STA 532 Homework9

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HW9 for STA-532

1.

(a)

Type I error $= P(Y \notin A(0) \mid \mu = 0)$ under $H : Y \sim N(0, 1)$

$$\begin{aligned} Pr(Y \notin A(0) \mid \mu = 0) &= Pr(Y < z_{\alpha(1-w)}) + Pr(Y > z_{1-\alpha w}) \\ &= \Phi(z_{\alpha(1-w)}) + 1 - \Phi(z_{1-\alpha w}) \\ &= \alpha(1-w) + 1 - (1 - \alpha w) = \alpha - \alpha w + 1 - 1 + \alpha w = \alpha \end{aligned}$$

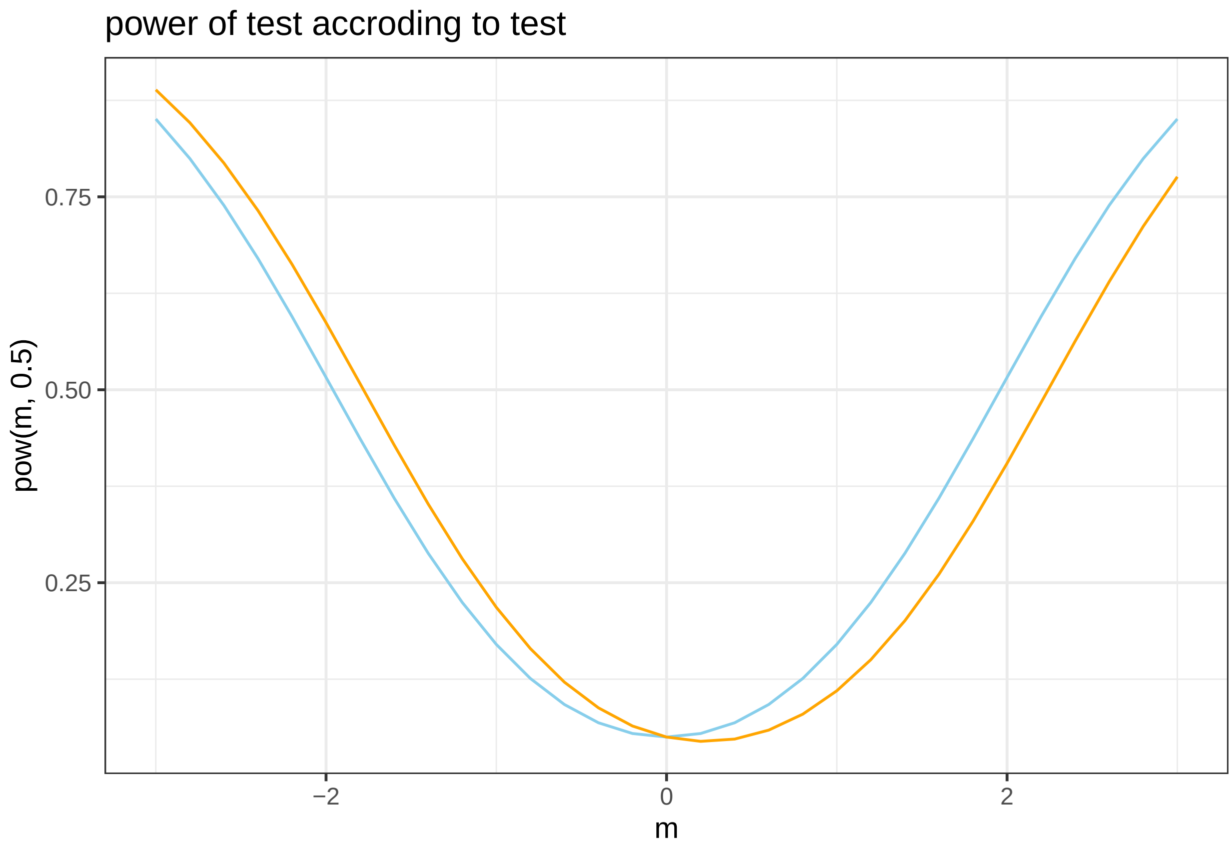
(b)

