Linear Algebra

Matrix Operations

Matrix Addition: $(A+B)_{ij} = A_{ij} + B_{ij}$

Scalar Multiplication: $(cA)_{ij} = cA_{ij}$

Matrix Multiplication: $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$

Note: $AB \neq BA$ in general (not commutative)

Transpose: $(A^T)_{ij} = A_{ji}$

Properties:

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(cA)^T = cA^T$

Trace: $\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$ (sum of diagonal elements)

Properties:

- $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
- $\mathrm{tr}(cA) = c\,\mathrm{tr}(A)$
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
- $\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i$ (sum of eigenvalues)

Determinants

2×2 Matrix:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

3×3 Matrix:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Properties:

- $\det(AB) = \det(A)\det(B)$
- $\det(A^T) = \det(A)$
- $\det(A^{-1}) = \frac{1}{\det(A)}$
- $\det(cA) = c^n \det(A)$ for $n \times n$ matrix
- If A has a row/column of zeros, then $\det(A) = 0$
- Swapping two rows/columns changes sign of determinant
- $\det(A) = \prod_{i=1}^{n} \lambda_i$ (product of eigenvalues)

Matrix Inverse

A matrix A is invertible if $\det(A) \neq 0$

Definition: $AA^{-1} = A^{-1}A = I$

2×2 Matrix Inverse:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Properties:

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(cA)^{-1} = \frac{1}{c}A^{-1}$

Eigenvalues and Eigenvectors

Definition: For a square matrix A:

$$A\mathbf{v} = \lambda\mathbf{v}$$

where λ is an eigenvalue and $\mathbf{v}$ is the corresponding eigenvector

Characteristic Equation:

$$\det(A - \lambda I) = 0$$

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$

For a 3×3 matrix:

$$-\lambda^3 + \text{tr}(A)\lambda^2 - (\text{sum of principal minors})\lambda + \det(A) = 0$$

Finding Eigenvectors:

Once λ is found, solve $(A - \lambda I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v}$

Properties:

- Sum of eigenvalues $= \text{tr}(A)$
- Product of eigenvalues $= \det(A)$
- Eigenvalues of $A^T =$ eigenvalues of A
- Eigenvalues of $A^{-1} = 1/\lambda_i$ (if A is invertible)
- Eigenvalues of $A^k = \lambda_i^k$
- If A is real and symmetric, all eigenvalues are real
- Eigenvectors corresponding to distinct eigenvalues are linearly independent

Diagonalization

A matrix A is diagonalizable if there exists an invertible matrix P such that:

$$P^{-1}AP = D$$

where D is a diagonal matrix with eigenvalues on the diagonal.

Construction:

- $P =$ matrix whose columns are eigenvectors of A
- $D =$ diagonal matrix with corresponding eigenvalues

$$P = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Matrix Powers: If $A = PDP^{-1}$, then

$$A^k = PD^kP^{-1} = P \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} P^{-1}$$

Conditions for Diagonalizability:

- A is diagonalizable if it has n linearly independent eigenvectors
- If A has n distinct eigenvalues, then A is diagonalizable
- Symmetric matrices are always diagonalizable

Similar Matrices

Matrices A and B are similar if there exists an invertible matrix P such that:

$$B = P^{-1}AP$$

Properties of Similar Matrices:

- Same eigenvalues
- Same determinant
- Same trace
- Same characteristic polynomial
- Same rank

Note: Eigenvectors transform as $\mathbf{w} = P^{-1}\mathbf{v}$

Orthogonal Matrices

A matrix Q is orthogonal if:

$$Q^TQ = QQ^T = I \quad \Rightarrow \quad Q^{-1} = Q^T$$

Properties:

- Columns (and rows) form an orthonormal set
- $\det(Q) = \pm 1$
- Preserves lengths: $\|Q\mathbf{x}\| = \|\mathbf{x}\|$
- Preserves inner products: $(Q\mathbf{x}) \cdot (Q\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

Symmetric Matrices

A matrix A is symmetric if $A^T = A$

Spectral Theorem:

