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Editor: Léo Sauvé, Algonquin College, 281 Echo Drive, Ottawa, Ontario, Canada K1S 1N3.

Managing Editor: F.G.B. Maskell, Algonquin College, 200 Lees Ave., Ottawa, Ontario, Canada K1S 0C5.

Typist-compositor: Nghi Chung.

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CONTENTS

ON SIX CONCYCLIC POINTS

JAN VAN DE CRAATS

A problem from the 1981 Hungarian Mathematical Olympiad reads as follows [1]:

Six points are given on a circle. Choosing any three of them (this can be done in 20 ways), the orthocentre of the triangle determined by these three points is connected by a straight line to the centroid of the triangle determined by the remaining three points. Prove that these 20 lines all go through a fixed point.

This problem provides an excellent opportunity to illustrate complex number methods in plane geometry.

Let the given circle be the unit circle in the complex plane, and, identifying points with their affixes, let $t_1, t_2, \ldots, t_6$ be the six given points on the circle. It will be convenient to denote by $s_{ij\ldots n}$ the point $t_i + t_j + \ldots + t_n$.

The *orthocentre* h_{123} of the triangle with vertices t_1, t_2, t_3 is

$$h_{123} = s_{123} = t_1 + t_2 + t_3$$

since, for example, $0, t_2, t_2+t_3, t_3$ are the vertices of a rhombus and $s_{123} - t_1 = t_2 + t_3$ is orthogonal to the side joining vertices t_2 and t_3 of the triangle. The *centroid* g_{456} of the triangle with vertices t_4, t_5, t_6 is

$$g_{456} = \tfrac{1}{3}s_{456} = \tfrac{1}{3}(t_4 + t_5 + t_6).$$

It is obvious that the point

$$p = \tfrac{1}{4}s_{123456} = \tfrac{1}{4}(t_1 + t_2 + \ldots + t_6)$$

lies on the line through h_{123} and g_{456}, since

$$p = \tfrac{1}{4}h_{123} + \tfrac{3}{4}g_{456}.$$

Because p is symmetric in $t_1, t_2, \ldots, t_6$, it is the required fixed point which solves our problem.

An easy generalization is obtained as follows. Consider the line

$$z = \lambda s_{123}, \qquad \lambda \in R,$$

which is the Euler line of the triangle with vertices t_1, t_2, t_3. (If $s_{123} = 0$, the line collapses into a point, the origin, but this will cause no algebraic difficulty in the sequel.) This Euler line contains the circumcentre 0, the orthocentre $h_{123} = s_{123}$, the centroid $g_{123} = \frac{1}{3}s_{123}$, and also the *nine-point-centre* $n_{123} = \frac{1}{2}s_{123}$ (see Morley [2]).

For any $\rho,\sigma \in R$, where $\rho + \sigma \neq 0$, the line through ρs_{123} and σs_{456} contains the point

$$q = \frac{\rho\sigma}{\rho+\sigma}\cdot s_{123456} = \frac{\sigma}{\rho+\sigma}\cdot\rho s_{123} + \frac{\rho}{\rho+\sigma}\cdot\sigma s_{456}.$$

Since q is symmetric in $t_1,t_2,\ldots,t_6$, it divides each of the 20 (if $\rho \neq \sigma$) or 10 (if $\rho = \sigma$) segments joining ρs_{ijk} and σs_{lmn} in the ratio $\rho : \sigma$. If $\rho + \sigma = 0$, then these segments are all parallel and of equal length, because then

$$\rho s_{ijk} - \sigma s_{lmn} = \rho s_{123456}.$$

In the original Olympiad problem, we have $\rho = 1$ and $\sigma = 1/3$, but now we see that we can have an infinity of variations. A trivial one is when $\rho = \sigma = 1/3$, giving the *barycentre*

$$b_{123456} = \tfrac{1}{6}s_{123456}$$

of the six points as the midpoint of each segment joining g_{ijk} to the "supplementary" centroid g_{lmn}.

Perhaps more impressive is the fact that the point $p = \frac{1}{4}s_{123456}$, which is the common point of the 20 lines mentioned in the Olympiad problem, is also the midpoint of the 10 segments joining "supplementary" nine-point-centres (take $\rho = \sigma = \frac{1}{2}$). Thus, in general we have no fewer than 30 remarkable lines through p.

As a final remark, we note that chains of theorems can be obtained as follows: From *seven* points $t_1,t_2,\ldots,t_7$ on the unit circle, we delete one point t_i. By one of the above processes, we get a point

$$q_i = \mu(s_{1234567} - t_i), \qquad \mu \in R,$$

which is symmetric in the six remaining t_j. Thus, for fixed $\mu \in R$ we have seven points q_i. If t ranges over the unit circle, then

$$z = \mu(s_{1234567} - t)$$

describes a circle with radius $|\mu|$ and centre $\mu s_{1234567}$, and the seven points q_i are on this circle.

From *eight* points t_i on the unit circle, we get eight points

$$q_i = \mu(s_{12345678} - t_i), \qquad \mu \in R,$$

all lying on the circle

$$z = \mu(s_{12345678} - t), \qquad |t| = 1.$$

And so on.

REFERENCES

1. M.S. Klamkin, "The Olympiad Corner: 29", this journal 7 (1981) 267-273.

2. Frank Morley and F.V. Morley, *Inversive Geometry*, Chelsea, New York, 1954, p. 186.

Mathematics Department, Leiden University, Postbus 9512, 2300 RA Leiden, The Netherlands.

* * *

ANOTHER PROOF OF THE A.M.-G.M. INEQUALITY

GEORGE TSINTSIFAS

Let $x_i > 0$, $i = 1,2,\ldots,n$. The familiar A.M.-G.M. inequality states that

$$(\Pi x_i)^{1/n} \le \frac{\Sigma x_i}{n},$$

where the products and sums, here and later, are for $i = 1,2,\ldots,n$, and this is equivalent to

$$\frac{1}{n^n} \ge \frac{\Pi x_i}{(\Sigma x_i)^n}. \tag{1}$$

While working on a geometrical problem, I came across the following proof of (1), which, as far as I know, is new.

For $x \ge 0$ we set

$$P(x) = \Pi(x+x_i), \qquad S(x) = \Sigma(x+x_i),$$

and let

$$f(x) = \frac{P(x)}{\{S(x)\}^n}.$$

Differentiating, we have

$$f'(x) = \frac{S(x)P'(x) - n^2P(x)}{\{S(x)\}^{n+1}}. \tag{2}$$

The logarithmic derivative of $P(x)$ is

$$\frac{P'(x)}{P(x)} = \Sigma\frac{1}{x+x_i},$$

and

$$S(x)\cdot\frac{P'(x)}{P(x)} = \Sigma(x+x_i)\cdot\Sigma\frac{1}{x+x_i} \ge n^2$$

follows easily from the fact that $a/b + b/a \geq 2$ when $a,b > 0$. Hence, from (2),

$$f'(x) = \frac{P(x)\{S(x)P'(x)/P(x) - n^2\}}{\{S(x)\}^{n+1}} \geq 0.$$

It follows that $f(x) \geq f(0)$, and we have the following generalization of the A.M.-G.M. inequality:

$$\frac{\Pi(x+x_i)}{\{\Sigma(x+x_i)\}^n} \geq \frac{\Pi x_i}{\{\Sigma x_i\}^n}. \tag{3}$$

The limit of the left member of (3) as $x \to \infty$ is $1/n^n$, and (1) follows.

Platonos 23, Thessaloniki, Greece.

*   *   *

THE PUZZLE CORNER

Puzzle No. 18: Deletion (14, 13)

In mathematics, seek
The FINE that saves relations,
And PRIME, the transformations
Bicontinuous, biunique.

Puzzle No. 19: Word deletion (10; 4, 6)

A seconds pendulum, to make:
Just 24.8 WHOLEs you take
Down to the SECOND of the bob.
The FIRST will then be right, if so's the job.

ALAN WAYNE
Holiday, Florida

Answer to Puzzle No. 16 [1982: 158]:

$$\begin{array}{r} 91860 \\ \underline{3} \\ 275580 \end{array}, \quad \text{ZERO} = 5041, \text{ SLEEP} = 98002;$$

$$\begin{array}{r} 97034 \\ \underline{6} \\ 582204 \end{array}, \quad \text{ZERO} = 2417.$$

Answer to Puzzle No. 17 [1982: 158]: ONE = 512, EIGHT = 24389, ELEVEN = 272621, THIRTEEN = 98469221 (not prime unless R = 6).

*   *   *

THE OLYMPIAD CORNER: 36

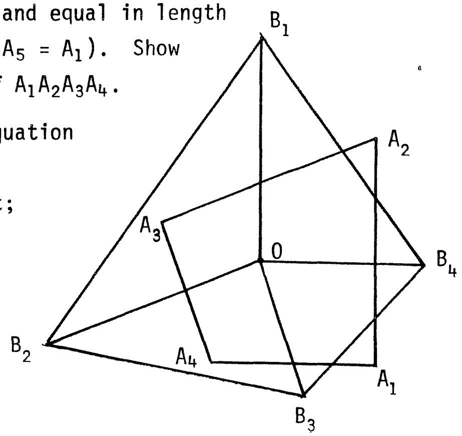
M.S. KLAMKIN

Several national Olympiads take place in May every year, in preparation for the International Mathematical Olympiad, which is scheduled this year for July in Hungary. I give below the problems set at the Canadian and U.S.A. 1982 Olympiads.

The 14th Canadian Mathematics Olympiad took place on 5 May 1982. The problems were set by the Canadian Mathematics Olympiad Committee, consisting of G. Butler (Chairman), P. Arminjon, C. Fisher, M.S. Klamkin, T. Lewis, A. Liu, J. Schaer, and J. Walker. Solutions will appear soon in this column.

