Point process likelihood

$$P(N_k \mid x_k, \theta) = (\lambda_k \cdot \Delta)^{N_k} \exp(-\lambda_k \cdot \Delta)$$

$$P(x_k \mid x_{k-1}, \theta) \sim N(\phi x_{k-1} + \alpha I_k, \sigma^2)$$

$$P(N_{1:k}, x_{0:k} \mid \theta) = P(N_{1:k} \mid x_{1:k}, \theta) \cdot P(x_{1:k} \mid \theta)$$

$$= \prod_{k=1}^{K} P(N_k \mid x_k, \theta) \; P(x_0) \prod_{k=1}^{K} P(x_k \mid x_{k-1}, \theta)$$

$$\log(P(N_{1:k}, x_{0:k} \mid \theta)) =$$

$$\mathcal{L}(N_{1:k}, x_{0:k} \mid \theta) =$$

$$-\frac{K+1}{2}\log 2\pi - (K+1)\log\sigma^2 - \sum_{k=1}^{K} \frac{(x_k - \phi x_{k-1} - \alpha I_k)^2}{2\sigma^2}$$

$$+ \frac{1}{2}\log(1-\phi^2) - \frac{x_0^2(1-\phi^2)}{2\sigma^2}$$

$$+ \sum_{k=1}^{K} \left[N_k(\mu + \beta x_k + \log\Delta) - \exp(\mu + \beta x_k) \cdot \Delta \right]$$

Posteriors of $x_{0:k}$ & θ.

$$P(x_{0:k} \mid N_{1:k}, \theta) \propto P(N_{1:k} \mid x_{0:k}, \theta) P(x_{0:k} \mid \theta)$$

Joint likelihood

$$P(\theta \mid x_{0:K}, N_{1:k}) \propto P(N_{1:k}, x_{0:k} \mid \theta) \cdot P(\theta)$$

Tensor G for $x_{0:K}$.

$$\frac{\partial L}{\partial x_0} = -\frac{\partial}{\partial x_0}\left(\frac{(x_1-\phi x_0-\alpha I_1)^2}{2\sigma^2}\right) - \frac{x_0(1-\phi^2)}{\sigma^2}$$

$$= \frac{\phi(x_1-\phi x_0-\alpha I_1)}{\sigma^2} - \frac{x_0(1-\phi^2)}{\sigma^2} = \frac{\phi x_1-\phi\alpha I_1-x_0}{\sigma^2}$$

$$\partial L/\partial x_k = -\frac{\partial}{\partial x_k}\left(\frac{(x_k-\phi x_{k-1}-\alpha I_k)^2}{2\sigma^2}\right)$$ for $k=[1,\dots,K-1]$

$$-\frac{\partial}{\partial x_k}\left(\frac{(x_{k+1}-\phi x_k-\alpha I_{k+1})^2}{2\sigma^2}\right)$$

$$+\beta N_k-\beta\exp(\mu+\beta x_k)\cdot\Delta$$

$$= -\frac{x_k-\phi x_{k-1}-\alpha I_k}{\sigma^2}+\frac{\phi(x_{k+1}-\phi x_k-\alpha I_{k+1})}{\sigma^2}$$

$$+\beta N_k-\beta\exp(\mu+\beta x_k)\cdot\Delta$$

$$=\frac{1}{\sigma^2}(\phi x_{k+1}-\phi^2 x_k-\phi\alpha I_{k+1}-x_k+\phi x_{k-1}+\alpha I_k)$$

$$+\beta N_k-\beta\exp(\mu+\beta x_k)\cdot\Delta$$

for $k=K$

$$\frac{\partial L}{\partial x_k} = \frac{-1}{\sigma^2}(x_k-\phi x_{k-1}-\alpha I_k)+\beta N_k-\beta\exp(\mu+\beta x_k)\cdot\Delta$$

Tensor G for $X_{0:k}$

Let $G(x)$ be the tensor as a function of current x, and can be derived from the Fisher Information matrix.

$$\{G(x)\}_{(1,1)} = E\left(-\frac{\partial^2 L}{\partial x_0^2}\right) = \frac{1}{\sigma^2}$$

