Piecewise Linear Orthogonal Approximation

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Abstract

We derive Sobolev-type inner products with respect to which hat functions on arbitrary triangulations of domains in $\mathbb{R}^d$ are orthogonal. Compared with linear interpolation, the resulting approximation schemes yield superior accuracy at little extra cost.

1 Introduction

Piecewise linear approximation of functions is a basic procedure in many numerical algorithms: It is used for rendering curves and surfaces in Computer Graphics, for visualizing and assessing scientific data, and for discretizing boundaries in FEM applications, to name just a few. In most cases, an approximation is determined by interpolating the given function at the vertices of a triangulation of the domain. The approximation error has the optimal order $O(h^2)$ for simplices of size h. However, in general, constants are suboptimal. Figure 1 shows a simple univariate example, the approximation of the function $f(x) = \sin(\pi x)$ on the interval $[-1, 1]$ using seven equally spaced knots. Standard linear interpolation *(left)* systematically overestimates the convex parts and underestimates the concave parts of the function. By contrast, the L^2-fit *(middle)* yields a smaller error, but in order to determine the coefficients, a linear system has to be solved. This overprices the approach for adaptive refinement or large data sets. Best approximation with respect to a suitably weighted Sobolev inner product *(right)* according to the results presented in Section 3 of this paper combines the simplicity of linear interpolation with improved accuracy.

The idea presented here goes back to [5, 4], where Sobolev type inner products are constructed with respect to which uniform univariate B-splines

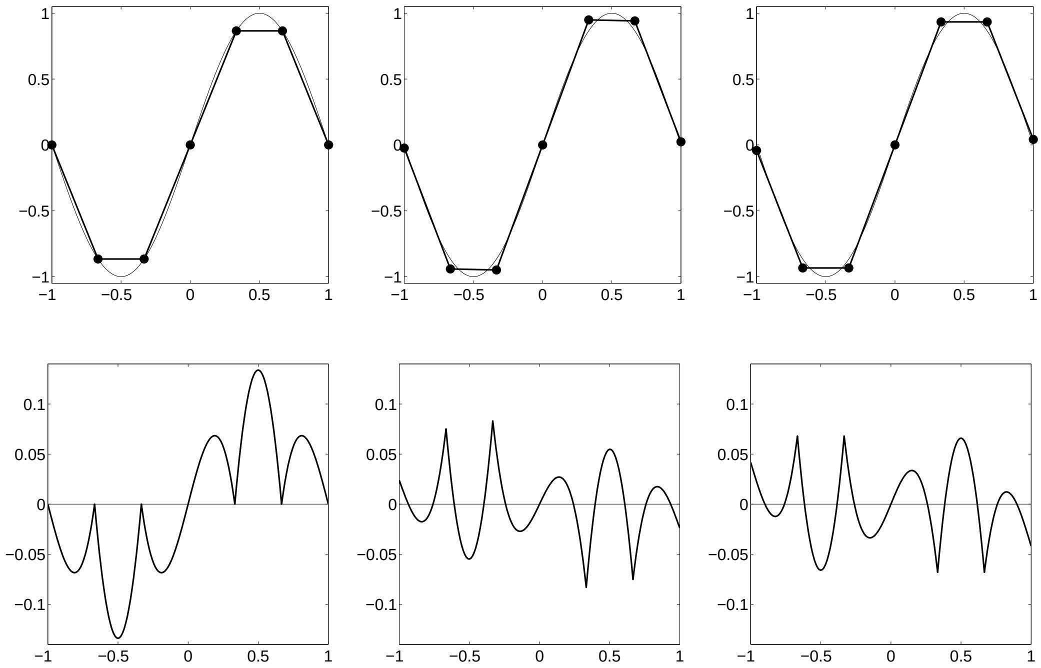

Figure 1: Linear interpolation *(left)*, L^2-approximation *(middle)*, and orthogonal approximation *(right)* of $f(x) = \sin(\pi x)$. The error *(bottom)* of linear interpolation is larger than the other two.

of arbitrary degree are orthonormal. We are going to construct an inner product involving function values and first derivatives with the property that d-variate hat functions, which are the canonical basis of the space of piecewise linear functions on a given triangulation, become orthogonal. Best approximation with respect to this inner product is

- reasonable by simultaneously minimizing the deviation of function values and gradients,
- explicit by avoiding the solution of a possibly large linear system.

The examples given in the last section show that, in typical 2d and 3d applications, linear interpolation requires about 50% more coefficients to comply with a given error maximum. Even though the methods do not differ by orders of magnitude, these potential savings might still be significant with respect to requested memory and speed of processing, in particular for large data sets.

The paper is organized as follows: After introducing basic concepts and notations in the next section, we specify weight matrices providing orthogonality of hat functions in $\mathbb{R}^d$. In Section 4, discrete variants of the inner product are derived. They avoid the possibly tedious integration of functions and gradients by resorting to polynomial interpolants for which all necessary data can be precomputed. A particularly convenient scheme is obtained when using quadratic interpolation. Here, the determination of coefficients of the best approximation boils down to forming linear combination of function values at the vertices and edge midpoints of the simplices. The numerical results given in the last section illustrate the potential benefits of the new method.

