Persistent Noise, Feedback, and Endogenous Optimism: A Rational Theory of Overextrapolation

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Abstract

I propose a noisy rational expectations model with persistent noise. Firms learn about economic conditions from signals, and the noise in the signals is persistent rather than *i.i.d.* over time. Firms rationally account for the persistence of noise and update their interpretations of signals based on ex post observations of true economic conditions. I show that this process gives rise to a novel mechanism by which optimism arises endogenously, which in turn amplifies or dampens the effects of underlying shocks. In particular, this model can generate the delayed overreaction in firms' expectations documented in the literature, when firms are better informed about idiosyncratic shocks relative to aggregate shocks. Moreover, strategic complementarity between firms and the resulting higher-order optimism further strengthen my mechanism. Finally, I distinguish empirically my rational theory of optimism from behavioral theories by exploiting the difference in the degree of overextrapolation between consensus and individual forecasts.

Keywords: Expectations, Information, Optimism, Informational Frictions, Overreaction.

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1. Introduction

How do agents form their expectations based on the information they have? Can waves of optimism and pessimism play a role in driving the economy? There are several strands of the macroeconomic literature that help us think about how to incorporate the role of expectations in business cycle models. The focus of most of such papers are mainly on exogenous shifts in expectations—e.g., sunspots, noise shocks in public signals, sentiment shocks—which are orthogonal to fundamentals and directly affect the economy.

Our approach in this paper, instead, is to capture optimism in an entirely different, yet rational way. Agents have signals that are informative about the economic condition, but they are uncertain about how to interpret these signals, and they are said to be optimistic when they interpret their signals in a more optimistic way. Agents are aware of the possibility of themselves having optimism or pessimism. Thus, they try to correct their optimism in a rational way and, if there is a strategic interaction between agents, try to learn others' optimism. In doing so, the optimism endogenously arises and is pinned down by the dynamic path of fundamentals, so it acts like an amplification/dampening mechanism rather than an autonomous driving force.[1]

Our mechanism and the endogenous optimism mainly come from the two assumptions we made about how agents form their expectations: persistent noise and the presence of feedback. Each of them is not new in the literature, but what brings about our mechanism is their novel interaction.

To illustrate what motivates these two assumptions, consider a real-world example where firms need to forecast the market demands for their products to set the prices or to make production plans. As in the literature, we assume that the firms receive noisy signals about the market demands at the beginning of each year. To fix the idea, suppose that these noisy signals come from consumer surveys they conduct, whose results are informative about the true market demands. First, we call into question the validity of the assumption of *i.i.d.* noise that is commonly maintained in the literature and depart from it by assuming that the noise terms are persistent to some degree. We provide an argument in favor of this departure soon in Section 2. Second, firms can receive feedback on their previous forecast by observing the true realization ex-post. For example, firms in the real world are likely to receive feedback on their previous forecasts later on as they can observe the true market demand ex-post.

Beyond these assumptions, we need to make one additional observation—there are many shocks in the real world that differ in the extent to which firms can be informed about them. For

[1] Similarly, Angeletos and Lian (2019) studies confidence multiplier which varies endogenously and amplifies demand shocks.

example, firms make decisions after receiving noisy information about some shocks that affect market demands, but realized market demands are also affected by some other shocks. Firms may be relatively well informed about shocks realized before they make forecasts or about shocks that are idiosyncratic to the firms, which then can be well reflected in the consumer survey. In contrast, firms are likely to be less well informed about shocks realized after making forecasts, or about aggregate shocks, such as shocks on aggregate demand.

The focus of this paper is the interplay between the persistent noise and the presence of feedback. To preview this interaction, suppose that there is a positive aggregate shock in 2020, but firms were not much informed about it through their consumer surveys. Then, the realized market demand is on average higher-than-expected to the firms, which makes them think that they were too pessimistic in interpreting the survey results. This in turn makes them over-optimistic in 2021 when they make new forecasts based on new survey results. Thus, the effect of the positive aggregate shock in 2020 is amplified in 2021. Theoretically, this interaction gives a novel propagation and amplification mechanism, and, empirically, it gives a new way to interpret forecast data.

In Section 2, we argue that it is more natural to assume persistent noise regardless of what real-world counterpart it models. We assume that firms observe noisy signals whose noise terms are auto-correlated. With such persistent noise terms, firms try to learn the noise terms in order to correct their ways of interpreting their signals. In this sense, they dynamically learn how to interpret the information they have. Thus, their beliefs about the noise terms affect how optimistic they are in interpreting the signals, which in turn affects their forecasts and hence their actions. We then formally define the notion of optimism. When agents underestimate the noise terms in their signals, they will overestimate the fundamentals for given values of the signals, so we call them optimistic.

In Section 3, we introduce a micro-founded macroeconomic model in which firm's output choices are made under dispersed information about their productivities. We characterize how firms dynamically learn persistent noise terms in their signals and how this novel channel of learning endogenously generates optimism, which either amplifies or dampens the underlying productivity shock. We assume that the productivity shock consists of two components. Firms receive noisy signals about the first "partly-observed" component and choose their output levels. However, the true productivity is also affected by the second "unobserved" component. This distinction formalizes the previous observation that there are many shocks in the real world that differ in the extent to which economic agents can be informed about them. We also assume that firms receive feedback on their previous forecast by observing the true productivity realization after making forecasts. The main finding of this section is that the effects of partly-observed shocks on the next-period outputs are dampened because they make firms pessimistic, while the effects of unobserved shocks are amplified because they instead make firms optimistic. We can further assume that firms are relatively well informed about idiosyncratic shocks so that partly-observed shocks correspond to idiosyncratic shocks while unobserved shocks correspond to aggregate shocks. In this case, the aggregate optimism fluctuates procyclically with the underlying aggregate shocks, and the aggregate output features delayed overreaction in response to aggregate productivity shocks as in Angeletos, Huo and Sastry (2020).

In Section 4, we introduce strategic complementarity into our baseline model. This gives incentive to firms to forecast others' optimism, others' beliefs about others' optimism, and so on (Higher-order optimism). We characterize how firms update their higher-order optimism and how this higher-order optimism in turn affects the output choices of firms through its effect on higher-order beliefs about the aggregate productivity. We first show that the main message of Section 3 continues to hold in a model with strategic complementarity—the effect of partly-observed shocks are dampened while the effect of unobserved shocks are amplified. Moreover, we show that higher-order optimism always moves in the same direction as the first-order optimism in response to underlying shocks and that the output choice is a weighted sum of higher-order optimism. These results together imply that the presence of strategic complementarity and resulting higher-order optimism strengthen our mechanism compared to the case without strategic complementarity. In other words, when firms observe better-than-expected economic conditions, they become optimistic not only about their signals (first-order optimism) but also about others' optimism (higher-order optimism). This result is in stark contrast to those in the literature which instead document that the higher the degree of strategic complementarity is, the less responsive the agents is to underlying shocks.

In Section 5, our focus is on how we can interpret forecast data in light of this framework. There are many empirical papers that use survey forecast data on, say, the inflation rate to directly measure agents' expectations. In this literature, forecasters are often assumed not to observe the previous inflation rates even ex-post and try to learn them dynamically from signals. This is partly because observing previous realization makes problem essentially static in their settings and makes it difficult to explain what we can observe in the dynamic forecast data. This paper gives a totally different way to interpret dynamic forecast data. We view this forecast data as outputs of agents' dynamically learning in which they can observe the previous inflation rate but try to learn how to interpret their own information; i.e., noise terms. Our model allows for forecasters observing previous realizations ex-post, while still being able to explain prominent empirical findings in the literature that have been explained using standard models. This includes the findings of Coibion and Gorodnichenko (2015), Kohlhas and Walther (2020), and Angeletos, Huo and Sastry (2020). There are, however, many other standard models, especially behavioral theories of overextrapolation combined with information frictions, can explain these findings. To distinguish our model from those models, we exploit the difference between the degrees of overextrapolation in consensus and individual forecasts. In the IBES dataset, analysts' expectations on earnings growth feature overextrapolation from past realizations only when we aggregate them to consensus forecasts. This is consistent with our model of the rational theory of overextrapolation.

The rest of the paper is organized as follows. Section 2 justifies our assumption of persistent noise terms and formalizes the notion of optimism. Section 3 develops a macroeconomic model without strategic complementarity. We characterize the learning of firms and see how our mechanism amplifies/dampens underlying productivity shocks. In Section 4 we use a simplified model to study further implications when there is strategic complementarity. We also show that these implications do not rely on our simplifying assumptions using a numerical example. Section 5 discusses about the new interpretation of dynamic forecast data and provides a series of empirical evidence that is suggestive of our model. Section 6 concludes.

2. Persistent Noise Terms and Optimism

In this section, we illustrate our main departure from the literature—persistent noise terms in signals—and how it naturally leads to a definition of optimism. The literature that investigates the role of expectations and information often postulates that agents receive noisy signals about the true state of nature (hereafter, fundamental) and that they know the stochastic relationships between the signals and fundamental. For example, signals are often modeled as fundamental plus a random noise term with a known distribution:

$$s_t = a_t + \xi_t$$

where s_t is the signal about the fundamental a_t at time t, and ξ_t is the noise term. This assumption is just a modeling device that captures the idea that we are partly informed about the true state of the world while preserving the tractability of models. Most of the papers in the literature, however, assume that noise is a random variable independent across time.[2] This

[2] One crucial reason for this assumption is that with persistent variables that cannot be observed perfectly, we have to tackle the infinite regress problem as in Townsend (1983). A large number of works have explored how to solve the infinite regress problem using either guess-and-verify, approximation, or the frequency domain technique. A partial list of these works includes Sargent (1991), Kasa (2000), Nimark (2017), Rondina and Walker (2018), and Huo and Takayama (2018). Even in those works, it is often assumed that only fundamentals are persistent, while noise follows *i.i.d.* Notable exceptions are Huo and Takayama (2015, 2018), but their main focuses are on the methodological contribution rather than on economic implications of persistent noise terms.

time-independence assumption simplifies the analysis a lot, and, combined with normality assumptions, it often leads to closed-form solutions.

Whether this *i.i.d.* assumption is a good or bad description of the real world depends on how we interpret the noisy signal; i.e., what the real-world counterpart of the noise is. There are two prominent interpretations in the literature, and we will argue in this section that whatever interpretation we adopt, *persistent* noise is more realistic assumption. We then discuss the implications of this persistence in the following sections.

First, we can literally interpret the signal as noise-ridden information about fundamental, and agents directly observe this signal. In the real world, information sources are always biased, and the bias is likely to be persistent across time. Agents, however, do not know the exact value of this bias. This gives rise to an additional component of noise, which makes the perceived noise also likely to be persistent.[3]

The second interpretation comes from the literature on rational inattention. Consider a slightly generalized version of the attention problem studied by Sims (2003) and Mackowiak and Wiederholt (2009).

$$\begin{aligned} \min_{b_0, b(\cdot), c_t(\cdot)} \quad & \mathbb{E}\left[\left(\mathbb{E}[a_t|s^t] - a_t\right)^2\right] \\ \text{s.t.} \quad & \mathcal{I}(\{s_t\};\{a_t\}) \leq \kappa \\ & a_t = \rho_a a_{t-1} + \varepsilon_t \\ & s_t = b_0 + b(L)\varepsilon_t + c_t(L)\tilde{\xi}_t \end{aligned}$$

where ε_t and $\tilde{\xi}_t$ follow independent Gaussian white noise processes.[4] The decision maker chooses a signal process s_t to forecast a_t, subject to a constraint on the information flow

[3] This makes no difference in static settings as it only adds another layer of uncertainty about the stochastic relationship. To see this point, consider a signal $s = a + \xi$ subject to bias, $\xi = bias + e$. An agent believes that the bias is distributed as $bias \sim F(\cdot)$ and that the (unbiased) error term is distributed as $e \sim G(\cdot)$. In static settings, it is isomorphic to the case in which the agent believes that the noise term ξ follows a *known distribution*

$$P(\xi \leq \hat{\xi}) = \int F(\hat{\xi} - e)dG(e).$$

In contrast, this additional layer of uncertainty is important in dynamic settings. If the bias in an information source is persistent, agents then have incentives to correct this bias over time. To see this clearly, consider a dynamic extension in which the agent uses the signal $s_t = a_t + \xi_t$ to form expectations about a_t for two periods $t = 0, 1$. Suppose we assume that $\xi_t = bias + e_t$ so that the bias is time-invariant. Assume further that the agent can observe the true realization of a_0 at the beginning of period 1. The agent then makes another prediction about a_1 after observing s_1. Then, the agent tries to correct the bias by comparing the previous signal s_0 with the realization a_0.

[4] This is a generalized version as we consider $c_t(\cdot)$ instead of a time invariant function $c(\cdot)$.

between $\{s_t\}$ and $\{a_t\}$, which sets an upper bound for

$$\mathcal{I}(\{s_t\};\{a_t\}) \equiv \lim_{T\to\infty} \frac{1}{T} I(s_1,\cdots,s_T;a_1,\cdots,a_T)$$

where $I(\cdot;\cdot)$ denotes the mutual information. They show that it is without loss to assume that the decision maker makes a forecast after observing a signal of the form "true state plus a time-independent noise term:"

$$s_t = a_t + \sigma \cdot \tilde{\xi}_t$$

where σ is a constant. In other words, such signals are optimal when agents can choose an information structure under the above constraint.

This result gives an elegant justification for the *i.i.d.* assumption. It is, however, not robust in the sense that it depends crucially on the precise form of the information constraint. For example, when we consider other forms of information constraints such as

$$I(s_t;a_t) \leq \kappa, \quad \forall t, \tag{1}$$

then we can easily show that agents can always be better off by inducing correlations in their signals.

Lemma 1 (Optimal Signal). *The signals with i.i.d. noise terms, $s_t = a_t + \sigma \cdot \tilde{\xi}_t$, cannot attain the minimum of the following attention problem*

$$\begin{aligned}
\min_{b_0, b(\cdot), c_t(\cdot)} \quad & \mathbb{E}\left[\left(\mathbb{E}[a_t|s^t] - a_t\right)^2\right] \\
\text{s.t.} \quad & I(s_t;a_t) \leq \kappa, \quad \forall t \\
& a_t = \rho_a a_{t-1} + \varepsilon_t \\
& s_t = b_0 + b(L)\varepsilon_t + c_t(L)\tilde{\xi}_t.
\end{aligned}$$

Instead, signals with persistent noise terms, $s_t = a_t + \sigma \cdot \tilde{\xi}$ where $\tilde{\xi} \sim \mathcal{N}(0,1)$ attain the minimum, which is equal to zero.

Proof. All proofs are in Appendix A. □

The intuition is simple, agents can make noise terms correlated across time periods *for free* under this information constraint.[5] By doing so, agents can dynamically learn and correct the

[5] This is costly in the original formulation of Sims (2003) and Mackowiak and Wiederholt (2009).

persistent component of noise.[6] There is, of course, no a priori reason that the information constraint in the real world is given by (1). However, the same is true that no reason favors the original information constraint of Sims (2003) and Mackowiak and Wiederholt (2009). Thus, Lemma 1 gives us a takeaway that unless the real-world information constraint is exactly the same as what Sims (2003) and Mackowiak and Wiederholt (2009) postulate, decision makers would optimally choose correlated noise terms in order to exploit their abilities to correct the persistent component of the noise terms over time.

Illustrative Example. We have argued that regardless of how we interpret the noise term, its counterpart in the real world is likely to be correlated across time periods. To further illustrate this argument, consider a forecaster who makes predictions of, say the annual US inflation rate. There are various information sources she can use to reach her prediction, but suppose that she only gets information from newspapers. There are N available newspapers, each of which is informative about the inflation rate. The forecaster chooses to subscribe to a subset of the newspapers while being subject to attention costs. Thus, these newspapers can be thought of as information sources in Myatt and Wallace (2004) and Pavan (2016). First note that if a specific newspaper gives a positively biased view of the inflation rate this year, then it is likely to give a view biased in the same direction next year. Consider a constraint on the attention cost which requires that the forecaster can read at most, say, three newspapers each year. Under this constraint, which is analogous to the cosntraint (1), the forecaster would optimally choose to read the same set of newspapers every year because she can at least partly correct the bias in the newspapers by herself.[7] Since the bias in each newspaper is persistent, and the forecaster would choose to read the same newspapers, it is as if she were receiving a noisy signal whose noise term is correlated across time periods. In contrast, under a constraint which is analogous to the original constraint in Sims (2003) and Mackowiak and Wiederholt (2009), if she reads three specific newspapers this year, then it would be strictly more costly (in terms of the cognitive cost) to read the same set of newspapers next year because she is able to get strictly more information from those newspapers. Thus, we reach a counterintuitive conclusion that it can be optimal to read three *new* newspapers every year.

Remark. To fix ideas, we talked about biased information sources and agents who try to learn this bias. In the real world, however, information sources are not only biased but also absent from a pre-specified way of interpreting them. For example, if you observe that the current unemployment rate is 5%, then how can you interpret this number as a signal about the current

[6] This is obvious when agents receive feedback later on so the objective function changes to $\mathbb{E}\left[(\mathbb{E}[a_t|s^t, a^{t-1}] - a_t)^2\right]$. However, this also holds without such feedback because agents can ultimately learn the precise value of $\tilde{\xi}$ after observing an infinite number of signal realizations.

[7] *"Better the devil you know than the devil you don't."*

inflation rate? It is indeed informative about the inflation rate to some degree, but it is not like observing a random variable which is distributed around the inflation rate as we assumed. Forecasters need to interpret information sources they have, but they are uncertain about how to interpret them. Thus, the forecaster in the previous example can also be seen as the one who tries to correct her way of interpreting newspapers. In this sense, "learning noise terms" in this paper also means "learning how to interpret information."

We have argued so far that it is natural to assume persistent noise terms. From now on, we model this persistence by simply assuming that agents observe a noisy signal about the fundamental a_t

$$s_t = a_t + \xi_t$$

whose noise term ξ_t is auto-correlated[8]

$$\xi_t = \rho \xi_{t-1} + \eta_t.$$

This is only a small departure from the literature and enables us to maintain tractability.

The persistence of noise terms naturally leads to a formal definition of optimism. A crucial difference from the model with *i.i.d.* noise terms is that agents try to learn the noise terms ξ_t in their signals. This affects how optimistic they are in interpreting the signals, which in turn affects their forecasts and hence their actions. We consider two information sets $\tilde{\Omega}_t$ and Ω_t, which are the information sets of an agent right before and right after, respectively, observing a signal s_t. From an outside observer's perspective, if an agent underestimates her noise term ξ_t, then for a given realization of her signal s_t, she would overestimate the fundamental a_t. Therefore, an agent is *optimistic* in interpreting her signal s_t if her belief about the noise term ξ_t is *lower* than the true value. This discussion leads to the following definition of (first-order) optimism.[9]

Definition 1. *An agent is said to be* ex-ante (ex-post, respectively) optimistic *if she underestimates the noise term in her signal:*

$$\mathbb{E}[\xi_t|\tilde{\Omega}_t] < \xi_t \quad (\mathbb{E}[\xi_t|\Omega_t] < \xi_t, \textit{ respectively})$$

The ex-ante (ex-post, respectively) optimism *of an agent is defined to be the extent to which she underestimates the noise term:*

$$\tilde{\mathcal{O}}_t \equiv \xi_t - \mathbb{E}[\xi_t|\tilde{\Omega}_t] \quad (\mathcal{O}_t \equiv \xi_t - \mathbb{E}[\xi_t|\Omega_t], \textit{ respectively})$$

[8] Throughout the paper, we will use the letter ξ to denote noise terms.

[9] In Section 4, we will define higher-order optimism when there are multiple agents who play a game with strategic complementarity.

Thus, ex-ante optimism captures how optimistic an agent is before observing a signal, and ex-post optimism captures how optimistic she is in interpreting a realized signal. In the next section, we will see how agents' dynamic learning about the persistent noise terms endogenously generates optimism, and the effect of optimism on equilibrium outcomes.

3. The Baseline Model

We start with a macroeconomic model in which firms' output choices are made under incomplete information about their productivity. The model structure is closely related to the island model in Angeletos and La'O (2009b) and Benhabib, Wang and Wen (2015), but the timeline is more similar to that in Kohlhas and Walther (2020). In this section, we consider specific parameter values under which we have no strategic complementarity between firms' decisions. This enables us to focus on how firms learn about persistent noise terms in their *own* signals. Compared to the benchmark case with *i.i.d.* noise terms—as is often the case in the literature—this novel channel of learning endogenously generates optimism and amplifies or dampens underlying shocks, depending on how much firms are informed about the shocks. In the next section, we consider the case with strategic complementarity and show that the presence of strategic complementarity always strengthens our mechanism.

Timeline. There is an infinite number of periods $t = 0, 1, \cdots$ and a representative household consisting of a continuum of workers. We use the island analogy of Lucas (1972) to capture the incompleteness of information in the real world. There is a continuum of islands $i \in [0, 1]$, each of which has its own labor market and own information set each period. Each island i is inhabited by a continuum of firms $j \in [0, 1]$, each of which specializes in the production of differentiated commodities. We will index these firms and their commodities by $(i, j) \in [0, 1] \times [0, 1]$. The timeline is as follows. First, at the beginning of each period, the household sends one worker to each island. Second, after observing noisy signals about island-specific productivity, firms commit to their output levels, and workers post wages at which they commit to supply any amount of labor. Third, the island-specific productivity realizes and firms' labor demand is determined by the committed level of output and the productivity. Finally, workers return to their home and commodity markets open. Prices adjust to clear the markets.

Household. A representative household consists of a continuum of workers who solve a team problem of jointly maximizing the household utility, which is given by

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t(\log C_t - \nu\cdot N_t)\right]$$

where $N_t = \int_0^1 N_{it}\,\mathrm{d}i$ is the total labor supply of its workers. For simplicity, we assume a unit intertemporal elasticity of substitution and a unit Frisch elasticity of labor supply, but none of our results qualitatively depend on this assumption. The consumption C_t has a nested structure. First, it is CES aggregation of the consumption bundle $\{C_{it}\}_{i\in[0,1]}$ from different islands,

$$C_t = \left(\int_0^1 C_{it}^{1-\frac{1}{\sigma}}\,\mathrm{d}i\right)^{\frac{\sigma}{\sigma-1}}$$

where $\sigma \le 1$ is the elasticity of substitution across consumption from different islands. Second, the consumption C_{it} from island i is also CES aggregation of the consumption bundle $\{C_{ijt}\}_{j\in[0,1]}$ from monopolistic firms in island i

$$C_{it} = \left(\int_0^1 C_{ijt}^{1-\frac{1}{\eta}}\,\mathrm{d}j\right)^{\frac{\eta}{\eta-1}} \tag{2}$$

where $\eta > 1$ is the elasticity of substitution across firms. We normalize the price index to one,

$$1 = P_t \equiv \left(\int_0^1 P_{it}^{1-\sigma}\,\mathrm{d}i\right)^{\frac{1}{1-\sigma}} \quad \text{where} \;\; P_{it} = \left(\int_0^1 P_{ijt}^{1-\eta}\,\mathrm{d}j\right)^{\frac{1}{1-\eta}}$$

where P_{ijt} is the price of the good from firm j in island i, P_{it} is the price index for goods in island i.

The budget constraint dictates that the total purchase of consumption goods and bonds cannot exceed the total income, which consists of profits, wage, and payment from the bond:[10]

$$\int_0^1\int_0^1 P_{ijt}C_{ijt}\,\mathrm{d}i\,\mathrm{d}j + B_{t+1} \le \int_0^1\int_0^1 \Pi_{ijt}\,\mathrm{d}i\,\mathrm{d}j + \int_0^1 W_{it}N_{it}\,\mathrm{d}i + (1+R_t)B_t$$

where B_t is the bond holding in period t, R_t is the gross interest rate between period t and $t+1$, Π_{ijt} is the profit of firm j in island i, W_{it} is the wage in island i, and N_{it} is the labor supply of its worker sent to island i. When workers jointly maximizing the utility of the household they belong to, they are subject to informational constraints. Workers sent to different islands, and their labor supply decisions are based on different information sets. After they return to

[10] We do not need to distinguish nominal terms and real terms because we normalize $P_t = 1$.

their home, all the information is shared, and the household makes consumption and saving decisions.

