# ODEs Sample Problem 2

This problem is question 2.2 on the 2023 Fall QE.

Solve the following second order differential equation:

$$\ddot{x} = x(t) + e^{-t} + \cos(t) \tag{1}$$
$$x(0) = 1, \quad \dot{x}(0) = 0 \tag{2}$$

Important note: Do not use Laplace methods.

## 1 Homogeneous Solution

We can rewrite the equation as the following to isolate x variables on one side:

$$\ddot{x} - x(t) = e^{-t} + \cos(t) \tag{3}$$
$$x(0) = 1, \quad \dot{x}(0) = 0 \tag{4}$$

Now, write the characteristic equation with the form:

$$ar^2 + br + c = 0 \tag{5}$$

For our case, we have:

$$r^2 - 1 = 0 \tag{6}$$

Solving for r, we find:

$$r = \pm 1 \tag{7}$$

Thus, the homogeneous solution is:

$$x_h(t) = C_1 e^t + C_2 e^{-t} \tag{8}$$

## 2 Particular Solution for $x'' - x = e^{-t}$

To find a particular solution for the non-homogeneous part $e^{-t}$, we can use the method of undetermined coefficients. Since $e^{-t}$ is already part of the homogeneous solution, we multiply by t to find a suitable form for the particular solution:

$$x_{p1}(t) = Ate^{-t} \tag{9}$$

Taking the first and second derivatives, we have:

$$\dot{x}_{p1}(t) = Ae^{-t} - Ate^{-t} \tag{10}$$
$$\ddot{x}_{p1}(t) = -2Ae^{-t} + Ate^{-t} \tag{11}$$

Substituting into the differential equation:

$$\ddot{x}_{p1} - x_{p1} = -2Ae^{-t} + Ate^{-t} - Ate^{-t} = -2Ae^{-t} \tag{12}$$

Setting this equal to $e^{-t}$, we find:

$$-2A = 1 \implies A = -\frac{1}{2} \tag{13}$$

Thus, the particular solution for this part is:

$$x_{p1}(t) = -\frac{1}{2}te^{-t} \tag{14}$$

# 3 Particular Solution for $x'' - x = \cos(t)$

Next, we find a particular solution for the non-homogeneous part $\cos(t)$. We can use the method of undetermined coefficients again, proposing a solution of the form:

$$x_{p2}(t) = B\cos(t) + C\sin(t) \tag{15}$$

Taking the first and second derivatives, we have:

$$\dot{x}_{p2}(t) = -B\sin(t) + C\cos(t) \tag{16}$$
$$\ddot{x}_{p2}(t) = -B\cos(t) - C\sin(t) \tag{17}$$

Substituting into the differential equation:

$$\ddot{x}_{p2} - x_{p2} = -B\cos(t) - C\sin(t) - B\cos(t) - C\sin(t) = -2B\cos(t) - 2C\sin(t) \tag{18}$$

Setting this equal to $\cos(t)$, we find:

$$-2B = 1 \implies B = -\frac{1}{2} \tag{19}$$
$$-2C = 0 \implies C = 0 \tag{20}$$

Thus, the particular solution for this part is:

$$x_{p2}(t) = -\frac{1}{2}\cos(t) \tag{21}$$

## 4 General Solution

Combining the homogeneous and particular solutions, we have the general solution:

$$x(t) = C_1 e^t + C_2 e^{-t} - \frac{1}{2} t e^{-t} - \frac{1}{2}\cos(t) \tag{22}$$

We can then apply the initial conditions to solve for $C_1$ and $C_2$. Using $x(0) = 1$:

$$1 = C_1 + C_2 - \frac{1}{2}\cos(0) \tag{23}$$

$$1 = C_1 + C_2 - \frac{1}{2} \tag{24}$$

$$C_1 + C_2 = \frac{3}{2} \tag{25}$$

Using $\dot{x}(0) = 0$:

$$\dot{x}(t) = C_1 e^t - C_2 e^{-t} - \frac{1}{2} e^{-t} + \frac{1}{2} t e^{-t} + \frac{1}{2}\sin(t) \tag{26}$$

$$0 = C_1 - C_2 - \frac{1}{2} \tag{27}$$

Solving the system of equations:

$$C_1 + C_2 = \frac{3}{2} \tag{28}$$

$$C_1 - C_2 = \frac{1}{2} \tag{29}$$

Adding the two equations, we find:

$$2C_1 = 2 \implies C_1 = 1 \tag{30}$$

Substituting back to find $C_2$:

$$1 + C_2 = \frac{3}{2} \implies C_2 = \frac{1}{2} \tag{31}$$

Thus, the final solution to the differential equation is:

$$x(t) = e^t + \frac{1}{2} e^{-t} - \frac{1}{2} t e^{-t} - \frac{1}{2}\cos(t) \tag{32}$$