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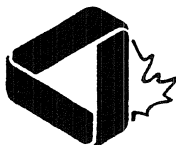
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Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

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RENSEIGNEMENTS GÉNÉRAUX

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire. Bien que principalement de nature éducative, elle sert aussi à ceux qui la lisent pour des raisons professionnelles, culturelles ou récréative.

Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

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THE OLYMPIAD CORNER

No. 143

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this number with a pre-Olympiad question set. These are the ten problems of Part A of the Interschool Mathematical Competition 1991 sponsored by the Singapore Mathematical Society. These were intended as short answer problems — no work required. For that reason we will not be publishing detailed solutions. Many thanks to Willie Yong, University of Singapore, for sending the materials to me.

SINGAPORE INTERSCHOOL MATHEMATICS COMPETITION 1991

Part A

1. A set of $n(n+1)/2$ distinct numbers is arranged at random in n rows so that for every $1 \leq i \leq n$, there are exactly i numbers in the i th row. Let M_i be the largest number in the i th row. Find the probability that $M_1 < M_2 < \cdots < M_n$.

2. Find three positive integers x , y and z such that $300 > x > y > z$ and $x^3 + y^4 = z^5$.

3. Find the coordinates of the centre of the circle that touches the circle $x^2 + y^2 = 1$ and the line $x = h$ at (h, k) , where $h, k > 1$.

4. Let a , b , c be real numbers satisfying $a + b + 1$, $a^2 + b^2 + c^2 = 1/2$. Find the maximum value of c .

5. Find the maximum and minimum number of “Friday the 13th”s that can occur in a year.

6. In a regular $(2n+1)$ -gon, three distinct vertices are chosen at random to form a triangle. What is the probability that the centre of the polygon is inside the triangle formed?

7. Find the total number of right-angled triangles whose sides have integral lengths less than 30.

8. Let $n = 2^s + 2^t$ where s and t are distinct positive integers. Find all integers i such that $0 \leq i \leq n$ and $\binom{n}{i}$ is odd.

9. Given that $ABCDEFGH$ is a regular heptagon, express AB in terms of x and y where $AC = x$ and $AD = y$.

10. Quadrilateral $ABCD$ is inscribed in a circle with centre O . Let the diagonals AC and BD meet at F , and let M be the midpoint of DC . Suppose that $\angle AFB$ is a

right-angle and that OM is a perpendicular bisector of CD . Find CD if $OM = a$ and the radius of the circle is 1.

* * *

As an Olympiad problem set we give the 14th Austrian–Polish Mathematics Competition written June 26–28, 1991 at Bad Ischl, Austria. Many thanks to Walther Janous, Ursulinengymnasium, Innsbruck, Austria, and to Marcin E. Kuczma, Warszawa, Poland, for both sending a copy of the problems our way.

14th AUSTRIAN–POLISH MATHEMATICS COMPETITION

Bad Ischl (Austria): June 26–28, 1991

Individual Contest — Day 1

1. Show that there are infinitely many integers $m \geq 2$ such that the equality $\binom{m}{2} = 3\binom{n}{4}$ holds for some integer $n = n(m) \geq 4$. Give the general form of all such m .

2. Determine all triples of real numbers (x, y, z) satisfying the system of equations

$$(x^2 - 6x + 13)y = 20, \quad (y^2 - 6y + 13)z = 20, \quad (z^2 - 6z + 13)x = 20.$$

3. Given are points A_1, A_2 in the plane. Determine all possible positions of a point A_3 with the following property: there exist an integer $n \geq 3$ and n points P_1, P_2, \dots, P_n such that the segments $P_1P_2, P_2P_3, \dots, P_{n-1}P_n, P_nP_1$ have equal length and their midpoints are $A_1, A_2, A_3, A_1, A_2, A_3, \dots$, in this order.

Individual Contest — Day 2

4. Let $P(x)$ be a real polynomial with $P(x) \geq 0$ for $x \in [0, 1]$. Prove that there exist polynomials $P_i(x)$ ($i = 0, 1, 2$) with $P_i(x) \geq 0$ for all real x and such that $P(x) = P_0(x) + xP_1(x) + (1-x)P_2(x)$.

5. Show that the inequality

$$x^2 + y^2 + z^2 + xy + yz + zx \geq 2(\sqrt{x} + \sqrt{y} + \sqrt{z})$$

holds for all positive numbers x, y, z with $xyz = 1$.

6. Inside a convex quadrilateral $ABCD$ there is a point P such that the triangles PAB, PBC, PCD, PDA have equal areas. Prove that the area of $ABCD$ is bisected by one of the diagonals.

Team Contest

7. For a given integer $n \geq 1$ determine the maximum value of the function

$$f(x) = \frac{x + x^2 + \dots + x^{2n-1}}{(1 + x^n)^2}$$

over $x \in (0, \infty)$ and find all $x > 0$ for which the maximum is attained.

8. Consider the system of simultaneous congruences

$$xy \equiv -1 \pmod{z}, \quad yz \equiv 1 \pmod{x}, \quad zx \equiv 1 \pmod{y}.$$

Find the number of triples (x, y, z) of distinct positive integers satisfying the system and such that one of x, y, z equals 19.

9. Let $A = \{1, 2, \dots, n\}$ with n a positive even integer. Suppose $g : A \rightarrow A$ is a function with $g(k) \neq k$, $g(g(k)) = k$ for $k \in A$. How many functions $f : A \rightarrow A$ are there such that $f(k) \neq g(k)$ and $f(f(f(k))) = g(k)$ for $k \in A$?

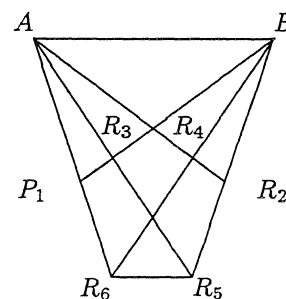
* * *

We now turn to readers' solutions to problems of the *13th Austrian-Polish Mathematics Competition 1990* [1992: 3–4].

1. Let A, B, P_1, \dots, P_6 be eight distinct points in the plane, the P_i 's all lying on the same side of line AB . Suppose the six triangles ABP_i ($1 \leq i \leq 6$) are similar. Show that P_1, \dots, P_6 lie on a circle.

Solutions by Pavlos Maragoudakis, student, University of Athens, Greece; and by D.J. Smeenk, Zaltbommel, The Netherlands.

We first construct points R_2, R_3, R_4, R_5, R_6 such that P_1, R_2, \dots, R_6 are all lying on the same side of line AB . We construct R_2 such that $\angle ABR_2 = \angle BAP_1$ and $\angle BAR_2 = \angle ABP_1$. It is clear that $\triangle ABP_1 \cong \triangle ABR_2$. Let $\angle AP_1B = \varphi$. We construct R_3, R_4, R_5, R_6 such that R_3 lies on BP_1 and $\angle R_3AB = \varphi$, R_4 lies on AR_2 and $\angle R_4BA = \varphi$. Then R_5 is the point of intersection of AR_3 and BR_2 . Finally R_6 is the point of intersection of AP_1 and BR_4 .



It is clear from the construction that triangles ABP_1, ABR_i ($2 \leq i \leq 6$) are similar.

Let P be a point of the plane of A, B, P_1 such that P, P_1 are lying on the same side of line AB and $\triangle ABP \cong \triangle ABP_1$. If $\angle APB = \angle AP_1B$ then $\triangle ABP \cong \triangle ABP_1$ and $P \equiv P_1$ or $P \equiv R_2$. If $\angle APB = \angle P_1AB$ then $\triangle ABP \cong \triangle ABP_3$ and $P \equiv R_3$ or $P \equiv R_4$. If $\angle APB = \angle P_1BA$ then $\triangle ABP \cong \triangle ABR_5$ and $P \equiv R_5$ or $P \equiv R_6$. So $\{P_1, \dots, P_6\} = \{P_1, R_2, \dots, R_6\}$.

Now $P_1R_2 \parallel R_3R_4 \parallel R_5R_6 \parallel AB$ and $P_1R_3R_4R_6$ is an isosceles trapezoid. Thus the points P_1, R_2, R_3, R_4 are concyclic.

Finally it is enough to prove that P_1, R_6, R_4, R_3 and R_2, R_5, R_3, R_4 are concyclic. This is true since $\angle R_3P_1R_5 + \angle R_3R_4R_6 = (\angle P_1AB + \angle ABP_1) + \varphi = 180^\circ$ and similarly $\angle R_4R_2R_5 + \angle R_4R_3R_5 = 180^\circ$.

2. Determine all triples (x, y, z) of positive integers such that

$$x^{(y^z)} \cdot y^{(z^x)} \cdot z^{(x^y)} = 1990^{1990}xyz.$$

Solution by Pavlos Maragoudakis, student, University of Athens, Greece.

We look for triples (x, y, z) of positive integers such that

$$x^{y^z-1} \cdot y^{z^x-1} \cdot z^{x^y-1} = 1990^{1990}.$$

Let $k = \min\{x, y, z\}$. Then $k^{3k^k-3} \leq 1990^{1990}$. If $k \geq 10$ then $k^{3k^k-3} \geq 10^{3 \cdot 10^{10}-1} > 10^{10^{10}} > 1990^{1990}$. So $k \leq 9$. But k divides 1990^{1990} , and $1990 = 199 \cdot 5 \cdot 2$. Thus $k = 1, 2, 4, 5$ or 8 . It is clear that if (x, y, z) is a solution then (y, z, x) and (z, x, y) are also solutions. There is no loss of generality in supposing that $x = k$.

Case 1. $x = 2^a$ $a = 1, 2, 3$. Then

$$2^{ay^2-a} \cdot y^{z^{2^a}-1} \cdot z^{2^{ay}-1} = 1990^{1990} = 2^{1990} 5^{1990} \cdot 199^{1990}. \quad (*)$$

We conclude that $199 \mid z$ or $199 \mid y$. Thus $z \geq 199$ or $y \geq 199$. If $z \geq 199$ then

$$y^{z^2} \geq 2^{199^2} \geq 2^{39601} \geq 2^{39600} = (2^{11})^{3600} > 1990^{1990}.$$

If $y \geq 199$ then $ay^2 - a \geq y^2 - 1 \geq 199^2 - 1 > 1990$, a contradiction since $(*)$ gives $ay^2 - a \leq 1990$.

Case 2. $x = 5$. Then

$$5^{y^2-1} y^{z^5-1} z^{5^y-1} = 2^{1990} 5^{1990} 199^{1990}. \quad (**)$$

Similarly to Case 1, we obtain a contradiction since $(**)$ implies that $y^2 - 1 \leq 1990$.

Case 3. $x = 1$. Then $y^{z-1} = 1990^{1990}$. So $z = 1991$, $y = 1990$.

Finally all solutions are $(1, 1990, 1991)$, $(1990, 1991, 1)$ and $(1991, 1, 1990)$.

3. Show that there are exactly two triples (x, y, z) of real numbers satisfying the system of equations

$$\begin{aligned} x + y^2 + z^4 &= 0 \\ y + z^2 + x^4 &= 0 \\ z + x^2 + y^4 &= 0. \end{aligned}$$

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by Pavlos Maragoudakis, student, University of Athens, Greece; and by Beatriz Margolis, Paris, France.