FOURTEENTH CANADIAN MATHEMATICS OLYMPIAD

5 May 1982 — Time: 3 hours



1. In the diagram, OB_i is parallel and equal in length to A_iA_{i+1} for $i = 1,2,3,4$ (with $A_5 = A_1$). Show that the area of $B_1B_2B_3B_4$ is twice that of $A_1A_2A_3A_4$.

2. If a,b,c are the roots of the equation $x^3 - x^2 - x - 1 = 0$,
   (i) show that a,b,c are all distinct;
   (ii) show that

$$\frac{b^{1982}-c^{1982}}{b-c}+\frac{c^{1982}-a^{1982}}{c-a}+\frac{a^{1982}-b^{1982}}{a-b}$$

is an integer.

3. Let R^n be the n-dimensional Euclidean space. Determine the smallest number $g(n)$ of points of a set in R^n such that every point in R^n is at irrational distance from at least one point in that set.

4. Let p be a permutation of the set $S_n = \{1,2,\ldots,n\}$. An element $j \in S_n$ is called a *fixed point* of p if $p(j) = j$. Let f_n be the number of permutations having no fixed point, and g_n the number of permutations with exactly one fixed point. Show that $|f_n - g_n| = 1$.

5. The altitudes of a tetrahedron ABCD are extended externally to points A', B', C', and D', respectively, where

$$AA' = k/h_a, \quad BB' = k/h_b, \quad CC' = k/h_c, \quad DD' = k/h_d.$$

Here, k is a constant and h_a denotes the length of the altitude of ABCD from vertex A, etc. Prove that the centroid of the tetrahedron A'B'C'D' coincides with the centroid of ABCD.

*

Now follow the problems set at the 11th U.S.A. Mathematical Olympiad, which took place on 4 May 1982. They were prepared by the Examination Committee of the U.S.A. Mathematical Olympiad, consisting of M.S. Klamkin (Chairman), A. Liu, and C. Rousseau. As in recent years, solutions are expected to be published in the November 1982 issue of *Mathematics Magazine*. A solution to Problem 2 below has already appeared in this journal in connection with Crux 639 [1982: 145].

ELEVENTH U.S.A. MATHEMATICAL OLYMPIAD

4 May 1982

1. In a party with 1982 persons, among any group of four there is at least one person who knows each of the other three. What is the minimum number of people in the party who know everyone else?

2. Let $S_r = x^r + y^r + z^r$. Find all integer pairs $\{m,n\}$ such that

$$\frac{S_{m+n}}{m+n} = \frac{S_m}{m}\cdot\frac{S_n}{n}$$

is meaningful and true for all real triples (x,y,z) with $S_1 = x+y+z = 0$, having given that two solutions are $\{m,n\} = \{3,2\}$ and $\{5,2\}$.

3. If point A_1 is in the interior of an equilateral triangle ABC and point A_2 is in the interior of triangle A_1BC, prove that

$$I.Q.(A_1BC) > I.Q.(A_2BC),$$

where the *isoperimetric quotient* of a figure F is defined by

$$I.Q.(F) = \frac{\text{Area }(F)}{\{\text{Perimeter }(F)\}^2}.$$

4. Prove that there exits a positive integer k such that $k\cdot 2^n + 1$ is composite for every positive integer n.

5. Given that A, B, and C are three interior points of a sphere S such that AB and AC are perpendicular to the diameter of S through A. Through A, B, and C, two spheres can be constructed which are both tangent to S. Prove that the sum of their radii is equal to the radius of S.

*

I now give solutions, comments, or corrections to various problems which have appeared earlier in this column.

6. [1981: 43] Determine all pairs (x,y) of integers satisfying the equation

$$x^3 + x^2y + xy^2 + y^3 = 8(x^2 + xy + y^2 + 1).$$

Comment.

A solution of this problem can be found in the solution of Crux 496 [1980: 323].

6. [1981: 44 ; 1982: 104] Find, with proof, the digit immediately to the left and the digit immediately to the right of the decimal point in the decimal expansion of the number

$$(\sqrt{2}+\sqrt{3})^{1980}.$$

II. *Solution by Stan Higgins, K.R. McLean, and many others. (Reprinted by permission from The Mathematical Gazette 66 (March 1982) 47.)*

Let $\alpha = \sqrt{3}+\sqrt{2}$ and $\beta = \sqrt{3}-\sqrt{2}$. Then $\alpha^2 = 5+2\sqrt{6} = a$, say, and $\beta^2 = 5-2\sqrt{6} = b$. So a and b are the roots of the equation $x^2 - 10x + 1 = 0$. For each non-negative integer n let $S_n = a^n + b^n$. Then $S_0 = 2$, $S_1 = 10$ and

$$S_n - 10S_{n-1} + S_{n-2} = 0.$$

So each S_n is an integer and $S_n + S_{n-2} \equiv 0 \pmod{10}$. Hence $S_{990} \equiv -S_0 \equiv 8 \pmod{10}$.

Thus $\alpha^{1980} + \beta^{1980} = S_{990}$ is an integer whose last digit is 8. Also β^{1980} is very small. In fact

$$\log_{10}(\beta^{1980}) = 1980\log_{10}\beta \approx -985.6$$

and so

$$(\sqrt{2}+\sqrt{3})^{1980} = \alpha^{1980} = \ldots 7.\underbrace{999\ldots9}_{\leftarrow\, 985\, \rightarrow}\ldots\quad.$$

1. [1981: 268] Six points are given on a circle. Choosing any three of them (this can be done in 20 ways), the orthocentre of the triangle determined by these three points is connected by a straight line to the centroid of the triangle determined by the remaining three points. Prove that these 20 lines all go through a fixed point.

Comment.

A solution of this problem appears in the article "On Six Concyclic Points" on page 160 of this issue.

1. [1981: 298] Which of the following two numbers is larger:

$$\sqrt[7]{7+\sqrt[7]{7}} - \sqrt[7]{7}, \qquad \sqrt[7]{7} - \sqrt[7]{7-\sqrt[7]{7}}\ ?$$

Solution.

By convexity, for $n \geq 1$, we have

$$\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2} \leq \sqrt[n]{\frac{a+b}{2}}.$$

Thus

$$\sqrt[7]{7 + \sqrt[7]{7}} + \sqrt[7]{7 - \sqrt[7]{7}} \leq 2\sqrt[7]{7},$$

and it follows that of the two given numbers the second is larger.

2. [1981: 298] Justify the following assertion: If the positive numbers $x_1, x_2, \ldots, x_n$ have product 1, then

$$\sum_{i=1}^{n} x_i^{n} \leq \sum_{i=1}^{n} x_i^{n+1}.$$

Solution.

From the A.M.-G.M. inequality, we have (all sums and products being for $i = 1, 2, \ldots, n$)

$$\Sigma x_i^n = \sqrt[n]{\Pi x_i}\cdot\Sigma x_i^n \leq \frac{\Sigma x_i}{n}\cdot\Sigma x_i^n \tag{1}$$

and, from the Power Mean inequality,

$$\frac{\Sigma x_i}{n} = \left\{\left(\frac{\Sigma x_i}{n}\right)^n\right\}^{1/n} \leq \left\{\frac{\Sigma x_i^n}{n}\right\}^{1/n}. \tag{2}$$

Now, from (1), (2), and the Power Mean inequality,

$$\Sigma x_i^n \leq n\left\{\frac{\Sigma x_i^n}{n}\right\}^{(n+1)/n} \leq \Sigma x_i^{n+1}.$$

3. [1981: 298] Determine the pairs (m,n) of natural numbers for which the equation

$$\frac{1 + \sin^2 nx}{1 + \sin^2 mx} = \sin nx$$

has real solutions.

Comment.

This problem was incorrectly stated in [1981: 298]. A solution to the above corrected version would be welcome.

4. [1981: 298] Show that

$$\sum_{k=1}^{n-1} \cot(k\pi/n)\cdot\cos^2(k\pi/n) = 0.$$

Solution.

Changing $\cos^2(k\pi/n)$ into $1 - \sin^2(k\pi/n)$, the desired result follows immediately from the known elementary sums

$$\sum_{k=1}^{n-1} \cot\frac{k\pi}{n} = 0 = \sum_{k=1}^{n-1} \sin\frac{2k\pi}{n}.$$

6. [1981: 298] If n is a given natural number, determine the largest and least values of the expression

$$\prod_{k=1}^{n} (2 - \cos^2\alpha_k) + \prod_{k=1}^{n} \cos^2\alpha_k.$$

Solution I.

Letting $x_k = \sin^2\alpha_k$, the given expression becomes

$$S \equiv \prod_{k=1}^{n} (1 + x_k) + \prod_{k=1}^{n} (1 - x_k), \qquad 0 \le x_k \le 1.$$

Since S is linear in each variable x_k, the maximum and minimum values are taken at the endpoints of the intervals $0 \le x_k \le 1$. Thus

$$2 \le S \le 2^n.$$

Solution II.

With the same notation as in solution I, we have

$$S = 2(1 + \Sigma x_1x_2 + \Sigma x_1x_2x_3x_4 + \ldots).$$

Hence

$$2 \le S \le 2\left(1 + \binom{n}{2} + \binom{n}{4} + \ldots\right) = 2^n.$$

The maximum is attained if and only if all $x_k = 1$; the minimum is attained if (but not only if) all $x_k = 0$ (for example, we could have $x_1 = 1$, $x_k = 0$ for $k \ge 2$).

7. [1981: 298] Show that, for nonnegative numbers a,b,c,d,

$$(a+c)(b+d)(2a+c+d)(2b+c+d) \ge 4cd(2a+c)(2b+d). \qquad (1)$$

Solution by Guy Even, student, The Hebrew Re'ally High School, Haifa, Israel.

