$\phi_0 = V_{t-1}^{-1}$, $\phi_1 = V_t^{-1}$

Then find bivariate $p(V_{t-1}, V_t \mid D_t)$

Let margin $p(\phi_0) \sim \text{gamma}(a, b)$

$\phi_1 = \phi_0 \eta / \beta$ where $\eta \sim \text{beta}(\beta a, (1-\beta)a)$ where $\beta \in (0,1)$ and $\phi_0 \perp \eta$

(a) $E(\phi_1 \mid \phi_0) = E(\phi_0 \eta / \beta \mid \phi_0) = \phi_0 \ E(\eta) \frac{1}{\beta}$

mean of beta$(a,b) = \frac{a}{a+b}$ $\Rightarrow$ $E(\eta) = \frac{\beta a}{a} = \beta$

$\Rightarrow E(\phi_1 \mid \phi_0) = \phi_0$

(b) $E(\phi_1) = E(E(\phi_1 \mid \phi_0)) = E(\phi_0) = \frac{a}{b}$

$E(\phi_0) = \frac{a}{b}$

$p(\phi_0, \phi_1) = \int p(\phi_0, \phi_1, \eta) d\eta$

$= \int p(\phi_1 \mid \phi_0, \eta) \, p(\phi_0, \eta) d\eta$

(c) $p(\phi_0, \eta) = p(\phi_0) p(\eta) \propto \phi_0^{a-1} e^{-b\phi_0} \times \eta^{\beta a - 1} (1-\eta)^{(1-\beta)a - 1}$

$p(\phi_0, \phi_1) = p_{\phi_0 \eta}\left(\phi_0, \frac{\phi_1 \beta}{\phi_0}\right) \left| \begin{matrix} 1 & -\frac{\phi_1 \beta}{\phi_0^2} \\ 0 & \frac{\beta}{\phi_0} \end{matrix} \right|$ by change of variable

$0 < \eta = \frac{\phi_1 \beta}{\phi_0} < 1 \Rightarrow 0 < \phi_1 < \frac{\phi_0}{\beta}$

$\Rightarrow p(\phi_0, \phi_1) \propto \phi_0^{a-1} e^{-b\phi_0} \times \left(\frac{\phi_1 \beta}{\phi_0}\right)^{\beta a - 1} \left(1 - \frac{\phi_1 \beta}{\phi_0}\right)^{(1-\beta)a - 1} \times \frac{\beta}{\phi_0}$

$\propto \phi_0^{a-1} e^{-b\phi_0} \times (\phi_1 \beta)^{\beta a - 1} \phi_0^{-\beta a + 1} \phi_0^{-(1-\beta)a + 1} (\phi_0 - \phi_1 \beta)^{(1-\beta)a - 1} \times \frac{\beta}{\phi_0}$

$\propto \phi_0^{a-1} e^{-b\phi_0} \phi_1^{\beta a - 1} \phi_0^{-a+2} \times \frac{1}{\phi_0} (\phi_0 - \phi_1 \beta)^{(1-\beta)a - 1}$

$\propto e^{-b\phi_0} \phi_1^{\beta a - 1} (\phi_0 - \phi_1 \beta)^{(1-\beta)a - 1}$

$p(\phi_1) = \int_0^\infty p(\phi_1, \phi_0) d\phi_0$

$\propto \phi_1^{\beta a - 1} \int_0^\infty e^{-b\phi_0} (\phi_0 - \phi_1 \beta)^{(1-\beta)a - 1} d\phi_0$

$\propto \phi_1^{\beta a - 1} \int_0^\infty e^{-b\phi_1 \beta} e^{-b(\phi_0 - \phi_1 \beta)} (\phi_0 - \phi_1 \beta)^{(1-\beta)a - 1} d\phi_0$

$\propto \phi_1^{\beta a - 1} e^{-\phi_1 \beta b} \int_0^\infty e^{-br} r^{(1-\beta)a - 1} dr$

$\sim$ kernel of gamma$(\beta a, \beta b)$

(d) Let $r = \phi_0 - \phi_1 \beta$ then

$p(\phi_1, r) = p_{\phi_0 \phi_1}(\phi_1, r + \phi_1 \beta) \left| \begin{matrix} 1 & -\beta \\ 0 & 1 \end{matrix} \right| = p_{\phi_0 \phi_1}(\phi_1, r + \phi_1 \beta)$

