Math 31
Curve Sketching

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Unfinished!

Contents

Increasing and Decreasing Functions

- If $f'(x) > 0$ for all values of x in a range (x_1, x_2), then $y = f(x)$ is increasing in the range (x_1, x_2)
- If $f'(x) < 0$ for all values of x in a range (x_1, x_2), then $y = f(x)$ is decreasing in the range (x_1, x_2)

Maxima and Minima

- If $f(c) \geq f(x)$ for all values of x, then the maxima is $x = c$
- If $f(c) \leq f(x)$ for all values of x, then the minima is $x = c$

Absolute maxima/minima is for the entire domain of $y = f(x)$

Local maxima/minima is for the just the range of an interval (a, b)

Fermat's Theorem

Critical numbers are points of a graph where a local min/max may occur.

f has a critical number at $x = c$ if either...

- $f'(c) = 0$
  (flat, horizontal portion of curve)
- $f'(c) = \text{DNE}$
  (divide by zero, e.g. cusp)

Find Absolute Max/Min of Continuous Function

You can get the critical number(s) of a function by solving for x when f' equals zero.

- Find $f(x)$ at the critical numbers
- Only if within a closed interval $[a, b]$, find $f(a)$ and $f(b)$ (since you can't derive end points)
- The largest of these values is the absolute max, the smallest the absolute min

First Derivative Test

- If $f'(x)$ changes from positive to negative at $x = c$, then $f(c)$ is a local maxima
- If $f'(x)$ changes from negative to positive at $x = c$, then $f(c)$ is a local minima
- If $f'(x)$ does not change sign at $x = c$, then there is no max/min at $x = c$

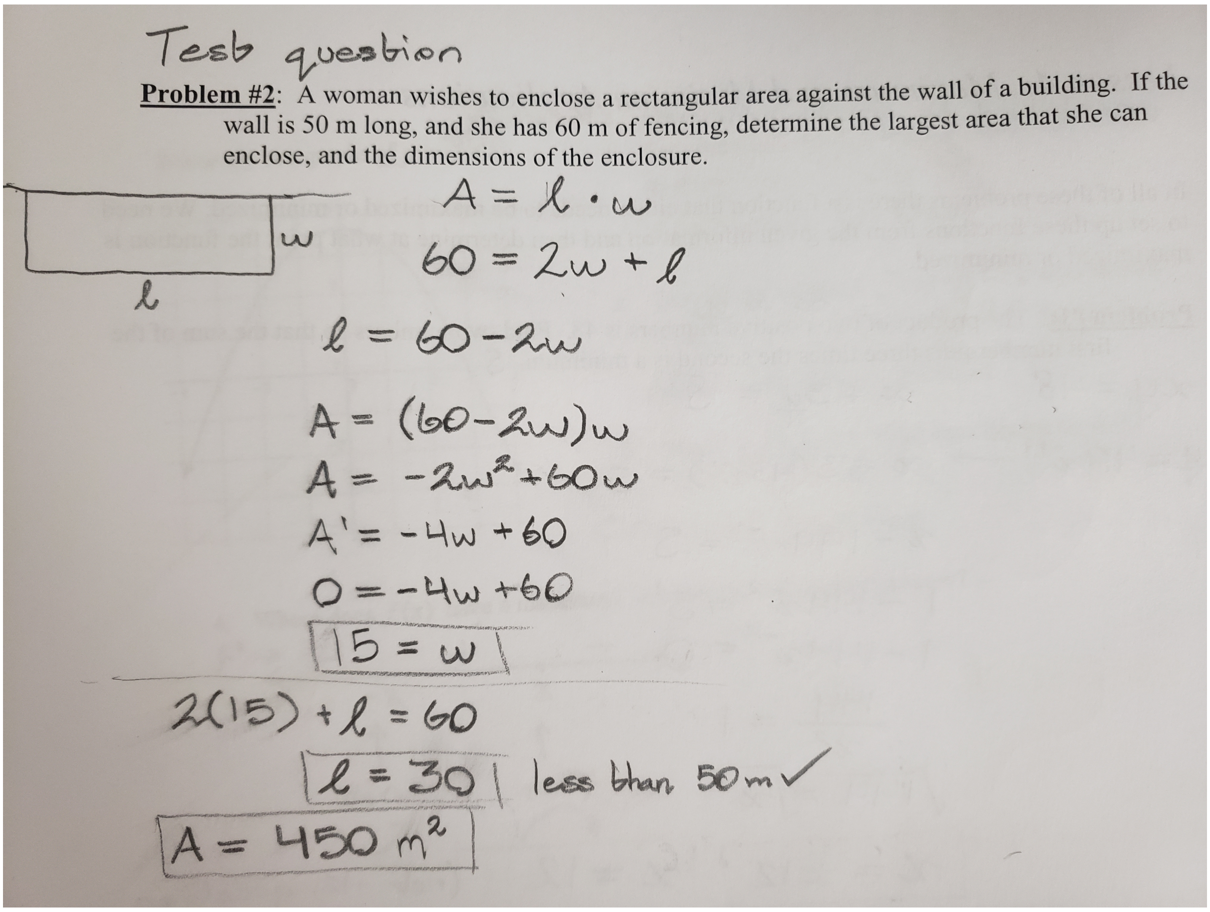
To find these things without a graphing calculator...

