STA 642: Fall 2018 – Homework #2 Exercises

1. Work through the AR(p) notes and especially the AR(2) examples as introduced in class. Among other things, this provides a foundation for a lot of basic/exploratory analysis of time series with AR models, as well as core theory and more building-blocks for general linear state space models. Read ahead into course notes and slides for the coming week(s), and get intimate with relevant sections of the P&W text.

   In particular, read and digest the material on AR(p) model order assessment using marginal likelihoods and information criteria (AIC, BIC) in P&W section 2.3.4. This relates to the question of selection of, and more broadly "inference on", the AR model order p, with more formal inferential/model-based ideas that complement the exploratory uses of ACF, PACF and other exploratory ideas in regression that you might use.

   The course code repository includes the Matlab function `arpcompare.m`. Open and review that function– it simply codes up the formal Bayesian marginal likelihood computation for model order p based on the conditional reference analysis, and also generates the AIC & BIC measures. It is used in examples in the class examples code for AR(p) models.

2. This excercise adds some theoretical structure to the stationary AR(1)$+$noise model (a.k.a. hidden Markov model). As part of this, the example here gets into ARMA models, and you may find the use of the backshift operator strategy useful (– though by no means necessary–) to explore parts of this question.

   In the stationary AR(1)$+$noise model– a first state space/hidden Markov model– we observe

   $$y_t = x_t + \nu_t \quad \text{where} \quad x_t \leftarrow AR(1|(\phi, v))$$

   with $\nu_t \sim N(0, w)$ and assuming $\nu_t \perp\!\!\!\perp \nu_r$ and $\nu_t \perp\!\!\!\perp \epsilon_r$ for all t, r. Clearly $y_t \sim N(0, q)$ with $q = s + w$ where $s = v/(1 - \phi^2)$.

   (a) Show that $y_t = \phi y_{t-1} + \eta_t$ where $\eta_t = \epsilon_t + \nu_t - \phi \nu_{t-1}$.
   (b) Show that the lag-1 correlation in the η_t sequence is $-\phi w/(w(1 + \phi^2) + v)$.
   (c) Find an expression for the lag$-k$ autocorrelation of the y_t process in terms of k, ϕ and the signal:noise ratio s/q. Comment on this result. (We already worked through this in class; do it again!)
   (d) Is y_t an AR(1) process? Is it Markov? Discuss and provide theoretical rationalisation.

3. This question relates to an alternative state space representation of an AR(p) model; this is a special case of the state space representation of ARMA models (P&W, top of page 75; note that the AR(p) is the special case when $q = 0, m = p$ in the notation there).

   Work this exercise explicitly in the case of $p = 2$; the linear algebra in this special case is easy and the special case illuminating of the more general AR(p) case. Provide solutions to the case of $p = 2$ for assessment. The general case is just a bit more linear algebra and will be given bonus credit, but is otherwise optional.

In the standard state space representation of the AR(p) model we have state vector $\mathbf{x}_t = (y_t, y_{t-1}, \ldots, y_{t-p+1})'$ and model equations $y_t = \mathbf{F}'\mathbf{x}_t$ and $\mathbf{x}_t = \mathbf{G}\mathbf{x}_{t-1} + \mathbf{F}\epsilon_t$ where

$$\mathbf{F} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{G} = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

with AR parameters $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)'$ and innovations $\epsilon_t \sim N(0, v)$.

Define the $p \times p$ symmetric matrix $\mathbf{A}$ by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \phi_2 & \phi_3 & \phi_4 & \cdots & \phi_{p-1} & \phi_p \\ 0 & \phi_3 & \phi_4 & \phi_5 & \cdots & \phi_p & 0 \\ \vdots & \vdots & \vdots & & \iddots & \iddots & \vdots \\ \vdots & \vdots & \vdots & \iddots & \iddots & & \vdots \\ 0 & \phi_{p-1} & \phi_p & 0 & \cdots & \cdots & 0 \\ 0 & \phi_p & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

(a) Verify that the matrix product $\mathbf{AG}$ is given by

$$\mathbf{AG} = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & \cdots & \phi_{p-1} & \phi_p \\ \phi_2 & \phi_3 & \phi_4 & \phi_5 & \cdots & \phi_p & 0 \\ \phi_3 & \phi_4 & \phi_5 & \phi_5 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \iddots & \iddots & \vdots \\ \vdots & \vdots & \vdots & \iddots & \iddots & & \vdots \\ \phi_{p-1} & \phi_p & 0 & 0 & \cdots & \cdots & 0 \\ \phi_p & 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

noting that this is also symmetric. $G'A' = AG$

(b) Show or deduce that:

i. For a proper AR(p) model in which $\phi_p \neq 0$, then $|\mathbf{A}| \neq 0$ so that $\mathbf{A}$ is non-singular.

