PATTERN RECOGNITION AND MACHINE LEARNING

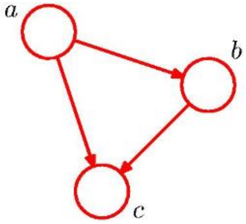
CHAPTER 8: GRAPHICAL MODELS

Graphs

Probabilistic graphical models offer several useful properties:

- They provide a simple way to visualize the structure of a probabilistic model and can be used to design and motivate new models.
- Insights into the properties of the model, including conditional independence properties, can be obtained by inspection of the graph.
- Complex computations, required to perform inference and learning in sophisticated models, can be expressed in terms of graphical manipulations, in which underlying mathematical expressions are carried along implicitly.

A graph comprises nodes (also called vertices) connected by links (also known as edges or arcs).



In a probabilistic graphical model, each node represents a random variable (or group of random variables), and the links express probabilistic relationships between these variables.

Graphs

In Bayesian networks, also known as directed graphical models, the links of the graphs have a particular directionality indicated by arrows.

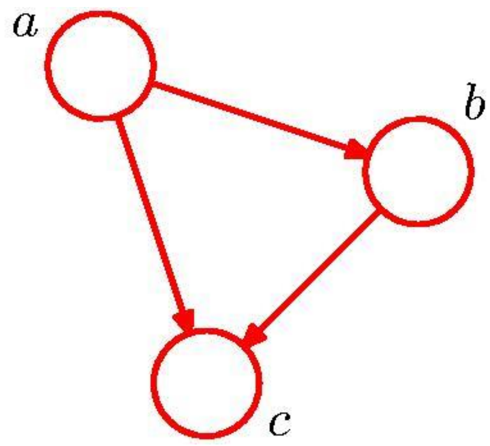
The other major class of graphical models are Markov random fields, also known as undirected graphical models, in which the links do not carry arrows and have no directional significance.

Directed graphs are useful for expressing causal relationships between random variables, whereas undirected graphs are better suited to expressing soft constraints between random variables.

For the purposes of solving inference problems, it is often convenient to convert both directed and undirected graphs into a different representation called a factor graphs.

Bayesian Networks

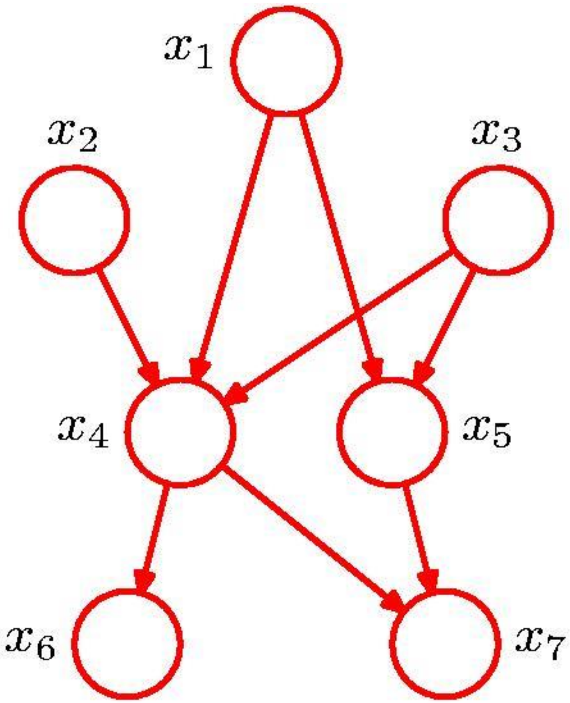
Directed Acyclic Graph (DAG)