Firms. Firm j in island i has a production function

$$Y_{ijt} = A_{it} \cdot N_{ijt}^{\theta}$$

where A_{it} is the common productivity of firms in island i, N_{ijt} is the firm's employment, and $\theta \in (0, 1]$ governs the decreasing return to scale in production. After commodity markets open and prices clear these markets, the firm's realized profit is

$$\Pi_{ijt} = P_{ijt}Y_{ijt} - W_{it}N_{ijt}.$$

Market Clearing. A key feature of the model is that decisions of firms and workers are made under imperfect information. After observing a signal for island-specific productivity, firms commit to the output level, Y_{ijt}, and workers post wages W_{it}. As will become evident in Lemma 2 below, this assumption makes firms choose higher output levels when they are more optimistic. Under this assumption, the labor market clearing is trivial: given the island-specific productivity, A_{it}, Firm j is island i demands

$$N_{ijt} = (Y_{ijt}/A_{it})^{1/\theta} \tag{3}$$

units of labor at the equilibrium wage W_{it}. After production takes place, goods markets open, and the price P_{ijt} adjusts to clear the market: $C_{ijt} = Y_{ijt}$ for all (i, j).

Shocks and Information. The only uncertainty is on the island-specific productivity, A_{it}, which follows an AR(1) process in log,[11]

$$a_{it} \equiv \log A_{it} = \rho_a a_{it-1} + \varepsilon_{it}.$$

We further assume that the innovation ε_{it} consists of two components,

$$\varepsilon_{it} = \varepsilon_{it}^{p} + \varepsilon_{it}^{u}$$

where $\varepsilon_{it}^{p} \sim \mathcal{N}\left(0, \sigma_p^2\right)$ and $\varepsilon_{it}^{u} \sim \mathcal{N}(0, \sigma_u^2)$ are independent across time periods and independent of each other. We do not restrict their correlation structure across islands in this section because

[11] Another way of modeling uncertainty is to assume island-specific preference shocks and aggregate demand shocks. But it is more difficult to take this route because it entails endogenous signals.

this correlation is not important without strategic interaction between firms in different islands. We define corresponding aggregate components of these variables as

$$a_t \equiv \int_0^1 a_{it}\,\mathrm{d}i, \;\; \varepsilon_t^p \equiv \int_0^1 \varepsilon_{it}^p\,\mathrm{d}i, \;\; \text{and} \;\; \varepsilon_t^u \equiv \int_0^1 \varepsilon_{it}^u\,\mathrm{d}i.$$

The difference between these two components is the extent to which firms are informed about them. The shock ε_{it}^p is called a *partly-observed* shock because, before making their decisions, firms and workers in island i receive a noisy signal

$$s_{it} = \rho_a a_{it-1} + \varepsilon_{it}^p + \xi_{it}. \tag{4}$$

Thus, they are at least partly informed about this shock when they make their decisions in period t. In contrast, the shock ε_{it}^u is called *unobserved* because it is not contained in firms' and workers' information set when they make their decisions at period t. Because the distinction between partly-observed shocks and unobserved shocks are important for our mechanism, we summarize it in Definition 2. This distinction formalizes the previous observation that there are many shocks in the real world that differ in the extent to which economic agents can be informed about them.

Definition 2. *Both partly-observed shocks ε_{it}^p and unobserved shocks ε_{it}^u drive the island-specific productivity. But when firms and workers make their decisions, they only receive signals about the partly-observed shocks while being completely uninformed about the unobserved shocks.*

Given a period-t information set Ω_{it} that we will soon specify, firms and workers in island i maximize their expected profits and expected household utility, respectively. Formally, firm j in island i chooses the level of Y_{ijt} that solves

$$\begin{aligned}
\max_{Y_{ijt}} \quad & \mathbb{E}\big[C_t^{-1} \cdot \Pi_{ijt} \big| \Omega_{it}\big] \\
\text{s.t.} \quad & \Pi_{ijt} = P_{ijt} Y_{ijt} - W_{it} N_{ijt} \\
& Y_{ijt} = \left(\frac{P_{ijt}}{P_{it}}\right)^{-\eta} \left(\frac{P_{it}}{P_t}\right)^{-\sigma} Y_t \\
& Y_{ijt} = A_{it} \cdot N_{ijt}^{\theta}
\end{aligned}$$

where the second constraint comes from the isoelastic demand relation. Also, the representative worker in island i chooses the level of W_{it} in a competitive way that solves

$$\max_{W_{it}} \quad \mathbb{E}\left[C_t^{-1}\frac{W_{it}}{P_t}N_{it} - \nu N_{it}\middle|\Omega_{it}\right]$$
$$\text{s.t.} \quad N_{it} = \begin{cases} 0 & \text{if } W_{it} > W'_{it} \\ \int_0^1 \left(\frac{Y_{ijt}}{A_{it}}\right)^{\frac{1}{\theta}} \mathrm{d}j & \text{if } W_{it} = W'_{it} \\ \infty & \text{if } W_{it} < W'_{it} \end{cases}$$

where W'_{it} is the wage level that other workers in island i choose. We should have $W'_{it} = W_{it}$ in the equilibrium.

Persistent Noise and Feedback. The framework so far is very similar to those typically used in the literature, except for the timing assumption. Now we introduce our two main assumptions. First, we assume that the noise terms in the signals are persistent:

$$\xi_{it} = \rho\xi_{it-1} + \eta_{it} \quad \text{where } \eta_{it} \sim \mathcal{N}\big(0, (1-\rho^2)\sigma_\eta^2\big)$$

where the innovation η_{it} is *i.i.d.* across time periods and across islands.[12] After observing the signal, firms form their beliefs about the island-specific productivity a_{it}.

Second, we assume that firms in island i receive feedback on their previous forecast by observing the true productivity realization a_{it} after making their decisions. Thus, the information set Ω_{it} that the firms' forecast is based on contains not only their signals up to date t but also the history of previous feedback[13]

$$\Omega_{it} = (\cdots, s_{it-2}, a_{it-2}, s_{it-1}, a_{it-1}, s_{it}).$$

This is a natural assumption in our setting. We assume that firms commit to the output levels, Y_{ijt}. When labor market opens, hence, they have to know the exact levels of their productivity A_{it} in order to compute the amount of labor needed to fulfill their commitment, which is given by equation (3). Likewise, workers can compute the island-specific productivity based on the labor demand of firms in their island. The main role of this assumption is to allow

[12] Whether these noise terms are correlated across islands is irrelevant in this section as we will assume away strategic interaction between islands.

[13] One might argue that it is also natural to assume that firms and workers can learn from the commodity markets. Indeed firms and workers can fully learn the aggregate shock ε_t^u from observing the prices and quantities in the commodity markets. However, we consider agents suffering a form of internal schizophrenia as in the vast majority of the literature. We think of the firms having two personalities. One choosing Y_{ijt} is inattentive and do not learn from the commodity markets, and another, who does not communicate with the former, adjusts the price to clear the commodity market. See Angeletos and La'O (2009a) for trade-offs in this modeling choice.

agents to receive feedback on their previous forecasts by observing the true realization ex post. However, it also plays an important role in making the analysis tractable and is exactly the same assumption that many papers adopt in order to simplify the learning to an essentially static one.[14] In general, firms dynamically learn the values of both a_{it} and ξ_{it}. The presence of feedback, however, essentially allows us to abstract from the dynamic learning about a_{it} and to focus on the dynamic learning about ξ_{it}. The logic of this section, however, would still hold insofar as firms observe the true productivity ex post with sufficiently high precision. Lastly, we define an additional information set that agents possess right before observing the signal,

$$\tilde{\Omega}_{it} = \Omega_{it} \backslash (s_{it}) = (\cdots, s_{it-2}, a_{it-2}, s_{it-1}, a_{it-1}).$$

Recall that we define ex-ante and ex-post optimism as

$$\begin{aligned}
\tilde{\mathcal{O}}_{it} &= \xi_{it} - \tilde{\mathbb{E}}_{it}[\xi_{it}] \quad \text{where } \tilde{\mathbb{E}}_{it}[\cdot] \equiv \mathbb{E}[\cdot|\tilde{\Omega}_{it}] \\
\mathcal{O}_{it} &= \xi_{it} - \mathbb{E}_{it}[\xi_{it}] \quad \text{where } \mathbb{E}_{it}[\cdot] \equiv \mathbb{E}[\cdot|\Omega_{it}].
\end{aligned}$$

We summarize the timeline of the model in Table 1. To emphasize the difference in the timing between receiving the signal s_{it} and receiving the feedback a_{it}, we think of each period as consisting of two stages. Finally, we define an equilibrium as follows where we write $\Omega \equiv (\Omega_{it})_i$ and $A \equiv (A_{it})_i$.

Definition 3. *A* rational expectations equilibrium *is a sequence of allocations* $\{C_{ijt}(\Omega, A), Y_{ijt}(\Omega_{it}), N_{ijt}(\Omega_{it}, A_{it})\}$ *and prices* $\{W_{it}(\Omega_{it}), P_{ijt}(\Omega, A)\}$ *such that (i) In stage 1, workers and firms maximize their expected objective functions based on the information they have; (ii) In stage 2, the representative household maximizes its utility, taking the prices as given; and (iii) All markets clear.*

Illustrative Example (Continued). To see that the main message of this section is not confined to our macroeconomic example, let us go back to the previous forecaster example. A forecaster i makes a forecast, or nowcast, $y_{it} = \mathbb{E}_{it}[a_{it}]$ about the inflation rate a_{it} each year. The forecaster is indeed partly informed about some shocks from the newspapers. However, there are many other shocks that affect the inflation rate while not being covered in the newspapers, or whose effects on the inflation rate are not even conceived by the forecaster. These are captured in the unobserved shock ε^u_{it}. Afterwards the forecaster can observe the realized value of the inflation rate.[15]

[14] Many papers use the strategy of giving intuition under the assumption that learning takes place within a period, and showing that the same intuition preserves to the general setting using numerical examples.

[15] Based on the literature on rational inattention such as Sims (2003) and Mackowiak and Wiederholt (2009), one might argue that even if the realized inflation rates are publicly revealed, the forecasters might not pay attention

Table 1: Timeline

		$\vdots$
		$\boxed{a_{it-1}} = \rho_a a_{it-1} + \varepsilon^p_{it-1} + \varepsilon^u_{it-1}$
period t	stage 1	$\boxed{s_{it}} = \rho_a a_{it-1} + \varepsilon^p_{it} + \xi_{it}$ where $\xi_{it} = \rho \xi_{it-1} + \eta_{it}$ Commit to Y_{ijt} and W_{it}
	stage 2	$\boxed{a_{it}} = \rho_a a_{it-1} + \varepsilon^p_{it} + \varepsilon^u_{it}$
		$\vdots$

Note: The variables in boxes are those observed by agents in island i. All the shocks (except for ξ_{it}) indexed by t are independent across time periods. All different types of shocks are independent of each other.

Optimality Conditions. The optimal wage choice of the representative worker in island i is given by

$$W_{it} = \nu \cdot \left(\mathbb{E}\left[C_t^{-1} \middle| \Omega_{it}\right]\right)^{-1}$$

which is a intratemporal optimality condition equating the marginal disutility from labor with the marginal utility from consumption. Log-linearizing this condition yields[16]

$$w_{it} = \mathbb{E}[c_t|\Omega_{it}] \tag{5}$$

where we use small letters to denote the log deviations from the steady state values. The higher the aggregate consumption workers expect, the higher the wage needed to compensate them. Next, consider the firm's optimization problem. Firm j in island i solves

$$\max_{Y_{ijt}} \; \mathbb{E}_{it}\left[Y_t^{-1}\left\{Y_{ijt}^{1-\frac{1}{\eta}} Y_t^{\frac{1}{\eta}} P_{it}^{1-\frac{\sigma}{\eta}} - W_{it}\left(\frac{Y_{ijt}}{A_{it}}\right)^{\frac{1}{\theta}}\right\}\right]$$

where we impose $C_t = Y_t$. The first order condition gives

$$\left(1 - \tfrac{1}{\eta}\right) Y_{ijt}^{-\frac{1}{\eta}} \mathbb{E}_{it}\left[Y_t^{-1+\frac{1}{\eta}} P_{it}^{1-\frac{\sigma}{\eta}}\right] = \frac{1}{\theta} Y_{ijt}^{\frac{1}{\theta}-1} \mathbb{E}_{it}\left[Y_t^{-1} W_{it} A_{it}^{-\frac{1}{\theta}}\right].$$

to them. However, it is unlikely that the forecasters who made a prediction for the inflation rate do not pay high attention to the realized value of it.

[16] Since Lucas (1972), the log linearization is frequently used in the literature as it allows a simple representation of the equilibrium with a signal extraction problem.

Log-linearizing, we have

$$\Big(1-\tfrac{1}{\eta}-\tfrac{1}{\theta}\Big)y_{ijt}=\mathbb{E}_{it}\Big[-\tfrac{1}{\eta}y_t+w_{it}-\tfrac{1}{\theta}a_{it}-\Big(1-\tfrac{\sigma}{\eta}\Big)p_{it}\Big].$$

The symmetry across firms in island i implies

$$P_{it}=P_{ijt}=\left(\frac{Y_{it}}{Y_t}\right)^{-\frac{1}{\sigma}},$$

which together with equation (5) shows that the equilibrium of our micro-founded model can be represented by the perfect Bayesian equilibrium of games with strategic complementarity as in Angeletos and La'O (2009a).

Lemma 2 (Representation). *Firms' equilibrium output choices, up to a log-linear approximation, are characterized by the solution to the following fixed-point problem:*

$$y_{ijt}=\mathbb{E}_{it}[(1-\alpha)a_{it}+\alpha y_t]$$

where the degree of strategic complementarity α is given by

$$\alpha=\frac{1/\sigma-1}{1/\theta+1/\sigma-1}\in[0,1).$$

In this section we maintain the following assumption in order to assume away strategic complementarity, $\alpha=0$, so that we can focus on how a firm learns about her own noise term.

Assumption 1 (No Strategic Complementarity). $\sigma=1$.

Remark. Before we proceed, it is worth discussing some modeling choices we have made. First, it makes no difference if we assume a general CRRA utility from consumption, $\frac{C_t^{1-\gamma}}{1-\gamma}$. We can follow the same steps to show that the equilibrium output choice is given by

$$y_{ijt}=\mathbb{E}_{it}[(1-\alpha)\tilde{a}_{it}+\alpha y_t]$$

where $\tilde{a}_{it}=\frac{1/\theta}{1/\theta+\gamma-1}a_{it}$ is the normalized productivity. With this normalization, we can get the same result as in the case with $\gamma=1$. Second, if we assume that the disutility from labor has a form $\nu\cdot\frac{N_t^{1+\varepsilon}}{1+\varepsilon}$ instead of $\nu\cdot N_t$, then we have an additional aggregate productivity term in the representation, $y_{ijt}=\mathbb{E}_{it}[(1-\alpha)a_{it}+\alpha y_t+\beta\cdot a_t]$ for some constant $\beta>0$. This only complicates the learning of firms without giving further intuition, so we assume constant marginal distutility from labor. Third, if we assume that firms commit to N_{ijt} instead of Y_{ijt}, then the optimal labor demand choice N_{ijt} is decreasing instead of increasing in $\mathbb{E}_{it}[A_{it}]$. This is because the CES aggregation (2) features diminishing marginal return, which implies that when productivity doubles, firms would increase their output less than double. Thus, optimism leads to lower outputs, which is the opposite of what we are trying to capture.

Benchmark: *i.i.d.* noise terms. We close this section with a benchmark case with *i.i.d.* noise terms as in the literature. This case is nested in the previous model with $\rho = 0$. Since we abstract from the dynamic learning of productivity, nothing left to learn dynamically, and the problem essentially becomes a repetition of static learning problems. Thus, what happened in a period has no effect on firms' decisions in the next period. In particular, it does not change firms' optimism in the next period. Therefore, shocks can affect tomorrow's outputs only through their effects on the productivity, regardless of whether they are partly-observed or unobserved. This is one of the reasons why Woodford (2003) does not adopt the assumption of Lucas (1972) that fundamental—monetary disturbances—becomes public information within a period. Given the fact that the monetary statistics are reported promptly, Woodford (2003) follows Sims (2003) to assume limited attention. In this paper, however, underlying shocks will have persistent effects even with firms observing the productivity within a period.

We can guess and verify the coefficients of a linear equilibrium. Proposition 1 summarizes the result.

Proposition 1 (Benchmark). *If noise terms in signals are i.i.d., $\rho = 0$, the optimal output choice of firms is given by*

$$y_{ijt+1} = \rho_a^2 a_{it-1} + \rho_a \varepsilon_{it}^p + \rho_a \varepsilon_{it}^u + \tilde{K}\varepsilon_{it+1}^p + \tilde{K}\eta_{it+1} \quad \text{where } \tilde{K} = \frac{\sigma_p^2}{\sigma_\eta^2 + \sigma_p^2} \in (0,1),$$

which can be aggregated into

$$y_{t+1} = \rho_a^2 a_{t-1} + \rho_a \varepsilon_t^p + \rho_a \varepsilon_t^u + \tilde{K}\varepsilon_{t+1}^p.$$

Thus, the effects of period-t shocks on period-$(t+1)$ outcomes are identical to their effects on the fundamental:

$$\frac{\partial y_{t+1}}{\partial \varepsilon_t^u} = \frac{\partial a_{t+1}}{\partial \varepsilon_t^u} \quad \textit{and} \quad \frac{\partial y_{t+1}}{\partial \varepsilon_t^p} = \frac{\partial a_{t+1}}{\partial \varepsilon_t^p}.$$

Note that a contemporaneous partly observed shock affects firms' decisions less than one-for-one, which reflects the fact that firms cannot fully identify this shock. On the other hand, a contemporaneous unobserved shock cannot affect their decisions as it is not in their information sets.

Persistent noise terms. Let us go back to our main case with persistent noise terms, $\rho \in (0, 1)$. We will characterize how firms update their beliefs about their noise terms and how this learning affects firms' forecasts of the productivity and hence their output choices. We write firms' belief about ξ_{it-1} right before observing s_{it} as $\xi_{it-1}|\tilde{\Omega}_{it} \sim \mathcal{N}(m_{it-1}, V_{t-1})$. After observing s_{it}, firms in island i make a forecast about a_{it} according to Bayes' rule:

Lemma 3 (Bayesian Updating). *Bayesian updating leads to the following forecast:*

$$\begin{aligned}
\mathbb{E}_{it}[a_{it}] &= \rho_a a_{it-1} + K_t\big(s_{it} - \rho_a a_{it-1} - \rho m_{it-1}\big) \quad \text{where } \; K_t = \tfrac{\sigma_p^2}{\rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2 + \sigma_p^2} \in (0,1) \\
&= \rho_a a_{it-1} + K_t\big(\varepsilon_{it}^p + \tilde{\mathcal{O}}_{it}\big) \\
&= \rho_a a_{it-1} + \varepsilon_{it}^p + \mathcal{O}_{it}
\end{aligned}$$

Since firms in island i know the value of a_{it-1} at this stage, it directly affects their forecasts. The contemporaneous partly-observed shock has an effect less than one-for-one (we will soon show that a positive realization of ε_{it}^p reduces $\mathcal{O}_{it}$), while the contemporaneous unobserved shock has no effect as in the benchmark. The difference is that now firms' forecasts also depend on how optimistic they are. Optimistic firms interpret their signals more optimistically, which leads them to make optimistic forecasts.

After making a forecast, firms in island i receive feedback at stage 2 by observing the true productivity, a_{it}, and revise their beliefs about the noise term by looking back on their previous forecasts. This learning can be characterized by the Kalman filter and the results are summarized in the next lemma.

Lemma 4 (Kalman Filter). *The law of motions for m_{it} and V_t are given by*

$$\begin{aligned}
m_{it} &= (\gamma_1(V_{t-1}) + \gamma_2(V_{t-1})) \cdot s_{it} + \gamma_3(V_{t-1}) \cdot \rho m_{it-1} - \gamma_1(V_{t-1}) \cdot \rho_a a_{it-1} - \gamma_2(V_{t-1}) \cdot a_{it} \\
V_t &= \frac{\sigma_p^2 \sigma_u^2 (\rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2)}{(\sigma_p^2 + \sigma_u^2)(\rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2) + \sigma_p^2 \sigma_u^2}
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1(V_{t-1}) &= \tfrac{\sigma_u^2(\rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2)}{(\sigma_p^2+\sigma_u^2)(\rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2) + \sigma_p^2\sigma_u^2} \\
\gamma_2(V_{t-1}) &= \tfrac{\sigma_p^2(\rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2)}{(\sigma_p^2+\sigma_u^2)(\rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2) + \sigma_p^2\sigma_u^2} \\
\gamma_3(V_{t-1}) &= \tfrac{\sigma_p^2\sigma_u^2}{(\sigma_p^2+\sigma_u^2)(\rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2) + \sigma_p^2\sigma_u^2}.
\end{aligned}$$

Note that $\gamma_1(V_{t-1}), \gamma_2(V_{t-1}), \gamma_3(V_{t-1}) \in (0, 1)$ *and* $\gamma_1(V_{t-1}) + \gamma_2(V_{t-1}) + \gamma_3(V_{t-1}) = 1$.

We can easily prove that there is a unique fixed point V such that $V_{t-1} = V$ implies $V_t = V$. Also we can show that the sequence $(V_t)_t$ converges to this fixed-point for any initial value of

$V_0 \geq 0$. Thus, we will consider a stationary environment in which $V_t = V$ for all t. We can then write the law of motion for m_{it} in a time-invariant form:

$$m_{it} = (\gamma_1 + \gamma_2) \cdot s_{it} + \gamma_3 \cdot \rho m_{it-1} - \gamma_1 \cdot \rho_a a_{it-1} - \gamma_2 \cdot a_{it} \quad \text{where } \gamma_i \equiv \gamma_i(V).$$

Also, we define the stationary Kalman gain as $K \equiv \frac{\sigma_p^2}{\rho^2 V + (1-\rho^2)\sigma_\eta^2 + \sigma_p^2} \in (0,1)$. We are now able to characterize the dynamics of optimism. Recall that we defined the optimism as the extent to which firms in island i underestimate ξ_{it}.

Proposition 2 (Optimism). *The law of motions for optimism $\tilde{\mathcal{O}}_{it} \equiv \xi_{it} - \tilde{\mathbb{E}}_{it}[\xi_{it}]$ and $\mathcal{O}_{it} \equiv \xi_{it} - \mathbb{E}_{it}[\xi_{it}]$ are given by*

$$\begin{aligned}
\tilde{\mathcal{O}}_{it+1} &= \gamma_3 \rho \tilde{\mathcal{O}}_{it} - \rho\gamma_1 \varepsilon_{it}^p + \rho\gamma_2 \varepsilon_{it}^u + \eta_{it+1} \\
\mathcal{O}_{it+1} &= \gamma_3 \rho \mathcal{O}_{it} - (1-K)\varepsilon_{it}^p + K\rho\gamma_2 \varepsilon_{it}^u + K\eta_{it+1}.
\end{aligned}$$

Thus, a positive realization of partly-observed shocks (unobserved shocks, respectively) makes firms pessimistic (optimistic, respectively) next period.

