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Primorial, Factorial and Multifactorial Primes

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About 2500 years ago, the great Greek geometer Euclid proved that there are infinitely many primes, as follows: suppose (for proof by contradiction) there are only finitely many primes, say $p_1, p_2, \ldots, p_n$. Let $N = p_1 p_2 \cdots p_n + 1$. Now N has some prime divisor p, but clearly p divides $N-1$ (all primes do), so p cannot divide N.

Like many of you, when we saw this proof we couldn't help but wonder about these numbers $p\#+1$. ($p\#$ denotes the product of the *primes* less than or equal to p, so $13\# = 13\times11\times7\times5\times3\times2$.) Are these numbers always prime? always composite? or just sometimes composite? ... And while we're wondering, what about the related forms $p\#-1$, $n!+1$ and $n!-1$?

Well, each of the forms $n!\pm1$ and $p\#\pm1$ has been frequently studied (see, for example references 1, 4–7, 9–10, 12). Primes of these forms are called (respectively) factorial primes and primorial primes (reference 13). For the convenience of the reader we have summarized what is currently known about the primality of these numbers in table 1 and simultaneously announce that we have completed the primality proofs for two more numbers of these forms—both indicated by an asterisk in table 1. In this article we first recall a generalization of the factorial function (which we here call $n!_k$) and then study the resulting primes $n!_k\pm1$. We end with a plea for an explanation of why one of these forms generates so few primes.

In what follows we make heavy use of the following two results, recalled here for the reader's convenience.

Wilson's theorem. Let p be an integer greater than 1. Then p is prime if and only if $(p-1)! \equiv -1 \pmod{p}$.

Fermat's theorem. Let p be a prime not dividing a. Then $a^{p-1} \equiv 1 \pmod{p}$.

Table 1. Factorial and primorial primes. See references 6 and 7.
(* denotes a newly shown prime.)

Form	n's for which the form is prime or [probable primes base 3]	Search limit
$n!+1$	1, 2, 3, 11, 27, 37, 41, 73, 77, 116, 154, 320, 340, 399, 427, 872, 1477 (4042 digits)	2662
$n!-1$	3, 4, 6, 7, 12, 14, 30, 32, 33, 38, 94, 166, 324, 379, 469, 546*, [974], [1963] (5614 digits)	2063
$p\#+1$	2, 3, 5, 7, 11, 31, 379, 1019, 1021, 2657, 3229, 4547, 4787, 11549, 13649, 18523 (8002 digits)	19051
$p\#-1$	3, 5, 11, 41, 89, 317, 337, 991, 1873, 2053, 2377*, [4093], [4297], [4583], [6569], [13033], [15877] (6845 digits)	16699

See any good elementary number theory text for the proofs.

We end this section by showing how Fermat's theorem can be used to test large numbers q for primality. Choose any small prime a. If a divides q, then you are done (q is composite). If not, then calculate $a^{q-1} \pmod{q}$. (This calculation can be done very quickly and easily, see reference 2.) If $a^{q-1} \not\equiv 1 \pmod{q}$, then by Fermat's theorem, q is composite. If instead $a^{q-1} \equiv 1 \pmod{q}$, then q is *probably* a prime (in other words, most, but not all, q's which satisfy this congruence are prime). So it is reasonable to call any number satisfying this congruence a *probable prime* (or more specifically, a probable prime base a). When looking for large primes you usually start by finding probable primes, then use more advanced techniques to prove these probable primes are actually primes. See references 3 and 11 for an explanation of how this is done.

The multifactorial function

Occasionally one runs across the double factorial function $n!!$ (in fact, it is a built-in function in the widely used symbolic processor *Mathematica* (reference 14)). Rather than multiplying all the integers less than n together, $n!!$ multiplies every other positive integer. For example $7!! = 7\times5\times3\times1$ and $8!! = 8\times6\times4\times2$. Let us give a more general definition.

Definition. Let n be an integer. Define the multifactorials as follows:

$n! = 1$ for $n \leqslant 1$, otherwise $n! = n\times(n-1)!$ (n factorial),

$n!! = 1$ for $n \leqslant 1$, otherwise $n!! = n\times(n-2)!!$ (n double-factorial),

$n!!! = 1$ for $n \leqslant 1$, otherwise $n!!! = n \times (n-3)!!!$ (n triple-factorial)

and in general

$n!_k = 1$ for $n \leqslant 1$, otherwise $n!_k = n \times (n-k)!_k$ (n k-factorial).

For example,

$$13!!! = 13!_3 = 13 \times 10 \times 7 \times 4 \times 1 \quad \text{and} \quad 23!_4 = 23 \times 19 \times 15 \times 11 \times 7 \times 3.$$

We end this section with a collection of trivial (but useful) relations between the factorial and the multifactorial functions.

Lemma 1. Let k and n be positive integers.

(a) $(kn)!_k = k^n n!$.

(b) $n! = n!_k (n-1)!_k \cdots (n-k+1)!_k$.

(c) $n! = n!!\,(n-1)!!$ (the special case $k = 2$ of part (b)).

The multifactorial primes

We used a Dubner processor to scan quickly numbers of the form $n!_k \pm 1$ for primality. (The Dubner processor is a custom-built processor for number theory which, for this type of calculation, is over 70 times as fast as a PC based on the Intel 486 processor. For information on an early version see reference 8.) We found many small primes. For example, $n!!!+1$ is prime for each of $n = 1, 2, 4, 5, 6, 7, 9, 10, 11, 17, \ldots$, including the 1997-digit prime $2076!!!+1$. There are many more of these primes than there are of the forms $n! \pm 1$ because of how much slower these numbers grow. (In fact, a quick estimate shows we should expect about $k-1$ times as many primes.) A more impressive example of what we found is the pair of 2151-digit twin primes $2846!!!! \pm 1$, the seventh-largest known pair of twin primes. (At the time of writing, the only larger known twin primes are $1706\,595 \times 2^{11\,235} \pm 1$, $1171\,452\,282 \times 10^{2490} \pm 1$, $571\,305 \times 2^{7701} \pm 1$, $75\,188\,117\,004 \times 10^{2298} \pm 1$, $663\,777 \times 2^{7650} \pm 1$ and $107\,570\,463 \times 10^{2250} \pm 1$ (see reference 15).) See table 2 for a summary of the results of our search.

Looking at tables 1 and 2, the reader will notice an obvious shortage of primes of the form $n!!+1$. In the next section we look for a reason why this form may have fewer primes.

Divisibility results

When trying to find out why the form $n!!+1$ has so many fewer primes, we began studying the divisibility properties of the multifactorials ± 1. Though the following theorem is phrased very generally, we first proved it for the case $(2n)!!+1$.

Theorem 2. Suppose $\gcd(p, k) = 1$ and $p \neq 1$. The integer p is prime if and only if p divides $[k(p-1)]!_k + 1$.

Table 2. Multifactorial primes.

Form	n's for which the form is prime or [probable primes base 3]	Search limit
$n!!+1$	1, 2, 518	2596
$n!!-1$	2, 4, 6, 8, 16, 26, 64, 82, 90, 118, 194, 214, 728, 842, 888, [2328]	2592
$n!!!+1$	1, 2, 4, 5, 6, 7, 9, 10, 11, 17, 24, 29, 39, 40, 57, 58, 59, 91, 155, 175, 245, 359, 372, 597, 864, 977, 1077, 1327, 2076	3988
$n!!!-1$	3, 4, 6, 8, 20, 26, 36, 50, 60, 114, 135, 138, 248, 315, 351, 429, 642	4461
$n!_4+1$	1, 2, 4, 6, 18, 38, 56, 94, 118, 148, 286, 1358, 1480, 1514, 2846, [2860]	3602
$n!_4-1$	3, 4, 6, 8, 12, 16, 22, 24, 54, 56, 98, 152, 156, 176, 256, 454, 460, 720, 750, 770, 800, 1442, 2846	4788
$n!_5+1$	1, 2, 4, 6, 9, 11, 13, 15, 17, 23, 26, 31, 32, 35, 36, 49, 52, 89, 92, 106, 120, 141, 149 173, 201, 280, 289, 353, 455, 483, 499, 543, 811, 866, 1010, 1126, 1557, 2358, 2411, 2435, 2485, 2491, 2772, 2851, 2937, 2996, [3642], [3777], [4123]	5425
$n!_5-1$	3, 4, 6, 7, 8, 12, 13, 14, 27, 28, 33, 35, 44, 50, 62, 64, 74, 88, 114, 140, 142, 242, 248, 262, 270, 284, 395, 473, 582, 600, 634, 707, 805, 882, 907, 1008, 1152, 1243, 1853, 2340, 2410, [3600], [3925]	2344

Proof. If p is prime, then by Fermat's theorem, $k^{p-1} \equiv 1 \pmod{p}$. By Wilson's theorem $(p-1)! \equiv -1 \pmod{p}$. Together these give

$$[k(p-1)]!_k \equiv k^{p-1}(p-1)! \equiv -1 \pmod{p}.$$

Adding 1 completes the proof. Conversely, suppose p is not prime. For $p > 4$ it is easy to show that $(p-1)! \equiv 0 \pmod{p}$, so $[k(p-1)]!_k$ is zero (not -1) modulo p. When $p = 4$, then $(3k)!_k+1$ is $6k^3+1$ and $4 \nmid 6k^3+1$.

Though theorem 2 applies to $n!!+1$ only when $n = 2(p-1)$ for some prime p, this happens often for small numbers. For example, if we scan $n \leqslant 200$, we first must throw out all the odd numbers n (other than 1), because if n is odd, $n!!+1$ is even. This leaves 100 even numbers, 25 of which are twice a prime -1. So only 75 possibilities are left. However, discarding these 25 possibilities is not enough to explain why $n!!+1$ has so many fewer primes than $n!!-1$. So we searched further. Three more of our results follow.

Theorem 3. Let p be an odd prime. Then

$$(p\pm 1)!! \equiv (-1)^{\frac{1}{2}(p-1)}(p-2)!! \equiv -(p-3)!! \pmod p$$

and

$$(p\pm 1)!!^2 \equiv [\tfrac{1}{2}(p-1)]!^2 \equiv (-1)^{\frac{1}{2}(p+1)} \pmod p.$$

Proof. We have

$$\begin{aligned}(p+1)!! &= (p+1)(p-1)!! \equiv (p-1)!! \pmod p\\ &\equiv (p-1)(p-3)!! \equiv -(p-3)!! \pmod p\end{aligned}$$

and

$$\begin{aligned}(p-1)!! &= (p-1)(p-3)\cdots 4\times 2\\ &= (p-1)(p-3)\cdots[p-(p-4)][p-(p-2)]\\ &\equiv (-1)(-3)\cdots[-(p-4)][-(p-2)] \pmod p\\ &\equiv (-1)^{\frac{1}{2}(p-1)}(p-2)!! \pmod p.\end{aligned}$$

Further,

$$\begin{aligned}(p\pm 1)!!^2 &\equiv (p-1)!!\,(p-1)!! \equiv (p-1)!!\,(-1)^{\frac{1}{2}(p-1)}(p-2)!! \pmod p\\ &\equiv (-1)^{\frac{1}{2}(p-1)}(p-1)! \equiv (-1)^{\frac{1}{2}(p+1)} \pmod p\end{aligned}$$

by Wilson's theorem. Finally,

$$[\tfrac{1}{2}(p-1)]!^2 = \left(\frac{(p-1)!!}{2^{\frac{1}{2}(p-1)}}\right)^2 \equiv (p-1)!!^2 \pmod p$$

by Fermat's theorem.

