Example 2 *Estimation of Generalized Learning Curve*

Consider the generalized learning curve (see Berndt, 1992, chapter 3)

$$C_t = C_1 N_t^{\alpha_c/R} Y_t^{(1-R)/R} \exp(u_t)$$

where

$$\begin{aligned} C_t &= \text{real unit cost at time } t \\ N_t &= \text{cumulative production up to time } t \\ Y_t &= \text{production in time } t \\ & \quad u_t \sim iid\ (0, \sigma^2) \\ \alpha_c &= \text{learning curve parameter} \\ R &= \text{returns to scale parameter} \end{aligned}$$

Intuition: Learning is proxied by cumulative production.

- If the learning curve effect is present, then as cumulative production (learning) increases real unit costs should fall.
- If production technology exhibits constant returns to scale, then real unit costs should not vary with the level of production.
- If returns to scale are increasing, then real unit costs should decline as the level of production increases.

The generalized learning curve may be converted to a linear regression model by taking logs:

$$\begin{aligned}
\ln C_t &= \ln C_1 + \left(\frac{\alpha_c}{R}\right)\ln N_t + \left(\frac{1-R}{R}\right)\ln Y_t + u_t \\
&= \beta_0 + \beta_1 \ln N_t + \beta_2 \ln Y_t + u_t \\
&= \mathbf{x}_t' \boldsymbol{\beta} + u_t
\end{aligned}$$

where

$$\begin{aligned}
\beta_0 &= \ln C_1 \\
\beta_1 &= \alpha_c / R \\
\beta_2 &= (1-R)/R \\
\mathbf{x}_t &= (1, \ln N_t, \ln Y_t)'.
\end{aligned}$$

The learning curve parameters may be recovered using

$$\begin{aligned}
\alpha_c &= \frac{\beta_1}{1+\beta_2} = g_1(\boldsymbol{\beta}) \\
R &= \frac{1}{1+\beta_2} = g_2(\boldsymbol{\beta})
\end{aligned}$$

Least squares gives consistent and asymptotically normal estimates

$$
\begin{aligned}
\hat{\boldsymbol{\beta}} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \xrightarrow{p} \boldsymbol{\beta} \\
\hat{\sigma}^2 &= n^{-1}\sum_{t=1}^{n}(y_t - \mathbf{x}_t'\hat{\boldsymbol{\beta}})^2 \xrightarrow{p} \sigma^2
\end{aligned}
$$

$$
\hat{\boldsymbol{\beta}} \overset{A}{\sim} N(\boldsymbol{\beta}, \hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1})
$$

Then from Slutsky's Theorem

$$
\begin{aligned}
\hat{\alpha}_c &= \frac{\hat{\beta}_1}{1+\hat{\beta}_2} \xrightarrow{p} \frac{\beta_1}{1+\beta_2} = \alpha_c \\
\hat{R} &= \frac{1}{1+\hat{\beta}_2} \xrightarrow{p} \frac{1}{1+\beta_2} = R
\end{aligned}
$$

provided $\beta_2 \neq -1$.

We can use the delta method to get the asymptotic distribution of $\hat{\eta} = (\hat{\alpha}_c, \hat{R})'$:

$$\begin{pmatrix} \hat{\alpha}_c \\ \hat{R} \end{pmatrix} = \begin{pmatrix} g_1(\hat{\boldsymbol{\beta}}) \\ g_2(\hat{\boldsymbol{\beta}}) \end{pmatrix}$$

$$\overset{A}{\sim} N\left(\mathbf{g}(\boldsymbol{\beta}), \left(\frac{\partial \mathbf{g}(\hat{\boldsymbol{\beta}})}{\partial \boldsymbol{\beta}'}\right) \hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1} \left(\frac{\partial \mathbf{g}(\hat{\boldsymbol{\beta}})}{\partial \boldsymbol{\beta}'}\right)'\right)$$

where

$$\frac{\partial \mathbf{g}(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}'} = \begin{pmatrix} \frac{\partial g_1(\boldsymbol{\beta})}{\partial \beta_1} & \frac{\partial g_1(\boldsymbol{\beta})}{\partial \beta_2} & \frac{\partial g_1(\boldsymbol{\beta})}{\partial \beta_3} \\ \frac{\partial g_2(\boldsymbol{\beta})}{\partial \beta_1} & \frac{\partial g_2(\boldsymbol{\beta})}{\partial \beta_2} & \frac{\partial g_2(\boldsymbol{\beta})}{\partial \beta_2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{1+\beta_2} & \frac{-\beta_1}{(1+\beta_2)^2} \\ 0 & 0 & \frac{-1}{(1+\beta_2)^2} \end{pmatrix}$$

Remark

Asymptotic standard errors for $\hat{\alpha}_c$ and $\hat{R}$ are given by the square root of the diagonal elements of

$$\left(\frac{\partial \mathbf{g}(\hat{\boldsymbol{\beta}})}{\partial \boldsymbol{\beta}'}\right) \hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1} \left(\frac{\partial \mathbf{g}(\hat{\boldsymbol{\beta}})}{\partial \boldsymbol{\beta}'}\right)'$$

where

$$\frac{\partial \mathbf{g}(\hat{\boldsymbol{\beta}})}{\partial \boldsymbol{\beta}'} = \begin{pmatrix} 0 & \frac{1}{1+\hat{\beta}_2} & \frac{-\hat{\beta}_1}{(1+\hat{\beta}_2)^2} \\ 0 & 0 & \frac{-1}{(1+\hat{\beta}_2)^2} \end{pmatrix}$$