Since x, y and z are real it follows that they must be non-positive. Because of the symmetry in the equations we may assume $0 \geq x \geq y \geq z$ or $0 \geq x \geq z \geq y$.

If $0 \geq x \geq y \geq z$ then $0 \leq x^2 \leq y^2 \leq z^2$ and $0 \leq x^4 \leq y^4 \leq z^4$. Subtracting the third equation from the first gives $0 = (x - z) + (y^2 - x^2) + (z^4 - y^4)$ where each bracketed quantity is nonnegative. It follows that $x = z = y$.

If $0 \geq x \geq z \geq y$ we obtain $0 \leq x^2 \leq z^2 \leq y^2$ and $0 \leq x^4 \leq z^4 \leq y^4$. Subtracting the second equation from the first gives $0 = (x - y) + (y^2 - z^2) + (z^4 - x^4)$ and again $x = y = z$ follows.

Now we obtain the equation $0 = x + x^2 + x^4 = x(1 + x + x^3)$. The second factor has one real root λ between -1 and 0 . The two solutions are therefore $(0, 0, 0)$ and $(\lambda, \lambda, \lambda)$.

Thus $M_n = (n^2 - 1)/3$.

[*Editor's note.* Both Maragoudakis and Yeung point out that this problem is also # 2 [1992: 34] of the Training Test for the 1991 U.S.S.R. I.M.O. Team.]

6. Let $P(x)$ be a polynomial with integer coefficients. Suppose that the integers x_1, x_2, \dots, x_n ($n \geq 3$) satisfy the conditions

$$P(x_i) = x_{i+1} \quad \text{for } 1 \leq i \leq n-1, \quad P(x_n) = x_1.$$

Show that $x_1 = x_3$.

Comment by Murray S. Klamkin, University of Alberta.

A solution to this problem is given in M.S. Klamkin, *U.S.A. Mathematical Olympiads 1972-1986*, M.A.A., Washington, D.C., 1988, Problem A/7, p. 20.

* * *

We now move to solutions from the readers to problems from the February 1992 number of the Corner. First are solutions to *Selection Questions For the 1990 Irish I.M.O. Team* [1992: 33].

1. Find all pairs of integers (x, y) such that $y^3 - x^3 = 91$.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by John Katsikis and Pavlos Maragoudakis, students, University of Athens, Greece; by Joseph Ling, The University of Calgary; by Beatriz Margolis, Paris, France; by Bob Prielipp, University of Wisconsin-Oshkosh; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Prielipp's solution.

It is easily seen that the equation $y^3 - x^3 = 91$ has no solution (x, y) where $x = 0$ or $y = 0$. Hence we shall assume that $x \neq 0$ and $y \neq 0$. The equation is equivalent to $(y - x)(y^2 + yx + x^2) = 91$. From $y^2 + yx + x^2 = (y + (1/2)x)^2 + (3/4)x^2 > 0$ we see that $y - x = 0$. Thus we obtain four cases:

$$y - x = 1, \quad y^2 + xy + x^2 = 91, \tag{1}$$

$$y - x = 7, \quad y^2 + yx + x^2 = 13, \tag{2}$$

$$y - x = 13, \quad y^2 + yx + x^2 = 7, \tag{3}$$

or

$$y - x = 91, \quad y^2 + xy + x^2 = 1. \tag{4}$$

In (1) replacing y by $x + 1$ in $y^2 + xy + x^2 = 91$ yields $(x, y) = (5, 6)$ or $(x, y) = (-6, -5)$. In (2) replacing y by $x + 7$ gives $(x, y) = (-3, 4)$ or $(x, y) = (-4, 3)$. In (3) and (4) proceeding as for (1), (2) yields in each case a quadratic equation with a negative discriminant.

It follows that

$$\{(x, y) : x \text{ and } y \text{ are integers and } y^3 - x^3 = 91\} = \{(5, 6), (-6, -5), (-3, 4), (-4, 3)\}.$$

2. Observe that, when the first digit of $x = 714285$ is moved to the end, we get $y = 142857$ and $y = x/5$. Find the smallest positive integer u such that if v is obtained from u by moving the first digit of u to the end, then $v = u/2$.

Solutions by John Katsikis and Pavlos Maragoudakis, students, The University of Athens, Greece; by Bob Prielipp, University of Wisconsin-Oshkosh; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Katsikis and Maragoudakis.

Let n be the number of digits of u and $u = x_n x_{n-1} \dots x_1$, where $x_n \neq 0$, and each x_i is a decimal digit. Then $v = x_{n-1} x_{n-2} \dots x_2 x_1 x_n$ and $u = 2v$ is equivalent to

$$10^{n-1}x_n + 10^{n-2}x_{n-1} + \dots + 10x_2 + x_1 = 2 \cdot 10^{n-1}x_{n-1} + \dots + 2 \cdot 10x_1 + 2x_n.$$

Thus

$$(10^{n-1} - 2)x_n = (2 \cdot 10^{n-1} - 10^{n-2})x_{n-1} + \dots + (2 \cdot 10 - 1)x_1$$

and

$$(10^{n-1} - 2)x_n = 19(10^{n-2}x_{n-1} + \dots + 10x_2 + x_1) \quad (1)$$

But $1 \leq x_n \leq 9$, so 19 divides $10^{n-1} - 2$. By calculating $10^k \equiv \text{mod } 19$ for $k = 1, 2, \dots$, we find that $10^{17} \equiv 2 \text{ mod } 19$, and no smaller power of 10 will do. Thus $n = 18$, and (1) becomes

$$(10^{17} - 2)x_{18} = 19(10^{16}x_{17} + \dots + 10x_2 + x_1).$$

From this

$$\frac{10^{17} - 2}{19}x_{18} = 10^{16}x_{17} + \dots + 10x_2 + x_1. \quad (2)$$

To make u least possible we take $x_{18} = 1$ and since $(10^{17} - 2)/19 = 5263157894736842$ we obtain $x_{17} = 0$, $x_{16} = 5, \dots$, $x_1 = 2$. This makes $u = 105263157894736842$.

[*Editor's note.* Wang points out that since $a_0 \equiv 2a_n \text{ mod } 10$ is even, there are five numbers u' with 18 digits satisfying $u' = 2v'$. Each can be obtained from the solution above by a cyclic permutation by 1, 13, 2 and 16 positions, respectively. He also points out that $x = 142857$ (the six digits after the decimal point in the expansion of $22/7$ as an approximation for π) is known to have the property that $2x$, $3x$, $4x$, $5x$ and $6x$ are all cyclic permutations of x .]

3. Let $2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, \dots$ be the sequence of non-squares (i.e., the sequence obtained from the natural numbers by deleting $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, $16 = 4^2$, etc.). Prove that the n th term of the sequence is

$$\left[n + \frac{1}{2} + \sqrt{n} \right].$$

(Note that, for x a real number, $[x]$ denotes the greatest integer z with $z \leq x$. Thus, for example, $[16/7] = 2$.)

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by Pavlos Maragoudakis, student, University of Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution, remark and proposed generalization.

Let $a_n = n + k$ denote the n th term of the sequence, where $k \geq 1$. Then $k = [\sqrt{n+k}]$ since the number of squares between 1 and $n+k$ is clearly $[\sqrt{n+k}]$ and $\sqrt{n+k}$ is not an integer, we obtain $\sqrt{n+k}-1 < k < \sqrt{n+k}$. From $k < \sqrt{n+k}$ we get $k^2 < n+k$, which implies $k^2 \leq n+k-1$, and $(k-1/2)^2 \leq n-3/4 < n$. Hence

$$k < \sqrt{n} + \frac{1}{2}. \quad (1)$$

On the other hand $\sqrt{n+k}-1 < k$ so $n+k < (k+1)^2$, which implies $n+k+1 \leq (k+1)^2$ or $k^2 + k \geq n$, $(k+1/2)^2 \geq n+1/4 > n$. Hence

$$\sqrt{n} - \frac{1}{2} < k. \quad (2)$$

From (1) and (2) we obtain $k = [1/2 + \sqrt{n}]$. Therefore $a_n = n + [1/2 + \sqrt{n}] = [n + 1/2 + \sqrt{n}]$, as claimed.

Remark. This question can be found as question 316, p. 14 of Book 4 of *1001 Problems in High School Mathematics*, created, collected and edited by E. Barbeau, M. Klamkin and W. Moser (published by the Canadian Mathematical Society). There the question is to show that $a_n = n + \{\sqrt{n}\}$ where $\{x\}$ denotes the integer closest to x .

Generalization. Let a_n denote the n th term of the sequence 2, 3, 5, 7, 10, 11, 12, ... obtained by deleting all the k th powers, $k = 2, 3, 4, \dots$. Find a formula for a_n .

This problem may well be hard.

4. Let $n \geq 3$ be a natural number. Prove that

$$\frac{1}{3^3} + \frac{1}{4^3} + \dots + \frac{1}{n^3} < \frac{1}{12}.$$

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We prove the stronger inequality $S_\infty < 1/12$ where $S_\infty = \sum_{k=3}^{\infty} 1/k^3$. Since $k^3 > k(k-1)(k+1)$ we have,

$$S_\infty < \sum_{k=3}^{\infty} \frac{1}{k(k-1)(k+1)}.$$

Now, using a partial fraction decomposition

$$\begin{aligned} \sum_{k=3}^N \frac{1}{k(k-1)(k+1)} &= \frac{1}{2} \sum_{k=3}^N \frac{1}{k-1} - \sum_{k=3}^N \frac{1}{k} + \frac{1}{2} \sum_{k=3}^N \frac{1}{k+1} \\ &= \frac{1}{2} \sum_{k=1}^{N-2} \frac{1}{k+1} - \sum_{k=2}^{N-1} \frac{1}{k+1} + \frac{1}{2} \sum_{k=3}^N \frac{1}{k+1} \\ &= \frac{1}{2} \left(\frac{1}{2} + \frac{1}{3} \right) - \frac{1}{3} - \frac{1}{N} + \frac{1}{2} \left(\frac{1}{N} + \frac{1}{N+1} \right). \end{aligned}$$

Thus

$$\sum_{k=3}^{\infty} \frac{1}{k(k-1)(k+1)} = \frac{1}{4} + \frac{1}{6} - \frac{1}{3} = \frac{1}{12},$$

as desired.

5. Let t be a real number and let

$$a_n = 2 \cos \left(\frac{t}{2^n} \right) - 1$$

($n = 1, 2, 3, \dots$). Let b_n be the product $a_1 \cdots a_n$. Find a formula for b_n which does not involve a product of n terms and deduce that

$$\lim_{n \rightarrow \infty} b_n = \frac{2 \cos t + 1}{3}.$$

Solutions by Wai Yin Chung (1992 Hong Kong I.M.O. team member), King's College, Hong Kong; and by Joseph Ling, The University of Calgary. We give Chung's solution.

