Alfred Tarski found his own solution to the problems of the correspondence theory. Alfred Tarski's theory of truth has two components. First, he defines a true staments as a statement that corresponds to reality. This is only a definition of "true statement" and not of "truth" in general. Of course, if one lists all true statements, one gets a definition of truth: truth is "snow is white" and "my name is Piero" and "the Earth is not the center of the universe" and "France won the 1998 world cup" and "..." But this is neither elegant nor practical (most languages have an infinite number of true statements). The second component to Tarski's theory is the idea that truth can only be defined relative to another language. Most languages include the word "true", but that leads to paradoxes like "I am lying" which is both true and false at the same time. The problem is simply that "true" is a word of the language and we are applying it to a statement of the language. Tarski realized that one can't define truth in a language through the language itself and avoid contradictions. So he defined only "truth in a language". One must use a "meta-language" to define truth in the "object language". Truth in the object language can then be defined recursively from the truth of elementary statements (the "sentential functions"). "For all sentences s in language L, s is true if and only if T(s) is true", where T(s) is a formula containing s and L's primitives. Alfred Tarski's theory of truth does not work well with ordinary languages, although it works wonders with the formal languages of mathematical logic. The problem with Tarski's theory is that it is not clear what he defined. He did not defined truth, but "truth in a language". By this, it is not clear if he indirectly acknowledged that the nature of truth is impossible or even pointless. A secondary problem is that his theory does not distinguish the linguistic theory from the metaphysical theory: explaining the word "true" is a linguistic matter, whereas explaining the nature of truth is a metaphysical matter. Tarski's theory is about the linguistic feature, and does not seem to even address the metaphysical question.

In this connection Church had the following to say concerning the period before 1925 (personal communication), "It is hard to comprehend now, but the implicit position of mathematicians of the day was that 'mathematical' theorems require proof, but logic is to be taken for granted."

Tarski's work on truth[35] represents a watershed in the understanding of what it means to say that a statement is true in formalized languages.[36] As is well-known, Tarski held a semantic conception of truth—based on the concept of satisfiability—aiming at capturing the Aristotelian intuition that:[37]

> The truth of a sentence consists in its agreement with (or correspondence to) reality.

For Tarski a definition of truth that captures such an intuition must *imply* all the equivalences of the kind:[38]

1) 'The cat is on the mat' is true if and only if the cat is on the mat.

What are the difficulties? One annoying problem arises in the semantics. Although the nature of the interpretations themselves is clear, the correct definition of truth is quite problematic. In effect mathematical practice does not seem to indicate unambiguously how the logical constants are to be understood. For example, if P is true and Q is undefined do we take the disjunction to be true or to be undefined? One might think that the meaning of the conditional or if-then connective would be especially problematic but this problem seems to have received a definitive solution (Ebbinghaus, 1969). It may very well be the case that mathematical practice is itself vague on some of these points so that more or less arbitrary connectives will have to be adopted – but one should always be reluctant to accept that kind of a conclusion.

If we use the term intuition to indicate the source of our knowledge of those mathematical truths which we do not logically deduce from others, then the process of axiomatization can be seen as an attempt to factor a mathematical science into its intuitive component and its logical component. The axioms of a science are often intended to represent a concentration or distillation of the intuitive component while the process of deduction is used to reconstitute the whole science from its concentrate. It may be a largely unexamined dogma of traditional mathematics that a science can always be divided into intuitive and logical components and then reconstituted without loss by recombination. Beth has referred to this view as one of Aristotle's most important discoveries.

In the 1930's, this idea was made precise by Gödel [45] and Gentzen [94]. They showed that we can embed classical logic in intuitionistic logic. In fact, they found a map sending any formula X of the propositional calculus to a new formula X°, such that X is provable classically if and only if X° is provable intuitionistically. (More impressively, this map also works for the predicate calculus.)

