Math 2021

1 Laplace Transforms

1.1

Let

$$\mathcal{L}[f] = \frac{1}{s^2(s^2+\omega^2)}$$

Find $f(t)$

$$\frac{1}{s^2(s^2+\omega^2)} = \frac{s}{s^3(s^2+\omega^2)} = FG \tag{1}$$

$$F = \frac{1}{s^3} \qquad H = \frac{s}{s^2+\omega^2} \tag{2}$$

Use

Laplace transform of convolution function

Given the convolution of two functions

$$h = (f \circ g)(t) = \int_0^\tau f(\tau)g(t-\tau)d\tau$$

Then the laplace transform is

$$\mathcal{L}[h] = \mathcal{L}\left[\int_0^\tau f(\tau)g(t-\tau)d\tau\right] = FG$$

$$\mathcal{L}^{-1}\left[F = \frac{1}{s^3}\right] \qquad \mathcal{L}^{-1}\left[G = \frac{s}{s^2+\omega^2}\right] \tag{3}$$

$$f = \frac{t^2}{2} \qquad g = cos(\omega t) \tag{4}$$

Conduct the convolution integral $h = \int_0^t f(\tau)g(t-\tau)d\tau$ using integration by parts $\int udv = uv - \int vdu$. Define u and v as

$$u \equiv \frac{\tau^2}{2} \rightarrow du = \tau d\tau \tag{5}$$

$$dv \equiv cos(\omega(t-\tau))d\tau \rightarrow v = \frac{-1}{\omega}sin(\omega(t-\tau)) \tag{6}$$

The uv term follows

$$uv = \frac{\tau^2}{2}\frac{-1}{\omega}sin(\omega(t-\tau))|_0^t \tag{7}$$

$$= \frac{-\tau^2}{2\omega}sin(0) - \frac{-0}{2\omega}sin(\omega t) \tag{8}$$

$$uv = 0 \tag{9}$$

The integral may need another round of integration by parts

$$-\int vdu = \int_0^t \frac{\tau}{\omega}sin(\omega(t-\tau))d\tau \tag{10}$$

$$u \equiv \frac{\tau}{\omega} \rightarrow du = \frac{d\tau}{\omega} \tag{11}$$

$$dv \equiv sin(\omega(t-\tau))d\tau \rightarrow v = \frac{-1}{\omega}cos(\omega(t-\tau)) \tag{12}$$

$$uv = \frac{-\tau}{\omega^2}cos(\omega(t-\tau))|_0^t = \frac{-t}{\omega^2} \tag{13}$$

$$-\int vdu = \int \frac{cos(\omega(t-\tau))}{\omega^2}d\tau = \frac{sin(\omega(t-\tau))}{\omega^3}|_0^t = -\frac{sin(\omega t)}{\omega^3} \tag{14}$$

$$= \frac{-t}{\omega^2} - \frac{1}{\omega^3}sin(\omega t) \tag{15}$$

Which give the final answer - I think, haven't double checked yet -

$$\boxed{f(t) = \frac{-t}{\omega^2} - \frac{1}{\omega^3}sin(\omega t)} \tag{16}$$

1.2

Using Laplace transform properties, solve the following integral equation for $y(t)$:

$$y(t) = t + \int_0^t y(\tau) sin(t-\tau) d\tau$$

Take the Laplace Transform for both sides of the equation, and for the intigral term use:

Laplace transform of convolution function

Given the convolution of two functions

$$h = (f \circ g)(t) = \int_0^\tau f(\tau) g(t-\tau) d\tau$$

Then the laplace transform is

$$\mathcal{L}[h] = \mathcal{L}\left[\int_0^\tau f(\tau) g(t-\tau) d\tau\right] = FG$$

$$\mathcal{L}\left[y(t)\right] = \mathcal{L}\left[t\right] + \mathcal{L}\left[\int_0^t y(\tau) sin(t-\tau) d\tau\right] \tag{17}$$

$$Y = \frac{1}{s} + Y\frac{1}{s^2+1} \tag{18}$$

$$Y = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} \tag{19}$$

Take the inverse transform

$$\boxed{y(t) = t + \frac{t^3}{6}} \tag{20}$$

2 Ordinary Differential Equations

2.1

Solve the following differential equation for $y(x)$:

