STA 532 Homework2

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HW2 for STA-532

1.

Let $\sigma^2 = 1/\lambda, \tau^2 = 1/k$. Then

$$
\begin{aligned}
P(Y \mid \theta) &= \sqrt(\frac{\lambda}{2\pi})exp\{-\frac{\lambda}{2}(Y-\theta)^2\} \\
P(\theta) &= \sqrt(\frac{k}{2\pi})exp\{-\frac{k}{2}(\theta-\mu)^2\}
\end{aligned}
$$

$$
\begin{aligned}
P(Y,\theta) &= P(Y \mid \theta)P(\theta) \\
&= \frac{\sqrt{\lambda k}}{2\pi}exp\{-\frac{1}{2}(\theta^2(\lambda+k) - 2\theta(\lambda Y + k\mu) + \lambda Y^2 + k\mu^2)\} \\
&= \frac{\sqrt{\lambda k}}{2\pi}exp\{-\frac{(\lambda+k)}{2}(\theta^2 - 2\theta\frac{\lambda Y + k\mu}{\lambda+k} + \frac{(\lambda Y + k\mu)^2}{(\lambda+k)^2}) - \frac{1}{2}(\lambda Y^2 + k\mu^2 - \frac{(\lambda Y + k\mu)^2}{\lambda+k})\} \\
&= \frac{\sqrt{\lambda k}}{2\pi}exp\{-\frac{(\lambda+k)}{2}(\theta - \frac{\lambda Y + k\mu}{\lambda+k})^2\}\times \\
&\sqrt{\frac{\lambda k}{2\pi(\lambda+k)}}exp\{-\frac{1}{\lambda+k}(\lambda^2Y^2 + \lambda kY^2 + \lambda k\mu^2 + k^2\mu^2 - \lambda^2Y^2 - 2\lambda kY\mu - k^2\mu^2)\} \\
&= \frac{\sqrt{\lambda k}}{2\pi}exp\{-\frac{(\lambda+k)}{2}(\theta - \frac{\lambda Y + k\mu}{\lambda+k})^2\} \times \sqrt{\frac{\lambda k}{2\pi(\lambda+k)}}exp\{-\frac{\lambda Y}{\lambda+k}(Y-\mu)^2\}
\end{aligned}
$$

and

$$
\begin{aligned}
P(Y) = \int P(Y,\theta)d\theta &= \sqrt{\frac{\lambda k}{2\pi(\lambda+k)}}exp\{-\frac{\lambda Y}{\lambda+k}(Y-\mu)^2\} \\
&= \sqrt{\frac{1}{2\pi(\sigma^2+\tau^2)}}exp\{-\frac{1}{2(\sigma^2+\tau^2)}(Y-\mu)^2\} \\
P(\theta \mid Y) = P(Y,\theta)/P(Y) &= \sqrt{\frac{\sigma^2+\tau^2}{2\pi\sigma^2\tau^2}}exp\{-\frac{1}{2}(1/\sigma^2+1/\tau^2)^{-1}(\theta - (1/\sigma^2+1/\tau^2)^{-1}(\frac{Y}{\sigma^2}+\frac{\mu}{\tau^2}))^2\}
\end{aligned}
$$

2.

Def. Random variable $Y_1, Y_2, \cdots, Y_n$ are independent if $Pr(Y_1 \in B_1, Y_2 \in B_2, \cdots Y_n \in B_n) = Pr(A_1 \in B_1) \times Pr(A_2 \in B_2) \times \cdots Pr(A_n \in B_n)$.

–> Let X,Y are independent, then by definition of independence,

$$
\begin{aligned}
&Pr(X \in A, Y \in B) = Pr(X \in B) \times Pr(Y \in B) \\
&\to \int_B \int_A P(x,y)dxdy = \int_A P_X(x)dx \int_B P_Y(y)dy \\
&\to \frac{d^2}{dxdy} \int_B \int_A P(x,y)dxdy = \frac{d^2}{dxdy} \int_A P_X(x)dx \int_B P_Y(y)dy \\
&\to P(x,y) = P_X(x)P_Y(y) \\
&Thus,\ P_{XY}(x,y) = P_X(x)P_Y(y)
\end{aligned}
$$

<– Let $P_{XY}(x,y) = P_X(x)P_Y(y)$, then

$$
\begin{aligned}
\int_A \int_B P_{XY}(x,y)dydx &= \int_A \int_B P_X(x)P_Y(y)dydx \\
&= \int_A P_X(x)dx \int_B P_Y(y)dy \\
\to Pr(X \in A, Y \in B) &= Pr(X \in A)Pr(Y \in B)
\end{aligned}
$$

3.

