Dynamics Sample Problem 1

This problem is question 2.1 from the Fall 2023 QE exam:

A uniform heavy thin bar has one end attached to a frictionless pivot (that cannot apply any moment). The bar is dropped from a stationary horizontal position. When it reaches vertical orientation, what is the velocity of its center of mass?

1 Determine solution method

Since this problem requires finding the velocity of an object, using energy methods will be most effective.

2 Identify knowns and unknowns

Knowns:

- The center of mass is located at the midpoint of the bar because it is uniform.
- The initial velocity of the bar is zero.
- The length of the bar is L.
- The mass of the bar is m.
- Gravitational acceleration is g.

Unknowns:

- The velocity of the center of mass when the bar reaches vertical orientation, denoted as v.

3 Conservation of energy

The total mechanical energy of the system is conserved since there are no non-conservative forces doing work (the pivot is frictionless).

$$E_{initial} = E_{final} \tag{1}$$

$$PE_{initial} + KE_{initial} = PE_{final} + KE_{final} \tag{2}$$

At the initial position, there is no kinetic energy since the bar is stationary. At the final position, there is no potential energy since the center of mass is at the lowest point. Then, the equation simplifies to:

$$PE_{initial} = KE_{final} \tag{3}$$

4 Expand and solve

The initial potential energy of the bar when it is horizontal is given by:

$$PE_{initial} = mgh \tag{4}$$

h is the vertical distance the center of mass falls. Since the center of mass is at the midpoint of the bar, $h = \frac{L}{2}$. Then:

$$PE_{initial} = mg\left(\frac{L}{2}\right) = \frac{mgL}{2} \tag{5}$$

The final kinetic energy of the bar when it is vertical is given by:

$$KE_{final} = \frac{1}{2}I\omega^2 \tag{6}$$

Where I is the mass moment of inertia of the bar. This is given by:

$$I = \frac{1}{3}mL^2 \tag{7}$$

Do not confuse this with the moment of inertia about the center of mass. Here, we use mass moment of inertia of the bar rotating about a pivot point at the end of the bar, not about its center of mass. Substitute in I into the kinetic energy equation.

$$KE_{final} = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\omega^2 = \frac{1}{6}mL^2\omega^2 \tag{8}$$

Now, substitute the expressions for initial potential energy and final kinetic energy into the conservation of energy equation.

$$\frac{mgL}{2} = \frac{1}{6}mL^2\omega^2 \tag{9}$$

Next, solve for ω.

$$3mgL = mL^2\omega^2 \tag{10}$$

$$3gL = L^2\omega^2 \tag{11}$$

$$\omega^2 = \frac{3g}{L} \tag{12}$$

$$\omega = \sqrt{\frac{3g}{L}} \tag{13}$$

Finally, find the linear velocity of the center of mass using the relationship between angular velocity and linear velocity.

$$v = \omega r \tag{14}$$

Where r is the distance from the pivot to the center of mass, which is $\frac{L}{2}$. Then:

$$v = \frac{L}{2}\sqrt{\frac{3g}{L}} \tag{15}$$

$$\boxed{v = \sqrt{\frac{3gL}{4}}} \tag{16}$$

Nice.