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Acute Triangulations of Polygons

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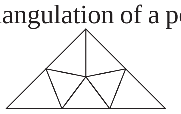
We prove that every n-gon can be triangulated into $O(n)$ acute triangles. We also present a short proof of the result that every polygon can be triangulated into right triangles.

1. INTRODUCTION

By a *triangulation* of a polygon, we mean a subdivision of the polygon into non-overlapping triangles in such a way that any two distinct triangles are either disjoint, have a single vertex in common, or have one entire edge in common. Every triangle can be divided into three obtuse triangles by the three line-segments each connecting a vertex to the centre of the inscribed circle of the triangle. Hence, every polygon can be triangulated into *obtuse* triangles. Baker, Grosse and Rafferty [1] proved that every polygon admits a triangulation into *non-obtuse* triangles. Bern, Mitchell and Ruppert [2] gave an algorithm for triangulating n-gons into $O(n)$ non-obtuse triangles.

An *acute* triangulation of a polygon is a triangulation whose triangles are all acute triangles. For example,  shows an acute triangulation of a right triangle. Any obtuse or right triangle can be triangulated into acute triangles, similarly. Hence every polygon admits a dissection into acute triangles. (In a *dissection*, vertices may appear within an edge of a sub-triangle.) Gerver [5] showed how to compute a dissection of a polygon with no angles larger than 72°, assuming all interior angles of the input measure at least 36°.

Now, does every polygon admit an acute triangulation? This is a tantalizing problem, and it seems not answered yet. In this paper, we prove the following theorem.

THEOREM 1. *Every polygon admits an acute triangulation.*

The key point for the proof is a *pivot* of a polygon, which is introduced in Section 2. In Section 3, we prove this theorem using the existence of a non-obtuse triangulation for a polygon. To be complete, we will present in Section 5, a short proof of the result that every polygon can be triangulated into *right* triangles.

How many triangles are necessary for an acute triangulation of an n-gon? Martin Gardner [4, pp. 39–42] proposed in 1960 a problem to ask how many acute triangles are necessary for an acute triangulation of an obtuse triangle. Wallace Manheimer [7] gave a solution that the number is seven. Cassidy and Lord [3] showed that a square can be triangulated into eight acute triangles, eight is the minimum number of acute triangles for a square, and the triangulation into eight acute triangles is unique in a sense. Maehara [6] showed that every quadrilateral can be triangulated into at most 10 acute triangles, and there is a concave quadrilateral that requires 10 acute triangles.

Concerning the number of triangles in our acute triangulation, we have the following.

THEOREM 2. *If a polygon can be triangulated into N non-obtuse triangles, then it can be triangulated into at most $2 \cdot 6^5 N$ acute triangles.*

Since every n-gon can be triangulated into $O(n)$ non-obtuse triangles by [2], we have the following.

COROLLARY 1. *Every n-gon can be triangulated into $O(n)$ acute triangles.* □

2. PIVOTS OF POLYGONS

The interior of a polygon Γ is denoted by Γ°, and the boundary of Γ is denoted by $\partial\Gamma$. A vertex of a polygon is called an *acute* (*right-angled*, or *obtuse*) *corner* if the interior angle at the vertex is acute (right-angled, or obtuse). Let $P \in \partial\Gamma$, that is, P is either a vertex of Γ or a point within an edge. When we trace $\partial\Gamma$ clockwise, the vertex we meet immediately before P and the vertex we meet immediately after P are called the *neighbouring vertices* or the *neighbouring corners* of P.

LEMMA 1. *Let Γ be a polygon, $P \in \partial\Gamma$. Let $A_1, A_2, \ldots, A_n, P$ be the cyclic sequence of the vertices of Γ and P. (Thus, A_1, A_n are the neighbouring vertices of P, and P itself may or may not be a vertex.) Suppose that* (1) *all the edges A_iA_{i+1} $(i = 1, 2, \ldots, n-1)$ are tangent to a circle with centre P, and* (2) *A_i, $i = 2, 3, \ldots, n-1$, are obtuse corners and*

$$45^\circ < \angle A_1 < 90^\circ, \qquad 45^\circ < \angle A_n < 90^\circ.$$

Then Γ is triangulated into acute triangles by the line-segments PA_i, $i = 2, \ldots, n-1$.

