Math 31

Limits and the Derivative

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Unfinished!

Contents

Factoring Brief Review

Differences of Square

$$x^2 - 4 = (x + 2)(x - 2)$$

Polynomial

$$2x^2 + 3x - 2$$
$$\longrightarrow (2x^2 + 4x)(-x - 2)$$
$$\longrightarrow 2x(x + 2) - 1(x + 2)$$
$$\longrightarrow (2x - 1)(x + 2)$$

Radical Fractions

- Multiply everything by monomial denominator
$$\frac{2}{\sqrt{2x}} \longrightarrow \frac{2\sqrt{2x}}{2x} \longrightarrow \frac{\sqrt{2x}}{x}$$

- Multiply everything by conjugate for polynomial denominators
$$\frac{3}{2+\sqrt{x}} \times \frac{2-\sqrt{x}}{2-\sqrt{x}} = \frac{6-3\sqrt{x}}{4-2\sqrt{2}+2\sqrt{x}-x} = \frac{6-3\sqrt{x}}{4-x}$$

Mixed Radicals

$$\sqrt{162} \longrightarrow \sqrt{9^2 \times 2} \longrightarrow \sqrt{9^2} \times \sqrt{2} \longrightarrow 9\sqrt{2}$$

Absolute Polynomial

$$|x - 1| = 3$$
$$x - 1 = 3,\ x = 4$$
$$x - 1 = -3,\ x = -2$$

Adding/Subtracting Fractions

Multiply both terms so that the denominators are the same, then add/subtract.
$$\frac{2}{x-1} - \frac{3}{x+3}$$
$$\longrightarrow \frac{2(x+3)}{(x-1)(x+3)} - \frac{3(x-1)}{(x-1)(x+3)}$$

$$\longrightarrow \frac{(2x+6)-(3x-3)}{(x-1)(x+3)}$$
$$= \frac{-x+3}{(x-1)(x+3)}$$

Piecewise Functions

Piecewise functions are functions with multiple inequalities/restrictions that dictate which function to use at specific x values.

When graphing...

- if an inequality is less/greater than a value, the plot point is not filled in

- if an inequality is less/greater than OR equal to a value, the plot point is filled in

- if x of different functions equal the same value, the graphs are continuous, and are filled in if one of the functions is inclusive

If the inequalities do not state a function for a specific x value (e.g. $x = 2$ for $2 < x < 2$) then that value **DNE**. (does not exist)

Rational Function

A function with a polynomial in the numerator and denominator.

Vertical Asymptotes

Zeros of the denominator of a rational function.

x may approach these values, but never touch them.

Point of Discontinuity

Any vertical asymptote (zeros of denominator) before simplifying a rational function.

These vertical asymptotes only applies to the unsimplified form; this makes it a point of discontinuity.

These points are gaps in a graph line, have no y value, and therefore make a graph discontinuous.

Horizontal Asymptotes

Horizontal asymptotes describe the ==trend== of a function.

    The graph line can cross over it fine, as opposed to vertical asymptotes.

    Determining Horizontal Asymptotes

- degree of numerator $<$ degree of denominator

  $\longrightarrow y = 0$

- degree of numerator $=$ degree of denominator

  $\longrightarrow y = \frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}}$

- degree of numerator $>$ degree of denominator

  $\longrightarrow$ Divergent (no horizontal asymptote)

Limits

$$\lim_{x \to a} f(x) = b$$

The limit of $f(x)$ as x approaches a is b.

A limit is the value of y as the x approaches a specific value, as opposed to equaling a specific value. This is useful for points of discontinuity, where the exact value doesn't exist, but the value approaching does.

For instance, if the point of discontinuity of $f(x)$ is $x = -1$, then...

$$f(-1) = \text{DNE}$$

$$\lim_{x \to -1} f(x) = -1$$

Properties

- $c = $ constant value

  $\lim_{x \to a} c = c$

  $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$

- $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$

- $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

- The rest of the rules can be summarized as limits have distributive property.

  e.g. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

Limits of Continuous Functions

Any Polynomial

$y = f(x)$ is continuous at every value of a.

$$\lim_{x \to a} f(x) = f(a)$$

Just substitute x in the function with a.

