Math 31
Limits and the Derivative

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Unfinished!

Contents

Factoring Brief Review

Differences of Square

$$x^2 - 4 = (x+2)(x-2)$$

Polynomial

$$2x^2 + 3x - 2$$
$$\longrightarrow (2x^2 + 4x)(-x - 2)$$
$$\longrightarrow \ 2x(x+2) - 1(x+2)$$
$$\longrightarrow (2x-1)(x+2)$$

Radical Fractions

- Multiply everything by monomial denominator
  $\frac{2}{\sqrt{2x}} \longrightarrow \frac{2\sqrt{2x}}{2x} \longrightarrow \frac{\sqrt{2x}}{x}$
- Multiply everything by conjugate for polynomial denominators
  $\frac{3}{2+\sqrt{x}} \times \frac{2-\sqrt{x}}{2-\sqrt{x}} = \frac{6-3\sqrt{x}}{4-2\sqrt{2}+2\sqrt{x}-x} = \frac{6-3\sqrt{x}}{4-x}$

Mixed Radicals

$$\sqrt{162} \longrightarrow \sqrt{9^2 \times 2} \longrightarrow \sqrt{9^2} \times \sqrt{2} \longrightarrow 9\sqrt{2}$$

Absolute Polynomial

$$|x-1| = 3$$

$$x - 1 = 3,\ x = 4$$
$$x - 1 = -3,\ x = -2$$

Adding/Subtracting Fractions

Multiply both terms so that the denominators are the same, then add/subtract.
$\frac{2}{x-1} - \frac{3}{x+3}$
$\longrightarrow \frac{2(x+3)}{(x-1)(x+3)} - \frac{3(x-1)}{(x-1)(x+3)}$

$\longrightarrow \frac{(2x+6)-(3x-3)}{(x-1)(x+3)}$
$= \frac{-x+3}{(x-1)(x+3)}$

Piecewise Functions

Piecewise functions are functions with multiple inequalities/restrictions that dictate which function to use at specific x values.

When graphing...

- if an inequality is less/greater than a value, the plot point is not filled in
- if an inequality is less/greater than OR equal to a value, the plot point is filled in
- if x of different functions equal the same value, the graphs are continuous, and are filled in if one of the functions is inclusive

If the inequalities do not state a function for a specific x value (e.g. $x = 2$ for $2 < x < 2$) then that value **DNE**. (does not exist)

Rational Function

A function with a polynomial in the numerator and denominator.

Vertical Asymptotes

Zeros of the denominator of a rational function.

x may approach these values, but never touch them.

Point of Discontinuity

Any vertical asymptote (zeros of denominator) before simplifying a rational function.

These vertical asymptotes only applies to the unsimplified form; this makes it a point of discontinuity.

These points are gaps in a graph line, have no y value, and therefore make a graph discontinuous.

Horizontal Asymptotes

Horizontal asymptotes describe the trend of a function.

The graph line can cross over it fine, as opposed to vertical asymptotes.

Determining Horizontal Asymptotes

- degree of numerator $<$ degree of denominator
  $\longrightarrow y = 0$
- degree of numerator $=$ degree of denominator
  $\longrightarrow y = \frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}}$
- degree of numerator $>$ degree of denominator
  $\longrightarrow$ Divergent (no horizontal asymptote)

Limits

$$\lim_{x \to a} f(x) = b$$

The limit of $f(x)$ as x approaches a is b.

A limit is the value of y as the x approaches a specific value, as opposed to equaling a specific value. This is useful for points of discontinuity, where the exact value doesn't exist, but the value approaching does.

For instance, if the point of discontinuity of $f(x)$ is $x = -1$, then...

$$f(-1) = \text{DNE}$$

$$\lim_{x \to -1} f(x) = -1$$

Properties

- $c =$ constant value
  $\lim_{x \to a} c = c$
  $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$
- $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$
- $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$
- The rest of the rules can be summarized as limits have distributive property.
  e.g. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

Limits of Continuous Functions

Any Polynomial

$y = f(x)$ is continuous at every value of a.

$$\lim_{x \to a} f(x) = f(a)$$

Just substitute x in the function with a.