$p(\phi_1) = \int p(\phi_1, r) dr$

$\propto \int e^{-b(r + \phi_1 \beta)} \phi_1^{\beta a - 1} r^{(1-\beta)a - 1} dr$

$\propto \phi_1^{\beta a - 1} e^{-\beta b \phi_1} \int \underbrace{r^{(1-\beta)a}}_{\text{kernel of gamma } (1-\beta)a,\, b} e^{-br} dr$

$\propto c \underbrace{\phi_1^{\beta a - 1} e^{-\beta b \phi_1}}_{\text{kernel of gamma}(\beta a, \beta b)}$

(e) we have defined that $r = \phi_0 - \phi_1 \beta$ at previous question

$\Rightarrow \phi_0 = r + \phi_1 \beta$

At previous question we could figure out that

$p(\phi_1, r) \propto \phi_1^{\beta a-1} e^{-\beta b \phi_1} \, r^{(1-\beta)a-1} e^{-br}$

and this can be factorized regarding ϕ_1, r as

$\phi_1^{\beta a-1} e^{-\beta b \phi_1} \propto p(\phi_1)$ and $r^{(1-\beta)a-1} e^{-br} \propto p(r)$ and this is kernel of gamma$((1-\beta)a, b)$

Thus we can conclude that $\phi_1 \perp\!\!\!\perp r$.

Moreover, at (d) we confirmed that $p(\phi_1 r) \propto p(\phi_1 \phi_0)$

That is $p(\phi_0 | \phi_1) = \dfrac{p(\phi_0, \phi_1)}{p(\phi_1)} \propto \dfrac{p(\phi_1, r)}{p(\phi_1)} = \dfrac{p(\phi_1) p(r)}{p(\phi_1)} = p(r)$

$\Rightarrow p(\phi_0 | \phi_1) \propto p(r)$

3. In the observational variance discount model of Section 4.3.7, prove that the beta-gamma evolution model of Equation (4.17) yields the posterior-to-prior gamma distributions of Equation (4.18).

$$v_t = \beta v_{t-1}/\gamma_t \quad \text{or, equivalently,} \quad \phi_t = \phi_{t-1}\gamma_t/\beta \tag{4.17}$$

where γ_t is a time t random "shock," with

$$(\gamma_t|\mathcal{D}_{t-1}) \sim Be(\beta n_{t-1}/2, (1-\beta)n_{t-1}/2),$$

independently of v_{t-1}, and $\beta \in (0,1]$. The specified parameter β acts as a discount factor, that is, the larger the value of β is, the smaller is the random "shock" to the observational variance at each time, with $\beta = 1$ leading to the constant variance model with $v_t = v$ for all t.

It is easy to see that, based on a time $t-1$ posterior $(v_{t-1}|\mathcal{D}_{t-1}) \sim IG(n_{t-1}/2, d_{t-1}/2)$, the implied prior for v_t following the evolution of equation (4.17) is $IG(\beta n_{t-1}/2, \beta d_{t-1}/2)$. Equivalently in terms of precision, the distribution evolves as

$$(\phi_{t-1}|\mathcal{D}_{t-1}) \sim G(n_{t-1}/2, d_{t-1}/2) \;\rightarrow\; (\phi_t|\mathcal{D}_{t-1}) \sim G(\beta n_{t-1}/2, \beta d_{t-1}/2) \tag{4.18}$$

For previous setting, we have checked that

In setting $p(\phi_0) \sim \text{gamma}(a, b)$ and $\phi_1 = \phi_0 \eta/\beta$ when $\eta \sim \text{Beta}(\beta a, (1-\beta)a)$ and $\beta \in (0,1)$ $\phi \perp\!\!\!\perp \eta$

$p(\phi_1) \sim \text{gamma}(\beta a, \beta b)$

If we replace $\phi_0 = \phi_{t-1}$, $\phi_1 = \phi_t$, $\eta = \eta_t$, $a = n_{t-1}$, $b = d_{t-1}$

and $p(\phi_0) \rightarrow p(\phi_{t-1}|D_{t-1})$, $p(\phi_1) \rightarrow p(\phi_t|D_{t-1})$ then we can easily check the evolution procedure that

$$\phi_{t-1}|D_{t-1} \sim G\left(\frac{n_{t-1}}{2}, \frac{d_{t-1}}{2}\right) \rightarrow \phi_t|D_{t-1} \sim G\left(\frac{\beta n_{t-1}}{2}, \frac{\beta d_{t-1}}{2}\right)$$