2 Preliminaries

Vectors, and in particular gradients, are always understood as column-vectors, components are indexed by superscripts, rows are separated by semi-colons, and the *Euclidean inner product* is denoted by parenthesis,

$$x = \begin{bmatrix} x^1 \\ \vdots \\ x^d \end{bmatrix} = [x^1; \dots ; x^d], \quad \nabla f(x) = \begin{bmatrix} \partial_{x^1} f(x) \\ \vdots \\ \partial_{x^d} f(x) \end{bmatrix}, \quad (x, y) := \sum_{i=1}^{d} x^i y^i.$$

The *bilinear form* induced by a symmetric matrix A with elements $a_{i,j}$ is denoted by

$$A(x, y) := (x, Ay) = \sum_{i,j} x^i \, a_{i,j} \, y^j .$$

Let $\mathcal{T}$ be a *finite triangulation* of a compact set $\Omega \subset \mathbb{R}^d$ consisting of *simplices* $T_i, i \in \mathcal{I}$, and *vertices* $v_j, j \in \mathcal{J}$. We assume that the simplices are not degenerate, i.e., the unsigned volume $|T_i|$ of T_i is positive for all i. The simplices sharing a vertex, and the vertices of a simplex are characterized by the index sets

$$\begin{aligned} \mathcal{I}_j &:= \{i \in \mathcal{I} : v_j \in T_i\}, \quad j \in \mathcal{J}, \\ \mathcal{J}_i &:= \{j \in \mathcal{J} : v_j \in T_i\}, \quad i \in \mathcal{I}, \end{aligned}$$

respectively. The number $\omega_j := \#\mathcal{I}_j$ of elements of $\mathcal{I}_j$ is called the *order* of v_j, while always $\#\mathcal{J}_i = d+1$. Obviously, $i \in \mathcal{I}_j$ if and only if $j \in \mathcal{J}_i$.

The space $\mathcal{L}$ of *piecewise linear functions* on $\mathcal{T}$ consists of all continuous functions on Ω which are linear on each simplex T_i. To each vertex v_j we associate a *hat function* $b_j \in \mathcal{L}$ which is defined by

$$b_j(v_k) = \delta_{j,k}, \quad j,k \in \mathcal{J}.$$

The set of hat functions is a basis of $\mathcal{L}$, and

$$\operatorname{supp} b_j = \bigcup_{i \in \mathcal{I}_j} T_i. \tag{1}$$

The *piecewise linear interpolant* to a function f on Ω is defined by

$$Lf := \sum_{j \in \mathcal{J}} f(v_j) b_j.$$

Given a (weakly) differentiable function f, we denote by

$$\mathbf{f} := [f;\, \nabla f] = \begin{bmatrix} f \\ \nabla f \end{bmatrix} \tag{2}$$

the vector consisting of the function and its gradient. The standard Sobolev inner product of first order on Ω is given by

$$\langle f, g \rangle_1 := \int_\Omega (\mathbf{f}, \mathbf{g}) = \sum_{i \in \mathcal{I}} \langle f, g \rangle_{T_i}, \quad \langle f, g \rangle_{T_i} := \int_{T_i} (\mathbf{f}, \mathbf{g}),$$

see, e.g., [1]. The space of all functions with finite norm $\|f\|_1 := \sqrt{\langle f, f \rangle_1}$ is denoted by $H^1(\Omega)$.

Now we generalize the concept in the following way: Let $\mathcal{W} := (W_i)_{i \in \mathcal{I}}$ be a sequence of symmetric positive definite $(d+1) \times (d+1)$-matrices. Then we define the *weighted Sobolev inner product*

$$\langle f, g \rangle_{\mathcal{T},\mathcal{W}} := \sum_{i \in \mathcal{I}} \langle f, g \rangle_{T_i, W_i}, \quad \langle f, g \rangle_{T_i, W_i} := \int_{T_i} W_i(\mathbf{f}, \mathbf{g}),$$

and the corresponding *weighted Sobolev norm*

$$\|f\|_{\mathcal{T},\mathcal{W}} := \sqrt{\langle f, f \rangle_{\mathcal{T},\mathcal{W}}}.$$

The standard Sobolev norm $\|\cdot\|_1$ and the weighted Sobolev norm $\|\cdot\|_{\mathcal{T},\mathcal{W}}$ are equivalent,

$$\sqrt{\lambda_{\min}}\,\|f\|_1 \;\le \|f\|_{\mathcal{T},\mathcal{W}} \le \sqrt{\lambda_{\max}}\,\|f\|_1,$$

where $\lambda_{\min}, \lambda_{\max}$ are bounds on the eigenvalues of all matrices W_i.

Given a function $f \in H^1(\Omega)$, the best approximation

$$Qf = \sum_{j\in\mathcal{J}} q_j b_j$$

in the space $\mathcal{L}$ of piecewise linear functions with respect to the weighted Sobolev norm is given by the solution $(q_k)_{k\in\mathcal{J}}$ of the Gramian system

$$\sum_{k\in\mathcal{J}} q_k \langle b_j, b_k\rangle_{\mathcal{T},\mathcal{W}} = \langle b_j, f\rangle_{\mathcal{T},\mathcal{W}}, \quad j \in \mathcal{J}.$$

Solving this linear system becomes trivial if the hat functions happen to be orthogonal with respect to the inner product: if

$$\langle b_j, b_k\rangle_{\mathcal{T},\mathcal{W}} = 0 \quad \text{for} \quad j \neq k,$$

then

$$q_j = \frac{\langle b_j, f\rangle_{\mathcal{T},\mathcal{W}}}{\langle b_j, b_j\rangle_{\mathcal{T},\mathcal{W}}}, \quad j \in \mathcal{J}.$$

3 Orthogonality

In this section, we specify matrices W_i such that hat functions become orthogonal. The key idea is to require orthogonality on each simplex individually. More precisely, we demand

$$\langle b_j, b_k\rangle_{T_i,W_i} = \delta_{j,k} \quad \text{for all} \quad j,k \in \mathcal{J}_i \text{ and } i \in \mathcal{I}. \tag{3}$$

The scaling $\langle b_j, b_j\rangle_{T_i,W_i} = 1$ is chosen for the sake of simplicity, and by no means necessary. In particular, it might make sense to set $\langle b_j, b_j\rangle_{T_i,W_i} = c_i$ with a constant c_i depending on the volume $|T_i|$. The generalization of the subsequent arguments, which yields similar but slightly more involved results, is left to the reader.