Theorem 3 becomes a divisibility result when p is a prime congruent to 3 (mod 4), as the following corollary shows.

Corollary 4. Let p be a prime congruent to 3 (mod 4).

(a) Either p divides $(p\pm 1)!!+1$, $(p-2)!!-1$ and $(p-3)!!-1$, or p divides $(p\pm 1)!!-1$, $(p-2)!!+1$ and $(p-3)!!+1$.

(b) p divides either $[\frac{1}{2}(p-1)]!+1$, or $[\frac{1}{2}(p-1)]!-1$.

(c) If $\gcd(p,k)=1$, then p divides one of

$$[\tfrac{1}{2}k(p-1)]!_k \pm 1 = k^{\frac{1}{2}(p-1)}[\tfrac{1}{2}(p-1)]! \pm 1.$$

Proof. Since $\frac{1}{2}(p+1)$ is even and the only solutions to $X^2 \equiv 1 \pmod{p}$ are $X \equiv \pm 1 \pmod{p}$, (a) and (b) are obvious by the previous result. Part (c) comes from lemma 1, Fermat's theorem and (b). When $k = 2$, (c) is contained in (a).

Examples. The first case of (a) holds for $p = 7$, 47, 59, 79 and 83; the second case holds for 11, 19, 23, 31, 43, 67, 71. The first case of (b) holds for 7, 11, 19, 43, 47, 67 and 79; the second case holds for 23, 31, 59, 71 and 83. Finally, for $p = 7$, the correct sign in (c) is, respectively, +, +, −, +, −, − for $k \equiv 1, 2, 3, 4, 5, 6 \pmod{7}$.

Extension lemma. Let p be a prime congruent to 3 (mod 4).

(a) Let $p > a > \frac{1}{2}p$. Then p divides one of $(2a)!!+1$ if and only if p divides one of $(2a-p)!!\pm 1$.

(b) Let $p > a$. Then p divides one of $(2a)!!+1$ if and only if p divides one of $[2(p-a-1)]!!\pm 1$.

Proof. (a) For any integers $2a > p > a$ (p odd) we can expand the appropriate double-factorials to see $(2a)!! \equiv (p-1)!!\,(2a-p)!! \pmod{p}$. After adding the requirement that p is prime we have $(p-1)!! \equiv \pm 1 \pmod{p}$ (theorem 3) so we are done. (b) By lemma 1, Fermat's theorem and Wilson's theorem we have

$$(2a)!!\,[2(p-a-1)]!! \equiv (p-1-a)!\,a! \equiv (-1)^{a+1} \pmod{p}.$$

Examples. When $p = 71$ a quick scan of the odd numbers less than p shows that p divides $n!!\pm 1$ for $n = 1$, 17, 45 and 57. By (a) of the previous lemma, p divides $n!!+1$ for $n = 72$, 88, 116 and 128 (just add p). By (b) p divides $n!!\pm 1$ for 68, 52, 24 and 12 (just subtract from $2p-2$). Finally, theorems 2 and 4 add that p divides $140!!+1$ and $70!!\pm 1$. These are *all* the values of n for which $n!!+1$ is divisible by 71.

Again, we first stated the results above in terms of the case which interested us most, $n!!\pm 1$, but later discovered they generalized to cover both $n!!\pm 1$, so cannot explain the differences between these two cases. Perhaps it has something to do with how often $n!!+1$ is divisible by a prime squared, or … .

Conclusion

Like the well-studied forms $n!\pm 1$, the new forms $n!_k\pm 1$ produce many primes. One form, however, $n!!+1$, does not seem to produce primes as it should. In the previous section we showed that Wilson's and Fermat's theorems lead to simple divisibility relations for these numbers, but none of them fully explain the dearth of primes. We would be pleased to hear from any reader who can explain this mystery, as well as from those of you who will expand our tables. Good hunting!

References

1. A. Borning, 'Some results for $k!+1$ and $2 \cdot 3 \cdot 5 \cdots P+1$', *Mathematics of Computation* **26** (1972), 567–570.

2. D. Bressoud, *Factorization and Primality Testing* (Springer-Verlag, New York, 1989), pp. 33–34.

3. J. Brillhart et al., *Factorizations of* $b^n \pm 1$, $b = 2, 3, 5, 6, 7, 10, 11, 12$ *up to High Powers* (American Mathematical Society, Providence, RI, 1988).

4. J. P. Buhler, R. E. Crandall and M. A. Peak, 'Primes of the form $M!+1$ and $2 \cdot 3 \cdot 5 \cdots P+1$' *Mathematics of Computation* **38** (1982), 639–643.

5. H. Dubner, 'The size of prime-factorials', *Journal of Recreational Mathematics* **19**(1987), 38–40.

6. H. Dubner, 'Factorial and primorial primes', *Journal of Recreational Mathematics* **19** (1987), 197–203.

7. H. Dubner, 'A new primorial prime', *Journal of Recreational Mathematics* **21** (1989), 276.

8. H. Dubner and R. Dubner, 'The development of a powerful low-cost computer for number theory applications', *Journal of Recreational Mathematics* **18** (1985–86), 81–86.

9. R. Guy, *Unsolved Problems in Number Theory* (Springer-Verlag, New York, 1981), Section A2.

10. R. Ondrejka, 'Ten extraordinary primes', *Journal of Recreational Mathematics* **18** (1985–86), 87–92.

11. P. Ribenboim, *The Book of Prime Number Records* (Springer-Verlag, New York, 1988).

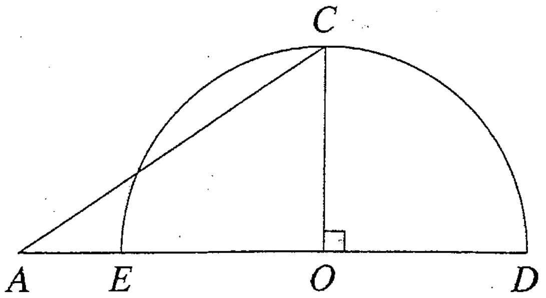
12. M. Templer, 'On the primality of $k!+1$ and $2 \cdot 3 \cdot 5 \cdots P+1$', *Mathematics of Computation* **34** (1980), 303–304.

13. D. Wells, *The Penguin Dictionary of Curious and Interesting Numbers* (Penguin, London, 1986).

14. S. Wolfram, *Mathematica, A System for Doing Mathematics by Computer* (Addison-Wesley, Reading, MA, 1988), p. 350.

15. S. Yates and C. Caldwell, 'The largest known primes.' Available from Professor C. Caldwell, Mathematics and Computer Science Department, University of Tennessee, Martin, TN 38238, USA.

O is the centre of the semicircle. Prove that the squares on AE, AC and AD are in arithmetic progression.



JOHN HALSALL
Exeter

Expansions into Iterated Square Roots

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For any non-negative real number x, we give below a special representation of $1+x$ as the limit of an infinite iteration of square roots which gives remarkable representations especially for the case of the non-negative integers.

Theorem . For every real number $x \geqslant 0$ it is the case that

$$1+x = \lim_{n\to\infty} \sqrt{1+x\sqrt{1+(1+x)\sqrt{1+(2+x)\sqrt{\cdots\sqrt{1+(n+x)\sqrt{1+(n+1+x)}}}}}}. \tag{1}$$

Proof. We consider a sequence of functions f_n each defined on $x \geqslant 0$, where

$$f_0(x) = 1 \quad \text{and} \quad f_n(x) = \sqrt{1+xf_{n-1}(1+x)} \quad (n = 1,2,3,\ldots). \tag{2}$$

From (2) it can readily be verified that

$$f_{n+2}(x) = \sqrt{1+x\sqrt{1+(1+x)\sqrt{1+(2+x)\sqrt{\cdots\sqrt{1+(n+x)\sqrt{1+(n+1+x)}}}}}}$$

and, therefore, to prove (1) it suffices to prove

$$1+x = \lim_{n\to\infty} f_n(x). \tag{3}$$

Since each function f_n is positive, from (2) it follows that

$$f_n(x) \geqslant 1 \tag{4}$$

for $n \geqslant 0$ and of course for $x \geqslant 0$.

Next, using induction on n we show that

$$f_n(x) \leqslant 1+x \tag{5}$$

for $n \geqslant 0$ and $x \geqslant 0$. From (2) it readily follows that $f_0(x) = 1 \leqslant 1+x$. Let us assume that $f_{n-1}(x) \leqslant 1+x$. But then again, from (2) and our assumption, it follows that

$$f_n(x) = \sqrt{1+xf_{n-1}(1+x)} \leqslant \sqrt{1+x(2+x)} = 1+x.$$

Thus (5) is established.

For every $x \geqslant 0$, from (5) we obtain

$$0 \leqslant (1+x)-f_n(x) = \frac{(1+x)^2-f_n^2(x)}{(1+x)+f_n(x)},$$

which, in view of (2) and (4), implies

$$0 \leqslant (1+x)-f_n(x) \leqslant \frac{x}{2+x}\{(2+x)-f_{n-1}(1+x)\}. \tag{6}$$

Applying the inequality given by (6) to $(2+x)-f_{n-1}(1+x)$ instead of $(1+x)-f_n(x)$, we derive

$$0 \leqslant (1+x)-f_n(x) \leqslant \frac{x}{2+x}\cdot\frac{1+x}{3+x}\{(3+x)-f_{n-2}(2+x)\}. \tag{7}$$

Applying again (a second time) the inequality given by (6) to $(3+x)-f_{n-2}(2+x)$ instead of $(1+x)-f_n(x)$, we derive

$$0 \leqslant (1+x)-f_n(x) \leqslant \frac{x}{2+x}\cdot\frac{1+x}{3+x}\cdot\frac{2+x}{4+x}\{(4+x)-f_{n-3}(3+x)\}. \tag{8}$$

Continuing this process $n-1$ times, we see that for every $x \geqslant 0$ it is the case that

$$0 \leqslant (1+x)-f_n(x) \leqslant \frac{x}{2+x}\cdot\frac{1+x}{3+x}\cdots\frac{n-1+x}{n+1+x}\{(n+1+x)-f_{n-n}(n+x)\}.$$

However, by (2) we have $f_0(x) = 1$ and therefore, from the above it follows that, for every $x \geqslant 0$,

$$0 \leqslant (1+x)-f_n(x) \leqslant \frac{x}{2+x}\cdot\frac{1+x}{3+x}\cdots\frac{n-1+x}{n+1+x}(n+x) = \frac{x(1+x)}{n+1+x}.$$

In short, we have derived

$$0 \leqslant (1+x)-f_n(x) \leqslant \frac{x(1+x)}{n+1+x} \tag{9}$$

for every $x \geqslant 0$ and every $n \geqslant 0$. Clearly,

$$\lim_{n\to\infty}\frac{x(1+x)}{n+1+x} = 0$$

for every $x \geqslant 0$, which in view of (9) implies

$$\lim_{n\to\infty} f_n(x) = 1+x$$

for every $x \geqslant 0$, which in turn, in view of (3), implies (1), as desired.