$$p(a, b, c) = p(c|a, b)p(a, b) = p(c|a, b)p(b|a)p(a)$$

$$p(x_1, \ldots, x_K) = p(x_K|x_1, \ldots, x_{K-1}) \ldots p(x_2|x_1)p(x_1)$$

Bayesian Networks

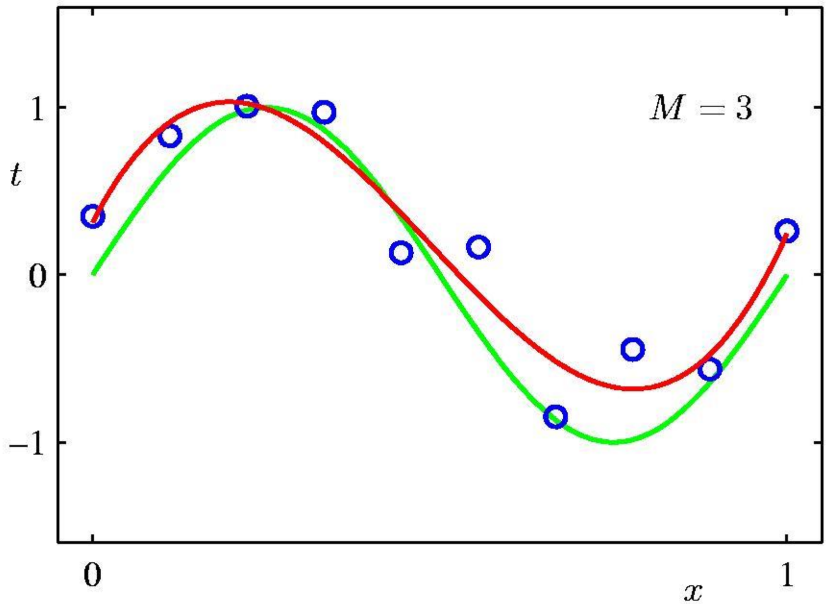


$$
\begin{aligned}
p(x_1, \ldots, x_7) \quad = \quad & p(x_1)p(x_2)p(x_3)p(x_4|x_1, x_2, x_3) \\
& p(x_5|x_1, x_3)p(x_6|x_4)p(x_7|x_4, x_5)
\end{aligned}
$$

General Factorization

$$p(\mathbf{x}) = \prod_{k=1}^{K} p(x_k|\mathrm{pa}_k)$$

Bayesian Curve Fitting (1)



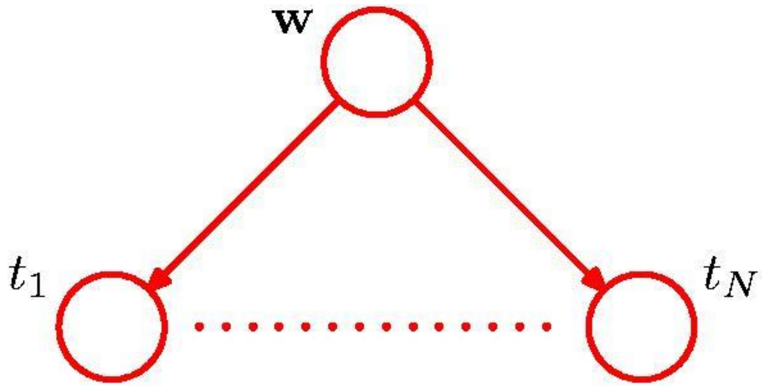
Polynomial

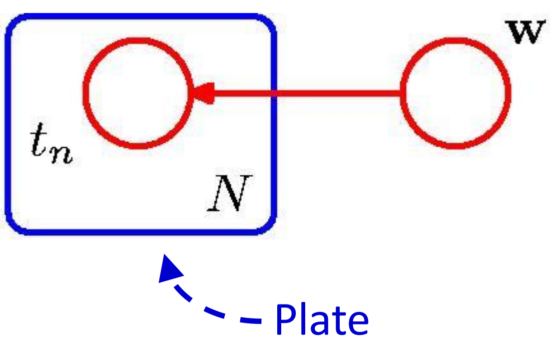
$$y(x, \mathbf{w}) = \sum_{j=0}^{M} w_j x^j$$

$$p(\mathbf{t}, \mathbf{w}) = p(\mathbf{w}) \prod_{n=1}^{N} p(t_n | y(\mathbf{w}, x_n))$$

Bayesian Curve Fitting (2)

$$p(\mathbf{t}, \mathbf{w}) = p(\mathbf{w}) \prod_{n=1}^{N} p(t_n | y(\mathbf{w}, x_n))$$

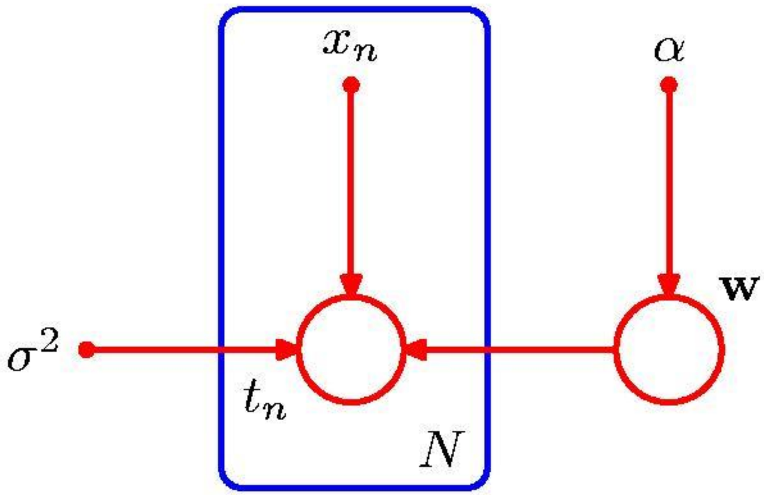




Bayesian Curve Fitting (3)

