Homework 4 Solutions

Problem 1

```
# Read in Data
school.1 <- read.table("~/Dropbox/UW/Teaching/CS&SS-STAT564/Solutions/HW4/school1.dat")[, 1]
school.2 <- read.table("~/Dropbox/UW/Teaching/CS&SS-STAT564/Solutions/HW4/school2.dat")[, 1]
school.3 <- read.table("~/Dropbox/UW/Teaching/CS&SS-STAT564/Solutions/HW4/school3.dat")[, 1]
```

```
# Create Functions for Obtaining Posterior Parameters
get.kappa.n <- function(y, kappa.0) {
  kappa.n <- kappa.0 + length(y)
  return(kappa.n)
}

get.mu.n <- function(y, mu.0, kappa.0, kappa.n) {

  mu.n <- (kappa.0*mu.0 + sum(y))/kappa.n
  return(mu.n)
}

get.nu.n <- function(y, nu.0) {
  nu.n <- nu.0 + length(y)
  return(nu.n)
}

get.sigma.sq.n <- function(y, mu.0, nu.0, sigma.sq.0, nu.n, kappa.n) {

  sigma.sq.n <- (nu.0*sigma.sq.0 +
                   sum((y - mean(y))^2) +
                   kappa.0*length(y)*(mean(y) - mu.0)^2/kappa.n)/nu.n

  return(sigma.sq.n)
}
```

```
# Set hyperparameters
mu.0 <- 5
sig.sq.0 <- 4
kappa.0 <- 1
nu.0 <- 2

# Compute posterior parameters
kappa.n.1 <- get.kappa.n(school.1, kappa.0)
kappa.n.2 <- get.kappa.n(school.2, kappa.0)
kappa.n.3 <- get.kappa.n(school.3, kappa.0)

mu.n.1 <- get.mu.n(school.1, mu.0, kappa.0, kappa.n.1)
mu.n.2 <- get.mu.n(school.2, mu.0, kappa.0, kappa.n.2)
mu.n.3 <- get.mu.n(school.3, mu.0, kappa.0, kappa.n.3)
```

```r
nu.n.1 <- get.nu.n(school.1, nu.0)
nu.n.2 <- get.nu.n(school.2, nu.0)
nu.n.3 <- get.nu.n(school.3, nu.0)

sig.sq.n.1 <- get.sigma.sq.n(school.1, mu.0, nu.0, sig.sq.0, nu.n.1, kappa.n.1)
sig.sq.n.2 <- get.sigma.sq.n(school.2, mu.0, nu.0, sig.sq.0, nu.n.2, kappa.n.2)
sig.sq.n.3 <- get.sigma.sq.n(school.3, mu.0, nu.0, sig.sq.0, nu.n.3, kappa.n.3)

# Take MC Samples
S <- 100000
sigma.sq.inv.samps.1 <- rgamma(S, nu.n.1/2, nu.n.1*sig.sq.n.1/2)
theta.samps.1 <- rnorm(S, mu.n.1,
                       sqrt((1/sigma.sq.inv.samps.1)/kappa.n.1))
tilde.y.samps.1 <- rnorm(S, theta.samps.1,
                       sqrt((1/sigma.sq.inv.samps.1)))

sigma.sq.inv.samps.2 <- rgamma(S, nu.n.2/2, nu.n.2*sig.sq.n.2/2)
theta.samps.2 <- rnorm(S, mu.n.2,
                       sqrt((1/sigma.sq.inv.samps.2)/kappa.n.2))
tilde.y.samps.2 <- rnorm(S, theta.samps.2,
                       sqrt((1/sigma.sq.inv.samps.2)))

sigma.sq.inv.samps.3 <- rgamma(S, nu.n.3/2, nu.n.3*sig.sq.n.3/2)
theta.samps.3 <- rnorm(S, mu.n.3,
                       sqrt((1/sigma.sq.inv.samps.3)/kappa.n.3))
tilde.y.samps.3 <- rnorm(S, theta.samps.3,
                       sqrt((1/sigma.sq.inv.samps.3)))
```

