# Math 525: Assignment 1

1. Sometimes it is useful to start with a small set of events $\mathcal{G}$ that is not necessarily a $\sigma$-algebra and "generate" a $\sigma$-algebra from it. The $\sigma$-algebra generated from $\mathcal{G}$ is

$$\sigma(\mathcal{G}) = \bigcap_{\substack{\mathcal{F} \text{ is a } \sigma\text{-algebra on } \Omega \\ \mathcal{G} \subset \mathcal{F}}} \mathcal{F}.$$

   That is, $\sigma(\mathcal{G})$ is the intersection of all $\sigma$-algebras containing $\mathcal{G}$. Show that...

   (a) $\sigma(\mathcal{G})$ is a $\sigma$-algebra.

   (b) $\sigma(\mathcal{G}) \subset \sigma(\mathcal{G}')$ whenever $\mathcal{G} \subset \mathcal{G}'$.

   (c) If $\mathcal{F}$ is a $\sigma$-algebra, then $\sigma(\mathcal{F}) = \mathcal{F}$.

   (d) If $\mathcal{F}$ is a $\sigma$-algebra and $\mathcal{G} \subset \mathcal{F}$, then $\sigma(\mathcal{G}) \subset \mathcal{F}$.

2. Show that the following two are algebras but not $\sigma$-algebras:

   (a) All finite subsets of $\mathbb{R}$ together with their complements.

   (b) All finite unions of intervals in $\mathbb{R}$ of the form $(a, b]$, $(-\infty, a]$, and $(b, \infty)$.

3. Prove the principle of inclusion-exclusion. That is, show that if $A_1, \ldots, A_n \in \mathcal{F}$, then

$$\begin{aligned} &\mathbb{P}(A_1 \cup \cdots \cup A_n) \\ &= \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} \mathbb{P}(A_1 \cap \cdots \cap A_n). \end{aligned}$$

   Note that when we write $\sum_{i<j}$, we mean $\sum_{i,j \in \{(i,j) \colon i<j\}}$ (and similarly for the sums involving more indices).

# Solutions

1. To simplify notation, let

$$\mathcal{M}(\mathcal{G}) = \{\mathcal{F} \colon \mathcal{F} \text{ is a } \sigma - \text{algebra on } \Omega \text{ and } \mathcal{G} \subset \mathcal{F}\}$$

so that

$$\sigma(\mathcal{G}) = \bigcap_{\mathcal{F} \in \mathcal{M}(\mathcal{G})} \mathcal{F}.$$

Note that as a trivial consequence of the definition, $\sigma(\mathcal{G}) \supset \mathcal{G}$.

(a) We claim that the intersection $\mathcal{F} = \cap_\alpha \mathcal{F}_\alpha$ of $\sigma$-algebras $\mathcal{F}_\alpha$ is itself a $\sigma$-algebra. Since $\sigma(\mathcal{G})$ is, by definition, the intersection of $\sigma$-algebras, the desired result follows from this claim. The claim is established by three points: *(i)* $\emptyset \in \mathcal{F}$ because $\emptyset \in \mathcal{F}_\alpha$ for each $\alpha$. *(ii)* Suppose $A \in \mathcal{F}$. Then, $A \in \mathcal{F}_\alpha$ for each $\alpha$ and hence $A^c \in \mathcal{F}_\alpha$ for each $\alpha$, from which it follows that $A^c \in \mathcal{F}$. *(iii)* Suppose $A_1, A_2, \ldots \in \mathcal{F}$. Then, $A_1, A_2, \ldots \in \mathcal{F}_\alpha$ for each $\alpha$ and hence $\cup_{n \geq 1} A_n \in \mathcal{F}_\alpha$ for each $\alpha$, from which it follows that $\cup_{n \geq 1} A_n \in \mathcal{F}$.

(b) Suppose $\mathcal{G} \subset \mathcal{G}'$. Since any $\sigma$-algebra which contains $\mathcal{G}'$ must also contain $\mathcal{G}$, we have $\mathcal{M}(\mathcal{G}) \supset \mathcal{M}(\mathcal{G}')$, from which the desired result follows.