The most-studied of these alternative systems are *weaker* than classical logic: they make it harder or even impossible to prove things we normally take for granted. One reason is that some logicians deny that certain familiar principles are actually valid. But there are also subtler reasons. One is that studying systems with rules of lesser strength allows for a fine-grained study of precisely which methods of reasoning are needed to prove which results. Another reason — the one that concerns us most here — is that dropping familiar rules and then adding them back in one at at time sheds light on the connection between logic and category theory.

For example, around 1907 Brouwer [49] began advocating 'intuitionism'. As part of this, he raised doubts about the law of excluded middle, which amounts to a rule saying that from $\neg\neg X$ we can deduce X. One problem with this principle is that proofs using it are not 'constructive'. For example, we may prove by contradiction that some equation has a solution, but still have no clue how to construct the solution. For Brouwer, this meant the principle was invalid.

Anyone who feels the law of excluded middle is invalid is duty-bound to study intuitionistic logic. But, there is another reason for studying this system. Namely: we do not really *lose* anything by dropping the law of excluded middle! Instead, we *gain* a fine-grained distinction: the distinction between a direct proof of X and a proof by contradiction, which yields merely $\neg\neg X$. If we do not care about this distinction we are free to ignore it, but there is no harm in having it around.

Each logical system comes with both a syntactic component, which among other things determines the notion of provability, and a semantic component, which determines the notion of logical validity. The logically valid formulas of a system are sometimes called the **theorems** of the system, especially in the context of first-order logic where Gödel's completeness theorem establishes the equivalence of semantic and syntactic consequence. In other settings, such as linear logic, the syntactic consequence (provability) relation may be used to define the theorems of a system.

A logical system is decidable if there is an effective method for determining whether arbitrary formulas are theorems of the logical system. For example, propositional logic is decidable, because the truth-table method can be used to determine whether an arbitrary propositional formula is logically valid.

First-order logic is not decidable in general; in particular, the set of logical validities in any signature that includes equality and at least one other predicate with two or more arguments is not decidable.[1] Logical systems extending first-order logic, such as second-order logic and type theory, are also undecidable.

A mathematician feels that he is saying something contentful when he asserts that aleph-two exists and yet

$$\exists x(x=a)$$

is logically true in standard logic. This points to the fact that standard logic presupposes that every individual constant denotes and it points to the absence of a means of expression of genuine existence. The myth that existential statements assert existence is hard to deflate until one realizes that such statements merely affirm that objects whose existence has been presupposed satisfy certain conditions. To be able to make existence as-

Most mathematicians, even those who do not regard themselves as platonists, take an intermediate view. The fact that logically true sentences are not written as proper axioms of mathematical theories is some evidence of this. Similar evidence derives from the fact that a non-independent set of axioms is regarded as redundant. An axiom deducible from the others is only redundant if one presupposes that what is obtained by logic is somehow obtained free of charge. The usual view that proper mathematical axioms are paid for by intuition (or postulation) while the consequences

Finally note that the following statement in analysis is undefined.

$$\exists x\left(x=\frac{1}{0}\right)$$

However, it follows by standard logic from the following sentence which is logically true in standard logic.

$$\forall y\exists x\left(x=\frac{1}{y}\right)$$

This points to two facts: first, that standard logic presupposes that each function is total and, second, that standard logic makes no pre-supposition.

Use of partial functions in mathematics is fairly widespread. Boole, for example, defined 'union' in such a way that $x+y$ is meaningful only when x and y are disjoint and his relative complementation is defined only when the second argument is a subset of the first (Kneale and Kneale, 1964, p. 410).

Later, yet another reason for being interested in intuitionistic logic became apparent: its connection to category theory. In its very simplest form, this connection works as follows. Suppose we have a set of propositions $X, Y, Z, \ldots$ obeying the laws of the intuitionistic propositional calculus. We can create a category C where these propositions are objects and there is at most one morphism from any object X to any object Y: a single morphism when X implies Y, and none otherwise!

A category with at most one morphism from any object to any other is called a **preorder**. In the propositional calculus, we often treat two propositions as equal when they both imply each other. If we do this, we get a special sort of preorder: one where isomorphic objects are automatically equal. This special sort of preorder is called a **partially ordered set**, or **poset** for short. Posets abound in logic, precisely because they offer a simple framework for understanding implication.

