Matrix Analysis

Part. 2

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February 21, 2020

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Properties I

Theorem

For $A \in \mathbb{R}^{m \times m}$ and some $x \neq 0$

1. $\lambda_i(A) = \lambda_i(A') \; \forall i \in \{1, \cdots, m\}$
2. $\nexists A^{-1}$ *iff* $\prod_{i=1}^{m} \lambda_i(A) = 0$
3. $diag(A)_i = \lambda_i(A) \; \forall i$ *if A is triangular*
4. $\lambda_i(BAB^{-1}) = \lambda_i(A) \; \forall i$ *if* $\exists B^{-1}$
5. $\lambda_i(A) = \pm 1 \; \forall i$ *if A is orthogonal.*

Properties II

Theorem

For $\lambda_x = \lambda(A)$ where $Ax = \lambda x$ and $x \neq 0$,

1. *If an integer $n \geq 1$, then $\lambda_x^n = \lambda(A^n)$.*
2. *If $\exists A^{-1}$, then $\lambda_x^{-1} = \lambda(A^{-1})$.*

Theorem

Suppose that the $m \times m$ matrix has r nonzero eigenvalues. Then, if A is symmetric, $rank(A) = r$.

Proof.

If $A = P\Lambda P'$ is the Spectral Decomposition of A,

$$rank(A) = rank(P\Lambda P') = rank(\Lambda) = r.$$

□

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Extremal Properties of Eigenvalues I

Theorem

Let A be a symmetric $m \times m$ matrix with ordered eigenvalues $\lambda_1 \leq \cdots \leq \lambda_m$. For any $x \in \mathbb{R}^m \backslash \{0\}$,

$$\lambda_1 \leq \frac{x'Ax}{x'x} \leq \lambda_m,$$

and λ_1 and λ_m are minimum and maximum values of $x'Ax/(x'x)$.

Proof.

$$\frac{x'A'x}{x'x} = \frac{x'P\Lambda P'x}{x'PP'x} = \frac{\sum_{i=1}^{m}(P'x)_i^2\lambda_i}{\sum_{i=1}^{m}(P'x)_i^2}$$

so the inequality comes. Next, if we take $x = e_1$ and $x = e_m$, then λ_1 and λ_m are verified as minimum and maximum respectively. $\square$

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Singular Value Decomposition I

Theorem

If X is an $n \times p$ matrix of rank $r > 0$, there exist orthogonal $n \times n$ and $p \times p$ matrices N and P, such that $X = NDP'$ and $D = N'X'P$ where the $n \times p$ matrix D is given by

a. Δ *if* $r = p = n$
b. $\begin{bmatrix} \Delta & (0) \end{bmatrix}$ *if* $r = n < p$
c. $\begin{bmatrix} \Delta \\ (0) \end{bmatrix}$ *if* $r = p < n$
d. $\begin{bmatrix} \Delta & (0) \\ (0) & (0) \end{bmatrix}$ *if* $r < n$ *and* $r < p$

where Δ is an $r \times r$ diagonal matrix with positive diagonal elements.

Singular Value Decomposition II

Corollary

Let X be an $n \times p$ matrix with rank $r > 0$. Then, there exists $n \times r$ and $p \times r$ orthogonal matrices N and P such that $X = N\Delta P'$ where Δ is an $r \times r$ diagonal matrix with positive diagonal elements.

Singular Value Decomposition III

Example

For $X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, X'X = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with eigenvalues $7, 3, 0$

$$\text{and } E := \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} 2^{-0.5} & 2^{-0.5} & 0 \\ 2^{-0.5} & -2^{-0.5} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Singular Value Decomposition IV

Example

Clearly, $rank(X) = rank(X'X) = 2$ with two singular values $7, 3$. So, since $N = XP\Delta^{-1}$,

$$N = X \begin{bmatrix} 2^{-0.5} & 2^{-0.5} \\ 2^{-0.5} & -2^{-0.5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{7}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{2}{\sqrt{14}} & \frac{-2}{\sqrt{6}} \\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

therefore $N'N = P'P = I$ and $X = N\Delta P'$.

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Spectral Decomposition I

Theorem

Let A be a symmetric $p \times p$ matrix. Then, there exist $P = \left[\begin{array}{ccc} e_1 & \cdots & e_p \end{array}\right], \Lambda = diag(\lambda_1, \cdots, \lambda_p)$ *such that* $A = P\Lambda P'$.