(c) Suppose $\mathcal{F}$ is a $\sigma$-algebra. Since $\mathcal{F} \in \mathcal{M}(\mathcal{F})$, it follows that $\sigma(\mathcal{F}) \subset \mathcal{F}$. Since we already know $\sigma(\mathcal{F}) \supset \mathcal{F}$, the desired result follows.

(d) If $\mathcal{G} \subset \mathcal{F}$, part (b) tells us $\sigma(\mathcal{G}) \subset \sigma(\mathcal{F})$. If $\mathcal{F}$ is a $\sigma$-algebra, part (c) tells us $\sigma(\mathcal{F}) = \mathcal{F}$. Combining these two points, the desired result follows.

2. Checking that these are algebras is straightforward, so I will just point out why they are not $\sigma$-algebras:

(a) Consider $\mathbb{N} = \{1, 2, \ldots\}$. Neither $\mathbb{N}$ nor $\mathbb{N}^c = \mathbb{R} \setminus \mathbb{N}$ are finite subsets of $\mathbb{R}$, and hence $\mathbb{N}$ is not in the algebra. However, we can write $\mathbb{N}$ as a countable union of elements of the algebra: $\mathbb{N} = \cup_{n \geq 1}\{n\}$.

(b) The set $(-\infty, 0)$ can be written as a countable union of elements of the algebra: $(-\infty, 0) = \cup_{n \geq 1}(-\infty, -1/n]$.

3. Suppose the claim holds for $n$. We attempt to establish it for $n+1$. Note that

$$\begin{aligned}
\mathbb{P}(A_1 \cup \cdots \cup A_{n+1}) &= \mathbb{P}((A_1 \cup \cdots \cup A_n) \cup A_{n+1}) \\
&= \mathbb{P}(A_1 \cup \cdots \cup A_n) + \mathbb{P}(A_{n+1}) - \mathbb{P}((A_1 \cup \cdots \cup A_n) \cap A_{n+1})
\end{aligned}$$

Let's handle each term in the sum separately. First, note that by our induction hypothesis,

$$\begin{aligned}
&\mathbb{P}(A_1 \cup \cdots \cup A_n) \\
&= \sum_{i \leq n} \mathbb{P}(A_i) - \sum_{i<j \leq n} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k \leq n} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} \mathbb{P}(A_1 \cap \cdots \cap A_n).
\end{aligned}$$

Similarly, applying our induction hypothesis,

$$\begin{aligned}
&\mathbb{P}((A_1 \cup \cdots \cup A_n) \cap A_{n+1}) = \mathbb{P}((A_1 \cap A_{n+1}) \cup \cdots \cup (A_n \cap A_{n+1})) \\
&\quad = \sum_{i \le n} \mathbb{P}(A_i \cap A_{n+1}) - \sum_{i<j\le n} \mathbb{P}(A_i \cap A_j \cap A_{n+1}) + \sum_{i<j<k\le n} \mathbb{P}(A_i \cap A_j \cap A_k \cap A_{n+1}) \\
&\qquad\qquad - \cdots + (-1)^{n+1}\mathbb{P}(A_1 \cap \cdots \cap A_n \cap A_{n+1}).
\end{aligned}$$

Now, we can simplify our expression for $\mathbb{P}(A_1 \cup \cdots \cup A_{n+1})$:

$$\begin{aligned}
&\mathbb{P}(A_1 \cup \cdots \cup A_{n+1}) \\
&\qquad = \sum_{i \le n+1} \mathbb{P}(A_i) - \sum_{i<j\le n+1} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k\le n+1} \mathbb{P}(A_i \cap A_j \cap A_k) \\
&\qquad\qquad - \cdots + (-1)^{n+2}\mathbb{P}(A_1 \cap \cdots \cap A_{n+1}).
\end{aligned}$$

To finish the proof, note that the claim holds trivially for $n = 1$ (base case).