Laplace Transform Sample Problem 2

This problem is question 1.2 from the Fall 2023 QE exam:

Find the continuous solution of the following integro-differential equation for z(t) using the properties of Laplace transforms:

$$z(t) - (1-t)e^t = \int_0^t z(t-\tau)z(\tau)d\tau \tag{1}$$

$$z(0) = 0 \tag{2}$$

1 Convolution Theorem

Using the convolution theorem, we can take the Laplace transform of the right side. Here, the integral is just the convolution of $z(t)$ with itself.

$$\int_0^t z(t-\tau)z(\tau)d\tau = z(t) * z(t) \tag{3}$$

$$\mathcal{L}[z(t) * z(t)] = Z(s) \cdot Z(s) = (Z(s))^2 \tag{4}$$

2 Laplace Transform of Left Side

We can first reqrite the left side of the equation:

$$z(t) - (1-t)e^t = z(t) - e^t + te^t \tag{5}$$

Now we can take the Laplace transform of each term:

$$\mathcal{L}[z(t)] = Z(s) \tag{6}$$

$$\mathcal{L}[e^t] = \frac{1}{s-1} \tag{7}$$

$$\mathcal{L}[te^t] = \frac{1}{(s-1)^2} \tag{8}$$

3 Combine and Solve for Z(s)

Putting it all together, we have:

$$(Z(s))^2 = Z(s) - \frac{1}{s-1} + \frac{1}{(s-1)^2} \tag{9}$$

$$(Z(s))^2 = Z(s) - \frac{s-2}{(s-1)^2} \tag{10}$$

$$(Z(s))^2 - Z(s) + \frac{s-2}{(s-1)^2} = 0 \tag{11}$$

Using the quadratic formula to solve for $Z(s)$:

$$Z(s) = \frac{1 \pm \sqrt{1 - 4\frac{s-2}{(s-1)^2}}}{2} \tag{12}$$

We can then expand the term under the square root.

$$Z(s) = \frac{1 \pm \sqrt{\frac{(s-1)^2 - 4(s-2)}{(s-1)^2}}}{2} \tag{13}$$

$$Z(s) = \frac{1 \pm \frac{\sqrt{s^2-2s+1-4s+8}}{s-1}}{2} \tag{14}$$

$$Z(s) = \frac{1 \pm \frac{\sqrt{s^2-6s+9}}{s-1}}{2} \tag{15}$$

$$Z(s) = \frac{1 \pm \frac{s-3}{s-1}}{2} \tag{16}$$

This gives us two possible solutions for $Z(s)$:

$$Z_+(s) = \frac{1 + \frac{s-3}{s-1}}{2} = \frac{s-2}{s-1} \tag{17}$$

$$Z_-(s) = \frac{1 - \frac{s-3}{s-1}}{2} = \frac{1}{s-1} \tag{18}$$

Z_+ is not valid since it is not continous at $t = 0$ (its inverse Laplace transform has a jump discontinuity). Thus, we take Z_-:

$$Z(s) = \frac{1}{s-1} \tag{19}$$

4 Inverse Laplace Transform

Taking the inverse Laplace transform gives:

$$z(t) = \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] = e^t \tag{20}$$

Nice.