Vibrations - QE Equation Sheet

Single Degree of Freedom (SDOF) Systems

Modeling and Equations of Motion

Translational System:

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

Rotational System:

$$I\ddot{\theta} + c_\theta\dot{\theta} + k_\theta\theta = M(t)$$

Standard Form:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f(t)}{m}$$

System Parameters

Natural Frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = 2\pi f_n \quad \text{(rad/s)}$$

Natural Period:

$$T_n = \frac{2\pi}{\omega_n} = \frac{1}{f_n} \quad \text{(s)}$$

Damping Ratio:

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n}$$

Critical Damping Coefficient:

$$c_c = 2\sqrt{km} = 2m\omega_n$$

Damped Natural Frequency:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \quad (\zeta < 1)$$

Equivalent Springs

Series:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots$$

Parallel:

$$k_{eq} = k_1 + k_2 + \cdots$$

Equivalent Dampers

Series:

$$\frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2} + \cdots$$

Parallel:

$$c_{eq} = c_1 + c_2 + \cdots$$

Free Vibration

Undamped Free Vibration ($c = 0$)

Equation of Motion:

$$m\ddot{x} + kx = 0$$

General Solution:

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

Or equivalently:

$$x(t) = C\cos(\omega_n t - \phi)$$

where $C = \sqrt{A^2 + B^2}$ and $\phi = \tan^{-1}(B/A)$

Using Initial Conditions:

If $x(0) = x_0$ and $\dot{x}(0) = v_0$:

$$x(t) = x_0\cos(\omega_n t) + \frac{v_0}{\omega_n}\sin(\omega_n t)$$

Damped Free Vibration ($c \neq 0$)

Underdamped ($\zeta < 1$):

$$x(t) = e^{-\zeta\omega_n t}(A\cos(\omega_d t) + B\sin(\omega_d t))$$

Or:

$$x(t) = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

With initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$:

$$x(t) = e^{-\zeta\omega_n t}\left[x_0\cos(\omega_d t) + \frac{v_0 + \zeta\omega_n x_0}{\omega_d}\sin(\omega_d t)\right]$$

Critically Damped ($\zeta = 1$):

$$x(t) = (A + Bt)e^{-\omega_n t}$$

With initial conditions:

$$x(t) = [x_0 + (v_0 + \omega_n x_0)t]e^{-\omega_n t}$$

Overdamped ($\zeta > 1$):

$$x(t) = Ae^{s_1 t} + Be^{s_2 t}$$

where $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

Logarithmic Decrement

For underdamped systems, ratio of successive peaks:

$$\delta = \ln\left(\frac{x(t)}{x(t + T_d)}\right) = \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}}$$

For n cycles:

$$\delta = \frac{1}{n}\ln\left(\frac{x_0}{x_n}\right)$$

Damping Ratio from Logarithmic Decrement:

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

For small damping ($\zeta \ll 1$):

$$\zeta \approx \frac{\delta}{2\pi}$$

Harmonic Forced Vibration

Harmonic Force Excitation

Equation of Motion:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega t)$$

or

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\cos(\omega t)$$

Steady-State Response:

$$x(t) = X\cos(\omega t - \phi)$$

Amplitude:

$$X = \frac{F_0/k}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = \frac{\delta_{st}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

where: - $r = \omega/\omega_n$ is the frequency ratio - $\delta_{st} = F_0/k$ is the static deflection

Magnification Factor:

$$M(r) = \frac{X}{\delta_{st}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

Phase Angle:

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

Resonance

Peak Response Frequency:

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2} \quad (\zeta < 1/\sqrt{2})$$

For small damping:

$$\omega_r \approx \omega_n$$

Peak Amplitude at Resonance:

$$X_{max} = \frac{F_0/k}{2\zeta\sqrt{1 - \zeta^2}} \approx \frac{F_0/k}{2\zeta} \quad (\zeta \ll 1)$$

