

# Vision: Generative Modeling - 2

EECE454 Intro. to Machine Learning Systems

Fall 2024

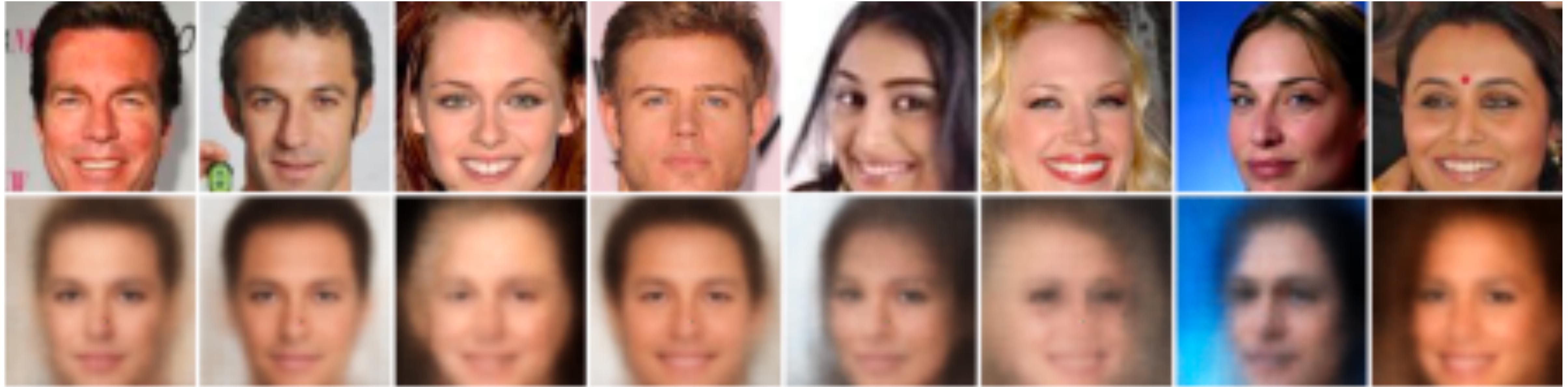
# Today

- Generative Adversarial Nets
- Diffusion Models

# Generative Adversarial Nets

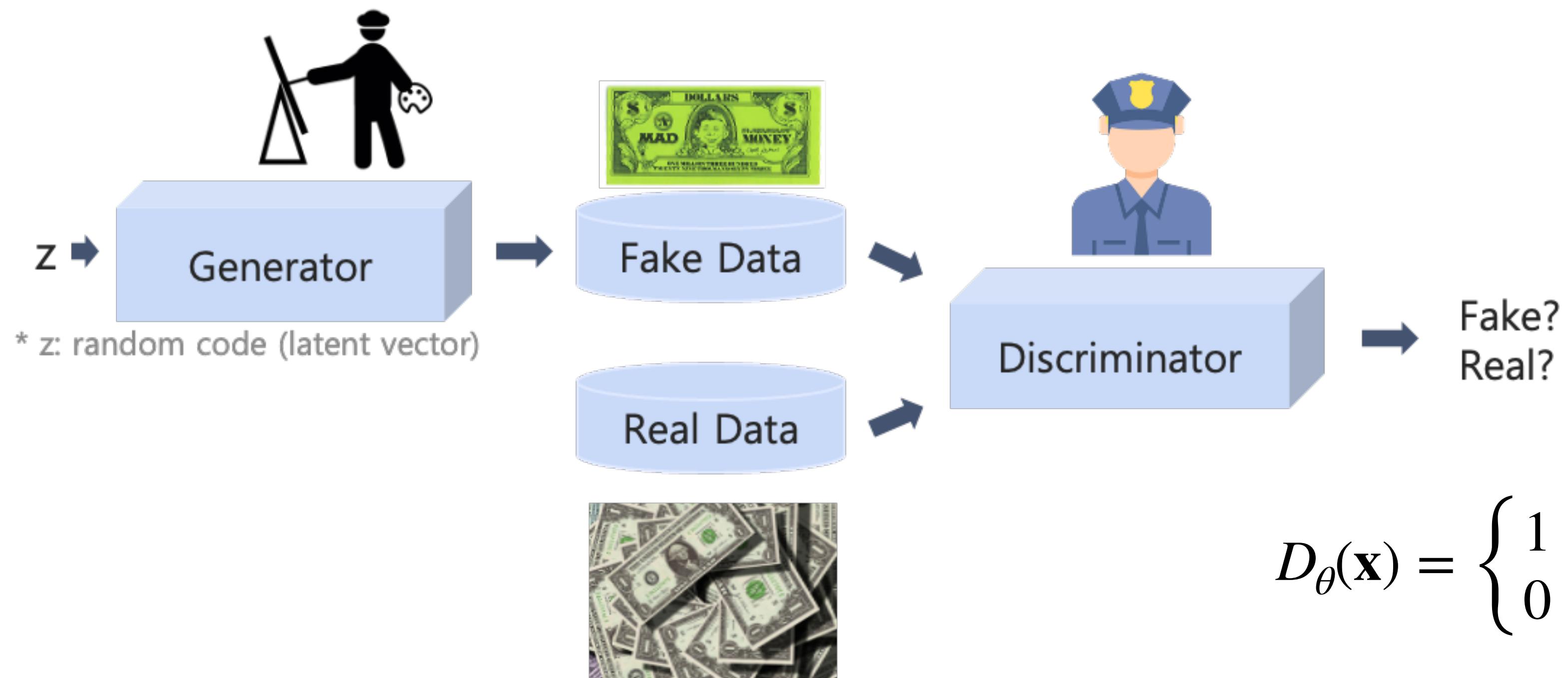
# Limitations of VAEs

- **Cons.** Known to be less “sharp,” with much noises
  - Clearly distinguishable from the real images
  - Question. Can we generate samples that are **undistinguishable** from real ones?



# Generative Adversarial Nets

- **Idea.** Explicitly train for “hard to distinguish” properties, by training a distinguisher together
  - View generative process as a **two-player game**
    - Generator. Tries to fool the discriminator
    - Discriminator. Tries to distinguish the real / fake images



