# Math 115A Homework 4

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## 1 Homework 4

1. (2.2 Problem 3) Let T: $\mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let $\beta$ be the standard ordered basis for $\mathbb{R}^2$ and $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$. Compute $[T]_\beta^\gamma$. If $\alpha = \{(1,2),(2,3)\}$, compute $[T]_\alpha^\gamma$.

   Standard ordered basis for $\mathbb{R}^2$ is (1,0), (0, 1).

   T(1, 0) = (1 - 0, 1, 2 * 1 + 0) = (1, 1, 2)

   a(1, 1, 0) + b(0, 1, 1) + c(2, 2, 3) = (1, 1, 2) $\frac{-1}{3}(1,1,0) + 0(0,1,1) + \frac{2}{3}(2,2,3) = (1,1,2)$

   T(0, 1) = (0 - 1, 0, 1) = (-1, 0, 1)

   a(1, 1, 0) + b(0, 1, 1) + c(2, 2, 3) = (-1, 0, 1) -1(1, 1, 0) + 1(0, 1, 1) + 0(2, 2, 3) = (-1, 0, 1)

   $$[T]_\beta^\gamma = \begin{pmatrix} \frac{-1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}$$

   T(1,2) = (-1, 1, 4) (-1, 1, 4) = a(1, 1, 0) + b(0, 1, 1) + c(2, 2, 3)

   a = $-\frac{7}{3}$ b = 2 c = $\frac{2}{3}$

   T(2, 3) = (-1, 2, 7) = a(1, 1, 0) + b(0, 1, 1) + c(2, 2, 3)

   a = $-\frac{11}{3}$ b = 3 c = $\frac{4}{3}$

   $$[T]_\alpha^\gamma = \begin{pmatrix} \frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}$$

2. (2.2 Problem 4) Define

   T: $M_{2\times 2}(R) \to P_2(\mathbb{R})$ by T$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ = (a + b) + (2d)x + b$x^2$

   Let $\beta = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$ and $\gamma = \{1, x, x^2\}$

   Compute $[T]_\gamma^\beta$

   T$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ = 1

   $a(1) + bx + cx^2 = 1$

   $a = 1, b = 0, c = 0$

   T$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ = 1 + $x^2$

   $a(1) + bx + cx^2 = 1 + x^2$

   $a = 1, b = 0, c = 1$

   T$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ = 0

   $a(1) + bx + cx^2 = 0$

   $a = 0, b = 0, c = 0$

   T$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ = 2x

   $a(1) + bx + cx^2 = 2x$

   $a = 0, b = 2, c = 0$

   $[T]_\gamma^\beta = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

3. (2.2 Problem 9) Let V be the vector space of complex numbers over the field R. Define T: $V \to V$ by $T(z) = \bar{z}$, where $\bar{z}$ is the complex conjugate of z. Prove that T is linear and compute $[T]_\beta$, where $\beta = \{1, i\}$.

   To show that T is linear, we must show that for two arbitrary elements that T preserves scalar multiplication and vector addition.

   Let x = a + bi and y = e + fi for some $a, b, e, f \in \mathbb{F}$

T(c(a + bi) + (e + fi)) = T((ca + e) + (cb + f)i) = (ca + e) - (cb + f)i

cT(a + bi) + T(e + fi) = c(a - bi) + (e - fi) = (ca + e) - (cb + f)i

Hence, T is linear.

$[T]_\beta$

$T(1) = 1 + 0i$

$T(i) = 0 - i$

$[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

4. (2.2 Problem 14) Let V and W be subspaces, and let T and U be nonzero linear transformations from V into W. If R(T) $\cap$ R(U) = {0}, prove that $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$.

   FSOC, assume that R(T) $\cap$ R(U) = {0}, but $\{T, U\}$ is a linearly dependent subset of $\mathcal{L}(V, W)$.

   Since T, U are nonzero linear transformations from V into W, $\exists c \in \mathbb{F}$ s.t. $cT + U = 0$ which equals $cT = U$.

   T is a nonzero transformation from V $\to$ W. So, $\exists v \in V$ and non-zero element w s.t $T(v) = w$.

