and 7.12, we have

$$\sum_{\text{cyclic}} m_a \leq 4R + r \quad \text{and} \quad 9r \leq \sum_{\text{cyclic}} h_a \,,$$

which yields

$$A \leq 2 \left[4R + r - 9r \right] = 8(R - 2r) \,.$$

References.

[1] D.S. Mitrinovic et. al., *Recent Advances in Geometric Inequalities*. Kluwer, Dordrecht 1989.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOE HOWARD, Portales, NM, USA; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; VEDULA N. MURTY, Dover, PA, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. Most solutions were not easily amended to show the stronger inequality above.

2705. [2002 : 53] *Proposed by Angel Dorito, Geld, ON.*

The interior of a rectangular container is **1** metre wide and **2** metres long, and is filled with water to a depth of $\frac{1}{2}$ metre. A cube of gold is placed flat in the tub, and the water rises to exactly the top of the cube without overflowing.

Find the length of the side of the cube.

Solution by Gavin Johnstone, student, Dame Alice Owen's School, Potters Bar, UK.

The volume of water in the tub is $(1)(2)(\frac{1}{2}) = 1$ m^3, and this is invariant. After the cube is placed in the tub,

vol. water = vol. water and cube − vol. cube .

Letting the side length of the cube be x, we have $1 = 2x - x^3$, and thus

$$x^3 - 2x + 1 = 0 \tag{1}$$
$$(x - 1)(x^2 + x - 1) = 0 \,.$$

Hence, $x = 1$ or $x = (-1 \pm \sqrt{5})/2$. The solution $x = (-1 - \sqrt{5})/2$ is negative and therefore inadmissible. The solution $x = (-1 + \sqrt{5})/2$, the Golden Ratio, is an acceptable side length for the cube. Whether $x = 1$ is acceptable depends on whether an exact fit is allowed for the cube in the tub. [*Editor's note:* The equation (1) is the same as was found for the solution of 2670 [2002 : 464].]

Also solved in essentially the same manner by AUSTRIAN IMO TEAM 2002; CHARLES ASHBACHER, Hiawatha, IA, USA; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; PAOLO CUSTODI, Fara Novarese, Italy; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; GERRY LEVERSHA, St. Paul's School, London, UK; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ROBERT MULLER, Landesschule Pforta, Schulpforte, Germany; VICTOR PAMBUCCIAN, ASU West, Phoenix, AZ, USA; JOEL SCHLOSBERG, student, New York University, NY, USA; ROBERT P. SEALY, Mount Allison University, Sackville, NB; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; Mª JESÚS VILLAR RUBIO, Santander, Spain; OLOV WILANDER, student, Christ's College, Cambridge, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer. There was one incorrect solution.

Strictly speaking, the depth of the container should be considered in deciding which solutions are possible lengths for a side of the cube. The depth is not given, but we are told that the water does not overflow when the cube is placed in the tub. Consequently, there are two cases: (1) depth ≥ 1, in which case both positive solutions are admissible; (2) $(\sqrt{5} - 1)/2 \leq$ depth < 1, in which case only the solution $(\sqrt{5} - 1)/2$ is admissible. If $1/2 <$ depth $< (\sqrt{5} - 1)/2$, then no solution is admissible, but this possibility seems to be ruled out by the wording of the problem. Only the Austrian IMO-Team and Natalio Guersenzvaig noted the different cases. All the other solutions assumed implicitly that the tub was deep enough for both cubes.

David Loeffler observes that the proposer's name, Angel Dorito, "is clearly a pseudonym, being an anagram of his stated address of Geld, Ontario".

2706. [2002 : 54] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that Γ_1 and Γ_2 are two circles having at least one point S in common. Take an arbitrary line ℓ through S. This line intersects Γ_k again at P_k (if ℓ is tangent to Γ_k, then $P_k = S$).

Let λ be a (fixed) real number, and let $R_\lambda = \lambda P_1 + (1 - \lambda)P_2$.

Determine the locus of R_λ as ℓ varies over all possible lines through S.

