Math 115A Homework 6

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1 Introduction

1. (6.1 Question 1)
   (a) True. That is the definition of an inner product space.
   (b) True. That is what we work with in general.
   (c) False. On the second component of the inner product, it violates the definition of linearity.
   (d) False. We can define as many inner products as we want on $\mathbb{R}^{\ltimes}$.
   (e) False. Theorem 6.2 doesn't state that dimension of the inner product space must be finite-dim.
   (f) False. Any matrix can have a conjugate transpose.
   (g) False. If let x be a zero vector, then y and z don't have to be equal.
   (h) True. y must be 0 in this case.

2. (6.1 Question 2) Let x = (2, 1 + i, i) and y = (2 - i, 2, 1 + 2i) be vectors in $\mathbb{C}^3$. Compute $\langle x \ , \ y \rangle$, $||x||, ||y||$, and $||x + y||$. Then verify both the Cauchy-Schwarz inequality and the triangle inequality.

$$\langle x \ , \ y \rangle = x\overline{y} = 2(2 - i) + (1 - i)2 - i(1 + 2i) = \tag{1}$$

$$4 - 2i + 2 - 2i - i + 2 \tag{2}$$

$$= 8 - 5i \tag{3}$$

$$||x|| = \sqrt{\langle x \ , \ x \rangle} = \sqrt{\sum_{n=1}^{3} x_n \bar{x_n}} \tag{4}$$

$$\sqrt{2^2 + (1+i)(1-i) + i(-i)} = \sqrt{4+2-1} = \sqrt{7} \tag{5}$$

$$||y|| = \sqrt{\langle y \ , \ y \rangle} = \sqrt{\sum_{n=1}^{3} y_n \bar{y_n}} \tag{6}$$

$$\sqrt{(2-i)(2+i) + 2^2 + (1+2i)(1-2i)} = \sqrt{4+1+4+1+4} = \sqrt{14} \tag{7}$$

$x + y = (2, 1+i, i) + (2-i, 2, 1+2i) = (4-i, 3+i, 1+3i)$

$$||x+y|| = \sqrt{16+1+9+1+1+9} = \sqrt{37} \tag{8}$$

Cauchy-Schwartz Inequality: $|\langle x \ , \ y \rangle| = |8+5i| = \sqrt{64+25} = \sqrt{89}$

$$||x|| \cdot ||y|| = \sqrt{7} * \sqrt{14} = \sqrt{98} \tag{9}$$

$\sqrt{89} \leqslant \sqrt{98}$

Triangle Inequality:

$$||x+y|| = \sqrt{37} \tag{10}$$

$$||x|| + ||y|| = \sqrt{7} + \sqrt{14} \tag{11}$$

$\sqrt{37} \leqslant \sqrt{7} + \sqrt{14}$

3. (6.1 Question 5) In C^2, show that $\langle x \ , \ y \rangle = xAy^*$ is an inner product, where

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \tag{12}$$

Compute $\langle x \ , \ y \rangle$ for $x = (1-i, 2+3i)$ and $y = (2+i, 3-2i)$.

To show that this is an inner product, we must verify the axioms.

Let x, y, z $\in C$

Addition:

$\langle x+z\ ,\ y\rangle = (x+z)Ay^* = xAy^* + zAy^* = \langle x\ ,\ y\rangle + \langle z\ ,\ y\rangle$

Scalar:

$\langle cx\ ,\ y\rangle = cxAy^* = c\langle x\ ,\ y\rangle$

Conjugate:

$\overline{\langle x\ ,\ y\rangle} = (xAy^*)^* = x^*A^*y$

$$A* = \begin{pmatrix} 1 & \overline{-i} \\ \bar{i} & 2 \end{pmatrix} = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} = A \tag{13}$$

$x^*A^*y = x^*A*y = yAx^* = \langle y\ ,\ x\rangle$

Positivity:

$\langle x\ ,\ x\rangle = xAx^*$

Let x = $\begin{pmatrix} x_1 & x_2 \end{pmatrix}$ and $x^* = \begin{pmatrix} \overline{x_1} \\ \overline{x_2} \end{pmatrix}$

$$xAx^* = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} \overline{x_1} \\ \overline{x_2} \end{pmatrix} = \begin{pmatrix} x_1 + ix_2 & -ix_1 + 2x_2 \end{pmatrix} \begin{pmatrix} \overline{x_1} \\ \overline{x_2} \end{pmatrix} \tag{14}$$

$$= \overline{x_1}x_1 - ix_2\overline{x_1} + ix_1\overline{x_2} + 2x_2\overline{x_2} = |x_1|^2 + 2|x_2|^2 \tag{15}$$

This equation can never be less than 0. It can only be 0 if both x_1, x_2 are 0.

