Stochastic Calculus: Lecture Notes

Contents

1 σ-Algebras

Definition 1.1. *Let Ω be a nonempty set. A* σ-algebra $\mathcal{F}$ *on Ω is a collection of subsets of Ω such that:*

1. $\Omega \in \mathcal{F}$;
2. *If $A \in \mathcal{F}$, then its complement $A^c = \Omega \setminus A \in \mathcal{F}$;*
3. *If $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$, then the union* $\displaystyle\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Exercise 1.1. *Let $\mathcal{F}$ be a σ-algebra on Ω. Show that:*

1. $\emptyset \in \mathcal{F}$.
2. *For any finite collection $A_1, \ldots, A_n \in \mathcal{F}$, also $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$ and $\bigcap_{i=1}^{n} A_i \in \mathcal{F}$.*
3. *For $A, B \in \mathcal{F}$, the difference $A \setminus B \in \mathcal{F}$.*
4. *The intersection of an arbitrary family $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ of σ-algebras is itself a σ-algebra.*

2 Measures, Probability

Definition 2.1. *Let Ω be a set and $\mathcal{F}$ a σ-algebra over Ω. A function $\mu : \mathcal{F} \to \mathbb{R}$ is called a measure if it satisfies the following properties:*

1. *Non-negativity: $\forall A \in \mathcal{F}, \mu(A) \geq 0$;*
2. *Null empty set: $\mu(\emptyset) = 0$;*
3. *Countable additivity: For every sequence $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{F}$,*

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

A probability measure P *is a measure with $P(\Omega) = 1$.*

3 Measurable Functions, Random Variables

Definition 3.1. *Let $(\Omega, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces, meaning that Ω and Y are sets equipped with respective σ-algebras $\mathcal{F}$ and $\mathcal{G}$. A function $f : \Omega \to Y$ is said to be measurable if for every $E \in \mathcal{G}$ the pre-image of E under f is in $\mathcal{F}$:*

$$f^{-1}(E) = \{x \in \Omega \mid f(x) \in E\} \in \mathcal{F}.$$

Definition 3.2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(Y, \mathcal{G})$ a measurable space. Then an $(Y, \mathcal{G})$-valued random variable is a measurable function*

$$X : \Omega \to Y.$$

X is $\mathcal{F}$-measurable. The notation $\sigma(X)$ denotes the σ-algebra generated by X.

4 Expectation, Indicator Functions

$$X \in L^1(\Omega) \quad \text{if} \quad E[|X|] < \infty.$$

$$E[X] = \int_\Omega X(\omega)\, dP(\omega).$$

For any event $A \in \mathcal{F}$, the *indicator function* is

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A. \end{cases}$$

One then has

$$P(A) = \int_\Omega \mathbf{1}_A \, dP = \mathbf{E}[\mathbf{1}_A].$$

5 Independence, Covariance

Two events $A, B \in \mathcal{F}$ are *independent* if

$$P(A \cap B) = P(A)P(B).$$

Random variables X and Y are independent if for all $A, B \in \mathcal{G}$:

$$P(X \in A \cap Y \in B) = P(X \in A)P(Y \in B).$$

$$\begin{aligned} E[XY] &= E[X]E[Y], \\ \mathrm{Cov}(X, Y) &= E[XY] - E[X]\,E[Y], \\ \rho(X, Y) &= \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}. \end{aligned}$$

A random variable X and a σ-algebra $\mathcal{F}$ are independent if $\sigma(X)$ and $\mathcal{F}$ are independent. This means:

$$P(X \in I \mid A) = P(X \in I), \quad \forall A \in \mathcal{F}, \text{ interval } I.$$

6 Conditional Expectation

Definition 6.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X : \Omega \to \mathbb{R}^n$ a random variable, and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra. A conditional expectation of X given $\mathcal{G}$, denoted $E[X \mid \mathcal{G}]$, is any $\mathcal{G}$-measurable function such that*

$$\int_A E[X \mid \mathcal{G}]\, dP = \int_A X \, dP, \quad \forall A \in \mathcal{G}.$$

