Laplace Transforms

Definition

The Laplace Transform and its inverse: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$

$\mathcal{L}^{-1}\{F(s)\} = f(t)$

Common Laplace Transforms

Function $f(t)$	**Transform** $F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
te^{at}	$\frac{1}{(s-a)^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t)$	1
$\delta(t-a)$	e^{-as}

Properties of Laplace Transforms

Linearity: $\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$

First Derivative: $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

Second Derivative: $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

n**-th Derivative:** $\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$

Integration: $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$

Time Shift (Second Shifting Theorem): $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s), \quad a \geq 0$ $\mathcal{L}\{g(t)u(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$

Frequency Shift (s-Shift Theorem): $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

Scaling: $\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0$

Multiplication by t**:** $\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\{t^nf(t)\} = (-1)^nF^{(n)}(s)$

Division by t**:** $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$

Convolution Theorem

Convolution of two functions: $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t f(t-\tau)g(\tau)\,d\tau$

Convolution Theorem: $\mathcal{L}\{(f * g)(t)\} = F(s) \cdot G(s)$

Inverse form: $\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = (f * g)(t)$

Initial and Final Value Theorems

Initial Value Theorem: $\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$ (provided the limit exists)

Final Value Theorem: $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$ (provided the limit exists and $f(t)$ has a final value; all poles of $sF(s)$ must have negative real parts except possibly a simple pole at $s = 0$)

Inverse Laplace Transforms

Partial Fraction Decomposition:

For a rational function $\dfrac{P(s)}{Q(s)}$ where $\deg(P) < \deg(Q)$:

Case 1: Simple Real Roots

If $Q(s) = (s-a_1)(s-a_2)\cdots(s-a_n)$ with distinct roots: $\frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_n}{s-a_n}$

Case 2: Repeated Real Roots

If $Q(s)$ has $(s-a)^n$ as a factor: Include terms: $\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}$

Case 3: Complex Conjugate Roots

If $Q(s)$ has a factor $(s-\alpha)^2 + \beta^2$: Include term: $\frac{As+B}{(s-\alpha)^2+\beta^2}$

Heaviside Cover-Up Method:

For a simple root at $s = a$, the coefficient A is: $A = \lim_{s\to a}(s-a)\frac{P(s)}{Q(s)}$

Or equivalently, substitute $s = a$ into $\dfrac{P(s)}{Q(s)}$ after "covering up" the factor $(s-a)$ in $Q(s)$.

Solving ODEs with Laplace Transforms

General Procedure:

1. Take the Laplace transform of both sides of the ODE
2. Apply the derivative properties using the initial conditions
3. Solve the resulting algebraic equation for $Y(s)$
4. Use partial fractions to decompose $Y(s)$ if necessary
5. Take the inverse Laplace transform to obtain $y(t)$

Example for second-order ODE:

Given: $y'' + ay' + by = f(t)$ with initial conditions $y(0)$ and $y'(0)$

Taking Laplace transforms: $s^2Y(s) - sy(0) - y'(0) + a[sY(s) - y(0)] + bY(s) = F(s)$

Solving for $Y(s)$: $Y(s) = \frac{F(s)+sy(0)+y'(0)+ay(0)}{s^2+as+b}$

Special Functions

Unit Step Function (Heaviside Function): $u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$

Used to "turn on" functions at time $t = a$

Dirac Delta Function (Unit Impulse): $\delta(t-a) = 0$ for all $t \neq a$ $\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$

Sifting Property: $\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a)$

In ODEs, $\delta(t-a)$ represents an instantaneous impulse at $t = a$

Solving Integral Equations

For equations of the form: $y(t) = f(t) + \int_0^t K(t-\tau)y(\tau)\,d\tau$

Solution Method:

1. Take Laplace transform of both sides
2. Use convolution theorem: $\mathcal{L}\{\text{integral term}\} = K(s)Y(s)$
3. Obtain: $Y(s) = F(s) + K(s)Y(s)$
4. Solve algebraically: $Y(s) = \dfrac{F(s)}{1-K(s)}$
5. Take inverse Laplace to find $y(t)$

Useful Trigonometric Identities

$\sin^2(\omega t) = \frac{1-\cos(2\omega t)}{2}$

$\cos^2(\omega t) = \frac{1+\cos(2\omega t)}{2}$

$\sin(\omega t)\cos(\omega t) = \frac{\sin(2\omega t)}{2}$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

Common Partial Fraction Results

$\frac{1}{s(s+a)} = \frac{1}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)$

$\frac{1}{(s+a)(s+b)} = \frac{1}{b-a}\left(\frac{1}{s+a} - \frac{1}{s+b}\right), \quad a \neq b$

$\frac{s}{s^2+\omega^2} \quad \xrightarrow{\mathcal{L}^{-1}} \quad \cos(\omega t)$

$\frac{\omega}{s^2+\omega^2} \quad \xrightarrow{\mathcal{L}^{-1}} \quad \sin(\omega t)$

$\frac{1}{(s+a)^2} \quad \xrightarrow{\mathcal{L}^{-1}} \quad te^{-at}$

$\frac{1}{s^2(s+a)} = \frac{1}{a^2}\left(\frac{1}{s} - \frac{1}{s+a}\right) - \frac{1}{a}\left(\frac{1}{s^2}\right)$