Homework 1

Jaden Ho

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1 Homework 1

1. Prove that the product of two consecutive integers is even.

   We want to prove that the product of two consecutive integers is even. Proof: Assume that n and n + 1 are consecutive integers in which n is even and n + 1 is odd. Given by the definition of an even integer: n, n + 1 can be defined as such:

   $$n = 2k, n + 1 = 2k + 1; k \in \mathbb{Z} \tag{1}$$

   Since n = 2k and n + 1 = 2k + 1,

   $$n * n + 1 = 2k * (2k + 1) = 4k^2 + 2k = 2(2k^2 + k) \tag{2}$$

   Hence, by the definition of an even integer, $2(2k^2+k)$ is an even integer. Thus, the product of two consecutive integers is even. □

2. Prove that if n is an odd integer, then $n^2 - 1$ is divisible by 8.

   We want to prove the if n is an odd integer, then $n^2 - 1$ is divisible by 8.
   Proof: Assume that n is an odd integer such that n can be defined as:

   $$n = 2k + 1, k \in \mathbb{Z} \tag{3}$$

   Then,

   $$n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k^2 + 4k \tag{4}$$

$$4k^2 + 4k = 4(k^2 + k) = 4k(k + 1) \tag{5}$$

Given that we proved that the product two consecutive integers, which were even and odd, above is even. We can determine that k * (k + 1) is an even integer. By definition of even integer:

$$k(k + 1) = 2q, q \in \mathbb{Z} \tag{6}$$

Then,

$$4k(k + 1) = 4(2q) = 8q \tag{7}$$

Since $n^2 - 1 = 8q$, $n^2 - 1$ is divisible by 8. Therefore, for any odd integer n, $n^2 + 1$ is divisible by 8. $\square$

3. Prove that if x is a real number such that $x^5 + 5x^3 + 7x \geqslant x^4 + x^2 + 4$ then $x \geqslant 0$.

   We want to prove that if x is a real number such that $x^5 + 5x^3 + 7x \geqslant x^4 + x^2 + 4$ then $x \geqslant 0$.
   Let's assume for the sake of contrapositive that $x < 0$, then $x^5 + 5x^3 + 7x < x^4 + x^2 + 4$.

   Proof: Suppose $x < 0$, then $x^5 < 0$. Also $5x^3 < 0$, because $x^3 < 0$ and $5 < 0$, meaning that their product would be less than 0. Similarly, 7x would also be < 0.

   The addition of negatives numbers is negative. Hence if $x < 0$, then $x^5 + 5x^3 + 7x < 0$.

   For the right hand side, x^4, x^2, and $4 > 0$, since the product of two negative numbers is positive. Since, the sum of positive numbers is positive, the right hand side of the inequality is positive.

   Hence, if $x < 0$, $x^5+5x^3+7x < 0$ and $x^4+x^2+4 > 0$, then $x^5+5x^3+7x < x^4 + x^2 + 4$. Thus, our original claim: if x is a real number such that $x^5 + 5x^3 + 7x \geqslant x^4 + x^2 + 4$, then $x \geqslant 0$. $\square$

4. If x is a non-zero rational number and y is an irrational number, prove that xy is irrational.

We want to prove that the product of a rational number and an irrational number is irrational.
Let x be a rational number such that $x = \frac{a}{b}$; for some integers, a and b; b $\neq$ 0. Let y $\in \mathbb{R} \setminus \mathbb{Q}$.

For the sake of contradiction, let's assume that if x is a non-zero rational number and y is a irrational number, but xy is a rational number such that $xy = \frac{c}{d}$; for some integers c and d; d $\neq$ 0. Since:

$$x * y = xy = \frac{a}{b} * y = \frac{c}{d} \tag{8}$$

$$y = \frac{cb}{ad} \tag{9}$$

Therefore, y is a rational number and an irrational number. However, that is impossible since a number cannot be rational and irrational at the same time. Thus, if x is a non-zero rational number and y is an irrational number, then xy would be irrational. □

5. Prove that $A \cup B \subseteq C$ if and only if $A \subseteq C$ and $B \subseteq C$.

   We want to prove that both directions are true for this iff statement.

