Heat Transfer Sample Problem 1

This is problem 1.1 on the 2023 QE.

A fluid of specific heat C_p, thermal conductivity k, and viscosity v flows steadily through a circular tube of diameter D and length L. The mass flow rate is $\dot{m}$ and the fluid enters with a mean temperature of $T_{m,i}$. The flow is turbulent and fully-developed over the entire length of the tube. The Nusselt number for this case is then

$$\overline{Nu}_D = \frac{\bar{h}D}{k} = 0.023 Re_D^{0.8} Pr^{0.4}$$

The tube surface (wall) is subjected to a heat flux that decreases linearly from inlet to outlet as: $q_s'' = 2q_o''\left(1 - \frac{x}{L}\right)$ where $q_o'' > 0$ is known.
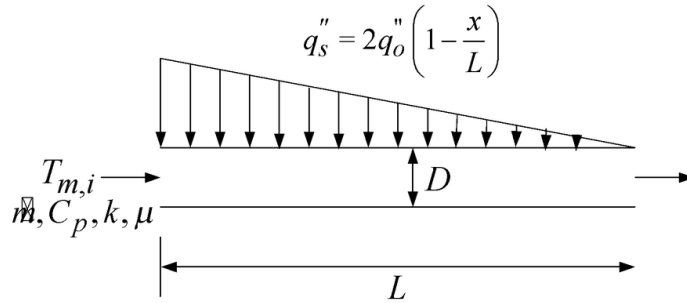
Part A

Derive an expression for variation of the mean fluid temperature $T_m(x)$ only in terms of known parameters. Hint: select a differential control volume and apply the energy balance equation.

Convective energy is given by the following equation:

$$E = \dot{m}C_p\Delta T \tag{1}$$

By conservation of energy, the total energy entering the system must equal the total energy leabing the system.

$$E_{in} + E_{wall} = E_{out} \tag{2}$$

By using a differntial control volume, we can express the terms in the energy balance as:

$$E_{in} = \dot{m}C_p T_m(x) \tag{3}$$
$$E_{out} = \dot{m}C_p T_m(x + dx) \tag{4}$$
$$E_{wall} = q''\pi D dx \tag{5}$$

Note for the wall energy term, we are using the heat flux q'' multiplied by the surface area of the pipe over the differential length Δx. The full energy balance is then:

$$\dot{m}C_pT_m(x) + q''\pi Ddx = \dot{m}C_pT_m(x + dx) \tag{6}$$

We can rearrange and solve for $\frac{dT_m}{dx}$ as follows:

$$q''\pi Ddx = \dot{m}C_pT_m(x + dx) - \dot{m}C_pT_m(x) \tag{7}$$

$$q''\pi Ddx = \dot{m}C_p\left[T_m(x + dx) - T_m(x)\right] \tag{8}$$

$$q''\pi Ddx = \dot{m}C_pdT_m \tag{9}$$

$$\frac{dT_m}{dx} = \frac{q''\pi D}{\dot{m}C_p} \tag{10}$$

We can then substitute $q'' = 2\ddot{q}_0(1 - \frac{x}{L})$ into the equation to get:

$$\frac{dT_m}{dx} = \frac{2\ddot{q}_0(1 - \frac{x}{L})\pi D}{\dot{m}C_p} \tag{11}$$

Then, integrating both sides from 0 to x gives:

$$\int_{T_{m,0}}^{T_m(x)} dT_m = \int_0^x \frac{2\ddot{q}_0(1 - \frac{x}{L})\pi D}{\dot{m}C_p}dx \tag{12}$$

$$T_m(x) - T_{m,0} = \frac{2\ddot{q}_0\pi D}{\dot{m}C_p}\left(x - \frac{x^2}{2L}\right) \tag{13}$$

$$\boxed{T_m(x) = T_{m,0} + \frac{2\ddot{q}_0\pi D}{\dot{m}C_p}\left(x - \frac{x^2}{2L}\right)} \tag{14}$$

Part B

Derive an expression for the total rate of heat transfer q to the fluid only in terms of known parameters.

