Point process likelihood

$$P(N_k \mid x_k, \theta) = (\lambda_k \cdot \Delta)^{N_k} \exp(-\lambda_k \cdot \Delta)$$

$$P(x_k \mid x_{k-1}, \theta) \sim \mathcal{N}(\phi x_{k-1} + \alpha I_k, \sigma^2)$$

$$P(N_{1:k}, x_{0:k} \mid \theta) = P(N_{1:k} \mid x_{1:k}, \theta) \cdot P(x_{1:k} \mid \theta)$$

$$= \prod_{k=1}^{K} P(N_k \mid x_k, \theta)\, P(x_0) \prod_{k=1}^{K} P(x_k \mid x_{k-1}, \theta)$$

$$\log(P(N_{1:k}, x_{0:k} \mid \theta)) =$$

$$\mathcal{L}(N_{1:k}, x_{0:k} \mid \theta) =$$

$$-\frac{K+1}{2}\log 2\pi - (K+1)\log\sigma^2 - \sum_{k=1}^{K} \frac{(x_k - \phi x_{k-1} - \alpha I_k)^2}{2\sigma^2}$$

$$+\frac{1}{2}\log(1-\phi^2) - \frac{x_0^2(1-\phi^2)}{2\sigma^2}$$

$$+ \sum_{k=1}^{K} \left[N_k(\mu + \beta x_k + \log\Delta) - \exp(\mu + \beta x_k) \cdot \Delta \right]$$

Posteriors of $x_{0:k}$ & θ.

$$P(x_{0:k} \mid N_{1:k}, \theta) \propto P(N_{1:k} \mid x_{0:k}, \theta)\, P(x_{0:k} \mid \theta)$$

Joint likelihood

$$P(\theta \mid x_{0:k}, N_{1:k}) \propto P(N_{1:k}, x_{0:k} \mid \theta) \cdot P(\theta)$$

Tensor G for $x_{0:K}$.

$$\frac{\partial \ell}{\partial x_0} = -\frac{\partial}{\partial x_0}\left(\frac{(x_1 - \phi x_0 - \alpha I_1)^2}{2\sigma^2}\right) - \frac{x_0(1-\phi^2)}{\sigma^2}$$

$$= \frac{\partial\left(x_1 - \phi x_0 - \alpha I_1\right)}{\sigma^2} - \frac{x_0(1-\phi^2)}{\sigma^2} = \frac{\phi x_1 - \phi\alpha I_1 - x_0}{\sigma^2}$$

$$\partial \ell / \partial x_k = -\frac{\partial}{\partial x_k}\left(\frac{(x_k - \phi x_{k-1} - \alpha I_k)^2}{2\sigma^2}\right) \quad \text{for } k = [1, \dots, K-1]$$

$$-\frac{\partial}{\partial x_k}\left(\frac{(x_{k+1} - \phi x_k - \alpha I_{k+1})^2}{2\sigma^2}\right)$$

$$+ \beta N_k - \beta \exp(\mu + \beta x_k)\cdot\Delta$$

$$= -\frac{x_k - \phi x_{k-1} - \alpha I_k}{\sigma^2} + \frac{\phi(x_{k+1} - \phi x_k - \alpha I_{k+1})}{\sigma^2}$$

$$+ \beta N_k - \beta \exp(\mu + \beta x_k)\cdot\Delta$$

$$= \frac{1}{\sigma^2}\left(\phi x_{k+1} - \phi^2 x_k - \phi\alpha I_{k+1} - x_k + \phi x_{k-1} + \alpha I_k\right)$$

$$+ \beta N_k - \beta \exp(\mu + \beta x_k)\cdot\Delta$$

for $k = K$

$$\frac{\partial L}{\partial x_k} = \frac{1}{\sigma^2}(x_k - \phi x_{k-1} - \alpha I_k) + \beta N_k - \beta \exp(\mu + \beta x_k)\cdot\Delta$$

Tensor G for $X_{0:k}$

Let $G(x)$ be the tensor as a function of current x, and can be derived from the Fisher Information matrix.

$$\{G(x)\}_{(1,1)} = E\left(-\frac{\partial^2 L}{\partial x_0^2}\right) = \frac{1}{\sigma^2}$$

$\{G(x)\}_{(i,i+1)}$ or $\{G(x)\}_{(i+1,i)}$ for $i = [1, \dots, k-1]$ is

$$E\left(-\frac{\partial^2 L}{\partial x_i \partial x_{i+1}}\right) = E\left(-\frac{\partial^2 L}{\partial x_{i+1} \partial x_i}\right)$$

$$= E\left(-\frac{\partial}{\partial x_{i+1}}\left(\frac{\partial L}{\partial x_i}\right)\right) = E\left(-\left(-\frac{\phi}{\sigma^2}\right)\right) = \phi/\sigma^2$$