The solution is based on successive applications of the A.M.-G.M. inequality. From

$$\frac{(2a+c)+d}{2} \ge \sqrt{d(2a+c)} \qquad \text{and} \qquad \frac{(2b+d)+c}{2} \ge \sqrt{c(2b+d)},$$

we get

$$(2a+c+d)(2b+c+d) \geq 4\sqrt{cd(2a+c)(2b+d)}. \qquad (2)$$

Similarly, from

$$a + c = \frac{(2a+c)+c}{2} \geq \sqrt{c(2a+c)} \quad \text{and} \quad b + d = \frac{(2b+d)+d}{2} \geq \sqrt{d(2b+d)},$$

we get

$$(a+c)(b+d) \geq \sqrt{cd(2a+c)(2b+d)}. \qquad (3)$$

Finally, (1) follows from (2) and (3). Equality holds when $a = b = 0$, and $cd = 0$ or $c = d$.

8. [1981: 298] Let G denote the geometric mean of the n positive numbers a_i and, for natural numbers k, let P_k denote the kth power mean, i.e.,

$$P_k = \left(\sum_{i=1}^{n} a_i^k/n\right)^{1/k}.$$

Show that

$$(n-1)G^n \leq nP_1P_{n-1}^{n-1} - P_n^n.$$

Solution by G.P. Henderson, Campbellcroft, Ontario.

The proposed inequality is equivalent to

$$\Sigma a_i \cdot \Sigma a_i^{n-1} - \Sigma a_i^n \geq n(n-1)\Pi a_i,$$

where the sums and product are for $i = 1,2,\ldots,n$, and this can be rewritten as

$$\Sigma a_i(S - a_i^{n-1}) \geq n(n-1)\Pi a_i, \qquad (1)$$

where $S = \Sigma a_j^{n-1}$. The product of the left side of (1), when expanded, contains $n(n-1)$ terms of the form $a_ia_j^{n-1}$, $j \neq i$, and the product of all these terms is $\Pi a_i^{n(n-1)}$. If we apply the A.M.-G.M. inequality to these $n(n-1)$ terms, we obtain (1).

9. [1981: 299] Show that, for an arbitrary pair (n,k) of natural numbers, there is a unique natural number $f(n,k)$ which satisfies the relation

$$(\sqrt{n+1} + \sqrt{n})^k = \sqrt{f(n,k)+1} + \sqrt{f(n,k)}. \qquad (1)$$

Solution.

It follows from (1) that also

$$(\sqrt{n+1} - \sqrt{n})^k = \sqrt{f(n,k)+1} - \sqrt{f(n,k)}. \qquad (2)$$

From (1) and (2), we therefore have uniquely

$$f(n,k) = \tfrac{1}{4}\{(\sqrt{n+1} + \sqrt{n})^k - (\sqrt{n+1} - \sqrt{n})^k\}^2, \tag{3}$$

and expanding the right side of (3) shows that $f(n,k)$ is an integer.

10. [1981: 299] For which real numbers x,y is the following inequality satisfied?

$$\sqrt[3]{\frac{x^3+y^3}{2}} \ge \sqrt{\frac{x^2+y^2}{2}}.$$

Solution.

The inequality holds only if $x+y \ge 0$. If $x = 0$ [$y = 0$], it holds for all $y \ge 0$ [$x \ge 0$]. Since equality holds for all pairs (x,y) with $x = y \ge 0$, we have only left to find the solutions for which $x+y \ge 0$, $xy \ne 0$, and $x \ne y$.

Raising both sides to the sixth power and rearranging, we obtain the equivalent inequality

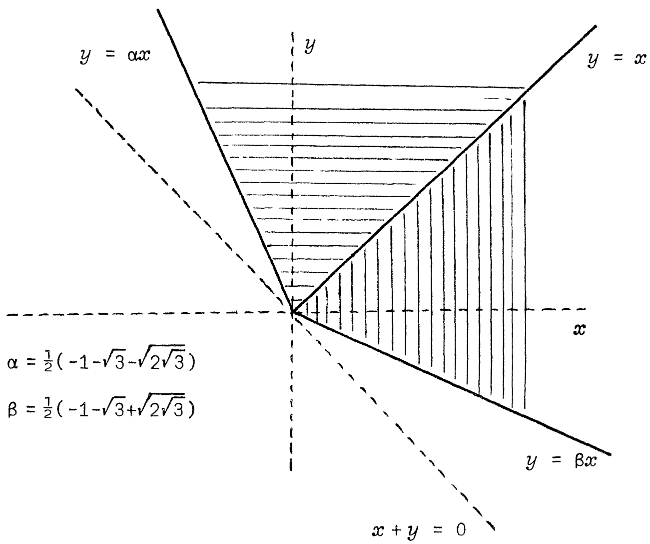
$$x^6 - 3x^4y^2 + 4x^3y^3 - 3x^2y^4 + y^6 \ge 0.$$

Dividing by $x^6 > 0$ and setting $t = y/x \ne 0$ or 1, we get

$$t^6 - 3t^4 + 4t^3 - 3t^2 + 1 \ge 0.$$

The left member of this inequality is divisible by $(t-1)^2 > 0$, and the division yields

$$t^4 + 2t^3 + 2t + 1 \ge 0.$$



Now, dividing by $t^2 > 0$ and then setting $u = t + 1/t$, we obtain

$$u^2 + 2u - 2 \geq 0,$$

from which

$$u = t + \frac{1}{t} \leq -1-\sqrt{3} \qquad \text{or} \qquad u = t + \frac{1}{t} \geq -1+\sqrt{3}. \tag{1}$$

The values of t that satisfy the first inequality in (1) are

$$t \leq \tfrac{1}{2}(-1-\sqrt{3}-\sqrt{2\sqrt{3}}) \approx -2.30 \qquad \text{and} \qquad -0.44 \approx \tfrac{1}{2}(-1-\sqrt{3}+\sqrt{2\sqrt{3}}) \leq t < 0,$$

and, since $t = y/x$, the corresponding points (x,y) are those in the second and fourth quadrants in the shaded portion of the figure. The second inequality in (1) is satisfied by all $t = y/x > 0$, which correspond to the points (x,y) in the first quadrant (this was obvious from the start by the Power Mean inequality).

Equality holds only along the three rays drawn solid in the figure:

$$y = \alpha x, \quad x \leq 0; \qquad y = x, \quad x \geq 0; \qquad y = \beta x, \quad x \geq 0.$$

2. [1982: 12] Prove that if a,b,c are the lengths of the sides of a scalene triangle and $a+b = 2c$, then the line joining the incentre and the circumcentre of the triangle is parallel to one side of the triangle.

Solution by Reuven R. Rottenberg, Technion-I.I.T., Haifa, Israel.

The problem is incorrect as stated. This is lengthy but not difficult to prove analytically, but it is convincing enough to draw a careful diagram with, say, $a = 5$, $b = 7$, $c = 6$ (so that $a+b = 2c$). It will then be obvious that the line joining the incentre and circumcentre is not parallel to any side of the triangle. But the problem becomes correct when "circumcentre" is replaced by "centroid", as we now show.

Let ABC be a scalene triangle with BC = a, etc., where $a+b = 2c$. If the bisector of angle C meets AB at E, then we have EB/AE = a/b, and so

$$\frac{c}{\text{AE}} = \frac{\text{AE} + \text{EB}}{\text{AE}} = \frac{a+b}{b} = \frac{2c}{b},$$

from which AE = $b/2$. Now the bisector of angle A meets CE at the incentre I, and we have

$$\frac{\text{CI}}{\text{IE}} = \frac{b}{b/2} = 2.$$

If G is the centroid of the triangle, then CG meets AB at its midpoint M, and CG/GM = 2. Finally,

$$\frac{\text{CI}}{\text{IE}} = \frac{\text{CG}}{\text{GM}} = 2$$

implies that IG is parallel to EM, that is, to AB.

2. [1982: 133] The fractional part $\{x\}$ of x is defined as the smallest nonnegative number such that $x - \{x\}$ is an integer. For example, $\{1.6\} = 0.6$ and $\{\pi\} = \pi - 3$. Show that

$$\lim_{n\to\infty}\{(2+\sqrt{3})^n\} = 1.$$

Comment.

This problem has already been solved in this column. See [1981: 113].

Editor's Note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

SWEET ARE THE USES OF MATHEMATICS

In his history of astronomy at the university of Jena, Germany, published there in 1937, O. Knopf mentions how the astronomer and mathematician Erhard Weigel tried to show that mathematics was the source of many virtues: the love of wisdom, good manners, sweet temper, discretion, truthfulness, moderation, chastity, and courage. Courage, for example, "is acquired extracting roots, because here one is asked to divide and has no divisor; one has, therefore, to venture into the void to find a divisor, which is the root from which the body of the product is formed". "He who divides is devout, and since he himself does not know the quotient, he—so to speak—raises his eye and asks the Lord that He may lead him to the sought after but hitherto hidden truth."

Erhard Weigel, born on 16 December 1625, became professor in Jena in 1653 and remained there until his death on 21 March 1699. He was not a "nut" but a highly respected teacher in his time (and even long after his death). He taught Leibniz mathematics, was an authority in his day on all questions of mechanics and technology, tried to reform the prevailing pedantic system of instruction, and suggested alternating intellectual and manual courses.

HAYO AHLBURG
Benidorm, Spain

* * *

A BORDERED PRIME MAGIC SQUARE

Composed of distinct primes, this pandiagonal sixth-order magic square has a fourth-order magic square in its center.

107	419	197	461	239	47
101	227	269	53	431	389
449	347	263	59	311	41
11	173	257	401	149	479
359	233	191	467	89	131
443	71	293	29	251	383

ALLAN WM. JOHNSON JR.
Washington, D.C.

