STA 532 Homework2

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HW3 for STA-532

1.

Let Y be a positive random variable. Use Jensen's inequality to relate

(a)

The function $f(y) = y^{p/q}$ is convex for $p/q > 1$

By Jensen's inequality

$$\begin{aligned}
&f(E(Y^q)) \leq E(f(Y^q)) \\
&\rightarrow [E(Y^q)]^{p/q} \leq E((Y^q)^{p/q}) \\
&\rightarrow E(Y^q)^{p/q} \leq E(Y^p) \\
&\rightarrow E(Y^q)^{1/q} \leq E(Y^p)^{1/p}
\end{aligned}$$

(b)

The function $f(y) = 1/y$ is concave function

By Jensen's inequality

$$\begin{aligned}
&E(f(Y)) \leq f(E(Y)) \\
&\rightarrow E(1/Y) \leq 1/E(Y)
\end{aligned}$$

(c)

The function $f(y) = logy$ is convex function

By Jensen's inequality

$$\begin{aligned}
&f(E(Y)) \leq E(f(Y)) \\
&\rightarrow logE(Y) \leq E(logY)
\end{aligned}$$

2.

Let $(w_1, w_2, w_3) \sim Dirichlet(\alpha_1, \alpha_2, \alpha_3)$

Then $P(w_1, w_2, w_3) = \frac{\Gamma(\sum \alpha_i)}{\prod \Gamma(\alpha_i)} \prod w_i^{\alpha_i - 1}$ where $0 \leq w_i < 1$ and $\sum w_i = 1$

Let denote $\sum \alpha_i = \alpha_0$

(a)

Derive the expected value and variance of $w_j for j \in \{1, 2, 3\}$

$$\begin{aligned}
E(w_1) &= \int w_1 P_{w_1}(w_1) dw_1 \\
&= \int w_1 (\int\int P(w_1, w_2, w_3) dw_2 dw_3) dw_1 \\
&= \int\int\int w_1 P(w_1, w_2, w_3) dw_1 dw_2 dw_3 \\
&= \frac{\Gamma(\alpha_0)}{\prod \Gamma(\alpha_i)} \int\int\int \underbrace{w_1^{\alpha_1+1-1} w_2^{\alpha_2-1} w_3^{\alpha_3-1}}_{kernel\ of\ dirichlet(\alpha_1+1,\alpha_2,\alpha_3)} dw_1 dw_2 dw_3 \\
&= \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \times \frac{\Gamma(\alpha_1+1)\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_0+1)} = \frac{\alpha_1}{\alpha_0} \\
& simliarly \quad E(w_2) = \frac{\alpha_2}{\alpha_0}, \quad E(w_3) = \frac{\alpha_3}{\alpha_0}
\end{aligned}$$

$$\begin{aligned}
E(w_1^2) &= \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \times \frac{\Gamma(\alpha_1+2)\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_0+2)} \\
&= \frac{\alpha_1(\alpha_1+1)}{\alpha_0(\alpha_0+1)} \\
\rightarrow var(w_1) &= E(w_1^2) - E(w_1)^2 \\
&= \frac{\alpha_1}{\alpha_0}(\frac{\alpha_1+1}{\alpha_0+1} - \frac{\alpha_1}{\alpha_0}) \\
&= \frac{\alpha_1}{\alpha_0}(\frac{\alpha_0\alpha_1+\alpha_0-\alpha_0\alpha_1-\alpha_1}{(\alpha_0\alpha_0+1)}) \\
&= \frac{\alpha_1(\alpha_2+\alpha_3)}{\alpha_0^2(\alpha_0+1)} \\
Likewise \quad var(w_2) &= \frac{\alpha_2(\alpha_1+\alpha_3)}{\alpha_0^2(\alpha_0+1)}, \quad var(w_3) = \frac{\alpha_3(\alpha_1+\alpha_2)}{\alpha_0^2(\alpha_0+1)}
\end{aligned}$$

(b)

Derive the covariance of w_1 and w_2, and explain intuitively the sign of the result.

