Math 115A Homework 5

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November 20, 2024

1 Introduction

1. Problem 2.5 3e) $\beta = \{x^2 - x, x^2 + 1, x - 1\}$ and $\beta' = \{5x^2 - 2x - 3, -2x^2 + 5x + 5, 2x^2 - x - 3\}$

   $$5x^2 - 2x - 3 = 5(x^2 - x) + 0(x^2 + 1) + 3(x - 1) \rightarrow \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$$

   $$-2x^2 + 5x + 5 = -6(x^2 - x) + 4(x^2 + 1) + 1(x - 1) \rightarrow \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix}$$

   $$2x^2 - x - 3 = 3(x^2 - x) + -1(x^2 + 1) + 2(x - 1) \rightarrow \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

   $$[Id]_{\beta'}^{\beta} = \begin{pmatrix} 5 & -6 & 3 \\ 0 & 4 & -1 \\ 3 & -1 & 2 \end{pmatrix}$$

2. Problem 2.5 4) Let T be the linear operator on $\mathbb{R}^{\nvDash}$ defined by $T(\begin{pmatrix} a \\ b \end{pmatrix}) = \begin{pmatrix} 2a + b \\ a - 3b \end{pmatrix}$, let β be the standard ordered basis for R^2, and let $\beta' = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\}$. Use Theorem 2.23 and the fact that $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ to find $[T]_{\beta'}$.

$\beta = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$

$\beta' = \{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\}$

$\mathrm{T}(\begin{pmatrix}1\\0\end{pmatrix}) = \begin{pmatrix}2*1+0\\1-3*0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} = 2\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix} : \begin{pmatrix}2\\1\end{pmatrix}$

$\mathrm{T}(\begin{pmatrix}0\\1\end{pmatrix}) = \begin{pmatrix}2*0+1\\0-3*1\end{pmatrix} = \begin{pmatrix}2\\-3\end{pmatrix} = 2\begin{pmatrix}1\\0\end{pmatrix} + -3\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix}$

$[T]_\beta = \begin{pmatrix}2 & 1\\1 & -3\end{pmatrix}$

Theorem 2.23 states that:

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q \tag{1}$$

$$[T]_{\beta'} = \begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\1 & -3\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix} \tag{2}$$

$$[T]_{\beta'} = \begin{pmatrix}2*2+-1(1) & 2*1+-1*-3\\-1*2+1*1 & -1*1+1*-3\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix} \tag{3}$$

$$[T]_{\beta'} = \begin{pmatrix}3 & 5\\-1 & -4\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix} = \begin{pmatrix}3*1+5*1 & 3*1+5*2\\-1*1+-4*1 & -1*1+-4*2\end{pmatrix} \tag{4}$$

$[T]_{\beta'} = \begin{pmatrix}8 & 13\\-5 & -9\end{pmatrix}$

3. Problem 2.5 6b)

$[L_A]_\beta = \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1+2\\2+1\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix} = 3\begin{pmatrix}1\\1\end{pmatrix} \rightarrow \begin{pmatrix}3\\0\end{pmatrix}$

$[L_A]_\beta = \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1-2\\2-1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} = -1\begin{pmatrix}1\\-1\end{pmatrix}$

$[\mathrm{L}_A]_\beta = \begin{pmatrix}3 & 0\\0 & -1\end{pmatrix}$

Q will just be the matrix form of the ordered basis vectors of β:
Q = $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

4. 5.1 Problem 2a) T$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a - b \\ 5a + 3b \end{pmatrix}$

   If we put T into matrix form we get: $\begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}$

   det(T) = (2 * 3) - (-1 * 5) = 11

   $\begin{pmatrix} 2-\lambda & -1 \\ 5 & 3-\lambda \end{pmatrix} = (2-\lambda)(3-\lambda) - (-1 * 5) = 6 - 2\lambda - 3\lambda + \lambda^2 + 5$
   $= \lambda^2 - 5\lambda + 11$

5. 5.1 Problem 3f) V = $M_{2\times 2}(R)$, T($\begin{pmatrix} -7a - 4b + 4c - 4d & b \\ -8a - 4b + 5c - 4d & d \end{pmatrix}$)

