MATH1231 Assignment (Applied Mathematics Flavour)

Q1.

The function $\boldsymbol{T}: \mathbb{R}^{\mathbf{3}} \to \mathbb{R}^{\mathbf{2}}$ is defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3x_2 - 3x_3 \\ -4x_1 + 2x_3 \end{pmatrix} \text{ for all } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3.$$

Show that $\boldsymbol{T}$ is linear.

Answer:

A transformation in the function $\boldsymbol{T}$ from vector space $\boldsymbol{V}_1$ to $\boldsymbol{V}_{\mathbf{2}}$ is only a linear transformation if it satisfies the vector addition condition

$$T(v_1 + v_2) = T(v_1) + T(v_2) \text{ where } \mathrm{v}_1, v_2 \in \mathbb{R}^3$$

and the vector scalar multiplication condition

$$T(\alpha v_1) = \alpha\, T(v_1) \text{ where } \alpha \in \mathbb{R} \text{ and } \mathrm{v}_1 \in \mathbb{R}^3$$

To show that $\boldsymbol{T}$ is linear, we will show that it satisfies both the vector addition and scalar multiplication conditions.

To show that $\boldsymbol{T}$ preserves vector addition, let $v_1, v_2 \in \mathbb{R}^3$ where $v_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $v_2 = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

where:

$$T(v_1) = \begin{pmatrix} -3x_2 - 3x_3 \\ -4x_1 + 2x_3 \end{pmatrix},$$

and

$$T(v_1) = \begin{pmatrix} -3y_2 - 3y_3 \\ -4y_1 + 2y_3 \end{pmatrix}.$$

For the function $T(v_1 + v_2)$, we can substitute the according vectors v_1 and v_2, leaving us with,

$$\begin{aligned} T(v_1 + v_2) &= T\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}, \\ &= \begin{pmatrix} -3(x_2 + y_2) - 3(x_3 + y_3) \\ -4(x_1 + y_1) + 2(x_3 + y_3) \end{pmatrix}. \end{aligned}$$

For $T(v_1) + T(v_2)$, we substitute the vectors v_1 and v_2 to get

$$\begin{aligned} T(v_1) &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \\ &= \begin{pmatrix} -3x_2 - 3x_3 \\ -4x_1 + 2x_3 \end{pmatrix}. \end{aligned}$$

and

$$\begin{aligned} T(v_2) &= \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \\ &= \begin{pmatrix} -3y_2 - 3y_3 \\ -4y_1 + 2y_3 \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} T(v_1) + T(v_2) &= \begin{pmatrix} -3x_2 - 3x_3 \\ -4x_1 + 2x_3 \end{pmatrix} + \begin{pmatrix} -3y_2 - 3y_3 \\ -4y_1 + 2y_3 \end{pmatrix}, \\ &= \begin{pmatrix} -3x_2 - 3x_3 - 3y_2 - 3y_3 \\ -4x_1 + 2x_3 - 4y_1 + 2y_3 \end{pmatrix}, \\ &= \begin{pmatrix} -3(x_2 + y_2) - 3(x_3 + y_3) \\ -4(x_1 + y_1) + 2(x_3 + y_3) \end{pmatrix}. \end{aligned}$$

Hence, $T(v_1 + v_2) = T(v_1) + T(v_2)$, showing that T satisfies the vector addition condition.

To show that $\boldsymbol{T}$ preserves vector scalar multiplication, let α be any arbitrary scalar value where $\alpha \in \mathbb{R}$. By also using the vector v_1 defined earlier we can multiply it with the scalar α giving us

$$\begin{aligned} T(\alpha v_1) &= T\left(\alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right), \\ &= T \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix}, \\ &= \begin{pmatrix} -3\alpha x_2 - 3\alpha x_3 \\ -4\alpha x_1 + 2\alpha x_3 \end{pmatrix}. \end{aligned}$$

Multiplying the scalar α with the entire function T gives us,

$$\begin{aligned} \alpha\, T(v_1) &= \alpha\, T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \\ &= \alpha \begin{pmatrix} -3x_2 - 3x_3 \\ -4x_1 + 2x_3 \end{pmatrix}, \\ &= \begin{pmatrix} -3\alpha x_2 - 3\alpha x_3 \\ -4\alpha x_1 + 2\alpha x_3 \end{pmatrix}. \end{aligned}$$

Therefore, $T(\alpha v_1) = \alpha\, T(v_1)$.

