CAT as a byproduct of pursuing AG through the applied lense of numerical integraion of PDEs and DAEs

Mathematics, like a polyhedron, is multi-faceted. The algebraists would get nowhere without the geometers first providing them with interesting objects to *co-ordinatize*, as Hermann Weyl put it. Without the algebraists' tools, the geometers would be unable to pass to higher dimensions, nor, without the analysts' assistance, could they discover the right non-commutative versions of earlier theories. Number theorists require the continuous to investigate the discrete, and analysts require the local to explore the global.

To write out the steps of this calculation in the usual linear form would be horrendous. We can think of the notation used in this process as both an iconic representation of the topological object, the tangle, while at the same time a symbolic representation of the algebra, a mapping between two algebraic entities. This is an excellent example of the phenomenon Emily Grosholz treats in her paper. Higher-dimensional algebra also throws light on the iconic aspects of very simple symbolism, where the letters of algebra and logic are zero-dimensional entities living in some higher-dimensional environment (see pp. 242-251 of Corfield 2003 for further details).

(1) Infinitesimal: the ratio of the infinitesimal change in the value of a function to the infinitesimal change in a function.
(2) Symbolic: the derivative of x^n is nx^{n-1}, the derivative of $\sin(x)$ is $\cos(x)$, the derivative of $f \circ g$ is $f' \circ g * g'$, *etc.*
(3) Logical: $f'(x) = d$ if and only if for every ϵ there is a δ such that when $0 < |\Delta x| < \delta$,

$$\left| \frac{f(x+\Delta x) - f(x)}{\Delta x} - d \right| < \delta.$$

(4) Geometric: the derivative is the slope of a line tangent to the graph of the function, if the graph has a tangent.
(5) Rate: the instantaneous speed of $f(t)$, when t is time.
(6) Approximation: The derivative of a function is the best linear approximation to the function near a point.
(7) Microscopic: The derivative of a function is the limit of what you get by looking at it under a microscope of higher and higher power.

37. The derivative of a real-valued function f in a domain D is the Lagrangian section of the cotangent bundle $T^*(D)$ that gives the connection form for the unique flat connection on the trivial $\mathbf{R}$-bundle $D \times \mathbf{R}$ for which the graph of f is parallel.

"God created integers, men made the remaining"
(KRONECKER)
. . . but it is surely the Devil who let them conceive partial differential equations!

ONCE UPON A TIME. . . a french professor of mathematics had two excellent students, far above the other ones and so good that whenever one was not the first at an examination, the second was and vice-versa. However, the minds of these two students were quite different: one was more attracted by the possibility to pass from explicit calculus on a white sheet of paper to high level abstract ideas, while the second hated to compute and was more concerned with logic through the use of computers. Of course, the professor was desperately dreaming to know who was the most intelligent student. It happened that, during a dark night when he was not able to sleep, the Devil spoke to him and offered (freely!) a way to select the best student on the basis of the following problem ([6], [9]).

DEVIL'S PROBLEM: Let u, v, y be 3 functions of the cartesian coordinates x^1, x^2, x^3 on euclidean space, related by the following system of 2 PDE where $\partial_{33}y = \partial^2 y/\partial_{x^3}\partial_{x^3}, \ldots$:

1872. However, most of the commentaries only insist on the words "group" and "geometry", while forgetting about the word "field" (F. Klein, le programme d'Erlangen, Collection "Discours de la méthode", Gauthier-Villars, Paris, 1974, § V, p. 17). In order to clear up this point, one can say that the problem sketched by Klein is to work out links in the following triangle:

$$\begin{tikzcd} & \text{differential geometry} & \\ \text{differential algebra} \arrow[ur] \arrow[rr, leftrightarrow] & & \text{algebraic geometry} \arrow[ul] \end{tikzcd}$$

Henceforth, it is only at the end of the last century that some mathematicians could have fulfilled the dream to find a Galois theory for systems of partial differential equations, simply called in the sequel "differential Galois theory".

