Partial Differential Equations

Classification of PDEs

General Second-Order Linear PDE:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

Classification:

- **Hyperbolic:** $B^2 - 4AC > 0$ (e.g., wave equation)
- **Parabolic:** $B^2 - 4AC = 0$ (e.g., heat equation)
- **Elliptic:** $B^2 - 4AC < 0$ (e.g., Laplace equation)

Three Fundamental PDEs

Wave Equation (Hyperbolic)

1D Wave Equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

where c is the wave speed

2D Wave Equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = c^2\nabla^2 u$$

3D Wave Equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$$

Heat (Diffusion) Equation (Parabolic)

1D Heat Equation:

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$$

where $\alpha = k/(\rho c_p)$ is the thermal diffusivity

2D Heat Equation:

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \alpha \nabla^2 u$$

3D Heat Equation:

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u$$

Laplace Equation (Elliptic)

2D Laplace Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\nabla^2 u = 0$$

3D Laplace Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Poisson Equation:

$$\nabla^2 u = f(x, y, z)$$

Solutions to Laplace equation are called *harmonic functions*.

Method of Separation of Variables

General Approach:

1. Assume solution has the form $u(x,t) = X(x)T(t)$ (or similar)
2. Substitute into PDE
3. Separate variables to get two ODEs
4. Solve each ODE with appropriate boundary conditions
5. Apply superposition principle to satisfy initial conditions

Heat Equation on Finite Domain

For: $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$, $0 < x < L$, $t > 0$

Boundary Conditions (Fixed Ends):

$$u(0,t) = 0, \quad u(L,t) = 0$$

Initial Condition:

$$u(x,0) = f(x)$$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\alpha(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

where

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Boundary Conditions (Insulated Ends):

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0$$

Solution:

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-\alpha(n\pi/L)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

where

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Wave Equation on Finite Domain

For: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, $0 < x < L$, $t > 0$

Boundary Conditions:

$$u(0,t) = 0, \quad u(L,t) = 0$$

Initial Conditions:

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x)$$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

where

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

D'Alembert's Solution (Wave Equation)

For infinite domain: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$

Initial Conditions:

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x)$$

D'Alembert's Solution:

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds$$

For zero initial velocity ($g(x) = 0$):

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2}$$

This represents two waves traveling in opposite directions at speed c.

Laplace Equation Solutions

Rectangular Domain

For: $\nabla^2 u = 0$ on rectangle $0 < x < a$, $0 < y < b$

Example Boundary Conditions:

$$u(0,y) = 0, \quad u(a,y) = 0, \quad u(x,0) = 0, \quad u(x,b) = f(x)$$

Solution:

$$u(x,y) = \sum_{n=1}^{\infty} B_n \frac{\sinh(n\pi y/a)}{\sinh(n\pi b/a)} \sin\left(\frac{n\pi x}{a}\right)$$

where

$$B_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

Circular Domain (Polar Coordinates)

For: $\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

On disk $r < R$ with boundary condition $u(R,\theta) = f(\theta)$:

Solution:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n [a_n \cos(n\theta) + b_n \sin(n\theta)]$$

where

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta)\, d\theta$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta)\, d\theta$$

2D Wave Equation (Rectangular Membrane)

For: $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

On rectangle $0 < x < a$, $0 < y < b$ with fixed boundaries:

Solution:

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$
$$\times \left[A_{mn} \cos(\omega_{mn} t) + B_{mn} \sin(\omega_{mn} t)\right]$$

where the frequencies are:

$$\omega_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

Heat Equation in Infinite Rod

For: $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$

Initial Condition: $u(x, 0) = f(x)$

Solution (using Fourier Transform):

$$u(x, t) = \frac{1}{\sqrt{4\pi \alpha t}} \int_{-\infty}^{\infty} f(\xi) e^{-(x-\xi)^2/(4\alpha t)}\, d\xi$$