ii. $\mathbf{AGA}^{-1} = \mathbf{G}'$ (*you can do this* without *trying to invert* $\mathbf{A}$).

iii. $\mathbf{AF} = \mathbf{F}$ and, as a result, $\mathbf{F}' = \mathbf{F}'\mathbf{A}^{-1}$.

(c) Hence show that an equivalent state space AR(p) form is given by $y_t = \mathbf{F}'\mathbf{z}_t$ and $\mathbf{z}_t = \mathbf{G}'\mathbf{z}_{t-1} + \mathbf{F}\epsilon_t$ based on a new $p \times 1$ state vector $\mathbf{z}_t = \mathbf{A}\mathbf{x}_t$ and where the state evolution matrix is $\mathbf{G}'$, i.e.,

$$\mathbf{G}' = \begin{pmatrix} \phi_1 & 1 & 0 & \cdots & 0 \\ \phi_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \phi_{p-1} & 0 & 0 & \cdots & 1 \\ \phi_p & 0 & 0 & \cdots & 0 \end{pmatrix}$$

(d) What is the interpretation of the elements of the transformed state vector $\mathbf{z}_t$?

$Y_t = F x_t$ and also $Y_t = F Z_t$

$x_t = G x_{t-1} + F \varepsilon_t$ $\quad Z_t = G' Z_{t-1} + F \varepsilon_t$

Both representation are state-space representation of y_t AR(p) process
It means that space-state representation of process is not unique and
Z_t is another state vector of y_t

$$Z_t = A x_t = \begin{pmatrix} y_t \\ \phi_2 y_{t-1} + \phi_3 y_{t-2} \cdots \phi_p y_{t-p+1} \\ \phi_3 y_{t-1} + \phi_4 y_{t-3} \cdots \phi_p y_{t-p+2} \\ \vdots \\ \phi_p y_{t-1} \end{pmatrix} \quad \text{when } p = 2, \ \begin{pmatrix} Z_t \\ Z_{t-1} \end{pmatrix} = \begin{pmatrix} y_t \\ \phi_2 y_{t-1} \end{pmatrix}$$

By above equations, we can interpret Z_t as weighted state vector of Y_t whose weights are given by estimated ϕ
This fact make it clear that unknown state underlying y_t's can be estimated with observation y_t

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \phi_2 & \phi_3 & \phi_4 & \cdots & \phi_{p-1} & \phi_p \\ 0 & \phi_3 & \phi_4 & \phi_5 & \cdots & \phi_p & 0 \\ \vdots & \vdots & \vdots & & \iddots & \iddots & \vdots \\ \vdots & \vdots & \vdots & \iddots & \iddots & & \vdots \\ 0 & \phi_{p-1} & \phi_p & 0 & \cdots & \cdots & 0 \\ 0 & \phi_p & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

2. This excercise adds some theoretical structure to the stationary AR(1)+noise model (a.k.a. hidden Markov model). As part of this, the example here gets into ARMA models, and you may find the use of the backshift operator strategy useful (– though by no means necessary–) to explore parts of this question.

In the stationary AR(1)+noise model– a first state space/hidden Markov model– we observe

$$y_t = x_t + \nu_t \quad \text{where} \quad x_t \leftarrow AR(1|(\phi, v))$$

with $\nu_t \sim N(0, w)$ and assuming $\nu_t \perp\!\!\!\perp \nu_r$ and $\nu_t \perp\!\!\!\perp \epsilon_r$ for all t, r. Clearly $y_t \sim N(0, q)$ with $q = s + w$ where $s = v/(1 - \phi^2)$.

(a) Show that $y_t = \phi y_{t-1} + \eta_t$ where $\eta_t = \epsilon_t + \nu_t - \phi \nu_{t-1}$.

(b) Show that the lag-1 correlation in the η_t sequence is $-\phi w/(w(1 + \phi^2) + v)$.

(c) Find an expression for the lag$-k$ autocorrelation of the y_t process in terms of k, ϕ and the signal:noise ratio s/q. Comment on this result. (We already worked through this in class; do it again!)