First, we can see that there is inertia in optimism as the firms can correct their optimism only through noisy learning.[17] Thus, a positive shock in the noise term, η_{it+1}, increases the firm's optimism and decays slowly over time. What is more interesting is the response of optimism to the underlying shocks. The partly observed shock and unobserved shock have the opposite effects on optimism. The intuition is simple. Suppose that the realized productivity $a_{it} = \rho_a a_{it-1} + \varepsilon_{it}^p + \varepsilon_{it}^u$ is greater than $\rho_a a_{it-1}$, and firms in island i observe this increase at the end of period-t. If this increase solely came from an increase in ε_{it}^u, then it would not be reflected in s_{it} at all, so this feedback a_{it} is *higher-than-expected* from the perspective of firms in island i. This makes them think they were too pessimistic in interpreting s_{it}, which in turn induces them to interpret s_{it+1} more optimistically next period. This is why a positive innovation in the unobserved shock increases the firms' optimism. In contrast, if the increase in the fundamental solely came from high ε_{it}^p, the agents rationally attribute this increase to both ε_{it}^p and ε_{it}^u when they observe a_{it}. However, the high realization of ε_{it}^p was fully reflected in s_{it}. Thus, the realized a_{it} is *lower-then-expected* to firms in island i. This makes them possess a more pessimistic belief next period.

Before turning to the analysis of firms' output choices, we will discuss comparative statics results for γ_1, γ_2, and γ_3 with respect to variance parameters, σ_p^2, σ_u^2, and σ_η^2. Lemma 5 summarizes the results.

Lemma 5 (Comparative Statics). *We have the following comparative statics.*

[17] In the discussion below, the optimism means both ex-ante and ex-post optimism.

(1) γ_1 is increasing in σ_u^2 and σ_η^2, while decreasing in σ_p^2

(2) γ_2 is increasing in σ_p^2 and σ_η^2, while decreasing in σ_u^2

(3) γ_3 is increasing in σ_p^2 and σ_u^2, while decreasing in σ_η^2

To understand this result, consider Part (2) first, which states how the effect of the unobserved shock on optimism, γ_2, depends on variance parameters. The main mechanism that changes the optimism is the rational confusion between various shocks. If the partly-observed shock is relatively more volatile, then firms misattribute an increase in the unobserved shock more to the partly-observed shock, so they underestimate their noise terms more. Thus, we get a larger effect of the unobserved shock on optimism. Following the same logic, σ_u^2 tends to decrease the effect of the unobserved shock on optimism. Moreover, since the optimism arises as firms overestimate or underestimate their ξ_{it}, and firms are more likely to do so when σ_η^2 is high, the effect of the unobserved shock on optimism tends to increase in σ_η^2. This explains Part (2), and we can apply the same argument for Part (1). For Part (3), note that γ_3 is the coefficient that determines the degree of inertia in optimism. This inertia is originated from firms' rational confusion between ξ_{it} and $(\varepsilon_{it}^p, \varepsilon_{it}^u)$, which makes them unable to fully correct their optimism. This explains why γ_3 is decreasing in the relative size of σ_η^2 compared to σ_p^2 and σ_u^2.

Combining the results so far, we can characterizes the dynamics of the output choice as in the next theorem, which is our first main result.

Theorem 1 (Output). *When noise terms in signals are persistent, $\rho > 0$, the optimal output choice of firms is given by*

$$y_{ijt+1} = \rho_a^2 a_{it-1} + (\rho_a - \rho K\gamma_1)\varepsilon_{it}^p + (\rho_a + \rho K\gamma_2)\varepsilon_{it}^u + K\varepsilon_{it+1}^p + K\eta_{it+1} + \rho\gamma_3 K\tilde{\mathcal{O}}_{it}$$

hence the aggregate output is

$$y_{t+1} = \rho_a^2 a_{t-1} + (\rho_a - \rho K\gamma_1)\varepsilon_t^p + (\rho_a + \rho K\gamma_2)\varepsilon_t^u + K\varepsilon_{t+1}^p + \rho\gamma_3 K \int_0^1 \tilde{\mathcal{O}}_{it}\,\mathrm{d}i.$$

Thus, the effects of partly-observed shocks (unobserved shocks, respectively) on the next period outcomes are dampened (amplified, respectively) compared to their effects on the productivity:

$$\frac{\partial y_{t+1}}{\partial \varepsilon_t^u} > \frac{\partial a_{t+1}}{\partial \varepsilon_t^u} \quad \text{and} \quad \frac{\partial y_{t+1}}{\partial \varepsilon_t^p} < \frac{\partial a_{t+1}}{\partial \varepsilon_t^p}.$$

Compared to the benchmark case in Proposition 1, this theorem establishes that the impact of the unobserved shock on the next period output is amplified through its impact on the agent's optimism.[18] At the same time, the impact of the partly-observed shock on the next period output is dampened as firms become pessimistic after a positive innovation in the partly-observed shock. A contemporaneous innovation η_{it+1} has a positive effect on the output since firms cannot fully distinguish it from other shocks, and its effect decays slowly over time as firms correct their optimism. A special case of interest is that with $\rho_a = 0$ (*i.i.d.* productivity). In this case, we can observe that optimism propagates the effect of unobserved shocks to the next period, while the effect of partly-observed shocks is negative next period. Note that shocks in this case cannot affect future outputs if we assume *i.i.d.* noise as in the literature. However, with persistent noise, those shocks can affect future outputs through their effects on firms' optimism.

Illustrative Example (Continued). Consider the inflation rate forecaster example again. Suppose that the forecaster predicted an inflation rate of 2% based on her reading of the newspapers. Suppose next that the true inflation rate turned out to be 1%. How does her way of interpreting the newspaper change? The fact that the inflation rate turned out to be *lower-than-expected* makes her think that she was *too optimistic* in interpreting the contents of the newspapers. So, the forecaster would rationally take this into account when she makes the prediction next time and would interpret the contents of the newspapers in a more pessimistic way.

Implication. The results so far might sound like "anything goes." It indeed implies that a shock can be either amplified or dampened.[19] Whether a given shock is amplified or dampened depends on how much firms are informed about that shock; i.e., the location of a given shock at the spectrum of *the degree of observability*, where at one extreme are fully-observed shocks, and at the other are unobserved shocks.

There are two ways to overcome this "anything goes" interpretation. First, we can use forecast data to measure the degree of observability of a shock of interest, then our theory disciplines its dynamic effects. This, however, is essentially an empirical question. Another way is to assume that the partly-observed shocks are more likely to be idiosyncratic, whereas the unobserved shocks are more likely to be common across firms. The assumption that agents are informed relatively well about idiosyncratic shocks and less about aggregate shocks is often

[18] Recall, however, that we have assumed away the sluggish response of expectations to the innovation in productivity. Thus, a correct interpretation of this result is that the presence of persistent noise terms amplifies (dampens, respectively) the effect of unobserved (partly-observed, respectively) shocks compared to the case with *i.i.d.* noise terms.

[19] Actually, with rational forecasts, the effect of one shock can be amplified precisely because the effect of another shock is dampened, and vice versa.

considered plausible in literature.[20] Based on this observation, we will make an additional assumption.

Assumption 2. *Firms make decisions based on noisy information about purely idiosyncratic shocks, but the true productivity also depends on aggregate shocks. That is,*

- *Partly-observed shocks are purely island specific:* $\varepsilon_{it}^{p} \overset{iid}{\sim} \mathcal{N}\big(0, \sigma_p^2\big)$ *across islands*
- *Unobserved shocks are common:* $\varepsilon_{it}^{u} \equiv \varepsilon_{t}^{aggr} \sim \mathcal{N}(0, \sigma_u^2)$

Under this assumption, partly-observed shocks still lead to rational confusion, but they are averaged out among the continuum of islands, $\varepsilon_t^p \equiv \int_0^1 \varepsilon_{it}^p \, \mathrm{d}i = 0$. Thus, we can aggregate the results in Proposition 2 and Theorem 1 as if the aggregate economy were driven only by unobserved shocks.

Corollary 1 (Aggregation). *Under Assumption 2, the aggregate output and aggregate optimism,* $\tilde{\mathcal{O}}_t = \int_0^1 \tilde{\mathcal{O}}_{it} \, \mathrm{d}i$*, follow*

$$
\begin{aligned}
y_{t+1} &= \rho_a^2 a_{t-1} + \beta_u \varepsilon_t^{aggr} + \beta_{\mathcal{O}} \tilde{\mathcal{O}}_t \\
\tilde{\mathcal{O}}_{t+1} &= \gamma_3 \rho \tilde{\mathcal{O}}_t + \rho \gamma_2 \varepsilon_t^{aggr}
\end{aligned}
$$

Thus, the aggregate shock has no contemporaneous effect on outcomes, while it has an amplified effect on the next period outcomes:

$$
\frac{\partial y_{t+1}}{\partial \varepsilon_t^{aggr}} > \frac{\partial a_{t+1}}{\partial \varepsilon_t^{aggr}} \quad \textit{while} \quad \frac{\partial y_{t+1}}{\partial \varepsilon_{t+1}^{aggr}} = 0 < \frac{\partial a_{t+1}}{\partial \varepsilon_{t+1}^{aggr}}.
$$

When firms make decisions based on noisy information about their idiosyncratic shocks while the economic condition also depends on an aggregate shock, the aggregate optimism fluctuates procyclically with the aggregate shock, and hence it has an amplified effect on the aggregate output after firms receive feedback on their previous forecasts. In contrast, the aggregate shock has no contemporaneous effect on the aggregate action and forecast. In other words, the aggregate shock that has a little effect on the contemporaneous expectation would be amplified later on when firms receive feedback. We can find suggestive evidence of this result in Angeletos, Huo and Sastry (2020). They show that, in response to aggregate shocks, agents' expectations underreact initially but overshoot later on. They attribute this *delayed overreaction* to a combination of dispersed information and behavioral over-extrapolation. However, this

[20] For a prominent example, Mackowiak and Wiederholt (2009) calibrate their model by matching the price changes observed in data and conclude that firms pay more attention to idiosyncratic conditions than to aggregate conditions. This is because idiosyncratic conditions are more volatile than aggregate conditions. The theory of Kohlhas and Walther (2020) also relies on this observation.

Table 2: Interaction between persistent noise and feedback

	Without feedback	With feedback
i.i.d. noise	No optimism	No optimism Static learning
Persistente noise	Higher-than-expected outcome $\Rightarrow$ pessimism	Higher-than-expected outcome $\Rightarrow$ optimism

finding can also be well understood using our result; expectations initially underreact due to the fact that agents are not informed about the aggregate shock—this part is identical to Angeletos, Huo and Sastry (2020)—and overshoot later on when they receive feedback and adjust their optimism. It is worth noting that the result of Corollary 1 does not rely on the exact form of Assumption 2. In Section 5 we will get qualitatively similar delayed overreaction as long as firms are informed relatively well about idiosyncratic shocks.

Remark. We conclude this section with a remark on the importance of the interaction between persistent noise terms and the presence of feedback in obtaining our mechanism. This is summarized in Table 2. First of all, if noise terms are *i.i.d.*, then the presence of feedback does not affect any qualitative results. This is because firms learn nothing ex post, so feedback does not affect firms' learning. Second, the presence of feedback is crucial to make our mechanism work. We essentially assume that firms observe two signals, s_{it} and a_{it}, and the second signal gives feedback on the forecast made with the first signal. One might argue that the second signal plays a redundant role in the sense that, even if firms only have the first signal, s_{it}, they can receive feedback from the future signal, s_{it+1}, and can learn the persistent noise terms. To formalize our mechanism, however, it is crucial to incorporate the second signal into the model.[21] To see this, suppose that firms in island i observe only the first signal, $s_{it} = \rho_a a_{it-1} + \varepsilon^p_{it} + \xi_{it}$, each period. The firms can indeed get feedback on s_{it} when they observe s_{it+1}, as this signal contains partial information about ξ_{it}. However, this feedback gives a result which is exactly the opposite of our previous intuition; with this feedback, higher-than-expected outcomes make firms *pessimistic*.[22] The reason is that, if firms observe a higher-than-expected signal at $t+1$ due to a positive innovation in ε^u_{it}, they partly attribute this surprise to a higher realization of ξ_{it+1}, which means that they become pessimistic. We conclude that what underlies our mechanism is the interaction between persistent noise terms and the presence of feedback.

[21] In general, we need another noisy signal about the true realization whose noise term is not much correlated with the first signal.

[22] Acharya, Benhabib and Huo (2019) document a similar result.

4. Strategic Complementarity

In this section, we illustrate how the introduction of strategic complementarity gives further insight. With strategic complementarity, firms have an incentive to forecast the actions of firms in other islands. To do so, they try to forecast other firms' optimism.[23] Similar to the literature on higher-order beliefs, firms in this model are not only concerned about others' optimism (second-order optimism), but also concerned about higher-order optimism—how other firms think about others' optimism, how other firms think about others' beliefs about others' optimism, and so on. We first characterize how firms update their higher-order optimism and how this higher-order optimism in turn affects the output choices of firms through its effects on higher-order beliefs about the productivity. The goal of this section is to show that the introduction of strategic complementarity and resulting higher-order optimism always work in the direction that strengthens the mechanism we document in the previous section.

The presence of higher-order optimism makes it difficult to solve the model due to the infinite regress problem of Townsend (1983), so we will make some simplifying assumptions to deliver some sharp analytical results. First, we consider a two-period version of the model where $t = 0, 1$. Second, we assume that productivity is *i.i.d.* across time periods (i.e., $\rho_a = 0$), focusing on how agents learn the noise terms. Third, we assume that noise terms are time-invariant, which we denote by ξ_i without t index, so agents in island i observe a signal of the form

$$s_{it} = \varepsilon^p_{it} + \xi_i$$

where ξ_i is independent across islands[24]

$$\xi_i \overset{iid}{\sim} \mathcal{N}\big(0, \sigma^2_\xi\big).$$

These assumptions are not essential for our results but simplify our exposition. In a numerical exercise in Section 4.1, we will show that the main message of this section does not rely on these simplifying assumptions. Last but not least, we assume that islands share one common productivity, which we denote by a_t without i index,

$$a_t = \varepsilon^p_t + \varepsilon^u_t$$

[23] This is analogous to the literature on higher-order beliefs, where agents try to forecast beliefs of other agents in order to forecast others' actions.

[24] We can alternatively allow the possibility that noise terms are positively correlated across agents. Then, agents try to learn this common component, which can generate additional channel through which underlying shocks affect the (higher-order) optimism.

Remark. The last assumption of common productivity requires further explanation. Our main goal in this section is to study the role of strategic complementarity on firms' learning and optimal output choices under incomplete information. However, if we assume that ε_{it}^{p} is a pure idiosyncratic shock, $\int_0^1 \varepsilon_{it}^{p}\,\mathrm{d}i = 0$, then the presence of strategic complementarity not only affects the learning of firms but also reduces the importance of productivity in choosing output. To see this clearly, consider the following two static examples, which use the notation of our model but are more similar to Woodford (2003) or Angeletos and La'O (2009a) in terms of the information structure.

Example 1. There is a continuum of agents $i \in [0,1]$ who share a common fundamental $a \sim \mathcal{N}(0, \sigma_a^2)$. Each agent i chooses an action y_i after observing a private signal $s_i = a + \xi_i$ where $\xi_i \sim \mathcal{N}(0, \sigma_\xi^2)$ is i.i.d. across agents. Agents' best response is assumed to be $y_i = (1-\alpha)\mathbb{E}_i a_i + \alpha \mathbb{E}_i y$ where $y = \int_0^1 y_j\,\mathrm{d}j$. We can show that the equilibrium action is given by

$$y_i = \frac{(1-\alpha)\sigma_a^2}{\sigma_\xi^2 + (1-\alpha)\sigma_a^2} s_i \quad \textit{and} \quad y = \frac{(1-\alpha)\sigma_a^2}{\sigma_\xi^2 + (1-\alpha)\sigma_a^2} a$$

Example 2. The only difference from Example 1 is that fundamental $a_i \sim \mathcal{N}(0, \sigma_a^2)$ is i.i.d. across agents, hence $\int_0^1 a_j\,\mathrm{d}j = 0$. In this case, we always have $y = 0$ and the equilibrium action is given by

$$y_i = (1-\alpha)\mathbb{E}_i a_i = \frac{(1-\alpha)\sigma_a^2}{\sigma_\xi^2 + \sigma_a^2} s_i$$

Thus, even under complete information ($\sigma_\xi^2 = 0$), when fundamental is purely idiosyncratic as in Example 2, a higher degree of strategic complementarity makes agents less responsive to the change in fundamental. This comparative static is neither our goal of this section nor the inertia documented by Woodford (2003) and Angeletos and La'O (2009a). Instead, what these papers document is that, in Example 1, a higher degree of strategic complementarity makes agents less responsive to the change in fundamental *only when* information is incomplete ($\sigma_\xi^2 > 0$). This is because the higher the degree of strategic complementarity is, the more weight agents put on the common prior. Thus, the role of our last assumption is that it enables us to isolate the effect of strategic complementarity on firms' learning.[25]

As before, all firms observe the realized productivity a_t at the end of each period, which depends also on the unobserved shock, ε_t^u. Thus, firms in island i have three different information sets

$$\Omega_{i0} = (s_{i0}), \;\; \tilde{\Omega}_{i1} = (s_{i0}, a_0), \;\; \text{and} \;\; \Omega_{i1} = (s_{i0}, a_0, s_{i1}).$$

[25] We can assume instead that $a_{it} = \varepsilon_{it}^{p} + \varepsilon_t^{u}$ where $\int_0^1 \varepsilon_{jt}^{p}\,\mathrm{d}j = 0$. In this case, however, a relevant comparative statics is changing α when agents' best response is given by $y_{it} = \mathbb{E}_{it} a_{it} + \alpha \mathbb{E}_{it} y_t$.

Table 3: Timeline

period 0	stage 1	$\boxed{s_{i0}} = \varepsilon_0^p + \xi_i$ $y_{i0} = (1-\alpha)\cdot\mathbb{E}_{i0}a_0 + \alpha\cdot\mathbb{E}_{i0}y_0$ Commit to Y_{ij0} and W_{i0}
	stage 2	$\boxed{a_0} = \varepsilon_0^p + \varepsilon_0^u$
period 1	stage 1	$\boxed{s_{i1}} = \varepsilon_1^p + \xi_i$ $y_{i1} = (1-\alpha)\cdot\mathbb{E}_{i1}a_1 + \alpha\cdot\mathbb{E}_{i1}y_1$ Commit to Y_{ij1} and W_{i1}
	stage 2	$\boxed{a_1} = \varepsilon_1^p + \varepsilon_1^u$

Note: The variables in boxes are those observed by agent i. The distribution of each shock is given by $\varepsilon_t^p \sim \mathcal{N}(0, \sigma_p^2)$, $\varepsilon_t^u \sim \mathcal{N}(0, \sigma_u^2)$ and $\xi_i \sim \mathcal{N}(0, \sigma_\xi^2)$. All the shocks indexed by i are independent across agents. All the shocks indexed by t are independent across time. All different types of shocks are independent of each other.

To introduce strategic complementarity, we depart from Assumption 1 and assume the following.

Assumption 3 (Strategic Complementarity). *The trade linkage is strong enough to induce strategic complementarity in output choices across islands:* $1/\sigma > 1$.

The inverse of the substitutability between goods from different islands, $1/\sigma$, governs the strength of a trade linkage. With a strong trade linkage, firms increase their output choices when they expect do so. At the same time, however, the log utility features diminishing marginal utility, which makes firms decrease their output choices when they expect other firms to increase their output levels. This is because then household is expected to have low marginal utility from consumption, increasing the equilibrium wage. When Assumption 3 holds, the first effect dominates the second, and the optimal output choices feature strategic complementarity across islands:

$$y_{ijt} = (1-\alpha)\mathbb{E}_{it}[a_t] + \alpha\mathbb{E}_{it}[y_t] \quad \text{where } \; y_t = \int_0^1 y_{jt}\,\mathrm{d}j$$

Here, the weight $\alpha = \frac{1/\sigma - 1}{1/\theta + 1/\sigma - 1} \in (0,1)$ is the degree of strategic complementarity and y_t is the average action of other firms. We summarize the timeline of the model in Table 3. We assume that this structure of the model is common knowledge among all firms and workers.

Higher-Order Optimism. One might argue that there is no point for firms to learn others' average optimism because *i.i.d.* noise terms of a continuum of firms are averaged out. However, this is not the case. Firms try to correct their optimism by observing their signals, so their optimism endogenously moves in the same direction on average. Higher-order optimism is about how firms think about this comovement, how firms think about others' beliefs about this comovement, and so on. The literature often postulates that optimism arises exogenously. For this optimism to affect the economy, those papers have to assume that optimism is correlated across economic agents. However, it is difficult to justify this correlation if we are silent about the origin of optimism. But our paper can give the answer: optimism is correlated across agents because they observe the same economic outcomes.

We will now formally define the higher-order optimism. Recall that we define the optimism of a firm in island i at time t as the extent to which this firm underestimates its noise term, ξ_{it}. Likewise, from a firm's perspective, other firms in different islands are expected to be optimistic in interpreting their signals if they are expected to underestimate their noise terms. As we consider an environment with strategic complementarity, we can call it the firm's optimism about others' optimism or the *second-order optimism*. We proceed in a similar manner to define (ex-post) higher-order optimism.[26]

Definition 4. *The* ex-post hth-order optimism *of firms in island i is defined recursively by*

$$\mathcal{O}_{it}^{h} \equiv \mathbb{E}\left[\int_0^1 \mathcal{O}_{jt}^{h-1}\, dj \middle| \Omega_{it}\right], \;\; h = 2, 3, 4, \cdots \quad \text{where } \mathcal{O}_{it}^{1} \equiv \mathcal{O}_{it}.$$

We also define the average ex-post higher-order optimism *by*

$$\mathcal{O}_{t}^{h} = \int_0^1 \mathcal{O}_{jt}^{h}\, dj.$$

Equipped with this definition, we will now solve for the equilibrium. It is well known that the period-0 equilibrium is unique.

Lemma 6 (Period-0 Equilibrium). *In period 0, there is a unique equilibrium, in which firms choose*

$$y_{i0} = \theta \cdot s_{i0} \quad \text{where } \theta = \tfrac{(1-\alpha)\sigma_p^2}{(1-\alpha)\sigma_p^2 + \sigma_\xi^2} \in (0,1)$$

and hence the aggregate output is $y_0 = \theta \cdot \varepsilon_0^p$.

[26] We can use ex-ante higher-order optimism instead, which is defined analogously. It turns out that ex-post higher-order optimism, however, is much easier to keep track of under the presence of strategic complementarity, so we focus only on ex-post ones in this section.

For a given value of α, high σ_p^2 or low σ_ξ^2 implies that the signal is more informative about the productivity. Firms then respond more to their signals. All else equal, the degree of strategic complementarity α reduces θ because it makes agents put more weights on higher-order beliefs, which are more anchored to their mean-zero prior.

What we are interested in, however, is not the period-0 equilibrium.[27] Our focus is on how firms learn their and others' noise terms and how this learning changes the effect of period-0 shocks on period-1 outcome. Note that we are able to characterize the period-1 equilibrium without calculating firms' higher-order optimism or higher-order beliefs about the fundamental. Starting with a guess of a linear equilibrium, we can calculate the first-order beliefs about the endogenous aggregate output y_1, which gives the updated linear best response. The fixed point of this guess-and-verify process gives a unique linear equilibrium for period 1. The result is summarized in Lemma 7, which corresponds to Theorem 1 for the case with strategic complementarity.

Lemma 7 (Period-1 Equilibrium). *In period 1, there is a unique linear equilibrium in which the equilibrium output is given by*[28]

$$y_1 = \gamma_p \varepsilon_0^p + \gamma_u \varepsilon_0^u + \gamma_p' \varepsilon_1^p$$

where

$$\begin{aligned}
\gamma_p &= -\frac{\sigma_u^2\,\sigma_\xi^2}{(1-\alpha)\sigma_p^2\,\sigma_u^2+\sigma_p^2\,\sigma_\xi^2+2\,\sigma_u^2\,\sigma_\xi^2} \\
\gamma_u &= \frac{\sigma_p^2\,\sigma_\xi^2}{(1-\alpha)\sigma_p^2\,\sigma_u^2+\sigma_p^2\,\sigma_\xi^2+2\,\sigma_u^2\,\sigma_\xi^2} \\
\gamma_p' &= \frac{(1-\alpha)\sigma_p^2\,\sigma_u^2+\sigma_p^2\,\sigma_\xi^2+\sigma_u^2\,\sigma_\xi^2}{(1-\alpha)\sigma_p^2\,\sigma_u^2+\sigma_p^2\,\sigma_\xi^2+2\,\sigma_u^2\,\sigma_\xi^2},
\end{aligned}$$

so $\gamma_u > 0 > \gamma_p$ *and* $\gamma_p' \in (0,1)$.

