STA 532 Homework1

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HW1 for STA-532

Exercise 1

Let assume $\mathbb{Y} = \mathbb{R}$

(a)

$$\begin{aligned} Pr(Y \in (a, b]) = Pr(a < Y \leq b) &= Pr(Y \leq b) - Pr(Y \leq a) \\ &= F(b) - F(a) \end{aligned}$$

(b)

$$\begin{aligned} Pr(Y \in (a, b)) = Pr(a < Y < b) &= Pr(Y < b) - Pr(Y \leq a) \\ &= Pr(Y \leq b) - Pr(Y = b) - Pr(Y \leq a) \\ &= F(b) - F(a) - Pr(Y = b) \end{aligned}$$

(c)

$$\begin{aligned} Pr(Y \in [a, b]) = Pr(a \leq Y \leq b) &= Pr(Y \leq b) - Pr(Y < a) \\ &= Pr(Y \leq b) - (Pr(Y \leq a) - Pr(Y = a)) \\ &= F(b) - F(a) + Pr(Y = a) \end{aligned}$$

Exercise 2

(a)

$$\begin{aligned} &y \in (0, 1) \rightarrow w = -log(y) \in (0, \infty) \\ &g(y) = -log(y) \; is \; monotonic \; and \; g^{-1}(w) = e^{-w} \end{aligned}$$

$$\begin{aligned} \rightarrow P_W(w) &= P_Y(g^{-1}(w)) \mid \frac{d}{dw} g^{-1}(w) \mid \\ &= 1 \times e^{-w} = e^{-w} \; where \; w \in (0, \infty) \end{aligned}$$

$$\int e^{-w} dw = 1 \rightarrow valid \; pdf$$

(b)

$$
\begin{aligned}
&y \in \mathbb{R} \rightarrow w = y^{-1} \in \mathbb{R} \\
&g(y) = 1/y \ is \ monotonic \rightarrow g^{-1}(w) = 1/w
\end{aligned}
$$

$$
\begin{aligned}
\rightarrow P_W(w) &= P_Y(1/w) \mid \frac{d}{dw}\frac{1}{w} \mid \\
&= 1/(\pi(1+w^{-2})) \times 1/w^{-2} = \frac{1}{\pi(1+w^2)}
\end{aligned}
$$

$$
\int P_W(w)dw = \int P_Y(y)dy = 1
$$

(c)

$$
\begin{aligned}
&y \in \mathbb{R} \rightarrow w = e^y \in (0, \infty) \\
&P_Y(y) = \frac{1}{\sqrt{(2\pi)}} \\
&g(y) = e^y \ is \ monotonic \rightarrow g^{-1}(w) = log(w)
\end{aligned}
$$

$$
\begin{aligned}
\rightarrow P_W(w) &= P_Y(log(w)) \mid \frac{d}{dw} log(w) \mid \\
&= \frac{1}{\sqrt{(2\pi)}} w^{-1} exp\{-\frac{1}{2}(log(w))^2\}
\end{aligned}
$$

(d)

$$
\begin{aligned}
&y \in \mathbb{R} \rightarrow w = y^2 \in [0, \infty) \\
&P_Y(y) = \frac{\Gamma((\nu+1)/2)}{\sqrt{(\nu\pi)}\Gamma(\nu/2)}(1+\frac{y^2}{\nu})^{-(v+1)/2} \\
&g(y) = y^2 \ is \ not \ monotonic
\end{aligned}
$$

$$
\begin{aligned}
\rightarrow F_W(w) &= Pr(W \leq w) \\
&= Pr(Y^2 \leq w) \\
&= Pr(-\sqrt{w} \leq Y \leq \sqrt{w}) \\
\rightarrow P_W(w) &= [P_Y(\sqrt{w}) + P_Y(\sqrt{-w}))] \times 1/2\sqrt{w} \\
&= \frac{\Gamma((\nu+1)/2)}{\sqrt{(\nu\pi)}\Gamma(\nu/2)}(1+\frac{w}{\nu})^{-(v+1)/2} \times w^{-\frac{1}{2}}
\end{aligned}
$$

Exercise 3

$0 = F(-\infty) < F(Y) < F(Y+\lambda) < F(\infty) = 1 \ for \ \lambda > 0$ which means strictly increasing CDF and F_Y^{-1} exists.

