

Math 31

Limits and the Derivative

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Unfinished!

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Factoring Brief Review

Differences of Square

$$x^2 - 4 = (x + 2)(x - 2)$$

Polynomial

$$\begin{aligned} 2x^2 + 3x - 2 \\ \longrightarrow (2x^2 + 4x)(-x - 2) \\ \longrightarrow 2x(x + 2) - 1(x + 2) \\ \longrightarrow (2x - 1)(x + 2) \end{aligned}$$

Radical Fractions

- Multiply everything by monomial denominator

$$\frac{2}{\sqrt{2x}} \longrightarrow \frac{2\sqrt{2x}}{2x} \longrightarrow \frac{\sqrt{2x}}{x}$$

- Multiply everything by conjugate for polynomial denominators

$$\frac{3}{2+\sqrt{x}} \times \frac{2-\sqrt{x}}{2-\sqrt{x}} = \frac{6-3\sqrt{x}}{4-2\sqrt{2}+2\sqrt{x}-x} = \frac{6-3\sqrt{x}}{4-x}$$

Mixed Radicals

$$\sqrt{162} \longrightarrow \sqrt{9^2 \times 2} \longrightarrow \sqrt{9^2} \times \sqrt{2} \longrightarrow 9\sqrt{2}$$

Absolute Polynomial

$$|x - 1| = 3$$

$$x - 1 = 3, x = 4$$

$$x - 1 = -3, x = -2$$

Adding/Subtracting Fractions

Multiply both terms so that the denominators are the same, then add/subtract.

$$\begin{aligned} \frac{2}{x-1} - \frac{3}{x+3} \\ \longrightarrow \frac{2(x+3)}{(x-1)(x+3)} - \frac{3(x-1)}{(x-1)(x+3)} \end{aligned}$$

$$\begin{aligned} &\longrightarrow \frac{(2x+6)-(3x-3)}{(x-1)(x+3)} \\ &= \frac{-x+3}{(x-1)(x+3)} \end{aligned}$$

Piecewise Functions

Piecewise functions are functions with multiple inequalities/restrictions that dictate which function to use at specific x values.

When graphing...

- if an inequality is less/greater than a value, the plot point is **not filled in**
- if an inequality is less/greater than **OR equal to** a value, the plot point is **filled in**
- if x of different functions equal the same value, the graphs are continuous, and are filled in if one of the functions is inclusive

If the inequalities do not state a function for a specific x value (e.g. $x = 2$ for $2 < x < 2$) then that value **DNE**. (**does not exist**)

Rational Function

A function with a polynomial in the numerator and denominator.

Vertical Asymptotes

Zeros of the denominator of a rational function.

x may approach these values, but never touch them.

Point of Discontinuity

Any vertical asymptote (zeros of denominator) **before simplifying** a rational function.

These vertical asymptotes only applies to the unsimplified form; this makes it a point of discontinuity.

These points are gaps in a graph line, have no y value, and therefore make a graph discontinuous.

Horizontal Asymptotes

Horizontal asymptotes describe the **trend** of a function.

The graph line can cross over it fine, as opposed to vertical asymptotes.

Determining Horizontal Asymptotes

- degree of numerator $<$ degree of denominator
→ $y = 0$
- degree of numerator $=$ degree of denominator
→ $y = \frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}}$
- degree of numerator $>$ degree of denominator
→ Divergent (no horizontal asymptote)

Limits

$$\lim_{x \rightarrow a} f(x) = b$$

The limit of $f(x)$ as x approaches a is b .

A limit is the value of y as the x approaches a specific value, as opposed to equaling a specific value. This is useful for points of discontinuity, where the exact value doesn't exist, but the value approaching does.

For instance, if the point of discontinuity of $f(x)$ is $x = -1$, then...

$$f(-1) = \text{DNE}$$

$$\lim_{x \rightarrow -1} f(x) = -1$$

Properties

- $c = \text{constant value}$

$$\lim_{x \rightarrow a} c = c$$

$$\lim_{x \rightarrow a} cf(x) = c \lim_{x \rightarrow a} f(x)$$

- $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$

- $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$

- The rest of the rules can be summarized as limits have distributive property.

$$\text{e.g. } \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

Limits of Continuous Functions

Any Polynomial

$y = f(x)$ is continuous at every value of a .

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Just substitute x in the function with a .

Any Rational Function

$y = \frac{f(x)}{g(x)}$ is continuous at every value of a as long as $g(x) \neq 0$. (cannot divide by zero)

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}, g(a) \neq 0$$

Just substitute x in the function with a , unless a makes the denominator equal to zero. If so, refer to the next section.

Any Radical Function

$y = \sqrt{f(x)}$ is continuous at every value of a as long as $f(x) \geq 0$. (cannot root negatives)

$$\lim_{x \rightarrow a} \sqrt{f(x)} = \sqrt{f(a)}, f(a) \geq 0$$

Just substitute x in the function with a , unless a makes the function equal to a negative. If so, refer to the next section.

Limits of Discontinuous Functions

Identically to finding points of discontinuity, simplify/rationalize the function in a limit if it does illegal math (divide by zero, root negatives) until it doesn't.

$$\begin{aligned} \lim_{x \rightarrow 4} \left(\frac{x^2 - 16}{x - 4} \right) \\ \lim_{x \rightarrow 4} \left(\frac{(x - 4)(x + 4)}{x - 4} \right) \\ \lim_{x \rightarrow 4} (x + 4) = 8 \end{aligned}$$

One-sided Limits

Limits of a function can be separated into the value of approaching **from the left** and **from the right**. This is denoted with a superscript on a .

- From the left ($x < a$): $\lim_{x \rightarrow a^-} f(x)$
- From the right ($x > a$): $\lim_{x \rightarrow a^+} f(x)$

This is only really relevant for graphs that end (such as $y = \sqrt{x}$, approaching from the side without a line is DNE) or piecewise functions.

Continuous or Discontinuous?

Continuous

Continuous functions have a left approaching limit and a right approaching limit **equal to one another**.

$$\begin{aligned} \text{if } \lim_{x \rightarrow a^-} f(x) &= \lim_{x \rightarrow a^+} f(x) \\ \text{then } \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} f(a) \end{aligned}$$

Discontinuous

Discontinuous functions have a left approaching limit and a right approaching limit **not equal to one another**.

$$\begin{aligned} \text{if } \lim_{x \rightarrow a^-} f(x) &\neq \lim_{x \rightarrow a^+} f(x) \\ \text{then } \lim_{x \rightarrow a} f(x) &= \text{DNE} \end{aligned}$$