Fluid Mechanics

Fluid Properties

Density and Specific Weight

Density:

$$\rho = \frac{m}{V}$$

Specific Weight:

$$\gamma = \rho g$$

Specific Gravity:

$$SG = \frac{\rho}{\rho_{H_2O}} = \frac{\gamma}{\gamma_{H_2O}}$$

Viscosity

Dynamic (Absolute) Viscosity:

$$\tau = \mu \frac{du}{dy}$$

Kinematic Viscosity:

$$\nu = \frac{\mu}{\rho}$$

Newton's Law of Viscosity: Fluids where $\tau \propto du/dy$ are Newtonian

Typical Values: - Water at 20°C: $\mu = 1.0 \times 10^{-3}$ Pa $\cdot$ s, $\nu = 1.0 \times 10^{-6}$ m^2/s - Air at 20°C: $\mu = 1.8 \times 10^{-5}$ Pa $\cdot$ s, $\nu = 1.5 \times 10^{-5}$ m^2/s

Surface Tension

$$\Delta P = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

For spherical droplet:

$$\Delta P = \frac{2\sigma}{R}$$

For cylindrical jet:

$$\Delta P = \frac{\sigma}{R}$$

Capillary Rise:

$$h = \frac{2\sigma \cos\theta}{\rho g r}$$

where θ is contact angle

Bulk Modulus

Compressibility:

$$E_v = -V\frac{dP}{dV} = \rho\frac{dP}{d\rho}$$

Speed of Sound:

$$c = \sqrt{\frac{E_v}{\rho}}$$

For ideal gas: $c = \sqrt{kRT}$

Fluid Statics

Pressure Variation

Hydrostatic Equation:

$$\frac{dP}{dz} = -\rho g$$

For incompressible fluid:

$$P_2 - P_1 = -\rho g(z_2 - z_1) = \rho gh$$

Absolute vs Gage Pressure:

$$P_{abs} = P_{gage} + P_{atm}$$

Manometry

U-Tube Manometer:

$$P_A - P_B = \rho_{fluid} gh$$

Differential Manometer:

$$P_A - P_B = (\rho_2 - \rho_1)gh$$

Forces on Submerged Surfaces

Horizontal Flat Surface:

$$F_R = P_c A = \rho g h_c A$$

where h_c is depth to centroid

Location of Center of Pressure:

$$y_{cp} = y_c + \frac{I_{xc}}{y_c A}$$

where I_{xc} is second moment of area about centroidal axis

Inclined Flat Surface:

Same formulas with y measured along inclined surface

Curved Surface:

Horizontal component: $F_H = \rho g h_c A_{projected}$

Vertical component: $F_V = \rho g V_{fluid}$

Buoyancy

Buoyant Force (Archimedes):

$$F_B = \rho_{fluid} g V_{displaced}$$

Acts upward through centroid of displaced volume (center of buoyancy)

Floating Body:

$$W = F_B$$

$$\rho_{body} V_{body} g = \rho_{fluid} V_{submerged} g$$

Stability:

Stable if metacenter is above center of gravity

Kinematics

Velocity Field

Eulerian Description:

$$\mathbf{V} = u(x, y, z, t)\mathbf{i} + v(x, y, z, t)\mathbf{j} + w(x, y, z, t)\mathbf{k}$$

Lagrangian Description:

Follow individual particles

Acceleration

Material (Substantial) Derivative:

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}$$

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

Local acceleration + Convective acceleration

Flow Classification

Steady: $\partial/\partial t = 0$

Uniform: $\partial/\partial s = 0$ along streamline

Incompressible: $\rho =$ constant

Streamlines, Pathlines, Streaklines

Streamline: Tangent to velocity at instant

For 2D: $\frac{dy}{dx} = \frac{v}{u}$

Pathline: Trajectory of fluid particle

Streakline: Locus of particles passing through a point

For steady flow: all three coincide

Vorticity and Circulation

Vorticity:

$$\boldsymbol{\omega} = \nabla \times \mathbf{V}$$

In 2D:

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Circulation:

$$\Gamma = \oint_C \mathbf{V} \cdot d\mathbf{s}$$

Irrotational Flow: $\nabla \times \mathbf{V} = 0$

Conservation Laws

Conservation of Mass (Continuity)

Differential Form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

Incompressible:

$$\nabla \cdot \mathbf{V} = 0$$

In Cartesian:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Integral Form (Control Volume):

$$\frac{\partial}{\partial t} \int_{CV} \rho \, dV + \int_{CS} \rho \mathbf{V} \cdot \mathbf{n} \, dA = 0$$

Steady Flow:

$$\sum \dot{m}_{in} = \sum \dot{m}_{out}$$

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

For incompressible:

$$Q = AV = \text{constant}$$

Momentum Equation

Differential Form (Navier-Stokes):

For incompressible, Newtonian fluid:

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla P + \mu \nabla^2 \mathbf{V} + \rho \mathbf{g}$$

x-component:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x$$

Euler's Equation (Inviscid):

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla P + \rho \mathbf{g}$$

Integral Form (Control Volume):

$$\sum \mathbf{F} = \frac{\partial}{\partial t} \int_{CV} \rho \mathbf{V} \, dV + \int_{CS} \rho \mathbf{V} (\mathbf{V} \cdot \mathbf{n}) \, dA$$

Steady Flow:

$$\sum \mathbf{F} = \sum \dot{m}_{out} \mathbf{V}_{out} - \sum \dot{m}_{in} \mathbf{V}_{in}$$

Energy Equation

First Law (Control Volume):

$$\dot{Q} - \dot{W}_s = \frac{\partial}{\partial t}\int_{CV} \rho e \, dV + \int_{CS} \rho \left(e + \frac{P}{\rho}\right) (\mathbf{V} \cdot \mathbf{n}) \, dA$$

where $e = u + V^2/2 + gz$ and $\dot{W}_s$ is shaft work

Bernoulli Equation

Along a Streamline:

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

or

$$\frac{P}{\gamma} + \frac{V^2}{2g} + z = \text{constant}$$

Assumptions: - Steady flow - Incompressible - Inviscid (frictionless) - Along a streamline

Extended Bernoulli (with losses):

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_L$$

With pump/turbine:

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_t + h_L$$

Pitot Tube:

$$V = \sqrt{\frac{2(P_0 - P)}{\rho}} = \sqrt{2gh}$$

where P_0 is stagnation pressure

Dimensional Analysis

Buckingham Pi Theorem

If physical phenomenon involves n variables and m fundamental dimensions:

Number of dimensionless groups $= n - m$

Important Dimensionless Numbers

Reynolds Number:

$$Re = \frac{\rho V L}{\mu} = \frac{VL}{\nu} = \frac{\text{inertial forces}}{\text{viscous forces}}$$

Froude Number:

$$Fr = \frac{V}{\sqrt{gL}} = \frac{\text{inertial forces}}{\text{gravity forces}}$$

Euler Number:

$$Eu = \frac{P}{\rho V^2} = \frac{\text{pressure forces}}{\text{inertial forces}}$$

Mach Number:

$$Ma = \frac{V}{c} = \frac{\text{flow velocity}}{\text{sound velocity}}$$

Weber Number:

$$We = \frac{\rho V^2 L}{\sigma} = \frac{\text{inertial forces}}{\text{surface tension forces}}$$

Drag Coefficient:

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$$

Lift Coefficient:

$$C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A}$$

Viscous Flow in Pipes

Flow Regimes

Laminar: $Re < 2300$

Transitional: $2300 < Re < 4000$

Turbulent: $Re > 4000$

Fully Developed Laminar Flow

Velocity Profile (Poiseuille Flow):

$$u(r) = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

where $u_{max} = \frac{R^2}{4\mu}\left(-\frac{dP}{dx}\right)$

Average Velocity:

$$V = \frac{u_{max}}{2} = \frac{R^2}{8\mu}\left(-\frac{dP}{dx}\right) = \frac{D^2}{32\mu}\left(-\frac{dP}{dx}\right)$$