Using the identity $\cos 2\theta = 2 \cos^2 \theta - 1$ we get

$$2 \cos \theta - 1 = \frac{2 \cos 2\theta + 1}{2 \cos \theta + 1}.$$

Then

$$b_n = \left(\frac{2 \cos t + 1}{2 \cos(t/2) + 1} \right) \left(\frac{2 \cos(t/2) + 1}{2 \cos(t/4) + 1} \right) \cdots \left(\frac{2 \cos(t/2^{n-1}) + 1}{2 \cos(t/2^n) + 1} \right) = \frac{2 \cos t + 1}{2 \cos(t/2^n) + 1}.$$

Now

$$\lim_{n \rightarrow \infty} b_n = \frac{2 \cos t + 1}{2 \cos 0 + 1} = \frac{2 \cos t + 1}{3}.$$

* * *

The next solutions are to problems from the *1988 Chinese Olympiad Training Camp, Test 1* [1992: 34].

1. Prove that

$$\frac{xyz(x + y + z + \sqrt{x^2 + y^2 + z^2})}{(x^2 + y^2 + z^2)(xy + yz + zx)} \leq \frac{3 + \sqrt{3}}{9}$$

for all positive real numbers x , y and z with equality holding if and only if $x = y = z$.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; C.J. Bradley, Clifton College, Bristol, United Kingdom; Wai Yin Chung and Leonhard Yeung (1992 Hong Kong I.M.O. team members); Murray S. Klamkin, University of Alberta; Joseph Ling, The University of Calgary; Pavlos Maragoudakis, student, University of Athens, Greece; and by Panos E. Tsaoussoglou, Athens, Greece. We give Chung and Yeung's solution.

By the Cauchy-Schwarz inequality

$$x + y + z \leq \sqrt{3} \sqrt{x^2 + y^2 + z^2}.$$

By the Arithmetic Mean–Geometric Mean inequality

$$x^2 + y^2 + z^2 \geq 3(xyz)^{2/3} \quad \text{and} \quad xy + yz + xz \geq 3(xyz)^{2/3}.$$

So

$$\begin{aligned} \frac{xyz(x + y + z + \sqrt{x^2 + y^2 + z^2})}{(x^2 + y^2 + z^2)(xy + yz + xz)} &\leq \frac{xyz(1 + \sqrt{3})\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)3(xyz)^{2/3}} \\ &= \frac{1 + \sqrt{3}}{3} \frac{(xyz)^{1/3}}{\sqrt{x^2 + y^2 + z^2}} \\ &\leq \frac{1 + \sqrt{3}}{3} \cdot \frac{1}{\sqrt{3}} = \frac{3 + \sqrt{3}}{9}, \end{aligned}$$

with equality if and only if $x = y = z$.

Generalization by Murray S. Klamkin, University of Alberta.

Let

$$P(x, y, z) = \frac{(xyz)^r(x + y + z + \sqrt{x^2 + y^2 + z^2})^s}{(x^2 + y^2 + z^2)^m(xy + yz + zx)^n},$$

where $r, s, m, n \geq 0$, $3r + s = 2(m + n)$ and $2m \geq s$. By the power mean inequality

$$(x + y + z + \sqrt{x^2 + y^2 + z^2})^s \leq (\sqrt{3} + 1)^s (x^2 + y^2 + z^2)^{s/2}.$$

Using the A.M.–G.M. mean inequality as above, $P(x, y, z) \leq (3 + \sqrt{3})^s / (3^{m+n})$, and with equality if and only if $x = y = z$. The given inequality corresponds to the special case $m = n = s = 1$. In a similar fashion one can construct any number of rational symmetric functions of degree zero in x, y, z (or in n variables) such that the maximum value is taken on if and only if all the variables are equal.

2. Determine the smallest value of the natural number $n > 3$ with the property that whenever the set $S_n = \{3, 4, \dots, n\}$ is partitioned into the union of two subsets, at least one of the subsets contains three numbers a, b and c (not necessarily distinct) such that $ab = c$. [Compare with Problem 11 by Morocco, [1988: 226] — E.T.H.W.].

Solution by Pei Fung Lam (1992 Hong Kong I.M.O. team member), Queen's College, Hong Kong.

We first show that $3^5 = 243$ has the property, then we will show it is the least solution.

Suppose S_{243} is partitioned into two subsets X_1, X_2 . Without loss of generality, let 3 be in X_1 . If $3^2 = 9$ is in X_1 , then we are done. Otherwise, 9 is in X_2 . If $9^2 = 81$ is in X_2 then we are done. Otherwise, 81 is in X_1 . If $81/3 = 27$ is in X_1 , then we are done. Otherwise 27 is in X_2 . Finally, either $3 \times 81 = 243$ is in X_1 or $9 \times 27 = 243$ is in X_2 . In either case we are done.

To show 243 is the smallest, we will show that S_{242} can be partitioned into two subsets, each of which does not contain products of its elements. Define C to be “prime” in S_{242} if C is not the product of elements of S_{242} . The “primes” in S_{242} consist of 4, 8, p , $2p$

where $p < 242$ is a usual prime number. Since the smallest “prime” in S_{242} is 3, no number in S_{242} is the product of more than four “primes”. Put all the “primes” and numbers that can be written as products of four “primes” in one subset X_1 , and let $X_2 = S_{242} \setminus X_1$.

No products in X_2 are in X_1 because numbers in X_2 have at least two “prime” factors, so their products can be written with at least four “prime factors”. Next looking at the products of 4, 8, p , $2p$ (p odd prime < 242), we see that a product of two “primes” cannot be factored into a product of four “primes”. So no products in X_1 are in X_2 .

3. A pharmacist has a number of ingredients some of which are “strong”. Using these ingredients he is to make 68 different medicines such that each medicine contains 5 different ingredients, at least one of which is “strong” and furthermore, for any 3 ingredients chosen, there is exactly one medicine containing them. Prove or disprove that one of the 68 medicines must contain at least 4 “strong” ingredients.

Solutions by C.J. Bradley, Clifton College, Bristol, United Kingdom; and by Pei Fung Lam (1992 Hong Kong I.M.O. team member), Queen’s College, Hong Kong. We give Lam’s solution.

Let n be the number of ingredients. If the medicines corresponding to the $\binom{n}{3}$ triples of ingredients are listed, then each of the 68 medicines is repeated $\binom{5}{3} = 10$ times. So we must have $\binom{n}{3} = 680$, which implies $n = 17$.

Suppose an ingredient X is used in k_X medicines. In each of these k_X medicines the four other ingredients yield $\binom{4}{2} = 6$ pairs. If $\binom{4}{2}k_X > \binom{16}{2}$ then some pair will be repeated in the k_X medicines. Then this pair and X will be used in more than one medicine. So $\binom{4}{2}k_X \leq \binom{16}{2}$, implying $k_X \leq 20$. Now $68 \times 5 = \sum_X k_X \leq 17 \times 20$, so $k_X = 20$ for each X .

Suppose a pair of ingredients X, Y are used in k_{XY} medicines. The $3k_{XY}$ other ingredients in these medicines must all be distinct (otherwise a triple will be repeated). So $3k_{XY} \leq 15$ implying $k_{XY} \leq 5$. Now $68\binom{5}{2} = \sum_{X,Y} k_{XY} \leq \binom{17}{2}5$ and there is equality, so $k_{XY} = 5$ for all X, Y . By hypothesis, any triple X, Y, Z of ingredients are used in exactly $k_{XYZ} = 1$ medicine.

Suppose there are m “strong” ingredients S_1, S_2, \dots, S_m and no medicine contains at least 4 “strong” ingredients. By the inclusion-exclusion principle, the number of medicines containing at least one “strong” ingredient is

$$\begin{aligned} 68 &= \sum k_{S_a} - \sum k_{S_a S_b} + \sum k_{S_a S_b S_c} - \dots \\ &= 20m - 5\binom{m}{2} + \binom{m}{3} - 0 + \dots \\ &= \frac{1}{6}m[(m-9)^2 + 56]. \end{aligned}$$

Since the last factor on the right is at least 56, it is easy to check that there is no solution.

Therefore one of the 68 medicines must contain at least 4 “strong” ingredients.

* * *

To finish this month’s number of the Corner we give solutions for the two problems of the *Training Test for the 1991 U.S.S.R. I.M.O. Team* [1992: 34].

1. Let $a_1 = 1$ and

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{4a_n}, \quad n \geq 1.$$

Prove that $\sqrt{2/(2a_n^2 - 1)}$ is a positive integer for $n > 1$.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; by Murray S. Klamkin, University of Alberta; by Pei Fung Lam (1992 Hong Kong I.M.O. team member), Queen's College, Hong Kong; by Pavlos Maragoudakis, student, University of Athens, Greece; by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We give the solution of Bradley, which is similar to that of Wildhagen.

Let $b_n = \sqrt{2/(2a_n^2 - 1)}$, $n > 1$. Now $a_2 = 1/2 + 1/4 = 3/4$ so $b_2 = 4$. Write $a_n = u_n/(2v_n)$, where $u_n, v_n \in \mathbb{N}$ and $\gcd(u_n, 2v_n) = 1$. Then $u_2 = 3$, $v_2 = 2$, and we obtain $u_{n+1} = u_n^2 + 2v_n^2$, $v_{n+1} = 2u_nv_n$, from the formula for a_{n+1} .

Also

$$b_n = 2v_n \sqrt{\frac{1}{u_n^2 - 2v_n^2}}.$$

Now

$$u_{n+1}^2 - 2v_{n+1}^2 = u_n^4 + 4u_n^2v_n^2 + 4v_n^4 - 8u_n^2v_n^2 = (u_n^2 - 2v_n^2)^2.$$

Hence if $u_n^2 - 2v_n^2 = 1$, then $u_{n+1}^2 - 2v_{n+1}^2 = 1$. But $u_2^2 - 2v_2^2 = 1$ and so $u_n^2 - 2v_n^2 = 1$ for all $n \geq 2$, by induction. Also, by induction, u_n, v_n are positive integers. It follows that $b_n = 2v_n$ is a positive integer for all $n > 1$.

2. Let n be a positive integer and S_n be the set of all permutations of $\{1, 2, \dots, n\}$. For $\sigma \in S_n$ let $f(\sigma) = \sum_{i=1}^n |i - \sigma(i)|$. Prove that

$$\frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) = \frac{n^2 - 1}{3}.$$

Solutions were sent in by Christopher J. Bradley, Clifton College, Bristol, U.K.; and by Joseph Ling, The University of Calgary. A solution is given earlier this number with problem 5 of the 13th Austrian-Polish Mathematics Competition.

* * *

Many thanks to Kin Li, Hong Kong University of Science and Technology, for collecting and forwarding the solutions by the 1992 Hong Kong I.M.O. team members.

* * *

That completes the column for March. The Olympiad season is nigh. Send me your national/regional Olympiad contests, as well as interesting pre-Olympiad contest material. And don't forget your nice solutions!

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

*To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **October 1, 1993**, although solutions received after that date will also be considered until the time when a solution is published.*

1821. *Proposed by Gerd Baron, Technische Universität, Vienna, and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Determine all pairs (a, b) of nonnegative real numbers such that the functional equation

$$f(f(x)) + f(x) = ax + b$$

has a unique continuous solution $f : \mathbb{R} \rightarrow \mathbb{R}$.