Based on (1), the limit of an infinite iteration of square roots can be defined in an obvious way, yielding remarkable representations of non-negative real numbers as the limits of infinite iterations of square roots. Indeed, (1) for $x = 1$ yields

$$2 = \sqrt{1+\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+6\sqrt{1+7\sqrt{1+\cdots}}}}}}}}.$$

Apocalypse Numbers

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The book called the Revelation (or Apocalypse) of John is the last book of the New Testament (except for the Syriac-speaking church, which has never accepted it). Various mystics have devoted much energy to deciphering the number 666, said by John the Apostle to designate the Number of the Beast, the Antichrist. (More recently, mystical individuals of the extreme fundamentalist right in the USA have noted that each word in the name Ronald Wilson Reagan has six letters.)

Over a year ago, I began a computer search for apocalypse numbers. These are Fibonacci numbers with precisely 666 digits. The sequence of numbers $(1, 1, 2, 3, 5, 8, \ldots)$, is called the *Fibonacci sequence* after the wealthy Italian merchant Leonardo Fibonacci of Pisa, and it plays important roles in mathematics and nature. These numbers are such that, after the first two, every number in the sequence equals the sum of the two previous numbers

$$F_n = F_{n-1} + F_{n-2}.$$

It turns out that the 3184th Fibonacci number is apocalyptic, having 666 digits. For numerologist readers, the apocalyptic number is:

116 724 374 081 495 541 233 435 764 579 214 184 068 974 717
443 439 437 236 331 282 736 262 082 452 385 312 960 682 327
210 312 278 880 768 244 979 876 073 455 971 975 198 631 224
699 392 309 001 139 062 569 109 651 074 019 651 076 081 705
393 206 023 798 479 391 897 000 377 475 124 471 344 025 467
950 768 706 990 550 322 971 334 370 940 093 654 442 411 815
206 857 904 041 043 400 568 568 081 194 379 503 001 967 669
356 633 792 347 218 656 896 136 583 990 327 918 167 352 721
163 581 650 359 577 686 552 293 102 708 827 224 247 109 476
382 115 427 568 268 820 040 258 504 986 113 408 773 333 220
873 616 459 116 726 497 198 698 915 791 355 883 431 385 556
958 002 121 928 147 052 087 175 206 748 936 366 171 253 380
422 058 802 655 291 403 358 145 619 514 604 279 465 357 644
672 902 811 711 540 760 126 772 561 572 867 155 746 070 260
678 592 297 917 904 248 853 892 358 861 771 163.

Here are some unsolved problems to consider.

1. Is the number shown here the only apocalyptic Fibonacci number?
2. Does there exist an apocalyptic prime number?
3. Is there any significance to the fact that the first four digits and last four digits (1167 and 1163) of the apocalypse number are both dates during the reign of Frederick I of Germany who intervened extensively in papal politics?
4. Is it just a coincidence that the keys of a piano appear to exhibit a segment of the Fibonacci sequence 1, 2, 3, 5, 8, ... ? There are two black notes, followed by three black notes. There are five black keys in an octave and eight white keys in an octave!

Comments on Questions 1 and 2

1. Binet's formula for the nth Fibonacci number is

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

For large n, $\{\frac{1}{2}(1-\sqrt{5})\}^n$ is small and F_n essentially grows by a factor $\frac{1}{2}(1+\sqrt{5})$ at each step. Now

$$\left(\frac{1+\sqrt{5}}{2}\right)^4 \approx 6.854 \quad \text{and} \quad \left(\frac{1+\sqrt{5}}{2}\right)^5 \approx 11.09,$$

so the apocalyptic number given by Dr Pickover will give four more apocalyptic numbers.

2. Bertrand's postulate says that, for every $n > 1$, there is a prime number between n and $2n$. So there will be a prime number between the given apocalyptic number and its double, and this will have 666 digits.

Editor

One Shape in Several Guises

L. SHORT, *Napier University*

The author lectures at Napier University, Edinburgh (known until 1992 as Napier Polytechnic) and is keenly interested in how simple mathematical ideas can shed light on different areas of mathematics. Of endless interest is the underlying (but elusive) unity of mathematics.

1. A variety of ideas

One often encounters an idea, an argument, or a result that seems familiar; one feels one has seen something similar in a different context. An interesting example of this is provided by the picture in figure 1, which crops up in a variety of situations, some related and some apparently not. In addition to their intrinsic interest and elementary nature, the results discussed suggest various extensions.

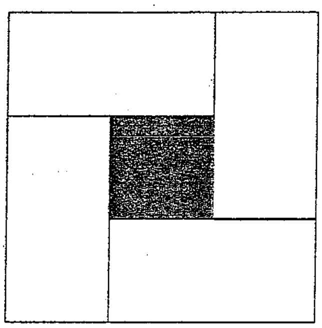


Figure 1

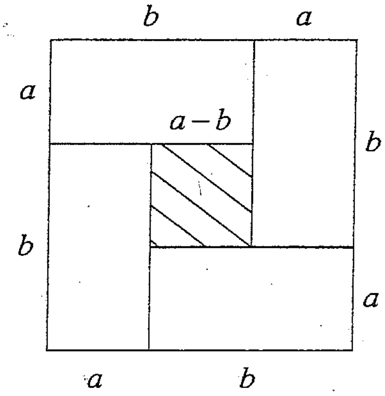


Figure 2. $(a+b)^2-(a-b)^2 = 4ab$

In figure 2 the algebraic identity

$$(a+b)^2-(a-b)^2 = 4ab \tag{1}$$

follows on noting that the difference of the areas of the two squares equals the sum of the areas of the four rectangles. Also (figure 3) the area of the larger square exceeding (or equal to) the sum of the areas of the rectangles easily yields the arithmetic mean–geometric mean inequality

$$\tfrac{1}{2}(a+b) \geqslant \sqrt{ab} \tag{2}$$

for two variables (see reference 11). In addition, we may note that the shaded area in figure 2 is zero if and only if $a = b$: hence only in this latter situation does the equality in (2) hold.

We can modify figure 3 in an interesting way. Relabelling a and b as x_1 and x_2 with $x_1 < x_2$, we consider x_1 to be increased and x_2 decreased

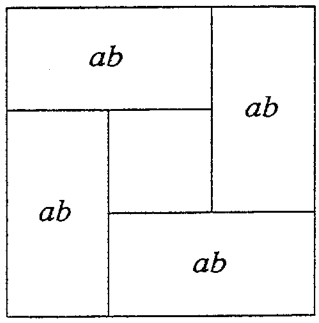


Figure 3. $(a+b)^2 \geqslant 4ab$, $\frac{1}{2}(a+b) \geqslant \sqrt{ab}$

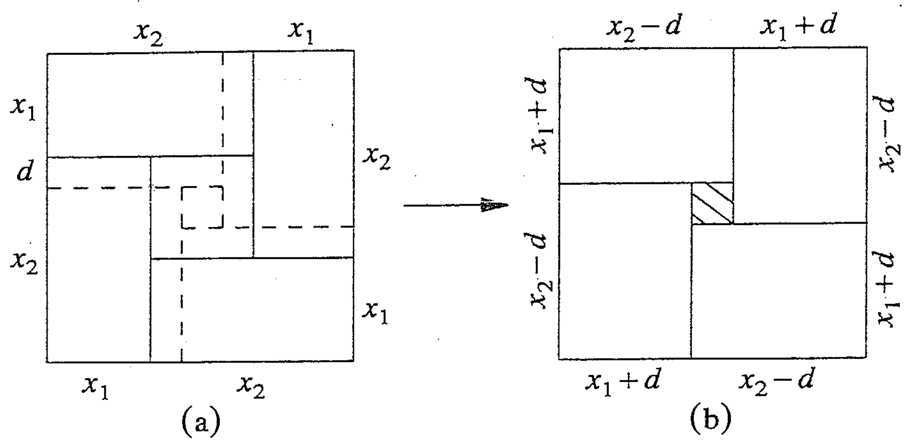


Figure 4. $x_1x_2 \leqslant (x_1+d)(x_2-d)$, with $0 \leqslant d \leqslant \frac{1}{2}(x_2-x_1)$

by the same (positive) amount d. This is indicated in figure 4 (a) by the dotted lines, and is redrawn as figure 4 (b). Since the total area of the rectangles has increased (why?), we obtain

$$4x_1x_2 \leqslant 4(x_1+d)(x_2-d)$$

i.e.

$$x_1x_2 \leqslant (x_1+d)(x_2-d) \quad (0 \leqslant d \leqslant \tfrac{1}{2}(x_2-x_1)). \tag{3}$$

Although we can use this basic idea to prove the general arithmetic mean–geometric mean inequality for n variables (see reference 7), the inequality (3) is of interest in its own right. It asserts that if, in a product of two (positive) numbers, we increase the smaller and decrease the larger by the same amount (within a certain limit), the product increases in value. For example

$$2\times10 \leqslant 3\times9 \leqslant 4\times8 \leqslant 5\times7 \leqslant 6\times6. \tag{4}$$

Figure 5 takes a 'discrete' view. If we define T_n to be the nth triangular number,

$$T_n = 1+2+3+\cdots+n, \tag{5}$$

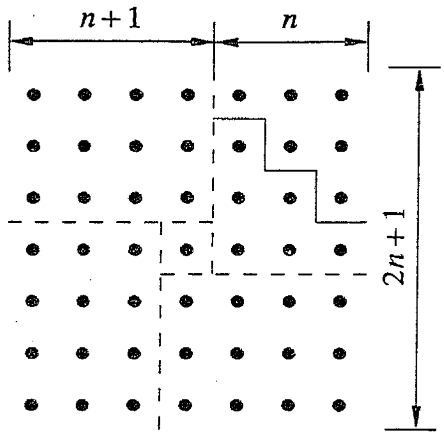


Figure 5. $8T_n + 1 = (2n+1)^2$

we obtain the result that 8 times a triangular number plus 1 is a square (since two triangular numbers form a rectangle, as indicated by the 'staircase' in figure 5):

$$8T_n + 1 = (2n+1)^2. \tag{6}$$

Algebraically (6) follows easily on summing the series (5), i.e.

