Linear Algebra by Hefferon
Chapter 2, combining Subspaces, exercise 4.35

(a) $W_1 \cap W_2 = \{\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} | c \in \mathbb{R}\}$, it has as basis $\langle\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\rangle$, so dimension 1.
$W_1 + W_2 = [\{\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}\} \cup \{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\}] = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$,
it has as basis $\langle\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\rangle$, so dimension 3.

(b) Lemma 1: The concatenation of the bases of U and W spans $U + W$.

Proof: $\langle\vec{\mu_1}, ..., \vec{\mu_j}\rangle$ be a basis of U and $\langle\vec{\omega_1}, ..., \vec{\omega_p}\rangle$ be a basis of W. Let $\vec{v} \in [U \cup W]$, $\vec{v} = c1.\vec{u_1} + ... + cm.\vec{u_m} + d1.\vec{w_1} + ... + dn.\vec{w_n}$, so $\vec{v} = c1.(\alpha_1.\vec{\mu_1} + ... + \alpha_j.\vec{\mu_j}) + ... + d1.(\gamma_1.\vec{\omega_1} + ... + \gamma_j.\vec{\omega_j})$, so $\vec{v} = k1.\vec{\mu_1} + ... + kj.\vec{\mu_j} + l1.\vec{\omega_1} + ... + lp.\vec{\omega_p}$. so v is a linear combination of basis vectors from U and W. QED.
(Basically, $\vec{v}$ is a linear combination of linear combinations.)

The concatenation of the expanded sequences is: $\langle\vec{\mu_1}, ..., \vec{\mu_j}, \vec{\beta_1}, ..., \vec{\beta_k}, \vec{\beta_1}, ..., \vec{\beta_k}, \vec{\omega_1}, ..., \vec{\omega_p}\rangle$ spans $U + W$ by Lemma 1. We can easily turn it into a basis: the duplicate $\vec{\beta}$ vectors can be removed since they are linearly dependent on themselves, resulting in $\langle\vec{\mu_1}, ..., \vec{\mu_j}, \vec{\beta_1}, ..., \vec{\beta_k}, \vec{\omega_1}, ..., \vec{\omega_p}\rangle$. The $\vec{\mu}$ and $\vec{\omega}$ vectors are linearly independent from $\vec{\beta}$ vectors. and they are also independent of each other.

Proof: WLOG, assume $\vec{\mu_1}$ is linearly dependent on $\{\vec{\omega_1}, .., \vec{\omega_i}\}$, then $\vec{\mu_1} \in U \cap W$, then $\vec{\mu_1}$ is a linear combination of $\vec{\beta}$ vectors since they are the basis of the intersection, but then the basis of U, $\langle\vec{\mu_1}, ..., \vec{\mu_j}, \vec{\beta_1}, ..., \vec{\beta_k}\rangle$ has linearly dependent vectors which is a contradiction. QED.

(c) Let $\mu = \langle\vec{\mu_1}, ..., \vec{\mu_m}\rangle$ be a basis of U, it has dimension m. $\omega = \langle\vec{\omega_1}, ..., \vec{\omega_m}\rangle$ be a basis of W, it has dimension n. $\beta = \langle\vec{\beta_1}, ..., \vec{\beta_k}\rangle$ be a basis of $U \cap W$, it has dimension k.
Every vector in $U \cap W$ is spanned by both μ and ω, so by the Exchange Lemma, we can replace some $\vec{\mu_x}$ and $\vec{\omega_y}$ by $\vec{\beta_1}$, obtaining two new bases that we call μ^1 and ω^1 signifying that they now include 1 basis vector from β.
By induction, we can replace all k vectors to get μ^k and ω^k, we can be sure that we will keep finding non-trivial linear combinations for a vector $\vec{\beta_i}$ at step i, because both bases still span β and $\vec{\beta_i}$ is linearly indepent from any $\vec{\beta}$ vectors in the bases, so it must be dependent on the $\vec{\mu}$ and $\vec{\omega}$ remaining

vectors.
μ^k contains a count of k, $\vec{\beta}$ vectors and $(j = m - k)$, $\vec{\mu}$ vectors remaining, similary ω^k contains $(p = n - k)$, $\vec{\omega}$ vectors.

With these new bases μ^k and ω^k, the problem is now identical to problem (a). From (a) we know that $\langle \vec{\mu_1}, ..., \vec{\mu_j}, \vec{\beta_1}, ..., \vec{\beta_k}, \vec{\omega_1}, ..., \vec{\omega_p} \rangle$ is a basis. It contains $j+k+p$ vectors, which is equal to $(j+k)+(p+k)-k = m+n-k$, so it's dimension $dim(U+W) = dim(U) + dim(W) - dim(U \cap W)$. QED.