This lemma shows that the intuition of the previous section is extended to a model with strategic complementarity; unobserved shocks are propagated to period 1 ($\gamma_u > 0$), while partly-observed shocks have negative effects on period-1 outcomes ($\gamma_p < 0$). A natural question that follows is whether the strategic complementarity and resulting higher-order optimism strengthen or weaken our mechanism. In order to answer this question and to fully understand the period-1 equilibrium, we need to keep track of higher-order optimism and its effects on higher-order beliefs about the productivity. We have assumed that the structure of the model is common knowledge, so firms take into account that other firms learn at the same time, that

[27] Actually, we do not even need to solve the period-0 equilibrium.

[28] The results below show that this is a unique equilibrium even if we allow for the possibility of a nonlinear equilibrium.

other firms also know that other firms learn at the same time, and so on. Thus, higher-order optimism endogenously arises and firms try to correct their higher-order optimism in a rational way, which in turn affects others' higher-order optimism. We first characterize the (ex-post) higher-order optimism in Lemma 8.

Lemma 8 (Higher-Order Optimism). *After observing $(s_{i0}, a_0, s_{i1})'$, higher-order optimism of firms in island i is given by*

$$\mathcal{O}_{i1} \equiv \xi_i - \mathbb{E}_{i1}[\xi_i] = Q\begin{pmatrix}\varepsilon_0^p & \xi_i & \varepsilon_0^u & \varepsilon_1^p\end{pmatrix}'$$
$$\mathcal{O}_{i1}^h \equiv \mathbb{E}_{i1}\left[\int_0^1 \mathcal{O}_{j1}^{h-1}\, dj\right] = QT^{h-1}\begin{pmatrix}\varepsilon_0^p & \xi_i & \varepsilon_0^u & \varepsilon_1^p\end{pmatrix}'$$

for some matrices $\underset{1\times 4}{Q}$ and $\underset{4\times 4}{T}$, where the sign of each element is

$$Q = \begin{pmatrix}- & + & + & -\end{pmatrix} \quad \textit{and} \quad QT^{h-1} = \begin{pmatrix}- & - & + & -\end{pmatrix}.$$

For future reference, we also note that the second element of Q, $Q_{1,2}$, is decreasing in σ_ξ^2 and increasing in σ_p^2 and σ_u^2.

First of all, the first-order optimism is increasing in ξ_i (see the sign of the second element of Q) because firms are unable to fully identify an increase in ξ_i as they rationally confuse it with changes in both ε_0^p and ε_0^u. It is then immediate that $Q_{1,2}$ is decreasing in σ_ξ^2 and increasing in σ_p^2 and σ_u^2 as in Part (3) of Lemma 5.

Second, the first-order optimism is decreasing in ε_0^p and increasing in ε_0^u as in the previous section (see the first and third elements of Q). More importantly, higher-order optimism always moves in the same direction as the first-order optimism in response to ε_0^p and ε_0^u (see the signs of the first and third elements of QT^{h-1}). To understand this, consider a firm in the *average* island i in the sense that $\xi_i = 0$. Suppose that there is a positive unit innovation in ε_0^u and that all other aggregate shocks remain zero. Then, a firm in the island i will observe higher-than-expected fundamental a_0 so her first-order optimism will be positive,

$$\mathcal{O}_{i1} = -\mathbb{E}_{i1}[\xi_i] > 0.$$

Since noise terms are symmetrically distributed around 0, this inequality means that she expects that the noise terms of other islands are on average $\mathcal{O}_{i1}$ units higher than her noise term. This in turn means that the first-order optimism of firms in other islands are on average $\mathcal{O}_{i1} \cdot Q_{1,2}$ units higher than her first-order optimism. At the same time, from her perspective, she is the one who is expected to have zero optimism (i.e., $\mathbb{E}_{i1}[\mathcal{O}_{i1}] = 0$). Thus, we can conclude that her second-order optimism is given by $\mathcal{O}_{i1}^2 = \mathcal{O}_{i1} \cdot Q_{1,2}$. This explains why the second-order optimism moves in the same direction as the first-order optimism in response to ε_0^u. In other words, firms who view a_0 as higher-than-expected will on average think that firms in other islands are likely to have higher noise terms than theirs, thereby being optimistic on average. Similar reasoning can be recursively applied to show that all the higher-order optimism is given by $\mathcal{O}_{i1}^h = \mathcal{O}_{i1} \cdot Q_{1,2}^{h-1}$, which also moves in the same direction. We can similarly see that higher-order optimism also moves in the same direction as the first-order optimism in response to ε_0^p.

A final observation is that, even though we start with the assumption that noise terms are *i.i.d.*, the fact that agents try to correct their optimism by observing their signals makes their optimism comove, which generates non-trivial average higher-order optimism, as we claimed before.

Next question is why this higher-order optimism is important. How does it affect the outcome in period 1? The answer is that it affects firms' higher-order beliefs, which in turn affect their output choices. We characterize this role of higher-order optimism in Lemma 9 and Corollary 2.

Lemma 9 (Higher-Order Beliefs). *Higher-order beliefs can be written as functions of ε_1^p and cumulative sums of higher-order optimism:*

$$\begin{aligned}\mathbb{E}_{i1}\bar{\mathbb{E}}_1^{h-1}[a_1] &= \varepsilon_1^p + \mathcal{O}_{i1} + \mathcal{O}_{i1}^2 + \cdots + \mathcal{O}_{i1}^h \quad (\textit{with } \ \mathbb{E}_{i1}a_1 = \varepsilon_1^p + \mathcal{O}_{i1}) \\ \bar{\mathbb{E}}_1^h[a_1] &= \varepsilon_1^p + \mathcal{O}_1 + \mathcal{O}_1^2 + \cdots + \mathcal{O}_1^h\end{aligned}$$

where we write $\bar{\mathbb{E}}_1[\cdot] = \int_0^1 \mathbb{E}_{i1}[\cdot]\,\mathrm{d}i$ and $\bar{\mathbb{E}}_1^h[\cdot] = \int_0^1 \mathbb{E}_{i1}\bar{\mathbb{E}}_1^{h-1}[\cdot]\,\mathrm{d}i$.

Corollary 2 (Outcome). *The aggregate output in period 1 is a weighted average of higher-order beliefs, and hence is a weighted sum of higher-order optimism:*

$$\begin{aligned}y_1 &= \sum_{h=1}^{\infty}(1-\alpha)\alpha^{h-1}\bar{\mathbb{E}}_1^h[a_1] \\ &= \varepsilon_1^p + \sum_{h=1}^{\infty}\alpha^{h-1}\mathcal{O}_1^h. \end{aligned} \tag{6}$$

It is well known that aggregate output is determined by higher-order beliefs. Thus, Lemma 9 naturally leads to Corollary 2, which gives how the aggregate output is determined by higher-order optimism. In Lemma 8, we characterize how underlying shocks affect higher-order optimism, which in conjunction with Corollary 2 characterizes how underlying shocks affect the aggregate output in period 1. This essentially gives the equivalent result of Lemma 7, but we have tracked higher-order optimism and higher-order beliefs to understand the mechanism behind it. Now we are ready to answer the main question of this section: do strategic complementarity and resulting higher-order optimism strengthen or weaken our mechanism?

With $\alpha = 0$, we get back to the case without strategic complementarity where we have $y_1 = \varepsilon_1^p + \mathcal{O}_1$. With $\alpha > 0$, we have additional higher-order optimism terms in equation (6). In Lemma 8, we have seen that these additional terms move in the same direction as the first-order optimism. Thus, we can conclude that the presence of strategic complementarity and resulting higher-order optimism always *strengthen* our mechanism relative to the case without strategic complementarity. In other words, when agents observe higher-than-expected outcomes, they become optimistic not only about their signals (first-order optimism) but also about others' optimism (higher-order optimism). Furthermore, higher α means higher coefficients on first-order and higher-order optimism terms in equation (6). Therefore, the response of y_1 to the underlying shocks ε_0^p and ε_0^u are even higher when we have stronger strategic complementarity. This discussion can be summarized in the following theorem, which is our second main result.

Theorem 2 (Strategic Complementarity). *The effects of period-0 shocks on period-1 outcome are increasing in the degree of strategic complementarity:*[29]

$$\frac{\partial}{\partial \alpha}\left(\frac{\partial y_1}{\partial \varepsilon_0^u}\right) > 0 \quad \text{and} \quad \frac{\partial}{\partial \alpha}\left|\frac{\partial y_1}{\partial \varepsilon_0^p}\right| > 0.$$

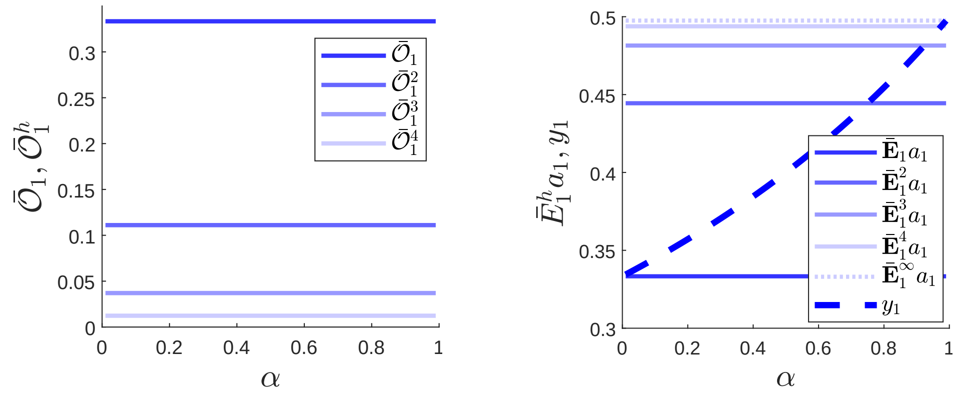
Thus, the strategic complementarity and resulting higher-order optimism always strengthen the amplification and dampening we documented in Section 3.

Remark. This theorem is in stark contrast to the results in Woodford (2003), Morris and Shin (2002), and Angeletos and Pavan (2007) which instead document that the higher the degree of strategic complementarity is, the less responsive the agents are to underlying shocks. This can be clearly seen in Example 1 where we have $\frac{\partial}{\partial \alpha}\left(\frac{\partial y}{\partial a}\right) < 0$. This is because higher strategic complementarity implies that the equilibrium actions of agents are more anchored to the common prior, so agents are less responsive to contemporaneous shocks. In our model, however, optimistic agents are on average expect that others are more optimistic than they are, so higher strategic complementarity makes agents more responsive to period 0 shocks.

Next, we provide a numerical example to illustrate our findings. We set $\sigma_u^2 = \sigma_\xi^2 = 1$ for the variance of the unobserved shock and noise terms, and $\sigma_p^2 = 2$ for the variance of the partly-observed shock.[30] We then change the degree of strategic complementarity α from 0 to 1. Figure 1 shows the result for unobserved shocks ε_0^u and Figure 2 for partly-observed

[29] Recall that $\frac{\partial \mathcal{O}_1^h}{\partial \varepsilon_0^u}$ is positive while $\frac{\partial \mathcal{O}_1^h}{\partial \varepsilon_0^p}$ is negative for all orders h.

[30] We set σ_p^2 higher than σ_u^2 according to the notion that agents are more concerned about volatile shocks; see footnote 20.



(a) Higher-Order Optimism (b) Higher-Order Beliefs and Outcome

Figure 1. Effects of a unit increase in unobserved shocks

shocks ε_0^p. These figures clearly illustrate our findings: (i) higher-order optimism moves in the same direction as the first-order optimism (increasing in the unobserved shock and decreasing in the partly-observed shock) (Lemma 8), (ii) higher-order beliefs are cumulative sums of higher-order optimism (Lemma 9), and (iii) the effects of underlying shocks are increasing in α (Theorem 2).

We conclude this section with comparative statics analyses. We have seen so far that higher-order optimism determines higher-order beliefs, which in turn determine outcomes. Thus, we will focus on how the values of underlying parameters change the effects of shocks on higher-order optimism. The results are summarized in Lemma 10.

Lemma 10 (Comparative Statics for Optimism). *The effect of the period-0 unobserved shock on higher-order optimism, $\partial \mathcal{O}_{i1}^h / \partial \varepsilon_0^u$, is*

(i) Increasing in σ_p^2

(ii) Decreasing in σ_u^2 if h is low (e.g., $h = 1$), while increasing in σ_u^2 if h is sufficiently high

(iii) Increasing in σ_ξ^2 if h is low (e.g., $h = 1$), while decreasing in σ_ξ^2 if h is sufficiently high.

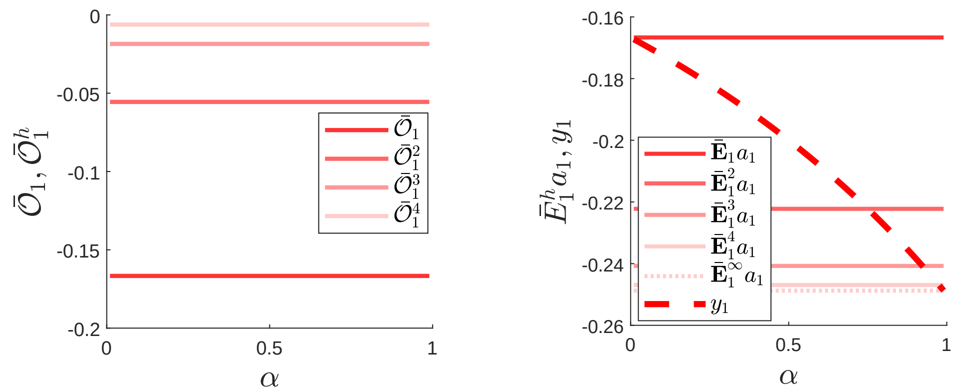
Similarly, the effect of period-0 partly-observed shock on higher-order optimism, $\left|\partial \mathcal{O}_{i1}^h / \partial \varepsilon_0^p\right|$ is

(i) Increasing in σ_u^2

(ii) Decreasing in σ_p^2 if h is low (e.g., $h = 1$), while increasing in σ_p^2 if h is sufficiently high

(iii) Increasing in σ_ξ^2 if h is low (e.g., $h = 1$), while decreasing in σ_ξ^2 if h is sufficiently high.

Recall that, in Lemma 5, we proved that the effect of the unobserved shock is increasing in the variance of the partly-observed shock and noise terms while decreasing in the variance of



(a) Higher-Order Optimism (b) Higher-Order Beliefs and Outcome

Figure 2. Effects of a unit increase in partly-observed shocks

the unobserved shock. This explains the first part of Lemma 10 for the first-order optimism ($h = 1$). For comparative statics for higher-order optimism, we discussed in Lemma 8 that the effect of the unobserved shock on higher-order optimism can be decomposed into[31]

$$\frac{\partial \mathcal{O}_{i1}^h}{\partial \varepsilon_0^u} = \frac{\partial \mathcal{O}_{i1}}{\partial \varepsilon_0^u} \cdot Q_{1,2}^{h-1}.$$

The first term reflects the fact that higher-order optimism increases precisely because the first-order optimism increases. In addition, for a given increase in the first-order optimism, the second term determines the increase in higher-order optimism. Recall that this second term is increasing in both σ_p^2 and σ_u^2 and decreasing in σ_ξ^2. Why do we have different comparative statics for the first and second terms? Lemma 8 tells us that the first term originates from firm i's rational confusion between ε_0^u and (ε_0^p, ξ_i), while the second term is originated from other firms' rational confusion between their noise terms and $(\varepsilon_0^p, \varepsilon_0^u)$. Thus, the first term is decreasing in the relative variance of ε_0^u, and the second term is decreasing in the relative variance of ξ_i. If h is low, then the effect of variance parameters on the first term dominates that on the second term so that h^{th}-order optimism has the same comparative statics as the first-order optimism. On the other hand, for sufficiently high h, the effect of variance on the second term dominates that on the first term and h^{th}-order optimism is increasing in σ_p^2 and σ_u^2 and decreasing in σ_ξ^2. This explains the first part of Lemma 10, and we can apply the same argument for the second part. Figure 3 illustrates the results. We use the same parameter values as in the previous numerical exercise and change the value of each variance one by one. We then calculate the effects of underlying shocks on higher-order optimism.

[31] Recall that we considered a unit innovation in ε_0^u, so we can interpret $\mathcal{O}_{i1}^h$ there as $\frac{\partial \mathcal{O}_{i1}^h}{\partial \varepsilon_0^u}$.

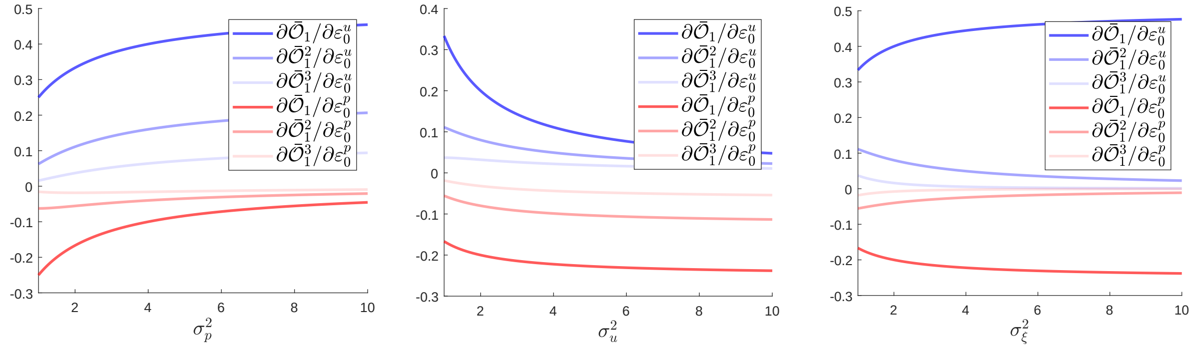


Figure 3. Comparative Statics with respect to Variance

The aggregate output in period 1 is a function of all orders of optimism, and variance parameters, $\sigma_p^2, \sigma_u^2, \sigma_\xi^2$, can have different effects depending on the order of optimism. Lemma 11, however, shows that the effects on the first-term above always dominate the effects on the second term when it comes to the aggregate output.

Lemma 11 (Comparative Statics for Outcome). *The effect of period-0 unobserved shocks on the period-1 aggregate outcome y_1 is (i) increasing in σ_p^2, (ii) decreasing in σ_u^2, and (iii) increasing in σ_ξ^2. Likewise, the effect of period-0 partly-observed shocks on the period-1 outcome is (i) increasing in σ_u^2, (ii) decreasing in σ_p^2, and (iii) increasing in σ_ξ^2.*

To sum up, the presence of strategic complementarity and resulting higher-order optimism strenghten our mechanism because of two facts. Higher-order beliefs are cumulative sums of higher-order optimism, and higher-order optimism always move in the same direction as the first-order optimism in response to underlying shocks.

4.1 Numerical Exercise: Infinite Period with Strategic Complementarity

In this section, we discuss the robustness of the results in the previous section. We relax the restrictive two-period assumptions and instead assume infinite periods. In order to prevent firms from full learning of their noise terms, we assume that noise terms follow an AR(1) process as in Section 3 with $\rho \in (0, 1)$ and $\sigma_\eta^2 > 0$. Except for these two assumptions, the model is the same as in Section 4.

We utilize the method of Woodford (2003) to solve for the equilibrium dynamics of the aggregate output, which exploits the fact that firms only need to track particular linear combinations of higher-order beliefs. The absence of endogenous signals permits us to do so; see Huo and Takayama (2015). We start from a guess that the relevant aggregate state can be summarized in

$$\mathbf{x}_t = \begin{pmatrix} \varepsilon_t^p & \varepsilon_t^u & F_t & y_t \end{pmatrix}'$$

where[32]

$$
\begin{aligned}
F_t &= \sum_{k=1}^{\infty}(1-\alpha)\alpha^{k-1}\overline{\mathbb{E}_{it}\xi_{it}}^{k} = (1-\alpha)\overline{\mathbb{E}_{it}\xi_{it}} + \alpha\bar{\mathbb{E}}_t F_t \\
y_t &= \sum_{k=1}^{\infty}(1-\alpha)\alpha^{k-1}\overline{\mathbb{E}_{it}\varepsilon^p_{it}}^{k} = (1-\alpha)\overline{\mathbb{E}_{it}\varepsilon^p_{it}} + \alpha\bar{\mathbb{E}}_t y_t
\end{aligned}
$$

with (and similarly for ε^p_{it})

$$
\overline{\mathbb{E}_{it}\xi_{it}} = \overline{\mathbb{E}_{it}\xi_{it}}^{1} = \int_0^1 \mathbb{E}_{jt}\xi_{jt}\,\mathrm{d}j \;\text{ and }\; \overline{\mathbb{E}_{it}\xi_{it}}^{k} = \int_0^1 \mathbb{E}_{jt}\overline{\mathbb{E}_{it}\xi_{it}}^{k-1}\,\mathrm{d}j
$$

and the expectation operators are based on the information set $\Omega_{it} = (\cdots, s_{it-1}, a_{t-1}, s_{it})$. Firms in island i then observe

$$
a_t = \begin{pmatrix}0 & 1 & 1 & 0 & 0\end{pmatrix}\mathbf{x}_{it} \;\text{ and }\; s_{it+1} = \begin{pmatrix}1 & 1 & 0 & 0 & 0\end{pmatrix}\mathbf{x}_{it+1}.
$$

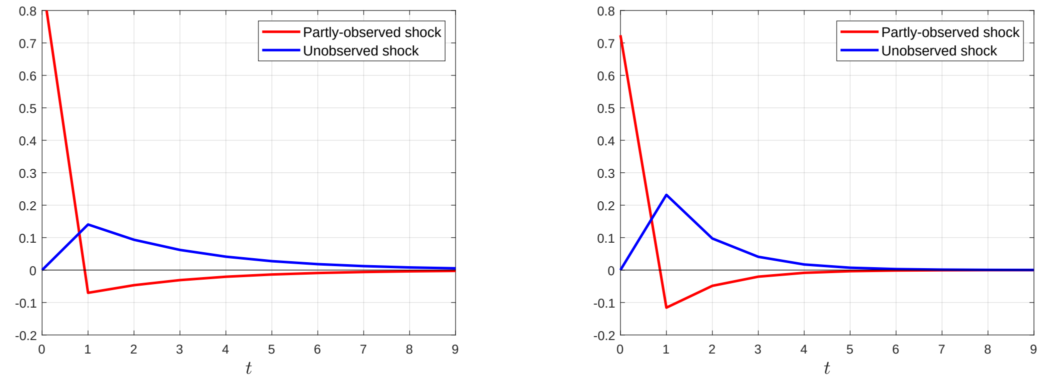
We will guess and verify that $\mathbf{x}_t$ evolves according to the following law of motion

$$
\mathbf{x}_t = \mathbf{M}\mathbf{x}_{t-1} + \mathbf{m}\begin{pmatrix}\varepsilon^p_t \\ \varepsilon^u_t\end{pmatrix}
$$

for some matrices $\mathbf{M} \in \mathbb{R}^{4\times 4}$ and $\mathbf{m} \in \mathbb{R}^{4\times 2}$. We can then solve for firms' signal extraction problem to obtain how firms update $\bar{\mathbb{E}}_{t+1}[\mathbf{x}_{t+1}]$ from $\bar{\mathbb{E}}_t[\mathbf{x}_t]$, taking the *perceived* law of motion assumed above as given. It turns out that $\begin{pmatrix}F_t \\ y_t\end{pmatrix}$ is a linear combination of $\bar{\mathbb{E}}_t[\mathbf{x}_t]$, thus we can calculate $\begin{pmatrix}F_t \\ y_t\end{pmatrix}$ as a function of the previous aggregate state $\mathbf{x}_{t-1}$ and innovation ε^p_t. This gives the *actual* law of motion of $\mathbf{x}_t$. The equilibrium is then characterized by a fixed point of mapping from the perceived law of motion to the actual law of motion. See **??** for further details. We use the same parameters as in Section 4 with $\sigma^2_\eta = 0.5$ and $\rho = 0.9$ and numerically calculate the trajectory of the economy after innovations in the underlying shocks. These parameters are arbitrary, but they are neither implausible nor qualitatively essential for the results below.

