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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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About Cyclic Quadrilaterals

JENS CARSTENSEN

Many theorems about cyclic quadrilaterals are not very well known. I shall prove three beautiful theorems below.

We start with the following:

Lemma 1. *Let* I *be the incentre of* $\triangle\mathrm{ABC}$ *and* D *the intersection of the bisector of angle* A *with the circumcircle. Then* $\mathrm{BD} = \mathrm{DC} = \mathrm{DI}$.

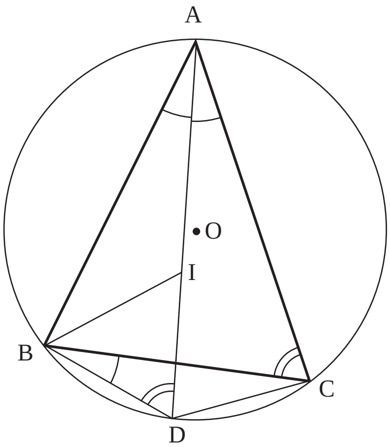

Figure 1.

Proof. The point D is the midpoint of the arc $\widehat{\mathrm{BC}}$ so that $\mathrm{DB} = \mathrm{DC}$. In figure 1 the indicated angles are equal, so $\triangle\mathrm{DIB}$ gives us

$$\begin{aligned}\angle\mathrm{DIB} &= 180^\circ - \angle\mathrm{IBD} - \angle\mathrm{BDI}\\ &= 180^\circ - (\tfrac{1}{2}\mathrm{B} + \tfrac{1}{2}\mathrm{A}) - \mathrm{C}\\ &= 180^\circ - (90^\circ - \tfrac{1}{2}\mathrm{C}) - \mathrm{C}\\ &= 90^\circ - \tfrac{1}{2}\mathrm{C} = \tfrac{1}{2}\mathrm{A} + \tfrac{1}{2}\mathrm{B}\\ &= \angle\mathrm{IBD}\,.\end{aligned}$$

Thus $\triangle\mathrm{DIB}$ is isosceles and $\mathrm{BD} = \mathrm{DI}$.

We now state a little-known theorem of cyclic quadrilaterals.

Theorem 1. *The diagonals* AC *and* BD *in the cyclic quadrilateral* ABCD *form four triangles* ($\triangle\mathrm{ABC}$, $\triangle\mathrm{BCD}$, $\triangle\mathrm{CDA}$ *and* $\triangle\mathrm{DAB}$). *The centres* P, Q, R *and* S *of the incircles of these triangles are the vertices of a rectangle (figure 2).*

Proof. Let M and N be the midpoints of the arcs $\widehat{\mathrm{BC}}$ and $\widehat{\mathrm{CD}}$ (figure 3). Since AN is the bisector of $\angle\mathrm{CAD}$, AN passes through Q and since DM is the bisector of $\angle\mathrm{BDC}$, DM passes through P.

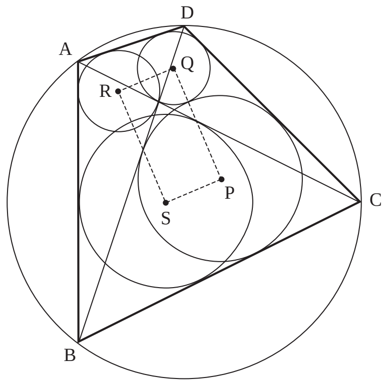

Figure 2.

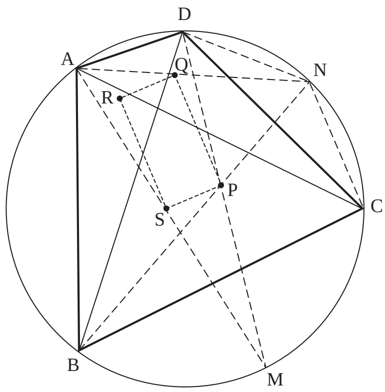

Figure 3.

In $\triangle\mathrm{ADC}$ we use the result of lemma 1 and obtain $\mathrm{ND} = \mathrm{NC} = \mathrm{NQ}$. Similarly in $\triangle\mathrm{BCD}$ the lemma yields $\mathrm{ND} = \mathrm{NC} = \mathrm{NP}$. But then $\triangle\mathrm{PQN}$ is isosceles and

$$\begin{aligned}\angle\mathrm{QPN} &= \tfrac{1}{2}(180^\circ - \angle\mathrm{PNQ}) = 90^\circ - \tfrac{1}{2}\angle\mathrm{PNQ}\\ &= 90^\circ - \tfrac{1}{2}\angle\mathrm{BNA}\,.\end{aligned}$$

In the same way we get

$$\angle\mathrm{SPM} = 90^\circ - \tfrac{1}{2}\angle\mathrm{DMA}\,.$$

Now, $\angle\mathrm{BPM}$ is an angle whose vertex is an interior point of the circle, so that

$$\angle\mathrm{BPM} = \tfrac{1}{2}(\widehat{\mathrm{BM}} + \widehat{\mathrm{ND}}) = \tfrac{1}{4}(\widehat{\mathrm{BC}} + \widehat{\mathrm{CD}})\,.$$

We have that

$$\angle\mathrm{QPN} + \angle\mathrm{QPS} + \angle\mathrm{SPB} = 180^\circ\,,$$

and thus

$$\begin{aligned}\angle QPS &= 180^\circ - \angle QPN - \angle SPB\\ &= 180^\circ - (90^\circ - \tfrac{1}{2}\angle BNA) - (\angle SPM - \angle BPM)\\ &= 90^\circ + \tfrac{1}{2}\angle BNA - \angle SPM + \angle BPM\\ &= 90^\circ + \tfrac{1}{2}\angle BNA - (90^\circ - \tfrac{1}{2}\angle DMA)\\ &\quad + \tfrac{1}{4}(\overset{\frown}{BC} + \overset{\frown}{CD})\\ &= \tfrac{1}{4}\overset{\frown}{AB} + \tfrac{1}{4}\overset{\frown}{DA} + \tfrac{1}{4}\overset{\frown}{BC} + \tfrac{1}{4}\overset{\frown}{CD}\\ &= \tfrac{1}{4}(\overset{\frown}{AB} + \overset{\frown}{BC} + \overset{\frown}{CD} + \overset{\frown}{DA}) = 90^\circ .\end{aligned}$$

In the same manner we see that the other angles in the quadrilateral PQRS are right angles.

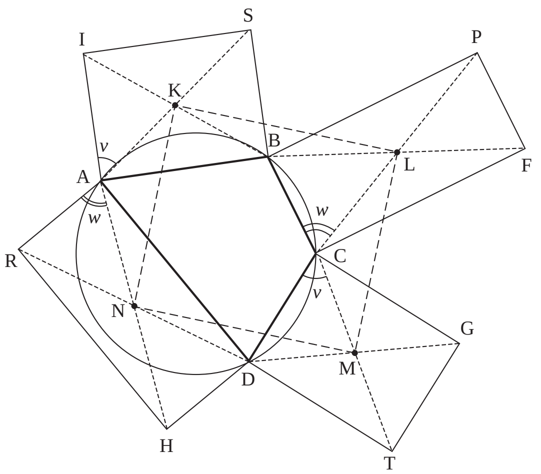

Figure 4.

We consider now a somewhat peculiar construction giving a surprising result: with the four sides of a cyclic quadrilateral ABCD as bases, erect rectangles ‘outward’ whose altitudes are equal to the opposite side of the quadrilateral, i.e. in figure 4, ADHR, ABSI, BCFP and CDTG are rectangles and

$$\begin{aligned} AI = BS = CD, &\qquad CF = BP = AD,\\ CG = DT = AB, &\qquad DH = AR = BC.\end{aligned}$$

The rectangles have centres K, L, M and N. We now prove a strange result.

Theorem 2. *The centres* K*,* L*,* M*,* N *of the rectangles erected on the sides of the cyclic quadrilateral* ABCD *as described form the vertices of a rectangle* KLMN *(figure 4).*

Proof. We see that CDTG and ABSI are congruent since their sides are pairwise equal. Also BCFP and DARH are congruent. Therefore,

$$\angle DCM = \angle IAK = v \quad \text{and}$$
$$\angle BCL = \angle RAN = w\,.$$

Since ABCD is a cyclic quadrilateral, $A + B = 180^\circ$. Figure 4 shows that

$$\angle RAI = 360^\circ - 90^\circ - 90^\circ - A = 180^\circ - A = C\,. \quad (1)$$

Further we have

$$\angle IAH = 360^\circ - \angle RAI - w \quad \text{and} \quad \angle LCD = 360^\circ - C - w\,.$$

Using (1) we get

$$\angle IAH = \angle LCD\,,$$

so

$$\angle IAH - v = \angle LCD - v\,,$$

and so

$$\angle KAN = \angle LCM\,.$$

We conclude that $\triangle KAN$ and $\triangle LCM$ are congruent, because their sides KA, AN and CM, LC are half the diagonals in congruent rectangles. But then KN = LM.

Similarly $\triangle KBL$ and $\triangle MND$ are congruent, so that KL = MN. We have now shown that KLMN is a parallelogram.

In this parallelogram, $\angle KLM = \angle KNM$, and further we have

$$\angle KLM = \angle BLC + \angle KLB + \angle CLM \quad \text{and}$$
$$\angle KNM = \angle AND - \angle KNA - \angle DNM\,.$$

Because $\angle KLM = \angle KNM$, addition of these equations gives

$$\begin{aligned}2\angle KLM = (\angle CLM - \angle KNA) + (\angle KLB - \angle DNM)\\ + (\angle BLC + \angle AND).\end{aligned}$$

The first parenthesis equals 0 because $\triangle KAN$ and $\triangle LCM$ are, as mentioned above, congruent. The second parenthesis equals 0 because $\triangle KBL$ and $\triangle MND$ are congruent. Thus

$$2\angle KLM = \angle BLC + \angle AND\,.$$

The angles on the right-hand side are the two supplementary angles between the diagonals of the congruent rectangles BCFP and ADHR, so that

$$2\angle KLM = 180^\circ \quad \text{or} \quad \angle KLM = 90^\circ\,.$$

We conclude that KLMN is a rectangle.

Our last theorem again involves the intersection of the diagonals.

Theorem 3. *Let* P *be the intersection of the diagonals in the cyclic quadrilateral* ABCD *and* O *the centre of the circumcircle (figure 5). If* O_1*,* O_2*,* O_3 *and* O_4 *denote the centres of the circumcircles of the triangles* $\triangle PAB$*,* $\triangle PBC$*,* $\triangle PCD$ *and* $\triangle PAD$*, then the lines* OP*,* O_1O_3 *and* O_2O_4 *are concurrent.*

Proof. We show that PO_1OO_3 is a parallelogram (figure 6). If O_1P intersects CD in X, we have

$$\angle CPX = \angle O_1PA = \angle O_1AP\,,$$

because $\triangle O_1AP$ is isosceles. Next

$$\angle AO_1P = 2\angle PBA = 2\angle PCD\,,$$

since $\angle AO_1P$ and $\angle PBA$ are angles with their vertex at the centre and at the circumference, respectively, of the

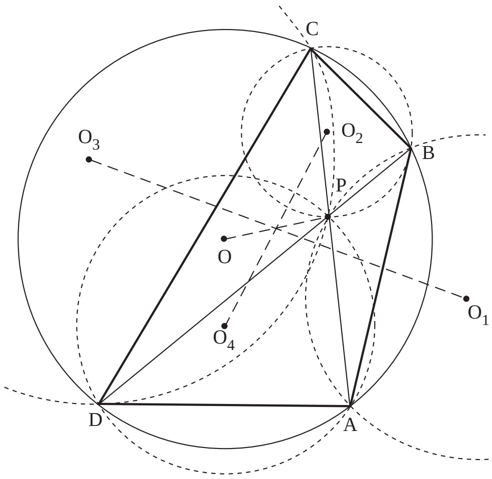

Figure 5.

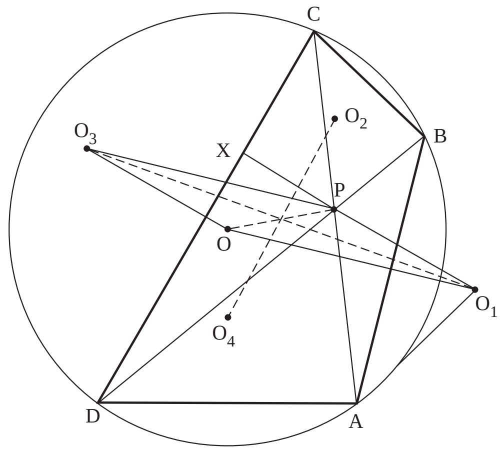

Figure 6.

circumcircle of $\triangle$PAB and since $\angle$PBA and $\angle$PCD subtend the same arc in the circumcircle of ABCD.

In $\triangle \mathrm{AO_1P}$ we see that

$$\begin{aligned}90^\circ &= \tfrac{1}{2}\angle \mathrm{AO_1P} + \tfrac{1}{2}\angle \mathrm{O_1AP} + \tfrac{1}{2}\angle \mathrm{O_1PA}\\ &= \angle \mathrm{PCD} + \tfrac{1}{2}\angle \mathrm{CPX} + \tfrac{1}{2}\angle \mathrm{CPX}\\ &= \angle \mathrm{PCD} + \angle \mathrm{CPX} = \angle \mathrm{PCX} + \angle \mathrm{CPX}\,.\end{aligned}$$

But in $\triangle$PCX this yields

$$\angle \mathrm{CXP} = 180^\circ - (\angle \mathrm{PCX} + \angle \mathrm{CPX}) = 90^\circ\,,$$

so that $\mathrm{O_1P} \perp \mathrm{CD}$.

Since also $\mathrm{OO_3} \perp \mathrm{CD}$ (the points O and $\mathrm{O_3}$ both lie on the perpendicular bisector of CD), we have that $\mathrm{OO_3} \| \mathrm{O_1P}$. Similarly, $\mathrm{OO_1} \| \mathrm{O_3P}$, so that $\mathrm{PO_1OO_3}$ is a parallelogram as claimed. Therefore the diagonals PO and $\mathrm{O_1O_3}$ intersect in their common midpoint.

