Systems and Controls

Laplace Transform for Control Systems

Definition:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Common Transforms:

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t\} = \frac{1}{s^2}, \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}, \quad \mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$

Derivatives:

$$\mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0)$$
$$\mathcal{L}\{\ddot{f}(t)\} = s^2F(s) - sf(0) - \dot{f}(0)$$

Final Value Theorem:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

(Valid only if $f(t)$ has a final value, i.e., system is stable)

Initial Value Theorem:

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

Transfer Functions

Definition:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\text{Output}}{\text{Input}}$$

assuming zero initial conditions

Standard Transfer Function Forms

First-Order System:

$$G(s) = \frac{K}{\tau s + 1}$$

where K is DC gain and τ is time constant

Second-Order System:

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where ω_n is natural frequency and ζ is damping ratio

Alternative form:

$$G(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

System Modeling

Mechanical System (Mass-Spring-Damper):

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

Transfer function:

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

RLC Circuit:

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = v(t)$$

Transfer function:

$$G(s) = \frac{Q(s)}{V(s)} = \frac{1}{Ls^2 + Rs + 1/C}$$

Mechanical-Electrical Analogies

Mechanical	Electrical
Force F	Voltage V
Velocity v	Current I
Mass m	Inductance L
Damper c	Resistance R
Spring k	$1/C$ (Capacitance)

Block Diagram Algebra

Series (Cascade):

$$G_{\text{eq}}(s) = G_1(s)G_2(s)$$

Parallel:

$$G_{\text{eq}}(s) = G_1(s) + G_2(s)$$

Feedback (Negative):

$$G_{cl}(s) = \frac{G(s)}{1 + G(s)H(s)}$$

where $G(s)$ is forward path and $H(s)$ is feedback path

Feedback (Positive):

$$G_{cl}(s) = \frac{G(s)}{1 - G(s)H(s)}$$

Unity Feedback: $H(s) = 1$

$$G_{cl}(s) = \frac{G(s)}{1 + G(s)}$$

Moving Summing Junctions and Pickoff Points

Moving summing junction past a block G: - Move forward: Divide by G - Move backward: Multiply by G

Moving pickoff point past a block G: - Move forward: Multiply by G - Move backward: Divide by G

Time Response Analysis

First-Order System Response

For $G(s) = \frac{K}{\tau s+1}$ with unit step input:

Time Response:

$$y(t) = K(1 - e^{-t/\tau})$$

Time Constant: τ - Time to reach 63.2% of final value - $y(\tau) = 0.632K$

Settling Time (2% criterion):

$$T_s = 4\tau$$

Second-Order System Response

For $G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ with unit step input:

Underdamped $(0 < \zeta < 1)$:

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi)$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \cos^{-1}(\zeta)$

Performance Specifications:

Rise Time (0% to 100%):

$$T_r \approx \frac{1.8}{\omega_n}$$

Peak Time:

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Percent Overshoot:

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

Settling Time (2% criterion):

$$T_s \approx \frac{4}{\zeta\omega_n}$$

Settling Time (5% criterion):

$$T_s \approx \frac{3}{\zeta\omega_n}$$

Damping Ratio from Overshoot:

$$\zeta = \frac{-\ln(M_p/100)}{\sqrt{\pi^2 + \ln^2(M_p/100)}}$$

For small overshoot:

$$\zeta \approx 1 - \frac{M_p}{100}$$

Dominant Pole Approximation

For higher-order systems, if one pole (or pair) is much closer to imaginary axis than others, system behaves approximately like first or second-order system with those dominant poles.

Steady-State Error

Error Definition

$$E(s) = R(s) - Y(s) = \frac{R(s)}{1 + G(s)H(s)}$$

Steady-State Error:

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

Error Constants (Unity Feedback)

Position Error Constant:

$$K_p = \lim_{s \to 0} G(s)$$

Velocity Error Constant:

$$K_v = \lim_{s \to 0} sG(s)$$

Acceleration Error Constant:

$$K_a = \lim_{s \to 0} s^2 G(s)$$

Steady-State Errors for Standard Inputs

Input	Type 0	Type 1	Type 2
Step: $R(s) = \frac{1}{s}$	$\frac{1}{1+K_p}$	0	0
Ramp: $R(s) = \frac{1}{s^2}$	∞	$\frac{1}{K_v}$	0
Parabolic: $R(s) = \frac{1}{s^3}$	∞	∞	$\frac{1}{K_a}$

System Type:

Number of integrators (poles at $s = 0$) in open-loop transfer function $G(s)H(s)$

For $G(s)H(s) = \frac{K(s+z_1)\cdots}{s^N(s+p_1)\cdots}$, Type $= N$

Stability Analysis

Characteristic Equation

For closed-loop system:

$$1 + G(s)H(s) = 0$$

or equivalently, denominator of $\frac{G(s)}{1+G(s)H(s)} = 0$

Stability Requirement:

All roots of characteristic equation must have negative real parts (left half-plane)

