Homework 2

Jaden Ho

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1 Homework 3

1. (2.1 Problem 1)

   (a) True. T(cx + y) = cT(x) + T(y) must be held for T to be a linear transformation.

   (b) False. We also must check if T(cx) = cT(x) for some $x \in V$.

   (c) False, T is one-to-one if and only if $T(x_1) = T(x_2) means x_1 = x_2$

   (d) True. T must contain the zero vector.

   (e) False, nullity(T) + rank(T) is equal to dim(V).

   (f) False. Counterexample: T(x, y) = (x, 0). Since set(1,0), (0, 1) is linearly independent. T will give you the set (1, 0), (0, 0) which is not linearly independent.

   (g) True. if $(U(v_i) = T(v_i))$ then U = T.

2. (2.1 Problem 6)
   To prove that T is a linear transformation, we must verify the linear transformation axiom.
   Let A, B $\in M_{m\times n}(F)$ and $c \in F$ where

$$tr(A) = \sum_{i=1}^{n} A_{ii} \quad (1)$$

   and

$$tr(B) = \sum_{i=1}^{n} B_{ii} \quad (2)$$

$$T(cA+B) = \sum_{i=1}^{n} cA_{ii} + \sum_{i=1}^{n} B_{ii} \tag{3}$$

$$= c\sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii} = cT(A) + T(B) \tag{4}$$

.

Hence, T is linear. □

3. (2.1 Problem 9b) $T(a_1, a_2) = (a_1, a_1^2)$ Let $T(a_1, a_2), T(b_1, b_2) \in T$ s.t. $a_1, b_1, a_2, b_2 \in \mathbb{R}$.

   Then $T(a_1, a_2) + T(b_1, b_2) = (a_1, a_1^2) + (b_1, b_1^2) = (a_1, a_1^2) + (b_1, b_1^2) = (a_1 + b_1, a_1^2 + b_1^2)$.
   $T(a_1 + b_1, a_2 + b_2) = (a_1 + b_1, (a_2 + b_2)^2) = (a_1 + b_1, a_2^2 + 2a_2b_2 + b_2^2)$

   Since, $T(a_1, a_2) + T(b_1, b_2) \neq T(a_1 + b_1, a_2 + b_2)$, T is not linear.

4. Let T: $\mathbb{Z}_2^2 \to \mathbb{Z}_2^2$ be the function $T(a, b) = (a, b^2)$. Prove that T is a linear transformation. (Characteristic field 2)

   Let x, y be arbitrary elements in $\mathbb{Z}_2^2$ and a be $\in \mathbb{Z}_2$. Let $x = (x_1, x_2) and y = (y_1, y_2)$ for some $x_1, x_2, y_1, y_2 \in \mathbb{Z}_2$

   To prove that T is a linear transformation, we must show that T(ax + y) = aT(x) + T(y).

   T(ax + y) $= T(a(x_1, x_2) + (y_1, y_2)) = T((ax_1 + y_1, ax_2 + y_2)) = (ax_1 + y_1, (ax_2 + y_2)^2) = (ax_1 + y_1, a^2x_2^2 + 2ax_2y_2 + y_2^2)$

   aT(x) + T(y) $= a(x_1, x_2^2) + (y_1, y_2^2) = (x_1 + y_1, a^2x_2^2 + y_2^2)$

   Since in $\mathbb{Z}_2$, 2 = 0, $2ax_2y_2 = 0$ because by property of the product of the identity element 0.

   So, $(ax_1 + y_1, a^2x_2^2 + 2ax_2y_2 + y_2^2) = (ax_1 + y_1, a^2x_2^2 + y_2^2)$.

   Hence, since T(ax + y) = aT(x) + T(y), T is linear. □

5. (2.1 Problem 11) Prove that $\exists$ a linear transformation T: $\mathbb{R}^2 \to \mathbb{R}^3$ s.t. T(1,1) = (1, 0, 2) and T(2,3) = (1, -1, 4). What is T(8,11)?

