HW2 -- Proof by Induction

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1) Prove that $9 \mid (4^{3n}+8)$ for every integer $n \geq 0$.

Proof. (Induction)

Base Case: For $n=0$, $(4^{3(0)}+8)=9$, $9|9=1$. It is divisible by 9. So result is true for $n=0$.

Now for $n=1$, $4^{3(1)}+8 = 72$, which is divisible by 9. Hence result is true for $n=1$.

Let the result be true for $n=m$, then:

$9 \mid 4^{3m}+8$, it means $\exists$ a positive.

Integer K such that: $(K \geq 0)$

$$4^{3m}+8 = 9K \Rightarrow 4^{3m} = 9K-8$$

Now, we will check for $n=m+1$.

$$4^{3(m+1)}+8 = 4^{3m} \cdot 4^3 + 8$$

$$4^{3(m+1)}+8 = (9K-8)64+8 \qquad (\because 4^{3m}=9K-8)$$

$$4^{3(m+1)}+8 = 9K \cdot 64 - 504$$

Since $(9K \cdot 64)$ is divisible by 9 and 504 is also divisible by 9, so $(9K \cdot 64 - 504)$ is divisible by 9.

So, $(4^{3(m+1)}+8)$ is also divisible by 9, hence result is true for $n=m+1$.

By mathematical induction, result is true for all $n \geq 0$.

$\therefore 9 \mid (4^{3n}+8)$ for every integer $n \geq 0$.

2) Prove that $\sum_{i=1}^{n} (8i-5) = 4n^2 - n$ for every positive integer $n$.

<u>Base Case:</u> $n = 1$.

$8 \cdot 1 - 5 = 3 = L.H.S.$

$4 \cdot 1^2 - 1 = R.H.S.$

$\therefore L.H.S. = R.H.S.$

Thus our statement is true for $n = 1$.

Let the expression be true for $n = K$.

$\therefore (8 \cdot 1 - 5) + (8 \cdot 2 - 5) + \ldots + (8 \cdot K - 5) = 4 \cdot K^2 - K$

$\Rightarrow 3 + 11 + 21 + \ldots + 8K - 5 = 4K^2 - K \quad — (I)$

Now, we will prove for $(K+1)$.

$\therefore 3 + 11 + 21 + \ldots + 8K - 5 + (8K + 8 - 5) = 4(K+1)^2 - (K+1)$

From (I) we can show:

$4K^2 - K + 8K + 3 = 4K^2 + 8K + 4 - K - 1$

$\Rightarrow 4K^2 + 7K + 3 = 4K^2 + 7K + 3$

$\therefore L.H.S. = R.H.S.$ for $(K+1)$

By mathematical induction we have proven that $\sum_{i=1}^{n} (8i-5) = 4n^2 - n$ for every positive integer $n$.

3) Prove that $1^2+2^2+3^2+4^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$ for every positive integer $n$.

Proof. (Induction)

Base Case: For $n=1$, the statement reduces to $1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{1\cdot 2\cdot 3}{6} = 1$ which is true.

Assuming the statement is true for $n=K$:

$$1^2+2^2+3^2+4^2+\ldots+K^2 = \frac{K(K+1)(2K+1)}{6}, \quad (eq_1)$$

We need to prove the statement must be true for $n=K+1$:

$$1^2+2^2+3^2+4^2+\ldots+(K+1)^2 = \frac{(K+1)(K+2)(2K+3)}{6} \quad (eq_2)$$

The left hand side of $eq_2$ can be written as:

$$1^2+2^2+3^2+4^2+\ldots+K^2+(K+1)^2.$$

In view of $eq_1$, this simplifies to:

$$(1^2+2^2+3^2+4^2+\ldots+K^2)+(K+1)^2 = \frac{K(K+1)(2K+1)}{6}+(K+1)^2$$

$$\text{R.H.S.} = \frac{K(K+1)(2K+1)+6(K+1)^2}{6}$$

$$= \frac{(K+1)[K(2K+1)+6(K+1)]}{6}$$

→ 3) cont...

$$= \frac{(K+1)(2K^2+7K+6)}{6}$$

$$= \frac{(K+1)(K+2)(2K+3)}{6}.$$

Thus the left-hand side of $eq_2$ is equal to the right-hand side of $eq_2$. This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer $n$.