41-st German Mathematical Olympiad 2002

4-th Round – Hamburg, May 5–8

Grade 10

First Day

1. If faces ABD, ACD, BCD of a tetrahedron $ABCD$ have right angles at vertex D, prove that the sum of the squares of areas of these three faces is equal to the square of area of the face ABC.

2. Find the greatest common divisor of all numbers of the form $n^8 - n^2$, where $n \in \mathbb{N}$.

3. Prove the following inequality for any real numbers a, b, c with $0 < a < b < c$:
$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt[3]{a}+\sqrt[3]{b}} < \frac{\sqrt{b}+\sqrt{c}}{\sqrt[3]{b}+\sqrt[3]{c}}.$$

Second Day

4. (a) Prove that a nonnegative integer n is even if and only if the number of ones in its base 3 representation is even.
   (b) Call a nonnegative integer WO (WithoutOnes) if its base 3 representation contains no ones. Show that every even nonnegative integer can be written as the sum of two WO-numbers.

5. Find all rational numbers x for which $4^x + 9^x + 16^x = 6^x + 8^x + 12^x$.

6. Let P be a point on the segment AB. Isosceles right triangles AO_1P and BO_1P with right angles at O_1 and O_2 are constructed on the same side of line AB. Describe the locus of midpoints of segments O_1O_2 as P moves along AB.

Grades 11-13

First Day

1. Find all real solutions (a, b) of the system
$$\begin{aligned} 2a^2 - 2ab + b^2 &= a, \\ 4a^2 - 5ab + 2b^2 &= b. \end{aligned}$$

2.

3. (a) Prove that for each natural number n there exists a natural number z which has exactly n positive divisors and which is divisible by n.

   (b) For each prime number n, find all numbers z with the property from (a).

Second Day

4.

5.

6. *(Grade 11)*

6. *(Grades 12-13)*