$$\begin{aligned}
cov(w_1, w_2) &= E[(w_1 - \alpha_1/\alpha_0)(w_2 - \alpha_2/\alpha_0)] \\
&= E(w_1 w_2) - 2\frac{\alpha_1\alpha_2}{\alpha_0^2} + \frac{\alpha_1\alpha_2}{\alpha_0^2} \\
&= E(w_1 w_2) - \frac{\alpha_1\alpha_2}{\alpha_0^2} \\
&= \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \times \frac{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\alpha_3)}{\Gamma(\alpha_0+2)} - \frac{\alpha_1\alpha_2}{\alpha_0^2} \\
&= \frac{\alpha_1\alpha_2}{\alpha_0(\alpha_0+1)} - \frac{\alpha_1\alpha_2}{\alpha_0^2} = -\frac{\alpha_1\alpha_2}{\alpha_0^2(\alpha_0+1)}
\end{aligned}$$

They must have negative relationship because $w_1 + w_2 + w_3 = 1$. If w_1 increase then w_2 must be reduced to satisfy $w_1 + w_2 + w_3 = 1$.

(c)

Derive the variance of w_1 and of $w_1 + w_2$

for $var(w_1)$, as we got at (a)

$$var(w_1) = \frac{\alpha_1(\alpha_2 + \alpha_3)}{\alpha_0^2(\alpha_0 + 1)}$$

and $w_1 + w_2 = 1 - w_3$. Thus

$$var(1 - w_3) = var(w_3) = \frac{\alpha_3(\alpha_1 + \alpha_2)}{\alpha_0^2(\alpha_0 + 1)}$$

(d)

Derive the distribution of w_1 and of $w_1 + w_2$

$$\begin{aligned}
P(w_1, w_2, w_3) &= \frac{\Gamma(\alpha_0)}{\prod(\Gamma(\alpha_i))} w_1^{\alpha_1 - 1} w_2^{\alpha_2 - 1} w_3^{\alpha_3 - 1} \\
&= \frac{\Gamma(\alpha_0)}{\prod(\Gamma(\alpha_i))} w_1^{\alpha_1 - 1} w_2^{\alpha_2 - 1} (1 - w_1 - w_2)^{\alpha_3 - 1} \\
&= P(w_1, w_2) \quad and \quad w_3 > 0 \rightarrow w_1 + w_2 < 1, w_1 > 0, w_2 > 0
\end{aligned}$$

Thus,

$$\begin{aligned}
P(w_1) &= \int P(w_1, w_2) dw_2 \\
&= \frac{\Gamma(\alpha_0)}{\prod(\Gamma(\alpha_i))} w_1^{\alpha_1 - 1} \int_0^{1 - w_1} w_2^{\alpha_2 - 1} (1 - w_1 - w_2)^{\alpha_3 - 1} dw_2 \\
Let \quad w_2 &= (1 - w_1)u \quad then \quad dw_2 = (1 - w_1)du \\
Then \quad P(w_1) &= \frac{\Gamma(\alpha_0)}{\prod \Gamma(\alpha_i)} w_1^{\alpha_1 - 1} (1 - w_1)^{\alpha_2 + \alpha_3 - 1} \int \underbrace{u^{\alpha_2 - 1} (1 - u)^{\alpha_3 - 1}}_{kernel\ of\ beta(\alpha_2, \alpha_3)} du \\
\rightarrow P(w_1) &= \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \times \frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_2 + \alpha_3)} w_1^{\alpha_1} (1 - w_1)^{\alpha_2 + \alpha_3 - 2} \\
&= \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\Gamma(\alpha_2 + \alpha_3)} w_1^{\alpha_1} (1 - w_1)^{\alpha_2 + \alpha_3 - 2}
\end{aligned}$$

Which is pdf of $beta(\alpha_1, \alpha_2 + \alpha_3)$. Thus $w_1 \sim beta(\alpha_1, \alpha_2 + \alpha_3)$.

Let $w_1 + w_2 = X$, $w_1 = Y$, That is $g_1(x, y) = x + y, g_2(x, y) = y$ which are inversible and 0<y<x<1.

