Exam 1 reference sheet

Equations

General

- **PAM** $$ \sum_{n=0}^{\infty} s_{i[n]} \left(t - nT \right) $$ $s_{i[n]}$: symbol; t: time; n: index; T: Period
- **Inner Product of signals** $$ \langle f, g \rangle = \int_0^T f(t) g^*(t) , dt $$ f, g: signals; T: interval of interest
- **Energy in Signal** $$ |f|^2 = \langle f, f \rangle = \mathcal{E} $$ f: signal; $\mathcal{E}$: Energy in Signal

Constellations

- **Bits per Symbol:** $$ k = \log_2 M \ M = 2^k $$ k: bits per symbol; M: number of symbols
- **BER:** $$ \text{BER} \propto d_{\text{min}}^2 $$ BER: Bit Error Rate; d_{min}: minimum distance between symbols
- **Energy per symbol** $$ E_s = \frac{1}{M} \sum_{m=1}^{M} |s_m|^2 $$
- **Energy per bit** $$ E_b = \frac{E_s}{k} $$
- **SNR per bit** $$ \gamma_b = 10 \log_{10} \left(\frac{E_b}{N_0} \right) $$ γ_b: SNR per bit; E_b: energy per bit; N_0: noise power spectral density
- **Symbol Rate** $$ R_s = \frac{1}{T_s} $$ R_s: Symbol Rate; T: Symbol Period
- **Bit Rate** $$ R_b = k R_s $$ R_b: Bit Rate; k: bits per symbol
- **Spectral Efficiency** $$ \eta = \frac{R_b}{W} \approx \frac{\log_2 M}{N} $$ η: Spectral Efficiency; R_b: Bit Rate; W: Bandwidth($\sim N$: number of real dimensions); M: number of symbols

Fourier Transforms

Frequency ω (rad/sec)

$$ X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega $$

Frequency f (Hz)

$$ X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt, \quad x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df $$

Inner Product

$$ \langle x, y \rangle = \int_{-\infty}^{\infty} x(t) y^(t) dt, \quad \langle X, Y \rangle = \int_{-\infty}^{\infty} X(f) Y^(f) df $$

Parseval's Theorem

$$ \langle x, y \rangle = \langle X, Y \rangle \ |x|^2 = \langle x, x \rangle \ |x| = |X| $$

Finite Energy Signal

$$ \mathcal{E}x = \int{-\infty}^{\infty} |x(t)|^2 , dt = \int_{-\infty}^{\infty} |X(f)|^2 , df $$

Finite Power Signals

$$ P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt $$

For one-sided signals:

$$ P_x = \frac{1}{T} \int_{0}^{T} |x(t)|^2 , dt $$

$$ \mathcal{P}_x \text{ finite } \longrightarrow \mathcal{E}_x = \infty \quad \mathcal{E}_x \text{ finite } \longrightarrow \mathcal{P}_x = 0 $$

WSS Signals

$$ \mu = E(x(t)), \quad r_{xx}(\tau) = E(x(t) x^*(t - \tau)) $$

- **Truncated FT**

  $$ X_T(f) = \int_{-T/2}^{T/2} x(t) e^{-j2\pi ft} dt $$

- **PSD**

  $$ S_x(f) = \lim_{T \to \infty} \frac{1}{T} E\left(|X_T(f)|^2\right) $$

Convolution

$$ y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau \ V(f) = G(f) * U(f) = \int_{-\infty}^{\infty} G(g) U(f - g) dg $$

Time Domain $\leftrightarrow$ Frequency Domain

$$ h(t) * x(t) \leftrightarrow H(f)X(f) \ u(t)v(t) \leftrightarrow U(f) * V(f) $$

Special Functions

- **Sinc function:** $$ \text{sinc}(u) = \frac{\sin(\pi u)}{\pi u} $$
- **Rectangular Pulse:** $$ \Pi(u) = 1_{[-1/2, 1/2]} $$
- **Triangular Pulse:** $$ \Lambda(u) = (1 - |u|) \cdot 1_{[-1, 1]} $$
- **Triangular and Rectangular Pulse:** $$ T\Lambda(t/T) = \Lambda(t/T) * \Lambda(t/T) $$

- **Fourier Transforms** $$ \Pi(t/T) \leftrightarrow T \text{sinc}(fT) \ \Lambda(t/T) \leftrightarrow T \text{sinc}^2(fT) $$

  $$ \text{sinc}(t/T) \leftrightarrow T\Pi(fT) \ \text{sinc}^2(t/T) \leftrightarrow T\Lambda(fT) $$