# Generative Adversarial Nets

- **Training.** Jointly train the generator and discriminator

- Objective. Minimax function

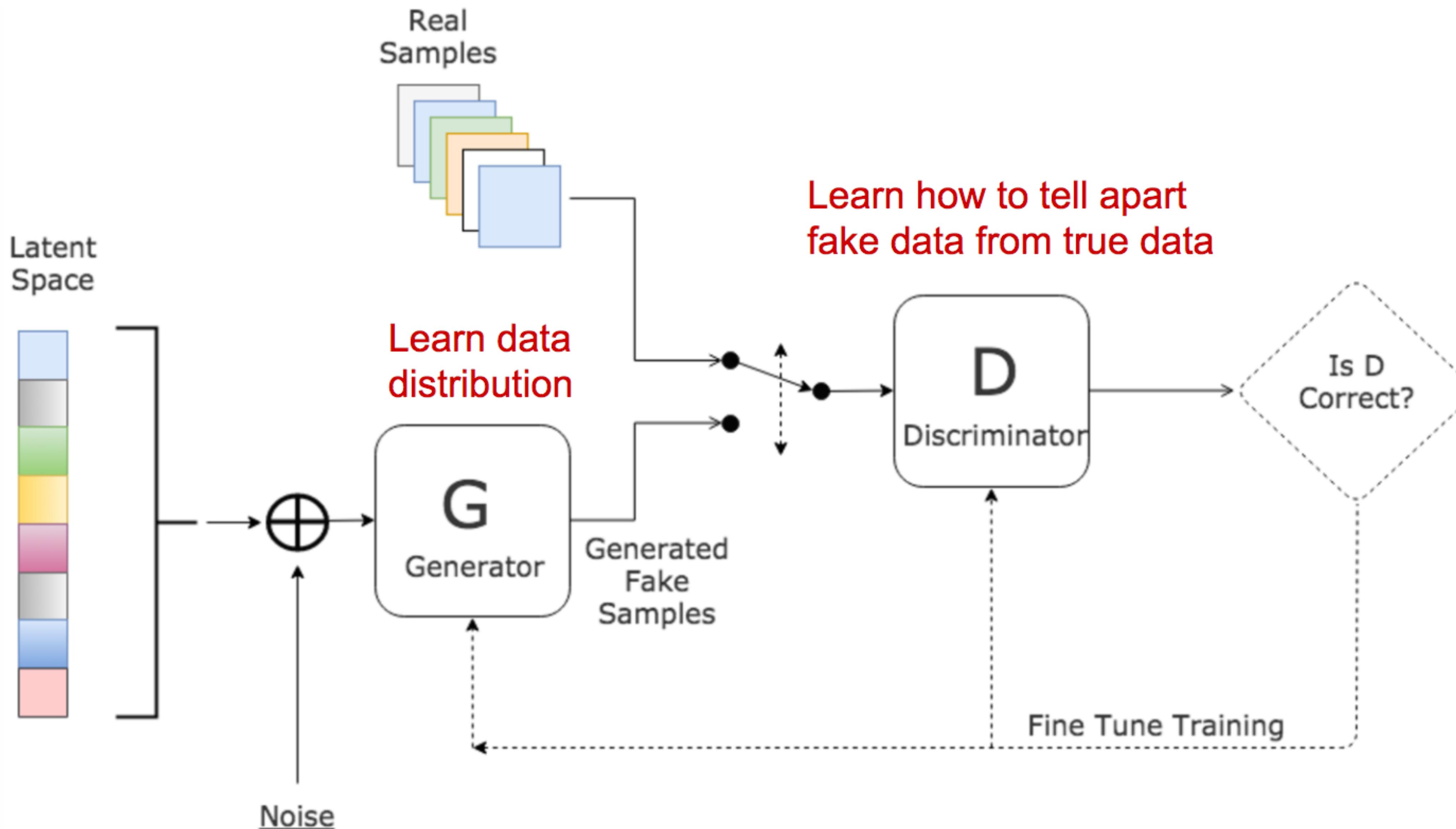
$$\min_{\theta_g} \max_{\theta_d} \left[ \mathbb{E}_{\mathbf{x} \sim \hat{p}} \log D_{\theta_d}(\mathbf{x}) + \mathbb{E}_{\mathbf{z} \sim p(z)} \log(1 - D_{\theta_d} \circ G_{\theta_g}(z)) \right]$$

Discriminator declares  
real image to be real      Discriminator declares  
fake image to be fake

- Discriminator outputs the likelihood of being real  $D_{\theta_d}(\mathbf{x}) \in [0,1]$
- This training objective is actually equivalent to the Jensen-Shannon divergence

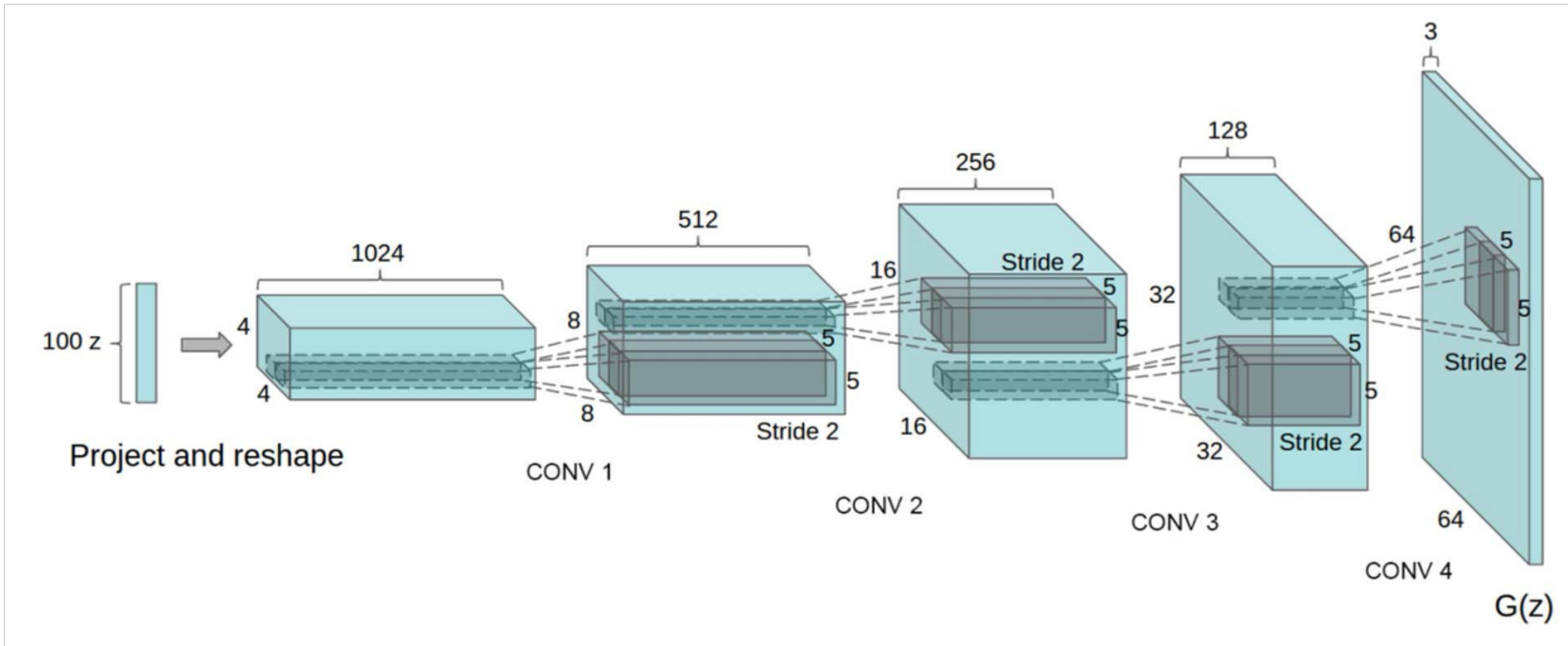
$$D \left( p_{\theta} \middle\| \frac{\hat{p} + p_{\theta}}{2} \right) + D \left( \hat{p} \middle\| \frac{\hat{p} + p_{\theta}}{2} \right)$$