Properties:

1. Linearity: $E[aX + bY \mid \mathcal{G}] = aE[X \mid \mathcal{G}] + bE[Y \mid \mathcal{G}]$
2. If X is $\mathcal{G}$-measurable:
$$\begin{aligned} E[XY \mid \mathcal{G}] &= X E[Y \mid \mathcal{G}], \\ E[X \mid \mathcal{G}] &= X \end{aligned}$$
3. Tower rule: $E[E[X \mid \mathcal{G}] \mid \mathcal{H}] = E[X \mid \mathcal{H}]$ if $\mathcal{H} \subset \mathcal{G}$
4. If X is independent of $\mathcal{G}$: $E[X \mid \mathcal{G}] = E[X]$
5. Total expectation: $E[E[X \mid \mathcal{G}]] = E[X]$

7 Stochastic Process

Definition 7.1. *Let T be a set of times. A stochastic process $\{X_t\}_{t \in T}$ is a collection of random variables on $(\Omega, \mathcal{F}, P)$.*

Definition 7.2. *A filtration is a family $\{\mathcal{F}_t\}$ of sub-σ-algebras of $\mathcal{F}$ with the property that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $0 \leq s < t$.*

A stochastic process $\{X_t\}$ is adapted to filtration $\{\mathcal{F}_t\}$ if X_t is $\mathcal{F}_t$-measurable for all t.

Example 7.1. *If T is discrete:*

- *X_0 is $\mathcal{F}_0$-measurable*
- *(X_0, X_1) is $\mathcal{F}_1$-measurable*
- *(X_0, X_1, X_2) is $\mathcal{F}_2$-measurable*

8 Martingale

Definition 8.1. *A process $\{X_t\}_{t \in T}$ is a* martingale *with respect to the filtration $\{\mathcal{F}_t\}$ if:*

1. *X_t is integrable for all t,*
2. *X_t is adapted to the filtration $\mathcal{F}_t$,*
3. *For all $s \leq t$, $E[X_t|\mathcal{F}_s] = X_s$.*

This implies:

$$E[X_t - X_s \mid \mathcal{F}_s] = 0$$

$$E[X_s] = E[X_t] = E[X_0], \quad \text{so the expectation is constant.}$$

Example 8.1. *$X_t = B_t^2 - t$.*

$$d(B_t^2) \;=\; 2B_t\,dB_t + dt \quad\Longrightarrow\quad dX_t = 2B_t\,dB_t.$$

Mean:

$$E[X_t] = E[B_t^2] - t = t - t = 0.$$

Martingale property: *for $s < t$,*

$$X_t - X_s \;=\; \int_s^t 2B_u\,dB_u,$$

so by independence $E[X_t - X_s \mid \mathcal{F}_s] = 0$ and hence

$$E[X_t \mid \mathcal{F}_s] = X_s.$$

Example 8.2. $X_t = B_t^3 - 3\int_0^t B_u\,du.$

$$d(B_t^3) = 3B_t^2\,dB_t + 3B_t\,dt \implies dX_t = 3B_t^2\,dB_t.$$

Mean: *by Itô isometry,*

$$E[X_t] = 0.$$

Martingale property: *for $s < t$,*

$$X_t - X_s = \int_s^t 3B_u^2\,dB_u,$$

so $E[X_t \mid \mathcal{F}_s] = X_s$.

Example 8.3. $X_t = \exp\bigl(B_t - \tfrac{1}{2}t\bigr).$

$$dX_t = X_t\,dB_t \quad (\textit{since } d(B_t - \tfrac{1}{2}t) = dB_t - \tfrac{1}{2}dt).$$

Mean:

$$E[X_t] = e^{-\frac{1}{2}t}\,E[e^{B_t}] = e^{-\frac{1}{2}t}\,e^{\frac{1}{2}t} = 1.$$

Martingale property: *for $s < t$,*

$$X_t - X_s = \int_s^t X_u\,dB_u,$$

so $E[X_t \mid \mathcal{F}_s] = X_s$.

