Mathematical Spectrum

2007/2008 Volume 40 Number 3

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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing. The copyright of all published material is vested in the Applied Probability Trust.

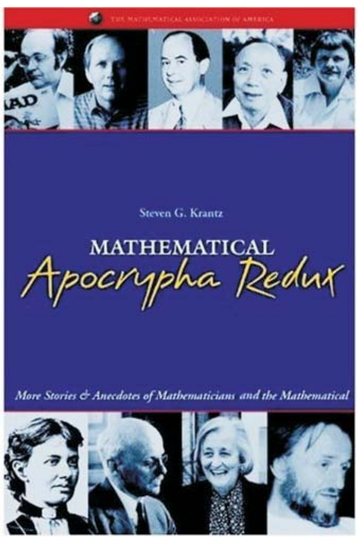
From the Editor

People Matter

Mathematics is about ideas, the ultimate in abstraction. What has it got to do with people? Well, for one thing, it has profoundly affected our lives. Without mathematics there would be no science or technology; we would still be in the Dark Ages! For another, behind mathematics there are mathematicians, who invented (or discovered?) it. What sort of people are they? Steven Krantz has compiled a follow-up volume to his *Mathematical Apocrypha* in which he continues with his anecdotes about mathematicians. It helps if you are a working mathematician who may recognize a good number of the names and mathematical terms, and if you are American. But even if you aren't, a dip into its pages will fill in odd minutes agreeably. The general impression is that mathematicians are a strange lot, so be warned if you are contemplating becoming one! Here is a taster to whet your appetite.

St Augustine in the fourth century AD advised that 'all good Christians should avoid contact with mathematicians, for they and others of empty prophecies are in league with the devil and lead us into darkness' (p. 267). Even if he was describing astrologers rather than those we would recognize as mathematicians, it can feel like that when you are struggling to understand a piece of mathematics or solve a problem.

Paul Erdős, who died in 1996, is a great subject for anecdotes, and features prominently. He was an itinerant mathematician. He never owned or rented a home, didn't have a driver's licence, and didn't have a credit card. He habitually would show up on the doorstep of a friend or collaborator anywhere in the world, declare that 'My brain is open', and expect to be fed, housed, and clothed. His motto was 'Another roof, another proof'. When he lost the sight in his right eye, he said 'Now I will have less distraction', a reminder of the great 18th century Swiss mathematician Leonhard Euler. When Erdős was detained by the police for loitering and asked to account for his activities, he offered up one of his maths papers – and



they accepted it! (pp. 251–252). When told that a friend of his had shot and killed his wife, he said 'Well, she was probably interrupting him when he was trying to prove a theorem' (p. 4). He even posed a problem in *Mathematical Spectrum* and offered a prize for a solution; no one claimed the prize!

Did you know that Karl Marx wrote a calculus book? *Das Kapital* wouldn't feed his family! (p. 182). Or that Albert Einstein played in string quartets with his friends? When he failed for the fourth time to get his entry right, the cellist said 'The problem with you, Albert, is that you simply cannot count' (p. 141).

Marilyn vos Savant has a popular column *Ask Marilyn* in a newspaper. A reader wrote in to ask what has become known as the *Monte Hall problem*. Monte Hall has a television game show *Let's Make a Deal*. You are a contestant on the show. You face the stage on which are three doors. Behind one door is a Cadillac, and behind the other two doors are goats. You are to choose a door, and you get as your prize whatever is behind that door. You choose door 2, and wait to see what is behind it.

While you are waiting, Monte Hall teases you by commanding that his assistant opens door 1, revealing a bleating goat standing there. Then he asks you, 'OK, one of the remaining doors – 2 or 3 – has a goat behind it and one has the desirable Cadillac. You have chosen door 2. Based on what I have just shown you, would you like to change your pick to door 3? What should you do?

According to Krantz, when Marilyn vos Savant gave her answer, over 2000 academic mathematicians wrote in to say she was wrong, some not too politely. But she was right! What advice did she give?

In case you are put off from an intention to study maths (or 'math' in American!), the story on page 119 might reassure you. The head at a local school in Arizona decided to remove all the controversial books from the library. When she had finished with her local version of the inquisition, nothing remained on the shelves but mathematics textbooks!

Reference

1 Steven G. Krantz, *Mathematical Apocrypha Redux: More Stories & Anecdotes of Mathematicians & the Mathematical* (Mathematical Association of America, Washington, DC, 2005).

Is this correct?

$$\begin{aligned}\lim_{x\to a}\frac{xf(a)-af(x)}{x-a} &= \lim_{x\to a}\frac{af(a)-af(x)}{x-a}\\ &= -a\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\\ &= -af'(a).\end{aligned}$$

Tehran, Iran **Seyamack Jafari**

On Triangular Circles

ALEKSANDER MATUSZOK

Ordinary circles

When we read the word *circle* most of us think of figure 1, a circle in the Euclidean metric. Only those who understand its definition as 'the set of all points whose distance, called the radius of the circle, from a point, called its centre, is constant and bigger than 0' and know the definition of the term 'distance' may imagine that an arbitrary circle does not have to look like this.

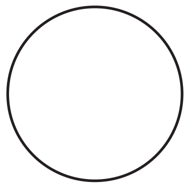


Figure 1

Distance or, in other words, a *metric* (see reference 1), is any function which satisfies the following three conditions (see reference 2):

- The distance between two points is greater than or equal to 0 and is equal to 0 if and only if these two points are equal.
- The distance from a point A to a point B is the same as the distance from B to A.
- Distance satisfies the so called *triangle inequality* condition which states that, for any three points A, B, and C, the sum of the distances from A to B and from B to C is greater than or equal to the distance from A to C.

Because of this definition, how a circle looks depends on the definition of distance (or metric, which means the same).

Natural distances and uncommon circles

Try to imagine that you find yourself in a city where all streets cross each other at an angle of $90°$. Let us say that we are in downtown Chicago, for example. We want to know how far we can go when we start our trip from a certain point, called the centre, if we have a given sum of money for a taxi. We assume that the taxi driver knows the city very well and will select the best possible way, and that there are no one-way streets. We may say that the range of such a trip defines a circle using a special distance which is generated by the fact that we find ourselves in Chicago. It will look like the one in figure 2. Here, C is the centre of this circle, from which all possible routes go reaching the points of our circle. The distance defined in this

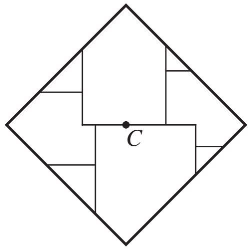


Figure 2

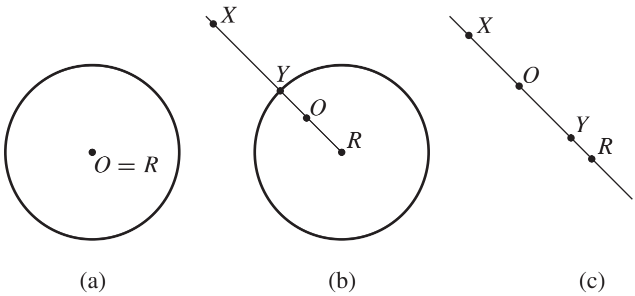


Figure 3

way is truly a distance function, which means that it satisfies the three given conditions. We call this the *taxi driver's metric*. So we have found a metric in which circles look like squares in the Euclidean metric. There are metrics in which not all circles look the same. There are also other metrics which produce square-like circles, for example the Tchebyshev metric (see reference 3).

The well-known saying 'all routes lead to Rome' was supposedly true in the Roman Empire. This saying also defines a distance function, so this metric is sometimes called the *Roman metric*. Fix a point R in a plane, called the centre of the metric. If you want to measure the distance between two points, you first check if these two points and the centre lie on the same line. If they do, the distance is simply the Euclidean distance between these two points. If they do not, apply the rule which speaks about Rome – to measure the distance first get to the centre from the first point, and then get from the centre to the second point. The Euclidean length of these two Euclidean segments is the distance in question. The circle with the centre R and a given radius will look exactly the same as a circle in the Euclidean metric. However, if we consider a circle in this metric with a different centre point, it will look quite different. In figure 3 we present examples of circles in the Roman metric. Figure 3(b) shows a circle with its centre at the point O for which the distance between O and the centre of the metric R is smaller than the radius r of the circle. It consists of a Euclidean circle with centre R and radius $r - OR$, but with the point Y replaced by X, where $OX = r$. Figure 3(c) shows a circle with centre O and radius r where $OR \geq r$. It consists of the two points, X and Y, where $OX = OY = r$.

Imagine that we have found ourselves in a jungle. All we know is that there is a river in this jungle (a straight line on a plane). To get from one point to another we must first reach the river

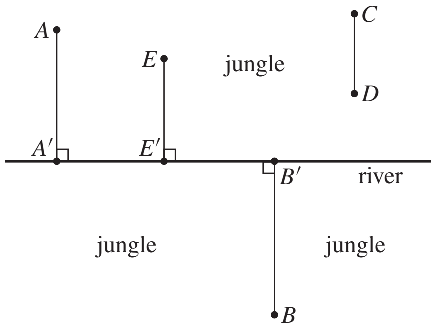


Figure 4 Here, distance$(A, B) = AA' + A'B' + B'B$, distance$(C, D) = CD$, distance$(A, E) = AA' + A'E' + EE'$, and distance$(A', E') = A'E'$.

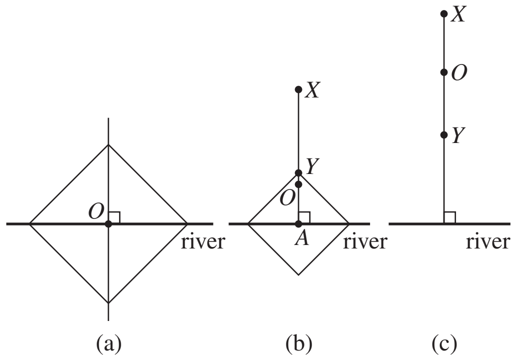


Figure 5

then move along the river and once we are directly opposite the final point, go directly to this point. If the two points are on the same line at right angles to the river and on the same side of the river, we do not go to the river at all. If the two points lie on the river, we move from one to the other along the river. The map of the jungle would look like figure 4. This metric is sometimes called the *river metric*. The reader can check that this is a metric. The reader can also try to sketch what circles look like in this metric when their centres lie on the river and when they lie off the river. If their centres lie on the river, they look exactly the same as circles in the taxi driver's metric, which means that they are square-like circles (see figure 5(a)). If the centre O lies off the river and the distance *OA* between the centre and the river is smaller than the radius r of the circle, then the circle consists of the square in figure 5(b), where $AY = r - OA$, with Y replaced by X, where $OX = r$. If the centre O lies off the river and the distance between the centre and the river is greater than or equal to the radius of the circle, then the circle consists of only the two points, X and Y, shown in figure 5(c), where $OX = OY = r$.

Circles which will change your point of view on circles

It turns out that circles may be even stranger. There are metrics in which they can become Euclidean polygons. How do we find such metrics? Let us fix a point on a plane, say O. We

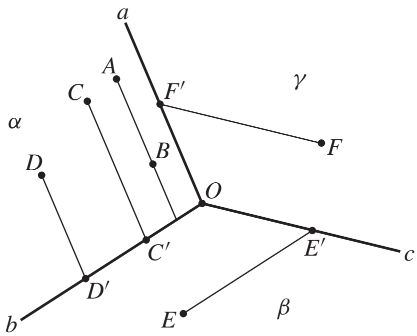


Figure 6

draw three half-lines, a, b, and c, from this point which split the plane into three angles, α, β, and γ, each less than $180\,^{\circ}$(see figure 6). We can say more descriptively that point O is a *spring* from which three *rivers* flow. In reference 4, I introduced this new metric by naming it the n*-spring*, where $n > 2$ is the number of lines going out from the point O or *rivers* flowing from a *spring* O. We define the distance between two points in the following way. If these points, say A and B, lie in the region of the angle α and lie on a Euclidean line parallel to the half-line a, then the distance from A to B is equal to the Euclidean length of the segment AB. If these points, now C and D, do not lie on such a line and do not lie on the half-line b then, if C' is the image of C in the projection to the half-line b along the half-line a, and D' is the image of D in the same projection, the distance from point C to D is equal to the Euclidean distance $CC' + C'D' + D'D$. If these points, A, B, C, and D, lie in the regions of the other angles, β or γ, then the distance is defined similarly: we use the projection to the half-line c along the half-line b in the case of the region of the angle β and the projection to the half-line a along the half-line c in the case of the region of the angle γ. If two points lie on the same half-line a, b, or c, then the distance is simply the Euclidean distance; for instance, the distance from D' to C' is equal to $D'C'$. If two points, D and E, lie in two different regions of two different angles, for example D is in the region of the angle α and E is in the region of the angle β, and E' is the image of the point E in the projection to the half-line c along the half-line b, then the distance from D to E is equal to the Euclidean distance $DD' + D'O + OE' + E'E$. If points D and F lie in the regions of the angles α and γ respectively, then the distance from D to F is equal to the Euclidean distance $DD' + D'O + OF' + F'F$, where the point F' is the image of the point F in the projection to the half-line a along the half-line c.

This new metric satisfies all three conditions. To prove the triangle inequality it is enough to consider the cases where

- all points lie in the same region,
- two points lie in the same region, the third lies in a different region,
- all three points lie in three different regions.