(d) Is y_t an AR(1) process? Is it Markov? Discuss and provide theoretical rationalisation.

(a) $y_t = x_t + V_t$, $x_t = \phi x_{t-1} + \varepsilon_t$

$y_{t-1} = x_{t-1} + V_{t-1}$, $x_{t-1} = y_{t-1} - V_{t-1}$

$\Rightarrow y_t = \phi y_{t-1} - \phi V_{t-1} + V_t + \varepsilon_t = \phi y_{t-1} + \eta_t$

(b) $Cov(\eta_t, \eta_{t-1}) = E(\varepsilon_t + V_t - \phi V_{t-1}, \varepsilon_{t-1} + V_{t-1} - \phi V_{t-2})$

$= -\phi E(V_{t-1}^2) = -\phi w$

$Var(\eta_t) = Var(\varepsilon_t) + Var(V_t) + \phi^2 Var(V_{t-1}) = v + w + \phi^2 w = (1+\phi^2)w + v$

$= Var(\eta_{t-1})$

$Corr(\eta_t, \eta_{t-1}) = \dfrac{Cov(\eta_t, \eta_{t-1})}{\sqrt{Var(\eta_t)}\sqrt{Var(\eta_{t-1})}} = \dfrac{-\phi w}{(1+\phi^2)w + v}$

(c) $x_t = \phi^k x_{t-k} + \varepsilon_t + \phi \varepsilon_{t-1} \cdots \phi^{k-1} \varepsilon_{t-k+1}$

$y_t = \phi^k x_{t-k} + \varepsilon_t \cdots \phi^{k-1} \varepsilon_{t-k+1} + V_t$.

$y_{t-k} = x_{t-k} + \varepsilon_{t-k} + V_{t-k}$

$Cov(y_t, y_{t-k}) = \phi^k Var(x_{t-k}) = \phi^k s$

$Var(y_t) = Var(y_{t-k}) = q.$ $\Rightarrow Corr(y_t, y_{t-k}) = \phi^k \frac{s}{q}$

(d) Let we have two conditional distribution of y_t given only y_{t-1} and given $y_{t-1} \cdots y_1$. If two distributions are same, y_t is Markov

$y_t = \phi y_{t-1} - \phi V_{t-1} + V_t + \varepsilon_t = \eta_t$

$P(y_t | y_{t-1}, v, w) \sim N(\phi y_{t-1}, (1+\phi^2)w + v)$

$P(y_t | y_{t-1}, \cdots y_1, v, w) \sim N(\phi y_{t-1}, (1+\phi^2)w + v)$ because y_t doesn't change at all by given $y_{t-2} \cdots y_1$

It means that y_t is Markov

By def of AR(1) process $y_t = \phi y_{t-1} + \eta_t$ where $\eta_t \perp\!\!\!\perp \eta_k$ for all $k \neq t$

But at (b) we found correlation between y_t, η_{t-1}. Thus y_t is not AR(1) process.

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Work this exercise explicitly in the case of $p = 2$; the linear algebra in this special case is easy and the special case illuminating of the more general AR(p) case. Provide solutions to the case of $p = 2$ for assessment. The general case is just a bit more linear algebra and will be given bonus credit, but is otherwise optional.

x_t = State vector (y_t, y_{t-1}), $F = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $G = \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix}$, $\phi = (\phi_1, \phi_2)'$

$\varepsilon_t \sim N(0, V)$

$y_t = F' x_t$ $\qquad A = \begin{pmatrix} 1 & 0 \\ 0 & \phi_2 \end{pmatrix}$

$x_t = G x_t$

$$\mathbf{F} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{G} = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \phi_2 & \phi_3 & \phi_4 & \cdots & \phi_{p-1} & \phi_p \\ 0 & \phi_3 & \phi_4 & \phi_5 & \cdots & \phi_p & 0 \\ \vdots & \vdots & \vdots & & ⋰ & ⋰ & \vdots \\ \vdots & \vdots & \vdots & ⋰ & ⋰ & & \vdots \\ 0 & \phi_{p-1} & \phi_p & 0 & \cdots & \cdots & 0 \\ 0 & \phi_p & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$x_t = (y_t \cdots y_{t-p+1})'$ $\quad \phi = (\phi_1 \cdots \phi_p)'$

