w.l.o.g. scalar RK . notational convenience.

$$\vec{k} = \begin{pmatrix} k_1 \\ \vdots \\ k_r \end{pmatrix} \in \mathbb{R}^r \qquad \vec{e} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$u_{n+1} = u_n + h\,\Psi(u_n)$$

$$(*)\ \begin{cases} \vec{k} = f(u_n \vec{e} + hA\vec{k}) \\ u_{n+1} = u_n + h\,\underbrace{\vec{b}^T\vec{k}}_{\Psi(u)} \end{cases}$$

$$e_{n+1} = e_n + \left(\Psi(u(t_n)) - \Psi(u_n)\right)h + \tau_n$$

error propagation.

Need to show:

① $(*)$ has a sol.

② $\Psi(\cdot)$ is Lip cont.

$\Rightarrow$ convergence following proof of Euler's method.

① mapping $T: \mathbb{R}^r \to \mathbb{R}^r$

$$T(\vec{k}) = f(u_n\vec{e} + hA\vec{k})$$

Find a fixed pt.

$\forall \vec{k}, \vec{k}' \in \mathbb{R}^r$

$$\|T(\vec{k}) - T(\vec{k}')\|_2^2 = \|f(u_n\vec{e} + hA\vec{k}) - f(u_n\vec{e} + hA\vec{k}')\|_2^2$$

$$\leq L^2 h^2 \|A(\vec{k} - \vec{k}')\|_2^2$$

$$\leq (Lh\|A\|_2)^2 \|\vec{k} - \vec{k}'\|_2^2$$

$Lh\|A\|_2 < \frac{1}{2} \Rightarrow T$ contraction mapping.

$\Rightarrow \vec{k}(u_n)$ exists & unique $\square$.

② $\Psi(u)$ Lip cont. $\Leftrightarrow \vec{k}(u)$ Lip. cont.

$\forall u, v \in \mathbb{R}.$

$$\|\vec{k}(u) - \vec{k}(v)\|_2^2 = \|f(u\vec{e} + hA\vec{k}(u)) - f(v\vec{e} + hA\vec{k}(v))\|_2^2$$

$$\leq L^2 \|(u-v)\vec{e} + hA(\vec{k}(u) - \vec{k}(v))\|_2^2$$

$$\leq L^2 r |u-v|^2 + L^2 h^2 \|A\|_2^2 \|\vec{k}(u) - \vec{k}(v)\|_2^2$$

$$Lh\|A\|_2 < \frac{1}{2}$$

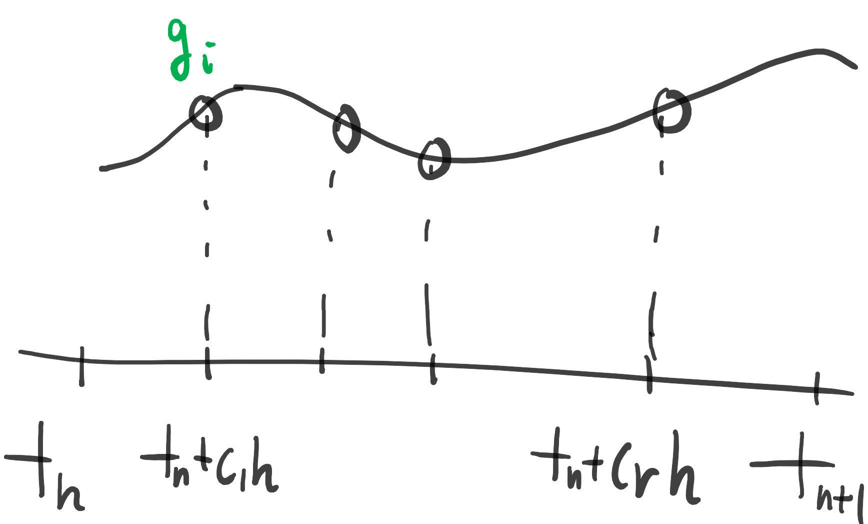
$$\|\vec{k}(u) - \vec{k}(v)\|_2^2 \leq 2L^2 r |u-v|^2$$

Lip const. $\vec{k}(\cdot)$ is bounded by $\sqrt{2r}\,L$ $\square$.

How to construct (implicit) RK methods of arbitrary order.

One possibility. quadrature. Collocation method.