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before January 1, 1983, although solutions received after that date will also be considered until the time when a solution is published.

MORE PROBLEMS ARE URGENTLY NEEDED

726. [1982: 78] *Correction.* Replace "circumdiameter" by "circumradius".

751. *Proposed by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.*

Who utters this sound?

$$\begin{array}{r} \text{CALM} \\ \text{BAA} \\ \underline{\text{BAA}} \\ 2395 \end{array}$$

The sum of this decimal cryptarithmic addition will answer the question.

752. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

For $i = 1,2,3$, let A_i be the vertices of a triangle with angles α_i, sides a_i, circumcenter O, and inscribed circle γ. The lines A_iO intersect γ in P_i and Q_i.

(a) Prove that

$$P_1Q_1 : P_2Q_2 : P_3Q_3 = f(\cos\alpha_1) : f(\cos\alpha_2) : f(\cos\alpha_3),$$

where $f(x)$ is a function to be determined.

(b) Prove or disprove that $a_2 = a_3$ if and only if $P_2Q_2 = P_3Q_3$.

753. *Proposed by H. Kestelman, University College, London, England.*

When an $n\times n$ matrix A is factorized as $\Omega T\Omega^{-1}$, with Ω unitary and T upper triangular, the diagonal elements $d_1, d_2, \ldots, d_n$ of T are such that

$$(t-d_1)(t-d_2)\ldots(t-d_n) = \det(tI-A).$$

Can the numbers d_r be permuted arbitrarily, given A, by suitably choosing Ω?

754. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $\sigma_n = A_0A_1\ldots A_n$ be an n-simplex in R^n and P an interior point of σ_n. For $i = 0,1,\ldots,n$, let A_i' be the intersection of the line A_iP with the $(n-1)$-simplex

$$\sigma_{n-1} = A_0A_1\ldots A_{i-1}A_{i+1}\ldots A_n.$$

Show that

$$\sum_{i=0}^{n} \overrightarrow{A_iA_i'} = \vec{0}$$

if and only if P is the centroid of σ_n.

755. *Proposed by Lászlo Csirmaz, Mathematical Institute, Hungarian Academy of Sciences.*

Find the locus of points with coordinates

$$(\cos A + \cos B + \cos C,\ \ \sin A + \sin B + \sin C)$$

(a) if A,B,C are real numbers with $A + B + C = \pi$;

(b) if A,B,C are the angles of a triangle.

756. *Proposed by Yang Lu and Zhang Jingzhong, China University of Science and Technology, Hefei, Anhui, People's Republic of China.*

Given three vertices A, B, and C of a parallelogram, find the fourth vertex D, using only a rusty compass.

757*. *Proposed by Dan Pedoe and Alfred Aeppli, University of Minnesota.*

Given only two distinct points A and B, prove or disprove that the midpoint of the segment AB can be found, using only a rusty compass.

758. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Find a necessary and sufficient condition on p,q,r so that the roots of the equation

$$x^3 + px^2 + qx + r = 0$$

are the vertices of an equilateral triangle in the complex plane.

759. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Given are four congruent circles intersecting in a point O, and a quadrilateral ABCD circumscribing these circles with each side of the quadrilateral tangent to two circles. Prove that quadrilateral ABCD is cyclic.

760. *Proposed by Jordi Dou, Barcelona, Spain.*

Given a triangle ABC, construct with ruler and compass, on AB and AC as bases, *directly* similar isosceles triangles ABX and ACY such that BY = CX. Prove that there are exactly two such pairs of isosceles triangles.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

618. [1981: 80; 1982: 82] *Proposed by J.A.H. Hunter, Toronto, Ontario.*

For i = 1,2,3, let I_i be the centres and r_i the radii of the three Malfatti circles of a triangle ABC. Calculate the sides a = BC, b = CA, and c = AB of the triangle in terms of the r_i.

II. *Comment by Dimitris Vathis, Chalcis, Greece.*

In response to the editor's request [1982: 84], I give a reference in the literature to this *inverse Malfatti problem*. The problem appears and is solved in Pallas [1]. Pallas, a professor of mathematics who died in 1981, first obtains, in much the same way, equation (4) of solution I. Although he does not solve (4) explicitly, as was done in solution I, he then explains how explicit expressions for the sides a,b,c can easily be found from the root of (4).

REFERENCE

1. Aristides Pallas, *Great Algebra*, Athens, 1957 (in Greek), pp. 103-104.

* * *

646. [1981: 178] *Proposed by J. Chris Fisher, University of Regina.*

Let M be the midpoint of a segment AB.

(a) What is the locus of a point the product of whose distances from A and B is the square of its distance from M; that is

$$\{X: XA \cdot XB = XM^2\}.$$

(b) In a circle γ through M and B, the three chords BP, BP', and BM satisfy

$$BP = BP' = \sqrt{2}BM.$$

Prove that the tangent to γ at M meets the lines BP and BP' (extended) in points X and Y, respectively, that are equidistant from M. Note that this fact suggests a construction for the locus of part (a), since X and Y satisfy

$$XA \cdot XB = YA \cdot YB = XM^2.$$

(c) What is the locus of a point for which the absolute value of the difference of its distances from A and B equals $\sqrt{2}$ times its distance from M; that is,

$$\{X: |XA - XB| = \sqrt{2}XM\}.$$

Solution by Jordi Dou, Barcelona, Spain.

(a) Since

$$XA^2 + XB^2 = 2XM^2 + \tfrac{1}{2}AB^2 \qquad (1)$$

and $XM^2 = XA\cdot XB$, we have $(XA - XB)^2 = \frac{1}{2}AB^2$. Thus

$$|XA - XB| = AB/\sqrt{2}$$

and the required locus is an equilateral hyperbola H with foci A and B.

(c) If $|XA - XB| = \sqrt{2}XM$, then $(XA - XB)^2 = 2XM^2$ and, from (1),

$$XA\cdot XB = \tfrac{1}{2}(XA^2 + XB^2 - 2XM^2) = (\tfrac{1}{2}AB)^2.$$

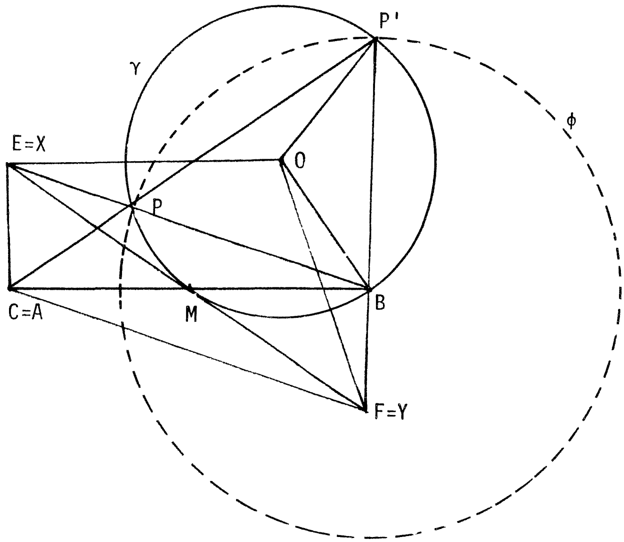
It follows that the required locus is a Bernoulli lemniscate L [1, p. 114, Ex. 4].

(b) Let O be the centre of γ and C the intersection of PP' and BM. Since γ is the inverse of line PP' with respect to circle ϕ with centre B and radius $\sqrt{2}$BM (see figure), we have

$$2BM^2 = BP^2 = BM\cdot BC;$$

hence $2BM = BC$ and $C = A$.

Now let E and F be points on BP and BP', respectively, such that BEAF is a parallelogram. Then M is the midpoint of EF. Since EP = EA = FB and EB = AF = FP', it follows that triangles OEB and OFP' are congruent, and so OE = OF. Thus OM $\perp$ EF,



so EF is tangent to γ at M, and E = X, F = Y, from which the desired result follows. □

For any circle γ_i through M and B that cuts circle ϕ in points P_i and P_i', the tangent to γ_i at M intersects BP_i and BP_i' in points X_i and Y_i on the hyperbola H.

It is very curious that a simple condition on products in part (a) leads to a constant difference $|XA - XB|$, while a simple condition on differences in part (c) leads to a constant product $XA \cdot XB$. Well, perhaps not so curious after all, for the hyperbloa H in (a) and the lemniscate L in (c) are inverses of each other with respect to the circle with diameter AB [1, p. 119, Ex. 11].

Also solved by BIKASH K. GHOSH, Bombay, India; J.T. GROENMAN, Arnhem, The Netherlands; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

REFERENCE

1. H.S.M. Coxeter, *Introduction to Geometry*, Second Edition, Wiley, 1969.

* * *

647. [1981: 178] *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

A cake (a rectangular parallelepiped) with icing on the top and the four sides is to be cut (using vertical cuts only) and shared by n persons.

(a) If the top is square, show how to cut the cake so that each person gets the same amounts of cake and icing as everybody else.

(b) Do the same for the general case of a rectangular top.

(c) In (b), is there a way to cut the cake so that each person's share is in one piece?