Typical Grammatical Results
Decidability of sentencehood
Unique decomposability
Equivalence of different grammars

Typical Proof-Theoretic Results
Normal Form Theorems
Derivability of Rules
Consistency Theorems
Decidability of Theoremhood
Theory-Completeness
Interpolation Theorems

Typical Semantic Results
Compactness (every unsatisfiable set has a finite unsatisfiable subset)
Skolem-Löwenheim (every interpretation is equivalent to a countable one)
Tarski-Vaught (every infinite interpretation has an equivalent countable submodel)
Functional Completeness of Sentential Logic
Representation Theorems

Typical Bridge Results
Logic-Completeness
Soundness
Logic-Incompleteness
Axiomatizability of Semantically Defined Theories

4.1.4. *Deductive excesses.* It has already been noted that many standard formulations of first-order logic include many axioms and rules which are not found in normal practice. This means that many of the deductions countenanced by standard systems do not correspond to normal deductions. For example, Kleene gives a 17 line deduction of $a=a$, a triviality to which normal practice would grant a single line, or two at most.

Apparently, normal practice admits of only two axiom schemes, the so-called laws of identity and excluded middle, and it is ironic that often both of these are absent from standard deductive systems.

Extended Hilbert Programs

(a) Arithmetical Predicativism.
(b) Theories of higher type functionals.
(c) Takeuti's "Hilbert-Gentzen finitist standpoint".
(d) Feferman's explicit mathematics.
(e) Martin-Löf's intuitionistic type theory.
(f) Constructive set theory (Myhill, Friedman, Beeson, Aczel).

Let $ML = \langle G, D, S\rangle$ be a mathematical logic with G for grammar, D for deductive system and S for semantics. The grammar G is often studied separately from D and S and purely *grammatical theorems* are proved – e.g., that every sentence is uniquely decomposable according to a certain mode of decomposition, that sentencehood in G is decidable and so on. When the deductive system D is studied apart from the semantics one gets purely *proof-theoretic* results – e.g., that every deduction is equivalent to a deduction which does not use a given rule of inference, that not all sentences are provable, and so on. When the semantics S is studied apart from the deductive system one gets purely *model-theoretic* or *semantic* results – e.g., that every infinite interpretation is equivalent to a countable one, that certain sets of sentences are or are not satisfiable, that all interpretations satisfying a given set of sentences are isomorphic and so on. Results which relate the deductive system to the semantics have been called *bridge results* by some of the members of the Berkeley School (but this terminology has not caught on). Examples of bridge results are completeness and soundness theorems. A deductive system can only be com-

From the time of Frege until the mid-twenties there had been no systematic comparison of the standard systems of deduction with deduction as found in practice. In the twenties Jaskowski began work on the question of whether deductive practice could be described with the same degree of mathematical rigor as the standard systems. Fitch (1952) is largely responsible for dissemination of Jaskowski's ideas in this country. Although Jaskowski's work was an important and large step toward codification of normal deductive practice, the full details are yet unknown. To many logicians the remaining questions may seem unimportant but it is possible that further research will unearth results which will have far reaching consequences in applied areas such as linguistics and pedagogy (cf. Corcoran, 1971a).

It should be clear that we are not claiming the existence of implications which can be derived in actual practice but which cannot be derived in standard logic. By the Gödel completeness theorem we know that if A is a set of first-order sentences and t is a first-order consequence of A then t can be formally derived from A in standard logic. The point is that the methods of getting from A to t in standard logic are not the methods actually used in practice. As Kreisel has pointed out more than once, that one can ride to work on a camel or a donkey does not imply that a camel is a donkey.

In Hilbert's approach to proof, there were many axioms and just one rule to deduce new theorems: *modus ponens*, which says that from X and 'X implies Y' we can deduce Y. Most of modern proof theory focuses on another approach, the 'sequent calculus', due to Gentzen [94]. In this approach there are few axioms but many inference rules.