Example

If we have a symmetric Σ, $\Sigma = P\Lambda P' := P\Lambda^{0.5}(P\Lambda^{0.5})' := \Sigma^{0.5}\Sigma^{0.5'}$.
So, $\Sigma^{-0.5} := (P\Lambda^{0.5})^{-1} = \Lambda^{-0.5}P'$. e.g., $\Sigma^{-0.5}(X - \mu) \sim N(0, I)$.

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Sqaure Matrix Norms I

Definition

A function $||\cdot||$ defined on any $m \times m$ matrix, real or complex, is a matrix norm if

a. $||A|| \geq 0$ and equality holds iff $A = (0)$
b. $||cA|| = |c| \cdot ||A||$ for any complex scalar c
c. $||A + B|| \leq ||A|| + ||B||$
d. $||AB|| \leq ||A|| \cdot ||B||$

all satisfy.

Remark A Matrix norm has the same conditions as those of a vector norm except the last condition.

Sqaure Matrix Norms II

Example

1. The **maximum column sum matrix norm** is given by

$$||A||_1 = \max_j \sum_{i=1}^{m} |a_{ij}|$$

2. The **maximum row sum matrix norm** is given by

$$||A||_\infty = \max_i \sum_{j=1}^{m} |a_{ij}|$$

3. The **spectral matrix norm(or maximum singular value $\sigma_{\max}$)** is given by

$$||A||_2 := \sigma_{\max}(A) = \sqrt{\lambda_{\max}(\bar{A}'A)} \left\{ = \sqrt{\lambda_{\max}(A'A)} \text{ if } A \text{ is real} \right\}$$

Sqaure Matrix Norms III

Theorem

For an $m \times m$ matrix norm $||\cdot||$, $||A||_C := ||C^{-1}AC||$ is also a matrix norm for an $m \times m$ matrix A and a nonsingular $m \times m$ matrix C.

Proof.

Since $||\cdot||$ is a matrix norm,

- a $||C^{-1}AC|| \geq 0$ and equality holds iff $A = (0)$ because $||A||_C = 0 \Leftrightarrow C^{-1}AC = (0) \Leftrightarrow A = (0)$.
- b $||cA||_C = |c| \cdot ||A||_C$ for any complex scalar c.
- c $||A + B||_C = ||C^{-1}(A + B)C|| \leq ||C^{-1}AC|| + ||C^{-1}BC|| = ||A||_C + ||B||_C$.
- d $||AB||_C = ||C^{-1}ABC|| = ||C^{-1}ACC^{-1}BC|| \leq ||C^{-1}AC|| \cdot ||C^{-1}BC|| = ||A||_C \cdot ||B||_C$.

□

Sqaure Matrix Norms IV

Definition

The **spectral radius** of an $m \times m$ matrix A, denoted by $\rho(A)$, is defined to be

$$\rho(A) = \max_{1 \le i \le m} |\lambda_i(A)|$$

Theorem

For any $m \times m$ matrix A and any matrix norm $\|\cdot\|$,

$$\rho(A) \le \|A\|.$$

Sqaure Matrix Norms V

Proof.

Let η be an eigenvalue of A such that $|\eta| = \rho(A)$ and x be its eigenvector. Therefore, $Ax1_m' = \eta x1_m'$ for $1_m = (1, \cdots, 1)'$. Then,

$$||A|| \cdot ||x1_m'|| \geq ||Ax1_m'|| = ||\eta x1_m'|| = \rho(A)||x1_m'||.$$

$\square$

Theorem

For any $m \times m$ matrix A and any scalar $\varepsilon > 0$, there exists a matrix norm $|| \cdot ||_{A,\varepsilon}$ such that

$$||A||_{A.\varepsilon} - \rho(A) < \varepsilon.$$

Sqaure Matrix Norms VI

Choose any vector or matrix norms $||\cdot||_a$ and $||\cdot||_b$.

Theorem

For a sequence of vectors $\{x_k\}_{k=1}^{\infty}$, $x_k \longrightarrow x$ as $k \longrightarrow \infty$ w.r.t. $||x||_a$ iff $x_k \longrightarrow x$ as $k \longrightarrow \infty$ w.r.t. $||x||_b$.

