$x_t \mid \phi_t \sim \text{poisson}(\phi_t)$

$\phi_{t-1} \mid D_{t-1} \sim \text{gamma}(a_{t-1}, a_{t-1}/m_{t-1})$

$\phi_t = \phi_{t-1} \eta_t / \beta$

$p(\phi_{t-1} \mid D_{t-1}) \propto \phi_{t-1}^{a_{t-1}-1}$

(a) In Exercise 4, we have proven that

$\phi_0 \sim gamma(a, b)$, $\eta \sim Be(\beta a, (1-\beta)a)$, then, $\phi_1 = \phi_0 \eta / \beta \sim gamma(\beta a, \beta b)$

In this case

$\phi_{t-1} | D_{t-1} \sim gamma(a_{t-1}, \frac{a_{t-1}}{m_{t-1}})$, $\eta_t | D_{t-1} \sim Beta(\beta a_{t-1}, (1-\beta) a_{t-1})$

Thus $p(\phi_t | D_{t-1}) \sim N(\beta a_{t-1}, \beta \frac{a_{t-1}}{m_{t-1}})$

(b) $E(x_t | D_{t-1}) = E(E(x_t | \phi_t, D_{t-1})) = E(\phi_t | D_{t-1})$

$$= \frac{\beta a_{t-1}}{\beta \frac{a_{t-1}}{m_{t-1}}} = m_{t-1}$$

(c) $$p(x_t | D_{t-1}) = \int p(x_t, \phi_t | D_{t-1}) \, d\phi_t$$

$$= \int p(x_t | \phi_t, D_{t-1}) \, p(\phi_t | D_{t-1}) \, d\phi_t$$

$$= \int p(x_t | \phi_t) \, p(\phi_t | D_{t-1}) \, d\phi_t$$

$$= \frac{1}{x_t!} \times \frac{1}{\Gamma(\beta a_{t-1})} \times \left(\frac{\beta a_{t-1}}{m_{t-1}}\right)^{\beta a_{t-1}} \int e^{-\phi_t} \phi_t^{x_t} e^{-\frac{\beta a_{t-1}}{m_{t-1}} \phi_t} \phi_t^{\beta a_{t-1} - 1} \, d\phi$$

$$= \frac{1}{x_t!} \times \frac{1}{\Gamma(\beta a_{t-1})} \times \left(\frac{\beta a_{t-1}}{m_{t-1}}\right)^{\beta a_{t-1}} \int \phi_t^{\beta a_{t-1} + x_t - 1} e^{-\phi_t (1 + \frac{\beta a_{t-1}}{m_{t-1}})} \, d\phi_t$$

$$= \frac{1}{x_t!} \times \frac{1}{\Gamma(\beta a_{t-1})} \times \left(\frac{\beta a_{t-1}}{m_{t-1}}\right)^{\beta a_{t-1}} \times \Gamma(\beta a_{t-1} + x_t) \left(1 + \frac{\beta a_{t-1}}{m_{t-1}}\right)^{-(\beta a_{t-1} + x_t)}$$

$$= \frac{\Gamma(\beta a_{t-1} + x_t)}{\Gamma(x_t + 1) \Gamma(\beta a_{t-1})} \left(\frac{\frac{\beta a_{t-1}}{m_{t-1}}}{1 + \frac{\beta a_{t-1}}{m_{t-1}}}\right)^{\beta a_{t-1}} \times \left(\frac{1}{1 + \frac{\beta a_{t-1}}{m_{t-1}}}\right)^{x_t}$$

$$= \frac{\Gamma(\beta a_{t-1} + x_t)}{\Gamma(x_t + 1) \Gamma(\beta a_{t-1})} \left(\frac{\beta a_{t-1}}{m_{t-1} + \beta a_{t-1}}\right)^{\beta a_{t-1}} \times \left(\frac{m_{t-1}}{m_{t-1} + \beta a_{t-1}}\right)^{x_t}$$

$$\sim nb\left(\beta a_{t-1}, \frac{\beta a_{t-1}}{m_{t-1} + \beta a_{t-1}}\right)$$

