Heat Transfer

Modes of Heat Transfer

Conduction

Fourier's Law:

$$q_x = -kA\frac{dT}{dx}$$

$$q''_x = -k\frac{dT}{dx}$$

where: - q_x = heat transfer rate (W) - q''_x = heat flux (W/m^2) - k = thermal conductivity (W/(m $\cdot$ K)) - A = cross-sectional area - Negative sign: heat flows from high to low temperature

Convection

Newton's Law of Cooling:

$$q = hA(T_s - T_\infty)$$

$$q'' = h(T_s - T_\infty)$$

where: - h = convection heat transfer coefficient (W/(m$^2 \cdot$ K)) - T_s = surface temperature - T_∞ = fluid temperature

Convective Energy Flow:

$$q_{conv} = \dot{m}c_p(T_{out} - T_{in})$$

where: - $\dot{m}$ = mass flow rate (kg/s) - c_p = specific heat capacity (J/(kg $\cdot$ K))

Radiation

Stefan-Boltzmann Law:

$$q = \epsilon\sigma A T_s^4$$

where: - ϵ = emissivity ($0 \le \epsilon \le 1$) - σ = Stefan-Boltzmann constant = 5.67×10^{-8} W/(m$^2 \cdot$ K^4) - T_s = absolute temperature (K)

Net Radiation Exchange:

$$q = \epsilon\sigma A(T_s^4 - T_{surr}^4)$$

Linearized Radiation:

$$q = h_r A(T_s - T_{surr})$$

where $h_r = \epsilon\sigma(T_s + T_{surr})(T_s^2 + T_{surr}^2)$

Heat Diffusion Equation

General Form (3D, Cartesian)

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

For constant k:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where $\alpha = k/(\rho c_p)$ is thermal diffusivity

Cylindrical Coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

Spherical Coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(k\sin\phi\frac{\partial T}{\partial \phi}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

One-Dimensional Steady-State Conduction

Plane Wall

Without Heat Generation:

$$q_x = \frac{kA(T_1 - T_2)}{L} = \frac{T_1 - T_2}{R_{cond}}$$

where $R_{cond} = L/(kA)$ is thermal resistance

Temperature Distribution:

$$T(x) = T_1 - \frac{T_1 - T_2}{L}x$$

With Uniform Heat Generation:

$$\frac{d^2 T}{dx^2} + \frac{\dot{q}}{k} = 0$$

Solution:

$$T(x) = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

Composite Wall

Series Resistance:

$$q = \frac{\Delta T_{overall}}{R_{total}} = \frac{T_1 - T_4}{R_1 + R_2 + R_3}$$

$$R_{total} = \sum R_i = \sum \frac{L_i}{k_i A}$$

With Convection:

$$R_{total} = R_{conv,1} + R_{cond,1} + R_{cond,2} + \cdots + R_{conv,2}$$

where $R_{conv} = 1/(hA)$

Overall Heat Transfer Coefficient:

$$q = UA\Delta T$$

$$\frac{1}{UA} = R_{total}$$

Cylindrical Systems

Hollow Cylinder (radial conduction):

$$q_r = \frac{2\pi Lk(T_1 - T_2)}{\ln(r_2/r_1)}$$

Thermal Resistance:

$$R_{cyl} = \frac{\ln(r_2/r_1)}{2\pi Lk}$$

With convection:

$$R_{conv,inner} = \frac{1}{h_i(2\pi r_1 L)}$$

$$R_{conv,outer} = \frac{1}{h_o(2\pi r_2 L)}$$

Critical Radius of Insulation:

$$r_{cr} = \frac{k}{h}$$

If $r_{outer} < r_{cr}$, adding insulation increases heat transfer

Spherical Systems

Hollow Sphere:

$$q_r = \frac{4\pi k r_1 r_2 (T_1 - T_2)}{r_2 - r_1}$$

Thermal Resistance:

$$R_{sph} = \frac{r_2 - r_1}{4\pi k r_1 r_2}$$

Extended Surfaces (Fins)

Fin Equation

General Fin Equation:

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

where: - $\theta = T - T_\infty$ - $m = \sqrt{hP/(kA_c)}$ - P = perimeter - A_c = cross-sectional area

Fin Solutions (Common Cases)

Very Long Fin ($L \to \infty$):

$$\theta(x) = \theta_b e^{-mx}$$

$$q_f = \sqrt{hPkA_c}\,\theta_b$$

Insulated Tip:

$$\theta(x) = \theta_b \frac{\cosh[m(L - x)]}{\cosh(mL)}$$

$$q_f = \sqrt{hPkA_c}\,\theta_b \tanh(mL)$$

Convection at Tip:

$$\theta(x) = \theta_b \frac{\cosh[m(L - x)] + (h/mk)\sinh[m(L - x)]}{\cosh(mL) + (h/mk)\sinh(mL)}$$

