Matrix Analysis

Part. 3

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February 8, 2020

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Consistency of a System of Equations I

Theorem

$X\beta = c$ is consistent iff $rank(\begin{bmatrix} X & c \end{bmatrix}) = rank(X)$.

Theorem

$X\beta = c$ is consistent iff $XX^{-}c = c$.

Proof.

Suppose that $X\beta^* = c$ for some solution β^*. Then, $XX^{-}c = XX^{-}X\beta^* = X\beta^* = c$. Conversely, if $XX^{-}c = c$, then $\beta = X^{-}c$. □

Consistency of a System of Equations II

Corollary

If the $n \times p$ matrix X has a full row rank, then $X\beta = c$ is consistent.

Proof.

Since $XX^- = I_n$, $XX^- c = c$ so this equation is consistent. $\square$

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Systems of Linear Equations I

Theorem

Suppose that $X'X\beta = X'y$ is a consistent system of equations. Then,

$$\beta = (X'X)^{-}X'y.$$

is a solution.

Proof.

Since $X'X(X'X)^{-}X'y = X'y$ because of the consistency of the equation,

$$X'X\beta = X'X(X'X)^{-}X'y = X'y$$

so β is a solution. $\square$

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One-way Anova I

Consider a model $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$ where $i = 1, \cdots, k$ and $j = 1, \cdots, n_i$. We can reformat this model as $\beta = (\mu, \tau_1, \cdots, \tau_k)'$ and

$$X = \begin{bmatrix} 1_{n_1} & 1_{n_1} & 0 & \cdots & 0 \\ 1_{n_2} & 0 & 1_{n_2} & \cdots & 0 \\ 1_{n_3} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1_{n_k} & 0 & 0 & \cdots & 1_{n_k} \end{bmatrix}; \; n \times (k+1).$$

Here, $rank(X) = k$ so is not full rank. Now, Let us solve a minimization equation $(X'X)\beta = X'y$ with the generalized inverse. Since X is a full row rank, we know that $\beta = (X'X)^{-}X'y$. Therefore, we can pick $\beta = (X'X)^{+}X'y = X^{+}y$ or $\beta = (X'X)^{-}X'y = X^{L}y$.

Rank Deficient Generalized LS Models I

Consider the regression model

$$y = X\beta + \varepsilon,$$

where $\varepsilon | X \sim N_n(0, \sigma^2 C)$ and C is p.d.. Suppose that X is not full column rank. Then, the GLS estimator is given by

$$\hat{\beta} = (X'C^{-1}X)^{-}X'C^{-1}y$$

since the solution equation is given as $X'C^{-1}X\beta = X'C^{-1}y$.

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Partitioned Matrices, Kim(2018) pp.191~ I

Theorem

Let $p \times p$ matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and suppose that A, A_{11} and A_{22} are nonsingular. Then,

$$A^{-1} = \begin{bmatrix} (A^{-1})_{11} & (A^{-1})_{12} \\ (A^{-1})_{21} & (A^{-1})_{22} \end{bmatrix} = \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -A_{11}^{-1}A_{12}(A^{-1})_{22} \\ -A_{22}^{-1}A_{21}(A^{-1})_{11} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix}.$$

Here, $(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} = A_{11}^{-1} + A_{11}^{-1}A_{12}(A^{-1})_{22}A_{21}A_{11}^{-1}$ and $(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} = A_{22}^{-1} + A_{22}^{-1}A_{21}(A^{-1})_{11}A_{12}A_{22}^{-1}$.

Proof.

Solve

$$AA^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} (A^{-1})_{11} & (A^{-1})_{12} \\ (A^{-1})_{21} & (A^{-1})_{22} \end{bmatrix} = \begin{bmatrix} I_{p_1} & (0) \\ (0) & I_{p_2} \end{bmatrix} = I_p.$$

□

Schur's Identity I

Define a block lower triangular matrix L and an arbitrary matrix M as

$$L = \begin{bmatrix} I_p & 0 \\ -D^{-1}C & I_q \end{bmatrix}, M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

After multiplication with the matrix L the Schur complement appears in the upper $p \times p$ block. The product matrix is

$$\begin{aligned} ML &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_p & 0 \\ -D^{-1}C & I_q \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & B \\ 0 & D \end{bmatrix} \\ &= \begin{bmatrix} I_p & BD^{-1} \\ 0 & I_q \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix}. \end{aligned}$$

Schur's Identity II

That is, we have effected a Gaussian decomposition

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_p & BD^{-1} \\ 0 & I_q \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I_p & 0 \\ D^{-1}C & I_q \end{bmatrix},$$

The first and last matrices on the RHS have determinant unity.