$$xy''(x) + y'(x) = 0$$

$$0 = xy'' + y' \tag{21}$$

$$-y' = x\frac{dy'}{dx} \tag{22}$$

$$\frac{-dx}{x} = \frac{dy'}{y'} \tag{23}$$

$$-ln(x) + c = ln(y') \tag{24}$$

$$\frac{c}{x} = \frac{dy}{dx} \tag{25}$$

$$dy = \frac{c}{x}dx \tag{26}$$

$$y = c_1 ln(x) + c_2 \tag{27}$$

2.2

Solve the following differential equation for $x(t)$, with $x(0) = 1$:

$$\dot{x} + x = sin(t)$$

First, the general solution

$$0 = \dot{x} + x \tag{28}$$
$$\dot{x} = -x \tag{29}$$
$$\frac{dx}{x} = -dt \tag{30}$$
$$ln(x) = -t + c \tag{31}$$
$$\boxed{x_h = ce^{-t}} \tag{32}$$

Second the particular solution. Select a function based off the forcing function and put it through diff. eq. Note: if the selected test function has a similar term to the homogeneous solution, then multiply that term by x. For this problem the test function is $x_p = Asin(t) + Bcos(t)$

$$sin(t) = \dot{x}_p + x_p \tag{33}$$
$$= \frac{d}{dt}\left[Asin(t) + Bcos(t)\right] + \left[Asin(t) + Bcos(t)\right] \tag{34}$$
$$= \left[Acos(t) - Bsin(t)\right] + \left[Asin(t) + Bcos(t)\right] \tag{35}$$
$$= sin(t)\left[A - B\right] + cos(t)\left[A + B\right] \tag{36}$$

Now match coefficients

$$\begin{bmatrix} A & -B & 1 \\ A & B & 0 \end{bmatrix} \rightarrow \begin{bmatrix} A & 0 & \frac{1}{2} \\ 0 & B & -\frac{1}{2} \end{bmatrix} \tag{37}$$

Plug these into the test function

$$\boxed{x_p = \frac{1}{2}sin(t) - \frac{1}{2}cos(t)} \tag{38}$$

Third find the initial condition for the linear combination $x_h + x_p$

$$x(t) = ce^{-t} + \frac{1}{2}sin(t) - \frac{1}{2}cos(t) \tag{39}$$
$$x_0 = 1 = ce^0 + \frac{1}{2}sin(0) - \frac{1}{2}cos(0) \tag{40}$$
$$1 = c - \frac{1}{2} \rightarrow c = \frac{3}{2} \tag{41}$$
$$\tag{42}$$

So the resulting solution is

$$\boxed{x(t) = \frac{3}{2}e^{-t} + \frac{1}{2}sin(t) - \frac{1}{2}cos(t)} \tag{43}$$

3 Fourier Transforms

3.1

Find the Fourier Cosine Series for the EVEN periodic extension of

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 2 & 1 < x < 2 \end{cases}$$

Cosine Transform

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$
$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Compute a_0

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \to \frac{2}{L}\left[\int_0^1 1dx + \int_1^2 2dx\right] && (44) \\ &= \frac{2}{L}\left[1 + 2(2-1)\right] && (45) \\ &= \frac{3}{L} && (46) \end{aligned}$$

Compute a_n

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx && (47) \\ &= \frac{2}{L}\left[\int_0^1 \cos\left(\frac{n\pi x}{L}\right)dx + 2\int_1^2 \cos\left(\frac{n\pi x}{L}\right)dx\right] && (48) \\ &= \frac{2}{L}\left[\left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^1 + 2\left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_1^2\right] && (49) \\ &= \frac{2}{n\pi}\left[\left[\sin\left(\frac{n\pi 0}{L}\right) - \sin\left(\frac{n\pi 1}{L}\right)\right] + 2\left[\sin\left(\frac{n\pi 1}{L}\right) - \sin\left(\frac{n\pi 2}{L}\right)\right]\right] && (50) \\ &= \frac{2}{n\pi}\left[\sin\left(\frac{n\pi 1}{L}\right) - 2\sin\left(\frac{n\pi 2}{L}\right)\right] && (51) \end{aligned}$$

