Dynamics and Vibrations - QE Equation Sheet

Kinematics of Particles

Rectilinear Motion

Position, Velocity, Acceleration:

$$v = \frac{dx}{dt} = \dot{x}, \quad a = \frac{dv}{dt} = \ddot{x}$$

Alternative forms:

$$a = v\frac{dv}{dx}, \quad v\,dv = a\,dx$$

Constant Acceleration:

$$v = v_0 + at$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

$$v^2 = v_0^2 + 2a(x - x_0)$$

Curvilinear Motion

Cartesian Coordinates:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$$

$$\mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}$$

Path Coordinates (Normal-Tangential):

$$\mathbf{v} = v\hat{\mathbf{e}}_t$$

$$\mathbf{a} = \dot{v}\hat{\mathbf{e}}_t + \frac{v^2}{\rho}\hat{\mathbf{e}}_n$$

where ρ is the radius of curvature

Polar Coordinates:

$$\mathbf{r} = r\hat{\mathbf{e}}_r$$

$$\mathbf{v} = \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{e}}_\theta$$

Cylindrical Coordinates:

$$\mathbf{v} = \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta + \dot{z}\hat{\mathbf{e}}_z$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{e}}_\theta + \ddot{z}\hat{\mathbf{e}}_z$$

Relative Motion

Translating Reference Frame:

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

Rotating Reference Frame:

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{rel}$$

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\omega} \times (\mathbf{v}_{B/A})_{rel} + (\mathbf{a}_{B/A})_{rel}$$

where $2\boldsymbol{\omega} \times (\mathbf{v}_{B/A})_{rel}$ is the Coriolis acceleration

Rigid Body Kinematics

General Planar Motion

Velocity:

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

In 2D:

$$v_B = v_A + \omega r_{B/A} \perp \mathbf{r}_{B/A}$$

Acceleration:

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A})$$

In 2D:

$$\mathbf{a}_B = \mathbf{a}_A + \alpha r_{B/A}(\perp \mathbf{r}_{B/A}) - \omega^2 r_{B/A}(\parallel \mathbf{r}_{B/A})$$

Rolling Without Slipping

For a wheel of radius R rolling on a fixed surface:

$$v_{\text{center}} = \omega R$$

$$a_{\text{center}} = \alpha R$$

Point of contact has zero velocity (instantaneous center)

Instantaneous Center of Zero Velocity (IC)

For planar motion, there exists a point IC such that $\mathbf{v}_{\text{IC}} = \mathbf{0}$ at that instant

Finding IC:

- Draw velocity vectors of two points
- Draw perpendiculars to each velocity
- IC is at the intersection

Using IC:

$$v_A = \omega \cdot r_{A/IC}$$

where $r_{A/IC}$ is distance from IC to point A

Note: IC generally has non-zero acceleration

Rotation About a Fixed Axis

$$\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{k}}$$
$$\boldsymbol{\alpha} = \ddot{\theta}\hat{\mathbf{k}}$$

For a point at distance r from the axis:

$$v = r\omega, \quad a_t = r\alpha, \quad a_n = r\omega^2$$

Mass Properties

Center of Mass

Discrete particles:

$$\mathbf{r}_G = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} = \frac{\sum m_i \mathbf{r}_i}{M}$$

Continuous body:

$$\mathbf{r}_G = \frac{1}{M} \int \mathbf{r}\, dm$$

In coordinates:

$$x_G = \frac{1}{M} \int x\, dm, \quad y_G = \frac{1}{M} \int y\, dm, \quad z_G = \frac{1}{M} \int z\, dm$$

Mass Moment of Inertia

Definition:

$$I = \int r^2 \, dm$$

where r is perpendicular distance from axis of rotation

Parallel Axis Theorem:

$$I_O = I_G + Md^2$$

where d is the distance between parallel axes through O and G

Radius of Gyration:

$$I = Mk^2 \quad \Rightarrow \quad k = \sqrt{\frac{I}{M}}$$

Common Moments of Inertia

Slender Rod (length L, mass m):

$$I_{\text{center}} = \frac{1}{12}mL^2, \quad I_{\text{end}} = \frac{1}{3}mL^2$$

Thin Disk/Cylinder (radius R, mass m):

$$I_{\text{center, perpendicular}} = \frac{1}{2}mR^2$$

$$I_{\text{center, diameter}} = \frac{1}{4}mR^2$$

Thin Ring (radius R, mass m):

$$I_{\text{center}} = mR^2$$

Sphere (radius R, mass m):

$$I_{\text{center}} = \frac{2}{5}mR^2$$

Rectangular Plate (sides $a \times b$, mass m):

$$I_{\text{center, perpendicular}} = \frac{1}{12}m(a^2 + b^2)$$

Thin Plate (perpendicular axis theorem):

$$I_z = I_x + I_y$$

Kinetics: Newton-Euler Equations

Particle Kinetics

Newton's Second Law:

$$\sum \mathbf{F} = m\mathbf{a}$$

In component form:

$$\sum F_x = ma_x, \quad \sum F_y = ma_y, \quad \sum F_z = ma_z$$

Rigid Body Kinetics (Planar Motion)