By change of variable formula.

$$
\begin{aligned}
P(X,Y) &= P(Y, X-Y)\begin{vmatrix} \frac{dw_1}{dX} & \frac{dw_1}{dY} \\ \frac{dw_2}{dX} & \frac{dw_2}{dY} \end{vmatrix} \\
&= \frac{\Gamma(\alpha_0)}{\prod P(\alpha_i)} Y^{\alpha_1 - 1}(X-Y)^{\alpha_2 - 1}(1-X)^{\alpha_3 - 1} \times \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \\
&= \frac{\Gamma(\alpha_0)}{\prod P(\alpha_i)} (1-X)^{\alpha_3 - 1}(X-Y)^{\alpha_2 - 1} Y^{\alpha_1 - 1} \\
\rightarrow P(X) &= \frac{\Gamma(\alpha_0)}{\prod P(\alpha_i)} (1-X)^{\alpha_3 - 1} \int_0^X (X-Y)^{\alpha_2 - 1} Y^{\alpha_1 - 1} dY \quad and\ Let\ Y = Xu \\
&= \frac{\Gamma(\alpha_0)}{\prod P(\alpha_i)} (1-X)^{\alpha_3 - 1} \int_0^1 (X(1-u))^{\alpha_2 - 1} (Xu)^{\alpha_1 - 1} X du \\
&= \frac{\Gamma(\alpha_0)}{\prod P(\alpha_i)} (1-X)^{\alpha_3 - 1} X^{\alpha_1 + \alpha_2 - 1} \times \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}
\end{aligned}
$$

3.

(a)

Show that if X and Y are independent then Cov(X,Y) = 0

By property of independence, E(XY) = E(X)E(Y)

$$
\begin{aligned}
Cov(X,Y) &= E[(X - E(X))(Y - E(Y))] \\
&= E(XY) - E(X)E(Y) = 0
\end{aligned}
$$

(b)

Show that if X = a + bY then Cor(X,Y) = 1 or -1

$$
\begin{aligned}
Cov(X,Y) &= Cov(a + bY, Y) \\
&= E[b(Y - E(Y))(Y - E(Y))] \\
&= bE((Y - E(Y))^2) \\
and\ var(X) &= b^2 var(Y) \\
Cor(X,Y) &= Cov(X,Y)/\sqrt{b^2 var(Y)^2} \\
&= sign(b) = 1\ or\ -1
\end{aligned}
$$

(c)

Let X_1,X_2,X_3 be three potentially correlated random variables.

i. Compute $Cov[a_1 + b_1 X_1, a_2 + b_2 X_2]$

$$
\begin{aligned}
Cov(a_1 + b_1 X_1, a_2 + b_2 X_2) &= E[(b_1 X_1 + a_1 - E(b_1 X_1 + a_1))(b_2 X_2 + a_2 - E(b_2 X_2 + a_2))] \\
&= E[b_1 b_2 (X_1 - E(X_1))(X_2 - E(X_2))] \\
&= b_1 b_2 E[(X_1 - E(X_1))(X_2 - E(X_2))] \\
&= b_1 b_2 Cov(X_1, X_2)
\end{aligned}
$$

ii. Compute $E(X_1 + X_2 + X_3)$ and $Var(X_1 + X_2 + X_3)$ using the definition of expectation and variance, and check that your answer matches the formula from class.

Let we have joint pdf of $X_1, X_2, X_3, P(X_1, X_2, X_3)$

$$\begin{aligned}
E(X_1 + X_2 + X_3) &= \int\int\int (X_1 + X_2 + X_3) P(X_1, X_2, X_3) dX_1 dX_2 dX_3 \\
&= \int\int\int X_1 P(X_1, X_2, X_3) dX_1 dX_2 dX_3 + \int\int\int X_2 P(X_1, X_2, X_3) dX_1 dX_2 dX_3 \int\int\int X_3 P(X_1, X_2, X_3 \\
&= \int X_1 P_{X_1}(X_1) dX_1 + \int X_2 P_{X_2}(X_2) dX_2 + \int X_3 P_{X_3}(X_3) dX_3 \\
&= E(X_1) + E(X_2) + E(X_3)
\end{aligned}$$