4. Consider the observational variance discount model of Section 4.3.7.

(a) Show that the time $t-1$ prior $(\phi_{t-1}|\mathcal{D}_{t-1}) \sim G(n_{t-1}/2, d_{t-1}/2)$ combined with the beta-gamma evolution model $\phi_t = \phi_{t-1}\gamma_t/\beta$ yields a conditional density $p(\phi_{t-1}|\phi_t, \mathcal{D}_{t-1})$ that can be expressed as $\phi_{t-1} = \beta\phi_t + \upsilon^*_{t-1}$, where

$$(\upsilon^*_{t-1}|\mathcal{D}_{t-1}) \sim G((1-\beta)n_{t-1}/2, d_{t-1}/2)$$

is independent of ϕ_t.

(b) Show further that $p(\phi_{t-1}|\phi_t, \mathcal{D}_T) \equiv p(\phi_{t-1}|\phi_t, \mathcal{D}_{t-1})$ for all $T \geq t$.

(c) Describe how this result can be used to recursively compute retrospective point estimates $E(\phi_t|\mathcal{D}_T)$ backwards in time, beginning at $t = T$.

(d) Describe how this result can similarly be used to recursively simulate a full trajectory of values of $\phi_T, \phi_{T-1}, \ldots, \phi_1$ from the retrospective smoothed posterior conditional on $\mathcal{D}_T$.

(a) As previous question we have shown this at Q3 (e)

(b) In this Discount Volatility Evolution set up, we have check that $\phi_{t-1} = \beta\phi_t + r_{t-1}$

which indicates that markovian structure in volatility

$\phi_{t-1} \leftarrow \phi_t \leftarrow \cdots \quad \phi_T$

(c) $\phi_t = \beta\phi_{t+1} + r^*_{t-1}$

similarly $E(\phi_{T-1}|D_T) = E(\beta\phi_T|D_T) + E(r^*_{T-1}|D_T)$

$\Rightarrow \quad E(\phi_{T-2}|D_T) = \beta E(\phi_{T-1}|D_T) + E(r_{T-2}|D_T)$

$\vdots$

$\Rightarrow E(\phi_t|D_T) = E(\beta\phi_{t+1}|D_T) + E(r_{t-1}|D_{t-1})$

Thus from $E(\phi_T|D_T)$, update estimate of $E(\phi_i|D_i)$ by above equation,

then we can estimate $E(\phi_t|D_T)$ $\quad 1 \le t \le T$

(d) Similar with above procedure, we can simulate trajectory values of $\phi_T \cdots \phi_1$

1. Sample ϕ_T from gamma$\left(\frac{n_T}{2}, \frac{n_T S_T}{2}\right)$

2. Sample r_{T-1} from gamma$\left(\frac{(1-\beta)n_{T-1}}{2}, \frac{n_{T-1}S_{T-1}}{2}\right)$

3. Calculate $\phi_{T-1} = \beta\phi_T + r_{T-1}$

4. Sample r_{T-2} from gamma$\left(\frac{(1-\beta)n_{T-2}}{2}, \frac{n_{T-2}S_{T-2}}{2}\right)$

5. Calculate $\phi_{T-2} = \beta\phi_{T-1} + r_{T-2}$

$\vdots$

Sample r_1 from gam

$\vdots$

$\phi_1 = \beta\phi_2 + r_1$

1. prior set up $\theta_{t-1} | V_{t-1}, D_{t-1} \sim N(m_t, \frac{V_{t-1}}{S_{t-1}} C_{t-1})$

$V_{t-1} | D_{t-1} \sim IG(\frac{n_{t-1}}{2}, \frac{n_{t-1} S_{t-1}}{2})$

2. update volatility $\theta_{t-1} | V_t, D_{t-1} \sim N(m_t, \frac{V_t}{S_{t-1}} C_{t-1})$

$V_t | D_{t-1} \sim IG(\beta \frac{n_{t-1}}{2}, \beta \frac{n_{t-1} S_{t-1}}{2})$

3. update state $\theta_{t-1} |$