Plug into the general function

3.2

Compute the complex Fourier transform of

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Complex Transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx && (52) \\ &= \frac{1}{\sqrt{2\pi}} \int_{0}^{1} x e^{-i\omega x} dx && (53) \end{aligned}$$

This will require integration by parts, so define u and dv as

$$\begin{cases} u \equiv x & du = dx \\ dv \equiv e^{-i\omega x} dx & v = \frac{i}{\omega} e^{-i\omega x} \end{cases} \tag{54}$$

Plugging in for uv and $-\int vdu$

$$\begin{aligned} uv &= \frac{i}{\omega}\left[e^{-i\omega x}\right]\Big|_0^1 = \frac{i}{\omega}\left[1 * e^{-i\omega 1} - 0 * e^{-i\omega 0}\right] && (55) \\ &= \frac{i}{\omega} e^{-i\omega} && (56) \\ -\int vdu &= -\int_0^1 \frac{i}{\omega} e^{-i\omega x} dx = \frac{i^2}{\omega^2} e^{-i\omega x}|_0^1 && (57) \\ &= \frac{-1}{\omega^2}\left[e^{-i\omega} - e^0\right] && (58) \\ &= \frac{-e^{i\omega}}{\omega^2} + \frac{1}{\omega^2} && (59) \end{aligned}$$

Which results in

$$\boxed{\frac{1}{2\pi}\left[\frac{i}{\omega} e^{-i\omega} + \frac{-e^{i\omega}}{\omega^2} + \frac{1}{\omega^2}\right]} \tag{60}$$

4 Partial Differential Equations

4.1

Solve the steady state heat equation for a membrane with radius $R = 1$. The initial temperature distribution around $R = 1$ is

$$f(\theta) = \begin{cases} 100 & 0 < \theta < \pi \\ 0 & -\pi < \theta < 0 \end{cases}$$

θ is the angle with respect to the positive x-axis. **Hint**: Given the solution to a similar problem is

$$g(x) = \sum_{n=0}^{\infty} \frac{4}{2n+1} sin(2n+1)x$$

$$\text{IC: } g(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

4.2

Given $u(x,t)$ satisfies the heat equation in an infinite rod,

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$$
$$u(0) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{else} \end{cases}$$

Find the solution $u(x,t)$

Heat Equation, Infinite Rod

For the Heat Equation

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x)$$

has the solution

$$u(x,t) = \frac{1}{\sqrt{4\pi\alpha t}}\int_{-\infty}^{\infty} f(\xi)e^{(x-\xi)^2/(4\alpha t)}d\xi$$

$$\frac{1}{\sqrt{4\pi 4t}}\int_{-1}^{1} 1 * e^{(x-\xi)^2/(4*4t)}d\xi \tag{61}$$
$$\frac{1}{4\sqrt{\pi t}}\int_{-1}^{1} e^{(x-\xi)^2/(16t)}d\xi \tag{62}$$
$$\frac{1}{4\sqrt{\pi t}}\left[\frac{-16t}{2(x-\xi)}e^{(x-\xi)^2/(16t)}\right]_{-1}^{1} \tag{63}$$
$$\frac{1}{4\sqrt{\pi t}}\left[\frac{-16t}{2(x-1)}e^{(x-1)^2/(16t)} - \left[\frac{-16t}{2(x+1)}e^{(x+1)^2/(16t)}\right]\right] \tag{64}$$
$$\frac{2t}{\sqrt{\pi t}}\left[\frac{-1}{(x-1)}e^{(x-1)^2/(16t)} + \frac{1}{(x+1)}e^{(x+1)^2/(16t)}\right] \tag{65}$$

5 Linear Algebra

5.1

Find the eigenvalues and eigenvectors for the matrix A

$$A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

$$|A - I\lambda| = 0 \tag{66}$$

$$\begin{vmatrix} -\lambda & 4 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 + 16 = 0 \tag{67}$$

$$(\lambda + i4)(\lambda - i4) = 0 \tag{68}$$

$$\lambda = \pm i4 \tag{69}$$

For the eigenvalue $\lambda = i4$

$$(A - I\lambda)V = 0 \tag{70}$$

$$\begin{bmatrix} -i4 & 4 \\ -4 & -i4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{71}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{72}$$