$$T_n = \tfrac{1}{2}n(n+1). \tag{7}$$

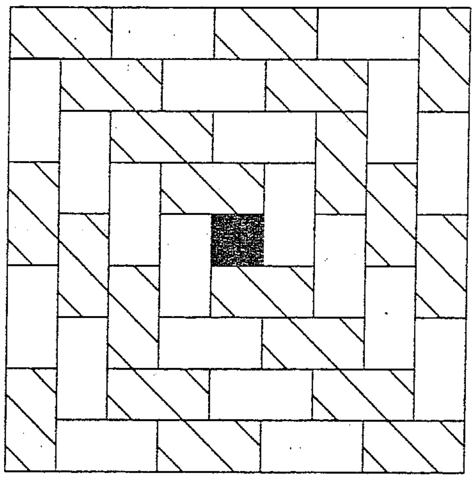
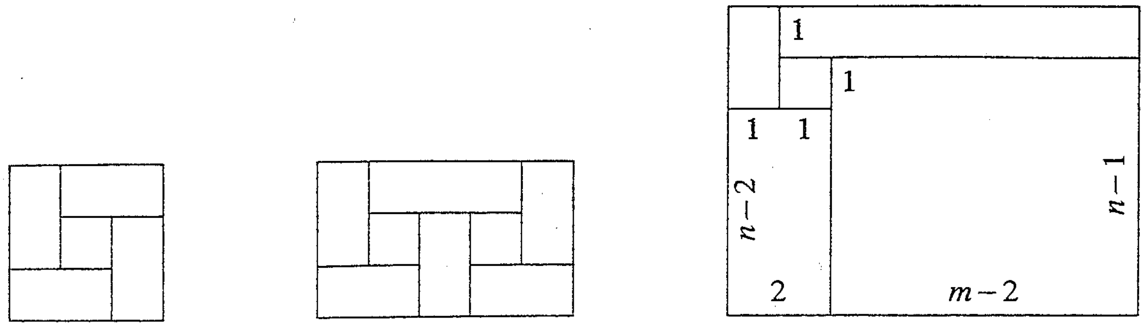


Figure 6. $1 + 2 \sum_{k=1}^{n} 4k = (2n+1)^2$

Figure 6 represents an elegant dissection of a square into rectangles; the resulting identity (see reference 13)

$$1 + 2 \sum_{k=1}^{n} 4k = (2n+1)^2 \tag{8}$$

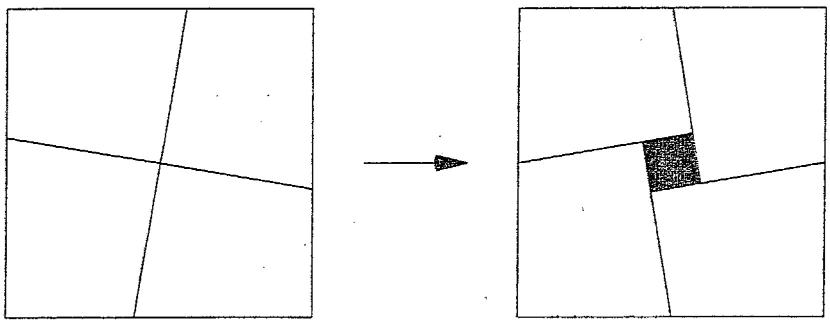
is readily seen to be equivalent to (6). We may note the presence of the basic square of figure 1 at the centre of figure 6.



(a) (b) (c)

Figure 7. (a), (b) Two simple tilings
(c) Tiling of an $m \times n$ rectangle $(m, n \geqslant 3)$ with five elements

A *simple* tiling of a given rectangle R by (smaller) rectangles is defined as one in which no connected set of two or more elements (tiles) forms a rectangle strictly inside R. Our basic shape, reproduced in figure 7 (a), thus constitutes a simple tiling of a 3×3 square; figure 7 (b) depicts a simple tiling of a 3×5 rectangle in which our basic shape plays no part. Although the latter does not play a fundamental role in the theory of simple tilings (see reference 5), the slightly more general shape of figure 7 (c) proves that a 5-element simple tiling of an $m \times n$ rectangle always exists, provided $m \geqslant 3$ and $n \geqslant 3$.



(a) (b)

Figure 8. A vanishing area

Along rather different lines, we have the shape of figure 8 (b), a slight generalization of figure 1, giving rise to an interesting little puzzle. The square of figure 8 (a) is cut along the indicated lines into four identical pieces which, when re-assembled as shown (which is not quite as easy as it looks), form a square of apparently the same size but with a 'square hole' at the centre. The question is, where has this area gone? What has happened can easily be seen by recalling the well-known dissection proof of Pythagoras's theorem depicted in figure 9.

Figure 10 is possibly 'stretching' matters a little too far, although figure 10 (h) does have some similarity to figures 1 and 8 (b). Figures 10 (a) and (b) are the two standard representations of the Möbius strip; in