PROOF. For each $1 < i < n$, the line A_iP bisects the obtuse angle at A_i. Hence $\angle A_{i-1}A_i P > 45^\circ$ and $\angle A_{i+1}A_iP > 45^\circ$. Therefore the triangles A_jPA_{j+1} are all acute triangles. □

COROLLARY 2. *If a polygon circumscribed to a circle has only obtuse corners, then the polygon is divided into acute triangles by the line-segments connecting the centre of the circle to the vertices of the polygon.* □

For a polygonal region Ω (possibly with holes), a point $P \in \Omega$ and an edge XY of Ω are said to be *facing to* each other *in* Ω, if the three points P, X, Y form a triangle contained in Ω and $\angle PXY$, $\angle PYX$ are both non-obtuse. In Figure 1, P is facing to XY, and is also facing to AB, BC in Ω, but P is not facing to CD, DE, EA, YZ, ZW, WX. Notice that if P and XY are facing to each other in Ω, there is a point F on the line-segment XY which is the foot of perpendicular from P to XY.

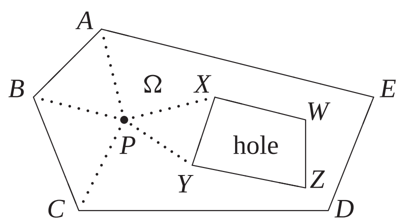


FIGURE 1. P is facing to XY in Ω.

A point P is called a *pivot* of a polygon Γ if either

- $P \in \Gamma^\circ$ and all edges of Γ are facing to P in Γ, or
- $P \in \partial\Gamma$, the edges not incident to P are all facing to P, and the neighbouring corners of P are both acute corners with angles $>45^\circ$.

See Figure 2.

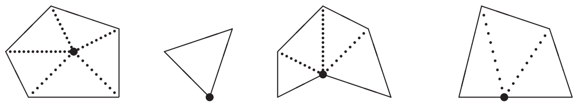


FIGURE 2. Pivots.

PROPOSITION 1. *If a polygon Γ has a pivot P, then it admits an acute triangulation in which the vertices newly introduced on the edges facing to P are the feet of the perpendiculars from P. If Γ has n vertices, then the number of triangles in this acute triangulation is at most* $6n$.

PROOF. Let us consider the case $P \in \partial\Gamma$. (In Figure 3(a), P is a vertex of the polygon $\Gamma = ABCDP$.) Take a small circle **O** with center P, and circumscribe to **O** a polygonal curve consisting of those line-segments that are parallel to the edges (non-incident to P) of Γ. For each acute or right-angled corner X of Γ that is not neighbouring to P, cut off the corresponding corner of the polygonal curve by a line perpendicular to PX and tangent to the circle **O**. (In Figure 3(a), the corner corresponding to the acute corner C is cut off by the line-segment C_1C_2.) Let Γ_1 be the polygon obtained by connecting both ends of this polygonal curve to the point P. (In Figure 3(a), $\Gamma_1 = A_1B_1C_1C_2D_1P$.) Then by Lemma 1, Γ_1 can be triangulated into acute triangles by the line-segments connecting P to the vertices of Γ_1. Note that for each acute or right-angled corner X of Γ that is not a neighbouring corner of P, there is a unique edge of Γ_1 that is facing to X in the region $\Gamma - \Gamma_1^\circ$. (In Figure 3(a), C_1C_2 is the unique edge facing to C in $\Gamma - \Gamma_1^\circ$.) Connect such an X to the endpoints of the unique edge by line-segments. Similarly, for each foot F of the perpendicular from P to an edge of Γ, there is a unique edge of Γ_1 that is facing to F in $\Gamma - \Gamma_1^\circ$. Connect F to the endpoints of the unique edge of Γ_1 by line-segments. Then the region $\Gamma - \Gamma_1^\circ$ is divided into triangles and convex quadrilaterals. Finally, divide each quadrilateral by the diagonal emanating from the obtuse corner of Γ. (In Figure 3(a), the quadrilateral $EBFB_1$ is divided by the diagonal BB_1. One may think that $\angle EBB_1$ does not look acute. But if the circle **O** centred at P becomes small, $\angle EBB_1$ becomes acute.) If the circle **O** is sufficiently small, this yields an acute triangulation of Γ.

The case $P \in \Gamma^\circ$ is similar. Figure 3(b) shows a case when P is an interior point of Γ.

The assertion on the number of triangles will be clear, see Figure 3(b). □

A polygonal decomposition $\mathcal{P} = \{\Lambda_1, \Lambda_2, \ldots, \Lambda_n\}$ of a polygon Γ is a decomposition of Γ into sub-polygons $\Lambda_1, \ldots, \Lambda_n$ such that for any $i \neq j$, Λ_i, Λ_j are either disjoint, have a single vertex in common, or have one entire edge in common. Using the existence of a non-obtuse triangulation, the following proposition is proved in the next section.

