LAPLACE EQUATION

MAIN FORMULAS FOR LAPLACE, FOURIER & PDE

$$\boxed{F(s) = L(f) = \int_0^{\infty} e^{-st} \cdot f(t)\, dt}$$

$f(t)$ is a function

$$e^{i\omega t} = \cos \omega t + i \sin \omega t$$

→ If $f'(t)$ is piece wise continuous for $t > 0$; then

$$L[f'] = s L[f] - f[0]$$

$$\boxed{\int u v' dt = uv| - \int u' v \, dt}$$

→ If f & f' is continuous and f'' is piece wise continuous, then

$$L[f''] = s^2 L[f] - s f[0] - f'[0]$$

In General;

$$\boxed{L[f^{n}] = s^{n} L[f] - s^{n-1} f(0) - s^{n-2} \cdot f'(0) + \dots f^{(n-1)}(0)}$$

LAPLACE TRANSFORM OF INTEGRALS

$$\boxed{L\left[\int_0^t f(\tau) \cdot d\tau\right] = \frac{F(s)}{s}}$$

$$\Rightarrow L^{-1}\left[\frac{1}{s} \cdot F(s)\right] = \int_0^t f(\tau) \cdot d\tau$$

INITIAL VALUE PROBLEM (IVP) : $\boxed{y'' + ay' + by = r(t)}$

If $y'' + ay' + by = r(t)$; then subsidiary equation is ;

$$\rightarrow \boxed{s^2 Y(s) - s\, y(0) - y'(0) + a[sY(s) - y(0)] + b\, Y(s) = R(s)} \rightarrow \text{Remember}$$

$$\Rightarrow y(s)[s^2 + as + b] = (s+a)\, y(0) + y'(0) + R(s)$$

$$\Rightarrow y(s) = \frac{1}{s^2 + as + b} \left[(s+a) \cdot y(0) + y'(0) + R(s)\right]$$

$$\boxed{\begin{aligned} Y(s) &= L[y(t)] \\ R(s) &= L[r(t)] \end{aligned}}$$

Now; INVERSE LAPLACE TRANSFORM is :

$$y(t) = L^{-1}[Y(s)]$$

SOLVE PDE with Laplace transform wrt (x, t)

sin (n

cos (m

FOURIER SERIES

Let there be a periodic function of $f(x) = f(x+p)$; where p

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)}$$; where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx \, dx$$

ORTHOGONALITY OF TRIGONOMETRIC FUNCTIONS OF FOURIER SERIES

Trigonometric systems $\{\cos nx\}$; $\{\sin nx\}$ are orthogonal within interval $[-\pi, \pi]$; if their inner product $= 0$

$$f \cdot g = \int_{-\pi}^{\pi} (f g)\, dx$$

$$\int_{-\pi}^{\pi} \cos nx \cdot \cos mx \, dx = 0 \quad ; \text{ if } m \neq n$$

$$\int_{-\pi}^{\pi} \sin nx \cdot \sin mx \, dx = 0 \quad ; \text{ if } m \neq n$$

$$\int_{-\pi}^{\pi} \sin nx \cdot \cos mx \, dx = 0 \quad ; \text{ if } m \neq n$$

CONVERGENCE AND SUM OF FOURIER SERIES

If 'f' is a periodic function and '$p = 2\pi$' & also it is piece wise continuous in $[-\pi, \pi]$ interval and $f(x)$ has a left hand & right hand derivative at each point ; then

① fourier series converges

② sum of series is $\left\{ \frac{f(x_0^{+0}) + f(x_0^{-0})}{2} \right\}$

FUNCTIONS OF ARBITRARY PERIOD ; $\boxed{p = 2L}$

$p \to$ period

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]}$$; where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

EVEN & ODD FUNCTIONS – HALF RANGE EXPANSIONS

EVEN FUNCTION : If $f(x) = f(-x)$; eg $\cos x$, x^2, $x \sin x$

$$\boxed{\int_{-L}^{L} f(x)\, dx = 2 \int_{0}^{L} f(x)\, dx}$$

ODD FUNCTION : If $g(-x) = -g(x)$; eg $\sin x$, x, x^3, $x \cos x$

$$\boxed{\int_{-L}^{L} g(x)\, dx = 0}$$

FOURIER COSINE SERIES

Fourier series of an even function with a period of '2L' is a fourier cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \text{; where} \qquad \left[\because b_n = 0\right]$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x).\cos\left(\frac{n\pi x}{L}\right)dx$$

