# STA532 - hw2 sample solution

Sheng Jiang (sj156@duke.edu)

1.(a) $Y|\theta \sim N(\theta, \sigma^2)$ admits the representation: $Y = \theta + \sigma Z$, $Z \sim N(0,1)$

Similarly, $\theta = \mu + \tau Z'$, with $Z' \sim N(0,1)$, $Z' \perp Z$ and $\mu$ fixed.

Then, $Y = \mu + \sigma Z + \tau Z'$, i.e., $Y \sim N(\mu, \sigma^2+\tau^2)$. $p_Y(y) = \frac{1}{\sqrt{2\pi(\sigma^2+\tau^2)}} e^{-\frac{1}{2}(y-\mu)^2/(\sigma^2+\tau^2)}$

(b) $p(\theta|Y) = \frac{p(Y|\theta)\,p(\theta)}{p(Y)} \propto p(Y|\theta)\,p(\theta) \propto e^{-\frac{1}{2}(y-\theta)^2/\sigma^2} e^{-\frac{1}{2}(\theta-\mu)^2/\tau^2}$

$\propto e^{-\frac{1}{2}(\sigma^{-2}+\tau^{-2})(\theta-\tilde{\theta})^2}$, where $\tilde{\theta} = (\sigma^{-2}+\tau^{-2})^{-1}(y/\sigma^2 + \mu/\tau^2)$

$\theta|Y \sim N\left(\frac{\sigma^{-2}}{\sigma^{-2}+\tau^{-2}}y + \frac{\tau^{-2}}{\sigma^{-2}+\tau^{-2}}\mu, (\sigma^{-2}+\tau^{-2})^{-1}\right)$

2. • $X \perp Y \Rightarrow p_{XY}(x,y) = p_X(x)\,p_Y(y)$

pf: By independence, $P(X \le x, Y \le y) = P(X \le x)\,P(Y \le y)$

Denote CDF of X as $F_X(x)$, CDF of Y as $F_Y(y)$, CDF of $(X,Y)$ as $F_{XY}(x,y)$

Then, $F_{X,Y}(x,y) = F_X(x)\,F_Y(y)$, and take derivative w.r.t. $x$ & $y$:

$p_{X,Y}(x,y) = p_X(x)\,p_Y(y)$ if corresponding PDF's exist.

• $p_{XY}(x,y) = p_X(x)\,p_Y(y) \Rightarrow X \perp Y$

pf: Pick any two events A & B,

$P(X\in A, Y \in B) = \int_{x\in A, y\in B} p_{X,Y}(x,y)\,dx\,dy = \int_{x\in A, y\in B} p_X(x)\,p_Y(y)\,dx\,dy$

$= \int_{x\in A} p_X(x)\,dx \int_{y\in B} p_Y(y)\,dy = P(X\in A)\,P(Y\in B)$

3. • $X \perp Y \Rightarrow p_{X|Y}(x|y) = p_X(x)$

pf: $X \perp Y$ implies $P(X\in A, Y\in B) = P(X\in A)\,P(Y\in B)$.

By Bayes rule $P(X\in A, Y\in B) = P(X\in A | Y\in B)\,P(Y\in B)$.

Let $A = (-\infty, x]$, $B = (-\infty, y]$, the above two give $F_X(x) F_Y(y) = F_{X|Y}(x|y) F_Y(y)$

take derivative w.r.t. $x$ & $y$: $p_X(x)\,p_Y(y) = p_{X|Y}(x|y)\,p_Y(y)$

if $p_Y(y) \neq 0$, $p_X(x) = p_{X|Y}(x|y)$

- $p_{X|Y}(x|y) = p_X(x) \Rightarrow X \perp Y.$

pf: By Bayes rule $P(X\in A, Y\in B) = P(X\in A \mid Y\in B)\, P(Y\in B)$,

$P(X\in A \mid Y\in B) = \int_A P(x \mid Y\in B)\,dx = \int_A P_X(x)\,dx = P(X\in A)$

then $P(X\in A, Y\in B) = P(X\in A)\, P(Y\in B)$

4. Note $\{\omega : g(X(\omega)) \in A\} = \{\omega : X(\omega) \in g^{-1}(A)\}$, where $\omega \in \Omega$, and $g^{-1}(\cdot)$ denote the pre-image of mapping $g(\cdot)$.

$P(U\in A, V\in B) = P(g(X)\in A, h(Y)\in B) = P(X\in g^{-1}(A), Y\in h^{-1}(B))$

$= P(X\in g^{-1}(A))\, P(Y\in h^{-1}(B)) = P(g(X)\in A)\, P(h(Y)\in B)$

$= P(U\in A)\, P(V\in B)$

5. $F_{Y_{(1)}}(y) = P(Y_{(1)} \le y) = P(\min_{1\le i\le n}\{Y_i\} \le y) = 1 - P(\min_i\{Y_i\} > y)$

$\overset{\text{by ind.}}{=} 1 - \prod_i P(Y_i > y) = 1 - \prod_i (1 - F_{Y_i}(y))$

$= 1 - (1 - F_Y(y))^n$ by iid.

pdf: $f_{Y_{(1)}}(y) = \frac{d}{dy} F_{Y_{(1)}}(y) = n\, p(y)\, (1 - F_Y(y))^{n-1}$

$F_{Y_{(n)}}(y) = P(Y_{(n)} \le y) = P(\max_i\{Y_i\} \le y) = \prod_i P(Y_i \le y)$ by ind.

