MATH 3012: Applied Combinatorics

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1 Proofs

1.1 Axioms

An argument in a proof is either an axiom or rests on an axiom. An axiom is an unproven statement, and different theorems can becom true or false depending on your choice of axioms.

The following axioms/assumptions are allowed

- The rules of algebra
  e.g. if x, y, z are real numbers and $x = y$, then $x + z = y + z$
- The set of integers is closed under addition, multiplication, and subtraction
- Every integer is either even or odd
- If x is an integer, there is no integer between x and x + 1
- The relative order of any two real numbers
  e.g. $\frac{1}{2} < 1$ or $4.2 \geq 3.7$
- The square of any number is greater than or equal to 0

1.2 Existential Instantiation

A law of logic that says if an object is known to exist, that object can be given a name as long as the name is not currently being used to denote something else.

Example
If n is an odd integer, $n = 2k + 1$ for some integer k.

1.3 Direct Proofs

In a direct proof of a conditional statement $p \to c$, the hypothesis p is assumed to be true and the conclusion c is proven as a direct result of the assumption.

If n is an odd integer, then n^2 is an odd integer.

Many theorems also have a universal quantifier such as

For every integer n, if n is odd then n^2 is odd.

1.3.1 Example

Theorem The square of every odd integer is also odd

Proof. Let n be an odd integer.
Since n is odd, $n = 2k + 1$ for some integer k.
Plug $n = 2 + 1$ into n^2 to get:

$$n^2 = (2k+1)^2 \tag{1}$$
$$= 4k^2 + 4k + 1 \tag{2}$$
$$= 2(2^2 + 2k) + 1 \tag{3}$$

Since k is an integer, $2^2 + 2$ is also an integer.
Since $n^2 = 2m + 1$, where $m = 2k^2 + 2$ is an integer, n^2 is odd. □

Two-Column Proof Format

n is odd	Assume p.
$n = 2k + 1$, for some $k \in \mathbf{Z}$	Definition of Odd
$n^2 = (2k+1)^2$	Square both sides
$n^2 = 4k^2 + 4k + 1$	expand $(2k+1)^2$
$n^2 = 2(2k^2 + 2k) + 1$	factor out 2
$w = 2k^2 + 2m$	define new variable
w is an integer	integers are closed under addition, multiplication, exponentiation
$n^2 + 2w + 1$, w is integer	substitute w
n^2 is odd	definition of odd

$\therefore$

Therefore, by direct proof, I have shown $p \to q$

1.3.2 Example

Prove that, if x and y are squares, then xy is a square
p: x and y are squares
q: xy is a square
prove $p \to q$