3. $p(x_{1:n}) = p(x_n | x_{n-1}) \cdots p(x_1 | x_0)\, p(x_0)$ where $x_n | x_{n-1} \sim N(\phi x_{n-1}, V)$

$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi V}} \exp\{-\frac{1}{2V}\sum_{i=1}^{n}(x_i - \phi x_{i-1})^2\} \times \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}x_0^2\}$ $\quad x_0 \sim N(0, S)$

$S = \frac{V}{1-\phi^2} = 1 \Rightarrow V = 1-\phi^2$

1. sample $x_1 \sim N(\phi x_0, 1-\phi^2)$
2. sample $x_t | x_{t-1} \sim N(\phi x_{t-1}, 1-\phi^2)$

$\phi = 10, 0.5, 0, -0.5, -10$

4.

Let $v^{-1} = \lambda$

5. $\phi | v, D_{t-1} \sim N(m_{t-1}, C_{t-1} \times (\frac{1}{\lambda S_{t-1}}))$

$\lambda | D_{t-1} \sim G(\frac{n_{t-1}}{2}, \frac{n_{t-1}S_{t-1}}{2})$

$P(\phi|\lambda, D_{t-1}) \propto \left(\frac{\lambda S_{t-1}}{C_{t-1}}\right)^{\frac{1}{2}} \exp\{-\frac{\lambda S_{t-1}}{2C_{t-1}}(\phi - m_{t-1})^2\}$

$P(\lambda|D_{t-1}) \propto \lambda^{\frac{n_{t-1}}{2}-1} \exp\{-\frac{n_{t-1}S_{t-1}}{2} \times \lambda\}$

$P(\phi, \lambda|D_{t-1}) \propto \lambda^{\frac{n_{t-1}+1}{2}-1} \exp\{-\frac{\lambda S_{t-1}}{2}[n_{t-1} + \frac{(\phi-m_{t-1})^2}{C_{t-1}}]\}$

(a) $P(\phi|D_{t-1}) = \int P(\phi|\lambda, D_{t-1}) P(\lambda|D_{t-1}) d\lambda$

$\propto \int \lambda^{\frac{n_{t-1}+1}{2}-1} \exp\{-\frac{\lambda S_{t-1}}{2}[n_{t-1} + \frac{(\phi-m_{t-1})^2}{C_{t-1}}]\}$

$= \Gamma(\frac{n_{t-1}+1}{2}) \times [\frac{1}{2}S_{t-1}(n_{t-1} + \frac{(\phi-m_{t-1})^2}{C_{t-1}})]^{-\frac{n_{t-1}+1}{2}} \underbrace{\int \frac{[S_{t-1}(n_{t-1} + \frac{(\phi-m_{t-1})^2}{C_{t-1}})]^{\frac{n_{t-1}+1}{2}}}{\Gamma(\frac{n_{t-1}+1}{2})} \lambda^{\frac{n_{t-1}+1}{2}-1} \exp\{-\frac{\lambda S_{t-1}}{2}(n_{t-1} + \frac{(\phi-m_{t-1})^2}{C_{t-1}})\}}_{1}$

$\propto [\frac{1}{2}S_{t-1}(n_{t-1} + \frac{(\phi-m_{t-1})^2}{C_{t-1}})]^{-\frac{n_{t-1}+1}{2}}$

$\propto [1 + \frac{(\phi-m_{t-1})^2}{n_{t-1}C_{t-1}}]^{-\frac{n_{t-1}+1}{2}} \Rightarrow \phi \sim T(n_{t-1}, m_{t-1}, C_{t-1})$

which has mean m_{t-1} and Variance C_{t-1}

(b) $x_t | \phi, \lambda, D_{t-1} \sim N(\phi x_{t-1}, \lambda^{-1})$.

$P(x_t|\phi, \lambda, D_{t-1}) \propto \lambda^{\frac{1}{2}} \exp\{-\frac{\lambda}{2}(x_t - \phi x_{t-1})^2\}$ Let $\frac{S_{t-1}}{C_{t-1}} = k$

$P(\phi|\lambda, D_{t-1}) \propto \left(\frac{\lambda S_{t-1}}{C_{t-1}}\right)^{\frac{1}{2}} \exp\{-\frac{\lambda k}{2}(\phi - m_{t-1})^2\}$

$P(x_t|\lambda, D_{t-1}) = \int P(x_t|\phi, \lambda, D_{t-1}) P(\phi|\lambda, D_{t-1}) d\phi$

$\propto \int \lambda \exp\{-\frac{\lambda}{2}(\phi^2(x_{t-1}^2+k) - 2\phi(x_t x_{t-1} + km_{t-1}) + x_t^2 + km_{t-1}^2)\} d\phi$