   (a) Proof 1: If $A \cup B \subseteq C$, then $A \subseteq C$ and $B \subseteq C$.

       FSOC, assume that if $A \cup B \subseteq C$, then $A \nsubseteq C$ or $B \nsubseteq C$. Let x be an element in A, but not in C. That means $x \in A \cup B$, then $x \in C$. However, that is a contradiction since x cannot be in or not be in a set at the same time. Similarly, can be said if you let x be an element in B, but not in C.
       Therefore, if $A \cup B \subseteq C$, then $A \subseteq C$ and $B \subseteq C$.

   (b) Proof 2: If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

       FSOC, assume that if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \nsubseteq C$. Let x be an element in A and B. That means that x has to be in A and B and C as well. However, we reached a contradiction. x cannot be in and not in C at the same time. Hence, if $A \subseteq C$ and

$B \subseteq C$, then $A \cup B \subseteq C$.
$\square$

6. Let f : A → B and g : B → C be functions. Prove that if g ∘ f is injective, then f is injective.

   We want to prove that if $g \circ f$ is injective, then f is injective.

   Assume that g ∘ f is injective. By the definition of injective, if $g(f(x_1)) = g(f(x_2))$, then $f(x_1) = f(x_2)$, $x_1, x_2 \in A$.
   Then $g \circ f(x_1) = g \circ f(x_2)$. Since $g \circ f$ is injective, then $x_1 = x_2$. $\square$

7. Suppose that A and B are non-empty sets, and f : A → B is a function.

   (A) Prove that f is injective if and only if there exists a function g : B → A such that g(f(x)) = x for all x ∈ A.

   We want to prove both propositions are true for this iff statement.
   Proof 1 (→):
   If f is injective, then there exists a function g: B → A such that g(f(x)) = x for all x ∈ A.
   Since f is injective, $f(x_1) = f(x_2)$, $x_1 = x_2 \in A$.
   Then $x_1 = g \circ f(x_1) = g \circ f(x_2)$ which equals x_2.

   Proof 2 (←):
   If there exists a function g : B → A such that g(f(x)) = x for all x ∈ A, then f is injective.
   Suppose that: $f(x_1) = f(x_2)$ for some $x_1, x_2 \in A$.
   If you apply g to both sides, I can get:

$$g \circ f(x_1) = x_1 \tag{10}$$

$$g \circ f(x_2) = x_2 \tag{11}$$

   Since $f(x_1) = f(x_2)$, we know that $x_1 = x_2$.
   This means that f(x) is injective. Therefore, there exists a function g : B → A such that g(f(x)) = x for all x ∈ A, then f is injective.

Hence, since we proved both propositions as true we can conclude that the entire proposition is true. □

(B) Prove that f is surjective if and only if there exists a function h : B → A such that f(h(y)) = y for all y ∈ B.

We want to prove both claims are true for this iff statement.
Proof 1: (→)
If f is surjective, then there exists a function h : B → A such that f(h(y)) = y for all y ∈ B.

Assume that f(a) is surjective, such that for every y ∈ B, there exists an a ∈ A. Let a = h(y)

Then for each y ∈ B, there exists an x ∈ A such that f(a) = y. Since we assumed f(a) is surjective, for all y ∈ B, f(h(y)) = f(a) = y.
Thus, if f is surjective then there exists a function h : B → A such that f(h(y)) = y for all y ∈ B.

Proof 2:(←)
If there exists a a function h : B → A such that f(h(y)) = y for all y ∈ B., then f is surjective.
Suppose y is an element in B. In addition, assume that for function h, that f(h(y)) = y.

Since y ∈ B, there exists b = h(y) ∈ A s.t. f(b) = y. Hence, for every $y \in B$, there exists an b ∈ A such that $f(b) = y$.

By definition of surjective functions, f(b) must be surjective.

Hence, both propositions have been proven. □

(C) Prove that if f is both injective and surjective, then there exists a unique function g : B → A such that g(f(x)) = x for all x ∈ A and f(g(y)) = y for all y ∈ B.