(a) Verify that the matrix product $\mathbf{AG}$ is given by

$$\mathbf{AG} = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & \cdots & \phi_{p-1} & \phi_p \\ \phi_2 & \phi_3 & \phi_4 & \phi_5 & \cdots & \phi_p & 0 \\ \phi_3 & \phi_4 & \phi_5 & \phi_5 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & ⋰ & ⋰ & \vdots \\ \vdots & \vdots & \vdots & ⋰ & ⋰ & & \vdots \\ \phi_{p-1} & \phi_p & 0 & 0 & \cdots & \cdots & 0 \\ \phi_p & 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$AG = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \vdots & \phi_2 & \phi_3 & \cdots & \phi_p \\ & \phi_3 & & ⋰ & \\ & \vdots & ⋰ & ⋰ & \\ 0 & \phi_p & \phi_p & & 0 \\ & & 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{bmatrix} 1 & \underset{1\times p-1}{0} \\ \underset{p-1\times 1}{0} & \underset{p-1\times p-1}{k} \end{bmatrix} \text{ where } k = \begin{pmatrix} \phi_2 & \cdots & \phi_p \\ \vdots & ⋰ & \vdots \\ \phi_p & \cdots & 0 \end{pmatrix} \qquad G = \begin{bmatrix} \underset{1\times p-1}{J} & \phi_p \\ \underset{p-1\times p-1}{I} & 0 \end{bmatrix} \text{ where } J = (\phi_1 \cdots \phi_{p-1})$$

$$AG = \begin{bmatrix} J & \phi_p \\ K & 0 \end{bmatrix} = \begin{pmatrix} \phi_1 & \cdots & \phi_{p-1} & \phi_p \\ \phi_2 & \cdots & \phi_p & \\ & ⋰ & & \\ \phi_p & \cdots & & \end{pmatrix}$$

when $p = 2$, $AG = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_2 & 0 \end{pmatrix}$

(b) Show or deduce that:

i. For a proper AR(p) model in which $\phi_p \neq 0$, then $|\mathbf{A}| \neq 0$ so that $\mathbf{A}$ is non-singular.
ii. $\mathbf{A}\mathbf{G}\mathbf{A}^{-1} = \mathbf{G}'$ (*you can do this* without *trying to invert* $\mathbf{A}$).
iii. $\mathbf{A}\mathbf{F} = \mathbf{F}$ and, as a result, $\mathbf{F}' = \mathbf{F}'\mathbf{A}^{-1}$.

1. $A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \phi_2 & \cdots & \phi_p \\ & \vdots & \iddots & \\ 0 & \phi_p & & \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & K_2 \end{pmatrix}$ $\quad |A| = |K_2|$

$|K_2| = \begin{vmatrix} \phi_2 & \cdots & \phi_p \\ \phi_3 & \iddots & 0 \\ \vdots & \phi_p & 0 \\ \phi_p & & 0 \end{vmatrix} = \phi_p |K_3|$ $\quad$ where $K_3 = \begin{pmatrix} \phi_3 & \cdots & \phi_p \\ \vdots & \iddots & \\ \phi_p & & \end{pmatrix}$

$|K_3| = \begin{vmatrix} \phi_3 & \cdots & \phi_p \\ \vdots & \iddots & 0 \\ \phi_p & & 0 \end{vmatrix} = \phi_p |K_4|$ $\quad$ where $K_4 = \begin{pmatrix} \phi_4 & \cdots & \phi_p \end{pmatrix}$ $\Rightarrow |A| = \phi_p^{p-1}$

$\vdots$

$|K_{p-1}| = \begin{vmatrix} \phi_{p-1} & \phi_p \\ \phi_{p-1} & 0 \end{vmatrix} = \phi_p^2 \Rightarrow |A| = \phi_p^{p-1}$ $\quad$ when $p=2$ $\quad |A| = \phi_2 \neq 0$

2. $A = \begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix}$ $\quad G = \begin{pmatrix} J & Q_p \\ I & 0 \end{pmatrix}$ $\quad AA' = \begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & K \cdot K \end{pmatrix}$

$AGA^{-1} = G'$

$\Rightarrow AG = G'A$ $\quad AG = \begin{pmatrix} J & Q_p \\ K & 0 \end{pmatrix}$ $\quad G'A = \begin{pmatrix} J' & I \\ Q_p' & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix} = \begin{pmatrix} J' & K \\ Q_p & 0 \end{pmatrix}$