   We know that (1, 1) and (2,3) can form a basis for $\mathbb{R}^2$ since it is LI and spans $\mathbb{R}^2$.

Let (x, y) $\in \mathbb{R}$ for some x,y $\in \mathbb{R}$. Then (x,y) = a(1,1) + b(2,3)
(x,y) = (a + 2b, a + 3b)
a + 2b = x
a + 3b = y
By solving systems of equations, you get:
b = x - y
a = 3x - 2y

Thus, (x,y) = (3x - 2y)(1,1) + (y - x)(2,3) T(x,y) = (3x - 2y)T(1,1) + (y - x)T(2,3) = (3x - 2y)(1, 0, 2) + (y - x)(1, -1, 4) = (2x - y, x - y, 2x)

Therefore, there exists a unique linear transformation T: $\mathbb{R}^2 \to \mathbb{R}^3$ s.t. T(x, y) = (2x - y, x - y, 2x)

Computing T(8, 11) = (2(8) - 11, 8 - 11, 2(8)) = (5, -3, 16).

$\square$

6. (2.1 Problem 12) Is there a linear transformation T: $\mathbb{R}^3 \to \mathbb{R}^2$ s.t. T(1,0,3) = (1,1) and T(-2, 0, -6) = (2,1).

   No. Since (-2, 0, -6) = -2(1, 0, 3), then T(-2, 0, -6) must be -2T(1, 0, 3) which is not (2, 1).

7. (2.1 Problem 14) Let V and W be vector spaces and T: $V \to W$ be linear.

   (a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

   $\to$) If T is one-to-one, then T carries linearly independent subsets of V onto linearly independent subsets of W.

   Assume T is one-to-one. WTS T carries L.I. subsets of V onto L.I. subsets of W.

   Let S be an linearly independent subset of V which = $\{v_1, v_2, ..., v_n\} \in V$. for ALL i = 1, 2, ... , n for some n $\in \mathbb{N} = \{0, 1, 2, ...\}$.

   To show that T(S) $\subseteq W$ is linearly independent, let us consider an arbitrary linear combination T(S) over S that equals 0. i.e. $a_1 T(v_1) + a_2 T(v_2) + ... + a_n T(v_n) = 0$ for some $a_i \in F$.

   WTS that $a_i (i = 0, 1, 2, ...n) = 0$

$$T(a_1v_1 + a_2v_2 + ... + a_nv_n) = 0 \tag{5}$$

Since the set S is LI, then $a_1v_1 + a_2v_2 + ... + a_nv_n = 0$ so T(0) = 0.

Since T is injective, we can only map only value of T onto 0. So, we can deduce $a_1T(v_1) + a_2T(v_2) + ... + a_nT(v_n) = 0$ s.t. the coefficients $a_i = 0$.

Hence, T(S) is LI, so we are done.

←) Conversely, if T carries L.I. subsets of V onto L.I. subsets of W, then T is one-to-one.

FSOC, suppose T is not one-to-one.

Then $\exists$ a nonzero v $\in V$ s.t. T(v) = 0.

Let S = v which is L.I. since v is nonzero.

Since T(v) = 0_W, v = $\{0_V\}$.

However, that is a contradiction since v cannot be a non-zero element if v is supposed to be linearly independent.

Hence, v = 0_V. $\square$

(b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is a subset of V. Prove that S is linearly independent if and only if T(S) is linearly independent.

Let S = $\{s_1, s_2, ...s_n\}$ for some n $\in \mathbb{N} := \{0, 1, 2, ...\}$

→) If S is linearly independent, then T(S) is linearly independent. Assume that S is linearly independent. WTS T(S) is also LI.

To show that the set T(S) is L.I., let us consider an arbitrary linear combination over T(S) that equals 0, i.e. . $a_1T(s_1) + a_2T(s_2) + ...a_nT(s_n) = 0$ with some fixed elements $T(s_i) \in S, a_i \in F$.

