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Encounters between binary and single stars and their effect on the dynamical evolution of stellar systems

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More than 1.4×10^4 computer-simulated encounters between binaries and field stars have been run for various preencounter relative velocities V_f and impact parameters p. At low values of V_f, the encounter causes the binary orbit to shrink and become more tightly bound with the two more massive of the three stars involved in the encounter remaining in the binary and the third being ejected from the binary at a relatively high velocity. As V_f increases, the amount of orbit shrinking decreases until, for V_f greater than a well-defined velocity V_E, the encounter tends to increase the semimajor axis of the binary and eventually to break it apart. We find that about 10% of the binaries with semimajor axes $a_0 > 10^3$ AU have broken apart in the solar neighborhood. In globular clusters almost all primordial binaries with $a_0 > 35$ AU have broken apart; this is true for binaries with $a_0 > 100$–1000 AU in open clusters and $a_0 > 1$–2 AU in galactic nuclei. The breakup of these loosely bound binaries has relatively little effect on the dynamical evolution of the stellar system. However, the shrinking of the more tightly bound binary orbits, which feeds kinetic energy into the cluster, will destroy an open cluster in a few times 10^8 yr unless it is either unusually massive or unusually deficient in binaries. While the lifetime of a globular cluster due to this mechanism is about 10^{11} yr, it is likely that binary stars limit the maximum density attainable in the nucleus of a globular cluster. The presence of the binaries transforms the familiar instability leading to a rapid shrinking of the core of the cluster and to the buildup of a high-density central cusp into one which causes an accelerated rate of decrease in the semimajor axes of the binaries in the core. Binaries are not likely to have any significant effect on the dynamical evolution of a dense galactic nucleus.

INTRODUCTION

SEVERAL investigators (cf. Aarseth 1968; Van Albeda 1968; Aarseth and Hills 1972) have found by accurate computer simulations of stellar clusters that binary stars play a prominent role in the evolution of open clusters. Although the computer-simulated clusters initially contain no binaries, they are eventually produced in multiple-star encounters. Further encounters between these newly formed binaries and the other stars in the cluster generally tend to decrease the separation of the binary components until in a well-evolved cluster the most tightly bound binaries may contain up to 90% of all the potential energy in the cluster. The increase in the binding energy of the binaries is largely at the expense of the binding energy of the cluster. Thus these encounters between binaries and field stars tend toward the destruction of the cluster.

In this paper we investigate the effect of preexisting binaries on the evolution of stellar clusters. Since about 70% of all stars are actually binary or multiple-star systems (Batten 1972) and the binding energy of these systems in either an open or a globular cluster is vastly greater than the binding energy of the cluster as a whole, it is evident that even a relatively small transfer of energy between the binaries and the cluster can profoundly affect the dynamical evolution of the cluster. Their effect on the evolution of the cluster may be anticipated to be more severe than that judged by computer-simulated clusters where the only binaries present are those formed by three-body encounters. It is not practical to computer simulate a cluster containing a realistic initial distribution of binary stars. Fortunately, the effect of a field star on a binary is predominantly tidal so that it can affect the binary strongly only during a very close encounter. Thus, the problem is reduced to studying single encounters between binaries and field stars. Our strategy is to find the average result of such an encounter as a function of impact parameter and preencounter velocity. We can then use this data to compute the lifetimes of stellar clusters due to energy transfer between the binaries and the other stars in the cluster.

In the next section we discuss the procedure used in computer-simulating encounters between binaries and field stars. In Sec. II we summarize the principal characteristics of such encounters for zero impact parameter while in Sec. III we discuss the variation of these characteristics with increasing impact parameter and use this to calculate the collision cross sections associated with these characteristics. In Sec. IV we calculate the lifetime of the binaries due to encounters with single stars. In Sec. V we discuss the dynamical properties of those binaries which have reached approximate statistical equilibrium. In Sec. VI we use the results of the previous sections to calculate the effect of binary stars on the dynamical evolution of stellar clusters.

I. COMPUTER SIMULATIONS OF ENCOUNTERS BETWEEN BINARIES AND FIELD STARS

The basic results of the computer simulations of 1.4×10^4 encounters are summarized in Table I and the several figures presented in this section.

The equations of motion of the three bodies involved in an encounter were numerically integrated using a fourth-order Runge–Kutta algorithm in which the step size in time was governed by the shortest distance between any pair of stars. The median accumulated error in the energy of a three-star system after the completion of an encounter was about 10^{-4} of the initial energy E_1, but in no case did this exceed 10^{-2} E_1. In each encounter the calculations were performed analytically, treating the binary as a single star, up to the point where the distance between the binary and the field star was 60 times the initial separation a_0 of the binary components. The subsequent evolution was then followed by integrating the exact three-body equations. These calculations were in turn terminated when any two of the three stars again became separated by more than $60a_0$.

In all cases the binary orbit was circular at the beginning of the integration. Three families of models were run. In family A all three stars have the same mass, $M_1=M_2=M_3=M_\odot$, and the initial semimajor axis of the binary is $a_0=10$ AU. Family B is similar

TABLE I. Encounter between a binary and a field star.

Series number	V_f (km/sec)	α	P/a_0	$\Delta E/E_0$	e	P.D.	P.E.	Total runs
Family A: masses = (1:1:1), $a_0=10$ AU, $V_c=11.53$ km/sec.								
A1	0	0	0	-0.475 ± 0.026	0.637 ± 0.019	0.00	0.89	170
A2	1	0.008	0	-0.493 ± 0.014	0.627 ± 0.013	0.00	0.89	444
A3	1	0.008	4	-0.489 ± 0.040	0.656 ± 0.026	0.00	0.86	83
A4	2	0.030	0	-0.486 ± 0.022	0.629 ± 0.014	0.00	0.86	279
A5	2	0.030	2	-0.365 ± 0.022	0.564 ± 0.016	0.00	0.87	224
A6	2	0.030	4	-0.341 ± 0.021	0.726 ± 0.013	0.00	0.83	219
A7	4	0.12	0	-0.449 ± 0.018	0.616 ± 0.010	0.00	0.92	418
A8	5	0.19	0	-0.400 ± 0.033	0.639 ± 0.035	0.00	0.93	138
A9	5	0.19	2	-0.215 ± 0.043	0.748 ± 0.027	0.00	0.88	67
A10	5	0.19	4	-0.283 ± 0.132	0.561 ± 0.048	0.00	0.83	12
A11	6	0.27	0	-0.328 ± 0.025	0.640 ± 0.010	0.00	0.97	307
A12	7.5	0.42	0	-0.031 ± 0.020	0.610 ± 0.030	0.00	0.99	87
A13	7.5	0.42	2	-0.050 ± 0.071	0.644 ± 0.042	0.00	0.40	38
A14	9	0.61	0	$+0.087\pm0.040$	0.597 ± 0.028	0.00	0.99	78
A15	9	0.61	2	-0.073 ± 0.071	0.640 ± 0.042	0.00	0.39	37
A16	10.9	0.89	0	$+0.191\pm0.049$	0.688 ± 0.021	0.00	1.00	75
A17	10.9	0.89	2	$+0.188\pm0.051$	0.482 ± 0.031	0.00	0.16	64
A18	10.9	0.89	2.75	$+0.130\pm0.015$	0.307 ± 0.01	0.00	0.12	120
A19	10.9	0.89	3.5	$+0.002\pm0.006$	0.154 ± 0.014	0.00	0.00	98
A20	12	1.08	0	$+0.238\pm0.035$	0.682 ± 0.015	0.015	1.00	198
A21	12	1.08	2	$+0.182\pm0.030$	0.477 ± 0.015	0.005	0.17	223
A22	12	1.08	3	$+0.028\pm0.018$	0.202 ± 0.016	0.00	0.01	106
A23	13	1.27	0	$+0.216\pm0.040$	0.694 ± 0.018	0.007	1.00	137
A24	13	1.27	2	$+0.192\pm0.032$	0.479 ± 0.017	0.005	0.18	192
A25	15.0	1.69	0	$+0.401\pm0.032$	0.774 ± 0.014	0.11	1.00	200
A26	15.2	1.74	0	$+0.427\pm0.047$	0.801 ± 0.017	0.09	1.00	92
A27	15.2	1.74	1	$+0.239\pm0.160$	0.773 ± 0.038	0.25	0.33	24
A28	15.2	1.74	2	$+0.152\pm0.030$	0.362 ± 0.019	0.06	0.08	200
A29	19.4	2.8	0	$+0.764\pm0.024$	0.920 ± 0.011	0.32	1.00	132
A30	19.4	2.8	1	$+0.310\pm0.048$	0.669 ± 0.015	0.25	0.03	199
A31	25	4.7	0	$+0.980\pm0.002$	0.714 ± 0.120	0.97	1.00	200
A32	25	4.7	0.2	$+0.979\pm0.009$	0.920 ± 0.041	0.90	0.50	100
A33	25	4.7	0.4	$+0.922\pm0.019$	0.932 ± 0.013	0.79	0.24	100
A34	25	4.7	0.6	$+0.708\pm0.045$	0.755 ± 0.025	0.55	0.18	100
A35	25	4.7	0.8	$+0.390\pm0.065$	0.700 ± 0.025	0.31	0.014	100
A36	25	4.7	1.0	$+0.221\pm0.058$	0.522 ± 0.023	0.16	0.00	100
A37	25	4.7	1.2	$+0.143\pm0.044$	0.409 ± 0.018	0.09	0.00	100
A38	25	4.7	1.5	$+0.019\pm0.030$	0.274 ± 0.014	0.01	0.00	100
A39	25	4.7	2.0	$+0.034\pm0.018$	0.160 ± 0.011	0.00	0.00	100
A40	34.6	9.0	0	$+0.974\pm0.010$	0.941 ± 0.050	0.88	0.14	137
A41	34.6	9.0	0.75	$+0.210\pm0.041$	0.543 ± 0.017	0.20	0.00	199
A42	34.6	9.0	1.50	$+0.010\pm0.019$	0.195 ± 0.010	0.00	0.00	149
A43	49.7	18.6	0	$+0.817\pm0.020$	0.901 ± 0.005	0.60	0.00	199
A44	49.7	18.6	1	$+0.141\pm0.028$	0.301 ± 0.17	0.00	0.00	122
A45	75	42.3	0	$+0.548\pm0.028$	0.732 ± 0.007	0.28	0.00	200
A46	100	75	0	$+0.421\pm0.048$	0.616 ± 0.016	0.27	0.00	100
A47	100	75	0.2	$+0.302\pm0.052$	0.611 ± 0.015	0.21	0.00	100
A48	100	75	0.4	$+0.289\pm0.052$	0.487 ± 0.025	0.20	0.00	100
A49	100	75	0.6	$+0.107\pm0.037$	0.325 ± 0.019	0.03	0.00	100
A50	110	91	0	$+0.405\pm0.041$	0.574 ± 0.015	0.18	0.00	100
A51	200	301	0	$+0.142\pm0.012$	0.374 ± 0.005	0.063	0.00	958
A52	200	301	0.5	$+0.152\pm0.029$	0.264 ± 0.017	0.07	0.00	162
A53	200	301	1.0	$+0.010\pm0.044$	0.065 ± 0.002	0.00	0.00	200
A54	200	301	1.5	$+0.004\pm0.002$	0.028 ± 0.001	0.00	0.00	200
A55	200	301	2.0	$+0.002\pm0.001$	0.016 ± 0.001	0.00	0.00	200
A56	500	1880	0	$+0.023\pm0.014$	0.186 ± 0.007	0.02	0.00	300