Corollary

For a sequence of matrices $\{A_k\}_{k=1}^{\infty}$, $A_k \longrightarrow A$ as $k \longrightarrow \infty$ w.r.t. $||A||_a$ iff $A_k \longrightarrow A$ as $k \longrightarrow \infty$ w.r.t. $||A||_b$.

Theorem

Let A be an $m \times m$ matrix and $||A|| < 1$ for some matrix norm $||\cdot||$. Then $A^k \longrightarrow (0)$ as $k \longrightarrow \infty$.

Sqaure Matrix Norms VII

Theorem

For an $m \times m$ matrix A, $A^k \to (0)$ as $k \to \infty$ iff $\rho(A) < 1$.

Proof.

First, suppose that $A^k \overset{k\to\infty}{\longrightarrow} (0)$. So, $A^k x \overset{k\to\infty}{\longrightarrow} 0$ for any eigenvector x of A corresponding to λ. Therefore, $\lambda^k x \overset{k\to\infty}{\longrightarrow} 0$ which leads to $|\lambda| < 1$ so $\rho(A) < 1$. Now if $\rho(A) < 1$, since we know that there exists a matrix norm $||\cdot||_{A,\varepsilon}$ such that $||A||_{A,\varepsilon} - \rho(A) < \varepsilon$ for any $\varepsilon > 0$, we can find $||A|| < 1$. So, $A^k \overset{k\to\infty}{\longrightarrow} (0)$. $\square$

Sqaure Matrix Norms VIII

Theorem

For any matrix norm $||\cdot||$,

$$||A^k||^{1/k} \stackrel{k\to\infty}{\longrightarrow} \rho(A).$$

Sqaure Matrix Norms IX

Proof.

$\lambda(A^k) = \lambda(A)^k$ and since $|\lambda^k| = |\lambda|^k$, $\rho(A)^k = \rho(A^k)$. And we know that $\rho(A^k) \leq \|A^k\|$ so $\rho(A) \leq \|A^k\|^{1/k}$. Since $\rho(A) < \|A^k\|^{1/k} + \varepsilon$ for any $\varepsilon > 0$, we show that $\exists K_\varepsilon > 0$ such that $\|A^k\|^{1/k} < \rho(A) + \varepsilon \;\; \forall k > K_\varepsilon$. This is equivalent with

$$\left\| \frac{A^k}{\{\rho(A) + \varepsilon\}^k} \right\|^{1/k} < 1 \Leftrightarrow \left\| \frac{A^k}{\{\rho(A) + \varepsilon\}^k} \right\| < 1.$$

Here, $\rho\left\{\frac{A}{\rho(A)+\varepsilon}\right\} = \frac{\rho(A)}{\rho(A)+\varepsilon} < 1$ so $\frac{A^k}{\{\rho(A)+\varepsilon\}^k} \overset{k\to\infty}{\to} (0)$. $\square$

Matrix norms induced by vector norms I

Definition

Given a norm $||\cdot||_\alpha$ on $\mathbb{R}^n$, and a norm $||\cdot||_\beta$ on $\mathbb{R}^m$, one can define a matrix norm on $\mathbb{R}^{m\times n}$ induced by these norms:

$$||A||_{\alpha,\beta} = \max_{x\neq 0} \frac{||Ax||_\beta}{||x||_\alpha}.$$

The matrix norm $||A||_{\alpha,\beta}$ is sometimes called a **subordinate norm**.

Remark Subordinate norms are consistent with the norms that induce them, giving

$$||Ax||_\beta \leq ||A||_{\alpha,\beta}||x||_\alpha.$$

Matrix norms induced by vector norms II

Remark Any induced operator norm is a sub-multiplicative matrix norm: $||AB|| \leq ||A||||B||$; this follows from

$$||ABx|| \leq ||A|| \cdot ||Bx|| \leq ||A|| \cdot ||B|| \cdot ||x||$$

and

$$\max_{||x||=1} ||ABx|| = ||AB||.$$

Matrix norms induced by vector norms III

Example

In the special cases of $p = 1, 2, \infty$ and a real-valued matrix A, the induced matrix norms can be computed or estimated by

$$||A||_{1,1} = \max_{||x||_1=1} ||Ax||_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|,$$

$$||A||_{\infty,\infty} = \max_{||x||_\infty=1} ||Ax||_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|,$$

$$||A||_{2,2} = \max_{||x||_2=1} ||Ax||_2 = \sigma_{\max}(A) = \sqrt{\lambda_{\max}(A'A)}.$$

Matrix norms induced by vector norms IV

Theorem

For any $\lambda(A)$, $|\lambda(A)| \leq ||A||$ for any subordinate matrix norm $||\cdot||$.