   Since $cT = U$, $U(v) = cw$.

   $$w = \frac{1}{c}(cw) = \frac{1}{c}U(v) \tag{1}$$

   Hence, w $\in$ $R(U)$. However, since $w \in R(T)$ and $w \in R(U)$, w $\in$ $R(T) \cap R(U)$ which means w = 0, which is a contradiction.

   Thus, $\{T, U\}$ is a linearly independent. □

5. (2.2 Problem 17) Let V and W be vector spaces such that dim(V) = dim(W), and let $T : V \to W$ be linear. Show that there exist ordered bases $\beta$ and $\gamma$ for V and W, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

   Suppose that nullity(T) = k. Let $\{x_1, x_2, ...x_k\}$ be a basis for the kernel for some arbitrary, but fixed $x_1, x_2, ..., x_k$ in the ker(T).

By the Dimension Theorem, rank(T) = n - k, s.t. n represents the dimension of V and W. There also must exist a basis for R(T) that has n - k elements.

Let $\gamma_{R(T)} = \{y_{k+1}, y_{k+2}, ..., y_n\}$ be a basis for the subspace $R(T)$. Then, the dimension of this basis is n - k.

Since $y_i \in$ R(T), for i in k + 1 to n, $\exists\ \beta \in V$ s.t. T$(x_i)$ = $y_i$. We know that $\{y_{k+1}, ..., y_n\}$ is linearly independent.

WTS that $\{x_{k+1}, ...x_n\}$ is also linearly independent. To show that this set of vectors is linearly independent, we will construct a linear combination that equals 0 and show the coefficients of the linear combination equal 0.

Suppose for the sake of contradiction that $\{x_{k+1}, ...x_n\}$ is linearly dependent from elements in $\{x_1, x_2, ...x_n\}$.

By the definition of linearly dependence, $\exists$ some vector $x_i$ in the set that can be written as a linear combination of the other elements in the set i.e. $x_i = a_1x_1 + a_2x_2 + ...a_kx_k$, for i in $\{1, 2, ..., k\}$.

If we apply linear transformation T on both sides we get $T(x_i) = T(a_1x_1 + a_2x_2 + ...a_kx_k)$

Since i is in the range of $\{1, 2, ..., k\}$, $x_i$ is in ker(T) so $T(x_i) = 0 = T(a_1x_1 + a_2x_2 + ...a_kx_k) = a_1y_1 + a_2y_2 + ...a_iy_i = 0$.

Also, i is in the range of 1 to k, and since $x_1, ..., x_k$ is linearly independent since it is a basis for N(T), all of the coefficients $a_1, ..., a_i$ must be zero. So we have:

$$x_j = 0 * x_1 + 0 * x_2 + ...0 * x_k \tag{2}$$

which contradicts our assumption that $x_j$ is L.D. from $\{x_1, x_2, ...x_n\}$. Hence, $\{x_1, x_2, ..., x_n\}$ is LI.

We now have the ordered basis for $\beta = \{x_1, x_2, ..x_n, \}$.

We want to construct $\gamma$. We can construct the following from N(T):

$\{T(x_1), T(x_2), ...T(x_k), \_\_\_\_\_\}$.

We want to fill out the rest of the basis so that the dimension of the basis is equal to $\beta$. By Replacement Theorem we can write:

$\{T(x_1), T(x_2), ...T(x_k), T(x_{k+1}), T(x_{k+2}), ..., T(x_n)\}$

To find out $[T]_\beta^\gamma$, we write the following equation below:

$$[T(x_i)]_\gamma = [T]_\beta^\gamma [x_i]_\beta \tag{3}$$

for i in i to n.

And we get the following matrix: $\begin{pmatrix} O & O \\ O & I \end{pmatrix}$ In which I represents the identity matrix from $n-k \times n-k$ and the rest of the matrix are all zeroes.

Hence, $[T]_\beta^\gamma$ is a diagonal matrix.

□

6. (2.3 Problem 3) Let g(x) = 3 + x. Let $T : P_2(R) \to P_2(R)$ and U: $P_2(R) \to R^3$ be the linear transformations respectively defined by

$$T(f(x)) = f'(x)g(x) + 2f(x) \tag{4}$$

and

$$U(a + bx + cx^2) = (a + b, c, a - b) \tag{5}$$

Let $\beta$ and $\gamma$ be the standard ordered bases of $P_2(R)$ and $R^3$, respectively.