Gaussian Pulse

$$ g_{\mu, \sigma}(u) = \frac{1}{2\sqrt{\pi}\sigma} \exp\left(-\frac{(u - \mu)^2}{2\sigma^2}\right) $$

$$ g_{\mu, \sigma}(t) \leftrightarrow \exp\left(-j2\pi \mu f - (2\pi)^2 \frac{\sigma^2}{2} f^2\right) $$

- **Properties:** $$ \int_{-\infty}^{\infty} g_{\mu, \sigma}(u) , du = 1 \ \int_{-\infty}^{\infty} u g_{\mu, \sigma}(u) , du = \mu \ \int_{-\infty}^{\infty} (u - \mu)^2 g_{\mu, \sigma}(u) , du = \sigma^2 $$

- **Convolution:** $$ g_{\mu_1, \sigma_1} * g_{\mu_2, \sigma_2} = g_{\mu, \sigma} $$ $$ \mu = \mu_1 + \mu_2, \quad \sigma = \sqrt{\sigma_1^2 + \sigma_2^2} $$

Shifting Properties

- **Time shift:** $$ x(t - t_0) \leftrightarrow e^{-j2\pi f t_0} X(f) $$

- **Frequency shift:** $$ e^{j2\pi f_0 t} x(t) \leftrightarrow X(f - f_0) $$

Impulse Functions

$$ e^{j 2 \pi f_0 t} \quad \leftrightarrow \quad \delta(f - f_0) $$ $$ \cos(2 \pi f_0 t) \quad \leftrightarrow \quad \frac{1}{2} \delta(f - f_0) + \frac{1}{2} \delta(f + f_0) $$ $$ \sin(2 \pi f_0 t) \quad \leftrightarrow \quad \frac{1}{2j} \delta(f - f_0) - \frac{1}{2j} \delta(f + f_0) $$

Important Identity

$$ cos(A)cos(B) = \frac{1}{2}cos(A+B) + \frac{1}{2}cos(A-B) $$

Symmetries of the Fourier Transform

$$ x^(t) \leftrightarrow X^(-f) $$ $$ x^(-t) \leftrightarrow X^(f) $$

1. **Zero-Phase**: $X(f)$ is purely real iff $x(t) = x^*(-t)$.

2. **Linear-Phase**: $\angle X(f)$ is linear if x has a point of symmetry in time: $$ X(f) = e^{-j \pi f T} X_R(f) \leftrightarrow x(t) = x^*(T - t) $$

3. $x(t)$ is real iff $X(f)$ is conjugate symmetric: $X(f) = X^*(-f)$:

   - $\text{Re} X(f)$ and $|X(f)|$ are even functions of f.
   - $\text{Im} X(f)$ and $\angle X(f)$ are odd functions of f.

4. $x(t)$ is purely imaginary iff $X(f) = -X^*(-f)$.

Correlation

- **Time-reversed Complex Conjugate** $$ \tilde{x}(t) = x^(-t) \ \tilde{x}(t) \leftrightarrow X^(f) $$

- **Deterministic Auto-Correlation** $$ r_{xx}(\tau) = \int_{-\infty}^{\infty} x(t + \tau) x^*(t) , dt \ r_{xx} \leftrightarrow |X(f)|^2 $$

- **Deterministic Cross-Correlation** $$ r_{xy}(t) = \int_{-\infty}^{\infty} x(t + \tau) y^*(t) , dt $$ $$ r_{xy} = x * \tilde{y} $$

Finite Power Spectrum

$$ r_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t + \tau) x^*(t) , dt $$

- **Deterministic power spectral density** (*power spectrum*): $$ S_x(f) = \lim_{T \to \infty} \frac{1}{T} |X_T(f)|^2 $$ = PSD for WSS signals

- **x is filtered by H to produce y** ($y = h * x$) $$ r_{yy} = h \ast \tilde{h} \ast x $$ $$ S_y(f) = |H(f)|^2 S_x(f) $$

- **Inner Product ~ correlation ~ convolution** $$ \langle x, \phi \rangle = \int_{-\infty}^{\infty} x(t) \phi^*(t) , dt \ = r_{x\phi}(0) \ = \tilde{\phi} * x \Big|_{t=0} $$