1822. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

AB and AC are tangent to a circle Γ at B and C respectively. Let D be a point on AB produced beyond B , and let P be the second intersection of Γ with the circumcircle of $\triangle ACD$. Let Q be the foot of the perpendicular from B to CD . Prove that $\angle DPQ = 2\angle ADC$.

1823. *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

A rectangular box is to be decorated with a ribbon that goes across the faces and makes various angles with the edges. If possible, the points where the ribbon crosses the edges are chosen so that the length of the closed path is a local minimum. This will ensure that the ribbon can be tightened and tied without slipping off. Is there always a minimal path that crosses all six faces just once?

1824. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle and M a point in its plane. We consider the circles with diameters AM , BM , CM and the circle containing and internally tangent to these three circles. Show that the radius P of this large circle satisfies $P \geq 3r$, where r is the inradius of $\triangle ABC$.

1825. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Suppose that the real polynomial $x^4 + ax^3 + bx^2 + cx + d$ has four positive roots. Prove that $abc \geq a^2d + 5c^2$.

1826. *Proposed by P. Penning, Delft, The Netherlands.*

(a) In a box we put two marbles, one black and one white. We choose one marble at random. If it is white, we put it back in the box, add an extra white marble to the box, shake the box, and draw again, continuing to replace the marble along with an extra white marble every time a white marble is chosen, until the black marble is chosen and the game ends. What is the average number of marbles chosen?

(b) What is the average number of marbles chosen if we add an extra white marble only after every *second* white marble that is chosen?

1827. *Proposed by Šefket Arslanagić, Trebinje, Yugoslavia, and D.M. Milošević, Pranjani, Yugoslavia.*

Let a, b, c be the sides, A, B, C the angles (measured in radians), and s the semi-perimeter of a triangle.

(i) Prove that

$$\sum \frac{bc}{A(s-a)} \geq \frac{12s}{\pi},$$

where the sums here and below are cyclic.

(ii)* It follows easily from the proof of *Cruz* 1611 (see [1992: 62] and the correction in this issue) that also

$$\sum \frac{b+c}{A} \geq \frac{12s}{\pi}.$$

Do the two summations above compare in general?

1828. *Proposed by T. W. O. Richards, Cheddar Gorge, Great Britain.*

In the last century, the English mathematician Arthur Cayley introduced a permutation problem, loosely based on the card game Treize, which he called **Mousetrap**. Suppose that the numbers $1, 2, \dots, n$ are written on n cards, one on each card. After shuffling (permuting) the cards, start counting the deck from the top card down. If the number on the card does not equal the count, then put that card at the bottom of the deck and continue counting. If the two are equal then put the card aside and start counting again from 1.

Let's say the game is **won** if all the cards have been put aside. In this case, form a new deck with the cards in the order in which they were set aside and play a new game with this deck. For example, if we start with $n = 5$ cards in the order 25143, we win:

$$25143 \rightarrow 3251 \rightarrow 3251 \rightarrow 513 \rightarrow 513 \rightarrow 51 \rightarrow 51 \rightarrow 1$$

and the new deck is 42351, which wins again:

$$42351 \rightarrow 3514 \rightarrow 351 \rightarrow 13 \rightarrow 3$$

but now our deck, 24513, puts aside no cards at all. Is there an arrangement (using more cards, if necessary) which will give you three or more consecutive wins?

1829. *Proposed by C.J. Bradley, Clifton College, Bristol, U.K.*

The quadrilateral $ABCD$ is inscribed in a circle with centre O . AD and BC meet at P . L and M are the midpoints of AD and BC respectively. Q and R are the feet of perpendiculars from O and P respectively to LM . Prove that $LQ = RM$.

1830. *Proposed by P. Tsaoussoglou, Athens, Greece.*

If $a > b > c > 0$ and $a^{-1} + b^{-1} + c^{-1} = 1$, prove that

$$\frac{4}{c^2} + \frac{1}{(a-b)b} + \frac{1}{(b-c)c} \geq \frac{4}{3}.$$

* * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1611. [1991: 43; 1992: 62] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle with angles A, B, C (measured in radians), sides a, b, c , and semiperimeter s . Prove that

$$(i) \sum \frac{b+c-a}{A} \geq \frac{6s}{\pi}; \quad (ii) \sum \frac{b+c-a}{aA} \geq \frac{9}{\pi}.$$

II. *Corrected solution by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*

On [1992: 62], Ian Goldberg's solution is incorrect, because his inequality

$$\frac{x_1}{y_1} + \frac{x_2}{y_2} + \frac{x_3}{y_3} \geq 3 \left(\frac{x_1 + x_2 + x_3}{y_1 + y_2 + y_3} \right) \quad (1)$$

is *false*. For example, put $x_1 = 1, x_2 = 2, x_3 = 3, y_1 = 2, y_2 = 2, y_3 = 3$.

But inequality (1) is indeed valid for $x_1 \geq x_2 \geq x_3 > 0$ and $0 < y_1 \leq y_2 \leq y_3$. For by Chebyshev's inequality,

$$\frac{x_1}{y_1} + \frac{x_2}{y_2} + \frac{x_3}{y_3} \geq \frac{1}{3}(x_1 + x_2 + x_3) \left(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} \right) \geq \frac{1}{3}(x_1 + x_2 + x_3) \cdot \frac{9}{y_1 + y_2 + y_3},$$

and (1) follows.

Now for the given problem we may assume that $A \leq B \leq C$, and then $a \leq b \leq c$. For part (i), clearly

$$b+c-a \geq c+a-b \geq a+b-c,$$

so the result follows by (1), as on [1992: 62]. For part (ii), it is also easy to prove that

$$\frac{b+c-a}{a} \geq \frac{c+a-b}{b} \geq \frac{a+b-c}{c},$$

and the result again follows by (1) as in Goldberg's solution. Finally, for the solution of *Crux* 1637 given on [1992: 63], since $0 < A \leq B \leq C \leq \pi/2$ implies

$$\sin B + \sin C \geq \sin A + \sin C \geq \sin A + \sin B$$

we may again use (1).

* * * *

1630. [1991: 79; 1992: 114] *Proposed by Isao Ashiba, Tokyo, Japan.*

Maximize

$$a_1a_2 + a_3a_4 + \cdots + a_{2n-1}a_{2n}$$

over all permutations a_1, a_2, \dots, a_{2n} of the set $\{1, 2, \dots, 2n\}$.

III. *Comment by Ignasi Mundet i Riera, Universitat de Barcelona, Catalunya, Spain.*

On [1992: 115], Walther Janous and Murray Klamkin asked for a generalization of this problem. Here is the solution I found. For a finite set U of real numbers, let $p(U) = \prod_{x \in U} x$.

Let $I = \{a_1, a_2, \dots, a_{kn}\}$, $a_i \in \mathbb{R}$, $0 < a_1 < a_2 < \cdots < a_{kn}$. Then the partition of I into k subsets U_1, U_2, \dots, U_k of n elements each that maximizes $\sum p(U_i)$ is

$$U_1 = \{a_1, a_2, \dots, a_n\}, \quad U_2 = \{a_{n+1}, a_{n+2}, \dots, a_{2n}\}, \quad \dots, \quad U_k = \{a_{n(k-1)+1}, \dots, a_{kn}\}.$$

First we show that this is true if $k = 2$. Suppose $k = 2$ and we have a partition $I = U_1 \cup U_2$, $|U_1| = |U_2| = n$, such that it's not true that one of U_1, U_2 contains a_1, a_2, \dots, a_n and the other a_{n+1}, \dots, a_{2n} . Now consider

$$P_1 = \{a_i \in U_1 \mid i \leq n\}, \quad Q_1 = \{a_i \in U_1 \mid i \geq n+1\},$$

$$P_2 = \{a_i \in U_2 \mid i \geq n+1\}, \quad Q_2 = \{a_i \in U_2 \mid i \leq n\}.$$

Clearly $|P_1| = |P_2| > 0$, $|Q_1| = |Q_2| > 0$, and (since the a_i 's are nonnegative)

$$p(P_1) < p(P_2), \quad p(Q_2) < p(Q_1).$$

Now, putting $V_1 = P_2 \cup Q_1$, $V_2 = P_1 \cup Q_2$, we have $|V_1| = |V_2| = n$ and $V_1 \cup V_2 = I$. Thus V_1, V_2 is a partition, and

$$\begin{aligned} [p(V_1) + p(V_2)] - [p(U_1) + p(U_2)] &= p(P_2)p(Q_1) + p(P_1)p(Q_2) - p(P_1)p(Q_1) - p(P_2)p(Q_2) \\ &= [p(P_2) - p(P_1)][p(Q_1) - p(Q_2)] > 0. \end{aligned}$$

So the partition $I = U_1 \cup U_2$ is not maximal. In fact the maximal partition must be

$$V_1 = \{a_{n+1}, a_{n+2}, \dots, a_{2n}\}, \quad V_2 = \{a_1, a_2, \dots, a_n\}.$$

Now the theorem is an easy consequence. First of all, a partition U_1, U_2, \dots, U_k that maximizes $\sum p(U_i)$ has to have the n greatest numbers in the same set. For suppose it doesn't, that is, there are two sets U_i, U_j each containing at least one of the n greatest numbers; then from above we see that we can make a different partition of the elements of $U_i \cup U_j$ with a greater sum, so our partition wouldn't be maximal. Suppose then $U_k = \{a_{(k-1)n+1}, \dots, a_{kn}\}$. Now the remaining $(k-1)n$ elements have to be separated into $k-1$ sets so that the sum is maximal. Again, the n greatest elements have to be together. Continuing, we construct the maximal partition as claimed.

A problem that I think is quite a bit more difficult is that of finding the *minimal* sum, in the case $k > 2$. I think that the solution depends in that case on the size of the numbers and there is not a general solution.

Solutions to the Janous-Klamkin question have since also been received from MURRAY S. KLAMKIN, University of Alberta; and MARCIN E. KUCZMA, Warszawa, Poland.

Kuczma notes that the result doesn't hold if negative a_i 's are allowed. For example, with $k = 3$, $n = 2$, and the set $\{-3, -2, -1, 1, 2, 3\}$,

$$(-3)(-2)(-1) + 1 \cdot 2 \cdot 3 < (-3)(-2) \cdot 3 + (-1) \cdot 1 \cdot 2.$$

* * * * *

1730. [1992: 76] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Prove that

$$\sum bc(s-a)^2 \geq \frac{sabc}{2},$$

where a, b, c, s are the sides and semiperimeter of a triangle, and the sum is cyclic over the sides.

I. Solution by Pavlos Maragoudakis, student, University of Athens, Greece.

We have

$$\sum bc(s-a)^2 = s^2 \sum bc + abc \sum a - 2abc \cdot 3s = s^2 \sum bc - 4abcs,$$

so it is enough to prove

$$s^2 \sum bc - 4sabc \geq \frac{sabc}{2},$$

or

$$s \sum bc \geq \frac{9abc}{2},$$

or

$$(a+b+c)(ab+bc+ca) \geq 9abc,$$

which is true since

$$a+b+c \geq 3\sqrt[3]{abc} \quad \text{and} \quad ab+bc+ca \geq 3(\sqrt[3]{abc})^2$$

by the A.M.-G.M. inequality.