Theorem

If $\exists D^{-1}$, then

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D| \times \left| A - BD^{-1}C \right|.$$

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The Kronecker Product I

Definition

For $m \times n$ matrix A and $p \times q$ matrix B, define $\otimes$ as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}; mp \times nq.$$

The Kronecker Product II

Theorem

Let A, B, C be any matrices and a, b be any two vectors. Then,

a. $a \otimes A = A \otimes a = aA$ for any scalar a
b. $(\alpha A) \otimes (\beta B) = \alpha\beta(A \otimes B)$ for any scalars α, β
c. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
d. $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$
e. $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$
f. $(A \otimes B)' = A' \otimes B'$
g. $ab' = a \otimes b' = b' \otimes a$

Prove for exercises.

The Kronecker Product III

Theorem

Pick any A, B, C, D so that AC, BD exist. Then,

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

Proof.

$$\Gamma := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \begin{bmatrix} c_{11}D & \cdots & c_{1c}D \\ \vdots & \ddots & \vdots \\ c_{n1}D & \cdots & c_{nc}D \end{bmatrix}$$

where $\Gamma_{ij} = \sum_{k=1}^{n} a_{ik}c_{kj}BD$ for $i = 1, \cdots, m$ and $j = 1, \cdots, c$. □

The Kronecker Product IV

Proof.

Here, $\Gamma_{ij} = (AC)_{ij}BD$ so

$$\Gamma = \begin{bmatrix} (AC)_{11}D & \cdots & (AC)_{1c}D \\ \vdots & \ddots & \vdots \\ (AC)_{m1}D & \cdots & (AC)_{mc}D \end{bmatrix} = AC \otimes BD.$$

$\square$

Theorem

For square matrices A, B, $tr(A \otimes B) = tr(A)tr(B)$.

Proof is easy for this one.

The Kronecker Product V

Theorem

Let A, B be $m \times n$, $p \times q$ matrices. Then, $(A \otimes B)^{-} = A^{-} \otimes B^{-}$ for any generalized inverses A^{-}, B^{-}.

Proof.

$$(A \otimes B)(A^{-} \otimes B^{-})(A \otimes B) = (AA^{-} \otimes BB^{-})(A \otimes B) = AA^{-}A \otimes BB^{-}B = A \otimes B.$$

□

The Kronecker Product VI

Theorem

Let $\{\lambda_i\}_{i\in I}$, $\{\mu_j\}_{j\in J}$ be eigenvalues of the $m\times m$ matrix A and $p\times p$ matrix B. Then the eigenvalues of $A\otimes B$ is given by $\{\lambda_i\mu_j\}_{(i,j)\in I\times J}$.

Theorem

$$|A\otimes B| = |A|^p|B|^m$$

Proof.

$$|A\otimes B| = \prod_{i\in I}\prod_{j\in J}(\lambda_i\mu_j) = \prod_{i\in I}\left(\lambda_i^p\prod_{j\in J}\mu_j\right) = \prod_{i\in I}(\lambda_i^p|B|) = |A|^p|B|^m.$$

□

The Kronecker Product VII

Theorem

Let A, B be $m \times n$, $p \times q$ matrices. Then,

$$rank(A \otimes B) = rank(A) rank(B).$$

Proof.

