Study Sheet

Based off example problems from 21 and 23

January 17, 2026

1 Vector Calculus

Observed Methods

- Work along a path
- Stokes theorem
- Divergence theorem
- Cylindrical bounds of integration
- Volume of a sphere

Line/Work Integral

Generally given a force field $F(x, y)$ and a curve through space $\mathcal{C}$, for which we want to compute

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$$

Which is the **work** of the vector field along the path. To simplify the integration,

$$d\vec{r}\frac{dt}{dt} \to \frac{d\vec{r}}{dt}dt \to \vec{v}dt$$

Before integrating, both F and $\mathcal{C}$ need to be *parameterized* in terms of t. Any definition for x works, but the algebra can be difficult. So the simplest method may be

$$\mathcal{C}(x, y) = r(t) \begin{cases} x \equiv t \\ y \equiv \mathcal{C}(x = t) \end{cases}$$

Plug these values into the vector field F.

$$F(x, y) = F(t) \begin{cases} x \equiv t \\ y \equiv \mathcal{C}(x = t) \end{cases}$$

Then the last step before integration is to take the derivative of $r(t)$ to get $v(t)$. Finally, the **line/work** integral can be computed via

$$\boxed{\int_{\mathcal{C}} \vec{F}(x, y) \cdot d\vec{r}(x, y) = \int_a^b F(t)v(t)dr(t)}$$

Divergence Theorem

The divergence of a vector field, $\nabla \cdot \vec{F}$, integrated over a volume, V is equal to the dot product between the vector field and the surface of the volume, $\vec{F} \cdot \vec{n}$ ($\vec{n}$ is the surface normal vector), integrated over the surface area of the volume, A.

$$\boxed{\int_V \nabla \cdot \vec{F} dV = \int_S \vec{F} \cdot \vec{n} dA}$$

Solving the **RHS** amounts to finding the surface-normal vector $\vec{n}$. The general method is to fist parameterize the given geometries defining the boundary of the volume.

$$S = f(x, y, z) \rightarrow z = f(x, y)$$
$$S = [f(u), f(v), f(u, v)] \begin{cases} x \equiv f(u) \\ y \equiv f(v) \end{cases}$$

With the surface parameterized, take the cross product between the two respective partial derivatives

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial S}{\partial u}\big|_{\hat{i}} & \frac{\partial S}{\partial u}\big|_{\hat{j}} & \frac{\partial S}{\partial u}\big|_{\hat{k}} \\ \frac{\partial S}{\partial v}\big|_{\hat{i}} & \frac{\partial S}{\partial v}\big|_{\hat{j}} & \frac{\partial S}{\partial v}\big|_{\hat{k}} \end{vmatrix} = \vec{n}$$

Take note that the direction of $\vec{n}$ needs to *face out of the surface*, and the cross product may have it facing the wrong direction.

2 Linear Algebra

Observed Methods

- Solutions to systems of ODE
- Matrix power
- Inverse
- adjudicate
- Co-factor
- Eig-stuff

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3 Laplace Transforms

Observed Methods

- Convolution transform
- Derivative transform
- Partial fractions
- Integration by parts

Convolution Theorem

The convolution of two functions is given by the integral

$$(f \circ g)(t) = h(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

The Laplace transform of a convolution is

$$\boxed{\mathcal{L}[h(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)] = FG}$$

The inverse of this theorem can be used if a function in Laplace space, H, can be observed as a product of two functions, F and G, for which the transforms are known. The inverse would be

$$H(s) = F(s)G(s)$$
$$\mathcal{L}[H] = \mathcal{L}[FG] = \int_0^t f(\tau)g(t-\tau)d\tau = h(t)$$

Derivatives of Laplace

Consider the n^{th} order differential

$$x^n = \frac{d^n x}{dt^n}$$

Taking the Laplace transform would give

$$\boxed{\mathcal{L}[x^n] = s^n X - s^{n-1}x_0 - s^{n-1}\dot{x}_0 - ... - sx_0^{n-2} - x_0^{n-1}}$$

where X is the Laplace transform of x, and each x_0^j is the initial condition for each derivative of x. The third order example would be

$$\mathcal{L}\,[\,\dddot{x}\,] = s^3X - s^2x_0 - s\dot{x}_0 - \ddot{x}_0$$

4 Fourier Transforms

Observed Methods

- Cosine Series
- Complex Transform
- Integration by parts
- General Series
- odd / even function cancellations

General Series Expansion

A function that is piecewise smooth, and has a period length $p = 2L$, can be approximated by a series of sines and cosines

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$
$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$$

Orthogonality of sin and cos

The product of trig functions are orthogonal on the interval $-\pi \leq x \leq \pi$, specifically:

$$\begin{cases} \int_{-\pi}^{\pi} \cos nx \cos mx \, dx & (n \neq m) \\ \int_{-\pi}^{\pi} \sin nx \sin mx \, dx & (n \neq m) \\ \int_{-\pi}^{\pi} \sin nx \cos mx \, dx & (\forall n, m) \end{cases}$$

Use this to reduce the number of coefficients that you need to compute in the Fourier Series, when $f(x)$ has trig terms

Transforms

Complex Transform

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dx$$

Sine Transform

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin wx \, dx$$
$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}(w) \sin wx \, dw$$

Cosine Transform

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos wx \, dx$$
$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}(w) \cos wx \, dw$$

5 Ordinary Diff. Eq.s

Observed Methods

- Undetermined coefficients
- Cauchy-Euler Eq
- Log properties

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6 Partial Diff. Eq.s

Observed Methods

- Infinite Rode (Heat Eq.)
- String (Wave Eq.)
- Separation of Variables

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7 General Math Techniques

Observed Methods

- Cross product
- Integration by Parts
- Partial Fractions
- Completing the square
- Quadratic equation