Any Rational Function

$y = \frac{f(x)}{g(x)}$ is continuous at every value of a as long as $g(x) \neq 0$. (cannot divide by zero)

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)},\ g(a) \neq 0$$

Just substitute x in the function with a, unless a makes the denominator equal to zero. If so, refer to the next section.

Any Radical Function

$y = \sqrt{f(x)}$ is continuous at every value of a as long as $f(x) \geq 0$. (cannot root negatives)

$$\lim_{x \to a} \sqrt{f(x)} = \sqrt{f(a)},\ f(a) \geq 0$$

Just substitute x in the function with a, unless a makes the function equal to a negative. If so, refer to the next section.

Limits of Discontinuous Functions

Identically to finding points of discontinuity, simplify/rationalize the function in a limit if it does illegal math (divide by zero, root negatives) until it doesn't.

$$\lim_{x \to 4} \left(\frac{x^2 - 16}{x - 4}\right)$$

$$\lim_{x \to 4} \left(\frac{(x-4)(x+4)}{x - 4}\right)$$

$$\lim_{x \to 4} (x + 4) = 8$$

One-sided Limits

Limits of a function can be seperated into the value of approaching from the left and from the right. This is denoted with a superscript on a.

- From the left ($x < a$): $\lim_{x \to a^-} f(x)$
- From the right ($x > a$): $\lim_{x \to a^+} f(x)$

This is only really relevant for graphs that end (such as $y = \sqrt{x}$, approaching from the side without a line is DNE) or piecewise functions.

Continuous or Discontinous?

Continuous

Continuous functions have a left approaching limit and a right approaching limit equal to one another.

$$\begin{aligned} \text{if} \quad \lim_{x \to a^-} f(x) &= \lim_{x \to a^+} f(x) \\ \text{then} \quad \lim_{x \to a} f(x) &= \lim_{x \to a} f(a) \end{aligned}$$

Discontinuous

Discontinuous functions have a left approaching limit and a right approaching limit not equal to one another.

$$\begin{aligned} \text{if} \quad \lim_{x \to a^-} f(x) &\neq \lim_{x \to a^+} f(x) \\ \text{then} \quad \lim_{x \to a} f(x) &= \text{DNE} \end{aligned}$$

Limits to Infinity

Limits of infinity either approach a value or DNE.

Rules

- Limits to infinity of normal numbers is often DNE. e.g.
  - $\lim_{n\to\infty} r^n = \text{DNE}$ ($\textit{iff}$ $|r| > 1$)
  - $\lim_{x\to\infty} 2^x = \text{DNE}$
  - $\lim_{x\to\infty} \frac{1}{x^{-3}} = \text{DNE}$
- Limits to infinity of fractions with variable denominators is often infinity small, so 0. e.g.
  - $\lim_{n\to\infty} r^n = 0$ ($\textit{iff}$ $|r| < 1$)
  - $\lim_{n\to\infty} \frac{1}{n} = 0$
- The limit of infinity does not exist.
  $\lim_{n\to\infty} 3^n = DNE$
- $\lim_{n\to\infty} (-1)^n = DNE$

Finding Limits to Infinity

- Any fraction with a variable in the denominator will be zero.

$$\lim_{x\to\infty} \frac{a}{x^b} = 0$$

- Because of this, multiply a limit by something in order to put a variable under the terms, making them equal zero. e.g.

$$\lim_{x\to\infty} \frac{6n+9}{3n-2}$$

$$\frac{6n+9}{3n-2} \times \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$\frac{6+\frac{9}{n}}{3-\frac{2}{n}} \longrightarrow \frac{6+0}{3-0}$$

$$\lim_{x \to \infty} \frac{6n+9}{3n-2} = 2$$

Derivatives

The derivative of a function gives the slope of a tangent line that just touches the point $(x, f(x))$.