I. *Solution to parts (a) and (b) by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

(a) The solution to this part is a piece of cake. If the top is a regular polygon (such as a square), divide the perimeter into n equal parts and make cuts from the points of division to the center of the polygon. In each case, the top of the piece is a triangle or a union of triangles with constant altitude (equal to the apothem of the polygon) and base(s) of total length p/n, where p is the perimeter of the polygon. Thus the pieces all have the same top area and the same portion of the perimeter of the original polygon, so all pieces have the same amounts of cake and icing. *Bon appétit.*

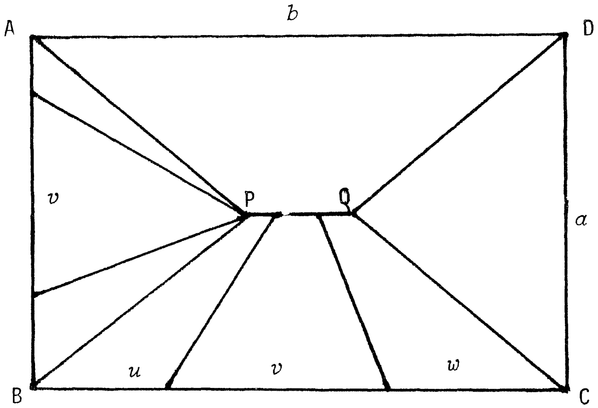
(b) If the top is star-like from an interior point P (i.e., if every point in the figure is visible from P), and if the perimeter is a union of straight segments (as in a rectangle), then divide each segment of the perimeter into n parts and make cuts from the extremities and the points of division to point P. Give one part from each group to each person. The cake has now been fairly divided, but the pieces in one person's share are, in general, not contiguous. *Bon appétit* again, and watch out for the crumbs.

II. *Solution by Leroy F. Meyers, The Ohio State University.*

All three parts of the problem are subsumed into the following generalization. Suppose that the perimeter of an $a \times b$ rectangle ABCD, where AB $= a \le b =$ BC, is subdivided into n segments, of lengths $v_1, v_2, \ldots, v_n$ (not necessarily all equal); then

$$v_1 + v_2 + \ldots + v_n = 2a + 2b.$$

We wish to partition the rectangular region into n polygonal pieces, of areas proportional to $v_1, v_2, \ldots, v_n$, so that, for each i, piece i meets the boundary of the rectangle in just the segment of length v_i. Let v be the length of one of the boundary segments.



If the segment is contained in a "short" side AB (of length a), as shown in the figure, then the piece corresponding to it will be a triangle whose altitude h is chosen so that the proper area is obtained, i.e.,

$$\tfrac{1}{2}vh = \frac{v}{2a+2b}\cdot ab, \qquad \text{or} \qquad h = \frac{ab}{a+b},$$

which is independent of v. Let PQ be the segment of the perpendicular bisector of AB and CD whose endpoints are at distance $ab/(a+b)$ from AB and CD. Then the length of PQ is the nonnegative number

$$b - \frac{2ab}{a+b} = \frac{b(b-a)}{a+b}.$$

In particular, if the rectangle is a square, then P = Q.

If the segment is contained in a "long" side BC (of length b), then the piece corresponding to it will be a trapezoid whose bases are segments of BC and PQ. If the given segment divides BC into three segments of lengths u, v, and w, as shown in the figure, then the corresponding segment of PQ will divide PQ into three segments of lengths proportional to u, v, and w. The segment of PQ corresponding to v will have length

$$\frac{v}{b}\cdot\frac{b(b-a)}{a+b} = \frac{v(b-a)}{a+b}.$$

Hence the area of the trapezoid corresponding to the segment of length v will be

$$\frac{1}{2}\cdot\frac{a}{2}\left(v + \frac{v(b-a)}{a+b}\right) = \frac{v}{2a+2b}\cdot ab,$$

as it should be.

Thus, to divide the cake into n pieces whose areas and boundary segments are proportional to $v_1, v_2, \ldots, v_n$, first find the points P and Q described earlier and cut the *icing* (not the cake) from P to Q. If an endpoint of a boundary segment lies on AB or CD, then cut the cake from that endpoint to P or Q, respectively. If an endpoint of a boundary segment lies on BC or DA, then find the point of PQ which divides PQ in the same ratio as that in which the endpoint of the segment divides BC or DA, and cut the cake from the point of BC or DA to the point of PQ. If the cuts from the two endpoints of a boundary segment go to distinct points of PQ, then cut the cake between these points. In any case, the piece of cake will be freed in this way, and the n pieces of cake can be cut independently of one another. □

It seems intuitively obvious (but I don't have a proof) that if the cake has a boundary in the shape of a simple rectifiable curve, then pieces whose areas and boundary segments are proportional to $v_1, v_2, \ldots, v_n$ can be cut out successively from the cake by merely making sure that adjacent segments leave no "holes" between the regions that correspond to them. The proof given above for a rectangular cake merely supplies a definite procedure for cutting the cake.

Also solved by JORDI DOU, Barcelona, Spain; BIKASH K. GHOSH, Bombay, India; and FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio. Partial solutions (parts (a) and (b) only) were received from J. WALTER LYNCH, Georgia Southern College, Statesboro, Georgia; and the proposer.

Editor's comment.

It is apparently not easy to divide a cake among mathematicians, especially when, as in our solution II, they may require different caloric intakes. It is somewhat easier to divide a Mars Bar (which has "icing" also on the bottom) equally among three mathematicians with just two vertical cuts [1]. But if we are to "let them eat cake" (a remark that was *not* made by Marie Antoinette. What she is *said* to have said is *"Qu'ils mangent de la brioche"*.), the difficulties can be very great. See, for example, the 17-page article [2], where the cakes don't even have icing, also [3] and [4], and the more than 20 references given therein.

Never mind the cake. We'll settle for tea and sympathy.

REFERENCES

1. John R. Ransom and W.A. Van Der Spek, Solutions to Problem 885 (proposed by R. Mark R. King), *Journal of Recreational Mathematics*, 13 (1980-81) 236-237.

2. L.E. Dubins and E.H. Spanier, "How to Cut a Cake Fairly", *American Mathematical Monthly*, 68 (1961) 1-17.

3. Walter Stromquist, "How to Cut a Cake Fairly", *ibid.*, 87 (1980) 640-644.

4. ________, "Addendum to 'How to Cut a Cake Fairly'", *ibid.*, 88 (1981) 613-614.

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648. [1981: 178] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Given are a triangle ABC, its centroid G, and the pedal triangle PQR of its incenter I. The segments AI,BI,CI meet the incircle in U,V,W; and the segments AG,BG,CG meet the incircle in D,E,F. Let ∂ denote the perimeter of a triangle and consider the statement

$$\partial PQR \le \partial UVW \le \partial DEF.$$

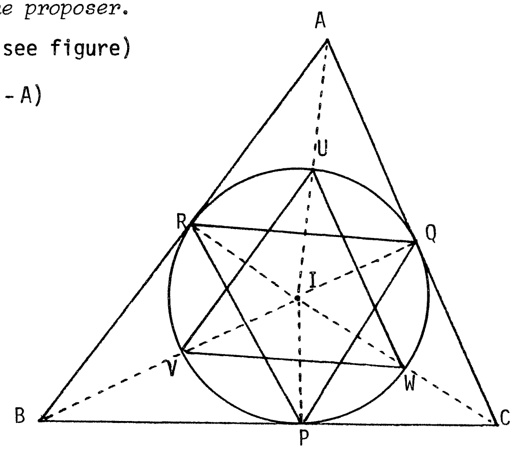
(a) Prove the first inequality.

(b)* Prove the second inequality.

I. *Solution to part (a) by the proposer.*

With r the inradius, we have (see figure)

$$QR^2 = r^2 + r^2 - 2r^2\cos(\pi - A)$$
$$= 2r^2(1 + \cos A)$$
$$= 4r^2\cos^2\frac{A}{2},$$



etc., and

$$VW^2 = r^2 + r^2 - 2r^2\cos\left(\pi - \frac{B+C}{2}\right)$$
$$= 2r^2\left(1 + \cos\frac{B+C}{2}\right)$$
$$= 4r^2\cos^2\frac{B+C}{4},$$

etc. Hence

$$\partial PQR = 2r\Sigma \cos \frac{A}{2}, \qquad \partial UVW = 2r\Sigma \cos \frac{B+C}{4},$$

and the required inequality is equivalent to

$$\Sigma \cos \frac{A}{2} \le \Sigma \cos \frac{B+C}{4}. \tag{1}$$

Now

$$2\Sigma \cos \frac{B+C}{4} \ge \Sigma 2 \cos \frac{B+C}{4} \cos \frac{B-C}{4} = 2\Sigma \cos \frac{A}{2},$$

and (1) follows, with equality just when A = B = C.

II. *Comment by M.S. Klamkin, University of Alberta.*

With brackets denoting area, we also have

$$[PQR] \le [UVW]. \tag{2}$$

For (see figure)

$$[IQR] = \tfrac{1}{2}r^2 \sin(\pi - A) = \tfrac{1}{2}r^2 \sin A, \text{ etc.}$$

and

$$[IVW] = \tfrac{1}{2}r^2 \sin\left(\pi - \frac{B+C}{2}\right) = \tfrac{1}{2}r^2 \cos \frac{A}{2}, \text{ etc.},$$

and (2) is equivalent to the known inequality [1]

$$\Sigma \sin A \le \Sigma \cos \frac{A}{2},$$

with equality just when the triangle is equilateral.

Part (a) was also solved by M.S. KLAMKIN, University of Alberta; and GEORGE TSINTSIFAS, Thessaloniki, Greece.

Editor's comment.

Two additional solutions to part (a), not recorded above, were received. Both used calculus; one was very long and the other was merely long. Is there any point in saying again that calculus should usually be avoided in proving triangle inequalities? It is easy enough (though the calculations can be and usually are lengthy) to show that a certain function $\phi(A,B,C)$, defined on the plane $A + B + C = \pi$, attains a relative extremum when A = B = C. But it is not so easy (and therefore usually omitted) to show that this extremum is absolute when the domain is restricted to $A,B,C > 0$.