I. *Solution by G. Tsintsifas, Thessaloniki, Greece.*

Let A and B be the opposite ends of the diameters through S in circles Γ_1 and Γ_2 respectively; denote the second intersection point of the circles by T (with $T = S$ when the circles are tangent). Since AT and BT both make right angels with ST, T must lie on AB. Let Q be the point of AB for which $Q = \lambda A + (1 - \lambda)B$. By definition, S is on $P_1 P_2$, so $P_1 P_2$ makes right angles with AP_1 and BP_2. Thus $AP_1 \| BP_2$ and so these lines are parallel to QR_λ (since QR_λ cuts transversals $P_1 P_2$ and AB proportionally). Therefore, also

$\angle QR_\lambda S$ is a right angle for all positions of R_λ, whose locus is consequently the circle with diameter SQ. Note that because O_1 is the mid-point of SA while O_2 is the mid-point of SB, the centre of the locus (which is the mid-point of SQ) must be $\lambda O_1 + (1 - \lambda)O_2$.

II. *Solution by David Loeffler, student, Trinity College, Cambridge, UK.*

The condition on R is equivalent to stating that the cross-ratio of the four points P_1, P_2, R, ∞ is constant and equal to $-\lambda$, where ∞ represents the point at infinity in the inversive plane. Let T be the second intersection of Γ_1 and Γ_2, which may coincide with S. Let us invert the diagram in some circle centred at S. Then the circles Γ_1 and Γ_2 become lines through the point T'. Since l passes through S it inverts into itself. This inversion has mapped ∞ into S, so since inversions preserve cross-ratios, $[P_1', P_2', R', S] = -\lambda$. This implies that the locus of R' is a fixed line through T', since P_1', P_2' and S are collinear. (Note that T' may be ∞). So the locus of R is a circle through S and T.

Also solved by MICHEL BATAILLE, Rouen, France (2 solutions); CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GERRY LEVERSHA, St. Paul's School, London, UK; TOSHIO SEIMIYA, Kawasaki, Japan; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

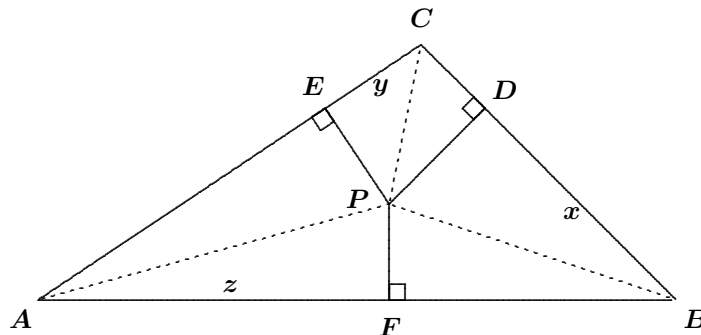**2707**. [2002 : 54] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let ABC be a triangle and P a point in its plane. The feet of the perpendiculars from P to the lines BC, CA and AB are D, E and F, respectively.

Prove that

$$\frac{AB^2 + BC^2 + CA^2}{4} \leq AF^2 + BD^2 + CE^2 \,,$$

and determine the cases of equality.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Let $AF = z$, $BD = x$, and $CE = y$ (signed lengths). [Also, $AB = c$, $BC = a$ and $CA = b$.] Then $FB = c - z$, $DC = a - x$, and $EA = b - y$. A well-known theorem (proved by applying the theorem of Pythagoras to the six triangles such as $\triangle PBD$) is that

$$x^2 + y^2 + z^2 \; = \; (a - x)^2 + (b - y)^2 + (c - z)^2\,,$$

so that

$$ax + by + cz \; = \; \tfrac{1}{2}(a^2 + b^2 + c^2)\,.$$

Hence,

$$\tfrac{1}{4}(a^2 + b^2 + c^2)^2 \; = \; (ax + by + cz)^2 \; \leq \; (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)\,,$$

where the last step follows by the Cauchy-Schwarz Inequality. Thus,

$$\tfrac{1}{4}(a^2 + b^2 + c^2) \; \leq \; (x^2 + y^2 + z^2)\,.$$

Equality holds when $a : b : c = x : y : z$, which is when D, E and F are the mid-points of the sides, making P the circumcentre of $\triangle ABC$.