Formulas

$$f'(x) = y' = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

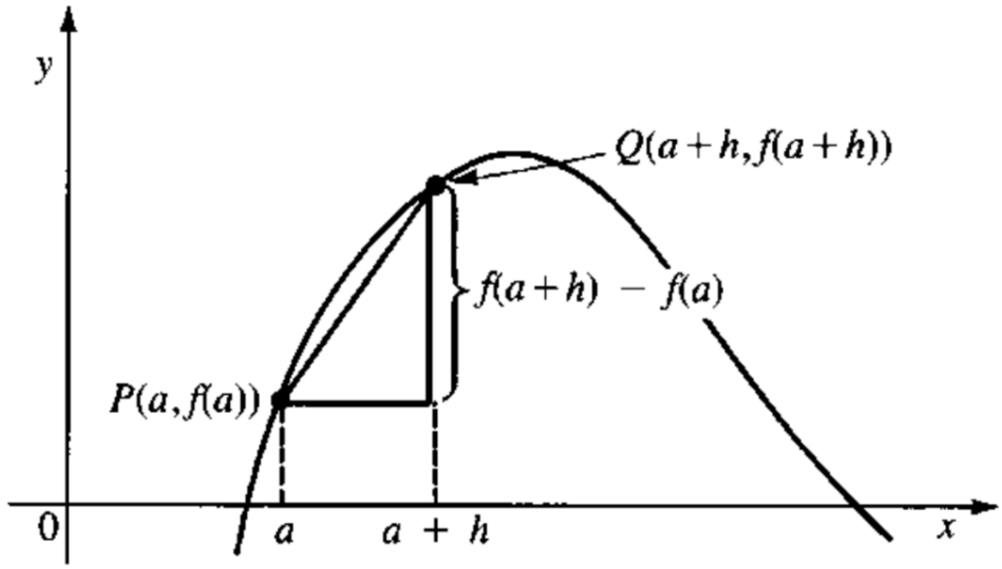


Figure 1: The limits formula calculates the slope of a secant line (between two points on curve) and shrinks said line (by h approaching zero) until it becomes a tangent line (the instantaneous slope of a point)

Limits Method

Slope at Specific Point

e.g. $f(x) = 3x^2 - 5x + 4$, find $f'(2)$

$$f(2) = 3(2)^2 - 5(2) + 4 = 6$$

$$\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \longrightarrow \lim_{h\to 0} \frac{[3(2+h)^2 - 5(2+h) + 4] - [6]}{h}$$

Expand and simplify until you are no longer dividing by zero.

$$\lim_{h\to 0} 3h + 7$$

Calculate the limit: subsitute h with 0

$$m = 7$$

General Expression

This is the actual derivative of a function. Inputting any value of x into this expression is equivalent to the previous step.

e.g. $f(x) = 3x^2 - 5x + 4$, find $f'(x)$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \longrightarrow \lim_{h\to 0} \frac{[3(x+h)^2 - 5(x+h) + 4] - [3x^2 - 5x + 4]}{h}$$

Some tears and bloodshed later...

$$f'(x) = 6x - 5$$

For instance, the previous section can be solved using this function.

$$f'(2) = 6(2) - 5 = 7$$

Slope to Equation

To get an equation such as $y = mx + b$ from just a slope (m) and a given point (x_1, y_1).

$$y - y_1 = m(x - x_1)$$

Differentiability

- If $f'(x)$ exists, then $f(x)$ is **differentiable**
- If $f(x)$ is differentiable, then $f(x)$ is continuous at point x
- Continuity does not imply differentiability

Non-differentiability Graphically

A point on a graph that is often non-differentiable due to the limit of said point not existing. This usually occurs from the left and right limit not being equal.

These, graphically, could be...

- **Cusp**: sharp peak on a graph, like the tip of a triangle
- **Crossing Point**: gap between two graph lines
- **Vertical Asymptote**
- **Point of Discontinuity/Hollow Point**
- **Vertical Separation**: point that graphs switch in piecewise functions
- **End Point**: graph line ends at a point

Other points that are non-differentiable could be...