Fin Performance

Fin Efficiency:

$$\eta_f = \frac{q_f}{q_{max}} = \frac{q_f}{hA_f\theta_b}$$

where A_f is total fin surface area

For fin with insulated tip:

$$\eta_f = \frac{\tanh(mL)}{mL}$$

Fin Effectiveness:

$$\epsilon_f = \frac{q_f}{q_{no\ fin}} = \frac{q_f}{hA_c\theta_b}$$

Overall Surface Efficiency:

$$\eta_o = 1 - \frac{A_f}{A_t}(1 - \eta_f)$$

where $A_t = A_b + A_f$ is total surface area

Total Heat Transfer with Fins:

$$q_{total} = \eta_o h A_t (T_b - T_\infty)$$

Transient Conduction

Lumped Capacitance Method

Biot Number:

$$Bi = \frac{hL_c}{k}$$

where $L_c = V/A_s$ is characteristic length

Validity: $Bi < 0.1$

Temperature Response:

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-t/\tau}$$

where $\tau = \frac{\rho V c_p}{hA_s} = \frac{\rho L_c c_p}{h}$ is time constant

Heat Transfer:

$$Q(t) = \rho V c_p (T_i - T(t))$$

Semi-Infinite Solid

Constant Surface Temperature:

$$\frac{T(x,t) - T_i}{T_s - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Surface Heat Flux:

$$q_s''(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

Constant Surface Heat Flux:

$$T(x,t) - T_i = \frac{2q_s''}{k}\sqrt{\frac{\alpha t}{\pi}}\exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{q_s'' x}{k}\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Heisler Charts / Analytical Solutions

For $Bi > 0.1$, use charts or one-term approximation:

Plane Wall:

$$\frac{T(x,t) - T_\infty}{T_i - T_\infty} = C_1 \exp(-\zeta_1^2 Fo)\cos(\zeta_1 x/L)$$

Cylinder:

$$\frac{T(r,t) - T_\infty}{T_i - T_\infty} = C_1 \exp(-\zeta_1^2 Fo) J_0(\zeta_1 r/r_o)$$

Sphere:

$$\frac{T(r,t) - T_\infty}{T_i - T_\infty} = C_1 \exp(-\zeta_1^2 Fo)\frac{\sin(\zeta_1 r/r_o)}{\zeta_1 r/r_o}$$

where: - $Fo = \alpha t/L_c^2$ is Fourier number - ζ_1 is first eigenvalue (from tables based on Bi) - C_1 is constant (from tables based on Bi)

Forced Convection

Dimensionless Numbers

Reynolds Number:

$$Re = \frac{\rho V L}{\mu} = \frac{VL}{\nu}$$

Prandtl Number:

$$Pr = \frac{\nu}{\alpha} = \frac{c_p \mu}{k}$$

Nusselt Number:

$$Nu = \frac{hL}{k}$$

Relationship:

$$Nu = f(Re, Pr)$$

External Flow

Flat Plate (Laminar, $Re_x < 5 \times 10^5$):

Local:

$$Nu_x = 0.332 Re_x^{1/2} Pr^{1/3}$$

Average:

$$\overline{Nu}_L = 0.664 Re_L^{1/2} Pr^{1/3}$$

Flat Plate (Turbulent, $Re_x > 5 \times 10^5$):

Local:

$$Nu_x = 0.0296 Re_x^{4/5} Pr^{1/3}$$

Average (mixed):

$$\overline{Nu}_L = (0.037 Re_L^{4/5} - 871) Pr^{1/3}$$

Cylinder in Cross Flow:

$$\overline{Nu}_D = C Re_D^m Pr^{1/3}$$

where C and m depend on Re_D (from tables)

For $Re_D = 0.4$ to 4×10^5:

$$\overline{Nu}_D = 0.3 + \frac{0.62 Re_D^{1/2} Pr^{1/3}}{[1 + (0.4/Pr)^{2/3}]^{1/4}} \left[1 + \left(\frac{Re_D}{282000}\right)^{5/8}\right]^{4/5}$$

Sphere:

$$\overline{Nu}_D = 2 + (0.4 Re_D^{1/2} + 0.06 Re_D^{2/3}) Pr^{0.4}$$

Internal Flow

Fully Developed Laminar (Circular Tube):

Constant surface temperature: $Nu_D = 3.66$

Constant heat flux: $Nu_D = 4.36$

Entry Length:

Hydrodynamic: $L_h/D \approx 0.05 Re_D$ (laminar)

Thermal: $L_t/D \approx 0.05 Re_D Pr$ (laminar)

Turbulent Flow ($Re_D > 10,000$):