$= F(y)^n$ by iid.

pdf: $f_{Y_{(n)}}(y) = \frac{d}{dy} F_{Y_{(n)}}(y) = n\, F(y)^{n-1}\, p(y)$

6. (a) $Y_i \overset{iid}{\sim} Exp(\lambda)$. $p(y) = \lambda e^{-\lambda y}$, $F_Y(y) = 1 - e^{-\lambda y}$. for $y > 0$.

$F_{Y_{(1)}}(y) = 1 - e^{-\lambda n y}$,

$f_{Y_{(1)}}(y) = \lambda n e^{-\lambda n y}$, $Y_{(1)} \sim Exp(n\lambda)$

(b) $Y_i \overset{iid}{\sim} Unif[0,1]$. $p(y) = 1(y\in[0,1])$, $F_Y(y) = y\, 1(0\le y\le 1) + 1(y>1)$

$$F_{Y_{(1)}}(y) = \begin{cases} 1, & y > 1 \\ 1-(1-y)^n, & y\in[0,1] \\ 0, & y \le 0 \end{cases}$$

$f_{Y_{(1)}}(y) = n(1-y)^{n-1}\mathbb{1}(0\le y\le 1)$ , $Y_{(1)} \sim \text{Beta}(1,n)$

(c). $F(y) = \sum_{t=1}^{y} p(t) = \sum_{t=1}^{y} \theta(1-\theta)^{t-1} = \theta \cdot \frac{1-(1-\theta)^y}{1-(1-\theta)} = 1-(1-\theta)^y$

$F_{Y_{(1)}}(y) = 1-(1-\theta)^{ny}$ for $y \in \{1,2,\cdots\}$

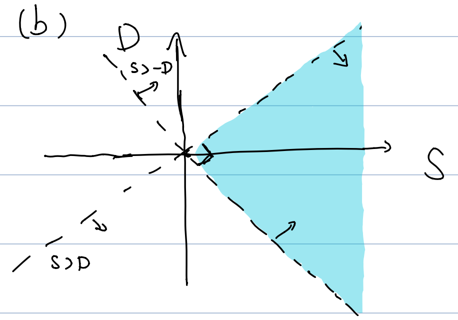
for $y = 1,2,\cdots$, $f_{Y_{(1)}}(y) = F_{Y_{(1)}}(y) - F_{Y_{(1)}}(y-1) = -(1-\theta)^{ny} + (1-\theta)^{n(y-1)} = (1-\theta)^{n(y-1)}(1-(1-\theta)^n)$ ,

$\therefore Y_{(1)} \sim \text{Geo}(1-(1-\theta)^n)$, supported on $\{1,2,\cdots\}$.

r.v. $Y_{(1)}$ is success until $Y_{(1)}$ th trial, with success probability $1-(1-\theta)^n$.

7. (a) $P(Y_i \le y) = \int_0^y e^{-t/\lambda}/\lambda \, dt = \int_0^{y/\lambda} e^{-x}\,dx = 1-e^{-y/\lambda}$

(b)




Note $S = D + 2Y_2 = -D + 2Y_1$.

$Y_1, Y_2 \overset{iid}{\sim} \text{Exp}(\frac{1}{\lambda})$ implies $D \overset{d}{=} -D$

$\therefore$ given $D$, $S$ takes values on $(|D|, +\infty)$

See shaded area in the left picture.

(c). • Note the linear transformation : $\begin{bmatrix} S \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$

Its determinant $\left| \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right| = -1-1 = -2$.

Apply the change-of-variable formula, for $s > 0$, $d \in \mathbb{R}$,

$$p_{S,D}(s,d) = p_{Y_1,Y_2}\left(\tfrac{s+d}{2}, \tfrac{s-d}{2}\right) \cdot \left| \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right|^{-1}$$

$$= \frac{1}{2}\frac{1}{\lambda^2} e^{-\frac{1}{\lambda}\left(\frac{s+d}{2} + \frac{s-d}{2}\right)} \mathbb{1}(s > |d|)$$

$$= \frac{1}{2\lambda^2} e^{-\frac{1}{\lambda}s} \mathbb{1}(s > |d|) ;$$

for $s \le 0$, $d \in \mathbb{R}$, $p_{S,D}(s,d) = 0$.

• Note $p_{D|S}(d|s) \propto p_{S,D}(s,d) \propto \mathbb{1}(|d| < s)$, then, $D|S \sim \text{unif}[-s, s]$.

(d). $p_D(t) = \int_{\mathbb{R}} p_{S,D}(s,t)\,ds = \int_{|t|}^{\infty} \frac{1}{2\lambda^2} e^{-\frac{1}{\lambda}s}\,ds = \frac{1}{2\lambda} e^{-\frac{|t|}{\lambda}}$

A sketch of the density function :