What does a circle look like in this metric? Let us consider only a circle radius r whose centre is point O. Such a circle is a Euclidean triangle (see figure 7) where $OK = OL = OM = r$.

How this triangle looks depends on how you draw lines from a point O. It is only necessary that the angles between these lines must be less than $180\,^{\circ}$. If you draw more lines from O, say

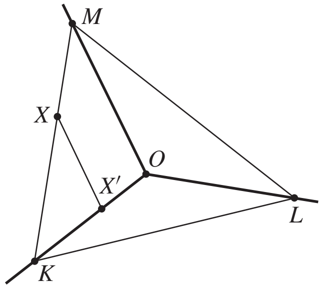


Figure 7 Here, $OK = OL = OM = XX' + X'O = KX' + X'O = r$.

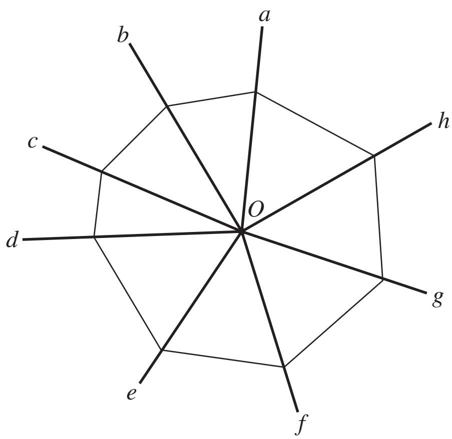


Figure 8

n lines where n is any number bigger than 2, you can define a distance in a similar way. This new n*-spring* metric produces polygon-like circles whose centre is O (see figure 8). The only constraint is that the angles between two adjacent half-lines must be less than $180°$ to enable a projection which is crucial in the definition of the n*-spring* metric to be defined.

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Aleksander Matuszok *is especially interested in geometry and computer graphics. Currently he works as a computer systems engineer at The National Fund for Environmental Protection and Water Management in Warsaw.*

Generalised Binet Formulae

PHILIP MAYNARD

Let $a, b \in \mathbb{R}^+$ be fixed. Then we consider a natural generalization of the Fibonacci sequence defined by

$$f_0^* := 0, \qquad f_1^* := 1, \qquad f_n^* := af_{n-1}^* + bf_{n-2}^* \quad \text{for } n \geq 2. \tag{1}$$

The case $a = b = 1$ gives the Fibonacci sequence f_n. In this case it is well known that

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right],$$

which is known as the *Binet formula*. We show that such expressions exist for the generalized Fibonacci sequence. Now (1) is just a linear second-order difference equation and could be solved from this point of view. However, in the following lemma we use simple mathematical induction.

Lemma 1 *Let $a, b \in \mathbb{R}^+$ be fixed. Then*

$$f_n^* := \frac{\alpha^n - \beta^n}{\sqrt{4b + a^2}},$$

where $\alpha = a/2 + \frac{1}{2}\sqrt{4b + a^2}$ and $\beta = a/2 - \frac{1}{2}\sqrt{4b + a^2}$.

Proof First, note that α and β are roots of the quadratic equation

$$x^2 = ax + b. \tag{2}$$

With f_n^* given by the formula

$$f_0^* = 0 \quad \text{and} \quad f_1^* = \frac{\alpha - \beta}{\sqrt{4b + a^2}} = 1,$$

we now proceed by induction by assuming the truth of the formula for all integers less than or equal to some positive integer k. Then we have

$$\begin{aligned} f_{k+1}^* &= af_k^* + bf_{k-1}^* \qquad \text{(by definition)} \\ &= \frac{a\alpha^k - a\beta^k + b\alpha^{k-1} - b\beta^{k-1}}{\sqrt{4b + a^2}} \qquad \text{(by the inductive hypothesis)} \\ &= \frac{\alpha^{k-1}(a\alpha + b) - \beta^{k-1}(a\beta + b)}{\sqrt{4b + a^2}} \\ &= \frac{\alpha^{k+1} - \beta^{k+1}}{\sqrt{4b + a^2}} \qquad \text{(by (2))}. \end{aligned}$$

This completes the induction and the proof.

We now consider the converse of Lemma 1. Thus, assume that we have a sequence defined by $s_n = c(\alpha^n - \beta^n)$ for $c, \alpha, \beta \in \mathbb{R}$. Note that α and β are the roots of the quadratic equation $(x-\alpha)(x-\beta) = 0$, i.e. $x^2 = (\alpha+\beta)x - \alpha\beta$. Then $s_0 = 0$ and $s_1 = c(\alpha - \beta)$. Put $a = \alpha + \beta$ and $b = -\alpha\beta$, then we obtain

$$s_n = c(\alpha^n - \beta^n) = c(\alpha^{n-2}(a\alpha + b) - \beta^{n-2}(a\beta + b)) = as_{n-1} + bs_{n-2}.$$

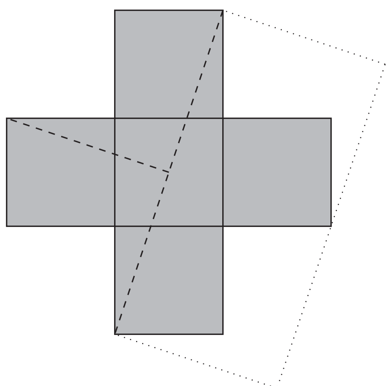
So s_n is a generalized Fibonacci sequence.

For example, let $a = 1$ and $b = 2$, which gives the sequence, $0, 1, 1, 3, 5, 11, 21, 43, \ldots$. From Lemma 1 it follows that $f_n^* = \frac{1}{3}(2^n - (-1)^n)$. In this example α and β, as defined in Lemma 1, are both integers. It is therefore interesting to investigate when $4b + a^2$ is a perfect square for positive integers a and b. Note, in this case, that α and β are both integers. This is because if $\sqrt{4b + a^2} = r$ then $a/2 \pm r/2$ is an integer. So a can be any integer and then take r to be any integer of the same parity as a such that $r > a$. Then set $b = \frac{1}{4}(r + a)(r - a)$. So a can be any integer and b is of the form $\frac{1}{4}(r + a)(r - a)$ for some integer r of the same parity as a and $r > a$.

In the Fibonacci case $a = b = 1$, a and b are not of this form. However, we have a nice example related to the $3: 4: 5$ triangle. Taking $b = 4$ and $a = 3$, we get $4b + a^2 = 5^2$; the sequence is $0, 1, 3, 13, 51, \ldots$. It follows that $\alpha = \frac{3}{2} + \frac{5}{2}$ and $\beta = \frac{3}{2} - \frac{5}{2}$, so $f_n^* = \frac{1}{5}(4^n - (-1)^n)$.

The author has spent much of his time researching in the area of combinatorics. However, he is especially interested in recreational mathematics and number theory.

Making five squares into two squares using two cuts



10 Shahid Azam Lane,
Makki Abad Avenue, Sirjan, Iran

Abbas Roohol Amini

Geometric Constructions Approximating π

THOMAS J. OSLER

1. Introduction

One of the three classical problems of antiquity is to *square the circle* (see reference 1, pp. 115–122). Given a circle of radius r, construct, using only straightedge and compass, the side x of a square having the same area. (Note that the straightedge must be unmarked.) Thus, to square the circle, we must construct the length (number) $\sqrt{\pi}r$. It is known that the only possible constructions are formed by adding, subtracting, multiplying, dividing, and taking the square root of previously found lengths. These constructions may only be performed a finite number of times. It is impossible to construct π, and thus the squaring of the circle is impossible.

It is the purpose of this article to show how to construct approximations to π geometrically of arbitrary accuracy related to Vieta's famous infinite product for π:

$$\frac{2}{\pi} = \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}}\cdots. \tag{1}$$

In Figure 1 we see the unit square. Angle L_1OL_∞ is $\pi/4$. Line OL_2 bisects angle L_1OL_∞ so that $\theta_2 = \pi/8$. Construct line OL_3 so that it bisects angle L_2OL_∞ making angle $\theta_3 = \pi/16$. Continuing, we construct line OL_4 so that it bisects angle L_3OL_∞ making angle $\theta_4 = \pi/32$, and so on.

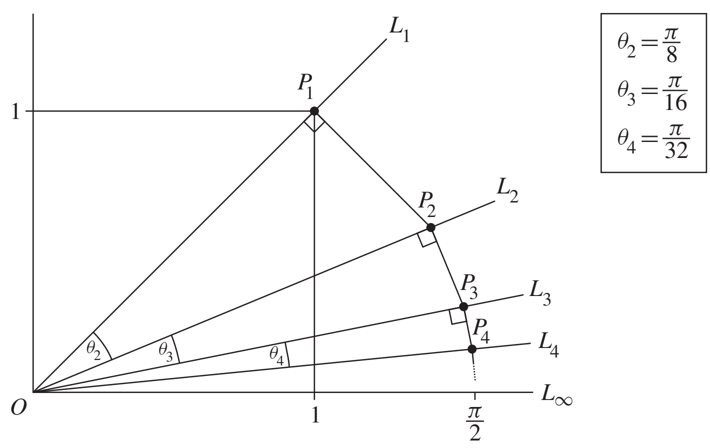


Figure 1 Constructing approximations to $\pi/2$.

From the corner P_1 of our unit square we construct a line perpendicular to line OL_1 meeting line OL_2 at P_2. From P_2 we construct a line perpendicular to line OL_2 meeting line OL_3 at P_3. We continue in this way forming additional points P_4, P_5, and so on. The lengths of the line segments OP_1, OP_2, $OP_3, \ldots$ converge to $\pi/2$. In fact, we will show that

$$\begin{aligned}
\overline{OP_1} &= \frac{1}{\sqrt{\frac{1}{2}}}, \\
\overline{OP_2} &= \frac{1}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}, \\
\overline{OP_3} &= \frac{1}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}}, \ldots.
\end{aligned}$$

Comparing these with (1) we see that the length of $\overline{OP_n}$ is the reciprocal of the first n factors of Vieta's product.

We now review a derivation of Vieta's product (1). Repeated use of a familiar trigonometric identity gives us

$$\begin{aligned}
\sin x &= 2\cos\frac{x}{2}\sin\frac{x}{2}, \\
\sin x &= 2^2\cos\frac{x}{2}\cos\frac{x}{2^2}\sin\frac{x}{2^2}, \\
\sin x &= 2^3\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\sin\frac{x}{2^3}.
\end{aligned}$$

Continuing this way and dividing by x, we get

$$\frac{\sin x}{x} = \frac{\sin(x/2^N)}{x/2^N}\prod_{k=1}^{N}\cos\frac{x}{2^k}, \tag{2}$$

or taking the reciprocal we get

$$\frac{x}{\sin x} = \frac{x/2^N}{\sin(x/2^N)}\prod_{k=1}^{N}\sec\frac{x}{2^k}. \tag{3}$$

Next we use a half-angle formula to replace $\cos(x/2^k)$ as follows:

$$\begin{aligned}
\cos\frac{x}{2} &= \sqrt{\tfrac{1}{2}+\tfrac{1}{2}\cos x}, \\
\cos\frac{x}{2^2} &= \sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\cos x}}, \\
\cos\frac{x}{2^3} &= \sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\cos x}}}.
\end{aligned}$$

Now (2) becomes

$$\frac{\sin x}{x} = \frac{\sin(x/2^N)}{x/2^N} \prod_{k=1}^{N} \underbrace{\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2} + \cdots + \tfrac{1}{2}\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\cos x}}}}}_{k \text{ radicals}} .$$

Finally we set $x = \pi/2$ and let N tend to infinity to get Vieta's product (1). Notice that (3) now takes the form

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} \sec \frac{\pi}{2^{k+1}}. \tag{4}$$

From Figure 1 we see that

$$\begin{aligned}
\overline{OP_1} &= \sec \frac{\pi}{4}, \\
\overline{OP_2} &= \overline{OP_1} \sec \frac{\pi}{8} = \sec \frac{\pi}{4} \sec \frac{\pi}{8}, \\
\overline{OP_3} &= \overline{OP_2} \sec \frac{\pi}{16} = \sec \frac{\pi}{4} \sec \frac{\pi}{8} \sec \frac{\pi}{16},
\end{aligned}$$

and so on. It is now clear from (4) that our constructions converge to $\pi/2$. Even though our constructions approach $\pi/2$ as a limiting case, we cannot say that we have constructed $\pi/2$ since limiting cases are not permitted in classical geometric construction.

Reference 2 is a classic book and is now available in an inexpensive paperback edition. A very readable discussion of constructible numbers, with complete proofs that require only a precalculus background, is found in Chapter 3. It is strongly recommended. New papers generalizing Vieta's product are given in references 3–6.

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***Tom Osler** is professor of mathematics at Rowan University and is the author of 95 mathematical papers. In addition to teaching university mathematics for the past 46 years, Tom has a passion for long distance running. He has been competing for the past 53 consecutive years. Included in his over 1950 races are wins in three national championships in the late 1960s at distances from 25 kilometers to 50 miles. He is the author of two running books.*

One Day Cricket Triangular Tournaments – Do Matches Count?

P. GLAISTER

Introduction

Over the last five years or so, the fixtures list for cricket matches between England and overseas touring sides has changed, with increasing numbers of one day internationals being scheduled. The pattern of one touring side per summer playing five test matches (each of five days duration), as well as three or five one day internationals, all against England, has been replaced by two touring teams, with two test series, and a one day triangular tournament held in between these.

