Numerical Methods of PDEs

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Outline

1. Numerical Partial Derivatives
2. Methods of time-dependent PDEs
   a. FTCS
   b. Crank-Nicolson
3. Example: Heat equation with FTCS
4. Tutorial: Elastic Beam

Numerical Derivatives

Forward difference:

$$f'(x) = \frac{f(x+h) - f(x)}{h} - h\frac{f''(x)}{2!} - h^2\frac{f'''(x)}{3!} - \ldots$$

Backward difference:

$$f'(x) = \frac{f(x) - f(x-h)}{h} + h\frac{f''(x)}{2!} - h^2\frac{f'''(x)}{3!} - \ldots$$

Combining backward and forward differences:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \mathcal{O}(h^2)$$

Nth-order Derivatives

We can take *n*th-order derivatives by repeated central differences, e.g.

$$\begin{aligned}\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{f(x+\frac{1}{2}h) - f(x-\frac{1}{2}h)}{h}\right) \\ &= \frac{1}{h}\left(\frac{f(x+h) - f(x)}{h} - \frac{f(x) - f(x-h)}{h}\right) \\ &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}\end{aligned}$$

In general, we can write

$$\frac{\partial^n w}{\partial x^n} = \frac{1}{\Delta x^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} w\left(x + \frac{(n-2i)\Delta x}{2}, t\right)$$

Partial Spatial Derivatives

We can extend our procedure for *n*th-order derivatives to partial spatial derivatives, e.g.

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial y}\left[\frac{f(x+h/2,y)-f(x-h/2,y)}{h}+\mathcal{O}(h^2)\right] \\
&= \frac{f(x+h/2,y+h/2)-f(x+h/2,y-h/2)+f(x-h/2,y-h/2)-f(x-h/2,y+h/2)}{h^2}+\mathcal{O}(h^2)
\end{aligned}
$$

We can reduce time-independent boundary value problems to solving a set of algebraic equations with these derivatives

FTCS Method

Forward-time centered-space (FTCS) method is an analog of Euler’s method for solving time-dependent ODEs.

We evolve the solution in time using Euler’s method while computing spatial derivatives using central differences.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

FTCS Stability

Using Fourier decomposition, we can show that FTCS solutions of the heat equation are numerically stable for

$$\Delta t \leq \frac{\Delta x^2}{2\alpha}$$

This bound comes from the fact that FTCS does not “mix” Fourier modes of the solution, so for the heat equation, we have

$$c_k(t + \Delta t) = \left[1 - \Delta t \frac{4\alpha}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right)\right]$$

Crank-Nicolson Method

Crank-Nicolson is an implicit version of the FTCS method. For a wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

Crank-Nicolson gives us the stable time-evolution rule

$$\psi(x, t+\Delta t) - \frac{1}{2}\Delta t \dot{\psi}(x, t+\Delta t) = \psi(x,t) + \frac{1}{2}\Delta t \dot{\psi}(x,t)$$

$$\dot{\psi}(x, t+\Delta t) - \Delta t \frac{v^2}{2\Delta x^2}\Big[\psi(x+\Delta x, t+\Delta t) + \psi(x-\Delta x, t+\Delta t) - 2\psi(x, t+\Delta t)\Big]$$

$$= \dot{\psi}(x,t) + \Delta t \frac{v^2}{2\Delta x^2}\Big[\psi(x+\Delta x, t) + \psi(x-\Delta x, t) - 2\psi(x,t)\Big]$$

Example: Heat Equation

Heat Equation

Heat diffusion is governed by the PDE

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad T(x=0,t) = T_a, \quad T(x=L,t) = T_b, \quad T(x,t=0) = T_0$$

Time part (Euler): $T(x,t_{i+1}) = T(x,t_i) + \Delta t \frac{\partial T(x,t_i)}{\partial t}$

Spatial part (central difference):

