

Math 525: Assignment 4

1. (Moment generating functions) Let X be a random variable with distribution function

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du$$

(we call X a *standard normal random variable*; you may have seen it before). Compute the third raw moment of X using the moment generating function.

Hint: If X is an absolutely continuous distribution function F with density f (i.e., $F(x) = \int_{-\infty}^x f(y)dy$) and if $g \geq 0$ is a Borel measurable function, then

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(u)f(u)du.$$

2. (Expectations) Let X be a nonnegative integrable random variable.

- (a) Show that $X = 0$ a.e. implies $\mathbb{E}X = 0$.
- (b) Show that $\mathbb{E}X = 0$ implies $X = 0$ a.e. **Hint:** consider sets of the form $A_n = \{X \geq 1/n\}$ where n is a positive integer.
- (c) Let E be a set of probability zero (i.e., $\mathbb{P}(E) = 0$). Use part (a) to show that $\mathbb{E}[XI_E] = 0$ where I_E is the indicator random variable on E .

3. (Expectations) Let X be a nonnegative random variable with distribution function F .

- (a) Show that, for any real number $x \geq 0$,

$$x = \int_0^{\infty} I_{[0,x)}(y)dy$$

where $I_{[0,x)}(y) = 1$ if $0 \leq y < x$ and $I_{[0,x)}(y) = 0$ otherwise. Use this to derive

$$\mathbb{E}X = \mathbb{E} \left[\int_0^{\infty} I_{[0,X)}(y)dy \right].$$

- (b) (Optional) Show, using the results of part (a), that

$$\mathbb{E}X = \int_0^{\infty} \mathbb{E}[I_{[0,X)}(y)] dy.$$

- (c) Show that $\mathbb{E}[I_{[0,X)}(y)] = \mathbb{P}(X > y)$.
- (d) Combine your findings in parts (b) and (c) to conclude

$$\mathbb{E}X = \int_0^{\infty} (1 - F(y)) dy.$$