In a triangular tournament, the three teams play each other three times, i.e. nine matches altogether, with one point awarded to the match winner. After all nine matches have been completed, the two teams with the most number of points play one *further* match, the final, and the winners of this are crowned champions. This is different from a standard 'round-robin' tournament in which the top team after nine matches have been played would be the winners of the competition with no further (final) match.

In one such series held recently the two teams going forward to the final were known before all nine matches had been completed. This was because the results of the remaining matches in the series could not affect the position of the team placed last in the final points table, rendering these matches 'dead', although they still took place! The competition was between England, Australia, and Bangladesh, and the finalists, Australia and England, were known after seven matches. After all that, the final was tied with scores level!

This led me to contemplate the likelihood of this happening. For example, how likely is it that the remaining two matches are dead, or how much more likely is it that the finalists are known after six matches than after five matches, or how likely is it that the ninth and final match is required? Obviously, a match that is 'dead', i.e. not required to determine the finalists, might be less appealing to some spectators, and hence a consideration of the odds of matches not being required beyond a certain point will be of interest.

In practice, a match can be 'drawn' if weather intervenes, or 'tied' if the scores are level once both innings have been completed. We shall assume that neither of these situations arises, so that each match has a winner and a loser, although it is normal in these cases to determine a winner based on some additional criteria, for example wickets lost. Also, as we shall see shortly, it is possible that two or, indeed, all three teams have the same points total and additional criteria, for example run rate, would be needed to decide who should go through to the final. However, for our problem it is not necessary to know which criteria would be used, merely that it is possible for any of the teams with the same points score to go through to the final depending on the actual criteria in place.

Before tackling the nine match series we consider first two simpler cases, a three and a six match series, in which the teams play each other once and twice respectively, which will make the nine match case easier to tackle. We note that the standard 'round-robin' problem with no final can be tackled in a similar manner.

Three match series

Suppose that the teams are denoted by A, B, and C, and for the first case we assume that the series comprises three matches, with A playing B, followed by B playing C, and finally A playing C. Each team plays the other teams once, and each team plays two matches altogether. There are $2^3 = 8$ possible combinations of results for the series, and these are represented in figure 1 where the team in bold face is the match winner, and beneath each combination is the points table after all three matches have taken place. For example, in the first case, A and B go through to the final, while in the second case, any two teams could go through depending on the criteria used to separate teams with the same number of points.

It is clear that it is not possible to determine after the first game which teams go through to the final. In this case the probability that the finalists are definitely known and cannot change after one match (out of a possible two combinations) is $0/2$ and we write

$$p_{3,1}(1) = \tfrac{0}{2} = 0,$$

where the subscript '3, 1' denotes a series comprising three teams playing each other once, i.e. three matches altogether.

Looking now at the results after two matches have been played, and the probability that the finalists are definitely known and cannot change, we need to consider the four possible combinations of results as shown in figure 2, together with the corresponding results tables.

A B
B C
A C

A	B	C
2	1	0

A B
B C
A **C**

A	B	C
1	1	1

A B
B **C**
A C

A	B	C
2	0	1

A B
B **C**
A **C**

A	B	C
1	0	2

A **B**
B C
A C

A	B	C
1	2	0

A **B**
B C
A **C**

A	B	C
0	2	1

A **B**
B **C**
A C

A	B	C
1	1	1

A **B**
B **C**
A **C**

A	B	C
0	1	2

Figure 1

A B
B C

A	B	C
1	1	0

A B
B **C**

A	B	C
1	0	1

A **B**
B C

A	B	C
0	2	0

A **B**
B **C**

A	B	C
0	1	1

Figure 2

In the last match, one point is added to the points total of either A or C. It is only in the second case in figure 2 that the result of the last match does not affect which teams go through. Therefore, the probability that the finalists are definitely known and cannot change after two matches is

$$p_{3,1}(2) = \tfrac{1}{4}.$$

Summarising, for a three match series, where the teams play each other once, the probability that the finalists are definitely known and remaining matches 'dead' is 0 after one match has been played, and $\frac{1}{4}$ after two matches have been played. This means that there is a 25% chance that the third and final match is 'dead' and its outcome irrelevant.

Six match series

With the teams again denoted by A, B, and C, we assume that the order of matches is as shown in figure 3, so the same pattern as a three match series which is repeated. This time there are $2^6 = 64$ possible combinations of results for the six match series, which are too many to list here but could easily be generated using a simple computer program.

Denoting by $n_{\mathrm{A}}(6)$ the number of points accrued by team A after all six matches have been played, we have

$$n_{\mathrm{A}}(6) + n_{\mathrm{B}}(6) + n_{\mathrm{C}}(6) = 6,$$

with $0 \le n_{\mathrm{A}}(6), n_{\mathrm{B}}(6), n_{\mathrm{C}}(6) \le 4$. Hence, the possible final points tables for $(\mathrm{A}, \mathrm{B}, \mathrm{C})$ are $(4, 2, 0)$, $(4, 1, 1)$, $(3, 2, 1)$, $(3, 3, 0)$, and $(2, 2, 2)$, and all permutations of these, and the number of ways of achieving each of these totals, or permutations thereof, are 6, 6, 36, 6, and 10 respectively, which readers may like to check.

After one, two, or three matches of the six match series, it is not possible to determine which teams go through. To see this we need only consider the eight possible combinations for the results after three matches, which are those shown in figure 1. In all cases the teams scheduled to go through to the final based on the results after three matches can change. This is because the outcome of the last three matches (which will be the same combinations as in figure 1) can add two points to any of the teams, and this is enough to allow for the possibility of a team going through when it would not necessarily have done so after three matches; thus,

$$p_{3,2}(3) = 0,$$

A	B
B	C
A	C
A	B
B	C
A	C

Figure 3

A B
B C
A C
A B

A	B	C
3	1	0

A B
B C
A C
A **B**

A	B	C
2	2	0

A B
B C
A **C**
A B

A	B	C
2	1	1

A B
B C
A **C**
A **B**

A	B	C
1	2	1

A B
B **C**
A C
A B

A	B	C
3	0	1

A B
B **C**
A C
A **B**

A	B	C
2	1	1

A B
B **C**
A **C**
A B

A	B	C
2	0	2

A B
B **C**
A **C**
A **B**

A	B	C
1	1	2

A **B**
B C
A C
A B

A	B	C
2	2	0

A **B**
B C
A C
A **B**

A	B	C
1	3	0

A **B**
B C
A **C**
A B

A	B	C
1	2	1

A **B**
B C
A **C**
A **B**

A	B	C
0	3	1

A **B**
B **C**
A C
A B

A	B	C
2	1	1

A **B**
B **C**
A C
A **B**

A	B	C
1	2	1

A **B**
B **C**
A **C**
A B

A	B	C
1	1	2

A **B**
B **C**
A **C**
A **B**

A	B	C
0	2	2

Figure 4

where the subscript '3, 2' denotes a series comprising three teams playing each other twice, i.e. six matches altogether. Clearly, if the outcome of the last three matches can change which teams go through to the final, so can the last four and last five matches; thus,

$$p_{3,2}(1) = p_{3,2}(2) = 0$$

as well.

Looking now at the results after four matches have been played, we need to consider the $2^4 = 16$ possible combinations of results after four matches as shown in figure 4, together with the corresponding points tables.

In the last two matches, either two teams add one point each or team C adds two points. It is only in two cases, the seventh and sixteenth with points totals $(2, 0, 2)$ and $(0, 2, 2)$ respectively, where it is impossible for the results of the remaining two matches to change which teams go through to the final based on the results after the first four matches. In the first case, A and C would go through because B can increase its points total only by one. Similarly, for the second case, B and C would go through.

Therefore, the probability that the finalists are definitely known and cannot change after four matches of a six match series is

$$p_{3,2}(4) = \tfrac{2}{16} = \tfrac{1}{8}.$$

Turning now to the situation after five matches have been played, there are now $2^5 = 32$ possible combinations of results to consider, with the remaining match to play being A versus C; hence,

$$n_{\mathrm{A}}(5) + n_{\mathrm{B}}(5) + n_{\mathrm{C}}(5) = 5,$$

with $0 \le n_{\mathrm{A}}(5), n_{\mathrm{C}}(5) \le 3$ and $0 \le n_{\mathrm{B}}(4) \le 4$. We leave readers to list all 32 possible combinations, which can be done quite easily from figure 4 by adding, in turn, one point to B or one point to C (because the fifth match is B versus C). What will be found is as follows.

Since the last match can increase the points total of either A or C by only one, then the only results tables after five matches for which the outcome of the final match cannot change the teams to go through to the final based on these results are $(3, 2, 0)$, $(2, 3, 0)$, $(3, 0, 2)$, $(2, 0, 3)$, $(0, 3, 2)$, $(0, 2, 3)$ (so all permutations of $(3, 2, 0)$), as well as $(2, 1, 2)$, and these occur in 1, 2, 1, 1, 2, 1, and 4 combinations respectively. Therefore, there is a total of 12 out of the 32 possible combinations of results where this occurs, and

$$p_{3,2}(5) = \tfrac{12}{32} = \tfrac{3}{8}.$$

Summarising, for a six match series, where the teams play each other twice, the probability that the finalists are definitely known and remaining matches 'dead' is $\frac{1}{8}$ after four matches have been played and $\frac{3}{8}$ after five matches have been played. This means that there is a $12.5\,\%$ chance that the last two matches are 'dead' with their outcome irrelevant, and a $37.5\,\%$ chance that the last match is 'dead' and its outcome irrelevant.

Finally, we consider the original problem of a nine match series where the teams play each other three times.

Nine match series

As for the six match series, the nine match series has the same pattern as a three match series repeated a further two times. The teams play each other three times, and each team plays six matches altogether. This time there are $2^9 = 512$ possible combinations of results, which are probably best generated using a computer! After all nine matches have been played, we have

$$n_{\mathrm{A}}(9) + n_{\mathrm{B}}(9) + n_{\mathrm{C}}(9) = 9,$$

with $0 \le n_{\mathrm{A}}(9), n_{\mathrm{B}}(9), n_{\mathrm{C}}(9) \le 6$. With a similar explanation as for the six match series, after either one, two, three, four, or five matches have been played the teams which will go through to the final based on these results can be changed by the results of the remaining matches, and we therefore have the following probabilities, which we leave readers to verify:

$$p_{3,3}(1) = p_{3,3}(2) = p_{3,3}(3) = p_{3,3}(4) = p_{3,3}(5) = 0.$$

As before, the subscript '$3, 3$' denotes a series comprising three teams playing each other three times, i.e. nine matches altogether.

So we are left with determining $p_{3,3}(6)$, $p_{3,3}(7)$, and $p_{3,3}(8)$, i.e. the probabilities that the finalists are known and cannot change after six, seven, and eight matches have been played respectively, so that the remaining matches are 'dead'. To make life easier for ourselves we use a simple computer program to run through all the possible combinations, and then get it to count how many have results tables for which the teams going through to the final, based on the results so far, cannot change.

Considering the three cases in turn, we look first at the $2^6 = 64$ possible combinations of results after six matches have been played. Here,

$$n_A(6) + n_B(6) + n_C(6) = 6,$$

with $0 \le n_A(6), n_B(6), n_C(6) \le 4$, and the results tables after these matches have been played for which the remaining three matches cannot change which teams go through to the final are just $(3, 3, 0)$, $(3, 0, 3)$, and $(0, 3, 3)$. For the remaining three matches a total of three points will be added, with at most two going to any one team, and the possible outcomes are the same as those given in figure 1.

After seven matches have been played, we have

$$n_A(7) + n_B(7) + n_C(7) = 7,$$

with $0 \le n_A(7), n_B(7) \le 5$ and $0 \le n_C(7) \le 4$, and in the remaining two matches two points will be added, one at most to either A or B, and two at most to C. This means that of the $2^7 = 128$ possible combinations of results after seven matches, those for which the finalists cannot change will have one of the following results tables: $(0, 3, 4)$ and all permutations of this, $(3, 1, 3)$, $(1, 3, 3)$, $(0, 5, 2)$, and $(5, 0, 2)$.

For the last case, after eight matches have been played we have

$$n_A(8) + n_B(8) + n_C(8) = 8,$$

with $0 \le n_A(8), n_C(8) \le 5$ and $0 \le n_B(8) \le 6$, and in the remaining match one point will either go to A or C. As one final exercise, readers can check that the results tables (of the $2^8 = 256$ possible combinations) for which the finalists cannot change are $(5, 0, 3)$, $(4, 4, 0)$, $(4, 3, 1)$, and all permutations of these three, as well as $(5, 1, 2)$, $(2, 1, 5)$, $(2, 6, 0)$, $(0, 6, 2)$, and $(3, 2, 3)$. (Note that in this case it is possibly easier to count those combinations for which the result of the remaining match may alter which teams go through to the final because the corresponding results tables are fewer in number, namely $(4, 2, 2)$ and all permutations of this, $(2, 5, 1)$, $(1, 5, 2)$, $(5, 2, 1)$, $(1, 2, 5)$, $(3, 3, 2)$, $(2, 3, 3)$, and $(1, 6, 1)$.)