$\propto \lambda^{\frac{1}{2}} \int \exp\{-\frac{\lambda(x_{t-1}^2+k)}{2}(\phi^2 - 2\phi(\frac{x_t x_{t-1} + km_{t-1}}{x_{t-1}^2+k}) + (\frac{x_t x_{t-1} + km_{t-1}}{x_{t-1}^2+k})^2) - \frac{\lambda}{2}(x_t^2 + km_{t-1}^2 - \frac{(x_t x_{t-1} + km_{t-1})^2}{x_{t-1}^2+k})\} d\phi$

$\propto \lambda^{\frac{1}{2}} \exp\{-\frac{\lambda}{2}(x_t^2 + km_{t-1}^2 - (\frac{x_t^2 x_{t-1}^2 + 2x_t(kx_{t-1}m_{t-1}) + k^2 m_{t-1}^2}{x_{t-1}^2+k}))\}$

$\propto \lambda^{\frac{1}{2}} \exp\{-\frac{\lambda}{2}(x_t^2(1 - \frac{x_{t-1}^2}{x_{t-1}^2+k}) - 2x_t(\frac{kx_{t-1}m_{t-1}}{x_{t-1}^2+k}) + km_{t-1}^2(1 - \frac{k}{x_{t-1}^2+k}))\}$

$\propto \lambda^{\frac{1}{2}} \exp\{-\frac{\lambda}{2}\frac{k}{x_{t-1}^2+k}(x_t^2 - 2x_t x_{t-1}m_{t-1} + x_{t-1}^2 m_{t-1}^2) \underbrace{- \frac{\lambda}{2}(km_{t-1}^2(1 - \frac{k}{x_{t-1}^2+k}) - \frac{k}{x_{t-1}^2+k}x_{t-1}^2 m_{t-1}^2)}_{0}\}$

$\Rightarrow x_t \sim N(x_{t-1}m_{t-1}, \lambda^{-1}(1 + \frac{x_{t-1}^2}{k}) = \lambda^{-1}(1 + \frac{C_{t-1}}{S_{t-1}}x_{t-1}^2) = \frac{v}{S_{t-1}}(S_{t-1} + C_{t-1}x_{t-1}^2)$

(c) $P(x_t|\lambda, D_{t-1}) \propto \lambda^{\frac{1}{2}} \exp\{-\frac{\lambda}{2}\frac{k}{x_{t-1}^2+k}(x_t - x_{t-1}m_{t-1})^2\}$

$P(x_t|D_{t-1}) \propto \int p(x_t|\lambda, D_{t-1}) p(\lambda|D_{t-1}) d\lambda$

$\propto \int \lambda^{\frac{n_{t-1}+1}{2}-1} \exp\{-\frac{\lambda}{2}[n_{t-1}S_{t-1} + \frac{k}{x_{t-1}^2+k}(x_t - x_{t-1}m_{t-1})^2]\}$

$\propto [n_{t-1}S_{t-1} + \frac{k}{x_{t-1}^2+k}(x_t - x_{t-1}m_{t-1})^2]^{-\frac{n_{t-1}+1}{2}}$

$\propto [1 + \frac{1}{n_{t-1}S_{t-1}} \times \frac{S_{t-1}}{C_{t-1}x_{t-1}^2 + S_{t-1}}(x_t - x_{t-1}m_{t-1})]^{-\frac{n_{t-1}+1}{2}}$

$\Rightarrow x_t \sim t(n_{t-1}, x_{t-1}m_{t-1}, C_{t-1}x_{t-1}^2 + S_{t-1})$

(d) $$P(\phi, \lambda \mid D_t) = P(\phi, \lambda \mid D_{t-1}, x_t) = \frac{P(\phi, \lambda, x_t \mid D_{t-1})}{P(x_t \mid D_{t-1})}$$

$$= \frac{P(x_t \mid \phi, \lambda, D_{t-1})\, P(\phi \mid \lambda, D_{t-1})\, P(\lambda \mid D_{t-1})}{P(x_t \mid D_{t-1})}$$

$$\propto \frac{\lambda^{\frac{1}{2}} \exp\{-\frac{\lambda}{2}(x_t - \phi x_{t-1})^2\} (\lambda k)^{\frac{1}{2}} \exp\{-\frac{\lambda k}{2}(\phi - m_{t-1})^2\} \lambda^{\frac{n_{t-1}}{2}} \exp\{-\frac{\lambda n_{t-1} s_{t-1}}{2}\}}{[1 + \frac{1}{n_{t-1}} \times \frac{1}{C_{t-1} x_{t-1}^2 + S_{t-1}} (x_t - x_{t-1} m_{t-1})]^{-\frac{n_{t-1}+1}{2}}}$$

$$\propto (\lambda k)^{\frac{1}{2}} \exp\{-\frac{\lambda k}{2}(\phi^2 - 2m_{t-1}\phi + \phi^2 \frac{x_{t-1}^2}{k} - 2\phi \frac{x_t x_{t-1}}{k}) \times \lambda^{\frac{n_{t-1}+1}{2}} \exp\{-\frac{\lambda}{2}(n_{t-1} S_{t-1} + k m_{t-1}^2 + x_t^2)\}$$