   Let $\beta = \{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\}$

   T($\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$) $= \begin{pmatrix} -7 * 1 + 0 + 4 * 1 - 0 & 0 \\ -8 * 1 + 0 + 5 * 1 - 0 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ -3 & 0 \end{pmatrix} = -3\begin{pmatrix} -3 & 0 \\ -3 & 0 \end{pmatrix} +$
   $0 + 0 + 0 \to \begin{pmatrix} -3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

   T($\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$) $= \begin{pmatrix} -1 * 1 + 0 + 0 + 0 & 2 \\ 8 - 4(2) + 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = 1\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \to$
   $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

   T($\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$) $= \begin{pmatrix} -7 * 1 + 4(2) & 0 \\ -8(1) + 5(2) & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = 1\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

   T($\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$) $= \begin{pmatrix} -7(-1) + -4(2) & 0 \\ -8(-1) - 4(2) & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = 1\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \to$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[\mathrm{T}]_\beta = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $[\mathrm{T}]_\beta$ is a diagonal matrix, the eigenvalues of T are $-3, 1, 1, 1$.

β is a basis consisting of eigenvectors of T.

6. 5.1 Problem 4cd)

   c) $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$ for F = C.

   $det(A - \lambda I) = det \begin{pmatrix} i-\lambda & 1 \\ 2 & -i-\lambda \end{pmatrix} = (i-\lambda)(-i-\lambda) - (2) = 0 =$
   $1 - \lambda i + \lambda i + \lambda^2 = \lambda^2 - 1 = 0$
   $\lambda = -1, 1$
   For $\lambda = -1$:
   $\begin{pmatrix} i-(-1) & 1 \\ 2 & -i-(-1) \end{pmatrix} = \begin{pmatrix} i+1 & 1 \\ 2 & -i+1 \end{pmatrix}$
   $\begin{pmatrix} i+1 & 1 & 0 \\ 2 & -i+1 & 0 \end{pmatrix}$
   rank = 1, so gemu = 1
   $(i+1)x + y = 0$
   $2x + (-i+1)y = 0$

   x $= -\frac{1}{2} + \frac{i}{2}$
   y $= 1$
   $E_{-1} = span(\begin{pmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \end{pmatrix})$
   For $\lambda = 1$:
   $\begin{pmatrix} i-(1) & 1 \\ 2 & -i-(1) \end{pmatrix} = \begin{pmatrix} i-1 & 1 \\ 2 & -i-1 \end{pmatrix}$

$\begin{pmatrix} i-1 & 1 & 0 \\ 2 & -i-1 & 0 \end{pmatrix}$

rank = 1, so gemu = 1

$(i-1)x + y = 0$

$2x + (-i-1)y = 0$

x = $\frac{1}{2} + \frac{i}{2}$

y = 1

$E_1 = span(\begin{pmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{pmatrix})$

Q = $\begin{pmatrix} -\frac{1}{2} + \frac{i}{2} & \frac{1}{2} + \frac{i}{2} \\ 1 & 1 \end{pmatrix}$

basis = $\{\begin{pmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{pmatrix}\}$

D = $Q^{-1}AQ = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

d) A = $\begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$

$\det(A - \lambda I) = \begin{pmatrix} 2-\lambda & 0 & -1 \\ 4 & 1-\lambda & -4 \\ 2 & 0 & -1-\lambda \end{pmatrix} =$

$0 + (-1)^{2+2}(1-\lambda)det(\begin{pmatrix} 2-\lambda & -1 \\ 2 & -\lambda-1 \end{pmatrix}) + 0$

$= (1-\lambda)((2-\lambda)(-\lambda-1) - (-1*2)) = \lambda^2 - \lambda = 0$

Eigenvalues: $\lambda = 0, 1$

$\lambda = 0$:

$\begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

rank = 2 so dim(ker(A)) = gemu = 1

x - $-\frac{1}{2} = 0$

y - 2z = 0

z is free

Let z = 1

basis for $E_0 = \text{span}(\begin{pmatrix} \frac{1}{2} \\ 2 \\ 1 \end{pmatrix})$