Dittus-Boelter equation:

$$Nu_D = 0.023 Re_D^{4/5} Pr^n$$

where $n = 0.4$ (heating) or $n = 0.3$ (cooling)

Gnielinski equation (more accurate):

$$Nu_D = \frac{(f/8)(Re_D - 1000)Pr}{1 + 12.7(f/8)^{1/2}(Pr^{2/3} - 1)}$$

where f is friction factor

Non-Circular Ducts:

Use hydraulic diameter:

$$D_h = \frac{4A_c}{P}$$

where A_c is cross-sectional area, P is wetted perimeter

Natural (Free) Convection

Dimensionless Numbers

Grashof Number:

$$Gr_L = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2}$$

where $\beta = 1/T$ for ideal gas (use absolute temperature)

Rayleigh Number:

$$Ra_L = Gr_L \cdot Pr = \frac{g\beta(T_s - T_\infty)L^3}{\nu\alpha}$$

Vertical Plate

Laminar ($Ra_L < 10^9$):

$$\overline{Nu}_L = 0.59 Ra_L^{1/4}$$

Turbulent ($Ra_L > 10^9$):

$$\overline{Nu}_L = 0.10 Ra_L^{1/3}$$

Full Range:

$$\overline{Nu}_L = \left[0.825 + \frac{0.387 Ra_L^{1/6}}{[1 + (0.492/Pr)^{9/16}]^{8/27}}\right]^2$$

Horizontal Plate

Hot surface facing up or cold surface facing down:

$$\overline{Nu}_L = 0.54 Ra_L^{1/4} \quad (10^4 < Ra_L < 10^7)$$

$$\overline{Nu}_L = 0.15 Ra_L^{1/3} \quad (10^7 < Ra_L < 10^{11})$$

Hot surface facing down or cold surface facing up:

$$\overline{Nu}_L = 0.27 Ra_L^{1/4} \quad (10^5 < Ra_L < 10^{10})$$

where $L = A_s/P$ is characteristic length

Horizontal Cylinder

$$\overline{Nu}_D = \left[0.60 + \frac{0.387 Ra_D^{1/6}}{[1 + (0.559/Pr)^{9/16}]^{8/27}}\right]^2$$

Sphere

$$\overline{Nu}_D = 2 + \frac{0.589 Ra_D^{1/4}}{[1 + (0.469/Pr)^{9/16}]^{4/9}}$$

Boiling and Condensation

Pool Boiling

Regimes: 1. Free convection boiling 2. Nucleate boiling 3. Transition boiling 4. Film boiling

Rohsenow Correlation (Nucleate Boiling):

$$q'' = \mu_l h_{fg} \left[\frac{g(\rho_l - \rho_v)}{\sigma}\right]^{1/2} \left[\frac{c_{p,l}(T_s - T_{sat})}{C_{sf} h_{fg} Pr_l^n}\right]^3$$

where C_{sf} and n depend on surface-fluid combination

Critical Heat Flux (CHF):

$$q''_{max} = 0.149 h_{fg} \rho_v \left[\sigma g(\rho_l - \rho_v)/\rho_v^2\right]^{1/4}$$

Film Condensation

Vertical Plate (Nusselt):

$$\overline{Nu}_L = 0.943 \left[\frac{g \rho_l (\rho_l - \rho_v) h'_{fg} L^3}{\mu_l k_l (T_{sat} - T_s)}\right]^{1/4}$$

where $h'_{fg} = h_{fg}[1 + 0.68 c_{p,l}(T_{sat} - T_s)/h_{fg}]$

Average Heat Transfer Coefficient:

$$\bar{h} = 0.943 \left[\frac{g\rho_l(\rho_l - \rho_v)k_l^3 h'_{fg}}{\mu_l L(T_{sat} - T_s)} \right]^{1/4}$$

Horizontal Tube:

$$\bar{h} = 0.729 \left[\frac{g\rho_l(\rho_l - \rho_v)k_l^3 h'_{fg}}{\mu_l D(T_{sat} - T_s)} \right]^{1/4}$$

Heat Exchangers

Overall Heat Transfer Coefficient

For tube:

$$\frac{1}{UA} = \frac{1}{h_i A_i} + \frac{\ln(r_o/r_i)}{2\pi k L} + \frac{1}{h_o A_o}$$

Including fouling:

$$\frac{1}{UA} = \frac{1}{h_i A_i} + \frac{R''_{f,i}}{A_i} + \frac{\ln(r_o/r_i)}{2\pi k L} + \frac{R''_{f,o}}{A_o} + \frac{1}{h_o A_o}$$

Log Mean Temperature Difference (LMTD)

$$q = UA\Delta T_{lm}$$

$$\Delta T_{lm} = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1/\Delta T_2)}$$

