Vector Calculus Sample Problem 2

This problem is question 5.2 on the Fall 2023 QE.

Consider the curve $\mathcal{C}$ parametized as:

$$\begin{cases} x = -1 + \cos(2t) \\ y = \sin(t) \\ z = pt \end{cases} \tag{1}$$

For $t \in (0, 2\pi)$ with $p > 0$. Denote with A the point on the helix a $t = 0$ and with B the point at $t = 2\pi$. Calculate the work done by the force $\vec{F} = yz\hat{x} + xz\hat{y} + xy\hat{z}$ along the curve $\mathcal{C}$ from point A to point B as a function of the parameter p.

1 Important Equations

The work done by a force $\vec{F}$ along a curve $\mathcal{C}$ is given by:

$$W = \int_{\mathcal{C}} F \cdot d\vec{r} \tag{2}$$

2 Find Points A and B

TO find the coordinates of points A and B, we can substitute their corresponding values of t into the equation for $\mathcal{C}$. At $t = 0$, we have:

$$A = (-1 + \cos(0), \sin(0), p \cdot 0) = (0, 0, 0) \tag{3}$$

At $t = 2\pi$, we have:

$$B = (-1 + \cos(4\pi), \sin(2\pi), p \cdot 2\pi) = (0, 0, 2\pi p) \tag{4}$$

3 Force along the Curve

To calculate the force along the curve, we substitute the parametric equations into the force equation:

$$F_x = yz = \sin(t) \cdot pt = pt\sin(t) \tag{5}$$

$$F_y = xz = (-1 + \cos(2t)) \cdot pt = pt(-1 + \cos(2t)) \tag{6}$$

$$F_z = xy = (-1 + \cos(2t)) \cdot \sin(t) = (-1 + \cos(2t))\sin(t) \tag{7}$$

4 Find $d\vec{r}$

To find $d\vec{r}$, we differentiate the parametric equations of the curve with the following formula:

$$d\vec{r} = \frac{d\vec{r}}{dt}dt \tag{8}$$

$$\frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}\right)dt \tag{9}$$

Differentiating each component, we have:

$$\frac{dx}{dt} = \frac{d}{dt}[-1 + \cos(2t)] = -2\sin(2t) \tag{10}$$

$$\frac{dy}{dt} = \frac{d}{dt}[\sin(t)] = \cos(t) \tag{11}$$

$$\frac{dz}{dt} = \frac{d}{dt}[pt] = p \tag{12}$$

5 Compute the Work Integral

Now we can compute the work integral. First, we need to find $F \cdot d\vec{r}$:

$$F \cdot d\vec{r} = (pt\sin(t)(-2\sin(2t)) + pt(-1 + \cos(2t))\cos(t) + (-1 + \cos(2t))\sin(t)p)\,dt \tag{13}$$

Thus, the work done by the force along the curve from point A to point B is given by:

$$W = \int_0^{2\pi} (pt\sin(t)(-2\sin(2t)) + pt(-1 + \cos(2t))\cos(t) + (-1 + \cos(2t))\sin(t)p)\,dt \tag{14}$$

Evaluating this integral, we have:

$$W = p\int_0^{2\pi} (-2t\sin(t)\sin(2t) + t(-1 + \cos(2t))\cos(t) + (-1 + \cos(2t))\sin(t))\,dt \tag{15}$$

To solve this integral, we can split it into three separate integrals:

$$W = p\left(\int_0^{2\pi} -2t\sin(t)\sin(2t)dt + \int_0^{2\pi} t(-1 + \cos(2t))\cos(t)dt + \int_0^{2\pi} (-1 + \cos(2t))\sin(t)dt\right) \tag{16}$$

Calculating each integral separately, we have:

$$\int_0^{2\pi} -2t\sin(t)\sin(2t)dt = \frac{8\pi}{3} \tag{17}$$

$$\int_0^{2\pi} t(-1 + \cos(2t))\cos(t)dt = -\frac{4\pi}{3} \tag{18}$$

$$\int_0^{2\pi} (-1 + \cos(2t))\sin(t)dt = 0 \tag{19}$$

Combining these results, we find:

$$W = p\left(\frac{8\pi}{3} - \frac{4\pi}{3} + 0\right) = \boxed{\frac{4\pi p}{3}} \tag{20}$$

Boom.