Routh-Hurwitz Stability Criterion

For characteristic equation:

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

Routh Array:

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\ s^{n-2} & b_1 & b_2 & b_3 \\ s^{n-3} & c_1 & c_2 & c_3 \\ \vdots & \vdots & \vdots & \vdots \\ s^0 & h_1 & & \end{array}$$

where:

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}, \quad b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}, \quad c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$$

Stability Criterion:

System is stable if and only if all elements in the first column have the same sign (no sign changes)

Number of RHP poles: = Number of sign changes in first column

Special Cases:

- If first element in row is zero but others aren't: Replace with small $\epsilon > 0$ and continue
- If entire row is zero: Indicates roots on imaginary axis (marginal stability)

Root Locus Method

Root Locus: Plot of closed-loop poles as gain K varies from 0 to ∞

For system: $1 + KG(s)H(s) = 0$

Root Locus Construction Rules

Rule 1 - Number of Branches:

Number of branches = Number of poles of $G(s)H(s) = n$

Rule 2 - Starting and Ending Points:

Loci start ($K = 0$) at poles of $G(s)H(s)$

Loci end ($K = \infty$) at zeros of $G(s)H(s)$

Rule 3 - Real Axis Segments:

Locus exists on real axis to the left of an odd number of real poles and zeros

Rule 4 - Asymptotes:

Number of asymptotes: $n - m$ (poles minus zeros)

Asymptote angles:

$$\theta_k = \frac{(2k+1)\pi}{n-m}, \quad k = 0, 1, 2, \ldots, (n-m-1)$$

Centroid (intersection point):

$$\sigma_a = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m}$$

Rule 5 - Breakaway/Break-in Points:

Solve: $\frac{dK}{ds} = 0$ or $\frac{d}{ds}[G(s)H(s)] = 0$

Points where locus leaves (breakaway) or enters (break-in) real axis

Rule 6 - Imaginary Axis Crossings:

Use Routh-Hurwitz to find: - Value of K at crossing - Frequency ω at crossing

Rule 7 - Angle of Departure/Arrival:

From complex pole:

$$\theta_d = 180\text{ř} - \sum \text{angles from zeros} + \sum \text{angles from other poles}$$

To complex zero:

$$\theta_a = 180\text{ř} + \sum \text{angles from zeros} - \sum \text{angles from poles}$$

Angle and Magnitude Conditions

Angle Condition:

$$\angle G(s)H(s) = (2k+1) \cdot 180\text{ř}, \quad k = 0, \pm 1, \pm 2, \ldots$$

Magnitude Condition:

$$K = \frac{1}{|G(s)H(s)|}$$

Frequency Response Analysis

Frequency Response

Substitute $s = j\omega$ into transfer function:

$$G(j\omega) = |G(j\omega)| e^{j\angle G(j\omega)}$$

Magnitude:

$$|G(j\omega)| = \sqrt{\text{Re}^2 + \text{Im}^2}$$

Phase:

$$\angle G(j\omega) = \tan^{-1}\left(\frac{\text{Im}}{\text{Re}}\right)$$

Bode Plots

Magnitude Plot: $20\log_{10}|G(j\omega)|$ (dB) vs $\log(\omega)$

Phase Plot: $\angle G(j\omega)$ (degrees) vs $\log(\omega)$

Bode Plot Construction

Constant K: - Magnitude: $20\log_{10}K$ dB (horizontal line) - Phase: 0ř if $K > 0$, -180ř if $K < 0$

Pole at origin $\frac{1}{s}$: - Magnitude: -20 dB/decade slope - Phase: -90ř

Zero at origin s: - Magnitude: $+20$ dB/decade slope - Phase: $+90$ř

First-order pole $\frac{1}{\tau s+1}$:

Break frequency: $\omega_b = 1/\tau$

- Magnitude: 0 dB for $\omega \ll \omega_b$, -20 dB/decade for $\omega \gg \omega_b$ - Phase: 0ř for $\omega \ll \omega_b$, -45ř at ω_b, -90ř for $\omega \gg \omega_b$

First-order zero $(\tau s + 1)$:

Break frequency: $\omega_b = 1/\tau$

- Magnitude: 0 dB for $\omega \ll \omega_b$, $+20$ dB/decade for $\omega \gg \omega_b$ - Phase: 0ř for $\omega \ll \omega_b$, $+45$ř at ω_b, $+90$ř for $\omega \gg \omega_b$

Second-order pole $\frac{1}{s^2/\omega_n^2+2\zeta s/\omega_n+1}$:

Break frequency: $\omega_b = \omega_n$

- Magnitude: 0 dB for $\omega \ll \omega_n$, -40 dB/decade for $\omega \gg \omega_n$ - Resonant peak for small ζ: $M_r \approx \frac{1}{2\zeta}$ at $\omega \approx \omega_n$ - Phase: 0ř for $\omega \ll \omega_n$, -90ř at ω_n, -180ř for $\omega \gg \omega_n$

