Solid Mechanics

Stress and Strain

Normal Stress

Axial Stress:

$$\sigma = \frac{P}{A}$$

where P is axial force and A is cross-sectional area

Sign Convention: - Tension: positive - Compression: negative

Shear Stress

Direct Shear:

$$\tau = \frac{V}{A}$$

where V is shear force and A is area

Torsional Shear Stress:

$$\tau = \frac{T\rho}{J}$$

where T is torque, ρ is radial distance, J is polar moment of inertia

For circular shaft:

$$\tau_{max} = \frac{Tr}{J} = \frac{16T}{\pi d^3}$$

Normal Strain

$$\epsilon = \frac{\Delta L}{L} = \frac{dL}{L}$$

Sign Convention: - Elongation: positive - Contraction: negative

Shear Strain

$$\gamma = \tan\theta \approx \theta \quad \text{(for small angles)}$$

where θ is the change in angle (in radians)

Hooke's Law

Uniaxial:

$$\sigma = E\epsilon$$

where E is Young's modulus (modulus of elasticity)

Shear:

$$\tau = G\gamma$$

where G is shear modulus

Relationship between E and G:

$$G = \frac{E}{2(1+\nu)}$$

where ν is Poisson's ratio

Poisson's Ratio

$$\nu = -\frac{\epsilon_{\text{lateral}}}{\epsilon_{\text{axial}}}$$

Typical values: 0.25 to 0.35 for most metals

Generalized Hooke's Law (3D)

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$
$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad \gamma_{yz} = \frac{\tau_{yz}}{G}, \quad \gamma_{xz} = \frac{\tau_{xz}}{G}$$

Axial Loading

Axial Deformation

$$\delta = \frac{PL}{AE}$$

For varying load, area, or material:

$$\delta = \int_0^L \frac{P(x)}{A(x)E(x)}\,dx$$

Statically Indeterminate Problems

For statically indeterminate systems:

1. Write equilibrium equations 2. Write compatibility (deformation) equations 3. Write force-deformation relationships 4. Solve system of equations

Example - Bar with supports at both ends:

Equilibrium: $R_A + R_B = P$

Compatibility: $\delta_A + \delta_B = 0$ (or specified gap/displacement)

Thermal Effects

Thermal Strain:

$$\epsilon_T = \alpha \Delta T$$

where α is coefficient of thermal expansion and ΔT is temperature change

Total Deformation (unconstrained):

$$\delta_T = \alpha L \Delta T$$

Thermal Stress (fully constrained):

$$\sigma_T = E \alpha \Delta T$$

Combined mechanical and thermal:

$$\delta = \frac{PL}{AE} + \alpha L \Delta T$$

Factor of Safety

$$n = \frac{\sigma_{\text{allowable}}}{\sigma_{\text{actual}}} = \frac{\sigma_{\text{fail}}}{\sigma_{\text{actual}}}$$

For design:

$$\sigma_{\text{allowable}} = \frac{\sigma_{\text{yield}}}{n}$$

Typical values: $n = 1.5$ to 3 (higher for uncertain loading)

Torsion

Angle of Twist

$$\phi = \frac{TL}{GJ}$$

For varying torque, geometry, or material:

$$\phi = \int_0^L \frac{T(x)}{G(x)J(x)}\,dx$$

Polar Moment of Inertia

Solid Circular Shaft:

$$J = \frac{\pi d^4}{32} = \frac{\pi r^4}{2}$$

Hollow Circular Shaft:

$$J = \frac{\pi(d_o^4 - d_i^4)}{32} = \frac{\pi(r_o^4 - r_i^4)}{2}$$

Thin-Walled Tube:

$$J \approx 2\pi r^3 t$$

where r is mean radius and t is wall thickness

Power Transmission

$$P = T\omega = 2\pi n T$$

where P is power (watts), T is torque ($\mathrm{N \cdot m}$), ω is angular velocity (rad/s), and n is rotational speed (rev/s)

In imperial units:

$$P(\mathrm{hp}) = \frac{T(\mathrm{lb \cdot in}) \cdot n(\mathrm{rpm})}{63,025}$$

Bending of Beams

Flexure Formula (Bending Stress)

$$\sigma = \frac{My}{I}$$

where: - M is bending moment at section - y is distance from neutral axis (positive for tension side) - I is second moment of area about neutral axis

Maximum Bending Stress:

$$\sigma_{max} = \frac{Mc}{I} = \frac{M}{S}$$

where c is distance to extreme fiber and $S = I/c$ is section modulus

Section Modulus

Rectangular Section ($b \times h$):

$$S = \frac{bh^2}{6}$$

Circular Section (diameter d):

$$S = \frac{\pi d^3}{32}$$

Hollow Circular Section:

$$S = \frac{\pi(d_o^4 - d_i^4)}{32d_o}$$

Second Moment of Area (Area Moment of Inertia)

Definition:

$$I = \int y^2 \, dA$$

Parallel Axis Theorem:

$$I_x = I_{\bar{x}} + Ad^2$$

where $I_{\bar{x}}$ is moment about centroidal axis and d is distance between axes

