1 LDA Likelihood

$$\begin{aligned} \pi_2 :=& P(C=2),\ C-1 \sim Bern(\pi_2),\ C \in \{1,2\} \\ P(C=c) =& \pi_2^{c-1}\pi_1^{2-c}, \quad \pi_1 := 1-\pi_2 \end{aligned}$$

$$\begin{aligned} f_1(\mathbf{x}_i) &:= P(\mathbf{x}_i|C_i=1) = |\det(2\pi\Sigma)|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mathbf{x}_i-\boldsymbol{\mu}_1)'\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu}_1)\right\} \\ f_2(\mathbf{x}_i) &:= P(\mathbf{x}_i|C_i=2) = |\det(2\pi\Sigma)|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mathbf{x}_i-\boldsymbol{\mu}_2)'\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu}_2)\right\} \end{aligned}$$

$$\begin{aligned} \log\frac{f_2(\mathbf{x}_i)}{f_1(\mathbf{x}_i)} &= -\frac{1}{2}\left\{(\mathbf{x}_i-\boldsymbol{\mu}_2)'\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu}_2) - (\mathbf{x}_i-\boldsymbol{\mu}_1)'\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu}_1)\right\} \\ &= -\frac{1}{2}\left\{\mathbf{x}_i'\Sigma^{-1}\mathbf{x}_i - \mathbf{x}_i'\Sigma^{-1}\boldsymbol{\mu}_2 - \boldsymbol{\mu}_2'\Sigma^{-1}\mathbf{x}_i + \boldsymbol{\mu}_2'\Sigma^{-1}\boldsymbol{\mu}_2 - \mathbf{x}_i'\Sigma^{-1}\mathbf{x}_i + \mathbf{x}_i'\Sigma^{-1}\boldsymbol{\mu}_1 + \boldsymbol{\mu}_1'\Sigma^{-1}\mathbf{x}_i - \boldsymbol{\mu}_1'\Sigma^{-1}\boldsymbol{\mu}_1\right\} \\ &= -\frac{1}{2}\left\{-\mathbf{x}_i'\Sigma^{-1}(\boldsymbol{\mu}_2-\boldsymbol{\mu}_1) - (\boldsymbol{\mu}_2-\boldsymbol{\mu}_1)'\Sigma^{-1}\mathbf{x}_i + \boldsymbol{\mu}_2'\Sigma^{-1}\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1'\Sigma^{-1}\boldsymbol{\mu}_1\right\} \\ &= -\frac{1}{2}\left\{-2\mathbf{x}_i'\boldsymbol{\beta} + (\boldsymbol{\mu}_2-\boldsymbol{\mu}_1)'\Sigma^{-1}\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2'\Sigma^{-1}(\boldsymbol{\mu}_2-\boldsymbol{\mu}_1)\right\} \\ &= -\frac{1}{2}(\boldsymbol{\mu}_1+\boldsymbol{\mu}_2-2\mathbf{x}_i)'\boldsymbol{\beta} = \left(\mathbf{x}_i - \frac{\boldsymbol{\mu}_1+\boldsymbol{\mu}_2}{2}\right)'\boldsymbol{\beta} \end{aligned}$$

for $\boldsymbol{\beta} := \Sigma^{-1}(\boldsymbol{\mu}_2-\boldsymbol{\mu}_1)$.

$$\begin{aligned} P(\mathbf{x}_i, c_i) = P(\mathbf{x}_i|c_i)P(c_i) &= \begin{cases} f_2(\mathbf{x}_i)\pi_2 & ,\ c_i = 2 \\ f_1(\mathbf{x}_i)\pi_1 & ,\ c_i = 1 \end{cases} \\ &= \{\pi_2 f_2(\mathbf{x}_i)\}^{c_i-1}\{\pi_1 f_1(\mathbf{x}_i)\}^{2-c_i} \end{aligned}$$

$$\begin{aligned} (\therefore)\quad L(\pi_2,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\Sigma) &= \prod_{i=1}^{n}\left[\{\pi_2 f_2(\mathbf{x}_i)\}^{c_i-1}\{\pi_1 f_1(\mathbf{x}_i)\}^{2-c_i}\right] \\ \ell(\pi_2,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\Sigma) &= \sum_{i=1}^{n}\log\left[\{\pi_2 f_2(\mathbf{x}_i)\}^{c_i-1}\{\pi_1 f_1(\mathbf{x}_i)\}^{2-c_i}\right] \\ &= \sum_{i=1}^{n}\left[(c_i-1)\left\{\log\pi_2 + \log f_2(\mathbf{x}_i)\right\} + (2-c_i)\{\log\pi_1 + \log f_1(\mathbf{x}_i)\}\right] \\ &= n_2\log\pi_2 + n_1\log\pi_1 + \sum_{i=1}^{n}\left\{\log f_1(\mathbf{x}_i) + (c_i-1)\log\frac{f_2(\mathbf{x}_i)}{f_1(\mathbf{x}_i)}\right\} \end{aligned}$$

$$\propto n_2\log\pi_2 + n_1\log\pi_1 + \sum_{i=1}^{n}\left\{-\frac{1}{2}\log|\Sigma| - \frac{1}{2}(\mathbf{x}_i-\boldsymbol{\mu}_1)'\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu}_1) + (c_i-1)\left(\mathbf{x}_i - \frac{\boldsymbol{\mu}_1+\boldsymbol{\mu}_2}{2}\right)'\boldsymbol{\beta}\right\}$$

$$= n_2\log\pi_2 + n_1\log\pi_1 + \sum_{i=1}^{n}(c_i-1)\left(\mathbf{x}_i - \frac{\boldsymbol{\mu}_1+\boldsymbol{\mu}_2}{2}\right)'\boldsymbol{\beta} - \frac{n}{2}\log|\Sigma| - \frac{1}{2}tr\left\{\Sigma^{-1}\sum_{i=1}^{n}(\mathbf{x}_i-\boldsymbol{\mu}_1)(\mathbf{x}_i-\boldsymbol{\mu}_1)'\right\}$$

$$= n_2 \log \pi_2 + n_1 \log \pi_1 + \sum_{i=1}^{n} (c_i - 1) \left(\mathbf{x}_i - \frac{\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2}{2} \right)' \boldsymbol{\beta} + \frac{n}{2} \log \left| \Sigma^{-1} \right| - \frac{n}{2} tr \left\{ \Sigma^{-1} S \right\}$$

where $S := \sum_{i=1}^{n} (\mathbf{x}_i - \boldsymbol{\mu}_1)(\mathbf{x}_i - \boldsymbol{\mu}_1)'/n$.