The basic idea was simple at first sight. First of all the reader must not forget that modern treatment of the classical Galois theory, namely *the existence of a bijective dual correspondence between intermediate fields and subgroups*, is very recent as it is based on results by Emile Artin (10). One hundred years ago, the aim was mainly to "know" about the roots of a given polynomial equation with the help of a minimum number of new symbolic quantities, being again roots of "simpler" (in a sense to precise) algebraic equations. Moreover, there was no need for knowing explicitly these quantities ($\sqrt{2} = 1,414, \ldots$).

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15.18

Numerical Algebraic Geometry Dictionary

Algebraic Geometry	example in 3-space	Numerical Analysis
variety	collection of points, algebraic curves, and algebraic surfaces	polynomial system + union of witness sets, see below for the definition of a witness point
irreducible variety	a single point, or a single curve, or a single surface	polynomial system + witness set + probability-one membership test
generic point on an irreducible variety	random point on an algebraic curve or surface	point in a witness set; a witness point is a solution of polynomial system on the variety and on a random slice whose codimension is the dimension of the variety
pure dimensional variety	one or more points, or one or more curves, or one or more surfaces	polynomial system + set of witness sets of same dimension + probability-one membership tests
irreducible decomposition of a variety	several pieces of different dimensions	polynomial system + array of sets of witness sets and probability-one membership tests

Table 2: Dictionary to translate algebraic geometry into numerical analysis

As a matter of fact, despite the fast progress of computer algebra during the last ten years, only a few steps have been done towards the use of symbolic computers for studying systems of partial differential equations (PDE) ([2], [13]). In particular, one must notice a few modern tentatives for dealing with algebraic PDE through differential algebraic techniques [19] or differential elimination techniques [16], [18] or exterior calculus ([1], Novosibirsk school), but these methods, being absolutely dependent on th coordinate system as they rely on old works ([6], [14], [18]), do not seem go far inside the intrinsic structure of the system. Despite this point, the methods have been applied with success to control theory during the la five years ([3], [12]).

The word "group" was introduced in mathematics by Evariste Galois in 1830, just two years before he died at the age of 21. This concept slowly passed from the field of pure algebra to that of differential geometry.

On another side, that is without any reference to applications, the *formal theory* of PDE has been developped during the period 1965-1975 by a few americans along the pioneering work of D. C. Spencer [17] and H. Goldschmidt [4]. These new techniques (homological algebra, commutative algebra, diagram chasing, jet theory,. . .), mixing together differential geometric and algebraic arguments, have been applied to the theory of *Lie pseudogroups*, namely groups of transformations solutions of systems of PDE, again algebraic in many useful situations (volume preserving, complex analytic or holomorphic, contact, symplectic transformations,. . .) ([8], [9]). However, this domain became no longer in fashion and disappeared from the mathematical scenery (See Mathematical Reviews 81 f:58046). As a byproduct, almost no applied mathematician took the risk to use these ideas as they appear quite difficult and sophisticated at first sight ([4], [7]).

After a short historical survey, the purpose of this communication is to convince the reader that *the formal theory of PDE is just the missing tool for studying linear and nonlinear systems of PDE by means of computer algebra*. In particular, it will provide new intrinsic guide-lines (δ-cohomology,

Abstract. Numerical analysis of time-integration algorithms has applied advanced algebraic techniques for more than fourty years. An explicit description of the group of characters in the Butcher–Connes–Kreimer Hopf algebra first appeared in Butcher's work on composition of integration methods in 1972. In more recent years, the analysis of structure preserving algorithms, geometric integration techniques and integration algorithms on manifolds have motivated the incorporation of other algebraic structures in numerical analysis. In this paper we will survey algebraic structures that have found applications within these areas. This includes pre-Lie structures for the geometry of flat and torsion free connections appearing in the analysis of numerical flows on vector spaces. The much more recent post-Lie and D-algebras appear in the analysis of flows on manifolds with flat connections with constant torsion. Dynkin and Eulerian idempotents appear in the analysis of non-autonomous flows and in backward error analysis. Non-commutative Bell polynomials and a non-commutative Faà di Bruno Hopf algebra are other examples of structures appearing naturally in the numerical analysis of integration on manifolds.