Thus, we can show that T is linear as T preserves the vector addition condition and preserves the scalar multiplication condition.

Q2.

Show that

$$S = \{x \in \mathbb{R}^4 : 4x_1 + 4x_2 - 9x_3 + 6x_4 = 0\}$$

is a subspace of $\mathbb{R}^4$.

Answer:

To show that $\boldsymbol{S}$ is a subspace of $\mathbb{R}^4$, we must first show that $\boldsymbol{S}$ is a vector space of its own.

To do this, we must prove that $\boldsymbol{S}$ satisfies the axiom of the existence of a zero vector, the axiom of closure under scalar multiplication, and the axiom of closure under addition.

I) To test if $\boldsymbol{S}$ satisfies the **axiom of the existence of a zero vector**, consider the zero vector

$$v_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then,

$$4(0) + 4(0) - 9(0) + 6(0).$$

Since

$$4(0) + 4(0) - 9(0) + 6(0) = 0,$$

$v_0 \in \boldsymbol{S}$ and therefore $\boldsymbol{S}$ satisfies the axiom of the existence of a zero vector.

II) To test if $\boldsymbol{S}$ satisfies the **axiom of closure under scalar multiplication**, let $\boldsymbol{\alpha}$ be a scalar multiple where $\boldsymbol{\alpha} \in \mathbb{R}$ and $\boldsymbol{v_1} \in \boldsymbol{S}$ where:

$$v_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \text{then } 4\mathrm{x}_1 + 4x_2 - 9x_3 + 6x_4 = 0.$$

Multiplying the vector v_1 by α gives us

$$\alpha v_1 = \alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \\ \alpha x_4 \end{pmatrix}.$$

Such that

$$\begin{aligned} \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \\ \alpha x_4 \end{pmatrix} &= 4\alpha x_1 + 4\alpha x_2 - 9\alpha x_3 + 6\alpha x_4, \\ &= \alpha(4x_1 + 4x_2 - 9x_3 + 6x_4), \\ &= \alpha \cdot 0, \\ &= 0. \end{aligned}$$

Therefore, $\boldsymbol{S}$ satisfies the axiom of closure under scalar multiplication.

III) To test if $\boldsymbol{S}$ satisfies the **axiom of closure under addition**, let $v_1, v_2 \in \boldsymbol{S}$ where:

$$v_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{and } v_2 = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

Then

$$4x_1 + 4x_2 - 9x_3 + 6x_4 = 0$$

and

$$4y_1 + 4y_2 - 9y_3 + 6y_4 = 0.$$

Consider $v_1 + v_2$ where:

$$\begin{aligned} v_1 + v_2 &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \\ &= \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{pmatrix}, \\ &= 4(x_1 + y_1) + 4(x_2 + y_2) - 9(x_3 + y_3) + 6(x_4 + y_4), \\ &= 4x_1 + 4y_1 + 4x_2 + 4y_2 - 9x_3 - 9y_3 + 6x_4 + 6y_4, \\ &= (4x_1 + 4x_2 - 9x_3 + 6x_4) + (4y_1 + 4y_2 - 9y_3 + 6y_4), \\ &= 0 + 0, \\ &= 0. \end{aligned}$$

Therefore, $\boldsymbol{S}$ satisfies the axiom of closure under addition.

Since $\boldsymbol{S}$ satisfies the axiom of the existence of a zero vector, the axiom of closure under scalar multiplication, and the axiom of closure under addition, $\boldsymbol{S}$ is a subspace of $\mathbb{R}^4$.

Q3.

Let $\mathbb{P}_n$ be the set of real polynomials of degree at most n. Show that

$$S = \{p \in \mathbb{P}_5 : p(6) = p(4)\}$$

is a subspace of $\mathbb{P}_5$.

Answer:

To show that $\boldsymbol{S}$ is a subset of $\mathbb{P}_5$, it must satisfy the three main axioms including the existence of a zero vector, closure under scalar multiplication, and closure under addition.

I) To test if $\boldsymbol{S}$ satisfies the axiom of the existence of a zero vector, let the polynomial $z(x) = 0$ be the zero polynomial.

Thus, for any x value,

$$P(x) = P(6) = P(4) = 0.$$

Therefore, $\boldsymbol{S}$ satisfies the axiom of the existence of a zero vector.