Let $\{\lambda_i\}_{i \in I}, \{\delta_j\}_{j \in J}$ be eigenvalues of AA', BB'. Since $(A \otimes B)(A \otimes B)'$ is symmetric,

$$rank(A \otimes B) = rank\left\{(A \otimes B)(A \otimes B)'\right\} = rank(AA' \otimes BB').$$

Since AA', BB', and their Kronecker product are all symmetric, $rank(AA' \otimes BB') = \#\{\lambda_i \delta_j \neq 0; i \in I, j \in J\} = \#\{\lambda_i \neq 0; i \in I\} \times \#\{\delta_j \neq 0; j \in J\} = rank(AA') rank(BB') = rank(A) rank(B)$. $\square$

An Example: Balanced One-way ANOVA I

For the balanced One-way ANOVA model with an intercept, we can find an LS estimator as

$$
\begin{aligned}
\hat{\beta} = (X'X)^{-}X'y &= \left\{ \begin{bmatrix} 1_k' \otimes 1_n' \\ I_k \otimes 1_n' \end{bmatrix} \begin{bmatrix} 1_k \otimes 1_n & I_k \otimes 1_n \end{bmatrix} \right\}^{-} \begin{bmatrix} 1_k' \otimes 1_n' \\ I_k \otimes 1_n' \end{bmatrix} y \\
&= \begin{bmatrix} 1_k' 1_k \otimes 1_n' 1_n & 1_k' \otimes 1_n' 1_n \\ 1_k \otimes 1_n' 1_n & I_k \otimes 1_n' 1_n \end{bmatrix}^{-} \begin{bmatrix} 1_k' \otimes 1_n' \\ I_k \otimes 1_n' \end{bmatrix} y \\
&= \begin{bmatrix} nk & n 1_k' \\ n 1_k & n I_k \end{bmatrix}^{-} \begin{bmatrix} 1_k' \otimes 1_n' \\ I_k \otimes 1_n' \end{bmatrix} y = \begin{bmatrix} 1/(nk) & 0' \\ 0 & \frac{I_k - P_{1_k}}{n} \end{bmatrix} \begin{bmatrix} 1_k' \otimes 1_n' \\ I_k \otimes 1_n' \end{bmatrix} y \\
&= \begin{bmatrix} (1_k' \otimes 1_k')/(nk) \\ (I_k \otimes 1_n')/n - (1_k 1_k' \otimes 1_n')/(nk) \end{bmatrix} y.
\end{aligned}
$$

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The *vec* Operator I

For an $m \times n$ matrix $A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}$, define the *vec* operator as

$$vec(A) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}.$$

Theorem

Let a, b be any vectors, while A, B are two matrices of the same size. Then,

a. $vec(a) = vec(a') = a$,
b. $vec(ab') = b \otimes a$,
c. $vec(\alpha A + \beta B) = \alpha vec(A) + \beta vec(B)$ *for scalars* α, β.

The *vec* Operator II

Theorem

Let A, B be $m \times n$, $m \times n$ matrices. Then,

$$tr(A'B) = vec(A)'vec(B).$$

Proof.

$$tr(A'B) = \sum_{i=1}^{n} a_i' b_i = \begin{bmatrix} a_1' & \cdots & a_n' \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = vec(A)'vec(B).$$

$\square$

The *vec* Operator III

Theorem

Let A, B, C be $m \times n$, $n \times p$, $p \times q$, respectively. Then, $vec(ABC) = (C' \otimes A)vec(B)$.

Proof.

Note that $B = \begin{bmatrix} B_1 & \cdots & B_p \end{bmatrix}$ can be written as $B = \sum_{i=1}^{p} B_i e_i'$. Thus,

$$
\begin{aligned}
vec(ABC) &= vec\left\{A\left(\sum_{i=1}^{p} B_i e_i'\right)C\right\} = \sum_{i=1}^{p} vec(AB_i e_i' C) \\
&= \sum_{i=1}^{p} vec\left\{AB_i(C'e_i)'\right\} = \sum_{i=1}^{p} C'e_i \otimes AB_i \\
&= (C' \otimes A)\sum_{i=1}^{p}(e_i \otimes b_i) = (C' \otimes A)vec(B).
\end{aligned}
$$

□

The *vec* Operator IV

Theorem

Let A, B, C, D be matrices of sizes such that $ABCD$ exists and is square. Then,

$$tr(ABCD) = vec(A')'(D' \otimes B)vec(C).$$

Proof.