Example 8.4. $X_t = \sin(B_t)$*, define*

$$M_t = \sin(B_t) + \tfrac{1}{2}\int_0^t \sin(B_u)\,du.$$

By Itô,

$$d\sin(B_t) = \cos(B_t)\,dB_t - \tfrac{1}{2}\sin(B_t)\,dt \implies dM_t = \cos(B_t)\,dB_t.$$

Mean: $E[M_t] = 0$. Martingale property: *for $s < t$,*

$$M_t - M_s = \int_s^t \cos(B_u)\,dB_u \implies E[M_t \mid \mathcal{F}_s] = M_s.$$

Equivalently,

$$\sin(B_t) + \tfrac{1}{2}\int_0^t \sin(B_u)\,du$$

is a martingale.

9 Brownian Motion

Definition 9.1. *The process B_t is a* Brownian motion *if:*

1. $B_0 = 0$,
2. B_t *is almost surely continuous,*

3. *B_t has independent increments,*

4. *$B_t - B_s \sim N(0, t-s)$ for $0 \leq s < t$.*

$$F_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Properties:

- $E[B_t] = 0$
- $E[B_t^2] = t = \mathrm{Var}(B_t)$
- B_t is a martingale
- $E[B_t \mid B_s] = B_s$

10 Itô Integral and Isometry

Given a simple adapted process $h_t = \sum_{i=0}^{n-1} h_i \mathbf{1}_{(t_i, t_{i+1}]}(t)$, define the Itô integral:

$$\int_0^T h_s \, dB_s = \sum_{i=0}^{n-1} h_i \big(B_{t_{i+1}} - B_{t_i}\big).$$

Itô isometry:

$$\mathbb{E}\left[\left(\int_0^T h_s \, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^T h_s^2 \, ds\right].$$

11 Stochastic Integral

$$\int_0^T W_s \, dW_s \quad \text{(not Riemann–Stieltjes)}$$

Elementary process:

$$\int_0^T h_t(\omega) \, dW_t(\omega) := \sum_{i=0}^{n-1} e_i(\omega)(W_{t_{i+1}} - W_{t_i})$$

General process:

$$\int_0^T X_t(\omega) dW_t(\omega) := \lim_{n\to\infty} \sum_{t_i} X_t^{(n)}(W_{t_{i+1}} - W_{t_i})$$

12 Itô's Lemma

Theorem 12.1 (Itô's Lemma)**.** *Let f be twice differentiable. Then for Brownian motion W_t:*

$$f(W_t) = f(0) + \frac{1}{2}\int_0^t f''(W_s)ds + \int_0^t f'(W_s)dW_s$$

Proof. Partition $[0,t]$ with $0 = t_0 < \cdots < t_n = t$:

$$f(W_t) = f(0) + \sum_{i=0}^{n-1} \big(f(W_{t_{i+1}}) - f(W_{t_i})\big)$$

Using Taylor's Theorem:

$$f(W_{t_{i+1}}) - f(W_{t_i}) = f'(W_{t_i})(W_{t_{i+1}} - W_{t_i}) + \frac{1}{2}f''(\theta_i)(W_{t_{i+1}} - W_{t_i})^2$$

with θ_i between W_{t_i} and $W_{t_{i+1}}$. □

Example 12.1. Problem. *Let*

$$X_t = B_t^2.$$

Find the process $a(s,\omega)$ such that

$$X_t = E[X_t] + \int_0^t a(s,\omega)\, dB_s.$$

Solution. *By Itô's formula,*

$$dX_t = d(B_t^2) = 2B_t\, dB_t + dt,$$

so

$$X_t = X_0 + \int_0^t (2B_s\, dB_s + ds) = 0 + t + \int_0^t 2B_s\, dB_s.$$

Hence

$$E[X_t] = t, \qquad a(s,\omega) = 2B_s.$$

Example 12.2. Problem. *Let*

$$X_t = B_t^3.$$

Find the process $a(s,\omega)$ such that

$$X_t = E[X_t] + \int_0^t a(s,\omega)\, dB_s.$$

Hint: *introduce $Y_t = t\,B_t$.*

Solution. *By Itô's formula,*

$$dX_t = d(B_t^3) = 3B_t^2\, dB_t + 3B_t\, dt.$$

Set

$$Y_t = tB_t, \quad dY_t = B_t\, dt + t\, dB_t \implies \int_0^t B_s\, ds = tB_t - \int_0^t s\, dB_s.$$