TABLE I (*continued*)

Series number	V_f (km/sec)	α	P/a_0	$\Delta E/E_0$	e	P.D.	P.E.	Total runs
Family B: masses = (1:1:3), a_0 = 10 AU, V_c = 8.60 km/sec.								
B1	0	0.00	0	-2.287 ± 0.114	0.661 ± 0.012	0.00	1.00	195
B2	1	0.014	0	-2.340 ± 0.040	0.628 ± 0.022	0.00	1.00	398
B3	2	0.054	0	-2.509 ± 0.129	0.662 ± 0.013	0.00	1.00	197
B4	4	0.22	0	-2.294 ± 0.110	0.675 ± 0.020	0.00	1.00	81
B5	7.5	0.76	0	-2.255 ± 0.099	0.681 ± 0.018	0.00	1.00	107
B6	10	1.35	0	-1.922 ± 0.105	0.690 ± 0.014	0.02	1.00	194
B7	15	3.05	0	-1.094 ± 0.125	0.727 ± 0.017	0.08	1.00	143
B8	20	5.41	0	-0.332 ± 0.034	0.792 ± 0.011	0.29	1.00	248
B9	25	8.46	0	$+0.238\pm0.047$	0.853 ± 0.018	0.59	1.00	100
B10	25	8.46	1	$+0.773\pm0.049$	0.730 ± 0.043	0.74	0.04	100
B11	25	8.46	2	$+0.288\pm0.052$	0.435 ± 0.024	0.18	0.00	100
B12	35	16.6	0	$+0.988\pm0.011$	0.920 ± 0.019	0.92	1.00	200
B13	60	48.7	0	$+1.00\ \pm0.00$	... ...	1.00	...	100
B14	60	48.7	1	$+0.360\pm0.057$	0.546 ± 0.025	0.26	0.00	100
B15	60	48.7	2	$+0.018\pm0.016$	0.156 ± 0.008	0.00	0.00	100
B16	100	135	0	$+1.00\ \pm0.00$	... ...	1.00	...	100
B17	150	305	0	$+0.891\pm0.021$	0.932 ± 0.006	0.68	0.00	100
B18	250	846	0	$+0.509\pm0.030$	0.708 ± 0.008	0.28	0.00	300
Family C: masses (3:3:1), a_0 = 10 AU, V_c = 30.51 km/sec.								
C1	0	0.00	0	-0.247 ± 0.026	0.567 ± 0.022	0.00	0.04	132
C2	5	0.027	0	-0.276 ± 0.033	0.584 ± 0.016	0.00	0.04	222
C3	10	0.107	0	-0.158 ± 0.023	0.598 ± 0.019	0.00	0.02	168
C4	15	0.240	0	-0.016 ± 0.028	0.608 ± 0.016	0.00	0.03	190
C5	20	0.43	0	$+0.327\pm0.050$	0.694 ± 0.038	0.00	0.04	23
C6	20	0.43	1	-0.101 ± 0.048	0.580 ± 0.025	0.00	0.09	85
C7	20	0.43	2	$+0.006\pm0.020$	0.201 ± 0.015	0.00	0.02	96
C8	15	0.67	0	$+0.349\pm0.014$	0.652 ± 0.010	0.00	0.01	195
C9	50	2.7	0	$+0.293\pm0.032$	0.558 ± 0.013	0.09	0.00	100
C10	50	2.7	1	$+0.056\pm0.022$	0.201 ± 0.017	0.00	0.00	100
C11	50	2.7	2	$+0.001\pm0.003$	0.038 ± 0.002	0.00	0.00	100
C12	100	10.7	0	$+0.101\pm0.030$	0.362 ± 0.013	0.03	0.00	100
C13	200	43.0	0	$+0.049\pm0.020$	0.235 ± 0.010	0.01	0.00	200
C14	400	172	0	$+0.029\pm0.017$	0.130 ± 0.007	0.01	0.00	200
C15	800	688	0	$+0.003\pm0.008$	0.069 ± 0.006	0.00	0.00	200

except that the mass of the field star is $M_3=3\ M_\odot$. In Family C the masses of the binary components are $M_1=M_2=3\ M_\odot$, while $M_3=1\ M_\odot$. Each family of models is subdivided into a number of series. In a given series the field star has a particular impact parameter p and preencounter velocity V_f with respect to the binary. A large number of runs was made in each series. Each run, a separate computer-simulated encounter between the binary and the field star, was specified by giving the binary a random new orbit orientation and phase in its orbit with respect to the impacting field star.

Table I gives the average postencounter results for all the runs in each series. It is particularly useful to give these results in dimensionless units. Columns 3 and 4 of the table tabulate V_f and p, which specify each series, in dimensionless units. Here, p is in units of the binary semimajor axis a_0 and V_f is in units of V_c, the minimum value of V_f required for the field star to be energetically capable of dissociating the binary. Here,

$$V_c=\left[\frac{GM_1M_2(M_1+M_2+M_3)}{M_3(M_1+M_2)a_0}\right]^{\frac{1}{2}}. \tag{1}$$

The ratio of the kinetic energy of the field star at infinity to the minimum kinetic energy required to dissociate the binary is just

$$\alpha\equiv(\text{K.E.})/(\text{K.E.})_0=(V_f/V_c)^2.$$

Column 5 of the table gives the average change $\langle\Delta E/E_0\rangle$, in the binding energy of the binary system in units of its initial binding energy, $E_0=GM_1M_2/2a_0$. If ΔE is positive, the semimajor axis of the binary has, on the average, increased while if it is negative the semimajor has, on the average, decreased. In particular, if $\langle\Delta E/E_0\rangle=1$, each computer run in the series terminates in the destruction of the binary system while if $\langle\Delta E/E_0\rangle=-1$, the average binding energy of the binaries after the encounter is twice as great as before the encounter. Column 6 gives the average eccentricity of the system after each encounter. The errors quoted for $\langle\Delta E/E_0\rangle$ and e are the standard errors of the mean. Column 7 gives P.D., the dissociation probability, the fraction of the binaries which are completely broken apart by the encounter. Column 8 gives P.E., the exchange probability, the fraction of the remaining undissociated binaries in which the field star is now a binary member and one of the original binary components has escaped to infinity. The last column gives the number of different encounters run in each series.

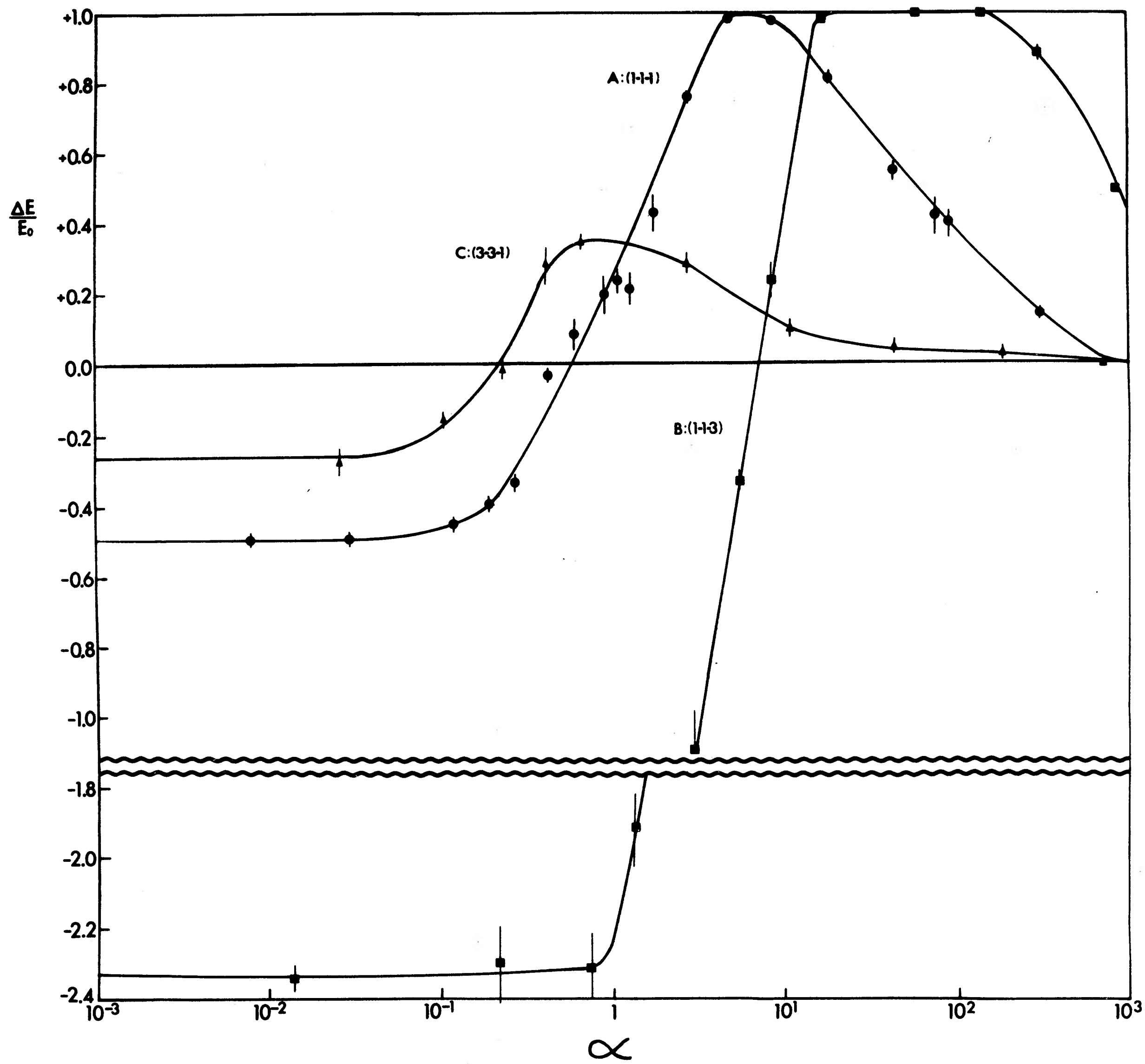

FIG. 1. The average fractional change in the binding energy of a binary due to an encounter with a field star at zero impact parameter. This is plotted as a function of the kinetic energy parameter $\alpha \equiv (V_f/V_c)^2$, where V_f is the velocity of the field star with respect to the binary prior to the encounter and V_c is the critical value of V_f at which the field star has just merely enough kinetic energy to break apart the binary. This is plotted for families A, B, and C.