Parallel Flow:

$$\Delta T_1 = T_{h,i} - T_{c,i}, \quad \Delta T_2 = T_{h,o} - T_{c,o}$$

Counter Flow:

$$\Delta T_1 = T_{h,i} - T_{c,o}, \quad \Delta T_2 = T_{h,o} - T_{c,i}$$

For other configurations:

$$q = UAF\Delta T_{lm,cf}$$

where F is correction factor and $\Delta T_{lm,cf}$ is for counter-flow

Effectiveness-NTU Method

Heat Capacity Rates:

$$C_h = (\dot{m}c_p)_h, \quad C_c = (\dot{m}c_p)_c$$

$$C_{min} = \min(C_h, C_c), \quad C_{max} = \max(C_h, C_c)$$

$$C_r = \frac{C_{min}}{C_{max}}$$

Maximum Possible Heat Transfer:

$$q_{max} = C_{min}(T_{h,i} - T_{c,i})$$

Effectiveness:

$$\epsilon = \frac{q}{q_{max}}$$

Number of Transfer Units:

$$NTU = \frac{UA}{C_{min}}$$

ϵ-NTU Relations:

Parallel flow:

$$\epsilon = \frac{1 - \exp[-NTU(1 + C_r)]}{1 + C_r}$$

Counter flow:

$$\epsilon = \frac{1 - \exp[-NTU(1 - C_r)]}{1 - C_r \exp[-NTU(1 - C_r)]}$$

For $C_r = 0$ (phase change):

$$\epsilon = 1 - \exp(-NTU)$$

Radiation Heat Transfer

Radiation Properties

Emissivity:

$$\epsilon = \frac{E}{E_b}$$

Absorptivity:

$$\alpha = \frac{G_{abs}}{G}$$

Reflectivity:

$$\rho = \frac{G_{ref}}{G}$$

Transmissivity:

$$\tau = \frac{G_{tr}}{G}$$

Energy Balance:

$$\alpha + \rho + \tau = 1$$

For opaque surface: $\tau = 0$, so $\alpha + \rho = 1$

Kirchhoff's Law:

$$\alpha = \epsilon$$

(for diffuse, gray surface)

View Factor

Definition:

F_{ij} = fraction of radiation leaving surface i that directly strikes surface j

Reciprocity:

$$A_i F_{ij} = A_j F_{ji}$$

Summation Rule:

$$\sum_{j=1}^{N} F_{ij} = 1$$

For flat or convex surface:

$$F_{ii} = 0$$

Radiation Exchange

Between Two Black Surfaces:

$$q_{12} = A_1 F_{12} \sigma (T_1^4 - T_2^4)$$

Between Two Gray, Diffuse Surfaces:

$$q_{12} = \frac{\sigma(T_1^4 - T_2^4)}{\frac{1-\epsilon_1}{A_1\epsilon_1} + \frac{1}{A_1 F_{12}} + \frac{1-\epsilon_2}{A_2\epsilon_2}}$$

Small Object in Large Enclosure:

$$q = A_1 \epsilon_1 \sigma (T_1^4 - T_2^4)$$

Radiation Shield:

Radiation shield between surfaces reduces heat transfer:

$$q_{with\ shield} = \frac{q_{without\ shield}}{2}$$

(for single shield with $\epsilon = \epsilon_1 = \epsilon_2$)

Finite Difference Method

Steady-State

Interior Node (2D):

$$T_{m,n} = \frac{T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1}}{4}$$

With Heat Generation:

$$T_{m,n} = \frac{T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} + \dot{q}\Delta x^2/k}{4}$$

Boundary with Convection:

$$\frac{T_{m-1,n} - T_{m,n}}{\Delta x/k} = h(T_{m,n} - T_\infty)$$

Transient (Explicit Method)

Fourier Number:

$$Fo = \frac{\alpha \Delta t}{(\Delta x)^2}$$

Stability Criterion (1D):

$$Fo \leq \frac{1}{2}$$

Interior Node (1D):

$$T_m^{p+1} = Fo(T_{m+1}^p + T_{m-1}^p) + (1 - 2Fo)T_m^p$$

where superscript p denotes time step

Quick Reference

Typical Values of h ($\mathbf{W/(m^2 \cdot K)}$): - Free convection (gases): 2-25 - Free convection (liquids): 50-1000 - Forced convection (gases): 25-250 - Forced convection (liquids): 100-20,000 - Boiling/Condensation: 2500-100,000

Thermal Conductivity ($W/(m \cdot K)$): - Metals: 15-400 - Non-metallic solids: 0.05-5 - Liquids: 0.1-0.7 - Gases: 0.01-0.1 - Insulation: 0.03-0.2