PROPOSITION 2. *For every polygon Γ, there is a polygonal decomposition $\mathcal{P} = \{\Lambda_1, \Lambda_2, \ldots, \Lambda_n\}$ of Γ such that (i) each Λ_i has a pivot P_i, and (ii) for any two polygons Λ_i, Λ_j with a common edge e, the edge e and the line-segment P_iP_j cross each other perpendicularly* (*see Figure* 4).

PROOF OF THEOREM 1. Let Γ be a polygon, and let $\mathcal{P} = \{\Lambda_1, \ldots, \Lambda_n\}$ be a polygonal decomposition of Γ satisfying the conditions (i), (ii) of Proposition 2. Then, each Λ_i admits an acute triangulation by Proposition 1, and by the condition (ii), these acute triangulations are consistent between adjacent polygons. Hence we can obtain an acute triangulation of Γ. □

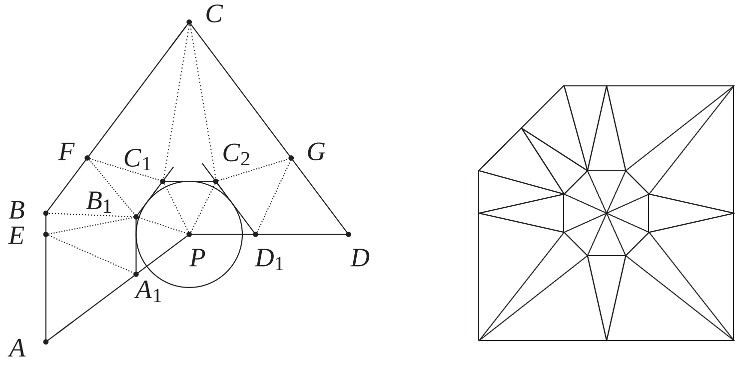


FIGURE 3. Acute triangulations by pivots.

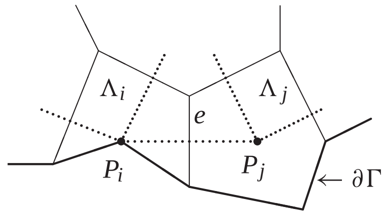


FIGURE 4. A polygonal decomposition in Proposition 2.

3. PROOF OF PROPOSITION 2

In this section, we use the fact [1, 2] that every polygon admits a non-obtuse triangulation. In Section 5, we also present a proof of this fact. To prove Proposition 2, we need another lemma.

Let $\mathcal{T}$ be a (not necessarily non-obtuse) triangulation of a polygon Γ. The number of triangles in $\mathcal{T}$ is called the *size* of $\mathcal{T}$, and it is denoted by $|\mathcal{T}|$. A vertex (edge) of $\mathcal{T}$ inside Γ is called an *inner* vertex (edge), while a vertex (edge) lying on the boundary of Γ is called an *outer* vertex (edge). A triangle of $\mathcal{T}$ that has exactly one outer edge is called a *side triangle* of $\mathcal{T}$, and a triangle that has two outer edges is called a *corner triangle*. For an outer edge e, the *opposite angle* $\theta(e)$ of e is the angle opposite to e in the unique triangle incident to e, see Figure 5. Let us denote by $\theta_{\min}(\mathcal{T})$, $\theta_{\max}(\mathcal{T})$, the minimum value and the maximum value of the angles $\theta(e)$ for all outer edges e of $\mathcal{T}$.

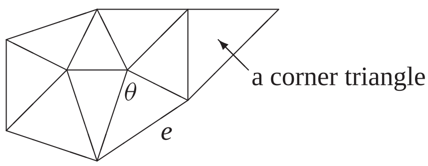


FIGURE 5. The opposite angle θ of an outer edge e.

Let ABC be a non-obtuse triangle, L, M, N be the midpoints of the edges AB, BC, CA, respectively, and Z be the circumcentre of ABC. (If ABC is a right triangle, Z coincides with one of L, M, N. If ABC is an acute triangle, Z is an interior point of the triangle, see Figure 6.) Then, by adding the line-segments connecting Z to A, B, C, L, M, N, the triangle ABC is divided into six (or four if ABC is a right triangle) right triangles. Let us call this operation to ABC the *basic subdivision*. Note that since Z is the circumcentre of ABC, we have

$$\angle BAC = \frac{1}{2}\angle BZC = \angle BZM = \angle MZC$$

by the inscribed angle theorem.

