Homework 2

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1 Homework 2

1. (Section 1.2 Question 1abcd)

   (a) True. The zero vector must be in a vector space.

   (b) False. The zero vector is unique.

   (c) False. Counterexample: x can be the zero vector which means a and b don't have to be equal to each other.

   (d) False. Scalar a can be zero which means you can have two arbitrary vectors x, y such that ax = ay.

2. (Section 1.2 Question 8) Show that $(a+b)(x+y) = ax + ay + bx + by$.
   Since a + b $\in$ F, we can use distributive property for vector spaces.
   $(a+b)x + (a+b)y = ax + bx + ay + by$ (VS 7 distributive property)
   $= ax + ay + bx + by$ (VS 1 commutativity of addition).

   Hence, $(a+b)(x+y) = ax + ay + bx + by$. $\square$

3. (Section 1.2 Question 13) Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \mathbb{R}$, define $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$ and $c(a_1, b_1) = (ca_1, a_2)$. Is V a vector space in $\mathbb{R}$ with these operations? Justify your answer.

   WTS if V is a vector space or not by checking the axioms.

   Commutative property:
   a + b = b + a

$(a_1, a_2) + (b_1, b_2) = (b_1, b_2) + (a_1, a_2)$
$(a_1 + b_1, a_1 b_1) = (b_1 + a_1, b_1 a_1)$

This set holds for commutative property.

Associativity of addition:
Let $c = (c_1, c_2)$.

Then, $(a + b) + c = (a_1 + b_1, a_2 + b_2) + (c_1, c_2) = (a_1 + b_1 + c_1, a_1 b_1 c_1)$

$a + (b + c) = (a_1, a_2) + (b_1 + c_1, b_2 + c_2) = (a_1 + b_1 + c_1, a_1 * (b_1 * c_1) = (a_1 + b_1 + c_1, a_1 b_1 c_1)$

This set holds under associativity of addition.

Additive identity: For $a + 0 = a$, there exists an element 0 such that $(a_1, a_2) + 0 = (a_1, a_2)$. If $0 = (0,\ 0)$, then $(a_1, a_2) + (0, 0) = (a_1 + b_1, 0)$

This set does not hold under additive identity since the sum of an element with the zero vector doesn't equal the element.

Additive Inverse: For every a, there exists a d such that a + d = 0. Let (0,0) be the zero vector. Let d = $(-a_1, -a_2)$
$(a_1, a_2) + (-a_1, -a_2) = (0, -a^2)$.

Since $(0, -a^2) \neq (0, 0)$, the set doesn't hold under additive inverse.

Scalar of 1: $1 * a = (1 * a_1, a_2) = (a_1, a_2) = a$

Associativity of Multiplication: Let i, j be scalars $\in \mathbb{R}$
(i * j) * a = (ija_1, a_2)
i * (j * a) = $i * (ja_1, a_2) = (ija_1, a_2)$

Since both match up, they hold under under associativity of multiplication.

Distributive Property: Let f be a scalar in $\mathbb{R}$. $f(a + b) = f * (a_1 + b_1, a_2 b_2) = (fa_1 + fb_1, a_2 b_2)$ $fa + fb = f(a_1, a_2) + f(b_1, b_2) = (fa_1, a_2) + (fb_1, b_2) = (fa_1 + fb_1, a_2 b_2)$

The set holds under distribution.

For distributive property for scalars, we will use i and j again.
$(i + j) * a = ((i + j)a_1, a_2^2)$
$(i + j) * a = ia + ja = i(a_1, a_2) + j(a_1, a_2) = (ia_1 + a_2) + (ja_1 + a_2) = ((i + j)a_1, a_2^2)$

Since both statements match, this set holds under distributive property.

Hence, V is not a vector space over $\mathbb{R}$ due to having no zero vector and no additive inverse.

$\square$

4. (Section 1.2 Question 17)
   Let V $= (a_1, a_2) : a_1, a_2 \in F$, where F is a field. Define addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in V$, define $c(a_1, a_2) = (a_1, 0)$. Is V a vector space with thesse operations?

   WTS that V is a vector space or not by verifying the VS axioms.
   Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$.
   Commutative Property: $a + b = (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) = (b_1 + a_1, b_2 + a_2) = b + a$. Hence, V holds under commutative property.