–> Let X and Y are independent, then by definition, $Pr(X \in A, Y \in B) = Pr(X \in A)Pr(Y \in B)$

and Conditional distribution of x given y, $Pr(X \in A \mid Y \in B) = \frac{Pr(X \in A, Y \in B)}{Pr(Y \in B)} = \frac{Pr(X \in A)Pr(Y \in B)}{Pr(Y \in B)} = Pr(X \in A)$.

$Pr(X \in A) = \int_A P_x(x)dx, Pr(X \in A \mid Y \in B) = \int_A P_{X|Y}(x \mid y)dx$

$\to \frac{d}{dx} Pr(X \in A) = \frac{d}{dx} P(X \in A \mid Y \in B)$

$\to P_X(x) = P_{X|Y}(x \mid y)$

<– Let $P_{X|Y}(x \mid y) = P_X(x)$

$$
\begin{aligned}
Pr(X \in A) = \int_A P_X(x) = \int_A P_{X|Y}(x \mid y)dx &= \int_A \frac{P(x,y)}{P(y)}dx \\
&= \frac{\int_A P(x,y)dx}{\int_R P(x,y)dx} \\
&= \frac{\int_B \int_A P(x,y)dxdy}{\int_B \int_A P(x,y)dxdy} \\
&= \frac{Pr(X \in A, Y \in B)}{Pr(X \in R, Y \in B)} \\
&= \frac{Pr(X \in A, Y \in B)}{Pr(Y \in B)} \\
\to Pr(X \in A)Pr(Y \in B) = Pr(X \in A, Y \in B)
\end{aligned}
$$

which means that X,Y are independent.

4.

Let assume that U,V are continous random variables.

For any u $u \in \mathbb{R}$ and $v \in \mathbb{R}$, let define

$$A = \{X : g(X) \leq u\} \quad and \quad B = \{Y : h(Y) \leq v\}$$

Then the joint CDF of U, and V is

$$\begin{aligned}
F_{U,V}(u,v) &= Pr(U \leq u, V \leq v) \\
&= Pr(g(X) \leq u, h(Y) \leq v) \\
&= Pr(X \in A, Y \in B) \\
&= Pr(X \in A)Pr(Y \in B) \quad by\ independence\ of\ X, Y
\end{aligned}$$

Then, joint pdf of U, and V is

$$\begin{aligned}
P_{U,V}(u,v) &= \frac{d^2}{dudv} F_{U,V}(U,V) \\
&= (\frac{d}{du} Pr(X \in A)) \times (\frac{d}{dv} Pr(Y \in B)) \\
&= (\frac{d}{du} Pr(g(X) \leq u)) \times (\frac{d}{dv} Pr(h(Y) \leq u)) \\
&= (\frac{d}{du} F_U(u)) \times (\frac{d}{dv} F_V(v))) \\
&= P_U(u)P_V(v)
\end{aligned}$$

By the result of Q2, U,V are independent.

In the case of discrete U and V, let define

$$A = \{X : g(X) = u\} \quad and \quad B = \{Y : h(Y) = v\}$$

Then joint pdf

$$\begin{aligned}
P_{U,V}(u,v) &= Pr(g(X) = u, h(Y) = v) \\
&= Pr(X \in A, Y \in B) \\
&= Pr(X \in A)Pr(Y \in B) \\
&= Pr(U = u)Pr(V = v) \\
&= P_U(u)P_V(v)
\end{aligned}$$

With same logic, U and V are independent.

5.

$$F_{y_n}(y) = Pr(Y_n \leq y) = Pr(Y_1, Y_2, \cdots, Y_n \leq y) = \prod_{i=1}^{n} F_{Y_i}(y) = F_Y(y)^n$$

because $Y_1, Y_2, \cdots, Y_n$ are iid.

$P_{Y_n}(y) = \frac{d}{dy} \prod_{i=1}^{n} F_Y(y) = nF_Y(y)^{n-1}P_Y(y)$

$$\begin{aligned}
&1 - F_{Y_1}(y) = Pr(Y_1 > y) = Pr(Y_1, Y_2, \cdots, Y_n > y) = \prod_{i=1}^{n}(1 - F_{Y_i}(y)) = (1 - F_Y(y))^n \\
&F_{Y_1}(y) = 1 - (1 - F_Y(y))^n \\
&P_{Y_1}(y) = \frac{d}{dy} F_{Y_1}(y) = n(1 - F_Y(y))^{n-1}P_Y(y)
\end{aligned}$$

6.

