

# Math 525: Assignment 1 Solutions

1. To simplify notation, let

$$\mathcal{M}(\mathcal{G}) = \{\mathcal{F} : \mathcal{F} \text{ is a } \sigma\text{-algebra on } \Omega \text{ and } \mathcal{G} \subset \mathcal{F}\}$$

so that

$$\sigma(\mathcal{G}) = \bigcap_{\mathcal{F} \in \mathcal{M}(\mathcal{G})} \mathcal{F}.$$

Note that as a trivial consequence of the definition,  $\sigma(\mathcal{G}) \supset \mathcal{G}$ .

- (a) We claim that the intersection  $\mathcal{F} = \bigcap_{\alpha} \mathcal{F}_{\alpha}$  of  $\sigma$ -algebras  $\mathcal{F}_{\alpha}$  is itself a  $\sigma$ -algebra. Since  $\sigma(\mathcal{G})$  is, by definition, the intersection of  $\sigma$ -algebras, the desired result follows from this claim. The claim is established by three points: (i)  $\emptyset \in \mathcal{F}$  because  $\emptyset \in \mathcal{F}_{\alpha}$  for each  $\alpha$ . (ii) Suppose  $A \in \mathcal{F}$ . Then,  $A \in \mathcal{F}_{\alpha}$  for each  $\alpha$  and hence  $A^c \in \mathcal{F}_{\alpha}$  for each  $\alpha$ , from which it follows that  $A^c \in \mathcal{F}$ . (iii) Suppose  $A_1, A_2, \dots \in \mathcal{F}$ . Then,  $A_1, A_2, \dots \in \mathcal{F}_{\alpha}$  for each  $\alpha$  and hence  $\bigcup_{n \geq 1} A_n \in \mathcal{F}_{\alpha}$  for each  $\alpha$ , from which it follows that  $\bigcup_{n \geq 1} A_n \in \mathcal{F}$ .
  - (b) Suppose  $\mathcal{G} \subset \mathcal{G}'$ . Since any  $\sigma$ -algebra which contains  $\mathcal{G}'$  must also contain  $\mathcal{G}$ , we have  $\mathcal{M}(\mathcal{G}) \supset \mathcal{M}(\mathcal{G}')$ , from which the desired result follows.
  - (c) Suppose  $\mathcal{F}$  is a  $\sigma$ -algebra. Since  $\mathcal{F} \in \mathcal{M}(\mathcal{F})$ , it follows that  $\sigma(\mathcal{F}) \subset \mathcal{F}$ . Since we already know  $\sigma(\mathcal{F}) \supset \mathcal{F}$ , the desired result follows.
  - (d) If  $\mathcal{G} \subset \mathcal{F}$ , part (b) tells us  $\sigma(\mathcal{G}) \subset \sigma(\mathcal{F})$ . If  $\mathcal{F}$  is a  $\sigma$ -algebra, part (c) tells us  $\sigma(\mathcal{F}) = \mathcal{F}$ . Combining these two points, the desired result follows.
2. Checking that these are algebras is straightforward, so I will just point out why they are not  $\sigma$ -algebras:
    - (a) Consider  $\mathbb{N} = \{1, 2, \dots\}$ . Neither  $\mathbb{N}$  nor  $\mathbb{N}^c = \mathbb{R} \setminus \mathbb{N}$  are finite subsets of  $\mathbb{R}$ , and hence  $\mathbb{N}$  is not in the algebra. However, we can write  $\mathbb{N}$  as a countable union of elements of the algebra:  $\mathbb{N} = \bigcup_{n \geq 1} \{n\}$ .
    - (b) The set  $(-\infty, 0)$  can be written as a countable union of elements of the algebra:  $(-\infty, 0) = \bigcup_{n \geq 1} (-\infty, -1/n]$ .
  3. Suppose the claim holds for  $n$ . We attempt to establish it for  $n + 1$ . Note that

$$\begin{aligned} \mathbb{P}(A_1 \cup \dots \cup A_{n+1}) &= \mathbb{P}((A_1 \cup \dots \cup A_n) \cup A_{n+1}) \\ &= \mathbb{P}(A_1 \cup \dots \cup A_n) + \mathbb{P}(A_{n+1}) - \mathbb{P}((A_1 \cup \dots \cup A_n) \cap A_{n+1}) \end{aligned}$$

Let's handle each term in the sum separately. First, note that by our induction hypothesis,

$$\begin{aligned} & \mathbb{P}(A_1 \cup \dots \cup A_n) \\ &= \sum_{i \leq n} \mathbb{P}(A_i) - \sum_{i < j \leq n} \mathbb{P}(A_i \cap A_j) + \sum_{i < j < k \leq n} \mathbb{P}(A_i \cap A_j \cap A_k) - \dots + (-1)^{n+1} \mathbb{P}(A_1 \cap \dots \cap A_n). \end{aligned}$$

Similarly, applying our induction hypothesis,

$$\begin{aligned} & \mathbb{P}((A_1 \cup \dots \cup A_n) \cap A_{n+1}) = \mathbb{P}((A_1 \cap A_{n+1}) \cup \dots \cup (A_n \cap A_{n+1})) \\ &= \sum_{i \leq n} \mathbb{P}(A_i \cap A_{n+1}) - \sum_{i < j \leq n} \mathbb{P}(A_i \cap A_j \cap A_{n+1}) + \sum_{i < j < k \leq n} \mathbb{P}(A_i \cap A_j \cap A_k \cap A_{n+1}) \\ & \quad - \dots + (-1)^{n+1} \mathbb{P}(A_1 \cap \dots \cap A_n \cap A_{n+1}). \end{aligned}$$

Now, we can simplify our expression for  $\mathbb{P}(A_1 \cup \dots \cup A_{n+1})$ :

$$\begin{aligned} & \mathbb{P}(A_1 \cup \dots \cup A_{n+1}) \\ &= \sum_{i \leq n+1} \mathbb{P}(A_i) - \sum_{i < j \leq n+1} \mathbb{P}(A_i \cap A_j) + \sum_{i < j < k \leq n+1} \mathbb{P}(A_i \cap A_j \cap A_k) \\ & \quad - \dots + (-1)^{n+2} \mathbb{P}(A_1 \cap \dots \cap A_{n+1}). \end{aligned}$$

To finish the proof, note that the claim holds trivially for  $n = 1$  (base case).