# Generative Adversarial Nets



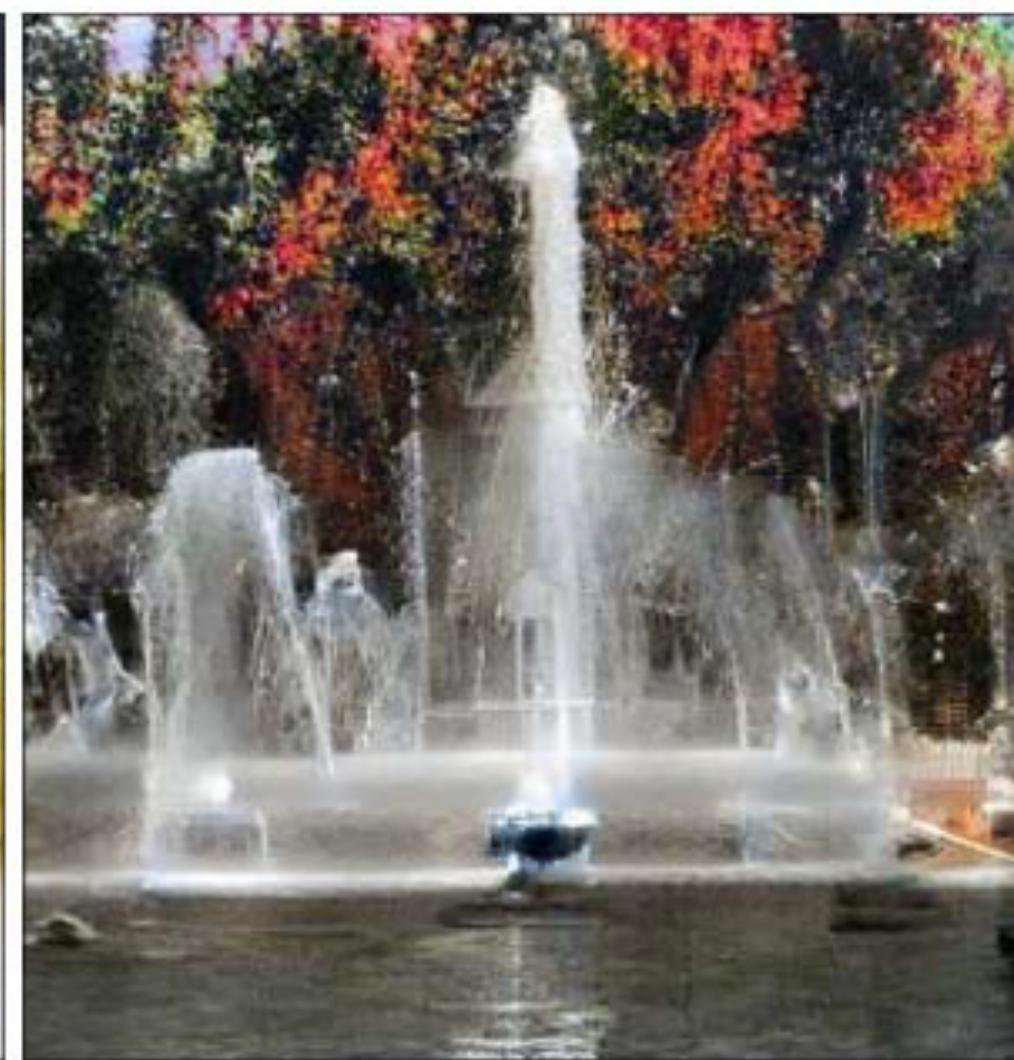
# Generative Adversarial Nets

- **Architecture.** Generator uses convolutional layers, of course.



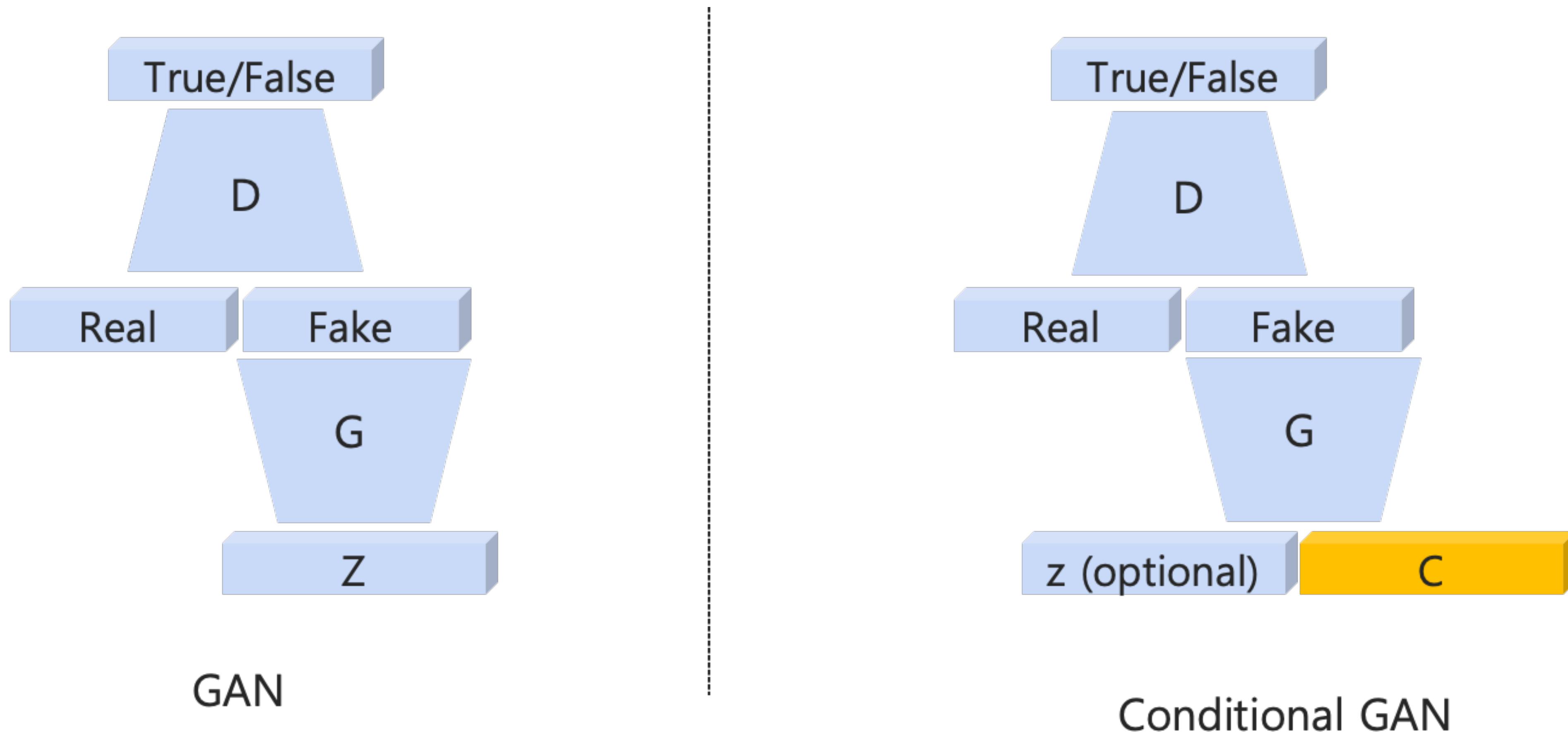
# Results

- Such training can give very sharp images



# Conditional GAN

- **Idea.** Add class/text information to the latent code
  - Generate realistic images under specific conditions