- **Vertical Line**: slope/derivative is undefined

Determine the Point Problem

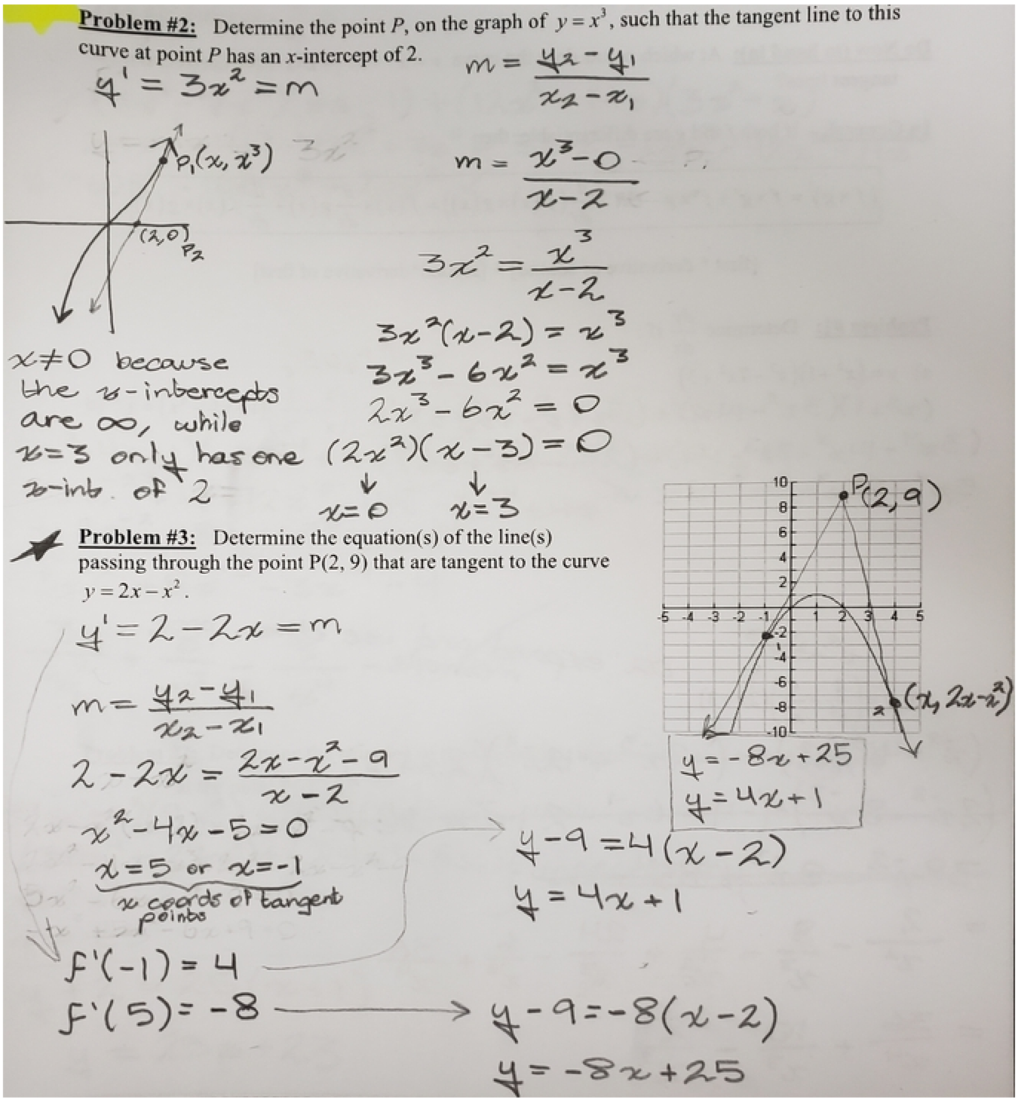
Recall this formula for calculating slope,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The derivative of a function calculates the slope of the tangent line touching point x on said function. You can replace m with the derivative then.

Replace the x's and y's with any given plot points. You can also give a point the coordinates of $(x, f(x))$ and solve.

These are example problems. You will likely be tested on questions similar to this.



Problem #2: Determine the point P, on the graph of $y = x^3$, such that the tangent line to this curve at point P has an x-intercept of 2.

$y' = 3x^2 = m$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$P_1(x, x^3)$

$(2,0)$ P_2

$m = \frac{x^3 - 0}{x - 2}$

$3x^2 = \frac{x^3}{x-2}$

$3x^2(x-2) = x^3$

$3x^3 - 6x^2 = x^3$

$2x^3 - 6x^2 = 0$

$(2x^2)(x-3) = 0$

$x = 0$ $x = 3$

$x \neq 0$ because the x-intercepts are ∞, while $x=3$ only has one x-int. of 2

Problem #3: Determine the equation(s) of the line(s) passing through the point P(2, 9) that are tangent to the curve $y = 2x - x^2$.