- **Matched Filter** $$ \psi(t) = \tilde{\phi}(t - T) \ \langle x, \phi \rangle = \psi * x \Big|_{t=T} $$ $$ \Psi(f) = e^{-j2\pi f T} \Phi^*(f) $$

Baseband

Phasors

- **Phasor** $$ x(t) = A \cos(2\pi f_c t + \theta) = i \cos 2\pi f_c t - q \sin 2\pi f_c t \ i = A \cos \theta, \quad q = A \sin \theta $$ $$ X = A e^{j\theta} = i + jq \ x(t) = \text{Re}\left[X e^{j 2\pi f_c t} \right] $$
- **Differentiator:** $$ Y = j2\pi f_c X $$
- **Integrator:** $$ Y = \frac{1}{j2\pi f_c} X $$

Baseband Equivalent

$$ x(t) = \text{Re}\left[x_{BB}(t) e^{j 2\pi f_c t} \right] $$ $$ x_{BB}(t) = A(t) e^{j\theta(t)}=i(t) + jq(t) \ x(t) = A(t) \cos(2\pi f_c t + \theta(t)) = i(t) \cos 2\pi f_c t - q(t) \sin 2\pi f_c t $$

Spectra

$$ X(f) = \frac{1}{2} X_{BB}(f - f_c) + \frac{1}{2} X_{BB}^*(-f - f_c) $$

Spectral Interpretation

$$ x(t) = \frac{1}{2} X e^{j2\pi f_c t} + \frac{1}{2} X^* e^{-j2\pi f_c t} $$ $$ X(f) = \frac{1}{2} X \delta(f - f_c) + \frac{1}{2} X^* \delta(f + f_c) $$

Systems

$$ h(t) = 2 \text{Re} \left[h_{BB}(t) e^{j2\pi f_c t} \right] $$ $$ y_{BB} = h_{BB} * x_{BB} $$ $$ Y_{BB}(f) = H_{BB}(f) X_{BB}(f) $$

Hilbert Transform

$$ H_{\text{Hilb}}(f) = -j , \text{sign}(f) = \begin{cases} -j & f > 0 \ j & f < 0 \end{cases} $$

$$ h_{\text{Hilbert}}(t) = \frac{1}{\pi t} $$

$$ x^{(H)}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau} , d\tau $$

- **Upper and Lower Sidebands** $$ x_U(t) = x(t) + jx^{(H)}(t) $$ $$ x_L(t) = x(t) - jx^{(H)}(t) $$

- **USSB and LSSB**

$$ x_{BB}^{(USSB)}(t) = x_{BB}(t) + jx_{BB}^{(H)}(t) $$ $$ x_{BB}^{(LSSB)}(t) = x_{BB}(t) - jx_{BB}^{(H)}(t) $$ $$ X_{BB}^{(USSB)}(f) = 2X_{BB}(f) \mathbf{1}{{f<0}} $$ $$ X{BB}^{(LSSB)}(f) = 2X_{BB}(f) \mathbf{1}_{{f<0}} $$

Discrete Hilbert Transform

$$ H_{DHT}(\omega) = \begin{cases} -j & 0 < \omega < \pi \ j & -\pi < \omega < 0 \end{cases} $$ $$ h_{DHT}[n] = \frac{2}{\pi n} 1_{{n \text{ odd}}} $$

Hermitian Transpose

$$ A^H = \text{Hermitian transpose} = \text{conjugate transpose of } A $$

- **Hermitian Matrix** $$ A = A^H $$

Inner Product for Vectors

$$ \langle u, v \rangle = v^H u = \sum_{i=1}^{n} u_i v^*_i $$

Outer Product for Vectors

$$ uv^H = \text{matrix with ijth component } u_i v^*_j $$

Properties of Hermitian Matrices

- A matrix A is Hermitian if $A = A^H$.
- For any vector a, $a^H A a$ is a real scalar.
- All eigenvalues of a Hermitian matrix are real.
- There exists an orthonormal set of eigenvectors that span the space.

Positive Semidefinite and Definite

- A Hermitian matrix A is **positive semidefinite** if $a^H A a \geq 0$ for all vectors a. This occurs iff all eigenvalues of $A \geq 0$.
- A Hermitian matrix A is **positive definite** if $a^H A a > 0$ for all $a \neq 0$. This occurs iff all eigenvalues of $A > 0$.