Input variables and explicit hyperparameters

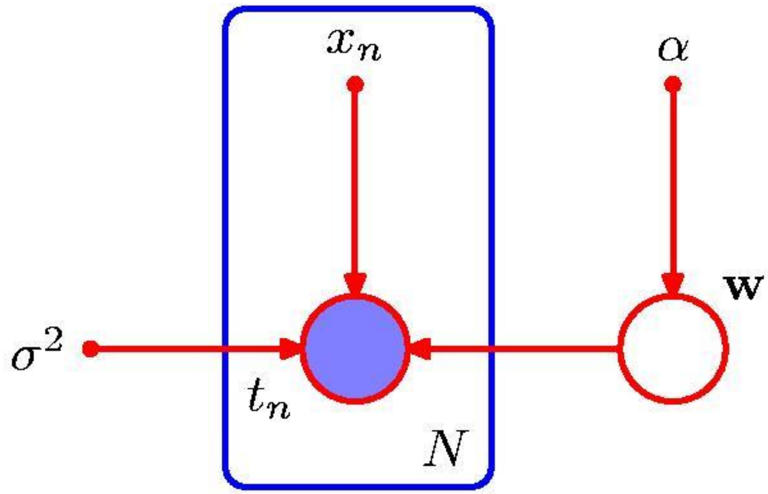
$$p(\mathbf{t}, \mathbf{w}|\mathbf{x}, \alpha, \sigma^2) = p(\mathbf{w}|\alpha) \prod_{n=1}^{N} p(t_n|\mathbf{w}, x_n, \sigma^2).$$



Bayesian Curve Fitting—Learning

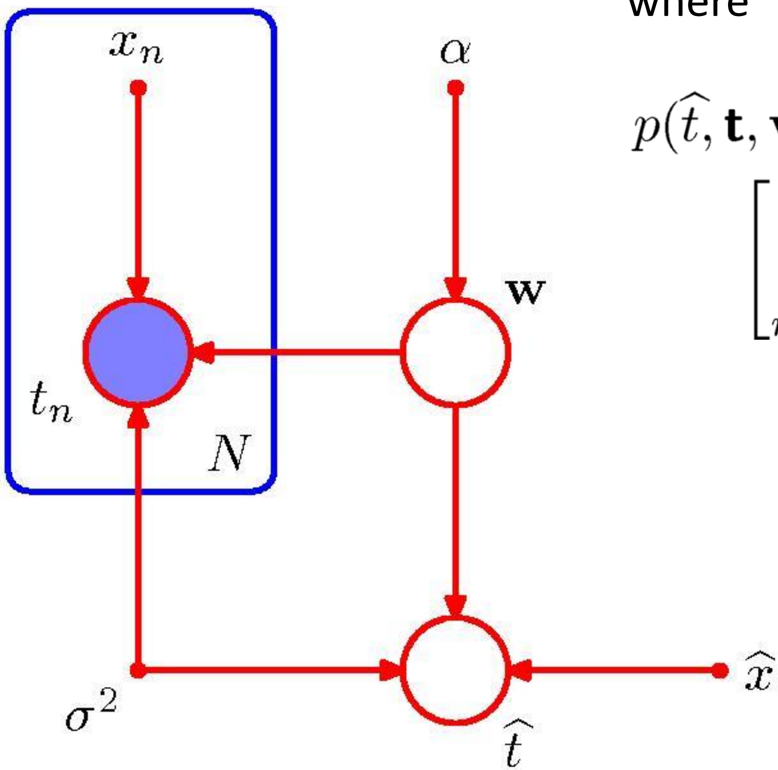
Condition on data

$$p(\mathbf{w}|\mathbf{t}) \propto p(\mathbf{w}) \prod_{n=1}^{N} p(t_n|\mathbf{w})$$



Bayesian Curve Fitting—Prediction

Predictive distribution: $p(\widehat{t}|\widehat{x}, \mathbf{x}, \mathbf{t}, \alpha, \sigma^2) \propto \int p(\widehat{t}, \mathbf{t}, \mathbf{w}|\widehat{x}, \mathbf{x}, \alpha, \sigma^2)\,\mathrm{d}\mathbf{w}$