We now reveal the results of the computer count! After six matches, for 6 out of the 64 possible combinations of results the remaining three matches will not be required; after seven matches, for 30 out of the 128 possible combinations of results the remaining two matches will not be required; and after eight matches, for 120 out of the 256 possible combinations of results the remaining match will not be required. This gives the following probabilities:

$$p_{3,3}(6) = \tfrac{6}{64} = \tfrac{3}{32}, \qquad p_{3,3}(7) = \tfrac{30}{128} = \tfrac{15}{64}, \qquad p_{3,3}(8) = \tfrac{120}{256} = \tfrac{15}{32}.$$

In other words, after six, seven, and eight matches have been played, there is (approximately) a 10%, 25%, and 50% chance respectively that the remaining matches are 'dead'. So I now have an idea of the risk I am taking in attending a match in a nine match triangular series where the finalists are already known and the outcome of the match irrelevant. In the series just completed, the finalists, Australia and England, were known after seven matches and the likelihood of this happening was $\frac{15}{64}$. You might think I'd be better off just attending the final, which I did, but that ended in a tie!

Standard round-robin tournament

Finally, we make a few remarks on the problem where the tournament winner is the team at the top of the table after the scheduled matches have been played, with no further (final) match being played, and the associated probabilities that matches beyond certain points are 'dead'. For example, after four matches of a six match series we see from figure 4 that it is in only the first and tenth cases for which the top team (A and B respectively) cannot be altered by the results of the remaining two matches. Hence, the probability that the top team is definitely known and cannot change after four matches of a six match series is $\frac{2}{16} = \frac{1}{8}$. The corresponding probability after five matches have been played, and the probabilities for series of three and nine matches, can be determined similarly. However, we note that for a triangular tournament with only three teams, if matches beyond a certain point are not required to determine which team is top, then the last two teams in the table will also be known. Hence, the probability that matches beyond a certain point are not required to determine which team comes top and wins in a standard 'round-robin' tournament will be the same as the probability that matches beyond a certain point are not required to determine which two teams take the last two positions in this standard problem. Moreover, by symmetry, the probability that the teams in the last two positions are known is the same as the probability that the top two teams are known; hence, this is the same as the probability that the two teams going through to the final are known in the problem we have been concerned with previously. For example, for a nine match series and our original problem, we have seen that the probability that the finalists are definitely known and remaining matches 'dead' is $\frac{15}{64}$ after seven matches have been played, and this is exactly the same as the probability that the bottom team, and also the top team, and hence the tournament winner, is known after the same number of matches of a standard 'round-robin' problem with no further, final match. Clearly, this will not be the case for a tournament with more than three teams.

Further work

We leave readers to consider, possibly during the 'lunch interval', variations such as a twelve match series, or incorporating drawn/tied matches where the points are shared, or even a one day competition between four teams! Of course this is all based on the assumption that each combination of results is equally likely to occur, and that each team is equally likely to win an individual match. If I knew the actual likelihoods of these, I'd be able to afford to attend as many matches as I liked!

Further applications include sets of round-robin matches from other sports, for example football leagues such as the Barclays Premiership in England where 20 teams play each other twice. The interest here is in which matches are 'dead' with respect to determining the winner, or the teams to be relegated or to compete in the 'play-offs'. There is the additional complication here, however, because the matches in each round are typically played simultaneously over a weekend, rather than sequentially with one match per weekend, otherwise the competition would take forever to complete!

Paul Glaister *lectures in mathematics at Reading University. In September 2005 the England cricket team beat Australia in a dramatic series of test matches, thereby regaining the Ashes after losing them 16 years earlier. There have been only two other occasions when he was as distraught and expectant as on that final day of the series. Happily, his two children were able to join in with the celebrations on this third occasion!*

The Better Bowler?

PETER STANLEY

The paradox of the bowlers' averages, with the surprising reversal in the runs per wicket performance indicator, is well known (see reference 1) and numerical examples can be put together without difficulty. However, it continues to intrigue. A graphical treatment is given below, which may be more illuminating than a treatment based on the relevant inequalities.

The scene (slightly remodelled for convenience) is set so. A county cricket team has two rival bowlers, A and B. In the first innings of a three-day match bowler A takes W_{A1} of the opposing team's wickets for R_{A1} runs, bowler B takes W_{B1} wickets for R_{B1} runs, and bowler A's 'average', R_{A1}/W_{A1}, is less than (i.e. better than) bowler B's average, R_{B1}/W_{B1}. In the second innings the corresponding figures are W_{A2}, R_{A2}, W_{B2}, and R_{B2}, and again R_{A2}/W_{A2} is less than R_{B2}/W_{B2}, i.e. A is the better bowler. However, on looking at the bowling performances for the match as a whole, it turns out that $(R_{B1}+R_{B2})/(W_{B1}+W_{B2})$ is less than $(R_{A1}+R_{A2})/(W_{A1}+W_{A2})$, i.e. B is the better bowler! How can this be so?

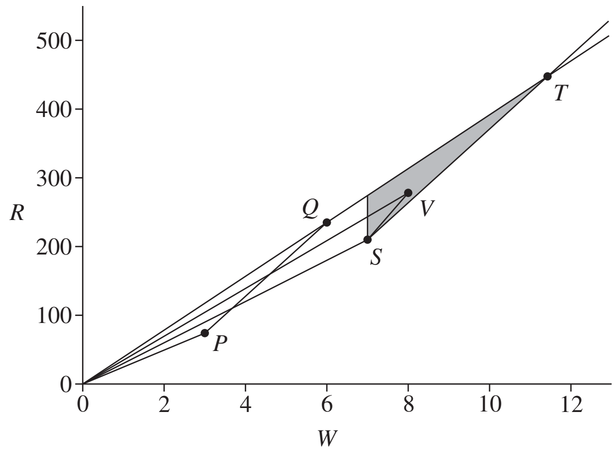


Figure 1 Graph for data in table 1. Point coordinates: $P=(W_{A1}, R_{A1})$, $Q=(W_{A1}+W_{A2}, R_{A1}+R_{A2})$, $S=(W_{B1}, R_{B1})$, and $V=(W_{B1}+W_{B2}, R_{B1}+R_{B2})$.

Table 1 Data for figure 1 ($R_{A1}/W_{A1} < R_{A2}/W_{A2}$).

	Bowler	Wickets W	Runs R	Average R/W
First innings	A	3	75	25
	B	7	210	30
Second innings	A	3	160	53.3
	B	1	70	70
Match	A	6	235	39.2
	B	8	280	35

Illustrative details are shown in figure 1 with coordinate axes W (wickets) and R (runs) and origin O; numerical values are shown in table 1. The coordinate points (W_{A1}, R_{A1}), $(W_{A1} + W_{A2}, R_{A1} + R_{A2})$, and (W_{B1}, R_{B1}) are denoted by P, Q, and S respectively. It can be seen that, in this case, bowler A's second innings performance is not as good as that in the first innings (i.e. the gradient of the line PQ is greater than that of the line OP). Bowler B's first innings performance, expressed as runs per wicket, is not as good as A's (i.e. the gradient of the line OS is greater than that of the line OP). A line is drawn through the point S parallel to the line PQ to intersect OQ (produced) at T. The triangle enclosed by the lines ST, QT, and that through S parallel to the R-axis is shaded. Subject to the restrictions that W and R are not continuous variables and that the sum $W_{A2} + W_{B2}$ cannot exceed 10, then any permitted point in the shaded triangle (e.g. V) will represent a second innings performance by bowler B which is inferior to A's second innings performance (i.e. SV is steeper than PQ) but for which B's overall match performance is better than A's (i.e. the gradient OV is less than the gradient OQ).

Figure 2 shows a case in which A's second innings performance is better than his first innings performance, i.e. the gradient of PQ is less than that of OP. Numerical values are shown in table 2. The paradox will follow if the representative point for B's second innings (and overall)

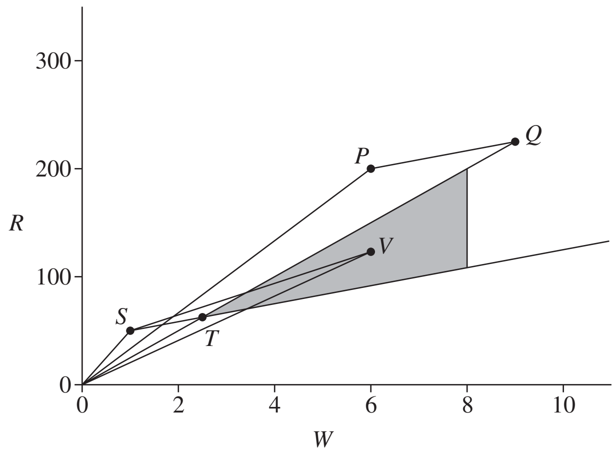


Figure 2 Graph for data in table 2. Point coordinates: $P = (W_{A1}, R_{A1})$, $Q = (W_{A1}+W_{A2}, R_{A1}+R_{A2})$, $S = (W_{B1}, R_{B1})$, and $V = (W_{B1} + W_{B2}, R_{B1} + R_{B2})$.

Table 2 Data for figure 2 ($R_{A1}/W_{A1} > R_{A2}/W_{A2}$).

	Bowler	Wickets W	Runs R	Average R/W
First innings	A	6	200	33.3
	B	1	50	50
Second innings	A	3	25	8.3
	B	5	75	15
Match	A	9	225	25
	B	6	125	20.8

performance falls within the shaded area between the lines ST (produced) and QT. (The third side of the shaded area in this case is a consequence of the $W_{A2} + W_{B2} \not> 10$ restriction.)

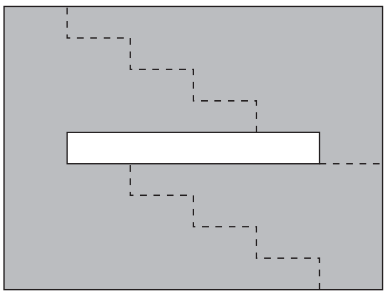
It is useful to see how this teaser can be clarified and 'explained' graphically, but as to who is the better bowler, opinions may still differ!

Reference

1 P. Stanley, Fair means or foul?, *Math. Gazette* **86** (2002), pp. 422–427.

Peter Stanley *retired from the School of Engineering at the University of Manchester some years ago. His principal research interests there were in the fields of Experimental Stress Analysis and Material Characterisation.*

Making a square from a holed rectangle



A 12×9 rectangle has an 8×1 rectangle cut from it (the centres being the same and the sides parallel). Make three cuts so as to construct a square.

10 Shahid Azam Lane,
Makki Abad Avenue, Sirjan, Iran **Abbas Roohol Amini**

If $\log_3 x = 5$, what is $\log_2 x$?

12 Pinewood Road, Midsomer Norton,
Bath BA3 2RG, UK **Bob Bertuello**

Euler's Inequality Revisited

ZHANG YUN

A quadrilateral ABCD is called a *double circle quadrilateral* if it has a circumcircle and an inscribed circle. In a previous article (see reference 1), we proved Euler's inequality for a double circle quadrilateral, $R \geq r\sqrt{2}$, where R and r denote the radii of the circumcircle and inscribed circle respectively. Here we shall find two expressions, involving the angles of such a quadrilateral, which lie between $r\sqrt{2}/R$ and 1. We denote the lengths of the sides AB, BC, CD, and DA by a, b, c, and d respectively, put $s = \frac{1}{2}(a+b+c+d)$, and denote the area of ABCD by Δ.

Theorem 1 *Let* ABCD *be a double circle quadrilateral. Then we obtain*

$$\frac{r\sqrt{2}}{R} \leq \frac{1}{2}\left(\sin\frac{\mathrm{A}}{2}\cos\frac{\mathrm{B}}{2} + \sin\frac{\mathrm{B}}{2}\cos\frac{\mathrm{C}}{2} + \sin\frac{\mathrm{C}}{2}\cos\frac{\mathrm{D}}{2} + \sin\frac{\mathrm{D}}{2}\cos\frac{\mathrm{A}}{2}\right) \leq 1.$$

In order to prove Theorem 1, we first prove the following lemma.