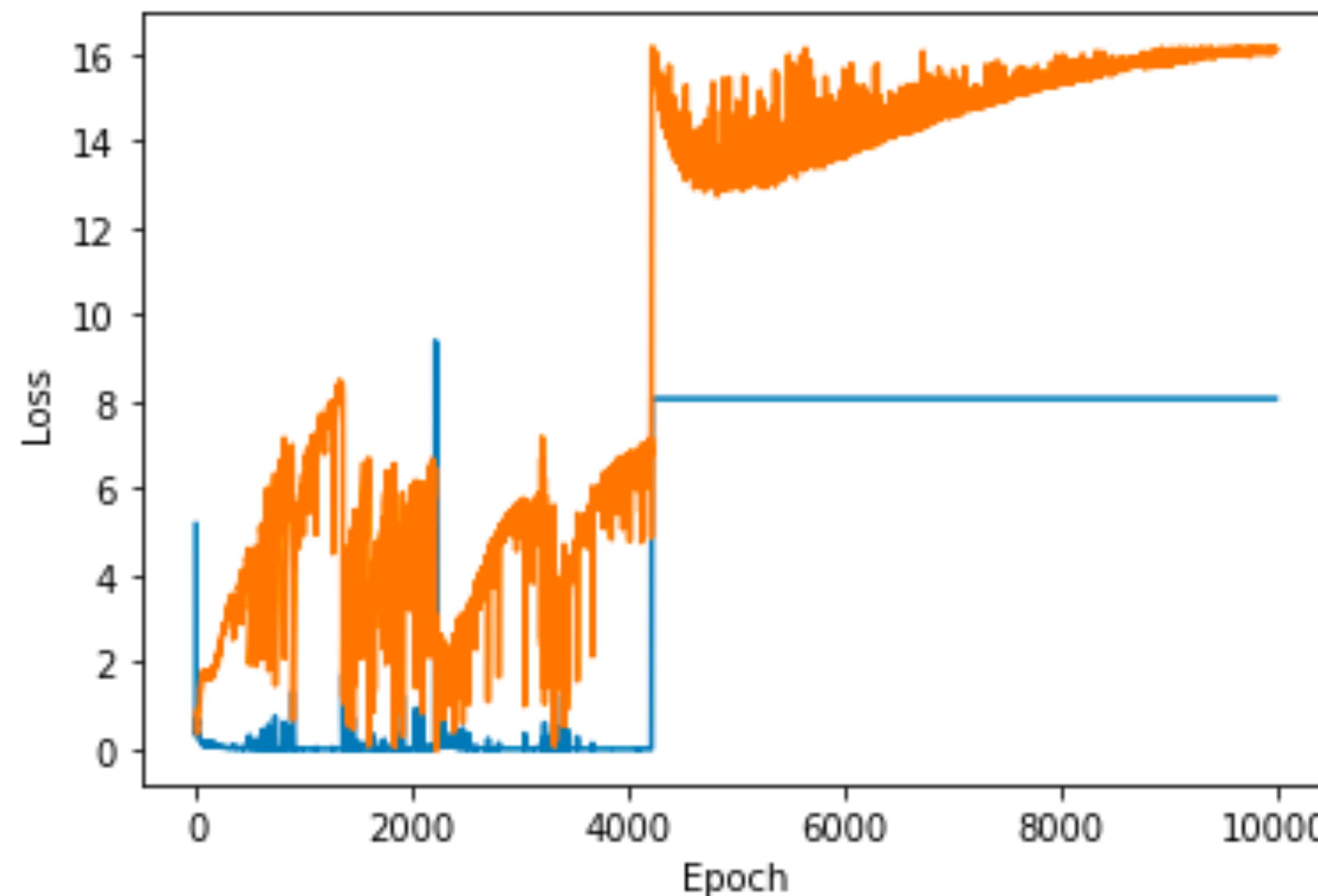


# Conditional GAN



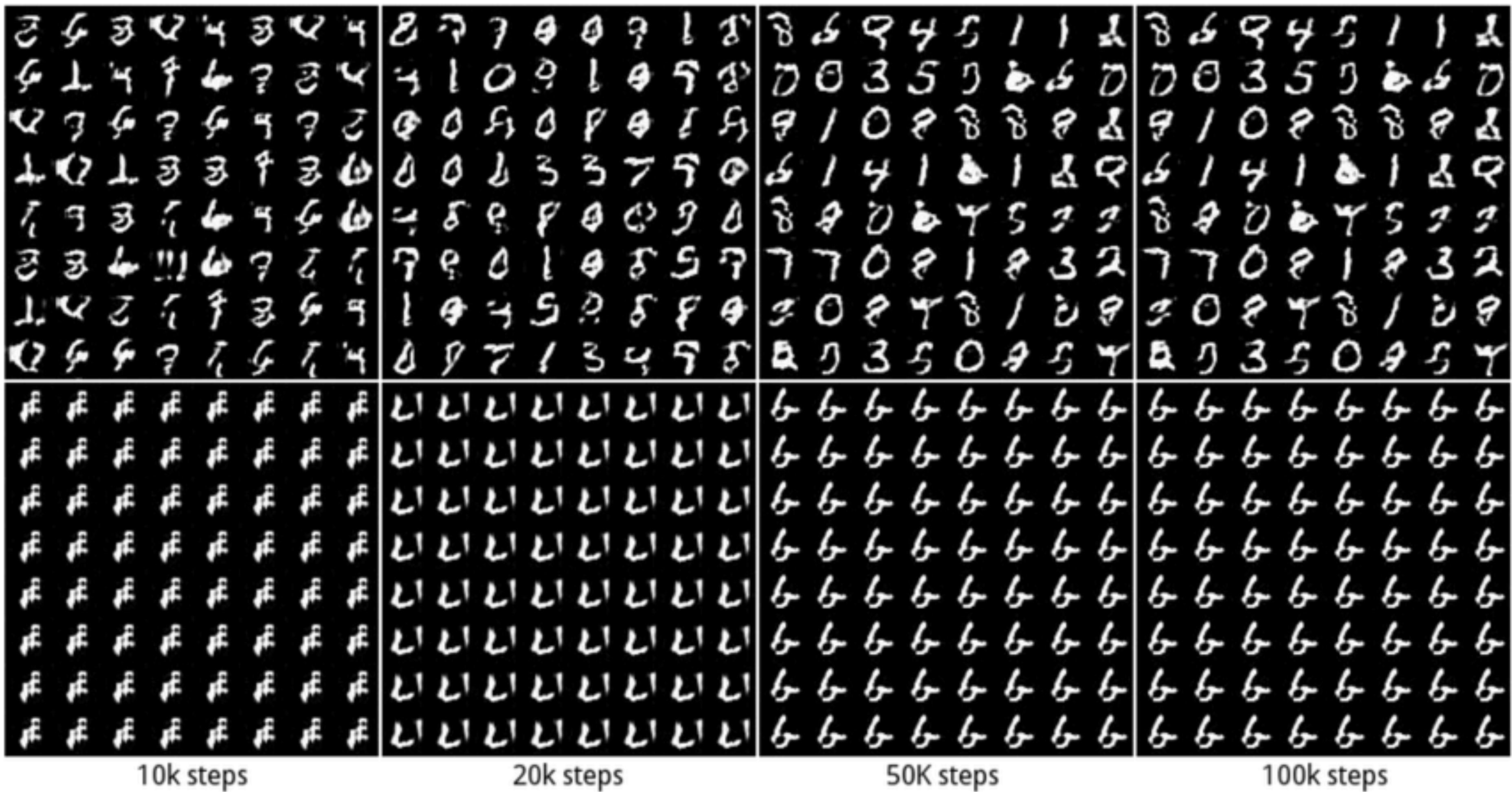
# Pitfalls

- Training GANs is known to be a **very unstable** procedure
  - If the discriminator works too well, the generator gives up learning
  - If the generator works too well, the discriminator cannot find meaningful patterns



# Pitfalls

- As a result, overfit to **few good** solutions (called “mode collapse”)