Exploiting (1), we find

$$\langle b_j, b_k\rangle_{\mathcal{T},\mathcal{W}} = \sum_{i\in\mathcal{I}} \langle b_j, b_k\rangle_{T_i,W_i} = \sum_{i\in\mathcal{I}_j\cap\mathcal{I}_k} \langle b_j, b_k\rangle_{T_i,W_i}.$$

We have $i \in \mathcal{I}_j \cap \mathcal{I}_k$ if and only if $j, k \in \mathcal{J}_i$. Hence, if (3) is satisfied, we obtain

$$\langle b_j, b_k\rangle_{\mathcal{T},\mathcal{W}} = \sum_{i\in\mathcal{I}_j\cap\mathcal{I}_k} \delta_{j,k} = \omega_j \delta_{j,k},$$

showing that the hat functions are orthogonal with respect to the weighted Sobolev inner product, and the norm of a hat function is given by the square root of the order of the corresponding vertex.

Given $i \in \mathcal{I}$, (3) provides $\binom{d+2}{2}$ linear conditions for the entries of W_i. Since W_i is assumed to be symmetric, the number of conditions coincides with the number of degrees of freedom suggesting that the problem to find an appropriate matrix W_i is well posed. We start with considering the orthogonality conditions on the unit simplex

$$T := \{x \in [0, 1]^d : (x, e) \le 1\},$$

where $e := [1; \ldots ; 1]$ is the vector of ones. The vertices of T are the origin e_0 and the unit vectors $e_1, \ldots , e_d$. The hat functions to the vertices $e_0, \ldots , e_d$ restricted to T are the linear Lagrange polynomials

$$l_j(x) := \begin{cases} 1 - (x, e) & \text{for } j = 0 \\ x^j & \text{for } j = 1, \ldots , d. \end{cases} \tag{4}$$

The corresponding vectors $\mathbf{l}_j$ according to (2) are given by

$$\mathbf{l}_j := [l_j;\ \nabla l_j] = \begin{cases} [1 - (x, e);\ -e] & \text{for } j = 0 \\ [x^j;\ e_j] & \text{for } j = 1, \ldots , d. \end{cases} \tag{5}$$

Now, a symmetric matrix W with

$$\langle l_j, l_k\rangle_{T,W} = \int_T W(\mathbf{l}_j, \mathbf{l}_k) = \delta_{j,k}, \quad j, k = 0, \ldots , d, \tag{6}$$

is sought. In the univariate case $d = 1$, an elementary computation shows that (6) is satisfied if we choose

$$W := \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix}.$$

This results coincides with the more general findings in [5]. In the multivariate case $d \geq 2$, symmetry suggests to treat all d coordinates in an equal way. That is, we assume that W has the form

$$W := \begin{bmatrix} w_0 & w_3 & w_3 & \cdots & w_3 \\ w_3 & w_1 & w_2 & \cdots & w_2 \\ w_3 & w_2 & w_1 & \cdots & w_2 \\ \vdots & \vdots & & \ddots & \vdots \\ w_3 & w_2 & w_2 & \cdots & w_1 \end{bmatrix}.$$

With this setting, we have

$$\begin{aligned} \langle l_0, l_1 \rangle_{T,W} &= \langle l_0, l_j \rangle_{T,W} \\ \langle l_1, l_1 \rangle_{T,W} &= \langle l_j, l_j \rangle_{T,W} \\ \langle l_1, l_2 \rangle_{T,W} &= \langle l_j, l_k \rangle_{T,W} \end{aligned}$$

for all $j, k = 1, \ldots, d$ with $j \neq k$. Hence, (3) is equivalent to the four equations

$$\langle l_0, l_0 \rangle_{T,W} = \langle l_1, l_1 \rangle_{T,W} = 1, \quad \langle l_0, l_1 \rangle_{T,W} = \langle l_1, l_2 \rangle_{T,W} = 0$$

for the four unknowns $w_0, \ldots, w_3$. To evaluate the integrals $\langle l_j, l_k \rangle_{T,W} = \int_T W(\mathbf{l}_j, \mathbf{l}_k)$, we use the formula

$$\int_T l_0^{\alpha_0} \cdots l_d^{\alpha_d} = \frac{\alpha_0! \cdots \alpha_d!}{(d + \alpha_0 + \cdots + \alpha_d)!}, \tag{7}$$

which can be found as Equation (2.3) in [3], and obtain the conditions

$$\begin{aligned} \langle l_0, l_0 \rangle_{T,W} &= \frac{2w_0}{(d+2)!} + \frac{w_1}{(d-1)!} + \frac{w_2}{(d-2)!} - \frac{2dw_3}{(d+1)!} = 1 \\ \langle l_1, l_1 \rangle_{T,W} &= \frac{2w_0}{(d+2)!} + \frac{w_1}{d!} + \frac{2w_3}{(d+1)!} = 1 \\ \langle l_0, l_1 \rangle_{T,W} &= \frac{w_0}{(d+2)!} - \frac{w_1}{d!} - \frac{(d-1)w_2}{d!} - \frac{(d-1)w_3}{(d+1)!} = 0 \\ \langle l_1, l_2 \rangle_{T,W} &= \frac{w_0}{(d+2)!} + \frac{w_2}{d!} + \frac{2w_3}{(d+1)!} = 0. \end{aligned}$$

Solving this system for $w_0, \dots, w_3$, we obtain the following result, which also comprises the univariate case $d = 1$:

Lemma 1 *For $d \in \mathbb{N}$, the functions $l_0, \dots, l_d$ according to* (4) *satisfy*

$$\langle l_j, l_k \rangle_{T,W} = \delta_{j,k}, \quad j, k = 0, \dots, d,$$

if

$$W := \begin{bmatrix} w_0 & 0 & 0 & \cdots & 0 \\ 0 & w_1 & w_2 & \cdots & w_2 \\ 0 & w_2 & w_1 & \cdots & w_2 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & w_2 & w_2 & \cdots & w_1 \end{bmatrix},$$

with

$$w_0 := (d+1)!, \quad w_1 := \frac{d!d}{d+2}, \quad w_2 := \frac{-d!}{d+2}.$$

In particular, we obtain

$$W = \frac{1}{2} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad W = \frac{1}{5} \begin{bmatrix} 120 & 0 & 0 & 0 \\ 0 & 18 & -6 & -6 \\ 0 & -6 & 18 & -6 \\ 0 & -6 & -6 & 18 \end{bmatrix}$$

for the bi- and trivariate case, respectively. For all $d \in \mathbb{N}$, the matrix W as defined above is strictly diagonally dominant and hence positive definite, making sure that W defines an inner product. Since $w_3 = 0$, this inner product can also be written in the form

$$\langle f, g \rangle_{T,W} = \int_T w_0 f g + \tilde{W}(\nabla f, \nabla g) \tag{8}$$

where

$$\tilde{W} := w_2 \big(E_d - (d+1) I_d \big)$$

is a $(d \times d)$-matrix defined in terms of the identity I_d and the matrix E_d of ones in $\mathbb{R}^{d \times d}$.

The result on the unit simplex T can be transferred to an arbitrary simplex T_i by a change of variables. Let $\mathcal{J}_i = \{\ell_0, \dots, \ell_d\}$ be the indices of the vertices of T_i. Then there exists an affine map $\mathcal{A}_i : x \mapsto A_i x + a_i$ with $\mathcal{A}_i(e_j) = v_{\ell_j}, j = 0, \dots, d$, so that $\mathcal{A}_i(T) = T_i$. The functions l_j on T and b_{ℓ_j} on T_i are related by $l_j(x) = b_{\ell_j}(y)$ and $\nabla l_j(x) = A_i^{\mathrm{t}} \nabla b_{\ell_j}(y)$, where $y = \mathcal{A}_i(x)$. Together,

$$\mathbf{l}_j(x) = M_i \, \mathbf{b}_{\ell_j}(y), \quad M_i := \begin{bmatrix} 1 & 0 \\ 0 & A_i^{\mathrm{t}} \end{bmatrix}.$$

By assumption, the volume $|T_i| = |\det A_i|$ of T_i is positive so that the matrix

$$W_i := \frac{1}{|T_i|} \, M_i^{\mathrm{t}} W M_i \tag{9}$$

is well defined. Since W is strictly diagonally dominant and M_i is invertible, W_i is positive definite. Now, using $dy = |\det A_i| \, dx$, we find

$$\langle b_{\ell_j}, b_{\ell_k} \rangle_{T_i, W_i} = \int_{T_i} W_i\big(\mathbf{b}_{\ell_j}(y), \mathbf{b}_{\ell_k}(y)\big) \, dy = \int_T W\big(\mathbf{l}_j(x), \mathbf{l}_k(x)\big) \, dx = \delta_{j,k}$$

for all $\ell_j, \ell_k \in \mathcal{J}_i$. That is, the matrices W_i provide orthonormality according to (3). We summarize our results as follows:

Theorem 2 *Let $\mathcal{W} := (W_i)_{i \in \mathcal{I}}$ be a sequence of matrices according to* (9). *Then*

$$\langle f, g \rangle_{\mathcal{T}, \mathcal{W}} := \sum_{i \in \mathcal{I}} \int_{T_i} W_i(\mathbf{f}, \mathbf{g})$$

defines a weighted Sobolev inner product with respect to which the hat functions are orthogonal,

$$\langle b_j, b_k \rangle_{\mathcal{T}, \mathcal{W}} = \omega_j \delta_{j,k}.$$

The best approximation of a function $f \in H^1(\Omega)$ with respect to the associated norm in the space $\mathcal{L}$ of piecewise linear functions is given by

$$Qf = \sum_{j \in \mathcal{J}} q_j b_j, \quad q_j = \frac{\langle b_j, f \rangle_{\mathcal{T}, \mathcal{W}}}{\omega_j}. \tag{10}$$

In view of (8), the weighted inner product on T_i can also be written in the form

$$\langle f, g\rangle_{T_i,W_i} = \frac{1}{|T_i|} \int_{T_i} w_0 fg + \tilde{W}_i(\nabla f, \nabla g), \quad \tilde{W}_i := A_i \tilde{W} A_i^{\mathrm{t}}. \tag{11}$$

