Binary Search Trees: AVL Trees

Daniel Kane

Department of Computer Science and Engineering
University of California, San Diego

Data Structures
Data Structures and Algorithms

Learning Objectives

- Understand what the height of a node is.
- State the AVL property.
- Show that trees satisfying the AVL property have low depth.

Learning Objectives

- Implement AVL trees.
- Understand the cases required for rebalancing algorithms.

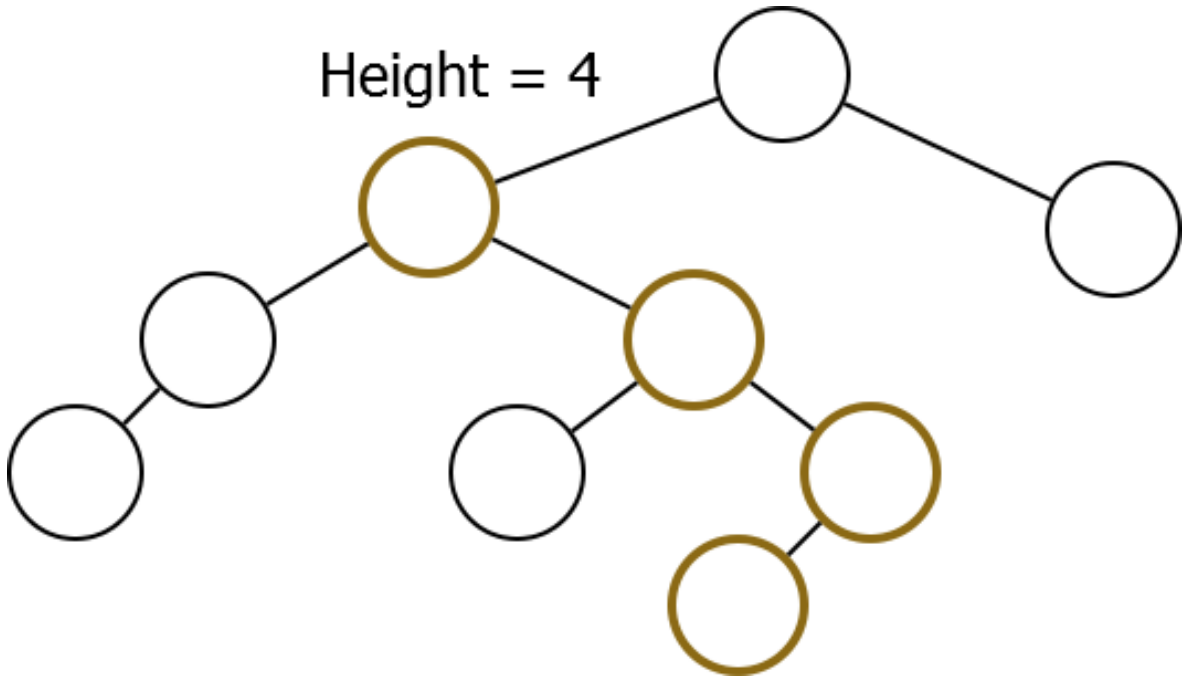
Balance

- Want to maintain balance.
- Need a way to measure balance.

Height

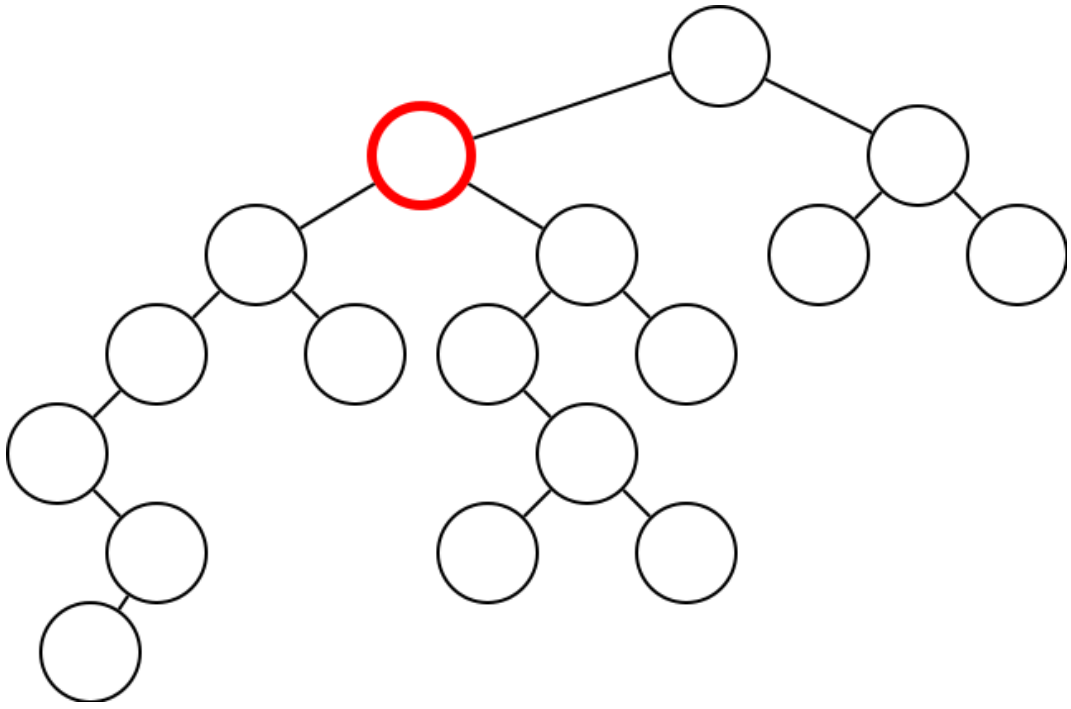
Definition

The height of a node is the maximum depth of its subtree.