$Pw(\mu) = (Y \notin A_0 \mid \mu)$ and $Y - \mu \sim N(0, 1)$

$$\begin{aligned} (Y \notin A_0 \mid \mu) &= Pr(Y < z_{\alpha(1-w)}) + Pr(Y > z_{1-\alpha w}) \\ &= Pr(Y - \mu < z_{\alpha(1-w)} - \mu) + Pr(Y - \mu > z_{1-\alpha w} - \mu) \\ &= \Phi(z_{\alpha(1-w)} - \mu) + 1 - \Phi(z_{1-\alpha w} - \mu) \end{aligned}$$

If w = 1/2 then $Pw(\mu) = \Phi(z_{\alpha/2} - \mu) + 1 - \Phi(z_{1-\alpha/2} - \mu)$. If w = 1/4 then $Pw(\mu) = \Phi(z_{3\alpha/4} - \mu) + 1 - \Phi(z_{1-\alpha/4} - \mu)$.

```r
a = 0.05
w = c(.5,0.25)
pow <- function(m,w){
  return(pnorm(qnorm(a*(1-w))-m) + 1 - pnorm(qnorm(1-a*w)-m))
}
m = seq(-3,3,by = 0.2)
ggplot() +
  geom_line(mapping = aes(x = m,y = pow(m,.5)), color = "skyblue") +
  geom_line(mapping = aes(x = m,y = pow(m,.25)), color = "orange") +
  labs(title = "power of test accroding to test")
```



When we see above graph, we can find that w = 1/4 locates at right side of w = 1/2. That means that if $\mu < 0$ power of test of w = 1/4 is larger than w = 1/2. Thus if we assume that μ locates below 0, we use w = 1/4, otherwise, w = 1/2.

2.

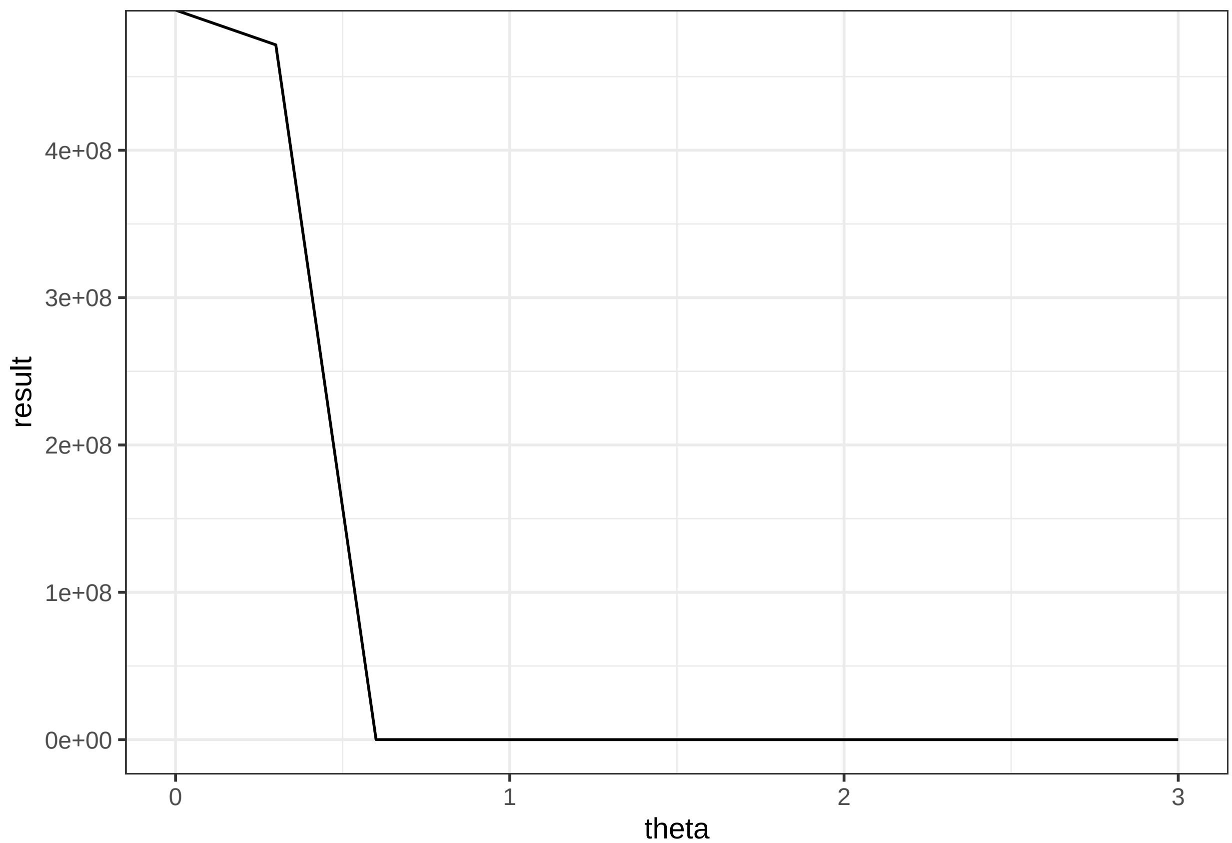
(a)