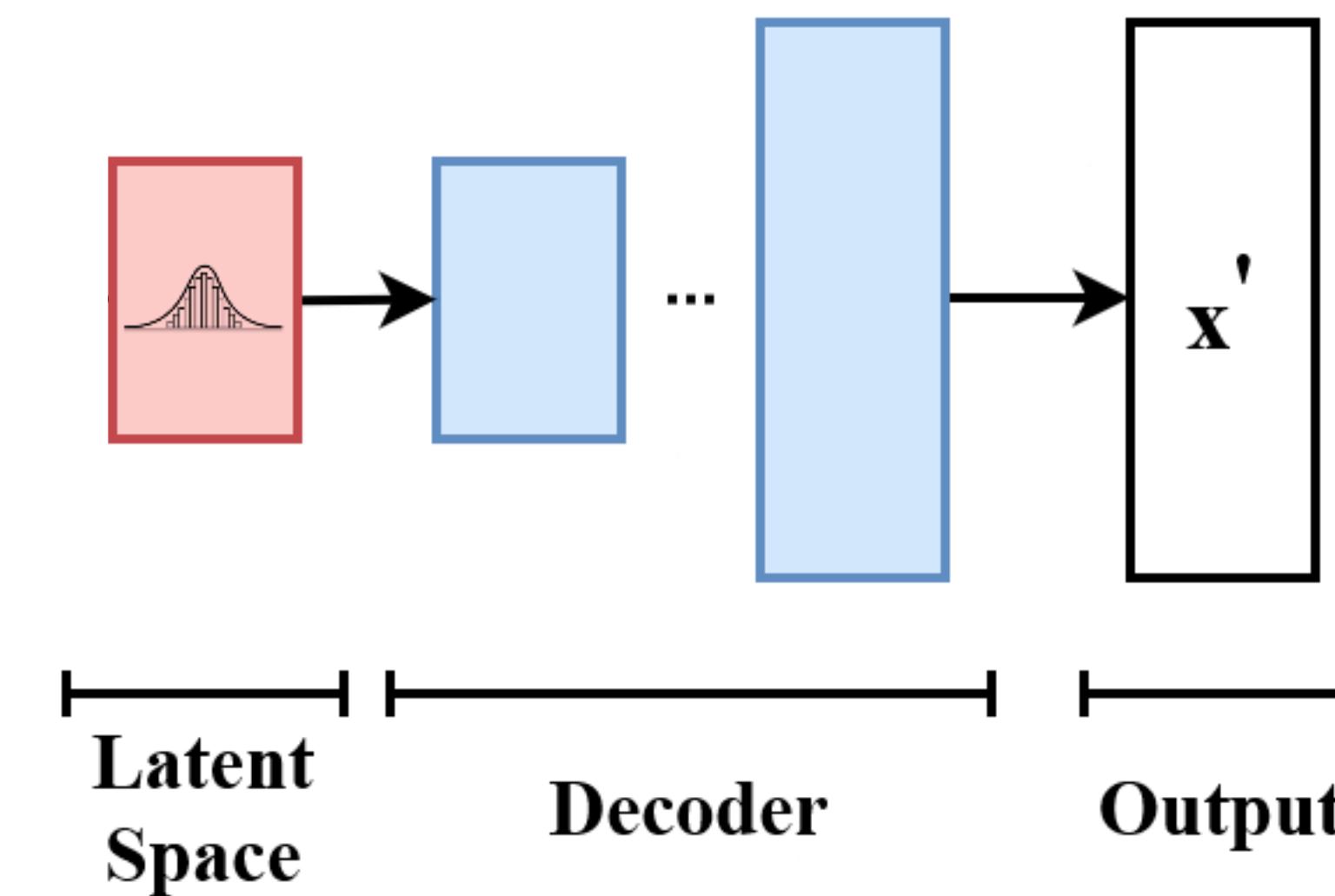
# Diffusion models

# VAEs

- **Recall: VAEs.** A decoder  $p_\theta(\mathbf{x} | \mathbf{z})$  that generates samples from a code  $\mathbf{z} \sim \mathcal{N}(0, I_k)$ , such that

$$p_{\text{data}}(\mathbf{x}) \approx p_\theta(\mathbf{x})$$

- Problem. To train such model, we needed a good inverse map  $p_\theta(\mathbf{z} | \mathbf{x})$

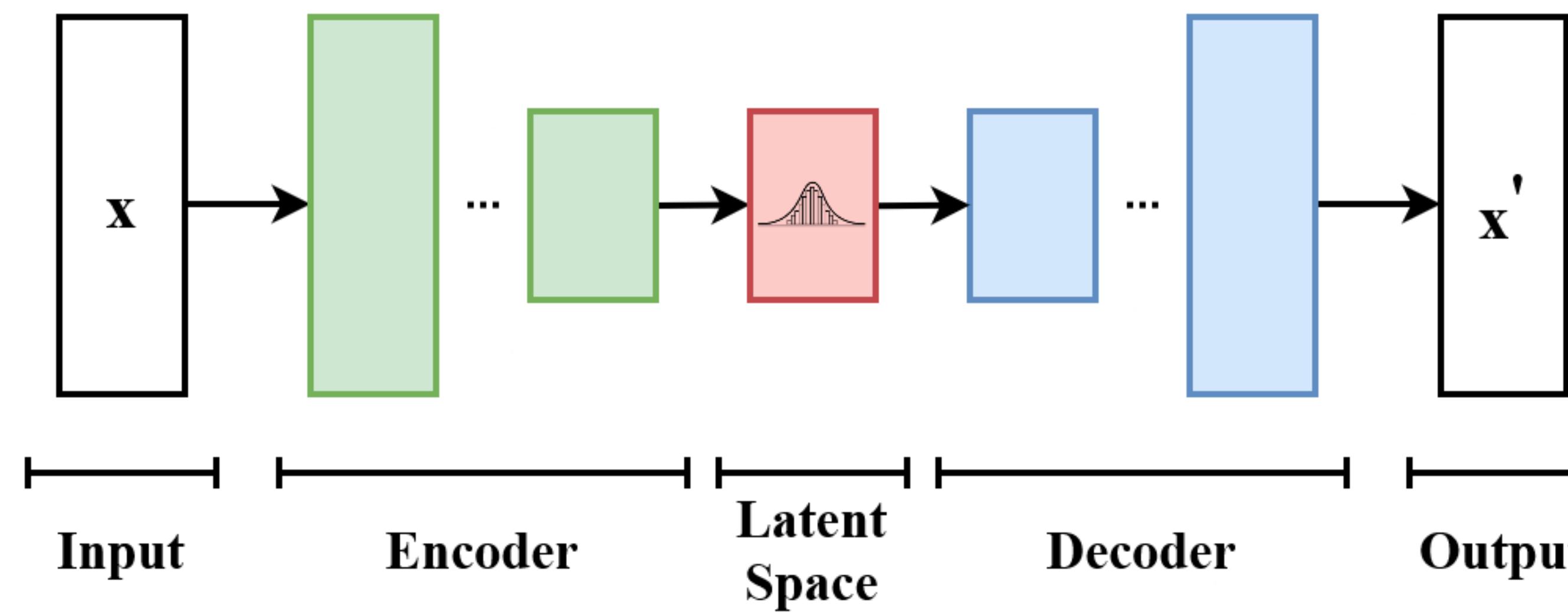


# VAEs

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$$p_{\text{data}}(\mathbf{x}) \approx p_\theta(\mathbf{x})$$

- Problem. To train such model, we needed a good inverse map  $p_\theta(\mathbf{z} | \mathbf{x})$
- Idea. **Jointly train** an encoder, which generates Gaussians from inputs
  - As the “distribution of images” is very complicated, maybe our neural nets have too low capacity to do this in a single forward...



# Diffusion models

- **Observation.** Another natural way to generate Gaussian-like distribution from inputs (i.e., encode)
  - Add Gaussian noise to the input, gradually:



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  - Add Gaussian noise to the input, gradually:

- Sample the data  $\mathbf{x}_t$  from the distribution

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N} \left( \mathbf{x}_t | \sqrt{\alpha_t} \mathbf{x}_{t-1}, (1 - \alpha_t)I \right)$$

- That is, we do  $\mathbf{x} \mapsto \sqrt{\alpha_t} \mathbf{x} + \sqrt{1 - \alpha_t} \epsilon$ ,  $\epsilon \sim \mathcal{N}(0, I)$

(We put scaling to preserve the  $\ell_2$  norm)

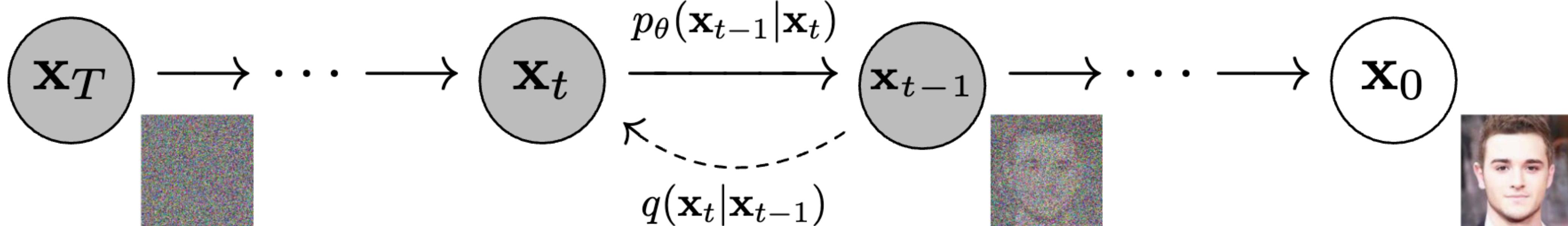


# Diffusion models

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  - Add Gaussian noise to the input, gradually:
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    - That is, we do  $\mathbf{x} \mapsto \sqrt{\alpha_t} \mathbf{x} + \sqrt{1 - \alpha_t} \epsilon$ ,  $\epsilon \sim \mathcal{N}(0, I)$   
(We put scaling to preserve the  $\ell_2$  norm)
- **Idea.** Let this be our (probabilistic) **encoder!**
  - Question. How can we train a decoder?

