Math 525: Lecture 4

January 16, 2018

1 Uniform distribution

Definition 1.1. We say X is *uniformly distributed* on $[a, b]$ (written $X \sim U[a, b]$) if it is a random variable with distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \leq x < b \\ 1 & \text{if } x \geq b. \end{cases}$$

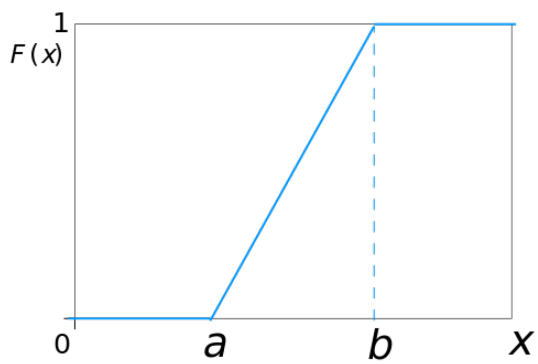

Figure 1: Uniform distribution

Intuitively, a uniform distribution tells us that any outcome in $[a, b]$ is "equally likely".

Remark 1.2. Actually, since F is continuous, no single outcome occurs with positive probability (recall that $\mathbb{P}(\{X = x\}) = F(x) - F(x-))$. What we really mean is that given two disjoint subsets A and B of $[a, b]$, X is equally likely to be in either of them.

Example 1.3. The position of the pointer on a gameshow wheel can be modelled as a random variable uniformly distributed on $[0, 2\pi)$.

How do we actually construct a random variable with a uniform distribution? There are a few ways to do this:

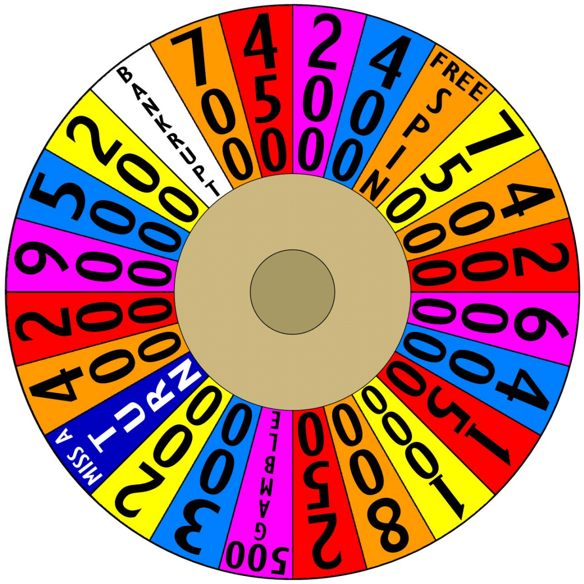

Figure 2: Game show wheel

Example 1.4. Let $\Omega = \mathbb{R}$ and

$$A_x = \{\omega \in \Omega \colon \omega \leq x\}$$

and define the probability measure $\mathbb{P} \colon \mathcal{B}(\mathbb{R}) \to [0,1]$ by

$$\mathbb{P}(A_x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \leq x < b \\ 1 & \text{if } x \geq b. \end{cases}$$

$(\Omega, \mathcal{B}(\mathbb{R}), \mathbb{P})$ is a probability space. Moreover, the random variable X defined by $X(\omega) = \omega$ is uniformly distributed on $[a,b]$.

Remark 1.5. You may ask, at this point, why have we not defined $\mathbb{P}$ for sets not of the form A_x? Remember that $\mathcal{G} = \{A_x\}_{x \in \mathbb{R}}$ generates $\mathcal{B}(\mathbb{R})$. It turns out that to define a probability measure uniquely, it is sufficient to define it on generating sets (see, e.g., Corollary 1.8 of Walsh, John B. *Knowing the odds: an introduction to probability*. Vol. 139. American Mathematical Soc., 2012). The proof of this fact uses something called the *monotone class theorem*, which is outside of the scope of this course.

We could have also taken a slightly different approach in defining a random variable that is uniformly distributed on $[a,b]$:

Example 1.6. Let $\Omega = [a,b]$ and $\mathcal{B}([a,b]) = \sigma(\{[a,x] \colon x \in \mathbb{R}\})$ (the Borel sets on $[a,b]$). Define A_x as before and the probability measure $\mathbb{P} \colon \mathcal{B}([a,b]) \to [0,1]$ by

$$\mathbb{P}(A_x) = \frac{x-a}{b-a}.$$

$(\Omega, \mathcal{B}([a,b]), \mathbb{P})$ is a probability space. Moreover, the random variable X defined by $X(\omega) = \omega$ is uniformly distributed on $[a,b]$.

The probability spaces in the last two examples are, for all intents and purposes, identical. The former allows for points outside of $[a, b]$ to be outcomes, but they occur with zero probability. The latter precludes them altogether.