If $\mathcal{T}$ is a non-obtuse triangulation of a polygon Γ, then by carrying out the basic subdivision to each triangle of $\mathcal{T}$, we get a refined triangulation, which is denoted by $\mathrm{sd}\mathcal{T}$. Note that $|\mathrm{sd}\mathcal{T}| \leq 6|\mathcal{T}|$, and that all triangles in $\mathrm{sd}\mathcal{T}$ are *right* triangles. From the above equality, we have

$$\theta_{\min}(\mathcal{T}) = \theta_{\min}(\mathrm{sd}\mathcal{T}), \qquad \theta_{\max}(\mathcal{T}) = \theta_{\max}(\mathrm{sd}\mathcal{T}).$$

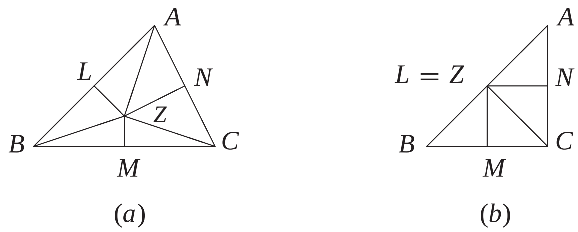


FIGURE 6. Basic subdivision.

LEMMA 2. *For every polygon Γ, there is a non-obtuse triangulation $\hat{\mathcal{T}}$ of Γ such that*

$$45^\circ < \theta_{\min} \qquad \text{and} \qquad \theta_{\max} < 90^\circ.$$

PROOF. Take a non-obtuse triangulation $\mathcal{T}$ of Γ. If $\mathcal{T}$ satisfies the condition of the lemma, we may put $\hat{\mathcal{T}} = \mathcal{T}$. If $\mathcal{T}$ has an outer edge e with opposite angle 90°, then we draw the perpendicular from the vertex of the opposite angle to the outer edge e. Then e is divided into two outer edges with opposite angles less than 90°. However, if e is an outer edge of a corner triangle with the other outer edge f, the perpendicular to e turns the opposite angle of f to the right angle. Hence, if $\mathcal{T}$ contains a right triangle as its corner triangle, then we first consider to eliminate all corner triangles by subdividing $\mathcal{T}$. To do this, we apply the basic subdivision to most triangles in $\mathcal{T}$. Suppose that ABC is a corner triangle of $\mathcal{T}$ with outer edges AB and BC. Thus, B is a corner vertex of Γ. If ABC is an acute triangle or B is a right-angled corner, then the basic subdivision to ABC yields no corner triangle. If ABC is a right triangle and one of AB or BC, say AB, is the hypotenuse, then the basic subdivision will yield a corner triangle that is similar to ABC. So, in this case, we divide ABC in the following way: let N be the midpoint of the inner edge AC (see Figure 7), and let F be the foot of the perpendicular from N to the hypotenuse AB. Add the line-segments NF, NB. Then ABC is divided into three right triangles, and this modification is consistent with the basic subdivision to the neighbouring triangle in $\mathcal{T}$. In this way, we have a non-obtuse triangulation $\mathcal{T}_1$ which has no corner triangle.

Now, to each outer edge of $\mathcal{T}_1$ with opposite angles 90°, draw the perpendicular from the vertex of its opposite angle, and let $\mathcal{T}_2$ be the resulting non-obtuse triangulation. Then $\theta_{\max}(\mathcal{T}_2)$ is less than 90°.

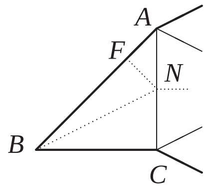


FIGURE 7. Elimination of a corner right triangle.

If $\theta_{\min}(\mathcal{T}_2) > 45°$, we may put $\hat{\mathcal{T}} = \mathcal{T}_2$. Suppose $\theta_{\min}(\mathcal{T}_2) \leq 45°$. Let XY be an outer edge of $\mathcal{T}_2$ with opposite angle $\leq 45°$, and let XYZ be the triangle with outer edge XY. Let L, M, N be the midpoints of the edges XY, XZ, YZ, respectively. Let Ω denote the set of interior points of the trapezoid $XMNY$ that are exterior to the three circles with diameter XM, MN, NY, see Figure 8(a). Since the opposite angle of XY is less than or equal to 45°, the point L is exterior to the circle with diameter MN, and any neighbourhood of L contains a point of Ω. Take a point $P \in \Omega$ near L, and divide XYZ into six triangles by the line-segments MN, PX, PM, PN, PY, PF, where F is the foot of perpendicular from P to XY, see Figure 8(b). Then these six triangles are all non-obtuse, and if P is sufficiently near L, then $\angle XPF$, $\angle YPF$ are both greater than 45°.