Thus

$$\int_0^t 3B_s\,ds = 3\Big(tB_t - \int_0^t s\,dB_s\Big),$$

and

$$X_t = \int_0^t \big(3B_s^2\,dB_s + 3B_s\,ds\big) = \int_0^t 3B_s^2\,dB_s + 3\Big(tB_t - \int_0^t s\,dB_s\Big).$$

Rearranging,

$$X_t = 3\int_0^t B_s^2\,dB_s + 3tB_t - 3\int_0^t s\,dB_s = 0 + \int_0^t \big(3B_s^2 + 3(t-s)\big)\,dB_s.$$

Therefore

$$E[X_t] = 0, \qquad a(s,\omega) = 3B_s^2 + 3\,(t-s).$$

Example 12.3. Problem. *Let*

$$X_t = e^{B_t}.$$

Find the process $a(s,\omega)$ such that

$$X_t = E[X_t] + \int_0^t a(s,\omega)\,dB_s.$$

Hint: *consider $Y_t = e^{B_t - \frac{1}{2}t}$.*

Solution. *By Itô's formula,*

$$d(e^{B_t}) = e^{B_t}\,dB_t + \tfrac{1}{2}e^{B_t}\,dt.$$

Let

$$Y_t = e^{B_t - \frac{1}{2}t},$$

then

$$dY_t = e^{B_t - \frac{1}{2}t}\,dB_t,$$

so

$$Y_t = 1 + \int_0^t e^{B_s - \frac{1}{2}s}\,dB_s.$$

Hence

$$e^{B_t} = e^{\frac{1}{2}t}Y_t = E[e^{B_t}] + \int_0^t e^{\frac{1}{2}t}e^{B_s - \frac{1}{2}s}\,dB_s.$$

Therefore

$$E[X_t] = e^{\frac{1}{2}t}, \qquad a(s,\omega) = e^{\frac{1}{2}t}e^{B_s - \frac{1}{2}s} = e^{B_s + \frac{1}{2}(t-s)}.$$

Example 12.4. Problem. *Let*

$$X_t = \sin(B_t).$$

Find the process $a(s,\omega)$ such that

$$X_t = E[X_t] + \int_0^t a(s,\omega)\,dB_s.$$

Hint: *consider $Y_t = e^{\frac{1}{2}t}\sin(B_t)$.*

Solution. *By Itô's formula,*

$$d\sin(B_t) = \cos(B_t)\,dB_t - \tfrac{1}{2}\sin(B_t)\,dt.$$

Set

$$Y_t = e^{\frac{1}{2}t}\sin(B_t), \quad dY_t = e^{\frac{1}{2}t}\cos(B_t)\,dB_t.$$

Thus

$$Y_t = \int_0^t e^{\frac{1}{2}s}\cos(B_s)\,dB_s,$$

and

$$\sin(B_t) = e^{-\frac{1}{2}t}Y_t = \int_0^t e^{-\frac{1}{2}(t-s)}\cos(B_s)\,dB_s.$$

Hence

$$E[X_t] = 0, \qquad a(s,\omega) = e^{-\frac{1}{2}(t-s)}\,\cos(B_s).$$

13 Itô Process

Definition 13.1. *A process satisfying*

$$dX_t = \mu_t dt + \sigma_t dW_t$$

is called an Itô process.

