Math 525: Assignment 9

1. Let $X, Y \sim U[0,1]$ be independent. Find the joint density of $\min(X,Y)$ and $\max(X,Y)$.
2. Let X and Y be random variables admitting the joint density
$$f_{XY}(x,y) = \begin{cases} \dfrac{1}{\pi}\exp\left(-\dfrac{(x^2+y^2)}{2}\right) & \text{if } xy \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$
Show that $X, Y \sim \mathcal{N}(0,1)$ but that X and Y are not independent. **Hint**: compute the marginal distribution.
3. Let U and V be independent $\mathcal{N}(0,1)$ random variables. Let $X = aU + bV$ and $Y = cU + dV$ where $a^2 + b^2 = 1$, $c^2 + d^2 = 1$, and $ac + bd = \rho$.
   (a) Show that $X \sim \mathcal{N}(0,1)$. **Hint**: what is the characteristic function of X?
   (b) Show that $\mathbb{E}[XY] = \rho$.
   (c) The *joint characteristic function* of X and Y is a function $\varphi : \mathbb{R}^2 \to \mathbb{C}$ given by
   $$\varphi_{X,Y}(t,s) = \mathbb{E}\left[\exp\left(itX + isY\right)\right]$$
   (of course, this can be generalized to the case of n random variables by $\varphi_{X_1,\ldots,X_n}(t_1,\ldots,t_n) = \mathbb{E}\left[\exp(it_1X_1 + \cdots + it_nX_n)\right]$). Show that
   $$\varphi_{X,Y}(t,s) = \exp\left(-\frac{t^2 + 2\rho ts + s^2}{2}\right).$$
   Hint: for a $Z \sim \mathcal{N}(\mu,\sigma)$, we know $\varphi_Z(t) = \exp(i\mu t - (\sigma t)^2/2)$.
   (d) Show that the joint density of X and Y is given by
   $$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{x^2 - 2\rho xy + y^2}{2\left(1-\rho^2\right)}\right).$$
   Hint: use the fact that if X and Y admit a joint density, it can be obtained as the *inverse Fourier transform* of their joint characteristic function (see page 152 of Walsh for details):
   $$f_{XY}(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{X,Y}(t,s)\exp\left(-i\left(tx + sy\right)\right)dtds.$$
   (e) Part (b) implies that if $\rho \neq 0$, then X and Y are not independent. Use your findings in part (d) to establish the converse. **Hint**: recall that $f_{XY} = f_X f_Y$ is a sufficient condition for independence.