Problem

What is the height of the selected node?



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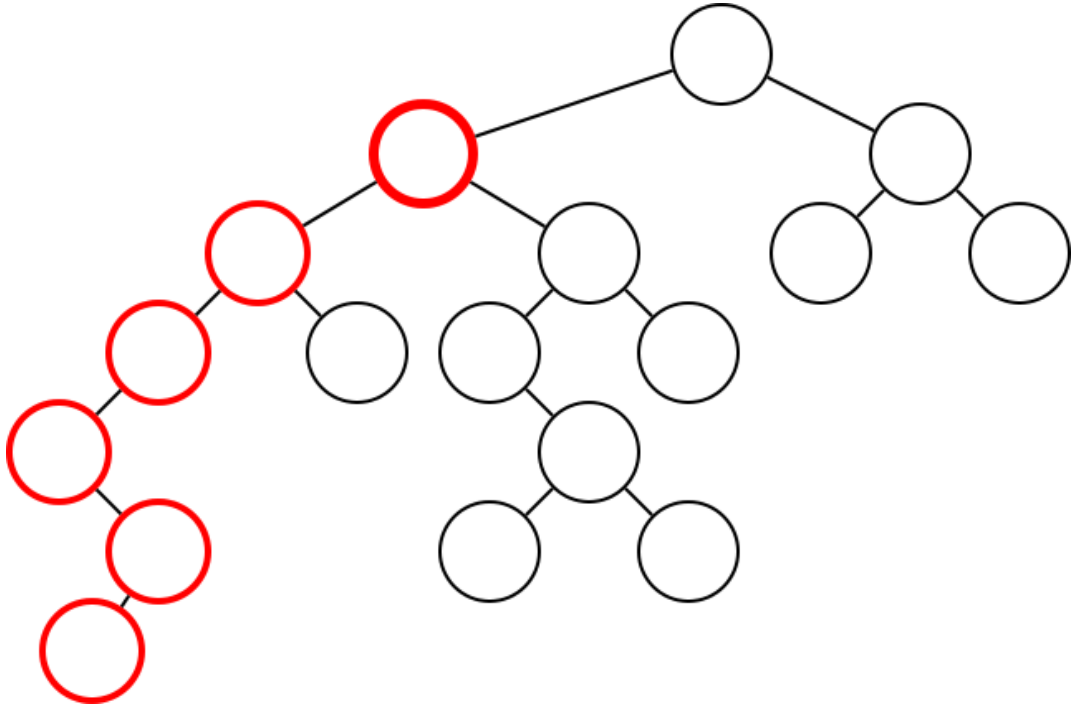


Recursive Definition

N.`Height` equals

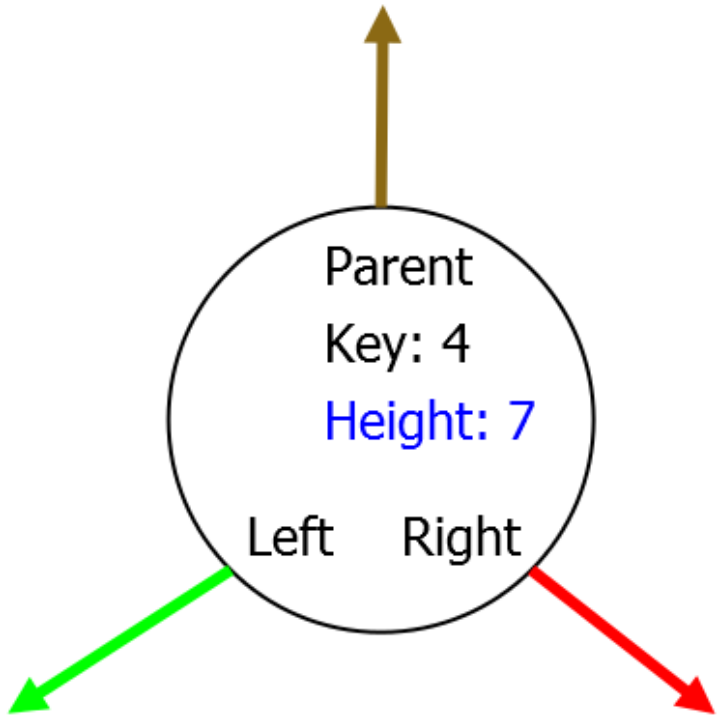
1 if N is a leaf,

$1 + \max(N.\texttt{Left.Height}, N.\texttt{Right.Height})$

otherwise.

Field

Add height field to nodes.