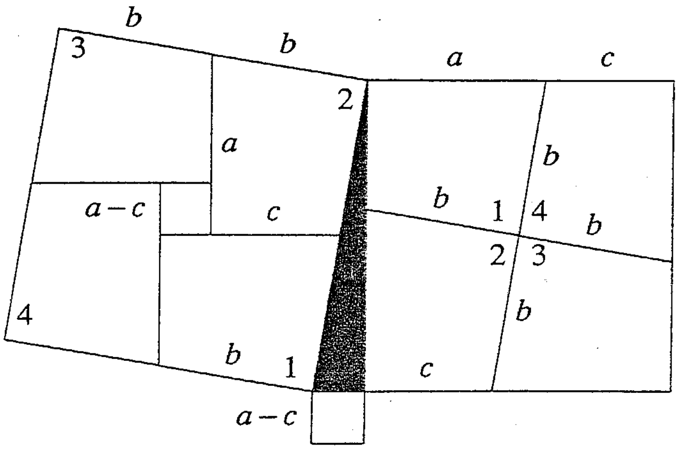


Figure 9. Pythagoras by dissection: $(a+c)^2+(a-c)^2 = 4b^2$

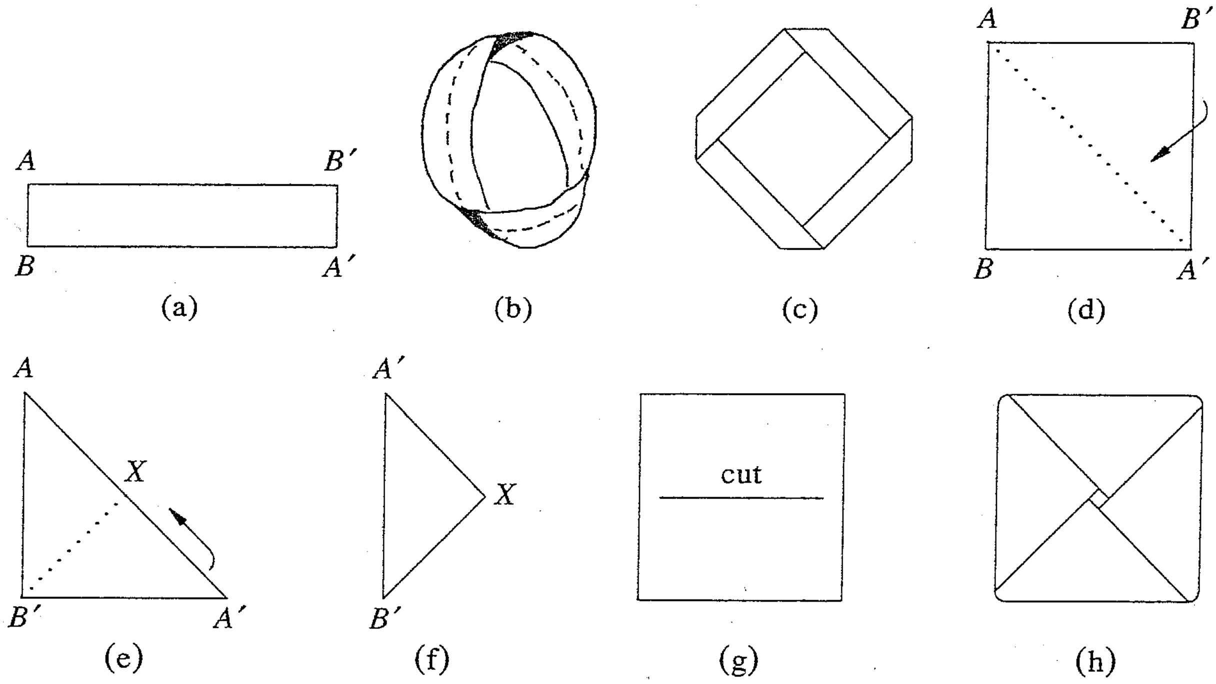


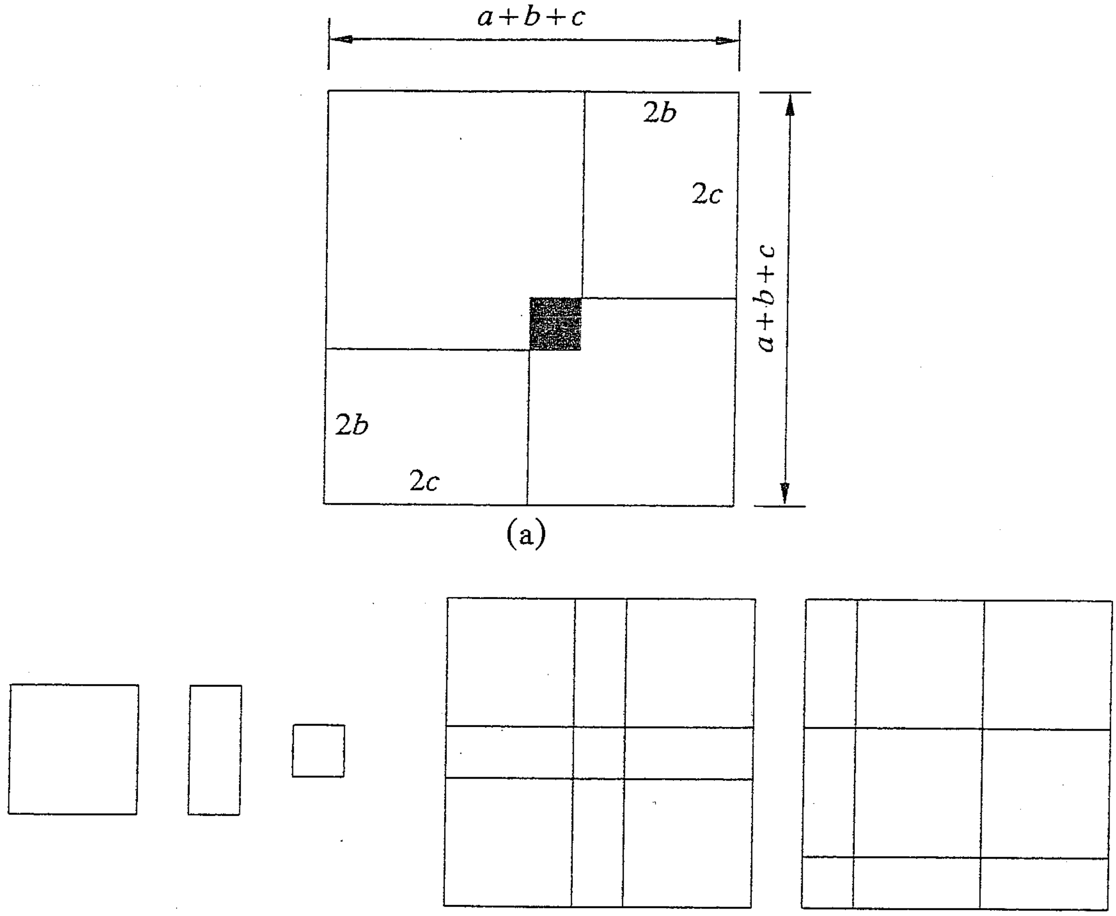
Figure 10. Flattening a Möbius strip

figure 10 (a) points A and A' are to be identified, as are the points B and B'. This produces the twist evident in figure 10 (b), but leaves a surface with only one side and one edge. Cutting along the dotted line of figure 10 (b) leaves the surface in one piece, although the cut has gone completely round the surface. (Although this is 'well known', it still comes as something of a surprise, and one really needs to make a model to convince oneself that this really happens.) Flattening this cut strip produces figure 10 (c), which begins to resemble figure 1; figure 10 (h) is merely figure 10 (c) but with different proportions. One may imagine this can be produced by starting with a shorter, fatter Möbius strip than in figure 10 (a); this is most easily achieved by modifying the above folding and cutting procedure. Folding as in figures 10 (d) and (e) produces figure

10(f) with the correct identifications; but to produce the cut, as in figure 10(b), great care is required. Firstly we must cut figure 10(d) *before* the foldings, as depicted in figure 10(g); the resulting figure 10(f) can then be opened out (with care) to produce figure 10(h). Again, one really requires a model to convince oneself of this. Further interesting aspects of this construction are discussed in reference 1.

2. Some further ideas

The reader can probably think of various generalizations of the ideas we have discussed in addition to the following.



(a)

(b) (c) (d)

Figure 11. Algebraic identities. (a) An algebraic identity? (b) The pieces: $x \times x$, $1 \times x$ and 1×1. (c), (d) $(2x+1)^2 = 4x^2+4x+1$

1. Many algebraic identities have simple pictorial interpretations, and many series can be summed in such a framework (see reference 12). The reader may care to try to deduce an algebraic identity from figure 11(a).

   An interesting attempt to teach the algebraic notion of factorization from a more geometric point of view arises naturally from such a perspective (see reference 3). The basic idea is to use suitable combinations of three different types of 'blocks', as depicted in figure

11 (b), to construct a rectangle as in figures 11 (c) and (d). The blocks represent the terms in the quadratic expansion and the sides of the rectangle the appropriate factors. In the simple instance given, the rectangle is a square. In addition, this view clarifies somewhat the troublesome notion of a *variable*. By forming different sets of starting pieces, one finds that the same factorization results; although the value of x can 'vary', the same final factorization results. In slightly more abstract language, the result of figures 11 (c) and (d) is an *identity* rather than an *equation*.

2. The construction of figure 8 can be applied to any rectangle, the size of the space appearing when the pieces are rearranged depending on the angle that the cuts make with the rectangle sides. The principle behind figure 8 is but one of several, giving rise to different classes of area-vanishing paradoxes; reference 6 contains a detailed discussion.

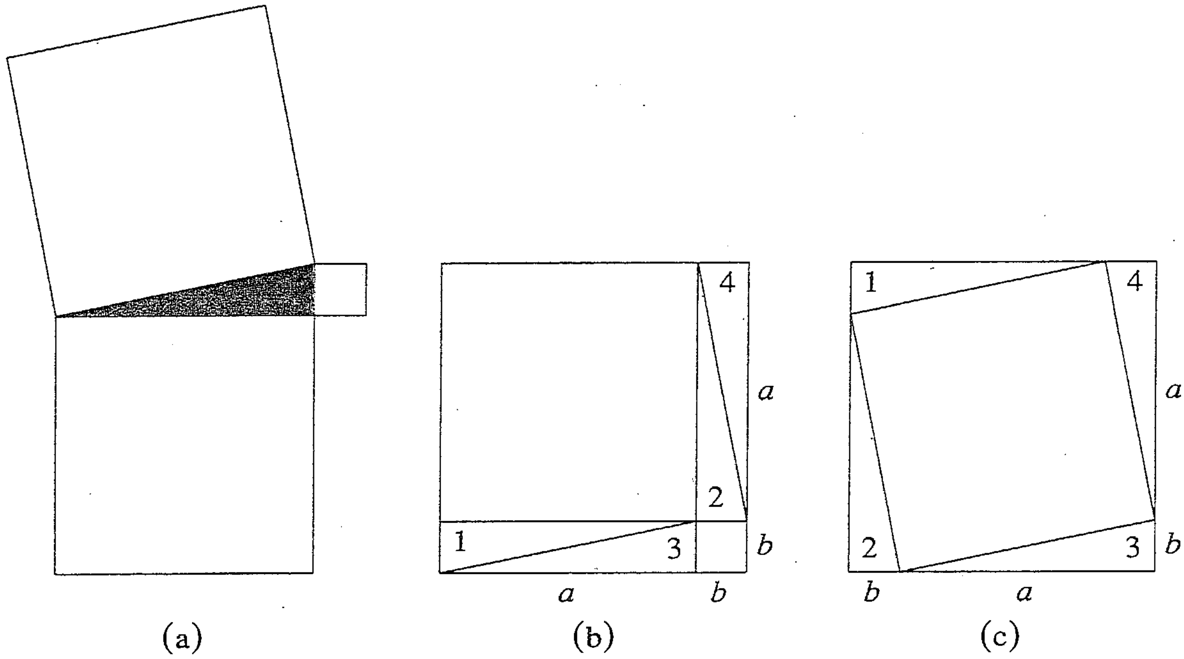


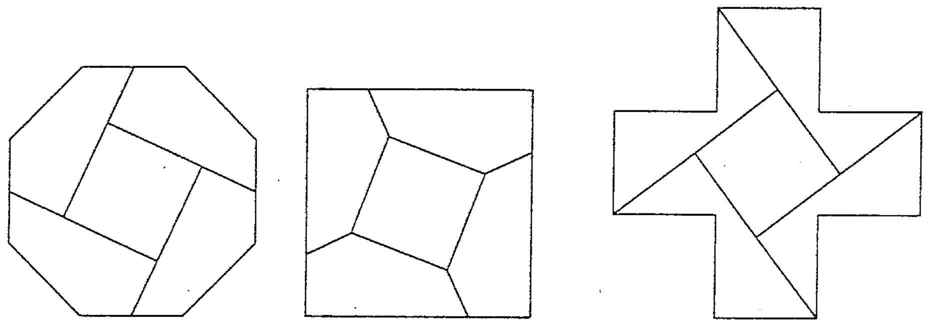
Figure 12. Pythagoras by completion

3. Returning to figure 8 (b); if we enlarge the central square until it touches the sides of the larger square, as in figure 12 (c), we obtain a second proof of Pythagoras's theorem. Here four replicas of the given triangle of figure 12 (a) are placed inside a square of side $a+b$. In the arrangement of figure 12 (b), the remainder of the square is made up of two squares of sides a and b. There is, however, an important difference between the proofs corresponding to figures 9 and 12. In the latter we see that, by equating the areas of the four triangles, the large square has an area equal to the sum of the areas of the two smaller squares. What the proof does *not* show is *how* to put the smaller squares together (ignoring the four triangles) to make the

larger one; this is exactly what figure 9 achieves. In figure 12 we show that the squares are of 'equal content' or 'equivalent by completion', whereas in figure 9 we show that they are 'of equal area' or 'equivalent by dissection'. One would hope that these concepts are equivalent, but is this so? Indeed, does either argument constitute a 'rigorous' proof? (The real importance of the concepts of equivalence by completion and dissection is that they are *not* equivalent concepts in *three* dimensions; this is briefly discussed below.)

4. Intuitively, two regions R_1 and R_2 are equivalent by dissection if R_1 can be cut into pieces which can be reassembled to form R_2. (This idea is behind the most familiar demonstration that a rectangle and a parallelogram, on the same base and between the same parallels, have equal area.)

   For polygonal regions R_1 and R_2 in the *Euclidean plane*, the following are equivalent:
   (a) R_1 and R_2 have equal area;
   (b) R_1 and R_2 are equivalent by dissection;
   (c) R_1 and R_2 are equivalent by completion.



(a) (b)

Figure 13. (a) Octagon $\rightarrow$ square. (b) Greek cross $\rightarrow$ square

In particular, any plane polygon can be dissected and reassembled to form another polygon of equal area. In figure 13 (a), a regular octagon is transformed into a square, both shapes bearing a marked similarity to our basic shape of figure 1. The reader may care to reassemble figure 13 (b), which again resembles figure 1, into a square. This can be done in one of two ways: to produce either a solid square, or a square that lacks a central Greek cross. Reference 9 is the classic source for such dissection problems.

The restrictions to Euclidean geometry and to two dimensions are both crucial. Although the results above may seem 'intuitively obvious', this is only because we are familiar with viewing objects in the Euclidean plane. In different geometries, and in three dimensions, the statement has to be amended. Most strikingly, a regular tetrahedron *cannot* be dissected and reassembled to form a cube of the same volume. Reference 12 contains an elementary discussion of these ideas, with reference 4 the definitive work.

5. Interpreting figure 2 as fitting four $a \times b$ rectangles into an $(a+b) \times (a+b)$ square, leaving a space in the centre, can we generalize the construction to three dimensions? Specifically, in view of the three-variable form of (2),

$$27abc \leqslant (a+b+c)^3,$$

can we always pack 27 $a \times b \times c$ blocks into a cube of side $a+b+c$ (see reference 2)?

6. We may note that, in (4), each *sum* has the same value $2+10 = 3+9 = 4+8 = \cdots$, and the maximum *product* occurs when each term has the *same* value 6. This is a particular instance of a very general result, used to great effect in reference 10 to solve maxima and minima problems *without* calculus.