$AG = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_p \\ \vdots & \vdots & \iddots & 0 \\ & & & 0 \\ \phi_p & \phi_p & \cdots & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad G'A = \begin{pmatrix} \phi_1 & \cdots & \phi_p \\ \vdots & & 0 \\ \vdots & \iddots & 0 \\ \phi_p & 0 & 0 \end{pmatrix}$ $\quad AG = G'A' \Rightarrow AG = G'A$

$\Rightarrow AGA^{-1} = G'$

$p=2$ $\quad AG = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_2 & 0 \end{pmatrix}$ $\quad G'A = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_2 & 0 \end{pmatrix}$

3. $F = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\quad AF = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = F$

$F' = F'A'$ $\quad A' = \begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix}$ $\quad K = \text{symm}$

$= F'A$

By properties of symmetric matrix

$F = A^{-1}F$ $\quad F' = F'A^{-1}$ $\quad (A^{-1})' = (A')^{-1}$

(c) $y_t = F' x_t$

$x_t = G x_{t-1} + F \varepsilon_t \quad \Rightarrow \quad A x_t = A G x_{t-1} + A F \varepsilon_t$

$\qquad\qquad\qquad\qquad\qquad\qquad = A G A^{-1} A x_{t-1} + A F \varepsilon_t$

$A x_t = Z_t \quad \Rightarrow \quad Z_t = A G A^{-1} Z_{t-1} + F \varepsilon_t.$

$\qquad\qquad\qquad\qquad\qquad\qquad = G' Z_{t-1} + F \varepsilon_t$

(d) $A x_t = Z_t$ means that

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \phi_2 & \cdots & \phi_p \\ & \phi_3 & \cdots \phi_p & 0 \\ & \vdots & \ddots & \vdots \\ 0 & \phi_p & 0 \cdots & 0 \end{pmatrix} \begin{pmatrix} y_t \\ \vdots \\ \\ y_{t-p+1} \end{pmatrix} = \begin{pmatrix} y_t \\ \phi_2 y_{t-1} + \phi_3 y_{t-2} \cdots \phi_p y_{t-p+1} \\ \phi_3 y_{t-2} \cdots\cdots \phi_p y_{t-p+2} \\ \\ \end{pmatrix}$$

$y_t \Rightarrow y_t$

$y_{t-1} \Rightarrow \phi_2 y_{t-1} + \phi_3 y_{t-2} \cdots \phi_p y_{t-p+1}$

$y_{t-2} \Rightarrow \phi_3 y_{t-1} + \phi_4 y_{t-a} \cdots \phi_p y_{t-p+2}$

when $p = 2$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & \phi_2 \end{pmatrix} \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} = \begin{pmatrix} y_t \\ \phi_2 y_{t-1} \end{pmatrix} = \begin{pmatrix} y_t \\ y_{t+1} - \phi_1 y_t \end{pmatrix}$

$=\quad ($

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \phi_2 & \cdots & \phi_p \\ & \phi_3 & \cdots \phi_p & 0 \\ & \vdots & \ddots & \vdots \\ 0 & \phi_p & 0 \cdots & 0 \end{pmatrix} \begin{pmatrix} y_t \\ \vdots \\ \\ y_{t-p+1} \end{pmatrix} = \begin{pmatrix} y_t \\ \phi_2 y_{t-1} + \phi_3 y_{t-2} \cdots \phi_p y_{t-p+1} \\ \phi_3 y_{t-2} \cdots\cdots \phi_p y_{t-p+2} \\ \\ \end{pmatrix}$$

$$Z_t = A x_t \Rightarrow \begin{pmatrix} Z_t \\ Z_{t-1} \\ \vdots \\ Z_{t-p+1} \end{pmatrix} = \begin{pmatrix} y_t \\ \phi_2 y_{t-1} + \phi_3 y_{t-2} \cdots \phi_p y_{t-p+1} \\ \phi_3 y_{t-1} + \phi_4 y_{t-2} \cdots \phi_p y_{t-p+2} \\ \vdots \\ \phi_p y_{t-1} \end{pmatrix}$$

$Z_{t-1} = y_{t+1} - \phi_1 y_t$

$Z_{t-2} = y_{t+2} - \phi_1 y_{t+1} - \phi_2 y_t$

$\vdots$

$Z_{t-p+1} = y_{t+p} - \phi_1 y_{t+p-1} \cdots \phi_p y_t$