The same line of reasoning shows that $\mathrm{PO_2OO_4}$ is a parallelogram, so that PO and $\mathrm{O_2O_4}$ intersect in their common midpoint. But then the three lines PO, $\mathrm{O_1O_3}$ and $\mathrm{O_2O_4}$ are concurrent through their common midpoint.

As a bonus we see that $\mathrm{O_1O_2O_3O_4}$ is a parallelogram.

The author is a teacher in a 'gymnasium' (or high school) in Copenhagen. He has published textbooks in mathematics and a great number of expository articles (or mathematical 'snapshots') in the magazine of the Danish Mathematics Teachers' Association.

Some Problems in the Theory of Runs

JOE GANI

Does bad (or good) luck come in runs?

1. Introduction

A friend runs an old motorcycle which he occasionally has trouble starting: a little attention to the carburettor usually solves the problem. But if he has trouble on two consecutive days, he takes it to a garage for an overhaul. If we represent 'no trouble' by 0, and 'trouble' by 1, then in any particular week, 0010010 will not entail an overhaul, while 0001100 with the run 11 on Thursday and Friday would result in an overhaul. This is a particular example of the theory of runs, in which we are interested in the probability of occurrence of runs of ones (or zeros) in a sequence of random events.

The problem may be posed generally as follows. Suppose we have a sequence of independent random variables

$$X_1, X_2, X_3, \ldots, X_{n-1}, X_n\,,$$

where each X_i may take the values 0 or 1, with 1 occurring with a probability $p > 0$, and 0 with a probability $q > 0$, such that $p + q = 1$. We shall later consider the case where the random variables are not independent. If we happen to know the values of the X_i, we may have a sequence such as

$$0, 0, 1, \ldots, 0, 1, 1\,, \qquad (1)$$

where 11 occurs for the first time at the end of the sequence.

We are now interested in the random length N of the sequence until, for the first time, the following events occur:

(a) a 1 arises,
(b) a run 11 of length 2 arises,
(c) a longer run, such as 1111, or a pattern such as 0110 arises.

Let us attempt to provide answers to these problems.

2. The geometric distribution

We begin with the simplest case (a), where the X_i, $i = 1, 2, 3, \ldots$, above are independent (Bernoulli) random variables. If the random length of the sequence until a 1 arises for the first time is N, so that $n-1$ zeros are followed by a final 1, then

$$\mathrm{P}\{N = n\} = q^{n-1}p, \qquad n = 1, 2, 3, \ldots . \tag{2}$$

Then N is said to have a geometric distribution; its probability generating function is

$$f(u) = \sum_{n=1}^{\infty} u^n q^{n-1} p = \frac{pu}{1-qu}, \qquad 0 < u \le 1 .$$

Setting $u = 1$, we see that the sum of the probabilities (2) is 1. Further details about this distribution may be found in reference 1, chapter 9, among other texts.

3. A Markov chain approach

We now consider the case (b) of the run 11. A convenient way of approaching the problem is through the use of Markov chains, as originally outlined in reference 2. A simple Markov chain is a sequence of random variables

$$Y_1, Y_2, Y_3, \ldots, Y_{n-1}, Y_n$$

such that the conditional probabilities $\mathrm{P}\{Y_{i+1} \mid Y_i\}$ are specified by a given matrix. The variable Y_i need not be univariate; we can, for example, consider the states $Y_i = (00)$, (01), (10), or (11). If we refer to the sequence (1) then, moving along it one integer at a time, we have the values

$$\begin{aligned} Y_1 &= (00) = (X_1\, X_2)\,, \\ Y_2 &= (01) = (X_2\, X_3)\,, \\ &\vdots \\ Y_{n-2} &= (01) = (X_{n-2}\, X_{n-1})\,, \\ Y_{n-1} &= (11) = (X_{n-1}\, X_n)\,, \end{aligned}$$

where the second integer in each upper bracket is the same as the first integer in the bracket below it. We note that, as a result, Y_{i+1} is dependent on Y_i, even if the X_i are themselves independent.

The matrix of conditional probabilities in the case where the X_i are independent random variables of the type described above can be written as

$$\begin{array}{cc} & Y_{i+1} \\ & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ Y_i \; \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \left[\begin{array}{ccc|c} q & p & 0 & 0 \\ 0 & 0 & q & p \\ q & p & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array} = \left[\begin{array}{c|c} \boldsymbol{P} & \boldsymbol{Q} \\ \hline 0 & 1 \end{array}\right], \tag{3}$$

where 11 is the absorbing state which terminates the sequence. Once this state is reached, the process stops, so that we remain in 11. Assuming that we do not start with 11, we may have as the row vector of initial probabilities

$$\boldsymbol{A} = [a_{00}\, a_{01}\, a_{10}]$$

with $a_{00} = \mathrm{P}\{\text{starting with } 00\}$, $a_{01} = \mathrm{P}\{\text{starting with } 01\}$, and $a_{10} = \mathrm{P}\{\text{starting with } 10\}$, where their sum is equal to 1.

Now let us examine how the state 11 can be reached for the first time when $N = n$. We begin with one of 00, 01, 10, which uses up two trials, and we must end up with a 1, which uses up a third trial, so we have $n-3$ trials left. Since we must avoid the state 11 until right at the end, this means that we must circulate among the states 00, 01, 10 for the remaining $n-3$ trials, i.e. stay within the matrix $\boldsymbol{P}$. It follows that

$$\mathrm{P}\{N = n\} = \boldsymbol{A}\,\boldsymbol{P}^{n-3}\,\boldsymbol{Q}\,, \qquad n = 3, 4, \ldots . \tag{4}$$

This is the matrix equivalent of the geometric distribution (2); note that $\mathrm{P}\{N = n\}$ is a scalar quantity, since $\boldsymbol{A}$ is a 1×3 matrix, $\boldsymbol{P}$ is a 3×3 matrix and $\boldsymbol{Q}$ is a 3×1 matrix.

The PGF of (4) is

$$\begin{aligned} \sum_{n=3}^{\infty} u^n \boldsymbol{A}\boldsymbol{P}^{n-3}\boldsymbol{Q} &= \sum_{n=3}^{\infty} u^3 \boldsymbol{A}(\boldsymbol{P}u)^{n-3}\boldsymbol{Q} \\ &= u^3 \boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}u)^{-1}\boldsymbol{Q}\,. \end{aligned} \tag{5}$$

We can readily check that when $u = 1$ this gives the sum of the probabilities as 1. For simplicity, let us take $\boldsymbol{P}$ and $\boldsymbol{Q}$ as in (3); then

$$\begin{aligned} (\boldsymbol{I} - \boldsymbol{P})^{-1}\boldsymbol{Q} &= \begin{bmatrix} p & -p & 0 \\ 0 & 1 & -q \\ -q & -p & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ p \\ 0 \end{bmatrix} \\ &= p^{-2} \begin{bmatrix} 1-pq & p & pq \\ q^2 & p & pq \\ q & p & p \end{bmatrix} \begin{bmatrix} 0 \\ p \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \end{aligned}$$

and

$$\boldsymbol{A}\,(\boldsymbol{I} - \boldsymbol{P})^{-1}\,\boldsymbol{Q} = [a_{00}\, a_{01}\, a_{10}] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1\,,$$

The advantage of this method is that if the X_i themselves form a Markov chain, we can solve problems of runs with little additional complication.

4. A sequence of Markov random variables

Suppose the random variables X_i now form a Markov chain, so that

$$\begin{aligned}
&\mathrm{P}\{X_{i+1}=0 \mid X_i=0\}=p_{00}>0, \\
&\qquad \mathrm{P}\{X_{i+1}=1 \mid X_i=0\}=p_{01}>0, \quad p_{00}+p_{01}=1\,, \\
&\mathrm{P}\{X_{i+1}=0 \mid X_i=1\}=p_{10}>0, \\
&\qquad \mathrm{P}\{X_{i+1}=1 \mid X_i=1\}=p_{11}>0, \quad p_{10}+p_{11}=1\,.
\end{aligned}$$

Then the matrix of probabilities for the Y_i in (3) will become

$$\left[\begin{array}{ccc|c} p_{00} & p_{01} & 0 & 0 \\ 0 & 0 & p_{10} & p_{11} \\ p_{00} & p_{01} & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{P} & \boldsymbol{Q} \\ \hline \boldsymbol{0} & 1 \end{array}\right]. \qquad (6)$$

The probability that the sequence reaches 11 for the first time when $N=n$ remains equal to (4), with the PGF as in (5), but with $\boldsymbol{P}$ and $\boldsymbol{Q}$ as in (6). As an exercise, take the initial probabilities to be $\frac{1}{3}$ each, with $p_{01}=p_{11}=p$, and $p_{00}=p_{10}=q$, and derive the matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ when the absorbing states are 01 and 10 respectively. Note that the structure of the matrices will be different for these two cases, and show that the probability of reaching 01 for the first time when $n=5$ is $qp(\frac{1}{3}+\frac{2}{3}q)$, while that of reaching 10 for the first time is $qp(\frac{1}{3}+\frac{2}{3}p)$.

For longer runs such as 1111 or patterns such as 0110, the theory remains the same. The Markov chain approach allows us to find the probability of the length N of the sequence required until the requisite pattern arises for the first time. For readers interested in these problems, the references will provide further information.

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2. J. C. Fu and M. V. Koutras, Distribution theory of runs: a Markov chain approach, *J. Amer. Statist. Assoc.* **89** (1994), pp. 1050–1058.

Joe Gani *is a retired applied probabilist, who is a visiting fellow at The Australian National University, Canberra. He has a variety of interests, among them inference in Markov chains, queueing and dam problems, epidemic modelling and the theory of runs.*

The Problem of the Pastilles

JULIA MOORE and DAVID K. SMITH

Suppose one is walking in a city with a regular rectangular grid plan of roads, and each intersection is marked with an instruction: 'turn left', 'turn right' or 'straight on'. If the instructions are varied, what is the longest closed path that can be found in such a city?

1. Introduction

Consider a city where all roads or streets run either north–south or east–west (or any equivalent pair of perpendicular directions), such as is found in parts of New York. The roads are broken into shorter sections between the cross-roads or junctions where they intersect. What kinds of routes on such roads are of interest to mathematicians? There are many problems defined on such grids. They have given their name to a way of measuring distance, 'the taxi-cab distance', which is the sum of the difference in the north–south co-ordinates and the east–west co-ordinates. It is possible to plan paths which go around such a city, starting and finishing at the same intersection, with specific conditions on the paths. For instance, the optimal postman tour is the shortest one in which each section of each road is included; the travelling salesman's problem is one where each intersection must be visited, again at shortest total distance. There are numerous books devoted to problems of paths on networks; a chapter of the Spode Group's book on decision mathematics (reference 1) has a basic introduction, and Jungnickel (reference 2) gives an up-to-date presentation.

In this article, we are concerned with finding the longest path which meets the following conditions. You can start at any intersection, and must return there at the end. You can choose any section of road that leaves your starting point as your first section. After that, you must choose a set of instructions, one for each intersection that you visit, which tell you what to do when you go through it. The choice at each intersection will be one of the three instructions: 'turn left', 'turn right' or 'straight on'. These will apply to you whenever you visit that point, so both the road that you have used to get there and the instruction will be needed to identify the road that you use to leave it. It will be possible to return to the start during the journey, so some rule to bring the path to an end will be essential. The tour stops if you reach the start and the instruction there is either impossible to obey or would make you retrace your path along a section that you have previously used in the same direction. The first condition is an obvious reason for stopping. The second will mean the path goes over the same sections again (and again).

Summary of the problem. What is the set of instructions, the starting point and first section of road which will yield that longest path on a chosen grid, which returns to the starting point and stops there because further progress is forbidden or impossible?

Unusually for a problem of finding a path in a network of roads, the orientation of the network is an important part of the problem.

This was presented as a challenge to sixth-formers on the web site of a French university (Université Catholique de l'Ouest, in Angers) where the instructions at the intersections were called 'pastilles', roughly translated as 'circular instruction sign'. As there is no ideal one-word translation, we will refer to the pastilles at each intersection. An extensive search of the literature on puzzles, networks and graphs, and on the web, has failed to find any other reference to the problem, and the university which originated the problem knew of none either. It may be known by some other name. A related problem is that faced by designers of mazes; some puzzle mazes have been published which challenge the participant to find a path through a maze where there are restrictions such as those on the pastilles on some junctions. Clearly the pastille problem is concerned with designing a special kind of maze like this, with the turns at each junction fixed. A source of information about all types of puzzle maze is B. J. Gilbert's web page (reference 3). Although this report is concerned with graph and network theory, we have tried to avoid use of technical language and vocabulary.

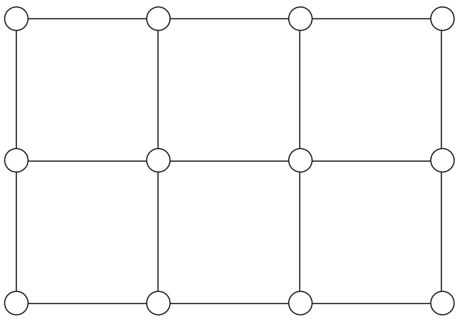


Figure 1. A rectangle with four roads north–south and three east–west; what pastilles should be placed at the 12 intersections? Where should the traveller begin the tour?

Suppose that the road layout is the rectangle shown in figure 1. What pastilles should be placed at each of the 12 intersections? Which should be the starting point? As there are 12 intersections, including those at the corners of the rectangle which are simply places where the direction must change, a naïve suggestion is that there will be 12 (starting points) multiplied by 3^{12} (for the three possible pastilles at each intersection), giving over six million possible solutions to explore. However, the symmetry of the problem means that there are only four 'different' starting points. They are a corner, the middle of the short side, one of the junctions on the long side and one of the central crossroads. Since it will not be possible to go 'straight on' at the corners of the rectangle, there will only be two possible pastilles at these. Finally, for every set of pastilles, there will be a symmetric set, with all the 'lefts' replaced by 'rights'. Use of these observations will reduce the number of solutions to be explored to 52 488, and many of these will not be feasible. But exhaustive search is not an ideal way to tackle such a problem, as we want to be able to find optimal tours in larger, more complex layouts of roads.

