جامعة بيروت العربية
BEIRUT ARAB UNIVERSITY

# Math 441
# Introduction to Complex Analysis

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- **Chapter 1: Complex Numbers** (Complex Numbers and Properties, Modulus, Argument, Forms of Complex Numbers, Euler’s Theorem, De Moivre’s Theorem).

- **Chapter 2: Complex Functions** (Main Complex Functions, Continuity, Differentiability, Holomorphic and Analytic Complex Functions, Taylor’s Series, Maclaurin’s Series, Harmonic Functions).

- **Chapter 3: Complex Integration** (Contour Integration, Liouville’s Theorem, Laurent Series, Cauchy Integral Theorem, Singularities, Residues, Cauchy’s Residue Theorem)

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# Chapter 1
# Complex Numbers

$$i^2 = -1$$

# Chapter 1: Complex Numbers

**Definition 1:** A **complex number** is a number of the form $z = a + ib$, such that $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. $a$ is called **the real part** of $z$ and denoted by $Re(z)$ and $b$ is called **the imaginary part** of $z$ and denoted by $Im(z)$.

Notice that $i = \sqrt{-1} \Longrightarrow i^2 = -1 \Longrightarrow i^3 = -i \Longrightarrow i^4 = 1$.

**Remark 1:** Let $\mathbb{C}$ be the set of all complex numbers; that is

$$\mathbb{C} = \left\{a + ib, \text{where } a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\right\}.$$

We see that:

1) $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

2) There is an isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$ defined by: $f(a + ib) = (a, b)$.

# Chapter 1: Complex Numbers

<u>**Remark 2:**</u> Let $z = a + ib$ and $w = c + id$ be two complex numbers. Then

1) $z + w = (a + c) + i(b + d)$.

2) $z - w = (a - c) + i(b - d)$.

3) $z.w = ac + adi + cbi + bdi^2 = (ac - bd) + i(ad + cb)$.

4) To compute $\frac{z}{w} = \frac{a+ib}{c+id}$, we multiply $z$ and $w$ by the conjugate of the denominator $w$.

5) $z = w$ if and only if $a = c$ and $b = d$.

<u>**Definition 2:**</u> A complex number $z = a + ib$ is said to be:

1) **Real** if $b = 0$.

2) **Pure imaginary** if $a = 0$.

# Chapter 1: Complex Numbers – Exercise

**Ex. 1:** Reduce the following complex numbers.

a) $i^5 + i^4 + i^3 + i^2 + i + 1.$

b) $i^{105} + i^{32} + i^{20} - i^{34}.$

c) $(1 + i)(2 - i).$

d) $(1 - i)(1 + 2i)(3 - i).$

e) $\frac{5+5i}{3-4i}$ .

f) $\frac{1+i}{2-2i} + \frac{2}{2+i}$ .

# Chapter 1: Complex Numbers – Exercise

**Ex. 2:**

a) Find the set of points $z \in \mathbb{C}$ for which $Re\left(\frac{1}{z}\right) = 0$.

b) Find the set of points $z \in \mathbb{C}$ for which $z^2 \in \mathbb{R}$.

c) Find the set of points $z \in \mathbb{C}$ for which $Im(z^2 + z) = 0$.

d) Find the set of points $z \in \mathbb{C}$ for which $Re(z^2) = 0$.

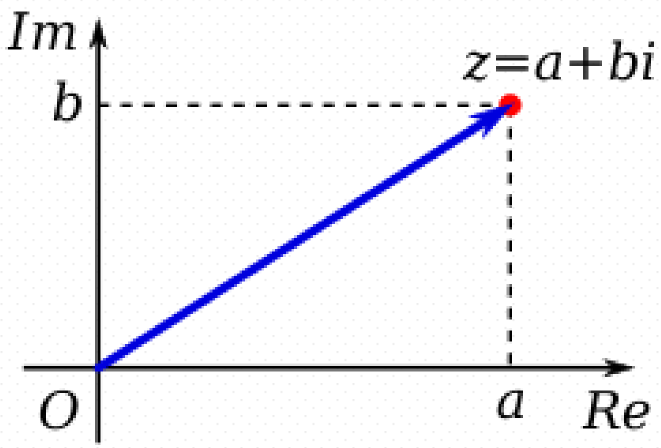
e) Find the set of points $z \in \mathbb{C}$ for which $z + \frac{1}{z} = 0$.

f) Find the set of points $z \in \mathbb{C}$ for which $Re(z) = Im(z)$.

# Chapter 1: Complex Numbers

**Definition 3:** Let $z = a + ib$ be a complex number. The **conjugate** of $z$ is the complex number $\bar{z} = a - ib$.

**Definition 4:** For any complex number $z = a + ib$ we associate a point $M$ on the $xoy$- plane with coordinates $M = (a, b)$. $M$ is called the **image** of $z$.



# Chapter 1: Complex Numbers

**Definition 5:** Let $z = a + ib$ be a complex number and let $M = (a, b)$ be its image on the $xoy$- plane. We define the **modulus** of $z$, denoted by $|z|$, by:

$$|z| = OM = \sqrt{a^2 + b^2}.$$

**Remark 3:** Let $z$ and $w$ be two complex numbers, then

a) $|z.w| = |z|.|w|$.

b) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ .

c) $|z^n| = |z|^n$.

d) $|z| = |-z| = |\bar{z}| = |-\bar{z}|$.