2 Existence of random variables

In the last lecture, we showed that the distribution function F of any random variable is nondecreasing and right-continuous with

$$\lim_{x \to -\infty} F(x) = 0 \qquad \text{and} \qquad \lim_{x \to \infty} F(x) = 1.$$

Today, we'll prove a "converse" of this fact.

Proposition 2.1. *Let $F : \mathbb{R} \to [0, 1]$ be nondecreasing and right-continuous with*

$$\lim_{x \to -\infty} F(x) = 0 \qquad \textit{and} \qquad \lim_{x \to \infty} F(x) = 1.$$

Then, there exists a random variable whose distribution function is F.

Note the subtlety here: in the previous lecture, we started out with a random variable and obtained a distribution function. The above proposition tells us we can go backwards: start with a distribution function and obtain a random variable.

Proof. We only consider the case in which F is a bijection. The general case is more challenging (see, e.g., Theorem 2.14 of Walsh, John B. *Knowing the odds: an introduction to probability.* Vol. 139. American Mathematical Soc., 2012).

Let $X \sim U[0, 1]$ and define $Y = F^{-1}(X)$. Since F is monotone, so too is F^{-1}. As a technical note, this implies that F^{-1} is Borel measurable and hence Y is indeed a random variable. Now, note that

$$\mathbb{P}(\{Y \leq y\}) = \mathbb{P}(\{F^{-1}(X) \leq y\}) = \mathbb{P}(\{X \leq F(y)\}) = \frac{F(y) - 0}{1 - 0} = F(y). \qquad \square$$

The proof above has a very important consequence for sampling from non-uniform distributions, as demonstrated below:

Example 2.2. You use a random number generator to generate n samples $U_1, \ldots, U_n \sim U[0, 1]$. You are given the distribution function F. Letting $X_i = F^{-1}(U_i)$, you obtain the samples $X_1, \ldots, X_n$, which all have the distribution function F.

This shows us that we can always turn the problem of sampling from a non-uniform distribution into one of sampling from a uniform distribution!

Example 2.3. Let U be a uniform random variable on $[0, 1]$. Let $X = U^2$. Let F denote the distribution function of X. Then, if $0 \leq x < 1$,

$$F(x) = \mathbb{P}(\{X \leq x\}) = \mathbb{P}(\{U^2 \leq x\}) = \mathbb{P}(\{U \leq \sqrt{x}\}) = \sqrt{x}.$$

3 Independence of random variables

In a previous lecture, we defined what it means for two events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to be independent (if $A, B \in \mathcal{F}$ and $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, we say A and B are independent). We extend this definition now to random variables.

Definition 3.1. Two random variables X and Y are *independent* if for all $x, y \in \mathbb{R}$,

$$\mathbb{P}(\{X \leq x, Y \leq y\}) = \mathbb{P}(\{X \leq x\})\mathbb{P}(\{Y \leq y\})$$

(i.e., the events $\{X \leq x\}$ and $\{X \leq y\}$ are independent).

Example 3.2. Let X be a random variable. Let $Y = X^2$. Suppose $0 < \mathbb{P}(\{X \leq a\}) < 1$ for some a. Then,

$$\mathbb{P}(\{X \leq a, Y \leq a^2\}) = \mathbb{P}(\{X \leq a, X^2 \leq a^2\}) = \mathbb{P}(\{X \leq a, X \leq a\}) = \mathbb{P}(\{X \leq a\}).$$

That is, X and Y are not independent.

The above definition concerns only sets of the form $\{X \leq x\}$ and $\{Y \leq y\}$. Can we extend it to other sets?

Proposition 3.3. *Let X and Y be independent random variables. Then,*

$$\mathbb{P}(\{X \in A\} \cap \{Y \in B\}) = \mathbb{P}(\{X \in A\})\mathbb{P}(\{Y \in B\})$$

whenever $A = (p, q]$ and $B = (r, s]$.

Proof. Note that

$$\begin{aligned}
\mathbb{P}\{X \in (-\infty, q], Y \in (r, s]\} &= \mathbb{P}\{X \leq q, r < Y \leq s\} \\
&= \mathbb{P}\{X \leq q, Y \leq s\} - \mathbb{P}\{X \leq q, Y \leq r\} \\
&= \mathbb{P}\{X \leq q\}\mathbb{P}\{Y \leq s\} - \mathbb{P}\{X \leq q\}\mathbb{P}\{Y \leq r\} \\
&= \mathbb{P}\{X \leq q\}\left(\mathbb{P}\{Y \leq s\} - \mathbb{P}\{Y \leq r\}\right) \\
&= \mathbb{P}\{X \leq q\}\mathbb{P}\{r < Y \leq s\}.
\end{aligned}$$

Now, use the same reasoning to get

$$\mathbb{P}\{p < X \leq q, r < Y \leq s\} = \mathbb{P}\{p < X \leq q\}\mathbb{P}\{r < Y \leq s\}. \qquad \square$$

Remark 3.4. The previous proposition can be extended more generally to the case of A and B in $\mathcal{B}(\mathbb{R})$ (see, e.g., Theorem 2.20 of Walsh, John B. *Knowing the odds: an introduction to probability.* Vol. 139. American Mathematical Soc., 2012). The proof uses, once again, the monotone class theorem.