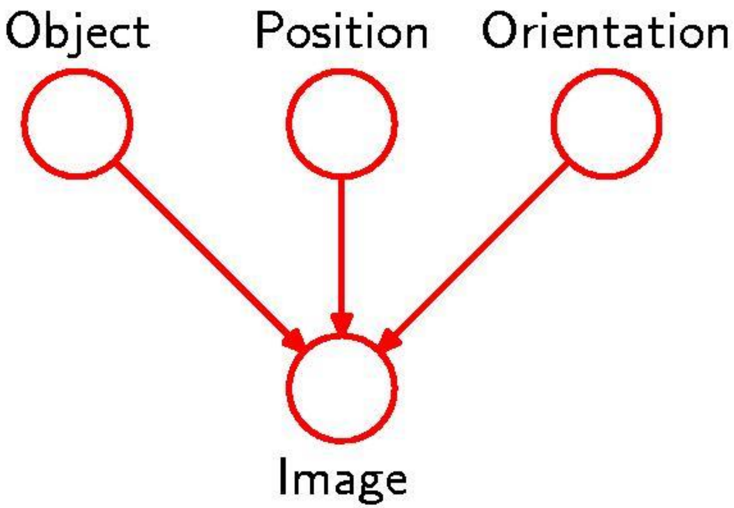


where

$$p(\widehat{t}, \mathbf{t}, \mathbf{w}|\widehat{x}, \mathbf{x}, \alpha, \sigma^2) = \left[\prod_{n=1}^{N} p(t_n|x_n, \mathbf{w}, \sigma^2)\right] p(\mathbf{w}|\alpha) p(\widehat{t}|\widehat{x}, \mathbf{w}, \sigma^2)$$

Generative Models

Causal process for generating images



Discrete Variables (1)

General joint distribution: $K^2 - 1$ parameters

$$p(\mathbf{x}_1, \mathbf{x}_2 | \boldsymbol{\mu}) = \prod_{k=1}^{K} \prod_{l=1}^{K} \mu_{kl}^{x_{1k} x_{2l}}$$

Independent joint distribution: $2(K - 1)$ parameters

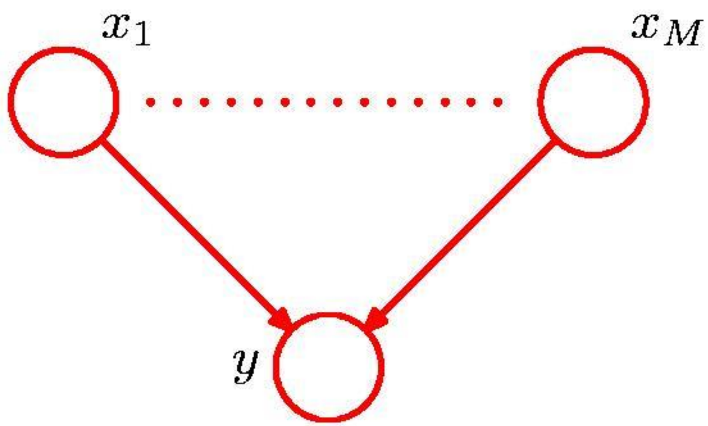
$$\hat{p}(\mathbf{x}_1, \mathbf{x}_2 | \boldsymbol{\mu}) = \prod_{k=1}^{K} \mu_{1k}^{x_{1k}} \prod_{l=1}^{K} \mu_{2l}^{x_{2l}}$$

Discrete Variables (2)

General joint distribution over M variables:
$K^M - 1$ parameters

M-node Markov chain: $K - 1 + (M - 1)K(K - 1)$ parameters

Parameterized Conditional Distributions



If $x_1, \ldots, x_M$ are discrete, K-state variables, $p(y = 1|x_1, \ldots, x_M)$ in general has $O(K^M)$ parameters.

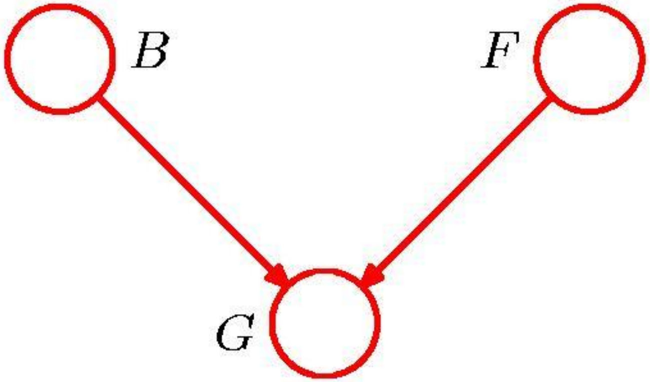
The parameterized form

$$p(y = 1|x_1, \ldots, x_M) = \sigma\left(w_0 + \sum_{i=1}^{M} w_i x_i\right) = \sigma(\mathbf{w}^{\mathrm{T}}\mathbf{x})$$

requires only $M + 1$ parameters

“Am I out of fuel?”