An excellent introduction to the sequent calculus is the book *Proofs and Types* by Girard, Lafont and Taylor, freely available online [44]. Here we shall content ourselves with some sketchy remarks. A 'sequent' is something like this:

$$X_1, \ldots, X_m \vdash Y_1, \ldots, Y_n$$

where X_i and Y_i are propositions. We read this sequent as saying that *all* the propositions X_i, taken together, can be used to prove at least *one* of the propositions Y_i. This strange-sounding convention gives the sequent calculus a nice symmetry, as we shall soon see.

So, why did Gentzen use sequents with a *list* of propositions on each side of the $\vdash$ symbol, instead just a single proposition? The reason is that this let him use only inference rules having the 'subformula property'. This says that every proposition in the sequent above the line appears as part of some proposition in the sequent below the line. So, a proof built from such inference rules becomes a 'tree' where all the propositions further up the tree are subformulas of those below.

This idea has powerful consequences. For example, in 1936 Gentzen was able prove the consistency of Peano's axioms of arithmetic! His proof essentially used induction on trees (Readers familiar with Gödel's second incompleteness theorem should be reassured that this sort of induction cannot itself be carried out in Peano arithmetic.)

of a rigorous (naturally primitive) rule of inference. It is already clear that in order to be taken as a rule of inference (relative to a given semantics) a rule must be sound and effective (Church, 1956). However, first, there are infinitely many rules which meet these two conditions but which would not be regarded as rigorous (see Corcoran, 1969) and, second, we seem to

which have been only inadequately modeled. Many mathematicians, Bourbaki for example, take the idea of an underlying logic very seriously and talk as if there were one logic underlying all branches of mathematics. We interpret such remarks to mean that the similarities among the underlying logics are so great that it is better to regard them as variants of one logic.

It is interesting to note that Bourbaki regarded Aristotle's theory as adequate. Of course, Bourbaki did not actually read the Aristotelian corpus and compare the 'model' therein described with the underlying logic of his own work. Had he done that, he would have seen several gaps between the model and the logic. In the first place there are *grammatical*

In certain cases this process seems to go through without a hitch. For example, we could give formalization of the theory of the ordering of the rationals by choosing the first order language whose only descriptive constant is < (analogue of 'less than') and write down axioms asserting 'no terminal elements', transitivity, strict trichotomy and density.

However, when we try this method on Peano arithmetic we are blocke because we cannot state mathematical induction in a first-order languag The usual way around this is to take all instances of mathematical indu

4.2.2. *Many-sorted logics.* Standard logics can account only for theories which have a single universe of discourse; yet many branches of mathematics are naturally formulated in a manner that presupposes two or more domains. For example, Hilbert's geometry has three domains: points, lines and planes. Many-sorted logics, i.e., logics which admit of multiple domains, have been worked out, but all within the standard framework (non-empty universes, all terms denoting, etc.). However, even Hilbert's geometry presupposed a free many-sorted logic. It is not meaningful to say that a plane is on a point. However, there may be no serious difficulties in constructing a faithful many-sorted free logic once a faithful one-sorted free logic has been given.

the underlying logic or logics of standard mathematical practice. Modern logic differs from traditional logic most conspicuously in scope but most importantly in method. Instead of aiming to characterize the properties of the underlying logic directly as did earlier logicians, modern logic constructs mathematical models as an intermediate step. The construction of mathematical models not only increases clarity and precision but it also relieves two pressures – the pressure to be right in every detail and the pressure to give an account of the ontological status of the subject. Today the value of idealized models is widely accepted and hardly any of the current logicians feel pressure to decide the relation between the logical and the mental, to give an account of propositions, to explicate the ground of logical consequence, etc.

13. Axiomatic Structure

In the next three chapters we will indicate how number, space, and time can be described by axioms; that is, by axioms for the natural numbers, the Euclidean plane, and the real line which describe these structures uniquely. In classical terminology, these axiom systems are *categorical*, in the sense that any two "models" of the axioms, taken within an inclusive set theory, are isomorphic—as in the case described in §4 for the reals (we will also note another "first order" version of these axioms where there can be non-isomorphic non-standard models). Thus these structures are closely attached to the traditional view that (say) the axioms of Euclidean geometry describe one specific object—physical space.