Of course, the affine map $\mathcal{A}_i$ is not uniquely determined because the correspondence of vertices of T and T_i used to define $\mathcal{A}_i$ admits permutation. However, we will now show that the resulting matrix W_i is independent of the labelling of indices in $\mathcal{J}_i = \{\ell_0, \dots, \ell_d\}$. Let $V_i = [v_{\ell_0}, \dots, v_{\ell_d}]$ denote the corresponding matrix of vertices of T_i. Then $\mathcal{A}_i(x) = A_i x + a_i$ is given by

$$a_i := v_{\ell_0}, \quad A_i := V_i K, \quad K := \begin{bmatrix} -e^{\mathrm{t}} \\ I_d \end{bmatrix},$$

where, as before, e is the vector of ones, and I_d is the identity in $\mathbb{R}^d$. Hence,

$$\tilde{W}_i = V_i K \tilde{W} K^{\mathrm{t}} V_i^{\mathrm{t}}.$$

A straightforward computation shows that

$$K\tilde{W}K^{\mathrm{t}} = w_2\big(E_{d+1} - (d+1)I_{d+1}\big)$$

is a $(d+1)\times(d+1)$-matrix with the same structure as $\tilde{W}$. Now, we consider a permutation of indices of the vertices of T_i. Then the corresponding matrix V_i' is related to V_i by a permutation of columns, i.e.,

$$V_i' = V_i \Pi$$

for some permutation matrix Π. Since $\Pi E_{d+1} \Pi^{\mathrm{t}} = E_{d+1}$ and $\Pi I_{d+1} \Pi^{\mathrm{t}} = I_{d+1}$, we obtain

$$\tilde{W}_i' = V_i \Pi K \tilde{W} K^{\mathrm{t}} \Pi^{\mathrm{t}} V_i^{\mathrm{t}} = \tilde{W}_i.$$

That is, the matrices $\tilde{W}_i$ and $\tilde{W}_i'$ corresponding to V_i and V_i' coincide. Further, also the scaling factor $1/|T_i|$ appearing in (11) is independent of the labelling of vertices, showing that the inner product $\langle\cdot,\cdot\rangle_{T_i,W_i}$, and hence also $\langle\cdot,\cdot\rangle_{\mathcal{T},\mathcal{W}}$ depends only on the geometry of the triangulation.

4 Discrete Variants

The orthogonal expansion (10) avoids the solution of a possibly large linear system, but compared with the standard interpolation technique, it is more expensive since it requires the integration of expressions depending on values and gradients of the given function f. To further increase efficiency, we now derive discrete variants of the weighted Sobolev inner product, which equally provide orthogonality. The idea is to replace the functions f, g by some polynomial interpolants p_i, q_i of degree n before computing the inner product $\langle f, g\rangle_{T_i,W_i}$ on the simplex T_i.

Given $n \in \mathbb{N}$, we denote by $\mathbb{P}_n^d$ the space of all polynomials of degree $\leq n$ in d variables. The dimension of $\mathbb{P}_n^d$ is $m := \binom{n+d}{n}$. Let $U = [u_1, \ldots, u_m]$ be a sequence of pairwise different points $u_\mu \in T$ in the unit simplex, and denote by $\varphi(U) := [\varphi(u_1); \ldots; \varphi(u_m)]$ the corresponding vector of values of a given function φ. Further, let us assume that U is chosen such that for any φ the interpolation problem

$$p(U) = \varphi(U)$$

has a unique solution $p \in \mathbb{P}_n^d$, see e.g. [6] for details on the solvability of multivariate interpolation problems. With $L^n := [l_1^n; \ldots; l_m^n]$ the vector of associated Lagrange polynomials, we have

$$p = \sum_{\mu=1}^{m} l_\mu^n \, \varphi(u_\mu) = \big(L^n, \varphi(U)\big).$$

Accordingly, for a simplex $T_i = \mathcal{A}_i(T)$, let

$$\begin{aligned} U_i &:= [u_{i,1}, \ldots, u_{i,m}], \qquad & u_{i,\mu} &:= \mathcal{A}_i(u_\mu) \\ L_i^n &:= [l_{i,1}^n; \ldots; l_{i,m}^n], \qquad & l_{i,\mu}^n(y) &:= l_\mu^n(x), \quad y = \mathcal{A}_i(x), \end{aligned}$$

denote the transformed interpolation points and associated Lagrange polynomials, respectively. Let $\varphi(x) = \varphi_i(y)$. Then $\varphi(U) = \varphi_i(U_i)$, and the polynomial

$$p_i := \big(L_i^n, \varphi_i(U_i)\big)$$

solves the interpolation problem $p_i(U_i) = \varphi_i(U_i)$. Moreover,

$$p(x) = p_i(y), \quad \nabla p(x) = A_i^{\mathrm{t}} \, \nabla p_i(y).$$

Now, we define the *discrete inner product* with respect to the points $\mathcal{U} := (U_i)_{i\in\mathcal{I}}$ and degree n by

$$[f,g]_{\mathcal{T},\mathcal{W}} := \sum_{i\in\mathcal{I}}[f,g]_{T_i,W_i}, \quad [f,g]_{T_i,W_i} := \langle L_i^n f(U_i), L_i^n g(U_i)\rangle_{T_i,W_i}.$$

That is, on each simplex, the given functions are replaced by their polynomial interpolants before the formerly defined weighted Sobolev inner product is computed. In this way, $[f,g]_{\mathcal{T},\mathcal{W}}$ depends only on the function values at the points in $\mathcal{U}$. Of course, positive definiteness has to be understood in the sense that $[f,f]_{\mathcal{T},\mathcal{W}} = 0$ only if f does vanish on all points in $\mathcal{U}$.

