Point process likelihood

$$P(N_k \mid x_k, \theta) = (\lambda_k \cdot \Delta)^{N_k} \exp(-\lambda_k \cdot \Delta)$$

$$P(x_k \mid x_{k-1}, \theta) \sim N(\phi x_{k-1} + \alpha I_k, \sigma^2)$$

$$P(N_{1:k}, x_{0:k} \mid \theta) = P(N_{1:k} \mid x_{1:k}, \theta) \cdot P(x_{1:k} \mid \theta)$$

$$= \prod_{k=1}^{K} P(N_k \mid x_k, \theta) \; P(x_0) \prod_{k=1}^{K} P(x_k \mid x_{k-1}, \theta)$$

$$\log(P(N_{1:k}, x_{0:k} \mid \theta)) =$$

$$\mathcal{L}(N_{1:k}, x_{0:k} \mid \theta) =$$

$$-\frac{K+1}{2}\log 2\pi - (K+1)\log\sigma^2 - \sum_{k=1}^{K} \frac{(x_k - \phi x_{k-1} - \alpha I_k)^2}{2\sigma^2}$$

$$+ \frac{1}{2}\log(1-\phi^2) - \frac{x_0^2(1-\phi^2)}{2\sigma^2}$$

$$+ \sum_{k=1}^{K} \left[N_k(\mu + \beta x_k + \log\Delta) - \exp(\mu + \beta x_k) \cdot \Delta \right]$$

Posteriors of $x_{0:k}$ & θ.

$$P(x_{0:k} \mid N_{1:k}, \theta) \propto P(N_{1:k} \mid x_{0:k}, \theta) P(x_{0:k} \mid \theta)$$

Joint likelihood

$$P(\theta \mid x_{0:K}, N_{1:k}) \propto P(N_{1:k}, x_{0:k} \mid \theta) \cdot P(\theta)$$

Tensor G for $x_{0:K}$.

$$\frac{\partial L}{\partial x_0} = -\frac{\partial}{\partial x_0}\left(\frac{(x_1 - \phi x_0 - \alpha I_1)^2}{2\sigma^2}\right) - \frac{x_0(1-\phi^2)}{\sigma^2}$$

$$= \frac{\phi(x_1 - \phi x_0 - \alpha I_1)}{\sigma^2} - \frac{x_0(1-\phi^2)}{\sigma^2} = \frac{\phi x_1 - \phi \alpha I_1 - x_0}{\sigma^2}$$

$$\partial L / \partial x_k = -\frac{\partial}{\partial x_k}\left(\frac{(x_k - \phi x_{k-1} - \alpha I_k)^2}{2\sigma^2}\right)$$ for $k = [1, \ldots, K-1]$

$$- \frac{\partial}{\partial x_k}\left(\frac{(x_{k+1} - \phi x_k - \alpha I_{k+1})^2}{2\sigma^2}\right)$$

$$+ \beta N_k - \beta \exp(\mu + \beta x_k) \cdot \Delta$$

$$= -\frac{x_k - \phi x_{k-1} - \alpha I_k}{\sigma^2} + \frac{\phi(x_{k+1} - \phi x_k - \alpha I_{k+1})}{\sigma^2}$$

$$+ \beta N_k - \beta \exp(\mu + \beta x_k) \cdot \Delta$$

$$= \frac{1}{\sigma^2}(\phi x_{k+1} - \phi^2 x_k - \phi \alpha I_{k+1} - x_k + \phi x_{k-1} + \alpha I_k)$$

$$+ \beta N_k - \beta \exp(\mu + \beta x_k) \cdot \Delta$$

for $k = K$

$$\frac{\partial L}{\partial x_k} = \frac{-1}{\sigma^2}(x_k - \phi x_{k-1} - \alpha I_k) + \beta N_k - \beta \exp(\mu + \beta x_k) \cdot \Delta$$

Tensor G for $x_{0:k}$

Let $G(x)$ be the tensor as a function of current x, and can be derived from the Fisher Information matrix.

