3.2 Kraft and McMillan Inequalities

length is simply the number of characters in a string given an alphabet ($\mathcal{A}$). given a set $\mathcal{S}$ with the encoding: $f : \mathcal{S} \to \mathcal{A}^*$. The set of word lengths then denoted as:

$$\{length(f(\mathcal{S})) : s \in \mathcal{S}\}$$

Theorem 1 (Kraft's Inequality)**.** *Let the set $\mathcal{S}$ of source words have m elements. Then, let the encoding alphabet $\mathcal{A}$ have n characters. A condition that there exists an* ***instantaneous uniquely decipherable*** *code $f : \mathcal{S} \to \mathcal{A}^*$ with lengths $\ell_1, \ell_2, \ldots, \ell_3$ is:*

$$\sum_{i=1}^{m} \frac{1}{n^{\ell_i}} \leq 1 \tag{1}$$

Theorem 2 (Mcmillan's Inequality)**.** *Let the set $\mathcal{S}$ of source words have m elements. Then, let the encoding alphabet $\mathcal{A}$ have n characters. A condition that there exists a* ***uniquely decipherable*** *code $f : \mathcal{S} \to \mathcal{A}^*$ with lengths $\ell_1, \ell_2, \ldots, \ell_3$ is:*

$$\sum_{i=1}^{m} \frac{1}{n^{\ell_i}} \leq 1 \tag{2}$$

Corollary. *If there is a uniquely decipherable code with given word lengths, then there is an* instantaneous *(uniquely decipherable) code with those word lengths.*

Remark. *The Inequalities give absolute limits on the size of the encoding words necessary to encode a 'vocabulary' $\mathcal{S}$ of source words with a certain size. These are independent of any probablistic considerations.*

There is a proof involved in this, but I have omitted it because I don't want to deal with it. That being said, since *any* uniquely decipherable code must satisfy the Inequality, an *instantaneous* one should as well.

3.3 Noiseless Coding Theorem