Any Rational Function

$y = \frac{f(x)}{g(x)}$ is continuous at every value of a as long as $g(x) \neq 0$. (cannot divide by zero)

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}, \ g(a) \neq 0$$

Just substitute x in the function with a, unless a makes the denominator equal to zero. If so, refer to the next section.

Any Radical Function

$y = \sqrt{f(x)}$ is continuous at every value of a as long as $f(x) \geq 0$. (cannot root negatives)

$$\lim_{x \to a} \sqrt{f(x)} = \sqrt{f(a)}, \ f(a) \geq 0$$

Just substitute x in the function with a, unless a makes the function equal to a negative. If so, refer to the next section.

Limits of Discontinuous Functions

Identically to finding points of discontinuity, simplify/rationalize the function in a limit if it does illegal math (divide by zero, root negatives) until it doesn't.

$$\lim_{x \to 4} \left(\frac{x^2 - 16}{x - 4} \right)$$
$$\lim_{x \to 4} \left(\frac{(x - 4)(x + 4)}{x - 4} \right)$$
$$\lim_{x \to 4} (x + 4) = 8$$

One-sided Limits

Limits of a function can be seperated into the value of approaching ==from the left== and ==from the right==. This is denoted with a superscript on a.

- From the left $(x < a)$: $\lim\limits_{x \to a^-} f(x)$

- From the right $(x > a)$: $\lim\limits_{x \to a^+} f(x)$

This is only really relevant for graphs that end (such as $y = \sqrt{x}$, approaching from the side without a line is DNE) or piecewise functions.

Continuous or Discontinous?

Continuous

Continuous functions have a left approaching limit and a right approaching limit ==equal to one another==.

$$\text{if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$
$$\text{then } \lim_{x \to a} f(x) = \lim_{x \to a} f(a)$$

Discontinuous

Discontinuous functions have a left approaching limit and a right approaching limit ==not equal to one another==.

$$\text{if } \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$
$$\text{then } \lim_{x \to a} f(x) = \text{DNE}$$

Limits to Infinity

Limits of infinity either approach a value or DNE.

Rules

- Limits to infinity of normal numbers is often DNE. e.g.

    - $\lim\limits_{n \to \infty} r^n = \text{DNE } (\textit{iff } |r| > 1)$
    - $\lim\limits_{x \to \infty} 2^x = \text{DNE}$
    - $\lim\limits_{x \to \infty} \frac{1}{x^{-3}} = \text{DNE}$

- Limits to infinity of fractions with variable denominators is often infinity small, so 0. e.g.

    - $\lim\limits_{n \to \infty} r^n = 0 \ (\textit{iff } |r| < 1)$
    - $\lim\limits_{n \to \infty} \frac{1}{n} = 0$

- The limit of infinity does not exist.
  $\lim\limits_{n \to \infty} 3^n = DNE$

- $\lim\limits_{n \to \infty} (-1)^n = DNE$

Finding Limits to Infinity

- Any fraction with a variable in the denominator will be zero.

$$\lim_{x \to \infty} \frac{a}{x^b} = 0$$

- Because of this, multiply a limit by something in order to put a variable under the terms, making them equal zero. e.g.

$$\lim_{x \to \infty} \frac{6n + 9}{3n - 2}$$

$$\frac{6n + 9}{3n - 2} \times \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$\frac{6 + \frac{9}{n}}{3 - \frac{2}{n}} \longrightarrow \frac{6 + 0}{3 - 0}$$

$$\lim_{x \to \infty} \frac{6n + 9}{3n - 2} = 2$$

Derivatives

The derivative of a function gives the slope of a tangent line that just touches the point $(x, f(x))$.