$\{G(x)\}_{(i,i)}$ for $i = [1, \dots, k-1]$

$$= E\left(-\partial^2 L / \partial x_i^2\right)$$

$$= E\left(-\left(\frac{-(1+\phi^2)}{\sigma^2} - \beta^2 \exp(\mu + \beta x_k) \cdot \Delta\right)\right)$$

$$= (1+\phi^2)/\sigma^2 + \beta^2 \exp(\mu + \beta x_k) \cdot \Delta$$

$$\{G(x)\}_{(k,k)} = E\left(-\frac{\partial^2 L}{\partial x_k^2}\right) = E\left(-\left(-\frac{1}{\sigma^2} - \beta^2 \exp(\mu + \beta x_k) \cdot \Delta\right)\right)$$

$$= 1/\sigma^2 + \beta^2 \exp(\mu + \beta x_k) \cdot \Delta$$

Tensor for Θ

$\Theta = (\mu, \gamma, \alpha)$ subject to transformation $\phi = \tanh(\gamma)$

$\nabla_\Theta l = \left(\frac{\partial l}{\partial \mu}, \frac{\partial l}{\partial \phi}, \frac{\partial l}{\partial \alpha}\right)$ $\quad \frac{\partial \phi}{\partial \gamma} = 1 - \phi^2$

$$\partial l / \partial \mu = \sum_{k=1}^{K} \left[N_k - \exp(\mu + \beta x_k) \cdot \Delta \right]$$

$$\partial l / \partial \gamma = \frac{\partial l}{\partial \phi} \cdot \frac{\partial \phi}{\partial \gamma} \qquad (1-\phi^2)$$

$$= \frac{\partial}{\partial \phi}\left(-\sum_{k=1}^{K} \frac{(x_k - \phi x_{k-1} - \alpha I_k)^2}{2\sigma^2} + \frac{1}{2}\log(1-\phi^2) - \frac{x_0^2(1-\phi^2)}{2\sigma^2} \right)$$

$$= \left[\sum_{k=1}^{K} \frac{x_{k-1}(x_k - \phi x_{k-1} - \alpha I_k)}{\sigma^2} - \frac{\phi}{1-\phi^2} + \frac{\phi x_0^2}{\sigma^2} \right] \cdot (1-\phi^2)$$

$$= (1-\phi^2) \sum_{k=1}^{K} x_{k-1}(x_k - \phi x_{k-1} - \alpha I_k) - \phi + \frac{\phi(1-\phi^2) x_0^2}{\sigma^2}$$

$$\frac{\partial l}{\partial \alpha} = \sum_{k=1}^{K} \left[\frac{(x_k - \phi x_{k-1} - \alpha I_k)}{\sigma^2} \cdot I_k \right]$$

$$\frac{\partial^2 l}{\partial \mu \partial \alpha} = \frac{\partial^2 l}{\partial \mu \partial \phi} = 0$$

Let $G(\Theta)$ be the tensor after integrating stochastic

$$\left[G(\Theta)\right]_{(1,1)} = E\left(-\frac{\partial^2 l}{\partial \mu^2}\right) = \sum_{k=0}^{K} \exp\left(\mu + \beta E(x_k) + \frac{\beta^2}{2} \mathrm{Var}(x_k)\right) \cdot \Delta$$

$$\left\{G(\Theta)\right\}_{(1,2),(1,3),(2,1),(3,1)} = 0$$

$$\left\{G(\Theta)\right\}_{(2,2)} = E\left(-\frac{\partial^2 l}{\partial \gamma^2}\right) = -2\phi^2 K(1-\phi^2) - \frac{1-\phi^2}{\sigma^2} \sum_{k=1}^{K} E(x_{k-1})^2$$

$$\left\{G(\Theta)\right\}_{(2,3),(3,2)} = E\left(-\frac{\partial^2 l}{\partial \gamma \partial \alpha}\right) = \frac{1-\phi^2}{\sigma^2} \sum_{k=1}^{K} E(x_{k-1}) I_k$$

$$\left[G(\Theta)\right]_{(3,3)} = E\left(-\frac{\partial^2 l}{\partial \alpha^2}\right) = -\sum_{k=1}^{K} \frac{I_k^2}{\sigma^2}.$$