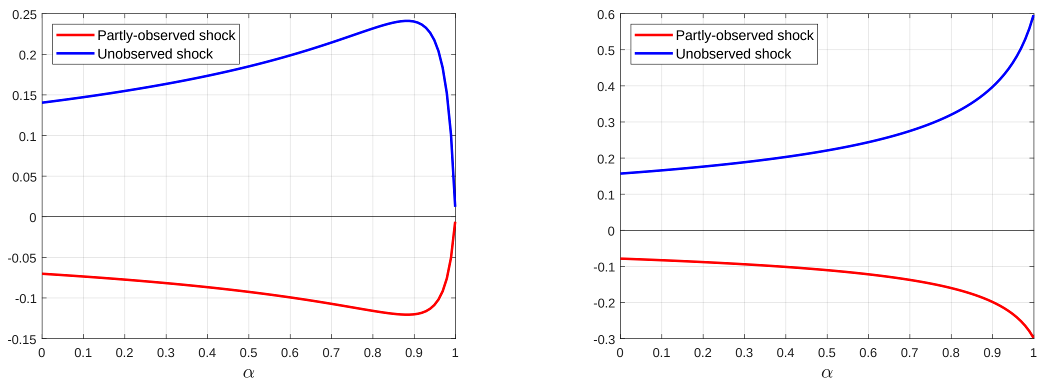
Figure 4 plots the impulse responses of aggregate output to positive innovations in partly-observed shock, ε^p_t, and unobserved shock, ε^u_t. Figure 4a corresponds to the case without strategic complementarity, and Figure 4b corresponds to the case with higher strategic complementarity, $\alpha = 0.8$.

[32] Note that $\mathbb{E}_{it}\left[\int_0^1 \xi_{jt}\,\mathrm{d}j\right]$ is always zero, but $\int_0^1 \mathbb{E}_{jt}\xi_{it}\,\mathrm{d}j$ is not.



(a) $\alpha = 0$ (b) $\alpha = 0.8$

Figure 4. Impulse response of aggregate output



(a) $\dfrac{\partial y_{t+1}}{\partial \varepsilon_t^p}$ and $\dfrac{\partial y_{t+1}}{\partial \varepsilon_t^u}$ (b) $\dfrac{\partial y_{t+1}}{\partial \varepsilon_t^p} \Big/ \dfrac{\partial y_t}{\partial \varepsilon_t^p}$ and $\dfrac{\partial y_{t+1}}{\partial \varepsilon_t^u} \Big/ \dfrac{\partial y_t}{\partial \varepsilon_t^p}$

Figure 5. Comparative static with respect to α

We can see that Lemma 7 continues to hold in this infinite horizon model: unobserved shocks are propagated to period 1, while partly-observed shocks have negative effects on the next period outcome. Also, comparing Figure 4a and Figure 4b, the effect of α is in line with Theorem 2: with higher degree of strategic complementarity, we seem to have stronger effects of period-t shocks on the period-$(t+1)$ outcome. However, if we plot $\frac{\partial y_{t+1}}{\partial \varepsilon_t^u}$ and $\frac{\partial y_{t+1}}{\partial \varepsilon_t^p}$ as a function of α in Figure 5a, then it turns out that this is not the case for very high values of α. In particular, both vanish as α converges to one. Does this mean that the intuition we obtain in the previous section is wrong? The answer is no. Note that the importance of underlying shocks goes to zero as α goes to 1, which can be clearly seen by $\lim_{\alpha\to 1} \frac{\partial y_t}{\partial \varepsilon_t^p} = 0$. For sufficiently high α, this force is dominant, so that the effects of period-t shocks on the period-$(t+1)$ output also go to zero. In Figure 5b, we plot the relative size of $\frac{\partial y_{t+1}}{\partial \varepsilon_t^u}$ and $\frac{\partial y_{t+1}}{\partial \varepsilon_t^p}$ compared to $\frac{\partial y_t}{\partial \varepsilon_t^p}$. Observe that these relative effects are indeed increasing in the degree of strategic complementarity, which is indeed in line with the result of Theorem 2. In this sense, we can conclude that the main message of Section 4 does not rely on the simplifying assumptions.

5. Implication on Forecast Survey Data

What is the empirical implication of our model? We depart from the macroeconomic model in this section and focus instead on an abstract model in which a_{it} can be any fundamental of interest. In Section 5.1, we first illustrate how our model provide an alternative interpretation of survey data, while explaining the prominent empirical findings in the literature in a unified way. In Section 5.2, we show that the gap between consensus-level overextrapolation and individual-level overextrapolation helps distinguish our rational theory of overextrapolation from other behavioral theories. This result is reminiscent of the result of Angeletos, Huo and Sastry (2020) that the gap between consensus-level underreaction and individual-level overreaction speaks to the role of information frictions.

Before going forward, we summarize the results of Section 3 and introduce a few notations that are used throughout this section. The fundamental follows an AR(1) process $a_{it+1} = \rho_a a_{it} + \varepsilon^p_{it+1} + \varepsilon^u_{it+1}$, and agents observe signals $s_{it+1} = \rho_a a_{it} + \varepsilon^p_{it+1} + \eta_{it+1}$. The optimism follows the low of motion $\tilde{\mathcal{O}}_{it+1} = \rho\gamma_3\tilde{\mathcal{O}}_{it} - \rho\gamma_1\varepsilon^p_{it} + \rho\gamma_2\varepsilon^u_{it} + \eta_{it+1}$. The forecast is then given by $\mathbb{E}_{it}[a_{it}] = \rho_a^2 a_{it-2} + (\rho_a - \rho K\gamma_1)\varepsilon^p_{it-1} + (\rho_a + \rho K\gamma_2)\varepsilon^u_{it-1} + K\varepsilon^p_{it} + K\eta_{it} + \rho\gamma_3 K\tilde{\mathcal{O}}_{it-1}$. Note that our timing convention implies that when agent i makes a forecast in period t, her information set $\Omega_{it} = (\cdot, s_{it-2}, a_{it-2}, s_{it-1}, a_{it-1}, s_{it})$ does not contain the realized fundamental a_{it}. Literature, however, often assumes that a_{it} is contained in the period-t information set, so we introduce a notation to make our results comparable to the literature $\mathbb{F}_{it}[\cdot] \equiv \tilde{\mathbb{E}}_{it+1}[\cdot] = \mathbb{E}\big[\cdot \mid (\cdots, s_{it-2}, a_{it-2}, s_{it-1}, a_{it-1}, s_{it}, a_{it})\big]$. We denote aggregate variables by either omitting i-index or using a bar over variables: $x_t = \int x_{it}\,\mathrm{d}i$ for $x \in \{a, \varepsilon^p, \varepsilon^u, \tilde{\mathcal{O}}\}$, $\overline{\mathbb{E}_{it}a_{it+k}} = \int_0^1 \mathbb{E}_{jt}a_{jt+k}\,\mathrm{d}j$, and $\overline{\mathbb{F}_{it}a_{it+k}} = \int_0^1 \mathbb{F}_{jt}a_{jt+k}\,\mathrm{d}j$. We write $\sigma_p^2 = \mathrm{Var}(\varepsilon^p_{it})$, $\sigma_u^2 = \mathrm{Var}(\varepsilon^u_{it})$, $\bar{\sigma}_p^2 = \mathrm{Var}(\varepsilon^p_t)$ and $\bar{\sigma}_u^2 = \mathrm{Var}(\varepsilon^u_t)$. Note that we always have $\bar{\sigma}_p^2 \le \sigma_p^2$ and $\bar{\sigma}_u^2 \le \sigma_u^2$ since the shocks may have idiosyncratic components which are canceled out when we aggregate them. This means that the degrees of commonality defined below are always less than or equal to one.

Definition 5. *We define* the degrees of commonality *of the partly-observed and unobserved shocks as*

$$\mathcal{C}_p = \frac{\bar{\sigma}_p^2}{\sigma_p^2} \text{ and } \mathcal{C}_u = \frac{\bar{\sigma}_u^2}{\sigma_u^2}, \text{ respectively.}$$

On the one hand, if a shock is fully idiosyncratic and hence always take zero value when we integrate it across agents, the degree of commonality is zero. On the other hand, if a shock is common then the degree of commonality is one. The relative size of $\mathcal{C}_p$ and $\mathcal{C}_u$ plays a central role in Section 5.1. A special case is where agents share a common fundamental a_t, such as the inflation rate of the economy or GDP growth. This necessarily implies $\mathcal{C}_p = \mathcal{C}_u = 1$. But, in the real world, even for such a common fundamental, forecast data can be better interpreted by a model with idiosyncratic fundamentals. To illustrate this, consider forecasters who form expectations about the US output growth. They have their own ways to view the US output growth, a_{it}, which is not necessarily the same as the true US output growth, a_t, even if it is unbiased, $\int_0^1 a_{it}\,\mathrm{d}i = a_t$. If this is the case, the feedback these forecasters receive is likely to be also in terms of their view of the US output growth, a_{it}, not in terms of the true US output growth, a_t. Thus, we assume hereafter that $\mathcal{C}_p$ and $\mathcal{C}_u$ are not necessarily equal to one even when we consider common fundamentals.

5.1 Empirical Findings in the Literature

Many empirical papers use survey panel data to directly measure agents' expectations. These papers often assume that forecasters do not observe previous realizations even ex post and dynamically learn fundamental from signals. This assumption is required because observing previous realization makes learning essentially static in their settings and makes it difficult to explain the dynamic pattern of forecast data. The literature often relies on rational inattention to justify this assumption. But, it is unlikely that the forecasters who made a prediction for a variable do not pay high attention to the realized value of it.

This paper gives a totally different way to interpret dynamic forecast data. This paper views the same survey data as outputs of forecasters' dynamic learning in which they can observe the previous inflation rate but try to learn how to interpret their own information; i.e., noise terms. Our model allows for forecasters observing previous realizations ex-post, while still being able to explain several empirical findings that have been explained using standard models. In this section, we first illustrate how our model explains the prominent empirical findings in the literature in a unified way. In particular, we will consider the empirical findings of Coibion and Gorodnichenko (2015) (hereafter, CG), Kohlhas and Walther (2020) (hereafter, KW), and Angeletos, Huo and Sastry (2020) (hereafter, AHS).

Coibion and Gorodnichenko (2015). We start with the finding of CG. They demonstrate the underreaction of consensus forecast by showing that forecast errors are positively correlated with forecast revisions. They run the following regression

$$a_{it+k} - \mathbb{E}_{it}a_{it+k} = \alpha_i + \delta(\overline{\mathbb{E}_{it}a_{it+k}} - \overline{\mathbb{E}_{it-1}a_{it+k}}) + \text{error}_{it}$$

and obtain a positive coefficient estimate, $\hat{\delta} > 0$. They explain this based on the gradual adjustment of average forecasts. The same result can be obtained in our model, but here it is based instead on the gradual adjustment of average optimism.

Proposition 3 (CG, 2015). *In our model, there exists a threshold $\lambda \in (0,1)$ such that $\frac{\mathcal{C}_p}{\mathcal{C}_u} > \lambda$ implies*

$$\mathrm{Cov}(a_{it+k} - \mathbb{E}_{it}a_{it+k}, \overline{\mathbb{E}_{it}a_{it+k}} - \overline{\mathbb{E}_{it-1}a_{it+k}}) > 0, \ \textit{for } k \geq 1,$$

This means that, unless the partly observed shocks are mostly averaged out, we can get the underreaction of consensus forecast as in the literature.

Kohlhas and Walther (2020). They show the coexistence of underreaction to new information and overextrapolation from recent realizations of the forecasted variable. The evidence of underreaction is the same as in CG, while, for the overextrapolation, they run the following regression for the US output growth:[33]

$$a_{t+k} - \overline{\mathbb{E}_{it+1}a_{it+k}} = \alpha + \gamma a_t + \text{error}_t$$

and obtain a negative coefficient estimate, $\hat{\gamma} < 0$. This directly implies that consensus forecast features overextrapolation to the recent realization. They show that this can happen if rational agents pay more attention to procyclical components of the variable. A simpler explanation is based on a behavioral overextrapolation model in which agents' perceived persistence of the output growth is higher than the true persistence. In the next proposition, we will argue that we can obtain the same result using our model, in which agents overextrapolate to recent realizations because of the endogenous change in optimism.

Proposition 4 (KW, 2020). *Suppose that agents in our model are relatively well informed about idiosyncratic components of shocks in the sense that $\mathcal{C}_p < \mathcal{C}_u$. Then, we have*

$$\mathrm{Cov}(a_{t+k} - \overline{\mathbb{E}_{it+1}a_{it+k}}, a_t) < 0, \ \textit{for } k \geq 1,$$

which is the result of KW. Also note that the contemporaneous effect of a_t on the forecast error is positive:

$$\mathrm{Cov}(a_{t+k} - \overline{\mathbb{E}_{it}a_{it+k}}, a_t) > 0.$$

As discussed in Assumption 2, it is often assumed in the literature that agents are indeed relatively well informed about idiosyncratic shocks. First, idiosyncratic shocks are more

[33] Their original specification has $a_{t+k} - \bar{\mathbb{E}}_t a_{t+k}$ on the left hand side. Under their timing convention, however, a_t is in the agents' information set when they form the expectation about a_{t+k}; so it should be $\bar{\mathbb{E}}_{t+1}a_{t+k}$ under our timing convention.

agent-specific, hence, it is easier to get information about them. Also, idiosyncratic shocks are likely to be more volatile than the aggregate shocks. Agents thus rationally pay more attention to the idiosyncratic shocks. This necessarily implies $\mathcal{C}_p < \mathcal{C}_u$. The first part of Proposition 4 says that our model predicts the finding of KW under this condition. The second part says that there is a reversal of covariance, $\mathrm{Cov}(a_{t+k} - \overline{\mathbb{E}_{it+1}a_{it+k}}, a_t) < 0 < \mathrm{Cov}(a_{t+k} - \overline{\mathbb{E}_{it}a_{it+k}}, a_t)$. This is reminiscent of the key implication of our model in Theorem 1 that a component that does not affect period-t expectation has a larger effect on period-$(t+1)$ expectation:

$$\frac{\partial E_{it+1}a_{t+1}}{\partial \varepsilon_t^p} < \frac{\partial a_{t+1}}{\partial \varepsilon_t^p} \quad \text{and} \quad \frac{\partial \mathbb{E}_{it+1}a_{t+1}}{\partial \varepsilon_t^u} > \frac{\partial a_{t+1}}{\partial \varepsilon_t^u}$$

and again comes from the presence of overextrapolation and information friction. This combination of overextrapolation and information friction is essential to explain the finding of KW in many papers. AHS explain it using the combination of behavioral overextrapolation and information friction. Our model and the model of KW essentially embed rational mechanisms of overextrapolation—persistent noise and feedback in our model and asymmetric attention in KW—into information friction models.

Angeletos, Huo and Sastry (2020). They document delayed overreaction of consensus forecasts—consensus forecasts initially underreact and overshoot later on. This finding is consistent with their model that combines incomplete information and behavioral over-extrapolation. Since my model provides a rational theory of overextrapolation of consensus forecast, we can obtain the same result. In our model, expectations initially underreact due to incomplete information and overshoot later on when agents receive feedback.[34]

5.2 Distinguish the Rational Theory from Behavioral Theories

Not only our model, but also many behavioral theories of overextrapolation can obtain Proposition 4. Moreover, as AHS pointed out, these theories combined with information friction can potentially explain Proposition 3 as well. How can we test our model against other behavioral overextrpolation models? Smoking gun evidence comes from exploiting the difference between the degrees of overextrapolation in consensus and individual forecasts. For example, consider

[34] They also show that behavioral over-extrapolation—misspecification in the stochastic process of *fundamental*—leads to the overreaction of individual forecasts as documented in Bordalo et al. (2020). Similarly, in our model, the misspecification in the stochastic process of *noise terms* leads to overreaction of individual forecasts. In particular, Proposition A.1 in Appendix states that when the perceived persistence of noise is greater than the true persistence, individual forecast errors are negatively correlated with forecast revisions, implying the individual-level overreaction.

the following two regressions.

$$a_{t+1} - \overline{\mathbb{E}_{it+1} a_{it+1}} = \beta_0 + \beta_{aggr} a_t + \text{error}_{t+1} \tag{7}$$

$$a_{it+1} - \mathbb{E}_{it+1} a_{it+1} = \beta_0 + \beta_{ind} a_{it} + \text{error}_{it+1} \tag{8}$$

Because a_{it} is contained in agent i's information set, our model can only generate overextrapolation at the consensus level, whereas in Proposition 5 we show that behavioral theories necessarily have the same coefficient for both regression specifications. We can compare the estimated coefficients of these regressions to distinguish our rational theory from behavioral theories.[35]

Proposition 5 (Consensus vs. Individual Overextrapolation). *Suppose that the fundamental follows an AR(1) process*

$$a_{it} = \rho_a a_{it-1} + \varepsilon_{it}$$

and agents receive signals about the fundamental with normally distributed noise terms

$$s_{it} = a_{it} + \eta_{it}.$$

For the following theories with behavioral elements

- *Extrapolation: Agents observe a_{it} (i.e., $\mathrm{Var}(\eta_{it}) = 0$) when they form expectations about a_{it+1}, but perceived AR(1) coefficient $\hat{\rho}_a$ is higher than the true one, ρ_a. We write $\mathbb{E}_{it+1}[\cdot] = \mathbb{E}[\cdot | \cdots, a_{it-1}, a_{it}]$.*
- *AHS: Perceived AR(1) coefficient, $\hat{\rho}_a$, is higher than the true one, ρ_a, and perceived precision of s_{it} is higher than the true one.*
- *Diagnostic expectation: $\mathbb{E}_{it} a_{i,t+k} = \mathbb{E}^{rational}_{it-1} a_{i,t+k} + g_k (s_{it} - \mathbb{E}^{rational}_{it-1} a_{it})$ with $g_k > K \cdot \rho^k$ where K is the Kalman gain.*

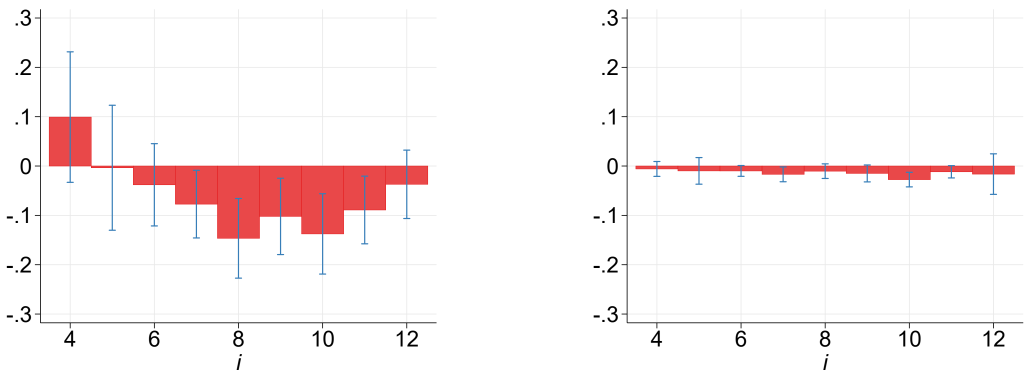
we always have

$$\widehat{\beta}_{aggr} = \widehat{\beta}_{ind}.$$

In our model without misspecification, suppose again that agents are relatively well informed about idiosyncratic components of shocks in the sense that $\mathcal{C}_p < \mathcal{C}_u$. Then, we have

$$\widehat{\beta}_{aggr} < 0 \quad \textit{and} \quad \widehat{\beta}_{ind} = 0.$$

[35] Consider an extended version of KW model with individual-specific fundamental. Agent i has the fundamental $y_{it} = \sum_j x_{ijt}$, where j-th component is determined by $x_{ijt} = a_j \theta_{it} + u_{ijt}$ where θ_{it} denotes a latent factor that follows an AR(1) process, $\theta_{it} = \rho_a \theta_{it-1} + \eta_{it}$. Agent i observes noisy signals $s_{ijt} = x_{ijt} + \varepsilon_{ijt}$. The shocks u_{ijt}, η_{it}, and ε_{ijt} are normally distributed, serially uncorrelated, and mutually independent. In this model, we have $\widehat{\beta}_{aggr} < \widehat{\beta}_{ind}$ if and only if $\frac{\mathrm{Var}(\int u_{ijt}\, di)}{\mathrm{Var}(u_{ijt})} > \frac{\mathrm{Var}(\int \eta_{it}\, di)}{\mathrm{Var}(\eta_{it})}$. But there is no reason to expect $\widehat{\beta}_{ind} = 0$.



(a) Aggregate Evidence: $\widehat{\beta}_{aggr}$ (b) Firm-Level Evidence: $\widehat{\beta}_{ind}$

Figure 6. Overextrapolation in Analyst Expectations

Gennaioli, Ma, and Shleifer (2016) document both $\widehat{\beta}_{aggr}$ and $\widehat{\beta}_{ind}$ for CFOs' and analysts' expectations on earnings growth, which is copied in Table 4, although their focus is not on comparing the coefficients. The coefficients in panel (A) correspond to $\widehat{\beta}_{aggr}$, and those in panel (B) correspond to $\widehat{\beta}_{ind}$. We can make two observations. First, analysts expectations feature a pattern consistent with Proposition 5; i.e., $\widehat{\beta}_{aggr} < 0$ and $\widehat{\beta}_{ind} \approx 0$. Second, CFOs expectations give higher extrapolation both at the consensus level and at the individual level. But the differences between two are approximately the same. These two observations are suggestive of the interpretation that analysts expectations are approximately rational but overextrapolate from past realizations once we aggregate them to consensus expectations, and that CFOs expectations are additionally subject to behavioral overextrapolation.[36] It is difficult, however, to formally map these estimates to the coefficients $\widehat{\beta}_{aggr}$ and $\widehat{\beta}_{ind}$ in Proposition 5 because Gennaioli, Ma, and Shleifer (2016) regress forecast errors of earning growth on past earnings per asset, not on past earnings growth. Thus, we redo their estimation using past earnings growth as independent variables. One should be cautious when choosing the length of time periods because our theory essentially implies initial underextrapolation and overextrapolation later on (See Proposition 4). Suppose that a_{it} denotes firm-level earning growth over one year starting from time t. We experiment with various values of the length of time periods between t and $t+1$, from four quarters ($i = 4$) to twelve quarters ($i = 12$).[37] Figure 6 shows the results for the regression specifications (7) and (8). Reassuringly, this again features a pattern consistent with Proposition 5 for intermediate values of i, $7 \leq i \leq 11$.

[36] Another interesting observation is that the differences between coefficients, $\widehat{\beta}_{aggr} - \widehat{\beta}_{ind}$, which measure the extra overextrapolation in the consensus forecasts, are almost identical for analyst expectation and CFO expectation.

[37] Because a_{it} denotes yearly earnings growth, we set $i \geq 4$ to ensure that there is no overlap between a_{it} and a_{it+1}.

Remark. The experience effects are studied by Malmendier and Nagel (2011, 2016) and subsequent papers. Recent evidence suggests longlasting effects of past personal experiences on expectations and behaviors. For example, personal lifetime experiences in the stock market affect future stock market investment behavior. This is inconsistent with traditional economic models in which the effect of personal experiences of events are not different from information about them *ceteris paribus*. The literature on experience effects emphasizes the longlasting neuropsychology effects as a key mechanism. The following corollary provides a natural explanation of experience effects through the lens of our model.