$y' = 2 - 2x = m$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$2 - 2x = \frac{2x - x^2 - 9}{x - 2}$

$x^2 - 4x - 5 = 0$

$\underbrace{x = 5 \text{ or } x = -1}_{x \text{ coords of tangent points}}$

$f'(-1) = 4$ $\rightarrow$ $y - 9 = 4(x-2)$

$y = 4x + 1$

$f'(5) = -8$ $\rightarrow$ $y - 9 = -8(x-2)$

$y = -8x + 25$

$P(2,9)$

$(x, 2x - x^2)$

$y = -8x + 25$

$y = 4x + 1$

Derivative Rules

The Power Rule

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$

- Multiply pre-existing coefficients with n
  e.g. $8x^2 \longrightarrow 16x$

- Convert fractions and radicals into exponent form to apply the power rule
  e.g. $\frac{4}{x^3} = 4x^{-3}$, $\sqrt{x^3} = x^{\frac{3}{2}}$

- The derivative of a variable with a degree of 1 equals 1 (since the power becomes zero)
  e.g. $4x^1 \longrightarrow 4(1x^{1-1}) \longrightarrow 4$

- The derivative of a constant is zero

The Sum and Difference Rule

If both f and g are differentiable,

$$(f+g)' = f' + g'$$

$$(f-g)' = f' - g'$$

In other words, replace every term with its derivative.

The Product Rule

If both f and g are differentiable,

$$(f \times g)' = f \times g' + f' \times g$$

In other words, (first)(derivative of second) + (second)(derivative of first)

The Quotient Rule

If both f and g are differentiable,

$$\left(\frac{f}{g}\right)' = \frac{f' \times g - f \times g'}{g^2}$$

With Respect To

$$\frac{\mathrm{d}y}{\mathrm{d}x}$$

The derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ is said to be "the derivative of y with respect to x."

Imagine it as actually being "the derivative of x when inside of y."

Or ask yourself, "inside of the function y, what is the derivative of x?"

This only becomes relevant when there are more variables than x and y, such as the chain rule below.

Chain Rules

The Chain Rule

If $y = f(u)$ and $u = g(x)$, then...

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)$$

e.g. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = u^2 + u$ and $u = x^3$.

- $\frac{\mathrm{d}y}{\mathrm{d}x} = (\frac{\mathrm{d}y}{\mathrm{d}u})(\frac{\mathrm{d}u}{\mathrm{d}x})$
- $\frac{\mathrm{d}y}{\mathrm{d}x} = (2u + 1)(3x^2)$

The Power/Chain Rule

The Power/Chain rule is the same as the Chain rule, but may be easier to understand.

If $y = u^n$ and $u = g(x)$, then...

$$\frac{\mathrm{d}y}{\mathrm{d}x} = nu^{n-1} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

In the simplest terms, treat the entire function as a variable and get the derivative of that. (imagine it as getting the derivative outside the brackets).

Then, multiply that by the derivative of the function inside the brackets.

e.g. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ of $y = (2x - 7x^2 + 9)^{-2}$.

- Let $u = 2x - 7x^2 + 9$
- $y = u^{-2}$
- $y = (nu^{n-1})(\frac{\mathrm{d}u}{\mathrm{d}x})$
- $\frac{\mathrm{d}y}{\mathrm{d}x} = (-2u^{-3})(2 - 14x)$

Product/Quotient and Chain Rules

For some questions you may need to multiple rules when there are multiple "functions" with exponents.

- Use product/quotient rules between the two "functions"
- In those rules, you need to get derivatives of functions. Use chain rule in these scenarios
- After simplifying both sides of the $+$ or $-$, try to factor out anything
- The last thing you should try is expanding and adding like terms

e.g. $f(x) = (x^2 - 1)^3(2 - 3x)^4$, what is $f'(x)$?

