Sta532 Hw9

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Exercise 1

(a) The type I error rate is

$$\begin{aligned}\mathsf{Pr}(Y \notin A(0) \mid \mu = 0) &= \mathsf{Pr}(Y \leqslant z_{\alpha(1-w)} \mid \mu = 0) + \mathsf{Pr}(Y \geqslant z_{1-\alpha w} \mid \mu = 0) \\ &= \Phi(z_{\alpha(1-w)}) + 1 - \Phi(z_{1-\alpha w}) \\ &= \alpha(1-w) + 1 - (1-\alpha w) = \alpha\end{aligned}$$

(b) The power function is given by

$$\begin{aligned}\mathsf{Pr}(Y \notin A(0) \mid \mu) &= \mathsf{Pr}(Y \leqslant z_{\alpha(1-w)} \mid \mu) + \mathsf{Pr}(Y \geqslant z_{1-\alpha w} \mid \mu) \\ &= \mathsf{Pr}(Y - \mu \leqslant z_{\alpha(1-w)} - \mu \mid \mu) + \mathsf{Pr}(Y - \mu \geqslant z_{1-\alpha w} - \mu \mid \mu) \\ &= \Phi(z_{\alpha(1-w)} - \mu) + 1 - \Phi(z_{1-\alpha w} - \mu)\end{aligned}$$

The plot for two power functions are presented in Figure 1 with $\alpha = 0.05$.

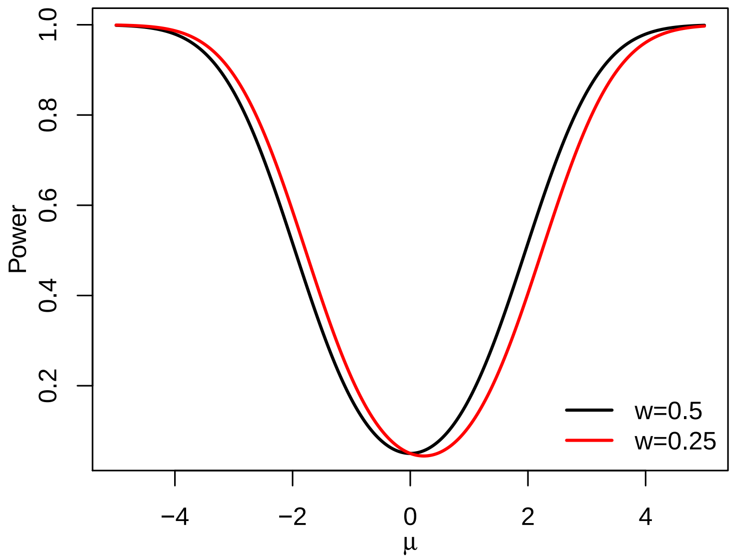

Figure 1: Power function

According to part **(a)**, the tests have the same type I error rate. Therefore, we would like to use the test with greater power. Clearly, we would use $w = 1/2$ when we believe the true parameter μ is larger than 0 and use $w = 1/4$ otherwise.

Exercise 2

(a) The CDF of P_θ is

$$\begin{aligned} F(y) &= \int_0^y \exp(-u/\theta)/\theta \mathrm{d}u \\ &= -\exp(-u/\theta)\Big|_0^y = 1 - \exp(-y/\theta) \end{aligned}$$

(b) For any θ, the type I error rate is

$$\mathsf{Pr}(Y > b \mid \theta) = \exp(-b/\theta)$$

To have a level-α test, we must have

$$\begin{aligned} &\mathsf{Pr}(Y > b \mid \theta) \leqslant \alpha \text{ for all } \theta < \theta_0 \\ \iff &\exp(-b/\theta) \leqslant \alpha \text{ for all } \theta < \theta_0 \\ \iff &\exp(-b/\theta_0) \leqslant \alpha \\ \iff &b \geqslant -\theta_0 \log \alpha \end{aligned}$$

Therefore, as long as $b \geqslant -\theta_0 \log \alpha$, we will have a level-$\alpha$ test.

The power function is given by

$$\mathsf{Pr}(Y > b \mid \theta) = \exp(-b/\theta)$$

To make the plot of the power function, let us take $b = -\theta_0 \log \alpha$ with $\theta_0 = 1$ and $\alpha = 0.05$. The plot is in Figure 2.

