Quantifying the Relation Between 'Linear System Solution Residual' and 'Particle Position Error'

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I Motivation

A very basic so far unanswered question is: "Given an imperfect linear system solver, how small should its solution's residual be so that a certain position accuracy is achieved". We will here only consider a linear system (not an LCP) at rest and also ignore any right hand side biasing. Such an artificial and idealized situation is of course not very telling, but we use it as a stepping stone on the path of quantifying increasingly realistic ones.

II Calculation

II.1 Impulse and Position

- Let $\begin{pmatrix} q \\ v \end{pmatrix}$ be the state of one particle of mass m.
- Let j be the perfect impulse and j' an imperfect impulse approximation.
- Applying an impulse j on one particle using an explicit integration step at time t and step h results in:

$$j = m(v^{t+h} - v^t)$$

$$v^{t+h} = v^t + \frac{j}{m}$$

$$q^{t+h} = q^t + v^{t+h}h = q^t + h(v^t + \frac{j}{m}) = [q^t + v^t h] + \frac{j}{m}h$$

- Similarly, applying j' gives

$$q'^{t+h} = [q^t + v^t h] + \frac{j'}{m}h$$

hence

$$q'^{t+h} - q^{t+h} = \frac{h}{m}(j' - j)$$

- Let the desired position accuracy be θ (e.g $\theta = 1mm$), then we have

$$|\frac{h}{m}(j' - j)| \leq \theta,$$

$$\underline{\underline{\underbrace{|j' - j|}_{\eta} \leq \theta . \frac{m}{h}}}$$

- Per example, with $m = 1, h = \frac{1}{60}$ and $\theta = 10^{-3}m.$, we need an impulse error η that is less that 10^{-2}:

$$\eta \leq 10^{-3} . \frac{1}{1/60} \leq 6.10^{-2} \leq 10^{-2}.$$

II.2 Solution Residual and Impulse Error

- Let us now search for the relation between η and the ($Ax = b$) linear system's solution residual $\underbrace{\rho_{j'}}_{\rho}$ that generates j' (while ρ_j generates j). Clearly

$$\rho_j = Aj - b = \vec{0}$$

$$\rho_{j'} = Aj' - b \neq \vec{0}$$

Now

$$b = Aj$$

$$Aj' - (Aj) = \rho$$

$$A(j' - j) = \rho$$

so

$$||A\eta|| = ||\rho||$$

- What is then the value $||\rho||$ for a given η, or at least some upper bound for it.

$$||A\eta|| \leq \underbrace{||A||}_{\text{some induced norm}} ||\eta||,$$

so

$$\underline{\underline{||\rho|| \leq ||A||.||\eta||}}$$

Let us work with the ∞-norm, which is induced. We then have for a linear system of n rows with i ranging over them:

$$||\eta||_\infty = max|\eta_i|, ||\rho||_\infty = max|\rho_i|$$

$$||A||_\infty = max||A^T_{i,*}||_1$$

which basically takes the maximum of row sums (themselves calculated using the 1-norm).

II.3 Specifically For a Particle Chain

- Let us focus on a particular and simple example where the matrix A is the standard $\Gamma = JM^{-1}J^T$ matrix.
- The constraint between any two particles is that their velocities should be equal: $v_x - v_y = 0$, the Jacobian of such a constraint has then the form:

$$J_{x,y} = (0, ..., \underbrace{1}_{\text{x}}, 0, ..., \underbrace{-1}_{\text{y}}, 0, ...)$$

- We then have the form for Γ:

$$\Gamma = \underbrace{\begin{pmatrix} 1 & -1 & & & \cdots \\ 0 & 1 & -1 & & \cdots \\ 0 & 0 & 1 & -1 & \cdots \end{pmatrix}}_{J} \underbrace{\begin{pmatrix} 1/m_1 & & & \\ & 1/m_2 & & \\ & & 1/m_3 & \\ & & & \ddots \end{pmatrix}}_{M^{-1}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ \vdots & \vdots & \ddots \end{pmatrix}}_{J^T}$$

- For a chain of particles with equal mass we get

$$\Gamma = \frac{1}{m} \begin{pmatrix} 2 & -1 & 0 & & \cdots \\ -1 & 2 & -1 & 0 & \cdots \\ & -1 & 2 & -1 & \cdots \\ & & \ddots & & \end{pmatrix}.$$

Clearly,

$$||\Gamma||_\infty = \frac{1}{m}[max(|2| + |-1|, |1| + |2| + |-1|, \cdots)]$$

$$\underline{\underline{||\Gamma||_\infty = \frac{4}{m}}}$$

- Going back to position accuracy θ, we have for a linear system of dimension n, stemming form a chain of n particles:

$$||\rho||_\infty \leq (\frac{4}{m}).max(|\theta_i.\frac{m}{h}|)$$

Taking the same value of θ for all particles, we have:

$$||\rho||_\infty \leq (\frac{4}{\cancel{m}})(\frac{\cancel{m}}{h}).\theta$$

$$\underline{\underline{||\rho||_\infty \leq \frac{4\theta}{h}}}$$

- In the equation above, θ is an error per particle. If we consider the last particle, the accumulated error Θ there is given by

$$\Theta = n\theta$$

and

$$||\rho||_\infty \leq \frac{4\Theta}{h.n}$$

- Finally, let us take the example with $h = 1/60$, $n = 100$, $\Theta = 10^{-3}(1 \text{ mm.})$. We get

$$||\rho||_\infty \leq \frac{4.10^{-3}}{(1/60).10^2}$$

$$||\rho||_\infty \leq 4.60.10^{-3}.10^2$$

$$||\rho||_\infty \leq (2,4).10^{-3}$$

$$\underline{\underline{||\rho||_\infty \leq 10^{-3}}}$$