Common Cross-Sections:

Rectangle ($b \times h$, about centroid):

$$I = \frac{bh^3}{12}$$

Circle (diameter d, about centroid):

$$I = \frac{\pi d^4}{64}$$

Hollow Circle (outer d_o, inner d_i):

$$I = \frac{\pi(d_o^4 - d_i^4)}{64}$$

Triangle (base b, height h, about base):

$$I = \frac{bh^3}{12}$$

Triangle (about centroid):

$$I = \frac{bh^3}{36}$$

Beam Deflection

Differential Equations:

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

$$\frac{dv}{dx} = \theta = \int \frac{M}{EI}\,dx$$

$$v = \iint \frac{M}{EI}\,dx\,dx$$

where v is deflection and θ is slope

Sign Convention: - Positive M: causes compression on top - Positive v: downward deflection - Positive θ: counterclockwise rotation

Common Beam Deflections

Simply Supported Beam - Point Load at Center:

$$v_{max} = \frac{PL^3}{48EI}$$

Simply Supported Beam - Uniform Load:

$$v_{max} = \frac{5wL^4}{384EI}$$

Cantilever Beam - Point Load at End:

$$v_{max} = \frac{PL^3}{3EI}$$

Cantilever Beam - Uniform Load:

$$v_{max} = \frac{wL^4}{8EI}$$

Singularity Functions (Macaulay Method)

Unit Step Function:

$$\langle x-a\rangle^0 = \begin{cases} 0 & x < a \\ 1 & x \geq a \end{cases}$$

Ramp Function:

$$\langle x-a\rangle^n = \begin{cases} 0 & x < a \\ (x-a)^n & x \geq a \end{cases}$$

Integration:

$$\int \langle x-a\rangle^n \, dx = \frac{\langle x-a\rangle^{n+1}}{n+1}$$

Point Load:

Use $\langle x-a\rangle^{-1}$ for point load at $x = a$

Shear and Moment Diagrams

Relationships

$$\frac{dV}{dx} = -w(x)$$

$$\frac{dM}{dx} = V(x)$$

$$M = \int V \, dx$$

where $w(x)$ is distributed load, V is shear force, and M is bending moment

Sign Conventions

Shear Force: - Positive: causes clockwise rotation of element

Bending Moment: - Positive: causes compression on top, tension on bottom (concave up)

Key Points

- V changes abruptly at point loads - V varies linearly under uniform load - M has maximum/minimum where $V = 0$ - M changes abruptly at concentrated moments

Transverse Shear Stress

Shear Formula

$$\tau = \frac{VQ}{Ib}$$

where: - V is shear force - $Q = \bar{y}' A'$ is first moment of area above (or below) the point - I is second moment of area of entire cross-section - b is width at the point of interest

Maximum Shear Stress (usually at neutral axis):

$$\tau_{max} = \frac{VQ_{max}}{Ib}$$

Shear Stress in Common Sections

Rectangular Section:

$$\tau_{max} = \frac{3V}{2A} = \frac{3V}{2bh}$$

Circular Section:

$$\tau_{max} = \frac{4V}{3A} = \frac{16V}{3\pi d^2}$$

I-Beam (web):

$$\tau_{web} \approx \frac{V}{A_{web}} = \frac{V}{t_w h_{web}}$$

Shear Flow

$$q = \frac{VQ}{I}$$

where q is shear flow (force per unit length)

Used for: - Built-up beams (fastener spacing) - Thin-walled sections - Composite beams

Combined Loading

Superposition

For linear elastic materials, stresses from different loads can be superimposed:

$$\sigma_{total} = \sigma_1 + \sigma_2 + \cdots$$

Common Combinations:

Axial + Bending:

$$\sigma = \frac{P}{A} \pm \frac{My}{I}$$

Bending in Two Planes:

$$\sigma = \frac{M_x y}{I_x} + \frac{M_y x}{I_y}$$

Torsion + Bending:

$$\tau_{max} = \sqrt{\left(\frac{Tr}{J}\right)^2 + \left(\frac{VQ}{Ib}\right)^2}$$

Stress Transformation

Plane Stress

For a state of stress σ_x, σ_y, τ_{xy}, the stress on a plane at angle θ:

Normal Stress:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

Shear Stress:

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

Principal Stresses

Principal Stresses:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

where σ_1 is maximum and σ_2 is minimum

Principal Angles:

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Two values of θ_p differ by 90ř

Maximum Shear Stress:

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_1 - \sigma_2}{2}$$

Angle for Maximum Shear:

$$\theta_s = \theta_p \pm 45\text{ř}$$

Mohr's Circle

Center:

$$C = \frac{\sigma_x + \sigma_y}{2}$$

Radius:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Construction: 1. Plot point $A(\sigma_x, \tau_{xy})$ and $B(\sigma_y, -\tau_{xy})$ 2. Draw circle with diameter AB 3. Principal stresses are intercepts on σ axis 4. Maximum shear stress is radius of circle