Lemma 1 *In a double circle quadrilateral* ABCD, *we have*

$$\sin\mathrm{A}\sin\mathrm{B} = \frac{r^2 + r\sqrt{r^2+4R^2}}{2R^2}.$$

Proof By reference 1, we have

$$\sin\mathrm{B} = \frac{2\Delta}{ab+cd}, \qquad \sin\mathrm{A} = \frac{2\Delta}{ad+bc}, \qquad \Delta = rs,$$
$$R = \frac{1}{4\Delta}\sqrt{(ab+cd)(ac+bd)(ad+bc)}.$$

Hence,

$$\frac{\sqrt{1+\sin\mathrm{A}\sin\mathrm{B}}}{\sin\mathrm{A}\sin\mathrm{B}} = \frac{\sqrt{(ab+cd)(ad+bc)}}{4\Delta^2}\sqrt{4\Delta^2+(ab+cd)(ad+bc)}.$$

By reference 1, we obtain

$$a+c = b+d = s, \qquad \sqrt{abcd} = \Delta, \qquad R \geq r\sqrt{2},$$

so that

$$\begin{aligned}\sqrt{4\Delta^2+(ab+cd)(ad+bc)} &= \sqrt{4abcd + (a^2bd+c^2bd) + (acb^2+acd^2)}\\ &= \sqrt{bd(a^2+2ac+c^2) + ac(b^2+2bd+d^2)}\\ &= \sqrt{s^2bd + s^2ac}\\ &= s\sqrt{ac+bd}.\end{aligned}$$

Hence,

$$\frac{\sqrt{1+\sin \mathrm{A} \sin \mathrm{B}}}{\sin \mathrm{A} \sin \mathrm{B}} = \frac{\sqrt{(ab+cd)(ad+bc)}}{4\Delta^2} s\sqrt{ac+bd} = \frac{Rs}{\Delta} = \frac{R}{r},$$

so that

$$R^2(\sin \mathrm{A} \sin \mathrm{B})^2 - r^2 \sin \mathrm{A} \sin \mathrm{B} - r^2 = 0$$

and

$$\sin \mathrm{A} \sin \mathrm{B} = \frac{r^2 + r\sqrt{r^2+4R^2}}{2R^2}.$$

Proof of Theorem 1 By Lemma 1 and Euler's inequality, $R \geq r\sqrt{2}$, we have

$$\sin \mathrm{A} \sin \mathrm{B} \geq \frac{2r^2}{R^2}.$$

Since ABCD is a double circle quadrilateral, $\mathrm{A}+\mathrm{C} = \mathrm{B}+\mathrm{D} = 180°$. Hence,

$$\begin{aligned}
&\frac{1}{2}\left(\sin\frac{\mathrm{A}}{2}\cos\frac{\mathrm{B}}{2} + \sin\frac{\mathrm{B}}{2}\cos\frac{\mathrm{C}}{2} + \sin\frac{\mathrm{C}}{2}\cos\frac{\mathrm{D}}{2} + \sin\frac{\mathrm{D}}{2}\cos\frac{\mathrm{A}}{2}\right)\\
&\quad = \frac{1}{2}\left(\sin\frac{\mathrm{A}}{2}\cos\frac{\mathrm{B}}{2} + \sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{A}}{2} + \cos\frac{\mathrm{A}}{2}\sin\frac{\mathrm{B}}{2} + \cos\frac{\mathrm{B}}{2}\cos\frac{\mathrm{A}}{2}\right) \qquad (1)\\
&\quad = \frac{1}{2}\left(\sin\frac{\mathrm{A}}{2} + \cos\frac{\mathrm{A}}{2}\right)\left(\sin\frac{\mathrm{B}}{2} + \cos\frac{\mathrm{B}}{2}\right)\\
&\quad = \sin\left(\frac{\pi}{4} + \frac{\mathrm{A}}{2}\right)\sin\left(\frac{\pi}{4} + \frac{\mathrm{B}}{2}\right)\\
&\quad \leq 1.
\end{aligned}$$

Also, by the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\sin\frac{\mathrm{A}}{2}\cos\frac{\mathrm{B}}{2} + \sin\frac{\mathrm{B}}{2}\cos\frac{\mathrm{C}}{2} + \sin\frac{\mathrm{C}}{2}\cos\frac{\mathrm{D}}{2} + \sin\frac{\mathrm{D}}{2}\cos\frac{\mathrm{A}}{2}\right)\\
&\quad = \frac{1}{2}\left(\sin\frac{\mathrm{A}}{2}\cos\frac{\mathrm{B}}{2} + \sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{A}}{2} + \cos\frac{\mathrm{A}}{2}\sin\frac{\mathrm{B}}{2} + \cos\frac{\mathrm{B}}{2}\cos\frac{\mathrm{A}}{2}\right)\\
&\quad \geq \frac{1}{2} \times 4\sqrt[4]{\sin^2\frac{\mathrm{A}}{2}\cos^2\frac{\mathrm{A}}{2}\sin^2\frac{\mathrm{B}}{2}\cos^2\frac{\mathrm{B}}{2}}\\
&\quad = 2\sqrt{\sin\frac{\mathrm{A}}{2}\cos\frac{\mathrm{A}}{2}\sin\frac{\mathrm{B}}{2}\cos\frac{\mathrm{B}}{2}}\\
&\quad = \sqrt{\sin \mathrm{A} \sin \mathrm{B}}\\
&\quad \geq \sqrt{\frac{2r^2}{R^2}}\\
&\quad = \frac{r\sqrt{2}}{R}.
\end{aligned}$$

Theorem 2 *We have*

$$\frac{r\sqrt{2}}{R} \leq \frac{1}{4}\left(\cos\frac{\mathrm{A}-\mathrm{B}}{2} + \cos\frac{\mathrm{B}-\mathrm{C}}{2} + \cos\frac{\mathrm{C}-\mathrm{D}}{2} + \cos\frac{\mathrm{D}-\mathrm{A}}{2}\right) \leq 1.$$

Proof We have

$$
\begin{aligned}
&\cos\frac{A-B}{2}+\cos\frac{B-C}{2}+\cos\frac{C-D}{2}+\cos\frac{D-A}{2}\\
&\quad=\cos\frac{A-B}{2}+\cos\frac{B-(180^\circ-A)}{2}\\
&\qquad+\cos\frac{180^\circ-A-(180^\circ-B)}{2}+\cos\frac{180^\circ-B-A}{2}\\
&\quad=\cos\frac{A-B}{2}+\sin\frac{A+B}{2}+\cos\frac{A-B}{2}+\sin\frac{A+B}{2}\\
&\quad=2\left(\sin\frac{A+B}{2}+\cos\frac{A-B}{2}\right)\\
&\quad=2\left(\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}+\cos\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}\right).
\end{aligned}
$$

By (1), Theorem 2 is in effect a restatement of Theorem 1.

Reference

1 Z. Yun, Some properties of a double circle quadrilateral, *Math. Spectrum* **37** (2004/2005), pp. 57–60.

Zhang Yun *is a senior teacher of mathematics at the first middle school of Xi'An City, Shan Xi Province, China. He is the author of over 140 mathematical papers. His research interests include elementary number theory, algebraic inequalities, and geometric inequalities. He likes to play table tennis.*

A 7x7 diagonal magic square

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

10 Shahid Azam Lane,
Makki Abad Avenue, Sirjan, Iran

Abbas Roohol Amini

A Square Wheel on a Round Track

PETER MORRIS

Some time ago I visited the Science Investigator Centre in Adelaide, Australia. One of the attractions was a car with square wheels running smoothly along a track made of curved segments. The following article describes how to construct a mathematical equation of such a track.

The problem here is to construct an expression for a curved track to be constructed on level ‘ground’ along which a square wheel can travel so that the axle or the centre of the square wheel remains at a constant height above the ground. For simplicity, I assume that the wheel is square with sides 2 units in length. An expression for the height, y, of the track in terms of θ (see figure 2, below) will be constructed.

When the wheel has rotated through an angle θ from its starting position at A, its point of contact with the track lies at (x, y). The height of the centre of the wheel above the ground represented by the x-axis is given by $l + y = \sqrt{2}$.

Consider the wheel shown in figures 1 and 2 starting its rotation from the ‘diamond’ position, i.e. where its diagonal is vertical, when vertex A is in contact with the ground under the track. Also, consider the track to be located on Cartesian axes with the origin, O, on the ground directly beneath the maximum point on the track.

Using the sine rule, we have

$$\frac{l}{\sin(\pi/4)} = \frac{\sqrt{2}}{\sin(3\pi/4 - \theta)} = \frac{\sqrt{2}}{\cos(\pi/4 - \theta)},$$

so that

$$l = \frac{1}{\cos(\pi/4 - \theta)} = \sec\left(\frac{\pi}{4} - \theta\right).$$

Hence,

$$y = \sqrt{2} - l = \sqrt{2} - \sec\left(\frac{\pi}{4} - \theta\right).$$

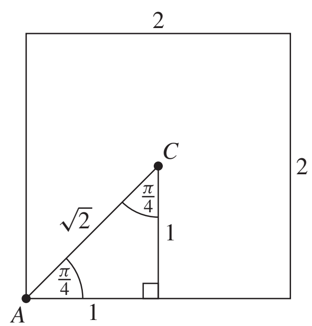


Figure 1 A square wheel.

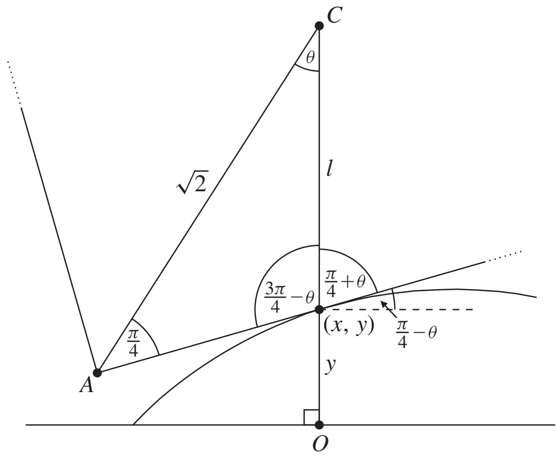


Figure 2 A square wheel on a curved track.

An expression for x in terms of θ can be found using the slope of the tangent to the curve at any point (x, y) given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\,\mathrm{d}\theta}{\mathrm{d}x/\,\mathrm{d}\theta} = \tan\left(\frac{\pi}{4} - \theta\right) \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \tan\left(\frac{\pi}{4} - \theta\right)\sec\left(\frac{\pi}{4} - \theta\right).$$

Hence,

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}y/\,\mathrm{d}\theta}{\tan(\pi/4 - \theta)} = \sec\left(\frac{\pi}{4} - \theta\right).$$

Therefore,

$$\begin{aligned} x &= \int \sec\left(\frac{\pi}{4} - \theta\right)\mathrm{d}\theta \\ &= -\ln\left(\sec\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right)\right) + c, \end{aligned}$$

with boundary condition $x = 0$ when $\theta = \pi/4$, which gives $c = 0$.

So, the coordinates of the curved track can be given in parametric form as follows:

$$x = -\ln\left(\sec\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right)\right) \quad \text{and} \quad y = \sqrt{2} - \sec\left(\frac{\pi}{4} - \theta\right).$$

To obtain a Cartesian equation for the track, write $\phi = \pi/4 - \theta$. Thus,

$$x = -\ln(\sec\phi + \tan\phi),$$

so that

$$\mathrm{e}^{-x} = \sec\phi + \tan\phi \quad \text{and} \quad \mathrm{e}^{x} = \frac{1}{\sec\phi + \tan\phi} = \sec\phi - \tan\phi.$$

Hence,

$$\mathrm{e}^{x} + \mathrm{e}^{-x} = 2\sec\phi,$$

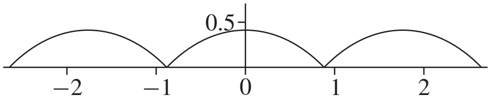


Figure 3

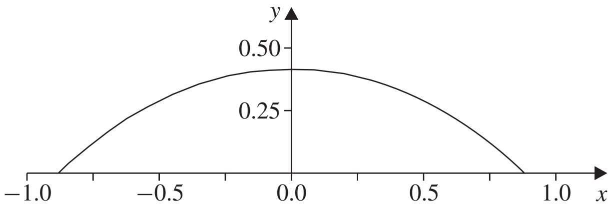


Figure 4

so

$$\sec\phi = \frac{e^x + e^{-x}}{2} = \cosh x.$$

Therefore,

$$y = \sqrt{2} - \cosh x.$$

Thus, the track, (see figure 3) is built up with segments, shown in figure 4, which are inverted catenaries, i.e. the curve formed by a uniform chain, fixed level at both ends, hanging freely in mid-air.

The arguments generated for a square wheel can be repeated to find the equation of the track for an equilateral triangle (see figures 5 and 6).

Thus, from the sine rule we obtain

$$\frac{l}{\sin(\pi/6)} = \frac{2\sqrt{3}/3}{\sin(5\pi/6 - \theta)},$$

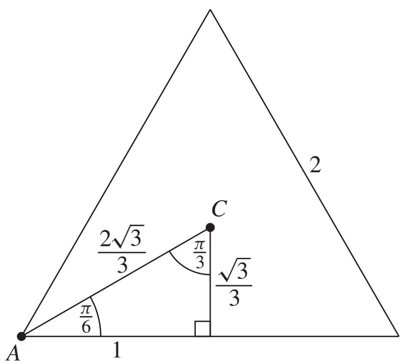


Figure 5 A triangular wheel.

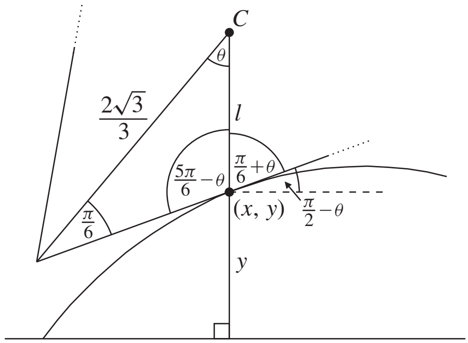


Figure 6 A triangular wheel on a curved track.

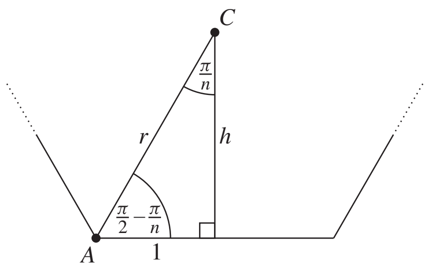


Figure 7 An n-gon wheel.

so that

$$l = \frac{\sqrt{3}}{3}\sec\left(\frac{\pi}{3}-\theta\right) \quad \text{and} \quad y = \frac{2\sqrt{3}}{3} - l = \frac{2\sqrt{3}}{3} - \frac{\sqrt{3}}{3}\sec\left(\frac{\pi}{3}-\theta\right).$$

We can find x in terms of the parametric angle, $\pi/3-\theta$, for the equilateral triangle, as before:

$$x = -\frac{\sqrt{3}}{3}\left(\ln\left(\sec\left(\frac{\pi}{3}-\theta\right)+\tan\left(\frac{\pi}{3}-\theta\right)\right)\right)$$

(where we take $x=0$ when $\theta=\pi/3$ to make the constant of integration zero).