Since this set holds for the inner product axioms, this is an inner product. □

x = (1 - i, 2 + 3i), y= (2 + i, 3 - 2i)

$$\langle x\ ,\ y\rangle = xAy^* \begin{pmatrix} 1-i & 2+3i \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} 2-i \\ 3+2i \end{pmatrix} = \tag{16}$$

$= \begin{pmatrix} 1 - i(1) + (2+3i)(-i) & (1-i)i + (2+3i)(2) \end{pmatrix}$

$= \begin{pmatrix} 4-3i & 5+7i \end{pmatrix} \begin{pmatrix} 2-i \\ 3+2i \end{pmatrix} = (4-3i)(2-i) + (5+7i)(3+2i) = 6 + 21i$

4. (6.1 Question 8) Provide reasons why each of the following is not an inner product on the given vector spaces.

(a) $\langle (a,b)\ ,\ (c,d)\rangle = ac - bd$ on $\mathbb{R}^2$.

If you set $a = b = c = d = 1$, you get 0, but (1, 1) is not the zero vector and thus, violates the positivity axiom, so it is not inner product space.

(b) $\langle A\ ,\ B\rangle = tr(A + B)$ on $M_{2\times 2}(R)$.

Counterexample:

Let A and B be the identity matrices in $M_{2\times 2}(R)$. $\langle 2I_2\ ,\ I_2\rangle = 3$ $2\langle I_2\ ,\ I_2\rangle = 4$.

The scalar axiom does not hold for this set. So it is not an inner product space.

(c) $\langle f(x)\ ,\ g(x)\rangle = \int_0^1 f'(t)g(t)dt$ on P(R).

Let f(x) = 1.

$\langle f(x)\ ,\ f(x)\rangle = 0 * 1 = 0 \neq f(x)$.

This set doesn't hold for the positivity axiom.

5. (6.1 Question 13) Suppose that $\langle \bullet\ ,\ \bullet\rangle_1$ and $\langle \bullet\ ,\ \bullet\rangle_2$ are two inner products on a vector space V. Prove that $\langle \bullet\ ,\ \bullet\rangle_1 + \langle \bullet\ ,\ \bullet\rangle_2$ is another inner product on V.

WTS $\langle \bullet\ ,\ \bullet\rangle_3 = \langle \bullet\ ,\ \bullet\rangle_1 + \langle \bullet\ ,\ \bullet\rangle_2$ is an inner product on V.

To show that $\langle \bullet\ ,\ \bullet\rangle_3$ is also an inner product, we must verify the inner product axioms.

Let x, y, z be in V.

Addition:

$\langle x + z\ ,\ y\rangle_3 = \langle x + z\ ,\ y\rangle_1 + \langle x + z\ ,\ y\rangle_2 =$

$\langle x\ ,\ y\rangle_1 + \langle z\ ,\ y\rangle_1 + \langle x\ ,\ y\rangle_2 + \langle z\ ,\ y\rangle_2 = \langle x\ ,\ y\rangle_3 + \langle z\ ,\ y\rangle_3$

This condition holds.

Scalar: $\langle cx\ ,\ y\rangle_3 = \langle cx\ ,\ y\rangle_1 + \langle cx\ ,\ y\rangle_2 = c\langle x\ ,\ y\rangle_1 + c\langle x\ ,\ y\rangle_2$

$= c\langle x\ ,\ y\rangle_3$

Positivity:

$\langle x\ ,\ x\rangle_3 = \langle x\ ,\ x\rangle_1 + \langle x\ ,\ x\rangle_2$.

Since $\langle \bullet\ ,\ \bullet\rangle_1$ and $\langle \bullet\ ,\ \bullet\rangle_2$ are inner product spaces, $\langle x\ ,\ x\rangle_1 + \langle x\ ,\ x\rangle_2$ is forced to be greater than 0.

Hence, $\langle \bullet\ ,\ \bullet\rangle_3$ is an inner product space. $\square$

6. (6.1 Question 16)

(a) Show that the vector space H with $\langle \cdot \ , \ \cdot \rangle$ defined on page 330 is an inner product space.

$$\langle f \ , \ g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}dt \tag{17}$$

To show that this is an inner product, we must verify the inner product axioms. Let $f, g, h \in H$.