Proof.

Let λ be an eigenvalue with an orthnormal eigenvector v. Since $||A|| = \max_{||x||=1} ||Ax||$, $||Av|| = ||\lambda v|| = |\lambda| \cdot ||v|| = |\lambda| \leq ||A||$. $\square$

Matrix norms induced by vector norms V

There is an important inequality for the case $p = 2$:

Theorem

$$||A||_2 = \sigma_{\max}(A) \leq ||A||_{\mathrm{F}} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}},$$

where $||A||_{\mathrm{F}}$ *is called* **Frobenius norm**.

When $p = 2$ we have an equivalent definition for $||A||_2$ as $\sup\{x^T Ay : x, y \in \mathbb{R}^n \text{ with } ||x||_2 = ||y||_2 = 1\}$ (Cauchy-Schwarz inequality).

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Matrix Norm Inequalities I

Let $||A||_p$ refers to the norm induced by the vector p-norm.

Theorem

For matrix $A \in \mathbb{R}^{m \times n}$ of rank r, the following inequalities hold:

1. $||A||_2 \le ||A||_F \le \sqrt{r}||A||_2$
2. $\frac{1}{\sqrt{n}}||A||_\infty \le ||A||_2 \le \sqrt{m}||A||_\infty$
3. $\frac{1}{\sqrt{m}}||A||_1 \le ||A||_2 \le \sqrt{n}||A||_1$.

Remark Another useful inequality between matrix norms is

$$||A||_2 \le \sqrt{||A||_1||A||_\infty},$$

which is a special case of **Hölder's inequality**.

Matrix Norm Inequalities II

Example

Consider an estimate of a covariance $\hat{\Sigma} := Z'Z/n := X'(I_n - \Pi_1)X/n$. Then, $||\hat{\Sigma}||_2 = \sigma_{\max}(\hat{\Sigma}) = \sqrt{\lambda_{\max}(\hat{\Sigma}^2)} = |\lambda_{\max}(\hat{\Sigma})| = \lambda_{\max}(\hat{\Sigma})$ provided that $\hat{\Sigma}$ is p.d.. Therefore,

$$\frac{1}{\sqrt{p}}||\hat{\Sigma}||_\infty \leq ||\hat{\Sigma}||_2 \leq \sqrt{p}||\hat{\Sigma}||_\infty,$$

and

$$\lambda_{\min}(\hat{\Sigma}) \leq \lambda_{\max}(\hat{\Sigma}) = ||\hat{\Sigma}||_2 \leq \sqrt{p}||\hat{\Sigma}||_\infty.$$

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The Moore-Penrose Generalized Inverse I

Definition

The Moore-Penrose inverse of the $m \times n$ matrix A is the $n \times m$ matrix, denoted by A^+, which satisfies the followings:

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(AA^+)' = AA^+$
4. $(A^+A)' = A^+A$

Theorem

A^+ for a matrix A always exists, and is unique.

The Moore-Penrose Generalized Inverse II

Proof.

First, if $A = (0)_{m\times n}$, it is trivial that $A^+ = (0)_{n\times m}$. For $A \neq (0)_{m\times n}$, so that $rank(A) > 0$, then by SVD, we know there exist orthogonal N, P and Δ such that $A = N\Delta P'$. If we put $A^+ = P\Delta^{-1}N'$, then this is a M-P inverse of A.

Now, suppose that there exist A_1^+, A_2^+ that are M-P inverses of A. Then, by the definition of M-P inverse,

$$AA_1^+A = AA_2^+A \Leftrightarrow A(A_1^+ - A_2^+)A = (0) \Leftrightarrow (A_1^+ - A_2^+)'A'A = (0),$$

which means $(A_1^+ - A_2^+)'P\Delta^2P' = (0)$ so $A_1^+ = A_2^+$. Therefore, A^+ is unique. $\square$

The Moore-Penrose Generalized Inverse III

Definition

Let A be an $m \times n$ matrix. Then M-P inverse of A is an unique $n \times m$ A^+ satisfying

a. $AA^+ = P_{R(A)}$
b. $A^+A = P_{R(A^+)}$

this is equivalent with the previous definition; the proof is on p.173 of Schott(1997).