Divide similarly all side triangles whose outer edges have opposite angles $\leq 45°$, and apply the basic subdivision to the remaining triangles of $\mathcal{T}_2$. Let $\hat{\mathcal{T}}$ be the resulting triangulation. Then $\hat{\mathcal{T}}$ is a non-obtuse triangulation with $45° < \theta_{\min}$, $\theta_{\max} < 90°$. □

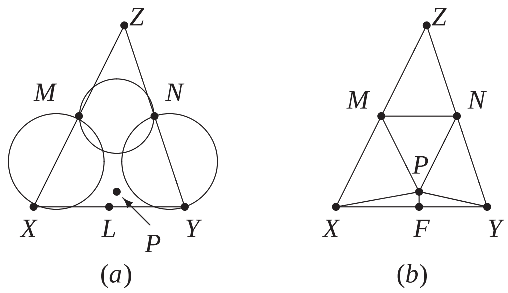
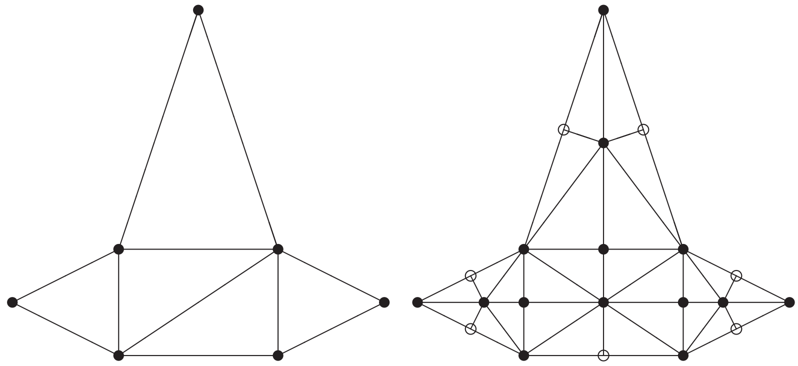


FIGURE 8. Make opposite angles of the outer edges be >45°.

REMARK. In the above proof, we have $|\mathcal{T}_1| \leq 6|\mathcal{T}|$, $|\mathcal{T}_2| \leq 2|\mathcal{T}_1|$, $|\hat{\mathcal{T}}| \leq 6|\mathcal{T}_2|$. Hence $|\hat{\mathcal{T}}| \leq 2 \cdot 6^2|\mathcal{T}|$.

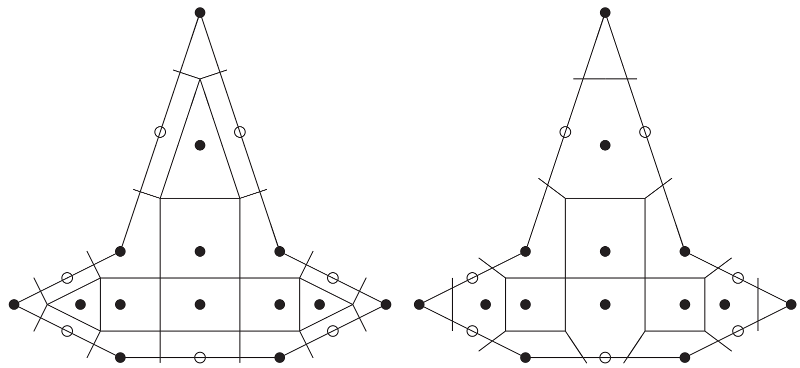
PROOF OF PROPOSITION 2. (1) *Outline*. Let Γ be a polygon, and let $\hat{\mathcal{T}}$ be a non-obtuse triangulation of Γ such that $45° < \theta_{\min}$ and $\theta_{\max} < 90°$ as in Lemma 2. Let M_i, $i = 1, 2, \ldots, n$, be the midpoints of the outer edges of $\hat{\mathcal{T}}$. (In Figure 9, (a) shows $\hat{\mathcal{T}}$, and (b) shows sd$\hat{\mathcal{T}}$. The midpoints M_i are denoted by $\circ$.) Let $\mathbf{S}$ denote the set of vertices of sd$\hat{\mathcal{T}}$. Then the 'constrained' Voronoi decomposition of Γ generated by $\mathbf{S} - \{M_1, M_2, \ldots, M_n\}$ gives a polygonal decomposition of Γ satisfying the conditions (i), (ii) of Proposition 2. (Figure 9(d) shows this decomposition.)