$\{G(x)\}_{(i,i+1)}$ or $\{G(x)\}_{(i+1,i)}$ for $i = \{1, \cdots, k-1\}$ is

$$E\left(-\frac{\partial^2 L}{\partial x_i \partial x_{i+1}}\right) = E\left(-\frac{\partial^2 L}{\partial x_{i+1} \partial x_i}\right)$$

$$= E\left(-\frac{\partial}{\partial x_{i+1}}\left(\frac{\partial L}{\partial x_i}\right)\right) = E\left(-\left(\frac{\phi}{\sigma^2}\right)\right) = -\frac{\phi}{\sigma^2}$$

$\{G(x)\}_{(i,i)}$ for $i = \{1, \cdots, k-1\}$

$$= E\left(-\partial^2 L / \partial x_i^2\right)$$

$$= E\left(-\left(\frac{-(1+\phi^2)}{\sigma^2} - \beta^2 \exp(\mu + \beta x_k) \cdot \Delta\right)\right)$$

$$= (1+\phi^2)/\sigma^2 + \beta^2 \exp(\mu + \beta x_k) \cdot \Delta$$

$$\{G(x)\}_{(k,k)} = E\left(-\frac{\partial^2 L}{\partial x_k^2}\right) = E\left(-\left(-\frac{1}{\sigma^2} - \beta^2 \exp(\mu + \beta x_k) \cdot \Delta\right)\right)$$

$$= 1/\sigma^2 + \beta^2 \exp(\mu + \beta x_k) \cdot \Delta$$

Hamiltonian Function of $x_{0:k}$ and its PDEs

$$H(x,p) = U(x) + K(p)$$

$$= -\overset{①}{\mathcal{L}(x)} + \overset{②}{\frac{1}{2}\log\left((2\pi)^D |G(x)|\right)} + \overset{③}{\frac{1}{2} p^T G(x)^{-1} p}$$

$-\log(\text{posterior}) + \text{normalizing constant} + \text{Kinetic energy}$

①
$$\frac{\partial \mathcal{L}}{\partial x_i} = \begin{cases} \dfrac{\phi x_1 - \phi \alpha I_1 - x_0}{\sigma^2} & i=0 \\ \dfrac{1}{\sigma^2}\left(\phi x_{i+1} - \phi^2 x_i - \phi \alpha I_{i+1} - x_i + \phi x_{i+1} + \phi \alpha I_{i+1}\right) + \beta N_k - \beta \exp(\mu + \beta x_k)\cdot\Delta & i=1,\cdots,k-1 \\ -\dfrac{1}{\sigma^2}\left(x_k - \phi x_{k-1} - \alpha I_k\right) + \beta N_k - \beta \exp(\mu+\beta x_k)\cdot\Delta & \end{cases}$$

$\nabla_x \mathcal{L}(x)$

②
$$\frac{\partial}{\partial x_i}\left(\frac{1}{2}\log\left((2\pi)^D |G(x)|\right)\right)$$

$$= \frac{\partial}{\partial x_i}\left(\frac{1}{2}\log|G(x)|\right) = \frac{1}{2}\operatorname{trace}\left(G(x)^{-1}\frac{\partial G(x)}{\partial x_i}\right)$$

$\dfrac{\partial G(x)}{\partial x_i}$ is a diagonal matrix:

$$\frac{\partial G(x)}{\partial x_0} = 0_{(k+1,k+1)}$$

$$\frac{\partial G(x)}{\partial x_i} = \beta^3 \exp(\mu+\beta x_i)\Delta \text{ at } (i,i) \text{ of the matrix for } i=1,\dots,k$$

Hence,

$$\frac{1}{2}\operatorname{trace}\left(G(x)^{-1}\cdot\frac{\partial G(x)}{\partial x_i}\right)$$

$$= \begin{cases} 0 & i=0 \\ \dfrac{1}{2}\left\{G(x)^{-1}\right\}_{(i,i)}\cdot\beta^3\exp(\mu+\beta x_i)\cdot\Delta & i=1,\dots,k \end{cases}$$