$$F(y) = \int_0^y P_\theta dx = \int_0^y e^{-x/\theta}/\theta = 1 - e^{-y/\theta}$$

(b)

Test stat : Y Level-α test: $Pr(Y > b \mid H) \leq \alpha \rightarrow e^{-b/\theta} < e^{-b/\theta_0} = \alpha$ under H. Then $e^{-b/\theta} < \alpha$ always for all $\theta < \theta_0$. b = $-\theta_0 log(\alpha)$

```
a = 0.05
theta0 = 2
theta = seq(0,3,.3)
result <- exp(-theta0*log(a)/theta)
ggplot()+
  geom_line(mapping = aes(x = theta, y = result))
```



(c)

$$
\begin{aligned}
Pr(\theta_0 \in C(Y) \mid H) &= Pr(y \in A(\theta_0) \mid H) \geq 1-\alpha \\
Thus\ C(Y) &= \{\theta_0 : H : \theta < \theta_0 is accpeted by y\} \\
&= \{\theta_0 : y \in A(\theta_0)\} \\
&= \{\theta_0 : y < -\theta_0 log(\alpha)\} \\
&= \{\theta_0 : \theta_0 > -\frac{y}{log(\alpha)}\} \quad beacuase\ -\log(\alpha)\ is\ positive \\
\rightarrow C(Y) &= (-\frac{y}{log(\alpha)}, \infty) \rightarrow C(y) = -\frac{y}{log(\alpha)}
\end{aligned}
$$

Then,

$$Pr(-\frac{y}{log(\alpha)} < \theta_0 \mid H) = Pr(y < -\theta_0 log(\alpha) \mid H) = Pr(Y < b \mid H) \geq 1-\alpha\ as\ shown\ Q2.(b)$$

3.

(a)

H: $\mu_A = \mu_B \rightarrow$ H:$\mu_A - \mu_B = 0$ Test stat: $\bar{Y}_A - \bar{Y}_B$ where $\bar{Y}_A \sim N(\mu_A, \sigma^2/4), \bar{Y}_B \sim N(\mu_B, \sigma^2/4)$. Then $\frac{\bar{Y}_A - \bar{Y}_B}{\sqrt{\sigma^2/4+\sigma^2/4}} \sim N(0,1)$, but we don't know σ^2 but they have same σ^2.
Thus we replace σ^2 with s_p^2 where $s_p^2 = (\sum(Y_{A,i} - \bar{Y}_A) + \sum(Y_{B,i} - \bar{Y}_B))/6$. Then $\frac{\bar{Y}_A - \bar{Y}_B}{s_p/\sqrt{(2)}} \sim t_6$ which is

null distribution. Let's call $\frac{\bar{Y}_A - \bar{Y}_B}{s_p/\sqrt{(2)}}$ as T and we know that T has contious t_6 distribution. If we define p-value function $Pv(T) = Pr(T \geq t \mid H)$, then as we have shown in the class, random $Pv(T)$ has a uniform distribution. Since Pv(T) follows uniform dist on [0,1], the minimum value of Pv(T) is 0. Thus 0 is the smallest p-value.