Stability Margins

Gain Margin (GM):

At phase crossover frequency ω_{pc} (where $\angle G(j\omega) = -180$ř):

$$\text{GM (dB)} = -20\log_{10}|G(j\omega_{pc})|$$

Stable if GM > 0 dB

Phase Margin (PM):

At gain crossover frequency ω_{gc} (where $|G(j\omega)| = 1$ or 0 dB):

$$\text{PM} = 180\text{ř} + \angle G(j\omega_{gc})$$

Stable if PM > 0ř

Typical Design Criteria: - GM ≥ 6 dB - PM ≥ 30ř to 60ř (typically 45ř for good performance)

Nyquist Stability Criterion

Nyquist Plot: Polar plot of $G(j\omega)H(j\omega)$ for $\omega : 0 \to \infty$

Nyquist Stability Criterion:

System is stable if:

$$Z = N + P = 0$$

where: - Z = number of closed-loop RHP poles - P = number of open-loop RHP poles - N = number of clockwise encirclements of $-1 + j0$ point

For stable open-loop system ($P = 0$): System is stable if Nyquist plot does not encircle $-1 + j0$ point

Simplified for Stable Open-Loop:

Stable if Nyquist plot passes to the left of $-1 + j0$ point

Controllers and Compensation

Proportional (P) Controller

$$G_c(s) = K_p$$

Effects: - Increases gain - Reduces steady-state error - Can destabilize system if too high - Does NOT eliminate steady-state error for step input (Type 0)

Proportional-Integral (PI) Controller

$$G_c(s) = K_p \left(1 + \frac{1}{T_i s}\right) = K_p \frac{T_i s + 1}{T_i s}$$

Effects: - Eliminates steady-state error for step input - Increases system type by 1 - May slow down response - Can reduce stability margins

Proportional-Derivative (PD) Controller

$$G_c(s) = K_p(1 + T_d s)$$

Effects: - Improves transient response - Increases damping - Improves stability margins - Does NOT affect steady-state error - Amplifies high-frequency noise

Proportional-Integral-Derivative (PID) Controller

$$G_c(s) = K_p \left(1 + \frac{1}{T_i s} + T_d s\right)$$

Practical form (with derivative filter):

$$G_c(s) = K_p + \frac{K_i}{s} + \frac{K_d s}{1 + \tau_f s}$$

Effects: - Combines benefits of P, I, and D - Zero steady-state error - Good transient response - Improved stability

Tuning Methods: - Ziegler-Nichols - Cohen-Coon - Trial and error - Software optimization

Lead Compensator

$$G_c(s) = K_c \frac{s+z}{s+p}, \quad z < p$$

Effects: - Increases phase margin - Improves transient response - Increases bandwidth - Used for phase lead

Maximum Phase Lead:

$$\phi_{max} = \sin^{-1}\left(\frac{p/z - 1}{p/z + 1}\right)$$

Occurs at $\omega_m = \sqrt{zp}$

Lag Compensator

$$G_c(s) = K_c \frac{s+z}{s+p}, \quad z > p$$

Effects: - Increases DC gain - Reduces steady-state error - Decreases bandwidth - May slow response

Lead-Lag Compensator

$$G_c(s) = K_c \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}$$

Combines benefits of both lead and lag compensation

State-Space Representation

State Equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

where: - $\mathbf{x}$ is state vector $(n \times 1)$ - u is input (scalar or vector) - y is output (scalar or vector) - $\mathbf{A}$ is system matrix $(n \times n)$ - $\mathbf{B}$ is input matrix $(n \times 1)$ - $\mathbf{C}$ is output matrix $(1 \times n)$ - $\mathbf{D}$ is feedthrough matrix (often 0)

Transfer Function from State-Space

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Stability from State-Space

System is stable if all eigenvalues of $\mathbf{A}$ have negative real parts:

$$\text{Re}(\lambda_i) < 0 \quad \text{for all } i$$

Controllability and Observability

Controllability Matrix:

$$\mathcal{C} = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}]$$

System is controllable if $\text{rank}(\mathcal{C}) = n$

Observability Matrix:

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}$$

System is observable if $\text{rank}(\mathcal{O}) = n$

Linearization

For nonlinear system $\dot{x} = f(x, u)$, linearize about equilibrium point (x_0, u_0):

$$\Delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{x_0,u_0} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0} \Delta u$$

where $\mathbf{A} = \left.\frac{\partial f}{\partial x}\right|_{x_0,u_0}$ (Jacobian)

Quick Reference

Typical Second-Order System: - Natural frequency: $\omega_n = \sqrt{k/m}$ - Damping ratio: $\zeta = c/(2\sqrt{km})$ - Settling time: $T_s \approx 4/(\zeta\omega_n)$ - Overshoot: $M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

Stability Check: 1. Closed-loop poles in LHP 2. Routh-Hurwitz: No sign changes 3. Nyquist: No encirclement of -1 4. Bode: GM > 0, PM > 0