Part a.

```r
summ <- rbind(c(mean(theta.samps.1),
                      quantile(theta.samps.1, c(0.025, 0.975)),
                mean(sqrt(1/sigma.sq.inv.samps.1)),
                      quantile(sqrt(1/sigma.sq.inv.samps.1), c(0.025, 0.975))),
                    c(mean(theta.samps.2),
                      quantile(theta.samps.2, c(0.025, 0.975)),
                      mean(sqrt(1/sigma.sq.inv.samps.2)),
                      quantile(sqrt(1/sigma.sq.inv.samps.2), c(0.025, 0.975))),
                    c(mean(theta.samps.3),
                      quantile(theta.samps.3, c(0.025, 0.975)),
                      mean(sqrt(1/sigma.sq.inv.samps.3)),
                      quantile(sqrt(1/sigma.sq.inv.samps.3), c(0.025, 0.975))))
row.names(summ) <- 1:3
colnames(summ) <- c("$\\mathbb{E}\\left[\\theta | \\boldsymbol y\\right]$",
                    "$\\theta | \\boldsymbol y_{\\left(0.025\\right)}$",
                    "$\\theta | \\boldsymbol y_{\\left(0.975\\right)}$",
                    "$\\mathbb{E}\\left[\\sigma | \\boldsymbol y\\right]$",
                    "$\\sigma | \\boldsymbol y_{\\left(0.025\\right)}$",
                    "$\\sigma | \\boldsymbol y_{\\left(0.975\\right)}$")
kable(summ, digits = 2)
```

$\mathbb{E}[\theta\|\boldsymbol{y}]$	$\theta\|\boldsymbol{y}_{(0.025)}$	$\theta\|\boldsymbol{y}_{(0.975)}$	$\mathbb{E}[\sigma\|\boldsymbol{y}]$	$\sigma\|\boldsymbol{y}_{(0.025)}$	$\sigma\|\boldsymbol{y}_{(0.975)}$
9.29	7.76	10.81	3.91	3.00	5.17
6.96	5.17	8.73	4.40	3.34	5.89
7.81	6.17	9.46	3.75	2.80	5.13

Part b.

```
comp.theta <- rbind(mean(theta.samps.1 < theta.samps.2 & theta.samps.2 < theta.samps.3),
                    mean(theta.samps.1 < theta.samps.3 & theta.samps.3 < theta.samps.2),
                    mean(theta.samps.2 < theta.samps.1 & theta.samps.1 < theta.samps.3),
                    mean(theta.samps.2 < theta.samps.3 & theta.samps.3 < theta.samps.1),
                    mean(theta.samps.3 < theta.samps.1 & theta.samps.1 < theta.samps.2),
                    mean(theta.samps.3 < theta.samps.2 & theta.samps.2 < theta.samps.1))
row.names(comp.theta) <- c("$\\text{Pr}\\left(\\theta_1 < \\theta_2 < \\theta_3 \\right)$",
                           "$\\text{Pr}\\left(\\theta_1 < \\theta_3 < \\theta_2 \\right)$",
                           "$\\text{Pr}\\left(\\theta_2 < \\theta_1 < \\theta_3 \\right)$",
                           "$\\text{Pr}\\left(\\theta_2 < \\theta_3 < \\theta_1 \\right)$",
                           "$\\text{Pr}\\left(\\theta_3 < \\theta_1 < \\theta_2 \\right)$",
                           "$\\text{Pr}\\left(\\theta_3 < \\theta_2 < \\theta_1 \\right)$")
kable(comp.theta, digits = 2)
```

$\text{Pr}\left(\theta_1 < \theta_2 < \theta_3\right)$	0.01
$\text{Pr}\left(\theta_1 < \theta_3 < \theta_2\right)$	0.00
$\text{Pr}\left(\theta_2 < \theta_1 < \theta_3\right)$	0.08
$\text{Pr}\left(\theta_2 < \theta_3 < \theta_1\right)$	0.67
$\text{Pr}\left(\theta_3 < \theta_1 < \theta_2\right)$	0.02
$\text{Pr}\left(\theta_3 < \theta_2 < \theta_1\right)$	0.22