$$\propto (\lambda k)^{\frac{1}{2}} \exp\{-\frac{\lambda(k + x_{t-1}^2)}{2} (\phi^2 - 2\phi \frac{k m_{t-1} + x_t x_{t-1}}{k + x_{t-1}^2} + \frac{(k m_{t-1} + x_t x_{t-1})^2}{(k + x_{t-1}^2)^2}) \times \lambda^{\frac{n_{t-1}-1}{2}} \exp\{-\frac{\lambda}{2}(n_{t-1} S_{t-1} + k m_{t-1}^2 + x_t^2 - \frac{(k m_{t-1} + x_t x_{t-1})^2}{k + x_{t-1}^2}\}$$

$$\propto (\lambda k)^{\frac{1}{2}} \exp\{-\frac{\lambda(k + x_{t-1}^2)}{2} (\phi - \frac{k m_{t-1} + x_t x_{t-1}}{k + x_{t-1}^2})^2\} \lambda^{\frac{n_{t-1}+1}{2}} \exp\{-\frac{\lambda}{2}(n_{t-1} S_{t-1} + k m_{t-1}^2 + x_t^2 - \frac{(k m_{t-1} + x_t x_{t-1})^2}{k + x_{t-1}^2}\}$$

$$\Rightarrow P(\phi \mid \lambda, D_t)\, P(\lambda \mid D_t)$$

$$P(\phi \mid \lambda, D_t) \sim N\left(\frac{k m_{t-1} + x_t x_{t-1}}{k + x_{t-1}^2}, \frac{v}{k + x_{t-1}^2}\right) \quad P(\lambda \mid D_t) \sim \mathrm{Gamma}\left(\frac{n_{t-1}+1}{2}, \frac{1}{2}(n_{t-1} S_{t-1} + k m_{t-1}^2 + x_t^2 - \frac{(k m_{t-1} + x_t x_{t-1})^2}{k + x_{t-1}^2})\right)$$

$$k = \frac{S_{t-1}}{C_{t-1}}$$

$$n_t = n_{t-1} + 1$$

$$n_t S_t = n_{t-1} S_{t-1} + k m_{t-1}^2 + x_t^2 - \frac{k^2 m_{t-1}^2 + 2k m_{t-1} x_t x_{t-1} + x_t^2 x_{t-1}^2}{k + x_{t-1}^2}$$

$$= n_{t-1} S_{t-1} + \frac{1}{k + x_{t-1}^2}(k^2 m_{t-1}^2 + k m_{t-1}^2 x_{t-1}^2 + k x_t^2 + x_t^2 x_{t-1}^2 - k^2 m_{t-1}^2 - 2k m_{t-1} x_t x_{t-1} - x_t^2 x_{t-1}^2)$$

$$= n_{t-1} S_{t-1} + \frac{k}{k + x_{t-1}^2}(x_t - m_{t-1} x_{t-1})^2$$

$$= n_{t-1} S_{t-1} + \frac{S_{t-1}}{S_{t-1} + C_{t-1} x_{t-1}^2}(x_t - m_{t-1} x_{t-1})^2$$

$$= S_{t-1}(n_{t-1} + \frac{e_t^2}{q_t})$$

$$\Rightarrow S_t = \frac{S_{t-1}}{n_t}(n_{t-1} + e_t^2 / q_t)$$

$$C_{t-1} - A_t^2 q_t = C_{t-1} - \frac{C_{t-1}^2 x_{t-1}^2}{q_t}$$

$$C_t = S_t\left(\frac{1}{k + x_{t-1}^2}\right) = r_t S_{t-1}\left(\frac{1}{k + x_{t-1}^2}\right) = r_t \frac{C_{t-1} S_{t-1}}{q_t}$$

$$= \frac{C_{t-1} S_{t-1}}{q_t}$$

$$= r_t (C_{t-1} - A_t^2 q_t)$$

$$m_t = \frac{k m_{t-1} + x_t x_{t-1}}{k + x_{t-1}^2} = \frac{S_{t-1} m_{t-1} + C_{t-1} x_t x_{t-1}}{S_{t-1} + C_{t-1} x_{t-1}^2}$$

$$m_t = m_{t-1} + A_t e_t = m_{t-1} + \frac{C_{t-1} x_{t-1} (x_t - m_{t-1} x_{t-1})}{S_{t-1} + C_{t-1} x_{t-1}^2}$$

$$= m_{t-1} + \frac{C_{t-1} x_t x_{t-1} - m_{t-1} C_{t-1} x_{t-1}^2}{S_{t-1} + C_{t-1} x_{t-1}^2} = \frac{S_{t-1} m_{t-1} + C_{t-1} x_t x_{t-1}}{S_{t-1} + C_{t-1} x_{t-1}^2}$$

(e) (i) That means m_t depend on x_t as much as adaptive coefficient relative to prior mean.

r_t indicate how much uncertanty changed by new observation

Along with r_t

(ii) updated mean m_t is adjusted from prior mean m_{t-1} by realized error of x_t multiplicated by adaptive coefficient.