$\lambda = 1$: $\begin{pmatrix} 2-1 & 0 & -1 \\ 4 & 1-1 & -4 \\ 2 & 0 & -1-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

rank = 1 so dim(ker(A)) = gemu = 2

x - z = 0

y, z are free

basis for $E_1 = \text{span}(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix})$

basis = $\{\begin{pmatrix} \frac{1}{2} \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\}$

Q = $\begin{pmatrix} \frac{1}{2} & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

D = $Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

7. 5.1 Problem 5ae)

   a) $V = \mathbb{R}^2$ and T(a,b) = $(-2a + 3b, -10a + 9b)$
      If we put T(a,b) in matrix form:
      Let A = $\begin{pmatrix} -2 & 3 \\ -10 & 9 \end{pmatrix}$
      To compute the eigenvalues we must put A into the form $(A - \lambda I)v = 0$

$$\begin{pmatrix} -2-\lambda & 3 \\ -10 & 9-\lambda \end{pmatrix} = (-2-\lambda)(9-\lambda) - 3(-10) = 0 \tag{5}$$

$$-18 + 2\lambda - 9\lambda + \lambda^2 + 30 = 0 \tag{6}$$

$$\lambda^2 - 7\lambda + 12 = 0 \tag{7}$$

$$(\lambda - 3)(\lambda - 4) = 0 \tag{8}$$

$\lambda = 3, 4$

$\lambda = 3$:

$$\begin{pmatrix} -2-3 & 3 \\ -10 & 9-3 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ -10 & 6 \end{pmatrix} \tag{9}$$

$-5x + 3y = 0$
$-10x + 6y = 0$

$x = \frac{1}{5}$, $y = \frac{1}{3} \rightarrow \begin{pmatrix} \frac{1}{5} \\ \frac{1}{3} \end{pmatrix}$

$\lambda = 4$:

$$\begin{pmatrix} -2-4 & 3 \\ -10 & 9-4 \end{pmatrix} = \begin{pmatrix} -6 & 3 \\ -10 & 5 \end{pmatrix} \tag{10}$$

$-6x + 3y = 0$
$-10x + 5y = 0$

$x = 1$, $y = 2 \rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} \frac{1}{5} & 1 \\ \frac{1}{3} & 2 \end{pmatrix} \tag{11}$$

$\beta = \{(\frac{1}{5}, \frac{1}{3}), (1, 2)\}$

e) $V = P_2(R)$ and $T(f(x)) = xf'(x) + f(2)x + f(3)$ $T(1) = 1 * 0 + 1 * x + 1 = 1 + x$

$T(x) = x * 1 + 2x + 3 = 3 + 3x$

$T(x^2) = x * 2x + 4x + 9 = 9 + 4x + 2x^2$

$[T]_\beta = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}$

To find eigenvalues, we put it into the form; $(A - I\lambda)v = 0$:

$\det \begin{pmatrix} 1-\lambda & 3 & 9 \\ 1 & 3-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{pmatrix} = 0$

LHS:

$$det(A - I\lambda) = 0 - 0 + (2-\lambda) * det(\begin{pmatrix} 1-\lambda & 3 \\ 1 & 3-\lambda \end{pmatrix} \tag{12}$$

$$(2-\lambda) * (1-\lambda)(3-\lambda) - 3(1) = (2-\lambda)(3 - \lambda - 3\lambda + \lambda^2 - 3) \tag{13}$$

$$(2-\lambda)(\lambda^2 - 4\lambda) = (2-\lambda)(\lambda)(\lambda - 4) = 0 \tag{14}$$

$\lambda = 0, 2, 4$

At $\lambda = 0$: $\begin{pmatrix} 1-0 & 3 & 9 \\ 1 & 3-0 & 4 \\ 0 & 0 & 2-0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}$