For the eigenvalue $\lambda = -i4$

$$(A - I\lambda)V = 0 \tag{73}$$

$$\begin{bmatrix} i4 & 4 \\ -4 & i4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{74}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{75}$$

5.2

Solve the system of differential equations

$$\begin{aligned}\dot{y}_1 &= 4y_2\\ \dot{y}_2 &= -4y_1\end{aligned}$$

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} = \begin{bmatrix}0 & 4\\ -4 & 0\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} \tag{76}$$

Where we can take the eigenvalues and eigenvectors from the last problem and plug them in as fallows

$$\vec{y} = c_1\vec{v}_1 e^{\lambda_1 t} + c_1\vec{v}_2 e^{\lambda_2 t} \tag{77}$$

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} = c_1\begin{bmatrix}1\\ i\end{bmatrix}e^{i4t} + c_2\begin{bmatrix}1\\ -i\end{bmatrix}e^{-i4t} \tag{78}$$

6 Vector Calculus

6.1

C is a curve formed at the crossing of $z = 2$ and $x^2 + y^2 = 2z$. For $\vec{F} = [3y, -xz, yz^2]$ Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$

Parametric Line integral

$$\int_c \vec{F}(x,y,z) \cdot d\vec{r}(x,y,z) = \int_c \vec{F}(t) \cdot \frac{d\vec{r}(t)}{dt} dt$$

The general process is to

- Choose functions $x(t)$ and $y(t)$
- Plug it into $z(x,y)$ which will likely be a given curve
- Combine the above two steps to get $\vec{r}(t)$, and take the derivative
- Replace x, y, z in $\vec{F}(x,y,z)$, to get $\vec{F}(t)$
- Take the dot product $\vec{F}(t) \cdot \frac{d\vec{r}(t)}{dt}$
- Reduce if possible, then integrate

Since the curve is circular, consider the parameterization for x and y

$$x \equiv r\cos t \quad y \equiv r\sin t \tag{79}$$

plugging this into the given equation for the curve

$$2z = x^2 + y^2 \rightarrow r^2\cos^2 t + r^2\sin^2 t \tag{80}$$

$$2z = r^2 \rightarrow z = \frac{r^2}{2} \tag{81}$$

With the intersection

$$z = \frac{r^2}{2} \rightarrow 2 = \frac{r^2}{2} \tag{82}$$

$$r = 2 \tag{83}$$

Construct the parameterized position vector $\vec{r}$, and take its derivative

$$\vec{r} = [x, y, z] \rightarrow \left[r\cos t, r\sin t, \frac{r^2}{2}\right] \tag{84}$$

$$\frac{d\vec{r}}{dt} = [-r\sin t, r\cos t, 0] \tag{85}$$

Plug the parameterized vector components into x, y, z for the vector $\vec{F}$

$$\vec{F}(x,y,z) = [3y, -xz, yz^2] \tag{86}$$

$$\vec{F}(t) = \left[3r\sin t, -r\cos t\frac{r^2}{2}, r\sin t\frac{r^4}{4}\right] \tag{87}$$

Compute the dot product $\vec{F}(t) \cdot \frac{d\vec{r}}{dt}$

$$\left[(3r\sin t), (\frac{r^2}{2} * -r\cos t), (\frac{r^4}{4} * r\sin t)\right] \cdot [-r\sin t, r\cos t, 0] \tag{88}$$

$$(-r\sin t * 3r\sin t) + \left(r\cos t * \frac{-r^3}{2}\cos t\right) + \left(0 * \frac{r^5}{4}\sin t\right) \tag{89}$$

$$\left(-3r^2\sin^2 t\right) + \left(\frac{-r^4}{2}\cos^2 t\right) + (0) \tag{90}$$

$$-12\sin^2 t + -8\cos^2 t \tag{91}$$

The line integral should be

$$\boxed{\int(-12\sin^2 t + -8\cos^2 t)dt} \tag{92}$$

6.2