# Chapter 1: Complex Numbers – Exercise

**Ex. 3:** Let $z = a + ib$ be a complex number. Show that

$$\frac{Re(z) + Im(z)}{\sqrt{2}} \leq |z| \leq |Re(z)| + |Im(z)|.$$

# Chapter 1: Complex Numbers – Exercise

**Ex. 4:** Prove that there is no complex number $z$ such that $|z| - z = i$.

# Chapter 1: Complex Numbers – Exercise

**Ex. 5:** Let $z_1$ and $z_2$ be two complex numbers. Show that

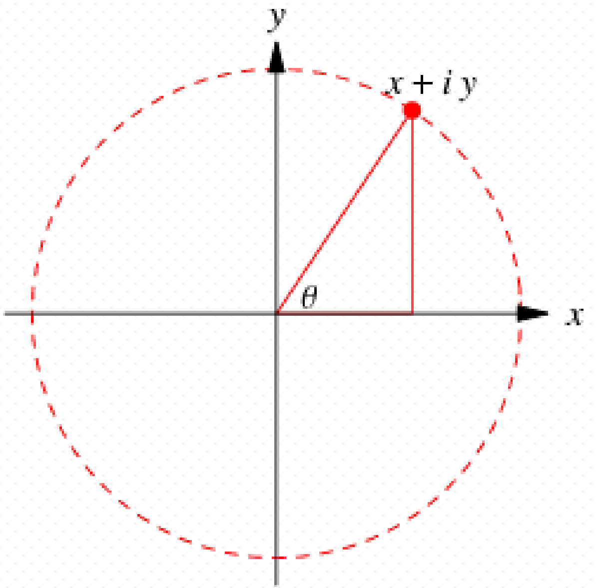
a) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$.

b) $|z_1 + z_2| \leq |z_1| + |z_2|$.

# Chapter 1: Complex Numbers

**Definition 6:** Let $z = a + ib$ be a complex number and let $M = (a, b)$ be its image on the $xoy$- plane. We define the **argument** of $z$, denoted by $\arg(z)$, as follows.

$$\arg(z) = \left(\vec{u}, \overrightarrow{OM}\right) + 2k\pi.$$




To find $\arg(z)$ we should do two steps.

1) Find $\tan^{-1}\left|\frac{b}{a}\right|$.

2) Study the quadrant where $M$ belongs.

a) If $M \in Q_1$ then $\arg(z) = \tan^{-1}\left|\frac{b}{a}\right| + 2k\pi$.

b) If $M \in Q_2$ then $\arg(z) = \pi - \tan^{-1}\left|\frac{b}{a}\right| + 2k\pi$.

c) If $M \in Q_3$ then $\arg(z) = \pi + \tan^{-1}\left|\frac{b}{a}\right| + 2k\pi$.

d) If $M \in Q_4$ then $\arg(z) = -\tan^{-1}\left|\frac{b}{a}\right| + 2k\pi$.

The **principal value** of the argument is the argument of $z$ restricted to the interval $]-\pi, \pi]$.

# Chapter 1: Complex Numbers

**Remark 4:** Let $z$ and $w$ be two complex numbers, then the following hold true.

a) $z$ is real $\Leftrightarrow \arg(z) = k\pi$.

b) $z$ is pure imaginary $\Leftrightarrow \arg(z) = \frac{\pi}{2} + k\pi$.

c) $\arg(z.w) = \arg(z) + \arg(w)$.

d) $\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$.

e) $\arg(\bar{z}) = -\arg(z)$.

f) $\arg(z^n) = n\arg(z)$.

# Chapter 1: Complex Numbers

**Definition 7:** Let $z = a + ib$ be a complex number (This form is called the algebraic form of $z$). Set $r = |z|$ and $\theta = \mathrm{Arg}(z)$. We define two additional forms of $z$ in the following way.

1) The **trigonometric form** of $z$ is

$$z = r(\cos\theta + \mathrm{i}\sin\theta).$$

2) The **exponential form** of $z$ is

$$z = re^{i\theta}.$$

Note that any form of $\theta$ works also.

Both previous forms could be denoted by $(r, \theta)$.

# Chapter 1: Complex Numbers – Exercise

**Ex. 6:** Find the modulus and the argument of the following complex numbers, then write their trigonometric and exponential forms.

a) $z = 5.$

b) $z = -2i.$

c) $z = -3 + 3i.$

d) $z = \frac{1+i}{1-i}\ .$

e) $z = (6 - 6i)^{10}.$

f) $z = (1 + i)(-2 - 2i).$

# Chapter 1: Complex Numbers – Exercise

**Ex. 7:** Describe or graph geometrically the regions of the $xoy$- plane determined in each of the following.

a) $|z+2| < 1.$

b) $|z+1| \geq 2.$

c) $Re(z)^2 + Im(z)^2 = 1.$

d) $0 < \arg(z) < \pi.$

e) $Im(z) \geq 2.$

f) $Re(z^2) \geq 0.$

# Chapter 1: Complex Numbers

**Theorem 1:** **(Euler's Theorem)** For any real number $\theta$ we have:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

**Proof:**

**Theorem 2:** **(De Moivre's theorem)** Let $z = r(\cos\theta + i\sin\theta)$. Then for any integer $n$, we have:

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)].$$

**Proof:**

# Chapter 1: Complex Numbers – Exercise

**Ex. 8:** Use De Moivre's theorem to reduce the following complex numbers.

a) $(2+2i)^4$.

b) $(1-i)^6$.

c) $(1+i)^9$.

d) $\left(1+i\sqrt{3}\right)^6$.

# Chapter 1: Complex Numbers – Exercise

**Ex. 9:**

a) Find all values of $\sqrt{i}$ and denote them by the $(r, \theta)$ form.

b) Find all values of $\sqrt[3]{-1}$ and denote them by the $(r, \theta)$ form.

c) Find all values of $\sqrt{-1-i}$ and denote them by the $(r, \theta)$ form.

d) Find all values of $\sqrt[3]{-8}$ and denote them by the $(r, \theta)$ form.

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# End of Chapter 1

$$i^2 = -1$$