x + 3y + 9z = 0
x + 3x + 4z = 0
2z = 0

z = 0, x + 3y = 0
x = -3y

$\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$

At $\lambda = 2$: $\begin{pmatrix} 1-2 & 3 & 9 \\ 1 & 3-2 & 4 \\ 0 & 0 & 2-2 \end{pmatrix} = \begin{pmatrix} -1 & 3 & 9 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} -1 & 3 & 9 & 0 \\ 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ -$R_1 \rightarrow R_1$ $\begin{pmatrix} 1 & -3 & -9 & 0 \\ 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $R_2 - R_1 \rightarrow R_2$ $\begin{pmatrix} 1 & -3 & -9 & 0 \\ 0 & 4 & 13 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\frac{1}{4}R_2 \rightarrow R_2$ $\begin{pmatrix} 1 & -3 & -9 & 0 \\ 0 & 1 & \frac{13}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $R_1 + 3R_2 \rightarrow R_1$ $\begin{pmatrix} 1 & 0 & \frac{3}{4} & 0 \\ 0 & 1 & \frac{13}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$x = -3/4t, y = -13/4t, z = t$

If $z = 4$, then x = -3, y = -13.

$$\begin{pmatrix} -3 \\ -13 \\ 4 \end{pmatrix}$$

At $\lambda = 4$: $\begin{pmatrix} 1-4 & 3 & 9 \\ 1 & 3-4 & 4 \\ 0 & 0 & 2-4 \end{pmatrix} = \begin{pmatrix} -3 & 3 & 9 \\ 1 & -1 & 4 \\ 0 & 0 & -2 \end{pmatrix}$

$$\begin{pmatrix} -3 & 3 & 9 & 0 \\ 1 & -1 & 4 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

-2z = 0, z = 0

-3x + 3y = 0
x - y = 0

x = 1, y = 1

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Matrix: $\begin{pmatrix} -3 & -3 & 1 \\ 1 & -13 & 1 \\ 0 & 4 & 0 \end{pmatrix}$

$\beta = \{-3 + x, -3 - 13x + 4x^2, 1 + x\}$

8. 5.1 Problem 9ab

   (a) Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

   $\rightarrow$ Suppose that T on a finite-dimensional vector space is invertible.

   FSOC, suppose that T is invertible, but zero is an eigenvalue.

   $det([T] - 0I) = det([T]) = 0$

   Which means that A is not invertible, but that is a contradiction. So, zero must not be an eigenvalue in this case.

   $\leftarrow$ If zero is not an eigenvalue of T, then T is invertible. Contrapositive: If T is not invertible, then zero is an eigenvalue.

To show that zero is eigenavalue, let us compute the determinant of the T - λI_n s.t. $\lambda = 0$

$det([T] - 0I) = det(T) = 0$.

Hence, it follows that 0 is an eigenvalue of T when T is not invertible.

$\square$

(b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}

$\rightarrow$ Suppose that scalar λ is an eigenvalue of T.

Let v be a vector $\in$ an arbitrary finite dimensional space V s.t. $T(v) = \lambda v$

WTS λ^{-1} is an eigenvalue of T^{-1}.

$T^{-1}(T(v)) = T^{-1}(\lambda v)$

$v = T^{-1}(\lambda v) = \lambda T^{-1}(v)$

$\lambda^{-1} v = T^{-1} v$

By definition, λ^{-1} is an eigenvalue of T^{-1}.

$\leftarrow$ Suppose that λ^{-1} is an eigenvalue of T^{-1}, then scalar λ is an eigenvalue of T.

Let v be an eigenvector of an arbitrary finite dimensional space V s.t. $\lambda^{-1} v = T^{-1} v$.

$T^{-1} v = \lambda^{-1} v$

$T(T^{-1} v) = T(\lambda^{-1} v)$

$v = \lambda^{-1} T(v)$

$\lambda v = T(v)$

By definition, λ is an eigenvalue of T. $\square$

9. (Section 5.1 Problem 10) Prove that the eigenvalues of an upper triangular matrix M are the diagonal entries of M.