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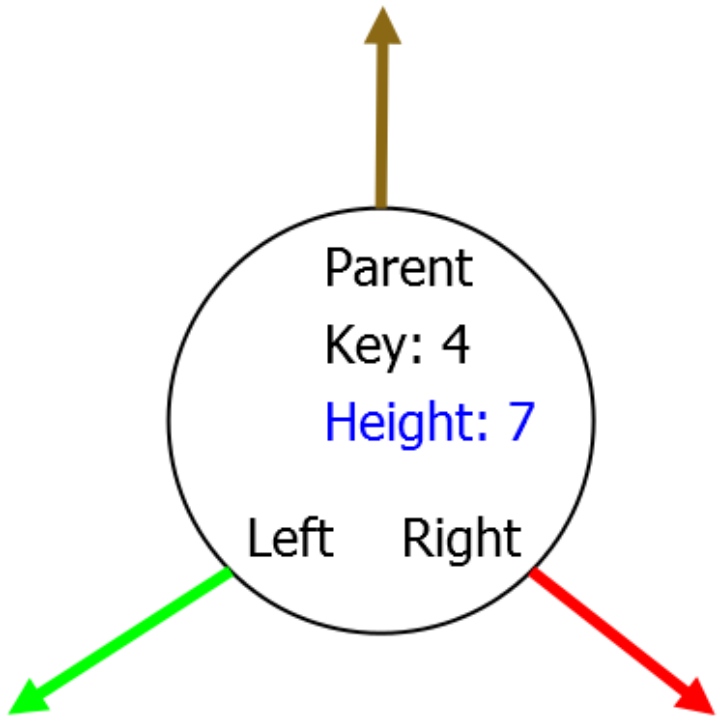

(Note: We'll have to work to ensure that this is kept up to date)

Balance

- Height is a rough measure of subtree size.
- Want size of subtrees roughly the same.
- Force heights to be roughly the same.

AVL Property

AVL trees maintain the following property:
For all nodes N,

$$|N.\texttt{Left.Height} - N.\texttt{Right.Height}| \leq 1$$

We claim that this ensures balance.

Idea

Need to show that AVL property implies $\mathrm{Height} = O(\log(n))$.

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Alternatively, show that large height implies many nodes.

Result

Theorem

Let N be a node of a binary tree satisfying the AVL property. Let $h = N.\texttt{Height}$. Then the subtree of N has size at least the Fibonacci Number F_h.

Recall

$$F_n = \begin{cases} 0, & n = 0\,, \\ 1, & n = 1\,, \\ F_{n-1} + F_{n-2}, & n > 1\,. \end{cases}$$

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$F_n \geq 2^{n/2}$ for $n \geq 6$.

Proof

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By induction on h.

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If $h = 1$, have one node.

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By induction on h.

If $h = 1$, have one node.

Otherwise, have one subtree of height $h - 1$ and another of height at least $h - 2$.

By inductive hypothesis, total number of nodes is at least $F_{h-1} + F_{h-2} = F_h$. □

Large Subtrees

So node of height h has subtree of size at least $2^{h/2}$.

In other words, if n nodes in the tree, have height $h \leq 2\log_2(n) = O(\log(n))$.

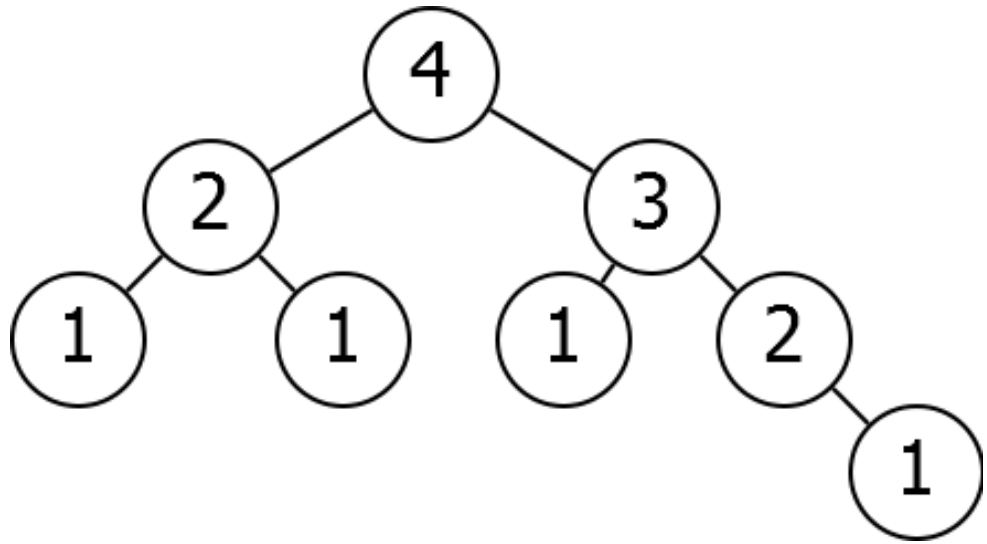
Conclusion

AVL Property

If you can maintain the AVL property, you can perform operations in $O(\log(n))$ time.