- $f'(x) = [(3(x^2 - 1)^2)(2x)](2 - 3x)^4 + [(4(2 - 3x)^3)(-3)](x^2 - 1)^3$
  Inside the square brackets is chain rule, outside the square brackets is product rule
- $f'(x) = 6x(x^2 - 1)^2(2 - 3x)^4 + -12(x^2 - 1)^3(2 - 3x)^3$
  Simplifying both sides of the +/-
- $f'(x) = 6(x^2 - 1)^2(2 - 3x)^3[x(2 - 3x) - 2(x^2 - 1)]$
  Factoring out
- $f'(x) = 6(x^2 - 1)^2(2 - 3x)^3[2x - 3x^2 - 2x^2 + 2]$
  Expanded
- $f'(x) = 6(x^2 - 1)^2(2 - 3x)^3[-5x^2 + 2x + 2]$
  Add like terms

Implicit Differentation

To get the derivative of equations where y is in the equation. (rather than $y = f(x)$, its could be like $x + y = c$)

- Get the derivative of each term like normal, all aforementioned rules still apply
- Everytime you get the derivative of y, append $\frac{\mathrm{d}y}{\mathrm{d}x}$ (aka. y') to it
- Solve for $\frac{\mathrm{d}y}{\mathrm{d}x}/y'$
  - Get any terms that include y' to one side, and simplify/factor until the equation is $y' = \ldots$

e.g. $x^2 + y^2 = 16$, what is y'?

$$2x + 2yy' = 0$$

$$y' = \frac{-2x}{2y} = \frac{-x}{y}$$

Higher Order Derivatives

The derivative of a derivative is denoted with increasing prime "ticks".

$$f''(x) = f'(f'(x)) \text{ (2nd derivative of } f(x))$$
$$f'''(x) = f'(f'(f'(x))) \text{ (3rd derivative of } f(x))$$
$$f^{(n)}(x) \text{ (}n\text{th derivative of } f(x))$$

e.g.

- $f(x) = x^8$
- $f'(x) = 8x^7$
- $f''(x) = 56x^6$
- $f'''(x) = 336x^5$
- $f^{(5)}(x) = 6720x^3$

Involving Implicit Differentiation

- Find y' like before
- When finding y'', subsitute any instance of y' with its actual value that you found
- When finding y'', replace any instance of the original function (if you find it) with the actual value you were given in the question

e.g. If $x^4 + y^4 = 16$, what is y''?

- Get y'
  $4x^3 + 4y^3y' = 0$
  $y' = \frac{-4x^3}{4y^3} = \frac{-x^3}{y^3}$
- Get y''. Notice how you still have to append y' to all y's.
  $y'' = \frac{(y^3)(-3x^2)-(-x^3)(3y^2y')}{(y^3)^2}$
  $y'' = \frac{-3x^2y^3+3x^3y^2y'}{y^6}$
- Notice how theres y', and we have it, so subsitute.
  $y'' = \frac{-3x^2y^3+3x^3y^2(\frac{-x^3}{y^3})}{y^6}$
  $y'' = \frac{-3x^2y^3-3x^6y^{-1}}{y^6}$
- When we factor out a value, we can see the original equation. Subsitute it with 16, since we were given that.
  $y'' = \frac{-3x^2y^{-1}(y^4+x^4)}{y^6}$
  $y'' = \frac{-3x^2y^{-1}(16)}{y^6}$
- Make sure you remember your exponent rules for these steps.
  $y'' = \frac{-48x^2}{y^7}$

Applications

Terms

- **Displacement** (s)
  position, direct line from start to current position

- **Average Velocity**
  Velocity over time. $v_{\text{avg}} = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1}$

- **Instantaneous Velocity**
  Velocity at a specific time. $v_{\text{inst}} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$

Velocity

The derivative of an equation for displacement will make it for velocity.

If $s = f(t)$ was a displacement equation, then...

$$f'(t) = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\Delta s}{\Delta t} = v$$

Acceleration

The 2nd derivative of an equation for displacement will make it for acceleration.

If $s = f(t)$ was a displacement equation, then...

$$f''(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\Delta v}{\Delta t} = a$$

Related Rates

TODO