   Let $c = (c_1, c_2)$.
   Associative Property: $(a + b) + c = (a_1 + b_1, a_2 + b_2) + (c_1, c_2) = (a_1 + b_1 + c_1, a_2 + b_2 + c_2) = (a_1, a_2) + (b_1 + c_1, b_2 + c_2) = a + (b + c)$

   Hence, V holds under associative property of addition.

   Additive Identity: $a + 0 = a$. Since by nature, (0, 0) is the zero vector, $(a_1, a_2) + (0, 0) = (a_1, a_2)$.

   Hence, V has the zero vector.

   Additive Inverse: $a + (-a) = 0 = (a_1, a_2) + (-a_1, -a_2) = (a_1 - a_1, a_2 - a_2) = (0, 0)$
   Hence, for every a, there exists an -a such that a + (-a) = 0.

   Scalar of One: $1 * a = 1 * (a_1, a_2) = (a_1, 0) \neq (a_1, a_2)$. Hence, a does not hold under scalar multiplication.

   V is not a vector field since it doesn't hold under scalar multiplication.
   $\square$

5. (Section 1.2 Question 21)
   $Z = (v, w) : v \in V, w \in W$
   V, W are vector spaces. Prove that Z is a vector space over F with the operations
   $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ and $c(v_1, w_1) = (cv_1, cw_1)$

   To prove that Z is a vector space over F, we need to prove that Z satisfies every vector space axiom.

Assume that V, W are vector spaces.

WTS that Z is also a vector space by verifying the axioms.
Proof:

Let $a = (x_1, y_1), b = (x_2, y_2)$ with $x_1, x_2 \in V, y_1, y_2 \in W$.
Since we know x and y are vector spaces, we can perform commutative property.
$x+y = (x_1, y_1)+(x_2, y_2) = (x_1+x_2, y_1+y_2) = (x_2+x_1, y_2+y_1) = y+x$.

Hence, Z holds for commutative property.

Let $c = (x_3, y_3)$.
$(a+b)+c = (x_1+x_2, y_1+y_2)+(x_3, y_3) = (x_1+x_2+x_3, y_1+y_2+y_3) = (x_1, y_1)+(x_2+x_3, y_2+y_3)$.

Hence, Z holds for associative property.

Next, we want to show the additive identity exists,
$b+0 = (x_2, y_2)+(0,0) = (x_2+0, y_2+0)$ Since x_2, y_2 are elements of a vector space:
$(x_2+0, y_2+0) = (x_2, y_2)$

Next, we want to show that for each element in Z, there's an additive inverse. Let f be an element in Z.
$a+f = 0$
$(x_1, y_1)+f = (0,0)$ $f = (0-x_1, 0-x_2) = (-x_1, -y_1)$.

Hence, for each element in Z, there's an additive inverse.

We want to show that 1a = a. $1 * (x_1, y_1) = (1 * x_1, 1 * y_1) = (x_1, y_1)$ (Scalar of one property for elements x_1, y_1)

Hence, scalar of one property holds for Z.

Let $i, j \in F$.

Next, we want to show that (ij)a = i(ja).

Since, elements in a, x_1, y_1, are in a vector space:

$(ij)a = (ij)(x_1, y_1) = (ijx_1, ijy_1) = i(jx_1, jy_1) = i(jx)$.

Hence, this vector space property holds for Z.

Next, we want to show that scalar distribution holds for Z.

$i(a+b) = i((x_1+y_1, x_2+y_2)) = (ix_1+iy_1, ix_2+iy_2) = (ix_1, ix_2) + (iy_1, iy_2) = ia + ib$. Hence, scalar distribution holds for Z.

Finally, we want to show that for pairs of elements in a field, scalar distribution holds for Z.

Let $j \in F$.

$(i+j)a = (i+j)(x_1, x_2) = ((i+j)x_1, (i+j)x_2) = (ix_1+jx_1, ix_2+jx_2) = (ix_1, ix_2) + (jx_1, jx_2) = ia + ja$.

Hence, for pairs in F, scalar distribution holds.

Since all the axioms hold for Z, Z is a vector space. □

6. (Section 1.2 Question 20)
   Let V denote the set of all real-valued functions f defined on the real line such that f(1) = 0. Prove that V is a subspace of P(x).

   WTS (V, + *) / $\mathbb{R}$ is a subspace of P(x).

   Suppose that f,g $\in$ V. WTS that (f + g) is $\in$ V.

   Proof: // (f + g)(1) = f(1) + g(1) = 0 + 0 = 0.

   Since the output is 0 and (f + g) is in V, V is closed under vector addition and contains the zero vector.