One of our calculus solvers made a valuable comment, however. He noted that in some triangles the centroid G lies outside the incircle, and one of the *segments* AG,BG,CG does not meet the incircle, so that triangle DEF does not exist. This was due to an editing error. The proposer's intent was that the *rays* AG,BG,CG meet the incircle for the first time in D,E,F, respectively, so triangle DEF always exists. With this correction, we have still left open for proof or disproof the inequality

$$\partial UVW \le \partial DEF,$$

and we may as well ask at the same time for a proof or disproof of the area relation

$$[UVW] \le [DEF].$$

REFERENCE

1. Murray S. Klamkin, "Notes on Inequalities Involving Triangles or Tetrahedrons", *Publikacije Elektrotehničkog Fakulteta Univerziteta U Beogradu*, No. 330 — No. 337 (1970) 1-15, esp. p. 7.

* * *

649. [1981: 178] *Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.*

The centroid and the circumcenter of a rectangle coincide. Are there other quadrangular laminae with this property?

Solution by Gali Salvatore, Perkins, Québec.

According to Coxeter [1], the centroid of a quadrangular lamina is the centre of the Wittenbauer parallelogram, whose sides join adjacent points of trisection of the sides of the quadrangle; the centroid of (equal masses at) the vertices of a quadrangle is the centre of the Varignon parallelogram, whose sides join adjacent midpoints of the sides of the quadrangle; and these two centroids coincide just when the quadrangle is a parallelogram.

If we are restricted to cyclic quadrangles, then the two centroids coincide (in the centre of the circle) only for a rectangle.

If, for a cyclic quadrangle ABCD, we require only that the centroid of the vertices coincide with the centre O of the circle, then ABCD is necessarily a rectangle. For let E and F be the midpoints of AB and CD, respectively. Since by hypothesis O is the centre of the Varignon parallelogram, E,O,F are collinerar, $OE = OF$ and $OE \perp AB$, $OF \perp CD$. Thus AB and CD are equal and parallel, so ABCD is a cyclic parallelogram, hence a rectangle.

The situation is different for a cyclic quadrangular lamina. If we require only that its centroid coincide with the centre of the circle, then the lamina is not necessarily rectangular. It could be, for example, an isosceles trapezoid ABCD in which the diagonals AC and BD meet in P at an acute angle of 60°, as shown in Figure 1. If STUV is the Wittenbauer parallelogram, it is clear from symmetry that the centre O lies on SU; hence O is the centroid if and only if it lies on TV, and this will follow from the following proof, where we show directly that O is the centroid.

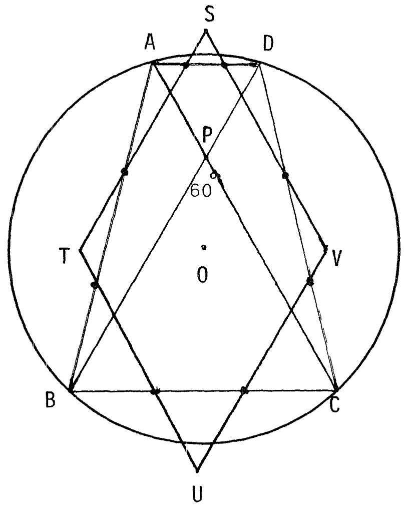


Figure 1

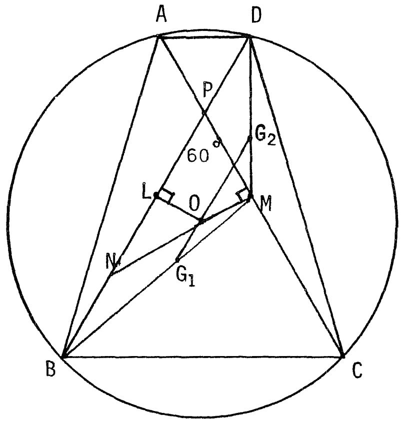


Figure 2

Let L and M be the midpoints of BD and AC, respectively, as shown in Figure 2, and extend MO to meet BD at N. Then $\angle LON = 60^\circ$ and we have $OM = OL = \frac{1}{2}ON$. If a line is drawn through O parallel to BD, it will therefore meet MB and MD in the centroids G_1 and G_2 of the triangular laminae ABC and CDA, respectively. Since $PL = PM = \frac{1}{2}PN$, then L is the midpoint of PN, and so $BN = PD$. Now, with the brackets denoting area, we have

$$\frac{[ABC]}{[CDA]} = \frac{BP}{PD} = \frac{ND}{BN} = \frac{OG_2}{OG_1}.$$

Thus $OG_1 \cdot [ABC] = OG_2 \cdot [CDA]$, and O is the centroid of the quadrangular lamina ABCD.

Also solved by BIKASH K. GHOSH, Bombay, India; LEROY F. MEYERS, The Ohio State University; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer. In addition, three incorrect solutions were received.

Editor's comment.

Our three incorrect solvers convinced themselves that, in a cyclic quadrangular lamina, the centroid and the circumcentre coincide if and only if the lamina is rectangular.

The following related question is of interest: How can one characterize the centroid of (a thin homogeneous wire around) the perimeter of a quadrangle? In particular (this question was raised by Meyers), if, for a cyclic quadrangle, the circumcentre and the centroid of the perimeter coincide, is the quadrangle necessarily rectangular?

REFERENCE

1. H.S.M. Coxeter, *Introduction to Geometry*, Second Edition, John Wiley & Sons, New York, 1969, pp. 216, 445, Ex. 3, 4, 5.

* * *

650. [1981: 179] *Proposed by Paul R. Beesack, Carleton University, Ottawa.*

(a) Two circular cylinders of radii r and R, where $0 < r \le R$, intersect at right angles (i.e., their central axes intersect at an angle of $\pi/2$). Find the arc length l of one of the two curves of intersection, as a definite integral.

(b) Do the same problem if the cylinders intersect at an angle γ, where $0 < \gamma < \pi/2$.

(c) Show that the arc length l in (a) satisfies

$$l \le 4r\int_0^{\pi/2} \sqrt{1+\cos^2\theta}\, d\theta < 5\pi r/2.$$

Solution by the proposer.

(a) Let the axes of the cylinders of radii r and R be chosen as the z- and y-axes of a rectangular coordinate system (O,x,y,z). The equations of the cylinders are then

$$x^2 + y^2 = r^2 \quad \text{and} \quad x^2 + z^2 = R^2.$$

Using cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, the parametric equations of the upper curve of intersection become

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = \sqrt{R^2-r^2\cos^2\theta}, \quad 0 \le \theta \le 2\pi.$$

Thus

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2 = r^2 + \frac{r^4\cos^2\theta\sin^2\theta}{R^2 - r^2\cos^2\theta}$$

$$= \frac{r^2(R^2-r^2\cos^4\theta)}{R^2 - r^2\cos^2\theta}$$

and, by symmetry, the required arc length is

$$l = 4r\int_0^{\pi/2}\sqrt{\frac{R^2-r^2\cos^4\theta}{R^2-r^2\cos^2\theta}}\, d\theta.$$

(b) Let the axis of the smaller cylinder still lie along the z-axis, and let the axis λ of the larger cylinder lie in the yz-plane and pass through the origin. This larger cylinder intersects the xy-plane in an ellipse E with semiminor axis of length R and semimajor axis of length $b = R\sec\gamma$. The equation of E is

$$\frac{x^2}{R^2} + \frac{y^2}{b^2} = 1,$$

so

$$y = -\sqrt{R^2-x^2}\sec\gamma$$

is the equation of the half of E for which $y \le 0$. Let $P(r\cos\theta, r\sin\theta, z)$ be a point on the upper curve of intersection of the cylinders, and let $Q(x_0, y_0, 0)$ be the point on E for which the line QP is parallel to λ. We then have

$$x_0 = r\cos\theta, \quad y_0 = -\sqrt{R^2-r^2\cos^2\theta}\cdot\sec\gamma.$$

Since the direction angles of λ (and QP) are $(\pi/2, \pi/2-\gamma, \gamma)$, its direction cosines are $(0, \sin\gamma, \cos\gamma)$, and the parametric equations of PQ are

$$x = x_0 = r\cos\theta,$$

$$y = y_0 + t\sin\gamma = t\sin\gamma - \sqrt{R^2-r^2\cos^2\theta}\cdot\sec\gamma,$$

$$z = t\cos\gamma.$$

The point P is given by that value of t for which $y = r\sin\theta$. Thus the parametric equations of the required curve of intersection are

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\sin\theta\cot\gamma + \sqrt{R^2-r^2\cos^2\theta}\cdot\csc\gamma.$$

By symmetry with respect to the yz-plane, half of the curve occurs for the range $-\pi/2 \le \theta \le \pi/2$. We now have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2 = r^2 + \left\{ r\cot\gamma\cos\theta + \csc\gamma\cdot\frac{r^2\cos\theta\sin\theta}{\sqrt{R^2 - r^2\cos^2\theta}}\right\}^2$$

$$= r^2\left[1 + \cos^2\theta\left\{\cot\gamma + \frac{r\csc\gamma\sin\theta}{\sqrt{R^2 - r^2\cos^2\theta}}\right\}^2\right].$$

The required arc length is therefore given by

$$l = 2r\int_{-\pi/2}^{\pi/2}\sqrt{1 + \cos^2\theta\left\{\cot\gamma + \frac{r\csc\gamma\sin\theta}{\sqrt{R^2 - r^2\cos^2\theta}}\right\}^2}\,d\theta.$$

(c) For the first inequality, it suffices to verify that, for $0 \le \theta \le \pi/2$ and $0 < r \le R$,

$$\frac{R^2 - r^2\cos^4\theta}{R^2 - r^2\cos^2\theta} \le 1 + \cos^2\theta.$$

This is equivalent to $r^2\cos^2\theta \le R^2\cos^2\theta$, which is clearly true. Finally, since $\sqrt{1+a} \le 1 + \frac{1}{2}a$ for $a \ge -1$, we have

$$\int_0^{\pi/2}\sqrt{1 + \cos^2\theta}\,d\theta \le \int_0^{\pi/2}(1 + \tfrac{1}{2}\cos^2\theta)d\theta = \frac{\pi}{2} + \frac{\pi}{8} = \frac{5\pi}{8},$$

from which the upper bound $5\pi r/2$ follows. □

The integral in (c), namely

$$\int_0^{\pi/2}\sqrt{1 + \cos^2\theta}\,d\theta = \sqrt{2}\int_0^{\pi/2}\sqrt{1 - \tfrac{1}{2}\sin^2\theta}\,d\theta = \sqrt{2}\,E(\pi/2,\ 1/\sqrt{2}) \approx 1.9100988,$$

is a complete elliptic integral of the second kind, so Crux 577 [1981: 250] is very much to the point in the present problem.

