HW3 - Proof by Contradiction and Relations $\boxed{JM}$

1) Prove that $\sqrt{6}$ is irrational.

Proof. (contradiction)

Assume that $\sqrt{6}$ is rational.

Then $\sqrt{6} = p/q$ where p and q are coprime integers.

$$\Rightarrow \sqrt{6}^2 = 6 = p^2/q^2$$

$$p^2 = 6q^2$$

Therefore p^2 is an <u>even number</u> since an even number multiplied by any other integer is also an even number. If p^2 is <u>even</u> then p must also be even since if p were odd, an odd number multiplied by an odd number would also be odd.

So we can replace p with $2K$ where K is an integer.

$$(2K)^2 = 2q^2$$

$$4K^2 = 6q^2$$

$$2K^2 = 3q^2$$

Now we see that $3q^2$ is even. For $3q^2$ to be even, q^2 must be even since 3 is odd and an odd times an even number is even. And by the same argument above, if q^2 is even then q is even.

→ 1) cont...

So both p and q are even which means both are divisible by 2. But that means they are <u>not</u> coprime, contradicting our assumption.

$\therefore$ By contradiction we have proven that $\sqrt{6}$ is irrational.

2) If $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.

Proof. (contradiction)

Assume that there exist $a, b \in \mathbb{Z}$ such that $a^2 - 4b - 2$ $\cancel{\neq}$ equals 0. Then:

$a^2 = 4b + 2 = 2(b+1)$, which means that a^2 is an even integer.

Because we know [n is even $\Leftrightarrow$ n^2 is even], this implies that a is also even.

Then, by definition, there exists $k \in \mathbb{Z}$ such that $a = 2k$. So we obtain:

$$(2k)^2 - 4b - 2 = 0 \iff 4k^2 - 4b - 2 = 0$$
$$\iff 2k^2 = 2b + 1.$$

So $2k^2$ is odd (since we know it can be written in the form $2h+1$ for some $h \in \mathbb{Z}$), but it is also even (since it can be written in the form $2l$ for some $l \in \mathbb{Z}$). This is our contradiction, since an integer cannot be even and odd at the same time.

$\therefore$ By contradiction we have proven that all $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.

3) Suppose $A \neq \emptyset$. Since $\emptyset \subseteq A \times A$, the set $R = \emptyset$ is a relation on A. Is R reflexive. Symmetric? Transitive? If a property doesn't hold, Say why.

Suppose $A \neq \emptyset$. Since, $\emptyset \subseteq A \times A$ and relation $R = \{\emptyset\}$ is a relation on A.

$\Rightarrow$ There is no $(x, x) \in R$ that can exist in R therefore we know it is vacuously reflexive.

So, R is reflexive on A.

$$\boxed{\begin{aligned} a \in A &\Rightarrow (a, a) \notin \emptyset \\ &\Rightarrow (a, a) \notin R \end{aligned}}$$

$\Rightarrow$ There is no (x, y) that exists in R therefore vacuously symmetric.

So, R is symmetric on A.

$$\boxed{\text{No } (x, y) \text{ so no } (y, x)}$$

$\Rightarrow$ There is no (x, y) that exists in R therefore vacuously transitive.

So, R is transitive on A.

$$\boxed{\text{No } (x, y) \text{ or } (y, z) \text{ or } (x, z)}$$

So, $R = \{\emptyset\}$ is an equivalence relation, it is reflexive, symmetric, and transitive, on A.

4) Define a relation R on $\mathbb{Z}$ as xRy if and only if $4 \mid (x+3y)$. Prove R is an equivalence relation. Describe its equivalence classes.

Reflexive: R is reflexive because for any $x \in \mathbb{Z}$ we have $4 \mid (x+3x)$, so xRx.

Symmetric: To prove this, suppose xRy. Then $4 \mid (x+3y)$, so $x+3y = 4a$ for some integer a. Multiplying by 3, we get $3x+9y = 12a$, which becomes $y+3x = 12a-8y$. Then $y+3x = 4(3a-2y)$, so $4 \mid (y+3x)$, hence yRx. Thus we've proven xRy implies yRx, so R is symmetric.

Transitive: To prove this, suppose xRy and yRz. Then $4 \mid (x+3y)$ and $4 \mid (y+3z)$, so $x+3y = 4a$ and $y+3z = 4b$ for some integers a and b. Adding these two equations produces:

$$x + 4y + 3z = 4a + 4b$$

OR

$$x + 3z = 4a + 4b - 4y = 4(a+b-y).$$

Consequently, $4 \mid (x+3z) \Rightarrow xRz$, and R is transitive.

→ 4) cont...

As R is reflexive, symmetric, and transitive, it is indeed an <u>equivalence relation</u>.

Describe its equivalence classes:

$[0] = \{x \in \mathbb{Z} : xR0\} = \{x \in \mathbb{Z} : 4 \mid (x + 3 \cdot 0)\} = \{x \in \mathbb{Z} : 4 \mid x\}$

$[0] = \{\ldots -4, 0, 4, 8, 12, 16 \ldots\}$

$[1] = \{x \in \mathbb{Z} : xR1\} = \{x \in \mathbb{Z} : 4 \mid (x + 3 \cdot 1)\} = \{x \in \mathbb{Z} : 4 \mid (x+3)\}$

$= \{\ldots -3, 1, 5, 9, 13, 17 \ldots\}$

$[2] = \{x \in \mathbb{Z} : xR2\} = \{x \in \mathbb{Z} : 4 \mid (x + 3 \cdot 2)\} = \{x \in \mathbb{Z} : 4 \mid (x+6)\}$

$= \{\ldots -2, 2, 6, 10, 14, 18 \ldots\}$

$[3] = \{x \in \mathbb{Z} : xR3\} = \{x \in \mathbb{Z} : 4 \mid (x + 3 \cdot 3)\} = \{x \in \mathbb{Z} : 4 \mid (x+9)\}$

$= \{\ldots -1, 3, 7, 11, 15, 19 \ldots\}$