Addition:

$$\langle f + h \ , \ g \rangle = \frac{1}{2\pi} \int_0^{2\pi} (f(t) + h(t))\overline{g(t)}dt \tag{18}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (f(t))\overline{g(t)}dt + \frac{1}{2\pi} \int_0^{2\pi} (h(t))\overline{g(t)}dt = \langle f \ , \ g \rangle + \langle h \ , \ g \rangle \tag{19}$$

Scalar:

$$\langle cf \ , \ g \rangle = \frac{1}{2\pi} \int_0^{2\pi} cf(t)\overline{g(t)}dt = \frac{c}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}dt = c\langle f \ , \ g \rangle \tag{20}$$

Conjugate:

$$\overline{\langle f \ , \ g \rangle} = \overline{\frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}dt} = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}dt = \frac{1}{2\pi} \int_0^{2\pi} g(t)f(t)dt \tag{21}$$

$= \langle g \ , \ f \rangle$

Positivity:

$$\langle f \ , \ f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)f(t)dt = \frac{1}{2\pi} \int_0^{2\pi} ||f(t)||^2 dt \tag{22}$$

For any function that is squared, it must be positive since the integral is bounded by positive integers. So $\langle f \ , \ f \rangle > 0$.

Hence, this is an IPS. $\square$

7. (6.1 Question 17) Let T be a linear operator on an inner product space V, and suppose that $||T(x)|| = ||x||$ for all x. Prove that T is one-to-one.

   Suppose that $||T(x)|| = ||x||$ for all x.

   WTS T is one-to-one.

   Suppose x, y $\in V$ s.t. $T(x) = T(y)$.

$$||T(x-y)|| = ||T(x) - T(y)|| = 0 \tag{23}$$

   From our assumption, we can say $||T(x-y)|| = ||x-y||$. So, $||x-y|| = 0$. By one of the theorems, the norm can only be 0 if and only if x - y = 0.

   So that forces x = y. So we are done. □

8. (6.1 Question 20b) $\langle x\ ,\ y\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k||x+i^k y||^2$ if F = C, where i = $\sqrt{-1}$

$$\frac{1}{4}\sum_{k=1}^{4} i^k||x+i^k y||^2 = \frac{1}{4}(i||x+iy||^2 - i||x-iy||^2 + ||x+y||^2) \tag{24}$$

$$\frac{i}{4}(||x+iy||^2 - ||x-iy||^2) + \frac{1}{4}(||x+y||^2 - ||x-y||^2) \tag{25}$$

$$\frac{i}{4}2(\langle x\ ,\ iy\rangle) + \langle iy\ ,\ x\rangle) + \frac{1}{4}2(\langle x\ ,\ y\rangle + \langle y\ ,\ x\rangle) \tag{26}$$

$$\frac{-1}{4}2(-\langle x\ ,\ y\rangle + \langle y\ ,\ x\rangle) + \frac{1}{4}4 * Re\langle x\ ,\ y\rangle \tag{27}$$

$$-\frac{1}{4}2(-2Im\langle x\ ,\ y\rangle) + Re\langle x\ ,\ y\rangle = Im\langle x\ ,\ y\rangle + Re\langle x\ ,\ y\rangle \tag{28}$$

   $= \langle x\ ,\ y\rangle$ □

9. (6.1 Question 24) Let V be a complex inner product space with an inner product $\langle \bullet\ ,\ \bullet\rangle$. Let [•, •] be the real-valued function s.t. [x, y] is the real part of the complex number $\langle \bullet\ ,\ \bullet\rangle$ for all x, y $\in$ V. Prove that [•, •] is an inner product for V, where V is regarded as a vector space over R. Prove furthermore, that [x, ix] = 0 for all $x \in V$.

   To prove that [•, •] is an inner product for V, we must verify the 3 axioms.

Linearity: Let x, y, z $\in V$ and $a \in F$. [ax + y, z] = Re[$\langle ax + y\ ,\ z\rangle$] = Re[a$\langle x\ ,\ z\rangle + \langle y\ ,\ z\rangle$] = a[x, z] + [y, z]

Conjugation: $\overline{[x, y]} = Re[\overline{\langle x\ ,\ y\rangle}] = Re[\langle y\ ,\ x\rangle]$ = [y, x]

Positivity: [x, x] = Re[$\langle x\ ,\ x\rangle$] = $\langle x\ ,\ x\rangle > 0$. If x $\neq$ 0.

Hence, the axioms are verified. $\square$

10. (6.2 Question 1)

    (a) False. It produces an orthogonal set, not always the orthonormal set.