The Cartesian equation is created by eliminating the parametric angle, i.e.

$$y = \frac{2\sqrt{3}}{3} - \frac{\sqrt{3}}{3}\cosh(x\sqrt{3}).$$

Generalizing the arguments for a regular n-gon wheel (see figure 7) gives the parametric equations

$$y = \operatorname{cosec}\left(\frac{\pi}{n}\right) - \cot\left(\frac{\pi}{n}\right)\sec\left(\frac{\pi}{n}-\theta\right),$$

$$x = -\cot\left(\frac{\pi}{n}\right)\ln\left(\sec\left(\frac{\pi}{n}-\theta\right)+\tan\left(\frac{\pi}{n}-\theta\right)\right),$$

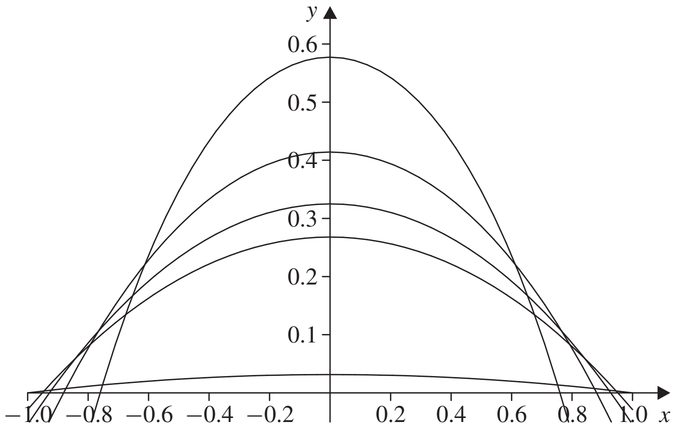


Figure 8 Graphs of tracks for $n = 3, 4, 5, 6, 50$.

and the Cartesian equation

$$y = \operatorname{cosec}\left(\frac{\pi}{n}\right) - \cot\left(\frac{\pi}{n}\right)\cosh\left(x\tan\left(\frac{\pi}{n}\right)\right).$$

Plotting these functions for $n = 3, 4, \ldots$ (see figure 8) shows that the track becomes flatter and flatter as the wheel becomes more and more circular. The limiting track for a circular wheel, of course, is the 'ground' itself.

__Peter Morris__ taught Mathematics in high schools in South Australia before emigrating to Sweden in 1993. He now lives and works as a teacher and coordinator in the International Baccalaureate programme at Saltsjobadens Samskola in Stockholm, Sweden. His interests are in Applied Mathematics and how real-world problem-solving skills can be taught to high school students.

Magic sums

Consecutive numbers beginning with $m + 1$ form an $n \times n$ magic square. What is its magic sum?

SS-Math-Hebron UNRWA,
PO Box 19149, Jerusalem, Israel

Muneer Jebreel Karama

On Constructing 4×4 Magic Squares with Pre-Assigned Magic Sum

HOSSEIN BEHFOROOZ

In this article, we will show that, for any given integer S, there exists a fourth-order magic square with a magic sum equal to that given integer S. A similar procedure to ours can be found in references 1 and 2. But here, the mathematics behind the procedure is explained and our created magic squares have more interesting properties.

Typically, a magic square is a square matrix of integers with the property that the sums along rows, columns, and two diagonals are all equal to a constant number, which is called the *magic sum.* We will use the following obvious property of magic squares in our construction.

Additive Property Adding a constant d to all entries of any $n \times n$ magic square with magic sum S results in a new magic square with magic sum

$$S + nd.$$

Suppose that I ask you to give me a positive integer, and you say '115'. Then I immediately write down the following magic square with magic sum $S = 115$.

27	32	21	35
22	34	28	31
37	23	30	25
29	26	36	24

Magic sum 115

In this magic square, not only is 115 the magic sum but it appears everywhere. For example,

$$\begin{array}{ll}
27+35+29+24=115, & 34+28+23+30=115, \\
27+32+22+34=115, & 32+21+34+28=115, \\
27+35+22+31=115, & 22+31+37+25=115, \\
32+21+26+36=115, & 27+21+37+30=115, \\
32+35+23+25=115, & 27+22+30+36=115, \\
32+34+25+24=115, & 32+22+25+36=115.
\end{array} \tag{1}$$

Figure 1 Melencolia.

There are relations between the squares of the entries too, i.e.

$$\begin{aligned}
&27^2 + 32^2 + 21^2 + 35^2 + 29^2 + 26^2 + 36^2 + 24^2 \\
&\quad = 22^2 + 34^2 + 28^2 + 31^2 + 37^2 + 23^2 + 30^2 + 25^2 \\
&\quad = 6808, \\
&27^2 + 22^2 + 37^2 + 29^2 + 32^2 + 34^2 + 23^2 + 26^2 \\
&\quad = 21^2 + 28^2 + 30^2 + 36^2 + 35^2 + 31^2 + 25^2 + 24^2 \\
&\quad = 6808.
\end{aligned} \tag{2}$$

If you change 115 to any other integer S, then there is a magic square with a magic sum equal to your chosen number S and the entries of this magic square will satisfy the set of relations similar to (1) and (2).

To construct these magic squares, we need a 'seed' magic square to use as a base or foundation and build up all other magic squares from this base by using the additive property. I prefer the magic squares with nonrepeated entries which satisfy the relations similar to (1) and (2).

In 1514, the German Renaissance painter and graphic artist *Albrecht Dürer* included a 4×4 magic square at the top-right-hand corner of his famous engraving entitled *Melencolia* (see Figure 1); see also references 3 and 4. This piece of art made this magic square very popular in Western countries. Since then, this magic square has been called the *Dürer magic square* with date 1514 (the two numbers in the bottom centre cells). The fact is that this magic square is from China and composed by Yang Hui (see reference 5, p. 528). Also, we can find it in Iranian literature without Dürer's name (see, for example, references 6 and 7). To construct the Dürer magic square, first write the sixteen numbers $1, 2, \ldots, 16$ in a 4×4 matrix as follows.

16	15	14	13
9	10	11	12
5	6	7	8
4	3	2	1

Then, switch the four pairs of entries $\{5, 9\}$, $\{8, 12\}$, $\{3, 15\}$, and $\{2, 14\}$; the result is a magic square with magic sum 34.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Dürer magic square with magic sum 34

The entries of the Dürer magic square have their own relations, i.e.

$$\begin{aligned}
&16+13+4+1 && 16+2+9+7\\
&\quad = 10+11+6+7 && \quad = 3+8+14+9\\
&\quad = 16+3+5+10 && \quad = 16+3+14+1\\
&\quad = 5+8+9+12 && \quad = 16+5+12+1\\
&\quad = 3+2+15+14 && \quad = 16+2+15+1\\
&\quad = 3+5+12+14 && \quad = 16+11+6+1\\
&\quad = 2+8+9+15 && \quad = 13+10+7+4\\
&\quad = 34, && \quad = 34,
\end{aligned}$$

$$\begin{aligned}
16^2+3^2+2^2+13^2 &= 4^2+15^2+14^2+1^2 = 438,\\
5^2+10^2+11^2+8^2 &= 9^2+6^2+7^2+12^2 = 310,\\
16^2+5^2+9^2+4^2 &= 13^2+8^2+12^2+1^2 = 378,\\
3^2+10^2+6^2+15^2 &= 2^2+11^2+7^2+14^2 = 370.
\end{aligned}$$

But wait, observe the following remarkable, linear, quadratic, and cubic forms of properties between the diagonal and nondiagonal entries:

$$\begin{aligned}
3+2+8+12+14+15+9+5 &= 16+10+7+1+13+11+6+4,\\
3^2+2^2+8^2+12^2+14^2+15^2+9^2+5^2 &= 16^2+10^2+7^2+1^2+13^2+11^2+6^2+4^2,\\
3^3+2^3+8^3+12^3+14^3+15^3+9^3+5^3 &= 16^3+10^3+7^3+1^3+13^3+11^3+6^3+4^3.
\end{aligned}$$

No wonder the Dürer magic square is seen by some to have religious or mystical significance! For more details on the Dürer magic square, see, for example, references 4, 7, and 8.

Despite these many interesting properties, the Dürer magic square is not useful for our purpose. Two consecutive numbers, 14 and 15, are in one line and that causes a problem.

(You will notice this problem later.) If we choose the Dürer magic square as a base, then sometimes we will have repeated numbers in our derived magic square. So let us go to the rich literature on magic squares with thousands of years of history and find the best one to use as a base. I found one from the Indian culture which is called the *Jaina magic square* (from the 11th or 12th century).

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Jaina magic square with magic sum 34

There are 880 different 4×4 magic squares with entries $1, 2, \ldots, 16$ (not including reflections or rotations of each square). In 1693, Frenicle published the list of all these squares, see references 9 (p. 187), 10 (pp. 119–121), and 11 (pp. 137–140). The Jaina magic square is one of the most famous ones in India and can be found at the top of the gate of Khajuraho in the Holy Temple of Jainism in India, see references 4, 12, and 13. This square can also be found in old Iranian books, see, for example, reference 14.

The distribution of the numbers $1, 2, \ldots, 16$ in this square is such that each row, column, and diagonal has only one of the numbers $\{1, 2, 3, 4\}$. This is true for the other three subsets $\{5, 6, 7, 8\}$, $\{9, 10, 11, 12\}$, and $\{13, 14, 15, 16\}$ too. We shall find out later why this is important in our construction. Now we have the seed (the Jaina magic square) and the tool (the additive property) and we are ready to present our procedure and the algorithm of the construction.

Procedure For any given integer S, first find the integers q and r such that $S - 34 = 4q + r$, with $r = 0, 1, 2$, or 3. Then, in the Jaina square, add q to all sixteen cells and r to the four cells which contain 13, 14, 15, and 16. The result is the following magic square with magic sum S.

$7+q$	$12+q$	$1+q$	$14+q+r$
$2+q$	$13+q+r$	$8+q$	$11+q$
$16+q+r$	$3+q$	$10+q$	$5+q$
$9+q$	$6+q$	$15+q+r$	$4+q$

Magic sum S

In this magic square the entries are not repeated because $r \geq 0$ is added to the largest four entries. Also, using matrix notation, it can be easily shown that the entries of this magic square satisfy the following sets of relations:

$$\begin{aligned}
&a_{11}+a_{14}+a_{41}+a_{44}=a_{22}+a_{23}+a_{32}+a_{33}=a_{11}+a_{12}+a_{21}+a_{22}=S,\\
&a_{12}+a_{13}+a_{22}+a_{23}=a_{11}+a_{14}+a_{21}+a_{24}=a_{21}+a_{24}+a_{31}+a_{34}=S,\\
&a_{12}+a_{13}+a_{42}+a_{43}=a_{11}+a_{13}+a_{31}+a_{33}=a_{12}+a_{14}+a_{32}+a_{34}=S,\\
&a_{11}+a_{21}+a_{33}+a_{43}=a_{12}+a_{22}+a_{34}+a_{44}=a_{12}+a_{21}+a_{34}+a_{43}=S,
\end{aligned} \tag{3}$$

and

$$\sum_{j=1}^{4} a_{1j}^2 + \sum_{j=1}^{4} a_{4j}^2 = \sum_{j=1}^{4} a_{2j}^2 + \sum_{j=1}^{4} a_{3j}^2 = \sum_{i=1}^{4} a_{i1}^2 + \sum_{i=1}^{4} a_{i2}^2 = \sum_{i=1}^{4} a_{i3}^2 + \sum_{i=1}^{4} a_{i4}^2 = A, \quad (4)$$

where

$$A = 748 + 8q^2 + 136q + 4qr + 2r^2 + 58r.$$

In the special case when $r = 0$, there are some additional relations too.

Example 1 Suppose that $S = 115$. Then

$$S - 34 = 81 = 4 \times 20 + 1.$$

In this case, $q = 20$ and $r = 1$. By adding 20 to all sixteen entries of the Jaina square (our base square) and 1 to the cells which contain 13, 14, 15, and 16, we will have the magic square with magic sum 115 with which we began. In this case, with remainder $r = 1$, the entries are nonrepeated integers $21, 22, 23, \ldots, 37$ with missing number 33. In general, for a nonzero remainder r, we expect a jump discontinuity with r consecutive missing numbers.

Example 2 Suppose that $S = 110$. Then $S - 34 = 76 = 4 \times 19$. In this case, $q = 19$ and $r = 0$. By adding 19 to all sixteen entries we will have the following magic square with magic sum 110. In this square, the entries are nonrepeated and consecutive integers with no jump discontinuity.

26	31	20	33
21	32	27	30
35	22	29	24
28	25	34	23

Magic sum 110

Example 3 Suppose that $S = 28$. Then $S - 34 = -6 = 4(-2) + 2$. In this case, $q = -2$ and $r = 2$. By adding $q = -2$ to all sixteen entries and $r = 2$ to the cells which contain 13, 14, 15, and 16, the result will be the following magic square with magic sum 28.