(2) *Details*. Let Λ_P be the union of those triangles of sd(sd$\hat{\mathcal{T}}$) = (sd)$^2\hat{\mathcal{T}}$ that have $P \in \mathbf{S}$ as a vertex. Then Λ_P is a polygon, and we have a polygonal decomposition $\mathcal{P} = \{\Lambda_P : P \in \mathbf{S}\}$ of Γ. (Figure 9(c) shows this decomposition.) Recall that by the basic subdivision to a non-obtuse triangle ABC of sd$\hat{\mathcal{T}}$, each edge of ABC is bisected, and new edges are drawn from the



(a) A non-obtuse triangulation with $45° < \theta_{\min}, \theta_{\max} < 90°$

(b) Basic subdivision



(c) Voronoi decomposition generated by $\bullet$ and $\circ$

(d) Voronoi decomposition generated by $\bullet$, which is a polygonal decomposition required in Proposition 2.

FIGURE 9. Proof of Proposition 2.

circumcentre of ABC to the midpoints of AB, BC, CA. Thus the new edges are perpendicular bisectors of the edges of $\mathrm{sd}\hat{\mathcal{T}}$. Hence, if an edge of $(\mathrm{sd})^2\hat{\mathcal{T}}$ on $\partial\Lambda_A$ (the boundary of the polygon Λ_A, $A \in \mathbf{S}$) is not incident to A, then the edge is facing to A, and the foot of the perpendicular from A to the edge is a vertex of $(\mathrm{sd})^2\hat{\mathcal{T}}$. Thus, each inner vertex P of $\mathrm{sd}\hat{\mathcal{T}}$ is a pivot of Λ_P. However, any outer vertex Q of $\mathrm{sd}\hat{\mathcal{T}}$ is not a pivot of Λ_Q since both the neighbouring corners of Q are right-angled and not acute, see Figure 9(c).

Let us modify the decomposition $\mathcal{P}$ of Γ around M_i, $i = 1, 2, \ldots, n$, in the following way: let XY be the outer edge of $\hat{\mathcal{T}}$ containing M_i, and let XM_iZ, ZM_iY be the two adjacent triangles of $\mathrm{sd}\hat{\mathcal{T}}$, see Figure 10. (Notice that Z is the circumcentre of the triangle in $\hat{\mathcal{T}}$ incident to XY.) Then XYZ is an isosceles triangle. Since $\hat{\mathcal{T}}$ satisfies the condition of Lemma 2, so does $\mathrm{sd}\hat{\mathcal{T}}$, and hence $\angle XZY = \angle XZM_i + \angle M_iZY > 45° + 45° = 90°$. Let L, N be the midpoints of XZ, YZ, respectively, and L', N' be the midpoints of XM_i, YM_i, respectively. Then $\angle XLM_i = \angle M_iNY = \angle XZY > 90°$. Let S, T be the points on XY such that $SL \perp XZ$ and $TN \perp YZ$. Then S lies between X and M_i, T lies between M_i and Y. And $\angle XSL = \angle YTN = \angle XZM_i > 45°$. Now, remove the polygon Λ_{M_i} from $\mathcal{P}$ and enlarge Λ_Z by attaching the trapezoid $SLNT$, enlarge Λ_X by attaching the triangle $LL'S$, and enlarge Λ_Y by attaching the triangle $NN'T$. Then we have a new decomposition $\mathcal{P}_1 = \{\Lambda_P : P \in \mathbf{S}_1\}$ with $\mathbf{S}_1 = \mathbf{S} - \{M_1, M_2, \ldots, M_n\}$. Now, each $P \in \mathbf{S}_1$ is a pivot of Λ_P, and if Λ_P and $\Lambda_{P'}$ $(P' \in \mathbf{S}_1)$ have an edge e in common, then e and the line-segment PP' cross each other perpendicularly. Hence $\mathcal{P}$ is a polygonal decomposition of Γ satisfying (i), (ii) of Proposition 2. □

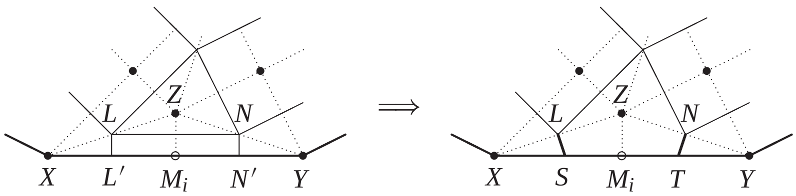


FIGURE 10. Modification around M_i.

