Mapping Robotics Transform Terminology to Abstract Linear Algebra

Motivation

My goal is to map the conventional robotics terminology for spatial transforms, as laid out in http://motion.cs.illinois.edu/RoboticSystems/CoordinateTransformations.html, to concept of abstract linear algebra. I do this because I never found a reference that does this explicitly, and it has been itching me for a long time. To keep it short the treatment, I only work in three dimensions, and ignore the two-dimensional case. I also only work with the rotation transform.

The essence of the idea is that all spatial vectors and matrices are in fact representations of linear algebraic vectors and maps. Since they are representations, they are always representations with respect to some basis.

Vectors and Frames

Let's identify a spatial vector $\mathbf{x}$ in some coordinate frame B with a representation of a linear algebraic vector v^x in some coordinate frame B.

$$\mathbf{x} \equiv \mathrm{Rep}_B(v^x)$$

where

$$\begin{aligned} \mathbf{x} &\in \mathbb{R}^3 && (1) \\ v^x &\in \mathbb{R}^3 && (2) \\ \mathrm{Rep}_B(v^x) &\in \mathbb{R}^3 && (3) \end{aligned}$$

Note that in the above, we should not be confused with the double role that $\mathbb{R}^3$ plays. It is an abstract linear algebraic space as in (2), as well as the abstract field $\mathbb{R}$ used to express the components of vector representations as in (3). For (1) any of the two roles is usable.

It follows that we also identify a spatial coordinate frame B, with a linear algebraic basis $\mathcal{B}^B$, whose i'th linear algebraic vector is $\mathcal{B}_i^B$

$$
\begin{aligned}
\mathcal{B}^B &= \left(\mathcal{B}^B_1, \mathcal{B}^B_2, \mathcal{B}^B_3\right) \\
B &= (B_1, B_2, B_3) \\
B_i &\equiv \mathrm{Rep}_A(\mathcal{B}^B_i)
\end{aligned}
$$

For this to work, there needs to be at least one special coordinate frame which does not depend on another frame. That is, whose vectors are not identified with representation with respect to some basis. This plays well with the robotics terminology of *privileged world frame* W. For this privileged frame we directly identify the spatial axes of W with linear algebraic vectors, without passing through a representation.

$$
\begin{aligned}
W &= (W_1, W_2, W_3) \\
W_i &\equiv \mathcal{B}^W_i \\
\mathcal{B}^W_i &= W_i
\end{aligned}
$$

where

$$W_i \in \mathbb{R}^3$$

Transforms

In linear algebra, it is a theorem that the action of a linear map on a space is fully determined by its action on the space's basis. We use this to identify rotations with certain vector representations, as we did for frames.

We identify the spatial rotation R acting in some frame B with the linear algebraic representation of the linear rotation map acting in the space of B.

We quote from our robotics reference: "A coordinate transform from frame A to frame B is expressed in the form T^B_A".

$$R \equiv R^B_B \qquad \equiv \mathrm{Rep}_{B\to B}(rot)$$

Note that we slightly modified Hefferon's notation for added clarity, using $\mathrm{Rep}_{X\to Y}$ instead of $\mathrm{Rep}_{X,Y}$.

This identification is consistent with the linear algebraic definition of map representation

$$\mathrm{Rep}_{X\to Y}(h) = \left[\mathrm{Rep}_Y(h(\mathcal{B}^X_1)) \;\mid\; \mathrm{Rep}_Y(h(\mathcal{B}^X_2)) \;\mid\; \mathrm{Rep}_Y(h(\mathcal{B}^X_2))\right]$$

As a consequence, to form the rotation matrix for R, we take the rotated i'th basis vector (expressed in the basis itself), as the i'th column vector of the matrix.

$$R \equiv R^B_B \qquad \equiv \mathrm{Rep}_{B\to B}(rot) = \left[\mathrm{Rep}_B(rot(\mathcal{B}^B_1)) \;\mid\; \mathrm{Rep}_B(rot(\mathcal{B}^B_2)) \;\mid\; \mathrm{Rep}_B(rot(\mathcal{B}^B_3))\right]$$

If we simplify notation, we have

$$
\begin{aligned}
B &= (B_1, B_2, B_3) \\
R &= [R(B_1) \;\mid\; R(B_2) \;\mid\; R(B_3)]
\end{aligned}
$$

Since the rotated axis are represented in B, the formation of the rotation matrix is trivial. We can start from imagining the identity coordinate frame (of course while using the same handedness as B), and then concatenate the rotated axes into a matrix. This will be the correct rotation matrix irrespective of B, again, because the representations are relative to B. In other words the rotated axes as coordinates that combine B's axes or as coordinates that combine the axis of the identity coordinate system, are the same. Technically, we are saying that

$$\text{Rep}_{B\to B}(rot) = \text{Rep}_{\mathcal{E}^3\to\mathcal{E}^3}(rot) \qquad = \text{Rep}_{\mathbb{R}^3\to\mathbb{R}^3}(rot)$$

Where $\mathcal{E}^n$ are the natural bases for vector spaces of dimension n as defined by Hefferon. This holds because we work exclusively with orthonormal coordinate frames and orthonormal (rotation) matrices, with compatible handedness. So for any such orthonormal frames M, N, we have:

$$\begin{aligned}\text{Rep}_{N\to N}(rot) &= \text{Rep}_{M\to N}(id)\ .\ \text{Rep}_{M\to M}(rot)\ .\ \text{Rep}_{N\to M}(id)\\ &= \text{Rep}_{M\to M}(rot)\end{aligned}$$

The geometrical meaning of this is that in analytic geometry, as long as the coordinate axes chosen to bootstrap the derivation are orthonormal and have the same handedness, the derived rotation matrix will have the same elements.

