STA 642 Homework5

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HW5 for STA-642

Exercise 3

Let $\phi_0 = v_{t-1}^{-1}, \phi_1 = v_t^{-1}$. Then find bivariate $P(\phi_0, \phi_t \mid D_t)$ given follow:

$P(\phi_0) \sim gamma(a,b), \phi_1 = \phi_0\eta/\beta$ where $\eta \sim beta(\beta a, (1-\beta)a),\ \beta \in (0,1)\ and\ \phi_0 \perp \eta$

(a)

$$
\begin{aligned}
&E(\phi_1 \mid \phi_0) = E(\phi_0\eta/\beta \mid \phi_0) = E(\eta)\phi_0/\beta \\
&mean\ of\ beta(a,b) = \frac{a}{a+b} \rightarrow E(\eta) = \frac{\beta a}{a} = \beta \\
&\rightarrow E(\phi_1 \mid \phi_0) = \phi_0
\end{aligned}
$$

(b)

$$
\begin{aligned}
&E(\phi_0) = a/b \\
&E(\phi_1) = E(E(\phi_1 \mid \phi_0)) = E(\phi_0) = a/b
\end{aligned}
$$

(c)

$$
\begin{aligned}
P(\phi_0, \eta) &= P(\phi_0)P(\eta) \quad (by\ independence) \\
&\propto \phi_0^{a-1}e^{-b\phi_0} \times \eta^{\beta a-1}(1-\eta)^{(1-\beta)a-1} \\
P(\phi_0, \phi_1) &= P_{\phi_0,\eta}(\phi_0, \frac{\phi_1\beta}{\phi_0}) \times \begin{vmatrix} 1 & -\frac{\phi_1\beta}{\phi_0^2} \\ 0 & \frac{\beta}{\phi_0} \end{vmatrix} \quad by\ change\ of\ variable \\
&and \quad 0 < \eta = \frac{\phi_1\beta}{\phi_0} < 1 \rightarrow 0 < \phi_1 < \frac{\phi_0}{\beta} \\
\rightarrow P(\phi_0, \phi_1) &\propto \phi_0^{a-1}e^{-b\phi_0} \times (\frac{\phi_1\beta}{\phi_0})^{\beta a-1}(1-\frac{\phi_1\beta}{\phi_0})^{(1-\beta)a-1} \times \frac{\beta}{\phi_0} \\
&\propto \phi_0^{a-1}e^{-b\phi_0} \times (\phi_1\beta)^{\beta a-1}\phi_0^{-\beta a+1}\phi_0^{-(1-\beta)a+1}(\phi_0 - \phi_1\beta)^{(1-\beta)a-1} \times \frac{\beta}{\phi_0} \\
&\propto \phi_0^{a-1}e^{-b\phi_0}\phi_1^{\beta a-1}\phi_0^{a-2} \times \frac{1}{\phi_0}(\phi_0 - \phi_1\beta)^{(1-\beta)a-1} \\
&\propto e^{-b\phi_0}\phi_1^{\beta a-1}(\phi_0 - \phi_1\beta)^{(1-\beta)a-1}
\end{aligned}
$$

(d)

Let $\nu = \phi_0 - \phi_1\beta$ then

$$\begin{aligned}
P(\phi_1, \nu) &= P_{\phi_0,\phi_1}(\phi_1, \nu + \phi_1\beta) \begin{vmatrix} 1 & -\beta \\ 0 & 1 \end{vmatrix} = P_{\phi_0,\phi_1}(\phi_1, \nu + \phi_1\beta) \\
P(\phi_1) &= \int P(\phi_1, \nu) d\nu \\
&\propto \int e^{-b(\nu+\phi_1\beta)} \phi_1^{\beta a - 1} \nu^{(1-\beta)a-1} d\nu \\
&\propto \phi_1^{\beta a - 1} e^{-\beta b \phi_1} \int \underbrace{\nu^{(1-\beta)a} e^{-b\nu}}_{kernel\ of\ gamma((1-\beta)a, b)} d\nu \\
&\propto C \underbrace{\phi_1^{\beta a - 1} e^{-\beta b \phi_1}}_{kernel\ of\ gamma(\beta a, \beta b)}
\end{aligned}$$

(e)

We have defined that $\nu = \phi_0 - \phi_1\beta$ at previous question $\rightarrow \phi_0 = \nu + \phi_1\beta$. At previous question, we could figure out that

$P(\phi_1, \nu) \propto \phi_1^{\beta a-1} e^{-\beta b \phi_1} \nu^{(1-\beta)a-1} e^{-b\nu}$.