5	10	−1	14
0	13	6	9
16	1	8	3
7	4	15	2

Magic sum 28

Notes on uniqueness

In our procedure, although q and r are unique, there is not a *unique* magic square with magic sum S for the following two reasons.

Reason 1 We can add r to the cells of one of the other three subsets

$$\{1, 2, 3, 4\}, \qquad \{5, 6, 7, 8\}, \qquad \text{or} \qquad \{9, 10, 11, 12\}.$$

However, this gives magic squares with repeated entries.

Reason 2 We can choose a different base. The following is a magic square from *Sheikh Bahaii* (see reference 14), who was an Iranian mathematician. The entries of this square can be partitioned into four disjoint subsets $\{1, 2, 3, 4\}$, $\{5, 6, 7, 8\}$, $\{9, 10, 11, 12\}$, and $\{13, 14, 15, 16\}$ such that, when we consider any one of these subsets, each row, column, and diagonal contains only one of the elements of this particular subset.

9	6	3	16
4	15	10	5
14	1	8	11
7	12	13	2

Sheikh Bahaii magic square with magic sum 34

Here again, by adding q to all sixteen entries and r to the largest four entries, we obtain a magic square with magic sum S. As before, the entries will be nonrepeated entries and, when $r \neq 0$, we expect a jump discontinuity with r missing numbers. More importantly, the entries of this magic square satisfy relations (3) and (4).

Final note

If we do not care about repetition or consecutiveness, then the procedure is simple. First consider any 4×4 magic square with magic sum T and then add $m = S - T$ to the locations of one of those four subsets in the Jaina magic square and you will get the desired magic square. For example, by using the locations of $\{13, 14, 15, 16\}$ we will obtain the output magic square with magic sun S from the input magic square with magic sum T.

Input

A	B	C	α
D	β	E	F
γ	G	H	I
J	K	δ	L

Magic sum T

Output

A	B	C	$\alpha + m$
D	$\beta + m$	E	F
$\gamma + m$	G	H	I
J	K	$\delta + m$	L

Magic sum S

Generalization

We can follow the above procedure and increase the order of the magic square from four to five or higher. The following is an example of 5×5 input and output.

Input

α	A	B	C	D
E	F	β	G	H
I	J	K	L	γ
M	δ	N	O	P
Q	R	U	θ	V

Magic sum T

Output

$\alpha + m$	A	B	C	D
E	F	$\beta + m$	G	H
I	J	K	L	$\gamma + m$
M	$\delta + m$	N	O	P
Q	R	U	$\theta + m$	V

Magic sum S

In a 3×3 matrix, it is not possible to consider three cells such that each row, column, and diagonal contains only one of those three cells. So, in general, our procedure does not work for 3×3 magic squares, except when S is a multiple of three. For example, if we consider the famous *Lo-Shu* magic square with magic sum $T = 15$, then for any integer S which is a multiple of three, we have $S - 15 = 3q$. By adding q to all nine entries of the Lo-Shu magic square, we obtain a 3×3 magic square with magic sum S.

Input

8	3	4
1	5	9
6	7	2

Lo-Shu magic square

Output

$8+q$	$3+q$	$4+q$
$1+q$	$5+q$	$9+q$
$6+q$	$7+q$	$2+q$

Magic sum S

Acknowledgment I would like to thank the anonymous referees for their comments and suggestions which improved the contents of this article.

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Hossein Behforooz *is Chair of the Mathematics Department at Utica College, New York. His research interests are in Approximation Theory, Spline Series, and Magic Squares. He writes 'Math is FUN, I love math, and I love to teach math'.*

Letters to the Editor

Dear Editor,

Rational approximation to square roots

I refer to Bor-Yann Chen's letter on this topic (see Volume 40, Number 1, pp. 39–40). While his method is interesting, if speed of iteration is considered important, the rational mean method (as described in my letter in Volume 39, Number 2, pp. 80–82) is much faster. That method uses the iteration formula

$$x_{r+1} = \frac{(n+1)Px_r + (n-1)x_r^{n+1}}{(n-1)P + (n+1)x_r^n},$$

where P is the number whose square root is required and n is the required root, in this case 2. Here, P may be an integer, a rational number, or a real number, but we will consider only rational numbers. Let $P = \frac{7}{3}$, as in Chen's letter, and let $x_r = a_r/b_r$ be the successive iterates. We will then have, after some manipulation,

$$x_{r+1} = \frac{a_{r+1}}{b_{r+1}} = \frac{21a_r b_r^2 + 3a_r^3}{7b_r^3 + 9a_r^2 b_r}.$$

With $x_0 = \frac{3}{2}$, as in Chen's letter, $a_1 = 333$ and $b_1 = 218$ giving $\sqrt{7/3}$ accurate to five decimal places. With $x_1 = \frac{333}{218}$, $a_2 = 443\,113\,443$ and $b_2 = 290\,085\,842$ giving $\sqrt{7/3}$ accurate to 17 decimal places, which is almost double the accuracy of Chen's fourth iterate, $\frac{36\,627}{23\,978}$.

Once again, the rational mean flexes its muscles!

Yours sincerely,

Bob Bertuello
(12 Pinewood Road
Midsomer Norton
Bath BA3 2RG
UK)

Dear Editor,

An extension of extracting a square root

This is an extension of Bob Bertuello's idea (see Volume 36, Number 1, p. 20), where we extract $r = \sqrt[m]{x}$. Bertuello proposed the following recurrence relation to find $r = \sqrt{x}$:

$$r_{n+1} = \frac{x - g^2}{r_n + g} + g,$$

where g is the initial trial and $r_0 = g$. This may be written as follows:

$$r_{n+1} = \frac{x + g r_n}{r_n + g}.$$

Similarly, to find $r = \sqrt[m]{x}$ we can use the relation

$$r_{n+1} = \frac{x + g r_n}{r_n^{m-1} + g}. \tag{1}$$

For example, to find $\sqrt[4]{20}$ a first guess could be $g = 2 = r_0$. Then we found the following unstable values (oscillating between $10.039\,22\ldots$ and $0.039\,532\ldots$):

$$\begin{aligned}
r_1 &= \frac{20 + 2 \times 2}{2^3 + 2} = 2.4,\\
r_2 &= 1.567\,24\ldots,\\
r_3 &= 3.954\,939\ldots,\\
r_4 &= 0.437\,039\ldots,\\
r_5 &= 10.018\,87\ldots,\\
r_6 &= 0.039\,733\ldots,\\
r_7 &= 10.039\,42\ldots,\\
r_8 &= 0.039\,53\ldots,\\
&\ \,\vdots\\
r_{2k+1} &= 10.039\,22\ldots, \qquad k \geq 4,\\
r_{2k} &= 0.039\,532\ldots.
\end{aligned}$$

This recurrence relation needs to be modified to result in values which converge to $r = \sqrt[m]{x}$. Suppose that we have two sequences $A_n = a_n/b_n$ and $B_n = c_n/d_n$, which satisfy

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{c_n}{d_n} = r_\infty,$$

say. Then some combined sequence $\{r_n\}$ might be appropriately selected with the following properties:

$$r_n \approx \frac{a_n}{b_n}, \qquad r_n \approx \frac{c_n}{d_n}, \qquad r_n \approx \frac{a_n + c_n}{b_n + d_n},$$

$$\lim_{n\to\infty} r_n = r_\infty = \lim_{n\to\infty} \frac{a_n + c_n}{b_n + d_n}.$$

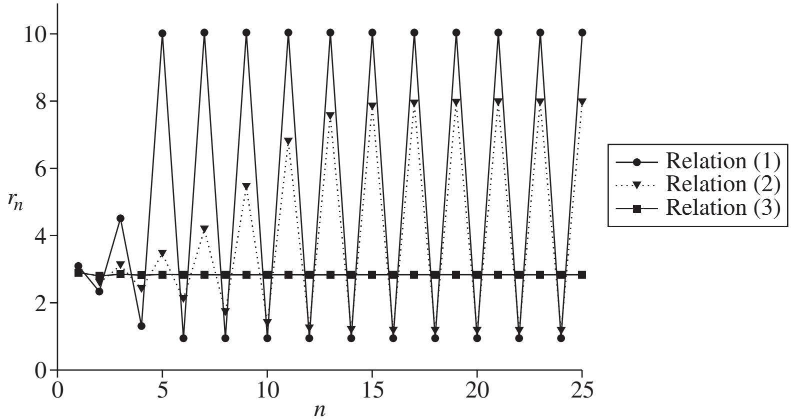


Figure 1

Using these properties, we can obtain some associated recurrence relations as follows:

$$r_{n+1} = \frac{x + gr_n + gr_n^2}{r_n^{m-1} + g + gr_n}, \tag{2}$$

$$r_{n+1} = \frac{x + gr_n + gr_n^2 + gr_n^3}{r_n^{m-1} + g + gr_n + gr_n^2}, \tag{3}$$

$$r_{n+1} = \frac{x + gr_n + gr_n^2 + gr_n^3 + gr_n^4}{r_n^{m-1} + g + gr_n + gr_n^2 + gr_n^3},$$

$$\vdots$$

$$r_{n+1} = \frac{x + gr_n \sum_{k=0}^{j} r_n^k}{r_n^{m-1} + g \sum_{k=0}^{j} r_n^k}, \quad \text{for all } j \in \mathbb{N}.$$

Then, we might use these relations for iteration with an initial guess of $g = r_0 = 2$ for $x = 20$ (see figure 1). Relation (3) yields a convergent value of $\sqrt[4]{20} = 2.114\,743\ldots$, whereas relations (1) and (2) only result in oscillating values as $n \to \infty$.

Yours sincerely,

Bor-Yann Chen

(National Ilan University
1 Shan-Lung Road
I-Lan 260
Taiwan)

Dear Editor,

Circles of best fit

In his article *Circles of Best Fit* (*Math. Spectrum*, Volume 39, Number 3), Guido Lasters gave a method for fitting a circle to a given curve at a particular point. Here we will give an alternative approach that will find a formula for the centre of the circle and its radius. Let the circle that fits the curve $y = f(x)$ at the point (x_0, y_0) be $(x-a)^2 + (y-b)^2 = r^2$. We have the following three requirements for the circle:

(i) it passes through the point (x_0, y_0),

(ii) its centre (a, b) lies on the normal to the curve at the point (x_0, y_0) and thus the circle has the same gradient as the curve at this point,

(iii) it has the same curvature, as expressed by the second derivative, as the curve at this point.

Requirement (i) implies that

$$(x_0 - a)^2 + (y_0 - b)^2 = r^2.$$

Requirement (ii) implies that

$$b - y_0 = \frac{-1}{f'(x_0)}(a - x_0).$$

Considering requirement (iii), the curvature of the curve at the point is $f''(x_0)$. Implicit differentiation on the equation of the circle gives $2(x-a) + 2(y-b)\,\mathrm{d}y/\mathrm{d}x = 0$, and thus $\mathrm{d}y/\mathrm{d}x = -(x-a)/(y-b)$. Implicit differentiation again gives

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + (y-b)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0,$$

and thus

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-1}{y-b}\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right) = \frac{-1}{y-b}\left(1 + \frac{(x-a)^2}{(y-b)^2}\right) = \frac{-r^2}{(y-b)^3}.$$

So requirement (iii) implies that

$$f''(x_0) = \frac{-r^2}{(y_0 - b)^3}.$$

From the equations obtained from the three requirements, we have

$$b = y_0 + \frac{(f'(x_0))^2 + 1}{f''(x_0)}, \qquad a = x_0 - \frac{f'(x_0)((f'(x_0))^2 + 1)}{f''(x_0)}, \qquad r^2 = \frac{((f'(x_0))^2 + 1)^3}{(f''(x_0))^2}.$$

These formulae naturally agree with the circles fitted to the curves in the examples in Lasters' article, but avoid each case having to be looked at separately from first principles.

Yours sincerely,

Paul Belcher
(Atlantic College
St Donat's Castle
Llantwit Major
Vale of Glamorgan CF61 1WF
UK)

Dear Editor,

Big numbers, happy ending

Most calculators display ten digits, so can be used to calculate the last five digits of the product of two large numbers. For example, a 10-digit calculator will not work out

$$98\,765\,432 \times 12\,345\,678, \tag{1}$$

but it will work out

$$65\,432 \times 45\,678 = 2\,988\,802\,896,$$

so the last five digits of (1) are 02 896.

In general, given two natural numbers m and n, we can express them in the form

$$m = 10^5 a + b \quad \text{and} \quad n = 10^5 c + d,$$

where b and d are their last five digits; whence,

$$mn = 10^5(10^5 ac + ad + bc) + bd,$$

and the last five digits of bd are the same as the last five digits of mn.

Using this, we can find the last *five* digits of 1249^{1249}, not merely the last four as asked for by Abbas Roohol Amini in Volume 39, Number 3, page 124; they are 61 249.