2. Systematic approaches to finding the cities which can be toured

A problem such as this can be approached in two ways. First, one can take examples of road networks and try to find the answer to the problem for each one. Alternatively, one can see what paths occur on a grid if one uses a simple set of pastilles, and then develop these paths. The latter approach is more attractive, because it is likely to be more generally useful and easier to extend and generalise.

Closed tours are only possible if there are pastilles with instructions to turn. The model we chose initially was to select a starting point, choose a first section and use the same pastille at each junction. This meant that the traveller walked around a unit square, either anti-clockwise or clockwise, as seen in figure 2. The road sections are numbered in sequence throughout this article, so the starting point can be identified.

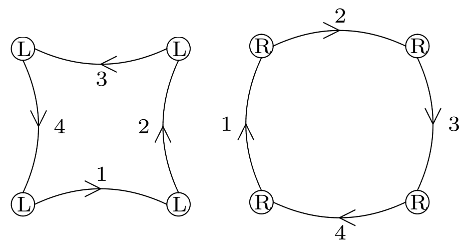


Figure 2. Two equivalent tours of a square. The arrows indicate the direction of travel along the sections; the sections themselves are roads which can be travelled in either direction.

These two paths are really mirror-images of one another. Rather than having to refer constantly to this symmetry, we assume that the first turn instruction which is followed will be 'right'. This may be at the end of the first section of road, or later if there are 'straight' (S) pastilles. Therefore the only path on the square is the right-hand one in figure 2. Given this basic tour, one may consider what happens when exactly one of the pastilles is changed. Obviously, this will take the path outside the square, so we use a simple rule to limit the exploration: all further pastilles should be 'right' turns until the path has completed a closed tour. Figures 3, 4, 5 and 6 show the four possibilities. All other changes are equivalent to one of these.

Several results follow from these. The first is concerned with the patterns which form blocks (figures 4, 5 and 6). In each of these, the perimeter road segments are used once and the interior segments are used twice. Are these tours the longest possible (maximal)? The answer is yes and it is the consequence of two theorems relating to the tours.

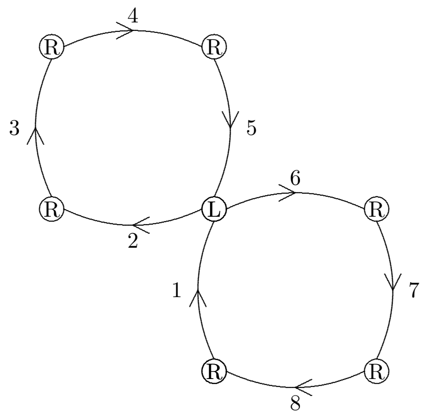


Figure 3. Changing the first instruction from R to L makes a tour around two 'kissing' squares. Equivalent tours result from changing the instruction on the second, third and starting intersections.

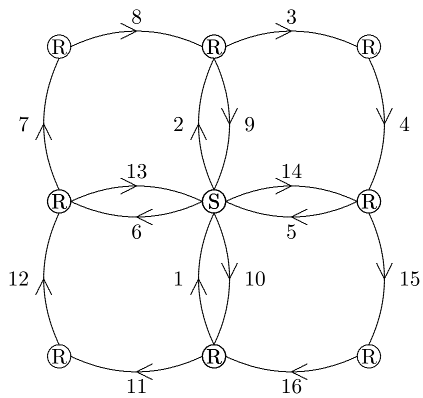


Figure 4. Changing the first instruction from R to S gives a tour of a square with three roads in each direction. This starts in the middle of one side.

Theorem 1. *In an optimal tour, no road segment is used more than once in each direction.*

Proof. Assume that there is at least one road segment which is used twice in the same direction. Suppose that one such repeated segment is the ith and it is repeated as the $(i + n)$th segment in the tour. Now consider the preceding segments for each; they must also be the same, so the $(i-1)$th and the $(i+n-1)$th segments are also used twice. Working backwards, we see that this means that the first and the $(n + 1)$th segments are repeated, which contradicts the stopping rule. Hence the assumption of a road segment being repeated is impossible, and the theorem is proved.

Theorem 2. *In an optimal tour on a rectangular grid, the road segments on the perimeter are only used once.*

Proof. Assume that there is a tour on a rectangular grid which uses a set of one or more road segments twice on the perimeter, once in each direction. Then there must be a segment in this set which starts with a turn. Without loss of generality suppose that it starts with a 'turn right'. The segment before is either an interior road or part of the perimeter. Then, when the segment is used in the reverse

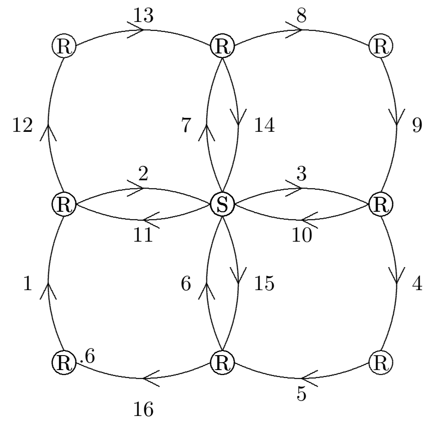


Figure 5. If the second instruction is changed from R to S, then a tour of a three-by-three square is found which starts at the corner.

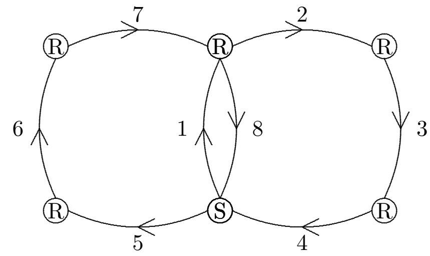


Figure 6. If the instruction on the starting intersection is changed from R to S, then a tour of a rectangle (a domino) with three roads north–south by two roads east–west results.

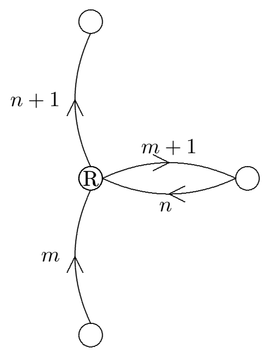


Figure 7. Segment m is either in the interior or on the perimeter of a rectangular pattern. Segment $m + 1$ is on the perimeter and is reused as segment n. The 'turn right' pastille means that the next segment ($n + 1$) will be outside the grid. The other pastilles do not affect this, and are left blank.

direction, at the finish there will be 'turn right'. But this is impossible if the segment is on the perimeter, since the segment which follows cannot be part of the pattern (see figure 7). Hence the original assumption cannot be true.

Using these two theorems one will have the following corollary.

Corollary. *If a tour on any grid uses every perimeter edge exactly once, and every interior edge twice, then this is the longest possible tour for that grid. Accordingly, we have found an optimal tour in these small rectangular shapes, since all the interior segments are used twice and all the perimeters are used once. But how can these results be extended to larger shapes?*

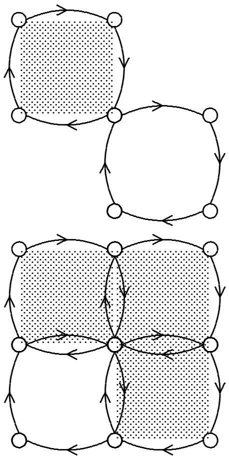
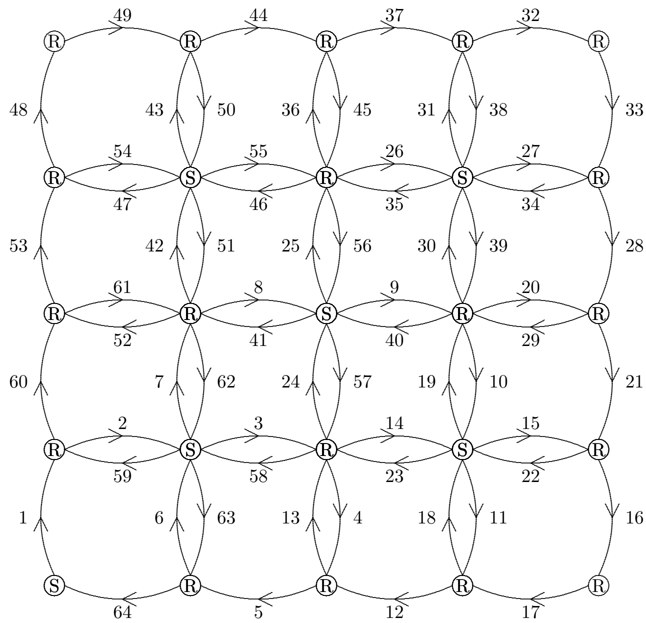


Figure 8. The shaded areas show the extension to the original square shape that is a consequence of the changes in figures 3, 4, 5 and 6.

3. Building with blocks

The extension comes from the simple patterns that we created by changing one pastille on the square. Instead of proceeding by trial and error, we treat these patterns, either as a whole or in part, as building blocks. The extra parts of the patterns seen in figures 3, 4, 5 and 6 are shown shaded in figure 8. The first three extensions can be used repeatedly, adding one 'square' to a corner of a pattern, or adding an 'L-shape' of three squares to a corner (two ways of doing this). Thus, larger patterns can be covered provided that they can be systematically tiled by these shapes. A single square and five L-shapes will tile a square pattern with five roads in each direction, shown in figure 9. However, the number of networks which can be tiled like this is comparatively small.

The initial exploration of modifications to the simple square produced a domino-like shape. This domino, which has three roads north–south by two roads east–west (figure 6), can be extended as a building block in its own right and not as an extension of the square. If one takes the first 'R' instruction and replaces it by 'L', then another domino will be added. This process can be extended indefinitely, so long as one is changing an 'R' flanked by two other 'R' pastilles (figure 10), and the three other pastilles of the domino either go on new junctions or onto existing 'R' pastilles. The two flanking pastilles are essential catalysts for the building block. They do not change when the domino is added. This domino appears to be the best building block for regular patterns, and is very good for others which are less regular. Dominoes can be stacked 'on top' of one another, or one can be placed 'on end' to butt on to a stack of two others. Stacking and butting mean that common edges are traversed twice, while the edges on the perimeter are traversed once. Dominoes can be used to tile any rectangular plan with an even number of squares (an odd number of roads) in at least one direction. With the earlier results about repetition of edges in an optimal tour, these dominoes allow optimal tours to be found in any such plan.



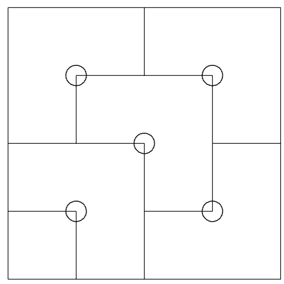


Figure 9. An optimal tour of a square, formed by changing 'R' pastilles to 'S' at the five junctions shown. This is effectively adding five 'L-shaped' pieces to an initial square.

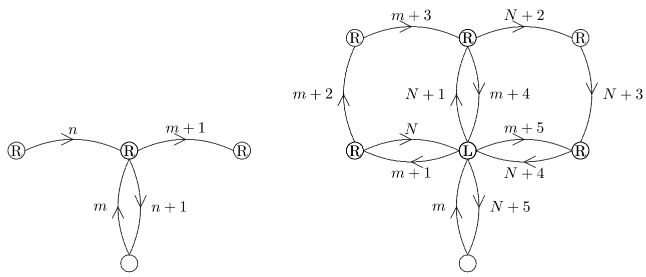


Figure 10. Fitting a domino piece where an 'R' pastille is flanked by two other 'R' pastilles. The central one is changed to an 'L' pastille. The unmarked pastille can be any kind. The right-hand diagram has assumed that $m < n$ and that there are no other changes in the street-plan. Since four extra segments are added to the path each time the 'L' pastille is visited, $N = n + 4$.

However, a four-by-four square cannot be covered; the best that dominoes can achieve is the tour shown in figure 11. (This tour is optimal; to show this, one has to examine all possible sets of pastilles for the square.) Similar incomplete tours are possible in all rectangular networks with even numbers of roads in the two directions, equivalent to an area tiled with an odd number of 'squares'.

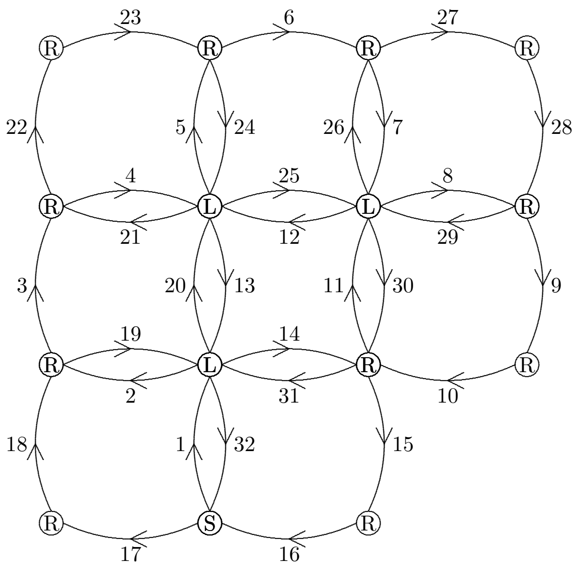


Figure 11. Walk of length 32 in town with 4×4 roads.