Quality Factor:

$$Q = \frac{1}{2\zeta} = \frac{X_{max}}{\delta_{st}}$$

Special Frequency Ratios

At $r = 1$ (forced at natural frequency):

$$X = \frac{F_0/k}{2\zeta}, \quad \phi = 90\text{ř}$$

At $r \ll 1$ (low frequency):

$$X \approx \frac{F_0}{k}, \quad \phi \approx 0\text{ř}$$

At $r \gg 1$ (high frequency):

$$X \approx \frac{F_0}{m\omega^2}, \quad \phi \approx 180\text{ř}$$

Base Excitation

Motion of Base:

$$y(t) = Y \sin(\omega t)$$

Equation of Motion:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

Relative Displacement: $z = x - y$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = m\omega^2 Y \sin(\omega t)$$

Absolute Displacement Amplitude:

$$X = Y\sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}$$

Displacement Transmissibility:

$$T_d = \frac{X}{Y} = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}$$

Force Transmissibility and Vibration Isolation

Force Transmitted to Base:

$$F_T = \sqrt{(kX)^2 + (c\omega X)^2}$$

Force Transmissibility:

$$T_F = \frac{F_T}{F_0} = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}$$

Note: $T_F = T_d$ (same formula)

Isolation Region:

$$T < 1 \quad \text{when} \quad r > \sqrt{2}$$

For good isolation:

- Choose $r > \sqrt{2}$ (typically $r > 3$ for practical systems)
- Use small damping (but enough for transient response)
- Lower the natural frequency (softer springs)

Percent Isolation:

$$\text{Isolation} = (1 - T) \times 100\%$$

Rotating Imbalance

Rotating Mass m_0 at Radius e with Angular Velocity ω:

Exciting Force:

$$F(t) = m_0 e\omega^2 \cos(\omega t)$$

Steady-State Amplitude:

$$X = \frac{m_0 e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

where m is total mass and $r = \omega/\omega_n$

Normalized Amplitude:

$$\frac{mX}{m_0 e} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Key Observations:

- At low speeds ($r \ll 1$): $X \approx 0$
- At resonance ($r = 1$): $X = \frac{m_0 e}{2m\zeta}$
- At high speeds ($r \gg 1$): $X \approx \frac{m_0 e}{m}$ (independent of damping)

General Forcing Functions

Impulse Response (Unit Impulse)

Impulse Response Function:

$$h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) \quad (t \geq 0)$$

For undamped system:

$$h(t) = \frac{1}{m\omega_n} \sin(\omega_n t)$$

Convolution Integral (Duhamel's Integral)

For arbitrary forcing $f(t)$:

$$x(t) = \int_0^t h(t-\tau) f(\tau)\, d\tau$$

$$x(t) = \frac{1}{m\omega_d} \int_0^t f(\tau) e^{-\zeta\omega_n (t-\tau)} \sin[\omega_d (t-\tau)]\, d\tau$$

Step Response

For step input $f(t) = F_0 u(t)$:

Undamped:

$$x(t) = \frac{F_0}{k}[1 - \cos(\omega_n t)]$$

Underdamped:

$$x(t) = \frac{F_0}{k}\left[1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)\right]$$

Or:

$$x(t) = \frac{F_0}{k}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \psi)\right]$$

where $\psi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$

Ramp Response

For ramp input $f(t) = F_0 \cdot t$:

$$x(t) = \frac{F_0}{k}\left[t - \frac{2\zeta}{\omega_n}\right] + \text{transient terms}$$

Frequency Response Methods

Transfer Function

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} = \frac{1/m}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Frequency Response Function (FRF):

$$H(j\omega) = H(s)|_{s=j\omega} = \frac{1/k}{(1-r^2) + j(2\zeta r)}$$

Magnitude:

$$|H(j\omega)| = \frac{1/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Phase:

$$\angle H(j\omega) = -\tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

Energy Methods

Rayleigh's Method

For finding natural frequency without solving EOM:

Principle: At maximum displacement, all energy is potential. At equilibrium, all energy is kinetic.

$$T_{max} = V_{max}$$

$$\frac{1}{2}m\omega_n^2 X^2 = \frac{1}{2}kX^2$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

For Distributed Systems:

$$\omega_n = \sqrt{\frac{V_{max}}{T_{max}}}$$

Rayleigh-Ritz Method

Assume displacement shape $x(y,t) = \phi(y)q(t)$

Use energy methods to find approximate ω_n

Multi-Degree-of-Freedom (MDOF) Systems

Equations of Motion (Matrix Form)

