Fluid Mechanics Sample Problem 1

This is problem 1.3 on the 2023 QE.

An incompressible, Newtonian, viscous fluid under a favorable pressure gradient is placed between two large parallel plates a distance h apart. Starting from Navier-Stokes, derive the volumetric flowrate through the plates per unit width.

The Navier-Stokes equation for steady, incompressible flow in the x-direction is given by:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

For flow between two large parallel plates, we can make the following assumptions:

- Steady flow: $\frac{\partial u}{\partial t}=0$
- Fully developed flow: $\frac{\partial u}{\partial x}=0$
- No velocity in the y-direction: $v=0$
- Constant pressure gradient: $\frac{\partial p}{\partial x}=-G$ (where G is a constant)
- No-slip boundary conditions at the plates: $u(0)=0$ and $u(h)=0$

With these assumptions, the Navier-Stokes equation simplifies to:

$$0=G+\mu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

Integrating this equation twice with respect to y gives:

$$\frac{\partial^2 u}{\partial y^2}=-\frac{G}{\mu}\implies\frac{\partial u}{\partial y}=-\frac{G}{\mu}y+C_1\implies u(y)=-\frac{G}{2\mu}y^2+C_1y+C_2 \tag{3}$$

Applying the no-slip boundary conditions:

$$u(0)=0\implies C_2=0 \tag{4}$$

$$u(h)=0\implies-\frac{G}{2\mu}h^2+C_1h=0\implies C_1=\frac{Gh}{2\mu} \tag{5}$$

Thus, the velocity profile is:

$$u(y)=\frac{G}{2\mu}\left(hy-y^2\right) \tag{6}$$

The volumetric flowrate per unit width (Q) is obtained by integrating the velocity profile across the gap between the plates:

$$Q=\int_0^h u(y)\,dy=\int_0^h\frac{G}{2\mu}\left(hy-y^2\right)dy \tag{7}$$

Calculating the integral:

$$Q = \frac{G}{2\mu}\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h = \frac{G}{2\mu}\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = \frac{Gh^3}{12\mu} \tag{8}$$

Thus, the volumetric flowrate through the plates per unit width is:

$$\boxed{Q = \frac{Gh^3}{12\mu}} \tag{9}$$