Hamiltonian function of $x_{0:k}$ and its PDEs

③: $\frac{\partial}{\partial x_i}\left(\frac{1}{2}p^T G(x)^{-1} p\right) = \frac{1}{2} p^T G(x)^{-1} \frac{\partial G(x)}{\partial x_i} G(x)^{-1} p$

$\therefore \nabla_x H(x,p) = \nabla_x (① + ② + ③)$

$\nabla_p H(x,p) = G(x)^{-1} p$

PDEs of the Hamiltonian system is therefore

$$\begin{cases} \dot{x} = \nabla_p H(x,p) \\ \dot{p} = -\nabla_x H(x,p) \end{cases}$$

Tensor for Θ

$\Theta = (\mu, \gamma, \alpha)$ subject to transformation $\phi = \tanh(\gamma)$

$\nabla_\Theta L = \left(\frac{\partial l}{\partial \mu}, \frac{\partial L}{\partial \phi}, \frac{\partial l}{\partial \alpha}\right)$ $\qquad \frac{\partial \phi}{\partial \gamma} = 1 - \phi^2$

$$\partial l / \partial \mu = \sum_{k=1}^{K} \left[N_k - \exp(\mu + \beta x_k) \cdot \Delta \right]$$

$$\partial l / \partial \gamma = \frac{\partial l}{\partial \phi} \cdot \frac{\partial \phi}{\partial \gamma} \qquad (1-\phi^2)$$

$$= \frac{\partial}{\partial \phi}\left(- \sum_{k=1}^{K} \frac{(x_k - \phi x_{k-1} - \alpha I_k)^2}{2\sigma^2} + \frac{1}{2}\log(1-\phi^2) - \frac{x_0^2(1-\phi^2)}{2\sigma^2} \right) \cdot$$

$$= \left[\sum_{k=1}^{K} \frac{x_{k-1}(x_k - \phi x_{k-1} - \alpha I_k)}{\sigma^2} - \frac{\phi}{1-\phi^2} + \frac{\phi x_0^2}{\sigma^2} \right] \cdot (1-\phi^2)$$

$$= (1-\phi^2) \sum_{k=1}^{K} x_{k-1}(x_k - \phi x_{k-1} - \alpha I_k) - \phi + \frac{\phi(1-\phi^2) x_0^2}{\sigma^2}$$

$$\frac{\partial l}{\partial \alpha} = \sum_{k=1}^{K} \left[\frac{(x_k - \phi x_{k-1} - \alpha I_k)}{\sigma^2} \cdot I_k \right]$$

$$\frac{\partial^2 l}{\partial \mu \partial \alpha} = \frac{\partial^2 l}{\partial \mu \partial \gamma} = 0$$

Let $G(\Theta)$ be the tensor after integrating states

$$[G(\Theta)]_{(1,1)} = E\left(-\frac{\partial^2 l}{\partial \mu^2}\right) = \sum_{k=0}^{K} \exp\left(\mu + \beta E(x_k) + \frac{\beta^2}{2} Var(x_k)\right) \cdot \Delta$$

$$[G(\Theta)]_{(1,2),(1,3),(2,1),(3,1)} = 0$$

$$[G(\Theta)]_{(2,2)} = E\left(-\frac{\partial^2 l}{\partial \gamma^2}\right) = 2\phi^2 + K(1-\phi^2) + \frac{1-\phi^2}{\sigma^2} \sum_{k=1}^{K} E(x_{k-1})^2$$

$$[G(\Theta)]_{(2,3),(3,2)} = E\left(-\frac{\partial^2 l}{\partial \gamma \partial \alpha}\right) = \frac{1-\phi^2}{\sigma^2} \sum_{k=1}^{K} E(x_{k-1}) I_k$$

$$[G(\Theta)]_{(3,3)} = E\left(-\frac{\partial^2 l}{\partial \alpha^2}\right) = \sum_{k=1}^{K} \frac{I_k^2}{\sigma^2}.$$