Corollary 3. *In our model, suppose again that agents are relatively well informed about idiosyncratic components of shocks in the sense that $\mathcal{C}_p < \mathcal{C}_u$. Then, when we run the following regression*

$$a_{it+1} - \mathbb{E}_{it+1} a_{it+1} = \beta_0 + \tilde{\beta}_{aggr} a_t + \widetilde{\text{error}}_{it+1},$$

we have

$$\widehat{\tilde{\beta}}_{aggr} = \widehat{\beta}_{aggr} < 0.$$

Suppose that you experience large negative stock market returns, $a_t < 0$ in your lives so far. Then Corollary 3 implies that you become pessimistic about the stock market, $a_{it+1} - \mathbb{E}_{it+1} a_{it+1} > 0$, being less likely to participate in it. Moreover, expectations adjust only for those who experience this negative shock, because this overextrapolation arises from agents evaluating their previous forecasts from the feedback they receive. Those who only read this negative shock do not have a chance to make a prediction, so they would not show overextrapolative behaviors. In other words, our model provides a novel reason that cognitive process of making a prediction and evaluating it affects the future expectation formation.

6. Conclusion

We start with two observations. First, noise in agents' signals is likely to be persistent regardless of the real-world counterpart of it. Second, agents in the real world receive feedback on their previous forecasts. With persistent noise and feedback, agents try to learn noise in their signals and others' noise, and optimism arises endogenously. With this additional channel of learning, feedback on previous forecasts affects the expectation about the noise, and shocks with different degrees of observability have different effects on the dynamics of aggregate outcomes through their differential effects on optimism. We obtain a novel mechanism through which *rational agents become overoptimistic after observing higher-than-expected outcomes of the economy* and this optimism amplifies/propagates underlying shocks. Here, optimism is not only about their own signals but also about others' optimism when there is strategic complementarity. Our model gives us a new way to interpret forecast dynamics in survey data—learning how to interpret information rather than learning fundamentals. This interpretation is consistent with many empirical findings in the literature.

Table 4: Tables 8 and 9 of Gennaioli, Ma, and Shleifer (2016)

A. Aggregate Evidence

	Realized − Expected Next 12m Earnings Growth	
	(1) Analyst	(2) CFO
Past 12m earnings/asset (%)	−0.0456	−0.0881
	(−3.68)	(−6.48)
Observations	106	57

B. Firm-Level Evidence

	Realized − Expected Next 12m Earnings Growth	
	(1) Analyst	(2) CFO
Past 12m earnings/asset (%)	−0.0080	−0.0511
	(−7.43)	(−5.14)
Firm fixed effects	Y	Y
Observations	103,930	606

Notes: In panel (A), the dependent variable is aggregate earnings growth in the next 12 months minus aggregate expectations of earnings growth in the next 12 months. Independent variables include aggregate earnings/asset in the four quarters prior to quarter $t-1$. In panel (B), the dependent variable is firm-level earnings growth in the next 12 months minus expectations of earnings growth in the next 12 months. Independent variables include firm-level earnings/asset in the four quarters prior to quarter $t-1$. t-statistics in parentheses. See Gennaioli, Ma, and Shleifer (2016) for details.

References

Acharya, Sushant, Jess Benhabib, and Zhen Huo (2019) "The Anatomy of Sentiment-driven Fluctuations," (April).

Angeletos, George Marios, Zhen Huo, and Karthik A Sastry (2020) "Imperfect Macroeconomic Expectations : Evidence and Theory," (April 2).

Angeletos, George Marios, and Jennifer La'O (2009a) "Noisy Business Cycles," *NBER Macroeconomics Annual*, 24.

——— (2009b) "Incomplete information, higher-order beliefs and price inertia," *Journal of Monetary Economics*, 56, S19–S37.

——— (2013) "Sentiments," *Econometrica*, 81 (2), 739–779.

Angeletos, George Marios, and Chen Lian (2019) "Confidence and the Propagation of Demand Shocks."

Angeletos, George Marios, and Alessandro Pavan (2007) "Efficient Use of Information and Social Value of Information," *Econometrica*, 75 (4), 1103–1142.

Benhabib, Jess, Pengfei Wang, and Yi Wen (2015) "Sentiments and Aggregate Demand Fluctuations," *Econometrica*, 83 (2), 549–585.

Bordalo, Pedro, Nicola Gennaioli, Yueran Ma, and Andrei Shleifer (2020) "Overreaction in Macroeconomic Expectations," (Marcg), 1–72.

Broer, Tobias, and Alexandre N Kohlhas (2019) "Forecaster (Mis-)Behavior," (February), 1–39.

Coibion, Olivier, and Yuriy Gorodnichenko (2012) "What can survey forecasts tell us about information rigidities?," *Journal of Political Economy*, 120 (1), 116–159.

——— (2015) "Information rigidity and the expectations formation process: A simple framework and new facts," *American Economic Review*, 105 (8), 2644–2678.

Huo, Zhen, and Naoki Takayama (2015) "Higher Order Beliefs, Confidence, and Business Cycles."

——— (2018) "Rational Expectations Models with Higher Order Beliefs," (February).

Kasa, Kenneth (2000) "Forecasting the Forecasts of Others in the Frequency Domain," *Review of Economic Dynamics*, 3, 726–756.

Kohlhas, Alexandre N, and Ansgar Walther (2020) "Asymmetric Attention," (September).

Lucas, Robert E (1972) "Expectations and the neutrality of money," *Journal of Economic Theory*, 4 (2), 103–124.

Mackowiak, Bartosz, and Mirko Wiederholt (2009) "Optimal Sticky Prices under Rational Inattention," *American Economic Review*, 99 (3), 769–803.

Morris, Stephen, and Hyun Song Shin (2002) "Social Value of Public Information," *American Economic Review*, 92 (5).

Myatt, David P, and Chris C Wallace (2004) "Adaptive play by idiosyncratic agents," *Games and Economic Behavior*, 48, 124–138.

Nimark, Kristoffer P (2017) "Dynamic Higher Order Expectations," (March).

Pavan, Alessandro (2016) "Attention, Coordination, and Bounded Recall," *Discussion Papers* (July).

Rondina, Giacomo, and Todd B Walker (2018) "Confounding Dynamics," (October).

Sargent, Thomas J (1991) "Equilibrium with signal extraction from endogenous variables," *Journal of Economic Dynamics and Control*, 15, 245–273.

Sims, Christopher A (2003) "Implications of rational inattention," *Journal of Monetary Economics*, 50, 665–690.

Townsend, Robert M (1983) "Forecasting the Forecasts of Others," *Journal of Political Economy*, 91 (4), 546–588.

Woodford, Michael (2003) "Imperfect Common Knowledge and the Effects of Monetary Policy," in *Knowledge, Information, and Expectations in Modern Macroeconomics: In Honor of Edmund S. Phelps*.

Appendix

A. Proofs for Sections 2–4

Proof of Lemma 1. Signals of the form $s_t = a_t + \sigma \cdot \tilde{\xi}_t$ give a strictly positive object function unless $\sigma = 0$, which violates the information constraint. Consider a signal of the form $s_t = a_t + \sigma \cdot \tilde{\xi}$ where σ is sufficiently high to satisfy the information constraint. Since we know unconditional distribution of a_t, with an infinite number of realizations of s_t, we can exactly learn the realization of $\tilde{\xi}$, thereby having $\mathbb{E}[a_t|s^t] = a_t$ for all t. $\square$

Proof of Lemma 2. We have

$$\left(1 - \frac{1}{\eta} - \frac{1}{\theta}\right) y_{ijt} = \mathbb{E}_{it}\left[-\frac{1}{\eta} y_t + \gamma y_t - \frac{1}{\theta} a_{it} + (1 - \frac{\sigma}{\eta})\frac{1}{\sigma}(y_{ijt} - y_t)\right]$$

or

$$\left(1 - \frac{1}{\sigma} - \frac{1}{\theta}\right) y_{ijt} = \mathbb{E}_{it}\left[\left(\gamma - \frac{1}{\sigma}\right) y_t - \frac{1}{\theta} a_{it}\right]$$

or equivalently

$$\begin{aligned} y_{ijt} &= \left(\frac{1}{\theta} + \frac{1}{\sigma} - 1\right)^{-1} \mathbb{E}_{it}\left[\frac{1}{\theta} a_{it} + \left(\frac{1}{\sigma} - \gamma\right) y_t\right] \\ &\equiv \mathbb{E}_{it}[(1-\alpha)\tilde{a}_{it} + \alpha y_t] \end{aligned}$$

which gives the desired result. $\square$

Proof of Proposition 1. Although we can prove this directly, this result can be seen as a special case of Theorem 1 with $\rho = 0$. $\square$

Proof of Lemma 3. Right before observing s_{it}, we have $\xi_{it-1}|\tilde{\Omega}_{it} \sim \mathcal{N}(m_{it-1}, V_{t-1})$, hence

$$\xi_{it}|\tilde{\Omega}_{it} \sim \mathcal{N}\big(\rho m_{it-1}, \rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2\big).$$

On the other hand, the prior belief of $\tilde{a}_{it} \equiv \rho_a a_{it-1} + \varepsilon_{it}^p$ is $\mathcal{N}\big(\rho_a a_{it-1}, \sigma_p^2\big)$. Thus, Bayesian updating gives

$$\mathbb{E}_{it}[a_{it}] = \mathbb{E}_{it}[\tilde{a}_{it}] = \rho_a \cdot a_{it-1} + K_t(s_{it} - \rho_a a_{it-1} - \rho m_{it-1})$$

where $K_t = \frac{\sigma_p^2}{\rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2 + \sigma_p^2} \in (0, 1)$. Also note that

$$s_{it} - \rho_a a_{it-1} - \rho m_{it-1} = \varepsilon_{it}^p + \xi_{it} - \rho m_{it-1} = \varepsilon_{it}^p + \tilde{\mathcal{O}}_{it}.$$

Finally, we have

$$\mathbb{E}_{it}[a_{it}] = \mathbb{E}_{it}[\tilde{a}_{it}] = \mathbb{E}_{it}[s_{it} - \xi_{it}] = s_{it} - \mathbb{E}_{it}[\xi_{it}] = \rho_a a_{it-1} + \varepsilon_{it}^p + \mathcal{O}_{it}. \qquad \square$$

Proof of Lemma 4. Consider the following state-space representation.

$$\boldsymbol{x}_t \equiv \begin{pmatrix} \tilde{a}_{it} \\ \xi_{it} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \rho_a a_{it-1} \\ \rho m_{t-1} \end{pmatrix}, \underbrace{\begin{pmatrix} \sigma_p^2 & 0 \\ 0 & \rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2 \end{pmatrix}}_{\Sigma} \right)$$

$$\boldsymbol{y}_t \equiv \begin{pmatrix} s_{it} \\ a_{it} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{G} \boldsymbol{x}_t + \begin{pmatrix} 0 \\ e_t^u \end{pmatrix}$$

The Kalman filter gives

$$\boldsymbol{x}_t | \boldsymbol{y}_t \sim \mathcal{N}\left(\begin{pmatrix} \rho_a a_{it-1} \\ \rho m_{t-1} \end{pmatrix} + K\left(\boldsymbol{y}_t - G \begin{pmatrix} \rho_a a_{it-1} \\ \rho m_{t-1} \end{pmatrix} \right), KRK' + (I - KG)\Sigma(I - KG)' \right)$$

where $K = \Sigma G'(G\Sigma G + R)^{-1}$ and $R = \text{Var}\begin{pmatrix} 0 \\ \varepsilon_t^u \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_u^2 \end{pmatrix}$. This gives $\xi_{it}|\tilde{\Omega}_{it+1} \sim \mathcal{N}(m_{it}, V_t)$ with

$$\begin{aligned} m_{it} &= (\gamma_1 + \gamma_2)s_{it} + \gamma_3 \rho m_{it-1} - \rho_1 \rho_a a_{it-1} - \gamma_2 a_{it} \\ V_t &= \gamma_3(\rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2) \end{aligned}$$

where

$$\begin{aligned} \gamma_1 &= \frac{\sigma_u^2(\rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2)}{(\sigma_p^2 + \sigma_u^2)(\rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2) + \sigma_p^2\sigma_u^2} \\ \gamma_2 &= \frac{\sigma_p^2(\rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2)}{(\sigma_p^2 + \sigma_u^2)(\rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2) + \sigma_p^2\sigma_u^2} \\ \gamma_3 &= \frac{\sigma_p^2\sigma_u^2}{(\sigma_p^2 + \sigma_u^2)(\rho^2 V_{t-1} + (1-\rho^2)\sigma_\eta^2) + \sigma_p^2\sigma_u^2} \end{aligned}$$

Thus, $\gamma_1, \gamma_2, \gamma_3 \in (0, 1)$ and $\gamma_1 + \gamma_2 + \gamma_3 = 1$. $\square$

Proof of Proposition 2. The law of motion for ex-ante optimism directly follows from Lemma 4. In the proof of Lemma 3, we have shown that

$$\mathcal{O}_{it} = K\tilde{\mathcal{O}}_{it} - (1-K)\varepsilon^p_{it},$$

which gives the law of motion for ex-post optimism. $\square$

Proof of Lemma 5. We have

$$V = \frac{\sigma_p^2\sigma_u^2(\rho^2 V + (1-\rho^2)\sigma_\eta^2)}{(\sigma_p^2+\sigma_u^2)(\rho^2 V + (1-\rho^2)\sigma_\eta^2) + \sigma_p^2\sigma_u^2}$$

or equivalently

$$\frac{1}{V} - \frac{1}{\rho^2 V + (1-\rho^2)\sigma_\eta^2} = \frac{1}{\sigma_p^2} + \frac{1}{\sigma_u^2}.$$

Since $\gamma_1 = \frac{V}{\sigma_p^2}$, we can write

$$\frac{1}{\gamma_1} - \frac{1}{\rho^2\gamma_1 + (1-\rho^2)\frac{\sigma_\eta^2}{\sigma_p^2}} = 1 + \frac{\sigma_p^2}{\sigma_u^2}.$$

The left hand side is then increasing in σ_η^2 and decreasing in σ_p^2. Moreover, as it can be alternatively written as

$$\frac{(1-\rho^2)\left(\frac{\sigma_\eta^2}{\sigma_u^2} - \gamma_1\right)}{\gamma_1\left(\rho^2\gamma_1 + (1-\rho^2)\frac{\sigma_\eta^2}{\sigma_p^2}\right)},$$

the left hand side is also decreasing in γ_1. Therefore, we can conclude that γ_1 is increasing in σ_u^2 and σ_η^2 while decreasing in σ_p^2. In a similar way, we can show that γ_2 is increasing in σ_p^2 and σ_η^2 while decreasing in σ_u^2. For the comparative statics for γ_3, define $W = \rho^2 V + (1-\rho^2)\sigma_\eta^2$. This implies $V = \rho^{-2}W - (\rho^{-2}-1)\sigma_\eta^2$ and $\gamma_3 \equiv \frac{V}{W} = \rho^{-2} - (\rho^{-2}-1)\frac{\sigma_\eta^2}{W}$. The last term $\frac{\sigma_\eta^2}{W}$ satisfies

$$\frac{(\rho^{-2}-1)\left(1-\frac{W}{\sigma_\eta^2}\right)}{\frac{W}{\sigma_\eta^2}\left(\rho^{-2}\frac{W}{\sigma_\eta^2} - (\rho^{-2}-1)\right)} = \frac{\sigma_\eta^2}{\sigma_p^2} + \frac{\sigma_\eta^2}{\sigma_u^2}.$$

The left hand side is increasing in $\frac{\sigma_\eta^2}{W}$. Thus, γ_3 is decreasing in σ_η^2 while increasing in σ_p^2 and σ_u^2. $\square$

Proof of Theorem 1. From Proposition 2 and the definition of optimism, we have

$$y_{it+1} = \rho_a a_{it} + K(s_{it+1} - \rho_a a_{it} - \rho m_{it})$$

$$
\begin{aligned}
&= \rho_a a_{it} + K(\varepsilon^p_{it+1} + \xi_{it+1} - \rho m_{it}) \\
&= \rho_a^2 a_{it-1} + (\rho_a - \rho K \gamma_1)\varepsilon^p_{it} + (\rho_a + \rho K \gamma_2)\varepsilon^u_{it} + K\varepsilon^p_{it+1} + K\eta_{it+1} + \rho\gamma_3 K \tilde{\mathcal{O}}_{it}.
\end{aligned}
$$

We can use the relationship between ex-ante and ex-post optimism to derive the second result. □

Proof of Lemma 6. Suppose that all agents except for i use a strategy of the form $y_{j0} = \theta s_{j0}$. Then, we can calculate the best response of i as

$$
\begin{aligned}
y_{i0} &= (1-\alpha)\mathbb{E}_{i0}[\varepsilon^p_0] + \alpha \mathbb{E}_{i0}[y_0] \\
&= (1-\alpha+\alpha\theta)\mathbb{E}_{i0}[\varepsilon^p_0] \\
&= (1-\alpha+\alpha\theta)\frac{\sigma_p^2}{\sigma_p^2 + \sigma_\xi^2} s_{i0}.
\end{aligned}
$$

Thus, the unique linear equilibrium is given by $y_{i0} = \theta s_{i0}$ where $\theta = \frac{(1-\alpha)\sigma_p^2}{(1-\alpha)\sigma_p^2 + \sigma_\xi^2} \in (0,1)$. For the general uniqueness, see Morris and Shin (2002). □

Proof of Lemma 7. Suppose that all agents except for i use a strategy of the form $y_{j1} = \theta_1 s_{j0} + \theta_2 a_0 + \theta_3 s_{j1}$, then these decisions aggregate into

$$
y_1 = \theta_1 \varepsilon^p_0 + \theta_2 a_0 + \theta_2 \varepsilon^p_1.
$$

Thus, the the best response of i in period 1 is given by

$$
\begin{aligned}
y_{i1} &= (1-\alpha)\mathbb{E}_{i1}\varepsilon^p_1 + \alpha \mathbb{E}_{i1} y_1 \\
&= (1-\alpha+\alpha\theta_3)\mathbb{E}_{i1}\varepsilon^p_1 + \alpha\theta_2 a_0 + \alpha\theta_1 \mathbb{E}_{i1}\varepsilon^p_0.
\end{aligned}
$$

Consider the following state-space representation.

$$
\begin{aligned}
\boldsymbol{x} &\equiv \begin{pmatrix} \varepsilon^p_0 \\ \varepsilon^p_1 \end{pmatrix} \sim \mathcal{N}(\mathbf{0}, \Sigma) \quad \text{where } \Sigma = \begin{pmatrix} \sigma_p^2 & 0 \\ 0 & \sigma_p^2 \end{pmatrix} \\
\boldsymbol{y} &\equiv \begin{pmatrix} s_{i0} \\ a_0 \\ s_{i1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{G} \begin{pmatrix} \varepsilon^p_0 \\ \varepsilon^p_1 \end{pmatrix} + \begin{pmatrix} \xi_i \\ \varepsilon^u_0 \\ \xi_i \end{pmatrix}.
\end{aligned}
$$

The Kalman filter gives

$$\mathbb{E}[\boldsymbol{x}|\boldsymbol{y}] = K\boldsymbol{y} \quad \text{where } K = \Sigma G'(G\Sigma G' + R)^{-1} \quad \text{with } R = \begin{pmatrix} \sigma_\xi^2 & 0 & \sigma_\xi^2 \\ 0 & \sigma_u^2 & 0 \\ \sigma_\xi^2 & 0 & \sigma_\xi^2 \end{pmatrix}.$$

Thus, we have

$$y_{i1} = \alpha\theta_1 \begin{pmatrix} 1 & 0 \end{pmatrix} K\boldsymbol{y} + (1-\alpha+\alpha\theta_3)\begin{pmatrix} 0 & 1 \end{pmatrix} K\boldsymbol{y} + \alpha\theta_2 a_0$$

Matching coefficient, we have

$$\begin{aligned} \theta_1 &= \begin{pmatrix} \alpha\theta_1 & 1-\alpha+\alpha\theta_3 \end{pmatrix} K \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \theta_2 &= \frac{1}{1-\alpha} \begin{pmatrix} \alpha\theta_1 & 1-\alpha+\alpha\theta_3 \end{pmatrix} K \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ \theta_3 &= \begin{pmatrix} \alpha\theta_1 & 1-\alpha+\alpha\theta_3 \end{pmatrix} K \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Let $B = \begin{pmatrix} \alpha\theta_1 & 1-\alpha+\alpha\theta_3 \end{pmatrix}$, then we can obtain B by

$$\begin{aligned} B &= \begin{pmatrix} 0 & 1-\alpha \end{pmatrix} + BK \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \end{pmatrix} + BK \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & \alpha \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1-\alpha \end{pmatrix} \left(I - K \begin{pmatrix} \alpha & 0 \\ 0 & 0 \\ 0 & \alpha \end{pmatrix} \right)^{-1}, \end{aligned}$$

which in turn gives the values for θ_1, θ_2 and θ_3. Thus, we can write

$$y_1 = \theta_1 \varepsilon_0^p + \theta_2(\varepsilon_0^p + \varepsilon_0^u) + \theta_3 \varepsilon_1^p \equiv \gamma_p \varepsilon_0^p + \gamma_u \varepsilon_0^u + \gamma_p' \varepsilon_1^p$$

where

$$\begin{aligned} \gamma_p &= -\frac{\sigma_u^2\,\sigma_\xi^2}{(1-\alpha)\sigma_p^2\,\sigma_u^2 + \sigma_p^2\,\sigma_\xi^2 + 2\,\sigma_u^2\,\sigma_\xi^2} \\ \gamma_u &= \frac{\sigma_p^2\,\sigma_\xi^2}{(1-\alpha)\sigma_p^2\,\sigma_u^2 + \sigma_p^2\,\sigma_\xi^2 + 2\,\sigma_u^2\,\sigma_\xi^2} \\ \gamma_p' &= \frac{(1-\alpha)\sigma_p^2\,\sigma_u^2 + \sigma_p^2\,\sigma_\xi^2 + \sigma_u^2\,\sigma_\xi^2}{(1-\alpha)\sigma_p^2\,\sigma_u^2 + \sigma_p^2\,\sigma_\xi^2 + 2\,\sigma_u^2\,\sigma_\xi^2}. \end{aligned}$$

□

Proof of Lemma 8. From the proof of Lemma 7, we can get

$$\begin{aligned}
\mathbb{E}_{it}[\xi_i] \equiv \mathbb{E}\left[\xi_i \middle| \begin{pmatrix} s_{i0} \\ a_0 \\ s_{i1} \end{pmatrix}\right] &= \mathbb{E}\left[s_{i0} - \varepsilon_0^p \middle| \begin{pmatrix} s_{i0} \\ a_0 \\ s_{i1} \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix} K\right] \begin{pmatrix} s_{i0} \\ a_0 \\ s_{i1} \end{pmatrix} \\
&= \underbrace{\left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix} K\right] \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}}_{L} \vec{\boldsymbol{z}}_i \quad \text{where } \vec{\boldsymbol{z}}_i \equiv \begin{pmatrix} \varepsilon_0^p \\ \xi_i \\ \varepsilon_0^u \\ \varepsilon_1^p \end{pmatrix}.
\end{aligned}$$

Note first that

$$\mathcal{O}_{i1} \equiv \xi_i - \mathbb{E}_{i1}\xi_i = \underbrace{\left[\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} - L\right]}_{Q} \vec{\boldsymbol{z}}_i.$$

Also note that

$$\begin{aligned}
\mathbb{E}_{i1}\left[\int_0^1 \vec{\boldsymbol{z}}_j \, dj\right] = \mathbb{E}_{i1}\left[\begin{pmatrix} \varepsilon_0^p \\ 0 \\ \varepsilon_0^u \\ \varepsilon_1^p \end{pmatrix}\right] &= \mathbb{E}_{i1}\left[\begin{pmatrix} s_{i0} - \xi_i \\ 0 \\ a_0 - s_{i0} + \xi_i \\ s_{i1} - \xi_i \end{pmatrix}\right] \\
&= \underbrace{\begin{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} - L \\ \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 & 1 & 0 \end{pmatrix} + L \\ \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix} - L \end{pmatrix}}_{T} \vec{\boldsymbol{z}}_i.
\end{aligned}$$