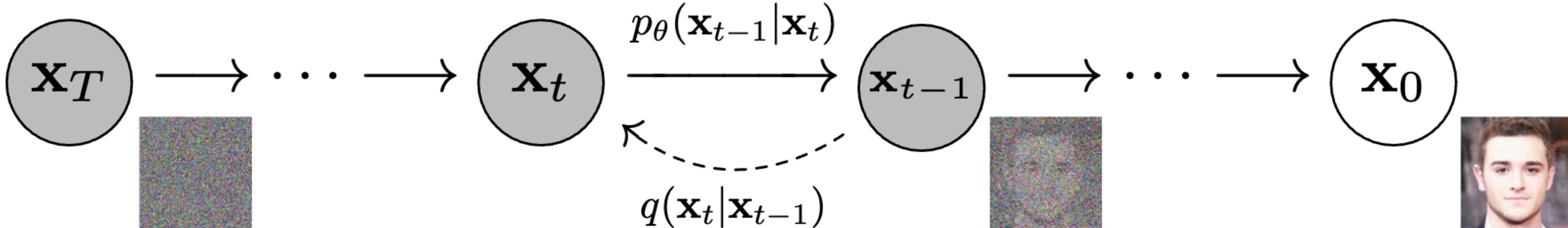
# Diffusion models

- **Decoder.** Train a reverse model  $p_{\theta}(\mathbf{x}_{t-1} | \mathbf{x}_t)$  which approximates  $q(\mathbf{x}_{t-1} | \mathbf{x}_t)$



# Diffusion models

- **Decoder.** Train a reverse model  $p_{\theta}(\mathbf{x}_{t-1} | \mathbf{x}_t)$  which approximates  $q(\mathbf{x}_{t-1} | \mathbf{x}_t)$ 
  - This reverse model will be parameterized as a **Gaussian**:
$$p_{\theta}(\mathbf{x}_{t-1} | \mathbf{x}_t) = \mathcal{N}(\mathbf{x}_{t-1} | \mu_{\theta,t}(\mathbf{x}_t), \Sigma_{\theta,t}(\mathbf{x}_t))$$
- That is, we train the mean predictor and variance predictor.
  - Dependent on the time  $t$



# Training a Diffusion Model

- **Training.** Suppose that we draw some sample sequence  $\mathbf{x}_0, \dots, \mathbf{x}_T$  using the forward diffusion:

$$q(\mathbf{x}_{0:T}) = q(\mathbf{x}_0) \prod_{t=1}^T q(\mathbf{x}_t | \mathbf{x}_{t-1})$$

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$$q(\mathbf{x}_{0:T}) = q(\mathbf{x}_0) \prod_{t=1}^T q(\mathbf{x}_t | \mathbf{x}_{t-1})$$

Then, train to maximize the log probability of generating the real image

$$\mathbb{E}_{q(\mathbf{x}_0)} [\log p_\theta(\mathbf{x}_0)]$$

where the **reverse diffusion process** is given as:

$$p_\theta(\mathbf{x}_{0:T}) = p_\theta(\mathbf{x}_T) \prod_{t=1}^T p_\theta(\mathbf{x}_{t-1} | \mathbf{x}_t).$$

# Evidence Lower Bound

- As in VAE, we use the Jensen's inequality.

$$\begin{aligned}\mathbb{E}_{q(\mathbf{x}_0)} [\log p_\theta(\mathbf{x}_0)] &= \mathbb{E}_{q(\mathbf{x}_0)} \left[ \log \left( \int p_\theta(\mathbf{x}_{0:T}) d\mathbf{x}_{1:T} \right) \right] \\ &= \mathbb{E}_{q(\mathbf{x}_0)} \left[ \log \left( \int q(\mathbf{x}_{1:T} | \mathbf{x}_0) \frac{p_\theta(\mathbf{x}_{0:T})}{q(\mathbf{x}_{1:T} | \mathbf{x}_0)} d\mathbf{x}_{1:T} \right) \right] \\ &= \mathbb{E}_{q(\mathbf{x}_0)} \left[ \log \mathbb{E}_{q(\mathbf{x}_{1:T} | \mathbf{x}_0)} \left[ \frac{p_\theta(\mathbf{x}_{0:T})}{q(\mathbf{x}_{1:T} | \mathbf{x}_0)} \right] \right] \\ &\geq \mathbb{E}_{q(\mathbf{x}_{0:T})} \left[ \log \frac{p_\theta(\mathbf{x}_{0:T})}{q(\mathbf{x}_{1:T} | \mathbf{x}_0)} \right]\end{aligned}$$

# Evidence Lower Bound

- The ELBO is further decomposed into:

$$\begin{aligned}\mathbb{E}_{q(\mathbf{x}_{0:T})} \left[ \log \frac{p_\theta(\mathbf{x}_{0:T})}{q(\mathbf{x}_{1:T} | \mathbf{x}_0)} \right] &= \mathbb{E}_q \left[ \log \frac{p_\theta(\mathbf{x}_T) \prod_{t=1}^T p_\theta(\mathbf{x}_{t-1} | \mathbf{x}_t)}{\prod_{t=1}^T q(\mathbf{x}_t | \mathbf{x}_{t-1})} \right] \\ &= \mathbb{E}_q \left[ \log p_\theta(\mathbf{x}_T) + \sum_{t=1}^T \log \frac{p_\theta(\mathbf{x}_{t-1} | \mathbf{x}_t)}{q(\mathbf{x}_t | \mathbf{x}_{t-1})} \right] \\ &= \mathbb{E}_q \left[ \log p_\theta(\mathbf{x}_T) + \sum_{t=2}^T \log \frac{p_\theta(\mathbf{x}_{t-1} | \mathbf{x}_t)}{q(\mathbf{x}_t | \mathbf{x}_{t-1})} + \log \frac{p_\theta(\mathbf{x}_0 | \mathbf{x}_1)}{q(\mathbf{x}_1 | \mathbf{x}_0)} \right]\end{aligned}$$

# Evidence Lower Bound

- Do additional conditioning

$$\begin{aligned} & \mathbb{E}_q \left[ \log p_\theta(\mathbf{x}_T) + \sum_{t=2}^T \log \frac{p_\theta(\mathbf{x}_{t-1} | \mathbf{x}_t)}{q(\mathbf{x}_t | \mathbf{x}_{t-1})} + \log \frac{p_\theta(\mathbf{x}_0 | \mathbf{x}_1)}{q(\mathbf{x}_1 | \mathbf{x}_0)} \right] \\ &= \mathbb{E}_q \left[ \log p_\theta(\mathbf{x}_T) + \sum_{t=2}^T \log \left( \frac{p_\theta(\mathbf{x}_{t-1} | \mathbf{x}_t)}{q(\mathbf{x}_{t-1} | \mathbf{x}_t, \mathbf{x}_0)} \cdot \frac{q(\mathbf{x}_{t-1} | \mathbf{x}_0)}{q(\mathbf{x}_t | \mathbf{x}_0)} \right) + \log \frac{p_\theta(\mathbf{x}_0 | \mathbf{x}_1)}{q(\mathbf{x}_1 | \mathbf{x}_0)} \right] \end{aligned}$$

$$\begin{aligned} q(\mathbf{x}_t | \mathbf{x}_{t-1}) &= q(\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{x}_0) \\ &= \frac{q(\mathbf{x}_t, \mathbf{x}_{t-1} | \mathbf{x}_0)}{q(\mathbf{x}_{t-1} | \mathbf{x}_0)} \\ &= \frac{q(\mathbf{x}_{t-1} | \mathbf{x}_t, \mathbf{x}_0)q(\mathbf{x}_t | \mathbf{x}_0)}{q(\mathbf{x}_{t-1} | \mathbf{x}_0)} \end{aligned}$$

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# Evidence Lower Bound

- Tidying up, we get

$$\begin{aligned} & \mathbb{E}_q \left[ \log \frac{p_\theta(\mathbf{x}_T)}{q(\mathbf{x}_T | \mathbf{x}_0)} + \sum_{t=2}^T \log \frac{p_\theta(\mathbf{x}_{t-1} | \mathbf{x}_t)}{q(\mathbf{x}_{t-1} | \mathbf{x}_t, \mathbf{x}_0)} + \log p_\theta(\mathbf{x}_0 | \mathbf{x}_1) \right] \\ &= \mathbb{E}_q[\log p_\theta(\mathbf{x}_0 | \mathbf{x}_1)] - \mathbb{E}_q D\left(q(\mathbf{x}_T | \mathbf{x}_0) \middle\| p(\mathbf{x}_T)\right) - \sum_{t=2}^T \mathbb{E}_q D\left(q(\mathbf{x}_{t-1} | \mathbf{x}_t, \mathbf{x}_0) \middle\| p_\theta(\mathbf{x}_{t-1} | \mathbf{x}_t)\right) \end{aligned}$$

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- **First term.** We know that this is the **squared loss** of the mean predictor.

