Lecture Exercise on Relations: Jordan Miguel

1) Let $A = \{1,2,3,4,5,6\}$ and consider the following equivalence relation on A:

$$R = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(2,3),(3,2),(3,2),(4,5),(5,4),(4,6),(6,4),(5,6),(6,5)\}.$$

List the equivalence classes of R.

$[1] = \{1\}$; $[2] = [3] = \{2,3\}$;

$[4] = [5] = [6] = \{4,5,6\}$.

---

2) Let $A = \{a, b, c, d, e\}$.

Suppose R is an equivalence relation on A. Suppose R has three equivalence classes. Also aRd and bRc. Write out R as a set.

$$R = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,d),(d,a),(b,c),(c,b)\}.$$

---

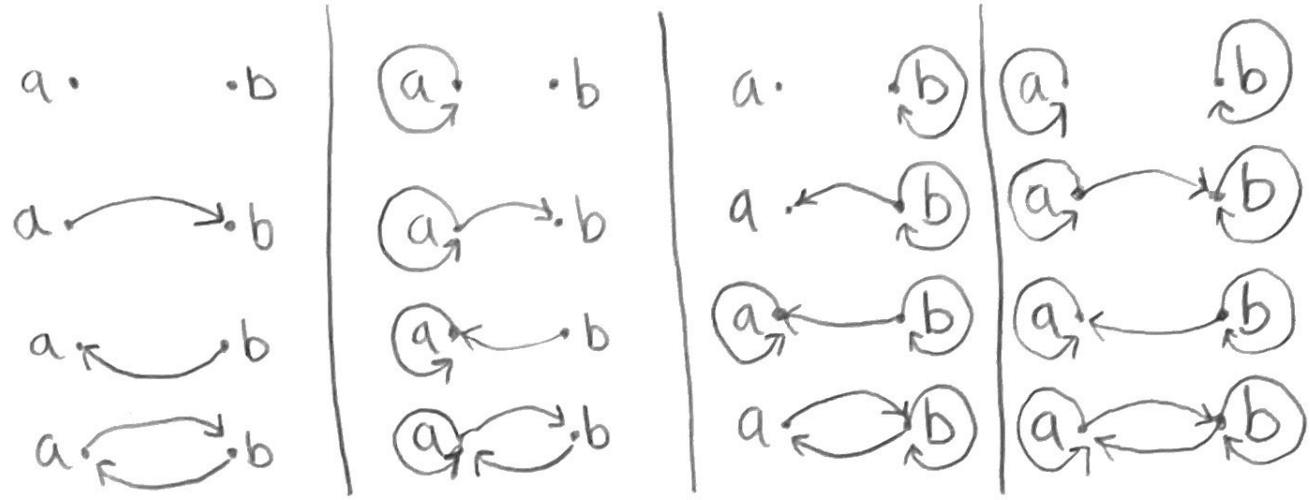
3) There are two different equivalence relations on the set $A = \{a, b\}$. Describe them.

$R = \{(a,a),(b,b)\}$ and $R = \{(a,a),(b,b),(a,b),(b,a)\}$.

3) cont.

16 possible relations

DIAGRAMS:



4) List all the partitions of the set $A = \{a, b\}$.

Defn. A partition of a set A is a set of non-empty subsets of A, such that the union of all subsets equal to A, and the intersection of any two different subsets is empty.

Partitions: are $2^n = 2^2 = 4$

Thus, $\boxed{\{\{a\}, \{b\}, \{a, b\}, \{\phi\}\}}$

5) Define a relation R on $\mathbb{Z}$ as xRy if and only if $4 \mid (x+3y)$. Prove R is an equivalence relation. Describe its equivalence classes.

Reflexive: R is reflexive because for any $x \in \mathbb{Z}$ we have $4 \mid (x+3x)$, so xRx.

Symmetric: To prove this, suppose xRy. Then $4 \mid (x+3y)$, so $x+3y=4a$ for some integer a. Multiplying by 3, we get $3x+9y=12a$, which becomes $y+3x=12a-8y$. Then $y+3x=4(3a-2y)$, so $4 \mid (y+3x)$, hence yRx. Thus we've proven xRy implies yRx, so R is symmetric.

Transitivity: To prove this, suppose xRy and yRz. Then $4 \mid (x+3y)$ and $4 \mid (y+3z)$, so $x+3y=4a$ and $y+3z=4b$ for some integers a and b. Adding these two equations produces $x+4y+3z = 4a+4b$, or $x+3z = 4a+4b-4y = 4(a+b-y)$.
Consequently, $4 \mid (x+3z) \Rightarrow xRz$, and R is transitive.

As R is reflexive, symmetric, and transitive, it is an equivalence relation.

Now we compute its equivalence classes.

$[0] = \{x \in \mathbb{Z} : xR0\} = \{x \in \mathbb{Z} : 4 \mid (x+3\cdot 0)\} = \{x \in \mathbb{Z} : 4 \mid x\}$

5) cont...

$[0] = \{\ldots -4, 0, 4, 8, 12, 16 \ldots\}$

$[1] = \{x \in \mathbb{Z} : x R 1\} = \{x \in \mathbb{Z} : 4 \mid (x + 3 \cdot 1)\} = \{x \in \mathbb{Z} : 4 \mid (x+3)\}$

$= \{\ldots -3, 1, 5, 9, 13, 17 \ldots\}$

$[2] = \{x \in \mathbb{Z} : x R 2\} = \{x \in \mathbb{Z} : 4 \mid (x + 3 \cdot 2)\} = \{x \in \mathbb{Z} : 4 \mid (x+6)\}$

$= \{\ldots -2, 2, 6, 10, 14, 18 \ldots\}$

$[3] = \{x \in \mathbb{Z} : x R 3\} = \{x \in \mathbb{Z} : 4 \mid (x + 3 \cdot 3)\} = \{x \in \mathbb{Z} : 4 \mid (x+9)\}$

$= \{\ldots -1, 3, 7, 11, 15, 19 \ldots\}$