

# Heat Transfer

## Modes of Heat Transfer

### Conduction

#### Fourier's Law:

$$q_x = -kA \frac{dT}{dx}$$

$$q_x'' = -k \frac{dT}{dx}$$

where: -  $q_x$  = heat transfer rate (W) -  $q_x''$  = heat flux (W/m<sup>2</sup>) -  $k$  = thermal conductivity (W/(m · K)) -  $A$  = cross-sectional area - Negative sign: heat flows from high to low temperature

### Convection

#### Newton's Law of Cooling:

$$q = hA(T_s - T_\infty)$$

$$q'' = h(T_s - T_\infty)$$

where: -  $h$  = convection heat transfer coefficient (W/(m<sup>2</sup> · K)) -  $T_s$  = surface temperature -  $T_\infty$  = fluid temperature

### Radiation

#### Stefan-Boltzmann Law:

$$q = \epsilon \sigma A T_s^4$$

where: -  $\epsilon$  = emissivity ( $0 \leq \epsilon \leq 1$ ) -  $\sigma$  = Stefan-Boltzmann constant =  $5.67 \times 10^{-8}$  W/(m<sup>2</sup> · K<sup>4</sup>) -  $T_s$  = absolute temperature (K)

#### Net Radiation Exchange:

$$q = \epsilon \sigma A (T_s^4 - T_{surr}^4)$$

#### Linearized Radiation:

$$q = h_r A (T_s - T_{surr})$$

where  $h_r = \epsilon \sigma (T_s + T_{surr})(T_s^2 + T_{surr}^2)$

## Heat Diffusion Equation

### General Form (3D, Cartesian)

$$\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

For constant  $k$ :

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

where  $\alpha = k/(\rho c_p)$  is thermal diffusivity

### Cylindrical Coordinates

$$\frac{1}{r} \frac{\partial}{\partial r} \left( k r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( k \frac{\partial T}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

### Spherical Coordinates

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( k r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial}{\partial \theta} \left( k \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left( k \sin \phi \frac{\partial T}{\partial \phi} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

## One-Dimensional Steady-State Conduction

### Plane Wall

Without Heat Generation:

$$q_x = \frac{kA(T_1 - T_2)}{L} = \frac{T_1 - T_2}{R_{cond}}$$

where  $R_{cond} = L/(kA)$  is thermal resistance

Temperature Distribution:

$$T(x) = T_1 - \frac{T_1 - T_2}{L} x$$

With Uniform Heat Generation:

$$\frac{d^2 T}{dx^2} + \frac{\dot{q}}{k} = 0$$

Solution:

$$T(x) = -\frac{\dot{q}}{2k} x^2 + C_1 x + C_2$$

## Composite Wall

### Series Resistance:

$$q = \frac{\Delta T_{overall}}{R_{total}} = \frac{T_1 - T_4}{R_1 + R_2 + R_3}$$

$$R_{total} = \sum R_i = \sum \frac{L_i}{k_i A}$$

### With Convection:

$$R_{total} = R_{conv,1} + R_{cond,1} + R_{cond,2} + \cdots + R_{conv,2}$$

where  $R_{conv} = 1/(hA)$

### Overall Heat Transfer Coefficient:

$$q = U A \Delta T$$

$$\frac{1}{UA} = R_{total}$$

## Cylindrical Systems

### Hollow Cylinder (radial conduction):

$$q_r = \frac{2\pi L k (T_1 - T_2)}{\ln(r_2/r_1)}$$

### Thermal Resistance:

$$R_{cyl} = \frac{\ln(r_2/r_1)}{2\pi L k}$$

### With convection:

$$R_{conv,inner} = \frac{1}{h_i(2\pi r_1 L)}$$

$$R_{conv,outer} = \frac{1}{h_o(2\pi r_2 L)}$$

### Critical Radius of Insulation:

$$r_{cr} = \frac{k}{h}$$

If  $r_{outer} < r_{cr}$ , adding insulation increases heat transfer

## Spherical Systems

### Hollow Sphere:

$$q_r = \frac{4\pi k r_1 r_2 (T_1 - T_2)}{r_2 - r_1}$$

### Thermal Resistance:

$$R_{sph} = \frac{r_2 - r_1}{4\pi k r_1 r_2}$$

## Extended Surfaces (Fins)

### Fin Equation

#### General Fin Equation:

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

where: -  $\theta = T - T_\infty$  -  $m = \sqrt{hP/(kA_c)}$  -  $P$  = perimeter -  $A_c$  = cross-sectional area