We observe the following: First, the hat functions are orthogonal also with respect to the discrete inner product,

$$[b_j, b_k]_{\mathcal{T},\mathcal{W}} = \omega_j \delta_{j,k}. \tag{12}$$

This follows immediately from the fact that the b_j are linear on each simplex and, by assumption, $n \geq 1$ so that

$$b_j(y) = \big(L_i^n(y), b_j(U_i)\big), \quad y \in T_i.$$

Second, using (11) and $\nabla l_\mu^n(x) = A_i^{\mathrm{t}}\, \nabla l_{i,\mu}^n(y)$, we find

$$\begin{aligned}
[f,g]_{T_i,W_i} &= \sum_{\nu,\mu=1}^{m} f(u_{i,\mu})g(u_{i,\nu}) \int_{T_i} W_i\big(\mathbf{l}_{i,\mu}^n(y), \mathbf{l}_{i,\nu}^n(y)\big)\, dy \\
&= \sum_{\nu,\mu=1}^{m} f(u_{i,\mu})g(u_{i,\nu}) \int_{T} W\big(\mathbf{l}_{\mu}^n(x), \mathbf{l}_{\nu}^n(x)\big)\, dx \\
&= G^n\big(f(U_i), g(U_i)\big),
\end{aligned}$$

where the $(m \times m)$-matrix G^n, defined by

$$G_{\mu,\nu}^n := \langle l_\mu^n, l_\nu^n\rangle_{T,W} = \int_T W(\mathbf{l}_\mu^n, \mathbf{l}_\nu^n),$$

is the Gramian of the Lagrange polynomials with respect to the weighted Sobolev inner product on the unit simplex. Notably, this matrix representing the discrete inner product on the simplex T_i in terms of the function values at the points U_i is *independent* of the geometry of T_i.

Third, to compute the best approximation

$$Q^n f = \sum_{j \in \mathcal{J}} q_j^n b_j, \quad q_j^n := \frac{[b_j, f]_{\mathcal{T},\mathcal{W}}}{\omega_j}, \tag{13}$$

of a function f with respect to the discrete inner product, we proceed as follows: Given an index $j \in J$ and a simplex T_i in the support of b_j, we choose the affine map $\mathcal{A}_i^j : T \to T_i$ such that $\mathcal{A}_i^j(e_0) = v_j$. This implies that the linear Lagrange polynomial l_0 on the unit simplex according to (4) corresponds to the hat function b_j on T_i,

$$l_0(x) = b_j(y), \quad x \in T,\ y = \mathcal{A}_i^j(x) \in T_i.$$

The transformed interpolation points are denoted by $U_i^j := \mathcal{A}_i^j(U)$. Then we obtain

$$[b_j, f]_{\mathcal{T},\mathcal{W}} = \sum_{i \in \mathcal{I}_j} G^n\big(b_j(U_i^j), f(U_i^j)\big) = \sum_{i \in \mathcal{I}_j} \big(R^n, f(U_i^j)\big),$$

where the vector

$$R^n := G^n\, l_0(U) = G^n\, b_j(U_i^j)$$

is independent of T_i. Together with (12), we obtain the following result.

Theorem 3 *The best approximation $Q^n f = \sum_j q_j^n b_j$ of a function f with respect to the discrete inner product $[\cdot,\cdot]_{\mathcal{T},\mathcal{W}}$ is given by the coefficients*

$$q_j^n := \frac{1}{\omega_j} \sum_{i \in \mathcal{I}_j} \big(R^n, f(U_i^j)\big),$$

where ω_j is the order of the vertex v_j, $f(U_i^j)$ are the function values at the points U_i^j in T_i, and R^n is a fixed vector of weights as defined above.

We note that for a given set U of interpolation points the vector $R^n = [r_1^n; \ldots; r_m^n]$ can be pre-computed conveniently using the representation

$$\begin{aligned} r_\mu^n &= \sum_{\nu=1}^m \langle l_\mu^n, l_\nu^n \rangle_{T,W}\, l_0(u_\nu) = \Big\langle l_\mu^n, \sum_{\nu=1}^m l_\nu^n l_0(u_\nu) \Big\rangle_{T,W} = \langle l_\mu^n, l_0 \rangle_{T,W} \\ &= w_0 \int_T l_\nu^n\, l_0 + w_2 \int_T (\nabla l_\nu^n, e), \end{aligned} \tag{14}$$

where, as before, e is the vector of ones.

In the linear case $n = 1$, the natural choice of interpolation points is the set of vertices, $U := [e_0, \ldots, e_d]$. Here, it is easily shown that $R^1 = [1; 0; \ldots; 0]$. That is, best approximation with respect to the discrete inner product boils down to standard linear interpolation. The quadratic case $n = 2$, which is more promising and in fact recommended for applications, is now discussed in some detail. Here, the natural choice of interpolation points is the set of vertices and edge midpoints. It is convenient to use double indices for the points in U,

$$U := [u_{j,k}]_{0\le j\le k\le d}, \quad u_{j,k} := (e_j + e_k)/2.$$

Accordingly, the components of the vectors L^2 and R^2 are now labelled $l^2_{j,k}$ and $r^2_{j,k}$, respectively. The Lagrange polynomials for the vertices and the edge midpoints are given by

$$l^2_{j,k} = \begin{cases} l_j(2l_j - 1) & \text{if } j = k \\ 4\, l_j l_k & \text{if } j < k, \end{cases} \tag{15}$$

respectively. We distinguish four different types of points: the origin (●), the remaining vertices (○), the midpoints of edges containing the origin (■), and the remaining midpoints (□). Here and below, the symbols in parenthesis correspond to the markers used in Figures 2, 4, and 7. For symmetry reasons, the weights $r^2_{j,k}$ coincide for all points within the same class. Application of the formulas (7), (14), and (15) yields

$$r^2_{j,k} = \begin{cases} r^2_{\bullet} := \frac{(3-d)(d^2+5d+2)}{(d+1)(d+2)(d+3)} & \text{if } 0 = j = k \\ r^2_{\circ} := \frac{-8}{(d+1)(d+2)(d+3)} & \text{if } 0 < j = k \\ r^2_{\blacksquare} := \frac{4(d^2+4d-1)}{(d+1)(d+2)(d+3)} & \text{if } 0 = j < k \\ r^2_{\square} := \frac{-4(d+5)}{(d+1)(d+2)(d+3)} & \text{if } 0 < j < k. \end{cases} \tag{16}$$