(a) $\beta = \{1, x, x^2\}$ $\gamma = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
U(1) = (1 + 0, 0, 1 - 0) = (1, 0, 1)
U(x) = (0 + 1, 0, 0 - 1) = (1, 0, -1)
$U(x^2) = (0, 1, 0)$ = (0, 1, 0)

$$[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

T(1) = (1)'(3 + x) + 2(1) = 2 $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

T(x) = (x)'(3 + x) + 2(x) = 3 + 3x $\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$

$T(x^2) = (x^2)'(3 + x) + 2(x^2) = 2x(3 + x) + 2x^2 = 6x + 4x^2$ $\begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix}$

$[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$
UT(1) = U(2) = (2, 0, 2)
UT(x) = U(3 + 3x) = (6, 0, 0)
$UT(x^2) = U(6x + 4x^2) = (6, 4, -6)$

$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$

$[U]_\beta^\gamma [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix} =$

$\begin{pmatrix} 1*2+0+0 & 1*3+1*3+0 & 1*0+1*6+0 \\ 0*2+0+1*0 & 0*3+0+1*0 & 0+0+1*4 \\ 1*2+-1*0+0 & 1*3+-1*3+0 & 1*0+(-1)*6+0 \end{pmatrix} =$

$\begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$

$\square$

(b) $h(x) = 3 - 2x + x^2$

$[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

U(h(x)) $= U(3 - 2x + x^2) = (3 - 2, 1, 3 - (-2)) = (1, 1, 5)$

$[U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

$[U(h(x))]_\gamma = [U]_\beta^\gamma [h(x)]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3*1+-2*1+0 \\ 0+0+1*1 \\ 1*3+-2*-1+0 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

$\square$

7. (2.3 Problem 9)

Define T$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a-b \\ b-a \end{pmatrix}$

and

$U\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c+d \\ -d-c \end{pmatrix}$ for some a,b,c,d $\in \mathbb{F}^2$.

UT($\begin{pmatrix} e \\ f \end{pmatrix}$) = U($\begin{pmatrix} e-f \\ f-e \end{pmatrix}$) $= \begin{pmatrix} e-f+(f-e) \\ -(f-e)-(f-e) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = T_0$

TU($\begin{pmatrix} e \\ f \end{pmatrix}$ = T($\begin{pmatrix} e+f \\ f-e \end{pmatrix}$) $= \begin{pmatrix} e+f-(-f-e) \\ f-e-(e+f) \end{pmatrix} = \begin{pmatrix} 2e+2f \\ -2e-2f \end{pmatrix} \neq T_0$

Let A and B be the matrix representations of the transformations of U and T, respectively:

A $= \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ B $= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

AB $= \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1(1)+1*(-1) & 1(-1)+1*1 \\ -1(1)+1(1) & -1(-1)+(-1)*1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

BA $= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1(1)+(-1*-1) & 1(1)+(-1*-1) \\ -1*1+(-1)*1 & -1(1)+(-1)*1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}$

8. (2.3 Problem 16a)

   Let V be a finite-dimensional vector space, and let T: V $\to$ V be linear. If rank(T) = rank($T^2$), prove that $R(T) \cap N(T) = \{0\}$. Deduce that $V = R(T) \bigoplus N(T)$.

   Assume that rank(T) = rank($T^2$) and $V = R(T) \bigoplus N(T)$.

   WTS $R(T) \cap N(T) = \{0\}$.

   Proof: Let v $\in N(T)$. T(v) = 0. So T$^2(v)$ =T(T(v)) = T(0) = 0 since T is linear. So we can deduce that N$(T) \subseteq N(T^2)$.

   By the Dimension Theorem,

$$dim(N(T)) = nullity(T) = dim(V) - rank(T) = dim(V) - rank(T^2) = nullity(T^2) \tag{6}$$

Since N(T) is a subspace, we can use Theorem 1.11, to state that if nullity(T) = $nullity(T^2)$, then T = $T^2$.

Let w $\in R(T) \cap N(T)$.
Given that w $\in R(T)$ implies that $\exists$ y s.t. T(y) = w.

T(T(y)) = T(w) = 0 because w is also in N(T).

