

# Fourier Series and Transforms

## Fourier Series

For a periodic function  $f(x)$  with period  $T = 2L$  (or  $f(x) = f(x + 2L)$ ):

### General Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

### Fourier Coefficients:

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \dots$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \dots$$

### Fourier Series for Period $2\pi$

For  $f(x)$  with period  $2\pi$  (i.e.,  $L = \pi$ ):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

### Coefficients:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

## Half-Range Expansions

**Fourier Cosine Series** (even extension on  $[0, L]$ ):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{2}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

**Fourier Sine Series** (odd extension on  $[0, L]$ ):

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

## Complex Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

**Complex Coefficients:**

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-in\pi x/L} dx$$

**Relationship to real coefficients:**

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}$$

## Properties of Fourier Series

**Even Function:**  $f(-x) = f(x)$

- All  $b_n = 0$  (only cosine terms)
- $a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

**Odd Function:**  $f(-x) = -f(x)$

- All  $a_n = 0$  (only sine terms)
- $b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

**Convergence:**

- At points where  $f$  is continuous, the series converges to  $f(x)$
- At jump discontinuities, the series converges to  $\frac{f(x^+) + f(x^-)}{2}$

**Parseval's Identity:**

$$\frac{1}{L} \int_{-L}^L |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

## Fourier Transform

For non-periodic functions defined on  $(-\infty, \infty)$ :

### Fourier Transform (Complex Form)

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

**Inverse Fourier Transform:**

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

### Fourier Cosine Transform

For functions defined on  $[0, \infty)$ :

$$F_c(\omega) = \int_0^{\infty} f(x) \cos(\omega x) dx$$

**Inverse:**

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(\omega) \cos(\omega x) d\omega$$

### Fourier Sine Transform

For functions defined on  $[0, \infty)$ :

$$F_s(\omega) = \int_0^{\infty} f(x) \sin(\omega x) dx$$

**Inverse:**

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\omega) \sin(\omega x) d\omega$$

## Common Fourier Transforms

Function $f(t)$	Transform $F(\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$e^{-at}u(t), a > 0$	$\frac{1}{a + i\omega}$
$e^{-a t }, a > 0$	$\frac{2a}{a^2 + \omega^2}$
$\text{rect}(t/a)$	$a \text{sinc}(a\omega/2)$
$e^{-at^2}$	$\sqrt{\frac{\pi}{a}} e^{-\omega^2/(4a)}$
$\cos(\omega_0 t)$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin(\omega_0 t)$	$i\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$

where  $\text{rect}(t/a) = \begin{cases} 1 & |t| < a/2 \\ 0 & |t| > a/2 \end{cases}$  and  $\text{sinc}(x) = \frac{\sin(x)}{x}$

## Properties of Fourier Transforms

**Linearity:**

$$\mathcal{F}\{af(t) + bg(t)\} = aF(\omega) + bG(\omega)$$

**Time Shifting:**

$$\mathcal{F}\{f(t - t_0)\} = e^{-i\omega t_0} F(\omega)$$

**Frequency Shifting:**

$$\mathcal{F}\{e^{i\omega_0 t} f(t)\} = F(\omega - \omega_0)$$

**Scaling:**

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

**Time Differentiation:**

$$\mathcal{F}\left\{\frac{d^n f}{dt^n}\right\} = (i\omega)^n F(\omega)$$

**Frequency Differentiation:**

$$\mathcal{F}\{t^n f(t)\} = i^n \frac{d^n F(\omega)}{d\omega^n}$$

**Integration:**

$$\mathcal{F}\left\{\int_{-\infty}^t f(\tau) d\tau\right\} = \frac{F(\omega)}{i\omega} + \pi F(0)\delta(\omega)$$

**Convolution Theorem:**

$$\mathcal{F}\{(f * g)(t)\} = F(\omega) \cdot G(\omega)$$

$$\mathcal{F}\{f(t) \cdot g(t)\} = \frac{1}{2\pi}(F * G)(\omega)$$

where  $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau) d\tau$

**Parseval's Theorem:**

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

**Symmetry Properties:**

If  $f(t)$  is real and even:  $F(\omega)$  is real and even

If  $f(t)$  is real and odd:  $F(\omega)$  is purely imaginary and odd

**Discrete Fourier Transform (DFT)**

For a discrete signal  $\{x_n\}$  with  $N$  samples:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-2\pi i k n / N}, \quad k = 0, 1, \dots, N-1$$

**Inverse DFT:**

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{2\pi i k n / N}, \quad n = 0, 1, \dots, N-1$$

**Fast Fourier Transform (FFT)**

The FFT is an efficient algorithm to compute the DFT with complexity  $O(N \log N)$  instead of  $O(N^2)$ .

Most efficient when  $N = 2^m$  (power of 2).

**Useful Integrals**

$$\int_0^{\pi} \sin(nx) \sin(mx) dx = \begin{cases} 0 & n \neq m \\ \pi/2 & n = m \end{cases}$$

$$\int_0^\pi \cos(nx) \cos(mx) dx = \begin{cases} 0 & n \neq m \\ \pi/2 & n = m \end{cases}$$

$$\int_0^\pi \sin(nx) \cos(mx) dx = 0 \text{ for all } n, m$$

$$\int_{-\pi}^\pi \sin(nx) \sin(mx) dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

$$\int_{-\pi}^\pi \cos(nx) \cos(mx) dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

## Special Functions

**Signum Function:**

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

**Unit Step Function:**

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

**Rectangle Function:**

$$\operatorname{rect}(x) = \begin{cases} 1 & |x| < 1/2 \\ 1/2 & |x| = 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

**Sinc Function:**

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x}$$

Note:  $\lim_{x \rightarrow 0} \operatorname{sinc}(x) = 1$

## Useful Trigonometric Identities

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\sin(x) \cos(x) = \frac{\sin(2x)}{2}$$

$$\sin(A) \sin(B) = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

$$\cos(A) \cos(B) = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$$

$$\sin(A) \cos(B) = \frac{1}{2} [\sin(A + B) + \sin(A - B)]$$

**Euler's Formula:**

$$e^{ix} = \cos(x) + i \sin(x)$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$