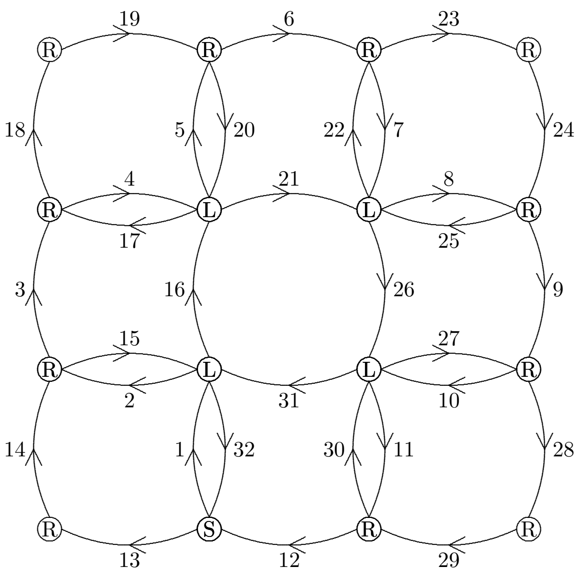


Figure 12. Walk of length 32 with centre sections covered once only.

4. Holes

Construction blocks such as dominoes are a very effective way of creating optimal tours of rectangular networks. They can be used for tours of irregular shapes. However, as noted earlier, they must be joined to two existing edges, so they cannot be used to find tours in all non-rectangular patterns.

Attempting to find an optimal path in the four-by-four square (figure 11) gives a pattern of pastilles which almost possesses fourfold symmetry. The pastilles on the perimeter are of type 'R' (except for the start) and three of the remaining pastilles are of type 'L'. Changing the fourth central pastille from 'R' to 'L' gives the tour in figure 12. Now all the road segments are used at least once, the perimeter is used exactly once, and the segments bordering the central square are also used exactly once. The result is pleasingly symmetric. In effect, this is a tour where the sides of the central square act as if they were on the outside, creating a 'hole' in the grid. There is another way of looking at this pattern. Suppose that one starts with three stacked dominoes. All the roads around the perimeter are used once only, and those inside are used twice. On either long side of this, there will be four 'R' pastilles. 'Behind' the two pastilles in the middle are two 'L' pastilles. The pattern in figure 12 comes from this by changing the central two pastilles on this long side to 'L', and adding 'R' pastilles when necessary.

Looking at the changes to the three stacked dominoes, it will be seen that the change has added some road sections and removed others. The changed roads define a 'T-shaped' set of four squares. This has three squares in a row, and one protruding. The last one is different, because it has cancelled one use of each of three sections of the existing path. These had been used twice, but now are used once. The process is illustrated in figure 13, and the result has been seen in figure 12. This T-shape can be used as a building block as well, and will create larger holes, such as that seen in figure 14. Further, the square with a square hole can be treated as a block by butting two together, with one 'R' pastille replaced by an 'L' pastille, as seen in figure 15.

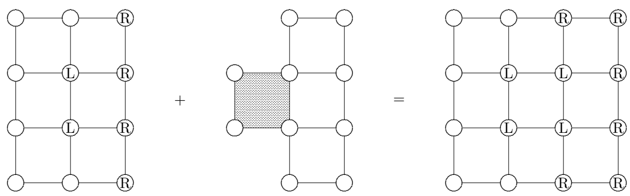


Figure 13. Adding a T-shape; the pastilles marked are essential. The shaded square acts to cancel some of the existing tour.

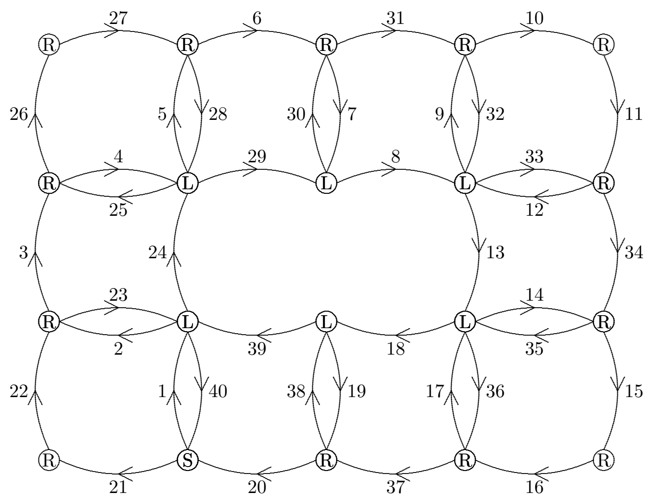


Figure 14. Walk with hole of size of a building block.

The building blocks can be joined together in many cases, and their value lies in the ability they give for identifying grids where a regular optimal tour can be found, and the form of that optimal tour. The simple rules of building a grid, and then changing one pastille, allow for analysis of many grids.

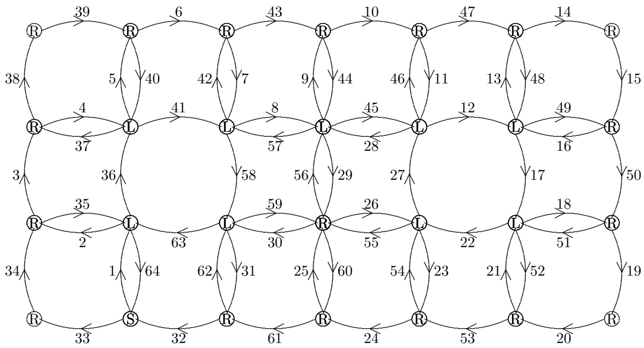


Figure 15. Walk of length 64 with two holes.

5. Conclusion and ideas for further work

The pastille problem is a deceptive tour-finding problem. Like other problems with graphs and networks, it is easy to state. Yet the results are interesting and in places profound. This introduction has produced some of the basic ideas that can be used for finding the answers to tours in general. In particular, we have shown two theorems concerning optimality in rectangular grids, and have established a number of 'building blocks' which can be used to create optimal tours in some grids.

There are unanswered questions. What tests can be used to prove optimality in irregular grids with holes? How can one find tours where some of the road segments can only be used in particular directions? Can the ideas be extended to grids in three or more directions? We have made some initial explorations into pastille problems with triangular and hexagonal grids, and these offer areas for further study.

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2. D. Jungnickel, *Graphs, Networks and Algorithms* (Springer, Berlin, 1999).
3. B. J. Gilbert, web site, available at http://www.bjgilbert.com/ .

Julia Moore *was a mature student at the University of Exeter who graduated in 2000. She enjoyed graph theory and networks and thought that the problem of the pastilles sounded an interesting final year project which would involve these topics. Since two of her children also graduated in the same year, she was able to perform consumer research on university graduation ceremonies!*

David Smith *teaches operational research at Exeter. Besides researching problems of graphs and networks, he enjoys supervising projects and investigating topics which combine serious mathematics with more light-hearted aspects of the subject. Books on the mathematics of puzzles and games take up more space on his shelves than those on graphs.*

A Geometric Approach to Pythagorean Triples

GUIDO LASTERS and DAVID SHARPE

A new look at Pythagorean triples.

Readers will be familar with Pythagoras' theorem for right-angled triangles, namely that the square of the hypotenuse is the sum of the squares of the other two sides—see figure 1.

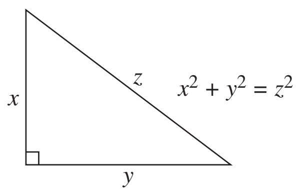


Figure 1. Pythagoras' theorem.

This leads naturally to the definition of a *Pythagorean triple* as a triple (x, y, z) of positive integers satisfying $x^2 + y^2 = z^2$. Examples of Pythagorean triples are (3, 4, 5), (4, 3, 5), (6, 8, 10), (5, 12, 13), (7, 24, 25), (11, 60, 61), and there are many others—in fact there are infinitely many.

Since ancient times, formulae have been known which describe all Pythagorean triples. In this article we shall use a simple geometrical approach to produce such formulae.

If (x, y, z) is a Pythagorean triple and if d is the highest common factor of x, y and z, then we can divide through the equation $x^2 + y^2 = z^2$ by d^2 to give

$$\left(\frac{x}{d}\right)^2 + \left(\frac{y}{d}\right)^2 = \left(\frac{z}{d}\right)^2,$$

and $(x/d, y/d, z/d)$ is a Pythagorean triple in which the three positive integers have highest common factor 1. Such Pythagorean triples are referred to as *primitive Pythagorean triples*. Examples are $(3, 4, 5)$, $(4, 3, 5)$, $(5, 12, 13)$, but not $(6, 8, 10)$. As we have pointed out, an arbitrary Pythagorean triple is just a multiple of a primitive one. Of course, if (x, y, z) is a Pythagorean triple, then so is (kx, ky, kz) for all positive integers k. Hence, we may as well consider just primitive Pythagorean triples. Because of the relation $x^2 + y^2 = z^2$, any common divisor of two of x, y, z will also divide the third so that, in a primitive Pythagorean triple, all pairs of the numbers x, y, z have highest common factor 1. Moreover, x, y cannot both be odd because, if they were, $x^2 \equiv 1 \pmod 4$ and $y^2 \equiv 1 \pmod 4$, whence $z^2 \equiv 2 \pmod 4$, yet a perfect square is congruent to 0 or 1 (mod 4). Thus, with a primitive Pythagorean triple (x, y, z), one of x, y is even and the other is odd, and z is thereby odd. Of course, if (x, y, z) is a primitive Pythagorean triple, then so is (y, x, z), so it is usual to stipulate that x is even and y is odd. However, we shall not do that here for reasons which should become clear.

So much for a general discussion of Pythagorean triples. We now look at the geometrical connection. Let (x, y, z) be a primitive Pythagorean triple. Then $x^2 + y^2 = z^2$, so

$$\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = 1$$

and $(x/z, y/z)$ are the coordinates of a point on the circle C centred at the origin with radius 1. Moreover, this point lies in the first quadrant and does not lie on the coordinate axes; and it has rational coordinates. Further, different primitive Pythagorean triples will produce different points, because the representation of the two rational number coordinates is in lowest terms, and this is possible in only one way.

Conversely, consider a point on the circle C, in the first quadrant, with rational coordinates $(a/b, c/d)$ say, where a, b, c, d are positive integers. Then

$$\left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 = 1,$$

so

$$(ad)^2 + (bc)^2 = (bd)^2$$

and (ad, bc, bd) is a Pythagorean triple. However, this correspondence is not well defined. For example $(\frac{3}{5}, \frac{4}{5})$ will give the Pythagorean triple $(15, 20, 25)$, but $(\frac{6}{10}, \frac{4}{5})$ will give $(30, 40, 50)$. Yet they are the same point! To avoid this difficulty, we divide through by the highest common factor of three numbers to produce a primitive Pythagorean triple. Thus $(\frac{3}{5}, \frac{4}{5})$ and $(\frac{6}{10}, \frac{4}{5})$ both give the same primitive Pythagorean triple $(3, 4, 5)$, and this will be true generally; if we multiply a, b by k (say), this multiplies the triple (ad, bc, bd) through by k, and similarly for the fraction c/d, so the process is well defined.

So now we have reverse processes (or 'mappings', to be precise). The primitive Pythagorean triple (x, y, z) gives rise to the point $(x/z, y/z)$ with rational coordinates on the circle C in the first quadrant, and the point $(a/b, c/d)$ gives rise to the primitive Pythagorean triple $(ad/k, bc/k, bd/k)$, where k is the highest common factor of ad, bc, bd. Moreover, these two mappings are inverses of one another:

$$(x, y, z) \mapsto \left(\frac{x}{z}, \frac{y}{z}\right) \mapsto \left(\frac{xz}{z}, \frac{yz}{z}, \frac{z^2}{z}\right) = (x, y, z)$$

and

$$\left(\frac{a}{b}, \frac{c}{d}\right) \mapsto \left(\frac{ad}{k}, \frac{bc}{k}, \frac{bd}{k}\right) \mapsto \left(\frac{ad/k}{bd/k}, \frac{bc/k}{bd/k}\right) = \left(\frac{a}{b}, \frac{c}{d}\right).$$

Thus, to find all primitive Pythagorean triples, we must find all points on the circle C in the first quadrant which have rational coordinates. There is a neat way to do this. Consider figure 2.

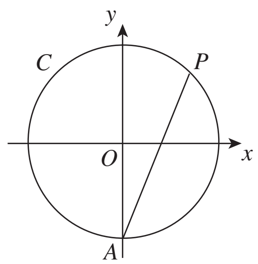


Figure 2.

The point A has coordinates $(0, -1)$, and $P = (x, y)$ is one of the points we are interested in. Since P has rational coordinates, as does A, the straight line AP will have a rational slope greater than 1. Conversely, consider a rational number greater than 1, which can be written as s/t where s, t are positive integers with $s > t$. We can suppose that the fraction s/t is written in lowest terms, so that s, t are coprime. The equation of the line through A with slope s/t is

$$y + 1 = \frac{s}{t}x,$$

and this meets C in the point (x, y), where

$$x^2 + \left(\frac{s}{t}x - 1\right)^2 = 1,$$

that is,

$$\left(1 + \frac{s^2}{t^2}\right)x^2 - \frac{2s}{t}x = 0.$$

The solution $x = 0$ gives the point A. The other solution is

$$x = \frac{2st}{s^2 + t^2},$$

and this is the x-coordinate of the point P where this line again meets the circle C. The y-coordinate of P is now given by

$$y = \frac{s}{t}x - 1 = \frac{s}{t}\frac{2st}{s^2 + t^2} - 1 = \frac{s^2 - t^2}{s^2 + t^2}.$$

Thus, P has rational coordinates

$$\left(\frac{2st}{s^2 + t^2}, \frac{s^2 - t^2}{s^2 + t^2}\right).$$

This describes all the points of C in the first quadrant with rational coordinates. Therefore, all the primitive Pythagorean triples are given by

$$\left(\frac{2st}{k}, \frac{s^2-t^2}{k}, \frac{s^2+t^2}{k}\right),$$

where k is the highest common factor of $2st, s^2-t^2$ and s^2+t^2.

Now s, t cannot both be even since they have highest common factor 1. If one of s, t is even and the other is odd, then s^2-t^2 and s^2+t^2 are both odd. Any prime number p which divides $2st, s^2-t^2$ and s^2+t^2 must therefore be odd and must divide $(s^2+t^2)+(s^2-t^2)$ and $(s^2+t^2)-(s^2-t^2)$, that is, $2s^2$ and $2t^2$, so p must divide s and t. But s, t are coprime. Hence $k=1$ and $(2st, s^2-t^2, s^2+t^2)$ is a primitive Pythagorean triple.