We note that because only linear operators appear in the equations of motion, any encounter between a binary and a field star in which all three stars have the same masses, irrespective of the specific masses and semimajor axis of the initial binary orbit, will have the same dimensionless output variables $\langle \Delta E/E_0 \rangle$, e, P.D., and P.E. as those found for the particular series of family A having the same dimensionless input parameters $\alpha \equiv (V_f/V_c)^2$ and P/a_0. Thus, family A gives the general case for any encounter between a binary and a field star in which the masses are in the ratio (1:1:1). Correspondingly, family B is the general case for mass ratios (1:1:3).

In most of the series listed in Table I $P/a_0=0$, so that in the absence of any gravitation deflection by the binary components the field star would pass through the center of mass of the binary. All four dimensionless output parameters listed in Table I approach zero at least as fast as a Gaussian at large values of P/a_0. The width of the Gaussian in P/a_0 increases with decreasing α due to the increasing importance of gravitational focusing in bringing about a close encounter between the binary and the field star. Since this behavior is quantitatively predictable, it is only necessary to determine for each output variable the width of the Gaussian in P/a_0 at a few values of α in

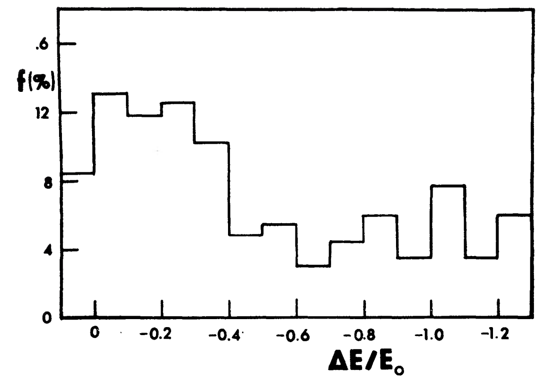

FIG. 2. The normalized distribution of $\Delta E/E_0$ among all the runs of series A4.

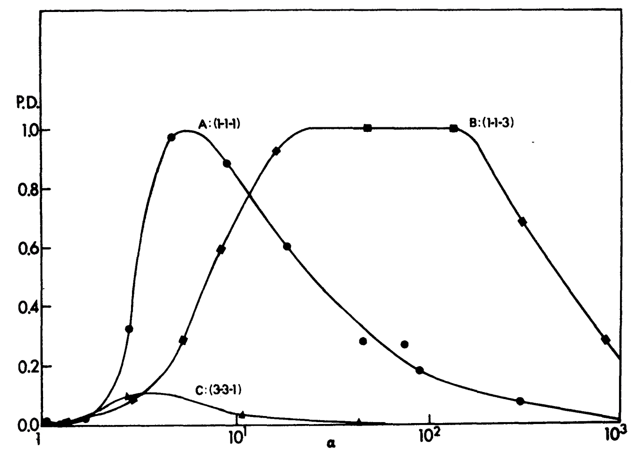

FIG. 3. The dissociation probability, the fraction of the binaries broken apart due to an encounter with a field star at zero impact parameter.

order to know its value at other values of α. This greatly simplifies our work by reducing the problem from a two-dimensional one in the variables $(\alpha, P/a_0)$ to a one-dimensional one, $(\alpha, 0)$. We first discuss the results for $(\alpha, 0)$. Then we discuss how the collision properties vary with increasing impact parameter. Finally, we calculate as a function of α the effective collision cross section associated with each of the four output parameters.

II. ENCOUNTERS AT ZERO IMPACT PARAMETERS

Figures 1, 3, 4, and 6 display each of the four dimensionless output variables as a function of α for $P/a_0 = 0$. Each point on these curves is the average for all the runs in each series.

A. $\Delta E/E_0$

The plots of $\Delta E/E_0$ vs α in Fig. 1 are particularly interesting. We note that as $\alpha \rightarrow 0$, which indicates free-fall of the two stars into each other, $\Delta E/E_0$ approaches asymptotically a finite negative value indicating either a net shrinking of the binary orbit or (and) the replacement of one of the initial binary components by a more massive field star. In family A, $\Delta E/E_0 \rightarrow -0.5$ indicating that the harmonic mean semimajor axis of the orbit after the encounter is about $\frac{2}{3}$ of its value before the encounter. The largest negative value of $\Delta E/E_0$ is in family B, where it approaches ~ -2.3 in this limit. In this family there is little shrinking of the orbit; the increase in the binding energy is principally due to the replacement of one of the binary components by the field star, which has three times its mass. In this case the final binding energy of the orbit is about three times its initial binding energy or the change in the binding energy is about twice the initial binding energy.

The average amount of orbit shrinking per encounter decreases with increasing α until eventually

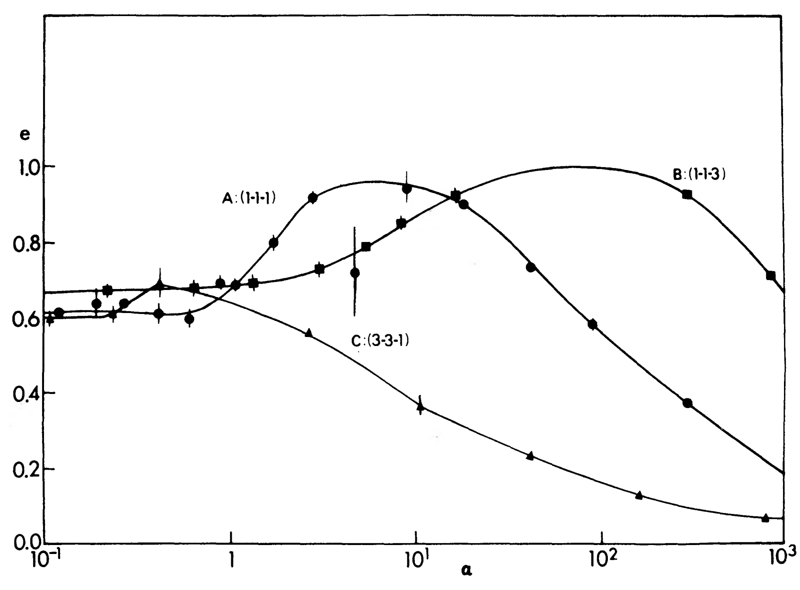

FIG. 4. The average eccentricity of a binary after an encounter with a field star at zero impact parameter.

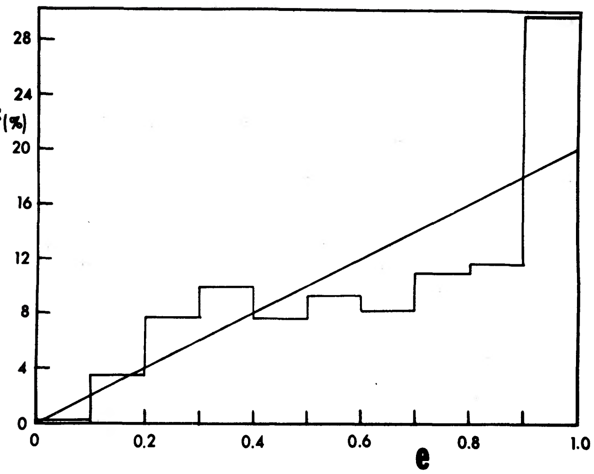

FIG. 5. The normalized distribution of eccentricities among all the runs of series A4.

the behavior reverses itself and the semimajor axis begins to increase as the result of an average encounter. The crossover occurs at $\alpha=0.57$, 7.0, and 0.22 for families A, B, and C, respectively. The value of $\Delta E/E_0$ continues to increase with advancing α until it reaches a distinct maximum of $+1.0$, $+1.0$, and $+0.36$ at $\alpha=6$, 60, and 0.8 for families A, B, and C, respectively. At the peak value nearly 100% of the binaries in families A and B are broken apart by the encounter. As α increases further, $\Delta E/E_0$ decreases toward zero, i.e., toward no change in the semimajor axis. This is due to the decreasing amount of time that the field star spends in the vicinity of the binary system during the encounter.

We should emphasize again that Fig. 1 gives the *average* value of $\Delta E/E_0$ for all the runs in each series. There is a considerable spread in $\Delta E/E_0$ between the individual runs in a given series; i.e., the result of a given encounter between a binary and a field star depends not only on the relative velocity of the two stars at infinity and on the impact parameter, but it also depends strongly on the orientation of the binary orbit with respect to the field star and on the initial phase of the binary in its orbit. We have averaged over these latter parameters by Monto Carlo sampling in constructing Fig. 1. For illustration, in Fig. 2 we have plotted the distribution of $\Delta E/E_0$ for all the individual runs of series A4. In a forthcoming publication (Hills and Lardas 1975) we shall give histograms showing the distribution of $\Delta E/E_0$ and the eccentricities for all the runs in each series listed in Table I.

B. Dissociation Probability

Figure 3 shows P.D., the dissociation probability, or the fraction of the binaries destroyed as the result of an encounter. The curves are qualitatively similar to the corresponding ones for $\Delta E/E_0$ for $\alpha > 1$ as one might expect.

C. Eccentricity

Figure 4 shows the average orbital eccentricity after an encounter while Fig. 5 shows the distribution of eccentricities for all the runs of one series, A4. For $\alpha<1$ the average eccentricity after the encounter in the three families is in the range $\langle e\rangle=0.60$–0.67 which is not far from the value of $\langle e\rangle=\frac{2}{3}$ for binaries in statistical equilibrium. This might not be surprising since for $\alpha<1$ a temporarily bound, triple-star system can form which gives the binary a better chance to approach statistical equilibrium. However, we can see from Fig. 5 that the distribution of eccentricities for these binaries deviates significantly from that for statistical equilibrium where the normalized distribution is $dn=2e\,de$. For $\alpha>1$ even the average eccentricity differs significantly from that for statistical equilibrium. For $\alpha>1$, the eccentricity decreases monotonically with increasing α in family C, while in families A and B the eccentricity peaks at a value of e significantly larger than the equilibrium value of $\frac{2}{3}$ before decreasing monotonically with further increases in α. The graph suggests that at least for the latter two families the field star selectively drags off one of the two binary components leaving it in a nearly radial orbit relative to its companion. This, indeed, seems to occur. For $\alpha<10$ the field star almost invariably suffers a close encounter with one of the binary components with the closest approach being much less than a_0. As a result, a substantial fraction of the momentum of the field star is transferred to this binary component with little direct effect on the other component.