   Let c be a scalar $\in \mathbb{R}$. cf = cf(1) = c * 0 = 0. Since cf is in V, V is closed under scalar multiplication.

   Hence, V is a subspace of all polynomials. □

7. (Section 1.3 8af)

   (a) $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2, a_3 = -a_2\}$
   Determine if W_1 is a subspace under $\in \mathbb{R}^3$.

   To show that W_1 is a subspace under $\mathbb{R}^3$, we need to show that the zero vector is in W_1 and W_1 is closed under scalar multiplication and vector addition.

   First, (0, 0, 0) : 0 = 3 * 0 and 0 = -0 = 0.

Next, assume that $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are in W_1. Then $x_1 = 3x_2$ and $x_3 = -x_2$ and $y_1 = 3y_2$ and $y_3 = -y_2$

The sum of the two vectors would be, $x + y = (x_1 + y_1, x_2 + y_2, x_3, y_3)$.

Which means: $x_1 + y_1 = 3(x_2 + y_2)$ and $x_3 + y_3 = -(x_2 + y_2)$

Since $x_1 + y_1 = 3x_2 + 3y_2 = 3(x_2 + y_2)$ and $x_3 + y_3 = -x_2 + (-y_2)$, then W_1 is closed under vector addition.

Next, let $c \in F$. Assume that $x = (x_1, x_2, x_3)$ is in W_1. Then, $cx = (cx_1, cx_2, cx_3)$ which would be $cx_1 = 3cx_2$ and $cx_3 = -cx_2$.

Since $cx = c*(x_1 = 3x_2) = cx_1 = 3cx_2$ and $c*(x_3 = -x_2) = cx_3 = -cx_2$ which means W_1 is closed under scalar multiplication.

Hence, since all of the requirements are met, W_1 is a subspace of $\mathbb{R}^3$.

(b) (1.3 8f)

$W_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

First, check for zero vector:

(0,0,0) $= 5 * 0 - 3 * 0^2 + 6 * 0^2 = 0 = 0$.

Next, assume that $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are in W_6. Then $5x_1^2 - 3x_2^2 + 6x_3^2 = 0$ and $5y_1^2 - 3y_2^2 + 6y_3^2 = 0$.

The sum of the 2 vectors x + y are: $5(x_1+y_1)^2-3(x_2+y_2)^2+6(x_3+y_3)^2 = 5x_1^2+10x_1y_1+5y_1^2-3x_2^2-6x_2y_2-3y_2^2+6x_3^2+6y_3^2+12x_3y_3$. However, since there are extra terms like $10x_1y_1, -6x_2y_2$, and $12x_3y_3$, it does not equal the sum of expanding x and y. Since the sum doesn't satisfy the equation, W_6 is not closed under addition.

Let c be $\in$ F. Assume that $x = (x_1, x_2, x_3)$ is in W_6. $c * x = c(5x_1^2 - 3x_2^2 + 6x_3^2 = 0) = 5cx_1^2 - 3cx_2^2 + 6cx_3^2 = 0$.

$(cx_1, cx_2, cx_3) = 5x_1^2 - 3x_2^2 + 6x_3^2 = 0$. Since these two equations match, W_6 is closed under scalar multiplication.

Hence, W_6 is not a subspace of $\mathbb{R}^3$.

8. (Section 1.3 11) Is the set $W = \{f(x) \in P(F) : f(x) = 0$ or $f(x)$ has degree n $\}$ a subspace of P(F) if $n \geqslant 1$?

Let $f(x) = 1 + x + x^2 + x^3 + ...x^{n-1} + x^n$ and $g(x) = 1 + x + x^2 + x^3 + ... + x^{n-1} - x^n$ s.t. $n > 1$. $f \in W$, $g \in W$.

Let $z(x) = f(x) + g(x) = 2 + 2x + 2x^2 + ...2x^{n-1}$. Since z(x) has a degree of n-1 instead of n, it is not in the set of W. Thus, W is not a subspace of P(F).

9. (Section 1.3 19) Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

$\rightarrow$) If $W_1 \cup W_2$ is a subspace, then $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.
Contrapositive: If $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$, then $W_1 \cup W_2$ is not a subspace.

Assume that $W_1 \nsubseteq W_2$, then $\exists x \in W_1$ s.t. $x \notin W_2$. Also, $\exists y \in W_2$ s.t. $x \notin W_1$.

WTS $W_1 \cup W_2$ is not a subspace.