References.

[1] **CRUX with MAYHEM** 23, No. 2 (March, 1997), p. 122.

[2] D. S. Mitronović, J. E. Pečarić and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989, p. 336.

Also solved by AUSTRIAN IMO TEAM 2002; MICHEL BATAILLE, Rouen, France; JOHN G. HEUVER, Grande Prairie, AB; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; GAVIN JOHNSTONE, student, Dame Alice Owen's School, Potters Bay, UK; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; GERRY LEVERSHA, St. Paul's School, London, UK; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; TOSHIO SEIMIYA, Kawasaki, Japan; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania. There was one incorrect solution. Two other solutions failed to treat the case of equality, and were considered to be incomplete.

The well-known theorem mentioned in the featured solution is discussed in [1]. Seimiya obtains the equation $ax + by + cz = \frac{1}{2}(a^2 + b^2 + c^2)$ by a different approach. He applies the Law of Cosines to each of the triangles BAP, CBP and ACP, eliminates the cosines using $AP\cos(\angle BAP) = AF$, $BP\cos(\angle CBP) = BD$ and $CP\cos(\angle ACP) = CE$, and adds the resulting three equations.

Heuver notes that this problem may be found in [2].

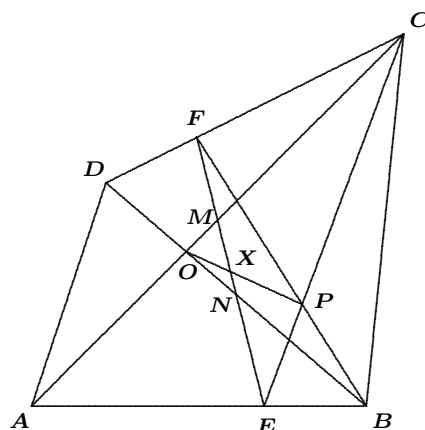**2708**. [2002 : 54] *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Suppose that

1. O is the intersection of diagonals AC and BD of quadrilateral $ABCD$,

2. $OA < OC$ and $OD < OB$,

3. M and N are the mid-points of AC and BD, respectively,

4. MN meets AB and CD at E and F, respectively, and

5. P is the intersection of BF and CE.

Prove that OP bisects the line segment EF.

Solution by Gerry Leversha, St. Paul's School, London, UK.

We use vectors. Taking point O as the origin, define the position vectors of points M and N as $\overrightarrow{m}$ and $\overrightarrow{n}$, respectively. Then the position vectors of points C and B can be taken as $c\,\overrightarrow{m}$ and $b\,\overrightarrow{n}$, where $b, c > 1$, and, since M and N are mid-points, the points A and D have position vectors $(2 - c)\,\overrightarrow{m}$ and $(2 - b)\,\overrightarrow{m}$, correspondingly.

The point E lies on both AB and MN, so that there exist λ and μ such that
$$\lambda(2 - c)\,\overrightarrow{m} + (1 - \lambda)b\,\overrightarrow{n} \;=\; \mu\,\overrightarrow{m} + (1 - \mu)\,\overrightarrow{n}\,.$$

Hence, by linear independence, we have $\lambda(2 - c) + (1 - \lambda)b = 1$, so that
$$\lambda \;=\; \frac{b - 1}{b + c - 2} \qquad \text{and} \qquad 1 - \lambda \;=\; \frac{c - 1}{b + c - 2}\,.$$

Therefore, the position vector of E is given by
$$(b + c - 2)\,\overrightarrow{e} \;=\; (2 - c)(b - 1)\,\overrightarrow{m} + b(c - 1)\,\overrightarrow{n}\,.$$