"...whatever naturality our mathematics does have is not due so much to l nature of the human mind. I mean the way in which our 'logical thinking' tied to illogical 'natural languages'." (Ruelle 1988, p. 266).

formal integrability) along which the computational aspect must follow patterns similar to the "old" ones ([6], [14]), though many results (the main theorem below is a good example) cannot even be proved by using classical methods.

As a general introduction to this theory is out of the scope of this short paper while details can be found in [17] and in the three books of the first-named author [9], [10], [11], we shall rather illustrate the algorithms it may provide along a *mathematical tale* (not far from a true story!) where the "puzzle" has been computed through MACSYMA on MULTICS and SUN by the second-named author. This example, taken from [6] and adapted from [9], has been presented for the first time under this form during an intensive course (april 17, 18, 19, 1989) at "Institut National de Recherche en Informatique et Automatique (INRIA, France)". Not only is this example showing how tricky PDE can be, but also it points out clearly two typical formal problems that can be asked about PDE. We also exhibit the modern solution in a rather intuitive and almost self-contained way, while comparing it with the one first proposed by M. Janet in 1920 [6]. A few similar examples are proposed to the reader at the end, as exercises. Finally, we present a few applications showing that the formal theory will fast become of increasing use in many branches of mathematical physics.

The most interesting feature of toposes is the presence in them of an object of truth values (or sub-object classifier). While in sets we have the usual set of truth values {true, false}, other toposes have more exotic ones. For instance, in the topos of presheaves over a category of two objects and a single arrow from one to the other, the truth values may be thought of as: never true, always true, eventually true. Elsewhere, we find truth value spaces and truth value procedures. For example, the object of truth values for the topos *Spaces*, like *Sets*, has two points *true* and *false*, but around *true* there is an infinitesimal fringe, so that *false* is the only thing not infinitesimally close to *true*, but so that it is close enough that any classically closed subset of the reals has a smooth classifying map taking all of it to *true* and its classical open complement to *false*. Possible to model "You couldn't say it's not red, but it's not exactly red either". There is also a graph of truth values in the topos of small directed graphs, which has two vertices anc

things living in an $(n + k)$-dimensional world. For example, in position $(1, 2)$ of the table flavoured with duals, we are dealing with lines and circles living in a 3-dimensional world. This is where knots, loops and tangles live. To find ways of distinguishing them, we need something algebraic which belongs to that same position. In the 1980s candidates were found, namely, representations of quantum groups. What is so unusual about this work is that the algebra has to be tailored to the dimension. Where ordinary algebraic topology was happy to use groups to pick up information in all dimensions, so-called *quantum topology* requires specific kinds of algebra for specific dimensions.

Homotopy continuation techniques may be used to approximate all isolated solutions of a polynomial system. More recent methods which form the crux of the young field known as numerical algebraic geometry may be used to produce a description of the complete solution set of a polynomial system, including the positive-dimensional solution components. There are four main topics in the

this process. Second, it allows stronger axioms to be employed and thus simpler proofs can be obtained. Brouwer realised this with his (classically false) assumption that all functions from the reals to the reals are continuous. It is also allows the Kock-Lawvere axiom in synthetic differential geometry, one that takes all curves to be infinitesimally linear.

AG and numerical analysis: backward propagation, refind quote.

In Numerical Algebraic Geometry we apply and integrate homotopy continuation methods to describe solution components of polynomial systems. One special, but important problem in Symbolic Computation concerns the approximate factorization of multivariate polynomials with approximate complex coefficients. Our algorithms to decompose positive dimensional solution sets of polynomial systems into irreducible components can be considered as symbolic-numeric, or perhaps rather as numeric-symbolic, since numerical interpolation methods are applied to produce symbolic results in the form of equations describing the irreducible components.

Here is the core of the story and perhaps of this book: Drach's thesis contains ideas of genius such as differential field, differential extension, differential adjunction . . . but *the main central result of the thesis is completely wrong and not even proved*. Moreover, *there is an additional subtle conceptual mistake* that we shall point out later on. But first, how was it possible for such a situation to occur?