### Fin Solutions (Common Cases)

#### Very Long Fin ( $L \rightarrow \infty$ ):

$$\theta(x) = \theta_b e^{-mx}$$

$$q_f = \sqrt{hPkA_c} \theta_b$$

#### Insulated Tip:

$$\theta(x) = \theta_b \frac{\cosh[m(L-x)]}{\cosh(mL)}$$

$$q_f = \sqrt{hPkA_c} \theta_b \tanh(mL)$$

#### Convection at Tip:

$$\theta(x) = \theta_b \frac{\cosh[m(L-x)] + (h/mk) \sinh[m(L-x)]}{\cosh(mL) + (h/mk) \sinh(mL)}$$

### Fin Performance

#### Fin Efficiency:

$$\eta_f = \frac{q_f}{q_{max}} = \frac{q_f}{hA_f\theta_b}$$

where  $A_f$  is total fin surface area

**For fin with insulated tip:**

$$\eta_f = \frac{\tanh(mL)}{mL}$$

**Fin Effectiveness:**

$$\epsilon_f = \frac{q_f}{q_{no\ fin}} = \frac{q_f}{hA_c\theta_b}$$

**Overall Surface Efficiency:**

$$\eta_o = 1 - \frac{A_f}{A_t}(1 - \eta_f)$$

where  $A_t = A_b + A_f$  is total surface area

**Total Heat Transfer with Fins:**

$$q_{total} = \eta_o h A_t (T_b - T_\infty)$$

## Transient Conduction

### Lumped Capacitance Method

**Biot Number:**

$$Bi = \frac{hL_c}{k}$$

where  $L_c = V/A_s$  is characteristic length

**Validity:**  $Bi < 0.1$

**Temperature Response:**

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-t/\tau}$$

where  $\tau = \frac{\rho V c_p}{h A_s} = \frac{\rho L_c c_p}{h}$  is time constant

**Heat Transfer:**

$$Q(t) = \rho V c_p (T_i - T(t))$$

### Semi-Infinite Solid

**Constant Surface Temperature:**

$$\frac{T(x, t) - T_i}{T_s - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

**Surface Heat Flux:**

$$q_s''(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

### Constant Surface Heat Flux:

$$T(x, t) - T_i = \frac{2q_s''}{k} \sqrt{\frac{\alpha t}{\pi}} \exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{q_s'' x}{k} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

### Heisler Charts / Analytical Solutions

For  $Bi > 0.1$ , use charts or one-term approximation:

#### Plane Wall:

$$\frac{T(x, t) - T_\infty}{T_i - T_\infty} = C_1 \exp(-\zeta_1^2 Fo) \cos(\zeta_1 x/L)$$

#### Cylinder:

$$\frac{T(r, t) - T_\infty}{T_i - T_\infty} = C_1 \exp(-\zeta_1^2 Fo) J_0(\zeta_1 r/r_o)$$

#### Sphere:

$$\frac{T(r, t) - T_\infty}{T_i - T_\infty} = C_1 \exp(-\zeta_1^2 Fo) \frac{\sin(\zeta_1 r/r_o)}{\zeta_1 r/r_o}$$

where: -  $Fo = \alpha t/L_c^2$  is Fourier number -  $\zeta_1$  is first eigenvalue (from tables based on  $Bi$ ) -  $C_1$  is constant (from tables based on  $Bi$ )