Formulas

$$f'(x) = y' = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
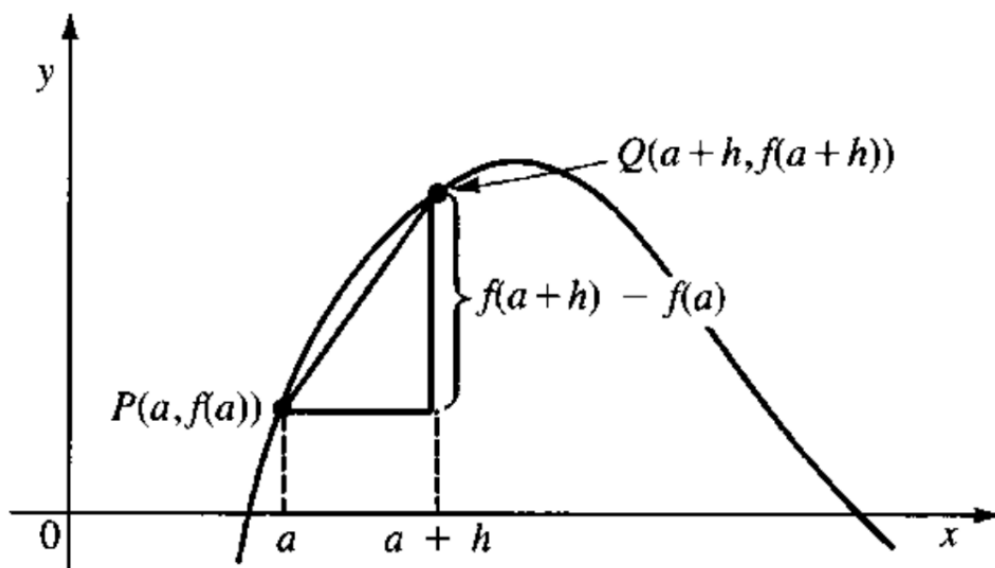

Figure 1: The limits formula calculates the slope of a secant line (between two points on curve) and shrinks said line (by h approaching zero) until it becomes a tangent line (the instantaneous slope of a point)

Limits Method

Slope at Specific Point

e.g. $f(x) = 3x^2 - 5x + 4$, find $f'(2)$

$$f(2) = 3(2)^2 - 5(2) + 4 = 6$$

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \longrightarrow \lim_{h \to 0} \frac{[3(2+h)^2 - 5(2+h) + 4] - [6]}{h}$$

Expand and simplify until you are no longer dividing by zero.

$$\lim_{h \to 0} 3h + 7$$

Calculate the limit: subsitute h with 0

$$m = 7$$

General Expression

This is the actual derivative of a function. Inputting any value of x into this expression is equivalent to the previous step.

e.g. $f(x) = 3x^2 - 5x + 4$, find $f'(x)$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \longrightarrow \lim_{h \to 0} \frac{[3(x+h)^2 - 5(x+h) + 4] - [3x^2 - 5x + 4]}{h}$$

Some tears and bloodshed later...

$$f'(x) = 6x - 5$$

For instance, the previous section can be solved using this function.

$$f'(2) = 6(2) - 5 = 7$$

Slope to Equation

To get an equation such as $y = mx + b$ from just a slope (m) and a given point (x_1, y_1).

$$y - y_1 = m(x - x_1)$$

Differentiability

- If $f'(x)$ exists, then $f(x)$ is **differentiable**

- If $f(x)$ is differentiable, then $f(x)$ is continuous at point x

- <mark>Continuity does not imply differentiability</mark>

Non-differentiability Graphically

A point on a graph that is often non-differentiable due to the limit of said point not existing. This usually occurs from the left and right limit not being equal.

These, graphically, could be...

- **Cusp**: sharp peak on a graph, like the tip of a triangle

- **Crossing Point**: gap between two graph lines

- **Vertical Asymptote**

- **Point of Discontinuity/Hollow Point**

- **Vertical Separation**: point that graphs switch in piecewise functions

- **End Point**: graph line ends at a point

Other points that are non-differentiable could be...

- **Vertical Line**: slope/derivative is undefined

Determine the Point Problem

Recall this formula for calculating slope,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The derivative of a function calculates the slope of the tangent line touching point x on said function. You can replace m with the derivative then.

Replace the x's and y's with any given plot points. You can also give a point the coordinates of $(x, f(x))$ and solve.

