<u>Thm</u>. No ERK is A-stable.

Pf: ERK. A lower triangular

w. 0 on the diagonal.

$A^r = 0$. $\Rightarrow$ A is Nilpotent

$$(I - zA)^{-1} = \sum_{j=0}^{r-1} (zA)^j$$

$$R(z) = 1 + \vec{b}^T \left(\sum_{j=0}^{r-1} (zA)^j \right) \vec{e} \cdot z$$

polynomial of z up order r.

No polynomials can be bounded on $\{\mathrm{Re}\, z \le 0\}$

$\square$

RAS for RK generally increases w.r.t order.

Compare $R(z)$ and e^z

Thm. RK of order p.

$$\left| R(z) - e^{z} \right| \sim O(z^{p+1})$$

Pf: LTE. $u' = \lambda u$. $u(t_n) = e^{\lambda t_n} u(0)$

$$\tau_n = u(t_{n+1}) - \left[u(t_n) + h \vec{b}^T \vec{k}(u(t_n)) \right]$$

$$= e^{\lambda t_n} \cdot u(0) \left(e^{\lambda h} - R(\lambda h) \right) \sim O(h^{p+1})$$

$\square$

Remark: $R(z) = e^{z} + O(z^{p+1}) \not\Rightarrow$ order p.

(think why)

Counter example

$$\begin{array}{c|ccc} & 0 & & \\ & 1 & 0 & \\ & 0 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{array}$$

Thm. All Gauss-Legendre methods are A-stable.

[Ise 4.3]

Why care about absolute stability at all?

Ex. linear system.

$$u' = Au \quad , \quad u \in \mathbb{R}^d .$$

$$u(0) = u_0$$

A diagonalizable. $A v_i = \lambda_i v_i$, $V = [v_1, \cdots, v_d]$

$$\tilde{u} = V^* u \longrightarrow \tilde{u}_i' = \lambda_i \tilde{u}_i$$

$\mathrm{Re}\,\lambda_i > 0$ exp. growth.

$\mathrm{Re}\,\lambda_i = 0$. difficult border line

$\mathrm{Re}\,\lambda_i < 0$ exp decay.

Some estimate on distribution of eigenvalue of A.

h small enough. $\lambda_i h \in RAS$ $\forall i$.

Ex. $u' = f(u)$, $u \in \mathbb{R}^d$, $f: \mathbb{R}^d \to \mathbb{R}^d$.

Given t_0. $f(v) \approx f(u(t_0)) + Df(u(t_0)) \cdot (v - u(t_0))$

$v \approx u(t_0)$

Around t_0. approx. linear eq. for v.

$$A = Df\left(u(t_0)\right) \in \mathbb{R}^{d \times d}$$

Stiff equation.

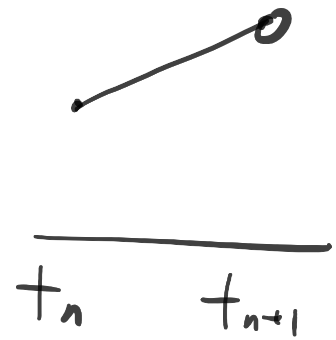
Backward differentiation formula (BDF).

LMM. Lag interpolation $\Rightarrow$ integration (quadrature)

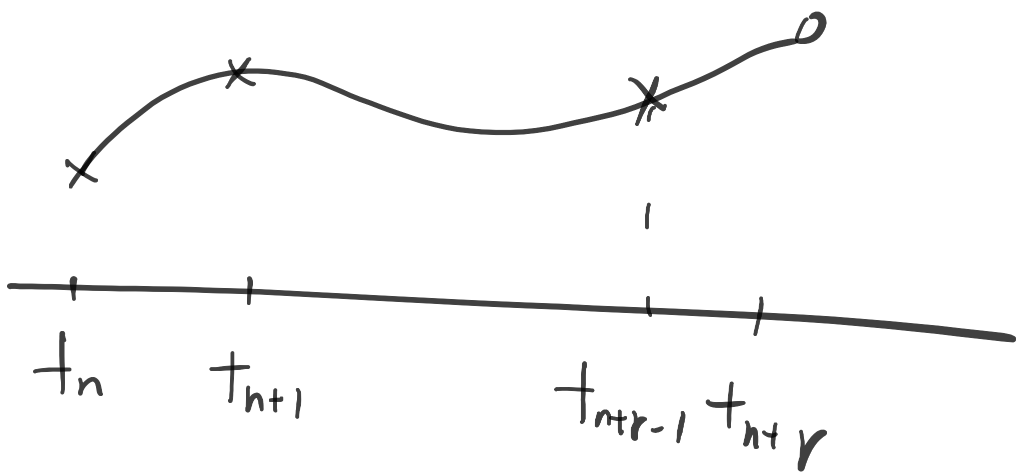
BDF. Lag " $\Rightarrow$ differentiation

Ex. Backward Euler.

$$u'(t) = f(u(t), t)$$



$$\frac{u_{n+1} - u_n}{h} = f(u_{n+1}, t_{n+1}).$$



$$\tilde{u}'(t_{n+r}) = f_{n+r}$$

polynomial.

