Linear Algebra Sample Problem 2

This problem is question 4.2 on the Fall 2023 QE.

Given the matrix $A = \begin{bmatrix} 0 & 1 \\ 1+a & b \end{bmatrix}$ calculate the eigenvalues and eigenvectors. Find the values for a and b for which the eigenvalues are real part negative.

1 Find the Eigenvalues

To find the eigenvalues of the matrix A, we need to solve the characteristic equation given by:

$$\det(A - \lambda I) = 0 \tag{1}$$

First, we compute $A - \lambda I$:

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ 1+a & b-\lambda \end{bmatrix} \tag{2}$$

Next, we compute the determinant:

$$\det(A - \lambda I) = (-\lambda)(b-\lambda) - (1)(1+a) = \lambda^2 - b\lambda - (1+a) \tag{3}$$

Setting the determinant equal to zero gives us the characteristic equation:

$$\lambda^2 - b\lambda - (1+a) = 0 \tag{4}$$

To find the eigenvalues, we can use the quadratic formula.

$$\lambda_{1,2} = \frac{b \pm \sqrt{b^2 + 4(1+a)}}{2} \tag{5}$$

2 Find the Eigenvectors

To find the eigenvectors corresponding to each eigenvalue, we solve the equation:

$$(A - \lambda I)\mathbf{v} = 0 \tag{6}$$

For the given matrix, this looks like:

$$\begin{bmatrix} -\lambda & 1 \\ 1+a & b-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \tag{7}$$

This leads to the system of equations:

$$-\lambda v_1 + v_2 = 0 \tag{8}$$
$$(1+a)v_1 + (b-\lambda)v_2 = 0 \tag{9}$$

From the first equation, we can express v_2 in terms of v_1:

$$v_2 = \lambda v_1 \tag{10}$$

We can then rewrite the v vector as:

$$v = \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \tag{11}$$

We can then sibstitute in the expression for λ to get the eigenvectors.

$$v_{1,2} = \begin{bmatrix} 1 \\ \frac{b \pm \sqrt{b^2+4(1+a)}}{2} \end{bmatrix} \tag{12}$$
$$= \begin{bmatrix} 2 \\ b \pm \sqrt{b^2+4(1+a)} \end{bmatrix} \tag{13}$$

3 Conditions for Negative Real Parts of Eigenvalues

For the eigenvalues to have negative real parts, we need to analyze the expression:

$$\lambda_{1,2} = \frac{b \pm \sqrt{b^2+4(1+a)}}{2} \tag{14}$$

To ensure that both eigenvalues have negative real parts, we need:

1. The real part of both eigenvalues must be negative, which implies:

$$b < 0 \tag{15}$$

2. The discriminant must be non-positive to ensure complex eigenvalues with negative real parts:

$$b^2 + 4(1+a) \le 0 \implies a \le -1 - \frac{b^2}{4} \tag{16}$$