Quiz 1 Sample Solution

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1. Since W is a positive r.v.,

$$EW = \int_0^\infty w\,p(w)\,dw \geq \int_c^\infty w\,p(w)\,dw \geq c \cdot \int_c^\infty p(w)\,dw = c\,P(W \geq c)$$

2. By Q1 and definition of variance,

$$P(|X-m| \geq \varepsilon) = P((X-m)^2 \geq \varepsilon^2) \leq \frac{E(X-m)^2}{\varepsilon^2} = \frac{V}{\varepsilon^2}$$

3. By linearity of expectation, $E(\bar{Y}) = E(\frac{1}{n}\sum_{i=1}^n Y_i) = \frac{1}{n}\sum_{i=1}^n EY_i = \mu$

By ind. of $\{Y_i\}$, $\operatorname{cov}(Y_i, Y_j) = 0$; note $V(Y_i) = \sigma^2 \quad \forall i = 1, \dots, n$,

then $V(\bar{Y}) = V(\frac{1}{n}\sum_{i=1}^n Y_i) = \frac{1}{n^2}\left(\sum_{i=1}^n V(Y_i) + \sum_{i \neq j} \operatorname{cov}(Y_i, Y_j)\right) = \frac{1}{n}\sigma^2$.

4. By chebyshev, $P(|\bar{Y}-\mu| > \varepsilon) \leq \frac{\sigma^2/n}{\varepsilon} \to 0$

5. $\sigma^2 = V(Y_i) = E[(Y_i - EY_i)^2] = E[Y_i^2] - (EY_i)^2 = m_2 - \mu^2$

6. $\hat{\sigma}^2 = \frac{1}{n}\sum Y_i^2 - (\frac{1}{n}\sum Y_i)^2$, which is a combination of consistent estimators for m_2 and μ.

By wLLN, $\frac{1}{n}\sum Y_i^2 \xrightarrow{P} EY_i^2 = m_2$; $\frac{1}{n}\sum Y_i \xrightarrow{P} EY_i = \mu$.

By cts mapping thm, $\hat{\sigma}^2 = \frac{1}{n}\sum Y_i^2 - (\frac{1}{n}\sum Y_i)^2 \xrightarrow{P} m_2 - \mu^2$

For an alternative approach, see hw6 Q1.

7. By CLT, $\sqrt{n}(\bar{Y}-\mu)/\sigma \xrightarrow{d} N(0,1)$

8. $\sqrt{n}(\bar{Y}-\mu)/\hat{\sigma} = \frac{\sqrt{n}(\bar{Y}-\mu)}{\sigma} \cdot \frac{\sigma}{\hat{\sigma}}$

By cts mapping thm, $\frac{\sigma}{\hat{\sigma}} \xrightarrow{p} 1$.

Note $\sqrt{n}(\bar{Y}-\mu) \xrightarrow{d} N(0,1)$, by Slutsky thm, $\frac{\sqrt{n}(\bar{Y}-\mu)}{\hat{\sigma}} \xrightarrow{d} N(0,1)$

9. $\{\mu \in (\bar{Y} - \frac{2\hat{\sigma}}{\sqrt{n}}, \bar{Y} + \frac{2\hat{\sigma}}{\sqrt{n}})\} = \{\sqrt{n}\,|\bar{Y}-\mu|/\hat{\sigma} \le 2\}$

From Q8, $\sqrt{n}\frac{(\bar{Y}-\mu)}{\hat{\sigma}} \xrightarrow{d} N(0,1)$, then $P(\sqrt{n}\,|\bar{Y}-\mu|/\hat{\sigma} \le 2) \gtrsim 0.95$.

That is, $P(\mu \in (\bar{Y} - \frac{2\hat{\sigma}}{\sqrt{n}}, \bar{Y} + \frac{2\hat{\sigma}}{\sqrt{n}})) \gtrsim 0.95$

10. Define $\hat{\theta} = a + b\bar{Y}$. $E\hat{\theta} = a + b\mu$, $V(\hat{\theta}) = b^2 V(\bar{Y}) = b^2 n^{-1}\sigma^2$

By Delta method, $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0, b^2\sigma^2)$

Similar to Q8, $\sqrt{n}\frac{(\hat{\theta}-\theta)}{b\,\hat{\sigma}} \xrightarrow{d} N(0,1)$

Based on the distribution theory, 95% CI for θ is $(\hat{\theta} - 2\frac{|b|\hat{\sigma}}{\sqrt{n}}, \hat{\theta} + 2\frac{|b|\hat{\sigma}}{\sqrt{n}})$.

For an alternative approach, note the map $f: x \mapsto a+bx$ is monotonic and bijective, then $\{\xi \in A\} = \{f(\xi) \in f(A)\}$.

$\{\mu \in (\bar{Y} - \frac{2\hat{\sigma}}{\sqrt{n}}, \bar{Y} + \frac{2\hat{\sigma}}{\sqrt{n}})\} = \{a + b\mu \in (a + b\bar{Y} - \frac{2|b|\hat{\sigma}}{\sqrt{n}}, a + b\bar{Y} + \frac{2|b|\hat{\sigma}}{\sqrt{n}})\}$

From Q9, $P(\mu \in (\bar{Y} - \frac{2\hat{\sigma}}{\sqrt{n}}, \bar{Y} + \frac{2\hat{\sigma}}{\sqrt{n}})) \approx 0.95$,

then 95% CI for $\theta = a + b\mu$ is $(a + b\bar{Y} - \frac{2|b|\hat{\sigma}}{\sqrt{n}}, a + b\bar{Y} + \frac{2|b|\hat{\sigma}}{\sqrt{n}})$.