$t_n, \ldots, t_{n+r}$.

implicit.

$r = 2.$

$$\tilde{u}(t) = u_n \cdot P_n(t) + u_{n+1} P_{n+1}(t) + u_{n+2} P_{n+2}(t)$$

$$P_n(t) = \frac{(t-t_{n+1})(t-t_{n+2})}{(t_n-t_{n+1})(t_n-t_{n+2})} = \frac{(t-t_{n+1})(t-t_{n+2})}{(-h)(-2h)}$$

uniform time stepping

Take derivative

$$P_n'(t_{n+2}) = \frac{2t_{n+2} - (t_{n+1}+t_{n+2})}{2h^2} = \frac{1}{2h}$$

$$P_{n+1}'(t_{n+2}) = -\frac{2}{h}, \quad P_{n+2}'(t_{n+2}) = \frac{3}{2h}$$

$$u_{n+2} \cdot \frac{3}{2} - 2u_{n+1} + \frac{1}{2}u_n = \underline{h f_{n+2}}$$

fix rhs.

Similarly BDF 3.

$$u_{n+3} - \frac{18}{11}u_{n+2} + \frac{9}{11}u_{n+1} - \frac{2}{11}u_n = \frac{6}{11}h f_{n+3}$$

$\cdots$ BDF n.

Root condition ?

Thm. No BDF is zero stable if $r > 6$.

BDF1 $= $ BE, ..., BDF6.

$r \le 6$, r-stage BDF is of order r.

Advantage of BDF is its relatively large stability region.

Stiffness is a matter of efficiency.

Ex. Prothero – Robinson example.

$$\begin{cases} \dot{u}(t) = -L\,(u(t) - \varphi(t)) + \varphi'(t) \\ u(0) = u_0 \end{cases}, \qquad \begin{array}{l} L \gg 1 \\ \varphi(t) \text{ is smooth} \end{array}$$

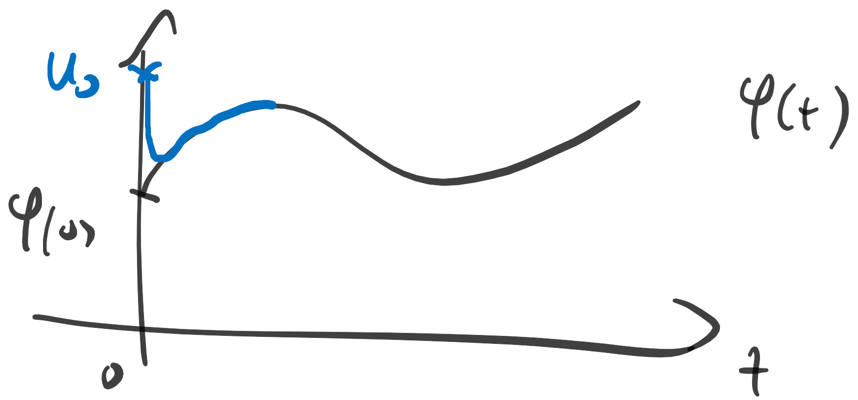
Duhamel's principle. exact sol.

$$u(t) = e^{-Lt} u_0 + \int_0^t e^{-L(t-s)} (L\varphi(s) + \varphi'(s))\, ds$$

$$= e^{-Lt} u_0 + \varphi(t) - \varphi(0) e^{-Lt}$$

$$= e^{-Lt} (u_0 - \varphi(0)) + \varphi(t)$$

If $u_0 = \varphi(0)$, $u(t) = \varphi(t)$



If $u_0 \neq \varphi(0)$, after initial period $\sim \frac{1}{L}$

$u(t) \approx \varphi(t)$

But explicit integrators blow up.

Theoretical challenge. error analysis.

ex. Euler.

$$\max_{0\le t_n\le T} \|e_n\| \le \left(\frac{1}{h}\max_n \|\tau_n\|\right)\frac{e^{LT}-1}{L} + e^{LT}\|e_0\|$$

error $\le \varepsilon$. $\quad \max \|\tau_n\| \le h e^{-LT} \varepsilon$.

Two challenges:

① Practical. Choose reasonably large time step while avoiding instability.

→ implicit scheme (best if A-stable)

② theoretical. for more special equations (rhs). more refined analysis tools.

For scalar eq. typically

$$|f'(u)| \gg |u'|$$

variation of rhs

variation of sol.

typically associated w. large Lip. constant.

For vectors. Sometimes easier to identify stiffness.

Jacobian $J = \nabla f(u) \in \mathbb{R}^{d \times d}$.

$$\gamma = \frac{|\lambda_{max}(J)|}{|\lambda_{min}(J)|}$$ eigenvalues of J.

$\gamma \gg 1$. might be stiff.