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t)$$

where $\mathbf{M}$ is mass matrix, $\mathbf{C}$ is damping matrix, $\mathbf{K}$ is stiffness matrix

Free Vibration (Undamped)

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$$

Assume harmonic solution:

$$\mathbf{x}(t) = \boldsymbol{\phi} e^{j\omega t}$$

Eigenvalue Problem:

$$(\mathbf{K} - \omega^2\mathbf{M})\boldsymbol{\phi} = \mathbf{0}$$

Characteristic Equation:

$$\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0$$

Solving gives n natural frequencies $\omega_1, \omega_2, \ldots, \omega_n$ and corresponding mode shapes $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n$

Orthogonality of Mode Shapes

$$\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_j = 0 \quad (i \neq j)$$
$$\boldsymbol{\phi}_i^T \mathbf{K} \boldsymbol{\phi}_j = 0 \quad (i \neq j)$$

Normalized Mode Shapes:

$$\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i = 1$$
$$\boldsymbol{\phi}_i^T \mathbf{K} \boldsymbol{\phi}_i = \omega_i^2$$

Modal Analysis

Modal Coordinate Transformation:

$$\mathbf{x}(t) = \boldsymbol{\Phi} \mathbf{q}(t) = \sum_{i=1}^{n} \boldsymbol{\phi}_i q_i(t)$$

where $\boldsymbol{\Phi} = [\boldsymbol{\phi}_1 \, \boldsymbol{\phi}_2 \, \cdots \, \boldsymbol{\phi}_n]$

Decoupled Modal Equations:

For proportional damping:

$$\ddot{q}_i + 2\zeta_i \omega_i \dot{q}_i + \omega_i^2 q_i = \frac{\boldsymbol{\phi}_i^T \mathbf{f}(t)}{\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i}$$

Each modal equation is a SDOF system!

Vibration Absorbers

Undamped Dynamic Absorber

Primary system with absorber attached:

Primary mass m_1 with stiffness k_1

Absorber mass m_2 with stiffness k_2

Tuning Condition (to eliminate vibration of primary mass):

$$\omega_2 = \sqrt{\frac{k_2}{m_2}} = \omega$$

where ω is the forcing frequency

Mass Ratio:

$$\mu = \frac{m_2}{m_1}$$

Effect: Primary mass amplitude becomes zero at tuned frequency, but two new resonance peaks appear nearby

Damped Dynamic Absorber

Adding damping c_2 to absorber provides:

- Reduces peak amplitudes at resonances
- Primary mass never reaches exactly zero amplitude
- Better performance over range of frequencies

Torsional Vibrations

Equation of Motion:

$$I\ddot{\theta} + c_t\dot{\theta} + k_t\theta = M(t)$$

Natural Frequency:

$$\omega_n = \sqrt{\frac{k_t}{I}}$$

Torsional Stiffness of Shaft:

$$k_t = \frac{GJ}{L}$$

where G is shear modulus, J is polar moment of inertia, L is length

For circular shaft: $J = \frac{\pi d^4}{32}$

Vibration Measurement and Testing

Measurement Techniques

From Free Decay:

- Natural frequency: $\omega_d = \frac{2\pi}{T_d}$
- Damping ratio: Use logarithmic decrement $\zeta = \frac{\delta}{2\pi}$

From Frequency Response:

- Natural frequency: Peak of magnitude plot
- Damping ratio: Half-power bandwidth method

Half-Power Bandwidth Method:

At points where $|H| = \frac{|H|_{max}}{\sqrt{2}}$:

$$\zeta = \frac{\omega_2 - \omega_1}{2\omega_n}$$

Fast Fourier Transform (FFT)

Convert time domain signal to frequency domain to identify:

- Natural frequencies (peaks in FFT)
- Dominant frequency components
- Harmonic content

$$X(f) = \text{FFT}[x(t)]$$

Quick Reference Formulas

Natural Frequency:

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{(Hz)}$$

Static Deflection Method:

$$\omega_n = \sqrt{\frac{g}{\delta_{st}}}$$

where δ_{st} is static deflection under weight

Amplitude at Resonance (underdamped):

$$X_{max} \approx \frac{F_0}{2\zeta k}$$

Transmissibility at $r = \sqrt{2}$:

$$T = 1 \quad \text{(crossover point)}$$