The Moore-Penrose Generalized Inverse IV

Theorem

For an $m \times n$ matrix A,

a. $(aA)^+ = a^{-1}A^+$ *for a scalar* $a \neq 0$,
b. $(A')^+ = (A^+)'$ *and* $(A^+)^+ = A$,
c. $A^+ = A^{-1}$ *if A is square and nonsingular,*
d. $(A'A)^+ = A^+(A')^+$ *and* $(AA')^+ = (A')^+A^+$,
e. $(AA^+)^+ = AA^+$ *and* $(A^+A)^+ = A^+A$,
f. $A^+ = (A'A)^+A' = A'(AA')^+$,
g. $A^+ = (A'A)^{-1}A'$ *and* $A^+A = I_n$ *if* $\text{rank}(A) = n$,
h. $A^+ = A'(AA')^{-1}$ *and* $AA^+ = I_m$ *if* $\text{rank}(A) = m$,
i. $A^+ = A'$ *if* $A'A = I_n$.

The Moore-Penrose Generalized Inverse V

Theorem

For any $m \times n$ matrix A,

$$rank(A) = rank(A^+) = rank(AA^+) = rank(A^+A).$$

Proof.

Recall that $rank(AB) \leq \min\{rank(A), rank(B)\}$. So,

$$\begin{aligned} rank(A) &= rank(AA^+A) \leq rank(AA^+) \leq rank(A^+) \\ rank(A^+) &= rank(A^+AA^+) \leq rank(AA^+) \leq rank(A) \end{aligned}$$

so $rank(A) = rank(A^+)$, and
$rank(A) \geq rank(AA^+) \geq rank(AA^+A) = rank(A)$ as well as
$rank(A^+) \geq rank(A^+A) \geq rank(A^+AA^+) = rank(A^+)$. □

The Moore-Penrose Generalized Inverse VI

Theorem

Let A be an $m \times m$ symmetric matrix. Then,

a. $(A^+)' = A^+$
b. $AA^+ = A^+A$
c. $A^+ = A$ *if* $A^2 = A$.

For the proof of (c),
$A^2 = A \Rightarrow A^+A^2 = A^+A \Rightarrow AA^+A = A^+A \Rightarrow A = A^+A \Rightarrow A = A^+$.

The Moore-Penrose Generalized Inverse VII

Theorem

Let $\Lambda = diag(\lambda_1, \cdots, \lambda_m)$*. Then,*

$$\Lambda^+ = diag(\phi_1, \cdots, \phi_m)$$

where

$$\phi_i = \begin{cases} 1/\lambda_i & \text{for } \lambda_i \neq 0 \\ 0 & \text{for } \lambda_i = 0 \end{cases}.$$

The Moore-Penrose Generalized Inverse VIII

Theorem

For an $m \times m$ symmetric A,

$$A^+ = P\Lambda^+ P'.$$

Proof.

Let $r = rank(A) > 0$. Then, by the Spectral Decomposition, $A = P\Lambda P' = P_r\Lambda_r P_r'$ where $\Lambda_r = diag(\lambda_1, \cdots, \lambda_r)$ and $P = \begin{bmatrix} P_r & P_{m-r} \end{bmatrix}$. So with this expression, we can show that $AA^+A = A, A^+AA^+ = A^+, (AA^+)' = AA^+, (A^+A)' = A^+A.$ □

The Moore-Penrose Generalized Inverse IX

Example

Consider a singular X. Even though $\nexists (X'X)^{-1}$, we can still find a projection matrix by

$$P_{R(X)} = XX^{+} = X(X'X)^{+}X'$$

by the definition. This is because

$$X^{+} = X^{+}XX^{+} = X^{+}(X^{+})'X' = (X'X)^{+}X'.$$

Example

Let $X \sim N_m(\partial, \Omega)$ with positive semidefinite Ω with the rank r. Since $\Omega = P_r \Lambda_r P_r'$, $\Omega^+ = P_r \Lambda_r^{-1} P_r'$. Then,

$$\Lambda_r^{-1/2} P_r' X \sim N_r(\Lambda_r^{-1/2} P_r' \partial, I_r)$$

so

$$X' \Omega^+ X \sim \chi_r^2(\Lambda_r^{-1/2} P_r' \partial).$$

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The Moore-Penrose Inverse of a Matrix Product I

Theorem

Let A be an $m \times n$ matrix, while P and Q are $h \times m$ and $n \times p$ matrices satisfying $P'P = I_m$ and $QQ' = I_n$. Then,

$$(PAQ)^+ = Q^+A^+P^+ = Q'A^+P'.$$

The Moore-Penrose Inverse of a Matrix Product II

Proof.