## Forced Convection

### Dimensionless Numbers

#### Reynolds Number:

$$Re = \frac{\rho V L}{\mu} = \frac{V L}{\nu}$$

#### Prandtl Number:

$$Pr = \frac{\nu}{\alpha} = \frac{c_p \mu}{k}$$

#### Nusselt Number:

$$Nu = \frac{hL}{k}$$

#### Relationship:

$$Nu = f(Re, Pr)$$

### External Flow

**Flat Plate (Laminar,  $Re_x < 5 \times 10^5$ ):**

Local:

$$Nu_x = 0.332 Re_x^{1/2} Pr^{1/3}$$

Average:

$$\overline{Nu}_L = 0.664 Re_L^{1/2} Pr^{1/3}$$

**Flat Plate (Turbulent,  $Re_x > 5 \times 10^5$ ):**

Local:

$$Nu_x = 0.0296 Re_x^{4/5} Pr^{1/3}$$

Average (mixed):

$$\overline{Nu}_L = (0.037 Re_L^{4/5} - 871) Pr^{1/3}$$

**Cylinder in Cross Flow:**

$$\overline{Nu}_D = C Re_D^m Pr^{1/3}$$

where  $C$  and  $m$  depend on  $Re_D$  (from tables)

For  $Re_D = 0.4$  to  $4 \times 10^5$ :

$$\overline{Nu}_D = 0.3 + \frac{0.62 Re_D^{1/2} Pr^{1/3}}{[1 + (0.4/Pr)^{2/3}]^{1/4}} \left[ 1 + \left( \frac{Re_D}{282000} \right)^{5/8} \right]^{4/5}$$

**Sphere:**

$$\overline{Nu}_D = 2 + (0.4 Re_D^{1/2} + 0.06 Re_D^{2/3}) Pr^{0.4}$$

**Internal Flow**

**Fully Developed Laminar (Circular Tube):**

Constant surface temperature:  $Nu_D = 3.66$

Constant heat flux:  $Nu_D = 4.36$

**Entry Length:**

Hydrodynamic:  $L_h/D \approx 0.05 Re_D$  (laminar)

Thermal:  $L_t/D \approx 0.05 Re_D Pr$  (laminar)

**Turbulent Flow ( $Re_D > 10,000$ ):**

Dittus-Boelter equation:

$$Nu_D = 0.023 Re_D^{4/5} Pr^n$$

where  $n = 0.4$  (heating) or  $n = 0.3$  (cooling)

Gnielinski equation (more accurate):

$$Nu_D = \frac{(f/8)(Re_D - 1000)Pr}{1 + 12.7(f/8)^{1/2}(Pr^{2/3} - 1)}$$

where  $f$  is friction factor

**Non-Circular Ducts:**

Use hydraulic diameter:

$$D_h = \frac{4A_c}{P}$$

where  $A_c$  is cross-sectional area,  $P$  is wetted perimeter

## Natural (Free) Convection

### Dimensionless Numbers

**Grashof Number:**

$$Gr_L = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2}$$

where  $\beta = 1/T$  for ideal gas (use absolute temperature)

**Rayleigh Number:**

$$Ra_L = Gr_L \cdot Pr = \frac{g\beta(T_s - T_\infty)L^3}{\nu\alpha}$$

### Vertical Plate

**Laminar ( $Ra_L < 10^9$ ):**

$$\overline{Nu}_L = 0.59Ra_L^{1/4}$$

**Turbulent ( $Ra_L > 10^9$ ):**

$$\overline{Nu}_L = 0.10Ra_L^{1/3}$$

**Full Range:**

$$\overline{Nu}_L = \left[ 0.825 + \frac{0.387Ra_L^{1/6}}{[1 + (0.492/Pr)^{9/16}]^{8/27}} \right]^2$$