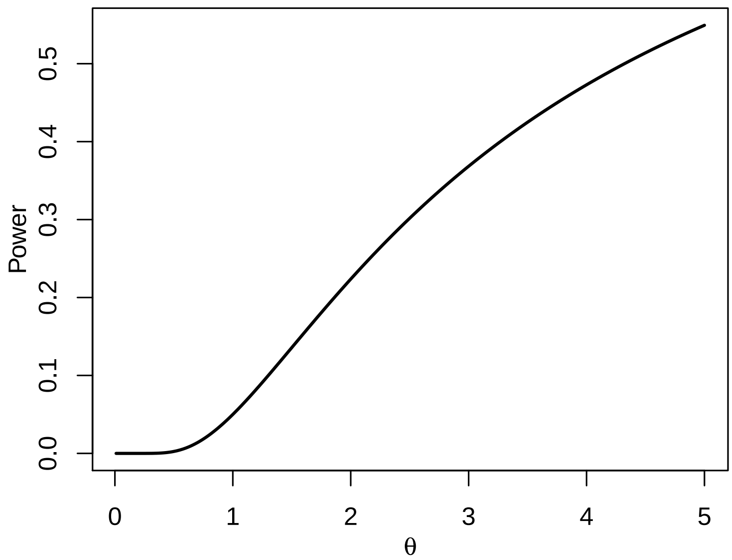

Figure 2: Power function

(c) We can construct a confidence region by inverting the tests we derived in part **(b)**. According to the duality between hypothesis tests and a confidence region, a $1-\alpha$ confidence region is given by the set of parameters for which the null is not rejected. If we choose $b = -\theta_0 \log \alpha$ in part **(b)**, then a confidence region is

$$\{\theta \mid Y \leqslant -\theta \log \alpha\} = \{\theta \mid \theta \geqslant -Y/\log \alpha\}$$

Therefore, a $1-\alpha$ confidence interval is $(-Y/\log\alpha, +\infty)$.

To reassure that the interval indeed has the nominal coverage, we have

$$\mathsf{Pr}(\theta \in (-Y/\log\alpha, +\infty) \mid \theta) = \mathsf{Pr}(Y \leqslant -\theta\log\alpha \mid \theta) = 1 - \exp(-(-\theta\log\alpha)/\theta) = 1-\alpha$$

Exercise 3

(a) The usual two-sample t-statistic assuming equal variances is

$$t(Y) = \frac{\bar{Y}_A - \bar{Y}_B}{s\sqrt{1/n_A + 1/n_B}}, \quad s = \frac{1}{n_A+n_B-2}\left(\sum_{i=1}^{n_A}(Y_{iA}-\bar{Y}_A)^2 + \sum_{i=1}^{n_B}(Y_{iB}-\bar{Y}_B)^2\right)$$

Under the null hypothesis, $t(Y) \sim t_{n_A+n_B-2}$ where $n_A = n_B = 4$. The p-value of this test is

$$\begin{aligned} p &= \mathsf{Pr}\left(t(Y) \leqslant -|t_{\text{obs}}(Y)| \mid H\right) + \mathsf{Pr}\left(t(Y) \geqslant |t_{\text{obs}}(Y)| \mid H\right) \\ &= F_{t_{n_A+n_B-2}}(-|t_{\text{obs}}(Y)|) + 1 - F_{t_{n_A+n_B-2}}(|t_{\text{obs}}(Y)|) \\ &= 2F_{t_{n_A+n_B-2}}(-|t_{\text{obs}}(Y)|) \end{aligned}$$

Since $t_{\text{obs}}(Y)$ can be arbitrarily small or large, the smallest possible p-value is 0.

(b) The total number of possible treatment assignments is $\binom{8}{4} = 70$. Under the randomization scheme, each treatment assignment is equally likely. Therefore the null distribution is a finitely supported, discrete distribution. Note that the null distribution is symmetric about 0. The p-value is given by

$$p = \mathsf{Pr}\left(t(Y) \leqslant -|t_{\text{obs}}(Y)| \mid H\right) + \mathsf{Pr}\left(t(Y) \geqslant |t_{\text{obs}}(Y)| \mid H\right)$$

When $t_{\text{obs}}(Y)$ is the most extreme value in all possible assignments, the smallest possible p-value is

$$p = 1/70 + 1/70 = 1/35$$

(c) To better illustrate the testing procedure, I attached the code below.

(1) Null distribution calculation

```
# load data
y <- c(7.5, 1.2, 7.5, 8.7, 3.2, 5.1, 6.2, 1.7)

# compute t statistic
# note that var() in R is calculated using sum_i (y_i-y_bar)^2/(n-1)
# input:     the indices of observations that are assigned with treatment A
# return:    t statistic
```

```
t.stat <- function(a.ix){
  y.a <- y[a.ix]
  y.b <- y[(1:8)[-a.ix]]

  s.sq <- (var(y.a) + var(y.b))/2
  return((mean(y.a) - mean(y.b))/sqrt(s.sq/2))
}

# calculate null distribution of the randomization test
# all possible treatment assignments
ix.combn <- combn(1:8, 4)

# calculate and plot the null distribution of the randomization test
random.null <- apply(ix.combn, 2, t.stat)

par(mfrow=c(1,2), mar = c(3,3,1,1), mgp = c(2,1,0))
library(MASS)
truehist(random.null, main = '', xlab = 't(Y)', nbins = 20)

# plot the null distributions of both tests together
x <- seq(-6,6,0.1)
plot(density(random.null), main = '', xlab = 't(Y)',
     col = 'red', lwd = 2, ylim = c(0, 0.38))
points(x, dt(x, df = 6), col = 'black', lwd = 2, type = 'l')
legend('topright', legend = c('t test', 'randomization test'),
       col = c(1,2), lwd = 2, bty = 'n')
```

The two null distributions are shown in Figure 3.