Sometimes, we have a certain rotation in mind given directly by the desired rotated axes. For example, we wish to rotate around the first axis such that the rotate second axis is aligned with the original unrotated third axis. When this is the case, the rotation matrix can be derived as explained above. In other cases though, the rotation in mind is more complex. Whatever that rotation is, formulae exist transform it into the rotated target axes we describe. For example, for an 'axis angle' type rotation, rotating around an axis e by and angle θ, the formulae are:

$$\begin{aligned}\text{Rep}_B(rot(\mathcal{B}_1^B)) &= \begin{bmatrix} e_1\,.\,e_1\,.\,(1-\cos_\theta)+1.\cos_\theta \\ e_2\,.\,e_1\,.\,(1-\cos_\theta)-e_3\,.\,\sin_\theta \\ e_3\,.\,e_1\,.\,(1-\cos_\theta)+e_2\,.\,\sin_\theta \end{bmatrix}\\ \text{Rep}_B(rot(\mathcal{B}_2^B)) &= \begin{bmatrix} e_1\,.\,e_2\,.\,(1-\cos_\theta)+e_3\,.\,\sin_\theta \\ e_2\,.\,e_2\,.\,(1-\cos_\theta)+1.\cos_\theta \\ e_3\,.\,e_2\,.\,(1-\cos_\theta)-e_1\,.\,\sin_\theta \end{bmatrix}\\ \text{Rep}_B(rot(\mathcal{B}_3^B)) &= \begin{bmatrix} e_1\,.\,e_3\,.\,(1-\cos_\theta)-e_2\,.\,\sin_\theta \\ e_2\,.\,e_3\,.\,(1-\cos_\theta)+e_0\,.\,\cos_\theta \\ e_3\,.\,e_3\,.\,(1-\cos_\theta)+1.\cos_\theta \end{bmatrix}\\ R \qquad \equiv \text{Rep}_{B\to B}(rot) &= \left[\text{Rep}_B(rot(\mathcal{B}_1^B))\ \middle|\ \text{Rep}_B(rot(\mathcal{B}_2^B))\ \middle|\ \text{Rep}_B(rot(\mathcal{B}_3^B))\right]\end{aligned}$$

An excellent reference for all rotation types and there formulae is "ROTATION: A review of useful theorems involving proper orthogonal matrices referenced to three dimensional physical space". A digital copy is freely available at https://my.mech.utah.edu/~brannon/public/rotation.pdf

Transform Hierarchies

The best way to think about transform hierarchies is to remember that all matrix columns are vector representations, and that a vector representation is a tuple of coordinates, each of which is the scale of its frames basis vector.

For example, the vector (with coordinates) $\begin{bmatrix}5\\6\\7\end{bmatrix}$ is only meaningful relative to some frame (B_1, B_2, B_3) , and the evaluation of that vector is $5.B_1 + 6.B_2 + 7.B_3$.

Now hierarchically, each of B_i is itself again the coordinates of some vector only meaningful to some frame A. For example, assuming $B_1 = \begin{bmatrix}-4\\-5\\-6\end{bmatrix}$, then the evaluation of B_1 is $-4.A_1 - 5.A_2 - 6.A_3$.

This continues until a privileged vector such as W is reached. Which is conventionally identified with the vector space $\mathbb{R}^3$ and a natural choice of basis. Hence

$$W_1 = \begin{bmatrix}1\\0\\0\end{bmatrix},\ W_2 = \begin{bmatrix}0\\1\\0\end{bmatrix},W_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

So a chain of spatial transforms such as

$$T_A^B U_B^C V_C^D x^D$$

is merely a chain of recursive linear combinations of vector coordinates.

Inverse Transforms

It turns out that for orthonormal transforms, the inverse of the formed matrix is equal to the transpose of that matrix. This allows to easily compute $\mathrm{Rep}_{B\to A}(t)$ from $\mathrm{Rep}_{A\to B}(t)$ and vice versa. Identically for T_A^B and T_B^A.

For example

$$(T_B^A) = (T_A^B)^{-1} = (T_A^B)^T$$

The identification of transforms with axis vector representations still holds. If T_A^B is given by

$$T_A^B \equiv \mathrm{Rep}_{A\to B}(t) \qquad = \left[\mathrm{Rep}_B(t(\mathcal{B}_1^A)) \ \mid\ \mathrm{Rep}_B(t(\mathcal{B}_2^A)) \ \mid\ \mathrm{Rep}_B(t(\mathcal{B}_3^A))\right]$$

then

$$T_B^A \equiv \mathrm{Rep}_{B\to A}(t) \qquad = \left[\mathrm{Rep}_A(t(\mathcal{B}_1^B)) \ \mid\ \mathrm{Rep}_A(t(\mathcal{B}_2^B)) \ \mid\ \mathrm{Rep}_A(t(\mathcal{B}_3^B))\right]$$

The nice property of orthonormal transformation matrices is that the representations of vectors above can easily be obtained from each other using matrix transposition.

Abstract Linear Algebra Concepts

We use Hefferon's free Linear Algebra textbook 'Linear Algebra. Second edition' as a reference for linear algebra concepts and notation.

- Definition of basis: p.110
- Definition of vector representation: p.113
- Definition of matrix representation: p.194
- Definition of linear map representation: p.195
- Derivation of the representation of a rotation matrix: p.197
- Definition of change of basis matrix: p.234
- Theorem for relating matrix representations: p.238
- Change of basis for a vector representation: p.234