1) $A = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$ Want: QR decomposition using Gram-Schimdt

$$q_1 = a_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \rightarrow e_1 = \frac{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}{\sqrt{1^2+1^2+0^2}} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$q_2 = a_2 - \text{proj}_{q_1}(a_2) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{\langle a_2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1\cdot 1 + 0\cdot 1 + 1\cdot 0}{\left(\sqrt{1^2+1^2+0^2}\right)^2} q_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$e_2 = \frac{q_2}{\|q_2\|} = \frac{(1/2, -1/2, 1)^T}{\sqrt{(1/2)^2 + (-1/2)^2 + 1}} = \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$$

$$q_3 = a_3 - \text{proj}_{q_1}(a_3) - \text{proj}_{q_2}(a_3)$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{q_1 \cdot a_3}{(\|q_1\|)^2} q_1 - \frac{q_2 \cdot a_3}{(\|q_2\|^2)} q_2$$

$$\text{proj}_{q_1}(a_3): \frac{1\cdot 0 + 1\cdot 1 + 0\cdot 1}{\left(\sqrt{0^2+1^2+1^2}\right)^2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$\text{proj}_{q_2}(a_3): \frac{1/2\cdot 0 + -1/2\cdot 1 + 1\cdot 1}{\left(\sqrt{(1/2)^2+(-1/2)^2+1^2}\right)^2}\begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} = \frac{1/2}{6/2}\begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/6 \\ -1/6 \\ 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/6 \\ -1/6 \\ 1/3 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$e_3 = \frac{a_3}{\|a_3\|} = \frac{(-2/3, 2/3, 2/3)^T}{\sqrt{3\left(\frac{2}{3}\right)^2}}$$

$$= \left(-\frac{\sqrt{3}}{3} \quad \frac{\sqrt{3}}{3} \quad \frac{\sqrt{3}}{3}\right)^T$$

$$Q = \begin{pmatrix} 1/\sqrt{2} & 1/6 & -\sqrt{3}/3 \\ 1/\sqrt{2} & -1/6 & \sqrt{3}/3 \\ 0 & 1/3 & \sqrt{3}/3 \end{pmatrix}$$

$$R = \begin{pmatrix} 2/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2\sqrt{3}/3 \end{pmatrix}$$

$$R = \begin{pmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & a_3 \cdot e_1 \\ 0 & a_2 \cdot e_2 & a_3 \cdot e_2 \\ 0 & 0 & a_3 \cdot e_3 \end{pmatrix}$$

$$a_3 \cdot e_1 = 0 \cdot \frac{1}{\sqrt{2}} + 1 \cdot \frac{1}{\sqrt{2}} + 1 \cdot 0$$

$$a_3 \cdot e_2 = 0 \cdot \frac{1}{\sqrt{6}} + 1 \cdot \frac{-1}{\sqrt{6}} + 1 \cdot \frac{2}{\sqrt{6}}$$

$$a_3 \cdot e_3 = 0 \cdot \frac{-\sqrt{3}}{3} + 1 \cdot \frac{\sqrt{3}}{3} + 1 \cdot \frac{\sqrt{3}}{3}$$

2.

a) $$A = \begin{array}{cc} a_1 & a_2 \\ \end{array}$$
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad q_1 = a_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{proj}_{q_1}(a_2) = \frac{a_2 \cdot q_1}{q_1 \cdot q_1} q_1 = \frac{2}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Method fails because the column vectors of A are linearly dependent.

b) $$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \|a_1\| = \sqrt{1+1} = \sqrt{2}$$

$$d = -\sqrt{2}$$

$$u = a_1 - d\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \sqrt{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}$$

$$H = I - \frac{2uu^T}{\|u\|^2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2\begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1+\sqrt{2} \end{bmatrix}}{\left(\sqrt{1^2+(1+\sqrt{2})^2}\right)^2}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2\begin{bmatrix} (1+\sqrt{2})^2 & 1+\sqrt{2} \\ 1+\sqrt{2} & 1 \end{bmatrix}}{4+\sqrt{2}}$$

$$H = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow \boxed{Q = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \quad R = \begin{bmatrix} -\sqrt{2} & -\sqrt{2} \\ 0 & 0 \end{bmatrix}}$$

$$\underset{Q}{\underset{\wr}{H}}A = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \underbrace{\frac{\sqrt{2}}{2}\begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix}}_{R}$$

3. $\{(x_i, y_i) \mid i = 1, 2, \ldots, m\}$ $m > 2$

$P(x) = be^{ax}$ $a, b = ?$

You cannot determine the values of a and b using the least-squares method. That's because P(x) will be exponentially-related which is something that the least-squares method would have trouble with.

5. a) Eigenvalues of A:

$-3.9616$, $0.3893$, $1.5492$, $6.1217$, $8.9044$

b) $k=2$

$$A_k = \begin{pmatrix} 7.4502 & 1.4775 & 0.0401 & 0.6901 & -0.0000 \\ 1.4775 & 7.3657 & -0.5067 & -1.1557 & 0.0575 \\ 0.0401 & -0.5067 & 0.6813 & -1.9703 & -0.0169 \\ 0.6901 & -1.557 & -1.9703 & -2.8876 & 0.0681 \\ 0 & 0.05 & -0.0169 & 0.0681 & 0.3904 \end{pmatrix}$$

$k=4$

$$A_k = \begin{pmatrix} 8.3964 & 1.0821 & -0.1536 & -0.0632 & -0.0000 \\ 1.0821 & 6.5961 & 0.5245 & 0.1903 & 0.0002 \\ -0.1536 & 0.5245 & -3.3392 & -1.6962 & 0.0003 \\ -0.0632 & 0.1903 & -1.6962 & 0.9573 & -0.0028 \\ 0 & 0.0002 & 0.0003 & -0.0028 & 0.3893 \end{pmatrix}$$

$k=6$

$$A_k = \begin{pmatrix} 8.7732 & 0.5903 & -0.0354 & -0.0022 & -0.0000 \\ 0.5903 & 6.2471 & 0.2393 & 0.0133 & 0.0000 \\ -0.0354 & 0.2393 & -3.9405 & -0.2899 & 0.0000 \\ -0.0022 & 0.0133 & -0.2899 & 1.5309 & -0.002 \\ 0 & 0.0000 & 0.0000 & -0.0002 & 0.3893 \end{pmatrix}$$

$k = 50$

$$A_k = \begin{pmatrix} 8.9044 & 0.0000 & -0.0000 & -0.0000 & -0.0000 \\ 0.0000 & 6.1217 & 0.0000 & 0.0000 & 0.0000 \\ -0.0000 & 0.0000 & -3.9616 & 0.0000 & 0.0000 \\ -0.0000 & 0.0000 & -0.0000 & 1.5462 & -0.0000 \\ 0 & 0.0000 & 0.0000 & -0.0000 & 0.3893 \end{pmatrix}$$

$k = 65$

$$A_k = \begin{pmatrix} 8.9044 & 0.0000 & -0.0000 & -0.0000 & 0.0000 \\ 0.0000 & 6.1217 & -0.0000 & 0.0000 & -0.0000 \\ 0.0000 & -0.0000 & -3.9616 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 0.0000 & 1.5462 & 0.0000 \\ 0 & -0.0000 & 0.0000 & 0.0000 & 0.3893 \end{pmatrix}$$