STA 642 Homework2

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HW2 for STA-642

Exercise 2

(a)

$$
\begin{aligned}
&y_t = x_t + v_t,\ x_t = \phi x_{t-1} + \epsilon_t \\
&y_{t-1} = x_{t-1} + v_{t-1},\ x_{t-1} = yt - 1 - v_{t-1} \\
&\rightarrow y_t = \phi y_{t-1} - \phi v_{t-1} + v_t + \epsilon_t = \phi y_{t-1} + \eta_t
\end{aligned}
$$

(b)

$$
\begin{aligned}
&Cov(\eta_t, \eta_{t-1}) = E(\epsilon_t + v_t - \phi v_{t-1}, \epsilon_{t-1}, v_{t-1}, -\phi_{t-2}) = -\phi E(v_{t-1}^2) = -\phi w \\
&Var(\eta_t) = Var(\epsilon_t) + Var(v_t) + \phi^2 Var(v_{t-1}) = v + w + \phi^2 w = (1+\phi^2)w + v = Var(\eta_{t-1}) \\
&Cor(\eta_t, \eta_{t-1}) = \frac{-\phi w}{(1+\phi^2)w + v}
\end{aligned}
$$

(c)

$$
\begin{aligned}
&x_t = \phi^k x_{t-k} + \epsilon_t + \phi\epsilon_{t-1} + \cdots \phi^{k-1}\epsilon_{t-k+1} \\
&y_t = \phi^k x_{t-k} + \epsilon_t + \phi\epsilon_{t-1} + \cdots \phi^{k-1}\epsilon_{t-k+1} + v_t \\
&y_{t-k} = x_{t-k} + \epsilon_{t-k} + v_{t-k} \\
&Cov(y_t, y_{t-k}) = \phi^k Var(x_{t-k}) = \phi^k s \\
&Var(y_t) = Var(y_{t-k}) = q \rightarrow Cor(y_t, y_{t-k}) = \phi^k s/q
\end{aligned}
$$

(d)

Let we have two conditional distribution of y_t given only y_{t-1} and given $y_{t-1} \cdots y_1$. If two distributions are same, y_t is markov

$$
\begin{aligned}
&y_t = \phi y_{t-1} + \eta_t \\
&P(y_t \mid y_{t-1}, v, w) \sim N(\phi y_{t-1}, (1+\phi^2)w + v) \\
&P(y_t \mid y_{t-1}, \cdots, y_1, v, w) \sim N(\phi y_{t-1}, (1+\phi^2)w + v)
\end{aligned}
$$

Because y_t does not change at all by given $y_{t-2} \cdots y_1$ and it means that y_t is Markov. By definition of AR(1) process, $y_t = \phi y_{t-1} + \eta_t$ where $\eta_t \perp \eta_k$ for all k $\neq$ t. But at (b), we found correlation between η_t, η_{t-1}. Thus y_t is not AR(1) process.

Exercise 3

(a)

When p = 2

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \phi_2 \end{bmatrix} \quad and \quad G = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}$$
$$\to AG = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_2 & 0 \end{bmatrix}$$

For more general case,

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \phi_2 & \phi_3 & \cdots & \phi_p & 0 \\ 0 & \phi_3 & \phi_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \phi_{p-1} & \phi_p & \cdots & \cdots & 0 \\ 0 & \phi_p & 0 & \cdots & \cdots & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & K \end{bmatrix} \quad where \quad K = \begin{bmatrix} \phi_2 & \phi_3 & \cdots & \cdots & \phi_p \\ \phi_3 & \phi_4 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \phi_{p-1} & \phi_p & \cdots & \cdots & 0 \\ \phi_p & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} J & \phi_p \\ I & 0 \end{bmatrix} \quad where \quad J = (\phi_1, \phi_2, \cdots, \phi_{0-1})$$

$$Then \quad AG = \begin{bmatrix} 1 & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} J & \phi_p \\ I & 0 \end{bmatrix} = \begin{bmatrix} J & \phi_p \\ K & 0 \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ \phi_2 & \phi_3 & \phi_4 & \cdots & \phi_p & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{p-1} & \phi_p & 0 & \cdots & 0 & 0 \\ \phi_p & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

(b)

i.

When p = 2

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \phi_2 \end{bmatrix} \quad then \quad | A |= \phi_2$$

if $\phi_2 \neq 0$then $| A | \neq 0$

For the general case,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & K_2 \end{bmatrix} \quad where \quad K_2 = \begin{bmatrix} \phi_2 & \cdots & \phi_p \\ \phi_3 & \cdots & 0 \\ \vdots & \ddots & 0 \\ \phi_p & \cdots & 0 \end{bmatrix}$$

$$Then \quad | A |=| K_2 | \quad and \quad | K_2 |= \phi_p | K_3 | \quad where \quad K_3 = \begin{bmatrix} \phi_3 & \cdots & \phi_p \\ \phi_4 & \cdots & 0 \\ \vdots & \ddots & 0 \\ \phi_p & \cdots & 0 \end{bmatrix}$$

$$| K_3 |= \phi_p | K_4 | \cdots \to | A |= \phi_p^p$$

Thus if $\phi_p \neq 0$then $\mid A \mid \neq 0$

ii.

We could know that A is symmetric and AG is also symmetric. Thus

$$AG = G^T A^T \rightarrow AG = G^T A \rightarrow AGA^{-1} = G^T$$

iii.

$$AF = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \phi_2 & \phi_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \phi_p & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = F$$

$$AF = F \rightarrow F = A^{-1}F \rightarrow F^T = F^T(A^{-1})^T \quad \rightarrow \quad F^T = F^T A^{-1}$$
$$for\ A\ is\ symmetric$$

(c)

$$\begin{aligned} & y_t = F^T x_t \\ & x_t = Gx_{t-1} + F\epsilon_t \\ & \rightarrow z_t = Ax_t = AGx_{t-1} + AF\epsilon_t \\ & = AGA^{-1}Ax_{t-1} + AF\epsilon_t \\ & = AGA^{-1}z_{t-1} + AF\epsilon_t \\ & = G^T z_{t-1} + AF\epsilon_t \\ & = G^T z_{t-1} + F\epsilon_t \end{aligned}$$

(d)

$$y_t = Fx_t, x_t = Gx_{t-1} + F\epsilon_t \quad and\ also\ y_t = Fz_t, z_t = G^T z_{t-1} + F\epsilon_t$$

Both representation are state-space representation of y_t, AR(p) process. It means that state-space representation of process is not unique and z_t is another state vector of y_t.

$$z_t = Ax_t = \begin{bmatrix} y_t \\ \phi_2 y_{t-1} + \phi_3 y_{t-2} + \cdots + \phi_p y_{t-p+1} \\ \phi_3 y_{t-1} + \phi_4 y_{t-2} + \cdots + \phi_p y_{t-p+2} \\ \vdots \\ \phi_p y_{t-1} \end{bmatrix} \quad when\ p = 2 \quad \begin{bmatrix} z_t \\ z_{t-1} \end{bmatrix} = \begin{bmatrix} y_t \\ \phi_2 y_{t-1} \end{bmatrix}$$

By above equations, we can interpret z_t as weighted state vector of y_t whose weights are given by estimated ϕ. This fact make it clear that unknown state underlying y_t's can be estimated with observation y_t