II. Generalization by Murray S. Klamkin, University of Alberta.

By letting $a = y + z$, $b = z + x$, $c = x + y$ where $x, y, z \geq 0$, we get $s = x + y + z$, and the inequality can be written

$$\sum (z+x)(x+y)x^2 \geq \frac{(x+y+z)(y+z)(z+x)(x+y)}{2}$$

(where the sum is cyclic over x, y, z), or

$$\sum \frac{x^2}{y+z} \geq \frac{x+y+z}{2},$$

or (putting $x+y+z=1$)

$$\sum \frac{x^2}{1-x} \geq \frac{1}{2}.$$

More generally, we will show that

$$\sum_{i=1}^m \frac{x_i^n}{1-x_i} \geq \frac{1}{(m-1)m^{n-2}} \quad (1)$$

where $x_1 + \dots + x_m = 1$, $x_i \geq 0$, and $n \geq 1$ or $n \leq 0$.

It suffices by Jensen's inequality to show that the function

$$F(x) = \frac{x^n}{1-x}$$

is convex for x in $[0, 1]$. After some calculations,

$$\begin{aligned} \frac{(1-x)^3 F''(x)}{x^{n-2}} &= n(n-1)(1-x)^2 + 2nx(1-x) + 2x^2 \\ &= (n-2)(n-1) \left(x - \frac{n}{n-1} \right)^2 + \frac{n}{n-1}, \end{aligned}$$

which is ≥ 0 for $0 \leq x \leq 1$ and for $n \geq 1$ (first line) or $n \leq 0$ (second line). Thus $F(x)$ is convex on $[0, 1]$ for $n \geq 1$ or $n \leq 0$.

Even though $F(x)$ is not convex for $0 < n < 1$, it appears that (1) is still valid for this range. This case is left as an open problem.

One can generalize still further by replacing (1) by

$$\sum \frac{w_i x_i^n}{1-x_i} \geq \frac{t^n}{1-t},$$

where the w_i are nonnegative weights with sum 1, $t = \sum w_i x_i$, and $n \geq 1$ or $n \leq 0$ (and still $\sum x_i = 1$). This follows as before by Jensen's inequality. As a special case of this, we have [putting $x_1 = 1 - a/s$, $w_1 = u/(u+v+w)$, etc.]

$$ubc(s-a)^n + vca(s-b)^n + wab(s-c)^n \geq \frac{abc[u(s-a) + v(s-b) + w(s-c)]^n}{(ua + vb + wc)(u+v+w)^{n-2}},$$

where the u, v, w are arbitrary nonnegative weights. From this we can obtain many cyclic triangle inequalities.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano,

Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; STEPHEN D. HNIDEI, student, University of Windsor; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; ANDY LIU, University of Alberta; JEAN-MARIE MONIER, Lyon, France; VEDULA N. MURTY, Dover, Pennsylvania; BOB PRIELIPP, University of Wisconsin-Oshkosh; TOSHIO SEIMIYA, Kawasaki, Japan; P. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Most solutions were similar to solution I. Janous further generalized (1) to

$$\sum_{i=1}^m x_i^n (1 - x_i)^l \geq \frac{(m-1)^l}{m^{l+n-1}}$$

where $x_1 + \cdots + x_m = 1$, $x_i \geq 0$, and either $l \geq 1$, $n \leq 0$ or $n \geq 1$, $l \leq 0$ or $l \leq 0$, $n \leq 0$. He also showed that the reverse inequality holds if $l \geq 1$, $n \geq 1$.

Can anyone decide (1) for the case $0 < n < 1$?

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1731*. [1992: 109] Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan.

Let P be a point within or on an isosceles right triangle and let c_1, c_2, c_3 be the lengths of the three concurrent cevians through P . Prove or disprove that c_1, c_2, c_3 form the sides of a nonobtuse triangle. [This problem was inspired by Murray Klamkin's problem 1631 [1992: 115].]

Solution by Marcin E. Kuczma, Warszawa, Poland.

True.

Let ABC be the triangle and AA' , BB' , CC' be the three cevians; $A = (0, 0)$, $B = (1, 0)$, $C = (0, 1)$, $B' = (0, y)$, $C' = (x, 0)$, $1 \geq x \geq y \geq 0$ (which may be assumed by symmetry); $x > 0$. Then $AA' \leq 1 \leq BB' \leq CC'$ and the condition under discussion is

$$f(P) := (AA')^2 + (BB')^2 - (CC')^2 \geq 0.$$

Now, $A' = (1 - t, t)$ where by Ceva's theorem $x \cdot \sqrt{2}t \cdot (1 - y) = (1 - x) \cdot \sqrt{2}(1 - t) \cdot y$, and so

$$t = \frac{(1 - x)y}{x + y - 2xy}$$

(the denominator is positive). Thus

$$\begin{aligned} f(P) &= (1 - t)^2 + t^2 + (1 + y^2) - (1 + x^2) \geq 1 - 2t + y^2 - x^2 \\ &= \frac{(1 + y^2 - x^2)(x + y - 2xy) - 2(1 - x)y}{x + y - 2xy} = \frac{x - y}{x + y - 2xy} \cdot g(x, y), \end{aligned}$$

where

$$g(x, y) = 1 + 2xy(x + y) - (x + y)^2.$$

Since $xy = (1-x)(1-y) + x + y - 1 \geq x + y - 1$, we get

$$g(x, y) \geq 1 + 2(x + y - 1)(x + y) - (x + y)^2 = (x + y - 1)^2 \geq 0,$$

and $f(P) \geq 0$ follows.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and TOSHIO SEIMIYA, Kawasaki, Japan.

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1732. [1992: 109] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $pal(n)$ be the n th palindromic number (i.e. $pal(1) = 1, \dots, pal(9) = 9, pal(10) = 11, pal(11) = 22$, etc.). Determine the set of all exponents α such that

$$\sum_{n=1}^{\infty} \frac{1}{[pal(n)]^{\alpha}}$$

converges.

Solution by Marcin E. Kuczma, Warszawa, Poland.

Let P_k be the set of all palindromic numbers in the interval $[100^{k-1}, 100^k)$. We can write

$$\sum_{n=1}^{\infty} \frac{1}{[pal(n)]^{\alpha}} = \sum_{k=1}^{\infty} Q_k \quad \text{where} \quad Q_k = \sum_{p \in P_k} \frac{1}{p^{\alpha}}.$$

Each of the numbers in P_k has either $2k-1$ or $2k$ digits. The first k are arbitrary, with the only constraint being that there is no leading zero, all the remaining digits are determined by reflection. Hence, P_k contains $2 \times 9 \times 10^{k-1}$ numbers. Therefore, we have the lower bound

$$Q_k > \frac{2 \times 9 \times 10^{k-1}}{(100^k)^{\alpha}} = \frac{9}{5} \times (10^{1-2\alpha})^k$$

and the upper bound

$$Q_k < \frac{2 \times 9 \times 10^{k-1}}{(100^{k-1})^{\alpha}} = 18 \times (10^{1-2\alpha})^{k-1}.$$

Thus if $\alpha \leq 1/2$ then for each k , $Q_k > 9/5$ and so $\sum_{n=1}^{\infty} [pal(n)]^{-\alpha}$ diverges. If $\alpha > 1/2$ then the sum is bounded above by

$$\sum_{n=1}^{\infty} \frac{1}{[pal(n)]^{\alpha}} = \sum_{k=1}^{\infty} Q_k \leq 18 \sum_{k=1}^{\infty} (10^{1-2\alpha})^{k-1} = \frac{18}{1 - 10^{1-2\alpha}}$$

and therefore converges. Therefore, the sum converges if and only if $\alpha > 1/2$.

Also solved by KEE-WAI LAU, Hong Kong; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

The proposer also asks whether this sum is rational, irrational, algebraic or transcendental when $\alpha = 1$.

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1733. [1992: 109] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with circumcenter O and such that $\angle A > 90^\circ$ and $AB < AC$. Let M and N be the midpoints of BC and AO , and let D be the intersection of MN with side AC . Suppose that $AD = (AB + AC)/2$. Find $\angle A$.

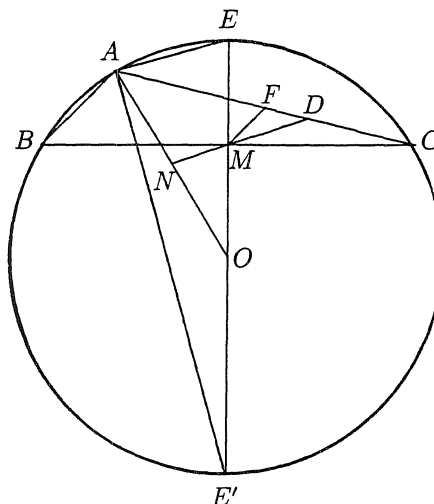
Solution by Jordi Dou, Barcelona, Spain.

Let F be the midpoint of AC and EE' be the diameter through M . If $AD = (AB + AC)/2$, from $AF = AC/2$ we have

$$FD = \frac{AB}{2} = FM.$$

Thus the external bisector of $\angle MFD$, and hence of $\angle BAC$, will be parallel to MD . Therefore $NM \parallel AE$, so M is the midpoint of OE and

$$\angle BAC = \angle BEC = 120^\circ.$$



Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; C. FESTAETS-HAMOIR, Brussels, Belgium; DAG JONSSON, Uppsala, Sweden; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

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1734. [1992: 109] *Proposed by Murray S. Klamkin, University of Alberta.*

Determine the minimum value of

$$\sqrt{(1 - ax)^2 + (ay)^2 + (az)^2} + \sqrt{(1 - by)^2 + (bz)^2 + (bx)^2} + \sqrt{(1 - cz)^2 + (cx)^2 + (cy)^2}$$

for all real values of a, b, c, x, y, z .

Solution by the proposer.

More generally, we determine the minimum value of the sum

$$S_n = \sum_{i=1}^n [(1 - a_i x_i)^2 + a_i^2 (T - x_i^2)]^m,$$

where $T = \sum_{i=1}^n x_i^2$, $0 < m \leq 1$, and for all real values of the a_i and x_i . [If all $x_i = 0$, then $S_n = n$ which will not be the minimum; thus we can assume $T \neq 0$.—Ed.]

Since

$$(1 - a_i x_i)^2 + a_i^2 (T - x_i^2) = T \left(a_i - \frac{x_i}{T} \right)^2 + \frac{T - x_i^2}{T},$$

we have

$$[(1 - a_i x_i)^2 + a_i^2 (T - x_i^2)]^m \geq \left(\frac{T - x_i^2}{T} \right)^m,$$

with equality if and only if $a_i = x_i/T$. Then (since $m \leq 1$)

$$\sum_{i=1}^n \left(\frac{T - x_i^2}{T} \right)^m \geq \sum_{i=1}^n \frac{T - x_i^2}{T} = n - 1.$$

Thus $\min S_n = n - 1$ and is taken on when all a_i and x_i equal 0 except for $a_1 x_1 = 1$, and permutations thereof.

For the given problem ($n = 3$, $m = 1/2$), the required minimum value is thus 2.

