BASKIN SCHOOL OF ENGINEERING
Department of Applied Mathematics

2019 First Year Exam, Take Home Part (September)

Due by 5:00 PM, Sunday, September 22, 2019

Instructions:

This is an individual takehome exam, not a group project. Please work individually on the questions. You are allowed to consult any material you wish, but do not communicate with any other individual regarding any aspect of the exam questions or any possible solution of them. Cheating will absolutely not be tolerated.

You are required to write a report using a word processing software (i.e., LaTeX or Microsoft Word). You are required to email your report as **one pdf file** to Prof. Hongyun Wang at `hongwang@soe.ucsc.edu`

by 5:00 PM, Sunday, September 22, 2019

Please organize and present the material in the best possible way. Be informative but concise. You should include a summary of your work at the beginning of the report, include and annotate all relevant figures and tables in the body of the report, write your conclusions in a separate section, and list your references (if any). Your report should consist of no more than 10 letter-size pages (typeset with 11pt or larger font and margins on all four sides of at least 1 inch), including all figures, tables, and appendices (but excluding the numerical codes); answers longer than 10 pages will lose credit for excess length. You must include your numerical codes (in Matlab or other programming languages) at the end of your report; the numerical codes do not count toward the page limit.

Exam Problem:

Question 1. Consider the partial differential equation (PDE) given as

$$\frac{\partial U}{\partial T} = a\frac{\partial^2 U}{\partial X^2} + bU, \quad 0 \le X \le L, \tag{1}$$

where a and b are positive constants. Let U^0 be the initial maximum value of U.

(i) Non-dimensionalize Eqn. (1) using

$$x = \frac{X}{L}, \quad u = \frac{U}{U^0} \tag{2}$$

to obtain

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \tilde{b}u, \ \ 0 \le x \le 1, \tag{3}$$

where t is the non-dimensional time. Find the explicit forms of t and $\tilde{b}$.

(ii) Write out the two linear finite difference equations that approximate the initial value problem of Eqn. (3) using the central differencing in space (CS), and respectively the forward Euler (i.e., explicit) and the backward Euler (i.e., implicit) in time (FT and BT). The two difference equations are FTCS and BTCS. For the source term $\tilde{b}u$, the discretization is fixed to be $\tilde{b}u_i^n$ for both BT and FT.

(iii) Use the von Neumann analysis to investigate the stability of each difference equation you found in (ii).

Question 2. Consider a specific case of Eqn. (3) by setting $\tilde{b} = 0$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \ \ 0 \le x \le 1. \tag{4}$$

The initial and boundary conditions on $u(x,t)$ are imposed as

$$\text{I.C:} \quad u(x,0) = \begin{cases} 2x & \text{if } \ 0 \le x \le 1/2, \\ 2(1-x) & \text{if } \ 1/2 \le x \le 1, \end{cases} \tag{5}$$

and

$$\text{B.C:} \ \ u(0,t) = 0, \ \ u(1,t) = 0, \ \ \forall t > 0. \tag{6}$$

(i) Show, by the method of separation of variables, that the analytical solution of Eqn. (4) satisfying Eqns. (5) and (6) is

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(\sin\frac{n\pi}{2})(\sin n\pi x)\exp(-n^2\pi^2 t). \tag{7}$$

(ii) Solve the initial-boundary PDE problem using the Forward Euler method in time (FT) and the centered in space (CS) using $\Delta x = 1/32$. Choose the timestep size Δt (and clearly mention its value in your report) according to the stability criteria from Question 1 (iii). Let $t_{\max}$ be the time (non-dimensional time in Eqns. (3) and (4)) it takes to numerically reach the desired steady state, which is to be detected by measuring the change of the temporal evolution of u in L_1,

$$\left\| \frac{\Delta u^n}{\Delta t} \right\|_1 \equiv \frac{1}{\Delta t N} \sum_{i=1}^{N} |u_i^n - u_i^{n-1}| < \epsilon, \tag{8}$$

where ϵ is to be set 10^{-6}. Numerically find $t_{\max}$. Compare (i.e., over-plot) your numerical solution with the analytical solution with the first 30 terms in Eqn. (7) at four time instances of $t/t_{\max} = [0.25, 0.5, 0.75, 1]$. For plots, use symbols for numerical solutions and colored solid curves for the analytical solutions.