Suppose that $\mathcal{O}_{i1}^{h-1} = QT^{h-2}\vec{\boldsymbol{z}}_i$ holds (This indeed holds for $h = 2$). Then, we have

$$\begin{aligned}
\mathcal{O}_{i1}^h &= \mathbb{E}_{i1}\left[QT^{h-2}\int_0^1 \vec{\boldsymbol{z}}_j \, dj\right] \\
&= QT^{h-1}\vec{\boldsymbol{z}}_i.
\end{aligned}$$

Thus, we can inductively show that

$$\mathcal{O}_{i1}^h = QT^{h-1}\vec{\boldsymbol{z}}_i.$$

After some algebra, we can write Q and QT^{h-1} as functions of underlying parameters:

$$Q = \begin{pmatrix} -\frac{\sigma_u^2\,\sigma_\xi^2}{\sigma_p^2\,\sigma_u^2+\sigma_p^2\,\sigma_\xi^2+2\,\sigma_u^2\,\sigma_\xi^2} & \frac{\sigma_p^2\,\sigma_u^2}{\sigma_p^2\,\sigma_u^2+\sigma_p^2\,\sigma_\xi^2+2\,\sigma_u^2\,\sigma_\xi^2} & \frac{\sigma_p^2\,\sigma_\xi^2}{\sigma_p^2\,\sigma_u^2+\sigma_p^2\,\sigma_\xi^2+2\,\sigma_u^2\,\sigma_\xi^2} & -\frac{\sigma_u^2\,\sigma_\xi^2}{\sigma_p^2\,\sigma_u^2+\sigma_p^2\,\sigma_\xi^2+2\,\sigma_u^2\,\sigma_\xi^2} \end{pmatrix}$$

$$QT^{h-1} = \begin{pmatrix} -\frac{(\sigma_p^2)^{h-1}\,(\sigma_u^2)^h\,\sigma_\xi^2}{\left(\sigma_p^2\,\sigma_u^2+\sigma_p^2\,\sigma_\xi^2+2\,\sigma_u^2\,\sigma_\xi^2\right)^h} & -\frac{(\sigma_p^2)^{h-1}\,(\sigma_u^2)^{h-1}\,\sigma_\xi^2\left(\sigma_p^2+2\,\sigma_u^2\right)}{\left(\sigma_p^2\,\sigma_u^2+\sigma_p^2\,\sigma_\xi^2+2\,\sigma_u^2\,\sigma_\xi^2\right)^h} & \frac{(\sigma_p^2)^h\,(\sigma_u^2)^{h-1}\,\sigma_\xi^2}{\left(\sigma_p^2\,\sigma_u^2+\sigma_p^2\,\sigma_\xi^2+2\,\sigma_u^2\,\sigma_\xi^2\right)^h} & -\frac{(\sigma_p^2)^{h-1}\,(\sigma_u^2)^h\,\sigma_\xi^2}{\left(\sigma_p^2\,\sigma_u^2+\sigma_p^2\,\sigma_\xi^2+2\,\sigma_u^2\,\sigma_\xi^2\right)^h} \end{pmatrix}.$$

Note that

$$\begin{aligned} Q_{1,2} &= \frac{\sigma_p^2\,\sigma_u^2}{\sigma_p^2\,\sigma_u^2+\sigma_p^2\,\sigma_\xi^2+2\,\sigma_u^2\,\sigma_\xi^2} \\ &= \frac{1}{1+\frac{\sigma_\xi^2}{\sigma_u^2}+2\frac{\sigma_\xi^2}{\sigma_p^2}} \end{aligned}$$

is decreasing in σ_ξ^2 and increasing in σ_u^2 and σ_p^2. □

Proof of Lemma 9. First, we have

$$\mathbb{E}_{i1}[a_1] = \mathbb{E}_{i1}[\varepsilon_{i1}^p] = \mathbb{E}_{i1}[s_{i1}-\xi_i] = \varepsilon_{i1}^p + \xi_i - \mathbb{E}_{i1}[\xi_i] = \varepsilon_{i1}^p + \mathcal{O}_{i1}.$$

Thus,

$$\bar{\mathbb{E}}_1 \varepsilon_1^p = \bar{\mathcal{O}}_1.$$

Suppose that we have

$$\begin{aligned} \mathbb{E}_{i1}\bar{\mathbb{E}}_1^{h-1}[a_1] &= \varepsilon_1^p + \mathcal{O}_{i1} + \mathcal{O}_{i1}^2 + \cdots + \mathcal{O}_{i1}^{h-1} \\ \bar{\mathbb{E}}_1^h[a_1] &= \varepsilon_1^p + \bar{\mathcal{O}}_1 + \bar{\mathcal{O}}_1^2 + \cdots + \bar{\mathcal{O}}_1^h \end{aligned}$$

for a given h. Then, we can obtain

$$\begin{aligned} \mathbb{E}_{i1}\bar{\mathbb{E}}_1^h[a_1] &= \mathbb{E}_{i1}\left[\varepsilon_1^p + \bar{\mathcal{O}}_1 + \bar{\mathcal{O}}_1^2 + \cdots + \bar{\mathcal{O}}_1^h\right] \\ &= \varepsilon_1^p + \mathcal{O}_{i1} + \mathcal{O}_{i1}^2 + \cdots + \mathcal{O}_{i1}^{h+1}. \end{aligned}$$

hence

$$\bar{\mathbb{E}}_1^{h+1}[a_1] = \varepsilon_1^p + \bar{\mathcal{O}}_1 + \bar{\mathcal{O}}_1^2 + \cdots + \bar{\mathcal{O}}_1^{h+1}.$$

Thus, we can inductively show Lemma 9. □

Proof of Lemma 10. We have shown in the proof of Lemma 8 that

$$\frac{\partial \bar{\mathcal{O}}_1^h}{\partial \varepsilon_0^u} = \frac{(\sigma_p^2)^h\,(\sigma_u^2)^{h-1}\,\sigma_\xi^2}{\left(\sigma_p^2\,\sigma_u^2 + \sigma_p^2\,\sigma_\xi^2 + 2\,\sigma_u^2\,\sigma_\xi^2\right)^h}, \quad \text{for all } h \geq 1$$

and that

$$\frac{\partial \bar{\mathcal{O}}_1^h}{\partial \varepsilon_0^p} = -\frac{(\sigma_p^2)^{h-1}\,(\sigma_u^2)^h\,\sigma_\xi^2}{\left(\sigma_p^2\,\sigma_u^2 + \sigma_p^2\,\sigma_\xi^2 + 2\,\sigma_u^2\,\sigma_\xi^2\right)^h}, \quad \text{for all } h \geq 1.$$

After some algebra, we can obtain the results. □

Proof of Lemma 11. Recall that, in the proof of Lemma 7, we have

$$y_1 = \gamma_p \varepsilon_0^p + \gamma_u \varepsilon_0^u + \gamma_p' \varepsilon_1^p$$

where

$$\gamma_p = -\frac{\sigma_u^2\,\sigma_\xi^2}{(1-\alpha)\sigma_p^2\,\sigma_u^2 + \sigma_p^2\,\sigma_\xi^2 + 2\,\sigma_u^2\,\sigma_\xi^2}$$
$$\gamma_u = \frac{\sigma_p^2\,\sigma_\xi^2}{(1-\alpha)\sigma_p^2\,\sigma_u^2 + \sigma_p^2\,\sigma_\xi^2 + 2\,\sigma_u^2\,\sigma_\xi^2}$$

Thus, the effect of ε_0^u on y_1 (i.e., γ_u) is increasing in σ_p^2, decreasing in σ_u^2, and increasing in σ_ξ^2. Likewise, the effect of ε_0^p on y_1 (i.e., $|\gamma_p|$) is increasing in σ_u^2, decreasing in σ_p^2, and increasing in σ_ξ^2. □

B. Proofs for Section 5

For future reference, we start with the following three lemmas.

Lemma A.1 (Covariance). *We have* $\mathrm{Cov}(\bar{\bar{\mathcal{O}}}_t, a_{t-1}) = -\frac{\rho V}{1-\rho\gamma_3\rho_a}\left(\frac{\bar{\sigma}_p^2}{\sigma_p^2} - \frac{\bar{\sigma}_u^2}{\sigma_u^2}\right)$.

Proof of Lemma A.1.

$$\begin{aligned}
\mathrm{Cov}(\bar{\bar{\mathcal{O}}}_t, a_{t-1}) &= \mathrm{Cov}\big((1-\rho\gamma_3 L)^{-1}(-\rho\gamma_1\varepsilon_{t-1}^p + \rho\gamma_2\varepsilon_{t-1}^u), (1-\rho_a L)^{-1}(\varepsilon_{t-1}^p + \varepsilon_{t-1}^u)\big)\\
&= \frac{-\rho\gamma_1\bar{\sigma}_p^2 + \rho\gamma_2\bar{\sigma}_u^2}{1-\rho\gamma_3\rho_a}\\
&= -\frac{\rho V}{1-\rho\gamma_3\rho_a}\left(\frac{\bar{\sigma}_p^2}{\sigma_p^2} - \frac{\bar{\sigma}_u^2}{\sigma_u^2}\right).
\end{aligned}$$
□

Lemma A.2 (Coefficients). *Let* $\Sigma \equiv \rho^2 V + (1-\rho^2)\sigma_\eta^2$, *then*

$$\frac{1}{V} = \frac{1}{\Sigma} + \frac{1}{\sigma_p^2} + \frac{1}{\sigma_u^2}.$$

Moreover, we have

$$K = \frac{\sigma_p^2}{\Sigma + \sigma_p^2} \quad \gamma_1 = \frac{\sigma_u^2 \Sigma}{\Psi} \quad \gamma_2 = \frac{\sigma_p^2 \Sigma}{\Psi} \quad \gamma_3 = \frac{\sigma_p^2 \sigma_u^2}{\Psi}$$

where $\Psi \equiv \Sigma(\sigma_p^2 + \sigma_u^2) + \sigma_p^2\sigma_u^2 = \frac{\sigma_p^2\sigma_u^2\Sigma}{V}$. *Thus,*

$$\gamma_1 = \frac{V}{\sigma_p^2} \quad \gamma_2 = \frac{V}{\sigma_u^2} \quad \gamma_3 = \frac{V}{\Sigma}.$$

Lemma A.3 (Variables). *We can write our variables of interest in terms of innovations and states:*

$$\begin{aligned}
Y_1 &\equiv \quad a_{t+1} - \bar{\mathbb{E}}_t a_{t+1} &&= \rho_a\big((1-K)\varepsilon_t^p + \varepsilon_t^u - K\rho\gamma_3\bar{\tilde{\mathcal{O}}}_{t-1} + \rho\gamma_1 K\varepsilon_{t-1}^p - \rho\gamma_2 K\varepsilon_{t-1}^u\big) + \varepsilon_{t+1}^p + \varepsilon_{t+1}^u \\
Y_2 &\equiv \quad a_{t+1} - \bar{\mathbb{E}}_{t+1} a_{t+1} &&= \varepsilon_{t+1}^u + (1-K)\varepsilon_{t+1}^p + \rho K\gamma_1\varepsilon_t^p - \rho K\gamma_2\varepsilon_t^u - \rho\gamma_3 K(\gamma_3\rho\bar{\tilde{\mathcal{O}}}_{t-1} - \rho\gamma_1\varepsilon_{t-1}^p + \rho\gamma_2\varepsilon_{t-1}^u) \\
X^{cg} &\equiv \bar{\mathbb{E}}_t a_{t+1} - \bar{\mathbb{E}}_{t-1} a_{t+1} &&\overset{\text{sgn}}{=} (\rho_a(1-K) - \rho K\gamma_1)\varepsilon_{t-1}^p + (\rho_a + \rho K\gamma_2)\varepsilon_{t-1}^u + K\varepsilon_t^p + (\rho\gamma_3 - \rho_a)K\bar{\tilde{\mathcal{O}}}_{t-1} \\
X^{kw} &\equiv \qquad a_t &&= \rho_a^2 a_{t-2} + \varepsilon_t^u + \varepsilon_t^p + \rho_a\varepsilon_{t-1}^p + \rho_a\varepsilon_{t-1}^u
\end{aligned}$$

Proof of Lemma A.3.

$$\begin{aligned}
Y_1 \equiv a_{t+1} - \bar{\mathbb{E}}_t a_{t+1} &= \rho_a(a_t - \bar{\mathbb{E}}_t a_t) + \varepsilon_{t+1}^p + \varepsilon_{t+1}^u \\
&= \rho_a((1-K)\varepsilon_t^p + \varepsilon_t^u - K\bar{\tilde{\mathcal{O}}}_t) + \varepsilon_{t+1}^p + \varepsilon_{t+1}^u \\
&= \rho_a\big((1-K)\varepsilon_t^p + \varepsilon_t^u - K\rho\gamma_3\bar{\tilde{\mathcal{O}}}_{t-1} + \rho\gamma_1 K\varepsilon_{t-1}^p - \rho\gamma_2 K\varepsilon_{t-1}^u\big) + \varepsilon_{t+1}^p + \varepsilon_{t+1}^u \\
Y_2 \equiv a_{t+1} - \bar{\mathbb{E}}_{t+1} a_{t+1} &= \varepsilon_{t+1}^u + (1-K)\varepsilon_{t+1}^p + \rho K\gamma_1\varepsilon_t^p - \rho K\gamma_2\varepsilon_t^u - \rho\gamma_3 K\bar{\tilde{\mathcal{O}}}_t \\
&= \varepsilon_{t+1}^u + (1-K)\varepsilon_{t+1}^p + \rho K\gamma_1\varepsilon_t^p - \rho K\gamma_2\varepsilon_t^u - \rho\gamma_3 K(\gamma_3\rho\bar{\tilde{\mathcal{O}}}_{t-1} - \rho\gamma_1\varepsilon_{t-1}^p + \rho\gamma_2\varepsilon_{t-1}^u) \\
X^{cg} \equiv \bar{\mathbb{E}}_t a_{t+1} - \bar{\mathbb{E}}_{t-1} a_{t+1} &= \rho_a(\bar{\mathbb{E}}_t a_t - \rho_a\bar{\mathbb{E}}_{t-1} a_{t-1}) \\
&\overset{\text{sgn}}{=} \Big(\rho_a^2 a_{t-2} + (\rho_a - \rho K\gamma_1)\varepsilon_{t-1}^p + (\rho_a + \rho K\gamma_2)\varepsilon_{t-1}^u + K\varepsilon_t^p + \rho\gamma_3 K\bar{\tilde{\mathcal{O}}}_{t-1}\Big) \\
&\quad - \rho_a\Big(\rho_a a_{t-2} + K(\varepsilon_{t-1}^p + \bar{\tilde{\mathcal{O}}}_{t-1})\Big) \\
&= (\rho_a(1-K) - \rho K\gamma_1)\varepsilon_{t-1}^p + (\rho_a + \rho K\gamma_2)\varepsilon_{t-1}^u + K\varepsilon_t^p + (\rho\gamma_3 - \rho_a)K\bar{\tilde{\mathcal{O}}}_{t-1} \\
X^{kw} \equiv a_t &= \rho_a^2 a_{t-2} + \varepsilon_t^u + \varepsilon_t^p + \rho_a\varepsilon_{t-1}^p + \rho_a\varepsilon_{t-1}^u
\end{aligned}$$

□

Proof of Proposition 3.

- Case 1: Common ε_t^p: $\mathrm{Var}(\varepsilon_t^p) = \sigma_p^2$

$$\begin{aligned}
\mathrm{Cov}(Y_2, X^{cg}) &\overset{\text{sgn}}{=} \rho K^2\gamma_1\sigma_p^2 - \rho\gamma_3 K\Big(\rho\gamma_3(\rho\gamma_3-\rho_a)K\,\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) - \rho\gamma_1(\rho_a(1-K)-\rho K\gamma_1)\sigma_p^2 + \rho\gamma_2(\rho_a+\rho K\gamma_2)\sigma_u^2\Big)\\
&\overset{\text{sgn}}{=} K\Sigma - \Big(\rho\gamma_3(\rho\gamma_3-\rho_a)K\,\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) - \rho\gamma_1(\rho_a(1-K)-\rho K\gamma_1)\sigma_p^2 + \rho\gamma_2(\rho_a+\rho K\gamma_2)\sigma_u^2\Big)\\
&\overset{\text{sgn}}{=} K\Sigma - \Big(\rho\gamma_3(\rho\gamma_3-\rho_a)K\,\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) - \rho V(\rho_a(1-K)-\rho K\gamma_1) + \rho V(\rho_a+\rho K\gamma_2)\Big)\\
&\overset{\text{sgn}}{=} K\Sigma - \rho\gamma_3(\rho\gamma_3-\rho_a)K\,\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) - \rho VK(\rho_a+\rho\gamma_2+\rho\gamma_1)\\
&\overset{\text{sgn}}{=} \Sigma - \rho\gamma_3(\rho\gamma_3-\rho_a)\,\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) - \rho V(\rho_a+\rho\gamma_2+\rho\gamma_1)
\end{aligned}$$

where $\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) = \dfrac{\rho^2(\gamma_1^2\sigma_p^2+\gamma_2^2\sigma_u^2)}{1-\gamma_3^2\rho^2}$. Since we always have $\Sigma > \rho V(\rho_a+\rho\gamma_2+\rho\gamma_1)$, or

$$1 > \rho\gamma_3(\rho_a+\rho-\rho\gamma_3),$$

a sufficient condition for $\mathrm{Cov}(Y, X) > 0$ is to have $\rho_a > \rho\gamma_3$.

Moreover, we can show that $Cov(Y_2, X^{cg}) > 0$ always holds.

- Case 2: Fully Idiosyncratic ε_t^p: $\mathrm{Var}(\varepsilon_t^p) = 0$ Then, since $\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_t) = \frac{\rho^2\gamma_2^2\sigma_u^2}{1-\rho^2\gamma_3^2}$,

$$\begin{aligned}
\mathrm{Cov}(Y_2, X^{cg}) &= -\rho\gamma_3 K\Big(\gamma_3\rho(\rho\gamma_3-\rho_a)K\,\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) + \rho\gamma_2(\rho_a+\rho K\gamma_2)\sigma_u^2\Big)\\
&\overset{\text{sgn}}{=} \gamma_3(\rho_a-\rho\gamma_3)K\,\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) - \gamma_2(\rho_a+\rho K\gamma_2)\sigma_u^2\\
&\overset{\text{sgn}}{=} (\rho_a-\rho\gamma_3)K\,\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) - \Sigma(\rho_a+\rho K\gamma_2).
\end{aligned}$$

This is linear in ρ_a, so it suffices to show $\mathrm{Cov} < 0$ when $\rho_a = 0$ and $\rho_a = 1$. The former is obvious, the latter is:

$$\begin{aligned}
\mathrm{Cov}(Y_2, X^{cg}) &\overset{\text{sgn}}{=} (1-\rho\gamma_3)K\,\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) - \Sigma(1+\rho K\gamma_2)\\
&\overset{\text{sgn}}{=} \frac{\rho^2\gamma_2^2\sigma_u^2K}{1+\rho\gamma_3} - \Sigma(1+\rho K\gamma_2)\\
&< 0.
\end{aligned}$$

- For the cases with Y_1 (maybe there is an easier proof...)

$$\begin{aligned}
\mathrm{Cov}(Y_1, X^{cg}) &\overset{\text{sgn}}{=} (1-K)K\bar{\sigma}_p^2 - K\rho\gamma_3(\rho\gamma_3-\rho_a)K\,\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) + K\rho\gamma_1(\rho_a(1-K)-\rho K\gamma_1)\bar{\sigma}_p^2 - K\rho\gamma_2(\rho_a+\rho K\gamma_2)\sigma_u^2\\
&\overset{\text{sgn}}{=} (1-K)\bar{\sigma}_p^2 - \rho\gamma_3(\rho\gamma_3-\rho_a)K\,\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) + \rho\gamma_1(\rho_a(1-K)-\rho K\gamma_1)\bar{\sigma}_p^2 - \rho\gamma_2(\rho_a+\rho K\gamma_2)\sigma_u^2
\end{aligned}$$

For $\bar{\sigma}_p^2 = \sigma_p$, we have

$$
\begin{aligned}
\mathrm{Cov}(Y_1, X^{cg}) &\overset{\text{sgn}}{=} (1-K)\sigma_p^2 - \rho\gamma_3(\rho\gamma_3 - \rho_a)K\,\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) + \rho\gamma_1(\rho_a(1-K) - \rho K\gamma_1)\sigma_p^2 - \rho\gamma_2(\rho_a + \rho K\gamma_2)\sigma_u^2 \\
&= (1-K)\sigma_p^2 - \rho\gamma_3(\rho\gamma_3 - \rho_a)K\,\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) + \rho V(\rho_a(1-K) - \rho K\gamma_1) - \rho V(\rho_a + \rho K\gamma_2) \\
&\overset{\text{sgn}}{=} \frac{1-K}{K}\sigma_p^2 - \rho\gamma_3(\rho\gamma_3 - \rho_a)\,\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) - \rho V(\rho_a + \rho\gamma_1 + \rho\gamma_2) \\
&\overset{\text{sgn}}{=} \mathrm{Cov}(Y_2, X^{cg})
\end{aligned}
$$

For $\bar{\sigma}_p^2 = 0$, we have

$$
\begin{aligned}
\mathrm{Cov}(Y_1, X^{cg}) &\overset{\text{sgn}}{=} \gamma_3(\rho_a - \rho\gamma_3)K\,\mathrm{Var}(\bar{\tilde{\mathcal{O}}}_{t-1}) - \gamma_2(\rho_a + \rho K\gamma_2)\sigma_u^2 \\
&\overset{\text{sgn}}{=} \mathrm{Cov}(Y_2, X^{cg})
\end{aligned}
\qquad \square
$$

Proof of Proposition 4.

$$
\begin{aligned}
\mathrm{Cov}(Y_2, X^{kw}) &\overset{\text{sgn}}{=} \gamma_1\bar{\sigma}_p^2 - \gamma_2\bar{\sigma}_u^2 - \gamma_3(-\rho\gamma_1\rho_a\bar{\sigma}_p^2 + \rho\gamma_2\rho_a\bar{\sigma}_u^2) - \rho\gamma_3^2\rho_a^2\,\mathrm{Cov}(\bar{\tilde{\mathcal{O}}}_{t-1}, a_{t-2}) \\
&\overset{\text{sgn}}{=} \frac{\bar{\sigma}_p^2}{\sigma_p^2} - \frac{\bar{\sigma}_u^2}{\sigma_u^2} < 0. \\
\mathrm{Cov}(Y_1, X^{kw}) &\overset{\text{sgn}}{=} (1-K)\bar{\sigma}_p^2 + \bar{\sigma}_u^2 - K\rho\gamma_3\rho_a^2\,\mathrm{Cov}(\bar{\tilde{\mathcal{O}}}_{t-1}, a_{t-2}) + K\rho\gamma_1\rho_a\bar{\sigma}_p^2 - K\rho\gamma_2\rho_a\bar{\sigma}_u^2 \\
&\overset{\text{Lemma A.1}}{=} (1-K)\bar{\sigma}_p^2 + \bar{\sigma}_u^2 + K\rho\gamma_3\rho_a^2\frac{\rho V}{1-\rho\gamma_3\rho_a}\left(\frac{\bar{\sigma}_p^2}{\sigma_p^2} - \frac{\bar{\sigma}_u^2}{\sigma_u^2}\right) + K\rho\rho_a V\left(\frac{\bar{\sigma}_p^2}{\sigma_p^2} - \frac{\bar{\sigma}_u^2}{\sigma_u^2}\right) \\
&= (1-K)\bar{\sigma}_p^2 + \bar{\sigma}_u^2 + \frac{K\rho\rho_a V}{1-\rho\gamma_3\rho_a}\left(\frac{\bar{\sigma}_p^2}{\sigma_p^2} - \frac{\bar{\sigma}_u^2}{\sigma_u^2}\right).
\end{aligned}
$$