(b)

H: no treatment effect
means $N_A(\mu_A, \sigma^2) = N_B(\mu_B, \sigma^2)$.

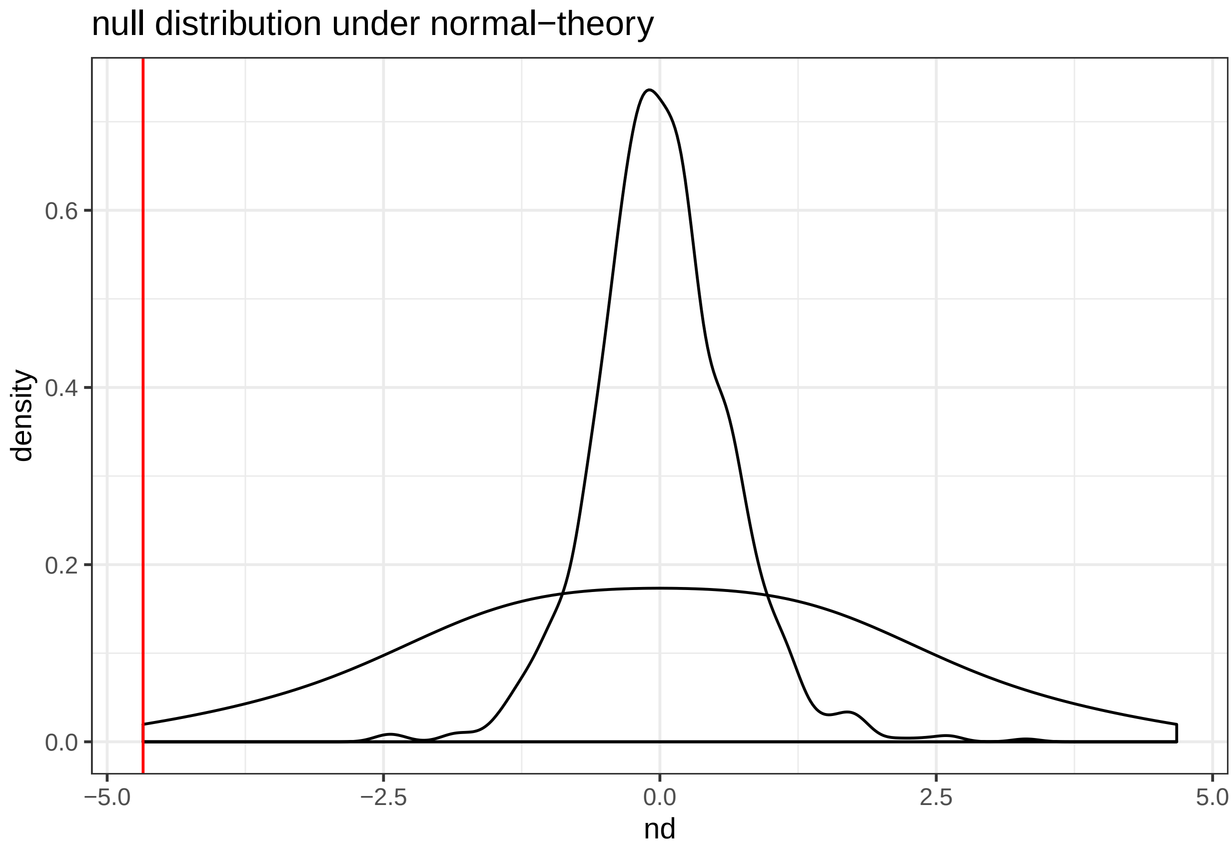
we have 4 observations for each, available randomization is ${}_8\mathrm{C}_4 = N$. By randomazation each permutation has same probability which means that if $(\bar{Y}_A - \bar{Y}_B)_{obs}$ is max or min of null dist, then p-value = 1/N. For each randomization, we can calculate test stat: $\bar{Y}_A - \bar{Y}_B$ and p-value = quantile of null dist that $\bar{Y}_A - \bar{Y}_B$ have on the null dist. If our observed assignment is minimum or maximum at null dist p-value = 1/N.

