

Vector Calculus Sample Problem 1

This problem is question 5.1 on the Fall 2023 QE.

Calculate the total flux of the vector field $\vec{V} = 3x\hat{x} + y^2\hat{y} + z^2\hat{z}$ across the sphere S of radius $R = 1$ centered at the origin. Note: superimposed har denotes unit vector.

1 Calculate Divergence

The divergence of a vector field is given by:

$$\nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \quad (1)$$

Calculating the partial derivatives, we have:

$$\frac{\partial V_x}{\partial x} = \frac{\partial(3x)}{\partial x} = 3 \quad (2)$$

$$\frac{\partial V_y}{\partial y} = \frac{\partial(y^2)}{\partial y} = 2y \quad (3)$$

$$\frac{\partial V_z}{\partial z} = \frac{\partial(z^2)}{\partial z} = 2z \quad (4)$$

Thus, the divergence of the vector field is:

$$\nabla \cdot \vec{V} = 3 + 2y + 2z \quad (5)$$

2 Apply Divergence Theorem

The Divergence Theorem states that the total flux of a vector field \vec{V} across a closed surface S is equal to the volume integral of the divergence of \vec{V} over the volume \mathcal{V} enclosed by S :

$$\Phi = \iint_S \vec{V} \cdot d\vec{A} = \iiint_{\mathcal{V}} (\nabla \cdot \vec{V}) dV \quad (6)$$

For our divergence, we have:

$$\Phi = \iiint_{\mathcal{V}} (3 + 2y + 2z) dV \quad (7)$$

Which we can split into three separate integrals:

$$\Phi = \iiint_{\mathcal{V}} 3dV + \iiint_{\mathcal{V}} 2ydV + \iiint_{\mathcal{V}} 2zdV \quad (8)$$

3 Evaluate Volume Integrals

To evaluate these integrals, we will use spherical coordinates:

$$x = r \sin \theta \cos \phi \quad (9)$$

$$y = r \sin \theta \sin \phi \quad (10)$$

$$z = r \cos \theta \quad (11)$$

The volume element in spherical coordinates is given by:

$$dV = r^2 \sin \theta dr d\theta d\phi \quad (12)$$

The limits of integration for a sphere of radius $R = 1$ are:

$$0 \leq r \leq 1 \quad (13)$$

$$0 \leq \theta \leq \pi \quad (14)$$

$$0 \leq \phi \leq 2\pi \quad (15)$$

3.1 First Integral

The first integral is:

$$\iiint_V 3dV = 3 \int_0^{2\pi} \int_0^\pi \int_0^1 r^2 \sin \theta dr d\theta d\phi \quad (16)$$

Evaluating this integral, we have:

$$= 3 \left(\int_0^{2\pi} d\phi \right) \left(\int_0^\pi \sin \theta d\theta \right) \left(\int_0^1 r^2 dr \right) = 3(2\pi)(2) \left(\frac{1}{3} \right) = 4\pi \quad (17)$$

3.2 Second Integral

The second integral is:

$$\iiint_V 2ydV = 2 \int_0^{2\pi} \int_0^\pi \int_0^1 (r \sin \theta \sin \phi) r^2 \sin \theta dr d\theta d\phi \quad (18)$$

Evaluating this integral, we have:

$$= 2 \left(\int_0^{2\pi} \sin \phi d\phi \right) \left(\int_0^\pi \sin^2 \theta d\theta \right) \left(\int_0^1 r^3 dr \right) = 2(0) \left(\frac{\pi}{2} \right) \left(\frac{1}{4} \right) = 0 \quad (19)$$

3.3 Third Integral

The third integral is:

$$\iiint_V 2zdV = 2 \int_0^{2\pi} \int_0^\pi \int_0^1 (r \cos \theta) r^2 \sin \theta dr d\theta d\phi \quad (20)$$

Evaluating this integral, we have:

$$= 2 \left(\int_0^{2\pi} d\phi \right) \left(\int_0^\pi \cos \theta \sin \theta d\theta \right) \left(\int_0^1 r^3 dr \right) = 2(2\pi)(0) \left(\frac{1}{4} \right) = 0 \quad (21)$$

4 Final Result

Combining the results of the three integrals, we have:

$$\Phi = 4\pi + 0 + 0 = 4\pi \quad (22)$$