It remains to consider the case when s, t are both odd, say $s=2u+1, t=2v+1$, where u, v are positive integers. Then

$$s^2-t^2=4(u^2-v^2+u-v)=4(u-v)(u+v+1),$$
$$s^2+t^2=4(u^2+v^2+u+v)+2,$$

so $2st$, s^2-t^2, s^2+t^2 are all even. If we divide through by 2, we obtain the triple

$$\big((2u+1)(2v+1),\ 2(u-v)(u+v+1),\ 2(u^2+v^2+u+v)+1\big).$$

Put

$$s'=u+v+1, \qquad t'=u-v.$$

Then

$$s'^2-t'^2=4uv+2u+2v+1=(2u+1)(2v+1),$$
$$s'^2+t'^2=2(u^2+v^2+u+v)+1,$$

so the triple is

$$(s'^2-t'^2, 2s't', s'^2+t'^2).$$

Moreover, one of s', t' is even and the other is odd because $s'+t'=2u+1$, which is odd. If s', t' have highest common factor ℓ (say), we can divide through by ℓ^2 to give the Pythagorean triple

$$(s''^2-t''^2, 2s''t'', s''^2+t''^2),$$

where s'', t'' are coprime and one is even and the other is odd. Hence, by our previous remarks, this Pythagorean triple is primitive. If we drop the double prime, we now obtain the primitive Pythagorean triple

$$(s^2-t^2, 2st, s^2+t^2),$$

where $s>t, s, t$ are coprime and one is even and the other is odd. It is now clear why we could not restrict ourselves to primitive Pythagorean triples (x, y, z) in which x is even and y is odd. For example, the point $(\frac{3}{5}, \frac{4}{5})$ on the circle C gives the Pythagorean triple (3, 4, 5) not (4, 3, 5), which comes from the point $(\frac{4}{5}, \frac{3}{5})$.

We have now proved the famous formula giving all primitive Pythagorean triples (x, y, z) with x even, namely

$$x=2st, \qquad y=s^2-t^2, \qquad z=s^2+t^2,$$

where s, t are coprime positive integers with $s>t$, one even and the other odd. This proof is unlikely to replace a more traditional number-theoretic argument—see reference 1, for example—but it does describe a nice connection between Pythagorean triples and points on the unit circle. And it has a kick in the tail in that the connection is not quite as straightforward as it may appear at first sight.

Reference

1. David M. Burton, *Elementary Number Theory,* 3rd edn (William C. Brown, Dubuque, Iowa, 1994).

Guido Lasters *lives in Tienen, Belgium, and* ***David Sharpe*** *teaches at the University of Sheffield, UK, and is Editor of* Mathematical Spectrum. *They have written a number of articles together with a geometrical flavour for* Mathematical Spectrum.

Spot the errors

(a) $-1=(-1)^3=(-1)^{2\times 3/2}=1^{3/2}=1$, so $-1=1$.

(b) $\cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{6}-\frac{\pi}{3}\right)$, so $\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}-\frac{\pi}{3}$, so $\frac{2\pi}{3}=\frac{2\pi}{6}$, and so $3=6$.

(c) $\int \sin x\cos x\,\mathrm{d}x=\frac{1}{2}\sin^2 x+c$ and $\int \sin x\cos x\,\mathrm{d}x=-\frac{1}{2}\cos^2 x+c$, so $\frac{1}{2}\sin^2 x+c=-\frac{1}{2}\cos^2 x+c$, and so $\sin^2 x+\cos^2 x=0$.

(d) $\lim\limits_{x\to a}\frac{xf(a)-af(x)}{x-a}=\lim\limits_{x\to a}\frac{af(a)-af(x)}{x-a}=a\lim\limits_{x\to a}\frac{f(a)-f(x)}{x-a}=-af'(a)$.

Seyamack Jafari
Iran

Distribution of Batting Scores and Opening Wicket Partnerships in Cricket

A. TAN and D. ZHANG

1. Introduction

The opening wicket partnership (i.e. the partnership between the pair of opening batsmen) is probably one of the most important factors on which the outcome of a cricket match depends (cf. reference 1, pp. 409–412). A solid opening stand lays the foundation upon which a large innings total may be built. A failed opening stand, on the other hand, often leads to a rapid collapse of later batsmen. The most prolific opening pair in the history of test cricket have been Greenidge and Haynes, who scored nearly 3000 runs more than their nearest rivals. Their feat occurred in the years of West Indian dominance in cricket. The most outstanding opening pair were, however, Hobbs and Sutcliffe, who registered 3249 runs from only 38 innings. In this article, we shall study the statistical distribution of the batting scores of a player. We shall then apply the results to analyse the Hobbs–Sutcliffe partnership.

2. The plot of batting scores

We first sort out all the batting scores of a player and draw them as ordinates in ranked order with the innings as abscissae. Thus, the highest score is at innings 1, the second highest score at innings 2, and so on. No distinction is made between scores when the batsman is dismissed (out) or unbeaten (not out). The resulting plot gives a clear picture of the batting record of a player. In this study, the data have been obtained from the internet (reference 2). Figure 1, for example, shows the batting record of Hobbs in test matches.

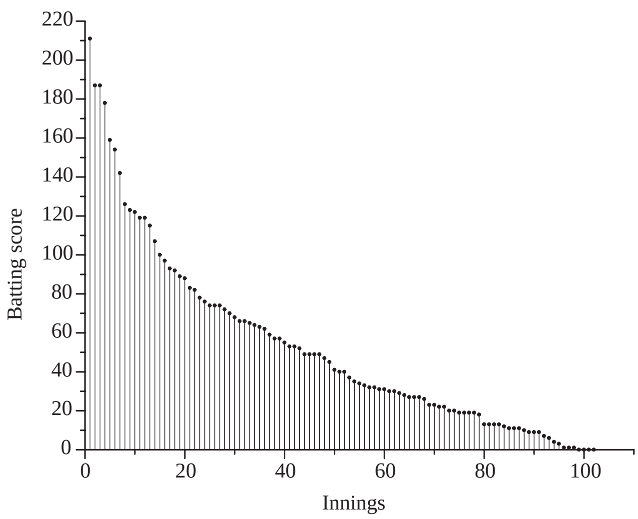


Figure 1. Ranked batting scores (Hobbs).

This plot, even though informative, is not very suitable for mathematical analysis.

3. The batting curve

In figure 2, the abscissae and the ordinates have been interchanged. The value of y at a particular x is now the number of times the score x has been equalled or exceeded. If the number of points is large enough, the discrete distribution tends to a continuous one. This is our batting curve for the player.

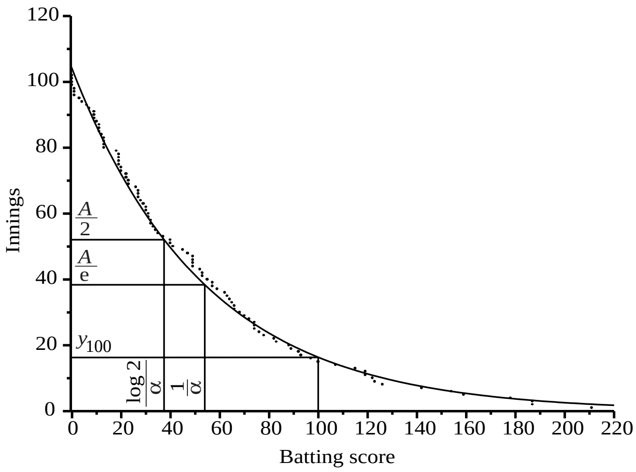


Figure 2. Batting curve (Hobbs); $y = A\mathrm{e}^{-\alpha x}$, $\hat{A} = 104$, $\hat{\alpha} = 0.0185$.

The batting curve thus obtained can conveniently be fitted by an exponential curve given by the equation

$$y = A\mathrm{e}^{-\alpha x}. \tag{1}$$

The amplitude A is generally close to, but may be slightly different from, the number n of innings batted. By analogy with the familiar radioactive decay phenomenon, α is called the decay constant, even though the terminology is not altogether appropriate.

The two constants A and α can be estimated from the data. Taking logarithms of both sides of (1) and summing over n, we get the normal equations from which the estimates $\hat{A}$ and $\hat{\alpha}$ can be obtained by elimination:

$$\log \hat{A} = \frac{\sum x \sum x \log y - \sum \log y \sum x^2}{\left(\sum x\right)^2 - n \sum x^2}$$

and

$$\hat{\alpha} = \frac{n \sum x \log y - \sum \log y \sum x}{\left(\sum x\right)^2 - n \sum x^2}.$$

Figure 2 shows the least-square-fit exponential curve thus obtained for Hobbs with $\hat{A} = 104$ and $\hat{\alpha} = 0.0185$. For convenience, we will now omit the circumflex over A and α.

Geometrically, the area under the batting curve obtained by integration is equal to the aggregate runs, R, scored by the player:

$$R = \int_0^\infty y \, \mathrm{d}x = \frac{A}{\alpha}.$$

The amplitude A, even though usually different from n, is the number of innings batted in this model. Thus, the mean score according to the model is given by

$$x_{\text{mean}} = \frac{R}{A} = \frac{1}{\alpha}. \tag{2}$$

The corresponding ordinate is obtained from (1): $y_{\text{mean}} = A/\mathrm{e}$. The median score is obtained when $y = y_{\text{median}} = A/2$. From (1), we get

$$x_{\text{median}} = \frac{\log 2}{\alpha}. \tag{3}$$

Clearly, from (2) and (3): $x_{\text{mean}} > x_{\text{median}}$ (cf. figure 2).

The numbers of half-centuries, centuries and double-centuries follow directly from (1) and are given by $y_{50} = A\mathrm{e}^{-50\alpha}$, $y_{100} = A\mathrm{e}^{-100\alpha}$ and $y_{200} = A\mathrm{e}^{-200\alpha}$, respectively. Among these, only y_{100} is marked in figure 2. Note that here half-centuries include centuries and the centuries include double-centuries.

4. The probability density

For an exponential distribution, the probability density is given by

$$\rho = \alpha \mathrm{e}^{-\alpha x}. \tag{4}$$

The value of ρ decreases with x, which indicates progressively smaller likelihoods of reaching higher scores. The density ρ is a one-parameter function of x. The parameter α, as we have seen earlier, gives the basic characteristics of the batting record. A batsman having a smaller value of α is expected to score higher than one with a larger α.

The probability density curve has the same shape as the batting curve and can be obtained from the latter by compressing the ordinate by a factor of α/A. The area under the probability density curve is the sum of the probabilities of reaching each score and is unity. This can be verified by integrating (4).

The probability that a certain score n will be equalled or exceeded is given by

$$P_n = \int_n^\infty \rho \, \mathrm{d}x = \mathrm{e}^{-n\alpha}. \tag{5}$$

Thus, the probabilities of making a half-century, a century and a double-century are respectively equal to $P_{50} = \mathrm{e}^{-50\alpha}$, $P_{100} = \mathrm{e}^{-100\alpha}$ and $P_{200} = \mathrm{e}^{-200\alpha}$. The numbers of half-centuries, centuries and double-centuries made are then $y_{50} = P_{50}A$, $y_{100} = P_{100}A$ and $y_{200} = P_{200}A$ respectively, the same results as obtained from (1) earlier.

5. Continuous and discrete distributions

Equation (4) shows that ρ is maximum for $x = 0$, which suggests that, among all possible scores, the probability of scoring 0 (a 'duck') is the highest. However, in order to calculate this probability, we need to bear in mind that the real nature of distribution of scores is actually discrete.

From (5), we find that the probability that a particular score n is equalled or exceeded is P_n and the probability that the score n is exceeded is P_{n+1}; whence the probability that the score n is made is equal to $Q_n = P_n - P_{n+1}$. Carrying out the integrations from (5), we get

$$Q_n = p(1-p)^n, \tag{6}$$

where $p = 1 - \mathrm{e}^{-\alpha}$. Equation (6) shows that the actual distribution of scores is a one-parameter geometric distribution, which is the discrete analogue of the exponential distribution (cf. reference 3, pp. 106–108).

The probability of scoring a duck follows from (6) by putting $n = 0$: $Q_0 = p = 1 - \mathrm{e}^{-\alpha}$. The number of ducks made by a player in his career, according to this model, is $Q_0 A$. This number can also be obtained directly from (1) as $y_0 - y_1$. Further, 0 is the mode of all scores in this model.

The various statistical quantities were calculated for Hobbs ($A = 104$; $\alpha = 0.0185$) and Sutcliffe ($A = 87$; $\alpha = 0.0187$) and entered in table 1. It is noticed that the decay constants for the two batsmen were very nearly the same. Consequently, many of the quantities belonging to the two are similar, only Hobbs batted on more occasions than Sutcliffe. On the whole, the agreement between the model and the actual numbers was quite close. Hobbs scored more ducks and fewer double-centuries than predicted, whereas Sutcliffe scored more half-centuries and centuries but fewer double centuries than predicted.

6. Opening wicket century and double-century stands

The probability of an opening wicket century partnership is calculated as follows. First, approximately 4% of all runs are made up of extras (reference 4). Assuming that both openers scored at the same rate, a century stand for the opening pair is realised when both openers score 48. Now the probability of a batsman reaching 48 is $P_{48} = \mathrm{e}^{-48\alpha}$. Hence, the probability of a century partnership for the opening wicket is given by the product of the P_{48} of each of the two openers:

$$P_{\text{century}} = P_{48}(\text{Opener 1}) \times P_{48}(\text{Opener 2}).$$

Similarly, the probability of a double-century stand between the openers is given by

$$P_{\text{double}} = P_{96}(\text{Opener 1}) \times P_{96}(\text{Opener 2}).$$

The numbers of century and double-century partnerships between two openers are then $P_{\text{century}}N$ and $P_{\text{double}}N$ respectively, where N is the number of times both players opened the innings together. Closely related to these numbers

Table 1. Batting statistics of Hobbs and Sutcliffe.