- Draw a number line with each critical number plotted

- Plug in any x value that is within the range between each critical number, and take note of its sign
  (you don't actually need $f(x)$ at the specific x, just the sign)
- Knowing the signs before and after a critical number will allow you to know if it is a local min or max

Max & Min Applications

- Find two equations that share variables
- Convert an equation with two variables into an equation with one
  - Choose any equation and isolate any variable to one side
  - Substitute this new equation into the other
- Only now get the derivative of this equation
- Set it to equal 0 and solve for either variable
- Now that you have the missing information, do not forget to fulfill the rest of the question criteria



Test question

Problem #2: A woman wishes to enclose a rectangular area against the wall of a building. If the wall is 50 m long, and she has 60 m of fencing, determine the largest area that she can enclose, and the dimensions of the enclosure.

w

ℓ

$A = \ell \cdot w$

$60 = 2w + \ell$

$\ell = 60 - 2w$

$A = (60-2w)w$

$A = -2w^2 + 60w$

$A' = -4w + 60$

$0 = -4w + 60$

$\boxed{15 = w}$

$2(15) + \ell = 60$

$\boxed{\ell = 30}$ less than 50m ✓

$\boxed{A = 450 \text{ m}^2}$

Review

Domain

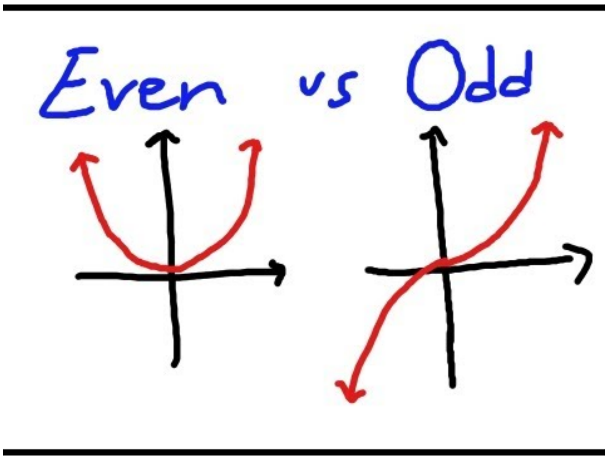
The domain is all possible x values of a function.

- Polynomial functions: all x values valid
- Rational functions: any x value that causes division by zero is invalid (NPV)
- Radical functions: any x value that causes rooting negatives is invalid

Intercepts

- **X-intercept**: $y = 0$, solve for x
- **Y-intercept**: $x = 0$, solve for y

Symmetry



Even Symmetry

$$f(-x) = f(x)$$

If the above is true, then the function is has **even symmetry**.