These are example problems. You will likely be tested on questions similar to this.

Problem #2: Determine the point P, on the graph of $y = x^3$, such that the tangent line to this curve at point P has an x-intercept of 2.

$$y' = 3x^2 = m$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$P_1(x, x^3)$

$(2,0) \; P_2$

$$m = \frac{x^3 - 0}{x - 2}$$

$$3x^2 = \frac{x^3}{x - 2}$$

$$3x^2(x - 2) = x^3$$

$$3x^3 - 6x^2 = x^3$$

$$2x^3 - 6x^2 = 0$$

$$(2x^2)(x - 3) = 0$$

$$x = 0 \qquad x = 3$$

$x \neq 0$ because the x-intercepts are ∞, while $x = 3$ only has one x-int. of 2

Problem #3: Determine the equation(s) of the line(s) passing through the point $P(2, 9)$ that are tangent to the curve $y = 2x - x^2$.

$$y' = 2 - 2x = m$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$2 - 2x = \frac{2x - x^2 - 9}{x - 2}$$

$$x^2 - 4x - 5 = 0$$

$$x = 5 \text{ or } x = -1$$

$\underline{x \text{ coords of tangent}}$ points

$$f'(-1) = 4$$

$$f'(5) = -8$$

$P_2(2, 9)$

$(x, 2x - x^2)$

$y = -8x + 25$

$y = 4x + 1$

$$y - 9 = 4(x - 2)$$

$$y = 4x + 1$$

$$y - 9 = -8(x - 2)$$

$$y = -8x + 25$$

Derivative Rules

The Power Rule

$$\text{If } f(x) = x^n, \text{ then } f'(x) = nx^{n-1}$$

- Multiply pre-existing coefficients with n
  e.g. $8x^2 \longrightarrow 16x$

- Convert fractions and radicals into exponent form to apply the power rule
  e.g. $\frac{4}{x^3} = 4x^{-3}$, $\sqrt{x^3} = x^{\frac{3}{2}}$

- The derivative of a variable with a ==degree of 1 equals 1== (since the power becomes zero)
  e.g. $4x^1 \longrightarrow 4(1x^{1-1}) \longrightarrow 4$

- ==The derivative of a constant is zero==

The Sum and Difference Rule

If both f and g are differentiable,

$$(f + g)' = f' + g'$$

$$(f - g)' = f' - g'$$

In other words, ==replace every term with its derivative==.

The Product Rule

If both f and g are differentiable,

$$(f \times g)' = f \times g' + f' \times g$$

In other words, (first)(derivative of second) + (second)(derivative of first)

The Quotient Rule

If both f and g are differentiable,

$$\left(\frac{f}{g}\right)' = \frac{f' \times g - f \times g'}{g^2}$$

With Respect To

$$\frac{\mathrm{d}y}{\mathrm{d}x}$$

The derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ is said to be "the derivative of y with respect to x."

Imagine it as actually being "the derivative of x when inside of y."

Or ask yourself, "inside of the function y, what is the derivative of x?"

This only becomes relevant when there are more variables than x and y, such as the chain rule below.

Chain Rules

The Chain Rule

If $y = f(u)$ and $u = g(x)$, then...

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)$$

e.g. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = u^2 + u$ and $u = x^3$.

- $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)$

- $\frac{\mathrm{d}y}{\mathrm{d}x} = (2u + 1)(3x^2)$

The Power/Chain Rule

The Power/Chain rule is the same as the Chain rule, but may be easier to understand.

If $y = u^n$ and $u = g(x)$, then...

$$\frac{\mathrm{d}y}{\mathrm{d}x} = nu^{n-1} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

In the simplest terms, treat the entire function as a variable and get the derivative of that. (imagine it as getting the derivative outside the brackets).

Then, multiply that by the derivative of the function inside the brackets.

e.g. Determine $\frac{dy}{dx}$ of $y = (2x - 7x^2 + 9)^{-2}$.