Assume that f: $A \to B$ is bijective such that $f(x_1) = f(x_2)$, then $x_1 = x_2$. In addition, for all y in B, there exists an x in A such that f(x) = y.

Proof:
Since we know that bijective functions are invertible, we can define g: $B \to A$ to be: $g(y) = f^{-1}(y)$. By definition of inverse, we can compute that $g(f(x)) = f^{-1}(f(x)) = x$. Hence, g(f(x)) for all x $\in A$.
Similarly, for y, we can compute that: $f(g(y)) = f(f^{-1}(y)) = y$.
Hence, if f is bijective, then there exists a unique function g : B $\to$ A such that g(f(x)) = x for all x $\in$ A and f(g(y)) = y for all y $\in$ B. $\square$

8. Prove that $\mathbb{C}$ is a field.
We want to prove that the set of complex numbers satisfies all the field axioms.

Let $x = a + bi$ and $y = c + di$ and $x = e + fi$, such that a, b, c, d, e, f are $\in \mathbb{R}$.

Proof: We proceed to prove by cases by proving axioms:

Commutative Property:
$x + y =$
$(a + bi) + (c + di) =$
$(a + c) + (b + d)i$ By the definition of community property under real numbers, we can switch the a,b,c,d like this: $(c + a) + (d + b)i$ which is equal to y + x.

$xy = (a+bi)*(c+di) = ac+adi+bci+(-1)*bd = (ac-bd)+(ad+bc)i =$
$c(a + bi) + d(-b + a) = c(a + bi) + di(a + bi)) =$
$(c + di)(a + bi) = yx$

Therefore, complex numbers hold under commutative property.

Associative Property: x + y + z =
(x + y) + z = ((a + bi) + (c + di)) + e + fi = (a + c + e) + (b + d + f)i = a + (c + e) + (b + (d + f))i
= (a + bi) + ((c + e) + (d + f)i) (By associative property of real numbers) = x + (y + z)

Identity Property:
Assume that for every x, there exists a complex number 0 + 0i in which

x + 0 = x. We know that for complex numbers that 0 + 0i can be written as zero. (a + bi) + (0 + 0i)
Since we know the product of real number and 0 is 0: = (a + bi) + (0 + 0i) = (a + 0) + (bi + 0) = a + bi

Assume that for every x, there exists a number 1 in which 1 * x = x. Based on the definition of distributive property for real numbers:
1 * (a + bi) = 1 * a + 1 * bi = a + bi

Inverse Identity:
For each element x in F, there exists elements s and t such that:
$x + s = 0$ and $x * t = 0$.

$$(a + bi) + (-a - bi) = 0 + 0i = 0 \tag{12}$$

Hence, the additive identity exists.

For the multiplicative identity, let t =

$$\frac{a}{a^2 + b^2} - \frac{bi}{a^2 + b^2} \tag{13}$$

such that a, b ∈ ℝ

$$x * t = a + bi * (\frac{a}{a^2 + b^2} - \frac{bi}{a^2 + b^2}) \tag{14}$$

$$= \frac{a * a}{a^2 + b^2} - i^2 * \frac{b * b}{a^2 + b^2} \tag{15}$$

$$= \frac{a^2 + b^2}{a^2 + b^2} = 1 \tag{16}$$

Hence, the multiplicative inverse exists.

Distribution property: Let's use x, y, z again.

$$x * (y + z) = (a + bi) * ((c + di) + (e + fi)) \tag{17}$$

By communative property:

$$(a + bi) * ((c + e) + (d + f)i) \tag{18}$$

Since we treat we know c + e and d + f ∈ ℝ, we can treat it as another complex number and use property of multiplication:

$$a * (c + e) + (b * (d + f))i \tag{19}$$

By definition of distribution in real numbers,

$$= a * c + a * e + (b * d + b * f)i \tag{20}$$

By commutative property:

$$= a * c + (b * d)i + a * e + (b * f)i \tag{21}$$

This can be simplified due to the property of multiplication for complex numbers to:

$$= xy + xz \tag{22}$$

Hence, this satisfies the distributivity property for fields.

Since, all the axioms are satisfied, the set of complex numbers is a field. □