    (b) True. Every non-zero finite-dimensional IPS has an orthonormal basis.

    (c) True. The orthogonal complement of any set can be a subspace of a vector space.

    (d) False. The basis has to be orthonormal.

    (e) True. That is the definition of an orthonormal basis.

    (f) False. {0} is orthogonal but not LI.

    (g) True. A set of orthogonal and non-zero vectors are LI. So naturally it follows that orthonormal sets are LI.

11. (6.2 Question 2cdi)

    c) V = $P_2(R)$ with the inner product $\langle f(x)\ ,\ \ g(x)\rangle = \int_0^1 f(t)g(t)dt$, S = $\{1, x, x^2\}$.

    $v_1 = w_1 = 1$

$$v_2 = w_2 - \frac{\langle w_2\ ,\ \ v_1\rangle}{||v_1||^2} = x - \frac{\langle x\ ,\ \ 1\rangle}{||1||^2}1 \tag{29}$$

$$\langle x\ ,\ \ 1\rangle = \int_0^1 x(1)dt = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} \rightarrow v_2 = x - \frac{1}{2} \tag{30}$$

$$v_3 = w_3 - \frac{\langle w_3\ ,\ \ v_1\rangle}{||v_1||^2}v_1 - \frac{\langle w_3\ ,\ \ v_2\rangle}{||v_2||^2}v_2 \tag{31}$$

$$x^2 - \frac{\langle x^2\ ,\ \ 1\rangle}{||1||^2} - \frac{\langle x^2\ ,\ \ x - \frac{1}{2}\rangle}{||x - \frac{1}{2}||^2} \tag{32}$$

$$\langle x^2\ ,\ \ 1\rangle = \int_0^1 x^2(1)dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} \rightarrow \frac{\frac{1}{3}}{1}1 = \frac{1}{3} \tag{33}$$

$$\langle x^2\ ,\ \ x - \frac{1}{2}\rangle = \int_0^1 x^2(x - \frac{1}{2})dx = \int_0^1 x^3 - \frac{1}{2}x^2 dx = \frac{x^4}{4} - \frac{x^3}{6}\Big|_0^1. \tag{34}$$

$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$
$||v_2||^2 = \langle x - \frac{1}{2}\ ,\ \ x - \frac{1}{2}\rangle$

$$\int_0^1 (x - \frac{1}{2})^2 dx = \int_0^1 x^2 - x + \frac{1}{4}dx = \frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{4}x\Big|_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} \tag{35}$$

$= \frac{1}{12}$

$$\frac{\langle w_3\ ,\ \ v_2\rangle}{||v_2||^2}v_2 = \frac{\frac{1}{12}}{\frac{1}{12}}(x - \frac{1}{2}) = x - \frac{1}{2} \tag{36}$$

$$v_3 = x^2 - \frac{1}{3} - (x - \frac{1}{2}) = x^2 - x + \frac{1}{6} \tag{37}$$

$$S' = \{1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}\} \tag{38}$$

$$||1|| = \sqrt{1} = 1 \tag{39}$$

$$||x - \frac{1}{2}|| = \sqrt{\langle x - \frac{1}{2}\ ,\ \ x - \frac{1}{2}\rangle} = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}} \tag{40}$$

$$||x^2 - x + \frac{1}{6}|| = \sqrt{\langle x^2 - x + \frac{1}{6}\ ,\ \ x^2 - x + \frac{1}{6}\rangle} \tag{41}$$

$$\int_0^1 (x^2 - x + \frac{1}{6})^2 = \frac{1}{180} \rightarrow \sqrt{\frac{1}{180}} = \frac{1}{6\sqrt{5}} \tag{42}$$

$$O = \{1, 2\sqrt{3}(x - \frac{1}{2}), 6\sqrt{5}(x^2 - x + \frac{1}{6})\} \tag{43}$$

d) V = span(S), where S = $\{(1, i, 0), (1 - i, 2, 4i)\}$

$v_1 = w_1 = (1, i, 0)$

$v_2 = w_2 - \frac{\langle v_2\ ,\ \ w_1\rangle}{||v_1||^2} w_1$

$\langle v_2\ ,\ \ w_1\rangle = \langle (1 - i, 2, 4i)\ ,\ \ (1, -i, 0)\rangle\ 1 + i(-i) + 0 = 2$