Remember

FOURIER ODD SERIES

Fourier series of an odd function with a period of '2L' is a fourier sine series

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \text{; where} \qquad \left[\because a_n = 0\right] \;\&\; a_0 = 0$$

$$b_n = \frac{2}{L}\int_0^L g(x).\sin\left(\frac{n\pi x}{L}\right)dx$$

Remember

If a function lies in bisector of 1st & 3rd quadrant; then it is ODD FUNCTION

FOURIER INTEGRAL

$$f(x) = \int_0^{\infty}\left[A(\omega).\cos(\omega x) + B(\omega).\sin(\omega x)\right]d\omega \text{ ; where}$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v).\cos(\omega v)\,dv \text{ ; \&}$$

interval $[-L, L]$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v).\sin(\omega v)\,dv$$

If $f(x)$ is EVEN FUNCTION; then $B(\omega) = 0$ & $A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v).\cos(\omega v)dv$

If $f(x)$ is ODD FUNCTION; then $A(\omega) = 0$ & $B(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v).\sin(\omega v)dv$

...RIER COSINE / SINE TRANSFORMATION

...: For an even function $f(x)$; it has a fourier cosine integral

$$f(x) = \int_0^{\infty} A(\omega) \cdot \cos(\omega x) \, d\omega \; ; \text{ where}$$

$$A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(v) \cdot \cos(\omega v) \, dv$$

set; $\hat{f}_c(\omega) = \sqrt{\frac{\pi}{2}} \, A(\omega)$; then

$$= \sqrt{\frac{\pi}{2}} \left[\frac{2}{\pi} \int_0^{\infty} f(v) \cdot \cos(\omega v) \, dv \right]$$

$$= \sqrt{\frac{\pi}{2}} \left[\frac{\sqrt{2}}{\sqrt{\pi}} \cdot \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} f(v) \cdot \cos(\omega v) \, dv \right]$$

$$\boxed{f_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cdot \cos(\omega x) \, dx}$$

$$\therefore \boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c \cdot \cos(\omega x) \, d\omega}$$

Sine : For an odd function $f(x)$; it has a fourier sine integral

$$\boxed{f_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cdot \sin(\omega x) \, dx}$$

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(\omega) \cdot \sin(\omega x) \, d\omega}$$

COSINE AND SINE TRANSFORM OF DERIVATIVES

$$\mathcal{F} \Longleftrightarrow \hat{f}$$

$$\boxed{\begin{aligned} F_c\{f'\} &= \omega F_s\{f\} - \sqrt{\tfrac{2}{\pi}}\, f(0) \\ F_s\{f'\} &= -\omega F_c\{f\} \end{aligned}}$$

& $F_c\{f''\} = -\omega^2 F_c\{f\} - \sqrt{\frac{2}{\pi}}\, f'(0)$

& $F_s\{f''\} = -\omega^2 F_s\{f\} + \sqrt{\frac{2}{\pi}}\, \omega \cdot f(0)$

COMPLEX FORM OF FOURIER TRANSFORM BY INTEGRATION

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) \cdot e^{-i\omega v} \, dv \right]}_{\hat{f}(\omega) \to \text{Complex fourier transform}} e^{i\omega x} \, d\omega$$

Fourier transform ; $\hat{f}(\omega) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) \cdot e^{-i\omega x} \, dx$

Inverse fourier transform ; $f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(\omega) \cdot e^{i\omega x} \, d\omega$

SIFTING PROPERTY of $\delta(t-a)$

$$\int_0 g(t).\ \delta(t-a).dt = g(a)$$

CONVOLUTION

$$\boxed{f(t).\ g(t) = \int_0^t f(\tau).\ g(t-\tau).d\tau.}$$

CONVOLUTION THEOREM :