Similarly, the position vector of F is given by
$$(b + c - 2)\,\overrightarrow{f} \;=\; (2 - b)(c - 1)\,\overrightarrow{n} + c(b - 1)\,\overrightarrow{m}\,.$$

The mid-point X of EF has position vector
$$\overrightarrow{x} \;=\; \frac{(b - 1)\,\overrightarrow{m} + (c - 1)\,\overrightarrow{n}}{b + c - 2}\,.$$

It remains to find the position vector of P. Since P lies on both BF and CE, there must exist α and β such that

$$\alpha(b+c-2)b\,\overrightarrow{n} + (1-\alpha)(2-b)(c-1)\,\overrightarrow{n} + (1-\alpha)c(b-1)\,\overrightarrow{m}$$
$$= \beta(b+c-2)c\,\overrightarrow{m} + (1-\beta)(2-c)(b-1)\,\overrightarrow{m} + (1-\beta)b(c-1)\,\overrightarrow{n}\,,$$

and therefore,

$$\alpha(b+c-2)b + (1-\alpha)(2-b)(c-1) \;=\; (1-\beta)b(c-1)$$

and

$$(1-\alpha)c(b-1) \;=\; \beta(b+c-2)c + (1-\beta)(2-c)(b-1)\,.$$

These in turn reduce to

$$(b^2 + 2bc - 3b - 2c + 2)\alpha + b(c-1)\beta \;=\; 2(b-1)(c-1)$$

and

$$(c^2 + 2bc - 3c - 2b + 2)\beta + c(b-1)\alpha \;=\; 2(b-1)(c-1)\,.$$

Eliminating β, we obtain

$$\alpha \;=\; \frac{c-1}{b+c-1}\,,$$

so that

$$1 - \alpha \;=\; \frac{b}{b+c-1}\,.$$

Thus, the position vector of the point P is given by

$$\overrightarrow{p} \;=\; \frac{bc[(b-1)\,\overrightarrow{m} + (c-1)\,\overrightarrow{n}]}{b+c-1}\,.$$

Comparing the expressions for $\overrightarrow{x}$ and $\overrightarrow{p}$, we see that the points O, X and P are collinear, which completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; G. TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Editorial comments. Janous has proved the more general result: If condition 3 is replaced by "M and N are the points dividing both segments AC and BD in ratio $\lambda : (1 - \lambda)$" then OP divides EF in ratio $\lambda : (1 - \lambda)$ too. (Here, $0 < \lambda < 1$.)

2709. $[2002:55]$ *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Suppose that

1. P is an interior point of $\triangle ABC$,

2. AP, BP and CP meet BC, CA and AB at D, E and F, respectively,

3. A' is a point on AD produced beyond D such that $DA' : AD = \kappa : 1$, where κ is a fixed positive number,

4. B' is a point on BE produced beyond E such that $EB' : BE = \kappa : 1$, and

5. C' is a point on CF produced beyond F such that $FC' : CF = \kappa : 1$.

Prove that $[A'B'C'] \leq \frac{(3\kappa+1)^2}{4}[ABC]$, where $[PQR]$ denotes the area of $\triangle PQR$.

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.

Victor Thébault investigated areas in a more general setting in the former Belgian journal *Mathesis* (1940), p. 67. In particular, he does not assume that AD, BE, and CF concur. We shall postpone that assumption until after having developed the relevant area formulas.

Lemma (Thébault). *Given $\triangle ABC$ with points D dividing BC internally in the ratio $(1-m) : m$, E dividing CA internally in the ratio $(1-n) : n$, and F dividing CA internally in the ratio $(1-p) : p$, we define A', B', C' as in the statement of problem 2709. Then*

$$\frac{[DEF]}{[ABC]} \;=\; mn + np + pm - (m+n+p) + 1 \qquad (1)$$

and

$$\frac{[A'B'C']}{[ABC]} \;=\; [mn + np + pm - (m+n+p) + 3](k+1)^2$$
$$-3k - 2 \,. \qquad (2)$$