- Let $u = 2x - 7x^2 + 9$

- $y = u^{-2}$

- $y = (nu^{n-1})(\frac{du}{dx})$

- $\frac{dy}{dx} = (-2u^{-3})(2 - 14x)$

Product/Quotient and Chain Rules

For some questions you may need to multiple rules when there are multiple "functions" with exponents.

- Use product/quotient rules between the two "functions"

- In those rules, you need to get derivatives of functions. Use chain rule in these scenarios

- After simplifying both sides of the $+$ or $-$, try to factor out anything

- The last thing you should try is expanding and adding like terms

e.g. $f(x) = (x^2 - 1)^3(2 - 3x)^4$, what is $f'(x)$?

- $f'(x) = [(3(x^2 - 1)^2)(2x)](2 - 3x)^4 + [(4(2 - 3x)^3)(-3)](x^2 - 1)^3$
  Inside the square brackets is chain rule, outside the square brackets is product rule

- $f'(x) = 6x(x^2 - 1)^2(2 - 3x)^4 + -12(x^2 - 1)^3(2 - 3x)^3$
  Simplifying both sides of the +/-

- $f'(x) = 6(x^2 - 1)^2(2 - 3x)^3[x(2 - 3x) - 2(x^2 - 1)]$
  Factoring out

- $f'(x) = 6(x^2 - 1)^2(2 - 3x)^3[2x - 3x^2 - 2x^2 + 2]$
  Expanded

- $f'(x) = 6(x^2 - 1)^2(2 - 3x)^3[-5x^2 + 2x + 2]$
  Add like terms

Implicit Differentation

To get the derivative of equations where y is in the equation. (rather than $y = f(x)$, its could be like $x + y = c$)

- Get the derivative of each term like normal, all aforementioned rules still apply

- Everytime you get the derivative of y, append $\frac{dy}{dx}$ (aka. y') to it

- Solve for $\frac{dy}{dx}/y'$

    - Get any terms that include y' to one side, and simplify/factor until the equation is $y' = ...$

e.g. $x^2 + y^2 = 16$, what is y'?

$$2x + 2yy' = 0$$

$$y' = \frac{-2x}{2y} = \frac{-x}{y}$$

Higher Order Derivatives

The derivative of a derivative is denoted with increasing prime "ticks".

$$f''(x) = f'(f'(x)) \text{ (2nd derivative of } f(x))$$
$$f'''(x) = f'(f'(f'(x))) \text{ (3rd derivative of } f(x))$$
$$f^{(n)}(x) \text{ (}n\text{th derivative of } f(x))$$

e.g.

- $f(x) = x^8$

- $f'(x) = 8x^7$

- $f''(x) = 56x^6$

- $f'''(x) = 336x^5$

- $f^{(5)}(x) = 6720x^3$

Involving Implicit Differentiation

- Find y' like before

- When finding y'', subsitute any instance of y' with its actual value that you found

- When finding y'', replace any instance of the original function (if you find it) with the actual value you were given in the question

e.g. If $x^4 + y^4 = 16$, what is y''?

- Get y'
  $$4x^3 + 4y^3 y' = 0$$
  $$y' = \frac{-4x^3}{4y^3} = \frac{-x^3}{y^3}$$

- Get y''. Notice how you still have to append y' to all y's.
  $$y'' = \frac{(y^3)(-3x^2) - (-x^3)(3y^2 y')}{(y^3)^2}$$
  $$y'' = \frac{-3x^2 y^3 + 3x^3 y^2 y'}{y^6}$$

- Notice how theres y', and we have it, so subsitute.
  $$y'' = \frac{-3x^2 y^3 + 3x^3 y^2 (\frac{-x^3}{y^3})}{y^6}$$
  $$y'' = \frac{-3x^2 y^3 - 3x^6 y^{-1}}{y^6}$$

- When we factor out a value, we can see the original equation. Subsitute it with 16, since we were given that.
  $$y'' = \frac{-3x^2 y^{-1} (y^4 + x^4)}{y^6}$$
  $$y'' = \frac{-3x^2 y^{-1} (16)}{y^6}$$

- Make sure you remember your exponent rules for these steps.
  $$y'' = \frac{-48x^2}{y^7}$$