Math 525: Lecture 10

February 6, 2018

1 Limits of expectations

Remember the moment generating function of a random variable X? It looked like

$$M(\theta) = \mathbb{E}\left[e^{\theta X}\right].$$

We claimed that $M(\theta)$ "generates the moments" in the sense that $M^{(k)}(0) = \mathbb{E}[X^k]$ for $k \in \mathbb{N}$. However, we did not justify this claim rigorously, as doing so required us to deal with limits under the expectation. In particular, we wanted to write

$$M(\theta) = \mathbb{E}\left[\lim_{n\to\infty} \sum_{n=0}^{N} \frac{\theta^n}{n!} X^n\right] = \lim_{n\to\infty} \mathbb{E}\left[\sum_{n=0}^{N} \frac{\theta^n}{n!} X^n\right] = \lim_{n\to\infty} \sum_{n=0}^{N} \frac{\theta^n}{n!} \mathbb{E}\left[X^n\right],$$

but we were unable to justify the second equality above.

Motivated by this (and other applications), we will talk today about limits of expectations. That is, given a sequence of random variables $(X_n)_n$, can we conclude that $\lim_n \mathbb{E}[X_n] = \mathbb{E}[\lim_n X_n]$? The answer is, "most of the time, but not always":

Example 1.1. Let $Y \sim U[0,1]$ and $X_n = nI_{[0,1/n]}(Y)$. Then,

$$\mathbb{E}\left[X_n\right] = n\mathbb{P}(Y \leq 1/n) = 1$$

but $X_n \to 0$ a.s.! That is, $1 = \lim_n \mathbb{E}[X_n] \neq \mathbb{E}[\lim_n X_n] = 0$.

We start out with a simple limit theorem for expectations. To simplify notation, let $x \wedge y = \min\{x, y\}$. For the remainder, it should be understood that all random variables are extended real valued.

Proposition 1.2. *Let X be a random variable such that $X \geq 0$ a.s. Then,*

$$\lim_{n\to\infty} \mathbb{E}\left[X \wedge n\right] = \lim_{n\to\infty} \mathbb{E}\left[X\right].$$

This is true even when X is not integrable (i.e., $\mathbb{E}[X] = \infty$).

Proof. Note that

$$X \wedge n = nI_{\{X=\infty\}} + (X \wedge n)\, I_{\{X<\infty\}}.$$

Therefore, if $\mathbb{P}\{X = \infty\} > 0$,

$$\mathbb{E}[X] \geq \mathbb{E}[X \wedge n] \geq \mathbb{E}\left[nI_{\{X=\infty\}}\right] = n\mathbb{P}\{X = \infty\} \to \infty \qquad \text{as} \qquad n \to \infty$$

and the claim is trivially true. Therefore, we may proceed assuming X is finite everywhere.

Note that

$$\underline{X}_k \wedge n = \underline{X}_k I_{\{\underline{X}_k \leq n\}} + nI_{\{\underline{X}_k > n\}}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}[X \wedge n] &\geq \mathbb{E}[\underline{X}_k \wedge n] \\
&\geq \mathbb{E}\left[\underline{X}_k I_{\{\underline{X}_k \leq n\}}\right] \\
&= \sum_{j=1}^{2^k n} \frac{j}{2^k} \mathbb{P}\left\{\underline{X}_k = \frac{j}{2^k}\right\}.
\end{aligned}$$

If we take limits of both sides of this inequality,

$$\lim_{n\to\infty} \mathbb{E}[X \wedge n] \geq \sum_{j=1}^{\infty} \frac{j}{2^k} \mathbb{P}\left\{\underline{X}_k = \frac{j}{2^k}\right\} = \mathbb{E}[\underline{X}_k].$$

Taking limits in the above inequality,

$$\lim_{n\to\infty} \mathbb{E}[X \wedge n] \geq \lim_{k\to\infty} \mathbb{E}[\underline{X}_k] = \mathbb{E}[X].$$

The reverse inequality is trivial: since $\mathbb{E}[X] \geq \mathbb{E}[X \wedge n]$ for all n,

$$\mathbb{E}[X] \geq \lim_{n\to\infty} \mathbb{E}[X \wedge n].$$

□

Recall that a nondecreasing sequence of real numbers $(a_n)_n$ can do one of two things: converge to a finite limit, or diverge to ∞. The Monotone Convergence Theorem (up next) is the analogue of this claim for random variables (or, more generally, measurable functions).