Also solved (usually with a lot more calculation!) by SEUNG-JIN BANG, University of California, Berkeley; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; and KEE-WAI LAU, Hong Kong. One incorrect solution was sent in.

Here are some details of Walther Janous's solution. Denoting the expression under consideration by f , Janous first shows that a necessary condition for f to be minimized is that $a = x/(x^2 + y^2 + z^2)$, etc. (see above). He obtains from this that $a^2 + b^2 + c^2 = (x^2 + y^2 + z^2)^{-1}$ and hence $x = a/(a^2 + b^2 + c^2)$, etc., so that $f \geq 2$ becomes the inequality

$$\sum \sqrt{\frac{b^2 + c^2}{a^2 + b^2 + c^2}} \geq 2. \quad (1)$$

Janous proves this using majorization and the concave function $g(w) = \sqrt{1-w}$, and then interprets it as the inequality

$$d_1 + d_2 + d_3 \geq 2d,$$

where d_1, d_2, d_3, d are the three face diagonals and the space diagonal of a rectangular box. (In fact he shows that

$$d\sqrt{6} \geq d_1 + d_2 + d_3 \geq 2d \quad (2)$$

which he then generalizes to n dimensions.) Janous ends by asking the readers for a similar but more general result, namely: for each $1 \leq k < l \leq n$, determine the minimum and maximum values of

$$\frac{\sum \sqrt{x_{i_1} + \cdots + x_{i_k}}}{\sum \sqrt{x_{i_1} + \cdots + x_{i_l}}},$$

where x_1, \dots, x_n are nonnegative real numbers satisfying $x_1 + \cdots + x_n = 1$, and the top and bottom sums are respectively over all k - and l -element subsets of $\{x_1, \dots, x_n\}$. (1), (2) is the case $k = n - 1$, $l = n$, where $n = 3$ (with $x_1 = a^2$, etc.).

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1735. [1992: 109] *Proposed by P. Penning, Delft, The Netherlands.*

In a conic (ellipse or hyperbola) with centre O , chords AB have the property that all triangles OAB have the same area. Find the locus of the midpoint of AB .

Solution by Christopher J. Bradley, Clifton College, Bristol, U.K..

Take first the case of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and let $A = (a \cos \theta, b \sin \theta)$ and $B = (a \cos \phi, b \sin \phi)$. [Let's assume that $\pi + \theta > \phi > \theta$ — *Ed.*] The area of OAB is evidently $\Delta = ab \sin(\phi - \theta)/2$, so for constant Δ the parameters θ, ϕ of A, B are related by

$$\sin(\phi - \theta) = \frac{2\Delta}{ab}.$$

If Δ is given greater than $ab/2$, there are no possible triangles and the locus is void. Otherwise let $\alpha = \arcsin(2\Delta/ab)$; then $\phi - \theta$ equals α or $\pi - \alpha$. Now the midpoint of AB has coordinates (X, Y) where

$$X = \frac{a}{2}(\cos \theta + \cos \phi) = a \cos \frac{\phi + \theta}{2} \cos \frac{\phi - \theta}{2},$$

$$Y = \frac{b}{2}(\sin \theta + \sin \phi) = b \sin \frac{\phi + \theta}{2} \cos \frac{\phi - \theta}{2}.$$

In the case $\alpha = \pi/2$ the locus of (X, Y) is

$$\frac{2x^2}{a^2} + \frac{2y^2}{b^2} = 1,$$

as $\cos[(\phi - \theta)/2] = \pm 1/\sqrt{2}$. Otherwise $\cos[(\phi - \theta)/2]$ equals δ or $\sqrt{1 - \delta^2}$ ($\delta^2 \neq 1/2$), and so the locus consists of a pair of concentric ellipses that are interior and similar to the given ellipse.

In the case of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

let $A = (\pm a \cosh \theta, b \sinh \theta)$ and $B = (\pm a \cosh \phi, b \sinh \phi)$ (with $+$ on the $x > 0$ branch and $-$ on the $x < 0$ branch, and no conditions placed on θ). Now $\Delta = ab \sinh |\theta \pm \phi|/2$, where the plus sign holds when the points are on different branches, and the minus sign when on the same branch. The midpoint of AB has coordinates $(\pm X, Y)$ where

$$X = \frac{a}{2}(\cosh \theta \pm \cosh \phi) = \begin{cases} a \cosh[(\theta + \phi)/2] \cosh[(\theta - \phi)/2] \\ a \sinh[(\theta + \phi)/2] \sinh[(\theta - \phi)/2] \end{cases},$$

$$Y = \frac{b}{2}(\sinh \theta + \sinh \phi) = b \sinh \frac{\theta + \phi}{2} \cosh \frac{\theta - \phi}{2},$$

and in the expression for X the plus sign holds (equalling a product of hyperbolic cosines) when the points are on the same branch, and the minus sign holds when the points are on opposite branches. The analysis is now very much like that for the ellipse; there are two cases depending upon whether $|\theta + \phi|$ or $|\theta - \phi|$ is constant, leading to a pair of concentric hyperbolas, one of which is similar to the given conic, the other to the conjugate of the given conic.

Comments by J. Chris Fisher. The affine transformation that takes $\triangle OAB$ to a triangle $OA'B'$ of the same area is called an *equiaffinity*. Its properties are discussed by H.S.M. Coxeter in his *Introduction to Geometry*, §13.4. In particular, the equiaffinity fixes a family of conics that are concentric and similar (or — in the case of hyperbolas — conjugate). It follows immediately that the midpoint of AB determines a conic of that family that is the locus called for in the problem. Furthermore, in the statement of the problem (and in the featured solution) the midpoint can be replaced by any point dividing AB in the ratio $\lambda : 1$ (for any fixed real number λ). The locus for $\lambda = 1$ (the original problem) is the conic of the concentric family that is tangent to AB . The question could be even further modified so that the given conic is a parabola, as long as the segment AB is constrained to move so that the area between it and the parabola remains constant.

Also solved by ENRICO AU-YEUNG, student, University of Waterloo; JORDI DOU, Barcelona, Spain; J. CHRIS FISHER, University of Regina; JUN-HUA HUANG, The 4th Middle School of Nanzian, Hunan, China; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAN PEDOE, Minneapolis, Minnesota; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Although all solvers got the essentials correct, many said “and so forth” a bit too soon. Léo Sauvé would probably have had a gentle reprimand ready for the occasion.

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1736. [1992: 110] *Proposed by Jordi Dou, Barcelona, Spain.*

Let A', B', C' be the feet of the altitudes of $\triangle ABC$, $K = AA' \cap B'C'$, and L, M the intersections of AB, AC with the perpendicular bisector of $A'K$. Prove that A, A', L, M are concyclic.

I. Solution by Shailesh Shirali, Rishi Valley School, India.

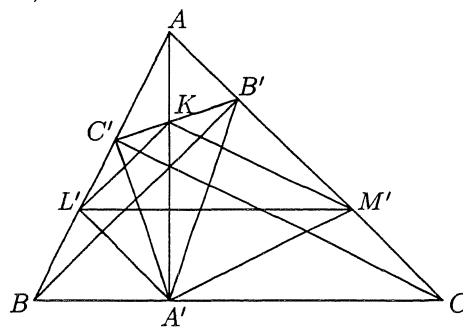
Let circle $KB'A'$ meet AC again at M' ; then, since $\angle AB'C' = \angle B$, etc., by the properties of cyclic quadrilaterals,

$$\angle KA'M' = \angle KB'A = B = \angle A'B'M' = \angle A'KM';$$

therefore segments $M'K$ and $M'A'$ are congruent.

Likewise, letting circle $KC'A'$ meet AB again at L' , segments $L'K$ and $L'A'$ are congruent. Therefore the

quadrilateral $L'KM'A'$ is a “kite”, and so the diagonal $L'M'$, which is an axis of symmetry of the kite, is the perpendicular bisector of the other diagonal KA' . It follows that the line



$L'M'$ coincides with the line LM of the problem. Finally,

$$\angle LA'M = \angle L'A'M' = \angle KA'M' + \angle L'A'K = B + C = 180^\circ - A,$$

and this shows that quadrilateral $ALA'M$ is cyclic.

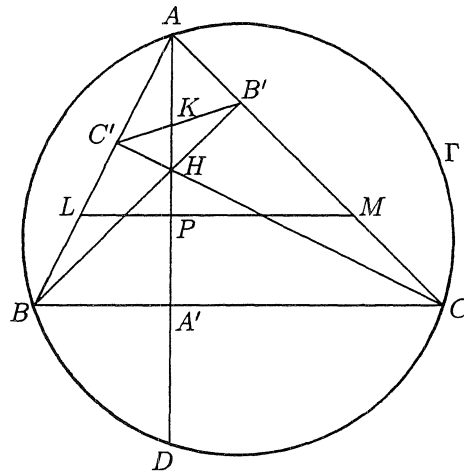
[*Editor's note.* This solution works if $\triangle ABC$ is acute. For obtuse triangles there are only minor changes.]

II. Solution by C. Festraets-Hamoir, Brussels, Belgium.

Soit Γ le cercle circonscrit au triangle ABC , et D le point d'intersection de Γ et AA' . H étant l'orthocentre du triangle ABC , on sait que $A'H = A'D$. Dans le quadrilatère complet $AB'HC'$, on a

$$(AH, KA') = -1.$$

[*Editor's note.* This is the cross ratio, i.e., A, H, K, A' form a harmonic quadruple, i.e., H and A are an inverse pair with respect to inversion in the circle having $A'K$ as diameter. See, e.g., Dan Pedoe's *Geometry: A Comprehensive Course*, §19.1.] P étant le milieu de KA' ,



$$PH \cdot PA = (PA')^2 (= (PK)^2) = (AA' - PA)^2$$

[e.g., H. Eves, *A Survey of Geometry*, p. 83, Theorem 2.8.5.], et

$$\begin{aligned} (AA')^2 &= 2AA' \cdot PA - (PA)^2 + PH \cdot PA \\ &= PA \cdot (2AA' - PA + PH) = PA \cdot (AA' + PA' + PH) \\ &= PA \cdot (AA' + A'H) = PA \cdot (AA' + A'D) = PA \cdot AD, \end{aligned}$$

ce qui donne

$$\frac{AD}{AA'} = \frac{AA'}{AP}. \quad (1)$$

Donc, l'homothétie de centre A et de rapport AP/AA' applique B sur L , C sur M , A' sur P et aussi, en vertu de (1), D sur A' . L'image du cercle Γ circonscrit au quadrangle $ABDC$ est un cercle circonscrit au quadrangle $ALA'M$.

[*Editor's note.* This solution works for acute triangles, but a similar proof works for arbitrary triangles if directed distances are used.]

III. Solution by the proposer.

Let π be the parabola tangent to lines AB , AC , BB' , CC' . Then: A' is the focus of π (because A' is the intersection of the circumcircles of the triangles $AC'C$ and $AB'B$ whose sides are tangent to π); $B'C'$ is the directrix of π (because the pairs of tangents

$B'B$, $B'C$ and $C'B$, $C'C$ are each orthogonal); and LM is tangent to π (because the point K symmetric to A' with respect to LM lies on the directrix). Therefore the circumcircle of ALM , whose sides are tangent to π , passes through A' .