If $L[f(t)] = F(s)$ &

$L[g(t)] = G(s)$; then

$$L(f * g) = F(s).\ G(s)$$

INTEGRATION OF LAPLACE TRANSFORM

$f(t)$.. $L[f(t)] = F(s)$

$$\therefore L\left[\frac{f(t)}{t}\right] = \int_0^\infty F(\tilde{s}).d\tilde{s}$$

$$\Rightarrow \boxed{L^{-1}\left[\int_s^\infty F(\tilde{s})\,d\tilde{s}\right] = \frac{f(t)}{t}}$$

DIFFERENTIATION OF LAPLACE TRANSFORM

$$L[f(t)] = F(s)$$

$$\therefore L[t.\ f(t)] = -F'(s)$$

$$\Rightarrow \boxed{L^{-1}[F'(s)] = -t.f(t)}$$

EQUATION SOLUTION : $\boxed{\frac{\partial^2 U}{\partial t^2} = c^2 \cdot \frac{\partial^2 U}{\partial x^2}}$

$\sin\left(\frac{n\pi x}{L}\right)$

$= B_n \cos(\lambda_n t) + D_n \sin(\lambda_n t)$; where $\boxed{\lambda_n = \frac{cn\pi}{L}}$

$u_n(x,t) = G_n(t) \cdot F_n(x)$

$\Rightarrow u_n(x,t) = \left[B_n \cos(\lambda_n t) + D_n \sin(\lambda_n t)\right] \cdot \sin\left(\frac{n\pi x}{L}\right)$

General solution is :

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[B_n \cos(\lambda_n t) + D_n \sin(\lambda_n t)\right] \cdot \sin\left(\frac{n\pi x}{L}\right)} ;$$

where $B_n = \frac{2}{L} \int_0^L f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx$ and

$$D_n = \frac{1}{\lambda_n} \cdot \frac{2}{L} \int_0^L g(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

HEAT EQUATION SOLUTION : $\boxed{\frac{\partial U}{\partial t} = c^2 \cdot \frac{\partial^2 U}{\partial x^2}}$

$$\boxed{u = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \cdot \sin\left(\frac{n\pi x}{L}\right) ; \text{ where}}$$

$$B_n = \frac{2}{L} \int_0^L f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

...ge of variables in double integral

In 1D ; $\int_a^b f(x)\,dx = \int_c^d f\{x(u)\}\cdot \frac{dx}{du}\cdot du$

In 2D ; $\iint f(x,y)\,dx\,dy = \iint f\{x(u,v);\, y(u,v)\}\cdot \underbrace{|J|}_{\text{Jacobian matrix}}\cdot du\,dv$

$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$

If polar coordinate ; assume; $x = r\cos\theta$
$y = r\sin\theta$

→ If function is not path independent ; then follow 3 steps :
a) $\vec{r}\cdot\vec{r}'$
b) $\vec{F}\cdot\vec{r}'$
c) $\int dt$

GREEN's theorem : $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C \vec{F}\cdot d\vec{r}$; where $\vec{F} = [F_1, F_2]$

SURFACE INTEGRAL OF A VECTOR FIELD

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_R \vec{F}\{\vec{r}(u,v)\}\cdot\vec{N}(u,v)\,du\,dv$$

DIVERGENCE THEOREM

$$\iiint_T \operatorname{div}\vec{F}\cdot dV = \iint_{S=\partial T} \vec{F}\cdot\vec{n}\,dA = \iint \vec{F}\cdot\{r(u,v)\}\cdot\vec{N}(u,v)\,du\,dv$$

STOKE's THEOREM

$$\iint_S (\operatorname{curl}\vec{F})\cdot\vec{n}\,dA = \oint_C \vec{F}\cdot d\vec{r} = \oint F\cdot\vec{r}'(t)\,dt$$

EIGEN VALUE & EIGEN VECTOR

$[A-\lambda I]\vec{X} = 0$ Let $A = \begin{bmatrix} a & b \\ c & d\end{bmatrix} \Rightarrow [A-\lambda I] = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda\end{bmatrix}$

Gradient : $\text{grad } f = \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

Directional Derivative :