Hamiltonian function of $\theta = (\mu, \gamma, \alpha)$ and its PDEs

$$H(\theta, p) = U(\theta) + K(p)$$

Let $p(\theta \mid x_{0:k}, N_{1:k})$ be the posterior density of θ

$$p(\theta \mid x_{0:k}, N_{1:k}) \propto p(x_{0:k}, N_{1:k} \mid \theta)\, p(\theta)$$

where $p(\theta) \sim MVN(0, \Sigma)$ & $\Sigma = 100 \cdot I_{3\times 3}$

or $p(\theta) = \pi(\mu)\,\pi(\phi)\,\pi(\alpha)$

for $\pi(\cdot) \sim N(0, 10^2)$

Let $\mathcal{L}(\theta) = \log(P(\theta \mid X_{0:k}, N_{1:k}))$

$$H(\theta, p) = -\mathcal{L}(\theta) + \frac{1}{2}\log((2\pi)^D \lvert G(\theta) \rvert) + \frac{1}{2} p^T G(\theta)^{-1} p$$

$$\frac{\partial L}{\partial \mu} = \sum_{k=1}^{K} [N_k - \exp(\mu + \beta x_k) \cdot \Delta]$$

1 $$\frac{\partial \mathcal{L}}{\partial \gamma} = \frac{\partial L}{\partial \phi} \cdot \frac{d\phi}{d\gamma} = (1-\phi^2) \sum_{k=1}^{K} x_{k-1}(x_k - \phi x_{k-1} - \alpha I_k) - \phi + \frac{\phi(1-\phi^2)\, x_0^2}{\sigma^2}$$

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \sum_{k=1}^{K} \left[\frac{x_k - \phi x_{k-1} - \alpha I_k}{\sigma^2} \cdot I_k \right]$$

$$\left(\frac{\partial G(\theta)}{\partial \mu}\right)_{(1,1)} = \sum_{k=0}^{K} \exp\left[\mu + \beta E(x_k) + \frac{\beta^2}{2} Var(x_k)\right] \cdot \Delta$$

$$(i \neq 1, j \neq 1) = 0$$

$\frac{\partial G(\theta)}{\partial \gamma}$:

$$E(x_k) = \alpha(I_k + \phi I_{k-1} + \cdots + \phi^{k-1} I_1)$$

$$\frac{\partial E(x_k)}{\partial \gamma} = \begin{cases} 0 & k = 0, 1 \\ (1-\phi^2)\alpha \cdot I_1 & k = 2 \\ (1-\phi^2)\alpha \cdot [I_2 + 2\phi I_1] & k = 3 \\ \vdots & \\ (1-\phi^2)\alpha \cdot [I_{k-1} + 2\phi I_{k-2} + \cdots + (k-1)\phi^{k-2} \cdot I_1] & k = K \end{cases}$$

$$Var(X_k) = \sigma^2 / 1-\phi^2$$

$$\partial Var / \partial\gamma = (1-\phi^2)\frac{d}{d\phi}\left(\sigma^2 / 1-\phi^2\right) = -2\phi\sigma^2 / (1-\phi^2)$$

$$\left\{\frac{\partial G(\theta)}{\partial\gamma}\right\}_{(1,1)} = \frac{\partial}{\partial\gamma}\left(\sum_{k=0}^{K} \exp\left[\mu+\beta E(X_k)+\frac{\beta^2}{2}Var(X_k)\right]\cdot\Delta\right)$$

$$= \sum_{k=0}^{K} \exp\left[\mu+\beta E(X_k)+\frac{\beta^2}{2}Var(X_k)\right]\cdot\Delta\cdot\left(\beta\frac{\partial E(X_k)}{\partial\gamma}+\frac{\beta^2}{2}\frac{\partial Var(X_k)}{\partial\gamma}\right)$$

$$\left\{\frac{\partial G(\theta)}{\partial\gamma}\right\}_{(1,2),(1,3),(2,1),(3,1)} = 0$$

$$\left\{G(\theta)\right\}_{(2,2)} = 2\phi^2+K(1-\phi^2)+\frac{(1-\phi^2)}{\sigma^2}\sum_{k=1}^{K} E(X_{k-1})^2$$

$$\left\{\frac{\partial G(\theta)}{\partial\gamma}\right\}_{(2,2)} = 4\phi-2K\phi-2\phi/\sigma^2\sum_{k=1}^{K} E(X_{k-1})^2+\frac{2(1-\phi^2)}{\sigma^2}\sum_{k=1}^{K} E(X_{k-1})\frac{\partial E(X_{k-1})}{\partial\gamma}$$