We will show that W_1 and W_2 are not closed under vector addition.

WTS (x + y) $\notin W_1 \cup W_2$.

FSOC, assume for some element z, that x + y = z s.t. z $\in W_1 \cup W_2$.

We need to prove by cases z being in W_1 and W_2.
Case 1: Assume that $z \in W_1$.

x + y = z
y = z - x
Since we know that the sum of z and -1 * x is in W_2, $y \in W_1$. However, that is a contradiction. y cannot be in and not in W_1.

Case 2: Assume that $z \in W_2$. x + y = z
x = z - y
Similarly to Case 1, x is in W_2. That is also a contradiction.

Hence, by cases, the union of $W_1 \cup W_2$ is not closed by addition.

$\leftarrow$) If $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then $W_1 \cup W_2$ is a subspace.

Let's prove by cases.

Case 1): Assume that W_1 and W_2 are subspaces and $W_1 \subseteq W_2$.
WTS $W_1 \cup W_2$ is a subspace. Since $W_1 \subseteq W_2$, then $W_1 \cup W_2 = W_2$.

Which means $W_1 \cup W_2$ is a subspace.

Case 2): Assume that W_1 and W_2 are subspaces and $W_2 \subseteq W_1$.
WTS $W_2 \cup W_1$ is a subspace. Since $W_2 \subseteq W_1$, then $W_2 \cup W_1 = W_1$.

Which means $W_2 \cup W_1$ is a subspace.

Hence, we have proved both sides. $\square$

10. (Section 1.3 20)

    Prove that if W is a subspace of a vector space V and $w_1, w_2, ..., w_n$ are in W, then $a_1w_1 + a_2w_2 + ... + a_nw_n \in W$ for any scalars $a_1 + a_2 + ... + a_n$.

    Assume that W is a subspace of vector space V and $w_1, w_2, ..., w_n$ are in W.

    WTS $a_1w_1 + a_2w_2 + ... + a_nw_n \in W$ for any scalars $a_1 + a_2 + ... + a_n$.

    Based on scalar multiplication principle for subspaces, $a_1w_1, a_2w_2, ..., a_nw_n$ are in W.

    If all of the scalars are zero, then you have the zero vector.

    Since we have $a_1w_1 + a_2w_2 + ... + a_nw_n$, we are adding vectors which satisfies the vector addition property of subspaces.

    Hence, $a_1w_1 + a_2w_2 + ... + a_nw_n \in W$. $\square$

11. (Section 1.3 21) Show that the set of convergent sequences (a_n) is a subspace of vector space V.

    WTS the set of convergent sequences is a subspace of V.
    Let a_n, b_n be sequences in the set of convergent sequences such that $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} b_n = L_2$.

    $a_n + b_n$: $\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L_1 + L_2$.

    The set is closed under vector addition.

    Let c be a scalar $\in$ F.
    ca_n: $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n =$ c*L_1.

    Since the scalar product of a convergent sequence also converges, then the set is closed under scalar multiplication.

$\lim_{n\to\infty} 0 = 0$. Since the zero vector is in the set, the set is a subspace of V.

Since all the properties of a subspace are verified for the set of convergent sequences, the set convergent sequences is a subspace of V. $\square$

12. (Section 1.3 23)

    Let W_1 and W_2 be subspaces of vector space V.

    (a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.
    Assume that $W_1 + W_2$ is a subspace.

    Proof: If W_1 and W_2 are subspaces, they both contain the zero vector. $0 \in W_1$, $0 \in W_2$.
    $0 + 0 = 0 \in W_1 + W_2$
    Let $w_1 \in W_1$ and $w_2 \in W_2$ and $s = w_1 + w_2$ and let t = $v_1 + v_2$ s.t. $v_1 \in W_1$ and $v_2 \in W_2$. Let s and t be in $W_1 + W_2$
    Then s + t = $(w_1 + w_2) + (v_1 + v_2) = (w_1 + v_1) + (w_2 + v_2)$.
    Since W_1 and W_2 are vector spaces, $w_1 + v_1 \in W_1$ and $w_2 + v_2 \in W_2$ due to closure of vector addition. Since that is the case, s + t is in both W_1 and W_2 which is $W_1 + W_2$.