D. Exchange Probability

Figure 6 shows the exchange probability (P.E.), the fraction of the undissociated, postencounter binaries in which the original field star is now one of the binary components. We note that this is almost a step function for families A and B with P.E.$\simeq$1 up to about the value of α where the peaks occur in $\Delta E/E_0$ and e, as shown in Figs. 1 and 4, and then it drops abruptly to zero for larger values of α. Thus, when α is small there is evidently very effective momentum transfer between the field star and that binary component which it encounters so that the field star is captured into the binary system and the original binary component is hurled out of the system in roughly the original direction of motion of the field star.

We note that for $\alpha<1$ in family A, the value of P.E. drops somewhat below unity. This can be attributed to the formation of a temporary triple-star system which gives the equal-mass stars a chance to lose memory of whether a particular star was originally one of the binary components or whether it was the field star. If a complete loss of memory occurs, P.E. would be $\frac{2}{3}$ since any one of the three stars would have an equal chance of being ejected. Since the minimum

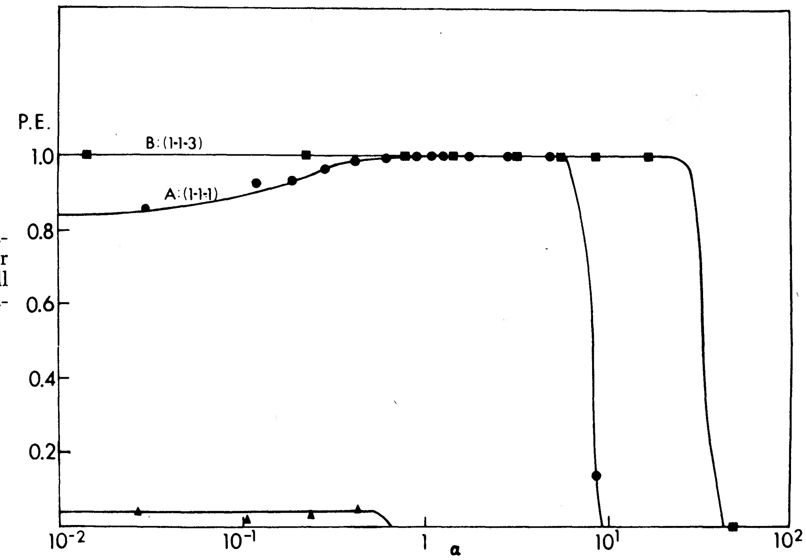

FIG. 6. The exchange probability, the probability that a binary which survives an encounter with a field star at zero impact parameter will have the original field star as one of its post-encounter components.

value of P.E. is about 0.9, it is evident that the loss of memory is only partial.

III. THE COLLISION CROSS SECTION

A. Energy Exchange

Figure 7 shows the dependence of $(\Delta E/E_0)$ on the impact parameter p for several values of α in family A. This tends to approach zero with increasing p at least as fast as a Gaussian. Assuming it is Gaussian,

$$\Delta E/E_0 = A\exp(-p^2/C^2), \tag{2}$$

where A is given as a function of α in Fig. 1 and the width parameter C is some function of α. We can then define an effective collision cross section which governs the rate at which energy is fed into the binary system due to encounters with field stars. This is

$$\sigma_E = A\int_0^\infty 2\pi p\exp(-p^2/C^2)dp = \pi AC^2. \tag{3}$$

From the data in Table I we can determine C and, hence, σ_E at several values of α. We would then like to use these values to derive an analytic expression relating σ_E/A to α. At very large values of α, σ_E/A approaches a constant since gravitational focusing is of little significance in bringing about a closer encounter between the binary system and the field star. In this limit the field star is not significantly affected by the binary (and vice versa) unless the orbit of one of the binary components happens to bring it unusually close to the path of the field star just as it is passing through the binary system. Numerically, in this limit, we find that $\sigma_E \rightarrow \sigma_E^0 \equiv \gamma_E(A\pi a_0^2)$, where $\gamma_E = 0.35 \pm 0.04$.

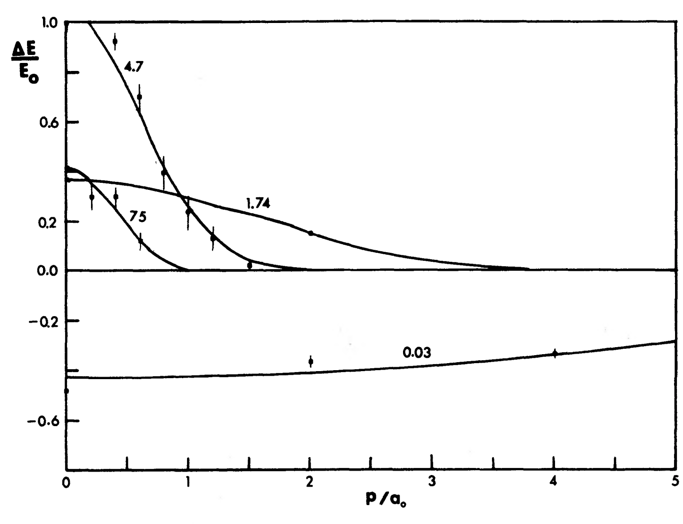

FIG. 7. The change in the binding energy of family A binaries as a function of the impact parameter p. This is plotted for several values of the energy parameter α.

At small values of α, gravitational focusing increases the effective collision cross section. The relationship between the impact parameter p and the closest approach r_0 of two stars of mass M_A and M_B in a hyperbolic orbit is

$$p^2 = r_0^2[1+2G(M_A+M_B)/(r_0 V_f^2)], \quad (4)$$

where V_f is the relative velocity of the two stars at infinity. Letting $M_A = M_1 + M_2$ and $M_B = M_3$, we can anticipate from Eq. 4 that the total collision cross section σ_E is related to its value, σ_E^0, in the high velocity limit by the equation

$$\sigma_E = \sigma_E^0[1+2G(M_1+M_2+M_3)/(r_0 V_f^2)], \quad (5)$$

where r_0 is now some characteristic length. Using the fact that $V_f^2 \equiv \alpha V_c^2$, we find that

$$\sigma_E = \sigma_E^0\left[1+\frac{\beta_E M_3(M_1+M_2)}{\alpha M_1 M_2}\right]. \quad (6)$$

From the data in Table I we find that $\beta_E \equiv (2a_0/r_0) = 3.8$ for all three families. Putting $\sigma_E^0 \equiv \pi a_0^2 \gamma_E A$ and defining

$$\alpha_0 \equiv \frac{\beta_E M_3(M_1+M_2)}{M_1 M_2} \quad (7)$$

we find that

$$\sigma_E^0 = \pi a_0^2 \gamma_E A\left(1+\frac{\alpha_0}{\alpha}\right). \quad (8)$$

In summary, we have found that the collision cross section for energy exchange between the binary and the field star is completely determined by three parameters: A, which we can find as a function of α by Fig. 1, and the two dimensionless constants γ and β, which are independent of α and, to the accuracy we can measure them, are quantitatively identical in all three families. In Fig. 8 we have plotted $|\sigma_E|/\pi a_0^2$ as a function of α for the three principal families. We note in the figure the large increase in $|\sigma_E|$ for small values of α due to the increasing effectiveness of gravitational focusing. At these small values of α the encounters with the field stars lead to a progressive decrease in the semimajor axis of the binary orbit. We note that for larger values of α, where the encounters with field stars tend to break up the binary, σ_E reaches a maximum value in the limited range of $(1.3\text{–}1.6)\pi a_0^2$ for all three families.

B. Cross Sections for Other Collision Processes

We can define collision cross sections of the kind given by Eqs. 3 and 8 for determining the rate of increase in the eccentricity of a binary orbit, the probability that a binary will be broken apart by the field stars, and the probability that a field star will replace one of the binary components. For these processes the value of A in Eq. 8 is given by the corresponding plots in Figs. 3, 4, and 6 while the values of γ and β for these processes, as determined numerically from the data in Table I, are given in Table II. These cross sections are of the same order as those previously calculated for the energy exchange $|\sigma_E|$.

For values of α sufficiently large that σ_E is positive, which indicates a tendency for the binaries to be dissociated by the encounters, the cross section governing the probability of a field star replacing one of the binary components, P.E., tends to be smaller than σ_E, while the cross section governing the rate of increase in the eccentricity, σ_e, tends to be somewhat larger than σ_E. Thus, if a binary system is in a stellar system where the velocity dispersion of the field stars is sufficiently large that σ_E is positive, the binary will likely be broken apart before it can exchange one of its components for a field star. However, its eccentricity may be significantly affected by encounters with the field stars before it is actually broken apart by these encounters.

If a binary system is tightly bound compared to the kinetic energy of the field stars, $\alpha \ll 1$, we find from our earlier work that $\sigma_E < 0$, which indicates that the binary becomes even more tightly bound through encounters with the field stars. For these cases we further find that both σ_e and $\sigma_{\text{P.E.}}$ tend to be somewhat larger than $|\sigma_E|$. Thus, if a group of binaries with $\alpha \ll 1$ are sufficiently old that encounters with field stars have had time to significantly decrease their semimajor axis, we can expect their distribution of eccentricities to be close to that required in statistical equilibrium and that most of them will not have the same component stars as they had originally. Considering Fig. 6, we further expect that in an encounter between a binary and a field star, the two most massive of the three stars involved will tend to be retained in the binary system. Thus, with advancing time the binary orbits will not only shrink, but the binary components will become progressively more massive. This general behavior has been found in the computer simulations of small stellar clusters (Aarseth and Hills 1972).

IV. LIFETIMES OF BINARY STARS

The average rate at which energy is fed into a binary system by interactions with field stars is given by

$$\frac{dE}{dt} = n_f \langle \sigma_E V_f \rangle E_0, \quad (9)$$

where n_f is the space density of field stars, E_0 is again the binding energy of the binary, and $\Gamma_E \equiv \langle \sigma_E V_f \rangle$ is the rate coefficient, the average value of the product of the energy cross section and the velocity of the field star with respect to the binary prior to the encounter.