$$\begin{aligned} p(G=1|B=1,F=1) &= 0.8 \\ p(G=1|B=1,F=0) &= 0.2 \\ p(G=1|B=0,F=1) &= 0.2 \\ p(G=1|B=0,F=0) &= 0.1 \end{aligned}$$



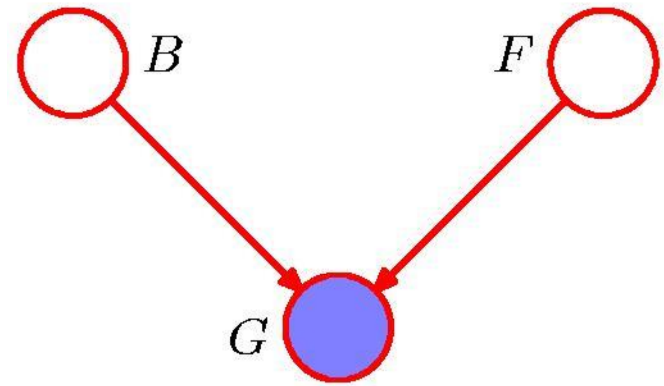
$$\begin{aligned} p(B=1) &= 0.9 \\ p(F=1) &= 0.9 \end{aligned}$$

and hence

$$p(F=0) = 0.1$$

B = Battery (0=flat, 1=fully charged)
F = Fuel Tank (0=empty, 1=full)
G = Fuel Gauge Reading (0=empty, 1=full)

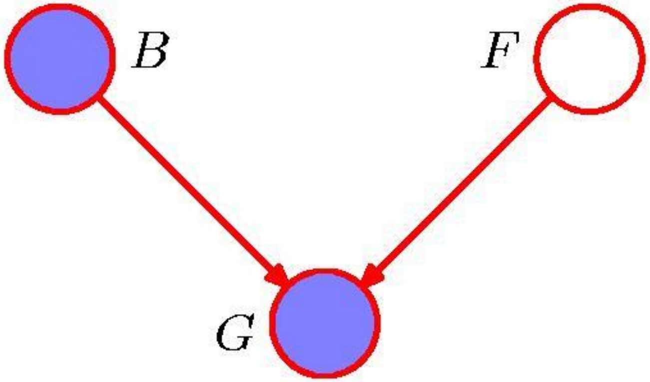
“Am I out of fuel?”



$$
\begin{aligned}
p(F=0|G=0) &= \frac{p(G=0|F=0)p(F=0)}{p(G=0)} \\
&\simeq 0.257
\end{aligned}
$$

Probability of an empty tank increased by observing $G = 0$.

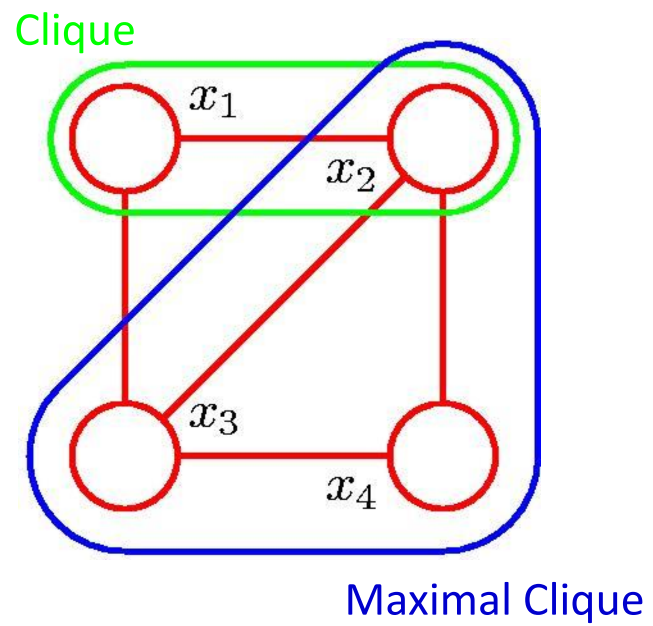
“Am I out of fuel?”



$$
\begin{aligned}
p(F=0|G=0,B=0) &= \frac{p(G=0|B=0,F=0)p(F=0)}{\sum_{F\in\{0,1\}} p(G=0|B=0,F)p(F)} \\
&\simeq 0.111
\end{aligned}
$$

Probability of an empty tank reduced by observing $B = 0$.
This referred to as “explaining away”.

