Jordan decomposition for A

$$AV = VJ \quad . \; V \text{ invertible.}$$

$$p(z) = \prod_{k=1}^{m} (z - z_k)^{\mu_k}$$

$$\sum \mu_k = r$$

$$A \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix}$$

$$J_k = \left.\begin{bmatrix} z_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & z_k \end{bmatrix}\right\} \mu_k$$

$$A = VJV^{-1}$$

$$\|A^n\| = \|VJ\cancel{V^{-1}V}JV^{-1}\cdots VJV^{-1}\| = \|VJ^nV^{-1}\|$$

$$\leq \underbrace{\|V\|\cdot\|V^{-1}\|}_{\kappa(V)\ \text{condition number.}}\cdot\|J^n\|$$

$$\|J^n\| = \left\| \begin{bmatrix} J_1^n & & \\ & \ddots & \\ & & J_k^n \end{bmatrix} \right\|$$

Ex. $J_1 = z_1 \quad (\mu_1 = 1) \quad J_1^n = z_1^n \quad |z_1| > 1$ blow up

≤ 1 good

Ex. $J_1 = \begin{bmatrix} z_1 & 1 \\ & z_1 \end{bmatrix} = z_1 I + R$ $\quad R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$R^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Nil-potent.

$J_1^n = (z_1 I + R)^n = z_1^n + n z_1^{n-1} R$

$|z_1| \geq 1$ blow up

$|z_1| < 1$ may grow transiently. good in the end.

More generally.

$$R = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

$$J_k^n = (z_k I + R)^n = \sum_{l=0}^{r-1} \binom{n}{l} z_k^{n-l} R^l$$

$\|J_k^n\|$ bound iff zero stability is satisfied.

$$\Updownarrow$$

$\|A^n\|$ bound by a (possible large) constant.

② $\tilde{u}_{n+1} = A\tilde{u}_n + h\tilde{f}_n$

$$\tilde{u}_1 = A\tilde{u}_0 + h\tilde{f}_0$$

$$\tilde{u}_2 = A(A\tilde{u}_0 + h\tilde{f}_0) + h\tilde{f}_1$$

$$= A^2\tilde{u}_0 + h(A\tilde{f}_0 + \tilde{f}_1)$$

$$\vdots$$

$$\tilde{u}_{n+1} = A^{n+1}\tilde{u}_0 + h\left(\sum_{j=0}^{n} A^{n-j}\tilde{f}_j\right)$$

discrete Duhamel's principle.

$\tilde{u}(t_{n+1})$ same way.

Duhamel's principle.

$$u'(t) = Au(t) + f(t), \quad u(0) = u_0$$

$$\Rightarrow \quad u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)} f(s)\,ds.$$

$$\|\tilde{e}_{n+1}\| \le \|A^{n+1}\| \, \|\tilde{e}_0\| + \sum_{j=0}^{n} \|A^{n-j}\| \left[h \|\tilde{f}(t_j) - \tilde{f}_j\| + \|\tilde{\tau}_j\| \right]$$

$$\|\tilde{f}(t_n) - \tilde{f}_n\| = \left\| \sum_{k=0}^{r} \beta_k \left(f(u(t_{n+k})) - f(u_{n+k}) \right) \right\|$$

diff. meaning of norm

$$\le \sum_{k=0}^{r} |\beta_k| \cdot L \cdot \|u(t_{n+k}) - u_{n+k}\| = L \sum_{k=0}^{r} |\beta_k| \cdot \|e_{n+k}\|$$

$$= L \sum_{k=0}^{r-1} |\beta_k| \, \|e_{n+k}\| + L |\beta_r| \cdot \|e_{n+r}\|$$

$$\le L \left(\sum_{k=0}^{r-1} |\beta_k| \right) \|\tilde{e}_n\| + L |\beta_k| \, \|\tilde{e}_{n+1}\|$$

$$\Rightarrow \|\tilde{e}_{n+1}\| \le K \|\tilde{e}_0\| + K \sum_{j=0}^{n} \left(h \|\tilde{f}(t_j) - f_j\| + \|\tilde{\tau}_j\| \right)$$

$\|A^n\| \le K$ for any n

$$\le K\|\tilde{e}_0\| + K \sum_{j=0}^{n} \left(hL \left(\sum_{s=0}^{r-1} |\beta_s| \right) \|\tilde{e}_j\| + hL\,|\beta_r|\, \|\tilde{e}_{j+1}\| + \|\tilde{\tau}_j\| \right)$$

$$= K\|\tilde{e}_0\| + KhL\,|\beta_r|\,\|\tilde{e}_{n+1}\| + \sum_{j=0}^{n} KhL \left(\sum_{s=0}^{r} |\beta_s| \right) \|\tilde{e}_j\| + K \sum_{j=0}^{n} \|\tilde{\tau}_j\|$$

If $KhL|\beta_r| < \frac{1}{2}$

$$\Rightarrow \|\tilde{e}_{n+1}\| \le 2K\|\tilde{e}_0\| + 2\sum_{j=0}^{n} KhL\,\|\beta\|_1\,\|\tilde{e}_j\| + 2K \sum_{j=0}^{n} \|\tilde{\tau}_j\|$$

bounded.

Discrete Gronwall inequality.

Thm. $0 \le \varepsilon_n \le C + L \sum_{j=0}^{n-1} \varepsilon_j$, $C, L > 0$

$$\Rightarrow \quad \varepsilon_n \le C e^{Ln}$$

Pf: RHS $\quad \zeta_n = C + L \sum_{j=0}^{n-1} \epsilon_j$.

$$\zeta_{n+1} = \zeta_n + L \epsilon_n \le (1+L)\zeta_n \quad . \ \zeta_0 = C \Rightarrow$$

$$\zeta_n \le (1+L)^n C \le C e^{Ln} \quad \square$$

Let $\quad k\|\tilde{e}_0\| + k\sum_{j=0}^{n}\|\tilde{\tau}_j\| \le C$

$$\|\tilde{e}_N\| \le \left(2k\|\tilde{e}_0\| + \frac{2k\bar{T}}{h}\tau\right) e^{2k\|\beta\|_1 L \underbrace{nh}_{=T}}$$

Proved

Dahlquist's first barrier

Highest attainable order of stable multistep methods

Theorem The order p of a stable linear r-step LMM method satisfies

$$p \le r + 2, \quad \text{if } r \text{ is even}$$
$$p \le r + 1, \quad \text{if } r \text{ is odd}$$
$$p \le r, \qquad \text{if } \frac{\beta_r}{\alpha_r} \le 0 \ \text{ (in particular if the method is explicit)}$$

[Hai] III.3