$D_{\vec{b}} f$ or $\frac{df}{ds}$ of a function $f(x, y, z)$ at a point 'P';

in the direction of a unit vector $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

$$\nabla f \text{ at } P = \left[\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P); \frac{\partial f}{\partial z}(P)\right]$$

$$\therefore D_{\vec{b}} f = \left[\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P); \frac{\partial f}{\partial z}(P)\right]\left(\frac{\vec{b}}{\sqrt{b_x^2 + b_y^2 + b_z^2}}\right)$$

Divergence : $\text{div } \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$

Curl of a vector :

$$\text{curl } \vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Line Integral

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b F\{\vec{r}(t)\} \cdot \frac{d\vec{r}}{dt} \cdot dt$$

$$= \int_a^b F\{\vec{r}(t)\} \cdot \vec{r}'(t) \cdot dt$$

Path independence of line integral :

Theorem 1 : Integration around a closed curve in D is always 'zero'.

Theorem 2 : $\vec{F} = \text{grad } f$

Theorem 3 : $\text{curl } \vec{F} = 0$. [i.e. If $\vec{F} \cdot d\vec{r}$ is exact; then $\text{curl } \vec{F} = 0$]

→ Eigen values & eigen vectors that are complex will always appear in complex conjugate pairs.

$X_1 = [\,\vdots\,]$ $X_2 = [\,\vdots\,]$

→ Symmetric matrix; if $A^T = A$

→ Skew symmetric matrix; if $A^T = -A$.

→ orthogonal matrix; if $A^T = A^{-1}$ or $\boxed{A^T A = 1}$

Determinant of an orthogonal matrix is $+1$ or -1

Complex Matrix

If we have complex vector $(\vec{u}, \vec{v}) = \overrightarrow{\vec{u}^T}\ \vec{v}$

If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$; such that $v_1 = a+ib$; then $\overrightarrow{v_1} = a - ib$

$$(\vec{v}, \vec{v}) = \overline{\vec{v}^T} . \vec{v}$$

$$= \overrightarrow{v_1}\, v_1 + \overline{v_2}\, v_2 + \dots - \overrightarrow{v_n}\, v_n$$

$$= |v_1|^2 + |v_2|^2 + \dots + |v_n|^2$$

Special complex variables

matrix is hermition if $\bar{A}^T = A$

matrix is skew hermition; if $\bar{A}^T = -A$

matrix is unitary ; if $\bar{A}^T = A^{-1}$

Diagonalization

Let A be a $(n \times n)$ matrix; & 'P' be a non singular $(n \times n)$ matrix;

then $P^{-1} A P$ is called similarity transform of A &

$\boxed{\hat{A} = P^{-1} A P \text{ is called similar to } A}$; $A = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$

↳ diagonal matrix

THEOREM 1 : If $\hat{A}$ is similar to A; then $\hat{A}$ and A; have same eigen values.

$$D = X^{-1} A X$$

$X \to$ ~~vector~~ matrix formed from eigen vectors..

ie $X = [x_1 \; x_2 \cdots x_n]$

First order ODE with constant coefficients

$$\vec{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

Try a solution ; $\vec{y} = \vec{v} e^{rt}$

r : eigen value of A

$\vec{v}$: eigen vector of A.

$$\boxed{\vec{y}(t) = c_1 \vec{v}^{(1)} e^{r_1 t} + \cdots c_n \vec{v}^{(n)} e^{r_n t}}$$

DIFFERENTIAL EQUATION (PDE)

WAVE EQUATION
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

HEAT EQUATION
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

LAPLACE EQUATION
$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots \frac{\partial^2 u}{\partial x_n^2} = 0$$

$$\nabla^2 u = 0 \quad \text{or} \quad \Delta u = 0$$

1D WAVE EQUATION
$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}} \;; \text{ where } c = \sqrt{\frac{T}{\rho}}$$

For boundary condition in space is : $u(0, t) = 0$

$u(L, t) = 0$

string is attached at two ends

Initial condition (in time)
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

$$u(x,t) = F(x).G(t)$$

$$\Rightarrow \frac{\partial u}{\partial t} = F(x).\dot{G}(t) \quad \& \quad \frac{\partial u}{\partial x} = \dot{F}(x).G(t)$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = F(x).\ddot{G}(t) \quad \Rightarrow \frac{\partial^2 u}{\partial x^2} = \ddot{F}(x).G(t) = F''(x).G(t)$$