Partial Differential Equations Sample Problem 1

This problem is question 6.1 on the 2023 QE.

Transverse vibrations of an idealized guitar string are described by the equation:

$$\rho \frac{\partial^2 u(x,t)}{\partial t^2} - T \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \tag{1}$$

with u the trasverse displacement, ρ the constant string density per unit length, and T the constant string tension. Assume fixed boundary condition given by $u(0,t) = u(L,t) = 0$. At $t = 0$, the string is plucvked at a point $x = 2L/3$ and released from rest. For simpicity, assume that the initial deformed shape is triangular and descirbed by $u(x,0) = 3Ax/(2L)$ for $0 < x < 2L/3$ and $u(x,0) = 3A(L-x)/L$ for $L/3 < x < L$. Determine the ensuing string vibrations using separation of variables.

1 Rewrite in Standard Form

First, we rewrite the equation in standard form to isolate the second time derivative.

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \cdot \frac{\partial^2 u}{\partial x^2} \tag{2}$$

For simplicity, we will let $c^2 = \frac{T}{\rho}$, so that the equation becomes:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3}$$

2 Separation of Variables

First, we will assume that the solution has the form:

$$u(x,t) = X(x)T(t) \tag{4}$$

We can then substitute this into our standard form equation.

$$X(x)T''(t) = c^2 X''(x)T(t) \tag{5}$$

Then, rearrange the equation to isolate the variables again and set them equal to a separation constant $-\lambda$.

$$\frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} = -\lambda \tag{6}$$

3 Solve the Spatial Equation

We will start by solving the X equation. First, rearrange it to make it equal to zero.

$$X'' + \frac{\lambda}{c^2}X = 0 \tag{7}$$

We will let $k^2 = \frac{\lambda}{c^2}$, so that the equation becomes:

$$X'' + k^2 X = 0 \tag{8}$$

The general solution to this equation is:

$$X(x) = A\cos(kx) + B\sin(kx) \tag{9}$$

We can then apply the boundary conditions to find the value of A. Applying the boundary condition at $x = 0$:

$$X(0) = A\cos(0) + B\sin(0) = A = 0 \tag{10}$$

Thus, the solution simplifies to:

$$X(x) = B\sin(kx) \tag{11}$$

Next, we apply the boundary condition at $x = L$:

$$X(L) = B\sin(kL) = 0 \tag{12}$$

For a non-trivial solution, we require that $\sin(kL) = 0$, which implies:

$$kL = n\pi \quad \text{for } n = 1, 2, 3, \dots \tag{13}$$

Thus, we have:

$$k = \frac{n\pi}{L} \tag{14}$$

We can substitute back into our spatial solution to get:

$$X_n(x) = B\sin\left(\frac{n\pi x}{L}\right) \tag{15}$$

4 Solve the Temporal Equation

Next, we will solve the T equation. First, rearrange it to make it equal to zero.

$$T'' + \lambda T = 0 \tag{16}$$

The general solution to this equation is:

$$T_n(t) = C\cos\left(\frac{n\pi ct}{L}\right) + D\sin\left(\frac{n\pi ct}{L}\right) \tag{17}$$

5 Combine Solutions

We can now combine the spatial and temporal solutions to get the general solution for $u(x,t)$:

$$u(x,t) = \sum_{n=1}^{\infty} \left[B_n \sin\left(\frac{n\pi x}{L}\right) \left(C_n \cos\left(\frac{n\pi ct}{L}\right) + D_n \sin\left(\frac{n\pi ct}{L}\right)\right)\right] \tag{18}$$

6 Apply Initial Conditions

We will now apply the initial conditions to determine the coefficients B_n, C_n, and D_n. The initial displacement condition is given by:

$$u(x,0) = \sum_{n=1}^{\infty} B_n C_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} \frac{3Ax}{2L} & 0 < x < \frac{2L}{3} \\ \frac{3A(L-x)}{2L} & \frac{2L}{3} < x < L \end{cases} \tag{19}$$

To find the coefficients $B_n C_n$, we can use the orthogonality of the sine functions:

$$B_n C_n = \frac{2}{L} \int_0^L u(x,0) \sin\left(\frac{n\pi x}{L}\right) dx \tag{20}$$

Calculating this integral piecewise, we have:

$$B_n C_n = \frac{2}{L} \left(\int_0^{\frac{2L}{3}} \frac{3Ax}{2L} \sin\left(\frac{n\pi x}{L}\right) dx + \int_{\frac{2L}{3}}^{L} \frac{3A(L-x)}{2L} \sin\left(\frac{n\pi x}{L}\right) dx \right) \tag{21}$$

Evaluating these integrals will yield the coefficients $B_n C_n$. The initial velocity condition is given by:

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n D_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right) = 0 \tag{22}$$

Since the initial velocity is zero, we have:

$$B_n D_n = 0 \quad \text{for all } n \tag{23}$$

Thus, we conclude that $D_n = 0$ for all n.

7 Final Solution

The final solution for the string vibrations is given by:

$$u(x,t) = \sum_{n=1}^{\infty} B_n C_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) \tag{24}$$