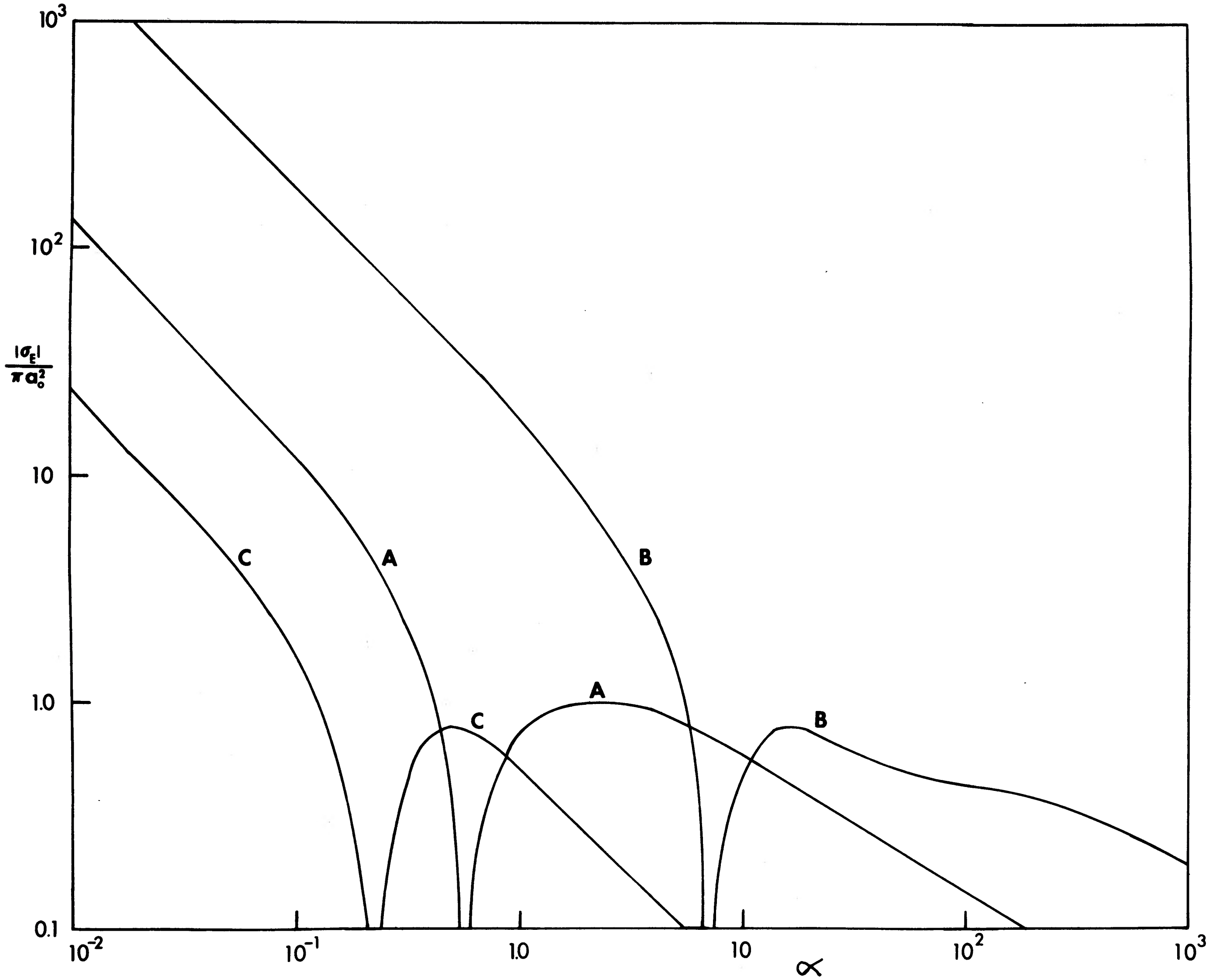

FIG. 8. The cross section for energy exchange, σ_E, as a function of the energy parameter α for families A, B, and C.

The effective lifetime of the binary orbit is then

$$\Upsilon_E=(n_f\langle\sigma_E V_f\rangle)^{-1}. \tag{10}$$

Using Eq. 5 this can be written as

$$\Upsilon_E=\{n_f\langle\pi a_0^2\gamma_E A\, V_f[1+\beta_E G(M_1+M_2+M_3)]/(a_0V_f^2)\rangle\}^{-1}. \tag{11}$$

Using Eq. 1 and $\alpha\equiv(V_f/V_c)^2$, this can also be expressed as

$$\Upsilon_E=\left[\frac{M_3(M_1+M_2)}{GM_1M_2(M_1+M_2+M_3)}\right]^{\frac{1}{2}}\times\left\{\pi a_0^{\frac{3}{2}}n_f\gamma_E A\alpha^{\frac{1}{2}}\left[1+\frac{\beta_E M_3(M_1+M_2)}{\alpha M_1M_2}\right]\right\}^{-1}. \tag{12}$$

Thus, for fixed values of M_1, M_2, M_3, and α, Υ_E scales simply as $\Upsilon_E\alpha(n_f a_0^{\frac{3}{2}})^{-1}$.

A. Limiting Cases

In the limit where

$$V_f^2\gg V_0^2\equiv\beta_E G(M_1+M_2+M_3)/a_0=V_c^2\left[\frac{M_3(M_1+M_2)}{M_1M_2}\right]\beta_E\sim V_c^2, \tag{13}$$

TABLE II. Collision constants.

Constant	$\frac{\Delta E}{E_0}$	e	P.E.	P.D.
γ	0.35	0.55	0.17	0.17
β	3.8	6.7	7.3	5.5

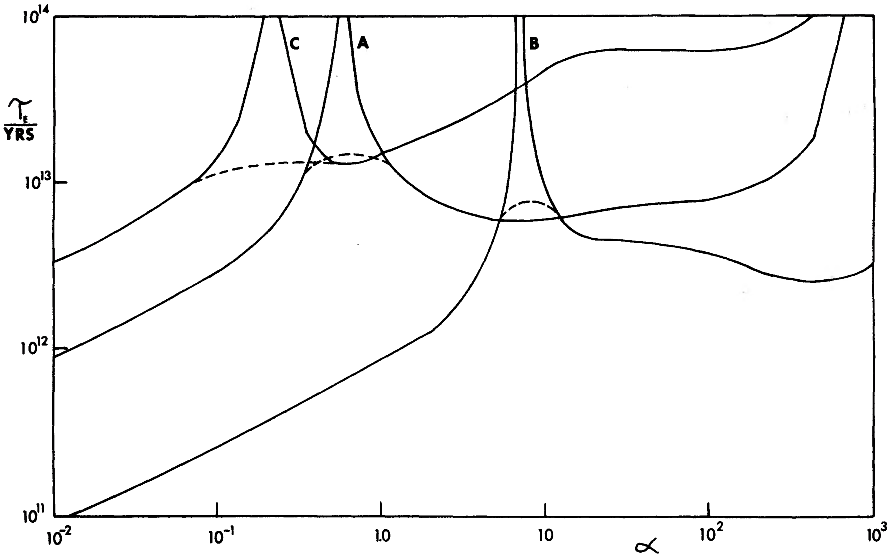

FIG. 9. The effective lifetimes of binary stars due to energy exchange as a function of the energy parameter α. This is plotted for families A, B, and C in a stellar system in which $n_f=1$ star pc^{-3}, $a_0=10$ AU, and $M_1=M_2=M_\odot$.

or equivalently $\alpha \gg 1$, Eq. 11 reduces to

$$\Upsilon_E \rightarrow (n_f \pi a_0^2 \gamma_E \langle A V_f \rangle)^{-1}. \tag{14}$$

Here $\Upsilon_E \rightarrow \alpha(a_0^2 V_f)^{-1}$ for a fixed α. In this limit the lifetime of the orbit is the time required to break up the binary.

In the other limit, $V_f^2 \ll V_0^2 \sim V_c^2$, the lifetime reduces to

$$\Upsilon_E \rightarrow [n_f \pi \gamma_E A_0 \beta_E G(M_1+M_2+M_3)\langle a_0 \rangle / V_f]^{-1}. \tag{15}$$

Here $A \rightarrow A_0$, a constant quantity which is independent of V_f or α, as we see in Fig. 1. In this limit the lifetime of the binary orbit is the time required for the semimajor axis of the orbit to shrink to half its initial value. In this limit $\Upsilon_E \alpha (V_f/a_0)$. Here, the dependence of Υ_E on velocity is just the reciprocal of its dependence in the opposite limit.

B. Numerical Calculations

Figure 9 shows Υ_E plotted as a function of α for families, A, B, and C. This is calculated using Eq. 12 for the particular case where $a_0=10$ AU, $n_f=1$ star pc^{-3}, and $M_1=M_2=M_\odot$ so that $M_3=1$, 3, and $\frac{1}{3}$ $M_\odot$ for families A, B, and C, respectively. We can use the data in this figure to find the lifetime of any other binary in any of the three families for any arbitrary binary mass and semimajor axis and for any arbitrary n_f since, for a given α, Υ_E scales simply as

$$\Upsilon_E \alpha [(M_1)^{-\frac{1}{2}} a_0^{-\frac{3}{2}} n_f^{-1}] \tag{16}$$

for each of the three families. The large increase in Υ_E shown in Fig. 9 as $A \rightarrow 0$ is somewhat artificial. It results from our having considered only the average value of $\Delta E/E_0$ at a given α and not the natural spread in the possible values of $\Delta E/E_0$ among the individual encounters. If we allowed for this spread, the effective lifetimes would probably be better represented by the dashed portions of the curves in Fig. 9.

Let us consider the effect of encounters on binary stars in four typical stellar systems: system I, the solar neighborhood, where $V_f=30$ km/sec and $n_f=0.1$ pc^{-3}; system II, an open cluster, with $V_f=1$ km/sec and $n_f=1$ pc^{-3}; system III, a globular cluster, with $V_f=15$ km/sec and $n_f=10^2$ pc^{-3}; system IV, a galactic nucleus, with $V_f=300$ km/sec and $n_f=10^4$ pc^{-3}. Figure 10 shows, as a function of semimajor axis a_0, the lifetimes of family A binaries with $M_1=M_2=M_\odot$ in each of the four typical stellar systems. Figure 11 shows similar plots for family B binaries with $M_1=M_2=M_\odot$ and $M_3=3$ $M_\odot$. We note that the lifetimes of a binary in Fig. 11 range from about the same to being about ten times shorter than the lifetime of the same binary in Fig. 10. If we had made a similar plot for family C binaries, the lifetimes would have been very much longer than those in Figs. 10 and 11. We can conclude that, as a general rule, a binary system will only be substantially affected by encounters with field stars having masses of the same order or greater than that of its two component stars.