The quadratic case $n = 2$ as described above has the advantage that it uses only function values at the vertices and the edge midpoints of the triangulation. Typically, these (and possibly further) data are also used for estimating the error maximum, say when computing an adaptively refined linear interpolant with prescribed accuracy. Hence, no extra function evaluations are required. Further, for a fine triangulation with simplices of size h and a smooth function f, let us compare the original approximation Qf according to (10) and the discrete approximation Q^2f according to (13). It is easily shown that the coefficients q_j and q_j^2 differ by terms of order $O(h^3)$, and thus also $\|Qf - Q^2f\|_\infty = O(h^3)$. Compared with the error $\|f - Qf\|_\infty = O(h^2)$ of the approximation itself, this difference is small. In particular, for small h, no substantial gain in accuracy can be expected when using cubic or even higher degree interpolation. Summarizing, we conclude that Q^2f is not more expensive than linear interpolation Lf and not less accurate than Qf.

5 Examples

In this section, we consider the piecewise linear approximation of univariate, bivariate, and trivariate functions using the discrete variant based on quadratic interpolation according to (16). Below, N_{int} and N_{app} denote the number of vertices used for linear interpolation and orthogonal approximation, respectively. The corresponding maximum errors, estimated by evaluation at the vertices, edge midpoints, and centers of the simplices, are denoted by $\Delta_{\text{int}} := \|f - Lf\|_\infty$ and $\Delta_{\text{app}} := \|f - Q^2f\|_\infty$. Throughout, solid lines are used for orthogonal approximation, while broken lines are used for linear interpolation.

5.1 Univariate case $d = 1$

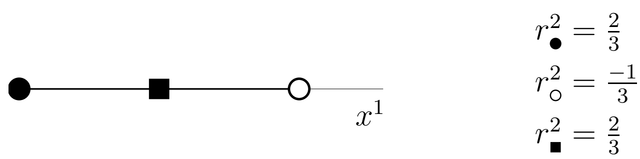

Figure 2: Unit interval with weights of discrete orthogonal approximation based on quadratic interpolation.

Figure 2 shows the weights used for orthogonal approximation in the univariate case. We consider the function $f(x) = \sin(\pi x)$ appearing already in the introduction. Figure 3 shows the results for equidistant knots. Asymptotically, the maximal error of linear interpolation is $\approx 50\%$ larger than the maximal error of orthogonal approximation, and accordingly, $\approx 23\%$ more coefficients are required to achieve a given maximal error.

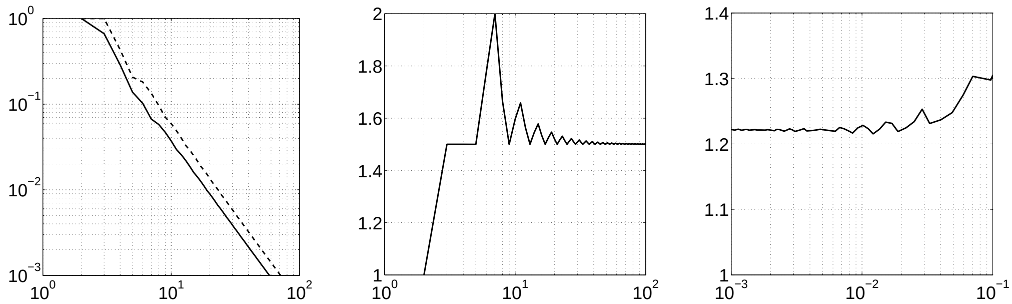

Figure 3: Univariate case. Errors $\Delta_{\text{int}}, \Delta_{\text{app}}$ *(left)* and ratio $\Delta_{\text{int}}/\Delta_{\text{app}}$ *(middle)* as a function of the number of vertices; ratio $N_{\text{int}}/N_{\text{app}}$ as a function of maximal error *(right)*.

5.2 Bivariate case $d = 2$

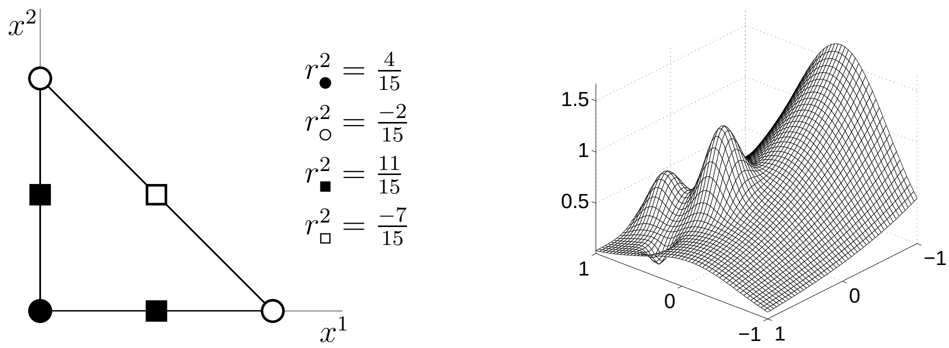

Figure 4: Unit triangle with weights of discrete orthogonal approximation based on quadratic interpolation *(left)* and Franke function *(right)*.

Figure 4 *(left)* shows the weights used for orthogonal approximation in the bivariate case. We consider the Franke function [2] on the domain $[-1, 1]^2$, see Figure 4 *(right)*. Figure 5 shows the results for a uniform partition, where the domain is split into pairs of right triangles, combining to squares of equal size. Asymptotically, the maximal error of linear interpolation is $\approx 50\%$ larger than the maximal error of orthogonal approximation, and accordingly, $\approx 50\%$ more coefficients are required to achieve a given maximal error.