"Co-locates" all functions of interest at the same set of points.



$f(u(t), t)$

Find $u(t_n + c_i h)$

$\to g_i := f(u(t_n+c_ih), t_n+c_ih)$

$0 \le c_1 < c_2 < \cdots < c_r \le 1$

Lagrange interpolation. w.r.t. $\{g_i\}_{i=1}^{r}$

$$P_i(t) = \prod_{j\neq i} \frac{t-(t_n+c_jh)}{h(c_i-c_j)}$$

$$P_i(t_n+c_kh) = \delta_{ik}$$

$$g(t) = \sum_{i=1}^{r} P_i(t)\, g_i \quad \approx f(u(t),t)$$

$$u(t_{n+1}) = u(t_n) + \int_{t_n}^{t_{n+1}} \underbrace{f(u(s),s)}_{g(s)}\, ds$$

As a scheme.

$$u_{n+1} = u_n + \sum_{i=1}^{r}\left(\int_{t_n}^{t_{n+1}} P_i(s)\, ds\right) g_i$$

Define $\quad b_i = \frac{1}{h} \int_{t_n}^{t_{n+1}} P_i(s)\,ds$

$$\longrightarrow \quad u_{n+1} = u_n + h \sum_{i=1}^{r} b_i g_i$$

need to show g_i's are k_i's.

To compute g_i

$$u(t_n + c_i h) = u(t_n) + \int_{t_n}^{t_n + c_i h} \underbrace{f(u(s), s)}_{\text{replace by } g(s)}\, ds$$

$$\tilde{u}(t_n + c_i h) = u_n + \sum_j \int_{t_n}^{t_n + c_i h} P_j(s)\, ds \; g_j$$

Define $a_{ij} = \frac{1}{h} \int_{t_n}^{t_n + c_i h} p_j(s)\,ds$

$$\tilde{u}(t_n + c_i h) = u_n + h \sum_j a_{ij} g_j$$

$$g_i := f\left(\tilde{u}(t_n + c_i h),\ t_n + c_i h\right)$$

$$= f\left(u_n + h \sum_j a_{ij} g_j,\ t_n + c_i h\right)$$

$$g_i \equiv k_i$$

Def. Collocation RK method. (fully implicit)

Given $0 \le c_1 < \cdots < c_r \le 1$ → only input

$$k_i = f\left(u_n + \sum_j a_{ij} k_j,\ t_n + c_i h\right)$$

$$u_{n+1} = u_n + h \sum_{i=1}^{r} b_i k_i$$

where.

$$a_{ij} = \int_0^{c_i} \tilde{p}_j(\theta)\, d\theta \qquad \tilde{p}_i(\theta) = \prod_{j \ne i} \frac{\theta - c_j}{c_i - c_j}$$

$$b_i = \int_0^1 \tilde{p}_i(\theta)\, d\theta.$$

Whether Kutta's condition is satisfied?

$$\sum_j a_{ij} = \int_0^{c_i} \sum_j \tilde{p}_j(\theta)\, d\theta$$

$$f(\theta) := \sum_j \tilde{p}_j(\theta), \quad f(c_i) = 1, \quad i = 1, \cdots, r$$

$$f(\theta) \in \mathbb{P}_{r-1} \Rightarrow f(\theta) \equiv 1.$$

$$\boxed{\sum_j a_{ij} = c_i}$$ Kutta's condition ✓.

Ex. $c_1 = 0.$

$$\begin{array}{c|c} 0 & a \\ \hline & b \end{array}$$

$a = 0.$ $b = \int_0^1 d\theta = 1.$

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

forward Euler.

$c_1 = 1.$

$a = 1$, $b = 1$

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

backward Euler.

Ex. $c_1 = \frac{1}{2}$. $a = \frac{1}{2}$ $b = 1.$

$$\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}$$

$$\begin{cases} k = f\left(u_n + \frac{1}{2}hk,\ t_n + \frac{1}{2}h\right) \\ u_{n+1} = u_n + hk \end{cases}$$

2nd order.

(exer)

Gauss-Legendre 1 (implicit midpoint rule)

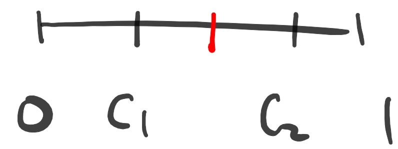
Ex. $c_1 = 0,\quad c_2 = 1.$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

trapezoidal. (exer).