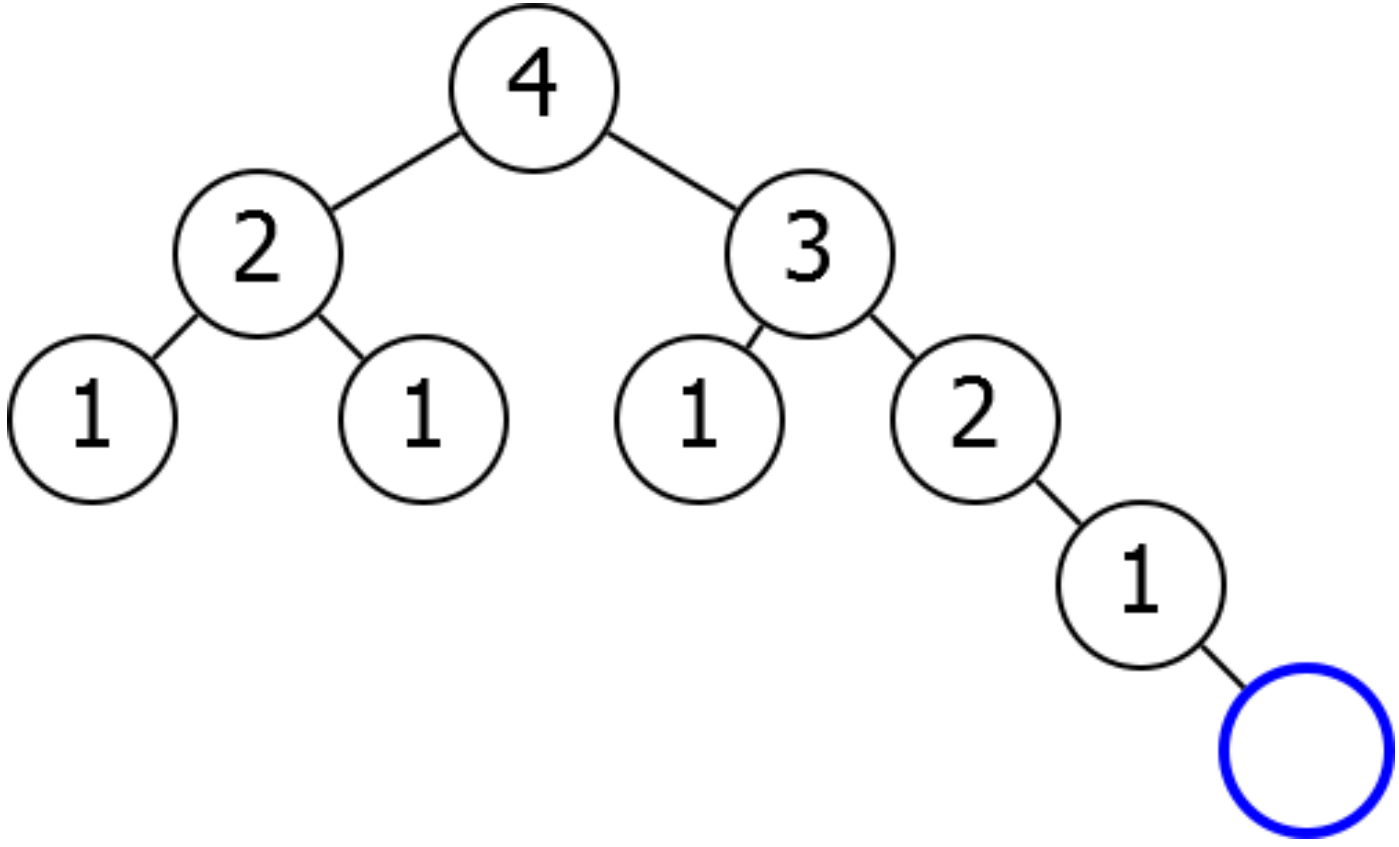
AVL Trees

Need ensure that children have nearly the same height.



Problem

Updates to the tree can destroy this property.