We note in Figs. 10 and 11 the anticipated abrupt increase in Υ_E as $A \rightarrow 0$. Again this is a somewhat artificial effect resulting from our having ignored the natural spread in $\Delta E/E_0$ at a given value of α. The dashed curves in the figures are probably more realistic representations of the lifetimes of the binaries near the singular points where $A \rightarrow 0$. Another peculiarity in A occurs as a_0 or α becomes very large. The nearly constant slope in A vs α shown in Fig. 1 at large values of α suggests that $A \rightarrow 0$ at a finite value of α. This would cause $\Upsilon_E \rightarrow \infty$ for α larger than this critical value. However, by physical reasoning we may expect A to remain finite at large values of α; although it may be too small to be reliably measured by any reasonable

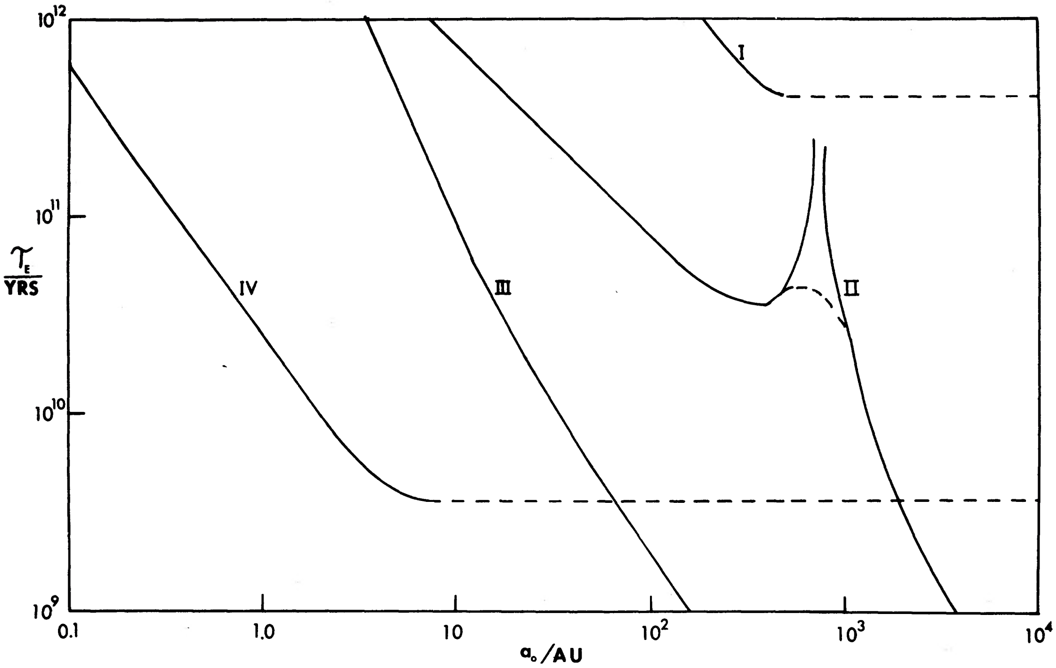

FIG. 10. The lifetime of family A binary stars due to energy exchange in each of the four typical stellar systems discussed in the text. This is plotted as a function of the semimajor axis a_0.

number of computer simulations. If A is small but finite at very large values of α, Υ_E could still be very short for weakly bound binaries with large a_0's since their geometric cross sections increase as πa_0^2 and Υ_E is inversely proportional to this factor at large values of α. We shall assume that Υ_E remains constant at the lowest determinable value of Υ_E for α very large. This assumed behavior is shown by the dashed line in Figs. 10 and 11.

Figures 12 and 13 show the times required to significantly change the eccentricities of the binary stars by encounters with field stars. This is calculated for

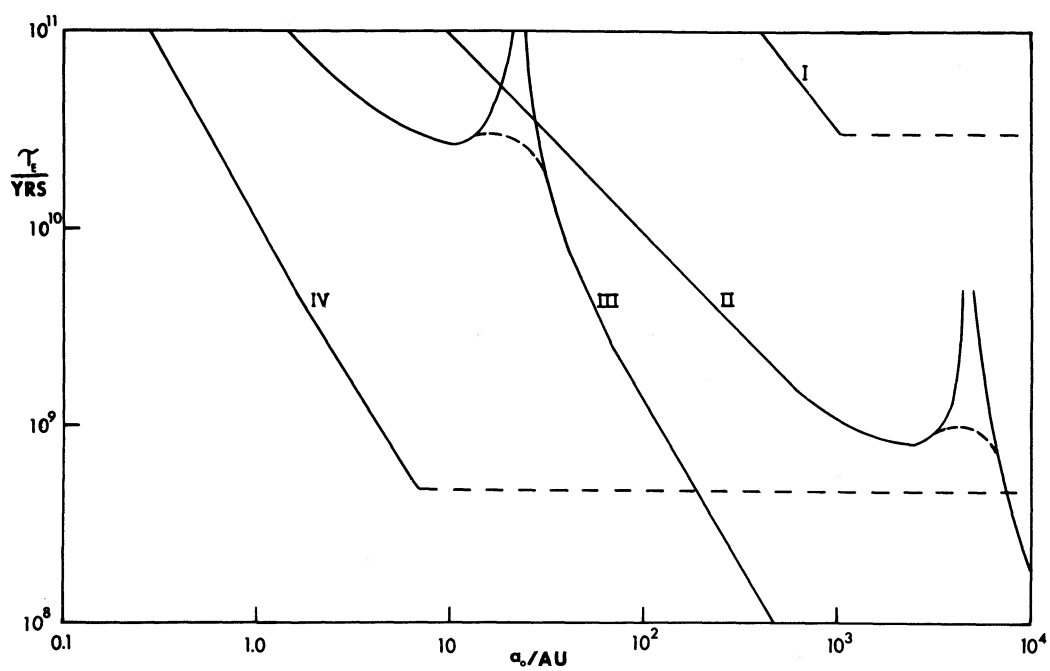

FIG. 11. The same as Fig. 10 except the results are for family B binaries.

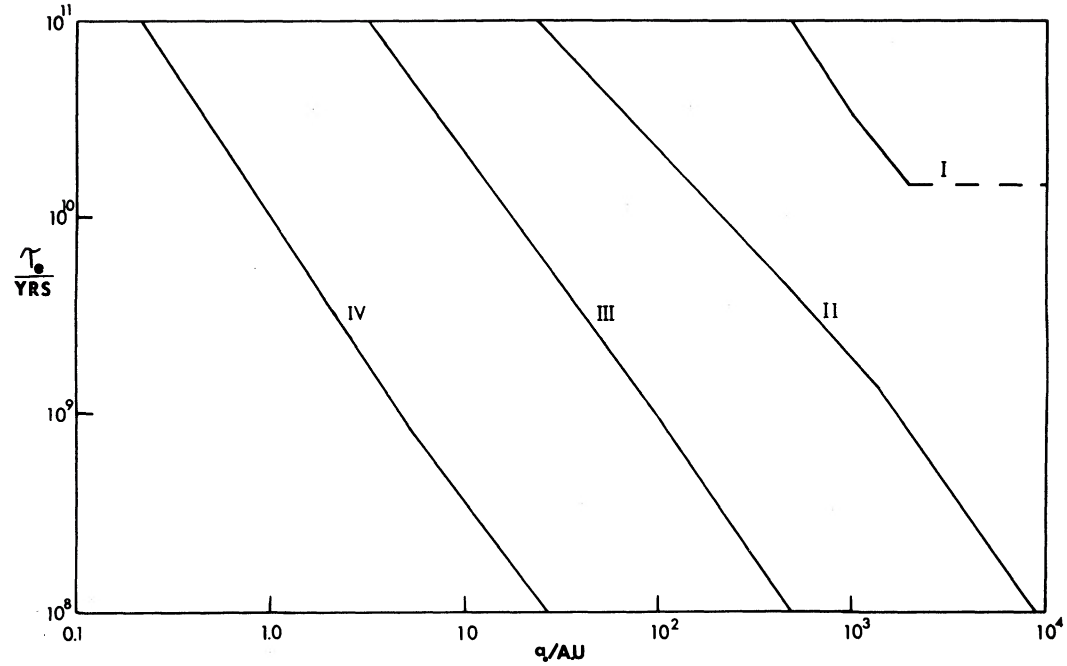

FIG. 12. The time required to randomize the eccentricities of the family A binaries in each of the four typical stellar systems as a function of the semimajor axes of the binaries.

binary families A and B in systems I–IV. The calculations were made using Eq. 11 with A being determined by Fig. 4, while $\gamma_e=0.55$ and $\beta_e=6.7$ as tabulated in Table II. The times plotted in the figures are essentially those required for the orbital eccentricities to become relaxed or thermalized by the encounters. After this time has elapsed we may expect the distribution of orbital eccentricities to follow that for statistical equilibrium, $dn=2e\,de$. We note that these relaxation times are typically two to four times shorter than the corresponding times required to change the semimajor axis, which were shown previously in Figs. 10 and 11. It seems to be significantly easier to change the angular momentum of a binary than to change its energy.

If we assume that systems I–IV are all 1.2×10^{10} yr old, we can determine for each system, by using Figs. 10 and 11, the permissible range of the semimajor axes of those primeval binary systems which could have survived to the present epoch. In comparing the four systems we note that the conditions for survival

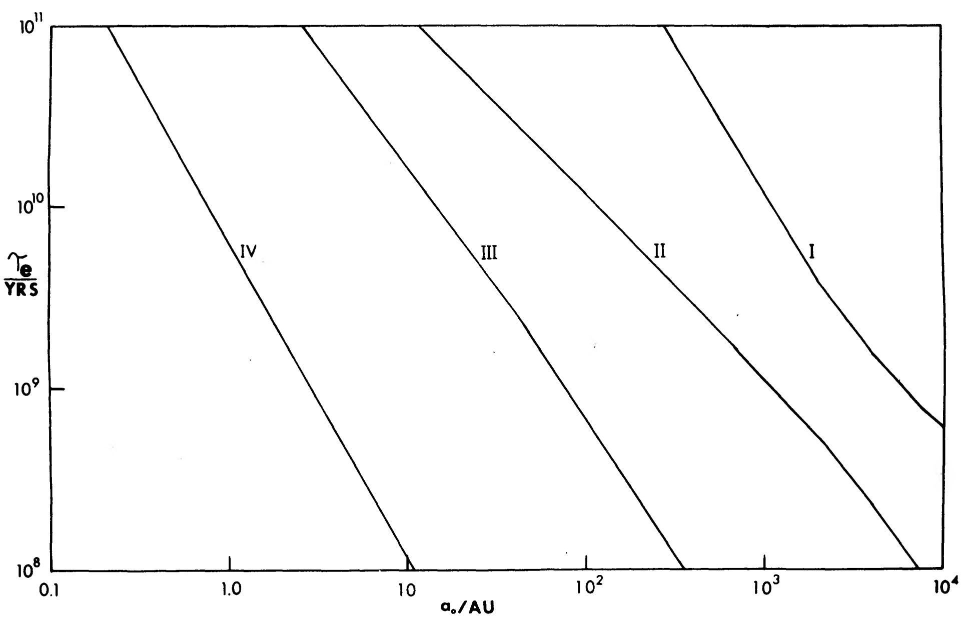

FIG. 13. This is the same as Fig. 12 except being for family B binaries.

are most favorable in the low-density environment of the solar neighborhood where we find that most binaries, regardless of their semimajor axes, have survived to the present epoch. Here, only about 10% of the binaries with $a_0 > 10^3$ AU have been broken apart, and the death rate is much less for binaries with smaller semimajor axes. We can expect similar survival probabilities for the comets in the Oort comet cloud. Similar qualitative conclusions have been reached by Cruz–González and Poveda (1971; see correction by Hénon 1972) from their computer simulation of the evolution of comet orbits due to stars passing through the solar neighborhood. Their study is the closest work to our own, but even here the method employed and the mass ratios used differ enough to make a direct quantitative comparison nearly impossible. Considering Figs. 12 and 13, we can expect that in the solar neighborhood the distribution of eccentricities of the surviving binaries with $a_0 \geq 10^3$ will be close to that for statistical equilibrium.