Ex. GL 2. 4th order

$$
\begin{array}{c|cc}
* & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\
* & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$



$$c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6} \qquad c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}$$

accuracy of quadrature.

$$\int_a^b f(x)\,dx \approx \sum_{l=1}^{r} f(x_l)\, \omega_l$$

nodes weights

weights can be computed via Lag poly.

$$P_k(x) = \prod_{l \neq k} \frac{x - x_l}{x_k - x_l}. \qquad P_k(x_l) = \delta_{kl}.$$

$$\omega_k = \int_a^b P_k(x)\,dx.$$

<u>Def</u>. $\left| \int_a^b f(x)dx - \sum_{l=1}^{r} f(x_l)\,\omega_l \right| \leq C(b-a)^{p+1} \sup_{a \leq x \leq b} |f^{(p)}(x)|$.

holds for all $f \in C^{(p)}([a,b])$, $c > 0$ indep. of a, b, f. Then quadrature is of order p.

Consistency of collocation RK methods.

Thm. Quadrature order p. r- stages (points)

$\Rightarrow$ LTE $O(h^{p+1})$. $\leftarrow$ only need to prove.

$\Rightarrow$ scheme order p.

Duhamel's principle.

Thm. $\dot{u}(t) = Au(t) + g(t)$, $u(0) = u_0$

$$\Rightarrow u(t) = e^{At} u_0 + \int_0^t e^{A(t-s)} g(s)\,ds.$$

Pf: $\dot{u}(t) = A e^{At} u(0) + g(t) + \int_0^t A e^{A(t-s)} g(s)\,ds$

$$= A\left(e^{At} u(0) + \int_0^t e^{A(t-s)} g(s)\,ds\right) + g(t)$$

$$= A u(t) + g(t)$$

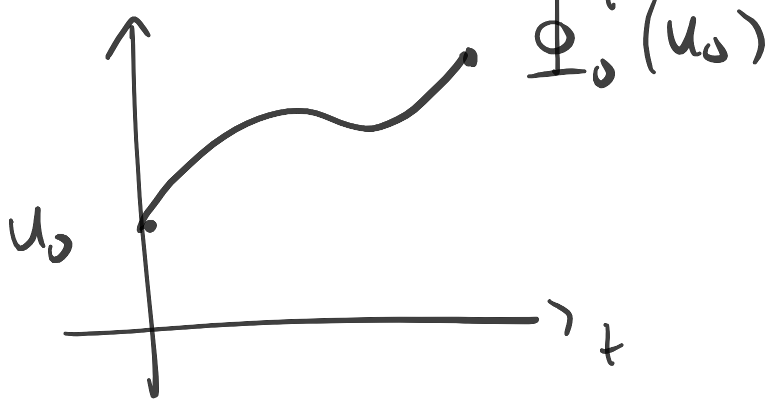
$u(0) = u_0$ $\square$.

$$\begin{cases} \frac{d}{dt} \Phi_s^t(w) = A\, \Phi_s^t(\omega) \\ \Phi_s^s(w) = w \end{cases} \longrightarrow \Phi_s^t(\omega) = e^{A(t-s)} w$$

Duhamel's principle.

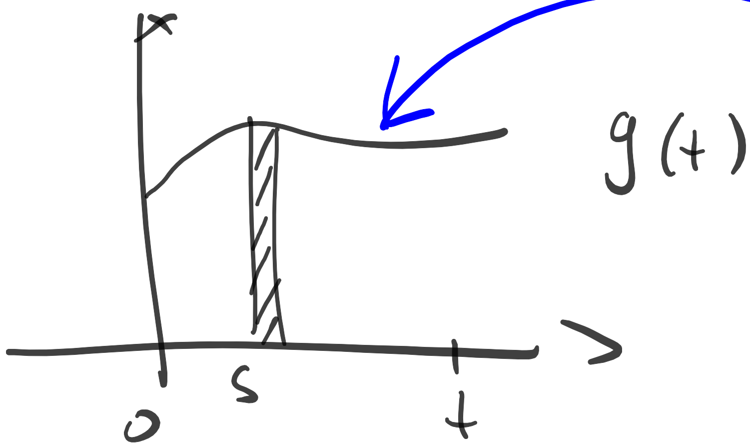
$$u(t) = \Phi_0^t(u_0) + \int_0^t \Phi_s^t(g(s))\, ds.$$

$g(t) \equiv 0$



$$\int_0^t \Phi_s^t(g(s))\, ds$$

$$\approx \sum_{i=0}^{N} \Phi_{s_i}^t(g(s_i))\,(s_{i+1} - s_i)$$



propagate for remaining time $(t-s)$

Thm (Alekseev – Gröbner)

Nonlinear version of Duhamel's principle.