(a)

$$\begin{aligned} F_U(u) &= Pr(U \leq u) \\ &= Pr(F_Y(Y) \leq u) \\ &= Pr(Y \leq F_Y^{-1}(u)) \\ &= F_Y(F_Y^{-1}(u)) = u \quad where\ u \in [0, 1] \end{aligned}$$

$$valid\ check: \int udu = 1$$

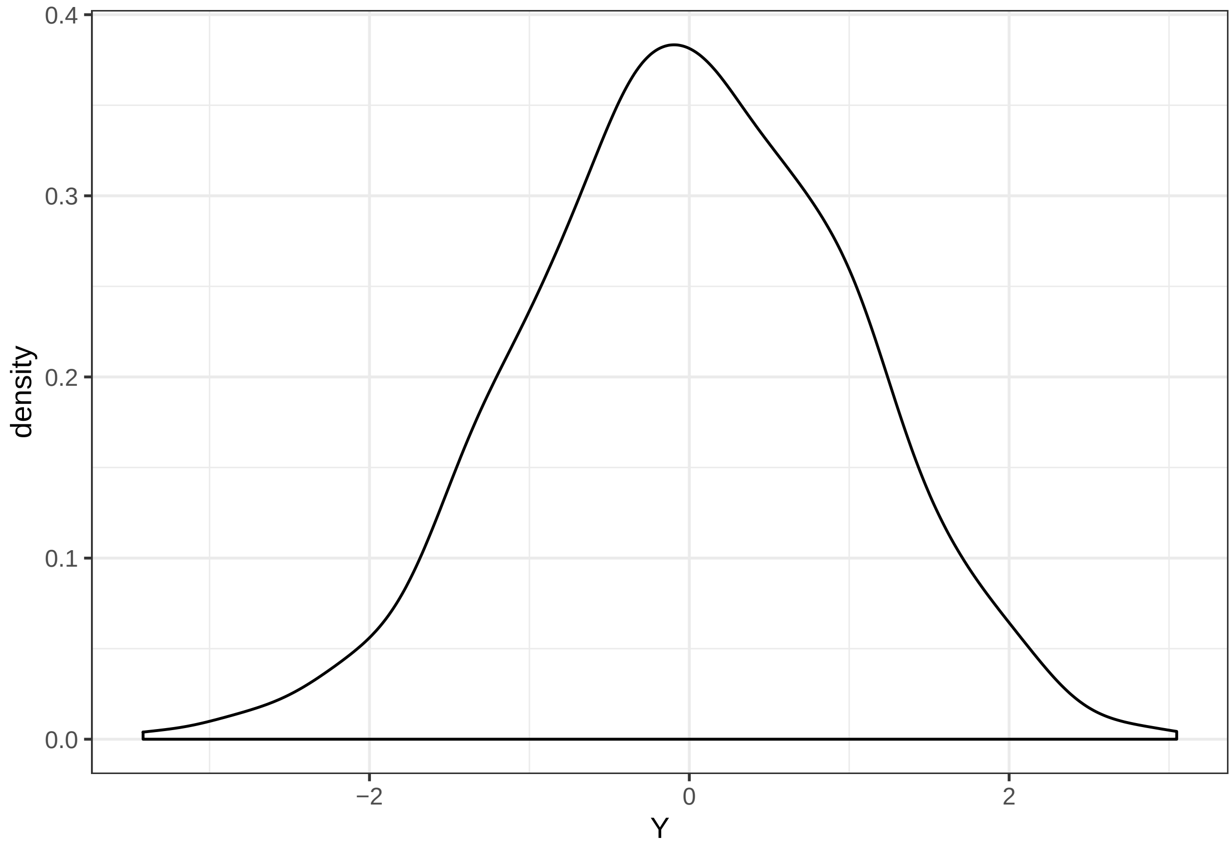
(b)

$$\begin{aligned} F_X(x) &= Pr(X \leq x) \\ &= Pr(F_Y^{-1}(U) \leq x) \\ &= Pr(U \leq F_Y(x)) \\ &= F_U(F_Y(x)) = F_Y(x) \end{aligned}$$

By Thm, X and Y are equivalent in distribution.

(c)

```
Y = qnorm(runif(1000))
library(ggplot2)
ggplot() + geom_density(aes(x = Y))
```



Exercise 4

Define $Pr(X \in A \mid Y = y)$

$$\begin{aligned}
Pr(X \in A \mid Y = y) &= \frac{P(\{w : X(w) \in A, Y(w) = y\})}{P(\{w : Y(w) = y\})} \\
&= \frac{P(\{w : X(w) \in A, Y(w) = y\})}{P(\{w : X \in \mathbb{R}, Y(w) = y\})} \\
&= \frac{\int_A P(x,y)dx}{\int_{\mathbb{R}} P(x,y)dx} \\
&= \frac{\int_A P(x,y)dx}{P(y)} = \int_A \frac{P(x,y)}{P(y)}dx = \int_A P_{X|Y}(x \mid y)dx
\end{aligned}$$