- Assuming that  $\Sigma = I$  for simplicity, we have:

$$\mathbb{E}_q[\log p_\theta(\mathbf{x}_0 | \mathbf{x}_1)] = -\frac{1}{2} \mathbb{E}_q \|\mathbf{x}_0 - \mu_{\theta,1}(\mathbf{x}_1)\|^2$$

# Evidence Lower Bound

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$$\begin{aligned} & \mathbb{E}_q \left[ \log \frac{p_\theta(\mathbf{x}_T)}{q(\mathbf{x}_T | \mathbf{x}_0)} + \sum_{t=2}^T \log \frac{p_\theta(\mathbf{x}_{t-1} | \mathbf{x}_t)}{q(\mathbf{x}_{t-1} | \mathbf{x}_t, \mathbf{x}_0)} + \log p_\theta(\mathbf{x}_0 | \mathbf{x}_1) \right] \\ & = \mathbb{E}_q[\log p_\theta(\mathbf{x}_0 | \mathbf{x}_1)] - \mathbb{E}_q D\left(q(\mathbf{x}_T | \mathbf{x}_0) \| p(\mathbf{x}_T)\right) - \sum_{t=2}^T \mathbb{E}_q D\left(q(\mathbf{x}_{t-1} | \mathbf{x}_t, \mathbf{x}_0) \| p_\theta(\mathbf{x}_{t-1} | \mathbf{x}_t)\right) \end{aligned}$$

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- **Second term.** This does not involve any learnable parameters.

- Thus, ignore!

# Evidence Lower Bound

$$-\frac{1}{2} \|\mathbf{x}_0 - \mu_{t,1}(\mathbf{x}_1)\|^2 - \sum_{t=2}^T \mathbb{E}_q D\left(q(\mathbf{x}_{t-1} | \mathbf{x}_t, \mathbf{x}_0) \| p_\theta(\mathbf{x}_{t-1} | \mathbf{x}_t)\right)$$

- **Third term.** First, we look at the LHS of the KL divergence.

- If we have the relationship

$$\mathbf{x}_{t-1} = \sqrt{\bar{\alpha}_{t-1}} \mathbf{x}_0 + \sqrt{1 - \bar{\alpha}_{t-1}} \boldsymbol{\epsilon}$$

$$\mathbf{x}_t = \sqrt{\alpha_t} \mathbf{x}_{t-1} + \sqrt{1 - \alpha_t} \boldsymbol{\epsilon}'$$

(we use the shorthands  $\bar{\alpha}_i = \alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_i$ )

Then the following relationship holds (exercise; use Bayes' theorem)

$$q(\mathbf{x}_{t-1} | \mathbf{x}_t, \mathbf{x}_0) = \mathcal{N}\left(\frac{\sqrt{\alpha_t}(1 - \bar{\alpha}_{t-1})}{1 - \bar{\alpha}_t} \mathbf{x}_t + \frac{\sqrt{\bar{\alpha}_{t-1}}(1 - \alpha_t)}{1 - \bar{\alpha}_t} \mathbf{x}_0, \frac{(1 - \alpha_t)(1 - \sqrt{\bar{\alpha}_{t-1}})}{1 - \bar{\alpha}_t} I\right)$$

# Evidence Lower Bound

$$-\frac{1}{2} \|\mathbf{x}_0 - \mu_{t,1}(\mathbf{x}_1)\|^2 - \sum_{t=2}^T \mathbb{E}_q D\left(q(\mathbf{x}_{t-1} | \mathbf{x}_t, \mathbf{x}_0) \| p_\theta(\mathbf{x}_{t-1} | \mathbf{x}_t)\right)$$

- Now, the KL-divergence between Gaussians can be written simply as:

$$D\left(\mathcal{N}(\mu_1, \sigma_1^2 I) \| \mathcal{N}(\mu_2, \sigma_2^2 I)\right) = \log \frac{\sigma_2}{\sigma_1} - \frac{d}{2} + \frac{d\sigma_1^2 + \|\mu_1 - \mu_2\|^2}{2\sigma_2^2}$$

- Plug this in to get the loss (ignoring the variance terms)

$$\begin{aligned} & \sum_{i=2}^T \left\| \mu_{\theta,t}(\mathbf{x}_t) - \frac{\sqrt{\alpha_t}(1 - \bar{\alpha}_{t-1})}{1 - \bar{\alpha}_t} \mathbf{x}_t + \frac{\sqrt{\bar{\alpha}_{t-1}}(1 - \alpha_t)}{1 - \bar{\alpha}_t} \mathbf{x}_0 \right\|^2 \\ &= \sum_{i=1}^T \|\mu_{\theta,t}(\mathbf{x}_t) - \mu_q(\mathbf{x}_t, \mathbf{x}_0)\|^2 \end{aligned}$$

# In a nutshell

- In a nutshell, training the reverse diffusion process is:
  - Sample an image  $\mathbf{x}_0$  from the dataset
  - Sample  $\mathbf{x}_1, \dots, \mathbf{x}_T$  using  $q(\cdot)$
  - Pick a time  $t$ :
    - Train  $\mu_{\theta,t}(\cdot)$  to minimize  $\|\mu_{\theta,t}(\mathbf{x}_t) - \mu_q(\mathbf{x}_t, \mathbf{x}_0)\|^2$
  - Repeat

# In a nutshell

- In a nutshell, training

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## Algorithm 1 Training

---

- Sample an image
- Sample  $\mathbf{x}_1, \dots,$
- Pick a time  $t:$ 
  - Train  $\mu_{\theta,t}(\cdot)$
- Repeat

```
1: repeat
2:    $\mathbf{x}_0 \sim q(\mathbf{x}_0)$ 
3:    $t \sim \text{Uniform}(\{1, \dots, T\})$ 
4:    $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ 
5:   Take gradient descent step on
      
$$\nabla_{\theta} \|\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_{\theta}(\sqrt{\bar{\alpha}_t} \mathbf{x}_0 + \sqrt{1 - \bar{\alpha}_t} \boldsymbol{\epsilon}, t)\|^2$$

6: until converged
```

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- This is typically reparametrized as a **noise prediction** (i.e., residual of the prediction)

# Prediction

- Generation is done by starting from a Gaussian distribution, then keep denoising...

