Laplace Transform Sample Problem 1

This problem is question 1.1 from the Fall 2023 QE exam:

Using Laplace methods, solve the following differntial equation.

$$y'''(t) + 2y''(t) + y'(t) = \delta(t) \tag{1}$$
$$y(0) = 0, \quad y'(0) = 0 \tag{2}$$

1 Take the Laplace transform of both sides

$$\mathcal{L}[y'''(t) + 2y''(t) + y'(t)] = \mathcal{L}[\delta(t)] \tag{3}$$

Using the linearity property of the Laplace transform, we can separate the left side.

$$\mathcal{L}[y'''(t)] + 2\mathcal{L}[y''(t)] + \mathcal{L}[y'(t)] = \mathcal{L}[\delta(t)] \tag{4}$$

Then we can apply the Laplace transform to each term.

$$\mathcal{L}[y'''(t)] = s^3Y(s) - s^2y(0) - sy'(0) - y''(0) = s^3Y(s) - y''(0) \tag{5}$$
$$\mathcal{L}[y''(t)] = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) \tag{6}$$
$$\mathcal{L}[y'(t)] = sY(s) - y(0) = sY(s) \tag{7}$$
$$\tag{8}$$

Now substituting these into the equation and cancelling out zeroes from intial conditions gives:

$$s^3Y(s) - y''(0) + 2s^2Y(s) + sY(s) = \mathcal{L}[\delta(t)] \tag{9}$$

The Laplace transform of the delta function is 1, so we have:

$$s^3Y(s) + 2s^2Y(s) + sY(s) - y''(0) = 1 \tag{10}$$

2 Solve for Y(s)

Factoring out $Y(s)$ on the left side:

$$Y(s)(s^3 + 2s^2 + s) - y''(0) = 1 \tag{11}$$

Rearranging to solve for $Y(s)$:

$$Y(s) = \frac{1 + y''(0)}{s^3 + 2s^2 + s} \tag{12}$$

Here, $y''(0) = 0$ since the jump in δ happens at $t = 0$ and the function is continuous. This leaves us with:

$$Y(s) = \frac{1}{s^3 + 2s^2 + s} \tag{13}$$
$$= \frac{1}{s(s^2 + 2s + 1)} \tag{14}$$
$$= \frac{1}{s(s+1)^2} \tag{15}$$

3 Partial fraction decomposition

We can decompose $Y(s)$ into partial fractions:

$$Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} \tag{16}$$

Multiplying both sides by the denominator $s(s+1)^2$ gives:

$$1 = A(s+1)^2 + Bs(s+1) + Cs \tag{17}$$

To start solving for the constants, we can choose $s = 0$:

$$1 = A(0+1)^2 + B(0)(0+1) + C(0) \tag{18}$$
$$A = 1 \tag{19}$$

Now, we can choose $s = -1$:

$$1 = A(-1+1)^2 + B(-1)(-1+1) + C(-1) \tag{20}$$
$$1 = 0 + 0 - C \tag{21}$$
$$C = -1 \tag{22}$$

Finally, choose $s = 1$:

$$1 = A(1+1)^2 + B(1)(1+1) + C(1) \tag{23}$$
$$1 = 4A + 2B + C \tag{24}$$
$$1 = 4(1) + 2B - 1 \tag{25}$$
$$1 = 3 + 2B \tag{26}$$
$$-2 = 2B \tag{27}$$
$$B = -1 \tag{28}$$

Thus, we have:

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2} \tag{29}$$

4 Take the inverse Laplace transform

Now we can take the inverse Laplace transform to find $y(t)$:

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] \tag{30}$$

Using known inverse Laplace transforms:

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1 \tag{31}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t} \tag{32}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] = te^{-t} \tag{33}$$

Substituting these back in gives:

$$y(t) = 1 - e^{-t} - te^{-t} \tag{34}$$

$$\boxed{y(t) = 1 - e^{-t}(1 - t)} \tag{35}$$

Nice.