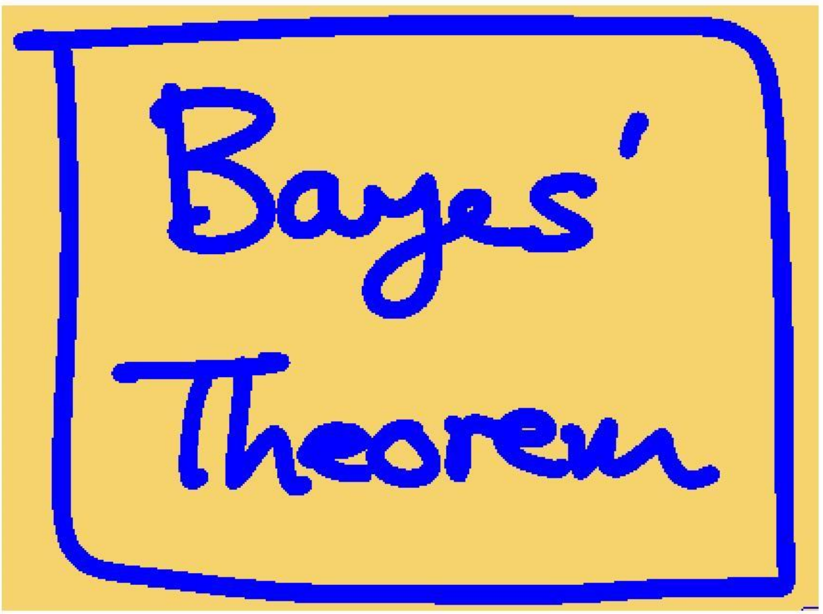
Cliques and Maximal Cliques



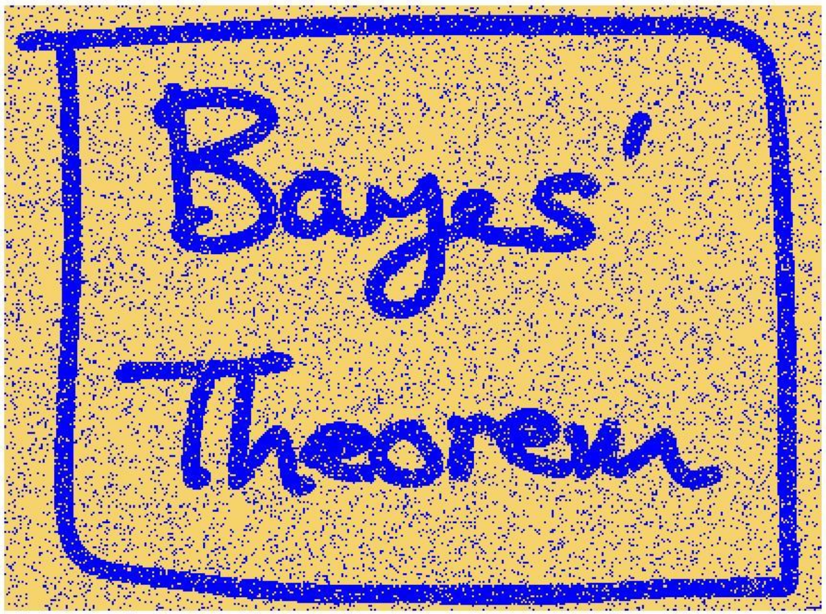
A clique is defined as a subset of the nodes in a graph such that there exists a link between all pairs of nodes in the subset.

A maximal clique is a clique such that it is not possible to include any other nodes from the graph in the set without it ceasing to be a clique

Illustration: Image De-Noising (1)