II) To test if $\boldsymbol{S}$ satisfies the axiom of closure under scalar multiplication, let α be a scalar multiple where $\alpha \in \boldsymbol{R}$ and suppose that P is an arbitrary polynomial where $P(x) = a_1x^5$.

Multiplying the scalar multiple and the polynomial P gives us,

$$\begin{aligned} \alpha P(x) &= \alpha(a_1x^5), \\ &= x^5(\alpha a_1). \end{aligned}$$

Hence, $\boldsymbol{S}$ satisfies the axiom of closure under scalar multiplication as $x^5(\alpha \cdot a_1) \in \mathbb{P}_5$.

III) To test if $\boldsymbol{S}$ satisfies the axiom of closure under addition, let P_1 and P_2 be two arbitrary polynomials such that $P_1, P_2 \in \mathbf{S}$, where:

$$P_1(x) = a_1x^5$$

and

$$P_2(x) = a_2x^5.$$

Adding these two polynomials gives us

$$\begin{aligned} P_1(x) + P_2(x) &= a_1x^5 + a_2x^5, \\ &= x^5(a_1 + a_2). \end{aligned}$$

Hence, $\boldsymbol{S}$ satisfies the axiom of closure under addition as $x^5(a_1 + a_2) \in \mathbb{P}_5$.

Therefore, from the proof in I, II and III, $\boldsymbol{S}$ is a subspace of $\mathbb{P}_5$ as $\boldsymbol{S}$ satisfies the three main axioms including the existence of a zero vector, closure under scalar multiplication, and closure under addition

Q4.

The air in a **80** cubic meter kitchen is initially clean, but when Laure burns her toast while making breakfast, smoke mixed with the room's air at a rate of **0.03** mg per second. An air conditioning system exchanges the mixture of air and smoke with air at a rate of **9** cubic metres per minute. Assume that the pollutant is mixed uniformly throughout the room and that burnt toast is taken outside after **72** seconds. Let $\boldsymbol{S(t)}$ be the amount of smoke in mg in the room at a time $\boldsymbol{t}$ (in seconds) after the toast first began to burn.

a) Find a differential equation obeyed by $\boldsymbol{S(t)}$
b) Find $\boldsymbol{S(t)}$ for $\boldsymbol{0 \leq t \leq 72}$ by solving the differential equation in (a) with an appropriate initial condition.
c) What is the level of pollution in mg per cubic meter after **72** seconds?
d) How long does it take for the level of pollution to fall to **0.008** mg per cubic meter after the toast is taken outside?

Answer:

a) To find a differential equation obeyed by $S(t)$, we must consider the rate of change of the amount of smoke in the room's air.

Firstly, let

$$\begin{aligned}\frac{\mathrm{d}S(\mathrm{t})}{dt} &= (\text{amount of smoke in}) - (\text{amount of smoke out}),\\ &= 0.03 - \frac{S(\mathrm{t})}{80}\cdot\frac{9}{60},\\ &= 0.03 - \frac{S(\mathrm{t})\cdot 3}{1600}.\end{aligned}$$

Thus, the differential equation obeyed by $S(t)$ is

$$\frac{\mathrm{d}S(\mathrm{t})}{dt} = 0.03 - \frac{S(\mathrm{t})\cdot 3}{1600}.$$

b) To solve the differential equation, we can rearrange our first order linear ODE in the form of

$$\frac{dy}{dx} + f(\mathrm{x})\cdot \mathrm{y} = g(\mathrm{x}).$$

This gives us,

$$\frac{\mathrm{d}S(\mathrm{t})}{dt} + \frac{S(\mathrm{t})\cdot 3}{1600} = 0.03.$$

Consider the integrating factor $e^{\int f(x)dx}$. Substituting our $f(x)$ and resolving the integration gives us

$$e^{\int \frac{3}{1600}dt} = e^{\frac{3t}{1600}}.$$

We then multiply the differential equation by the integrating factor, giving us:

$$e^{\frac{3t}{1600}}\cdot\frac{\mathrm{d}S(\mathrm{t})}{dt} + e^{\frac{3t}{1600}}\cdot\frac{S(\mathrm{t})\cdot 3}{1600} = 0.03\cdot e^{\frac{3t}{1600}}.$$