Since T is linear, we can arrange the equation as such: $T(a_1s_1 + a_2s_2 + ... + a_ns_n) = 0$

Since S is LI, $a_i = 0$. By scalar property of linear transformations we get:

$$a_1T(s_1) + a_2T(s_2) + ...a_nT(s_n) = 0 \tag{6}$$

And since $a_i = 0$, T(S) is LI.

$\leftarrow$) Conversely, if T(S) is LI then S is LI. WTS that S is LI.

To show that the set S is L.I., let us consider an arbitrary linear combination over S that equals 0, i.e. . $a_1 s_1 + a_2 s_2 + ...a_n s_n = 0$ with some fixed elements $s_i \in S, a_i \in F$.

T$(a_1 s_1 + a_2 s_2 + ...a_n s_n) = 0$

$a_1 T(s_1) + a_2 T(s_2) + ...a_n T(s_n) = 0$

Since we assumed that T(S) is LI, the coefficients $a_i = 0$. Since S share the same coefficients as T(S), we can deduce S is LI.

$\square$

(c) Suppose $\beta = \{v_1, v_2, ..., v_n\}$ is a basis for V and T is one-to-one and onto. Prove that T$(\beta) = \{T(v_1), T(v_2), ..., T(v_n)\}$ is a basis for W.

Since T is one-to-one and β, then T$(\beta) = \{T(v_1), T(v_2), ..., T(v_n)\}$ is also LI.

Since T is onto, T(V) = span(T(β)) = W. Hence, T(β) is a basis for W. $\square$

8. (2.1 Problem 16) Let $T : P(R) \to P(R)$ be defined by $T(f(x)) = f'(x)$. Recall that T is a linear. Prove that T is onto, but not one-to-one.

WTS that T is onto, but T is not one-to-one.

Proof by counterexample: Let f(x) = 2x + 1 and g(x) = 2x for some f(x), g(x) $\in P(R)$

T(f(x)) = T(g(x)) = 2, however f(x) $\neq$ g(x). Hence, T is not one-to-one.

To show that T is onto, we must show for all arbitary function h(x) $\in$ P(R), that there exists an element t(x) s.t. T(h(x)) = t(x) for some t(x) in P(R). Let's introduce a fixed, but arbitrary function for t(x).

Let t(x) = $a_1 + a_2 x + a_3 x^2 ... + a_n x^n$ for some arbitrary elements $a_1, a_2, ...a_n \in \mathbb{R}$

$$\int t(x) = a_1 x + \frac{a_2}{2} x^2 + \frac{a_3}{3} x^3 + ... + \frac{a_n}{n+1} x^{n+1} \tag{7}$$

Let h(x) be this function below.

$$h(x) = a_1 x + \frac{a_2}{2} x^2 + \frac{a_3}{3} x^3 + ... + \frac{a_n}{n+1} x^{n+1} \tag{8}$$

T(h(x)) = t(x).

Hence, since we got the general form of onto transformations, T is a onto transformation but not a one-to-one transformation. □

9. (2.1 Problem 17) Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear.

   (a) Prove that if $dim(V) < dim(W)$, then T cannot be onto.
   FSOC assume that $dim(V) < dim(W)$, but T is onto. Then rank(T) = dim(W).
   Then by Dimension Theorem, dim(V) = nullity(T) + rank(T)

   $dim(W) > dim(V) = dim(W) > nullity(T) + rank(T)$
   $= dim(W) > dim(W) + nullity(T)$
   Which means that the dimension of the null space must be negative. However, that is a contradiction since nullity(T) cannot be a negative number.

   Hence, T cannot be onto with those conditions. □

   (b) Prove that if $dim(V) > dim(W)$, then T cannot be one-to-one.
   FSOC assume that $dim(V) > dim(W)$, but T is one-to-one.

   Then nullity(T) must equal 0 by Theorem 2.4. $dim(V) > dim(W) = rank(T) + nullity(T) > dim(W)$
   $rank(T) > dim(W)$

   However, that is a contradiction since rank(T) is a subspace of W and cannot be greater in dimension than W.

