ODEs Sample Problem 1

This problem is question 2.1 from the Fall 2023 QE exam:

Solve the follwoing differential equation:

$$x^2y'' + axy' + by = 0 \tag{1}$$
$$a = -1.5, \quad b = -1.5 \tag{2}$$

1 Cauchy-Euler Equation

This is a Cauchy-Euler equation, where we can assume the solution is of the form:

$$y = x^r \tag{3}$$

We can then find the ffirst two derivatives.

$$y' = rx^{r-1} \tag{4}$$
$$y'' = r(r-1)x^{r-2} \tag{5}$$

Then, substitute these into the original equation.

$$x^2(r(r-1)x^{r-2}) + ax(rx^{r-1}) + b(x^r) = 0 \tag{6}$$
$$r(r-1)x^r + arx^r + bx^r = 0 \tag{7}$$
$$x^r[r(r-1) + ar + b] = 0 \tag{8}$$

2 Characteristic Equation

Since $x^r \neq 0$, we can set the term in brackets to zero:

$$r(r-1) + ar + b = 0 \tag{9}$$
$$r^2 - r + ar + b = 0 \tag{10}$$
$$r^2 + (a-1)r + b = 0 \tag{11}$$

This is the characteristic equation. Substituting in the values for a and b:

$$r^2 + (-1.5-1)r - 1.5 = 0 \tag{12}$$

We can multiply through by 2 to eliminate the fraction:

$$2r^2 - 5r - 3 = 0 \tag{13}$$

Using the quadratic formula to solve for r:

$$r = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)} \tag{14}$$

$$r = \frac{5 \pm \sqrt{25 + 24}}{4} \tag{15}$$

$$r = \frac{5 \pm 7}{4} \tag{16}$$

This gives us two roots:

$$r_1 = \frac{12}{4} = 3 \tag{17}$$

$$r_2 = \frac{-2}{4} = -0.5 \tag{18}$$

3 solution

Since we have two distinct real roots, the general solution is given by:

$$y = C_1 x^{r_1} + C_2 x^{r_2} \tag{19}$$

Substituting in the values for r_1 and r_2:

$$y = C_1 x^3 + C_2 x^{-0.5} \tag{20}$$