I submitted this problem just a few weeks after it (at least part (a), and part (b) with $\gamma = 45^\circ$) was proposed to me by a practising engineer. He works for a heating firm and must give cost estimates on *welding* sheet metal pipes together in a T-joint or a ⊬-joint. The cost is, of course, directly proportional to the arc length.

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651. [1981: 179] *Proposed by Charles W. Trigg, San Diego, California.*

It is June, the bridal month, and LOVE is busting out all over. So THEY obey the biblical injunction to go forth and multiply, resulting paradoxically in a cryptarithmic *addition* which you are asked to investigate with averted eyes.

Find out in how many ways

$$\begin{array}{c} \text{THEY} \\ \underline{\text{MADE}} \\ \text{LOVE} \end{array}$$

and in which way their LOVE was greatest.

Solution by Allan Wm. Johnson Jr., Washington, D.C.

It is obvious that $Y = 0$, so the problem reduces to solving the cryptarithmic addition

$$\text{THE} + \text{MAD} = \text{LOV}, \tag{1}$$

in which all nine nonzero digits appear. Metchette [1] discusses this problem and notes that our proposer wrote an article on it in 1962 [2]. Metchette observes that, in any solution, interchanging T and M, H and A, or E and D produces another solution with the same LOV, so the solutions can conveniently be partitioned into groups of eight, corresponding to these interchanges, the solution in which $T > M$, $H > A$, and $E > D$ being called the *basic* solution of the group. Using a Microkit 8/16 processor, Metchette showed that (1) has 42 basic solutions (which he did not give), and so there are $8 \cdot 42 = 336$ ways of making LOVE (more than in Dr. Alex Comfort's *The Joy of Sex* and the *Kamasutra* combined!). On a Radio Shack Model I TRS-80 microcomputer, I reproduced these 42 basic solutions and found that the greatest LOV occurs in the basic solutions

$$746 + 235 = 981 \quad \text{and} \quad 657 + 324 = 981,$$

so the greatest LOVE occurs in

$$\begin{array}{r} 6570 \\ \underline{3247} \\ 9817 \end{array}$$

and in the seven other variations of its group. (The smallest LOVE was 4596 and the average was approximately 7746.69.)

It is evident that LOVE cannot exceed 9817 because, as Metchette proves, $L + O + V$ must be a multiple of 9, therefore at most 18, whence $LOV \le 981$ and $LOVE \le 9817$.

The 42 basic solutions are listed below, in order of increasing LOVE.

THEY	+ MADE	= LOVE	THEY	+ MADE	= LOVE
2860	1736	4596	5960	1426	7386
2950	1735	4685	6590	1249	7839
3590	1279	4869	5690	2149	7839
3680	1278	4958	6580	1348	7928
3870	1627	5497	5760	2436	8196
4390	1289	5679	4670	3527	8197
3490	2189	5679	6950	1425	8375
3940	1824	5764	5960	2416	8376
3780	2168	5948	5290	3179	8469
4870	1527	6397	5930	2713	8643
3970	2517	6487	7390	1259	8649
4390	2189	6579	6590	2149	8739
4930	1823	6753	6570	2347	8917
3940	2814	6754	5670	3247	8917
4780	2158	6938	6750	2435	9185
5860	1436	7296	5760	3426	9186

THEY	+ MADE	= LOVE	THEY	+ MADE	= LOVE
5860	3416	9276	7380	2168	9548
7840	1524	9364	7480	2158	9638
7830	1623	9453	6580	3148	9728
6280	3178	9458	7460	2356	9816
6830	2713	9543	6570	3247	9817

Also solved by RICHARD V. ANDREE, University of Oklahoma; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; and ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec. Incomplete solutions were submitted by CLAYTON W. DODGE, University of Maine at Orono; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Comment by Edith Orr.

The editor has asked me, as an expert in the field, to comment on the incomplete solutions received.

Dodge found that the greatest LOVE was 9817 but he did not report, as the problem required, on the total number of ways of making LOVE. Newly married when the problem was published, he probably found the greatest LOVE on his honeymoon and could not wait to send us his answer. Wilke found only 264 ways of making LOVE (from 33 basic solutions). Still, it's not bad for one who lives in the Bible Belt. The proposer found only 328 ways of making LOVE (from 41 basic solutions). This is very good for a man of his age, but he had done better than that in his 1962 article, when he was 20 years younger. Which proves something or other.

So now we know. Greater LOVE hath no man than 9817. But for most of us happiness is seated in the mean, 7746.69, a figure which I have not checked because, with *that* fractional part, it has just got to be right.

REFERENCES

1. Stewart Metchette, "Distinct-Digit Sums", *Journal of Recreational Mathematics*, 11 (1978-79) 279-283.

2. Charles W. Trigg, "Solutions of ABC + DEF = GHK with Distinct Digits", *Recreational Mathematics Magazine*, February 1962, pp. 35-36.

* * *

652. [1981: 179] *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

Let R, r, s represent respectively the circumradius, the inradius, and the semiperimeter of a triangle with angles α, β, γ. It is well known that

$$\Sigma\sin\alpha = \frac{s}{R},\qquad \Sigma\cos\alpha = \frac{R+r}{R},\qquad \Sigma\tan\alpha = \frac{2rs}{s^2-4R^2-4Rr-r^2}.$$

As for half angles, it is easy to prove that $\Sigma\tan(\alpha/2) = (4R+r)/s$. Find similar expressions for $\Sigma\cos(\alpha/2)$ and $\Sigma\sin(\alpha/2)$.

Comment by M.S. Klamkin, University of Alberta.

In an important paper on triangle inequalities mentioned recently in this journal [1982: 68] Anders Bager tacitly implied (on page 21) that at the present time there are no simple R-r-s representations for the following triangle functions:

$$\Sigma\sin\frac{\alpha}{2},\qquad \Sigma\sin\frac{\beta}{2}\sin\frac{\gamma}{2},\qquad \Sigma\csc\frac{\alpha}{2},\qquad \Sigma\csc\frac{\beta}{2}\csc\frac{\gamma}{2}; \tag{1}$$

$$\Sigma\cos\frac{\alpha}{2},\qquad \Sigma\cos\frac{\beta}{2}\cos\frac{\gamma}{2},\qquad \Sigma\sec\frac{\alpha}{2},\qquad \Sigma\sec\frac{\beta}{2}\sec\frac{\gamma}{2}. \tag{2}$$

However, consideration of $(\Sigma\sin(\alpha/2))^2$ shows that, if an R-r-s expression can be found for $\Sigma\sin(\alpha/2)$, then R-r-s expressions can also be found for the remaining functions of (1); and a similar statement can be made for the functions of (2). Now such expressions *can* be found, but they are not simple, and we conjecture that they cannot be expressed as rational functions of R,r,s. For example, we show how to find an R-r-s expression for $\Sigma\cos(\alpha/2)$; a similar technique will yield an R-r-s expression for $\Sigma\sin(\alpha/2)$.

Let

$$t^3 - ut^2 + vt - w = 0 \tag{3}$$

be the cubic equation whose roots are $\cos\alpha$, $\cos\beta$, $\cos\gamma$. It is known that

$$u = \Sigma\cos\alpha = \frac{R+r}{R},\qquad v = \Sigma\cos\beta\cos\gamma = \frac{r^2+s^2-4R^2}{4R^2},\qquad w = \Pi\cos\alpha = \frac{s^2-(2R+r)^2}{4R^2}.$$

With these values of u,v,w, the values of $\cos\alpha$, $\cos\beta$, $\cos\gamma$ can be found from (3), in terms of radicals, and substituted in the right member of

$$\Sigma\cos\frac{\alpha}{2} = \Sigma\sqrt{\tfrac{1}{2}(1+\cos\alpha)}.$$

A solution was received from JACK GARFUNKEL, Flushing, N.Y. Comments were received from V.N. MURTY, Pennsylvania State University, Capitol Campus; and the proposer.

Editor's comment.

Garfunkel found complicated explicit expressions for $\Sigma\sin(\alpha/2)$ and $\Sigma\cos(\alpha/2)$, but these were not exclusively R-r-s expressions since they included the sum $\Sigma\sqrt{r^2+(s-a)^2}$.

Murty did not find explicit expressions for $\Sigma\sin(\alpha/2)$ and $\Sigma\cos(\alpha/2)$, except in the special case when the triangle is isosceles (and these were complicated enough), but he did find the following lower bounds:

$$\Sigma \sin\frac{\alpha}{2} \geq 1 + \frac{r}{R}, \qquad \Sigma \sin\frac{\beta}{2}\sin\frac{\gamma}{2} \geq \frac{r}{2R}\left(\frac{r}{R} + \frac{5}{2}\right),$$

$$\Sigma \cos\frac{\alpha}{2} \geq \frac{s}{R}, \qquad \Sigma \cos\frac{\beta}{2}\cos\frac{\gamma}{2} \geq \frac{s^2}{2R^2} - \frac{r}{4R} - 1,$$

equality being attained in each case just when the triangle is equilateral.