$\frac{\langle v_2\ ,\ \ w_1\rangle}{||v_1||^2} w_1\ =\ \frac{1-3i}{2}(1, i, 0)$

$$(\frac{1 - 3i}{2}, \frac{3 + i}{2}, 0) \tag{44}$$

$$w_2 = (1 - i, 2, 4i) - (\frac{1 - 3i}{2}, \frac{3 + i}{2}, 0) = (\frac{1 + i}{2}, \frac{1 + i}{2}, 4i) \tag{45}$$

$$S' = \{(1, i, 0), (\frac{1 + i}{2}, \frac{1 + i}{2}, 4i)\} \tag{46}$$

$\sqrt{\langle w_1\ ,\ \ w_1\rangle} = \sqrt{1^2 + i(-i) + 0} = \sqrt{2}$

$\sqrt{\langle w_2\ ,\ \ w_2\rangle} = \sqrt{\frac{1+i}{2}\frac{1-i}{2} + \frac{1+i}{2}\frac{1-i}{2} + 4i * -4i} = \sqrt{68}$

$$O = \{\frac{1}{\sqrt{2}}(1, i, 0), \frac{1}{2\sqrt{17}}(1 + i, 1 - i, 4i)\} \tag{47}$$

i) V = span(S) with the inner product $\langle f\ ,\ \ g\rangle = \int_0^\pi f(t)g(t)dt, S = \{sin(t), cos(t), 1, t\}$

$u_1 = w_1 = sin(t)$

$$u_2 = w_2 - \frac{\langle w_2\ ,\ \ u_1\rangle}{||u_1||^2} u_1 \tag{48}$$

$$\langle sin(t)\ ,\ \ cos(t)\rangle = \int_0^\pi cos(t)sin(t)dt \tag{49}$$

$u = sin(t), du = cos(t)dt$

$$= \int u^n du = \frac{u^2}{2} = \frac{sin^2(x)}{2}\Big|_0^\pi = 0 \tag{50}$$

$$u_2 = cos(t) - 0 = cos(t) \tag{51}$$

$$u_3 = w_3 - \frac{\langle w_3 \ , \ \ u_1 \rangle}{||u_1||^2} - \frac{\langle w_3 \ , \ \ u_2 \rangle}{||u_2||^2} \tag{52}$$

$$\langle w_3 \ , \ \ u_1 \rangle = \langle 1 \ , \ \ sin(t) \rangle = \int_0^\pi sin(t)dt = -cos(t)\Big|_0^\pi = -cos(\pi) + cos(0) = 2 \tag{53}$$

$$\langle u_1 \ , \ \ u_1 \rangle = \langle sin(t) \ , \ \ sin(t) \rangle = \int_0^\pi sin^2(t)dt = \frac{1}{2}\int_0^\pi 1 - cos(2t)dt = \tag{54}$$

$$\frac{1}{2}(t - \frac{sin(2t)}{2}\Big|_0^\pi) = \frac{1}{2}(\pi - 0) = \frac{\pi}{2} \tag{55}$$

$$\frac{2}{\frac{\pi}{2}}sin(t) = \frac{4}{\pi}sin(t) \tag{56}$$

$$\langle w_3 \ , \ \ u_2 \rangle = \langle 1 \ , \ \ cos(t) \rangle = \int_0^\pi cos(t)dt = sin(t)\Big|_0^\pi = 0 - 0 = 0 \tag{57}$$

$$v_3 = 1 - \frac{4}{\pi}sin(t) - 0 = 1 - \frac{4}{\pi}sin(t) \tag{58}$$

$$u_4 = w_4 - \frac{\langle w_4 \ , \ \ u_1 \rangle}{||u_1||^2} - \frac{\langle w_4 \ , \ \ u_2 \rangle}{||u_2||^2} - \frac{\langle w_4 \ , \ \ u_3 \rangle}{||u_3||^2} \tag{59}$$

$$\langle w_4 \ , \ \ u_1 \rangle = \langle t \ , \ \ sin(t) \rangle = \int_0^\pi tcos(t)dt \tag{60}$$

$$-tcos(t) + sin(t)\Big|_0^\pi = -\pi(-1) + 0 - 0 = \pi \tag{61}$$

$$\langle w_4\ ,\ \ u_2\rangle = \langle t\ ,\ \ cos(t)\rangle = \int_0^\pi tcos(t)dt = tsin(t)+cos(t)\Big|_0^\pi = 0+(-1)-1 = -2 \tag{62}$$

$$\langle w_4\ ,\ \ u_3\rangle = \langle t\ ,\ \ 1-\frac{4}{\pi}sin(t)\rangle = \int_0^\pi t(1-\frac{4}{\pi}sin(t)dt \tag{63}$$