We know that $tr(ABCD) = vec(A')'vec(BCD)$. Therefore,

$$tr(ABCD) = vec(A')'(D' \otimes B)vec(C).$$

□

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Hadamard Product I

Definition

For $m \times n$ matrices A, B, define $\odot$ as

$$A \odot B = \begin{bmatrix} a_{11}b_{11} & \cdots & a_{1n}b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}b_{m1} & \cdots & a_{mn}b_{mn} \end{bmatrix},$$

that is, an elementwise product of the identically sized matrices.

Remark In `R`, `A*B` is the Hadamard product for $m \times n$ matrices A, B.

Hadamard Product II

Theorem

For the identically sized matrices A, B, C,

a. $A \odot B = B \odot A$,
b. $(A \odot B) \odot C = A \odot (B \odot C)$
c. $(A + B) \odot C = A \odot C + B \odot C$
d. $(A \odot B)' = A' \odot B'$
e. $A \odot (0) = (0)$
f. $A \odot 1_m 1_n' = A$,
g. $A \odot I_m = diag(a_{11}, \cdots, a_{mm})$ *if A is square,*
h. $C(A \odot B) = (CA) \odot B = A \odot (CB)$ *and* $(A \odot B)C = (AC) \odot B = A \odot (BC)$ *if A, B, C are square and C is diagonal,*
i. $ab' \odot cd' = (a \odot c)(b \odot d)'$ *for the identically sized vectors a, c and b, d.*

Hadamard Product III

*pp.*268 ~ of Schott(1997)

Theorem

For $m \times n$ matrices A, B, $rank(A \odot B) \leq rank(A)rank(B)$.

Theorem

For $m \times n$ matrices A, B and $m \times 1$, $n \times 1$ vectors x, y,

1. $1'_m(A \odot B)1_n = tr(AB')$,
2. $x'(A \odot B)y = tr(D_x A D_y B')$

where D_x denotes a diagonal matrix with $x = (x_1, \cdots, x_m)$.

Theorem

Let A, B be symmetric matrices. Then,

1. *$A \odot B$ is semi-p.d. if A, B are semi-p.d..*
2. *$A \odot B$ is p.d. if A, B are p.d..*

Hadamard Product IV

Theorem

Let A, B be symmetric matrices. If B is p.d. and A is semi-p.d. with positive diagonal elements, then $A \odot B$ is p.d.

Theorem

If A is an $m \times m$ p.d. matrix, then

$$|A| \leq \prod_{i=1}^{m} a_{ii}$$

with equality iff A is a diagonal matrix.

Hadamard Product V

Proof.

By the constrained Rayleigh Quotient in *pp*.105 $\sim$ 106 of Schott(1997), if we partition A as $A = \begin{bmatrix} A_1 & \cdots & A_i & A_{i+1} & \cdots & A_m \end{bmatrix} = \begin{bmatrix} A_{\sim i} & A_{i+1\sim} \end{bmatrix}$,

$$\lambda_i = \min_{x \in S_h; ||x||_2 = 1} x' A x$$

where $S_h = \{A_{\sim i} x; x \in \mathbb{R}^i\}$ for $i = 1, \cdots, m$. So, we know that $\lambda_i \leq a_{ii}$ for $x = e_i$. Therefore, $\prod_{i=1}^m \lambda_i \leq \prod_{i=1}^m a_{ii}$ since A is p.d.. $\square$

The extremal properties of $\lambda(A \odot B)$ is given in *pp*.274 $\sim$ of Schott(1997).

Hadamard Product VI

Corollary

Let B be an $m \times m$ nonsingular matrix. Then,

$$|B|^2 \leq \prod_{i=1}^{m} \left(\sum_{j=1}^{m} b_{ij}^2 \right)$$

with equality iff the rows of B are orthogonal.

This is obvious because $|BB'| = |B| \cdot |B|' = |B|^2$ and $(BB')_{ii} = \sum_{j=1}^{m} b_{ij}^2$.

Theorem

Let A, B be $m \times m$ semi-p.d. matrices. Then,

$$|A| \prod_{i=1}^{m} b_{ii} \leq |A \odot B|.$$