Ordinary Differential Equations

Classification of ODEs

Order: Highest derivative present in the equation

Linearity: An ODE is *linear* if it can be written as:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

If $g(x) = 0$, the equation is *homogeneous*; otherwise, it is *non-homogeneous*.

First-Order ODEs

Separable Equations

Form: $\dfrac{dy}{dx} = f(x)g(y)$

Solution Method:

$$\frac{dy}{g(y)} = f(x)dx$$

$$\int \frac{dy}{g(y)} = \int f(x)dx + C$$

Linear First-Order ODEs

Standard form: $\dfrac{dy}{dx} + P(x)y = Q(x)$

Integrating Factor Method:

$$\mu(x) = e^{\int P(x)dx}$$

General Solution:

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right]$$

Or equivalently:

$$\mu(x) \cdot y = \int \mu(x)Q(x)dx + C$$

Exact Equations

Form: $M(x,y)dx + N(x,y)dy = 0$

Exactness Condition:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If exact, there exists a function $F(x,y)$ such that:

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

Solution: $F(x,y) = C$

Finding $F(x,y)$:

$$F(x,y) = \int M(x,y)dx + g(y)$$

where $g(y)$ is determined by setting $\frac{\partial F}{\partial y} = N$

Bernoulli Equations

Form: $\frac{dy}{dx} + P(x)y = Q(x)y^n$

Solution Method: Substitute $v = y^{1-n}$ to transform into a linear ODE

For $n \neq 0, 1$:

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

Second-Order Linear ODEs

General form: $a(x)y'' + b(x)y' + c(x)y = g(x)$

Standard form: $y'' + p(x)y' + q(x)y = f(x)$

Homogeneous Equations with Constant Coefficients

Form: $ay'' + by' + cy = 0$

Characteristic Equation:

$$ar^2 + br + c = 0$$

General Solution depends on the roots:

Case 1: Two distinct real roots r_1, r_2 ($b^2 - 4ac > 0$):

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Case 2: Repeated real root r $(b^2 - 4ac = 0)$:

$$y(x) = (c_1 + c_2 x)e^{rx}$$

where $r = -\frac{b}{2a}$

Case 3: Complex conjugate roots $r = \alpha \pm i\beta$ $(b^2 - 4ac < 0)$:

$$y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

where $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{4ac-b^2}}{2a}$

Non-Homogeneous Equations: Method of Undetermined Coefficients

Form: $ay'' + by' + cy = g(x)$

General Solution:

$$y(x) = y_h(x) + y_p(x)$$

where y_h is the homogeneous solution and y_p is a particular solution

Form of Particular Solution y_p:

If $g(x)$ **is:**	**Try** y_p:
$P_n(x)$ (polynomial of degree n)	$x^s(A_n x^n + \cdots + A_1 x + A_0)$
$ke^{\alpha x}$	$x^s A e^{\alpha x}$
$k\cos(\beta x)$ or $k\sin(\beta x)$	$x^s(A\cos(\beta x) + B\sin(\beta x))$
$e^{\alpha x}P_n(x)$	$x^s e^{\alpha x}(A_n x^n + \cdots + A_0)$
$e^{\alpha x}[P(x)\cos(\beta x) + Q(x)\sin(\beta x)]$	$x^s e^{\alpha x}[(A_n x^n + \cdots)\cos(\beta x)$ $+(B_n x^n + \cdots)\sin(\beta x)]$

where s = multiplicity of the root in the characteristic equation (usually $s = 0, 1,$ or 2)

Variation of Parameters

For: $y'' + p(x)y' + q(x)y = f(x)$

Given homogeneous solutions y_1 and y_2:

Particular Solution:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where:

$$u_1 = -\int \frac{y_2 f}{W} dx, \quad u_2 = \int \frac{y_1 f}{W} dx$$

Wronskian:

$$W = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

Cauchy-Euler Equations

Form: $ax^2y'' + bxy' + cy = 0$

Trial Solution: $y = x^r$

Characteristic Equation:

$$ar(r-1) + br + c = 0$$
$$ar^2 + (b-a)r + c = 0$$

General Solution depends on roots:

Case 1: Two distinct real roots r_1, r_2:

$$y(x) = c_1x^{r_1} + c_2x^{r_2}$$

Case 2: Repeated real root r:

$$y(x) = (c_1 + c_2 \ln|x|)x^r$$

Case 3: Complex conjugate roots $r = \alpha \pm i\beta$:

$$y(x) = x^\alpha[c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|)]$$

Systems of First-Order Linear ODEs

Form: $\mathbf{x}' = A\mathbf{x}$ where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and A is an $n \times n$ matrix

General Solution:

$$\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_ne^{\lambda_n t}$$

where λ_i are eigenvalues and $\mathbf{v}_i$ are corresponding eigenvectors of A

Special Cases for 2×2 Systems

For $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

Case 1: Two distinct real eigenvalues λ_1, λ_2:

$$\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t}$$

Case 2: Repeated eigenvalue λ: Find generalized eigenvector $\mathbf{w}$ satisfying $(A - \lambda I)\mathbf{w} = \mathbf{v}$

$$\mathbf{x}(t) = c_1\mathbf{v}e^{\lambda t} + c_2(\mathbf{w}t + \mathbf{v})e^{\lambda t}$$

Case 3: Complex eigenvalues $\lambda = \alpha \pm i\beta$:

If $\lambda_1 = \alpha + i\beta$ has eigenvector $\mathbf{v} = \mathbf{a} + i\mathbf{b}$:

$$\mathbf{x}(t) = c_1 e^{\alpha t}[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)] + c_2 e^{\alpha t}[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)]$$

Higher-Order ODEs

An n-th order ODE can be converted to a system of first-order ODEs.

For: $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$

Define: $x_1 = y$, $x_2 = y'$, $x_3 = y''$, $\ldots$, $x_n = y^{(n-1)}$

System:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = f(t, x_1, x_2, \ldots, x_n) \end{cases}$$

Nonlinear Systems

Critical Points (Equilibrium Points)

For $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, critical points satisfy $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$

Linearization Near Critical Points

For a system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ near a critical point $\mathbf{x}^*$:

Jacobian Matrix:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\mathbf{x}=\mathbf{x}^*}$$

The linearized system is: $\mathbf{u}' = J\mathbf{u}$ where $\mathbf{u} = \mathbf{x} - \mathbf{x}^*$

Stability Classification (2D Systems)

Based on eigenvalues λ_1, λ_2 of Jacobian J:

- **Node (Stable):** Both $\lambda_i < 0$ (real)
- **Node (Unstable):** Both $\lambda_i > 0$ (real)

- **Saddle Point:** $\lambda_1 < 0 < \lambda_2$ (real, opposite signs) - Unstable
- **Spiral (Stable):** $\mathrm{Re}(\lambda_i) < 0$ (complex)
- **Spiral (Unstable):** $\mathrm{Re}(\lambda_i) > 0$ (complex)
- **Center:** $\mathrm{Re}(\lambda_i) = 0$ (purely imaginary) - Neutrally stable

Useful Formulas

Reduction of Order:

If y_1 is a known solution to $y'' + p(x)y' + q(x)y = 0$, then a second solution is:

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2(x)} dx$$

Abel's Formula (for Wronskian):

For $y'' + p(x)y' + q(x)y = 0$:

$$W(x) = W(x_0) e^{-\int_{x_0}^{x} p(t)dt}$$