$$\int_0^\pi t - \frac{4}{\pi}\int_0^\pi tsin(t)dt = \frac{t^2}{2}\Big|_0^\pi - \frac{4}{\pi}(-tcos(t)+sin(t))\Big|_0^\pi \tag{64}$$

$$\frac{\pi^2}{2} - \frac{4}{\pi} * \pi = \frac{\pi^2}{2} - 4 \tag{65}$$

$$||v_1||^2 = \frac{\pi}{2} \tag{66}$$

$$||v_2||^2 = \langle cos(t)\ ,\ \ cos(t)\rangle = \int_0^\pi cos^2(t)dt = \frac{1}{2}(1+cos(2t)dt \tag{67}$$

$$= \frac{1}{2}(t+\frac{sin(2t)}{2})\Big|_0^\pi = \frac{\pi}{2} \tag{68}$$

$$||v_3||^2 = \langle 1-\frac{4}{\pi}sin(t)\ ,\ \ 1-\frac{4}{\pi}sin(t)\rangle = \int_0^\pi (1-\frac{4}{\pi}sin(t))^2dt = \pi - \frac{8}{\pi} \tag{69}$$

$$t - \frac{\pi}{\frac{\pi}{2}}sin(t) - \frac{-2}{\frac{\pi}{2}}cos(t) - \frac{\frac{\pi^2}{2}-4}{\pi-\frac{8}{\pi}}(1-\frac{4}{\pi}sin(t)) \tag{70}$$

$$t - 2sin(t) + \frac{4}{\pi}cos(t) - \frac{\pi}{2} + 2sin(t) = t + \frac{4}{\pi}cos(t) - \frac{\pi}{2} \tag{71}$$

$$S' = \{sin(t), cos(t), 1-\frac{4}{\pi}sin(t), t+\frac{4}{\pi}cos(t)-\frac{\pi}{2}\} \tag{72}$$

$$\frac{u_1}{||u_1||} = \frac{sin(t)}{\sqrt{\frac{\pi}{2}}} = \sqrt{\frac{2}{\pi}}sin(t) \tag{73}$$

$$\frac{u_2}{||u_2||} = \frac{cos(t)}{\sqrt{\frac{\pi}{2}}} = \sqrt{\frac{2}{\pi}}cos(t) \tag{74}$$

$$\frac{u_3}{||u_3||} = \frac{1-\frac{4}{\pi}sin(t)}{\sqrt{\pi-\frac{8}{\pi}}} = \sqrt{\frac{\pi}{\pi^2-8}} \tag{75}$$

$$||u_4||^2 = \int_0^{\pi}(t+\frac{4}{\pi}cos(t)-\frac{\pi}{2})^2dt = \frac{pi^4-96}{12\pi} \tag{76}$$

$$\frac{u_4}{||u_4||} = \frac{t+\frac{4}{\pi}cos(t)-\frac{\pi}{2}}{\sqrt{\frac{\pi^4-96}{12\pi}}} = \sqrt{\frac{12\pi}{\pi^4-96}}(t+\frac{4}{\pi}cos(t)-\frac{\pi}{2}) \tag{77}$$

$$O = \{\frac{\sqrt{2}sin(t)}{\sqrt{pi}}, \frac{\sqrt{2}cos(t)}{\sqrt{pi}}, \frac{\pi-4sin(t)}{\sqrt{\pi^3-8\pi}}, \sqrt{\frac{12\pi}{\pi^4-96}}(t+\frac{4}{\pi}cos(t)-\frac{\pi}{2})\} \tag{78}$$

12. (6.2 Question 4) Let S = $\{(1,0,i),(1,2,1)\}$ in C^3. Compute $S^{\perp}$.

$$(1,0,i)*(x,y,z) = 0, (1,2,1)*(x,y,z) = 0 \tag{79}$$

 $x + -i * z = 0 \rightarrow x = iz$
 $x = i, z = 1$
 $x + 2y + z = 0 = i + 2y + 1 = 0$
 $y = \frac{-i-1}{2}$
 $S^{\perp} = \{i, \frac{-i-1}{2}, 1\}$

13. (6.2 Question 6) Let V be an inner product space and let W be a fin-dim subspace of V. If x $\notin W$, prove that there exists $y \in V$ s.t. $y \in W^{\perp}$, but $\langle x\ ,\ y\rangle \neq 0$.

 Suppose y $\in V$. By Theorem 6.6, we can express x = u + z, with $u \in W$ and $z \in W^{\perp}$.