### Horizontal Plate

**Hot surface facing up or cold surface facing down:**

$$\overline{Nu}_L = 0.54Ra_L^{1/4} \quad (10^4 < Ra_L < 10^7)$$

$$\overline{Nu}_L = 0.15 Ra_L^{1/3} \quad (10^7 < Ra_L < 10^{11})$$

**Hot surface facing down or cold surface facing up:**

$$\overline{Nu}_L = 0.27 Ra_L^{1/4} \quad (10^5 < Ra_L < 10^{10})$$

where  $L = A_s/P$  is characteristic length

## Horizontal Cylinder

$$\overline{Nu}_D = \left[ 0.60 + \frac{0.387 Ra_D^{1/6}}{[1 + (0.559/Pr)^{9/16}]^{8/27}} \right]^2$$

## Sphere

$$\overline{Nu}_D = 2 + \frac{0.589 Ra_D^{1/4}}{[1 + (0.469/Pr)^{9/16}]^{4/9}}$$

## Boiling and Condensation

### Pool Boiling

**Regimes:** 1. Free convection boiling 2. Nucleate boiling 3. Transition boiling 4. Film boiling

**Rohsenow Correlation (Nucleate Boiling):**

$$q'' = \mu_l h_{fg} \left[ \frac{g(\rho_l - \rho_v)}{\sigma} \right]^{1/2} \left[ \frac{c_{p,l}(T_s - T_{sat})}{C_{sf} h_{fg} Pr_l^n} \right]^3$$

where  $C_{sf}$  and  $n$  depend on surface-fluid combination

**Critical Heat Flux (CHF):**

$$q''_{max} = 0.149 h_{fg} \rho_v \left[ \sigma g(\rho_l - \rho_v) / \rho_v^2 \right]^{1/4}$$

### Film Condensation

**Vertical Plate (Nusselt):**

$$\overline{Nu}_L = 0.943 \left[ \frac{g \rho_l (\rho_l - \rho_v) h'_{fg} L^3}{\mu_l k_l (T_{sat} - T_s)} \right]^{1/4}$$

where  $h'_{fg} = h_{fg} [1 + 0.68 c_{p,l} (T_{sat} - T_s) / h_{fg}]$

**Average Heat Transfer Coefficient:**

$$\bar{h} = 0.943 \left[ \frac{g\rho_l(\rho_l - \rho_v)k_l^3 h'_{fg}}{\mu_l L (T_{sat} - T_s)} \right]^{1/4}$$

**Horizontal Tube:**

$$\bar{h} = 0.729 \left[ \frac{g\rho_l(\rho_l - \rho_v)k_l^3 h'_{fg}}{\mu_l D (T_{sat} - T_s)} \right]^{1/4}$$

## Heat Exchangers

**Overall Heat Transfer Coefficient**

**For tube:**

$$\frac{1}{UA} = \frac{1}{h_i A_i} + \frac{\ln(r_o/r_i)}{2\pi k L} + \frac{1}{h_o A_o}$$

**Including fouling:**

$$\frac{1}{UA} = \frac{1}{h_i A_i} + \frac{R''_{f,i}}{A_i} + \frac{\ln(r_o/r_i)}{2\pi k L} + \frac{R''_{f,o}}{A_o} + \frac{1}{h_o A_o}$$

**Log Mean Temperature Difference (LMTD)**

$$q = UA\Delta T_{lm}$$

$$\Delta T_{lm} = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1/\Delta T_2)}$$

**Parallel Flow:**

$$\Delta T_1 = T_{h,i} - T_{c,i}, \quad \Delta T_2 = T_{h,o} - T_{c,o}$$

**Counter Flow:**

$$\Delta T_1 = T_{h,i} - T_{c,o}, \quad \Delta T_2 = T_{h,o} - T_{c,i}$$

**For other configurations:**

$$q = UAF\Delta T_{lm,cf}$$

where  $F$  is correction factor and  $\Delta T_{lm,cf}$  is for counter-flow

## Effectiveness-NTU Method

### Heat Capacity Rates:

$$C_h = (\dot{m}c_p)_h, \quad C_c = (\dot{m}c_p)_c$$

$$C_{min} = \min(C_h, C_c), \quad C_{max} = \max(C_h, C_c)$$

$$C_r = \frac{C_{min}}{C_{max}}$$

### Maximum Possible Heat Transfer:

$$q_{max} = C_{min}(T_{h,i} - T_{c,i})$$

### Effectiveness:

$$\epsilon = \frac{q}{q_{max}}$$

### Number of Transfer Units:

$$NTU = \frac{UA}{C_{min}}$$

### $\epsilon$ -NTU Relations:

Parallel flow:

$$\epsilon = \frac{1 - \exp[-NTU(1 + C_r)]}{1 + C_r}$$

Counter flow:

$$\epsilon = \frac{1 - \exp[-NTU(1 - C_r)]}{1 - C_r \exp[-NTU(1 - C_r)]}$$

For  $C_r = 0$  (phase change):

$$\epsilon = 1 - \exp(-NTU)$$

## Radiation Heat Transfer

### Radiation Properties

#### Emissivity:

$$\epsilon = \frac{E}{E_b}$$

#### Absorptivity:

$$\alpha = \frac{G_{abs}}{G}$$

**Reflectivity:**

$$\rho = \frac{G_{ref}}{G}$$

**Transmissivity:**

$$\tau = \frac{G_{tr}}{G}$$

**Energy Balance:**

$$\alpha + \rho + \tau = 1$$

For opaque surface:  $\tau = 0$ , so  $\alpha + \rho = 1$

**Kirchhoff's Law:**

$$\alpha = \epsilon$$

(for diffuse, gray surface)

**View Factor**

**Definition:**

$F_{ij}$  = fraction of radiation leaving surface  $i$  that directly strikes surface  $j$

**Reciprocity:**

$$A_i F_{ij} = A_j F_{ji}$$

**Summation Rule:**

$$\sum_{j=1}^N F_{ij} = 1$$

**For flat or convex surface:**

$$F_{ii} = 0$$

**Radiation Exchange**

**Between Two Black Surfaces:**

$$q_{12} = A_1 F_{12} \sigma (T_1^4 - T_2^4)$$

**Between Two Gray, Diffuse Surfaces:**

$$q_{12} = \frac{\sigma (T_1^4 - T_2^4)}{\frac{1-\epsilon_1}{A_1 \epsilon_1} + \frac{1}{A_1 F_{12}} + \frac{1-\epsilon_2}{A_2 \epsilon_2}}$$

**Small Object in Large Enclosure:**

$$q = A_1 \epsilon_1 \sigma (T_1^4 - T_2^4)$$

### Radiation Shield:

Radiation shield between surfaces reduces heat transfer:

$$q_{with\ shield} = \frac{q_{without\ shield}}{2}$$

(for single shield with  $\epsilon = \epsilon_1 = \epsilon_2$ )

## Finite Difference Method

### Steady-State

#### Interior Node (2D):

$$T_{m,n} = \frac{T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1}}{4}$$

#### With Heat Generation:

$$T_{m,n} = \frac{T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} + \dot{q}\Delta x^2/k}{4}$$

#### Boundary with Convection:

$$\frac{T_{m-1,n} - T_{m,n}}{\Delta x/k} = h(T_{m,n} - T_{\infty})$$

### Transient (Explicit Method)

#### Fourier Number:

$$Fo = \frac{\alpha \Delta t}{(\Delta x)^2}$$

#### Stability Criterion (1D):

$$Fo \leq \frac{1}{2}$$

#### Interior Node (1D):

$$T_m^{p+1} = Fo(T_{m+1}^p + T_{m-1}^p) + (1 - 2Fo)T_m^p$$

where superscript  $p$  denotes time step

## Quick Reference

**Typical Values of  $h$  (W/(m<sup>2</sup> · K)):** - Free convection (gases): 2-25 - Free convection (liquids): 50-1000 - Forced convection (gases): 25-250 - Forced convection (liquids): 100-20,000 - Boiling/Condensation: 2500-100,000

**Thermal Conductivity ( $\text{W}/(\text{m} \cdot \text{K})$ ):** - Metals: 15-400 - Non-metallic solids: 0.05-5 - Liquids: 0.1-0.7 - Gases: 0.01-0.1 - Insulation: 0.03-0.2