Replace all instances of x in the function with $-x$.

If it makes no difference, than it has even symmetry.

Odd Symmetry

$$f(-x) = -f(x)$$

If the above is true, then the function is has **odd symmetry**.

Replace all instances of x in the function with $-x$.

If the only difference is all signs flipped (you could factor out -1) than it has odd symmetry.

No Symmetry

If the above two are both not true, then no symmetry is possible.

Limits to Infinity of Asymptotes

Vertical

$y = f(x)$ has a vertical asymptote of $x = a$ if...

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty$$

The limit exists even if left and right are different signs, as long as its infinity.

Determining Sign of Asymptote Limit

Knowing whether an asymptote approaches positive or negative infinity is important when curve sketching.

- Find any vertical asymptotes (solve for x when denominator $= 0$)
- Create two limits for x approaching each vertical asymptote — approaching from the left and right
- Factor denominator if possible (not required, but makes things easier)
- Determine the sign of limit by finding sign of numerator and denominator

Determine The Sign

This is the hard part, since you don't really calculate anything.

- To test a limit of x approaching the VA from the right, substitute all x in the function with a value slightly larger than x
  (e.g. if $\lim_{x \to 2^+}$, then set something like $x = 2.1$)
- To test a limit of x approaching the VA from the left, substitute all x in the function with a value slightly smaller than x
  (e.g. if $\lim_{x \to 2^-}$, then set something like $x = 1.9$)

e.g. Determine the vertical asymptote equation of $f(x) = \frac{1}{x}$.

Vertical asymptote is $x = 0$

$$\lim_{x \to 0^+} \frac{1}{x} = \frac{1}{0.1} = \frac{+}{+} = +\infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = \frac{1}{-0.1} = \frac{+}{-} = -\infty$$

Horizontal

This is the identical to "Finding Limits to Infinity" from Unit 1. It is the same for limits approaching positive or negative infinity.

- Any fraction with a variable to any power in the denominator will be zero.

$$\lim_{x\to\infty} \frac{a}{x^b} = 0$$

- Multiply a limit by something in order to put a variable (to the power of the highest existing power) under the terms, making them equal zero.

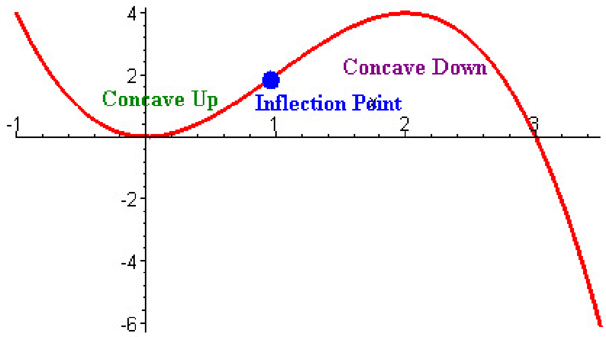
$$\lim_{x\to\infty} \frac{6n+9}{3n-2}$$

$$\frac{6n+9}{3n-2} \times \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$\frac{6+\frac{9}{n}}{3-\frac{2}{n}} \longrightarrow \frac{6+0}{3-0}$$

$$\lim_{x\to\infty} \frac{6n+9}{3n-2} = 2$$

Concavity



Concave Up

If the graph lies above the tangents of all of its points.

If $f''(x) > 0$ for all x in an interval, $y = f(x)$ is concave upwards in that interval.

Concave Down

If the graph lies below the tangents of all of its points.

If $f''(x) < 0$ for all x in an interval, $y = f(x)$ is concave downwards in that interval.

Point of Inflection

If $f''(x)$ changes sign ($+$ to -, - to $+$) at $x = c$, then a **point of inflection** occurs at $x = c$.

Second Derivative Test

- If $f'(c) = 0$ AND $f''(c) < 0$, then the local max is at $x = c$
- If $f'(c) = 0$ AND $f''(c) > 0$, then the local min is at $x = c$