Hagen-Poiseuille Equation:

$$\Delta P = \frac{32\mu LV}{D^2} = \frac{128\mu LQ}{\pi D^4}$$

Friction Factor:

$$f = \frac{64}{Re}$$

Turbulent Flow

Friction Factor:

Smooth pipes (Blasius):

$$f = \frac{0.316}{Re^{1/4}} \quad (Re < 10^5)$$

Colebrook equation (implicit):

$$\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{\epsilon/D}{3.7} + \frac{2.51}{Re\sqrt{f}}\right)$$

where ϵ is roughness height

Moody Chart: Graphical representation of f vs Re for various ϵ/D

Power Law Velocity Profile:

$$u(r) = u_{max}\left[1 - \frac{r}{R}\right]^{1/n}$$

where $n \approx 7$ for turbulent flow

Head Loss

Major Loss (Darcy-Weisbach):

$$h_L = f\frac{L}{D}\frac{V^2}{2g}$$

Minor Losses:

$$h_L = K_L\frac{V^2}{2g}$$

where K_L is loss coefficient (from tables)

Common values: - Pipe entrance (sharp): $K_L = 0.5$ - Pipe exit: $K_L = 1.0$ - 90° elbow: $K_L \approx 0.9$ - Gate valve (open): $K_L \approx 0.2$

Total Head Loss:

$$h_L = f\frac{L}{D}\frac{V^2}{2g} + \sum K_L\frac{V^2}{2g}$$

Pipe Systems

Pipes in Series:

$$Q = Q_1 = Q_2 = Q_3$$

$$h_L = h_{L1} + h_{L2} + h_{L3}$$

Pipes in Parallel:

$$Q = Q_1 + Q_2 + Q_3$$

$$h_L = h_{L1} = h_{L2} = h_{L3}$$

Three-Reservoir Problem:

Use continuity and energy equations at junction

External Flow

Boundary Layer

Boundary Layer Thickness (δ):

Height where $u = 0.99U_\infty$

Displacement Thickness:

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

Momentum Thickness:

$$\theta = \int_0^\infty \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$$

Flat Plate

Laminar ($Re_x < 5 \times 10^5$):

Boundary layer thickness:

$$\delta = \frac{5x}{\sqrt{Re_x}}$$

Local skin friction coefficient:

$$C_{f,x} = \frac{0.664}{\sqrt{Re_x}}$$

Average skin friction coefficient:

$$\overline{C}_f = \frac{1.328}{\sqrt{Re_L}}$$

Turbulent ($Re_x > 5 \times 10^5$):

Boundary layer thickness:

$$\delta = \frac{0.37x}{Re_x^{1/5}}$$

Local skin friction coefficient:

$$C_{f,x} = \frac{0.059}{Re_x^{1/5}}$$

Average (with turbulent from leading edge):

$$\overline{C}_f = \frac{0.074}{Re_L^{1/5}}$$

Drag Force:

$$F_D = \overline{C}_f \left(\frac{1}{2}\rho V^2\right) A_s$$

where A_s is surface area

Flow Over Cylinder and Sphere

Drag:

$$F_D = C_D \left(\frac{1}{2}\rho V^2\right) A_f$$

where A_f is frontal area

Typical C_D Values:

Sphere: - $Re < 1$: $C_D = 24/Re$ (Stokes flow) - $10^3 < Re < 2 \times 10^5$: $C_D \approx 0.4$ - $Re > 3 \times 10^5$: $C_D \approx 0.1$ (turbulent, drag crisis)

Cylinder: - $Re < 1$: $C_D \approx 10/Re$ - $10^3 < Re < 2 \times 10^5$: $C_D \approx 1.2$

Terminal Velocity:

When $F_D = W - F_B$:

$$V_t = \sqrt{\frac{2(W - F_B)}{\rho C_D A_f}}$$

Potential Flow

Velocity Potential

For irrotational flow:

$$\mathbf{V} = \nabla \phi$$

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}$$

Laplace Equation:

$$\nabla^2\phi = 0$$

Stream Function

For 2D incompressible flow:

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}$$

Lines of constant ψ are streamlines

Volume Flow Rate:

Between two streamlines:

$$Q = \psi_2 - \psi_1$$

Elementary Flows

Uniform Flow:

$$\phi = Ux, \quad \psi = Uy$$

Source/Sink:

$$\phi = \frac{m}{2\pi}\ln r, \quad \psi = \frac{m}{2\pi}\theta$$

where m is strength (positive for source)

Vortex:

$$\phi = \frac{\Gamma}{2\pi}\theta, \quad \psi = -\frac{\Gamma}{2\pi}\ln r$$

where Γ is circulation

Doublet:

$$\phi = -\frac{\Lambda\cos\theta}{r}, \quad \psi = \frac{\Lambda\sin\theta}{r}$$

Superposition:

Complex potential flows can be created by superposition of elementary flows

Compressible Flow

Speed of Sound

$$c = \sqrt{\left.\frac{\partial P}{\partial\rho}\right|_s}$$

For ideal gas:

$$c = \sqrt{kRT}$$

Mach Number Relations

Subsonic: $Ma < 1$

Sonic: $Ma = 1$

Supersonic: $Ma > 1$

Hypersonic: $Ma > 5$

Isentropic Relations

Stagnation Properties:

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} Ma^2$$

$$\frac{P_0}{P} = \left(1 + \frac{k-1}{2} Ma^2\right)^{k/(k-1)}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{k-1}{2} Ma^2\right)^{1/(k-1)}$$

Critical Conditions ($Ma = 1$):

$$\frac{T^*}{T_0} = \frac{2}{k+1}$$

$$\frac{P^*}{P_0} = \left(\frac{2}{k+1}\right)^{k/(k-1)}$$

Normal Shock

Across shock:

$$\frac{P_2}{P_1} = 1 + \frac{2k}{k+1}(Ma_1^2 - 1)$$

$$\frac{\rho_2}{\rho_1} = \frac{(k+1)Ma_1^2}{(k-1)Ma_1^2 + 2}$$

$$Ma_2^2 = \frac{(k-1)Ma_1^2 + 2}{2kMa_1^2 - (k-1)}$$

Shock always: $Ma_1 > 1$, $Ma_2 < 1$

Open Channel Flow

Froude Number

$$Fr = \frac{V}{\sqrt{gy}}$$

where y is flow depth

Subcritical: $Fr < 1$ (tranquil)

Critical: $Fr = 1$

Supercritical: $Fr > 1$ (rapid)

Manning Equation

$$V = \frac{1}{n} R_h^{2/3} S^{1/2}$$

where: - n = Manning roughness coefficient - $R_h = A/P$ = hydraulic radius - S = channel slope

Flow Rate:

$$Q = \frac{1}{n} A R_h^{2/3} S^{1/2}$$

Specific Energy

$$E = y + \frac{V^2}{2g}$$

Critical Depth:

For rectangular channel:

$$y_c = \left(\frac{q^2}{g}\right)^{1/3}$$

where $q = Q/b$ is discharge per unit width

Quick Reference

Common Fluid Properties (at 20°C, 1 atm):

Water: - $\rho = 998$ kg/m^3 - $\mu = 1.0 \times 10^{-3}$ Pa $\cdot$ s - $\nu = 1.0 \times 10^{-6}$ m^2/s

Air: - $\rho = 1.20$ kg/m^3 - $\mu = 1.8 \times 10^{-5}$ Pa $\cdot$ s - $\nu = 1.5 \times 10^{-5}$ m^2/s

Standard Atmosphere: - $P = 101.325$ kPa $= 1$ atm - $T = 15$ř$C = 288$ K - $\rho = 1.225$ kg/m^3