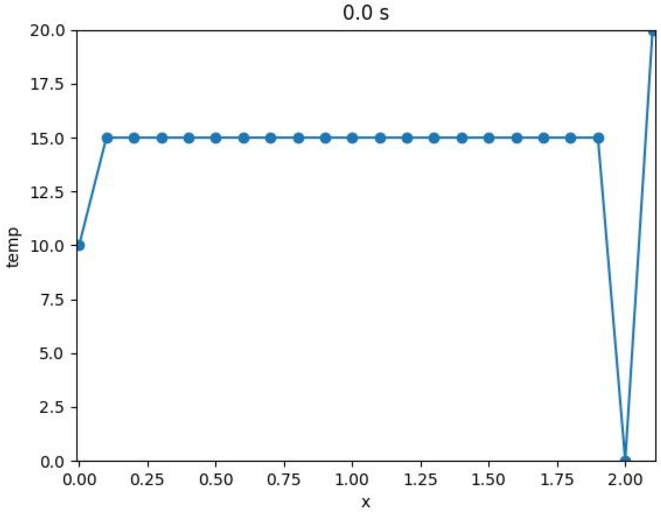
$$\frac{\partial T(x,t_i)}{\partial t} = \alpha \frac{\partial^2 T(x,t_i)}{\partial x^2} = \frac{\alpha}{\Delta x^2} \left[T(x_{j+1},t_i) - 2T(x_j,t_i) + T(x_{j-1},t_i) \right]$$

Heat Equation $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$

- The iterative solution to the heat equation

$$T(x_j, t_{i+1}) = T(x_j, t_i) + \alpha \frac{\Delta t}{(\Delta x)^2} \left[T(x_{j+1}, t_i) - 2T(x_j, t_i) + T(x_{j-1}, t_i)\right]$$

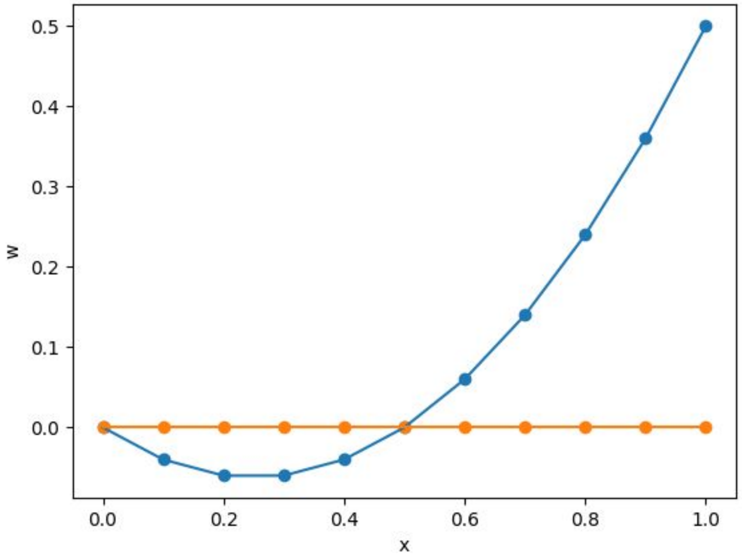
- Example script solves with fixed-temperature BCs



Tutorial: Elastic Beams

Background: Elastic Beam Equation

The transverse displacements w(x,t) of a beam are governed by the PDE



$$0 = \rho_s h \frac{\partial^2 w}{\partial t^2} + B \frac{\partial^4 w}{\partial x^4} + P \frac{\partial^2 w}{\partial x^2}$$

Assignment

- Examine the time evolution of an elastic beam, given some boundary conditions and some initial conditions
- Clamped boundary conditions:

$$\frac{\partial^n w}{\partial x^n} = 0.$$

- In this problem, the 2 leftmost and 2 rightmost points will be fixed at w(x,t)=0.

Assignment

1. Fill in function to evaluate *n*th-order spatial derivatives
2. Fill in functions to evolve the system using FTCS
3. Compare FTCS solution's stability with pre-filled Crank-Nicolson solution
4. Animate your solution using the given code

$$\frac{\partial^n w}{\partial x^n} = \frac{1}{\Delta x^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} w\left(x + \frac{(n-2i)\Delta x}{2}, t\right)$$

$$0 = \frac{\partial^2 w}{\partial t^2} + c^2 \frac{\partial^4 w}{\partial x^4}$$