	Hobbs		Sutcliffe	
	Model	Actual	Model	Actual
Aggregate runs	5621.62	5410	4652.41	4555
Mean score	54.05	53.04	53.48	54.23
Median score	37.47	40	37.07	38
Mode of scores	0	0, 13, 19	0	3, 20, 29
Number of ducks	1.91	4	1.61	2
Number of 50s	41.24	43	34.15	39
Number of 100s	16.35	15	13.41	16
Number of 200s	2.57	1	2.07	0
Number of 300s	0.40	0	0.32	0
Probability of making a duck	1.84%	3.92%	1.88%	2.38%
Probability of reaching 48	41.15%	46.08%	40.75%	47.62%
Probability of reaching 96	16.93%	15.69%	16.61%	20.24%
Probability of making 100	15.72%	14.71%	15.41%	19.05%

are the number of times both openers scored half-centuries and the number of times both scored centuries in the same innings. Following the same argument, they are respectively $P_{50}(\text{Opener 1}) \times P_{50}(\text{Opener 2}) \times N$ and $P_{100}(\text{Opener 1}) \times P_{100}(\text{Opener 2}) \times N$.

These numbers for the Hobbs–Sutcliffe partnership were calculated using the figures from table 1 and entered in table 2. It is of interest to note that the actual numbers far exceed the model predictions which are based on the assumption of random scores. This indicates that the batting scores of Hobbs and Sutcliffe must have had substantial positive correlation. This correlation is not surprising, however. Batsmen often register big scores together on friendly pitches or against weaker bowling. Likewise, they may also bat poorly at the same time against strong bowlers. The latter is particularly true for the opening batsmen, who have to 'see the shine off the new ball'.

Table 2. Century and double-century stands between Hobbs and Sutcliffe.

	Model	Actual
Number of century stands	6.37	15
Number of double-century stands	1.07	2
Number of times both scored 50s in the same innings	5.92	14
Number of times both scored 100s in the same innings	0.92	3
Probability of both reaching 50 in the same innings	15.57%	36.84%
Probability of both reaching 100 in the same innings	2.42%	7.89%

7. Mean and median values of the opening wicket partnership

Estimating the opening wicket partnership is not a straightforward exercise because it depends on the batting curves of both openers and also on how their scores are correlated. Interestingly, we notice that in all the four leading opening partnerships in test cricket (Greenidge–Haynes, Slater–Taylor, Lawry–Simpson and Hobbs–Sutcliffe), the decay constants of the two openers were very nearly equal. Without appreciable error, we may assume them to be exactly equal. Further, the amplitudes of the two batting curves are necessarily equal to N, the number of times they opened the innings together. Thus, the batting curves for the purpose of partnership considerations may be assumed to be the same. The partnership depends upon the overlap of the scores of the two openers and is ended when either batsman is dismissed.

We shall consider three cases of correlation between the batting scores of the two openers. In the first scenario, we assume a maximum positive correlation, in which case the batting curves overlap entirely and the partnership can proceed to the fullest extent. While this is an idealised case which can never be realised in practice, it serves to provide an upper limit to the results of the partnership. In this case, the mean and the median values of the partnership W are respectively twice the mean and median scores of the openers plus extras. Substituting x_{mean} and x_{median} from (2) and (3), we obtain:

$$W_{\text{mean}} = 2\frac{c}{\alpha} \quad \text{and} \quad W_{\text{median}} = 2(\log 2)\frac{c}{\alpha}.$$

Here the constant c takes the extra runs into account: $c = 1.04$.

In the second case, we assume a maximum negative correlation between the batting scores of the openers. In this extreme case, the highest score of one opener coincides with the lowest score of the other. Consequently, many stands end in their infancy. However, this is another hypothetical case, which cannot be realised in practice, but which provides the lower limit for the results of the partnership. In figure 3, ABC and OBE are the batting curves having maximum negative correlation. The aggregate runs scored is twice the overlap area OAB plus extras. Further, in this case there is no distinction between the mean and the median partnerships. Here

$$\text{Area OAB} = \text{Area OABD} - \text{Area OBD}. \tag{7}$$

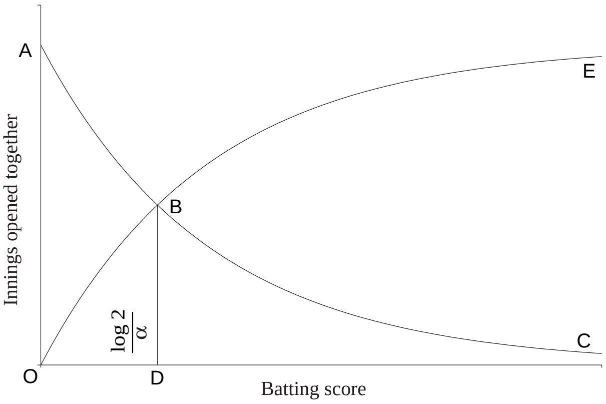


Figure 3. Maximum negative correlation.

The two areas on the right-hand side of (7) are calculated as follows:

$$\text{Area OABD} = \int_0^{(\log 2)/\alpha} N\mathrm{e}^{-\alpha x}\,\mathrm{d}x = \frac{N}{2\alpha}$$

and

$$\text{Area OBD} = \int_0^{(\log 2)/\alpha} N(1-\mathrm{e}^{-\alpha x})\,\mathrm{d}x = \frac{N}{2\alpha}(2\log 2 - 1)\,.$$

Thus,

$$\text{Area OAB} = \frac{N}{\alpha}(1-\log 2)\,.$$

The aggregate of runs scored is $2cN(1-\log 2)/\alpha$ whence, upon division by N,

$$W_{\text{mean}} = W_{\text{median}} = \frac{2c}{\alpha}(1-\log 2)\,.$$

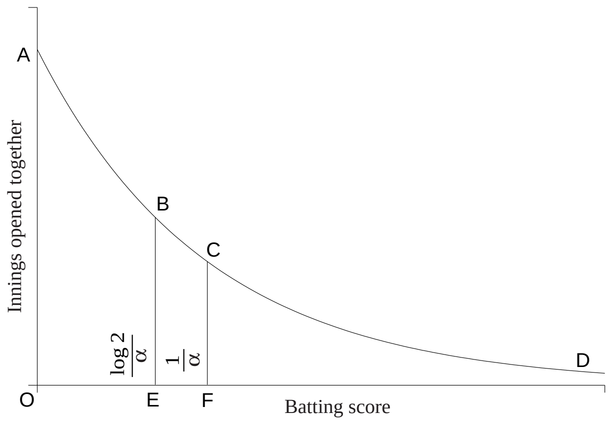


Figure 4. No correlation.

Finally, we consider the case when there is no correlation between the batting scores of the two openers. In figure 4, ABCD is the batting curve of either opener, OF is the mean score $1/\alpha$ and OE is the median score $(\log 2)/\alpha$. For the mean and the median partnerships, the aggregate runs respectively apply to twice the areas OACF and OABE plus extras. As before, we calculate the areas as follows:

$$\text{Area OACF} = \int_0^{1/\alpha} N\mathrm{e}^{-\alpha x}\,\mathrm{d}x = \frac{N}{\alpha}\Big(1-\frac{1}{\mathrm{e}}\Big)$$

and

$$\text{Area OABE} = \int_0^{(\log 2)/\alpha} N\mathrm{e}^{-\alpha x}\,\mathrm{d}x = \frac{N}{2\alpha}\,.$$

Multiplying by $2c$, we obtain the aggregate runs; whereupon, by division by N, we get

$$W_{\text{mean}} = \frac{2c}{\alpha}\Big(1-\frac{1}{\mathrm{e}}\Big) \quad \text{and} \quad W_{\text{median}} = \frac{c}{\alpha}\,.$$

Table 3. Mean and median values of the Hobbs–Sutcliffe partnership.

	Model partnership		Actual partnership	
Correlation	Partnership	Value	Partnership	Value
Maximum positive	Mean	118.83		
correlation	Median	77.52	Mean	85.50
No correlation	Mean	67.97	Median	65.50
	Median	55.91		
Maximum negative	Mean	34.31		
correlation	Median	34.31		

The mean and the median values of the Hobbs–Sutcliffe partnership are calculated for all three cases of correlations and entered in table 3. Here, we have used the mean value of α for the two batsmen: $\alpha = 0.0186$. Also shown in the table are the mean and the median values of their actual partnership, which lie between the cases of maximum positive correlation and no correlation. The numbers indicate that there was a very significant correlation between the batting scores of these two great openers, which only confirms our earlier findings. In fact, the correlation coefficient between the Hobbs and Sutcliffe scores (when they batted together) was 0.448. Similar studies of the Greenidge–Haynes and Lawry–Simpson partnerships also reveal positive correlations between the batting scores, even though the magnitudes of their correlations were not as large.

References

1. M. Manley, *A History of West Indies Cricket* (Andre Deutsch, London, 1988).
2. *Cricinfo,* available at http://www.cricinfo.com.
3. M. Evans, N. Hastings and B. Peacock, *Statistical Distributions* (John Wiley, New York, 2000).
4. K. A. P. Sandiford, private communication.

***A. Tan** is a faculty member at Alabama Agricultural and Mechanical University, where*
***D. Zhang** is a graduate student.*

Mathematics in the Classroom

Resources for teaching statistics

I recently received a book (reference 1) for review. On opening the book, my attention was immediately grabbed by the first article entitled 'Teaching Statistics: More Data, Less Lecturing' and I knew at once that this was a book with which I could entirely empathise.

The book has evolved from the simple observation that there are many students taking a statistics course in a mathematics department which does not have a statistician on the staff. This has forced some mathematicians to become teachers of statistics, often with little formal education in the discipline. Under these conditions it is not surprising that more theory is being delivered than data models presented.

Aware of this, the American Statistical Association in conjunction with the Mathematical Association of America has produced a series of STATS workshops which have focussed on methods for teaching statistics which use activities, software and projects. This book has arisen out of those workshops and offers samplers-with-commentary in which a project is described to a limited extent but sufficient to enable the reader to know if the project is something that he or she wants to know more about.

Some of the ideas to be found in the book are quite novel and are brought to your attention as worthy of further investigation, especially if you want to develop your skills of statistical thinking. Here are some examples of what the book holds.

Weights of race boat crew members

The recent Olympic success of Steve Redgrave has raised our awareness of the feats of boat crews. Questions that could be investigated include:

- How does the weight of British oarsmen compare to other Olympic teamsmen over the years that Steve Redgrave has been winning medals?
- How does the weight of a crew affect boat performance?

Interestingly, we are given the example of a comparison of the 1992 Olympic rowing team and the men in the Cambridge–Oxford boat race. The former had a mean weight of 196 lbs whilst the latter had a mean weight of 181.4 lbs. Successful rowers, we are told, are lean, heavy and tall. Useful web sites are cited, such as http://home.hia.no/∼stephens/rowing.htm and http://users.ox.ac.uk/∼rowing/.

Pence and the central limit theorem

Although students initially think that all distributions are normal, they soon encounter shapes which are not. The age of coins in circulation, for example, is decidedly not normal. A sample of the ages of 25 twenty-pence coins which pass through your pocket will likely provide a histogram which is far from normal. If each member of a class can be encouraged to collect a set of such data and then calculate the mean age of five coins, these means can be plotted as a histogram and conclusions drawn about the shape of their distribution and their mean and standard deviation (relative to the original mean and standard deviation of the population of ages of a single coin).

This can be repeated by changing to the mean of 10 coins and then 25 coins so that changes in the shape, centre and spread of the histogram due to the change in sample size can be seen. At this point students are usually able to state what they think the central limit theorem says.

Real and perceived distances

As one of the most important aspects of data analysis is the study of relationships between variables, this activity explores the ideas of correlation and regression by producing pairs of values which should be related.

The class as a group is taken to a pre-chosen spot armed with clipboard and 50 ft tape measure. They are then asked to guess (independently) the distances from the reference point to 12 landmarks. The group is then divided into teams whose task it is to measure the actual distance to each of the 12 landmarks. Scatterplots are then drawn up to show the relationships between the guesses and the actual distances, and the calibration of the student as a measuring instrument can also be achieved.

Table 1. The lengths of reigns of British kings and queens.

Ruler	Reign	Ruler	Reign	Ruler	Reign	Ruler	Reign
William I	21	Edward III	50	Edward IV	6	George I	13
William II	13	Richard II	22	Mary I	5	George II	33
Henry I	35	Henry IV	13	Elizabeth I	44	George III	59
Stephen	19	Henry V	9	James I	22	George IV	10
Henry II	35	Henry VI	39	Charles I	24	William IV	7
Richard I	10	Edward IV	22	Charles II	25	Victoria	63
John	17	Edward V	0	James II	3	Edward VII	9
Henry III	56	Richard III	2	William III	13	George V	25
Edward I	35	Henry VII	24	Mary II	6	Edward VIII	1
Edward II	20	Henry VIII	38	Anne	12	George VI	15

Designing experiments

This is a useful activity for introducing the concepts involved in experimental design. The class is divided into teams, and each team is given two balsa wood aeroplanes, a paperclip and scissors. Students pick two factors at two levels, each using the materials provided (e.g. presence and absence of paperclip on wings, length of wings, angle of take-off, etc.), that could affect the mean flight distance of the planes. After choosing the factors, they plan and conduct an experiment to find out which level of each factor maximises the flight distance. Avoid conducting the experiment on a windy day though!

Reigns of British kings and queens

The lengths of reign (to the nearest year) of 40 British rulers since William the Conqueror in 1066 forms an interesting data set; see table 1. Producing a histogram and superimposing a normal distribution makes an interesting exercise.

Finally

These activities provide just a taster of what the book contains. It also gives advice on evaluating introductory statistics textbooks and suggests you can check for a good book by looking for data in the exercises—preferably real data. In conclusion it gives a list of World Wide Web addresses for accessing such data. Definitely a good resource for all those involved in teaching statistics.