(d) $P(\phi_t|D_t) = P(\phi_t|D_{t-1}, x_t)$

$$\propto P(\phi_t, x_t|D_{t-1})$$

$$\propto P(x_t|\phi_t)P(\phi_t|D_{t-1})$$

$$\propto \phi_t^{x_t} e^{-\phi_t} \times \phi_t^{\beta a_{t-1}-1} e^{-\phi_t \frac{\beta a_{t-1}}{m_{t-1}}}$$

$$\propto \underbrace{\phi_t^{\beta a_{t-1}+x_t-1}} e^{-\phi_t(1+\frac{\beta a_{t-1}}{m_{t-1}})}$$

$\Rightarrow P(\phi_t|D_t) \sim \text{gamma}(\beta a_{t-1}+x_t, 1+\frac{\beta a_{t-1}}{m_{t-1}})$

$\Rightarrow a_t = \beta a_{t-1}+x_t, \quad \frac{a_t}{m_t} = 1+\frac{\beta a_{t-1}}{m_{t-1}} \Rightarrow \frac{m_t}{a_t} = \frac{m_{t-1}}{m_{t-1}+\beta a_{t-1}}$

$\Rightarrow m_t = \left(\frac{\beta a_{t-1}+x_t}{m_{t-1}+\beta a_{t-1}}\right) m_{t-1}$

m_t positively depends on x_t which means that as x_t increase, m_t also increase which means estimate of mean is corrected by new observation x_t

we can interpret this as correction procedure. By observing new value, we adjust our estimate of phi. In this context, a_{t-1} plays a role as how much we give on previous estimate (as prior sample number). Thus if a_{t-1} is large, correction from new observation become weak and we give more weight on prior estimate.

(e) As we have shown HW 5, exercise 4 (e)

$P(\phi_0|\phi_1)$ is defined by $\phi_0 = \beta\phi_1 + r$ where $r \sim \text{gamma}((1-\beta)a, b)$ with $\phi_1 \perp\!\!\!\perp r$

when we adjust above result, we can get

$\phi_{t-1} = \beta\phi_t + r_t$ where $r_t \sim \text{gamma}((1-\beta)a_{t-1}, \frac{a_{t-1}}{m_{t-1}})$

which indicates first order markovian structure

Since ϕ_t has first order markovian structure

$\phi_{t-1} \leftarrow \phi_t \leftarrow \cdots \phi_T$

we can simulate retrospective distribution for $\phi_{t-1}|\phi_t, D_T$ as follow:

From $\phi_T|D_T \sim \text{gamma}(a_T, \frac{a_T}{m_T})$

Smooth retrospective distribution for $t = T-1 \cdots 1$

From beginning $m_T^* = m_T$, $a_T^* = a_T$

$E(\phi_{T-1}|D_T) = m_{T-1}^* = \beta E(\phi_T|D_T) + (1-\beta)m_{T-1} = \beta m_T^* + (1-\beta)m_{T-1}$

$a_{T-1}^* = (1-\beta)a_{T-1} + \beta a_T^*$

Then $P(\phi_{T-1}|\phi_T, D_T) \sim \text{gamma}(a_{T-1}^*, \frac{a_{T-1}^*}{m_{T-1}^*})$

From above result, by similar procedure

we can simulate retrospective distribution $P(\phi_{t-1}|\phi_t, D_T) \sim \text{gamma}(a_{t-1}, \frac{a_{t-1}}{m_{t-1}^*})$

(e) we can simulate full posterior distribution of $\phi_1 \cdots \phi_T$

from the fact that $\phi_{t-1} = \beta\phi_t + r_t$ $\quad r_t \sim \text{gamma}((1-\beta)a_{t-1}, \frac{a_{t-1}}{m_{t-1}})$

First. sample $\phi_T^{(s)}$ from $\phi_T | D_T \sim \text{gamma}(a_T, \frac{a_T}{m_T})$

Second. sample $r_T^{(s)}$ from $\text{gamma}(a_{T-1}, \frac{a_{T-1}}{m_{T-1}})$

Third. Calculate $\phi_{T-1}^{(s)} = \beta\phi_T^{(s)} + r_T^{(s)}$

Forth. Then sample $r_{T-1}^{(s)}$ from $\text{gamma}(a_{T-2}, \frac{a_{T-2}}{m_{T-2}})$

Fifth, calculate $\phi_{T-2}^{(s)} = \beta\phi_{T-1}^{(s)} + r_{T-1}^{(s)}$

$\vdots$

iterate above procedure

we can get $(\phi_1 \cdots \phi_T)^{(s)}$ simulate full joint posterior distribution