Then y must be in N($T^2$) = N(T).

Since T(w) = y = 0, we can conclude that R(T) $\cap$ N(T) = {0}

9. (2.4 Problem 2)

   c) T: $\mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$
   If we put it in matrix form we get:
   $\begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix}$ -1R1 + R3 $\to$ R3 $\begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 4 & 2 \end{pmatrix}$ -4R2 + R3 $\to$ R3
   $\begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ 1/3R1 $\to$ R1, 1/2R3 $\to$ R3 $\begin{pmatrix} 1 & 0 & -2/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 2/3R3 +
   R1 $\to$ R1 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ By using RREF, I got the identity matrix which means the rank of A is 3. So the matrix is invertible.

   e) T: $M_{2\times 2}(R) \to P_2(R)$ defined by T$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2$

   To show whether or not T is invertible, let's create use the standard basis $\beta = \{e_{11}, e_{12}, e_{21}, e_{22}\}$ and create a basis for $P_2(R)$ which is $\gamma = \{1, x, x^2\}$

   T($\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$) $= 1 + 0x + 0x^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

   T($\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$) $= 1 + 0x + 1x^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

   T($\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$) $= 0 + 0x + 0x^2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

T($\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$) $= 1 + 0x + 1x^2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

However, since $[T]_\beta^\gamma$ is not a square matrix, it is not invertible. So T is not invertible.

f) T: $M_{2\times 2}(R) \to M_{2\times 2}(R)$ defined by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$ To show whether or not T is invertible, let's create use the standard basis $\{e_{11}, e_{12}, e_{21}, e_{22}\}$.

T($\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$) $= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 1e_{11} + 1e_{22} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

T($\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$) $= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1e_{11} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

T($\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$) $= \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = 1e_{21} + 1e_{22} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

T($\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$) $= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1e_{22} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

[T]$_\beta$ $= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

Cofactor along 4th column: det([T]$_\beta$) = (-1) * 0 + (1) * 0 + (-1) * 0 + 1 * 1 * det $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ cofactor along 3rd row: (-1) * 1 *

$\det(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$ + 0 + 0 = (-1 * 1) = -1

Since $\det([T]_\beta)$ = -1 $\neq$ 0, $[T]_\beta$ is invertible.

10. (2.4 Problem 3)

    (a) $F^3$ and $P_3(F)$ cannot be isomorphic since their ordered bases do not have the same count of elements. In order for $F^3$ and $P_3(F)$ to be isomorphic, they must be invertible. To be invertible, you must have dimensions of the same size. And that is not the case for this example.

    (b) $F^4$ and $P_3(F)$ are isomorphic as long as the are in the same field F and have equal dimensions.

    (c) $M_{2\times 2}(R)$ and $P_3(R)$ are isomorphic. Since you can create an ordered basis of 4 elements for both $M_{2\times 2}(R)$ and $P_3(R)$, they have the same dimension which means they are isomorphic.

    (d) $V = \{A \in M_{2\times 2}(R) : tr(A) = 0\}$ and $R^4$ cannot be isomorphic. $R^4$ has an ordered basis consisting of 5 elements, while V has 4 elements in the ordered basis. Thus, their dimensions are not equal to each other so they cannot be isomorphic.

11. (2.4 Problem 4) Let A and B be $n \times n$ invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

    WTS AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

    To show that AB is invertible, we must show that AB times $B^{-1}A^{-1}$ is the identity matrix.

    Given that A and B are invertible:

    $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I_n)A^{-1} = AA^{-1} = I_n$

    Hence, AB is invertible and $B^{-1}A^{-1}$ is the inverse of AB. □

12. (2.4 Problem 14) Let V = $\{\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix}$: a, b, c $\in$ F$\}$

    Construct an isomorphism from V to $F^3$.

    $$\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = a\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =$$

We can set the basis as $\{\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\}$.

By Theorem 2.21, any finite-dim vector space V with ordered basis $\beta, \phi_\beta$ is an isomorphism.

Then $\phi_\beta = (a, b, c)$ and is isomorphic.

13. (2.4 Problem 16) Let B be an $n \times n$ invertible matrix. Define $\Phi : M_{n\times n}(F) \to M_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that $\Phi$ is an isomorphism.