Carol Nixon

References

1. T. L. Moore (ed.), *Teaching Statistics: Resources for Undergraduate Instructors* (MAA, Washington, 2000).

Computer Column

Mathematics on the web with MathML

The web began as a means for scientists at the CERN particle accelerator to share their ideas, but today it has grown beyond anything they could have dreamed of. Now you can do everything from finding out the latest football scores to ordering a pizza or booking a flight to Australia via the web. It has even spread to mobile phones via WAP, so people can go out on a Saturday night safe in the knowledge that they won't need to wait a second longer than they have to for the lottery results. However, one thing it is surprisingly difficult to do on the web—despite its origins as a scientific tool—is display mathematics. Until very recently, mathematicians have either had to do what they could with standard text or create pictures of their equations for the web. In the last year or two, special plug-ins have become available for web browsers to do proper justice to mathematical notation, but mathematics has still not been supported directly in HTML, the language of web pages.

That is all set to change with the introduction of an extension to HTML called MathML. The original HTML, or HyperText Mark-up Language, was a subset of a more general language called the eXtensible Mark-up Language (XML), which was itself a subset of an even more general language called Standard Generalised Mark-up Language (SGML). With XML, you can define your own descriptive labels (or 'tags') for your data, in a Document Type Definition (DTD). Essentially, HTML is just one particular DTD, with tags defined so that you can store text headings, paragraphs, hypertext links, pictures and so on. If you chose, with the right DTD you could store a database or spreadsheet in XML just as easily. Now, the World Wide Web Consortium (W3C), which oversees the development of the web, has devised a new DTD called—you guessed it—Mathematical Mark-up Language (MathML). This, taken together with HTML, allows mathematical documents to be described and, with the right browser, put up on the web.

MathML comes with two sets of tags: presentational and semantic. The presentational tags are only concerned with how mathematical expressions should be laid out on the page, whereas the content tags try to describe what the expressions actually mean. To see the difference, imagine that you want to describe the equation

$$x^2 - 6x + 9 = 0,$$

perhaps as a prelude to a brilliant proof that its only solution in real numbers is $x = 3$. Appendix A shows how you can represent it using presentational tags, while Appendix B gives one way of doing it using semantic tags.

All the things in angled brackets, like `<math>`, are tags, and usually come in pairs, with a starting tag (`<math>`) and a finishing tag (`</math>`). In the presentational version in Appendix A, the `<mrow>` tags enclose a set of items that should all be typeset along a line, while `<msup>` tags are for sets where each item needs to be set as a superscript of the one before it. The `<mn>`, `<mi>` and `<mo>` tags enclose numbers, variables and operators respectively. (Note the special `⁢` operator, which is needed because numbers and variables must always have an operator between them.) As you can see, the presentational tags simply describe where to put the variables, numbers and operators, without saying any more about them. In addition, anything enclosed between `<mo>` tags is treated as an operator, without worrying what it is.

By contrast, the semantic tags shown in Appendix B simply describe the objects in the equation and the relations between them, leaving it up to the browser to decide how they should be displayed. The <cn> and <ci> tags denote numbers and variables much as before, but the operators are handled quite differently. For example, the <apply> tags define a section where a subsequent tag like <power/> or <plus/> then defines the operator separating the objects. (These tags, with a slash at the end of their name, are exceptions to the rule that tags must come in pairs.) The default operator is equals, which is why there is no <equals/> tag.

The great advantage of the semantic system is that, because it describes what the expression actually *is* mathematically, not how you have chosen to write it down, it is much easier to search automatically through documents for a particular equation or theorem. If the documents had been written using presentational tags, then documents where the same equation had been laid out according to different visual conventions could not be found by a single search. In addition, it means that, once you have told your browser how to display the various operators, you won't have to struggle with other people's ways of writing things down. Unfortunately, though, because it places greater demands on browser software, most implementations of MathML just handle presentational tags at the moment. It is to be hoped that this will change in the future.

Of course, there is much more to MathML than there is space for here; it has been designed to handle anything that mathematicians can think of to throw at it, and some publishers of scientific journals (such as Springer) are working on using it for everything they publish. At the moment, there is very little software that supports MathML, but there are many projects in the pipeline. For example, MOZILLA, an offshoot of the NETSCAPE web browser, handles MathML, as do computer algebra programs like MATHEMATICA, MATHCAD and MAPLE. In addition, a program called MathWriter is being developed for writing and editing MathML visually (without having to resort to writing out all the tags). Watch this space!

Appendix A. Presentational tags

```
<math mode='display'>
   <mrow>
      <mrow>
         <msup>
            <mi>x</mi>
            <mn>2</mn>
         </msup>
         <mo>-</mo>
         <mrow>
            <mn>6</mn>
            <mo>&InvisibleTimes;</mo>
            <mi>x</mi>
         </mrow>
         <mo>+</mo>
         <mn>9</mn>
      </mrow>
      <mo>=</mo>
      <mn>0</mn>
   </mrow>
</math>
```

Appendix B. Semantic tags

```
<math mode='display'>
   <apply>
      <plus/>
      <apply>
         <minus/>
         <apply>
            <power/>
            <ci>x</ci>
            <cn>2</cn>
         </apply>
         <apply>
            <times/>
            <cn>6</cn>
            <ci>x</ci>
         </apply>
      </apply>
      <cn>9</cn>
   </apply>
   <cn>0</cn>
</math>
```

Peter Mattsson

Web links

MathML: http://www.w3c.org/math/
MOZILLA: http://www.mozilla.org
MATHCAD: http://www.mathcad.com
MATHEMATICA: http://www.mathematica.com
MAPLE: http://www.maplesoft.com
Stilo MathWriter: http://www.stilo.com

MATHEMATICA

A WORLD OF NUMBERS . . . AND BEYOND

October 6, 2001 – May 5, 2002

The mid-20th century, classic exhibition, Mathematica, will be showing once more at the Exploratorium, San Fransisco, from 6 October. Designed in 1961 by the world renown modernist designers Charles and Ray Eames during the era of Sputnik, their aim was to let the fun of mathematics and science out of the bag. Mathematica: A World of Numbers and Beyond is the centerpiece of a larger look at both mathematics and design on view at the Exploratorium until 5 May 2002. With Exploratorium additions, expect everything from a pool shark demonstrating the mathematics of billiards to finding your way through a floor maze! Look for special events, demonstrations and lectures throughout the run of the exhibition. More information is available from the Exploratorium website, http://www.exploratorium.edu/.

Letters to the Editor

Dear Editor,

Spiralling to a limit

Farshod Arjomandi asks about the limits of the sequences $\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2$ and $\frac{1}{2}y_n + y_{n+1} + y_{n+2} + y_{n+3} = \frac{3}{2}$ in *Mathematical Spectrum*, Vol. 33, No. 3. The existence of these limits follows immediately from consideration of the auxilliary equation $\frac{1}{2} + t + t^2 + t^3 = 0$. There is a root between $-\frac{1}{2}$ and -1. The remaining two roots are complex (easily seen as there is no turning point) and their product lies between $\frac{1}{2}$ and 1 hence their modulus is less than 1. The general solution involves powers of the auxilliary roots which tend to 0 as $n \to \infty$ since each has a modulus less than 1; and a constant term $\frac{4}{7}$ (for x) and $\frac{3}{7}$ (for y).

Alternatively, the fact that x_n and y_n are bounded and -1 is not an auxilliary root leads to the required result. A third proof avoids the use of the auxilliary equation and involves a discussion of upper and lower bounds of the two sequences, which can be shown to be equal.

Yours sincerely,
ALASTAIR SUMMERS
(57 Conduit Road,
Stamford,
Lincolnshire,
PE9 1QL.)

Dear Editor,

Calculating square roots

Readers may be interested in the following method of finding an approximate value of a square root, using only the four operations $+$, $-$, $\times$, $\div$. (We can suppose that the square root button on our calculator has jammed!) As an illustration, suppose we want to find the square root of 123.456, which we denote by a^2. Thus we want to find a. Put $b^2 = 121$, the perfect square nearest to a^2, so that $b = 11$.

Step 1. Calculate

$$r_1 = \frac{(123.456 - 121)^2}{2(123.456 + 121)} = 0.0123374678.$$

Step 2. Calculate

$$\begin{aligned} r_2 &= 2(123.456 + 121) - 0.0123374678 \\ &= 488.899662532. \end{aligned}$$

Step 3. Calculate

$$r_3 = \frac{(123.456 - 121)^2}{488.899662532} = 0.01233777902.$$

Step 4. Calculate

$$\begin{aligned} r_4 &= \frac{(123.456 - 121) - 0.01233777902}{2 \times 11} \\ &= 0.11107555549. \end{aligned}$$

Step 5. Calculate

$$r_5 = 11 + 0.11107555549 = 11.11107555549.$$

This is our approximation to $\sqrt{123.456}$.

Yours sincerely,
M. E. HARE
(7 Culpins Close,
Spalding,
Lincolnshire,
PE11 2JL.)

[Ed: If these steps are carried out using general terms a, b, then r_5 is seen to differ from a by

$$\frac{(a - b)^6}{3b(3a^2 + b^2)(a^2 + 3b^2)},$$

which is less than

$$\frac{1}{3b(3a^2 + b^2)(a^2 + 3b^2)},$$

since $|a - b| < 1$.]

Dear Editor,

The ladder problem (Problem 33.2)

It is possible to give expressions which yield at least rational values to the various lengths, and these can then be made integral.

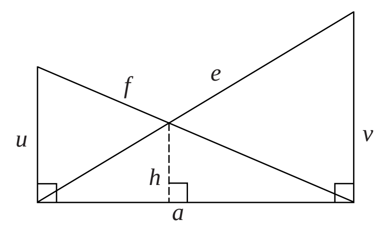


Figure 1.

Denote the lengths as in figure 1. It is known that the sides of a Pythagorean triangle are completely described by $\kappa(m^2 - n^2)$, $\kappa(2mn)$, $\kappa(m^2 + n^2)$, where κ, m, n are positive integers with $m > n$, $\gcd(m, n) = 1$ and exactly one of m and n is odd.

One can easily deduce from figure 1 by considering similar triangles that

$$\frac{1}{h} = \frac{1}{u} + \frac{1}{v}\,.$$

Now $a,\ v,\ e$ are sides of a right-angled triangle, so we put $a = \lambda(m_1^2 - n_1^2),\ v = \lambda(2m_1, n_1),\ e = \lambda(m_1^2 + n_1^2)$. Similarly, we put $a = \mu(m_2^2 - n_2^2),\ u = \mu(2m_2n_2)$, $f = \mu(m_2^2 + n_2^2)$. Hence $a = \lambda(m_1^2 - n_1^2) = \mu(m_2^2 - n_2^2)$. We will take $\lambda = m_2^2 - n_2^2,\ \mu - m_1^2 - n_1^2$. This gives the various values

$$\begin{aligned} a &= (m_1^2 - n_1^2)\,(m_2^2 - n_2^2),\\ u &= 2m_2n_2(m_1^2 - n_1^2),\\ v &= 2m_1n_1(m_2^2 - n_2^2),\\ e &= (m_1^2 + n_1^2)\,(m_2^2 - n_2^2),\\ f &= (m_1^2 - n_1^2)\,(m_2^2 + n_2^2),\\ h &= \frac{2m_1m_2n_1n_2(m_1^2 - n_1^2)(m_2^2 - n_2^2)}{m_1n_1(m_2^2 - n_2^2) + m_2n_2(m_1^2 - n_1^2)}\,. \end{aligned}$$

For example, $m_1 = 2,\ n_1 = 1,\ m_2 = 3,\ n_2 = 2$ yields $e = 25,\ f = 39,\ u = 36,\ v = 20,\ h = \frac{90}{7}$. To make all values integral we multiply by 7 to give $e = 175,\ f = 273$, $u = 252,\ v = 140,\ h = 90$.

Yours sincerely,

K. R. S. Sastry
(Jeevan Sandhya,
Doddakalsandra Post,
Raghuvana Halli,
Bangalore 560 062,
India.)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

34.1 Given positive integers n and k, what is the smallest positive integer s such that, whenever $x_1, \ldots, x_n$ are non-negative real numbers such that $x_1^k + \cdots + x_n^k = 1$, we can always choose s of the x_is whose sum is at least 1.

(Submitted by Hassan Shah Ali, Tehran)

34.2 Simplify the expression

$$\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C}\,,$$

where A, B, C are the angles of a triangle.

(Submitted by J. A. Scott, Chippenham)

34.3 For a positive integer n, $S'(n)$ denotes the smallest positive integer m such that n divides $m!!$, where

$$m!! = \begin{cases} m(m-2)(m-4)\cdots 4\cdot 2 & \text{if } m \text{ is even,}\\ m(m-2)(m-4)\cdots 3\cdot 1 & \text{if } m \text{ is odd.} \end{cases}$$

($m!!$ is called m *double factorial* and S' is called the *Smarandache double factorial function*.) For a prime number p, determine the number of positive integers x such that $S'(x) = p$.

(Submitted by Gilbert Johnson, Church Rock, New Mexico, USA)

34.4 For real numbers $a,\ b,\ c$, determine when

$$(b-c)^2 + (c-a)^2 + (a-b)^2$$

and

$$(2a-b-c)^2 + (2b-c-a)^2 + (2c-a-b)^2$$

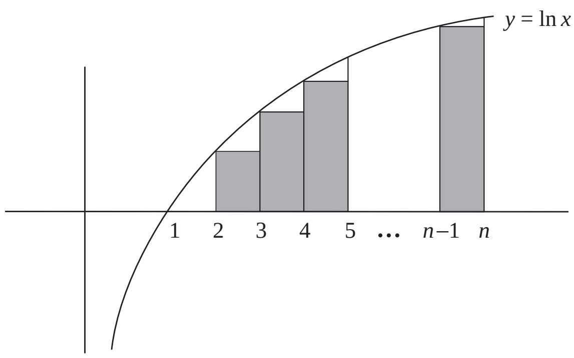
are equal, and generalize.