The proposer showed that $\Sigma \sin(\alpha/2)$ is a solution of the quartic equation

$$x^4 = \frac{2R-r}{2R}x^2 + \frac{2r}{R}x + \frac{s^2-4R^2-4Rr}{4R^2},$$

and that $\Sigma \cos(\alpha/2)$ is a solution of

$$x^4 = \frac{4R+r}{R}x^2 + \frac{2s}{R}x + \frac{s^2}{4R^2}.$$

* * *

653. [1981: 179] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

For every triangle ABC, show that

$$\Sigma \cos^2\frac{B-C}{2} \geq 24\Pi \sin\frac{A}{2},$$

where the sum and product are cyclic over A,B,C, with equality if and only if the triangle is equilateral.

I. *Solution by S.C. Chan, Singapore.*

The following inequality is known to hold whenever $\alpha + \beta + \gamma = 180^\circ$ [1, p. 25]:

$$\Sigma \cos^2(\beta-\gamma) \geq 24\Pi \cos\alpha. \tag{1}$$

With

$$\alpha = 90^\circ - \frac{A}{2}, \qquad \beta = 90^\circ - \frac{B}{2}, \qquad \gamma = 90^\circ - \frac{C}{2},$$

(1) reduces to the required

$$\Sigma \cos^2\frac{B-C}{2} \geq 24\Pi \sin\frac{A}{2}. \tag{2}$$

II. *Solution by Jack Garfunkel, Flushing, N.Y.*

We establish the following chain of inequalities, which greatly sharpens (2):

$$\Sigma \cos^2\frac{B-C}{2} \geq \Sigma \cos\frac{C-A}{2}\cos\frac{A-B}{2} \geq 3\Pi \cos\frac{B-C}{2} \geq 24\Pi \sin\frac{A}{2}. \tag{3}$$

The first inequality in (3) follows from the well-known

$$x^2 + y^2 + z^2 \geq yz + zx + xy.$$

The second inequality in (3) is equivalent to

$$\Sigma \sec\frac{B-C}{2} \ge 3,$$

which is clearly true since $\sec\theta \ge 1$ for $-\pi/2 < \theta < \pi/2$. Finally, the third inequality in (3) follows immediately from the first inequality in Crux 585 [1981: 303].

III. *Solution by Bob Prielipp, University of Wisconsin-Oshkosh.*

From Newton's formulas [2]

$$\cos\frac{B-C}{2} = \frac{b+c}{a}\cdot\sin\frac{A}{2}, \quad \text{etc.}$$

and the half-angle formulas of trigonometry, the proposed inequality is equivalent to

$$\sum\frac{(b+c)^2}{a(s-a)} \ge 24. \tag{4}$$

Now

$$\frac{(b+c)^2}{a(s-a)} = \frac{(2s-a)^2}{a(s-a)} \ge 8 \iff (2s-3a)^2 \ge 0, \quad \text{etc.}$$

Hence (4) holds, with equality if and only if $a = b = c = 2s/3$.

IV. *Solution by M.S. Klamkin, University of Alberta.*

In a recent article in this journal [1982: 62], Murty referred to *Blundon's fundamental inequality*

$$(r^2+s^2)^2 + 12Rr^3 - 20Rrs^2 + 48R^2r^2 - 4R^2s^2 + 64R^3r \le 0, \tag{5}$$

which is a necessary and sufficient condition for the existence of an R-r-s triangle (a triangle with circumradius R, inradius r, and semiperimeter s). Blundon had pioneered in developing a direct method of establishing triangle inequalities by first converting them into an R-r-s representation and then comparing them with one of the following two "best quadratic inequalities", which follow from (5):

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr} \le s^2 \le 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}.$$

The method usually enables us to strengthen a proposed inequality. For more information, see [3] and [4]. We apply Blundon's method to the present problem.

The inequality proposed here is easily seen to be equivalent to

$$3 + \Sigma\cos(B-C) \ge 48\Pi\sin\frac{A}{2}. \tag{6}$$

With

$$\Sigma\cos B\cos C = \frac{r^2+s^2-4R^2}{4R^2}, \qquad \Sigma\sin B\sin C = \frac{s^2+r^2+4Rr}{4R^2}, \qquad \Pi\sin\frac{A}{2} = \frac{r}{4R},$$

we find that (6) is equivalent to

$$22Rr - r^2 - 4R^2 \le s^2. \tag{7}$$

Hence (7) will follow from

$$22Rr-r^2-4R^2 \le 2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}.$$

This last inequality is equivalent to

$$3R(R-2r) \ge (R-2r)\sqrt{R^2-2Rr}. \tag{8}$$

Since $R-2r \ge 0$ for every triangle, it is now easy to show by simple algebra that (8) holds, with equality if and only if $R = 2r$, that is, if and only if the triangle is equilateral. □

It is easy to verify that (8) remains true if $2R(R-2r)$ is added to its right member. This strengthens (2) and (7) to

$$\Sigma\cos^2\frac{B-C}{2} \ge 1 + 16\Pi\sin\frac{A}{2} \tag{9}$$

and

$$s^2 \ge 14Rr - r^2. \tag{10}$$

Coincidentally, these last two inequalities are equivalent to that of Crux 644 [1982: 154] and to the other equivalent forms given in my solution to that problem [1982: 157]. That (9) and (10) are sharper than (2) and (7) is seen explicitly in the inequalities

$$\Sigma\cos^2\frac{B-C}{2} \ge 1 + 16\Pi\sin\frac{A}{2} \ge 24\Pi\sin\frac{A}{2},$$

the second of which follows from $8\Pi\sin(A/2) \le 1$ [1, p. 20]; and in the inequalities

$$s^2 \ge 14Rr-r^2 \ge 22Rr-r^2-4R^2,$$

the second of which is a consequence of $R \ge 2r$.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; ROLAND H. EDDY, Memorial University of Newfoundland; J.T. GROENMAN, Arnhem, The Netherlands; V.N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh (second solution); and the proposer. Incomplete solutions were received from BIKASH K. GHOSH, Bombay, India; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

Editor's comment.

Our solutions I and II are very neat, but their authors did not establish the condition under which equality holds, as the problem required (nor did reference [1], appealed to by the first solver). It is not clear whether this omission was due to forgetfulness or to astuteness (because the required condition did not fall out easily from their approach to the problem).

REFERENCES

1. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.

2. William H. Beyer (Ed.), *CRC Standard Mathematical Tables*, 26th Edition, CRC Press, 1981, p. 144.

3. W.J. Blundon, "Inequalities Associated with the Triangle", *Canadian Mathematical Bulletin*, 8 (1965) 615-626.

4. R. Frucht and M.S. Klamkin, "On Best Quadratic Triangle Inequalities", *Geometriae Dedicata*, 2 (1973) 341-348.

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654. [1981: 179] *Proposed by Randall J. Covill, Maynard, Massachusetts.*

Suppose that some extraterrestrials have three hands and a total of thirteen fingers on each of the two sides of their symmetric bodies. Each hand has one or more fingers. How many different types of gloves are necessary to outfit those extraterrestrials?

Solutions were received form CLAYTON W. DODGE, University of Maine at Orono; BIKASH K. GHOSH, Bombay, India; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer. A comment was received from FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio.

Editor's comment.

Several solvers pointed out that this problem is not well posed and that several answers are possible, depending on the interpretation given. The solutions received, which had answers ranging from 18 to 16382, were of little mathematical interest and there seems to be no point in publishing any of them, especially since the recently released film *E.T.* shows that extraterrestrials don't even wear gloves.

The editor regrets that he has allowed this problem to be published without more careful editing.

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655. [1981: 180] *Proposed by Kaidy Tan, Fukien Teachers' University, Foochow, Fukien, China.*

If $0 < a,b,c,d < 1$, prove that

$$(\Sigma a)^3 > 4bcd\Sigma a + 8a^2bcd\Sigma(1/a),$$

where the sums are cyclic over a,b,c,d.

Solution by M.S. Klamkin, University of Alberta.

We sharpen the inequality to

$$(\Sigma a)^3 > \frac{9}{2}bcd\Sigma a + 8\Sigma bcd > 4bcd\Sigma a + 8a^2bcd\Sigma(1/a). \tag{1}$$

The second inequality in (1) is equivalent to

$$\frac{1}{2}bcd\Sigma a + 8(1-a)\Sigma bcd > 0,$$

which is clearly true, and the first is the sum of

$$\frac{1}{2}(\Sigma a)^3 > \frac{9}{2}bcd\Sigma a \tag{2}$$

and

$$\frac{1}{2}(\Sigma a)^3 \ge 8\Sigma bcd. \tag{3}$$

Inequality (2) is equivalent to $(\Sigma a)^2 > 9bcd$, and this follows from

$$\left(\frac{\Sigma a}{3}\right)^2 > \left(\frac{b+c+d}{3}\right)^2 \ge (bcd)^{2/3} > bcd;$$

and, with $p_1 = (\Sigma a)/4$ and $p_3 = (\Sigma bcd)/4$, inequality (3) follows from the Maclaurin inequality $p_1 \ge p_3^{1/3}$ [1].

Also solved by the proposer. One other solution, at least partly incorrect, was received.

Editor's comment.

The incorrect solution received was very long and partly incomprehensible. It claimed, among other things, that equality holds when $a = b = c = d = 3/2$, which a simple calculation easily disproves.

REFERENCE

1. G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, London, 1952, p. 52.

* * *

MATHEMATICAL CLERIHEWS

Jacques Bernoulli
Gave us, truly,
At an early date,
A lemniscate.

Étienne Pascal,
Not Blaise, *et al.*,
First worked upon
The limaçon.

ALAN WAYNE
Holiday, Florida