[*Editor's note.* For some remarks relevant to this lovely proof, see the comments following Dou's solution of his problem *Cruz* 1597 [1992: 26–27].]

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; C.J. BRADLEY, Clifton College, Bristol, U.K.; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; and D.J. SMEENK, Zaltbommel, The Netherlands.

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1737. [1992: 110] Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.

Let $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ describe the Fibonacci sequence. A point of the plane is a *Fibonacci point* if both of its coordinates are Fibonacci numbers. In the *Fibonacci Quarterly* Vol. 28 (1990) pp. 22–27, Clark Kimberling calls a hyperbola (with axes parallel to the coordinate axes) a *Fibonacci hyperbola* if it contains an infinity of Fibonacci points.

(a) Consider the sequence $\phi_0, \phi_1, \phi_2, \dots$ of Fibonacci points (listed by distance from the origin) in the first quadrant which lie on the Fibonacci hyperbola $x^2 + xy - y^2 + 1 = 0$. Prove that the area of the triangle $\phi_{n-1}\phi_n\phi_{n+1}$ is constant over all positive integers n .

(b)* What about for other Fibonacci hyperbolas?

I. Combination of solutions to part (a) by R.P. Sealy, Mount Allison University, Sackville, New Brunswick; and Shailesh Shirali, Rishi Valley School, India.

The Fibonacci points ϕ_n are given by

$$\phi_n = (F_{2n}, F_{2n+1}).$$

We have

$$\begin{aligned} F_{2n}^2 + F_{2n}F_{2n+1} - F_{2n+1}^2 + 1 &= F_{2n}(F_{2n} + F_{2n+1}) - F_{2n+1}^2 + 1 \\ &= F_{2n}F_{2n+2} - F_{2n+1}^2 + 1 = 0, \end{aligned}$$

thus the Fibonacci point (F_{2n}, F_{2n+1}) is on the curve. We next show that no other Fibonacci points are on the curve. Since the graph is increasing in the first quadrant, the only possibilities are points of the form (F_{2n+1}, F_{2n+2}) . After substituting, we have

$$\begin{aligned} F_{2n+1}^2 + F_{2n+1}F_{2n+2} - F_{2n+2}^2 + 1 &= F_{2n+1}(F_{2n+1} + F_{2n+2}) - F_{2n+2}^2 + 1 \\ &= F_{2n+1}F_{2n+3} - F_{2n+2}^2 + 1 = 2; \end{aligned}$$

therefore there are no other Fibonacci points on the curve.

We now show that the area of the triangle $\phi_n\phi_{n+1}\phi_{n+2}$ is equal to $1/2$ for all $n \geq 0$. We have that $\text{area}(\phi_n\phi_{n+1}\phi_{n+2})$ is

$$\begin{aligned}
\frac{1}{2} \begin{vmatrix} F_{2n} & F_{2n+2} & F_{2n+4} \\ F_{2n+1} & F_{2n+3} & F_{2n+5} \\ 1 & 1 & 1 \end{vmatrix} &= \frac{1}{2} \begin{vmatrix} F_{2n} & F_{2n+2} - F_{2n} & F_{2n+4} - F_{2n+2} \\ F_{2n+1} & F_{2n+3} - F_{2n+1} & F_{2n+5} - F_{2n+3} \\ 1 & 0 & 0 \end{vmatrix} \\
&= \frac{1}{2} \begin{vmatrix} F_{2n+2} - F_{2n} & F_{2n+4} - F_{2n+2} \\ F_{2n+3} - F_{2n+1} & F_{2n+5} - F_{2n+3} \end{vmatrix} = \frac{1}{2} \begin{vmatrix} F_{2n+1} & F_{2n+3} \\ F_{2n+2} & F_{2n+4} \end{vmatrix} \\
&= \frac{1}{2} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \begin{vmatrix} F_{2n-1} & F_{2n+1} \\ F_{2n} & F_{2n+2} \end{vmatrix} = \frac{1}{2} \begin{vmatrix} F_{2n-1} & F_{2n+1} \\ F_{2n} & F_{2n+2} \end{vmatrix} \\
&= \dots = \frac{1}{2} \begin{vmatrix} F_1 & F_3 \\ F_2 & F_4 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = \frac{1}{2},
\end{aligned}$$

as claimed.

II. Comment by the editor.

No solutions to part (b) were received, except that Sealy reports that the hyperbola $x^2 + xy - y^2 - 1 = 0$ contains infinitely many Fibonacci points of the form $\phi_n = (F_{2n+1}, F_{2n+2})$ with area of the triangle $\phi_n\phi_{n+1}\phi_{n+2}$ again equal to $1/2$. In the paper mentioned in the problem statement, Clark Kimberling shows that the Fibonacci hyperbolas are precisely

$$x^2 + (-1)^{n+1} L_n xy + (-1)^n y^2 \pm F_n^2 = 0, \quad n = 1, 2, 3, \dots,$$

where $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ is the Lucas sequence. The Fibonacci points lying on these hyperbolas are always of the form

$$(F_k, F_{k+n}), (F_{k+h}, F_{k+n+h}), (F_{k+2h}, F_{k+n+2h}), \dots$$

The proposer conjectures that the areas of the corresponding triangles $\phi_{n-1}\phi_n\phi_{n+1}$ are constant for each of these hyperbolas. Kimberling (in a letter to the editor) notes that in fact, if $\{a_n\}$ and $\{b_n\}$ are any arithmetic progressions of positive integers with the same common difference, then

$$\begin{vmatrix} 1 & 1 & 1 \\ F_{a_n} & F_{a_{n+1}} & F_{a_{n+2}} \\ F_{b_n} & F_{b_{n+1}} & F_{b_{n+2}} \end{vmatrix}$$

seems to be constant. Can any reader prove this (or supply a reference)?

Part (a) also solved by CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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1738. [1992: 110] *Proposed by Shiko Iwata, Gifu, Japan.*

There are four points A_1, A_2, A_3, A_4 in a plane. Let H_i be the orthocenter of the triangle formed by excluding A_i from these four points. Show that the areas of the quadrilaterals $A_1A_2A_3A_4$ and $H_1H_2H_3H_4$ are equal.

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

The area of $\triangle XYZ$ is denoted by $[XYZ]$, where we take $[XYZ] = -[YXZ]$. [That is, $[XYZ]$ is a *signed* area, positive if XYZ is counterclockwise, say, and negative if XYZ is clockwise. — *Ed.*] Furthermore, if T is an arbitrary point in the plane of $\triangle XYZ$ then

$$[XYZ] = [XYT] + [Yzt] + [ZXT]. \quad (1)$$

As $A_2H_1 \parallel A_1H_2$ we have

$$[H_1H_2A_1] = [A_2H_2A_1];$$

as $A_2H_3 \parallel A_3H_2$ we have by (1)

$$\begin{aligned} [H_2H_3A_1] &= [H_2H_3A_2] + [H_3A_1A_2] + [A_1H_2A_2] \\ &= [A_3H_3A_2] + [H_3A_1A_2] + [A_1H_2A_2]; \end{aligned}$$

and as $A_1H_3 \parallel A_3H_1$ we have

$$[H_3H_1A_1] = [H_3A_3A_1].$$

Thus by (1)

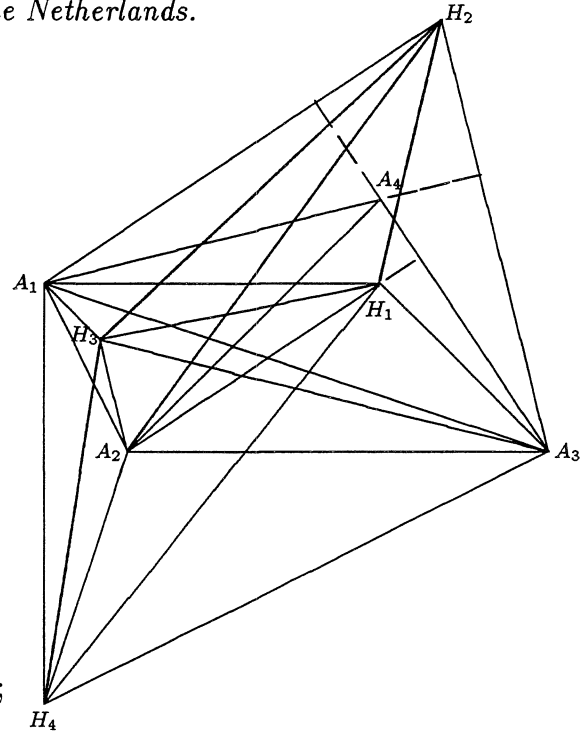
$$\begin{aligned} [H_1H_2H_3] &= [H_1H_2A_1] + [H_2H_3A_1] + [H_3H_1A_1] \\ &= [A_2H_2A_1] + [A_3H_3A_2] + [H_3A_1A_2] + [A_1H_2A_2] + [H_3A_3A_1] \\ &= ([A_1A_2H_2] + [A_2A_1H_2]) + ([A_1A_2H_3] + [A_2A_3H_3] + [A_3A_1H_3]) \\ &= 0 + [A_1A_2A_3] = [A_1A_2A_3]. \end{aligned} \quad (2)$$

In the same way $[H_1H_3H_4] = [A_1A_3A_4]$ and so

$$\text{area}(H_1H_2H_3H_4) = \text{area}(A_1A_2A_3A_4).$$

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; C.J. BRADLEY, Clifton College, Bristol, U.K.; JORDI DOU, Barcelona, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; DAN PEDOE, Minneapolis, Minnesota; IGNASI MUNDET I RIERA, Barcelona, Catalunya, Spain; and the proposer. A partial solution in the case that A_1, A_2, A_3, A_4 are concyclic was sent in by MURRAY S. KLAMKIN, University of Alberta.

The stronger result (2), that the areas of triangles $A_1A_2A_3$ and $H_1H_2H_3$ (for example) are equal, was also given by Bellot Rosado and López Chamorro, by Pedoe and by Riera. Pedoe in fact located this result as Crux 717 [1983 : 64]! He also proved that if $A_1A_2A_3A_4$ is concyclic then triangles $A_1A_2A_3$ and $H_1H_2H_3$ are congruent. It follows that



in this situation $A_1A_2A_3A_4$ and $H_1H_2H_3H_4$ are congruent, which (as Bellot and López observe) is the content of a problem from the 1984 Balkanic Mathematical Olympiad; see [1985: 243], where more references are given.

Some solvers point out that the quadrilateral $A_1A_2A_3A_4$ ought to be assumed convex, else it isn't clear what "area" means.

Janous replaced the orthocentres by the centres of gravity G_1, G_2, G_3, G_4 of the four triangles, and obtained:

$$\text{area}(G_1G_2G_3G_4) = \frac{1}{9}\text{area}(A_1A_2A_3A_4).$$

* * * *

1739. [1992: 110] *Proposed by Richard K. Guy, University of Calgary.*

Express 19 as the sum of two cubes of positive rational numbers in two different ways.

Solution by Hayo Ahlburg, Benidorm, Spain.

Two approaches come to mind immediately, especially since 19 is a small number.