On considering the other three systems, we find that the majority of primordial binaries with $a_0 > a_E = 1$–2, 35, and 100–1000 AU in systems IV, III, and II, respectively, have been eliminated by encounters with field stars. In systems II and IV most of the primeval binaries with $a_0 > a_E$ have been broken apart by the encounters. In system II all the binaries with $a_0 > a_A$, the critical semimajor axis where A goes from being negative to being positive, have broken apart while those having originally $a_A > a_0 > a_E$ have had their semimajor axes decreased to less than a_E by the encounters. From Figs. 12 and 13 we find that binaries with $a_0 > a_e \simeq 0.9$, 15, and 150 AU in systems IV, III, and II, respectively, have reached statistical equilibrium with respect to their distribution of eccentricities.

We note again for emphasis that in the above calculations we have assumed for simplicity that all four systems are 1.2×10^{10} yr old. Most open clusters are a good deal younger than this. However, by using Eq. 16 the reader can easily scale the graphical results to any younger cluster.

We have just found that primordial binaries with $a_0 > a_E$ have been eliminated from the stellar systems. However, there will still be binaries with $a_0 > a_E$ in these systems due to their production in three-body encounters. The steady-state number of these objects should be close to that predicted by statistical equilibrium. We shall discuss these in the next section.

V. BINARIES IN STATISTICAL EQUILIBRIUM

Their distribution in semimajor axis a_0 and eccentricity e is given by

$$dn_b = 3\pi\left(\frac{3}{2\pi}\right)^{\frac{1}{2}} n_{1f} n_{2f} \left[\frac{G(M_1+M_2)}{\langle V^2\rangle a_0}\right]^{\frac{3}{2}} \times[\exp(a_{\exp}/a_0)] 4\pi a_0^2 da_0 2e\, de, \quad (17)$$

where

$$a_{\exp} = \frac{3}{2}\left(\frac{GM_1M_2}{\langle mV^2\rangle}\right). \quad (18)$$

This was derived in a straightforward fashion as guided by the discussion in Jeans (1919). This gives the number of binaries per unit volume having components with masses M_1 and M_2. Here, n_{1f} is the number of unbound stars of mass M_1 per unit volume while n_{2f} is the corresponding quantity for M_2. Also $\langle V^2\rangle$ is the mean-squared velocity of all stars in the cluster and m is the mean mass of all stars in the system with the binaries being counted as single stars.

Equation 17 diverges both as $a_0 \to \infty$, where $dn_b \alpha (a_0^{\frac{1}{2}} da_0)$ or $n_b \alpha (a_0^{\frac{3}{2}})$ and as $a_0 \to 0$, where n_b increases exponentially. In practice, as is well known, these divergences do not create any serious problems since a_0 cannot exceed the radius R of the system and statistical equilibrium is not valid for a_0 smaller than some critical value a_E for which the relaxation time, in this case the time required for the binaries with $a_0 = a_E$ to reach statistical equilibrium, just equals the age of the cluster. In the previous section we derived this relaxation time τ_E as a function of a_0 in four typical stellar systems.

The two divergences or sinks in Eq. 17 are likely the controlling factors which determine qualitatively the behavior of the curves in Fig. 1 showing the change in the semimajor axis of a binary due to an encounter with a third star. If a binary has a small α or correspondingly a small a_0, the encounter tends to nudge the binary towards the nearest of the two divergences, which in this case causes a decrease in its semimajor axis. For these binaries the exponential Boltzmann factor dominates causing them to become more strongly bound. If a_0 is initially large, the binary is nudged toward the second divergence which causes its semimajor axis to increase. The second divergence is due to the phase-space volume available to a binary increasing monotonically with a_0.

A. Frequency of Captured Binaries in Typical Systems

By numerically integrating Eq. 17 we can easily find for a stellar system in statistical equilibrium the number of binaries with semimajor axes in some interval a_1–a_2. This work shows that the number of such binaries is almost negligibly small for semimajor axes a_0 greater than some critical value a_b and enormously large for a_0 less than a_b. We find that for our four standard systems $a_b \simeq 0.03 a_{\exp}$. Thus the enormous increase in dn_b/da_0 in Eq. 17 occurs where the Boltzmann factor is $\exp(a_{\exp}/a_0) \sim 10^{10}$. The value of the remaining terms in the equation is only weakly dependent on a_0 (proportional to $a_0^{\frac{1}{2}}$). Thus if dn_b/da_0 is some value f at $a_0 = a_b$, it is $10^{10} f$ at $a_0 = \frac{1}{2} a_b$ and $10^{20} f$ at $a_0 = \frac{1}{3} a_b$. Thus the sudden increase in the number of binary stars below some critical $a_0=a_b$ can easily be understood.

The numerical integrations show that $a_b=0.085$ AU$=18$ R$_\odot$ in the solar neighborhood, $a_b=160$ AU in an open cluster, $a_b=0.4$ AU$=86$ R$_\odot$ in a typical globular cluster, and $a_b=7\times10^{-4}$ AU$=0.15$ R$_\odot$ in a typical galactic nucleus. The extremely small value of a_b in the galactic nucleus immediately rules out its having any significant number of binaries composed of normal stars under conditions of statistical equilibrium. In fact, even if we invoked a binary composed of white dwarfs, neutron stars, or black holes, we find that if $a_0=0.15$ R$_\odot$ and $M_1=M_2=$M$_\odot$ the two stars would spiral together by the emission of gravitational waves in only 4×10^4 yr. Thus we can rule out binaries composed of *any* stars existing in a galactic nucleus under conditions of statistical equilibrium. The above data also indicates that only relatively close spectroscopic binaries can exist in statistical equilibrium in the solar neighborhood and in globular clusters. Only in open clusters can there exist conventional visual binaries in statistical equilibrium.

In order for binaries with $a_0\simeq a_b$ to have reached statistical equilibrium, we find from the work in the previous section of this paper that the ages of systems I, II, III, and IV would have to be 2.0×10^{16} yr, 3.3×10^{10} yr, 2.0×10^{12} yr, and 2.3×10^{14} yr, respectively. Since these times are much greater than the age of the universe, in none of these systems do we expect to find any significant number of binaries with semimajor axes $a_0>a_E$, the minimum semimajor axis of those binaries which have reached statistical equilibrium. Almost all the binaries in each system are primordial with $a_0<a_E$; these have not had enough time to reach statistical equilibrium.

B. Rate of Binary Formation in Three-Star Encounters

Although we have found that this is not important in our four standard systems, we note, for the sake of completeness, that this is easily computed from our previous work. Because of detailed balancing, in statistical equilibrium there are as many binaries of a particular kind created per unit time in three-star encounters as there are those destroyed in single encounters with field stars. Thus the rate of binary star formation is clearly

$$\frac{dn_b}{dt}=\frac{dn_b}{T_{\rm P.D.}}, \qquad (19)$$

where dn_b is given by Eq. 17 and $T_{\rm P.D.}$ is the mean binary lifetime between breakups resulting from single encounters with field stars. Here, $T_{\rm P.D.}$ is given by Eq. 12, while $\beta=\beta_{\rm P.D.}=5.5$ and $\gamma=\gamma_{\rm P.D.}=0.17$ (Table II) and A is given by Fig. 3.

We note that since $A=0$ for $\alpha<1$, $T_{\rm P.D.}=\infty$ for these binaries. Thus no tightly bound binaries with $\alpha\equiv(V_f/V_c)^2<1$ can form directly by three-body encounters. For a given group of three stars we see from Eq. 1 that V_c depends only on the semimajor axis a_0 of the binary. Since V_f is limited by the escape velocity from the system, there is clearly a lower limit on the semimajor axis of a binary formed in a three-body encounter in a stellar system. A more tightly bound binary can only be produced by the shrinking due to encounters with single stars of the semimajor axis of a more loosely bound binary originally having $\alpha>1$. This behavior is illustrated by computer simulations of smaller clusters (Aarseth and Hills 1972) in which one finds that a long-lived binary usually forms with a semimajor axis such that $\alpha\simeq1$ assuming that V_f is the rms velocity in the cluster. The binary then shrinks due to further encounters with field stars.

A more detailed analysis of binary star formation in stellar systems is in preparation (Hills and Moore 1975).

VI. THE EFFECT OF BINARIES ON THE DYNAMICAL EVOLUTION OF A STELLAR CLUSTER

We found in the previous section that very few binaries form by capture in open clusters and almost none form this way in the solar neighborhood, in globular clusters, or in galactic nuclei. Consequently, we may expect primordial binaries to affect the dynamical evolution of a stellar cluster more than captured binaries. In this section we shall investigate their influence.

The interactions between binaries and field stars within a cluster may affect its dynamics in one of two quite different ways: If the energy parameter, $\alpha\equiv(V_f/V_c)^2$, for a binary is greater than a critical value α_c we see again from Fig. 1 that $(\langle\Delta E/E_0\rangle)>0$, which indicates that the binary tends to be broken apart by encounters with field stars. When this occurs the potential energy which had been binding together the binary is transferred to the cluster as a whole. Thus the breakup of the binary increases the binding energy of the cluster. At the other extreme, if $\alpha<\alpha_c$, the binary orbit tends to shrink in encounters with field stars. This increases the kinetic energy and decreases the binding energy of the cluster which contributes to its destruction. We shall show that the latter binaries with $\alpha<\alpha_c$ affect the cluster more severely than those with $\alpha>\alpha_c$.

