Fourier Series and Transforms Sample Problem 1

This problem is question 3.1 on the Fall 2023 QE.

A triangle wave $x(t)$ with period $T = 2\pi$ is described by the following equation:

$$x(t) = \begin{cases} t + \frac{\pi}{2}, & -\pi \le t < 0 \\ -t + \frac{\pi}{2}, & 0 \le t < \pi \end{cases} \tag{1}$$

Compute the Fourier series representation of the signal. Provide a general expression for all the tterms in the infinite series.

1 General Form

The general solution for the Fourier series of a periodic function with period 2π is given by:

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)] \tag{2}$$

Where:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t)\, dt \tag{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x(t) \cos(nt)\, dt \tag{4}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x(t) \sin(nt)\, dt \tag{5}$$

2 Compute a_0

We already know that:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t)\, dt \tag{6}$$

Since we are dealing with a piecewise function, we can split the integral into two parts:

$$a_0 = \frac{1}{2\pi} \left[\int_{-\pi}^{0} \left(t + \frac{\pi}{2}\right) dt + \int_{0}^{\pi} \left(-t + \frac{\pi}{2}\right) dt \right] \tag{7}$$

Let’s compute the first integral:

$$\int_{-\pi}^{0} \left(t + \frac{\pi}{2}\right) dt = \left[\frac{t^2}{2} + \frac{\pi t}{2} \right]_{-\pi}^{0} \tag{8}$$

$$= (0 + 0) - \left(\frac{(-\pi)^2}{2} + \frac{\pi(-\pi)}{2} \right) = 0 - \left(\frac{\pi^2}{2} - \frac{\pi^2}{2} \right) = 0 \tag{9}$$

Now, let's compute the second integral:

$$\int_0^\pi \left(-t + \frac{\pi}{2}\right) dt = \left[-\frac{t^2}{2} + \frac{\pi t}{2}\right]_0^\pi \tag{10}$$

$$= \left(-\frac{\pi^2}{2} + \frac{\pi^2}{2}\right) - (0 + 0) = 0 \tag{11}$$

Thus, we have:

$$a_0 = \frac{1}{2\pi}(0 + 0) = 0 \tag{12}$$

3 Compute b_n

Next, we compute b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\sin(nt)\,dt \tag{13}$$

Again, we split the integral:

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(t + \frac{\pi}{2}\right)\sin(nt)\,dt + \int_0^\pi\left(-t + \frac{\pi}{2}\right)\sin(nt)\,dt\right] \tag{14}$$

Calculating the first integral using integration by parts:

$$\int_{-\pi}^{0}\left(t + \frac{\pi}{2}\right)\sin(nt)\,dt = \left[-\frac{t + \frac{\pi}{2}}{n}\cos(nt)\right]_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0}\cos(nt)\,dt \tag{15}$$

$$= -\frac{\frac{\pi}{2}}{n} + \frac{\pi}{2n} + \frac{1}{n^2}[\sin(nt)]_{-\pi}^{0} = 0 \tag{16}$$

Now, calculating the second integral:

$$\int_0^\pi\left(-t + \frac{\pi}{2}\right)\sin(nt)\,dt = \left[-\frac{-t + \frac{\pi}{2}}{n}\cos(nt)\right]_0^\pi + \frac{1}{n}\int_0^\pi\cos(nt)\,dt \tag{17}$$

$$= -\frac{-\pi + \frac{\pi}{2}}{n} + \frac{\frac{\pi}{2}}{n} + \frac{1}{n^2}[\sin(nt)]_0^\pi = 0 \tag{18}$$

Thus, we have:

$$b_n = \frac{1}{\pi}(0 + 0) = 0 \tag{19}$$

Note a shortcut here: since the triangle wave is an even function, all b_n coefficients (which correspond to sine terms) will be zero.So we don't have to integrate if we can identify this.

4 Compute a_n

Finally, we compute a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\cos(nt)\,dt \tag{20}$$

Splitting the integral:

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(t+\frac{\pi}{2}\right)\cos(nt)\,dt + \int_{0}^{\pi}\left(-t+\frac{\pi}{2}\right)\cos(nt)\,dt\right] \tag{21}$$

Calculating the first integral using integration by parts:

$$\int_{-\pi}^{0}\left(t+\frac{\pi}{2}\right)\cos(nt)\,dt = \left[\frac{t+\frac{\pi}{2}}{n}\sin(nt)\right]_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}\sin(nt)\,dt \tag{22}$$

$$= 0 + \frac{1}{n^2}[\cos(nt)]_{-\pi}^{0} = \frac{1}{n^2}(1-\cos(n\pi)) = \frac{1}{n^2}(1-(-1)^n) \tag{23}$$

Now, calculating the second integral:

$$\int_{0}^{\pi}\left(-t+\frac{\pi}{2}\right)\cos(nt)\,dt = \left[\frac{-t+\frac{\pi}{2}}{n}\sin(nt)\right]_{0}^{\pi} - \frac{1}{n}\int_{0}^{\pi}\sin(nt)\,dt \tag{24}$$

$$= 0 + \frac{1}{n^2}[\cos(nt)]_{0}^{\pi} = \frac{1}{n^2}(\cos(n\pi)-1) = \frac{1}{n^2}((-1)^n-1) \tag{25}$$

Thus, we have:

$$a_n = \frac{1}{\pi}\left[\frac{1}{n^2}(1-(-1)^n) + \frac{1}{n^2}((-1)^n-1)\right] = \frac{2}{\pi n^2}(1-(-1)^n) \tag{26}$$

5 Final Fourier Series Representation

Combining all the coefficients, we have:

$$x(t) = \sum_{n=1}^{\infty} a_n\cos(nt) = \sum_{n=1}^{\infty}\frac{2}{\pi n^2}(1-(-1)^n)\cos(nt) \tag{27}$$