$$\begin{aligned}
Var(X_1 + X_2 + X_3) &= E((X_1 + X_2 + X_3 - E(X_1 + X_2 + X_3))^2) \\
&= E[((X_1 - E(X_1)) + (X_2 - E(X_2)) + (X_3 - E(X_3))^2] \\
&= E[(X_1 - E(X_1))^2] + E[(X_2 - E(X_2))^2] + E[(X_3 - E(X_3))^2] \\
&+ 2E[(X_1 - E(X_1))(X_2 - E(X_2))] + 2E[(X_2 - E(X_2))(X_3 - E(X_3))] + 2E[(X_1 - E(X_1))(X_3 - E(X_3) \\
&= Var(X_1) + Var(X_2) + Var(X_3) + 2\sum_{i \neq j} Cov(X_i, X_j)
\end{aligned}$$

Which is corresponding with formula from class.

4.

(a)

Compute the expectation and variance of $Y_1, Y_2 \; and \; Y_3$

$$E(Y_1) = E(Y_2) = E(Z + X_1) = E(Z + X_2) = E(Z) + E(X_1) = E(Z) + E(X_2) = 0$$
$$E(Y_3) = E(Z^2 + X_3) = E(Z^2) + E(X_3) = Var(Z) + 0 = 1$$

$$Var(Y_1) = Var(Y_2) = Var(Z + X_1) = Var(Z + X_2)$$
$$= Var(Z) + Var(X_1) = Var(Z) + Var(X_2) = 2$$

$$\begin{aligned}
Var(Y_3) &= Var(Z^2 + X_3) \\
&= Var(Z^2) + Var(X_3) \\
&= E[(Z^2 - E(Z^2))^2] + 1 \\
&= E[Z^4 - 2Z^2 + 1] + 1 \\
&= 3 - 2 + 1 + 1 = 3
\end{aligned}$$

(b)

$$
\begin{aligned}
Cov(Y_1, Y_2) &= Cov(Z + X_1, Z + X_2) \\
&= Cov(Z + X_1, Z) + cov(Z + X_1, X_2) \\
&= Cov(Z, Z) + Cov(X_1, Z) + Cov(Z, X_2) + Cov(X_1, X_2) \\
&= Var(Z) = 1 \\
Cov(Y_1, Y_3) &= Cov(Z + X_1, Z^2 + X_3) \\
&= Cov(Z, Z^2) + Cov(Z, X_3) + Cov(X_1, Z^2) + Cov(X1, X_3) \\
&= E(Z(Z^2 - 1)) + E(ZX_3) + E(X_1(Z^2 - 1)) \quad funtions\ are\ also\ independent \\
&= E(Z^3 - Z) = 0 = Cov(Y_2, Y_3)
\end{aligned}
$$

Thus, Covariance matrix is

$$
\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}
$$

(c)

Are $Y_$1 and Y_3 independent?

No, because $P(Y_3 \mid Y_1) \neq P(Y_3)$, for

$$
\begin{aligned}
P(Y_3 \mid Y_1) &= P(Z^2 + X_3 \mid Y_1) \\
&= P(Z^2 + X_3 \mid Z + X_1) \\
&\neq P(Z^2 + X_3)
\end{aligned}
$$

because $Z^2 + X_3$ is affected by Z value determined by Y_1

5.

For two jointly distributed random variables X and Y and functions f and g, show that

(a)

E[E[f(Y)|X]] = E[f(Y)]

$$
\begin{aligned}
E(f(Y) \mid X) &= \int f(Y) P_{Y|X(Y|X)} dY \\
&= \int f(Y) \frac{P(X, Y)}{P_X(X)}) dY \\
&= \frac{1}{P_X(X)} \int f(Y) P(X, Y) dY \\
E[E(f(Y) \mid X)] &= \int E(f(Y) \mid X) P_X(X) dX \\
&= \int \frac{1}{P_X(X)} \int f(Y) P(X, Y) dY P_X(X) dX \\
&= \int \int f(Y) P(X, Y) dX dY \\
&= E[f(Y)]
\end{aligned}
$$

(b)

$$
\begin{aligned}
E[f(X)g(X,Y) \mid X] &= \int f(X)g(X,Y)P(Y \mid X)dY \\
&= f(X)\int g(X,Y)P(Y \mid X)dY \\
&= f(X)E[g(X,Y) \mid Y]
\end{aligned}
$$