A. Binaries with $\alpha>\alpha_c$

We shall show that the breakup of all such binaries within a cluster cannot significantly affect the total binding energy E_c of the cluster. If the mass of the cluster is $M=Nm$, where N is the total number of stars in the cluster and m is their average mass,

$$E_c=\tfrac{1}{2}|W|=\tfrac{1}{2}\left[\frac{G(Nm)^2}{2R}\right], \qquad (20)$$

where the effective radius of the cluster, R, is the harmonic mean distance between all stars in the cluster.

We shall assume that the mean velocity of a field star with respect to a binary prior to an encounter is $V_f=2^{\frac{1}{2}}\langle V^2\rangle^{\frac{1}{2}}$, where $\langle V^2\rangle$ is the mean-squared velocity in the cluster. By the virial theorem this yields

$$V_f \simeq (GNm/R)^{\frac{1}{2}}. \tag{21}$$

If we require that $\alpha\equiv(V_f/V_c)^2 \geq \alpha_c$, we find from Eqs. 1 and 21 that the semimajor axis of the binary has to satisfy the condition

$$a_0 \geq a_c \equiv \alpha_c \frac{R}{N}\left[\frac{M_1M_2(M_1+M_2+M_3)}{mM_3(M_1+M_2)}\right], \tag{22}$$

where M_1 and M_2 are the masses of the binary components and M_3 is the mass of the field star. If $M_1=M_2=M_3\equiv M_0$, the average mass of an individual star in the system, the average mass m of the binaries and single stars combined is

$$m=M/N=M_0(1+f_b), \tag{23}$$

where f_b is the fraction of the N stars which are binaries. Using Eqs. 22 and 23 and noting that $\alpha_c \simeq 0.6$ for $M_1=M_2=M_3$ (Fig. 1), we find that the critical semimajor axis is

$$a_c=\frac{9}{10}\frac{R}{N(1+f_b)}. \tag{24}$$

Let f_b' be the fraction of the stars in the system which are binaries with $a_0>a_c$ and let $f_c\equiv\langle a_0\rangle/a_c$, where $\langle a_0\rangle$ is the harmonic mean semimajor axis of these binaries. The total binding energy of all binaries with $\alpha>\alpha_c$ is then

$$E_b'=\frac{f_b'NGM_0^2}{2f_ca_c}. \tag{25}$$

From Eqs. 20, 24, and 25 we find that the ratio of the total binding energy of these binaries to that of the cluster as a whole is

$$E_b'/E_c=\frac{20}{9}\frac{(f_b'/f_c)}{(1+f_b)}. \tag{26}$$

Realistically, we may expect that $f_b\equiv\frac{1}{2}$, $f_b'<\frac{1}{4}$, and $f_c>2$, so that $E_b'/E_c<0.2$. Thus the breakup of the loosely bound binaries with $\alpha>\alpha_c$ can have relatively little effect on a stellar cluster.

B. Binaries with $\alpha<\alpha_c$

We shall show that these binaries have a profoundly disturbing effect on a stellar cluster. It is clear that the total binding energy of these binaries is much greater than the binding energy of a typical cluster. Thus, the potential for disruption is very large. The basic question is whether or not close encounters between these tightly bound binaries and other cluster members are frequent enough to influence the dynamical evolution of the cluster.

The rate at which energy is fed into the cluster by these encounters is given by

$$\frac{dE_c}{dt}=n_fV_f\int_0^\infty E_0\sigma_E N_b' da_0, \tag{27}$$

where $n_f=N/(\frac{4}{3}\pi R^3)$ is the space density of the field stars and N_b' is the total number of binaries in the cluster with semimajor axis in the range $(a_0-\frac{1}{2}\delta a_0)$ to $(a_0+\frac{1}{2}\delta a_0)$. Here, E_0 is the average binding energy of each binary in this semimajor interval and σ_E is the cross section for the exchange given by Eq. 6. Putting in the appropriate expressions for E_0 and σ_E yields

$$\frac{dE_c}{dt}=-n_fV_f\pi\gamma_E\left(\frac{GM_1M_2}{2}\right) \cdot\int_0^\infty a_0N_b'A\left[1+\frac{\beta_EG(M_1+M_2+M_3)}{a_0V_f^2}\right]da_0. \tag{28}$$

Dividing Eq. 20 by Eq. 28, using Eq. 21, and letting $n_f=N/(\frac{4}{3}\pi R^3)$, we find that the characteristic lifetime of the cluster due to the kinetic energy being fed into it by these binary stars is

$$\tau_s\equiv\left(\frac{E_c}{dE_c/dt}\right) =\frac{2}{3}\left(\frac{m^2}{M_1M_2}\right)\left[\frac{N^{\frac{1}{2}}R^{\frac{3}{2}}}{(Gm)^{\frac{1}{2}}\gamma_E\int a_0N_b'AYda_0}\right], \tag{29}$$

where

$$Y\equiv1+\beta_E(R/a_0)(M_1+M_2+M_3)/(Nm). \tag{30}$$

If we use Eq. 22 to evaluate (R/N) in Eq. 30, we find that

$$Y=1+\left[\left(\frac{\beta_E}{\alpha_c}\right)\left(\frac{a_c}{a_0}\right)\right]\left[\frac{M_3(M_1+M_2)}{M_1M_2}\right]. \tag{31}$$

Since $\beta_E=3.8$ and $\alpha_c\simeq0.6$, it is evident that the second term in this equation is very much larger than unity for $a_0<a_c$. Thus, in determining the disruptive effect of binaries with $a_0<a_c$, we can ignore the factor of unity in Eq. 30. In addition, in the limit where $a_0\ll a_c$,

$A \rightarrow A_0$, a constant, as is seen in Fig. 1. Thus, we can pull $A = A_0$ out of the integral in Eq. 30. With these two simplifications, Eqs. 29 and 30 reduce to

$$\tau_s = \frac{2}{3}\left(\frac{1}{\gamma_E A_0 \beta_E}\right)\left\{\left(\frac{N^{\frac{3}{2}}}{N_b}\right)\left[\frac{R^{\frac{3}{2}}}{(Gm)^{\frac{1}{2}}}\right]\right\} \times \left[\frac{m^3}{M_1 M_2 (M_1+M_2+M_3)}\right]. \quad (32)$$

Here,

$$N_b = \int_0^{a_c} N_b' da_0,$$

the total number of binaries with $a_0 < a_c$. We note that the lifetime of the cluster does not depend on the semimajor axes of the individual binaries, but it depends only on the total number of binaries with $a_0 < a_c$. This is due to the fact that as we reduce a_0 the decrease in the encounter cross section is just compensated by the increased energy gained per encounter. If we let $M_1 = M_2 = M_3 = m$, $N = M/m$, and $f_b \equiv N_b/N$, the fraction of the stars in the cluster which are binaries with $a_0 < a_c$, Eq. 32 reduces to

$$\tau_s = \frac{2}{9}\left(\frac{1}{f_b \gamma_E A_0 \beta_E}\right)\left(\frac{M}{m}\right)\left[\frac{R^{\frac{3}{2}}}{(GM)^{\frac{1}{2}}}\right]. \quad (33)$$

From Table II we find that $\gamma_E = 0.35$ and $\beta_E = 3.8$. From Fig. 1 we find that averaged over the three families $|A_0| = 1.0$. Thus,

$$\tau_s = 0.17(M/m)[R^{\frac{3}{2}}/(GM)^{\frac{1}{2}}]/f_b$$

$$= \frac{2.5\times10^6 \text{ yr}}{f_b}\left[\left(\frac{M}{\mathrm{M}_\odot}\right)^{\frac{1}{2}}\left(\frac{\mathrm{M}_\odot}{m}\right)\left(\frac{R}{\mathrm{pc}}\right)^{\frac{3}{2}}\right]. \quad (34)$$

This quantity is simply related to the relaxation time τ_R in the cluster. If we use the virial theorem to express R in this equation in terms of M and $\langle V^2 \rangle$, we find that

$$(\tau_s/\tau_R) = \{[1.3 \ln(N/2)]/f_b\}, \quad (35)$$

where we take τ_R to be the reference time (Spitzer and Härm 1958). Here, $N \equiv M/N$, the total number of stars in the system. For a typical open cluster $\tau_s = 2\times10^8$ yr if $M = 250$ $\mathrm{M}_\odot$, $R = 2$ pc, $m = 1$ $\mathrm{M}_\odot$, and $f_b = \frac{1}{2}$. This is very close to the empirically determined mean lifetimes of open clusters (Wielen 1971). Thus, binaries may be a major, if not the chief, cause of mortality of open clusters. Only unusually massive open clusters or those with relatively few binaries can live as long as M67 and NGC 188.

For a typical globular cluster with $M = 2\times10^5$ $\mathrm{M}_\odot$, $R = 5$ pc, $m = 0.5$ $\mathrm{M}_\odot$, and (we assume) $f_b = \frac{1}{3}$, $\tau_s = 7.5\times10^{10}$ yr. Thus, binaries may severely affect the dynamical evolution of a globular cluster, but they are not likely to disrupt the cluster. However, binaries may severely limit the maximum density attainable in the center of a globular cluster. It is well known that the core of a cluster made up of single stars tends to run away in the sense that as the stars in the core interact with each other, some are ejected into a halo. This causes the remaining stars to lose energy which results in the core shrinking and increasing its density. This in turn accelerates the rate of ejection and the consequent shrinking. Thus, the core attains a very high density. But with binaries present in the system, the increasing density only increases the rate of encounters between the binaries and the field stars, which increases the kinetic energy in the core causing it to expand rather than to shrink. Unless globular clusters are terribly deficient in binaries, this mechanism will significantly decrease the rate of shrinking of the core and may eventually limit the maximum stellar density attainable in the core. We can see that the presence of the binaries transforms the instability from one causing the monotonic shrinking of the core of the cluster to one causing the monotonic shrinking of the orbits of the binary stars in the core. A more detailed discussion of this has been prepared (Hills 1975).

Considering Eq. 34 we find that binaries are not likely to be important in the dynamical evolution of a galactic nucleus. Since, in these systems a_c is, at most, only a few solar radii, the binaries are also ineffective in limiting the density in the center of a galaxy.

In this paper we have only considered encounters between binaries and single stars. If the binary frequency is high, encounters between binary stars may be important in the dynamical evolution of the stellar system; although we note that unless the two binary stars in an encounter have approximately the same semimajor axis, the encounter is approximated quite well by an equivalent encounter between the more loosely bound binary and a single star having the same total mass as the tightly bound binary.

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