Condition:

$$\int_0^t (\sigma_s^2 + |\mu_s|)ds < \infty$$

Chain rule:

$$d(f(t,X_t)) = f_t\,dt + f_X\,dX_t + \frac{1}{2}f_{XX}(dX_t)^2$$

Multivariate:

$$d(f(X_{1,t},\ldots,X_{d,t})) = \sum_i f_{X_i}dX_{i,t} + \frac{1}{2}\sum_{i,j} f_{X_iX_j}dX_{i,t}dX_{j,t}$$

14 Quadratic Variation

$$\langle X\rangle_t = \lim_{n\to\infty}\sum_{j=1}^n \left[X_{j/n} - X_{(j-1)/n}\right]^2$$

$$\langle X,Y\rangle_t = \lim_{n\to\infty}\sum_{j=1}^n \left[X_{j/n} - X_{(j-1)/n}\right]\left[Y_{j/n} - Y_{(j-1)/n}\right]$$

$$\begin{aligned}
(dW_t)^2 &= dt,\\
[dW_t, dW_t] &= dt,\\
[W_t, W_t] &= t,\\
[t,t] &= 0, \quad [t, W_t] = 0
\end{aligned}$$

$$dX_t = \mu dt + \sigma_t dW_t \Rightarrow d\langle X_t\rangle = \sigma_t^2 dt$$

15 Itô's Isometry

Let $X_t \in L^2[0,T]$. Then:

$$E\left[\left(\int_0^T X_t\,dW_t\right)^2\right] = E\left[\int_0^T X_t^2\,dt\right]$$

Proof sketch.

$$E\left[\left(\int_0^T X_t\,dW_t\right)^2\right] = E\left[\left(\sum_{i=0}^{n-1} e_i(W_{t_{i+1}} - W_{t_i})\right)^2\right]$$

Expanding the square:

$$= \sum_{i=0}^{n-1} E\left[e_i^2(W_{t_{i+1}} - W_{t_i})^2\right] + 2\sum_{0\le i<j\le n-1} E\left[e_i e_j (W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j})\right]$$

Because Brownian increments are independent and have mean zero, the cross terms vanish:

$$\mathbb{E}[(W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j})] = 0 \quad \text{for } i \neq j$$

Therefore:

$$E\left[\left(\int_0^T X_t\,dW_t\right)^2\right] = \sum_{i=0}^{n-1} E\left[e_i^2\right](t_{i+1} - t_i) = E\left[\int_0^T X_t^2\,dt\right]$$

For general $X_t \in L^2([0,T])$, we approximate using a sequence of such simple processes $X_t^{(n)} \to X_t$ in L^2, and apply the isometry to the limit by dominated convergence.

16 Integral Martingale

$$Y_t := \int_0^t X_u\,dW_u$$

is a martingale if $X_t \in L^2[0,T]$.

Proof:

$$\begin{aligned} E[Y_t \mid \mathcal{F}_s] &= E\left[\int_0^s X_u\,dW_u + \int_s^t X_u\,dW_u \mid \mathcal{F}_s\right] \\ &= \int_0^s X_u\,dW_u = Y_s \end{aligned}$$

17 Geometric Brownian Motion

$$dX_t = \mu X_t\,dt + \sigma X_t\,dW_t \Rightarrow \frac{dX_t}{X_t} = \mu dt + \sigma dW_t$$

Apply Itô's Lemma to $\log(X_t)$, then integrate:

$$X_T = X_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T\right)$$

If $\mu = 0$, then:

$$X_T = X_0 \exp\left(\sigma W_T - \frac{\sigma^2}{2} T\right), \quad E[X_T] = X_0$$

Example 17.1. Problem. *Let $X \sim N(\mu, \sigma^2)$. Using the fact that if*

$$Y_t = \exp\Big(-\tfrac{1}{2}\sigma^2 t + \sigma B_t\Big)$$

is a martingale, compute

$$E\big[e^X\big].$$

Solution. *Write $X = \mu + \sigma B_1$, so*

$$e^X = \exp\big(\mu + \sigma B_1\big) = \exp\Big(\mu + \tfrac{1}{2}\sigma^2\Big) \exp\Big(-\tfrac{1}{2}\sigma^2 + \sigma B_1\Big).$$