The first surprise is the examiners of the thesis: Emile Picard, Henri Poincaré and Gaston Darboux! Then reading the report (appendix) on the thesis, one first discovers that only Picard, the appointed referee, read it, as

Darboux just confirmed in a few lines the judgement of Picard, while Poincaré did not even appear. As we already said, Picard was not at all a differential geometer. Also, one must not forget that Lie was completely isolated and only understood by a few friends, including Klein. We may even say that Picard was far from understanding properly the concept of Lie pseudogroups. A few months after Drach had got his doctorate, Vessiot started to discover the mistake (appendix). Then, counterexamples were provided by Cartan, and the story finishes with a letter of Paul Painlevé clearing the omission of Picard (appendix)! But Drach was also a friend of Emile Borel, very influential at this time, and went on using the wrong results of his thesis for many years without correcting them. Of course, no other mathematician would have dared to point out the mistake . . . a fact showing up Picard's omission. Even Cartan wrote on the Drach–Vessiot theory, as he called it, only as late as in 1938 (24)! The funniest part of this "affair" was that, in 1904, Painlevé gave a review on the irreducibility of PDE at the International Congress of Mathematicians using the wrong theorem in his name and not even quoting Drach! . . . *Exit* Drach; but Vessiot spent almost 30 years trying to correct this mistake! (See 84 for a complete bibliography.)

As we already said, continuum mechanics, in its modern macroscopic presentation, started in 1822 with the understanding of what is today called a stress tensor. Soon after this arose the problem of determining the stress–strain relation in general media such as crystals. Any linear relation between a small stress tensor and a small strain tensor, both with six independent components as they are 3×3 symmetric tensors, should lead to $6 \times 6 = 36$ constitutive coefficients. George Green discovered in 1837 that, using the methods of thermodynamics, one could prove, in accordance with experiments, that the latter 6×6 matrix was again itself symmetric, a fact leading to only 21 coefficients. This was the birth of *energetics*. In the case of homogeneous isotropic media, one only has two independent coefficients, namely the Young and Poisson coefficients or the two Lamé coefficients.

When establishing the dynamic equations for stress, one must not forget that the dynamic study is transformed into a static study through the introduction of inertial forces, that is to say one must work with space–time and not add time to space. Hence, using the fundamental theorems of mechanics, one first gets as many divergence equations as the number of translations and *then* the usual symmetry of the stress tensor quoted by as many equations as the number of rotations. Thereafter, the conjecture of

More precisely, the purpose of the Cosserats was to exhibit the dynamic equations for stress through a mathematical "procedure" mixing group theory and variational calculus in such a way that group theory could give the *geometry* while variational calculus could provide the *physics*. In this situation, they discovered that only the knowledge of the rigidifying group of Euclidian motions should allow one to define the strain tensor in Lagrangian or Eulerian variables, as a basic set of independent differential invariant expressions involving the displacement vector and its first-order partial derivatives. Now, a 2-tensor may not be a strain tensor (*field*) unless it satisfies the *compatibility conditions* (*field equations*) that say that it comes from a displacement (*potential*). However, the work of the Cosserats was forgotten till 1950 and, today, mechanicians just keep in mind that certain media may not have a symmetric stress tensor because of some this reason, mechanicians have completely missec which was to create a mathematical "machine" s *any* Lie group of transformations could produce gc potentials, with as many potentials and as many equations as the number of group parameters. At t

equations as the number of group parameters. At this time, Special Relativity imposes on us the Poincaré group, and it becomes more difficult to speak about concepts such as "small translations", "small rotations" and "moving frames" . . . because of the lack of any intuitive support. The same comment could be made for any enlargement of the group and we arrive at the idea of Hermann Weyl for linking electromagnetism with continuum mechanics (115). In fact, to the *pure fields* (deformation, electromagnetic field) that satisfy *field equations* (compatibility conditions, first set of Maxwell equations), expressing that they come from *potentials* (displacements, electromagnetic potential), correspond the *dual fields* (stress, electromagnetic induction) satisfying the *dual field equations* (dynamic equations, second set of Maxwell equations). The Cosserats had no time to solve this problem because of the death of François, and Weyl was not able to succeed in introducing the conformal group with 15 parameters, that is to say $1 + 4$ parameters more than the Poincaré group, corresponding to absolute temperature and the electromagnetic 4-potential, according to an old idea of Ernest Mach and Gabriel Lippmann (1, 66, 68).