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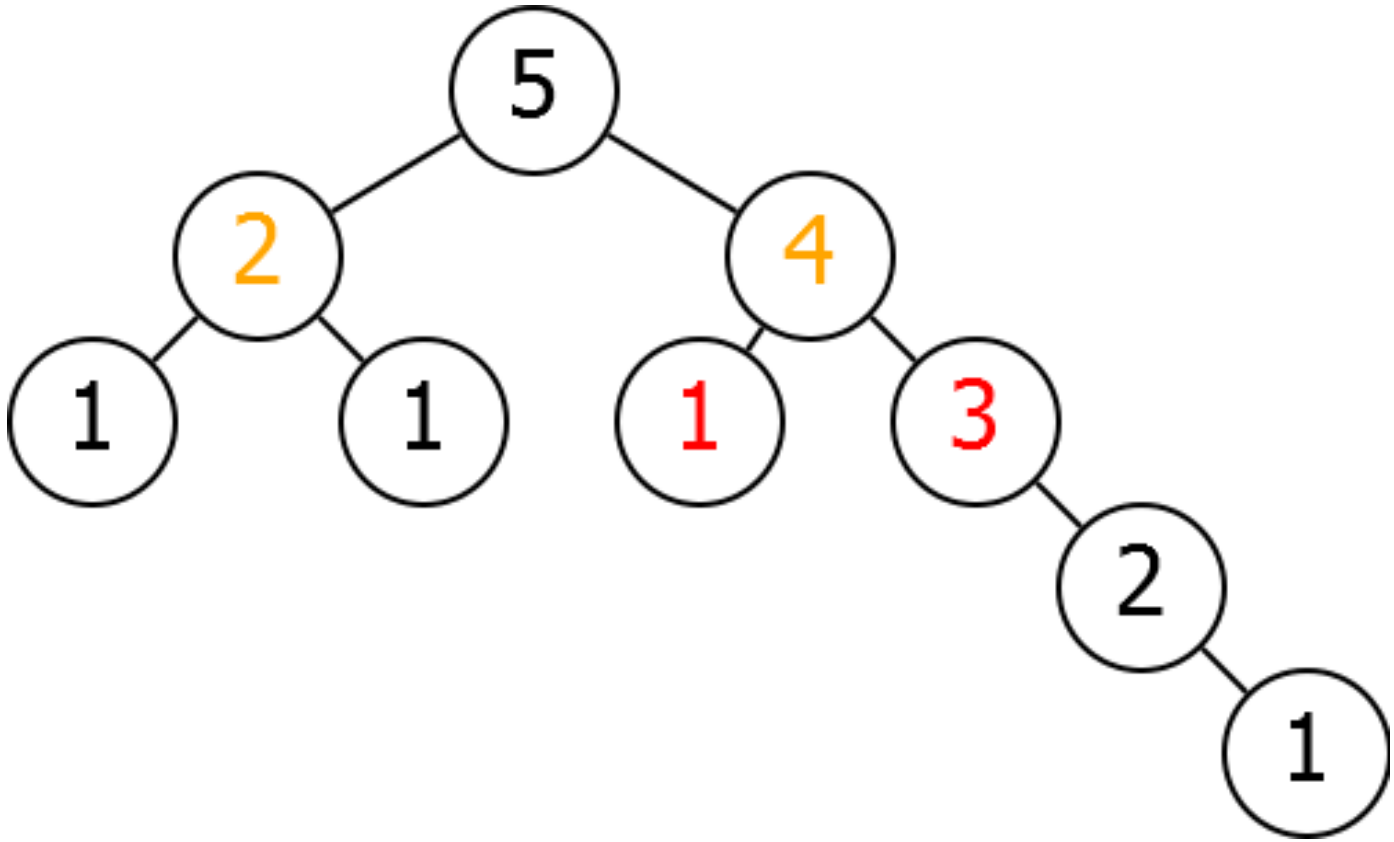
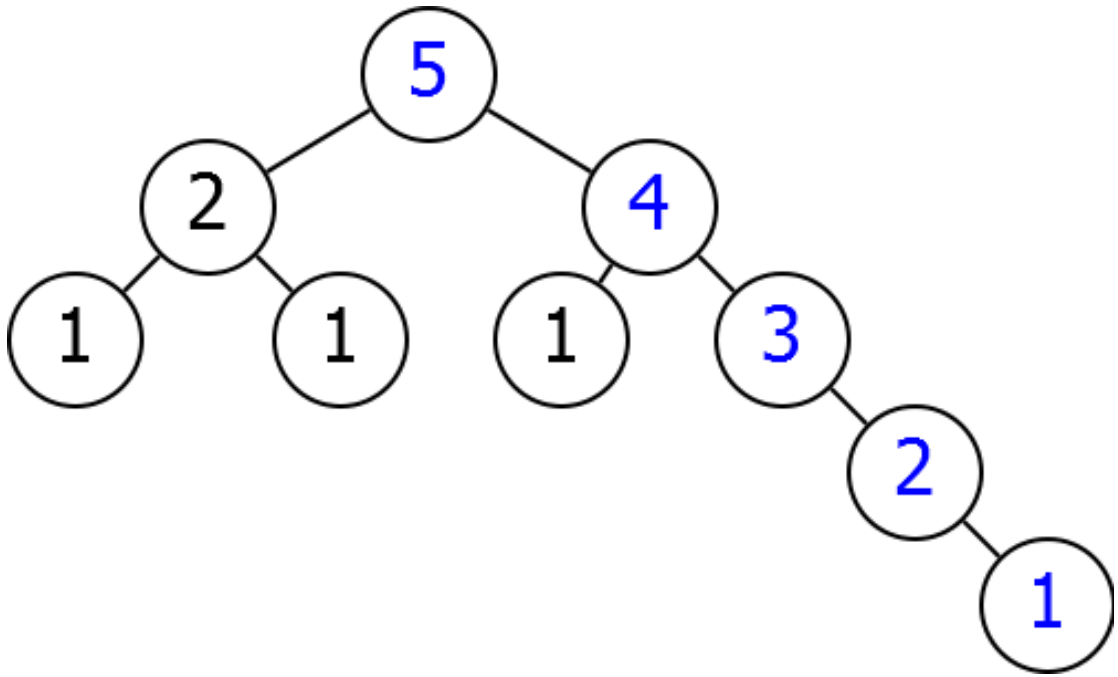


Need to correct this.

Errors

Heights stay the same except on the insertion path.



Only need to worry about this path.

Insertion

We need a new insertion algorithm that involves rebalancing the tree to maintain the AVL property.

Idea

AVLInsert(k, R)

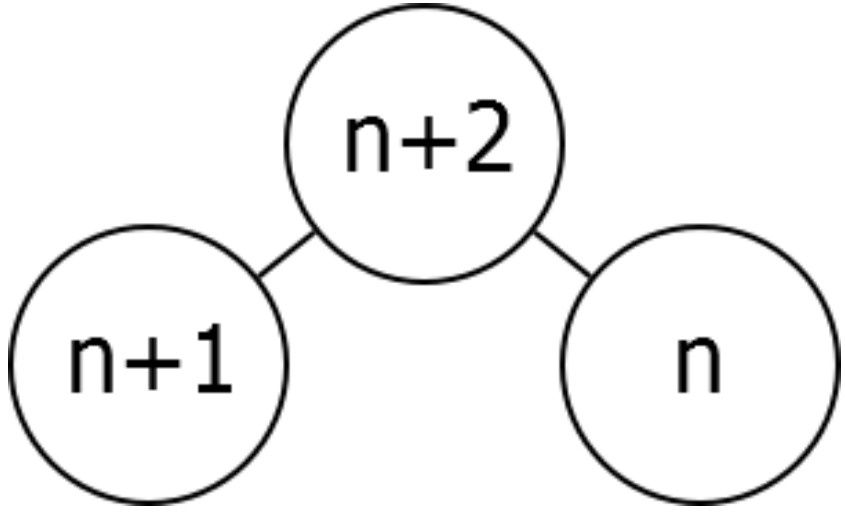
Insert(k, R)
$N \leftarrow$ Find(k, R)
Rebalance(N)