Part c.

```
comp.tilde.y <- rbind(mean(tilde.y.samps.1 < tilde.y.samps.2 & tilde.y.samps.2 < tilde.y.samps.3),
                      mean(tilde.y.samps.1 < tilde.y.samps.3 & tilde.y.samps.3 < tilde.y.samps.2),
                      mean(tilde.y.samps.2 < tilde.y.samps.1 & tilde.y.samps.1 < tilde.y.samps.3),
                      mean(tilde.y.samps.2 < tilde.y.samps.3 & tilde.y.samps.3 < tilde.y.samps.1),
                      mean(tilde.y.samps.3 < tilde.y.samps.1 & tilde.y.samps.1 < tilde.y.samps.2),
                      mean(tilde.y.samps.3 < tilde.y.samps.2 & tilde.y.samps.2 < tilde.y.samps.1))
row.names(comp.tilde.y) <- c("$\\text{Pr}\\left(\\tilde{y}_1 < \\tilde{y}_2 < \\tilde{y}_3 \\right)$",
                             "$\\text{Pr}\\left(\\tilde{y}_1 < \\tilde{y}_3 < \\tilde{y}_2 \\right)$",
                             "$\\text{Pr}\\left(\\tilde{y}_2 < \\tilde{y}_1 < \\tilde{y}_3 \\right)$",
                             "$\\text{Pr}\\left(\\tilde{y}_2 < \\tilde{y}_3 < \\tilde{y}_1 \\right)$",
                             "$\\text{Pr}\\left(\\tilde{y}_3 < \\tilde{y}_1 < \\tilde{y}_2 \\right)$",
                             "$\\text{Pr}\\left(\\tilde{y}_3 < \\tilde{y}_2 < \\tilde{y}_1 \\right)$")
kable(comp.tilde.y, digits = 2)
```

$\text{Pr}\left(\tilde{y}_1 < \tilde{y}_2 < \tilde{y}_3\right)$	0.11
$\text{Pr}\left(\tilde{y}_1 < \tilde{y}_3 < \tilde{y}_2\right)$	0.11
$\text{Pr}\left(\tilde{y}_2 < \tilde{y}_1 < \tilde{y}_3\right)$	0.18

$\text{Pr}\left(\tilde{y}_2 < \tilde{y}_3 < \tilde{y}_1\right)$	0.27
$\text{Pr}\left(\tilde{y}_3 < \tilde{y}_1 < \tilde{y}_2\right)$	0.14
$\text{Pr}\left(\tilde{y}_3 < \tilde{y}_2 < \tilde{y}_1\right)$	0.20

Part d.

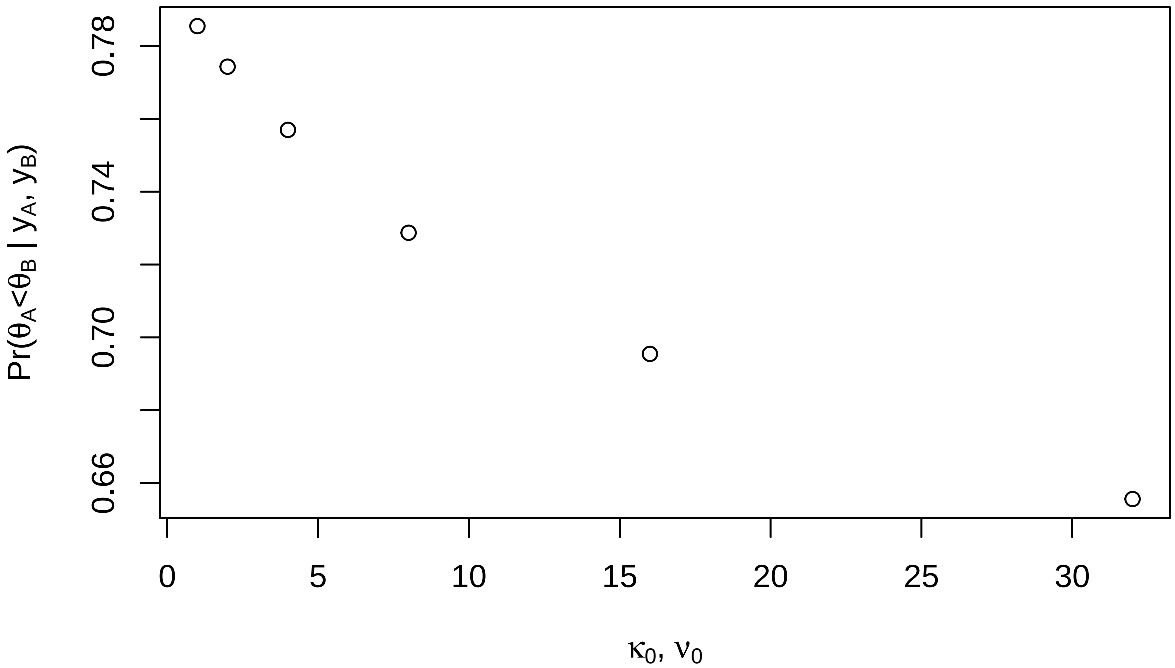
```
comp <- rbind(mean(theta.samps.1 > theta.samps.2 & theta.samps.1 > theta.samps.3),
              mean(tilde.y.samps.1 > tilde.y.samps.2 & tilde.y.samps.1 > tilde.y.samps.3))
row.names(comp) <- c("$\\text{Pr}\\left(\\theta_1 > \\theta_2\\text{ \\& }\\theta_1 > \\theta_3\\right)$
                     "$\\text{Pr}\\left(\\tilde{y}_1 > \\tilde{y}_2\\text{ \\& }\\tilde{y}_1 > \\tilde{
kable(comp, digits = 2)
```