$$\left\{G(\theta)\right\}_{(2,3)} = \left\{G(\theta)\right\}_{(3,2)} = \frac{1-\phi^2}{\sigma^2}\sum_{k=1}^{K} E(X_{k-1})\cdot I_k$$

$$\left\{\frac{\partial G(\theta)}{\partial\gamma}\right\}_{(2,3),(3,2)} = \frac{-2\phi}{\sigma^2}\sum_{k=1}^{K} E(X_{k-1})I_k+\frac{1-\phi^2}{\sigma^2}\sum_{k=1}^{K}\frac{\partial E(X_{k-1})}{\partial\gamma}\cdot I_k$$

$$\left\{G(\theta)\right\}_{(3,3)} = \sum_{k=1}^{K} I_k^2/\sigma$$

$$\left\{\frac{\partial G(\theta)}{\partial\gamma}\right\}_{(3,3)} = 0$$

$\frac{\partial G(\theta)}{\partial \alpha}$:

$$\begin{cases} E(X_k) = \alpha(I_k + \phi I_{k-1} + \cdots + \phi^{k-1} I_1) \\ Var(X_k) = \sigma^2 / 1-\phi^2 \end{cases}$$

$$\begin{cases} \frac{\partial E(X_k)}{\partial \alpha} = I_k + \phi I_{k-1} + \cdots + \phi^{k-1} I_1 \\ \frac{\partial Var(X_k)}{\partial \alpha} = 0 \end{cases}$$

$$\left\{\frac{\partial G(\theta)}{\partial \alpha}\right\}_{(1,1)} = \frac{\partial}{\partial \alpha}\left\{\sum_{k=0}^{K} \exp\left[\mu + \beta E(X_k) + \frac{\beta^2}{2} Var(X_k)\right] \cdot \Delta\right\}$$

$$= \sum_{k=0}^{K} \exp\left[\mu + \beta E(X_k) + \beta^2/2\, Var(X_k)\right] \cdot \Delta \cdot \left(\beta \frac{\partial E(X_k)}{\partial \alpha} + \frac{\beta^2}{2} \frac{\partial Var(X_k)}{\partial \alpha}\right)$$

$$\left\{\frac{\partial G(\theta)}{\partial \alpha}\right\}_{(1,2),(1,3),(2,1),(3,1)} = 0$$

$$\left\{\frac{\partial G(\theta)}{\partial \alpha}\right\}_{(2,2)} = \frac{\partial}{\partial \alpha}\left\{2\phi^2 + K(1-\phi^2) + \frac{1-\phi^2}{\sigma^2} \sum_{k=1}^{K} E(X_{k-1})^2\right\}$$

$$= \frac{2(1-\phi^2)}{\sigma^2} \sum_{k=1}^{K} E(X_{k-1}) \cdot \frac{\partial E(X_{k-1})}{\partial \alpha}$$

$$\left\{\frac{\partial G(\theta)}{\partial \alpha}\right\}_{(2,3),(3,2)} = \frac{\partial}{\partial \alpha}\left\{\frac{1-\phi^2}{\sigma^2} \sum_{k=1}^{K} E(X_{k-1}) I_k\right\}$$

$$= \frac{(1-\phi^2)}{\sigma^2} \sum_{k=1}^{K} \frac{\partial E(X_{k-1})}{\partial \alpha} \cdot I_k$$

$$\left\{\frac{\partial G(\theta)}{\partial \alpha}\right\}_{(3,3)} = 0$$

Hamiltonian function of θ and its PDEs

$$\frac{\partial H(\theta, p)}{\partial \theta_i} = -\frac{\partial \mathcal{L}(\theta)}{\partial \theta_i} + \frac{1}{2}\mathrm{tr}\left(G(\theta)^{-1}\frac{\partial G(\theta)}{\partial \theta_i}\right) - \frac{1}{2}p^T G(\theta)^{-1}\frac{\partial G(\theta)}{\partial \theta_i}G(\theta)^{-1}p$$

$$\frac{\partial H(\theta, p)}{\partial p_i} = \left[G(\theta)^{-1}p\right]_i$$