Since $Y_1 = \exp(-\frac{1}{2}\sigma^2 + \sigma B_1)$ is a martingale with $E[Y_1] = 1$, it follows that

$$E\big[e^X\big] = \exp\Big(\mu + \tfrac{1}{2}\sigma^2\Big) E[Y_1] = \exp\Big(\mu + \tfrac{1}{2}\sigma^2\Big).$$

18 Change of Measure (Randon-Nikodym)

$$P(A) = \int_A dP, \quad Q(A) = \int_A f\, dP, \quad f = \frac{dQ}{dP}$$

Suppose p, q are measures on $(\Omega, \mathcal{F})$. q is absolutely continuous with respect to p, written $q \ll p$, if for every $E \in \mathcal{F}$, if $p(E) = 0$, then $q(E) = 0$. p and q are mutually absolutely continuous or equivalent if $p \ll q$ and $q \ll p$.

$$p(E) = 0 \iff q(E) = 0$$

Theorem 18.1 (Randon-Nikodym)**.** *Suppose p, q are σ-finite measures on $(\Omega, \mathcal{F})$ with $q \ll p$. Then there exists a function f such that for every E.*

$$q(E) = \int_E f\, dp$$

The function f is called the Randon-Nikodym derivative of q with respect to p and is denoted

$$f = \frac{dq}{dp}$$

19 Girsanov's Theorem

Theorem 19.1 (Girsanov)**.** *Let $u \in L^2([0, T])$ be deterministic. Then the process:*

$$X_t = \underbrace{\int_0^t u(s)\, ds}_{\textit{drift}} + \underbrace{W_t}_{\textit{Brownian motion}}, \quad 0 \le t \le T$$

is a Brownian motion with respect to the probability measure Q given by

$$dQ = \exp\left(\int_0^T u(s)\,ds) - \frac{1}{2}\int_0^T u(s)^2\,ds\right) dP$$

$X_t = at + W_t$	$X_t = at + W_t$
W_t is a P-Brownian motion	W_t is not a Q-Brownian motion
X_t is not a P-Brownian motion	X_t is a Q-Brownian motion

Table 1: Girsanov's Theorem

For constant u(s),

$$dQ = \exp\left(-aW_t - \frac{1}{2}a^2 t\right) dP$$

Example 19.1. Problem. *Compute*

$$E\Big[\big(B_t + t^2\big)\ \exp\Big(-\int_0^t s\,dB_s\ -\ \tfrac{1}{2}\int_0^t s^2\,ds\Big)\Big].$$

Solution.

$$\frac{dQ}{dP} = \exp\Big(-\int_0^t s\,dB_s - \tfrac{1}{2}\int_0^t s^2\,ds\Big).$$

By Girsanov, under Q the process

$$X_t = B_t + \int_0^t s\,ds = B_t + \tfrac{t^2}{2}$$

is a Q–Brownian motion. Hence

$$E\big[(B_t + t^2)\exp(\cdots)\big] = \int_\Omega (B_t + t^2)\,dQ = E^Q\big(B_t + t^2\big) = E^Q\Big(X_t + \tfrac{t^2}{2}\Big) = \tfrac{t^2}{2}.$$

Example 19.2. Problem. *Compute*

$$E\Big[\big(B_t + t^2\big)\ \exp\Big(-2\int_0^t s\,dB_s\Big)\Big].$$

Solution.

$$\frac{dQ}{dP} = \exp\Big(-2\int_0^t s\,dB_s - \tfrac{1}{2}\int_0^t (2s)^2\,ds\Big).$$

Under Q,

$$X_t = B_t + \int_0^t 2s\,ds = B_t + t^2$$

is a Q–Brownian motion, so

$$E\big[(B_t + t^2)\exp(-2\int_0^t s\,dB_s)\big] = \int_\Omega (B_t + t^2)\,dQ = E^Q(X_t) = 0.$$