$$\{G(x)\}_{(1,1)} = E\left(-\frac{\partial^2 L}{\partial x_0^2}\right) = \frac{1}{\sigma^2}$$

$\{G(x)\}_{(i,i+1)}$ or $\{G(x)\}_{(i+1,i)}$ for $i = \{1, \cdots, k-1\}$ is

$$E\left(-\frac{\partial^2 L}{\partial x_i \partial x_{i+1}}\right) = E\left(-\frac{\partial^2 L}{\partial x_{i+1} \partial x_i}\right)$$

$$= E\left(-\frac{\partial}{\partial x_{i+1}}\left(\frac{\partial L}{\partial x_i}\right)\right) = E\left(-\left(\frac{\phi}{\sigma^2}\right)\right) = -\frac{\phi}{\sigma^2}$$

$\{G(x)\}_{(i,i)}$ for $i = \{1, \cdots, k-1\}$

$$= E\left(-\partial^2 L / \partial x_i^2\right)$$

$$= E\left(-\left(\frac{-(1+\phi^2)}{\sigma^2} - \beta^2 \exp(\mu + \beta x_k) \cdot \Delta\right)\right)$$

$$= (1+\phi^2)/\sigma^2 + \beta^2 \exp(\mu + \beta x_k) \cdot \Delta$$

$$\{G(x)\}_{(k,k)} = E\left(-\frac{\partial^2 L}{\partial x_k^2}\right) = E\left(-\left(-\frac{1}{\sigma^2} - \beta^2 \exp(\mu + \beta x_k) \cdot \Delta\right)\right)$$

$$= 1/\sigma^2 + \beta^2 \exp(\mu + \beta x_k) \cdot \Delta$$

Tensor for θ

$\Theta = (\mu, \gamma, \alpha)$ subject to transformation $\phi = \tanh(\gamma)$

$\nabla_\Theta L = \left(\frac{\partial l}{\partial \mu}, \frac{\partial l}{\partial \phi}, \frac{\partial l}{\partial \alpha}\right)$ $\qquad \frac{\partial \phi}{\partial \gamma} = 1 - \phi^2$

$$\partial l / \partial \mu = \sum_{k=1}^{K} \left[N_k - \exp(\mu + \beta x_k) \cdot \Delta \right]$$

$$\partial l / \partial \gamma = \frac{\partial l}{\partial \phi} \cdot \frac{\partial \phi}{\partial \gamma} \qquad (1-\phi^2)$$

$$= \frac{\partial}{\partial \phi}\left(-\sum_{k=1}^{K} \frac{(x_k - \phi x_{k-1} - \alpha I_k)^2}{2\sigma^2} + \frac{1}{2}\log(1-\phi^2) - \frac{x_0^2(1-\phi^2)}{2\sigma^2} \right) \cdot$$

$$= \left[\sum_{k=1}^{K} \frac{x_{k-1}(x_k - \phi x_{k-1} - \alpha I_k)}{\sigma^2} - \frac{\phi}{1-\phi^2} + \frac{\phi x_0^2}{\sigma^2} \right] \cdot (1-\phi^2)$$

$$= (1-\phi^2) \sum_{k=1}^{K} x_{k-1}(x_k - \phi x_{k-1} - \alpha I_k) - \phi + \frac{\phi(1-\phi^2) x_0^2}{\sigma^2}$$

$$\frac{\partial l}{\partial \alpha} = \sum_{k=1}^{K} \left[\frac{(x_k - \phi x_{k-1} - \alpha I_k)}{\sigma^2} \cdot I_k \right]$$

$$\frac{\partial^2 l}{\partial \mu \partial \alpha} = \frac{\partial^2 l}{\partial \mu \partial \phi} = 0$$