Let's express the inner product of x and z.

$$\langle x \ , \quad z\rangle = \langle u \ , \quad z\rangle + \langle z \ , \quad z\rangle = 0 + ||z||^2 \tag{80}$$

Since x $\notin W$, we know that $z \neq 0$. This is because if z = 0, then x = u $\in W$ contradicting our assumption that x $\notin W$

Thus, $\langle x \ , \quad z\rangle > 0$. □

14. (6.2 Question 9) Let W = span($\{(i,0,1)\}$ in C^3. Find orthonormal bases for W and W$^\perp$.

$$W = \frac{(i,0,1)}{||(i,0,1)||} = \frac{(i,0,1)}{\sqrt{i * -i + 0 + 1 * 1}} = (\frac{i}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}) \tag{81}$$

for W$^\perp$:

$(a,b,c)(i,0,1) = 0$

$ai + c = 0$

$(1,0,-i), (0,1,0)$ are 2 solutions for this.

$$W^\perp = \{\frac{(1,0,-i)}{\sqrt{1^2 + 0 + -i * i}} + \frac{(0,1,0)}{\sqrt{1^2}}\} = \{(\frac{1}{\sqrt{2}}, 0, -\frac{i}{\sqrt{2}}), (0,1,0)\} \tag{82}$$

15. (6.2 Question 17) Let T be a linear operator on an inner product space V. If $\langle T(x) \ , \quad y\rangle = 0$ for all x, y $\in V$, prove that $T = T_0$. In fact, prove this result of the equality holds for all x and y in some basis for V.

Suppose $\langle T(x) \ , \quad y\rangle = 0 \forall y \in V$. WTS T = T_0.

Since the inner product is zero, T(x) must be 0 for all cases to equal 0. So T(x) = 0 and we are done.

Prove this result of the equality holds for all x and y in some basis for V. Let $v_k = v_1, ..., v_k$ be a basis for V. Then for an arbitrary x $\in V$, we can write that x = $\sum_{i=1}^{k} a_i v_i$.

Suppose y $\in V$

$\langle T(x) \ , \quad y\rangle = 0$ by our assumption for all x and y. Then, let us choose y = T(x) to get $\langle T(x) \ , \quad T(x)\rangle = ||T(x)||^2 = 0$.

This deduces that T(x) = 0 and we are done. □

16. (6.2 Question 18) Let V = $C([-1,1])$. Suppose that W_e and W_o denote the subspaces of V consisting of the even and odd functions, respectively. Prove that $W_e^{\perp} = W_o$, where the inner product on V is defined by

$$\langle f\ ,\ \ g\rangle = \int_{-1}^{1} f(t)g(t)dt \tag{83}$$

To prove that $W_e^{\perp} = W_o$, we must show that both subspaces are subsets of each other.

$\subseteq$: Let g be an arbitrary function $\in W_o$.

WTS g is orthogonal to any function in W_e.

Let f be an even function.

$$\langle f\ ,\ \ g\rangle = \int_{-1}^{1} f(t)g(t)dt \tag{84}$$

We know that for any even * odd function, the symmetry of the integral will make the integral 0. In short, $\langle f\ ,\ \ g\rangle = 0$

U-substitution:

u = -t
du = -dt

$$\int_{1}^{-1} f(-u)g(-u)du = \int_{1}^{-1} f(u)*-g(u)du = -\int_{-1}^{1} f(u)g(u)du = -\langle f\ ,\ \ g\rangle \tag{85}$$

= 0. Since f is orthogonal to g, this implies that $W_o \subseteq W_e^{\perp}$.

Now let's prove the other subset equality.

Suppose h $\in W_e^{\perp}$.

Define f $\in W_e$ and g $\in W_o$.

Observe h = f + g.

$h \in W_e^{\perp} \to 0 = \langle h\ ,\ \ f\rangle = \langle f+g\ ,\ \ f\rangle = \langle f\ ,\ \ f\rangle + \langle g\ ,\ \ f\rangle = ||f||^2 + 0$ (it's 0 from converse proof).