First, $Q^+A^+P^+ = Q'(QQ')^+A^+(P'P)^+P' = Q'A^+P'$. Now, let $D := PAQ$. Then,

a. $DD^+D = DQ'A^+P'D = PAA^+AQ = PAQ = D$,
b. $D^+DD^+ = Q'A^+P'DQ'A^+P' = Q'A^+AA^+P' = Q'A^+P' = D^+$,
c. $(DD^+)' = (D^+)'D' = P(A^+)'QQ'A'P' = P(A^+)'A'P' = P(AA^+)'P' = PAA^+P' = DD^+$,
d. $(D^+D)' = D'(D^+)' = Q'A'P'(Q'A^+P')' = Q'A'(A^+)'Q = Q'(A^+A)'Q = Q'A^+AQ = D^+D$.

$\square$

The Moore-Penrose Inverse of a Matrix Product III

Theorem

Let A, B be $m \times p, p \times n$ matrices. If $rank(A) = rank(B) = r$, then $(AB)^+ = B^+ A^+$.

Proof.

Since A, B are full ranks, we know that $A^+ = (A'A)^{-1}A'$ and $B^+ = B'(BB')^{-1}$. Then,

- a $AB(B^+A^+)AB = ABB'(BB')^{-1}(A'A)^{-1}A'AB = AB$,
- b $(B^+A^+)AB(B^+A^+) = B^+A^+$,
- c $(ABB^+A^+)' = (AI_pA^+)' = I_m$ so symmetric,
- d $(B^+A^+AB)' = (B^+I_pB)' = I_n$ so symmetric.

□

The Moore-Penrose Inverse of a Matrix Product IV

Theorem

Let A be an $m \times p$ matrix and B be a $p \times n$ matrix. Then each of the following conditions are necessary and sufficient for $(AB)^+ = B^+A^+$.

- a $A^+ABB'A' = BB'A'$ and $BB^+A'AB = A'AB$.
- b A^+ABB' and $A'ABB^+$ are symmetric.
- c $A^+ABB'A'ABB^+ = BB'A'A$.
- d $A^+AB = B(AB)^+AB$ and $BB^+A' = A'AB(AB)^+$.

Theorem

If we define $B_1 = P_{R(A^+)}B = A^+AB$ and $A_1 = AP_{R(B_1)} = AB_1B_1^+$, then $AB = A_1B_1$ and $(AB)^+ = B_1^+A_1^+$.

The Moore-Penrose Inverse of a Matrix Product V

Theorem

Let A, B be $m \times p$, $p \times n$ matrices. If we define $B_1 = P_{R(A^+)}B = A^+AB$ and $A_1 = AP_{R(B)} = ABB^+$, then $(AB)^+ = B_1^+ A_1^+$.

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The Moore-Penrose Inverse of Partitioned Matrices I

Theorem

Let the $m \times n$ matrix A be partitioned as $A = \begin{bmatrix} U & V \end{bmatrix}$ where U, V are $m \times n_1$, $m \times n_2$, and $n = n_1 + n_2$. Then,

$$A^+ = \begin{bmatrix} U^+ - U^+V(C^+ + W) \\ C^+ + W \end{bmatrix}$$

where $C = (I_m - UU^+)V$, $M = \{I_{n_2} + (I_{n_2} - C^+C)V'(U^+)'U^+V(I_{n_2} - C^+C)\}^{-1}$ and $W = (I_{n_2} - C^+C)MV'(U^+)'U^+(I_m - VC^+)$. Additionally,

$$A^+ = \begin{bmatrix} U^+ \\ V^+ \end{bmatrix}$$

iff $U'V = (0)$.