4. PROOF OF THEOREM 2

Let Γ be a polygon, and let $\mathcal{T}$ be a non-obtuse triangulation of Γ with size N. Then by the proof of Lemma 2, we can make a non-obtuse triangulation $\hat{\mathcal{T}}$ of Γ such that $45° < \theta_{\min}, \theta_{\max} < 90°$. By the remark after the proof of Lemma 2, we have $|\hat{\mathcal{T}}| \leq 2 \cdot 6^2 N$. By the proof of Proposition 2, we can make from this $\hat{\mathcal{T}}$, a polygonal decomposition $\mathcal{P} = \{\Lambda_i : i \in I\}$ of Γ, and

$$\sum_{i \in I} \#(\text{edges of } \Lambda_i) \leq |(\mathrm{sd})^2 \hat{\mathcal{T}}| \leq 6^2 |\hat{\mathcal{T}}| \leq 2 \cdot 6^4 N.$$

Our acute triangulation of Γ is obtained by applying Proposition 1 to each Λ_i of $\mathcal{P}$. Hence the number of triangles in our acute triangulation of Γ is at most

$$6 \sum_{i \in I} \#(\text{edges of } \Lambda_i) \leq 2 \cdot 6^5 N.$$

This proves Theorem 2. □

5. NON-OBTUSE TRIANGULATIONS

Since the proofs of the existence of a non-obtuse triangulation for a polygon in [1] and [2] are long, we present here a short proof.

LEMMA 3. *Let ABC be a triangle with acute or right-angled corner A. Then, for any n points $P_1, P_2, \ldots, P_n$ on the edge BC, there is a non-obtuse triangulation $\mathcal{T}$ of ABC such that the vertices of $\mathcal{T}$ that lie on BC are $P_1, P_2, \ldots, P_n$.*

PROOF (SEE FIGURE 11). We may suppose $\angle B < 90°$. Through each P_i, draw a line ℓ_i parallel to AB, and draw the perpendiculars from P_i to AB, and draw the perpendicular from the intersection of ℓ_i and AC to AB. Draw also the perpendicular from C to AB. Then ABC is divided into right triangles and rectangles. Divide each rectangle by a diagonal. □

The next lemma was obtained in [1].

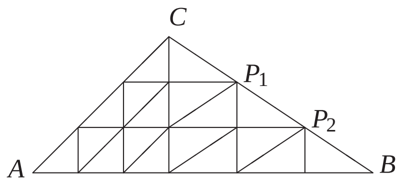


FIGURE 11. Proof of Lemma 3.

LEMMA 4. *Let* $A = (1,0), B = (1,1), C = (0,1), X = (x,0), Y = (0,y)$, *where* $0 < x \le y < 1$. *Then the pentagon* $ABCYX$ *can be triangulated into non-obtuse triangles without introducing new vertices within the edges* XA, AB, BC, CY.

PROOF. Since $x \le y$, we have $\angle BXY \le \angle BYX$. If $\angle BYX \le 90°$, then the diagonals BX, BY divide the pentagon into three non-obtuse triangles. Suppose that $\angle BYX$ is obtuse. (In this case, $x < 1/2$.) Then there is a point $P = (x, y_1)$, $0 < y_1 < y$, such that $\angle BPY = 90°$, see Figure 12. Draw the line-segments BY, PY, PB, PA, PX and draw the perpendicular from P to XY. Then the pentagon is triangulated into six non-obtuse triangles. □

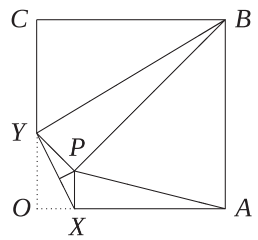


FIGURE 12. Lemma 4.

A *lattice point* in the Euclidean plane is a point whose coordinates are both integers. The lines $x = i$ and $y = j$ (i, j are integers) are the *lattice lines*, and a unit square whose vertices are lattice points is a *lattice cell*. An edge of a polygon that lies on a lattice line is called a *lattice edge* of the polygon.