Let $G(\theta)$ be the tensor after integrating states

$$[G(\theta)]_{(1,1)} = E\left(-\frac{\partial^2 l}{\partial \mu^2}\right) = \sum_{k=0}^{K} \exp\left(\mu + \beta E(x_k) + \frac{\beta^2}{2} Var(x_k)\right) \cdot \Delta$$

$$[G(\theta)]_{(1,2),(1,3),(2,1),(3,1)} = 0$$

$$[G(\theta)]_{(2,2)} = E\left(-\frac{\partial^2 l}{\partial \gamma^2}\right) = -2\phi^2 - K(1-\phi^2) - \frac{1-\phi^2}{\sigma^2} \sum_{k=1}^{K} E(x_{k-1})^2$$

$$[G(\theta)]_{(2,3),(3,2)} = E\left(-\frac{\partial^2 l}{\partial \gamma \partial \alpha}\right) = \frac{1-\phi^2}{\sigma^2} \sum_{k=1}^{K} E(x_{k-1}) I_k$$

$$[G(\theta)]_{(3,3)} = E\left(-\frac{\partial^2 l}{\partial \alpha^2}\right) = -\sum_{k=1}^{K} \frac{I_k^2}{\sigma^2}.$$

Hamiltonian Function of $x_{0:k}$ and its PDEs

$$H(x,p) = U(x) + K(p)$$

$$= -\mathcal{L}(x) + \frac{1}{2}\log\left((2\pi)^D |G(x)|\right) + \frac{1}{2} p^T G(x)^{-1} p$$

(1) $-\log(\text{posterior})$ + (2) normalizing constant + (3) Kinetic energy

(1)

$$\frac{\partial \mathcal{L}}{\partial x_i} = \begin{cases} \dfrac{\phi x_1 - \phi \alpha I_1 - x_0}{\sigma^2} & i=0 \\ \dfrac{1}{\sigma^2}\left(\phi x_{i+1} - \phi^2 x_i - \phi \alpha I_{i+1} - x_i + \phi x_{i-1} + \phi \alpha I_{i-1}\right) \\ \quad + \beta N_k - \beta \exp(\mu + \beta x_k)\cdot\Delta & i = 1, \cdots, k-1 \\ -\dfrac{1}{\sigma^2}\left(x_k - \phi x_{k-1} - \alpha I_k\right) + \beta N_k - \beta \exp(\mu + \beta x_k)\cdot\Delta \end{cases}$$

$\nabla_x \mathcal{L}(x)$

(2) $\dfrac{\partial}{\partial x_i}\left(\frac{1}{2}\log\left((2\pi)^D |G(x)|\right)\right)$

$$= \frac{\partial}{\partial x_i}\left(\frac{1}{2}\log|G(x)|\right) = \frac{1}{2}\operatorname{trace}\left(G(x)^{-1}\frac{\partial G(x)}{\partial x_i}\right)$$

$\dfrac{\partial G(x)}{\partial x_i}$ is a diagonal matrix:

$$\frac{\partial G(x)}{\partial x_0} = O_{(k+1,\,k+1)}$$

$$\frac{\partial G(x)}{\partial x_i} = \beta^3 \exp(\mu + \beta x_i)\Delta \text{ at } (i,i) \text{ of the matrix for } i = 1, \dots, k$$

Hence,

$$\frac{1}{2}\operatorname{trace}\left(G(x)^{-1}\cdot\frac{\partial G(x)}{\partial x_i}\right)$$

$$= \begin{cases} 0 & i=0 \\ \frac{1}{2}\left\{G(x)^{-1}\right\}_{(i,i)}\cdot \beta^3 \exp(\mu + \beta x_i)\cdot\Delta & i = 1, \dots, k \end{cases}$$

Hamiltonian function of $x_{0:k}$ and its PDEs

③: $\frac{\partial}{\partial x_i}\left(\frac{1}{2}p^T G(x)^{-1} p\right) = \frac{1}{2} p^T G(x)^{-1} \frac{\partial G(x)}{\partial x_i} G(x)^{-1} p$

$\therefore \nabla_x H(x,p) = \nabla_x (① + ② + ③)$

$\nabla_p H(x,p) = G(x)^{-1} p$

PDEs of the Hamiltonian system is therefore.

$$\begin{cases} \dot{x} = \nabla_p H(x,p) \\ \dot{p} = -\nabla_x H(x,p) \end{cases}$$