Rebalancing

If

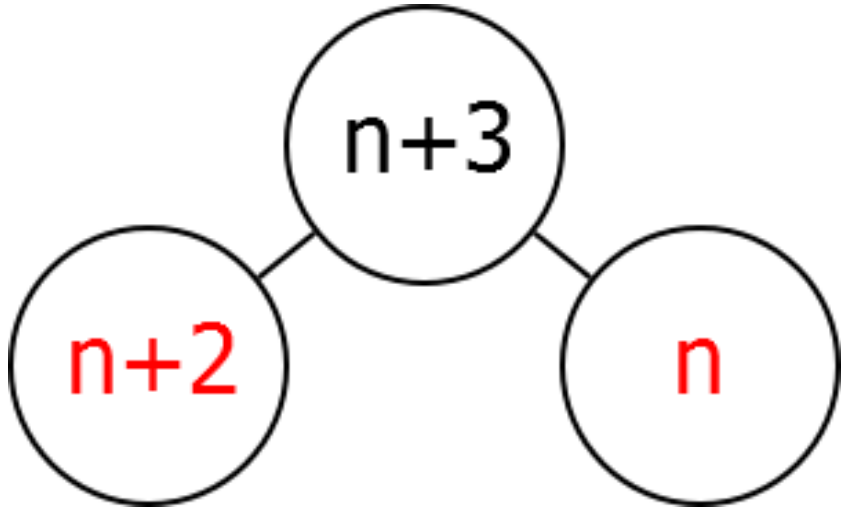
$$|N.\texttt{Left.Height} - N.\texttt{Right.Height}| \leq 1$$

fine.



Problem

Difficulty if heights differ by more.



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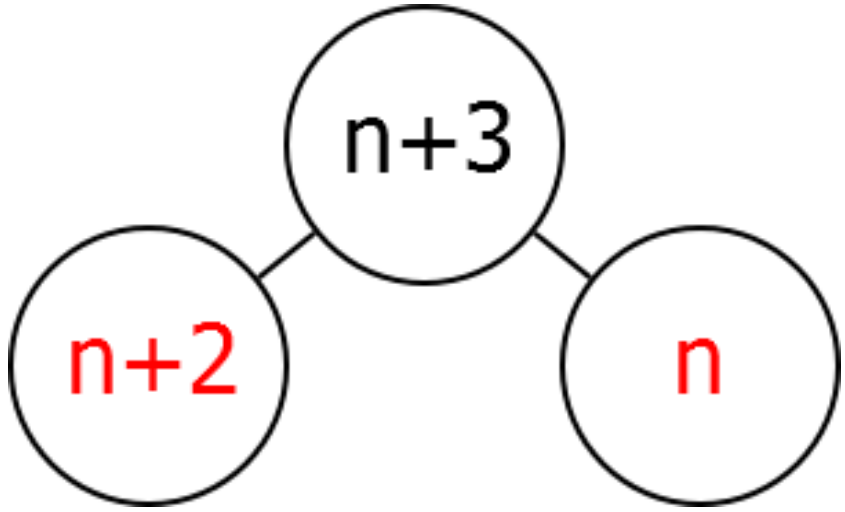


Never more than 2.

Code

Rebalance(N)

```
P ← N.Parent
if N.Left.Height > N.Right.Height+1:
  RebalanceRight(N)
if N.Right.Height > N.Left.Height+1:
  RebalanceLeft(N)
AdjustHeight(N)
if P ≠ null:
  Rebalance(P)
```

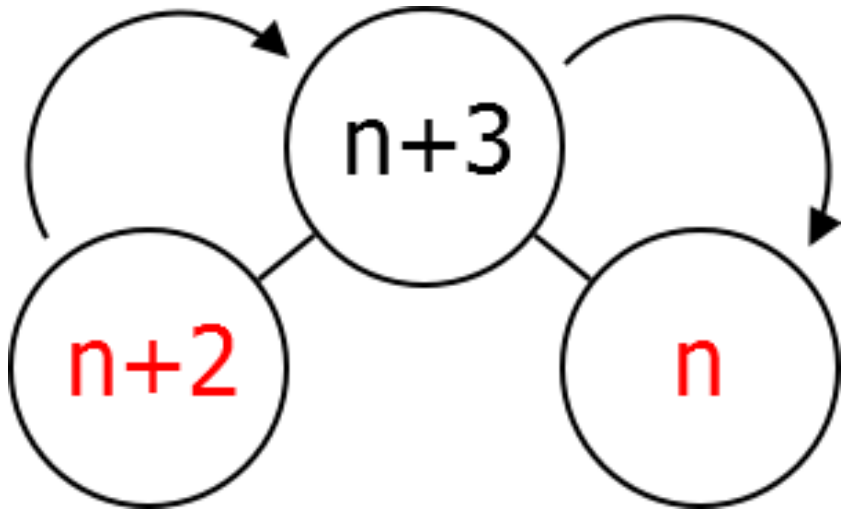
Adjust Height

AdjustHeight(N)

$$N.\texttt{Height} \leftarrow 1 + \max(N.\texttt{Left.Height}, N.\texttt{Right.Height})$$