Equation of Motion:

$$\sum \mathbf{F} = m\mathbf{a}_G$$

$$\sum M_G = I_G\alpha$$

About a Fixed Point O:

$$\sum M_O = I_O\alpha$$

About Point P (not fixed, not at G):

$$\sum M_P = I_G\alpha + \mathbf{r}_{G/P} \times m\mathbf{a}_G$$

Component Form (2D):

$$\sum F_x = m(a_G)_x$$

$$\sum F_y = m(a_G)_y$$

$$\sum M_G = I_G\alpha$$

Work and Energy

Work

Definition:

$$W = \int \mathbf{F} \cdot d\mathbf{r}$$

Constant Force:

$$W = F \cdot d\cos\theta$$

Spring Force:

$$W = -\frac{1}{2}k(x_2^2 - x_1^2)$$

Weight:

$$W = -mg(y_2 - y_1) = -mg\Delta h$$

Moment/Torque:

$$W = \int M \, d\theta$$

For constant moment:

$$W = M(\theta_2 - \theta_1)$$

Kinetic Energy

Particle:

$$T = \frac{1}{2}mv^2$$

Rigid Body (General Planar Motion):

$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$$

Rigid Body (Rotation about Fixed Axis):

$$T = \frac{1}{2}I_O\omega^2$$

Rolling Body:

$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2 = \frac{1}{2}(m + \frac{I_G}{R^2})v_G^2$$

Potential Energy

Gravitational:

$$V_g = mgh$$

where h is height above reference datum

Elastic (Spring):

$$V_e = \frac{1}{2}kx^2$$

where x is deformation from unstretched position

Work-Energy Theorem

$$T_1 + \sum U_{1-2} = T_2$$

or

$$T_1 + V_1 + W_{\text{nc}} = T_2 + V_2$$

where W_{nc} is work by non-conservative forces

Conservation of Energy:

If only conservative forces do work:

$$T_1 + V_1 = T_2 + V_2$$

$$E = T + V = \text{constant}$$

Power

Definition:

$$P = \frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v}$$

For rotation:

$$P = M\omega$$

Efficiency:

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}}$$

Impulse and Momentum

Linear Impulse-Momentum

Linear Momentum:

$$\mathbf{L} = m\mathbf{v}$$

Impulse-Momentum Principle:

$$\int_{t_1}^{t_2} \sum \mathbf{F}\, dt = m\mathbf{v}_2 - m\mathbf{v}_1$$

Conservation of Linear Momentum:

If $\sum \mathbf{F}_{\text{ext}} = 0$:

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = \text{constant}$$

Angular Impulse-Momentum

Angular Momentum about Point O:

For a particle:

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$$

For a rigid body:

$$\mathbf{H}_O = I_O \boldsymbol{\omega}$$

For a rigid body about G:

$$\mathbf{H}_G = I_G \boldsymbol{\omega}$$

For a rigid body about arbitrary point O:

$$\mathbf{H}_O = I_G \boldsymbol{\omega} + \mathbf{r}_{G/O} \times m\mathbf{v}_G$$

Angular Impulse-Momentum Principle:

$$\int_{t_1}^{t_2} \sum M_O \, dt = (H_O)_2 - (H_O)_1$$

Conservation of Angular Momentum:

If $\sum M_O = 0$:

$$H_O = \text{constant}$$

Impact and Collisions

Coefficient of Restitution:

$$e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$$

where subscripts A and B denote two bodies, and subscripts 1 and 2 denote before and after impact

Types of Collisions:

- $e = 1$: Perfectly elastic (kinetic energy conserved)
- $e = 0$: Perfectly plastic (bodies stick together)
- $0 < e < 1$: Real collisions (some energy lost)

For direct central impact:

Conservation of momentum:

$$m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$$

Restitution:

$$(v_B)_2 - (v_A)_2 = -e[(v_B)_1 - (v_A)_1]$$

Special Topics

Systems of Particles

Center of Mass Motion:

$$\sum \mathbf{F}_{\text{ext}} = M\mathbf{a}_G$$

Total Linear Momentum:

$$\mathbf{L}_{\text{total}} = M\mathbf{v}_G$$

Angular Momentum about G:

$$\mathbf{H}_G = \sum (\mathbf{r}_i - \mathbf{r}_G) \times m_i\mathbf{v}_i$$

Variable Mass Systems

Thrust Equation (Rocket):

$$\sum F_{\text{ext}} = m\frac{dv}{dt} - v_{\text{rel}}\frac{dm}{dt}$$

where v_{rel} is exhaust velocity relative to rocket

Constraints and Dependent Motion

Cable Constraint:

For a system with cables/ropes, write constraint equation based on constant total length:

$$L_{\text{total}} = \sum L_i = \text{constant}$$

Differentiate to relate velocities:

$$\sum \frac{dL_i}{dt} = 0$$

Differentiate again to relate accelerations:

$$\sum \frac{d^2L_i}{dt^2} = 0$$

Rolling Constraint:

For pure rolling (no slip):

$$v = R\omega, \quad a = R\alpha$$

Gears:

For meshing gears:

$$r_1\omega_1 = r_2\omega_2$$

$$r_1\alpha_1 = r_2\alpha_2$$

Problem-Solving Strategies

Choosing the Right Approach

Use Newton-Euler when:

- Need to find forces/reactions
- Acceleration is needed
- Problem involves time explicitly

Use Work-Energy when:

- Only interested in velocities (not time)
- Forces do work over a distance
- Springs are involved

Use Impulse-Momentum when:

- Forces act over time intervals
- Collisions/impacts occur
- Conservation principles apply

Common Steps

For Newton-Euler Problems:

1. Draw free body diagram (FBD)
2. Choose coordinate system
3. Write kinematic relationships
4. Apply $\sum F = ma$ and $\sum M = I\alpha$
5. Solve system of equations

For Energy Problems:

1. Identify datum for potential energy
2. Write $T_1 + V_1$ at initial state
3. Write $T_2 + V_2$ at final state
4. Calculate work by non-conservative forces
5. Apply $T_1 + V_1 + W_{nc} = T_2 + V_2$