This is linear in $\bar{\sigma}_p^2$ (note: V and γ's depend on σ_p^2, not $\bar{\sigma}_p^2$), so it suffices to show $\mathrm{Cov} > 0$ when $\bar{\sigma}_p^2 = 0$ and $\bar{\sigma}_p^2 = \sigma_p^2$. The latter is obvious, the former is:

$$
\begin{aligned}
\mathrm{Cov}(Y_1, X^{kw}) &\overset{\text{sgn}}{=} \bar{\sigma}_u^2 - \frac{K\rho\rho_a V}{1-\rho\gamma_3\rho_a}\frac{\bar{\sigma}_u^2}{\sigma_u^2} \\
&\overset{\text{sgn}}{=} 1 - \frac{K\rho\rho_a\gamma_2}{1-\rho\gamma_3\rho_a} \\
&\overset{\text{sgn}}{=} 1 - \rho\rho_a(\gamma_3 + K\gamma_2) > 0.
\end{aligned}
$$

Finally, $a_{t+1} - \bar{\mathbb{E}}_t a_{t+1} = \rho_a(a_t - \bar{\mathbb{E}}_t a_t) + \varepsilon_{t+1}^p + \varepsilon_{t+1}^u$, so $\mathrm{Cov}(Y_1, X^{kw}) \overset{\text{sgn}}{=} \mathrm{Cov}(LY_2, X^{kw})$. $\square$

Proposition A.1 (Misspecification). *When agent i thinks that her noise term follows an AR(1) process, $\xi_{it} = \hat{\rho}\xi_{it-1} + \eta_{it}$ where $\eta_{it} \sim \mathcal{N}\big(0, (1-\hat{\rho}^2)\sigma_\xi^2\big)$, while the truth is $\xi_{it} = \rho\xi_{it-1} + \eta_{it}$*

where $\eta_{it} \sim \mathcal{N}\big(0, (1-\rho^2)\sigma_\xi^2\big)$*, we have*[38]

$$\mathrm{Cov}(a_{it+h} - \mathbb{E}_{it}a_{it+h}, \mathbb{E}_{it}a_{it+h} - \mathbb{E}_{it-1}a_{it+h}) < 0 \iff \rho < \hat{\rho}.$$

Proof. We can ignore the volatility from $\varepsilon^p, \varepsilon^u$. Modulo this volatility, we have

$$\begin{aligned}
Y_t &= a_{it+h} - \mathbb{E}_{it}a_{it+h} \\
&= \rho_a^h(a_t - \mathbb{E}_{it}a_{it}) + \varepsilon_{t,t+h} \\
&= \rho_a^h\Big(\big(\rho_a a_{it-1} + \varepsilon_{it}^p + \varepsilon_{it}^u\big) - \big(\rho_a a_{it-1} + K(\varepsilon_{it}^p + \xi_{it} - \hat{\rho}m_{t-1})\big)\Big) + \varepsilon_{t,t+h} \\
&= \rho_a^h\Big(- K\xi_{it} + K\hat{\rho}((\gamma_1+\gamma_2)\xi_{it-1} + \gamma_3\hat{\rho}m_{t-2})\Big) \\
X_t &= \mathbb{E}_{it}a_{it+h} - \mathbb{E}_{it-1}a_{it+h} \\
&= \rho_a^h(\mathbb{E}_{it}a_{it} - \rho_a\mathbb{E}_{it-1}a_{it-1}) \\
&= \rho_a^h\Big(\big(\rho_a a_{it-1} + K(\varepsilon_{it}^p + \xi_{it} - \hat{\rho}m_{t-1})\big) - \rho_a\big(\rho_a a_{it-2} + K(\varepsilon_{it-1}^p + \xi_{it-1} - \hat{\rho}m_{t-2})\big)\Big) \\
&= \rho_a^h\Big(\rho_a(\varepsilon_{it-1}^p + \varepsilon_{it-1}^u) + K\varepsilon_{it}^p + K\xi_{it} - K\hat{\rho}m_{t-1} - \rho_a K\varepsilon_{it-1}^p - \rho_a K\xi_{it-1} + \rho_a K\hat{\rho}m_{t-2}\Big) \\
&= \rho_a^h\Big(K\xi_{it} - K\hat{\rho}((\gamma_1+\gamma_2)\xi_{it-1} + \gamma_3\hat{\rho}m_{t-2}) - \rho_a K\xi_{it-1} + \rho_a K\hat{\rho}m_{t-2}\Big)
\end{aligned}$$

Thus, as $m_t = \gamma_3\hat{\rho}m_{t-1} + (\gamma_1+\gamma_2)\xi_{it}$,

$$\begin{aligned}
\tilde{Y}_t \equiv \frac{Y_t}{K\rho_a^h\hat{\rho}(\gamma_1+\gamma_2)} &= -\frac{1}{\hat{\rho}(\gamma_1+\gamma_2)}\xi_{it} + \xi_{it-1} + \frac{\gamma_3\hat{\rho}}{\gamma_1+\gamma_2}m_{t-2} \\
&= -\frac{1}{\hat{\rho}(\gamma_1+\gamma_2)}\xi_{it} + \xi_{it-1} + \gamma_3\hat{\rho}\xi_{it-2} + (\gamma_3\hat{\rho})^2\xi_{it-3} + \cdots \\
\tilde{X}_t \equiv \frac{X_t}{K\rho_a^h\hat{\rho}(\gamma_1+\gamma_2)} &= \frac{1}{\hat{\rho}(\gamma_1+\gamma_2)}\xi_{it} + \frac{\rho_a - \hat{\rho}\gamma_3}{\gamma_1+\gamma_2}m_{t-2} - \left(1 + \frac{\rho_a}{\hat{\rho}(\gamma_1+\gamma_2)}\right)\xi_{it-1}
\end{aligned}$$

so, for $\theta \equiv \frac{1}{\hat{\rho}(\gamma_1+\gamma_2)}$, $\beta = \gamma_3\hat{\rho}$, and $\delta = \rho_a - \hat{\rho}\gamma_3$,

$$\frac{\mathbb{E}\big[\tilde{Y}_t\tilde{X}_t\big]}{\sigma_\eta^2} = \begin{pmatrix}-\theta & 1 & \beta & \beta^2 & \beta^3 & \cdots\end{pmatrix}\begin{pmatrix}1 & \rho & \rho^2 & \rho^3 & \cdots \\ \rho & 1 & \rho & \rho^2 & \cdots \\ \rho^2 & \rho & 1 & \rho & \cdots \\ \rho^3 & \rho^2 & \rho & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}\theta \\ -1-\rho_a\theta \\ \delta \\ \delta\beta \\ \delta\beta^2 \\ \delta\beta^3 \\ \vdots\end{pmatrix} = (1) + (2) + (3)$$

[38] We have normalized the variance of innovation to have $\mathrm{Var}(\xi_{it}) = \sigma_\xi^2$.

$$\text{where } (1) = \delta\beta\left(\sum_{j,k\geq 0}\rho^{|j-k|}\beta^{j+k}\right) = \delta\beta\left(\frac{1}{1-\beta^2} + \sum_{t\geq 1}\rho^t 2\frac{\beta^t}{1-\beta^2}\right) = \delta\beta\left(\frac{1}{1-\beta^2} + \frac{2}{1-\beta^2}\frac{\rho\beta}{1-\rho\beta}\right)$$
$$= \delta\beta\frac{1+\rho\beta}{(1-\beta^2)(1-\rho\beta)}$$
$$(2) = -\theta\left(\theta - \rho(1+\rho_a\theta) + \frac{\delta\rho^2}{1-\beta\rho}\right) + \left(\rho\theta - 1 - \rho_a\theta + \frac{\delta\rho}{1-\beta\rho}\right)$$
$$(3) = \theta\frac{\beta\rho^2}{1-\beta\rho} - (1+\rho_a\theta)\frac{\beta\rho}{1-\beta\rho}$$

Finally, we can show that

$$\frac{\partial\Big((1)+(2)+(3)\Big)}{\partial\rho} \geq 0 \qquad \square$$

Proof of Proposition 5.

- For our model,

  Proof. Note first that

$$z_{t+1} \equiv a_{t+1} - \bar{\mathbb{E}}_{t+1}a_{t+1} = \rho\gamma_3 z_t + \kappa_{t+1} \quad \text{where } \kappa_{t+1} = -\rho(\gamma_3 + K\gamma_2)\varepsilon^u_t + \varepsilon^u_{t+1} + (1-K)\varepsilon^p_{t+1}$$
$$a_t = \rho_a a_{t-1} + \mu_t \quad \text{where } \mu_t = \varepsilon^p_t + \varepsilon^u_t.$$

  Thus,

$$\begin{aligned}
\mathrm{Cov}(z_{t+1}, a_t) &= \frac{1}{1-\rho\gamma_3\rho_a}\Big(\mathrm{Cov}(\kappa_{t+1},\mu_t) + \rho\gamma_3\,\mathrm{Cov}(z_t,\mu_t) + \rho_a\,\mathrm{Cov}(\kappa_{t+1}, a_{t-1})\Big)\\
&= \frac{1}{1-\rho\gamma_3\rho_a}\Big(-\rho(\gamma_3+K\gamma_2)\bar{\sigma}_u^2 + \rho\gamma_3((1-K)\bar{\sigma}_p^2 + \bar{\sigma}_u^2)\Big)\\
&= \frac{\rho}{1-\rho\gamma_3\rho_a}\Big(\gamma_3(1-K)\bar{\sigma}_p^2 - K\gamma_2\bar{\sigma}_u^2\Big)\\
&= \frac{\rho K V}{1-\rho\gamma_3\rho_a}\left(\frac{\bar{\sigma}_p^2}{\sigma_p^2} - \frac{\bar{\sigma}_u^2}{\sigma_u^2}\right)
\end{aligned}$$

  Finally, we have $\mathrm{Var}(a_t) = \frac{\bar{\sigma}_p^2+\bar{\sigma}_u^2}{1-\rho_a^2}$. $\square$

- For extrapolation,

  Proof.

$$\mathrm{Cov}(a_{t+1} - \bar{E}_{t+1}a_{t+1}, a_t) = \mathrm{Cov}(\rho a_t + u_{t+1} - \hat{\rho}a_t, a_t) = (\rho - \hat{\rho})\,\mathrm{Var}(a_t)$$

39

$$\mathrm{Cov}(a_{it+1} - E_{it+1}a_{it+1}, a_{it}) = \mathrm{Cov}(\rho a_{it} + u_{it+1} - \hat{\rho} a_{it}, a_{it}) = (\rho - \hat{\rho})\,\mathrm{Var}(a_{it}).$$

$\square$

- For diagnostic expectations with Kalman gain g_0, we have

$$\widehat{\beta}_{aggr} = \widehat{\beta}_{ind} = \frac{(1-g_0)\rho(1-\rho^2)(1-K)}{1-\rho^2(1-K)} \overset{\text{sgn}}{=} (1-g_0)$$

Proof. We have

$$\begin{aligned} a_{it+1} - \mathbb{E}_{it+1}a_{it+1} &= a_{it+1} - \mathbb{E}^{NRE}_{it}a_{it+1} - g_0(z_{it+1} - \mathbb{E}^{NRE}_{it}a_{it+1}) \\ &\equiv (1-g_0)(a_{it+1} - \mathbb{E}^{NRE}_{it}a_{it+1}) \end{aligned}$$

Thus,

$$\begin{aligned} \widehat{\beta}_{ind} &= (1-g_0)\frac{\mathrm{Cov}(a_{it+1} - \mathbb{E}^{NRE}_{it}a_{it+1}, a_{it})}{\mathrm{Var}(a_{it})} \\ &= (1-g_0)\rho\frac{\mathrm{Cov}(a_{it} - \mathbb{E}^{NRE}_{it}a_{it})}{\mathrm{Var}(a_{it})} \\ &= (1-g_0)\rho\left(1 - \frac{K}{1-\rho^2(1-K)}\right) \\ &= \frac{(1-g_0)\rho(1-\rho^2)(1-K)}{1-\rho^2(1-K)} \end{aligned}$$

where the penultimate equality uses the fact that

$$\begin{aligned} \mathbb{E}^{NRE}_{it}a_{it} &= \mathbb{E}^{NRE}_{it-1}a_{it} + K(z_{it} - \mathbb{E}^{NRE}_{it-1}a_{it}) \\ &= Kz_{it} + \rho(1-K)\mathbb{E}^{NRE}_{it-1}a_{it-1} \\ &= K\sum_{h\geq 0}\rho^h(1-K)^h z_{it-h} \end{aligned}$$

hence

$$\mathrm{Cov}(\mathbb{E}^{NRE}_{it}a_{it}, a_{it}) = K\sum_{h\geq 0}\rho^h(1-K)^h\underbrace{\mathrm{Cov}(z_{it-h}, a_{it})}_{=\rho^h\,\mathrm{Var}(a_{it})}.$$

Second, we have

$$a_{t+1} - \bar{\mathbb{E}}_{t+1}a_{t+1} \equiv (1-g_0)(a_{t+1} - \bar{\mathbb{E}}^{NRE}_{it}a_{t+1})$$

hence

$$\begin{aligned}
\widehat{\beta}_{aggr} &= (1-g_0)\frac{\text{Cov}(a_{t+1}-\bar{\mathbb{E}}_t^{NRE}a_{t+1}, a_t)}{\text{Var}(a_t)} \\
&= (1-g_0)\rho\frac{\text{Cov}(a_t-\bar{\mathbb{E}}_t^{NRE}a_t, a_t)}{\text{Var}(a_t)} \\
&= (1-g_0)\rho\left(1-\frac{K}{1-\rho^2(1-K)}\right) \\
&= \frac{(1-g_0)\rho(1-\rho^2)(1-K)}{1-\rho^2(1-K)}
\end{aligned}$$

where the penultimate equality uses the fact that

$$\bar{\mathbb{E}}_t^{NRE}a_t = K\sum_{h\geq 0}\rho^h(1-K)^h a_{t-h}$$

hence

$$\text{Cov}(\bar{\mathbb{E}}_t^{NRE}a_t, a_t) = K\sum_{h\geq 0}\rho^h(1-K)^h \underbrace{\text{Cov}(a_{t-h}, a_t)}_{=\rho^h\,\text{Var}(a_t)}.$$

□

- For AHS (2020),

$$\begin{aligned}
a_{it} &= \rho a_{it-1} + \varepsilon_{it} && \text{(perceived: } \hat{\rho}\text{)} \\
z_{it} &= a_{it} + \frac{1}{\sqrt{\tau}}u_{it}, && \text{(perceived: } \hat{\tau}\text{)}
\end{aligned}$$

we have

$$\widehat{\beta}_{aggr} = \widehat{\beta}_{ind} = \rho - \hat{K}\rho - \frac{\hat{K}\hat{\rho}(1-\hat{K})}{1-\hat{\rho}\rho(1-\hat{K})}.$$

Proof. As in above, we have

$$\mathbb{E}_{it}a_{it} = \hat{K}\sum_{h\geq 0}\hat{\rho}^h(1-\hat{K})^h z_{it-h}$$

where $\hat{K}\in(0,1)$ is a function of $\hat{\rho}$ and $\hat{\tau}$. We then have

$$a_{it+1} - \mathbb{E}_{it+1}a_{it+1} = \sum_{h\geq 0}\rho^h\varepsilon_{i,t+1-h} - \hat{K}\sum_{h\geq 0}\hat{\rho}^h(1-\hat{K})^h z_{i,t+1-h}$$

so

$$
\begin{aligned}
\mathrm{Cov}(a_{it+1} - \mathbb{E}_{it+1} a_{it+1}, a_{it}) &= \sum_{h \geq 0} \rho^h \,\mathrm{Cov}(\varepsilon_{i,t+1-h}, a_{it}) - \hat{K} \sum_{h \geq 0} \hat{\rho}^h (1-\hat{K})^h \,\mathrm{Cov}(z_{i,t+1-h}, a_{it}) \\
&= \rho \,\mathrm{Var}(\varepsilon_{it}) + \rho^3 \,\mathrm{Var}(\varepsilon_{it}) + \rho^5 \,\mathrm{Var}(\varepsilon_{it}) + \cdots \\
&\quad - \hat{K}\rho \,\mathrm{Var}(a_{it}) - \hat{K}\hat{\rho}(1-\hat{K}) \,\mathrm{Var}(a_{it}) - \hat{K}\hat{\rho}^2(1-\hat{K})^2 \rho \,\mathrm{Var}(a_{it}) - \cdots \\
&= \mathrm{Var}(a_{it}) \left(\rho - \hat{K}\rho - \frac{\hat{K}\hat{\rho}(1-\hat{K})}{1-\hat{\rho}\rho(1-\hat{K})} \right).
\end{aligned}
$$

where the last equality uses the fact that

$$
(1-\rho^2)\,\mathrm{Var}(a_{it}) = \mathrm{Var}(\varepsilon_{it}).
$$

Thus, we have

$$
\widehat{\beta}_{ind} = \rho - \hat{K}\rho - \frac{\hat{K}\hat{\rho}(1-\hat{K})}{1-\hat{\rho}\rho(1-\hat{K})}.
$$

We have

$$
\begin{aligned}
\mathrm{Cov}(a_{t+1} - \bar{\mathbb{E}}_{t+1} a_{t+1}, a_t) &= \sum_{h \geq 0} \rho^h \,\mathrm{Cov}(\varepsilon_{t+1-h}, a_t) - \hat{K} \sum_{h \geq 0} \hat{\rho}^h (1-\hat{K})^h \,\mathrm{Cov}(a_{t+1-h}, a_t) \\
&= \rho \,\mathrm{Var}(\varepsilon_t) + \rho^3 \,\mathrm{Var}(\varepsilon_t) + \rho^5 \,\mathrm{Var}(\varepsilon_t) + \cdots \\
&\quad - \hat{K}\rho \,\mathrm{Var}(a_t) - \hat{K}\hat{\rho}(1-\hat{K}) \,\mathrm{Var}(a_t) - \hat{K}\hat{\rho}^2(1-\hat{K})^2 \rho \,\mathrm{Var}(a_t) - \cdots \\
&= \mathrm{Var}(a_t) \left(\rho - \hat{K}\rho - \frac{\hat{K}\hat{\rho}(1-\hat{K})}{1-\hat{\rho}\rho(1-\hat{K})} \right).
\end{aligned}
$$

where the last equality uses the fact that

$$
(1-\rho^2)\,\mathrm{Var}(a_t) = \mathrm{Var}(\varepsilon_t).
$$

Thus, we have

$$
\widehat{\beta}_{aggr} = \rho - \hat{K}\rho - \frac{\hat{K}\hat{\rho}(1-\hat{K})}{1-\hat{\rho}\rho(1-\hat{K})}. \qquad \square
$$

- Kohlhas and Walther (2020):

$$
\begin{aligned}
y_{it} &= \sum_j x_{ijt} \\
x_{ijt} &= a_j \theta_{it} + b_j u_{ijt} \\
\theta_{it} &= \rho \theta_{it-1} + \eta_{it}
\end{aligned}
$$

$$z_{ijt} = x_{ijt} + q_j \cdot \varepsilon_{ijt}$$

Then, we have $\widehat{\beta}_{aggr} < \widehat{\beta}_{ind}$ if and only if

$$\frac{\mathrm{Var}(u_{jt})}{\mathrm{Var}(\eta_t)} > \frac{\mathrm{Var}(u_{ijt})}{\mathrm{Var}(\eta_{it})}.$$

Proof. Starting from (3), we have

$$\begin{aligned}
\mathbb{E}_{it}[\theta_{it}] &= \mathbb{E}_{it-1}[\theta_{it}] + \sum_j g_j(z_{ijt} - \mathbb{E}_{it-1} z_{ijt}) \\
&= \mathbb{E}_{it-1}[\theta_{it}] + \sum_j g_j(z_{ijt} - a_j \mathbb{E}_{it-1}\theta_{it}) \\
&= \rho(1 - \sum_j g_j a_j)\mathbb{E}_{it-1}\theta_{it-1} + \sum_j g_j(a_j\theta_{it} + b_j u_{ijt} + q_j \varepsilon_{ijt}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\theta_{it} - \mathbb{E}_{it}\theta_{it} &= (1 - \sum_j g_j a_j)(\rho\theta_{it-1} + \eta_{it}) - \rho(1 - \sum_j g_j a_j)\mathbb{E}_{it-1}\theta_{it-1} - \sum_j g_j b_j u_{ijt} - \sum_j g_j q_j \varepsilon_{ijt} \\
&= \underbrace{\rho(1 - \sum_j g_j a_j)}_{\Gamma}(\theta_{it-1} - \mathbb{E}_{it-1}\theta_{it-1}) + \underbrace{(1 - \sum_j g_j a_j)\eta_{it} - \sum_j g_j b_j u_{ijt} - \sum_j g_j q_j \varepsilon_{ijt}}_{\zeta_{it}} \\
&= \sum_{h \geq 0} \Gamma^h \zeta_{it-h}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
y_{it+1} - \mathbb{E}_{it+1} y_{it+1} &\equiv (\sum_j a_j)(\theta_{it+1} - \mathbb{E}_{it+1}\theta_{it+1}) \\
&= (\sum_j a_j) \sum_{h \geq 0} \Gamma^h \zeta_{i,t+1-h}.
\end{aligned}$$

and

$$y_{it} = (\sum_j a_j) \sum_{h \geq 0} \rho^h \eta_{it-h} + \sum_j b_j u_{ijt}.$$

Thus, the covariance between them is

$$\mathrm{Cov} = -(\sum_j a_j)\Gamma \sum_j g_j b_j^2 \,\mathrm{Var}(u_{ijt}) + (\sum_j a_j)^2 \Gamma \underbrace{\mathrm{Cov}\left(\sum_{h \geq 0} \Gamma^h \zeta_{it-h}, \sum_{h \geq 0} \rho^h \eta_{it-h} \right)}_{=(1 - \sum_j g_j a_j)\frac{1}{1 - \Gamma\rho}\mathrm{Var}(\eta_{it})}$$

Thus,

$$\widehat{\beta}_{ind} = \frac{(\sum_j a_j)\Gamma\Big(\frac{(\sum_j a_j)(1-\sum_j g_j a_j)}{1-\Gamma\rho}\operatorname{Var}(\eta_{it}) - \sum_j g_j b_j^2 \operatorname{Var}(u_{ijt})\Big)}{\frac{(\sum_j a_j)^2}{1-\rho^2}\operatorname{Var}(\eta_{it}) + \sum_j b_j^2 \operatorname{Var}(u_{ijt})}.$$

On the other hand, we have

$$\begin{aligned} y_{t+1} - \bar{\mathbb{E}}_{t+1} y_{t+1} &= (\sum_j a_j)\sum_{h\geq 0}\Gamma^h \zeta_{t+1-h} \\ y_t &= (\sum_j a_j)\sum_{h\geq 0}\rho^h \eta_{t-h} + \sum_j b_j u_{jt}. \end{aligned}$$

hence

$$\widehat{\beta}_{aggr} = \frac{(\sum_j a_j)\Gamma\Big(\frac{(\sum_j a_j)(1-\sum_j g_j a_j)}{1-\Gamma\rho}\operatorname{Var}(\eta_t) - \sum_j g_j b_j^2 \operatorname{Var}(u_{jt})\Big)}{\frac{(\sum_j a_j)^2}{1-\rho^2}\operatorname{Var}(\eta_t) + \sum_j b_j^2 \operatorname{Var}(u_{jt})}.$$

Both beta hats are of the form (impose $\operatorname{Var}(u_{jt}) = \operatorname{Var}(u_{j't})$)

$$\widehat{\beta} = \frac{b - a\kappa}{d + c\kappa} \quad \text{where } \kappa = \frac{\operatorname{Var}(u)}{\operatorname{Var}(\eta)} > 0$$

wherer $a, b, c, d > 0$. We can easily show that $\widehat{\beta}$ is decreasing in κ. So we have $\widehat{\beta}_{aggr} < \widehat{\beta}_{ind}$ if and only if

$$\frac{\operatorname{Var}(u_{jt})}{\operatorname{Var}(\eta_t)} > \frac{\operatorname{Var}(u_{ijt})}{\operatorname{Var}(\eta_{it})}. \qquad \square$$

$\square$