So f = 0 $\to h = g \in W_o$. $\square$

17. (6.3 Question 2c) c) V = $P_2(R)$ with

$$\langle f(x)\ ,\ \ h(x)\rangle = \int_0^1 f(t)h(t)dt \tag{86}$$

,g(f) = f(0) + f'(1)

g(f) = $\langle f(x)\ ,\ \ h(x)\rangle$ for all f $\in P_2(R)$.
let f(x) = $ax^2 + bx + c$ be an arbitrary element in $P_2(R)$
let h(x) = $dx^2 + ex + f$ also be an arbitrary element in $P_2(R)$.

$$\langle f(x)\ ,\ \ h(x)\rangle = \int_0^1 f(x)g(x)dx = \int_0^1 (ax^2+bx+c)(dx^2+ex+f)dx \tag{87}$$

$$= \frac{ad}{5} + \frac{ae+bd}{4} + \frac{af+cd}{3} + (bf+ce)2 + ce \tag{88}$$

f(0) + f'(1) = $2a + b + c$

$$= \frac{ad}{5} + \frac{ab'+ba'}{4} + \frac{af+cd}{3} + \frac{bd+ce}{2} + cd \tag{89}$$

$$\frac{d}{5} + \frac{e}{4} + \frac{f}{3} = 2 \rightarrow 12d + 15e + 20f = 120 \tag{90}$$

$$\frac{d}{4} + \frac{e}{3} + \frac{f}{2} = 1 \rightarrow 3d + 4e + 6f = 12 \tag{91}$$

$$\frac{d}{3} + \frac{e}{2} + \frac{f}{1} = 1 \rightarrow 2d + 3e + 6f = 6 \tag{92}$$

(d, e, f) = (-210, -204, 33)

So, h(x) = $210x^2 - 204x + 33$.

18. (6.3 Question 3ac) a) V = R^2, $T(a,b) = (2a+b, a-3b), x = (3,5)$
$\langle x\ ,\ \ T(y)\rangle = \langle T*(x)\ ,\ \ y\rangle$

$((2a+b, a-3b)\cdot(3,5)) = 6a + 3b + 5a - 15b = 11a - 12b$

$T^*(x) = (11, -12)$

c) V = $P_1(R)$ with $\langle f(x)\ ,\ \ g(x)\rangle = \int_{-1}^1 f(t)g(t)dt$, $T(f) = f' + 3f$, $f(t) = 4 - 2t$

$f'(t) = -2$

T(f) = f' + 3f = -2 + 3(4- 2t) = -2 + 12 - 6t = 10 - 6t

19. (6.3 Question 6) Let T be a linear operator on an inner product space V. Let $U_1 = T + T^*$ and $U_2 = TT^*$. Prove that $U = U_1^*$ and $U_2 = U_2^*$.

    Proof:

    To prove the equality, let's find the value of U_1^*

    $U_1^* = (T + T^*)^*$

    We can use Theorem 6.11, which tells us about properties of adjoints of linear transformations.

    $T^* + (T^*)^* = T + T^* = U_1$

    Similarly for the proof that $U_2 = U_2^*$:

    $U_2^* = (TT^*)^* = T^*T = U_2$

    $\square$

20. (6.3 Question 8) Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Prove that if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

    Suppose T is invertible.

    WTS T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

    Since T is invertible, then $TT^{-1} = T^{-1}T = I_V$.

    If we take the adjoint of this equation we get:

    $(TT^{-1})^* = (T^{-1}T)^* = I_V^*$

    $(T^{-1})^*T^* = T^*(T^{-1})^* = I_V$

    Thus, T^* is invertible.

    Since we proved that T^* is invertible, WTS $(T^*)^{-1} = (T^{-1})^*$.

    $T^*(T^*)^{-1} = (T^*)^{-1}T^* = I_V$

    From what we've computed earlier:

    $(T^{-1})^*T^* = T^*(T^{-1})^* = I_V$

    So $(T^*)^{-1} = (T^{-1})^*$.

    $\square$

21. (6.3 Question 12a) Let V be an inner product space an let T be a linear operator on V. Prove that $R(T^*)^\perp = N(T)$.

To show that $R(T^*)^\perp = N(T)$, we must show that for any arbitrary element, if it is $R(T^*)^\perp$, then it is in $N(T)$ and we have to show the converse of that is also true.

$\rightarrow$: Suppose $y \in N(T)$ and T(y) = 0 for some $y \in V$. Suppose x is an arbitrary element $\in W$.

We have the property that:

$$\langle x \ , \ \ T(y) \rangle = \langle T^*(x) \ , \ \ y \rangle = 0 \tag{93}$$

This means that y is orthogonal to all vectors $T^*(x)$. So we can deduce that $y \in R(T^*)^\perp$.

$\leftarrow$: Suppose $y \in R(T^*)^\perp$.

Then,

$$\langle y \ , \ \ T^*(x) \rangle = \langle T(y) \ , \ \ x \rangle = 0 \tag{94}$$

To make this equation true for all x, T(y) = 0. So $y \in N(T)$.

Hence, $N(T) = R^*(T)$. $\square$