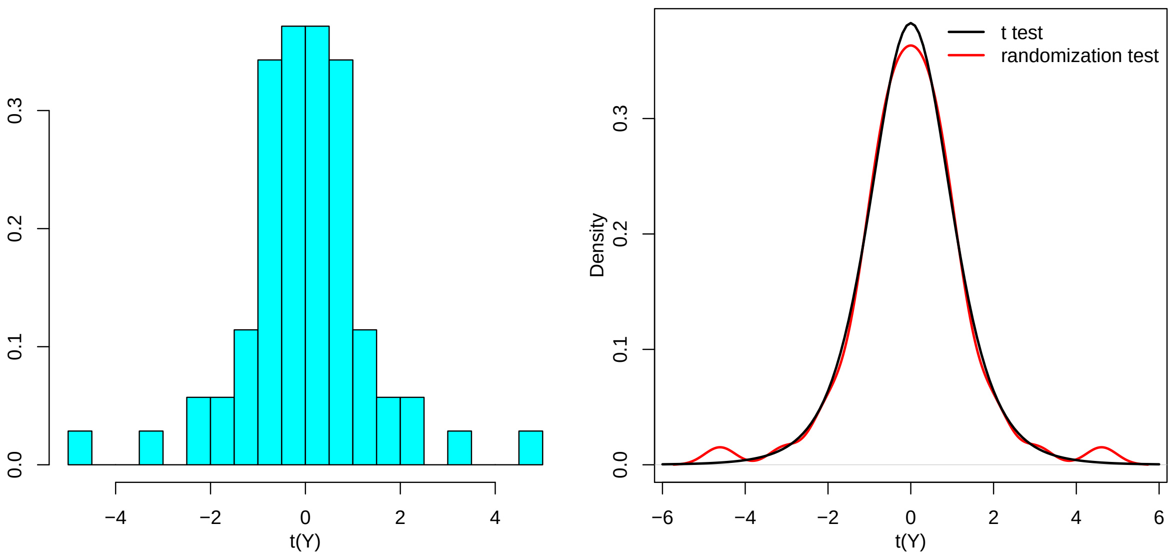

Figure 3: Null distributions of two tests

(2) ***p*-value calculation**

```
# treatment A assignment
a.ix <- c(2,5,6,8)

# compute test statistic
t.obs = t.stat(a.ix)

# p-value for t test, see part (a)
2*pt(-abs(t.obs), df = 6)

# p-value for randomization test, see part (b)
(sum(random.null <= -abs(t.obs)) + sum(random.null >= abs(t.obs)))/choose(8,4)
```

The p-value for t test is 0.00366 and the p-value for randomization test is 0.02857 (1/35).

The t test we used here assumes that samples come from normal distributions with equal variances. In the randomization test, we do not have this distributional assumption. It only requires that the treatment assignments are random.

Exercise 4

(a) Under the null hypothesis, we have

$$T(Y) = \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{n-1}$$

Therefore, $T(Y)$ is a pivotal quantity and is used as the test statistic. The null hypothesis is a simple hypothesis and we may construct a equal-tailed, two-sided rejection region. To find the critical values, we have

$$\begin{aligned} &\mathsf{Pr}(T(Y) < c_1 \mid \sigma^2 = \sigma_0^2) = \alpha/2, \quad \mathsf{Pr}(T(Y) > c_2 \mid \sigma^2 = \sigma_0^2) = \alpha/2 \\ \Longrightarrow\ & c_1 = \chi^2_{n-1,\alpha/2}, \quad c_2 = \chi^2_{n-1,1-\alpha/2} \end{aligned}$$

where $\chi^2_{d,q}$ denotes the qth quantile of χ^2_d distribution. Therefore, the test statistic is $T(Y)$ and the rejection region is $(0, \chi^2_{n-1,\alpha/2}) \cup (\chi^2_{n-1,1-\alpha/2}, +\infty)$.

(b) Again, we can construct a $1-\alpha$ confidence interval by inverting the tests we have in part **(a)**. The confidence interval is given by

$$\begin{aligned} &\{\sigma^2 \mid \chi^2_{n-1,\alpha/2} \leqslant (n-1)S^2/\sigma^2 \leqslant \chi^2_{n-1,1-\alpha/2}\} \\ =\ &\{\sigma^2 \mid (n-1)S^2/\chi^2_{n-1,1-\alpha/2} \leqslant \sigma^2 \leqslant (n-1)S^2/\chi^2_{n-1,\alpha/2}\} \end{aligned}$$

Therefore, a $1-\alpha$ confidence interval is $\Big((n-1)S^2/\chi^2_{n-1,1-\alpha/2}, (n-1)S^2/\chi^2_{n-1,\alpha/2}\Big)$.