Proposition 1.3 (Monotone Convergence Theorem)**.** *Consider a sequence of random variables $(X_n)_n$ satisfying $0 \leq X_1 \leq X_2 \leq \cdots$ a.s. and $X_n \to X$ a.s. Then,*

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

As before, this is true even when X is not integrable (i.e., $\mathbb{E}[X] = \infty$).

Proof. As usual, we can ignore the "a.s." (though you should convince yourself carefully that this is the case).

Taking expectations of the inequality, we get $0 \leq \mathbb{E}X_1 \leq \mathbb{E}X_2 \leq \cdots$. Therefore, $\lim_n \mathbb{E}[X_n] \leq \mathbb{E}[X]$ (note that $\lim_n \mathbb{E}[X_n]$ may be infinite). Therefore, if we can establish the reverse inequality $\lim_n \mathbb{E}[X_n] \geq \mathbb{E}[X]$, we will be done. By Proposition 1.2, $\mathbb{E}[X_n] = \lim_N \mathbb{E}[X_n \wedge N]$ and $\mathbb{E}[X] = \lim_N \mathbb{E}[X \wedge N]$ and hence it is sufficient to establish

$$\lim_n \mathbb{E}[X_n \wedge N] \geq \mathbb{E}[X \wedge N] \qquad \text{for all } N.$$

Fix N and let $Y = X \wedge N$ and $Y_n = X_n \wedge N$. Trivially, Y and Y_N are integrable. Let $\epsilon > 0$ and

$$A_{\epsilon,n} = \{Y - Y_n \geq \epsilon\}.$$

Note that $A_{\epsilon,1} \supset A_{\epsilon,2} \supset \cdots$ is a decreasing sequence of sets (since $Y_1 \leq Y_2 \leq \cdots$) and hence by continuity of the probability measure,

$$\mathbb{P}(A_{\epsilon,n}) \to \mathbb{P}(\cap_n A_{\epsilon,n}) \qquad \text{as} \qquad n \to \infty.$$

Moreover, since $Y_n \to Y$ a.s., $\cap_n A_{\epsilon,n} = \emptyset$. Note that

$$Y - Y_n \leq N I_{A_{\epsilon,n}} + \epsilon I_{A^c_{\epsilon,n}}$$

and hence

$$\mathbb{E}[Y] - \mathbb{E}[Y_n] = \mathbb{E}[Y - Y_n] \leq \mathbb{E}\left[N I_{A_{\epsilon,n}} + \epsilon I_{A^c_{\epsilon,n}}\right] \leq N\mathbb{P}(A_{\epsilon,n}) + \epsilon\mathbb{P}(A^c_{\epsilon,n}) \leq N\mathbb{P}(A_{\epsilon,n}) + \epsilon.$$

Taking limits,

$$\mathbb{E}[Y] - \lim_n \mathbb{E}[Y_n] \leq \epsilon.$$

Taking $\epsilon \to 0$ gives us the desired inequality. □

The monotone convergence theorem was for nonnegative increasing sequences. What about decreasing sequences?

Corollary 1.4. *Consider a sequence of random varibles $(X_n)_n$ satisfying $X_1 \geq X_2 \geq \cdots \geq 0$ a.s. and $X_n \to X$ a.s. If X_1 is integrable, then*

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

Proof. Let $Y_n = X_1 - X_n$ and note that Y_n increases to $Y = X_1 - X$. Therefore, by the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \mathbb{E}[Y_n] = \mathbb{E}[Y].$$

Now, $\mathbb{E}[Y] = \mathbb{E}[X_1] - \mathbb{E}[X]$ due to integrability. Similarly, $\mathbb{E}[Y_n] = \mathbb{E}[X_1] - \mathbb{E}[X_n]$. Plugging these into the above and simplifying, we obtain the desired result. □

It is not possible to remove the integrability condition from the above:

Example 1.5. Let $Y \sim U[0,1]$ and $X_n = \frac{1}{nY}$. Then,

$$\mathbb{E}\left[X_n\right] = \int_0^1 \frac{1}{ny} dy = \frac{1}{n} \int_0^1 \frac{1}{y} dy = \frac{1}{n} \lim_{y \downarrow 0} \left(\log 1 - \log y\right) = \infty.$$

However, $X_n \downarrow 0$ a.s.