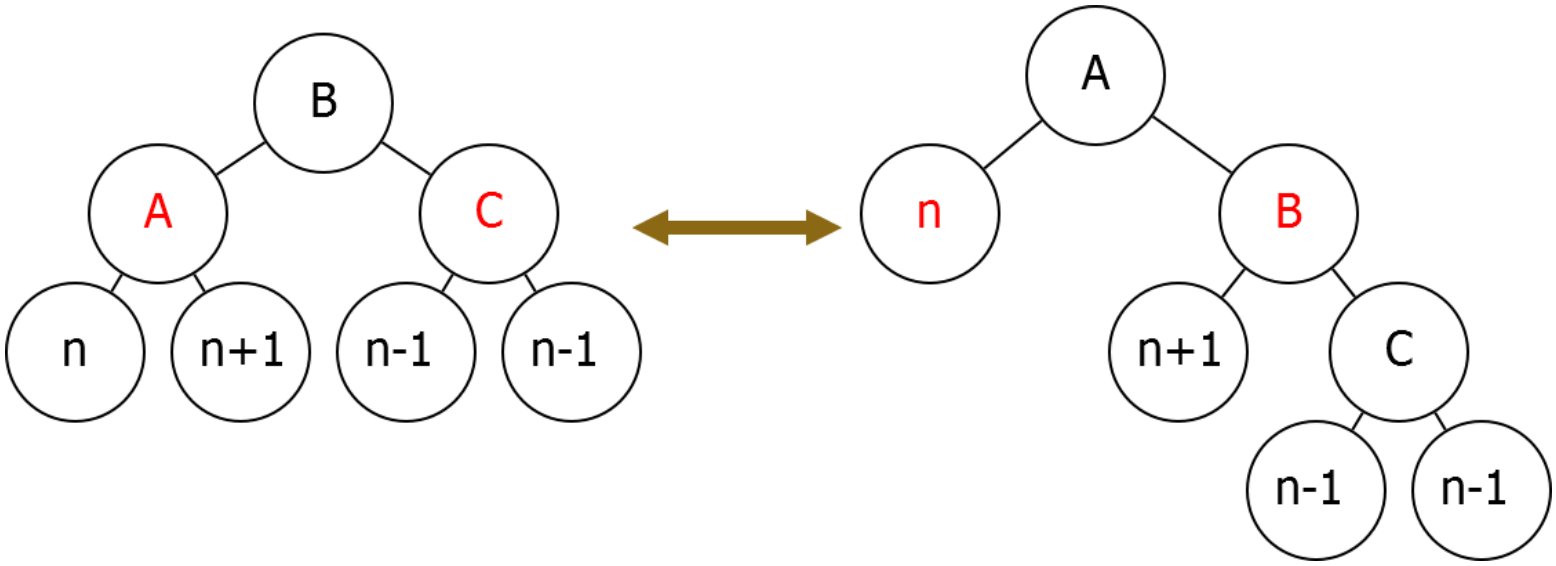
Rebalancing

If left subtree too heavy, rotate right:



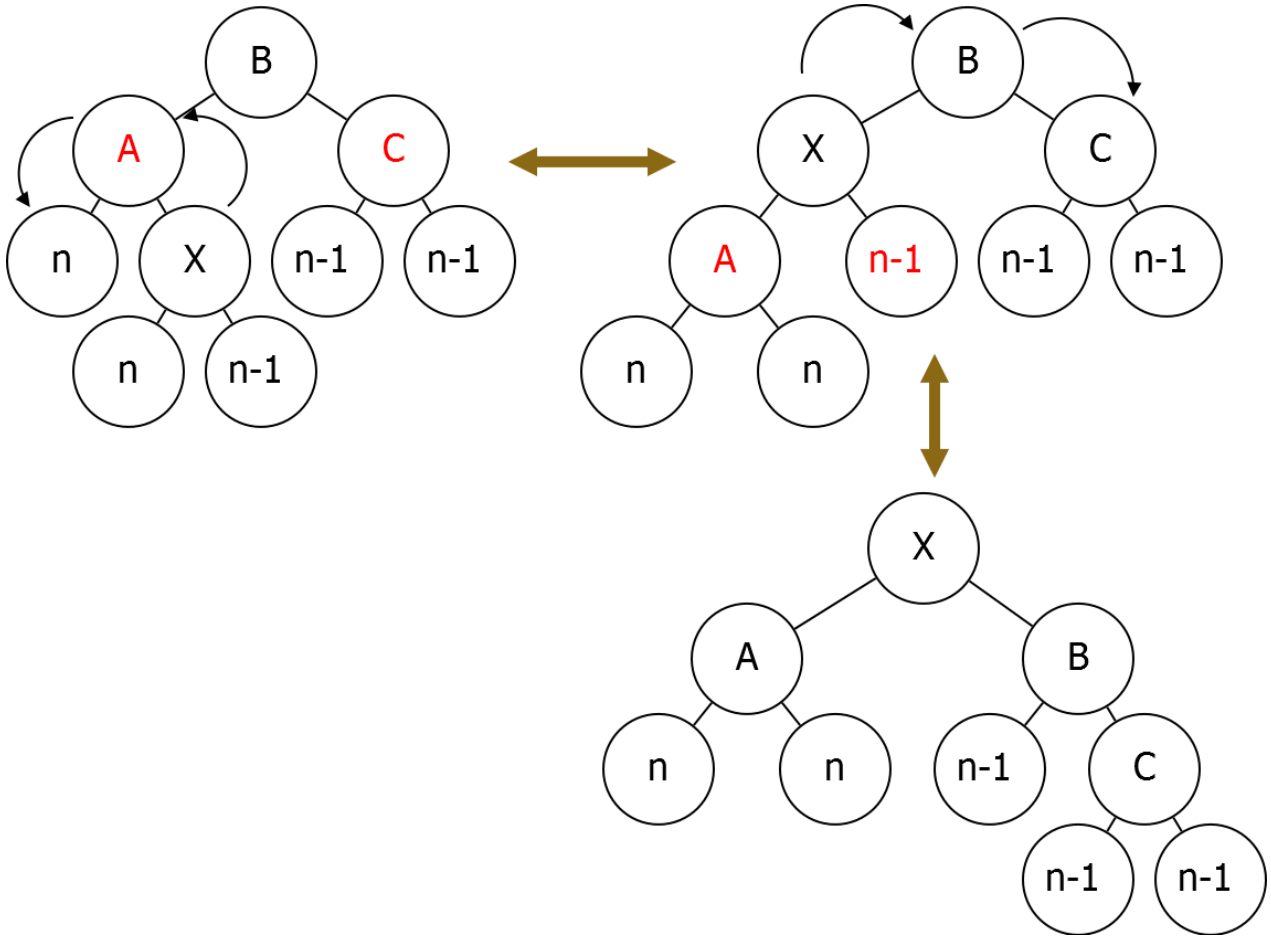
Bad Case

Doesn't work in this case.



Fix

Must rotate left first.



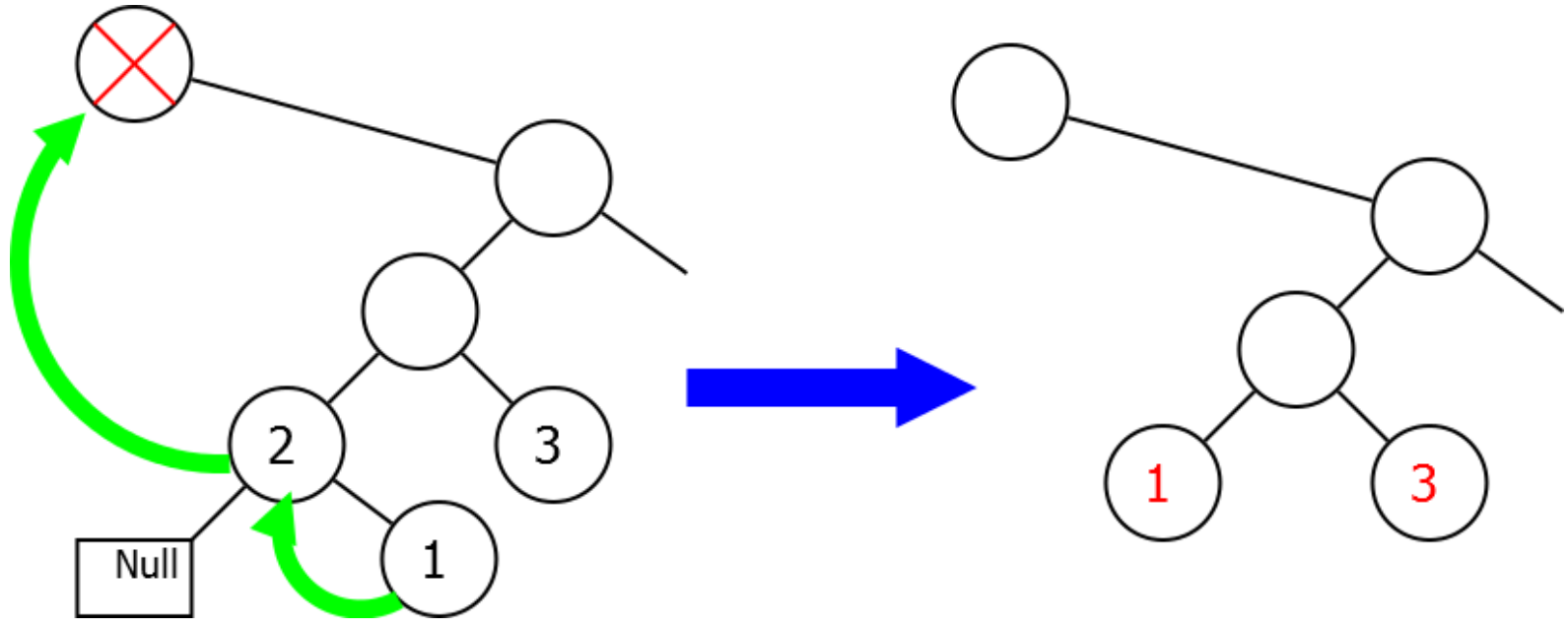
Rebalance

RebalanceRight(N)

$M \leftarrow N$.Left
if M.Right.Height $>$ M.Left.Height:
  RotateLeft(M)
RotateRight(N)
AdjustHeight on affected nodes

Delete

Deletions can also change balance.



New Delete

AVLDelete(N)

Delete(N)
$M \leftarrow$ Parent of node replacing N
Rebalance(M)

Conclusion

Summary

AVL trees can implement all of the basic operations in $O(\log(n))$ time per operation.