Yours sincerely,

Bob Bertuello
(12 Pinewood Road,
Midsomer Norton,
Bath BA3 2RG,
UK)

Dear Editor,

Identical digits

Quite by chance I came across the following ‘coincidence’:

$$817^3 = \mathbf{54}5\,\mathbf{338}\,513,$$
$$816^3 = \mathbf{54}3\,\mathbf{338}\,496.$$

Five out of nine, indeed five out of the first six digits in these consecutive cubes are identical. Exploring further with a little algebra and a calculator, I hit on a method to compute other similar examples. Using the method I then found

$$2\,581\,988\,898^3 = \mathbf{17\,213\,259\,3}2\mathbf{7\,045\,183\,24}5\,01\mathbf{3}\,606\,792,$$
$$2\,581\,988\,897^3 = \mathbf{17\,213\,259\,3}0\mathbf{7\,045\,183\,24}4\,57\mathbf{3}\,810\,273.$$

These consecutive cubes are identical in 18 of the first 19 digits and in over 65% of all the digits. May I challenge readers to improve on this. You will need a programme for multiplying large numbers as a normal electronic calculator will not deal with 29-digit numbers!

Yours sincerely,

Alastair Summers
(57 Conduit Road
Stamford
Lincolnshire PE9 1QL
UK)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

40.9 Solve the simultaneous equations

$$\begin{aligned} 2008\log[x] + \{\log y\} &= 0, \\ 2008\log[y] + \{\log z\} &= 0, \\ 2008\log[z] + \{\log x\} &= 0, \end{aligned}$$

where $[\cdot]$ and $\{\cdot\}$ denote the integer part and the fractional part respectively, and the logs are to base 10.

(Submitted by Mihály Bencze, Brasov, Romania)

40.10 Let $n > 7$ be an integer such that $n-1$ and $n+1$ are both prime. Show that $n^2(n^2-4)(n^2-9)$ is divisible by 2 721 600.

(Submitted by Roger Cook, Pembroke, UK)

40.11 For a positive integer n, prove the identity

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

(a) algebraically, (b) graphically.

(Submitted by Paul Levrie, Karel de Grote – Hogeschool, Antwerpen, Belgium)

40.12 Show that, if a 5-digit number is divisible by 41, then so is every cyclic permutation of its digits, for example

$$28\,577 = 41 \times 697, \qquad 57\,728 = 41 \times 1408.$$

(Submitted by Bob Bertuello, Midsomer Norton, Bath, UK)

Solutions to Problems in Volume 40 Number 1

40.1 Prove that

$$(\sqrt{3}+1)\sin\left\{\frac{5\pi}{6}\right\} - (\sqrt{3}-1)\cos\left\{\frac{5\pi}{6}\right\} = 2(\sin 2 + \cos 2),$$

where $\{\cdot\}$ denotes the fractional part.

Solution by Bor-Yann Chen, University of California, Irvine, USA

The left-hand side is equal to

$$\begin{aligned}
&2\left[\left(\sin\frac{\pi}{3}+\cos\frac{\pi}{3}\right)\sin\left(\frac{5\pi}{6}-2\right)-\left(\sin\frac{\pi}{3}-\cos\frac{\pi}{3}\right)\cos\left(\frac{5\pi}{6}-2\right)\right]\\
&\quad=2\left[\cos\left(\frac{\pi}{3}-\frac{5\pi}{6}+2\right)+\sin\left(\frac{5\pi}{6}-2-\frac{\pi}{3}\right)\right]\\
&\quad=2\left[\cos\left(2-\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{2}-2\right)\right]\\
&\quad=2(\sin 2+\cos 2).
\end{aligned}$$

Also solved by Henry Ricardo (Evers College, New York, USA) and M. A. Khan, (Lucknow, India).

40.2 A student chooses between two exam strategies. In the first, he answers three questions. If at least two answers are correct, then he passes, otherwise he fails. In the second, he answers an unlimited number of questions until he answers two consecutive questions correctly, in which case he passes, or two consecutive questions incorrectly, in which case he fails. If the probability of his answering each question correctly is the same and these are independent, which strategy is preferable?

Solution by Ian Smith, Massachusetts Institute of Technology, USA

Denote by p the probability of answering a given question correctly. The probability of the student passing the exam in the three-question strategy is

$$p^3+3p^2(1-p)=3p^2-2p^3.$$

In the second strategy, denote the probability of his passing on the nth question by p_n. Then he passed the $(n-1)$th question and, counting back, they then alternate fail–pass. Hence,

$$p_n=\begin{cases} p^2p^{(n-2)/2}(1-p)^{(n-2)/2} & \text{when } n \text{ is even,}\\ p^2(1-p)p^{(n-3)/2}(1-p)^{(n-3)/2} & \text{when } n \text{ is odd.}\end{cases}$$

The probability of his passing is

$$\begin{aligned}
&\sum_{\substack{n=2\\ n\text{ even}}}^{\infty}p_n+\sum_{\substack{n=3\\ n\text{ odd}}}^{\infty}p_n\\
&\quad=p^2[1+p(1-p)+p^2(1-p)^2+\cdots]\\
&\qquad+p^2(1-p)[1+p(1-p)+p^2(1-p)^2+\cdots]\\
&\quad=\frac{p^2(2-p)}{1-p(1-p)}.
\end{aligned}$$

Now

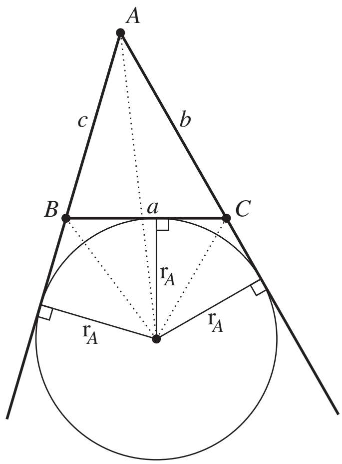
$$3p^2-2p^3-\frac{p^2(2-p)}{1-p(1-p)}=\frac{p^2(1-p)^2(1-2p)}{1-p(1-p)},$$

so the first strategy is better than the second if $0<p<\frac{1}{2}$; if $\frac{1}{2}<p<1$ then the second strategy is better than the first. If $p=0,\ \frac{1}{2}$ or 1, then the two strategies give the same probability of passing.

40.3 Let a, b, c be the length of the sides of a triangle ABC and let r_A, r_B, r_C be the radii of its escribed circle. Prove that

$$\frac{a^2}{\mathrm{r}_B\mathrm{r}_C} + \frac{b^2}{\mathrm{r}_C\mathrm{r}_A} + \frac{c^2}{\mathrm{r}_A\mathrm{r}_B} \ge 4.$$

Solution by M. A. Khan, Lucknow, India



We have

$$\Delta = \tfrac{1}{2}b\mathrm{r}_A + \tfrac{1}{2}c\mathrm{r}_A - \tfrac{1}{2}a\mathrm{r}_A,$$

where Δ denotes the area of triangle ABC. Hence,

$$\mathrm{r}_A = \frac{\Delta}{s-a},$$

where $s = \frac{1}{2}(a+b+c)$. Similarly,

$$\mathrm{r}_B = \frac{\Delta}{s-b}, \qquad \mathrm{r}_C = \frac{\Delta}{s-c},$$

and the required inequality becomes

$$a^2(s-b)(s-c) + b^2(s-c)(s-a) + c^2(s-a)(s-b) \ge 4\Delta^2.$$

Since $\Delta^2 = s(s-a)(s-b)(s-c)$, this reduces to

$$\frac{a^2}{s(s-a)} + \frac{b^2}{s(s-b)} + \frac{c^2}{s(s-c)} \ge 4,$$

or

$$a\left(\frac{1}{s-a} - \frac{1}{s}\right) + b\left(\frac{1}{s-b} - \frac{1}{s}\right) + c\left(\frac{1}{s-c} - \frac{1}{s}\right) \ge 4,$$

or

$$\frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} - \frac{a+b+c}{s} \ge 4,$$

or

$$\left(\frac{s}{s-a}-1\right)+\left(\frac{s}{s-b}-1\right)+\left(\frac{s}{s-c}-1\right)-2\geq 4,$$

or

$$\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c}\geq 9.$$

Since $s=(s-a)+(s-b)+(s-c)$, this is equivalent to

$$\frac{s-b}{s-a}+\frac{s-c}{s-a}+\frac{s-c}{s-b}+\frac{s-a}{s-b}+\frac{s-a}{s-c}+\frac{s-b}{s-c}\geq 6.$$

This inequality follows from the arithmetic–geometric mean inequality.

Also solved by Mihály Bencze (Brasov, Romania).

40.4 You are given a straight line with equation $ax+by+c=0$ and a point (p,q) not on the line. The line is then 'altered' to go through (p,q) by adjusting either a alone, b alone, or c alone. If s is the percentage change required in a if we alter a alone, t is the percentage change required in b if we alter b alone, and u is the percentage change required in c if we alter c alone, show that

$$\frac{1}{s}+\frac{1}{t}+\frac{1}{u}+\frac{1}{100}=0.$$

Solution by M. A. Khan, Lucknow, India

We have

$$\left(a+\frac{sa}{100}\right)p+bq+c=0,$$

or

$$\frac{100}{s}=\frac{-ap}{ap+bq+c}.$$

Similarly,

$$\frac{100}{t}=\frac{-bq}{ap+bq+c},\qquad \frac{100}{u}=\frac{-c}{ap+bq+c}.$$

Hence,

$$\frac{100}{s}+\frac{100}{t}+\frac{100}{u}=-1.$$

which gives the result.

Also solved by Bor-Yann Chen (University of California, Irvine, USA).

Correction to the Solution of Problem 39.9

Bruce Shawyer has written to point out that the solution to Problem 39.9 published in Volume 40 Number 2 is incomplete in that the consecutive odd primes may not differ by 2. The problem is to show that the sum of two consecutive odd primes p and q (say) is the product of at least three primes. Since

$$p<\frac{p+q}{2}<q$$

and $\frac{1}{2}(p+q)$ is an integer, $\frac{1}{2}(p+q)$ cannot be prime and so must be the product of at least two (not necessarily distinct) primes. Thus, $p+q$ is the product of at least three primes, one of which is 2. Other readers have provided a similar solution.

Reviews

Hesiod's Anvil: Falling and Spinning Through Heaven and Earth. By Andrew J. Simoson. MAA, Washington, DC, 2007. Hardback, 344 pages, $48.95 (ISBN 0-88385-336-8).

This is an interesting, innovative, and unusual book on the mathematics of motion. The author looks at how motion has been viewed in the past by poets, storytellers, and scientists and then proceeds to investigate the problems using mathematics. The mathematics used would be understandable to a good first-year university student with a solid grounding in mechanics and calculus. The book provides excellent examples on how to set up mathematical models and the author shows a willingness and an enthusiasm to explore a variety of models of varying complexity within each chapter. There are fifteen chapters in total. The format of each chapter is a preamble, followed by several writers' thoughts on motion, each accompanied by the appropriate mathematical investigation of a suitably constructed model. The chapters conclude with a set of exercises for the reader to attempt, following on from the ideas that have been pursued. There are comments on selected exercises at the end of the book. There is also a short appendix giving definitions and some formulae to assist with the mathematics used in the book. The writers that the author comments on are wonderfully varied, ranging from Homer to Stephen Hawking in time, Archimedes to Zeno alphabetically, and including the styles of writing of Leonardo da Vinci, Isaac Newton, Jules Verne, and Terry Pratchett. It is refreshing that the author does not feel at all constrained in the items that he wishes to comment on and how he can find worthwhile mathematics to investigate and solve in each case. This is a book that I could wholeheartedly recommend as it creates enjoyment, amusement, and thought. It is a book to be savoured, say a chapter at a time. I think that the author can be very pleased that he is passing on well the concept of telling stories in the mathematical classroom.

Atlantic College **Paul Belcher**

Other books received

An Introduction to Categorical Data Analysis. By Alan Agresti. John Wiley, Chichester, 2nd edition, 2007. Hardback, 372 pages, £52.95 (ISBN 978-0-471-22618-5).

Combinatorics, Complexity, and Chance: A Tribute to Dominic Welsh. Edited by Geoffrey Grimmett and Colin McDiamond. Oxford University Press, 2007. Hardback, 350 pages, £39.50 (ISBN 978-0-19-857127-8).

Differential Geometry and Its Applications. By John Oprea. MAA, Washington, DC, 2007. Hardback, 510 pages, $54.95 (ISBN 978-0-88385-748-9).

Empirical Methods in Short-Term Climate Prediction. By Huug van den Dool. Oxford University Press, 2006. Hardback, 288 pages, £49.95 (ISBN 0-19-920278-8).

Hypersolids and Quasi-Hypersolids. By Patrick Taylor. Nattygrafix, Ipswich, 2006. Paperback, 80 pages, £6.00 (ISBN 0-9516701-7-4).

Models for Intensive Longitudinal Data. By Theodore A. Walls and Joseph L. Schafer. Oxford University Press, 2006. Hardback, 288 pages, £38.99 (ISBN 0-19-517344-4).

The Porous Medium Equation: Mathematical Theory. By Juan Luis Vazquez. Oxford University Press, 2006. Hardback, 646 pages, £65.00 (ISBN 0-19-856903-3).

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POPULAR LECTURES 2008

This year's lectures will take place in London during July and in Birmingham during September.

The Lecturers will be:

Dr Tadashi Tokieda
(University of Cambridge)

‘Toy Models’

Abstract to follow

Dr Reidun Twarock
(The University of York)

‘Know Your Enemy – Viruses under the Mathematical Microscope’

Mathematics can help us understand the structure of viruses and the principles responsible for their formation. Can this knowledge be used to find their Achilles' heel and develop new strategies for anti-viral drug design?

Once details are finalised, they will appear on the LMS website (www.lms.ac.uk).

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