Vibrations

Laplace Transform

Definition:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

Common transforms:

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\dot{f}\} = sF(s) - f(0)$$

$$\mathcal{L}\{\ddot{f}\} = s^2F(s) - sf(0) - \dot{f}(0)$$

Final Value Theorem:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \quad \text{(stable systems)}$$

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Transfer Functions

Definition:

$$G(s) = \frac{Y(s)}{U(s)}$$

Standard forms:

$$G_1(s) = \frac{K}{\tau s + 1}$$

$$G_2(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Mechanical system (mass–spring–damper):

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

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Block Diagrams

Series:

$$G_{\text{eq}} = G_1 G_2$$

Parallel:

$$G_{\text{eq}} = G_1 + G_2$$

Feedback (negative):

$$G_{cl}(s) = \frac{G(s)}{1 + G(s)H(s)}$$

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Closed-Loop Response

Characteristic equation:

$$1 + G(s)H(s) = 0$$

Second-order step response:

$$T_s \approx \frac{4}{\zeta\omega_n}$$

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

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Steady-State Error

Error:

$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

Error constants (unity feedback):

$$K_p = \lim_{s\to 0} G(s)$$

$$K_v = \lim_{s\to 0} sG(s)$$

$$K_a = \lim_{s\to 0} s^2 G(s)$$

Steady-state error:

$$e_{ss} = \begin{cases} \frac{1}{1+K_p} & \text{step} \\ \frac{1}{K_v} & \text{ramp} \\ \frac{1}{K_a} & \text{parabolic} \end{cases}$$

System type:

$$\text{Type} = \#\text{ of integrators in } G(s)$$

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Root Locus

Rules:

- Number of branches = number of poles
- Locus starts at poles, ends at zeros
- Asymptotes: $n - m$

Asymptote angles:

$$\theta_k = \frac{(2k+1)\pi}{n-m}$$

Centroid:

$$\sigma = \frac{\sum p_i - \sum z_i}{n-m}$$

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Frequency Response

Frequency substitution:

$$s = j\omega$$

Magnitude (dB):

$$20\log_{10}|G(j\omega)|$$

Phase:

$$\angle G(j\omega)$$

Stability margins:

$$\text{Gain margin, Phase margin}$$

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Controllers

Proportional (P):

$$G_c(s) = K$$

Proportional–Integral (PI):

$$G_c(s) = K\left(1 + \frac{1}{T_i s}\right)$$

Proportional–Derivative (PD):

$$G_c(s) = K(1 + T_d s)$$

PID:

$$G_c(s) = K\left(1 + \frac{1}{T_i s} + T_d s\right)$$

Effects:

- P: faster response
- I: zero steady-state error
- D: improved damping

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State Space

State equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + \mathbf{D}u$$

Transfer function:

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Stability:

$$\Re(\lambda_i(\mathbf{A})) < 0$$

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