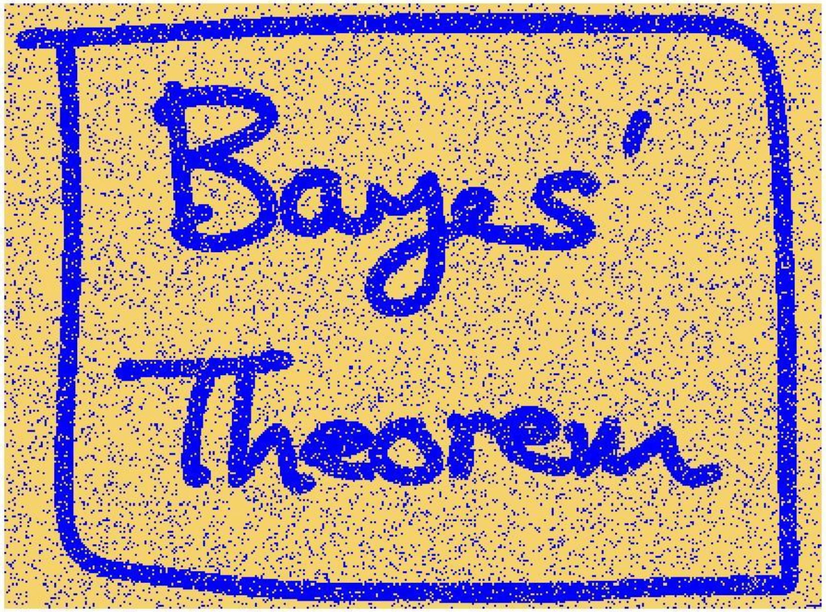


Original Image

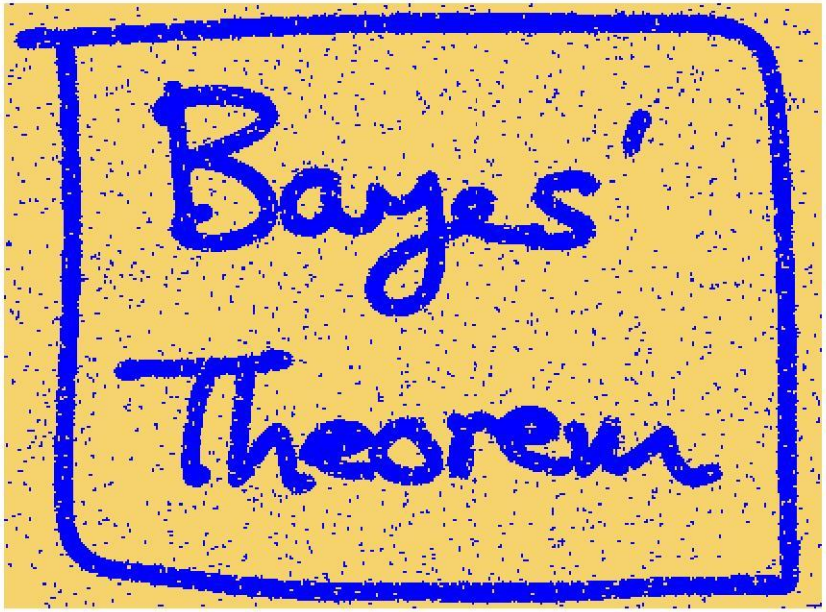


Noisy Image

Illustration: Image De-Noising (2)