$\text{Pr}\left(\theta_1 > \theta_2 \text{ \& } \theta_1 > \theta_3\right)$	0.89
$\text{Pr}\left(\tilde{y}_1 > \tilde{y}_2 \text{ \& } \tilde{y}_1 > \tilde{y}_3\right)$	0.47

Problem 2

```
n.a <- n.b <- 16

y.bar.a <- 75.2
s.a <- 7.3

y.bar.b <- 77.5
s.b <- 8.1

mu.0 <- 75
sig.sq.0 <- 100

kappa.0 <- nu.0 <- 2^seq(0, 5, by = 1)

kappa.n.a <- kappa.0 + n.a
kappa.n.b <- kappa.0 + n.b

mu.n.a <- (kappa.0*mu.0 + n.a*y.bar.a)/kappa.n.a
mu.n.b <- (kappa.0*mu.0 + n.b*y.bar.b)/kappa.n.b

nu.n.a <- nu.0 + n.a
nu.n.b <- nu.0 + n.b

sig.sq.n.a <- (nu.0*sig.sq.0 + (n.a - 1)*s.a^2 + ((kappa.0*n.a)/kappa.n.a)*(y.bar.a - mu.0)^2)/nu.n.a
sig.sq.n.b <- (nu.0*sig.sq.0 + (n.b - 1)*s.b^2 + ((kappa.0*n.b)/kappa.n.b)*(y.bar.b - mu.0)^2)/nu.n.b

ests <- numeric(length(kappa.0))
# Compute posterior theta_a < theta_b
S <- 100000
for (i in 1:length(ests)) {
  sigma.sq.inv.samps.a <- rgamma(S, nu.n.a[i]/2, nu.n.a[i]*sig.sq.n.a[i]/2)
```

```
  theta.samps.a <- rnorm(S, mu.n.a[i],
                         sqrt((1/sigma.sq.inv.samps.a)/kappa.n.a[i]))
  sigma.sq.inv.samps.b <- rgamma(S, nu.n.b[i]/2, nu.n.b[i]*sig.sq.n.b[i]/2)
  theta.samps.b <- rnorm(S, mu.n.b[i],
                         sqrt((1/sigma.sq.inv.samps.b)/kappa.n.b[i]))
  ests[i] <- mean(theta.samps.a < theta.samps.b)
}
plot(kappa.0, ests,
     xlab = expression(paste(kappa[0], ", ", nu[0], sep = "")),
     ylab = expression(paste("Pr(", theta[A], "<", theta[B], " | ", y[A], ", ", y[B], ")"), sep = ""))
```



From the plot, we see that $\Pr(\theta_A < \theta_B|\boldsymbol{y}_A, \boldsymbol{y}_B)$ is not very sensitive to the choice of κ_0 and ν_0 when $\kappa_0 = \nu_0$. As κ_0 and ν_0 increase, i.e. our prior belief that $\theta_A = \theta_B$ and $\sigma_A = \sigma_B$ increases in certainty, $\Pr(\theta_A < \theta_B|\boldsymbol{y}_A, \boldsymbol{y}_B)$ decreases slowly. Someone with a strong prior belief that $\theta_A = \theta_B$, i.e. $\kappa_0 = \nu_0 = 32$, would still conclude that $\theta_A < \theta_B$ after seeing the data.