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## Algorithm 2 Sampling

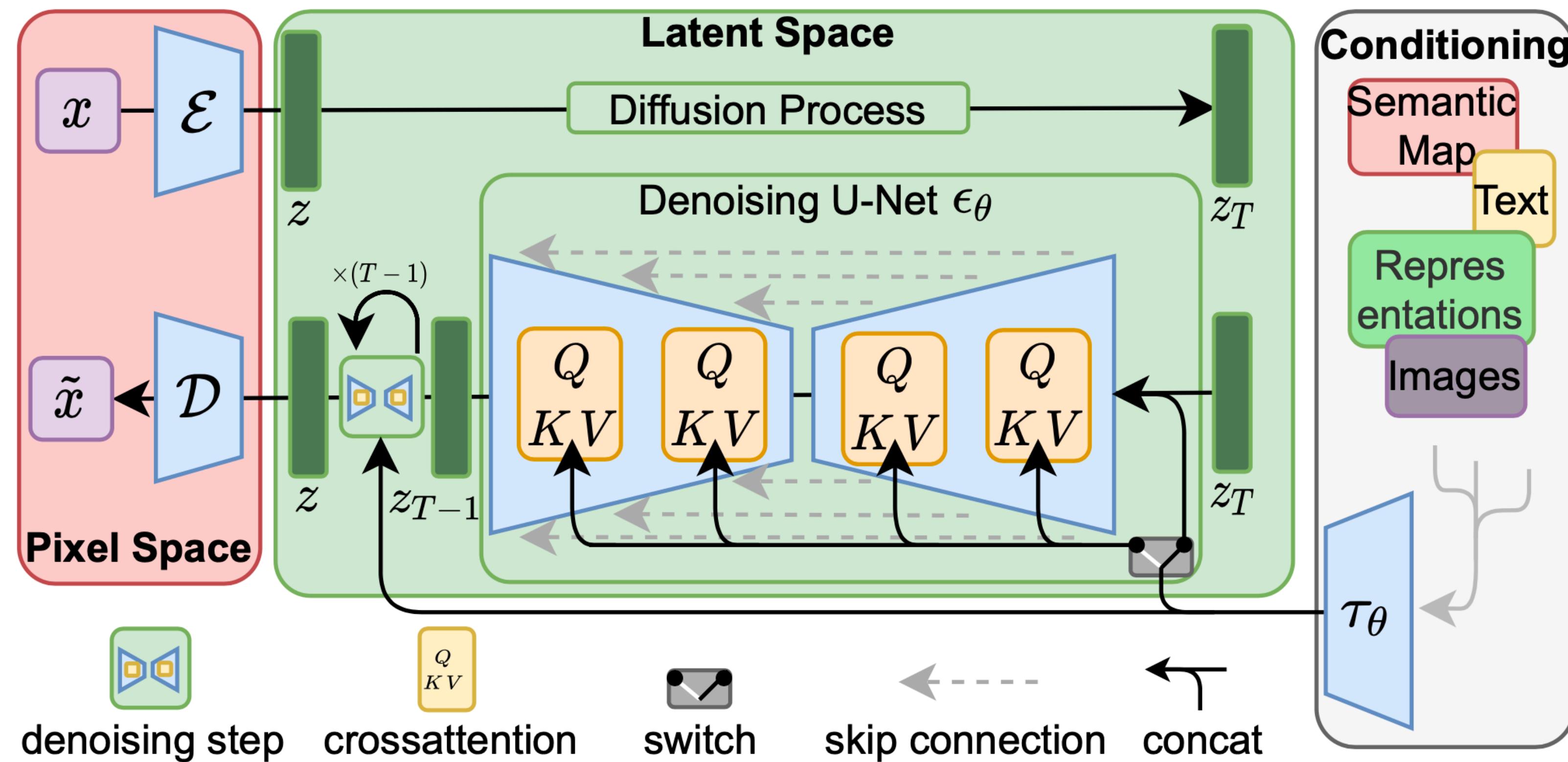
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- 1:  $\mathbf{x}_T \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$
- 2: **for**  $t = T, \dots, 1$  **do**
- 3:    $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$  if  $t > 1$ , else  $\mathbf{z} = \mathbf{0}$
- 4:    $\mathbf{x}_{t-1} = \frac{1}{\sqrt{\alpha_t}} \left( \mathbf{x}_t - \frac{1-\alpha_t}{\sqrt{1-\bar{\alpha}_t}} \boldsymbol{\epsilon}_\theta(\mathbf{x}_t, t) \right) + \sigma_t \mathbf{z}$
- 5: **end for**
- 6: **return**  $\mathbf{x}_0$

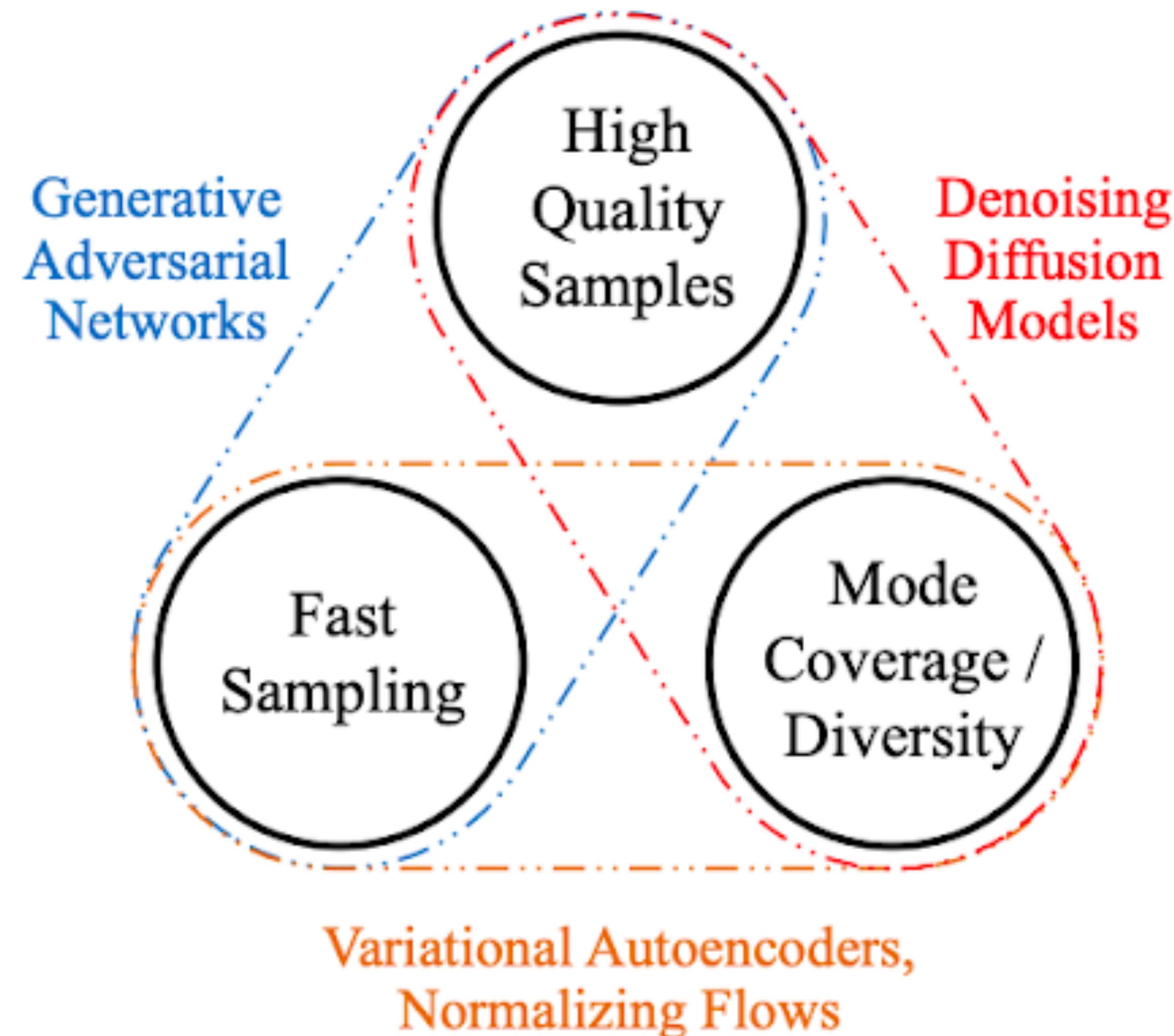
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# Latent Diffusion

- We use diffusion in some latent space.
  - Combine with the ideas of VAE
- Plus, we do some conditioning



# Pros & Cons



# More references

- <https://huggingface.co/blog/annotated-diffusion>
- <https://lilianweng.github.io/posts/2021-07-11-diffusion-models/>
- <https://arxiv.org/abs/2403.18103>

Cheers