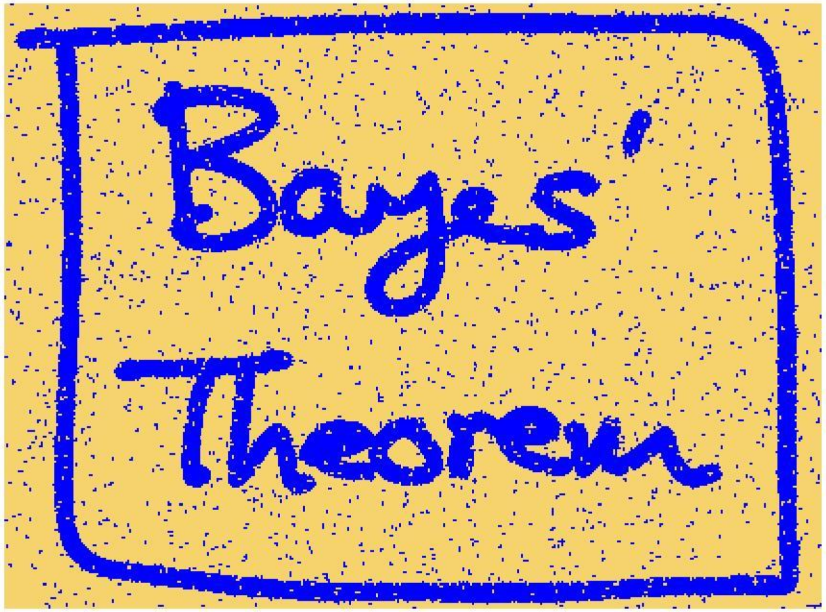


Noisy Image

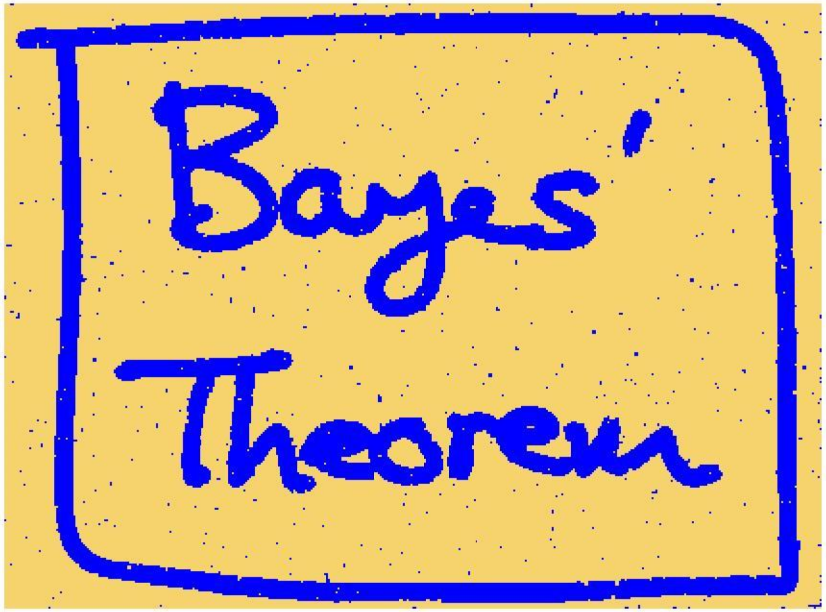


Restored Image (ICM)

Illustration: Image De-Noising (3)



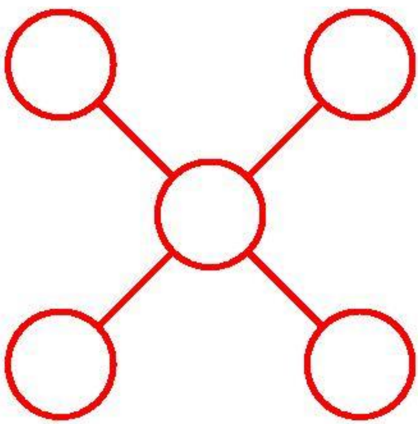
Restored Image (ICM)



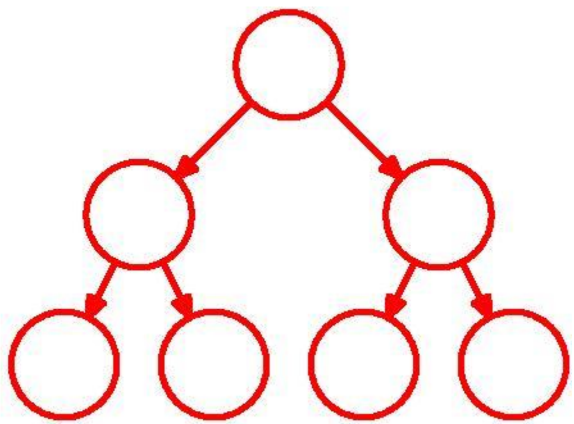
Restored Image (Graph cuts)

Trees

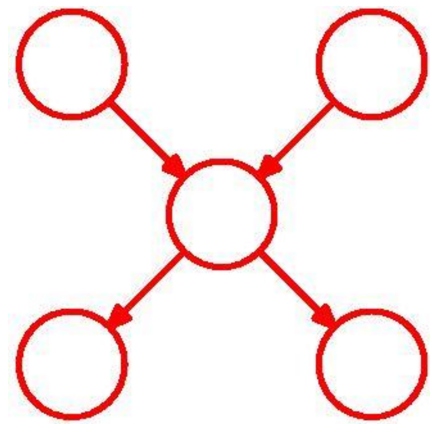
Undirected Tree



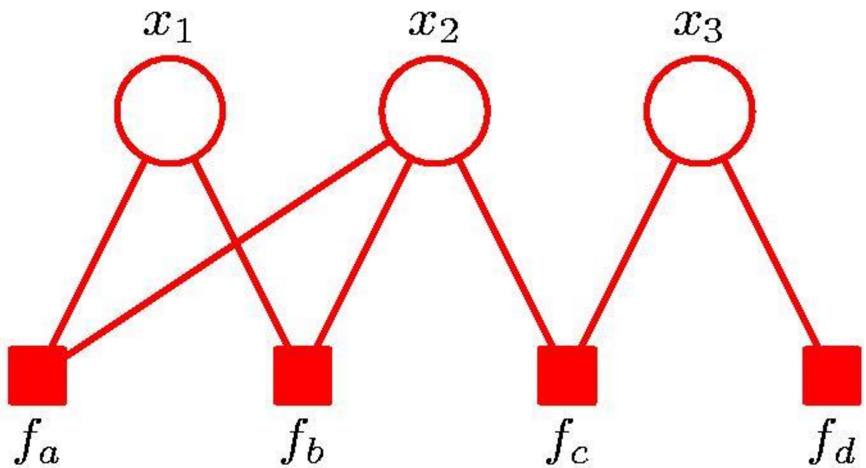
Directed Tree



Polytree



Factor Graphs



$$p(\mathbf{x}) = f_a(x_1, x_2) f_b(x_1, x_2) f_c(x_2, x_3) f_d(x_3)$$

$$p(\mathbf{x}) = \prod_s f_s(\mathbf{x}_s)$$