Using result from above question, $P_{Y_1}(y) = n(1 - F_Y(y))^{n-1} P_Y(y)$

(a)

$$P(y) = \lambda e^{-\lambda y},\ F_Y(y) = 1 - e^{\lambda y} \rightarrow P_{Y_1}(y) = ne^{-\lambda(n-1)y}\lambda e^{-\lambda y} = \lambda n e^{-\lambda n y}$$

(b)

$$P(y) = 1, F_Y(y) = y \rightarrow P_{Y_1}(y) = n(1-y)^{n-1}$$

(c)

$$\begin{aligned}
&P(y) = 1, F_Y(y) = \sum_{i=1}^{y} \theta(1-\theta)^{i=1} = 1 - (1-\theta)^y \\
&1 - F_{Y_1}(y) = (1-\theta)^{ny} \\
&F_{Y_1}(y) = 1 - (1-\theta)^{ny} \\
&P_{Y_1}(y) = F_{Y_1}(y) - F_{Y_1}(y-1) = (1-\theta)^{n(y-1)}(1 - (1-\theta)^n)
\end{aligned}$$

7.

(a)

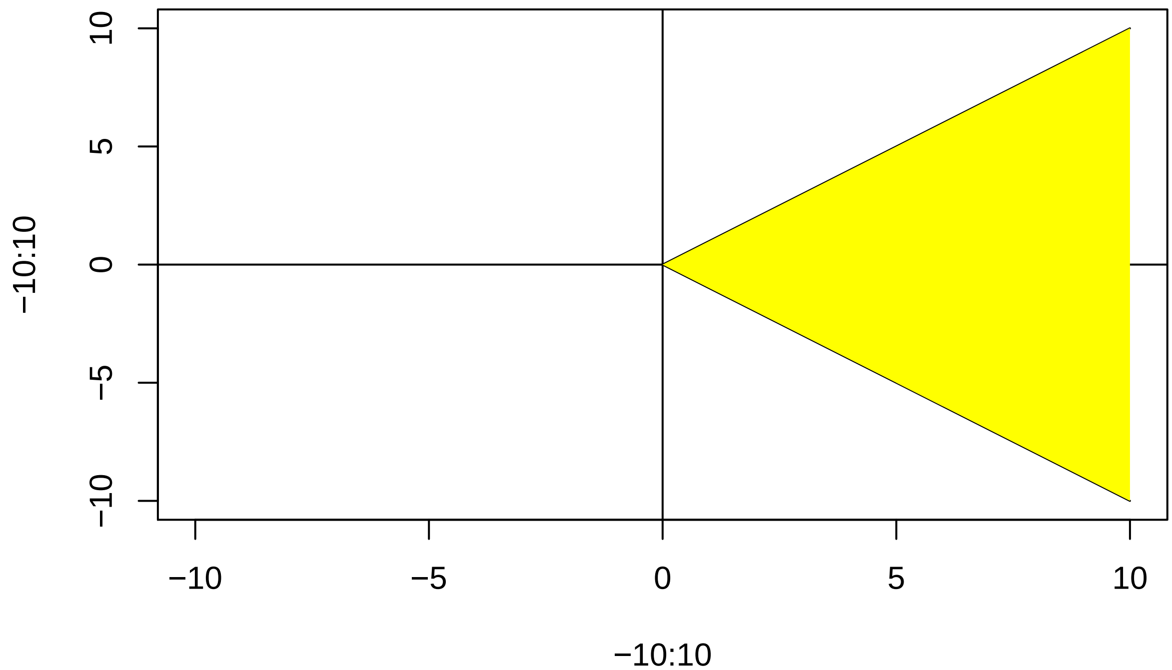
Let $\theta = 1/\lambda$, then P(y) $= \theta e^{-\theta y}$ $F(y) = \int_0^y \theta e^{-\theta \hat{y}} d\hat{y} = 1 - e^{-\theta y} \rightarrow Pr(Y_i \leq y) = 1 - e^{\theta y}$

(b)

$$\begin{aligned}
Y_1 &= \frac{S+D}{2} \in (0, \infty) \\
Y_2 &= \frac{S-D}{2} \in (0, \infty) \\
\rightarrow -S < D &< S \quad and \quad S > 0
\end{aligned}$$

```
x = 0:10
y_high = 0:10
y_low = 0:-10

plot(-10:10,-10:10, type = "n")
lines(x, y_high)
lines(x, y_low)
abline(h = 0, v = 0)
polygon(c(x,rev(x)), c(y_high,rev(y_low)), col = "yellow", border = NA,)
```