This is a convolution with the heat kernel (fundamental solution):

$$K(x, t) = \frac{1}{\sqrt{4\pi \alpha t}} e^{-x^2/(4\alpha t)}$$

Sturm-Liouville Problems

General Form:

$$\frac{d}{dx}\left[p(x) \frac{dy}{dx}\right] + [q(x) + \lambda r(x)] y = 0$$

with boundary conditions at endpoints

Properties:

- Eigenvalues λ_n are real
- Eigenfunctions $y_n(x)$ are orthogonal with weight function $r(x)$:

$$\int_a^b y_m(x)y_n(x)r(x)\,dx = 0 \quad (m \neq n)$$

- Eigenfunctions form a complete basis

Common Examples:

For heat/wave equation with fixed ends:

$$\frac{d^2X}{dx^2} + \lambda X = 0, \quad X(0) = X(L) = 0$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

For heat/wave equation with insulated ends:

$$\frac{d^2X}{dx^2} + \lambda X = 0, \quad X'(0) = X'(L) = 0$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$

Maximum Principle

For Heat Equation:

The maximum and minimum values of $u(x,t)$ in a domain occur either:

- At the initial time $t = 0$, or
- On the spatial boundary

For Laplace Equation:

Harmonic functions achieve their maximum and minimum values on the boundary of the domain (no local extrema in the interior).

Uniqueness and Well-Posedness

Well-Posed Problem (Hadamard):

1. Solution exists

2. Solution is unique
3. Solution depends continuously on initial/boundary data

Heat Equation: Well-posed with initial condition and boundary conditions

Wave Equation: Well-posed with initial position, initial velocity, and boundary conditions

Laplace Equation: Well-posed with boundary conditions (Dirichlet, Neumann, or mixed)

Boundary Conditions

Dirichlet BC: Specifies value of u on boundary

$$u|_{\partial\Omega} = f$$

Neumann BC: Specifies normal derivative on boundary

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = g$$

Robin (Mixed) BC: Linear combination

$$\alpha u + \beta \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = h$$

Periodic BC:

$$u(0,t) = u(L,t), \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t)$$

Bessel Functions

Arise in problems with cylindrical symmetry.

Bessel's Equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

General Solution:

$$y(x) = c_1 J_n(x) + c_2 Y_n(x)$$

where $J_n(x)$ is Bessel function of the first kind and $Y_n(x)$ is Bessel function of the second kind.

Properties:

- $J_n(0) = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}$

- $Y_n(x) \to -\infty$ as $x \to 0^+$
- $J_n(x)$ and $Y_n(x)$ oscillate with decreasing amplitude as $x \to \infty$

Orthogonality:

$$\int_0^a x J_n(\alpha_m x) J_n(\alpha_k x)\, dx = 0 \quad (m \neq k)$$

where α_m and α_k are zeros of J_n.

Legendre Polynomials

Arise in problems with spherical symmetry.

Legendre's Equation:

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

Solutions: Legendre polynomials $P_n(x)$ for integer n

First Few Legendre Polynomials:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Orthogonality:

$$\int_{-1}^{1} P_m(x)P_n(x)\, dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}$$

Green's Functions

Solution to PDE can be expressed using Green's function $G(\mathbf{x}, \mathbf{x}_0)$:

For Poisson equation $\nabla^2 u = f$:

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}_0)\, d\mathbf{x}_0 + \text{boundary terms}$$

Green's Function satisfies:

$$\nabla^2 G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0)$$

3D Free Space:

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|}$$

Useful Formulas

Energy Method (for uniqueness):

Define energy: $E(t) = \frac{1}{2} \int_\Omega u^2 \, dx$

Show $\frac{dE}{dt} \leq 0$ to prove uniqueness.

Separation Constant:

When separating variables, if spatial part gives negative eigenvalues, use $-\lambda$ to get:

$$X'' + \lambda X = 0$$

This gives sinusoidal solutions for $\lambda > 0$.