The total rate of heat transfer to the fluid can be found by integrating the heat flux over the surface area of the pipe:

$$q = \int_0^L q''\pi Ddx \tag{15}$$

$$q = \int_0^L 2\ddot{q}_0(1 - \frac{x}{L})\pi Ddx \tag{16}$$

$$q = 2\ddot{q}_0\pi D\int_0^L (1 - \frac{x}{L})dx \tag{17}$$

$$q = 2\ddot{q}_0\pi D\left[x - \frac{x^2}{2L}\right]_0^L \tag{18}$$

$$\boxed{q = \ddot{q}_0\pi DL} \tag{19}$$

Part C

Derive an expression for variation of the surface (wall) temperature $T_s(x)$ only in terms of known parameters.

The surface temperature can be found using the convective heat transfer equation:

$$q'' = h(T_s - T_m) \tag{20}$$

Rearranging for T_s gives:

$$T_s = \frac{q''}{h} + T_m \tag{21}$$

We also know from the problem statement that the Nusselt number is:

$$Nu = \frac{hD}{k} = 0.023 Re^{0.8} Pr^{0.4} \tag{22}$$

Rearranging for h gives:

$$h = \frac{0.023 k Re^{0.8} Pr^{0.4}}{D} \tag{23}$$

We can then substitute h, q'', and our expression for T_m into the equation for T_s.

$$T_s = \frac{2\ddot{q}_0 (1 - \frac{x}{L})}{\frac{0.023 k Re^{0.8} Pr^{0.4}}{D}} + \left[T_{m,0} + \frac{2\ddot{q}_0 \pi D}{\dot{m} C_p} \left(x - \frac{x^2}{2L} \right) \right] \tag{24}$$

$$\boxed{T_s(x) = \frac{2\ddot{q}_0 (1 - \frac{x}{L})D}{0.023 k Re^{0.8} Pr^{0.4}} + T_{m,0} + \frac{2\ddot{q}_0 \pi D}{\dot{m} C_p} \left(x - \frac{x^2}{2L} \right)} \tag{25}$$

Part D

Derive an expression for the axial location x_max at which the surface (wall) temperature is maximum.

To find the location where the surface temperature is maximum, we need to take the derivative of $T_s(x)$ and set it equal to zero.

$$\frac{dT_s}{dx} = \frac{d}{dx} \left[\frac{2\ddot{q}_0 (1 - \frac{x}{L})D}{0.023 k Re^{0.8} Pr^{0.4}} + T_{m,0} + \frac{2\ddot{q}_0 \pi D}{\dot{m} C_p} \left(x - \frac{x^2}{2L} \right) \right] \tag{26}$$

$$\frac{dT_s}{dx} = \frac{-2\ddot{q}_0 D}{0.023 k Re^{0.8} Pr^{0.4} L} + \frac{2\ddot{q}_0 \pi D}{\dot{m} C_p} \left(1 - \frac{x}{L} \right) \tag{27}$$

Setting the derivative equal to zero and solving for x gives:

$$0 = \frac{-2\ddot{q}_0 D}{0.023 k Re^{0.8} Pr^{0.4} L} + \frac{2\ddot{q}_0 \pi D}{\dot{m} C_p}\left(1 - \frac{x}{L}\right) \tag{28}$$

$$\frac{2\ddot{q}_0 D}{0.023 k Re^{0.8} Pr^{0.4} L} = \frac{2\ddot{q}_0 \pi D}{\dot{m} C_p}\left(1 - \frac{x}{L}\right) \tag{29}$$

$$\frac{\dot{m} C_p}{0.023 k Re^{0.8} Pr^{0.4} L \pi} = 1 - \frac{x}{L} \tag{30}$$

$$\frac{x}{L} = 1 - \frac{\dot{m} C_p}{0.023 k Re^{0.8} Pr^{0.4} L \pi} \tag{31}$$

$$\boxed{x_{max} = L\left(1 - \frac{\dot{m} C_p}{0.023 k Re^{0.8} Pr^{0.4} L \pi}\right)} \tag{32}$$