Every real symmetric matrix can be orthogonally diagonalized:

$$A = QDQ^T$$

where Q is orthogonal and D is diagonal

Properties:

- All eigenvalues are real
- Eigenvectors corresponding to distinct eigenvalues are orthogonal
- Can always find an orthonormal basis of eigenvectors

Matrix Exponential

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

If A is diagonalizable: $A = PDP^{-1}$

$$e^{At} = Pe^{Dt}P^{-1} = P \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

Properties:

- $e^{A \cdot 0} = I$
- $\frac{d}{dt} e^{At} = Ae^{At}$
- $e^{A(t+s)} = e^{At}e^{As}$ (if A commutes with itself)
- $(e^{At})^{-1} = e^{-At}$
- $\det(e^{At}) = e^{\text{tr}(A)t}$

For 2×2 Matrices:

Case 1: Distinct Real Eigenvalues λ_1, λ_2

$$e^{At} = \frac{1}{\lambda_2 - \lambda_1} \left[(e^{\lambda_2 t})(A - \lambda_1 I) - (e^{\lambda_1 t})(A - \lambda_2 I) \right]$$

Case 2: Repeated Eigenvalue λ

$$e^{At} = e^{\lambda t}[I + (A - \lambda I)t]$$

Case 3: Complex Eigenvalues $\lambda = \alpha \pm i\beta$

$$e^{At} = e^{\alpha t}\left[\cos(\beta t)I + \frac{\sin(\beta t)}{\beta}(A - \alpha I)\right]$$

Jordan Canonical Form

Every square matrix is similar to a Jordan canonical form J:

$$A = PJP^{-1}$$

where J is block diagonal with Jordan blocks:

Jordan Block:

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$

Special Matrices

Identity Matrix: $I_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Diagonal Matrix: $D_{ij} = 0$ for $i \neq j$

Upper Triangular: $A_{ij} = 0$ for $i > j$

Lower Triangular: $A_{ij} = 0$ for $i < j$

For triangular matrices: $\det(A) = \prod_i A_{ii}$ (product of diagonal elements)

Positive Definite Matrix:

A symmetric matrix A is positive definite if:

- $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$
- All eigenvalues are positive
- All leading principal minors are positive

Linear Systems

For $A\mathbf{x} = \mathbf{b}$:

Cramer's Rule (if $\det(A) \neq 0$):

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i is A with column i replaced by $\mathbf{b}$

Matrix Solution:

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Vector Spaces

Linear Independence:

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent if:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0} \quad \Rightarrow \quad c_1 = c_2 = \cdots = c_n = 0$$

Basis: A linearly independent set that spans the vector space

Dimension: Number of vectors in a basis

Rank: Dimension of column space = dimension of row space

Nullity: Dimension of null space (kernel)

Rank-Nullity Theorem:

$$\text{rank}(A) + \text{nullity}(A) = n$$

where n is the number of columns of A

Inner Products and Norms

Inner Product (Dot Product):

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = \sum_{i=1}^{n} u_i v_i$$

Norm (Length):

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

Unit Vector: $\|\mathbf{v}\| = 1$

Orthogonal Vectors: $\mathbf{u} \cdot \mathbf{v} = 0$

Orthonormal Set: Vectors that are pairwise orthogonal and each has norm 1

Cauchy-Schwarz Inequality:

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

Triangle Inequality:

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

Gram-Schmidt Orthogonalization

To convert basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ to orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$:

$$\mathbf{w}_1 = \mathbf{v}_1, \quad \mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}$$

$$\mathbf{w}_k = \mathbf{v}_k - \sum_{j=1}^{k-1} (\mathbf{v}_k \cdot \mathbf{u}_j)\mathbf{u}_j, \quad \mathbf{u}_k = \frac{\mathbf{w}_k}{\|\mathbf{w}_k\|}$$

Useful Formulas

2×2 Eigenvalue Formula:

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$\lambda = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}(A)^2 - 4\det(A)}}{2}$$

Cayley-Hamilton Theorem:

Every matrix satisfies its own characteristic equation. If $p(\lambda) = \det(A - \lambda I)$, then $p(A) = 0$.