In this first chapter, we have deliberately followed a different order of axiomatics, emphasizing those systems of axioms (linear order, group, metric space) which have many essentially different models. This use of axioms is historically more recent than the categorical axiomatization of geometry. In particular, it allows for the view that the formal systems studied in Mathematics come in a great variety and are intended primarily to help organize and understand selected aspects of the "real world" without being necessarily exact descriptions of a part of that unique world. For example, our presentation allows that the first step in the formalization of space could be the description of figures and chunks of space as models of metric space and not as subsets of Euclidean space. This is by no means the conventional view.

tion as axioms, i.e., to take a scheme which is really an infinite number of axioms. This move has been felt to be unsatisfactory for two reasons. In the first place the resulting infinite set of axioms is still not equivalent to mathematical induction. In the second place, one feels that there is something illegitimate about taking an infinite number of sentences as axioms.

The only condition that standard logic places on a set of axioms is that it be recursive, i.e., that it admit of a mechanical decision procedure. From an epistemological consideration of the notion of an axiomatized science it is clear that decidability is a necessary condition but it is not at all clear that decidability should be regarded as sufficient also. This situation is especially suspicious when one realizes that the infinite axioms sets in current formalizations all resulted from a reluctance to accept second-order logic as logic. The feeling that second-order logic is really set theory in disguise loses some of its force when one reflects that if it *is* set theory then it is far weaker than any theory that has ever been advanced as a theory of sets (second-order logic presupposes at worst only sets of rank one and it does not even presuppose the axiom of choice for those sets).

tween the model and the logic. In the first place there are *grammatical deficiencies* in the Aristotelian language. In the second place there are *deductive deficiencies* in Aristotle's model; *modus ponens*, conditionalization, generalization, reductio and other familiar inference rules are absent from his system. Likewise the semantics for connectives, function symbols, etc. is absent, so there are *semantic deficiencies* as well. In this case the deductive and semantic deficiencies are engendered by the grammatical deficiencies but this is generally not true, i.e., generally deductive and semantic deficiencies are not predicated on grammatical deficiencies. It should be clear that grammatical deficiencies are often not purely grammatical – they are called grammatical because they can be thought of as 'originating' in the grammar but they are usually recognized as a mismatch between the expressive power of the underlying logic and that of the model. For example, Russell's type distinctions were designed to overcome a grammatical deficiency in Frege's model.

$*54{\cdot}43.\ \vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54{\cdot}26 . \supset \vdash :. \alpha = \iota{}^{\backprime}x . \beta = \iota{}^{\backprime}y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$
$[*51{\cdot}231] \qquad \equiv . \iota{}^{\backprime}x \cap \iota{}^{\backprime}y = \Lambda .$
$[*13{\cdot}12] \qquad \equiv . \alpha \cap \beta = \Lambda \quad (1)$
$\vdash . (1) . *11{\cdot}11{\cdot}35 . \supset$
$\vdash :. (\exists x, y) . \alpha = \iota{}^{\backprime}x . \beta = \iota{}^{\backprime}y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \quad (2)$
$\vdash . (2) . *11{\cdot}54 . *52{\cdot}1 . \supset \vdash . \text{Prop}$

From this proposition it will follow, when arithmetical addition has been defined, that $1+1=2$.

nection, several others have written on elusive phenomena. Hasenjaeger (1972) has pointed out that the logics so far developed are capable of treating naturally only interpretations whose universes of discourse are composed of discrete objects and, consequently, that treatment of continuous domains always entails considerable logical difficulties. He suggests the need for a logic which involves a 'truly continuous ontology'.

To see the next point it is necessary to notice that there are predicates which are true of some things, false of others and not applicable to still others. For example, the predicate 'prime' is true of two, false of four and not applicable to pi. This means that each predicate has a range of applicability within which it holds true or false and outside of which it does not hold at all. Thus a sentence can fail to be true without being false and it can fail to be false without being true. Yet the following is logically true in standard logic.

$$\forall x(Px \vee \sim Px)$$

This reflects the fact that standard semantics presupposes universal ranges of applicability for all predicates.