Note following ex.

$$\begin{cases} \dot{u}_1 = -L_1 u_1 \\ \dot{u}_2 = -L_2 u_2 \end{cases} \qquad \begin{cases} i\dot{u}_1 = -L_1 u_1 \\ i\dot{u}_2 = -L_2 u_2 \end{cases}$$

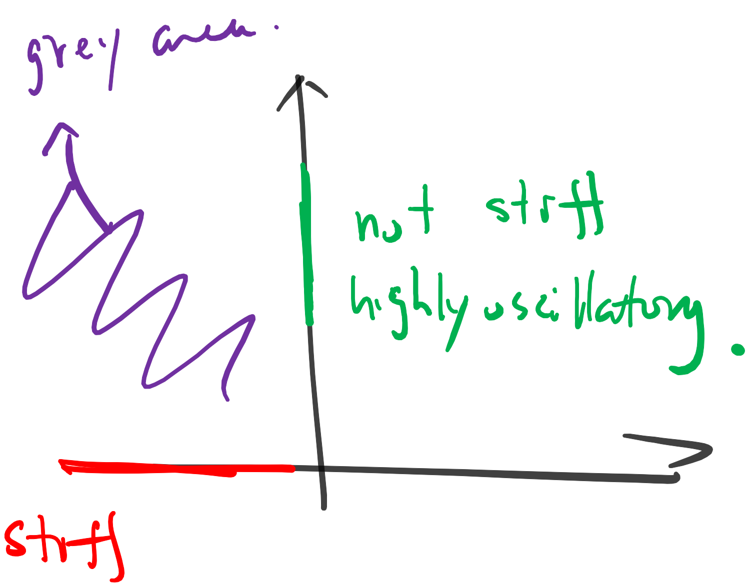
$0 < L_1 \ll L_2$, $L_1, L_2 \in \mathbb{R}$.

stiff

not a stiff eq.

must use a small time step to resolve u_2.

Still very difficult.



Theoretical analysis. P-R example.

$$u' = -L\,(u-\varphi) + \varphi' = f(u)$$

Backward Euler.

$$u_{n+1} = u_n + h\, f(u_{n+1})$$

$$u(t_{n+1}) = u(t_n) + h\, f(u(t_{n+1})) + \tau_n$$

$$\Rightarrow \quad u(t_{n+1}) - u_{n+1} = u(t_n) - u_n - Lh\big(u(t_{n+1}) - u_{n+1}\big) + \tau_n$$

$$\Rightarrow \|e_{n+1}\|^2 = \big(u(t_{n+1}) - u_{n+1}\big)^2$$

$$= \left(u(t_{n+1}) - u_{n+1}\right)\left(u(t_n) - u_n\right) + \tau_n \left(u(t_{n+1}) - u_{n+1}\right)$$

$$- L_h \|e_{n+1}\|^2 .$$

$$\leq \left(u(t_{n+1}) - u_{n+1}\right)\left(u(t_n) - u_n\right) + \tau_n \left(u(t_{n+1}) - u_{n+1}\right)$$

$$\leq \|e_{n+1}\| \cdot \|e_n\| + \|\tau_n\| \cdot \|e_{n+1}\|$$

$$\Rightarrow \|e_{n+1}\| \leq \|e_n\| + \|\tau_n\| \quad \leftarrow \text{no Lip const!}$$

$$\leq \cdots \leq \|e_0\| + \|\tau_1\| + \cdots + \|\tau_n\|$$

$$\leq \|e_0\| + \frac{T}{h} \max_{1 \leq i \leq n} \|\tau_i\|$$

Ex. more general RK method.

$$\vec{k} = f(u_n\vec{e} + hA\vec{k}, t_n + \vec{c}h)$$

$$= -L(u_n\vec{e} + hA\vec{k}) + L\varphi(t_n + \vec{c}h) + \varphi'(t_n + \vec{c}h)$$

$$u_{n+1} = u_n + h\,\vec{b}^T\vec{k}$$

$$\vec{k} = (I + LhA)^{-1}\left[-Lu_n\vec{e} + L\varphi(t_n + \vec{c}h) + \varphi'(t_n + \vec{c}h)\right]$$

$$u_{n+1} = R(z)\,u_n + \vec{b}\,Lh\,(I + LhA)^{-1}\left(L\varphi(t_n + \vec{c}h) + \varphi'(t_n + \vec{c}h)\right)$$

$$z = -Lh, \quad R(z) = 1 + z\,\vec{b}^T(I - zA)^{-1}\vec{e}$$

$$u(t_{n+1}) = R(z)\, u(t_n) + \qquad\qquad + \tau_n$$

$$e_{n+1} = u(t_{n+1}) - u_{n+1} = R(z) \cdot e_n + \tau_n$$

$$\|e_{n+1}\| \le |R(z)|\, \|e_n\| + \|\tau_n\|$$

$$\le |R(z)|^2 \|e_{n-1}\| + |R(z)|\, \|\tau_{n-1}\| + \|\tau_n\|$$

$$\le \cdots \le |R(z)|^{n+1} \|e_0\| + \frac{1 - |R(z)|^n}{1 - |R(z)|} \max_{1 \le i \le n} \|\tau_i\|$$

$|R(z)| < 1$

forget initial error

finite memory.