Example 19.3. Problem. *Compute*

$$E\Big[(B_t - t)^2 \, \exp\Big(\int_0^t e^{-s}\, dB_s\Big)\Big].$$

Solution.

$$\frac{dQ}{dP} = \exp\Big(-\tfrac{1}{2}\int_0^t e^{-2s}\, ds + \int_0^t e^{-s}\, dB_s\Big).$$

Under Q,

$$X_t = B_t - \int_0^t e^{-s}\, ds = B_t - 1 + e^{-t}$$

is a Q–Brownian motion. Hence

$$E\big[(B_t - t)^2\, e^{\int_0^t e^{-s} dB_s}\big] = \int_\Omega (B_t - t)^2\, e^{\frac{1}{2}\int_0^t e^{-2s} ds}\, dQ = e^{\frac{1}{2}\int_0^t e^{-2s} ds}\, E^Q\big[(B_t - t)^2\big].$$

Since

$$B_t - t = X_t + (1 - e^{-t} - t),$$

and $E^Q[X_t^2] = t$,

$$E^Q\big[(B_t - t)^2\big] = t + \big(1 - e^{-t} - t\big)^2.$$

Finally,

$$\int_0^t e^{-2s} ds = \tfrac{1}{2}\big(1 - e^{-2t}\big),$$

so the result is

$$\exp\Big(\tfrac{1}{4}(1 - e^{-2t})\Big)\,\big[t + (1 - e^{-t} - t)^2\big].$$

20 Stopping Times

Definition 20.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$. A random time*

$$\tau : \Omega \to [0, \infty]$$

is called a stopping time *(with respect to $(\mathcal{F}_t)$) if for every $t \geq 0$ the event $\{\tau \leq t\} \in \mathcal{F}_t$.*

20.1 Doob's Optional Stopping Theorem

Theorem 20.1 (Discrete–time OST)**.** *Let $(X_n)_{n\geq 0}$ be a martingale with respect to a filtration $(\mathcal{F}_n)$ and let τ be a stopping time. If one of the following holds:*

1. *τ is bounded, i.e. there exists $N < \infty$ such that $\tau(\omega) \leq N$ for all $\omega \in \Omega$;*
2. *$\sup_n |X_n(\omega)| < K < \infty$ for all n, ω (i.e. the martingale is uniformly bounded) and $\tau < \infty$ a.s.;*
3. *$\tau < \infty$ a.s. and there is $K < \infty$ such that $|X_n - X_{n-1}| < K$ for all n, ω,*

then X_τ is integrable and

$$E[X_\tau] = E[X_0].$$

Theorem 20.2 (Continuous–time OST)**.** *Let $(M_t)_{t\geq 0}$ be a right-continuous $\mathcal{F}_t$-martingale and τ a stopping time. If either*

- *τ is bounded, i.e. $\exists N < \infty$ such that $\tau \leq N$,*
- *or $\exists c > 0$ such that $\sup_{t\geq 0} E[|M_t|] \leq c$,*

then

$$E[M_\tau] = E[M_0].$$

Moreover, the stopped process $M_{t\wedge\tau}$ is itself a martingale.

Example 20.1 (Two-Barrier Hitting Probability for Brownian Motion)**.** *Let $(B_t)_{t\geq 0}$ be standard Brownian motion and fix two barriers $+U$ and $-D$ (with $U, D > 0$). Define the stopping time*

$$\tau = \inf\{t \geq 0 : B_t = +U \text{ or } B_t = -D\}.$$

Since $B_{t\wedge\tau}$ is bounded by $\max\{U, D\}$, by the optional-stopping theorem

$$E[B_{t\wedge\tau}] = E[B_0] = 0.$$

But $B_{t\wedge\tau}$ takes the value $+U$ with probability p and $-D$ with probability $1 - p$, so

$$p\,U \;-\; (1-p)\,D \;=\; 0 \quad\Longrightarrow\quad p \;=\; \frac{D}{U+D}.$$

Thus the probability of hitting the upper barrier first is

$$\boxed{p = \frac{D}{U+D}\,.}$$