This can be factorized regarding ϕ_1, ν as

$\phi_1^{\beta a-1} e^{-\beta b\phi_1} \propto P(\phi_1)$ and $\nu^{(1-\beta)a-1} e^{-b\nu} \propto P(\nu)$ and this is kernel of $gamma((1-\beta)a, b)$. Thus we can conclude that $\phi_1 \perp \nu$.

Moreover, at (d), we have confirmed that $\phi_1, \nu \propto P(\phi_1, \phi_0)$. This is $P(\phi_0 \mid \phi_1) = \frac{P(\phi_0,\phi_1)}{P(\phi_1)} \propto \frac{P(\phi_1,\nu)}{P(\phi_1)} = \frac{P(\phi_1)P(\nu)}{P(\phi_1)} = P(\nu) \rightarrow P(\phi_0 \mid \phi_1) \propto P(\nu)$.

Exercise 4

For previous setting, we have checked that, in setting where

$$\begin{aligned}
&P(\phi_0) \sim gamma(a, b) \\
&\phi_1 = \phi_0 \eta / \beta \\
&\eta \sim beta(\beta a, (1-\beta)a) \\
&\beta \in (0, 1) \\
&\phi_1 \perp \eta \\
\rightarrow &P(\phi_1) \sim gamma(\beta a, \beta b)
\end{aligned}$$

If we replace $\phi_0 = \phi_{t-1}, \phi_1 = \phi_t, \eta = \eta_t, a = n_{t-1}, b = d_{t-1}$ and $P(\phi_0) \rightarrow P(\phi_{t-1} \mid D_{t-1}), P(\phi_1) \rightarrow P(\phi_t \mid D_{t-1})$ then we can easily check the evolution procedure that

$\phi_{t-1} \mid D_{t-1} \sim gamma(n_{t-1}/2, d_{t-1}/2) \rightarrow \phi_t \mid D_{t-1} \sim gamma(\beta n_{t-1}/2, \beta d_{t-1}/2)$

Exercise 5

(a)

At previous question Q3 (e), we have shown this.

(b)

In this discount volatility evolution set up, we have check that $\phi_{t-1} = \beta\phi_t + \nu_{t-1}$ which indicates Markovian structure in volatility

$\phi_{t-1} \leftarrow \phi_t \leftarrow \cdots \phi_T$ for all T $\geq$ t.

(c)

$$\begin{aligned}
&\phi_t = \beta\phi_{t+1} + \nu_t^* \\
simliarly, \quad &E(\phi_{T-1} \mid D_T) = E(\beta\phi_T \mid D_T) + E(\nu_{T-1}^* \mid D_T) \\
\rightarrow &E(\phi_{T-2} \mid D_T) = \beta E(\phi_{T-1} \mid D_T) + E(\nu_{T-2} \mid D_T) \\
&\vdots \\
\rightarrow &E(\phi_t \mid D_t) = \beta E(\phi_{t+1} \mid D_T) + E(\nu_t^* \mid D_t)
\end{aligned}$$

it means that from $E(\phi_T \mid D_T)$, by updating estimate of $E(\phi_i \mid D_i)$ by above equation, we can estimate $E(\phi_t \mid D_T)$ for $1 \leq t \leq T$.

(d)

Similar with above procedure, we can estimate trajectory values of $\phi_T, \phi_{T-1}, \cdots \phi_1$.

$$\begin{aligned}
&1.\ Sample\ \phi_T\ from\ gamma(n_T/2, n_T s_T/2) \\
&2.\ Sample\ \nu_{T-1}\ from\ gamma((1-\beta)n_{T-1}/2, n_{T-1}s_{T-1}/2) \\
&3.\ Calculate\ \phi_{T-1} = \beta\phi_T + \nu_{T_1} \\
&4.\ sample\ \nu_{T-2}\ from\ gamma((1-\beta)n_{T-2}/2, n_{T-2}s_{T-2}/2) \\
&5.\ Calculate\ \phi_{T-1} = \beta\phi_T + \nu_{T_1} \\
&\vdots \\
&\quad Sample\ \nu_1\ from\ gamma((1-\beta)n_1/2, n_1 s_1/2) \\
&\quad Calculate\ \phi_1 = \beta\phi_2 + \nu_1
\end{aligned}$$