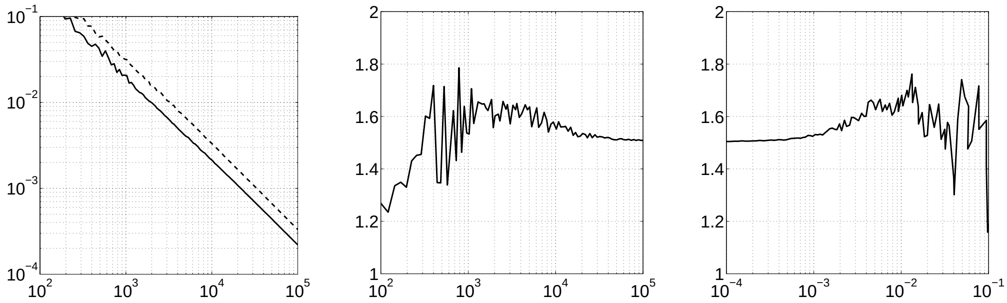

Figure 5: Bivariate case, uniform partition. Errors $\Delta_{\text{int}}, \Delta_{\text{app}}$ *(left)* and ratio $\Delta_{\text{int}}/\Delta_{\text{app}}$ *(middle)* as a function of the number of vertices; ratio $N_{\text{int}}/N_{\text{app}}$ as a function of maximal error *(right)*.

Figure 6 shows the results for adaptive refinement, where insufficient triangles are split at the midpoint of the longest edge. The pattern is less clear than in the uniform case, but also here, the potential savings are significant.

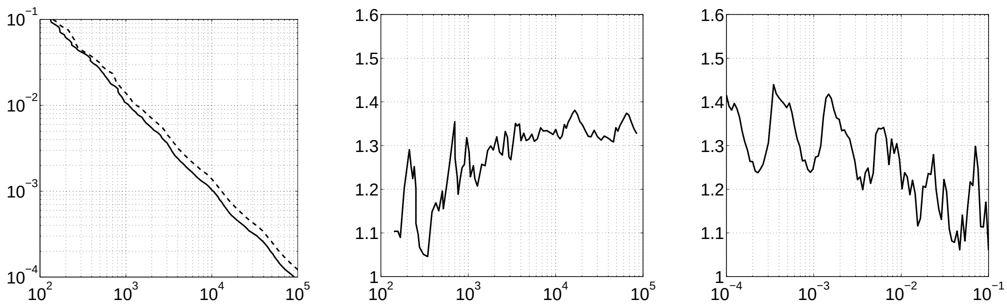

Figure 6: Bivariate case, adaptive refinement. Errors $\Delta_{\text{int}}, \Delta_{\text{app}}$ *(left)* and ratio $\Delta_{\text{int}}/\Delta_{\text{app}}$ *(middle)* as a function of the number of vertices; ratio $N_{\text{int}}/N_{\text{app}}$ as a function of maximal error *(right)*.

5.3 Trivariate case $d = 3$

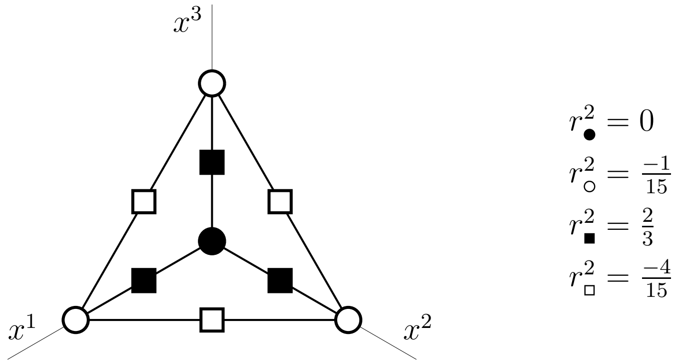

Figure 7: Unit tetrahedron with weights of discrete orthogonal approximation based on quadratic interpolation.

Figure 7 shows the weights used for orthogonal approximation in the trivariate case. We consider the function $f(x) = \exp(x^2 - y^2 - 2z^2)$. Figure 8 shows the results for a type-4-partition [7] of a uniform hexahedral grid. Asymptotically, the maximal error of linear interpolation is $\approx 28\%$ larger than the maximal error of orthogonal approximation, and accordingly, $\approx 45\%$ more coefficients are required to achieve a given maximal error.

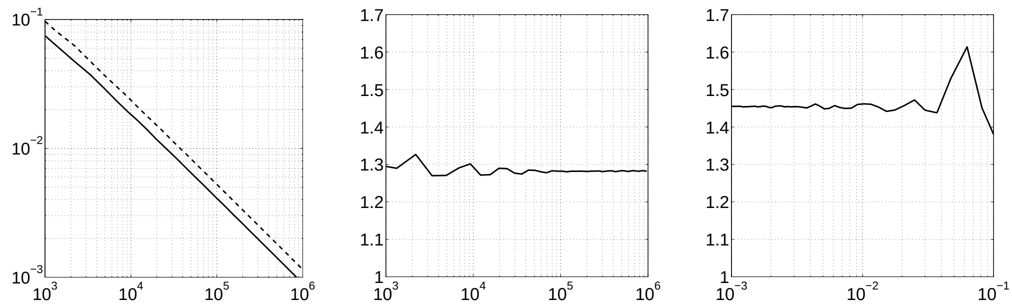

Figure 8: Trivariate case. Errors $\Delta_{\text{int}}, \Delta_{\text{app}}$ *(left)* and ratio $\Delta_{\text{int}}/\Delta_{\text{app}}$ *(middle)* as a function of the number of vertices; ratio $N_{\text{int}}/N_{\text{app}}$ as a function of maximal error *(right)*.

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