   Hence, T cannot be one-to-one under these circumstances. □

10. (2.1 Problem 22) Let T: $\mathbb{R}^3 \to \mathbb{R}$ be linear. Show that there exist scalars a, b, and c such that T(x,y,z) = ax + by + cz for all (x,y,z) $\in \mathbb{R}^3$. Can you generalize this result for T: $F^n \to F$? State and prove an analogous result for T: $F^n \to F^m$.

    To show that ∃ scalars a,b,c s.t. T(x,y,z) = ax + by + cz ∀ (x,y,z) $\in \mathbb{R}$, we will let a = (1, 0, 0), b = (0, 1, 0), c = (0, 0, 1).
    T(x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1))
    Since T is linear, T(x(1, 0, 0)) + T(y(0, 1, 0)) + T(z(0, 0, 1)) = ax +

by + cz.

To generalize this result, T(x) = $x_1v_1 + x_2v_2 + ... + x_nv_n$ for some $x_1, x_2, ..., x_n \in F$ and $v_1, v_2, ..., v_n \in F^n$.

Prove an analogous result for T: $F^n \to F^m$.

Based on the previous results, we had that:

$T(x_1, x_2, ...x_n) = x_1T(e_1) + x_2T(e_2) + ...x_nT(e_n)$ s.t e_i represent the tuples. If we put this in matrix form, we get:

$$\text{T: } F^n \to F^m := \begin{pmatrix} a_{11}.....a_{1n} \\ a_{21}.....a_{2n} \\ ...... \\ .... \\ .... \\ a_{m1}.....a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} x_1a_{11}.....x_na_{1n} \\ x_1a_{21}.....x_na_{2n} \\ ...... \\ .... \\ .... \\ x_1a_{m1}.....x_na_{mn} \end{pmatrix}$$

This conforms that each component of the output is a linear combination of the input vectors and there are m-tuples.

□

11. (2.1 Problem 38) Prove that if V and W are vector spaces over $\mathbb{Q}$, then any additive function from V into W is a linear transformation.

    Assume that V and W are vector spaces over $\mathbb{Q}$.

    WTS any additive function from V to W is a linear transformation.

    Let $c = \frac{a}{b}$ for some a, b $\in \mathbb{Z}, b \neq 0$.

    Let T be the transformation from V to W. Since T is additive function, T(x + y) = T(x) + T(y) for some x,y $\in V$

    Now we must prove linearity for scalars.

$$T(cx) = T(\frac{a}{b}x) = T(\frac{1}{b}ax) = T(\frac{1}{b}x + \frac{1}{b}x + ... + \frac{1}{b}x) \tag{9}$$

$$aT(\frac{1}{b}x) = \frac{a}{b}T(x) = cT(x) \tag{10}$$

    Hence, T is linear. □

12. (2.1 Problem 39) Let T: $C \to C$ be the function defined by T(z) = $\bar{z}$. Prove that T is additive but not linear.

    To show that T is additive but not linear we must check the two conditions for linear transformations.

    Let z = a + bi and y = c + di, for some a, b, c, d $\in$ F and x, y $\in \mathbb{C}$.

    T(z + y) = $\overline{(a + c) + (b + d)i}$ = (a + c) - (b + d)i

    T(z) + T(y) = $\overline{a + bi}$ + $\overline{c + di}$ = a - bi + c - di = (a + c) - (b + d)i

    Hence, T(z + y) = T(z) + T(y)

    Let x be a scalar in F. xT(z) = x(a - bi) = ax - bxi T(xz) = ax - bxi

    Proof by counterexample:
    Let x = ei for some e $\in F$ s.t. $e \neq 0$.
    T(xz) = T(ei(a + bi)) = T(aei - be) = -be - aei xT(z) = xT(a + bi) = ei(a - bi) = aei + be

    Since xT(z) $\neq$ T(xz) in this instance, T is not linear. □