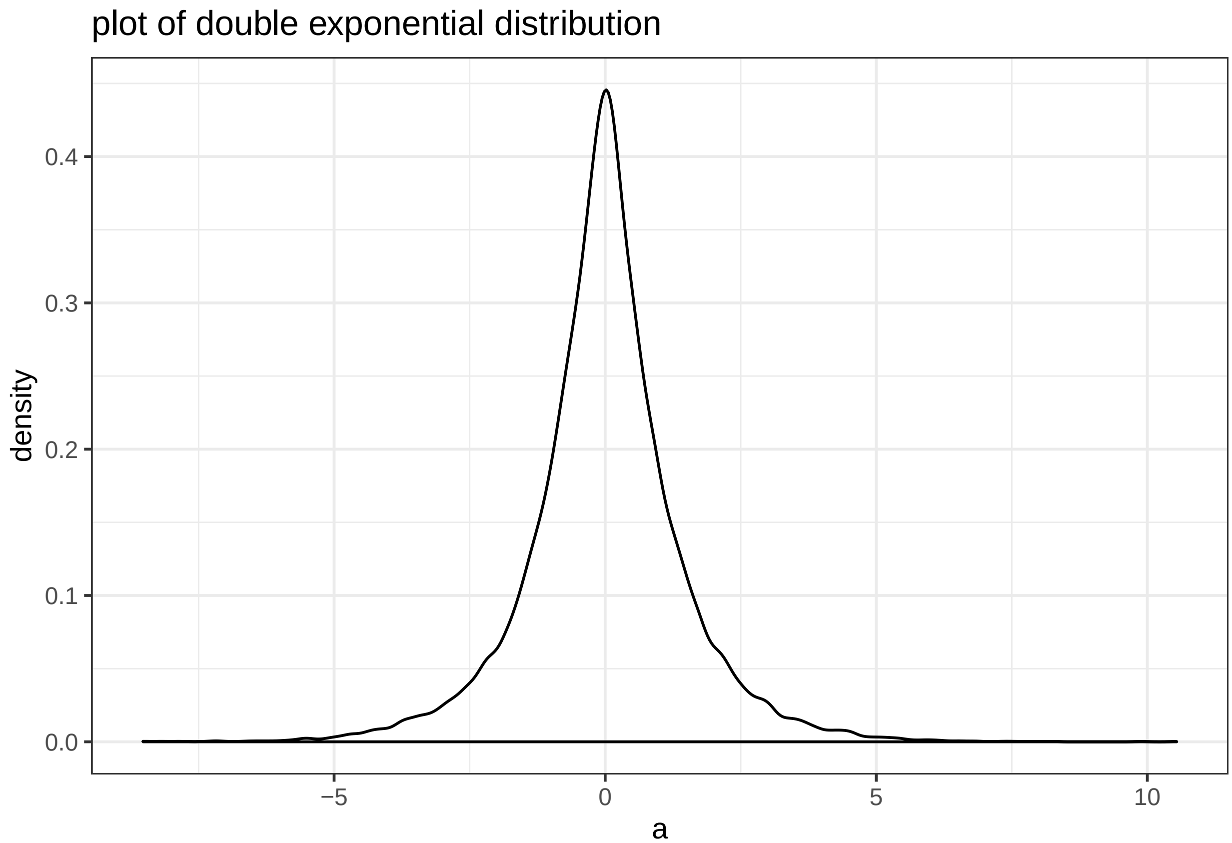
(c)

$$
\begin{aligned}
&P_{Y_1,Y_2} = (y_1, y_2) = \theta^2 e^{-\theta(y_1+y_2)} \\
&P_{S,D} = P_{Y_1,Y_2}\left(\frac{S+D}{2}, \frac{S-D}{2}\right) \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} \\
&= \theta^2 e^{-\theta S} \times 1/2 \\
&P_S(s) = \int_{-S}^{S} \theta^2 e^{-\theta S}/2dD = S\theta^2 e^{-\theta S} \\
&P(D \mid S) = 1/2S
\end{aligned}
$$

(d)

$$
P(D) = \int_{|D|}^{\infty} P(S,D)dS = \int_{|D|}^{\infty} \theta^2 e^{-\theta S} dS = -\frac{1}{2}\theta e^{-\theta S} \mid_{|D|}^{\infty} = \frac{1}{2}\theta e^{-\theta|D|}
$$

```
data = data.frame(a = c(rexp(10000,1),-rexp(10000,1)))
ggplot(data = data, aes(x=a)) + geom_density() +
  labs(title = "plot of double exponential distribution")
```


plot of double exponential distribution
0.4
0.3
0.2
0.1
0.0
density
−5
0
5
10
a