Math 7334 Operator Theory

Instructor: Michael Loss

Lectures: TTh 1:35-2:55pm

Location: Skiles 154

Office hours: Thursday 12 - 1:30 or by appointment

Textbook: Theory of linear operators in Hilbert space by N.I. Akhiezer and I.M. Glazman, Dover.

A number of mathematical and physical problems can be formulated and actually solved through the methods of operator theory.
The prime example is that certain evolution equations can be viewed as an initial value problem of an `ordinary differential equation'
involving a linear operator on a Banach space or Hilbert space. To stay with a concrete example, consider the heat equation on some domain
with some boundary conditions. Associated with this problem is a linear, unbounded operator, the generator of the heat flow.
If this operator is selfadjoint then it follows from the spectral theorem that this evolution problem has a solution global in time.
The notion of selfadjointness, therefore, captures an essential part of the nature of this problem, in particular features like boundary
conditions must enter into the picture. One difficulty is that in most of the interesting examples one has to deal with unbounded operators.

Thus, the course will evolve as follows. We will review Hilbert space theory, in particular the projection lemma, the Riesz representation
theorem and the uniform boundedness principle. We will emphasize the examples, such as standard L^2 theory, as well as the Hilbert space of
almost periodic functions. We will then talk about linear operators, at first only bounded ones and work out the spectral theory of completely continuous
operators in detail. These are important since they are a direct generalization of matrices. Many practical problems can be reduced to this case.

The next step up is to deal in detail with bounded operators, in particular the theory of bounded selfadjoint operators culminating in the spectral theorem. Here the notion of operator valued function becomes important.

The fun starts with unbounded operators. The difference is now that the domain is a defining part of the operator. The notion of closed is a relaxation of the notion of continuity and is crucial for doing any kind of analysis. The notion of selfadjointness is also tricky; there are symmetric operators that are not selfadjoint and for which the spectral theorem does not hold. These developments are not discussed for generalization's sake but many of the interesting applications require unbounded operators. The theory was invented in large parts because of quantum mechanics and if time permits we will talk about this topic.

I will follow the textbook, which is a classic, very closely. This textbook is good and cheap. Sometimes I will post additional notes here.

Here is a rough syllabus with an approximate number of lectures devoted to each topic:

Review of Hilbert space theory: Projection lemma, Riesz representation theorem, orthonormal systems and basis, examples of Hilbert spaces such as the space of almost periodic functions, weak and strong convergence, weak compactness. about 4 Lectures

Linear operators: Bounded operators, completely continuous operators, projection operators, unitary operators with examples such as the Fourier transform. 4 Lectures

General concepts for linear operators: Closed operators, invariant subspaces, resolvent and spectrum, symmetric and selfadjoint operators, with examples such as multiplication operators and differential operators, singular integrals. 4 Lectures

Spectral analysis of completely continuous operators: eigenspaces and eigenvalues, spectral theorem for selfadjoint completely continuous operators, proof of the fundamental theorem of almost periodic functions, if time permits. 4 Lectures

Spectral analysis of unitary and selfadjoint operators: Bochner's theorem, spectral theorem for unitary operators, spectral theorem for unitary operators, Cayley transforms, examples. 5 Lectures

Extension of operators: Adjoint operators, symmetric operators versus selfadjoint operators, deficiency indices, semi-bounded operators. 5 Lectures

Semigroup theory: If time permits, I will add a discussion about semigroup theory, in particular the theorem of Hille-Yosida concerning generators of contraction semigroups. 4 Lectures.

GRADES: There will be no tests but occasionally some homework which you are required to hand in. The grade will be awarded according to correctly solved homework problems.

Synopsis on Hilbert spaces
Homework 1

L^2-spaces
Homework 2

The Neumann series
Homework 3

The Fredholm alternative
Homework 4 **Solutions**

Spectral Theorem for compact operators
Homework 5 **Solutions**

The Fourier transform

Closed operators

Basic theorem for self adjoint operators

The Laplace operator as a self-adjoint operator

The Kato-Rellich theorem

Symmetric and self-adjoint extensions