# Laplace Transforms

## Definition

The Laplace Transform and its inverse: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$
$\mathcal{L}^{-1}\{F(s)\} = f(t)$

## Common Laplace Transforms

| **Function** $f(t)$ | **Transform** $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $t$ | $\frac{1}{s^2}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $te^{at}$ | $\frac{1}{(s-a)^2}$ |
| $t^n e^{at}$ | $\frac{n!}{(s-a)^{n+1}}$ |
| $\sin(\omega t)$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cos(\omega t)$ | $\frac{s}{s^2+\omega^2}$ |
| $e^{at}\sin(\omega t)$ | $\frac{\omega}{(s-a)^2+\omega^2}$ |
| $e^{at}\cos(\omega t)$ | $\frac{s-a}{(s-a)^2+\omega^2}$ |
| $\sinh(\omega t)$ | $\frac{\omega}{s^2-\omega^2}$ |
| $\cosh(\omega t)$ | $\frac{s}{s^2-\omega^2}$ |
| $u(t-a)$ | $\frac{e^{-as}}{s}$ |
| $\delta(t)$ | $1$ |
| $\delta(t-a)$ | $e^{-as}$ |

## Properties of Laplace Transforms

**Linearity:** $\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$

**First Derivative:** $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

**Second Derivative:** $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

**$n$-th Derivative:** $\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$

**Integration:** $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$

**Time Shift (Second Shifting Theorem):** $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s), \quad a \geq 0$ $\mathcal{L}\{g(t)u(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$

**Frequency Shift (s-Shift Theorem):** $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

**Scaling:** $\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0$

**Multiplication by $t$:** $\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\{t^nf(t)\} = (-1)^nF^{(n)}(s)$

**Division by $t$:** $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$

## Convolution Theorem

**Convolution of two functions:** $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t f(t-\tau)g(\tau)\,d\tau$

**Convolution Theorem:** $\mathcal{L}\{(f * g)(t)\} = F(s) \cdot G(s)$

**Inverse form:** $\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = (f * g)(t)$

## Initial and Final Value Theorems

**Initial Value Theorem:** $\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$ (provided the limit exists)

**Final Value Theorem:** $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$ (provided the limit exists and $f(t)$ has a final value; all poles of $sF(s)$ must have negative real parts except possibly a simple pole at $s = 0$)

## Inverse Laplace Transforms

**Partial Fraction Decomposition:**

For a rational function $\dfrac{P(s)}{Q(s)}$ where $\deg(P) < \deg(Q)$:

**Case 1: Simple Real Roots**

If $Q(s) = (s-a_1)(s-a_2)\cdots(s-a_n)$ with distinct roots: $\frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_n}{s-a_n}$

**Case 2: Repeated Real Roots**

If $Q(s)$ has $(s-a)^n$ as a factor: Include terms: $\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}$

**Case 3: Complex Conjugate Roots**

If $Q(s)$ has a factor $(s-\alpha)^2 + \beta^2$: Include term: $\frac{As+B}{(s-\alpha)^2+\beta^2}$

**Heaviside Cover-Up Method:**

For a simple root at $s = a$, the coefficient $A$ is: $A = \lim_{s \to a}(s-a)\frac{P(s)}{Q(s)}$

Or equivalently, substitute $s = a$ into $\dfrac{P(s)}{Q(s)}$ after "covering up" the factor $(s-a)$ in $Q(s)$.

## Solving ODEs with Laplace Transforms

**General Procedure:**

1. Take the Laplace transform of both sides of the ODE
2. Apply the derivative properties using the initial conditions
3. Solve the resulting algebraic equation for $Y(s)$
4. Use partial fractions to decompose $Y(s)$ if necessary
5. Take the inverse Laplace transform to obtain $y(t)$

**Example for second-order ODE:**

Given: $y'' + ay' + by = f(t)$ with initial conditions $y(0)$ and $y'(0)$

Taking Laplace transforms: $s^2Y(s) - sy(0) - y'(0) + a[sY(s) - y(0)] + bY(s) = F(s)$

Solving for $Y(s)$: $Y(s) = \frac{F(s)+sy(0)+y'(0)+ay(0)}{s^2+as+b}$

## Special Functions

**Unit Step Function (Heaviside Function):** $u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$

Used to "turn on" functions at time $t = a$

**Dirac Delta Function (Unit Impulse):** $\delta(t-a) = 0$ for all $t \neq a$ $\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$

**Sifting Property:** $\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a)$

In ODEs, $\delta(t-a)$ represents an instantaneous impulse at $t = a$

## Solving Integral Equations

For equations of the form: $y(t) = f(t) + \int_0^t K(t-\tau)y(\tau)\,d\tau$

**Solution Method:**

1. Take Laplace transform of both sides
2. Use convolution theorem: $\mathcal{L}\{\text{integral term}\} = K(s)Y(s)$
3. Obtain: $Y(s) = F(s) + K(s)Y(s)$
4. Solve algebraically: $Y(s) = \dfrac{F(s)}{1-K(s)}$
5. Take inverse Laplace to find $y(t)$

## Useful Trigonometric Identities

$\sin^2(\omega t) = \frac{1-\cos(2\omega t)}{2}$

$\cos^2(\omega t) = \frac{1+\cos(2\omega t)}{2}$

$\sin(\omega t)\cos(\omega t) = \frac{\sin(2\omega t)}{2}$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

## Common Partial Fraction Results

$\frac{1}{s(s+a)} = \frac{1}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)$

$\frac{1}{(s+a)(s+b)} = \frac{1}{b-a}\left(\frac{1}{s+a} - \frac{1}{s+b}\right), \quad a \neq b$

$\frac{s}{s^2+\omega^2} \quad \xrightarrow{\mathcal{L}^{-1}} \quad \cos(\omega t)$

$\frac{\omega}{s^2+\omega^2} \quad \xrightarrow{\mathcal{L}^{-1}} \quad \sin(\omega t)$

$\frac{1}{(s+a)^2} \quad \xrightarrow{\mathcal{L}^{-1}} \quad te^{-at}$

$\frac{1}{s^2(s+a)} = \frac{1}{a^2}\left(\frac{1}{s} - \frac{1}{s+a}\right) - \frac{1}{a}\left(\frac{1}{s^2}\right)$