Fourier Series and Transforms Sample Problem 2

This problem is question 3.2 on the Fall 2023 QE.

The rectangular function is defined as:

$$\mathrm{rect}(t/a) = \begin{cases} 0, & |t| > \frac{a}{2} \\ \frac{1}{2}, & |t| = \frac{a}{2} \\ 1, & |t| < \frac{a}{2} \end{cases} \tag{1}$$

With $a > 0$. Compute the Fourier transform of the signal and sketch the magnitude of the spectrum signal, as a function of the parameter a.

1 Fourier Transform Definition

The Fourier transform $X(f)$ of a time-domain signal $x(t)$ is defined as:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \tag{2}$$

Where $\omega = 2\pi f$ is the angular frequency.

2 Compute the Fourier Transform of rect(t)

To compute the Fourier transform of the rectangular function, we substitute $x(t) = \mathrm{rect}(t)$ into the Fourier transform definition:

$$X(f) = \int_{-\infty}^{\infty} \mathrm{rect}(t/a) e^{-j\omega t}\, dt \tag{3}$$

Since $\mathrm{rect}(t)$ is non-zero only in the interval $[-\frac{a}{2}, \frac{a}{2}]$, we can limit the integration bounds. Furthermore, within this interval , $\mathrm{rect}(t/a) = 1$. Thus, we have:

$$X(f) = \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-j\omega t}\, dt \tag{4}$$

Evaluating this integral, we get:

$$X(f) = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\frac{a}{2}}^{\frac{a}{2}} = \frac{1}{-j\omega}\left(e^{-j\omega\frac{a}{2}} - e^{j\omega\frac{a}{2}}\right) \tag{5}$$

Simplifying the expression using Euler's formula, we find:

$$X(f) = \frac{2\sin\left(\frac{\omega a}{2}\right)}{\omega} = \frac{2\sin\left(\pi f a\right)}{2\pi f} = a\,\mathrm{sinc}(fa) \tag{6}$$

For context, Euler's formula is:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{7}$$