(c)

null dist under H for normal-theory test is t_6

```
x = c("B","A","B","B","A","A","B","A")
y = c(7.5, 1.2, 7.5, 8.7, 3.2, 5.1, 6.2, 1.7)
data <- cbind.data.frame(x,y)
#normal
idx <- data$x == "A"
Y_A <- mean(data[idx,2])
Y_B <- mean(data[!idx,2])
sp <- sum((data[idx,2] - Y_A)^2+(data[!idx,2] - Y_B)^2)/nrow(data)
tt <- (Y_A-Y_B)#/(sqrt(sp/6))
set.seed(532)
nd <- sqrt(sp/6)*rt(1000,6)
#randomization test
perm <- permutations(2,8,c("A","B"),set = T,repeats.allowed = T)
idx <- apply(perm,1,function(x){sum(x == "A")}) == 4
perm <- perm[idx,]
Td <- rep(NA,nrow(perm))
for(i in 1:nrow(perm)){
  data <- cbind.data.frame(x = perm[i,],y)
  idx <- data$x == "A"
  Y_A <- mean(data[idx,2])
  Y_B <- mean(data[!idx,2])
  Td[i] <- Y_A-Y_B
}

ggplot() +
  geom_density(mapping = aes(x = nd)) +
  geom_density(mapping = aes(x = Td)) +
  geom_vline(mapping = aes(xintercept = tt), color = "red") +
  labs(title = "null distribution under normal-theory")
```



4.

$(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$

(a)

Test stat: $t(y) = s^2$ H: $\sigma^2 = \sigma_0^2$ since $(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$, $Pr(\chi^2_{\alpha/2} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{1-\alpha/2}) = 1-\alpha$, where $\chi^2_{\alpha/2}$ *and* $\chi^2_{1-\alpha/2}$ are quantile of χ^2_{n-1}.

Level-α test: $Pr(t(y) \notin A(\sigma_0^2) \mid H) \leq \alpha \rightarrow Pr(t(y) \in A(\sigma_0^2) \mid H) \geq 1-\alpha$
$\rightarrow Pr(s^2 \in A(\sigma_0^2) \mid H) \geq 1-\alpha$. we showed $Pr(\chi^2_{\alpha/2} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{1-\alpha/2}) = 1-\alpha \rightarrow Pr(\frac{\sigma_0^2}{n-1}\chi^2_{\alpha/2} < s^2 < \frac{\sigma_0^2}{n-1}\chi^2_{1-\alpha/2}) = 1-\alpha$.

Thus we can find that acceptance region $A(\sigma_0^2)$ for H:$\sigma^2 = \sigma_0^2$ is $(\frac{\sigma_0^2}{n-1}\chi^2_{\alpha/2}, \frac{\sigma_0^2}{n-1}\chi^2_{1-\alpha/2})$. If $s^2 > \frac{\sigma_0^2}{n-1}\chi^2_{1-\alpha/2}$ or $s^2 < \frac{\sigma_0^2}{n-1}\chi^2_{\alpha/2}$, then reject H. Otherwise, accpet H.

(b)

$Pr(S^2 \in A(\sigma_0^2) \mid H) = Pr(\sigma_0^2 \in c(y) \mid H) = 1 - \alpha$
$where\ c(y)\ is\ confidence\ region$

$$Pr(s^2 \in A(\sigma_0^2) \mid H) = Pr(\chi^2_{\alpha/2} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{1-\alpha/2})$$
$$= Pr(\frac{s^2}{n-1}\chi^2_{\alpha/2} < \sigma_0^2 < \frac{s^2}{n-1}\chi^2_{1-\alpha/2})$$
$$= Pr(\sigma_0^2 \in c(y) \mid H)$$
$$\rightarrow c(y) = (\frac{s^2}{n-1}\chi^2_{\alpha/2}, \frac{s^2}{n-1}\chi^2_{1-\alpha/2})$$