Sign Convention for Mohr's Circle: - σ: tension positive (right), compression negative (left) - τ: plot as given (up for positive, down for negative) - Angles on circle are 2θ (double physical angle) - Rotate counterclockwise on circle for counterclockwise rotation of element

Pressure Vessels

Thin-Walled Cylindrical Vessel

Thin-walled criterion: $r/t \geq 10$

Hoop Stress (Circumferential):

$$\sigma_1 = \frac{pr}{t}$$

Longitudinal Stress:

$$\sigma_2 = \frac{pr}{2t}$$

Maximum Shear Stress:

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{pr}{4t}$$

where p is internal pressure, r is mean radius, t is wall thickness

Thin-Walled Spherical Vessel

Stress (same in all directions):

$$\sigma = \frac{pr}{2t}$$

Thick-Walled Cylinders

Lamé Equations:

$$\sigma_r = A - \frac{B}{r^2}$$

$$\sigma_\theta = A + \frac{B}{r^2}$$

where constants A and B are determined from boundary conditions

For internal pressure p_i and external pressure $p_o = 0$:

$$\sigma_r = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right)$$

$$\sigma_\theta = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r^2}\right)$$

Column Buckling

Euler Buckling Load

Critical Load:

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2}$$

where: - E is Young's modulus - I is minimum second moment of area - L is column length - K is effective length factor

Effective Length Factor K: - Both ends pinned: $K = 1.0$ - Both ends fixed: $K = 0.5$ - One fixed, one pinned: $K = 0.7$ - One fixed, one free: $K = 2.0$

Critical Stress:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(KL/r)^2}$$

where $r = \sqrt{I/A}$ is radius of gyration

Slenderness Ratio:

$$\frac{KL}{r}$$

Euler formula valid for slender columns (high slenderness ratio)

Johnson Parabola (Intermediate Columns)

For intermediate slenderness ratios:

$$\sigma_{cr} = \sigma_y\left[1 - \frac{1}{4}\left(\frac{KL/r}{(KL/r)_c}\right)^2\right]$$

where $(KL/r)_c = \sqrt{2\pi^2 E/\sigma_y}$ is transition slenderness ratio

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Eccentric Loading

Secant Formula:

$$\frac{P}{A} = \frac{\sigma_{max}}{1 + (ec/r^2)\sec[(L/2r)\sqrt{P/(EA)}]}$$

where e is eccentricity of load

Energy Methods

Strain Energy

Axial Loading:

$$U = \int_0^L \frac{P^2}{2AE}\,dx = \frac{P^2 L}{2AE}$$

Torsion:

$$U = \int_0^L \frac{T^2}{2GJ}\,dx = \frac{T^2 L}{2GJ}$$

Bending:

$$U = \int_0^L \frac{M^2}{2EI}\,dx$$

Shear:

$$U = \int_0^L \frac{V^2}{2GA}\,dx$$

Total Strain Energy:

$$U = U_{axial} + U_{torsion} + U_{bending} + U_{shear}$$

Castigliano's Theorem

First Theorem (displacement):

$$\delta_i = \frac{\partial U}{\partial P_i}$$

where δ_i is displacement at point where force P_i is applied

Second Theorem (force):

$$P_i = \frac{\partial U}{\partial \delta_i}$$

For finding deflections:

If no force at desired location, apply dummy load Q:

$$\delta = \left.\frac{\partial U}{\partial Q}\right|_{Q=0}$$

Principle of Virtual Work

$$\sum P_i \delta_i = \int_0^L \frac{M\,\delta M}{EI}\,dx$$

Used for deflection calculations in complex structures

Composite Beams

Transformed Section Method:

For beam with two materials (1 and 2):

Transform material 2 to equivalent material 1:

$$n = \frac{E_2}{E_1}$$

Width of material 2 becomes: $b_{2,\text{equiv}} = nb_2$

Solve as single-material beam with transformed geometry

Stress in Each Material:

$$\sigma_1 = \frac{My}{I_{transformed}}$$

$$\sigma_2 = n\frac{My}{I_{transformed}}$$

Curved Beams

For beams with significant initial curvature:

Curved Beam Formula:

$$\sigma = \frac{M(R-r)}{Aer}$$

where: - M is bending moment - R is radius to neutral axis - r is radius to point of interest - A is cross-sectional area - $e = R - r_c$ is eccentricity (r_c is radius to centroid)

Stress Concentrations

Stress Concentration Factor:

$$K_t = \frac{\sigma_{max}}{\sigma_{nominal}}$$

Common sources: - Holes - Notches - Fillets - Changes in cross-section - Keyways

For static loading with ductile materials, K_t can often be ignored (yielding redistributes stress)

For fatigue or brittle materials, K_t is critical

Material Properties

Common Values:

Material	E (GPa)	G (GPa)	ν
Steel	200	80	0.30
Aluminum	70	26	0.33
Concrete	25-30	-	0.15-0.20
Timber	10-15	-	-