    To show that $\Phi$ is an isomorphism, let $\Psi$ be the map that follows $\Psi : M_{n\times n}(F) \to M_{n\times n}(\mathrm{F})$ by $\Psi(A) = BAB^{-1}$.

    We will show that $\Psi = \Phi^{-1}$

    $\Phi(\Psi(A)) = \Phi(BAB^{-1}) = B^{-1}(BAB)B^{-1} = I_nAI_n = A$

    $\Psi(\Phi(A)) = \Phi(B^{-1}AB) = BB^{-1}ABB^{-1} = I_nAI_n = A$

    Since $\Psi(\Phi(A)) = \Phi(\Psi(A)) = A$, we can conclude that $\Psi$ is $\Phi^{-1}$ which means $\Phi$ is invertible.

    We must show that $\Psi$ is linear. Let C, D be fixed, but arbitrary matrices $\in M_{n\times n}(F)$ and scalar $a \in F$. To show that $\Phi$ is linear, we must verify that $\Phi$ preserves scalar multiplication and vector addition. Let's construct a transformation as such:
    $\Psi(aC+D) = B(aC+D)B^{-1} = (aBC+BD)B^{-1} = aBCB^{-1}+BDB^{-1}$

    $a\Psi(C) + \Psi(D) = aBCB^{-1} + BDB^{-1}$

    Since, $\Psi(aC + D) = a\Psi(C) + \Psi(D)$, we have shown that the invertible transformation of $\Phi$ is linear.

    $\Phi$ is invertible and its inverse $\Psi$ is linear, then $\Phi$ is an isomorphism. $\square$

14. (2.4 Problem 17) Let V and W be finite-dimensional vector spaces and T: V $\to$ W be an isomorphism. Let $V_0$ be a subspace of V.

    (a) Prove that $T(V_0)$ is a subspace of W.

        To prove that $T(V_0)$ is a subspace of W, we must verify the subspace axioms by using arbitrary, but fixed elements $w_1, w_2 \in T(V_0)$ and scalar $c \in F$.

        WTS that $cw_1 + w_2 \in T(V_0)$. Let $v_1, v_2$ be arbitrary, but fixed elements in $V_0$ s.t. $T(v_1) = w_1, T(v_2) = w_2$.

Since $V_0$ is a subspace of V, $cv_1 + v_2 \in V$. Hence, $T(cv_1 + v_2) = cT(v_1) + T(v_2) = cw_1 + w_2 \in T(V_0)$.

$T(V_0)$ is closed under vector addition and scalar muliplication.

Since T is an isomorphism, $T(0) = 0 \in T(V_0)$.

$T(V_0)$ contains the zero vector.

Hence, $T(V_0)$ is a subspace of W. $\square$

(b) Prove that $dim(V_0) = dim(T(V_0))$

FSOC, assume that $dim(V_0) \neq dim(T(V_0))$.

Since $V_0, W_0$ are finite dimensional, we can use the dimension theorem:

$$nullity(T(V_0)) + rank(T(V_0)) = dim(V_0) \tag{7}$$

Since $T : V \to W$ is an isomorphism and $V_0$ is a subspace of V, $nullity(T(V_0)) = 0$.

So, $rank(T(V_0)) = dim(V_0)$.

From the Corollary from Theorem 2.17, we are given that $rank(T(V_0)) = dim(R(T(V_0)) = dim(T(V_0))$.

So, $dim(V_0) = dim(T(V_0))$.

However, that contradicts our assumption that $dim(V_0) \neq dim(T(V_0))$.

Hence, in this scenario: $dim(V_0) = dim(T(V_0))$ $\square$

15. (2.4 Problem 19)

(a) Compute $[T]_\beta$

$\beta = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$

$T(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$T(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$T(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$T(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) Verify that $L_A\phi_\beta(M) = \phi_\beta T(M)$ for A $= [T]_\beta$ and M $= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\mathrm{T}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = 1 * E_{11} + 3 * E_{12} + 2 * E_{21} + 4 * E_{22}$$

$$\phi_\beta T(M) = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$$

$$L_A\phi_\beta(M) = L_A\phi_\beta(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$$

Hence, $\phi_\beta T(M) = L_A\phi_\beta(M)$