LEMMA 5. *Let* Γ *be a polygon no two edges of which cut the same lattice cell, and no vertex of which lies inside a lattice cell. Then there is a non-obtuse triangulation of* Γ *such that the vertices newly introduced within the lattice edges of* Γ *are the lattice points on the lattice edges.*

PROOF. Since no two edges cut the same lattice cell, the lattice lines divide Γ into squares (lattice cells), pentagons (as in Lemma 4), right triangles, and trapezoids. Each square can be divided into two right triangles by a diagonal. Since each of our trapezoids is one that is obtained by cutting a square by a line, it can be divided into two non-obtuse triangles by one of its diagonals. Each pentagon can be triangulated into non-obtuse triangles as in Lemma 4. Hence Γ admits a non-obtuse triangulation such that the newly introduced vertices within the lattice edges of Γ are the lattice points on the edges. □

Let σ be a lattice cell and P be an interior point of σ. The disk of radius $\sqrt{10}/2$ centred at the centre of σ is covered by 13 lattice cells. The union of these 13 cells is called the two-*neighbourhood* of P and denoted by $N_2(P)$, see Figure 13. (The prefix '2' comes from the fact that every point in $N_2(P)$ can be reached from P by crossing at most two lattice lines.) Then $N_2(P)$ is a polygon with 20 edges, and all lattice points on the boundary of $N_2(P)$ are the vertices of the polygon $N_2(P)$. Let us call a (boundary) vertex of $N_2(P)$ where the interior angle is 270° a *concave corner*.

LEMMA 6. *Let Γ be a polygon such that the minimum distance between non-adjacent edges is greater than four. Let P be a vertex of Γ such that the interior angle of Γ at P is greater than* 90°, *and suppose that P lies inside a lattice cell. Then the polygon $\Gamma \cap N_2(P)$ can be triangulated into non-obtuse triangles without introducing new vertices within the lattice edges of the polygon $\Gamma \cap N_2(P)$.*

PROOF. Since the distances between non-adjacent edges of Γ are all greater than four, only the two edges emanating from P intersect $N_2(P)$. Since the interior angle of Γ at P is greater than 90°, Γ contains at least one concave corner of $N_2(P)$. Let A, B be the intersection points of the boundary of Γ and the boundary of $N_2(P)$, and let $C_1, C_2, \ldots, C_k$ be those concave corners of $N_2(P)$ that lie in Γ, with $A, C_1, C_2, \ldots, C_k, B$ in counter-clock-wise order on the boundary of $\Gamma \cap N_2(P)$. Connect each C_i to P by a line-segment, and connect C_i to C_{i+1} for $i = 1, 2, \ldots, k-1$, by line-segments. Connect C_1 to A and connect C_k to B, and draw the perpendiculars from C_1 to PA and from C_k to PB, see Figure 13. If there appear trapezoids and/or squares, divide them by suitable diagonals. Then we can get a desired non-obtuse triangulation of $\Gamma \cap N_2(P)$. □

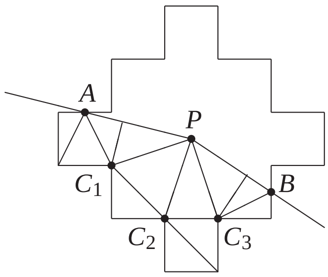


FIGURE 13. A non-obtuse triangulation of $\Gamma \cap N_2(P)$.

THEOREM 3. *Every polygon admits a non-obtuse triangulation.*

PROOF. Let Γ be a polygon. By suitably cutting off each acute or right-angled corner, we have a polygon Γ_1 whose interior angles are all greater than 90°. If Γ_1 admits a non-obtuse triangulation, then so does Γ by Lemma 3. Since the scale is irrelevant, we may suppose that the minimum distance between non-adjacent edges of Γ_1 is greater than 10. Now, slide and rotate Γ_1, if necessary, so that each vertex of Γ_1 lies inside a lattice cell. This is clearly possible. Then, for any two distinct vertices P, Q of Γ_1, their two-neighbourhoods $N_2(P)$ and $N_2(Q)$ are disjoint. For each vertex P of Γ_1, the polygon $\Gamma_1 \cap N_2(P)$ admits a non-obtuse triangulation as in Lemma 6. Let Γ_2 denote the remaining part $\Gamma_1 - \bigcup_P N_2(P)$. Then Γ_2 is a polygon satisfying the condition of Lemma 5. Hence it admits a non-obtuse triangulation as in Lemma 5. Then, for each vertex P, the non-obtuse triangulation of Γ_2 and that of $\Gamma_1 \cap N_2(P)$ are consistent in their common boundary. Hence Γ_1 admits a non-obtuse triangulation. □

For a non-obtuse triangulation $\mathcal{T}$ of a polygon, sd$\mathcal{T}$ is a triangulation of the polygon into right triangles. Hence we have the following.

COROLLARY 3. *Every polygon can be triangulated into right triangles.* □

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