Using the product rule of differentiation, we get:

$$\frac{d}{dt}\left(e^{\frac{3t}{1600}} \cdot S(t)\right) = 0.03 \cdot e^{\frac{3t}{1600}}.$$

Since our goal is to solve for $S(t)$, we must eliminate $\frac{d}{dt}$ by integrating both sides, giving us:

$$\int \frac{d}{dt}\left(e^{\frac{3t}{1600}} \cdot S(t)\right) dt = \int 0.03 \cdot e^{\frac{3t}{1600}}\, dt,$$

$$e^{\frac{3t}{1600}} \cdot S(t) = 16 \cdot e^{\frac{3t}{1600}} + C \text{ (where } C \in \mathbb{R}).$$

Dividing both sides by $e^{\frac{3t}{1600}}$ gives us:

$$S(t) = 16 + C \cdot e^{-\frac{3t}{1600}}.$$

Since there is initially no smoke in the air, we can conclude that $S(t) = 0$ and $t = 0$. Substituting these values into the equation allows us to find C.

$$S(t) = 16 + C \cdot e^{-\frac{3t}{1600}},$$
$$0 = 16 + C \cdot e^{-\frac{3(0)}{1600}},$$
$$C = -16.$$

Substituting C gives us:

$$S(t) = 16 - 16 \cdot e^{-\frac{3t}{1600}} \text{ (for } 0 \leq \text{t} \leq 72).$$

c) The level of pollution after 72 seconds can be found by substituting t as 72 into $S(t)$ and dividing it by the total volume of the room, 80 m^3.

Substituting t as 72 into $S(t)$ gives us:

$$S(72) = 16 - 16 \cdot e^{-\frac{3(72)}{1600}},$$
$$\approx 2.020545413\; mg \;(10\; s.f.).$$

Dividing this value by the total volume of the room, 80 m^3, gives us

$$\frac{2.020545413}{80} \approx 0.02525681766\; \frac{mg}{m^3}\; (10\; s.f.).$$

d) When the toast is taken outside of the kitchen, there is no longer smoke production, and we require the use of a new differential equation as the previous only considers $0 \leq t \leq 72$. This can be denoted by the equation

$$\begin{aligned}\frac{\mathrm{d}S(\mathrm{t})}{dt} &= -(\text{amount of smoke out}),\\ &= -\frac{S(t)\cdot 3}{1600}.\end{aligned}$$

Therefore,

$$\frac{d * S(t)}{S(t)} = -\frac{3}{1600}dt.$$

By first rearranging the elements of this equation and once again integrating both sides of the equation, we can determine the value of $\boldsymbol{S(t)}$, giving us:

$$\begin{aligned}\frac{d * S(t)}{S(t)} &= -\frac{3}{1600}\cdot dt,\\ dt &= -\frac{d\cdot S(t)\cdot 1600}{S(t)\cdot 3},\\ \int dt &= \int -\frac{1600}{S(t)\cdot 3}\cdot d\cdot S(t),\\ t &= -\frac{1600}{3}\cdot \ln\big(S(t)\big) + C \text{ (where } C \in \mathbb{R}\text{)..}\end{aligned}$$

We can determine the constant C by substituting $S(t) = 0.02525681766\frac{mg}{m^3}$ $\big(\text{from part (c)}\big)$ and $t = 0$, giving us:

$$\begin{aligned}0 &= -\frac{1600}{3}\cdot \ln(0.02525681766) + C,\\ C &= \frac{1600}{3}\cdot \ln(0.02525681766).\end{aligned}$$

Therefore, the full equation of is

$$t = -\frac{1600}{3}\cdot \ln\big(S(t)\big) + \frac{1600}{3}\cdot \ln(0.02525681766).$$

Then, we can find the amount of time it takes for the level of pollution to fall to $0.008\ \frac{mg}{m^3}$ by substituting $S(t) = 0.008$, resulting in:

$$\begin{aligned}t &= -\frac{1600}{3}\cdot \ln(0.008) + \frac{1600}{3}\cdot \ln(0.02525681766),\\ &= 613.1491114 \text{ seconds } (10\ s.f.),\\ &\approx 613 \text{ seconds.}\end{aligned}$$

Hence, it takes approximately 613 seconds for the pollution levels in to fall to $0.008\ \frac{mg}{m^3}$.