The Moore-Penrose Inverse of Partitioned Matrices II

Theorem

Let $m \times n$ matrix A be given by

$$A = \begin{bmatrix} A_{11} & (0) & \cdots & (0) \\ (0) & A_{22} & \cdots & (0) \\ \vdots & \vdots & \ddots & \vdots \\ (0) & (0) & \cdots & A_{rr} \end{bmatrix},$$

where A_{ii} is $m_i \times n_i$ and $\sum_{i=1}^r m_i = m$, $\sum_{i=1}^r n_i = n$. Then,

$$A^+ = \begin{bmatrix} A_{11}^+ & (0) & \cdots & (0) \\ (0) & A_{22}^+ & \cdots & (0) \\ \vdots & \vdots & \ddots & \vdots \\ (0) & (0) & \cdots & A_{rr}^+ \end{bmatrix}.$$

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The Moore-Penrose Inverse of a Sum I

Theorem

Let U be an $m \times n_1$ matrix and V be an $m \times n_2$ matrix. Then,

$$(UU' + VV')^+ = \{I_m - (C^+)'V'\}(U^+)'KU^+(I_m - VC^+) + (CC')^+,$$

where $K = I_m - U^+V(I_{m_2} - C^+C)M(U^+V)'$, and C, M are defined as in 41.

Theorem

Suppose U, V are both $m \times n$ matrices. If $UV' = (0)$, then

$$(U + V)^+ = U^+ + (I_n - U^+V)(C^+ + W),$$

where C, W are as given in 41.

The Moore-Penrose Inverse of a Sum II

Theorem

If U, V are $m \times n$ matrices such that $UV' = (0)$, $U'V = (0)$, then

$$(U+V)^+ = U^+ + V^+.$$

Theorem

Suppose that $U_1, \cdots, U_k$ is a sequence of $m \times n$ matrices such that $U_i U_j' = (0)$ and $U_i' U_j = (0)$ for any $i \neq j$. Then,

$$\left(\sum_{i=1}^{k} U_i \right)^+ = \sum_{i=1}^{k} U_i^+.$$

Contents

Other Generalized Inverses I

Definition

We would denote any matrix that satisfies the first condition of the four M-P conditions by A^{-}, that is,

$$AA^{-}A = A.$$

Other Generalized Inverses II

Theorem

Let A be an $m \times n$ matrix and let A^- be a generalized inverse of A. Then,

- a $(A')^- = (A^-)'$,
- b *if* α *is a nonzero scalar,* $(\alpha A)^- = \alpha^{-1} A^-$,
- c *if* A *is square and nonsingular,* $A^- = A^{-1}$ *uniquely,*
- d *if* B, C *are nonsingular,* $(BAC)^- = C^{-1} A^- B^{-1}$,
- e $rank(A) = rank(AA^-) = rank(A^- A) \leq rank(A^-)$,
- f $rank(A) = m$ *iff* $AA^- = I_m$,
- g $rank(A) = n$ *iff* $A^- A = I_n$.

Other Generalized Inverses III

Theorem

Let A, B, C be matrices of sizes $p \times m$, $m \times n$, $n \times q$ respectively. If $\mathrm{rank}(ABC) = \mathrm{rank}(B)$, then $B^- = C(ABC)^- A$.

Proof.

pp.195-196 of Schott(1997). □

Other Generalized Inverses IV

Definition

Denote the matrix which satisfies conditions 1 and 3 of M-P inverse by A^L, that is,

$$AA^LA = A$$
$$(AA^L)' = AA^L$$

and it is called the least sqaures inverse.

Other Generalized Inverses V

Theorem

Let A be an $m \times n$ matrix. Then,

- a *for any least quares inverse, A^L of A, $AA^L = AA^+$,*
- b $A^L = (A'A)^- A'$.

Other Generalized Inverses VI

Proof.

a. $AA^L = AA^+AA^L = (A^+)'A'(A^L)'A' = (A^+)'(AA^LA)' = (A^+)'A' = AA^+$

b. First,

$$\begin{aligned} A(A'A)^-A'A &= AA^+A(A'A)^-A'A = (A^+)'A'A(A'A)^-A'A \\ &= (A^+)'A'A = AA^+A = A \end{aligned}$$

so first condition satisfies. Next,

$$\begin{aligned} \{A(A'A)^-A'\}' &= \{A(A'A)^-A'(A^+)'A'\}' = \{A(A'A)^-A'AA^+\}' = (AA^+)' \\ &= AA^+ \end{aligned}$$

so second condition also satisfies since AA^+ is symmetric. Note that $A(A'A)^-A'A = A$.

□