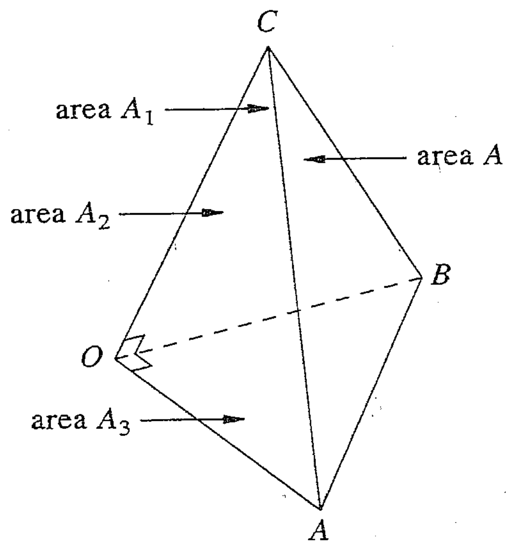


Figure 14. Pythagoras in three dimensions: $A^2 = A_1^2 + A_2^2 + A_3^2$

7. Does the three-dimensional form of figure 1 give rise to any 'interesting' generalizations of the results we have discussed? In particular, can the three-dimensional form of Pythagoras's theorem (see reference 8 and figure 14) be obtained from generalizations of figures 9 or 12? (As we have seen above, results in three dimensions can be rather different from their two-dimensional counterparts.)

References

1. S. Barr, *Experiments in Topology* (John Murray, London, 1965).
2. E. R. Berlekamp, J. H. Conway and R. K. Guy, *Winning Ways*, Volume 2 (Academic Press, New York, 1982).

3. J. K. Bidwell, A physical model for factoring quadratic polynomials, *Math. Teacher* **65** (1972), 201–205.
4. V. G. Boltianskii, *Hilbert's Third Problem* (Wiley, New York, 1978).
5. F. R. K. Chung, E. N. Gilbert, R. L. Graham, J. B. Shearer and J. H. van Lint, Tiling rectangles with rectangles, *Math. Mag.* **55** (1982), 286–291.
6. M. Gardner, *Mathematics, Magic and Mystery* (Dover, New York, 1956).
7. H. Henng, Letters to the Editor, *Math. Mag.* **59** (1986), 255.
8. L. Hull, H. Perfect and J. Heading, Pythagoras in higher dimensions—three approaches, *Math. Gaz.* **62** (1978), 206–211.
9. H. Lindgren, *Geometric Dissections* (Van Nostrand, Princeton NJ, 1964).
10. I. Niven, *Maxima and Minima without Calculus* (Mathematical Association of America, Washington DC, 1981).
11. D. Schattschneider, Proof without words—arithmetic mean–geometric mean inequality, *Math. Mag.* **59** (1986), 11.
12. L. Short, Some pictorial aspects of mathematics, Napier University preprint.
13. S. Wakin, Proof without words—domino = 2 squares: concentric squares, *Math. Mag.* **60** (1987), 327.

The 1993 Puzzle

In Volume 25 Number 3 we posed the problem to express the numbers 1 to 100 in terms of the digits 1, 9, 9, 3 in order using only $+$, $-$, $\times$, $\div$, $\surd$, ! and concatenation (i.e. writing 19, for example). All numbers except 65, 67, 68 and 70 yielded to readers' efforts. Among the most attractive were:

$$47 = -1+[(\sqrt{9}!)!\div(9+3!)],$$
$$59 = -1+[(\sqrt{9}!)!\div(9+3)]$$

from Khalid Khan, a student at the London School of Economics.

With reference to the article by Caldwell and Dubner in the present volume (pp. 1–7), our typesetter submits the following solutions:

$$65 = -1+(\sqrt{9}!)!!+\sqrt{9}\times 3! = -1+(\sqrt{9}!)!!+(\sqrt{9}!)!_3,$$
$$67 = +1+(\sqrt{9}!)!!+\sqrt{9}\times 3! = +1+(\sqrt{9}!)!!+(\sqrt{9}!)!_3,$$
$$68 = -1+(\sqrt{9}!)!!+(\sqrt{9}!)!!!+3,$$
$$70 = +1+(\sqrt{9}!)!!+(\sqrt{9}!)!!!+3.$$

Computer Column

MIKE PIFF

Rodent control

Our next task is to provide a modest mouse driver for our Modula-2 system. We provide a minimal number of functions which are just adequate for us to be able to get information from the mouse and show or hide its icon. The definition module is as follows.

```
DEFINITION MODULE Mouse;
PROCEDURE ShowMouse;
PROCEDURE HideMouse;
PROCEDURE GetMousePosition
  (VAR x,y:INTEGER);
PROCEDURE GetMouseState
  (VAR left,right:BOOLEAN);
END Mouse.
```

The MS-DOS implementation uses an interrupt to communicate with the mouse, with the parameters and function numbers passed in the registers. We could provide more functions, for instance, to change the icon, put the mouse pointer at a specified position or set the sensitivity of the mouse, making each visible as usual in the definition module. The appropriate descriptions of these functions may be found in any good DOS manual, for instance the *DOS Programmer's Reference* by Terry Dettmann, published by QUE Corporation.

```
IMPLEMENTATION MODULE Mouse;
FROM System IMPORT Trap, AX, BX, CX, DX;
CONST
  MouseInt=51;
  showfn=1; hidefn=2; statefn=3;
PROCEDURE ShowMouse;
BEGIN
  AX:=showfn;
  Trap(MouseInt);
END ShowMouse;
PROCEDURE HideMouse;
BEGIN
  AX:=hidefn;
  Trap(MouseInt);
END HideMouse;
PROCEDURE GetMousePosition
  (VAR x,y:INTEGER);
BEGIN
  AX:=statefn;
  Trap(MouseInt);
  x:=CX; y:=DX;
END GetMousePosition;
PROCEDURE GetMouseState
  (VAR left,right:BOOLEAN);
VAR
  temp:INTEGER;
BEGIN
  AX:=statefn;
  Trap(MouseInt);
  temp:=BX;
  left:=ODD(temp);
  right:=ODD(temp DIV 2)
END GetMouseState;
END Mouse.
```

Prizes for Student Contributors

The Editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £25 for letters, solutions to problems, and other items.

Letters to the Editor

Dear Editor,

On two diophantine equations

I particularly admire Paul Young's elegant idea for obtaining integral solutions of $x^3+y^3 = z^2$ (see *Mathematical Spectrum*, Volume 25, Number 2, page 57). The same idea can be used to generate natural-number solutions to the more general diophantine equations

$$x^{2n+1} \pm y^{2n+1} = z^2.$$

Here is how it works. The natural numbers $x = a$, $y = b$ and $a \geqslant b$ yield

$$x^{2n+1} \pm y^{2n+1} = a^{2n+1} \pm b^{2n+1}. \tag{1}$$

If this result is c^2 for some natural number c then $(x, y, z) = (a, b, c)$ is a solution. Otherwise we can write (1) as λc^2 where λ does not contain an integral square factor. Then $(x, y, z) = (\lambda a, \lambda b, \lambda^{n+1} c)$ is a solution. For an illustration, suppose that $n = 2$ and that we desire a solution of

$$x^5+y^5 = z^2. \tag{2}$$

If $x = 3$, $y = 2$ then $x^5+y^5 = 275 = 11(5)^2$ with $a = 3$, $b = 2$, $\lambda = 11$. This yields a solution of (2) where $(x, y, z) = (33, 22, 6655)$. In general, for any two natural numbers a and b with $a \geqslant b$ and $\lambda = a^{2n+1} \pm b^{2n+1}$, we always have a solution of (1) given by

$$(x, y, z) = (\lambda a, \lambda b, \lambda^{n+1} c).$$

Secondly, on reading about the Ramanujan problem (*Mathematical Spectrum* Volume 24, Number 3, page 79 and Volume 25, Number 1, page 26), I asked myself if the simultaneous equations

$$x+\sqrt{y} = a, \qquad \sqrt{x}+y = b \tag{3}$$

can have more than one solution (x, y) in accordance with the understanding that $\sqrt{\alpha}$ stands for the non-negative square root of the non-negative real number α. The answer is in the affirmative if $a = b$. If $a = b$, the attractive diophantine equation

$$x+\sqrt{y} = \sqrt{x}+y \tag{4}$$

has an infinity of neat rational-number solutions, not just the obvious $x = y = r \geqslant 0$ but others too. To see this, without loss of generality let us assume that $x \geqslant y$ and recast (4) as

$$(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y}-1) = 0. \tag{5}$$

Now $\sqrt{x}-\sqrt{y} = 0$ yields $x = y = r$, a non-negative rational number. If $x \neq y$ then $\sqrt{x}+\sqrt{y}-1 = 0$ has an infinity of rational-number solutions given by

$$x = \frac{m^2}{n^2}, \qquad y = \frac{(n-m)^2}{n^2} \quad (n \geqslant m \geqslant \tfrac{1}{2}n). \tag{6}$$

To obtain a pair of simultaneous equations $x+\sqrt{y} = \sqrt{x}+y = a$ which has one solution rational and another irrational we may put $m = 2$ and $n = 3$ in (6). This gives $x = \frac{4}{9}$, $y = \frac{1}{9}$, $a = \frac{7}{9}$ as the rational solution. An irrational solution $x = y = \frac{1}{18}(23 \pm \sqrt{333})$ is obtained by solving the equation

$$x+\sqrt{x} = \tfrac{7}{9},$$

which is quadratic in $\sqrt{x}$.

For appropriately chosen values of a and b, the simultaneous equations $x+\sqrt{y} = a$ and $\sqrt{x}+y = b$ could have up to four real solutions *without* the use of the signs $\pm$ and $\sqrt{x}$ and $\sqrt{y}$ as David Singmaster had to do with the Ramanujan problem. However, I am unable to resolve the related question:

For what distinct non-negative real numbers a and b can the simultaneous equation $x+\sqrt{y} = a$, $\sqrt{x}+y = b$ have

- two rational solutions,

or

- four real solutions with at least one rational solution?

Yours sincerely,

K. R. S. Sastry
(Box 21862, Addis Ababa,
Ethiopia)

Dear Editor,

On fast convergence to π

An iteration function with third-order convergence to π was introduced by P. Glaister (*Mathematical Spectrum*, Volume 23, Number 2, pages 33–35) and generalized to fifth-order by J. Mooney (*Mathematical Spectrum*, Volume 24, Number 4, pages 120–121). These functions may be further generalized to have fixed points of arbitrary odd order.

Glaister showed that iteration of

$$F_3(x) = x - \tan x$$

has third-order convergence to π (that is, $F_3'(\pi) = F_3''(\pi) = 0$ and $F_3'''(\pi) = -2 \neq 0$) and that F_3 is obtained by applying Newton's method to $f(x) = \sin x$. Recall that Newton's method is in general a second-order procedure. It is third-order in this case because $f''(\pi) = 0$.

Mooney gives a fifth-order generalization of F_3, namely,

$$F_5(x) = x - \tan x + \tfrac{1}{3}\tan^3 x, \tag{1}$$

and points out that the first four derivatives of (1) vanish, while $\mathrm{D}^5 F_5(\pi) = 24$. Thus F_5 is monotonically fifth-order converging to π, and the iteration converges for all $x_0 \in (\frac{2}{3}\pi, \frac{4}{3}\pi)$.