It was precisely at this time, in 1922, that Cartan created the theory of generalized spaces, while referring to the Cosserat brothers and Weyl (22, 23). The 1930 exchange of letters between Einstein and Cartan that followed may be considered as quite unfair! Indeed, Einstein, who met Cartan in Paris in 1922, did not even quote Cartan nor read his papers on the generality of solutions of field equations; Cartan did not say a single word about the fact

that Janet had completely solved this problem 10 years before, although he knew it perfectly well! Nevertheless, a new concept of *torsion* was added to that of *curvature*, but the choice of such a word has deeply misled mechanicians who tried to relate it to couple-stress, a *nonsense* similar to the one already stated, as couple-stress is related to rotations through the principle of virtual moves while torsion is a 2-form with value in the Lie algebra of the subgroup of translations. These ideas led Weyl to gauge theory in order to overcome a dead end with the conformal group. Later on, in 1954, non-Abelian gauge theory was introduced by C. N. Yang and R. L. Mills, still with a reference to the $U(1)$-Abelian electromagnetic model (122). They were in fact, completely forgetting that electromagnetism is one of the best tested macroscopic theories and that photoelasticity experiments can be seen in any civil engineering laboratory. Of course, no reference to Maxwell in the literature on gauge theory ever mentions that the laws of photoelasticity were obtained by Maxwell too.

invariance of Maxwell's equatio search for the program initi separation of his main pa "electrodynamic" part. Then, in equations of *General Relativity* *tensor calculus* developed, after mathematicians Curbastro Gre problem was that of a lever or a laboratory, after short computations of relativistic dynamics, there seems to be a couple appearing when they are viewed from another moving laboratory. The "energy current" of Von Laue and Tolman (103) looks like an artificial "virtue". It would have been fairer to say that a first generalization of the concept of *vector* is that of *tensor*, but that the concept of *torsor*, as a second and different kind of generalization, was not well understood.

was solved by the Cosserat brothers in 1909 (28). In particular, they discovered that stress may not be symmetric and that both sets of equations were first order, contradicting at once the exhibition of the Einstein tensor in General Relativity. Indeed it is an *apparent nonsense* to relate stress (or impulsion-energy in the 4-dimensional formalism)—which, by definition, has

to do with translations through the principle of virtual work—with curvature that can be considered as a 2-form with value in the Lie algebra of the subgroup of rotations.

It was a momentous occasion when we discovered in 1983 that a unique mathematical tool, namely the Spencer sequence, although fitting perfectly with Continuum Mechanics, Gauge Theory and General Relativity, at the same time provided contradictions between these three theories.

published later on his *Disquisitiones Arithmeticae*. His main theorem on this question, found in 1801, is that a regular polygon with n sides can be constructed with a ruler and a compass *if and only if* $n = 2^\lambda p_1 \cdots p_k$ where λ is an integer $\geqslant 0$ and $p_1, \ldots, p_k$ are prime numbers of the form $F_i = 2^{2^i} + 1$ called

*i*th Fermat number. It is remarkable that today, even with the help of powerful computers, the only such prime numbers n are 2, 3, 5, 17, 257, 65 537. As for the Fermat numbers, Fermat conjectured that they were prime for any i, a fact proved to be wrong by Euler in 1732. But the only prime Fermat numbers known today are the above ones already found by Fermat himself.

known to mechanicians. In this situation the Spencer sequence can be read:

rigid motion → displacement → deformation → dislocation

and deformation is nothing other than a section of the first Spencer bundle killed by the Spencer operator following it. The case of Lie groups provides a good framework for constructing a dynamic on Lie groups according to the

ideas of Garett Birkhoff (16) and V. Arnold (8,9), together with its generalization for continuous media. In particular, *it explains why the ambient space does not appear in the case of rigid body dynamics*. A careful review of the