Decision Theory

Decision Rule

- **Decision Rule**: An algorithm to make decisions by partitioning decision space D into regions ${Λ_m}^M_{m=1}$. Select H_m if $\mathbf{r} \in Λ_m$.

Probability Model

- **Likelihood Function**: $P(\mathbf{r} | \mathbf{s}_m)$
- **Prior Model**: $\pi_m = P(\mathbf{s}_m)$
- **Posterior Distribution**: $P(\mathbf{s}_m | \mathbf{r})$ calculated as: $$ P(\mathbf{s}_m | \mathbf{r}) = \frac{P(\mathbf{r} | \mathbf{s}*m) \pi_m}{\sum*{k=1}^M P(\mathbf{r} | \mathbf{s}_k)\pi_k} $$

Decision Rules

- **MAP (Maximum A-Posteriori)**: Chooses the hypothesis with the maximum posterior probability. $$ \hat{m}_{MAP} = \text{arg max}_m , P(\mathbf{s}_m | \mathbf{r}) $$
- **ML (Maximum Likelihood)**: Maximizes the likelihood function. $$ \hat{m}_{ML} = \text{arg max}_m , P(\mathbf{r} | \mathbf{s}_m) $$
- **LS (Least-Squares)**: Minimizes the Euclidean distance between the received signal and the transmitted symbol. $$ \hat{m}_{LS} = \text{arg min}_m , |\mathbf{r} - \mathbf{s}_m| $$

MAP vs ML

- **MAP** incorporates prior probabilities and is optimal in minimizing the probability of error.
- **ML** ignores prior probabilities and is optimal when hypotheses are equally probable.
- If all hypotheses are equally probable, MAP and ML are equivalent.

Communication Channels

- **Binary Symmetric Channel (BSC)**: A binary vector is transmitted, and errors are symmetric with probability p.
  - **ML decision** minimizes the Hamming distance between the received and transmitted vectors: $$ \hat{m}_{ML} = \text{arg min}*m , d*{\text{Ham}}(\mathbf{r}, \mathbf{s}_m) $$
- **AWGN Channel**: The received signal is modeled as: $$ \mathbf{r} = \mathbf{s}_m + \mathbf{n} $$ where $\mathbf{n}$ is Gaussian noise. The likelihood function is: $$ P(\mathbf{r} | \mathbf{s}_m) \propto \exp\left(-\frac{1}{2\sigma^2} |\mathbf{r} - \mathbf{s}_m|^2 \right) $$
  - **ML decision** minimizes the Euclidean distance.

Matched Filter

- **Correlation Receiver**: The matched filter compares the received signal with the transmitted signal using: $$ \langle \mathbf{r}, \mathbf{s}_m \rangle = \int r(t) s_m^*(t) , dt $$
  - **MF Decision Rule**: Select the hypothesis that maximizes the real part of the correlation: $$ \hat{m} = \text{arg max}_m , \text{Re} \langle \mathbf{r}, \mathbf{s}_m \rangle $$

Optimum Decision Rule for AWGN

- **MAP for AWGN** can be implemented as a matched filter with bias terms to account for different symbol probabilities and energies: $$ \hat{m}_{MAP} = \text{arg max}_m , \left[\text{Re} \langle \mathbf{r}, \mathbf{s}_m \rangle + b_m \right] $$ where b_m is a bias term.

Summary

- **MAP Decision Rule** minimizes the probability of error.
- **MAP** = **ML** for equiprobable symbols
- **ML** = **LS** for AWGN channels
- **LS** = **MF**, for equal energy symbols

Acronyms/Abbreviations

Short	Long
ML	Maximum Likelihood
MAP	Maximum A-Posteriori
SNR	Signal to Noise Ratio
BER	Bit Error Rate
PSK	Phase Shift Keying
FSK	Frequency Shift Keying
QPSK	Quadrature Phase Shift Keying
QAM	Quadrature Amplitude Modulation
LPI	Low Probability of Intercept
LPD	Low Probability of Detect
LOS	Line of Sight
NLOS	Non-Line of Sight
ISI	Intersymbol Interference
PAM	Pulse Amplitude Modulation
USB	Upper Sideband
LSB	Lower Sideband
USSB	Upper Single Sideband
LSSB	Lower Single Sideband
DSB	Double Sideband
QOS	Quality of Service
AWGN	Additive White Gaussian Noise
PSD	Power Spectral Density
LS	Least Square
WSS	Wide Sense Stationary
BSC	Binary Symmetric Channel

Short	Long
MF	Matched Filter