Let M be an $n \times n$ upper triangular matrix i.e.

$$\text{M} = \begin{pmatrix} m_{11} & m_{12} & \ldots. & m_{1n} \\ 0 & m_{22} & \ldots\ldots\ldots\ldots & \\ \ldots.. & \ldots.. & m_{33} & \ldots\ldots\ldots \\ 0 & \ldots\ldots\ldots.. & \ldots. & m_{nn} \end{pmatrix}$$

To find the eigenvalues of M, we must find the determinant of $M - \lambda I_n$:

M = $\begin{pmatrix} m_{11}-\lambda & m_{12} & .... & m_{1n} \\ 0 & m_{22}-\lambda & .... & \\ ..... & ..... & m_{33}-\lambda & .... \\ 0 & .... & .... & m_{nn}-\lambda \end{pmatrix}$

Since $M - \lambda I_n$ is upper-triangular, the determinant is:

$$(m_{11}-\lambda)(m_{22}-\lambda).....(m_{nn}-\lambda) = 0 \tag{15}$$

And the roots are the diagonals of the upper triangular matrix M so we are done.

□

10. (Section 5.1 Problem 13a) Prove that similar matrices have the same characteristic polynomial.

    Similar matrices - Matrices with the same determinants and eigenvalues

    Let A be a matrix in $M_{n\times n}(F)$ for some n ∈ ℕ.

    The characteristic polynomial of A is

    $$\chi_A(\lambda) = det(A - \lambda I_n) \tag{16}$$

    To show that a similar matrix of A has the same characteristic polynomial as A, it is sufficient to show that it has the same characteristic polynomial.

    Let B be a similar matrix to A and let Q be an invertible matrix s.t. $A = Q^{-1}BQ$

    $$\chi_A(\lambda) = det(A - \lambda I_n) = det(Q^{-1}BQ - Q^{-1}\lambda I_n Q) \tag{17}$$

    $$= det(Q^{-1}(B - \lambda I_n)Q) = det(Q^{-1})det(B - \lambda I_n)det(Q) = det(B - \lambda I_n) \tag{18}$$

    Thus, the similar matrix has the same characteristic polynomial as the original matrix.

    □

11. (Section 5.1 18a) Let T be the linear operator on $M_{n\times n}(R)$ defined by T(A) = A^T.

    Show that ± 1 are the only eigenvalues of T.

    T(A) = λA

$$A^T = \lambda A \tag{19}$$

$$(A = \lambda A^T)^T \to A = \lambda A^T = \lambda^2 A \tag{20}$$

    If $A = \lambda^2 A$, then λ can only be ± 1. So we are done. $\square$

12. (Section 5.1 18c) Let T be the linear operator on $M_{n\times n}(R)$ defined by T(A) = A^T. Find an ordered basis β for $M_{2\times 2}(R)$ s.t. $[T]_\beta$ is a diagonal matrix.

    Let $\beta = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$

    $\mathrm{T}(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

    $\mathrm{T}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

    $\mathrm{T}(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

    $\mathrm{T}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$

    $[T]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

13. (Section 5.2 3c) V = R^3 and T is defined as $T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_1 \\ 2a_3 \end{pmatrix}$

Let us use the standard basis of R^3: $\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}$

T($\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$) $= \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

T($\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$) $= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

T($\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$) $= \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

In matrix form, this matrix can be represented as: $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

det(T - λI) $= \begin{pmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} = 0$

$$(2-\lambda)(\lambda^2 - (-1)) = (2-\lambda)(\lambda^2 + 1) = 0 \tag{21}$$

T doesn't split over R so it is not diagonalizable.

14. (Section 5.2 3d) V = $P_2(R)$ and T is defined by $T(f(x)) = f(0) + f(1)(x + x^2)$

Let β be the standard ordered basis of $P_2(R)$.

T(1) = 1 + 1(x + x^2) = 1 + x + x^2 $\rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

T(x) = 0 + 1(x + x^2) = x + x^2 $\rightarrow \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

T(x^2) = 0 + 1(x + x^2) = x + x^2 $\rightarrow$ $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

So $[\mathrm{T}]_\beta$ = $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Characteristic polynomial: det($[\mathrm{T}]_\beta$ - λI) = 0

$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} = (1{-}\lambda)((1{-}\lambda)(1{-}\lambda){-}1 = (1{-}\lambda)(\lambda^2{-}2\lambda) = 0$

$\lambda(1-\lambda)(\lambda-2) = 0$

$\lambda = 0, 1, 2$

It is diagonalizable with D = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\lambda = 1$:

$\begin{pmatrix} 1-1 & 0 & 0 \\ 1 & 1-1 & 1 \\ 1 & 1 & 1-1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

x + z = 0
x + y = 0
z is free

z = 1, x = -1, y = 1

$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = 2$:

$\begin{pmatrix} 1-2 & 0 & 0 \\ 1 & 1-2 & 1 \\ 1 & 1 & 1-2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

x = 0
- y + z = 0
z is free

z = 1, x = 0, y = 1

$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = 0$:

$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

x = 0
y + z = 0
z is free

z = 1, x = 0, y = -1

$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

Ordered basis can be $\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \}$

and $[\mathrm{T}]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

15. (Section 5.2 7)

A = $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

$$A^k = QD^kQ^{-1} \tag{22}$$

det(A - λI) = $\begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix}$ = (1 - λ)(3 - λ) - (4 * 2) =
3 - $\lambda - 3\lambda + \lambda^2$ - 8 = $\lambda^2 - 4\lambda - 5 = 0$

$\lambda = -1, 5$

$\lambda = -1$:

det(A + I) = $\begin{pmatrix} 1+1 & 4 \\ 2 & 3+1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}$

2x + 4y = 0

x = -2, y = 1: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

λ = 5:

det(A - 5I) = $\begin{pmatrix} 1-5 & 4 \\ 2 & 3-5 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}$

-x + y = 0

x = 1, y = 1 $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Q = $\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$ D = $\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$

$$Q^{-1} = \frac{1}{det(Q)} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \tag{23}$$

Answer:

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}^k = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^k & 0 \\ 0 & 5^k \end{pmatrix} \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \tag{24}$$

16. (Section 5.2 8) Suppose that $A \in M_{n\times n}(F)$ has two distinct eigenvalues, λ_1 and λ_2 and that dim(E_{λ_1}) = n - 1. Prove that A is diagonalizable.

    WTS dim(E_{λ_1}) + dim(E_{λ_2}) = n and A is diagonalizable.

    Since λ_1 and λ_2 are distinct e-vals, the eigenbases of both eigenvalues are LI so it follows that the intersection of the two eigenspaces E_{λ_1} and E_{λ_2} = {0}.

    We know that dim(E_{λ_1}) = n - 1, and dim(E_{λ_2}) $\geqslant$ 1.

    Knowing that

    dim(E_{λ_1}) + $dim(E_{\lambda_2}) \leqslant n$.

    Since dim(E_{λ_1}) = n - 1, we can rewrite the equation as this:

$$n - 1 + dim(E_{\lambda_2}) \leqslant n \tag{25}$$

    So $dim(E_{\lambda_2}) \leqslant 1$.

    Since dim(E_{λ_2}) $\geqslant$ 1 and $dim(E_{\lambda_2}) \leqslant 1$,

$$dim(E_{\lambda_2}) = 1$$

Since the eigenvectors of both eigenspaces are LI, their union will form a basis for $M_{n\times n}(F)$, we can conclude that A is diagonalizable.

□

17. (Section 5.2 10) Let T be a linear operator on a finite-dimensional vector space V with the distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ and corresponding multiplicities $m_1, m_2, ..., m_k$. Suppose that β is a basis for V such that $[\mathrm{T}]_\beta$ is an upper triangular matrix. Prove that the diagonal entries of $[\mathrm{T}]_\beta$ are $\lambda_1, \lambda_2, ..., \lambda_k$ and that each λ_i occurs m_i times. (1 ⩽ i ⩽ k).

    Proof:

    The characteristic polynomial of T is independent of the choice of basis β. For upper-triangular matrices, we know the determinant is the product of the diagonals so our characteristic polynomial is:

$$det([T]_\beta - \lambda I) = \chi_T(\lambda) = \prod_{i=1}^{k}(\lambda_i - \lambda)^{m_i} \tag{26}$$

    for i = 1, 2, 3,..., k.

    We know that the characteristic polynomial of T will split. If we set $\lambda = 0$, we get $\prod_{i=1}^{k}(\lambda)^{m_i}$. By the assumption we made that T has distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ with respective almus of $m_1, m_2, ..., m_k$, we can deduce that each eigenvalue λ_i occurs exactly m_i times on the diagonal entries of $[T]_\beta$. So we are done. □