    Let c be a scalar $\in F$. Since scalar multiplication is closed in both W_1 and W_2, $cw_1 \in W_1$ and $cw_2 \in W_2$, so $cs = cw_1 + cw_2 \in W_1 + W_2$.
    Hence, $W_1 + W_2$ contains both W_1 and W_2. $\square$

    (b) Prove that any subspace that contains both W_1 and W_2 must also contain $W_1 + W_2$.
    Contrapositive: If a subspace doesn't contain $W_1 + W_2$, then it doesn't contain W_1 or W_2.

    FSOC, let's assume that if subspace doesn't contain $W_1 + W_2$, but it contains both W_1 and W_2.
    Let S a subspace which doesn't contain the subset $W_1 + W_2$ but contains W_1 and W_2.
    Let $w_1 \in W_1$ and let $w_2 \in W_2$.
    Since, $W_1 \subseteq S$ and $W_2 \subseteq S$, $w_1 + w_2 \in S$. This means that there

are values in S such that $w_1 + w_2 \in W_1 + W_2$ due to closure in vector addition.

However, that is a contradiction since $W_1 + W_2 \in S$, but we assumed that $W_1 + W_2 \notin S$

Hence, for any subset that contains both W_1 and W_2 must also contain $W_1 + W_2$. $\square$

13. Let F be a field. Prove that the set $W = \{A \in M_{n \times n}(F) | A^T = -A\}$ of skew symmetric matrices is a subspace of $M_{n \times n}(F)$.

    Proof:
    Let arbitrary matrices $A, B \in W$.

    WTS that the set W is a subspace of $M_{n \times n}(F)$..

    Case 1: Zero vector

    If A = 0, then the transpose of A is equal to the scalar multiplication of -1 with A.

    Case 2: Since $A^t = -A$ and $B^t = -B$, then $(A + B)^t = A^t + B^t = -A - B$.

    Since, vector addition still maintains the condition of the set, this set is closed under vector addition.

    Let $c \in F$. Since $A^t = -A$, then $(cA)^t = c(A^t) = c(-A) = -cA$.

    Since, scalar multiplication still maintains the condition of the set, this set is closed under scalar multiplication.

    Hence, the set is closed under scalar multiplication and vector addition so it is a subspace of $M_{n \times n}(F)$. $\square$

14. (Section 1.4 1)

    (a) True. The zero vector can be possible if all the scalars are zero.

    (b) False. The empty set spans the zero vector.

    (c) True, the span of a subset equals the intersection of all subspaces of V that contain S.

    (d) False, you cannot multiply zero to the equation.

    (e) True, you can add a multiple of one equation with another

(f) False, it can have infinite or zero solutions.

15. (Section 1.4 10) Show that if

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

The transpose of the three matrices above is its original self. Hence, they are symmetric matrices.

WTS that the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices. Let S be an arbitrary symmetric 2×2 matrix such that

$$S = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

$$S = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Since S is a linear combination of the three symmetric matrices and is symmetric, we can conclude that the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices. $\square$

16. (Section 1.4 12) Show that a subset W of a vector space V is a subspace of V if and only if span(W) = W.

$\leftarrow$) If span(W) = W, then W is a subspace of vector space V.

Assume that W is also the span of W.
WTS that W is also a subspace of vector space V.

Proof: If W = $\varnothing$, then W contains 0.

Let x be an element in W s.t., $x = c_1w_1 + c_2w_2 + c_3w_3 + ...c_nw_n$ s.t. $c_1, c_2, c_3, ..., c_n \in F$ and $w_1, w_2, ..., w_n \in W$.

And let $y = a_1y_1 + a_2y_2 + a_3y_3 + ...a_ny_n$ s.t. $a_1, a_2, a_3, ..., a_n \in F$ and $y_1, y_2, ..., y_n \in W$.

If $c_1, c_2, c_3, ..., c_n = 0$, then the zero vector is in W.

x + y = $c_1w_1 + a_1y_1 + ... + c_nw_n + a_ny_n$

Let c be a scalar $\in$ F. dx = $(dc_1)w_1 + (dc_2)w_2 + ... + (dc_n)w_n$

x + y, and dx are linear combinations of vectors in W, so x + y and dx are in span(W). Additionally the zero vector is in W. Hence, span(W) is a subspace of V.

$\rightarrow$) If W is a subset of vector space V is also a subspace of V, then span(W) = W.

Assume that W is a subspace of V.

WTS span(W) = W.

Proof:
We know that the span of W will also be a subspace of V since it is closed under vector addition and scalar multiplication.

Hence, since span(W) is the linear combinations of W, $W \subseteq span(W)$. Hence, W = span(W).

$\square$