Proof. For the proof we introduce barycentric (areal) coordinates with $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$ from which we obtain

$$D(0,m,1-m)\,,\; E(1-n,0,n)\,,\; F(p,1-p,0)\,,$$
$$A'(-k,m(1+k),(1-m)(1+k))\,,\text{ etc.}$$

Therefore,

$$\frac{[DEF]}{[ABC]} \;=\; \begin{vmatrix} 0 & m & 1-m \\ 1-n & 0 & n \\ p & 1-p & 0 \end{vmatrix}$$
$$=\; mn + np + pm - (m+n+p) + 1\,, \quad \text{and}$$

$$\frac{[A'B'C']}{[ABC]} \;=\; \begin{vmatrix} -k & m(1+k) & (1-m)(1+k) \\ (1-n)(1+k) & -k & n(1+k) \\ p(1+k) & (1-p)(1+k) & -k \end{vmatrix}$$
$$=\; [mn + np + pm - (m+n+p) + 3](k+1)^2 - 3k - 2\,,$$

as claimed.

From (1) and (2) we can eliminate m, n, p to obtain

$$[A'B'C'] = (k+1)^2[DEF] + k(2k+1)[ABC]\,. \qquad (3)$$

We now turn to our problem 2709 and assume that the cevians AD, BE, CF concur; by Ceva's Theorem we have

$$
\begin{aligned}
\frac{1-m}{m} \cdot \frac{1-n}{n} \cdot \frac{1-p}{p} &= 1\,, \text{ or equivalently} \\
(1-m)(1-n)(1-p) &= mnp\,, \text{ or} \qquad (4) \\
mn + np + pm - (m+n+p) &= 2mnp - 1
\end{aligned}
$$

Thus (from the last line of (4)), when the given cevians are concurrent, equations (1) and (2) become

$$
\begin{aligned}
\frac{[DEF]}{[ABC]} &= 2mnp \\
\frac{[A'B'C']}{[ABC]} &= 2(1+k)^2(mnp+1) - 3k - 2\,.
\end{aligned}
$$

In the special case when our cevians are the medians of $\triangle ABC$, we have $m = n = p = \frac{1}{2}$ so that the last formula becomes

$$[A'B'C'] = \frac{(3k+1)^2}{4}[ABC]\,.$$

(Compare a note by Thébault, *Mathesis* (1939) page 311.)

To finish the problem we must show that the above value of $[A'B'C']$, when D, E, F are mid-points, is its maximum value (as the intersection point of the cevians ranges over the triangle's interior). Note that formula (3) tells us that to maximize $[A'B'C']$ we need only find the maximum of $[DEF]$. [*Editor's comment.* Seimiya stated in his solution that this is known: The mid-point triangle is the "cevian triangle" of largest area. Bellot's argument is so simple, however, we shall continue with it.] We just saw that when the cevians concur, $[DEF] = 2mnp[ABC]$, so the problem reduces to finding the maximum of the product mnp or, more conveniently from (4), of

$$(mnp)^2 = mnp(1-m)(1-n)(1-p)\,.$$

The product is composed of three pairs of factors such as $m(1-m)$, numbers that have a constant sum 1. Such a product achieves its maximum when both factors are equal, which means $m = \frac{1}{2}$. Similarly, $n = \frac{1}{2}$ and $p = \frac{1}{2}$, so that DEF is the mid-point triangle of $\triangle ABC$, and the argument is complete.

2710. [2002 : 56] *Proposed by Jaroslav Švrček, Palacký University, Olomouc, Czech Republic.*

Determine the point P on the semicircle Γ, constructed externally over the side AB of the square $ABCD$, such that $AP^2 + CP^2$ is maximal.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Without loss of generality, assume the side length of the square is two. Consider a rectangular coordinates system with the mid-point of AB as the origin so that $A = (-1, 0)$, $C = (1, -2)$ and $P = (\cos\theta, \sin\theta)$, $0 \le \theta \le \pi$. Then

$$\begin{aligned} AP^2 + CP^2 &= (1+\cos\theta)^2 + \sin^2\theta + (1-\cos\theta)^2 + (2+\sin\theta)^2 \\ &= 8 + 4\sin\theta\,. \end{aligned}$$

This is maximal when $\theta = \pi/2$ and so P is the mid-point of $\overset{\frown}{AB}$.