As Mooney added a term to F_3 to get fifth-order convergence, we also add terms to F_5 to get even higher-order convergence. For instance,

$$F_7(x) = x - \tan x + \tfrac{1}{3}\tan^3 x - \tfrac{1}{5}\tan^5 x \tag{2}$$

is seventh-order converging to π since the first six derivatives of (2) vanish at $x = \pi$, while $D^7F_7(\pi) = -720$. In general, the first $2p+2$ derivatives of

$$F_{2p+3}(x) = x - \sum_{k=0}^{p} (-1)^k \frac{\tan^{2k+1} x}{2k+1} \tag{3}$$

vanish at $x = \pi$ and

$$D^{2p+3}F_{2p+3}(\pi) = (-1)^{p+1}(2p+2)!\,.$$

That is, iteration of F_{2p+3} is $(2p+3)$th-order converging to π. The reader may recognize the sum in (3) as the $(2p+1)$th-degree Maclaurin polynomial for $\arctan u$ evaluated at $u = \tan x$. Indeed, as p goes to infinity, equation (3) becomes

$$\begin{aligned} F_\infty(x) &= x - (\tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \tfrac{1}{7}\tan^7 x + \cdots) \\ &= x - \arctan(\tan x). \end{aligned}$$

Now the Maclaurin series for $\arctan x$ converges for $-1 \leqslant x \leqslant 1$. Thus the series for $\arctan(\tan x)$ converges for $-1 \leqslant \tan x \leqslant 1$ or $\tfrac{1}{4}(4k-1)\pi \leqslant x \leqslant \tfrac{1}{4}(4k+1)\pi$ for all integers k. Moreover, since $\tan x = \tan(x - k\pi)$, it follows that F_∞ is identically $k\pi$ in this range. For $k = 1$, we have $F_\infty(x) = \pi$ for $\tfrac{3}{4}\pi \leqslant x \leqslant \tfrac{5}{4}\pi$.

Evidently, adding more terms to F_{2p+3} yields faster convergence to π, but the range of convergence shrinks as p gets large. In the limit, this range approaches that of F_∞, namely, $[\tfrac{3}{4}\pi, \tfrac{5}{4}\pi]$.

Of the three functions examined by Mooney, perhaps the most interesting is

$$H_3(x) = x + \sin x,$$

which he observes is cubically convergent to π with $H_3'''(\pi) = 1$ and interval of convergence $(0, 2\pi)$. Additional terms may be appended to H_3 to produce higher-order iterations. For instance,

$$H_5(x) = x + \sin x + \tfrac{1}{6}\sin^3 x$$

is fifth-order converging to π with $D^5H_5(\pi) = 9$. Note that the interval of convergence is still $(0, 2\pi)$. This process of adding terms may be continued indefinitely, and in fact, the coefficients of each new term are precisely the coefficients of the Maclaurin series for $\arcsin x$. In the limit, one obtains

$$\begin{aligned} H_\infty(x) &= x + (\sin x + \tfrac{1}{6}\sin^3 x + \tfrac{3}{40}\sin^5 x + \tfrac{5}{112}\sin^7 x + \cdots) \\ &= x + \arcsin(\sin x), \end{aligned}$$

which converges for $-1 \leqslant \sin x \leqslant 1$, that is, for all x. In particular, the series converges for $-\tfrac{1}{2}\pi \leqslant x \leqslant \tfrac{1}{2}\pi$. Using this and the fact that $\sin x = \sin(x - 2k\pi) = \sin([2k+1]\pi - x)$, we can show that H_∞ is piecewise linear and, in fact,

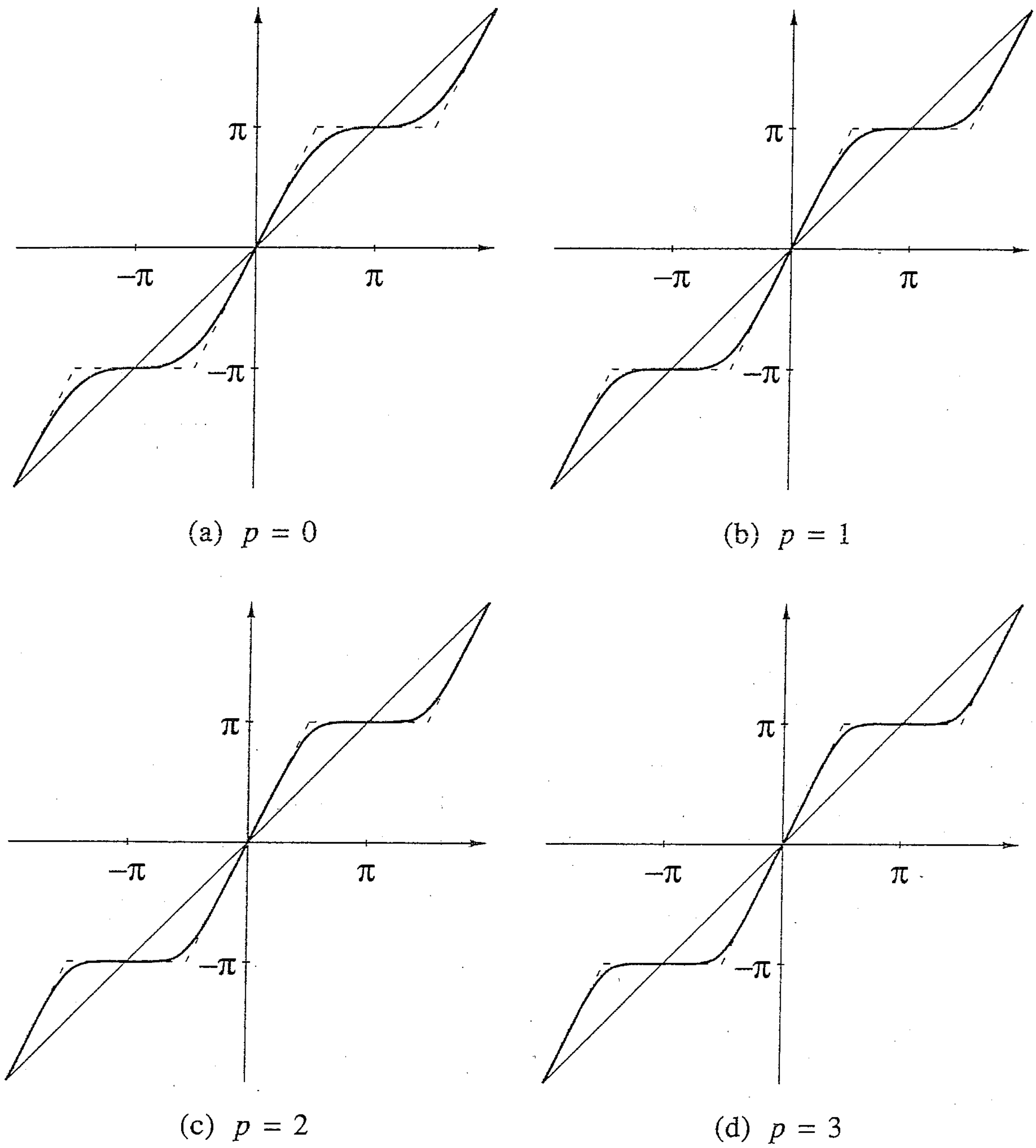


Figure 1. The first four members of the family H_{2p+3}

$$H_\infty(x) = \begin{cases} 2x - 2k\pi & (\text{if } \tfrac{1}{2}(4k-1)\pi \leqslant x \leqslant \tfrac{1}{2}(4k+1)\pi), \\ (2k+1)\pi & (\text{if } \tfrac{1}{2}(4k+1)\pi \leqslant x \leqslant \tfrac{1}{2}(4k+3)\pi). \end{cases}$$

Observe that each odd multiple of π is attracting under iteration of H_∞. Indeed, for all $x_0 \in (2k\pi, 2[k+1]\pi)$, the iteration not only converges, but is eventually equal to $(2k+1)\pi$.

Yours sincerely,

T. R. SCAVO
(117 E. 24th Place,
Eugene, OR 97405, USA)

Dear Editor,

From stamps to Diophantine equations

It might interest your readers to know that theorem 3 in the article by Ian Tame (*Mathematical Spectrum* Volume 25, Number 4, pages 110–112) was proved by J. J. Sylvester in the 1884 issue of *Educational Times* **41**, 21. This fact does not, of course, diminish my admiration of the author for having rediscovered it and for describing it so elegantly.

Yours sincerely,
STEVEN VAJDA
(University of Sussex,
Falmer, Brighton)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues, and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

26.1 (Submitted by the Iranian International Mathematical Olympiad Committee)
A collection of 1993^n positive rational numbers has the property that, if any one of them is removed, the remaining ones can be divided into 1992 equal collections whose products are equal. Prove that all 1993^n numbers are the same.

26.2 (Submitted by G. Lasters, Tienen, Belgium)
Find an equation for the bisector of angle BAC, where A, B and C are the points representing the complex numbers a, b and c in the Argand diagram.

26.3 (Submitted by Farshid Arjmandi, Tehran, Iran)
Let X be a finite non-empty set for which there is a mapping $f: X \to X$ such that $f^{1993} = \mathrm{Id}_X$. Prove that the number of elements $x \in X$ such that $f(x) \neq x$ is a multiple of 1993.

Solutions to Problems in Volume 25 Number 3

25.7 By an oversight, this problem also appeared as 25.5: a solution was given in Volume 25 Number 4. Further solutions were received from Polly Shaw (Dame Allan's Girls' School, Newcastle upon Tyne) and Thomas Womach (Winchester College).

25.8 Show that the simultaneous equations

$$x^2+y = a, \qquad x+y^2 = b,$$

where a and b are fixed, can have at most one solution in which x and y are integers when $a \neq b$. Show also that, if there is a repeated (double) solution with $x = \alpha$ and $y = \beta$, then $4\alpha\beta = 1$, and find a and b if the equations have a triple solution.

Solution by the proposer

Let there be an integer solution $x = m$, $y = n$. Then, if (x, y) is a different integer solution,

$$x^2+y = m^2+n, \qquad x+y^2 = m+n^2.$$

Eliminate y to give

$$(x^2-m^2)^2-2n(x^2-m^2)+(x-m) = 0.$$

Since $x \neq m$, we can divide by $x-m$ to give

$$(x+m)(x^2-m^2-2n) = -1.$$

We are dealing with integers, so either

$$x+m = -1 \quad \text{and} \quad x^2-m^2-2n = 1$$

or vice versa. The former case gives $m = n$, the latter $m+n = 1$. But

$$a-b = (m^2+n)-(m+n^2) = (m-n)(m+n-1),$$

so in either case $a = b$. Hence, when $a \neq b$ there can be at most one simultaneous solution in integers.

