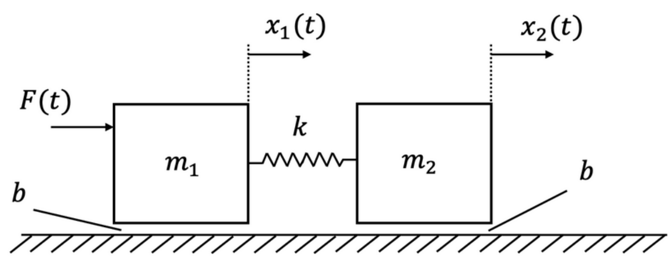
Vibrations Sample Problem 1

This is problem 2.3 on the 2023 QE.

Consider the following mechanical system, two masses, m_1 and m_2, slide on a surface with friction coefficient b. The two masses are connected by a spring with constant k. The force input to mass m_1 is $F(t)$, and x_1 and x_2 are displacements of the two masses.



(a) Obtain the differential equations describing the system.
(b) Find a single expression relating $x_2(t)$ to $x_1(t)$ in either the time or frequency/Laplace domain.
(c) It is known that $m_1 = m_2 = 1$ kg, $b = 2$ N•s/m, and $k = 1$ N/m. When $x_1(t) = 2u(t)$, $u(t)$ is a unit step input, compute $x_2(t)$. Consider zero initial conditions. *Hint: You may find Inverse Laplace Transforms helpful.*

Part A

First, we need to consider the forces on m_1.

$$\text{constant force: } F(t) \tag{1}$$
$$\text{spring force: } -k(x_1 - x_2) \tag{2}$$
$$\text{viscous damping friction force: } -b\dot{x}_1 \tag{3}$$

Note that we use viscous damping instead of Coulomb (kinetic) friction because the problem provides b in units of $\frac{N \cdot m}{s}$. It is also used more often in modelling problems like this one.

Using Newton's second law, we have:

$$F_{\text{total}} = m_1\ddot{x}_1 \tag{4}$$
$$m_1\ddot{x}_1 = F(t) - k(x_1 - x_2) - b\dot{x}_1 \tag{5}$$
$$\boxed{F(t) = m_1\ddot{x}_1 + b\dot{x}_1 + k(x_1 - x_2)} \tag{6}$$

Now, consider the forces on m_2.

$$\text{spring force: } k(x_1 - x_2) \tag{7}$$
$$\text{viscous damping friction force: } -b\dot{x}_2 \tag{8}$$

Using Newton's second law again, we have:

$$F_{\text{total}} = m_2\ddot{x}_2 \tag{9}$$
$$m_2\ddot{x}_2 = k(x_1 - x_2) - b\dot{x}_2 \tag{10}$$
$$\boxed{0 = m_2\ddot{x}_2 + b\dot{x}_2 - k(x_1 - x_2)} \tag{11}$$

Part B

To find a single expression that relates x_1 and x_2, we need to take the Laplace transform of both equations from Part A.

$$\text{mass 1 eq: } F(s) = m_1 s^2 X_1(s) + bsX_1(s) + k(X_1(s) - X_2(s)) \tag{12}$$
$$\text{mass 2 eq: } 0 = m_2 s^2 X_2(s) + bsX_2(s) - k(X_1(s) - X_2(s)) \tag{13}$$

We can then rearrange the mass 2 equation to solve for $\frac{X_2}{X_1}$.

$$k(X_1(s) - X_2(s)) = m_2 s^2 X_2(s) + bsX_2(s) \tag{14}$$
$$kX_1(s) = X_2(s)(m_2 s^2 + bs + k) \tag{15}$$
$$\boxed{\frac{X_2(s)}{X_1(s)} = \frac{k}{m_2 s^2 + bs + k}} \tag{16}$$

Part C

We now know that:

$$m_1 = m_2 = 1kg \tag{17}$$
$$b = 2\frac{N \cdot m}{s} \tag{18}$$
$$k = 1\frac{N}{m} \tag{19}$$
$$x_1(t) = 2u(t) \rightarrow \text{u is a unit step} \tag{20}$$

Using these values, we can solve for an expression for $x_2(t)$. First start by rearranging the equation from Part B to solve for $X_2(s)$.

$$X_2(s) = X_1(s) \cdot \frac{k}{m_2 s^2 + bs + k} \tag{21}$$

$$X_2(s) = \frac{2}{s} \cdot \frac{1}{s^2 + 2s + 1} \tag{22}$$

$$X_2(s) = \frac{2}{s(s+1)^2} \tag{23}$$

Now we can use partial fraction decomposition to simplify $X_2(s)$.

$$\frac{2}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} \tag{24}$$

$$2 = A(s+1)^2 + Bs(s+1) + Cs \tag{25}$$

$$2 = A(s^2 + 2s + 1) + B(s^2 + s) + Cs \tag{26}$$

$$\tag{27}$$

To start solving for the coefficients, let $s = 0$.

$$2 = A(0 + 0 + 1) + B(0 + 0) + C(0) \tag{28}$$

$$A = 2 \tag{29}$$

Next, let $s = -1$.

$$2 = A(1 - 2 + 1) + B(1 - 1) + C(-1) \tag{30}$$

$$2 = 0 + 0 - C \tag{31}$$

$$C = -2 \tag{32}$$

Now, we can solve for B by substituting A and C back into the equation.

$$2 = 2s^2 + 4s + 2 + Bs^2 + Bs - 2s \tag{33}$$

$$0 = (2 + B)s^2 + (2 + B)s \tag{34}$$

$$B = -2 \tag{35}$$

Thus, we have:

$$X_2(s) = \frac{2}{s} - \frac{2}{s+1} - \frac{2}{(s+1)^2} \tag{36}$$

Now we can take the inverse Laplace transform to find $x_2(t)$.

$$x_2(t) = \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{2}{s+1} - \frac{2}{(s+1)^2}\right\} \tag{37}$$

$$\boxed{x_2(t) = 2 - 2e^{-t} - 2te^{-t}} \tag{38}$$