1) If you have a list (or want to make up a list) of the smallest solutions of

$$a^3 + b^3 = c^3 + d^3 = N,$$

look for any $N = 19n^3$. Such a list with all 101 solutions for $N < 5000000$ was compiled by C.E. Britton and published in *Scripta Mathematica*, vol. xxv, no. 2, July 1960, pages 165–166. Since I don't have this available, I made up my own list of all ten solutions for $N < 100000$, and conveniently find that already the second solution can be used in our problem:

$$1^3 + 12^3 = 9^3 + 10^3 = 1729 = 7 \cdot 13 \cdot 19 \quad (\text{Ramanujan})$$

$$2^3 + 16^3 = 9^3 + 15^3 = 4104 = 6^3 \cdot 19.$$

Dividing by 6^3 , we have our first solution

$$\left(\frac{1}{2}\right)^3 + \left(\frac{8}{3}\right)^3 = \left(\frac{3}{2}\right)^3 + \left(\frac{5}{2}\right)^3 = 19.$$

By inspection of Britton's table, someone might check whether there are several more in this range. At least one is given below.

2) In Fermat's famous notes to Diophantus (with commentary by C.G. Bachet), where note IV is the "last problem", note IX explains how Fermat for the first time was able to partition the sum of two cubes into two different cubes, and this in an *infinite number of ways*. He employs three formulas (given by Vieta) successively, and this can be done indefinitely. While we could start with either one of the two sums given above, this leads to rather unwieldy numbers very quickly. However, just using Vieta's first formula, I get a still reasonably small new solution.

$$A^3 - B^3 = x^3 + y^3 \quad \text{has the solution} \quad x = \frac{A(A^3 - 2B^3)}{A^3 + B^3}, \quad y = \frac{B(2A^3 - B^3)}{A^3 + B^3}.$$

At first glance, we conveniently see that $3^3 - 2^3 = 19$, so using $A = 3$, $B = 2$ we get $x = 33/35$, $y = 92/35$, giving us the third pair of cubes summing to 19.

The question of finding sums of two cubes having certain values has, as Mordell wrote, “attracted the attention of many mathematicians over a period of years as can be seen from the second volume of Dickson’s *History of the Theory of Numbers*” (pp. 550–561). I just notice that Dickson quotes in the following section (p. 562) Fauquembergue (1906) with his

$$19 = \left(\frac{8}{3}\right)^3 + \left(\frac{1}{3}\right)^3 = \left(\frac{5}{2}\right)^3 + \left(\frac{3}{2}\right)^3 = \left(\frac{92}{35}\right)^3 + \left(\frac{33}{35}\right)^3 = \left(\frac{27323}{10386}\right)^3 + \left(\frac{9613}{10386}\right)^3 = \dots$$

Also solved by HARVEY L. ABBOTT, University of Alberta; CHARLES ASHBACHER, Cedar Rapids, Iowa; SAM BAETHGE, Science Academy, Austin, Texas; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; J.L. BRENNER, Palo Alto, California; C. FESTAETS-HAMMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; DAVID E. MANES, State University of New York, Oneonta; J.A. MCCALLUM, Medicine Hat, Alberta; JEAN-MARIE MONIER, Lyon, France; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; SHAILESH SHIRALI, Rishi Valley School, India; P. TSAOUSSOGLU, Athens, Greece; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Several solvers gave the Dickson reference, or referred to Sierpinski’s *Theory of Numbers* (Theorem 10, page 82 or Theorem 12, page 383) or other number theory texts, for the result that there are infinitely many solutions. Klamkin and Wilke also sent in computer-generated solutions which were hundreds of digits long!

The proposer’s solution used the number 4104 as Ahlburg did in his first approach. The proposer originally suggested the problem (a “Crux 1729” candidate) as the answer Ramanujan might have given if Hardy’s taxi number had been 19 instead of 1729.

Konečný offers the additional “solution”

$$\left(\frac{2353}{1111}\right)^3 + \left(\frac{2353}{1111}\right)^3 = 19,$$

which works if, in his words, “you have a sense of humour and a cheap calculator”! (The continued fraction expansion of $\sqrt[3]{9.5}$ will reveal the spectacular reason for this “solution”.)

* * * * *

1740. [1992: 110] Proposed by Dan Pedoe, Minneapolis, Minnesota. (Dedicated in memoriam to Joseph Konhauser.)

In triangle ABC the points N, L, M , in that order on AC , are respectively the foot of the perpendicular from B onto AC , the intersection with AC of the bisector of

$\angle ABC$, and the midpoint of AC . The angles ABN , NBL , LBM and MBC are all equal. Determine the angles of $\triangle ABC$. [Some comments on the origin of this proposal will be given when a solution is published.]

I. *Solution by Michael Parmenter, Memorial University of Newfoundland.*

Setting $\theta = \angle ABN$, we note that $\angle BAC = \pi/2 - \theta$ and $\angle BCA = \pi/2 - 3\theta$. By the law of sines,

$$BM = \frac{AM \sin(\pi/2 - \theta)}{\sin 3\theta} = \frac{AM \cos \theta}{\sin 3\theta}$$

and

$$BM = \frac{CM \sin(\pi/2 - 3\theta)}{\sin \theta} = \frac{CM \cos 3\theta}{\sin \theta}.$$

Since $AM = CM$, this means that

$$\frac{\cos \theta}{\sin 3\theta} = \frac{\cos 3\theta}{\sin \theta}.$$

Thus $\sin \theta \cos \theta = \sin 3\theta \cos 3\theta$, so $\sin 2\theta = \sin 6\theta$. The solutions for this are given by $6\theta = 2k\pi + 2\theta$ and $6\theta = (2k - 1)\pi - 2\theta$, i.e. $\theta = k\pi/2$, $\theta = (2k - 1)\pi/8$, where k is an integer. The only solution fitting the conditions of the problem is $\theta = \pi/8$, yielding

$$\angle A = \frac{3\pi}{8}, \quad \angle B = \frac{\pi}{2}, \quad \angle C = \frac{\pi}{8}.$$

II. *Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

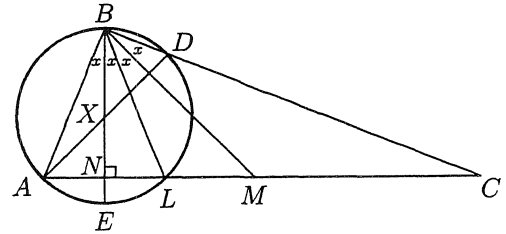
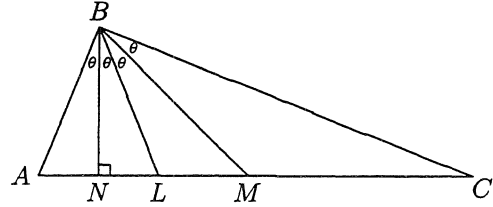
The altitude and BO (O the circumcenter) are isogonal in any triangle [that is, $\angle ABN = \angle OBC$]; therefore in our case the median BM passes through O . If $O \neq M$, BM would be perpendicular to AC , the median and the altitude would coincide and the condition of the problem would not hold. So $O = M$ and the triangle is right-angled at B . But then $\angle ABN = \pi/8$, $\angle BAC = 3\pi/8$ and $\angle BCA = \pi/8$.

III. *Solution by P. Penning, Delft, The Netherlands.*

The four equal angles at B are denoted by x . Then $\angle ABC = 4x$, $\angle CAB = 90^\circ - x$ and $\angle BCA = 90^\circ - 3x$. Draw the circle through A , B , L . Its centre O lies on BN . Let BN intersect the circle in E (BE is a diameter). The side BC intersects the circle in D . Since

$$\text{arc } BD = \text{arc } BLE - 3 \text{ arc } EL,$$

we have $\angle BAD = 90^\circ - 3x = \angle BCA$. So triangles DBA and ABC are similar, and they are divided in exactly the same way by the three lines through B . So the intersection X of BE and AD must be the midpoint of AD , since M is the midpoint of AC . The diameter BE divides the chord AD into two equal parts, which is possible in two cases



only: either the chord and diameter are perpendicular (which is clearly not the case), or the chord is a diameter itself and the point of intersection is the centre of the circle, which must be the case here. So X coincides with O and angle B must be 90° , and $x = 22.5^\circ$. So $\angle CAB = 67.5^\circ$ and $\angle BCA = 22.5^\circ$.

IV. *Solution by Toshio Seimiya, Kawasaki, Japan.*

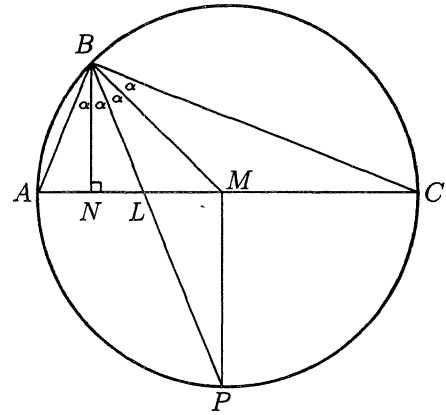
We put

$$\angle ABN = \angle NBL = \angle LBM = \angle MBC = \alpha.$$

Let P be the intersection of BL with the circumcircle of $\triangle ABC$ other than B ; then P is the midpoint of arc AC . Therefore we get $MP \perp AC$, and thus $MP \parallel BN$, so that

$$\angle MPB = \angle NBL = \angle MBP.$$

Consequently we get $MB = MP$. As M is the intersection of the perpendicular bisectors of BP and AC , M is the center of the circumcircle of $\triangle ABC$, and thus AC is a diameter. We get $4\alpha = \angle ABC = 90^\circ$, and therefore $\alpha = 22.5^\circ$. As $\angle ABC = 90^\circ$ and $BN \perp AC$, we get $\angle BAC = \angle CBN = 3\alpha = 67.5^\circ$ and $\angle BCA = \angle ABN = \alpha = 22.5^\circ$.



Also solved by HAYO AHLBURG, Benidorm, Spain; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; SAM BAETHGE, Science Academy, Austin, Texas; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; JORDI DOU, Barcelona, Spain; RICHARD K. GUY, University of Calgary; DAVID HANKIN, Brooklyn, New York; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; MARCIN E. KUCZMA, Warszawa, Poland; ANDY LIU, University of Alberta; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; J.A. MCCALLUM, Medicine Hat, Alberta; D.J. SMEENK, Zaltbommel, The Netherlands; L.J. UPTON, Mississauga, Ontario (four solutions!); A.W. WALKER, Toronto, Ontario; and the proposer.

Smeenk, Walker, and the proposer submitted solutions similar to I, II, and IV respectively. The other solutions were usually a bit more complicated.

Ahlburg mentions that connecting the first, second, and fifth vertices of a regular octagon results in the triangle in question.

Janous has seen the problem before, but does not know its origin.

The proposer attached the following note (in tribute to the late Joseph Konhauser) to the problem statement: "I was asked to take over Joe's class at Macalester College when his heart gave way. He died under surgery soon after. The above problem, new to me, is a searching one, and appeared in his second (ungraded) Problem Set, which I took over. I do not know whether Joe originated this problem, but it is a reflection of his geometric excellence. He was a fine teacher."

* * * * *

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