Show that $Pr(X \in A \mid Y = y)P_Y(y)dy = Pr(X \in A)$

$$
\begin{aligned}
\int Pr(X \in A \mid Y = y) P_Y(y) dy &= \int \int_A P_{X|Y}(x \mid y) dx P_Y(y) dy \\
&= \int \int_A P_{X|Y}(x \mid y) P_Y(y) dx dy \\
&= \int_A \int P_{X|Y}(x \mid y) P_Y(y) dy dx \quad (by\ Fubini's\ theorem) \\
&= \int_A \int \frac{P(x,y)}{P_Y(y)} P_Y(y) dy dx \\
&= \int_A \int P(x,y) dy dx \\
&= \int_A P_X(x) dx = Pr(X \in A)
\end{aligned}
$$

Since we integrate(sum) all probability $X \in A$ given Y = y over y, It returns the probability that $X \in A$. Assume that there are 5 people and a cup of water exists. After 5 minutes, probability that cup is empty is sum of probability that one of people drank that water. In mathematical explanation, Let $\mathbb{Y} = \{1, 2, \cdots, 5\}$ and $P(\{w : Y(w) = i\}) = \frac{1}{5}$.

$$
\begin{aligned}
Pr(X \in A) &= P(\{w : X(w) \in A\}) \\
&= P(\cup_{i=1}^{5} \{w : X(w) \in A, y(w) = i\}) \\
&= \sum_{i=1}^{5} P(\{w : X(w) \in A, y(w) = i\}) \\
&= \sum_{i=1}^{5} P(\{w : X(w) \in A\})/5 = P(\{w : X(w) \in A\})
\end{aligned}
$$

Exercise 5

$B_\epsilon = (y - \epsilon, y]$ and $\lim_{\epsilon \to 0} B_\epsilon = \{y\}$

$$
\begin{aligned}
\lim_{\epsilon \to 0} Pr(X \in A \mid Y \in B_\epsilon) &= Pr(X \in A \mid Y = y) \\
&= \frac{P(X \in A, Y = y)}{P(Y = y)} \\
&= \frac{P(X \in A, Y = y)}{P_Y(y)} \\
&= \int_A \frac{P(x,y)}{P_Y(y)} = \int_A P_{X|Y}(x \mid y) dx
\end{aligned}
$$

Exercise 6

$$
\begin{aligned}
P(x) = b^a x^{a-1} e^{-bx} / \Gamma(a) &\to \int_0^\infty P(x) dx = \frac{b^a}{\Gamma(a)} \int x^{a-1} e^{-bx} dx = 1 \\
&\to \int x^{a-1} e^{-bx} dx = \frac{\Gamma(a)}{b^a}
\end{aligned}
$$

$$
P_{Y|X}(y \mid x) = x^c y^{c-1} e^{-xy} / \Gamma(c)
$$

$$
\begin{aligned}
P_Y(y) &= \int P(x,y)dx \\
&= \int P(y \mid x)P(x)dx \\
&= \int x^c y^{c-1} e^{-xy}/\Gamma(c) \times b^a x^{a-1} e^{-bx}/\Gamma(a) dx \\
&= \frac{b^a y^{c-1}}{\Gamma(a)\Gamma(c)} \int x^{a+c-1} e^{-x(b+y)} dx \\
&= \frac{b^a y^{c-1}}{\Gamma(a)\Gamma(c)} \times \frac{\Gamma(a+c)}{(b+y)^{a+c}} \\
&= \frac{\Gamma(a+c)}{\Gamma(a)\Gamma(c)} \times (\frac{b}{b+y})^a \times (\frac{y}{b+y})^{c-1} \times \frac{1}{b+y}
\end{aligned}
$$

$$
\begin{aligned}
P_{X|Y}(x \mid y) &= \frac{P(x,y)}{P_Y(y)} \\
&= \frac{P_{Y|X}(y \mid x)P(x)}{P_Y(y)} \\
&= \frac{\frac{1}{\Gamma(a)\Gamma(c)} \times x^{a+c-1} e^{-x(b+y)}}{\frac{\Gamma(a+c)}{\Gamma(a)\Gamma(c)} \times (\frac{b}{b+y})^a \times (\frac{y}{b+y})^{c-1} \times \frac{1}{b+y}} \\
&= \frac{1}{\Gamma(a+c)} \times \frac{1}{b+y} \times (\frac{bx}{b+y})^a \times (\frac{yx}{b+y})^{c-1} \times e^{-x(b+y)}
\end{aligned}
$$