Vector Calculus

Vector Operations

Vector Addition: $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

Scalar Multiplication: $c\mathbf{v} = (cv_1, cv_2, cv_3)$

Dot Product (Scalar Product):

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}||\mathbf{v}|\cos\theta$$

Properties:

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (commutative)
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributive)
- $\mathbf{u} \cdot \mathbf{v} = 0 \Leftrightarrow \mathbf{u} \perp \mathbf{v}$

Cross Product (Vector Product):

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Properties:

- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ (anti-commutative)
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$
- $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$
- $\mathbf{u} \times \mathbf{v} = \mathbf{0} \Leftrightarrow \mathbf{u}$ parallel to $\mathbf{v}$

Scalar Triple Product:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Geometric interpretation: Volume of parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$

Vector Triple Product:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

Magnitude: $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Unit Vector: $\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$

Vector Functions and Curves

Position Vector: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

Velocity: $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}$

Acceleration: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \frac{d^2\mathbf{r}}{dt^2}$

Speed: $|\mathbf{v}(t)| = \left|\frac{d\mathbf{r}}{dt}\right|$

Unit Tangent Vector: $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

Arc Length:

$$s = \int_a^b |\mathbf{r}'(t)|\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Gradient, Divergence, and Curl

Del Operator (Nabla):

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Gradient

For scalar field $f(x, y, z)$:

$$\nabla f = \text{grad } f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Properties:

- ∇f points in direction of maximum rate of increase of f
- $|\nabla f|$ gives the magnitude of that maximum rate
- ∇f is perpendicular to level surfaces $f(x, y, z) = c$

Directional Derivative:

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$

where **u** is a unit vector in the desired direction

Divergence

For vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$:

$$\nabla \cdot \mathbf{F} = \text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Interpretation: Divergence measures the "outflow" of a vector field from an infinitesimal region

Curl

For vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$:

$$\nabla \times \mathbf{F} = \text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Interpretation: Curl measures the rotation or circulation of a vector field

Laplacian

For scalar field f:

$$\nabla^2 f = \Delta f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

For vector field $\mathbf{F}$:

$$\nabla^2 \mathbf{F} = (\nabla^2 P)\mathbf{i} + (\nabla^2 Q)\mathbf{j} + (\nabla^2 R)\mathbf{k}$$

Vector Identities

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \quad \text{(divergence of curl is zero)}$$

$$\nabla \times (\nabla f) = \mathbf{0} \quad \text{(curl of gradient is zero)}$$

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla \cdot (f\mathbf{F}) = f(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot (\nabla f)$$

$$\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F}$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$$

Line Integrals

Scalar Line Integral:

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt$$

Vector Line Integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

In component form:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$$

Work: $W = \int_C \mathbf{F} \cdot d\mathbf{r}$

Path Independence:

$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if $\mathbf{F}$ is conservative

Conservative Vector Field:

$\mathbf{F}$ is conservative if:

- $\mathbf{F} = \nabla f$ for some scalar potential function f
- $\nabla \times \mathbf{F} = \mathbf{0}$
- $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent
- $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C

Finding Potential Function:

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} = \nabla f$, then:

$$f(x, y, z) = \int P\,dx + g(y, z)$$

where $g(y, z)$ is determined by matching $\frac{\partial f}{\partial y} = Q$ and $\frac{\partial f}{\partial z} = R$

Fundamental Theorem for Line Integrals:

If $\mathbf{F} = \nabla f$, then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Surface Integrals

Parametric Surface: $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$

Normal Vector:

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$$

Surface Area Element:

$$dS = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du\, dv$$

Scalar Surface Integral:

$$\iint_S f\, dS = \iint_D f(\mathbf{r}(u, v)) \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du\, dv$$

Vector Surface Integral (Flux):

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du\, dv$$

For surface $z = g(x, y)$:

$$\mathbf{n}\, dS = \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \right) dx\, dy$$

Fundamental Theorems

Green's Theorem

For region D in the xy-plane with boundary curve C (counterclockwise):

$$\oint_C P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Or in vector form:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$$

Area by Green's Theorem:

$$\text{Area} = \oint_C x\, dy = -\oint_C y\, dx = \frac{1}{2} \oint_C x\, dy - y\, dx$$

Stokes' Theorem

For surface S with boundary curve C:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

"Circulation around C equals flux of curl through S"

Divergence Theorem (Gauss' Theorem)

For solid region E with boundary surface S (outward normal):

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E (\nabla \cdot \mathbf{F}) \, dV$$

"Flux through S equals total divergence in E"

Coordinate Systems

Cylindrical Coordinates

Conversion:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$
$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x)$$

Volume Element: $dV = r \, dr \, d\theta \, dz$

Gradient:

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z$$

Divergence:

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial (rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

Laplacian:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Coordinates

Conversion:

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$
$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \phi = \cos^{-1}(z/\rho), \quad \theta = \tan^{-1}(y/x)$$

where $\rho \geq 0$, $0 \leq \phi \leq \pi$, $0 \leq \theta < 2\pi$

Volume Element: $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Gradient:

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{1}{\rho\sin\phi}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta$$

Divergence:

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho^2}\frac{\partial (\rho^2 F_\rho)}{\partial \rho} + \frac{1}{\rho\sin\phi}\frac{\partial (\sin\phi \, F_\phi)}{\partial \phi} + \frac{1}{\rho\sin\phi}\frac{\partial F_\theta}{\partial \theta}$$

Laplacian:

$$\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right) + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$$

Special Functions and Identities

Position Vector: $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$|\mathbf{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$\nabla r = \frac{\mathbf{r}}{r}$$

$$\nabla \cdot \mathbf{r} = 3$$

$$\nabla \times \mathbf{r} = \mathbf{0}$$

$$\nabla \left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$$

$$\nabla^2 \left(\frac{1}{r}\right) = 0 \quad (r \neq 0)$$