Also solved by the AUSTRIAN IMO TEAM; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; PIERRE BORNSZTEIN, Pontoise, France; PAOLO CUSTODI, Fara Novarese, Italy; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GAVIN JOHNSTONE, student, Dame Alice Owen's School, Potters Bay, UK; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; GERRY LEVERSHA, St. Paul's School, London, UK; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; VICTOR PAMBUCCIAN, ASU West, Phoenix, AZ, USA; JAWAD SADEK, NW Missouri State University, Maryville, MO, USA; ROBERT P. SEALY, Mount Allison University, Sackville, NB; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; KIRSTIN STROKORB, Winckelmann-Gymnasium, Stendal, Germany; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2711⋆. [2002 : 56] *Proposed by Catherine Shevlin, Wallsend upon Tyne, UK.*

Two circles, centres O_1 and O_2, of radii R_1 and R_2 ($R_1 > R_2$), respectively, are externally tangent at P. A common tangent to the two circles, not through P, meets O_1O_2 produced at Q, the circle with centre O_1 at A_1 and the circle with centre O_2 at A_2.
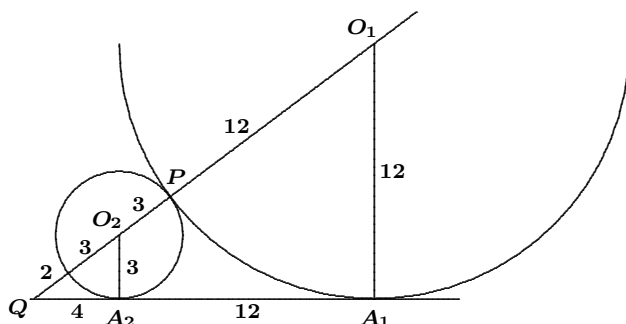
Prove or disprove that there exist simultaneously integer triangles QO_1A_1 and QO_2A_2.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.
A parameterization that works is

$$\begin{aligned} R_1 &= u^2(u^2-v^2) & R_2 &= v^2(u^2-v^2) \\ O_2Q &= v^2(u^2+v^2) & O_1Q &= u^2(u^2+v^2) \end{aligned}$$

where u and v are coprime and of opposite parity.

Another parameterization is

$$R_1 = (u+v)^2 (2uv) \qquad R_2 = (u-v)^2 (2uv)$$
$$O_2Q = (u-v)^2 (u^2v^2) \qquad O_1Q = (u+v)^2 (u^2+v^2)$$

where u and v are coprime and of opposite parity.

Apart from scale factors, the above two parameterizations are complete, as can be seen, since we require

$$O_2Q = \frac{R_2(R_1+R_2)}{(R_1-R_2)},$$

together with the usual Pythagorean triples.

Also solved by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; GERRY LEVERSHA, St. Paul's School, London, UK; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; DANIEL REISZ, Voncelles, France; Mª JESÚS VILLAR RUBIO, Santander, Spain (2 solutions); PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU, Bucharest, Romania. There was one incorrect submission.

2712. [2002 : 56] *Proposed by Antreas P. Hatzipolakis, Athens, Greece; and Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given $\triangle ABC$, let Y and Z be the feet of the altitudes from B and C. Suppose that the bisectors of $\angle BYC$ and $\angle BZC$ meet at X. Prove that $\triangle BXC$ is isosceles.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

The circle with diameter BC passes through Y and Z. The bisector of $\angle BYC$ intersects the circle in the point X such that the arc BX equals the arc CX. The bisector of $\angle BZC$ intersects the circle in the same point. Therefore, $\triangle BXC$ is an isosceles right triangle.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; AUSTRIAN IMO TEAM 2002; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College,

Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; GEOFFREY A. KANDALL, Hamden, CT, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; GERRY LEVERSHA, St. Paul's School, London, UK; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; JAWAD SADEK, NW Missouri State University, Maryville, MO, USA; JOEL SCHLOSBERG, student, New York University, NY, USA; TOSHIO SEIMIYA, Kawasaki, Japan; IFTIMIE SIMION, Roslyn H.S., N.Y. City State; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposers.
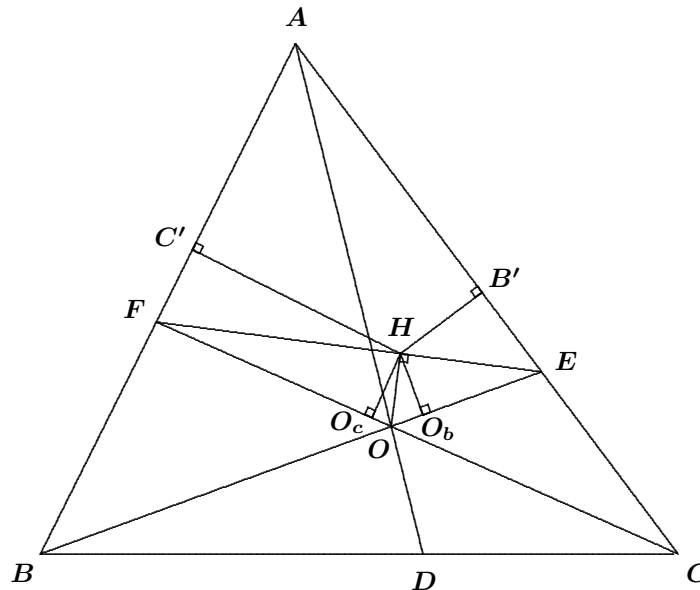
Editorial comments. Several solvers (Jefferys, Kunchev, Seimiya, Specht, Zvonaru and Ioniţă) noted that the claim of the problem is not true if either of the angles B and C of the triangle ABC is greater than $90°$. Seimiya noted that the point X does not exist for a right triangle ABC.

2713. [2002 : 110] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that O is an interior point of $\triangle ABC$, and that AO, BO and CO meet BC, CA and AB at D, E and F, respectively. Let H be the foot of the perpendicular from D to EF.

Prove that the feet of the perpendiculars from H to AF, FO, OE and EA are concyclic.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

If the line EF meets the line BC at D', then it is known that the points D and D' are harmonic conjugates of B and C, correspondingly, and since $\angle D'HD = 90°$, the point H lies on the Apolonian circle with diameter DD'. Hence HD is a bisector of the angle BHC and therefore

$$\angle FHB = \angle CHE \,. \tag{1}$$

If the line EF is parallel to the line BC, then D is the mid-point of BC and HD is a median, altitude and bisector in $\triangle BHC$, so that the equality (1) still holds.

Let C', O_c, O_b and B' be the feet of the perpendiculars from H to AF, FO, OE and EA. Then the points H, O_b, E and B' are concyclic and so are the points H, O_b, B and C'. Hence,

$$\angle B'O_bH \ = \ \angle B'EH \qquad \text{and} \qquad \angle C'O_bH \ = \ \angle C'BH \, .$$

Thus,

$$\angle B'O_bC' \ = \ \angle B'EH + \angle C'BH \, .$$

Similarly,

$$\angle B'O_cC' \ = \ \angle B'CH + \angle C'FH \, .$$

Then

$$
\begin{aligned}
\angle B'O_bC' - \angle B'O_cC' \ &= \ (\angle B'EH + \angle C'BH) - (\angle B'CH + \angle C'FH) \\
&= \ (\angle B'EH - \angle B'CH) - (\angle C'FH - \angle C'BH) \\
&= \ \angle CHE - \angle FHB \\
&= \ 0;
\end{aligned}
$$

the last equality follows from (1). Therefore,

$$\angle B'O_bC' = \angle B'O_cC' \, ,$$

which shows that the points C', O_c, O_b and B' are concyclic.

> *Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; G. TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*