Proposition 1.6 (Fatou's Lemma)**.** *Let $(X_n)_n$ be a sequence of random variables with $X_n \geq 0$ a.s. Then,*

$$\liminf_{n \to \infty} \mathbb{E}\left[X_n\right] \geq \mathbb{E}\left[\liminf_{n \to \infty} X_n\right].$$

Proof. Trivially,

$$\mathbb{E}\left[X_n\right] \geq \mathbb{E}\left[\inf_{j \geq n} X_j\right].$$

Letting $Y_n = \inf_{j \geq n} X_j$ and applying limit inferiors to both sides of the above inequality,

$$\liminf_n \mathbb{E}\left[X_n\right] \geq \liminf_n \mathbb{E}\left[Y_n\right].$$

Note, in particular, that Y_n is a nondecreasing sequence. Therefore, by the Monotone Convergence Theorem,

$$\lim_n \mathbb{E}\left[Y_n\right] = \mathbb{E}\left[\lim_n Y_n\right] = \mathbb{E}\left[\lim_n \inf_{j \geq n} X_j\right] = \mathbb{E}\left[\liminf_n X_n\right].$$

□

Proposition 1.7. *Let $(X_n)_n$ be a sequence of random variables dominated by some integrable random variable Y (i.e., $\mathbb{E}|Y| < \infty$ and $|X_n| \leq Y$) such that $X_n \to X$ a.s. Then,*

$$\mathbb{E}\left[X_n\right] \to [X].$$

Proof. First, we handle the a.s. case. Indeed, suppose $X_n \to X$ a.s. Then, $Y - X_n \geq 0$ a.s. and $Y - X_n \to Y - X$ a.s. By Fatou's lemma,

$$\begin{aligned}\mathbb{E}\left[Y\right] - \limsup_n \mathbb{E}\left[X_n\right] = \liminf_n \mathbb{E}\left[Y - X_n\right] \geq \mathbb{E}\left[\liminf_n \left(Y - X_n\right)\right] &= \mathbb{E}\left[Y\right] - \mathbb{E}\left[\limsup_n X_n\right] \\ &= \mathbb{E}\left[Y\right] - \mathbb{E}\left[\lim_n X_n\right].\end{aligned}$$

Therefore,

$$\limsup_n \mathbb{E}\left[X_n\right] \leq \mathbb{E}\left[\lim_n X_n\right].$$

An identical argument with $Y + X_n \geq 0$ yields

$$\liminf_n \mathbb{E}\left[X_n\right] \geq \mathbb{E}\left[\lim_n X_n\right].$$

Combining the two inequalities above, the desired result follows. □

Corollary 1.8. *Let $(X_n)_n$ be a sequence of random variables dominated by a real number (i.e., $|X_n| \leq M$) such that $X_n \to X$ a.s. Then,*

$$\mathbb{E}\left[X_n\right] \to \left[X\right].$$

Proof. Take $Y = M$ in the Dominated Convergence Theorem. $\square$

Corollary 1.9. *Let $(X_n)_n$ be a sequence of random variables satisfying the requirements of the Dominated Convergence Theorem. Then, $X_n \xrightarrow{\mathbb{L}^1} X$.*

Proof. Note that $|X_n - X| \leq 2Y$ and $|X_n - X| \to 0$. Therefore, $\mathbb{E}|X_n - X| \to 0$ by the Dominated Convergence Theorem, and hence the sequence converges in $\mathbb{L}^1$. $\square$

Conversely, if we have a sequence $(X_n)_n$ converging to some X in $\mathbb{L}^1$,

$$|\mathbb{E}X_n - \mathbb{E}X| = \mathbb{E}\left[|X_n - X|\right] \to 0$$

and hence it is trivially the case that $\mathbb{E}X_n \to \mathbb{E}X$.