Hamiltonian function of $\Theta = (\mu, \gamma, \alpha)$ and its PDEs

$$\mathcal{H}(\theta, p) = U(\theta) + K(p)$$

Let $p(\Theta \mid x_{0:k}, N_{1:k})$ be the posterior density of Θ

$$p(\Theta \mid x_{0:k}, N_{1:k}) \propto p(x_{0:k}, N_{1:k} \mid \theta)\, p(\theta)$$

where $p(\theta) \sim MVN(0, \Sigma)$ & $\Sigma = 100 \cdot I_{3\times 3}$

or $p(\Theta) = \pi(\mu)\,\pi(\gamma)\,\pi(\alpha)$

for $\pi(\cdot) \sim N(0, 10^2)$

Let $\mathcal{L}(\Theta) = \log(P(\Theta \mid X_{0:k}, N_{1:k}))$

$$\mathcal{H}(\theta, p) = -\mathcal{L}(\theta) + \frac{1}{2}\log((2\pi)^D |G(\theta)|) + \frac{1}{2}p^T G(\theta)^{-1} p$$

$\nabla_\Theta \mathcal{L}(\Theta) =$

$$\left(\sum_{k=1}^{K} [N_k - \exp(\mu + \beta x_k)\cdot\Delta],\ (1-\phi^2)\sum_{k=1}^{K} x_{k-1}(x_k - \phi x_{k-1} - \alpha I_k)\cdot\phi + \frac{\alpha(1-\phi) x_0^2}{\sigma^2},\ \frac{1}{\sigma^2}\sum_{k=1}^{K} [(x_k - \phi x_{k-1} - \alpha I_k)\cdot I_k] \right)$$

$$\left[\frac{\partial G(\theta)}{\partial \mu}\right]_{(1,1)} = \sum_{k=0}^{K} \exp(\mu + \beta E(x_k) + \frac{\beta^2}{2} Var(x_k))\cdot\Delta$$

$(i,j) = 0 \qquad (i,j) \neq (1,1)$

$$\left[\frac{\partial G(\theta)}{\partial \gamma}\right]_{(1,1),(1,2),(1,3),(2,1)(2,3)} = 0$$

$$\left[\frac{\partial G(\theta)}{\partial \gamma}\right]_{(2,2)} = \frac{\partial}{\partial \gamma}\left(2\phi^2 + K(1-\phi^2)\right) + \frac{1-\phi^2}{\sigma^2}\sum_{k=1}^{K} E(X_{k-1})^2$$

$$= (1-\phi^2)\left(4\phi - 2K\phi - \frac{2\phi}{\sigma^2}\sum_{k=1}^{K} E(X_{k-1})^2\right)$$

$$\left[\frac{\partial G(\theta)}{\partial \gamma}\right]_{(2,3),(3,2)} = \frac{\partial}{\partial \gamma}\left(\frac{1-\phi^2}{\sigma^2}\sum_{k=1}^{K} E(X_{k-1}) I_k\right)$$

$$= (1-\phi^2)\left(\frac{-2\phi}{\sigma^2}\sum_{k=1}^{K} E(X_{k-1}) I_k\right)$$

$$\left[\frac{\partial G(\theta)}{\partial \gamma}\right]_{(3,3)} = 0 \qquad \frac{\partial G(\theta)}{\partial \alpha} = O_{3\times 3}$$

Hamiltonian function of θ and its PDEs

$$\frac{\partial H(\theta, p)}{\partial \theta_i} = -\frac{\partial \mathcal{L}(\theta)}{\partial \theta_i} + \frac{1}{2} \mathrm{tr}\left(G(\theta)^{-1} \frac{\partial G(\theta)}{\partial \theta_i} \right) - \frac{1}{2} p^T G(\theta)^{-1} \frac{\partial G(\theta)}{\partial \theta_i} G(\theta)^{-1} p$$

$$\frac{\partial H(\theta, p)}{\partial p_i} = \left[G(\theta)^{-1} p \right]_i$$