(iii) Let Eqn. (4) be approximated at (x_i, t^n) by the difference equation

$$\theta \left(\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} \right) + (1-\theta) \left(\frac{u_i^n - u_i^{n-1}}{\Delta t} \right) = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}, \quad n > 1, \tag{9}$$

where $u_i^n = u(x_i, t^n)$. Consider the truncation error (TE) defined as the residual term when the exact solution is substituted into (9). Show that the truncation error (TE) at (x_i, t^n) is $\mathcal{O}(\Delta t + (\Delta x)^2)$, i.e., TE is given as

$$A\Delta t + B(\Delta x)^2 + \mathcal{O}((\Delta t)^p + (\Delta x)^q) \tag{10}$$

where A and B are terms including temporal and spatial derivatives of u.
Find the explicit forms of A and B, as well as p and q in H.O.T.
Find θ_{opt} that will improve the order of TE from $\mathcal{O}(\Delta t + \Delta x^2)$ to $\mathcal{O}((\Delta t)^p + (\Delta x)^q)$.

(iv) Show that the difference scheme in Eqn. (9) is unstable for $\theta = 1$ and unconditionally stable for $\theta = 0$ by performing the von Neumann stability analysis.

(v) Implement Eqn. (9) using $\theta = 0$ and $\Delta x = 1/32$. Since this becomes an implicit method, you will need to invert a matrix to solve the resulting linear system $\mathbf{Ax} = \mathbf{b}$. Write your own code to implement the solution of $\mathbf{Ax} = \mathbf{b}$. Use any numerical linear solver of your choice to do this. Clearly identify your choice with justification. Note that you are not allowed to invert the matrix directly using any existing routines (e.g., no use of Matlab's $\texttt{inv}(\mathbf{A}) * \mathbf{b}$ or $\mathbf{A} \backslash \mathbf{b}$).
Compare (i.e., over-plot) this implicit numerical solution with the solutions from (i) and (ii) at four time instances of $t/t_{\max} = [0.25, 0.5, 0.75, 1]$. Use time step $\Delta t = \frac{C}{2}(\Delta x)^2$ and choose three values of $C = 0.1, 1.0$, and 10. For each case of Δt, report the number of steps, N_{steady}, it takes for the numerical solution to reach a steady state.

Question 3. In this question, we seek the numerical solution of Eqn. (4) using the Crank-Nicolson method with N internal grid points over $0 \le x \le 1$.

(i) Write out the Crank-Nicolson method. In each step of the Crank-Nicolson method, denote the unknown $\mathbf{u}^{n+1} = (u_1^{n+1}, \dots, u_N^{n+1})^T$ as $\mathbf{x}$ and write out a linear system $\mathbf{Ax} = \mathbf{b}$ based on the Crank-Nicolson method. Clearly identify the matrix $\mathbf{A}$ and the right-hand side vector $\mathbf{b}$. Then, apply the Jacobi iteration method to the resulting linear system to obtain the Jacobi iteration algorithm of the form

$$\mathbf{x}^{(k+1)} = \mathbf{J}\mathbf{x}^{(k)} + \mathbf{c}, \quad k = 0, 1, 2, \dots, \tag{11}$$

where $\mathbf{x}^{(0)} = \mathbf{u}^n$ and the matrix $\mathbf{J}$ is associated with $\mathbf{A}$ according to the Jacobi method. Determine Eqn. (11) by identifying $\mathbf{J}$ and $\mathbf{c}$ explicitly.

(ii) Use $N = 32$ and $\Delta t = 5 \times 10^{-4}$ in the numerical discretization. Identify the relation between N and Δx, and select Δx according to the given N. In each time step, solve the Jacobi iteration in Eqn. (11) to calculate $\mathbf{u}^{n+1}$, given $\mathbf{u}^n$. Use

$$\left\|\frac{\Delta \mathbf{x}^{(k+1)}}{\Delta t}\right\|_1 \equiv \frac{1}{\Delta t N} \sum_{i=1}^{N} |x_i^{(k+1)} - x_i^{(k)}| < \epsilon \tag{12}$$

with $\epsilon = 10^{-6}$ as the convergence criterion for the Jacobi iteration. Evolve the numerical solution $\mathbf{u}^n$ forward in time until a steady state is achieved using the same criteria in Eqn. (8) for detecting the steady state.

Report the number of time steps, N_{steady}, it takes for the numerical solution to reach such a steady state. Monitor the history of Jacobi iterations at *each and every* time step and report the maximum number of Jacobi iterations $N_{JacobMax}$ in any time step during the entire temporal evolutions of $\mathbf{u}^n$ to reach the steady state.

Report the error between the numerical solution and the exact solution in Eqn. (7), measured at $t^n = t_{\max}$ as

$$\text{Error} = ||\Delta E^n||_1 \equiv \frac{1}{N} \sum_{i=1}^{N} |u_i^n - u_{exact}(x_i, t^n)|. \tag{13}$$

(iii) Investigate the convergence rate $N_{JacobMax}$ of the Jacobi iteration algorithm for each $\Delta t = [5 \times 10^{-5}, 5 \times 10^{-4}, 5 \times 10^{-3}]$. Provide a mathematical explanation on the differing convergence behaviors for different Δt. Also report the Error defined in Eqn. (13) for each case of Δt.