Problem Set 5

These problems have been selected from the Course Textbook by Cover and Thomas.

8.1 *Differential entropy.* Evaluate the differential entropy $h(X) = -\int_{-\infty}^{\infty} f(x)\ln(f(x))\mathrm{d}x$ for the following:

(a) The exponential density, $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$.

(b) The Laplace density, $f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}$, $\lambda > 0$.

(c) The sum of X_1 and X_2, where X_1 and X_2 are independent normal random variables with mean μ_1 and μ_2 and variance σ_1^2 and σ_2^2, respectively.

8.7 *Differential entropy bound on discrete entropy.* Let X be a discrete random variable on the set $\mathcal{X} = \{a_1, a_2, \ldots\}$ with $\Pr(X = a_i) = p_i$. Show that

$$H(p_1, p_2, \ldots) \leq \frac{1}{2}\log\left(2\pi e\left(\sum_{i=1}^{\infty} p_i i^2 - \left(\sum_{i=1}^{\infty} i p_i\right)^2 + \frac{1}{12}\right)\right).$$

Moreover, for every permutation σ,

$$H(p_1, p_2, \ldots) \leq \frac{1}{2}\log\left(2\pi e\left(\sum_{i=1}^{\infty} p_{\sigma(i)} i^2 - \left(\sum_{i=1}^{\infty} i p_{\sigma(i)}\right)^2 + \frac{1}{12}\right)\right).$$

Hint: Construct a random variable X' such that $\Pr(X' = i) = p_i$. Let U be uniformly distributed over $(0, 1]$ and let $Y = X' + U$, where X' and U are independent. Use the maximum entropy bound on Y to obtain the bounds in the problem. This bound is due to Massey (unpublished) and Willems (unpublished).

8.8 *Channel with uniformly distributed noise.* Consider an additive channel with input $X \in \mathcal{X} = \{0, \pm 1, \pm 2\}$ and output $Y = X + Z$, where Z uniformly distributed over the interval $[-1, 1]$. Thus the input of the channel is a discrete random variable, whereas the output of the channel is continuous. Calculate the capacity $C = \max_{p(x)} I(X;Y)$ of this channel.

8.10 *Shape of the typical set.* Let $X_1, X_2, \ldots$ be i.i.d., each with density $f(x)$, where

$$f(x) = ce^{-x^4}.$$

Let $h(X) = -\int_{-\infty}^{\infty} f(x)\log(f(x))\mathrm{d}x$. Describe the geometric shape (or form) of the typical set $A_\epsilon^{(n)} = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : 2^{-n(h(X)+\epsilon} \leq f_n(x_1, \ldots, x_n) \leq 2^{-n(h(X)-\epsilon)}\right\}$, where $f_n(x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i)$.

9.3 *Output power constraint.* Consider an additive white Gaussian noise channel with an expected *output* power constraint P. Thus, $Y = X + Z$, where $Z \sim \mathcal{N}(0, \sigma^2)$, Z is independent of X and $E(Y^2) \leq P$. Find the channel capacity.

9.4 *Exponential noise channel.* Consider the memoryless channel $X \mapsto Y$, where $Y = X + Z$ and where Z is exponentially distributed with mean μ and independent of X. Assume that $X \geq 0$, and that we have a mean constraint on the signal, i.e., $E(X) \leq \lambda$. Show that the channel capacity is $C = \log(1 + \lambda/\mu)$.

9.5 *Fading channel.* Consider an additive fading channel $X \mapsto Y$, where $Y = XV + Z$, where Z is additive noise, V is multiplicative noise and Z and V are independent of each other and of X. Argue that knowledge of the fading factor V improves capacity by showing that

$$I(X; Y \mid V) \geq I(X; Y).$$

9.6 *Parallel channels and water-filling.* Consider a pair of Gaussian channel:

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \text{ where } \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}\right),$$

subject to the power constraint $E(X_1^2 + X_2^2) \leq 2P$. Assume that $\sigma_1^2 > \sigma_2^2$. At what power does the channel stop behaving like a single channel with a noise variance σ_2^2, and begin behaving like a pair of channels?

9.7 *Multipath Gaussian channel.* Consider a Gaussian noise channel with power constraint P, where the signal takes two different paths and the received noisy signals are added together at the antenna. More precisely, $X \mapsto Y$, where

$$Y = \underbrace{X + Z_1}_{\text{path 1}} + \underbrace{X + Z_2}_{\text{path 2}}.$$

(a) Find the capacity of this channel if Z_1 and Z_2 are jointly normal with covariance matrix

$$K_Z = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}.$$

(b) What is the capacity for $\rho = 0$, $\rho = 1$, $\rho = -1$?

9.9 *Vector Gaussian channel.* Consider the vector Gaussian channel $X \mapsto Y$, where $Y = X + Z$, with $X = (X_1, X_2, X_3)$, $Z = (Z_1, Z_2, Z_3)$, $Y = (Y_1, Y_2, Y_3)$, $E(X_1^2 + X_2^2 + X_3^2) \leq P$, and

$$Z \sim \mathcal{N}\left(0, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}\right).$$

Find the capacity. The answer may be surprising.