The following list summarizes our classification of gaps between mathematical practice (real and putative) and existing mathematically defined logics.

I. Existence of underlying logics not yet modeled
II. Inadequate models of underlying logics
  Kinds of inadequacies: Excesses
    Defects
  Locations of inadequacies: Grammar
    Deductive System
    Semantics
    Postulate Systems
III. Existence of Phenomena not yet treated mathematically
IV. Purely Experimental Models

From the above and simila· ·tuations we can identify as a possible postulational excess the fact that standard logic admits of infinite axiom sets. In addition, some mathematicians regard mathematical induction as a non-logical rule of inference, not as a scheme or second-order sentence. Since non-logical rules are not admitted in standard formulations, we can identify their absence as a possible postulational defect.

For example, here is a proof of the proposition (A ⇒ B ⇒ C) ⇒ (A ∧ B ⇒ C).

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{[y : A \wedge B]}\,(\text{A})}{A}\,(\wedge\text{E}) \qquad \overline{[x : A \Rightarrow B \Rightarrow C]}\,(\text{A})}{B \Rightarrow C}\,(\Rightarrow\text{E}) \qquad \dfrac{\overline{[y : A \wedge B]}\,(\text{A})}{B}\,(\wedge\text{E})}{\dfrac{C}{A \wedge B \Rightarrow C}\,(\Rightarrow\text{I},y)}\,(\Rightarrow\text{E})}{(A \Rightarrow B \Rightarrow C) \Rightarrow (A \wedge B \Rightarrow C)}\,(\Rightarrow\text{I},x)$$

Rules for Falsity

$$\dfrac{\begin{matrix}[x : \neg P] \\ \vdots \\ \bot\end{matrix}}{P}\ (\text{reductio ad absurdum, RAA}/x) \qquad \dfrac{\bot}{P}\ (\text{ex falso quodlibet, EFQ})$$

Reductio ad absurdum (RAA) is an interesting rule. It embodies proofs by contradiction. It says that if by assuming that P is false we can derive a contradiction, then P must be true. The assumption x is discharged in the application of this rule. This rule is present in classical logic but not in **intuitionistic** (constructive) logic. In intuitionistic logic, a proposition is not considered true simply because its negation is false.

Excluded Middle

Another classical tautology that is not intuitionistically valid is the **the law of the excluded middle**, P ∨ ¬P. We will take it as an axiom in our system. The Latin name for this rule is *tertium non datur*, but we will call it *magic*.

$$\dfrac{}{P \vee \neg P}\ (\text{magic})$$

In mathematics, a **complete Boolean algebra** is a Boolean algebra in which every subset has a supremum (least upper bound). Complete Boolean algebras are used to construct Boolean-valued models of set theory in the theory of forcing. Every Boolean algebra A has an essentially unique completion, which is a complete Boolean algebra containing A such that every element is the supremum of some subset of A. As a partially ordered set, this completion of A is the Dedekind–MacNeille completion.