(Submitted by J. MacNeill, University of Warwick)

Solutions to Problems in Volume 33 Number 2

33.5 Prove that, for all integers $n > 1$, $\ln(n-1)! < n\ln n - n + 1$.

Solution by Daniel Lamy (Nottingham High School)



For a positive integer $n > 1$, the area of the shaded region is less than the area under the curve $y = \ln x$ between $x = 1$ and $x = n$, so

$$\sum_{r=1}^{n-1} \ln r < \int_1^n \ln x \, \mathrm{d}x \, ,$$

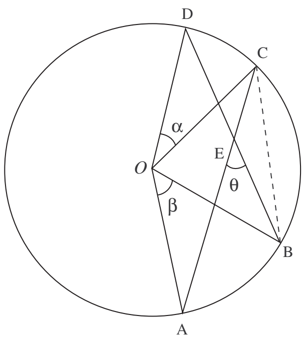
so

$$\ln 1 + \ln 2 + \cdots + \ln(n-1) < [x \ln x - x]_1^n$$

and so

$$\ln(n-1)! < n \ln n - n + 1.$$

33.6



Express θ in terms of α and β. (O is the centre of the given circle.)

Solution by Daniel Lamy

$$\angle\mathrm{CBD} = \tfrac{1}{2}\angle\mathrm{COD} = \tfrac{1}{2}\alpha \, ,$$
$$\angle\mathrm{ACB} = \tfrac{1}{2}\angle\mathrm{AOB} = \tfrac{1}{2}\beta \, ,$$

and

$$\theta = \angle\mathrm{AEB} = \angle\mathrm{CBE} + \angle\mathrm{ECB} = \tfrac{1}{2}(\alpha + \beta).$$

33.7 The 'complement' of a positive integer a is defined to be $10^n - a$, where n is the number of digits in the decimal representation of a. For example, the complement of 975 is 25. Determine the number of n-digit positive integers which are divisible by their complements.

Solution by Daniel Lamy

The complement of a is $10^n - a$. Suppose that this divides a. Then

$$10^n - a = \frac{a}{m} \quad \text{for some positive integer } m,$$

so $m10^n = a(m+1)$, so

$$\frac{10^n}{m+1} = \frac{a}{m}$$

and the complement of a is a divisor of 10^n not 10^n itself. Conversely, consider a positive divisor of 10^n not 10^n itself, which can be written as $10^n/(m+1)$ for some positive integer m. Its complement is

$$10^n - \frac{10^n}{m+1} = \frac{m10^n}{m+1} \, ,$$

which is divisible by $10^n/(m+1)$. Hence the number of positive integers which are divisible by their complements is the number of positive divisors of 10^n excluding 10^n itself. But $10^n = 2^n \times 5^n$ and has $(n+1)^2$ positive divisors, so the number of positive integers which are divisible by their complements is $(n+1)^2 - 1$.

33.8 Let m, n be positive integers with $\gcd(m, (n-1)!) = 1$, and let $a_1, \ldots, a_n$ be integers whose sum is not divisible by m. Prove that there is a permutation $b_1, \ldots, b_n$ of $a_1, \ldots, a_n$ such that none of $b_1, b_1 + b_2, b_1 + b_2 + b_3, \ldots, b_1 + b_2 + \cdots + b_n$ is divisible by m.

Solution by Daniel Lamy

Suppose that all the sums of $n - 1$ of $a_1, \ldots, a_n$ are divisible by m. Then, for $1 \le k \le n$,

$$a_k \equiv a_1 + \cdots + a_n \pmod m,$$

so, adding together any $n - 1$ of these, we have

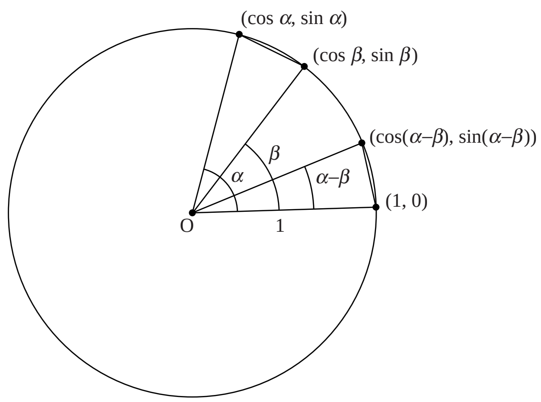
$$(n-1)(a_1 + \cdots + a_n) \equiv 0 \pmod m.$$

But $m, n - 1$ are coprime, so

$$a_1 + \cdots + a_n \equiv 0 \pmod m,$$

which is not so. Hence there is a k such that $a_1 + \cdots + a_{k-1} + a_{k+1} + \cdots + a_n$ is not divisible by m. We can now repeat this argument with $a_1 + \cdots + a_{k-1} + a_{k+1} + \cdots + a_n$ in place of $a_1 + \cdots + a_n$. The result follows after $n - 1$ steps.

Deriving a trigonometric formula



Use this diagram to derive the well-known trigonometric formula

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Guido Lasters
Tienen, Belgium

Reviews

Niels Henrik Abel and his Times. By ARILD STUBHAUG. Springer, Berlin, 2000. Pp. 580. Hardback £27.00 (ISBN 3-540-66834-9).

This book is principally the sad story of a young Norwegian mathematical genius who died just when his dazzling gifts came to be universally acknowledged. But a biography is incomplete unless it tells the reader something about the setting in which the subject's life was lived, and many aspects of Abel's times are here painstakingly recreated.

Niels Henrik Abel (1802–1829) was the second son in a large family. The mother came from a wealthy merchant family, while the father, Søren Georg Abel (1772–1820), was a Lutheran pastor deservedly popular with his parishioners. However, his interest shifted from parish to national affairs, and his career was ruined when he got involved in a bitter and pointless dispute. Heartbroken, he now drank himself into an early grave. His widow was by then also an alcoholic, so the family's condition suddenly changed from comfort to penury.

At the age of 13, Niels Henrik was sent to the Christiania[1] Cathedral School where he was supported by a scholarship and where his mathematical gifts were discovered by his young teacher Bernt Michael Holmboe. Abel not only read voraciously, he also tackled ever more difficult problems. In his last year at school he even took on the famous problem of the solution of polynomial equations by radicals, which means that addition, subtraction, multiplication, division and the extraction of nth roots are the only permitted operations on the coefficients. It was well known that equations up to degree 4 could be solved under this constraint, but the question of the quintic had been open for 300 years. Abel now failed in his search for a solution, but when he returned to the problem three years later he showed, sensationally, that the long-sought solution of the general quintic did not, in fact, exist.

In 1811 a new university was founded in Christiania, the first in Norway. Abel entered it in 1821 and, being without means, he was given a free room in the university residence. However, there were no scholarships, so some of the professors paid for his subsistence.

It was not long before Abel had exhausted the university's teaching ability. The university therefore awarded him a two-year foreign travel grant, and in September 1825 he set out for the Continent. In Berlin he met A. L. Crelle (1780–1855) an engineer with a passion for mathematics who was about to launch a new mathematical journal; an invitation to contribute resulted in Volume 1 containing seven papers by Abel.

At this time, in a letter to Holmboe, Abel remarked on the few cases in which the sums of standard series had actually been rigorously obtained; and he went on to rectify this deficiency in a long paper mainly on power series $\sum_{n=0}^{\infty} a_n x^n$.

Abel next travelled to Paris where he presented his favourite paper to the French Academy on 30 October 1826; A. L. Cauchy and A. M. Legendre were appointed referees. Abel had hoped for a quick decision, but in fact no more was heard of the paper in his lifetime. Troubled also by a persistent cough he consulted a doctor, to be told that he had tuberculosis.

In May 1827 Abel returned to Christiania, practically penniless and subsisting on small loans from his friends, but mathematically fertile as ever.

So-called elliptic integrals are similar to those which measure arc length along the circumference of an ellipse. Noting that the relation

$$u = \int_0^x \frac{1}{\sqrt{1-s^2}}\,\mathrm{d}s, \qquad (1)$$

i.e. $u = \sin^{-1} x$, is best explored in the inverse form $x = \sin u$, Abel divined that the same observation applies when the integrand in (1) is of the type occurring in elliptic integrals. Abel called the resulting functions of u elliptic functions, and the study of these transformed the subject.

Abel had a standing invitation to spend the university vacations with the family that employed his fiancée as governess; and despite his deteriorating health he insisted on accepting the Christmas 1828 invitation. He now had a severe lung hæmorrhage while waiting for the sledge to take him back to Christiania, and he died three months later, in April 1829, at the age of 26.

Abel always aspired to a permanent position which would provide him with the security he had never known and enable him to marry his patient fiancée. So the final irony was that two days after his death Abel was offered a prestigious permanent post in Berlin.

Next, here are some glimpses into the rest of the book:

Abel and several friends who also had travel grants did not spend all their time on academic pursuits; they did a great deal of sightseeing, rightly regarding it as part of their education. The whole of Part VI of the book is a perceptive account of their travels. An incident related by the gentle Abel, without comment, is that the coach in which he was travelling ran over a young boy and continued on its way without stopping (p. 421).

A very different attitude, even towards animals, had been promoted by Søren Georg Abel, namely 'to handle them properly and humanely, submitting them neither to blows nor excessive labour, for heartlessness towards animals can easily lead to cruelty to people' (p. 87).

In Christiania University there was a student society rather like a modern Students' Union. In Abel's time a burning issue was whether the students should adopt a uniform (pp. 261–264).

There is an appendix on Abel's collected works and on the extraordinary story of his great Paris paper (pp. 548–552).

After reading a biography one is often left wondering

[1] Christiania was renamed Oslo in 1925.

what happened to some of the more minor characters, but this author is kinder to his readers: a good deal of the appendix is devoted to the later lives of these figures.

Finally, some general remarks:

- The illustrations are very good.
- Unfortunately the English text is often unidiomatic.
- A general index, not confined to names, would be a great help.
- This is a monumental work; it will surely be the standard biography of Abel for many years to come.

University of Sheffield H. BURKILL

Introducing Statistics. By GRAHAM UPTON AND IAN COOK. Oxford University Press, 2001. Pp. 500. Paperback £17.50 (ISBN 0-19-914801-5).

Written to coincide with the 2001 single A-level Mathematics syllabus, this new edition of a familiar text provides a comprehensive backup to the standard textbooks written specifically for certain syllabuses. The way it is set out and its sheer volume make it initially daunting, but it provides a good reference, and this, alongside extensive examples and exam-type questions, makes it a useful revision aid.

The inclusion of biographies, and suggestions for practical work that involves both calculator and computer exercises, adds novel interest to the book. While it was not sufficiently detailed for my needs as a student studying Further Mathematics, it aims for a wide-ranging cover of statistics which is both useful and accessible.

Student, Solihull Sixth Form College JOHN ACKRILL

Introducing Pure Mathematics. By ROBERT SMEDLEY AND GARRY WISEMAN. Oxford University Press, 2001. Pp. 550. Paperback £20.00 (ISBN 0-19-914803-1).

This is a new edition of an already available book. It covers a lot of the basics needed for the core pure mathematics of an A-level course and has two opening chapters on algebra and geometry which are designed to support those students with a grade C at GCSE by preparing them for the rigours of A-level. It is set out clearly in chapters whose titles indicate which topics are being addressed, so it is easy to find your way around the text.

Each topic is broken down into smaller, manageable-size chunks, this allowing easy access for the user who just wants to cover specific topics. Each is presented clearly with examples, explanations and exercises which guide you through the topics under consideration. Answers are also provided to check work that has been done, so progress can easily be monitored.

The language used throughout the text is simple and the vocabulary used is not too challenging, making the book easy to read and understand.

Having learned my pure mathematics from three relatively small and self-contained textbooks which have been focussed directly on the three compulsory pure maths modules of the syllabus that I am following, I feel that learning solely from this textbook would be a daunting task and not easily done. But as an extra aid to classroom teaching it provides an excellent overview of topics to enhance understanding and gives an additional welcome source for revision.

Student, Solihull Sixth Form College SARAH RUSSELL

Other books received

Mathematical Olympiads: Problems and Solutions from Around the World 1998–1999. Edited by TITU ANDREESCU AND ZUMING FENG. MAA, Washington, 2000. Pp. 280. Paperback $28.50 (ISBN 0-88385-642-5).

This book is a continuation of *Mathematical Contests 1997–1998*. It contains solutions to challenging problems from algebra, geometry, combinatorics and number theory featured in the earlier book, together with selected questions (without solutions) from 30 national and regional Olympiads given during 1999.

Proofs Without Words II. By ROGER B. NELSEN. MAA, Washington, 2000. Pp. 142. Paperback $24.00 (ISBN 0-88385-721-9).

This book will appeal to all who think that mathematics should be done without the use of words.

Foundation GCSE Mathematics. By MARK BINDLEY. Oxford University Press, 2001. Pp. viii+488. Paperback £10.00 (ISBN 0-19-914793-0).

Do Brilliantly: AS Maths. By JOHN BERRY, TED GRAHAM AND ROGER WILLIAMSON. Collins, Glasgow, 2001. Pp. 128. Paperback £5.99 (ISBN 0-00-710702-1).

Do Brilliantly: GCSE Maths. By PAUL METCALF. Collins, Glasgow, 2001. Pp. 128. Paperback £5.99 (ISBN 0-00-710490-1).

Instant Revision: GCSE Maths. By PAUL METCALF. Collins, Glasgow, 2001. Pp. iv+124. Paperback £3.99 (ISBN 0-00-710972-5).

Total Revision: GCSE Maths. By PAUL METCALF. Collins, Glasgow, 2001. Pp. vi+346. Paperback £9.99 (ISBN 0-00-711202-5).

Do Brilliantly: KS3 Maths. By KEVIN EVANS AND KEITH GORDON. Collins, Glasgow, 2001. Pp. vi+122. Paperback £5.99 (ISBN 0-00-711211-4).

Total Revision: KS3 Maths. By KEVIN EVANS AND KEITH GORDON. Collins, Glasgow, 2001. Pp. vi+250. Paperback £9.99 (ISBN 0-00-711208-4).

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