Fourier Series and Transforms

Fourier Series

For a periodic function $f(x)$ with period $T = 2L$ (or $f(x) = f(x + 2L)$):

General Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

Fourier Coefficients:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \ldots$$

Fourier Series for Period 2π

For $f(x)$ with period 2π (i.e., $L = \pi$):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

Coefficients:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

Half-Range Expansions

Fourier Cosine Series (even extension on $[0, L]$):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Fourier Sine Series (odd extension on $[0, L]$):

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Complex Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

Complex Coefficients:

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx$$

Relationship to real coefficients:

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}$$

Properties of Fourier Series

Even Function: $f(-x) = f(x)$

- All $b_n = 0$ (only cosine terms)
- $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

Odd Function: $f(-x) = -f(x)$

- All $a_n = 0$ (only sine terms)
- $b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

Convergence:

- At points where f is continuous, the series converges to $f(x)$
- At jump discontinuities, the series converges to $\frac{f(x^+)+f(x^-)}{2}$

Parseval's Identity:

$$\frac{1}{L} \int_{-L}^{L} |f(x)|^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Fourier Transform

For non-periodic functions defined on $(-\infty, \infty)$:

Fourier Transform (Complex Form)

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

Inverse Fourier Transform:

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega$$

Fourier Cosine Transform

For functions defined on $[0, \infty)$:

$$F_c(\omega) = \int_0^{\infty} f(x) \cos(\omega x) \, dx$$

Inverse:

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(\omega) \cos(\omega x) \, d\omega$$

Fourier Sine Transform

For functions defined on $[0, \infty)$:

$$F_s(\omega) = \int_0^{\infty} f(x) \sin(\omega x) \, dx$$

Inverse:

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\omega) \sin(\omega x) \, d\omega$$

Common Fourier Transforms

Function $f(t)$	**Transform** $F(\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$e^{-at}u(t)$, $a > 0$	$\frac{1}{a + i\omega}$
$e^{-a\|t\|}$, $a > 0$	$\frac{2a}{a^2 + \omega^2}$
$\text{rect}(t/a)$	$a \, \text{sinc}(a\omega/2)$
e^{-at^2}	$\sqrt{\frac{\pi}{a}} e^{-\omega^2/(4a)}$
$\cos(\omega_0 t)$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin(\omega_0 t)$	$i\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$

where $\text{rect}(t/a) = \begin{cases} 1 & |t| < a/2 \\ 0 & |t| > a/2 \end{cases}$ and $\text{sinc}(x) = \frac{\sin(x)}{x}$

Properties of Fourier Transforms

Linearity:

$$\mathcal{F}\{af(t) + bg(t)\} = aF(\omega) + bG(\omega)$$

Time Shifting:

$$\mathcal{F}\{f(t - t_0)\} = e^{-i\omega t_0} F(\omega)$$

Frequency Shifting:

$$\mathcal{F}\{e^{i\omega_0 t} f(t)\} = F(\omega - \omega_0)$$

Scaling:

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

Time Differentiation:

$$\mathcal{F}\left\{\frac{d^n f}{dt^n}\right\} = (i\omega)^n F(\omega)$$

Frequency Differentiation:

$$\mathcal{F}\{t^n f(t)\} = i^n \frac{d^n F(\omega)}{d\omega^n}$$

Integration:

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(\tau)\,d\tau\right\} = \frac{F(\omega)}{i\omega} + \pi F(0)\delta(\omega)$$

Convolution Theorem:

$$\mathcal{F}\{(f * g)(t)\} = F(\omega) \cdot G(\omega)$$

$$\mathcal{F}\{f(t) \cdot g(t)\} = \frac{1}{2\pi}(F * G)(\omega)$$

where $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau$

Parseval's Theorem:

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$$

Symmetry Properties:

If $f(t)$ is real and even: $F(\omega)$ is real and even

If $f(t)$ is real and odd: $F(\omega)$ is purely imaginary and odd

Discrete Fourier Transform (DFT)

For a discrete signal $\{x_n\}$ with N samples:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-2\pi i k n/N}, \quad k = 0, 1, \ldots, N-1$$

Inverse DFT:

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{2\pi i k n/N}, \quad n = 0, 1, \ldots, N-1$$

Fast Fourier Transform (FFT)

The FFT is an efficient algorithm to compute the DFT with complexity $O(N \log N)$ instead of $O(N^2)$.

Most efficient when $N = 2^m$ (power of 2).

Useful Integrals

$$\int_0^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 0 & n \neq m \\ \pi/2 & n = m \end{cases}$$

$\int_0^\pi \cos(nx)\cos(mx)\,dx = \begin{cases} 0 & n \neq m \\ \pi/2 & n = m \end{cases}$

$$\int_0^\pi \sin(nx)\cos(mx)\,dx = 0 \text{ for all } n, m$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

Special Functions

Signum Function:

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

Unit Step Function:

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Rectangle Function:

$$\operatorname{rect}(x) = \begin{cases} 1 & |x| < 1/2 \\ 1/2 & |x| = 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

Sinc Function:

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x}$$

Note: $\lim_{x \to 0} \operatorname{sinc}(x) = 1$

Useful Trigonometric Identities

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\sin(x)\cos(x) = \frac{\sin(2x)}{2}$$

$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

Euler's Formula:

$$e^{ix} = \cos(x) + i\sin(x)$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$