Vibrations

Modeling and Equations of Motion

Single degree of freedom (SDOF):
$$m\ddot{x} + c\dot{x} + kx = f(t)$$

Rotational form:
$$I\ddot{\theta} + c_\theta \dot{\theta} + k_\theta \theta = M(t)$$

Equivalent parameters:
$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2\sqrt{km}}$$

—

Free Vibration

Undamped $(c = 0)$:
$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

Damped $(0 < \zeta < 1)$:
$$x(t) = e^{-\zeta \omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right)$$
$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Critical damping:
$$c_c = 2\sqrt{km}$$

Logarithmic decrement:
$$\delta = \frac{1}{n}\ln\left(\frac{x(t)}{x(t + nT_d)}\right) = \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}}$$

—

Forced Vibration (Harmonic)

Harmonic force:
$$f(t) = F_0\cos\omega t$$

Steady-state response amplitude:
$$X(\omega) = \frac{F_0/k}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \quad r = \frac{\omega}{\omega_n}$$

Phase angle:
$$\tan\phi = \frac{2\zeta r}{1 - r^2}$$

Resonance frequency:
$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$$

—

General Forcing

Impulse response function:
$$h(t) = \frac{1}{m\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t)$$

Convolution integral:
$$x(t) = \int_0^t h(t - \tau)f(\tau)\,d\tau$$

Step response:
$$x(t) = \frac{F_0}{k}\left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}}\sin\omega_d t\right)\right]$$

—

Frequency Response and Transmissibility

Force transmissibility:
$$T_F = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

Motion transmissibility:
$$T_X = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

Vibration isolation region:
$$r > \sqrt{2}$$

—

Two-Degree-of-Freedom Systems

Matrix form:
$$\mathbf{M\ddot{x}} + \mathbf{C\dot{x}} + \mathbf{Kx} = \mathbf{f}(t)$$

Undamped eigenvalue problem:
$$(\mathbf{K} - \omega^2\mathbf{M})\phi = 0$$

Natural frequencies:
$$\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$$

Mode shape orthogonality:
$$\phi_i^T\mathbf{M}\phi_j = 0, \quad \phi_i^T\mathbf{K}\phi_j = 0 \quad (i \neq j)$$

Modal coordinates:
$$\mathbf{x} = \boldsymbol{\Phi}\mathbf{q}$$

—

Vibration Absorbers

Primary mass response (undamped absorber):
$$\omega_a = \sqrt{\frac{k_a}{m_a}}$$

Tuned absorber condition:
$$\omega_a = \omega_n$$

—

Vibration Measurement

Log decrement (experimental):
$$\zeta \approx \frac{\delta}{2\pi} \quad (\zeta \ll 1)$$

Frequency from FFT peak:
$$\omega = 2\pi f$$

—