[Hai] I.14.

$$\begin{cases} \dot{u}(t) = f(u(t), t) \\ u(0) = u_0 \end{cases} \longrightarrow \begin{cases} \frac{d}{dt}\Phi_s^t(w) = f(\Phi_s^t(w), t) \\ \Phi_s^s(w) = w \end{cases}$$

$$\begin{cases} \dot{v}(t) = f(v(t), t) + g(v(t), t) \\ v(0) = u_0 \end{cases}$$

$$\Rightarrow v(t) = u(t) + \int_0^t \left[\frac{\partial}{\partial w}\Phi_s^t(w)\right]\Bigg|_{w=v(s)} g(v(s), s)\, ds$$

(exer) use A-G to recover Duhamel's principle

i.e. $f(u(t), t) = A\,u(t)$

Pf. (Convergence of collocation RK).

LTE.

$$\begin{cases} \dot{u}(t) = f(u(t), t) \\ u(t_n) = u(t_n) \end{cases}$$

numerical scheme

$$\begin{cases} \dot{v}(t) = \tilde{f}(v(t), t) \leftarrow \text{Lag. interp.} \\ v(t_n) = u(t_n) \qquad \text{at } t_n + c_i h \end{cases}$$

$$\dot{v}(t) = \tilde{f}(v(t), t) = f(v(t), t) + \underbrace{\left(\tilde{f}(v(t), t) - f(v(t), t)\right)}_{\substack{\shortparallel \\ g(v(t), t)}}$$

interpolatory: $g(v(t_n + c_i h), t_n + c_i h) = 0$.

$$v(t_{n+1}) = u(t_{n+1}) + \int_{t_n}^{t_{n+1}} \left[\frac{\partial \Phi_s^{t_{n+1}}(w)}{\partial w}\right]\Bigg|_{w = v(s)} g(v(s), s)\, ds$$

$$= u(t_{n+1}) + \underset{\text{quadrature}}{\underset{\uparrow}{0}} + \underset{\text{error of quadrature.}}{\underset{\uparrow}{O(h^{p+1})}} \qquad \square.$$

Gauss quadrature. orthogonal polynomials.

Consider $[-1,1]$, $L^2([-1,1]) = \{ f \mid \int_{-1}^{1} |f(x)|^2 dx < \infty \}$.

$f : [-1,1] \to \mathbb{R}$

Inner product, $f, g \in L^2([-1,1])$

$$(f, g) := \int_{-1}^{1} f(x)\, g(x)\, dx.$$

$$|(f,g)|^2 \le \int_{-1}^{1} f^2(x)\,dx \int_{-1}^{1} g^2(x)\,dx < \infty .$$

If $(f,g)=0$. Orthogonal.

$P_0(x) = 1$

$P_1(x) = x$ $\longrightarrow$ Legendre polynomial.

$P_2(x) = \frac{1}{2}(3x^2-1)$ orthogonal w.r.t.

$\cdots$ inner product

$P_K(1) = 1$ (convention)

$$\int_{-1}^{1} f(x)g(x)\,dx$$

Remark: generalize $\int_{-1}^{1} \omega(x) f(x) g(x)\, dx$.

$\omega(x) > 0$. also an inner product.

Legendre: $\omega(x) \equiv 1$.

In general. 3-term recurrence.

$$\begin{cases} l P_l(x) = (2l-1)\, x\, P_{l-1}(x) - (l-1) P_{l-2}(x) \\ P_{-1}(x) = 0, \quad P_0(x) = 1 \end{cases}$$

Gauss quadrature on $[-1,1]$.

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{l=1}^{r} f(x_l)\, w_l .$$

$\{x_l\}_{l=1}^{r}$: roots of $P_r(x)$

Legendre poly of order r.

$\{w_l\}_{l=1}^{r}$: weights. obtained from Lag. poly.