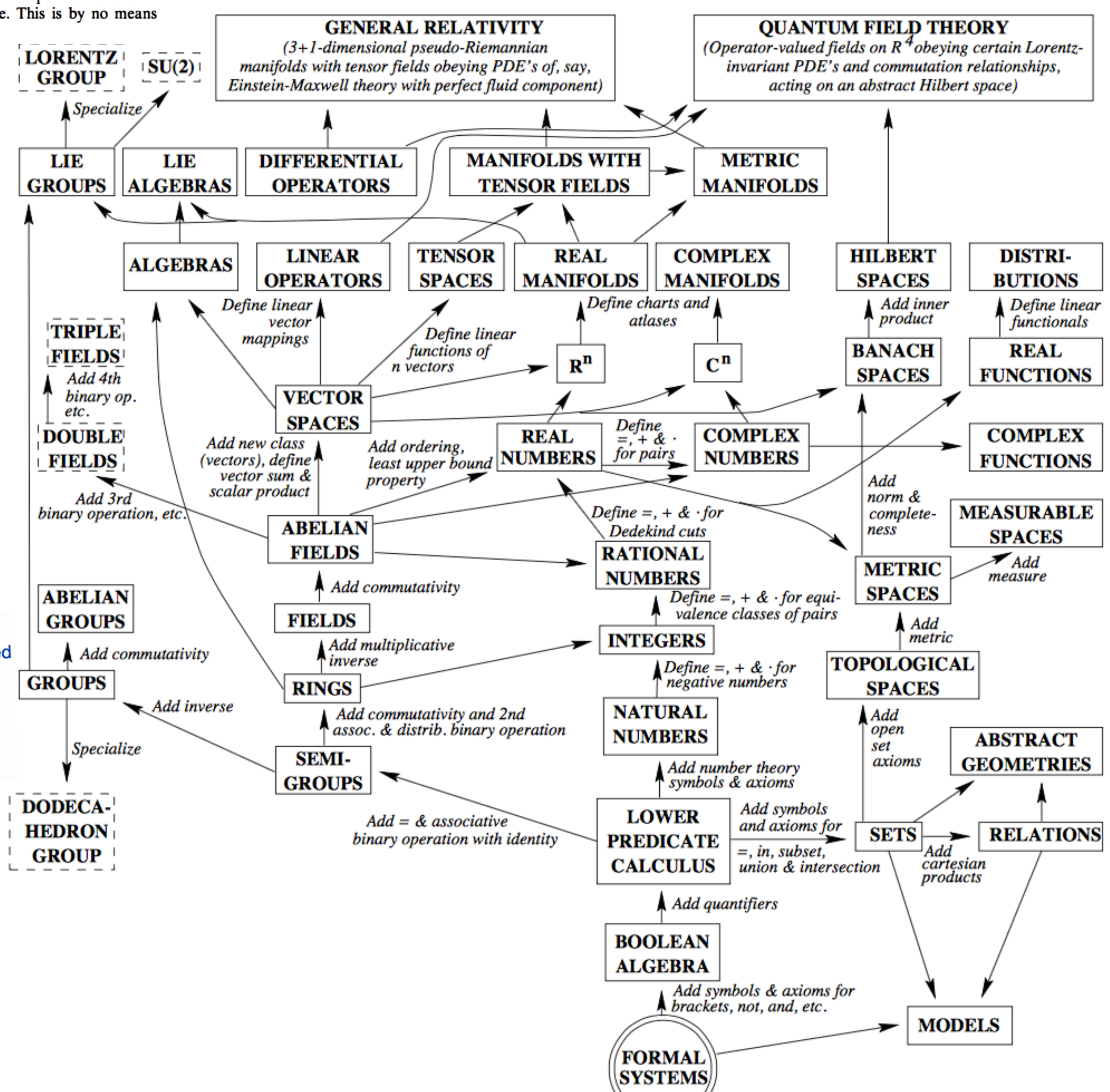

FIG. 1. Relationships between various basic mathematical structures. The arrows generally indicate addition of new symbols and/or axioms. Arrows that meet indicate the combination of structures — for instance, an algebra is a vector space that is also a ring, and a Lie group is a group that is also a manifold.

After the two-element Boolean algebra, the simplest Boolean algebra is that defined by the power set of two atoms:

∧	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

∨	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

x	0	a	b	1
¬x	1	b	a	0

Starting with the propositional calculus with κ sentence symbols, form the Lindenbaum algebra (that is, the set o sentences in the propositional calculus modulo tautology). This construction yields a Boolean algebra. It is in fac free Boolean algebra on κ generators. A truth assignment in propositional calculus is then a Boolean algebra homomorphism from this algebra to the two-element Boolean algebra.

The **Axiom of Reducibility** was introduced by Bertrand Russell in the early 20th century as part of his ramified theory of types. Russell devised and introduced the Axiom in an attempt to manage the contradictions he had discovered in his analysis of set theory.[1]