From $x^2+y = a$ and $x+y^2 = b$ we obtain

$$x+(a-x^2)^2 = b$$

or

$$x^4-2ax^2+x+(a^2-b) = 0. \qquad (1)$$

If this has a repeated root α, then we can differentiate to give

$$4\alpha^3-4a\alpha+1 = 0, \qquad (2)$$

so

$$4\alpha^3-4(\alpha^2+\beta)\alpha+1 = 0,$$

whence $4\alpha\beta = 1$.

For a triple root, we can differentiate (2) again to give

$$12\alpha^2-4a = 0,$$

whence $a = 3\alpha^2$. The roots of (1) add up to zero, so they must be α, α, α and -3α. Also, the sum of their products taken three at a time is -1, i.e.

$$\alpha^3-3\alpha^3-3\alpha^3-3\alpha^3 = -1,$$

i.e.

$$8\alpha^3 = 1,$$

so

$$\alpha = \tfrac{1}{2} \quad \text{or} \quad \tfrac{1}{2}\exp\tfrac{2}{3}\pi i \quad \text{or} \quad \tfrac{1}{2}\exp\tfrac{4}{3}\pi i.$$

Hence

$$a = 3\alpha^2 = \tfrac{3}{4} \quad \text{or} \quad \tfrac{3}{4}\exp\tfrac{4}{3}\pi i \quad \text{or} \quad \tfrac{3}{4}\exp\tfrac{2}{3}\pi i.$$

From (1),

$$a^2 - b = -3\alpha^4,$$

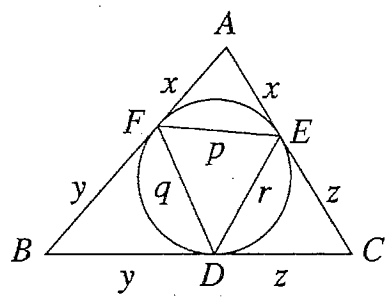
so $b = a^2 + 3\alpha^4$, which gives the corresponding values of b to be

$$\tfrac{3}{4} \quad \text{or} \quad \tfrac{3}{4}\exp\tfrac{2}{3}\pi i \quad \text{or} \quad \tfrac{3}{4}\exp\tfrac{4}{3}\pi i.$$

25.9 Show that

$$\frac{p^2}{a(b+c-a)} = \frac{q^2}{b(c+a-b)} = \frac{r^2}{c(a+b-c)}$$

(where $BC = a$, $CA = b$ and $AB = c$).



Solution by Sammy and Jimmy Yu (7th and 6th graders at Vermillion Middle School, South Dakota, USA)

Applying the cosine rule to triangles AEF and ABC, we have

$$p^2 = 2x^2(1-\cos A) = 2x^2\left(1 - \frac{b^2+c^2-a^2}{2bc}\right)$$
$$= \frac{x^2(c+a-b)(a+b-c)}{bc}.$$

Now $z+x = b$, $x+y = c$ and $y+z = a$, so $x = \tfrac{1}{2}(b+c-a)$. Hence

$$p^2 = \frac{(b+c-a)^2(c+a-b)(a+b-c)}{4bc},$$

so that

$$\frac{p^2}{a(b+c-a)} = \frac{(b+c-a)(c+a-b)(a+b-c)}{4abc}.$$

Similarly

$$\frac{q^2}{b(c+a-b)} = \frac{(b+c-a)(c+a-b)(a+b-c)}{4abc} = \frac{r^2}{c(a+b-c)}.$$

The result follows.

Also solved by Sinefakopoulos Achilleas (University of Athens).

Reviews

Visions of Symmetry: Notebooks, Periodic Drawings and Related Work of M. C. Escher. By DORIS SCHATTSCHNEIDER. W. H. Freeman and Company, New York, 1992. Paperback £19.95 (ISBN 0-7167-2352-2).

Escher's work is as famous as it is characteristic; most mathematics departments have at least one of his puzzling pictures on display. Dr Schattschneider examines Escher's 'regular division of the plane' patterns. These are repeated shapes (representing a whole range of fabulous beasts) which join together without leaving any gaps. For mathematicians it is interesting to note that the division is not limited to the Euclidean plane: there are examples of work in both hyperbolic and spherical geometries.

Escher's own notes are used for the first time to explain the history and development of this series of paintings; from a casual interest in the Moorish decorations he found in Spain in 1922 to a full-blown obsession with the subject in 1937–8 and the years that followed. His system of classification, based on his own theories of the underlying symmetries, is explored and sheds light on how Escher achieved the baffling effects of the final pictures.

Symmetry is a particularly accessible part of mathematics, but one to which little formal attention had been paid. Although he was primarily an artist, Escher's 'layman's theory' shows a keen mathematical mind at work, and this theory predates similar work at the forefront of crystallography. Indeed, the beauty and simplicity of his work has made it invaluable in conveying ideas of symmetry across a wide variety of disciplines.

Plentiful colour copies of both early sketches and finished paintings make the book a delight to flick through, while the detailed notes and text allow a more rigorous study to be made.

St Catherine's College, Cambridge — DAVID BRACKIN

Another Fine Math You've Got Me Into. By IAN STEWART. W. H. Freeman, New York, 1992. Pp. xi+269. Paperback £10.95 (ISBN 0-7167-2341-7)

Taken from a selection of columns in *Pour la Science*, the French edition of *Scientific American*, this book is the sequel to the excellent *Game, Set and Math* (reviewed in Volume 23 Number 3 of *Mathematical Spectrum*. Each chapter is self-contained and introduces a subject area, usually from discrete mathematics, through the use of a story. Invariably the story involves a problem, which is where the mathematics comes in.

Starting from the basics, each chapter develops into a fully fledged investigation, sometimes ending in unsolved problems. Even readers with the most basic mathematical knowledge can feel that they are at the frontiers of current mathematical research.

My favourite chapter is the final one, which explores the shape of the optimal sofa—the largest shape that can be carried through a corridor of fixed width. Anyone who has ever moved any furniture will appreciate that this is not a trivial problem! Another chapter deals with how to represent all the intersections of n

sets on a Venn diagram in an aesthetically pleasing way. Venn himself got stuck on five sets. The answer is deceptively simple—once you see it, that is.

Other topics tackled include applying group theory to bell-ringing, the thermodynamics of a curve, the Knight's tour of unusual surfaces (including a Klein bottle), the problem of the lion, the llama and the lettuce, or how to apply the tower of Hanoi problem to finding the average distance between two points on a Sierpinski gasket.

Although this book is very easy reading, it gives a good grounding, mostly in discrete mathematics, and I thoroughly enjoyed reading it.

Student at the London School of Economics KHALID KHAN

Slices of Mathematical Pie.. Compiled by BERNARD ATKIN. The Mathematical Association, Leicester, 1991. Paperback £3.00.

It is impossible to say how many schoolchildren over the years have been given an interest in mathematics by *Mathematical Pie*. Its modest format hides a potent weapon in rescuing mathematics from the classroom and putting it in the realm of the imagination. This modest volume contains selections from the first 100 issues. A wonderful present to give to a youngster—or to treat yourself to.

University of Sheffield D. W. SHARPE

The Road to Infinity. By S. A. KNIGHT. Book Guild, Lewes, 1992. Pp. 202. Paperback £12.50 (ISBN 0-86332-683-8).

This is a highly enjoyable look at a topic that has excited lively debate among mathematicians from Greek times to the present day, the role of the infinitely large and the infinitely small in mathematics. The reader is confronted with the various paradoxes that are such a fascinating aspect of any study of the infinite, all of them being explained in simple yet elegant language. Using personal reminiscences, humour and cartoons, the author presents the views of Pythagoras, Zeno, Galileo, Newton, Leibniz, Dedekind and Cantor in such a way that the complexities of their contributions are readily understood.

The book is written for the intelligent lay person with a high-school knowledge of mathematics, who is eager to learn of the mysteries that lurk just beneath the surface of our number system, whose existence is so often taken for granted. The author calls his work an entertainment, and this it certainly is. He has an engaging style of exposition and an impish sense of humour, and delights in personalizing his account whenever possible—this all makes for an exhilarating read. Other attractive features are an excellent cover design, clear illustrations, apt quotations at the head of each chapter, the placing of material in historical perspective and amusing one-liners (*infinity is where things happen that don't t!*). There are several genuinely funny cartoons, my favourite of which shows William Tell consoling his son (prior to taking aim at the apple perched on the young boy's head) with the advice, 'Don't worry, son. Shut your eyes and think of Zeno'.

Alas, my generally enthusiastic response to this highly worthwhile enterprise has to be tempered somewhat by the large number of errors spattering the later pages of the book: spelling mistakes, misprints, a lack of distinction between equality and approximation, factual errors, mathematical inaccuracies and conceptual howlers. This is a great pity, for almost all of these could easily have been corrected by any competent mathematician. Some of the errors do have their own humorous contributions to make, even so. While berating the other authors for using the limit $n \to \infty$, the present one, in deriving an expression for e, himself employs those 'well known' limits, so beloved of weak students, $n-1 \to n$, $n-2 \to n$, ..., as $n \to \infty$! The printer, clearly unaccustomed to working with mathematical manuscripts, in a praiseworthy enthusiasm to use the customary ... to convey the idea of 'and so on to infinity' continues them beyond the text and right into the margin! The discovery of continuous, nowhere differentiable functions is ascribed to Weirestruss (sic), a name perhaps more suited to the inventor of support functions! The author, however, does make the general comment that 'the road to infinity is paved with contradictions', maybe even more than *he* had bargained for!

University of Sheffield ROGER WEBSTER

Other books received

The Moscow Puzzles; 359 Mathematical Recreations. By BORIS A. KORDEMSKY. Dover, New York, 1992. Pp. 256. Paperback £8.95 (ISBN 0-486-27078-5).

This is a reprint of a famous book of mathematical puzzles. If you feel that your enthusiasm for mathematics may be draining away, try this book.

Optimization Algorithms for Networks and Graphs. By JAMES R. EVANS AND EDWARD MINIEKA. Marcel Dekker, New York, 1992. Pp. 480. Hardback $59.75.

This is a second edition, revised and expanded, of a book which first appeared in the 1970s, and includes a computer package, NETSOLVE. It is suitable for final-year undergraduates and postgraduate students.

Geometry: Axiomatic Developments and Problem Solving. By EARL PERRY. Marcel Dekker, New York, 1992. Pp. 376. $45.00 (ISBN 0-8247-8727-7).

Mathematics and Logic. By MARK KAC AND STANISLAW M. ULAM. Dover, New York, 1992. Pp. ix + 170. Paperback £6.95 (ISBN 0-486-67085-6).

This wide-ranging survey is a re-issue of a book first published in 1968.

Abstract Algebra and Solutions by Radicals. By JOHN E. MAXFIELD AND MARGARET W. MAXFIELD. Dover, New York, 1992. Pp. xi + 209. Paperback £7.95 (ISBN 0-486-67121-6).

This attractively produced, student-friendly book is a reprint with corrections of one first published in 1971. It is suitable for undergraduates, starting with the definition of a group and reaching the fundamental theorem of Galois theory.

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