4 Independence of multiple events

Let's generalize the concept of independence to families of random variables.

Definition 4.1. We say the family $\{A_\alpha\}_{\alpha\in\mathcal{A}} \in \mathcal{F}$ is independent if for each positive integer $k \leq n$ and $\{A_{i_1}, \ldots, A_{i_k}\} \subset \{A_\alpha\}_{\alpha\in\mathcal{A}}$,

$$\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdots \mathbb{P}(A_{i_k}).$$

The notion of independence above is stronger than requiring each pair of events to be independent:

Example 4.2. Toss two fair coins at the same time. Let A be the event that the first coin is heads, B be the event that the second coin is heads, and C be the event that the first and second coins disagree (i.e., one is heads and the other is tails). Note that $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = 1/2$.

Obviously, A and B are independent. To see that A and C are independent, note that

$$\mathbb{P}(A \cap C) = 1/4 = \mathbb{P}(A)\mathbb{P}(C).$$

Similarly, B and C are independent. This establishes that the three events are **pairwise** independent. However, note that $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) \neq 0$. Therefore, the events A, B, C are not independent, despite being pairwise independent.

Definition 4.3. We say the family of random variables $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ are independent if for each positive integer $k \leq n$, $\{X_{i_1}, \ldots, X_{i_k}\} \subset \{X_\alpha\}_{\alpha\in\mathcal{A}}$, and $x_1, \ldots, x_k$,

$$\mathbb{P}(\{X_{i_1} \leq x_1, \ldots, X_{i_k} \leq x_n\}) = \mathbb{P}(\{X_{i_1} \leq x_1\}) \cdots \mathbb{P}(\{X_{i_k} \leq x_n\}).$$

Definition 4.4. Let $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ be an independent family of random variables, each having the same distribution function F. In this case, we say the family is *independent and identically distributed* (i.i.d.).

Exercise 4.5. Let X and Y be i.i.d. integer-valued random variables (i.e., $\mathbb{P}(\{X \text{ is an integer}\}) = 1$ and similarly for Y). Let $p_n = \mathbb{P}(\{X = n\})$. Then,

$$\mathbb{P}(\{X = Y\}) = \sum_{n=-\infty}^{\infty} \mathbb{P}(\{X = n\})\mathbb{P}(\{Y = n\}) = \sum_{n=-\infty}^{\infty} p_n^2.$$

Similarly,

$$\mathbb{P}(\{X \leq Y\}) = \sum_{n=-\infty}^{\infty} \mathbb{P}(\{X = n\}) \sum_{m=n}^{\infty} \mathbb{P}(\{Y = n\}) = \sum_{n=-\infty}^{\infty} p_n \sum_{m=n}^{\infty} p_m.$$

For example, suppose

$$p_n = \frac{6}{\pi^2}\frac{1}{n^2} \text{ if } n > 0 \qquad \text{and} \qquad p_n = 0 \text{ otherwise}$$

(you can check that $\sum_{n=1}^{\infty} p_n = 1$). Then,

$$\mathbb{P}(\{X = Y\}) = \sum_{n=1}^{\infty} p_n = \frac{36}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{36}{\pi^4}\frac{\pi^4}{90} = \frac{36}{90} = \frac{2}{5}.$$

and

$$\mathbb{P}(\{X \leq Y\}) = \sum_{n=1}^{\infty} p_n \sum_{m=n}^{\infty} p_m = \frac{36}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{m=n}^{\infty}\frac{1}{m^2} = \frac{36}{\pi^4}\frac{7\pi^4}{360} = \frac{7}{10}.$$

5 Types of distributions

Definition 5.1. Let X be a random variable.

1. X has a *discrete distribution* if we can find a countable subset $\{x_n\}_n \subset \mathbb{R}$ for which
$$\sum_{n=1}^{\infty} \mathbb{P}\{X = x_n\} = 1.$$
2. X has a *continuous distribution* if its distribution function F is continuous.
3. X has an *absolutely continuous distribution* if its distribution function can be written
$$F(x) = \int_{-\infty}^{x} f(x)dx$$
for some integrable function f.

There are random variables that do not fall into these categories:

Example 5.2. Consider flipping a coin. If the coin is heads, you receive one dollar. Otherwise, you receive Y dollars, where $Y \sim U[0,1]$. This random variable is
$$X = I_{\{\text{tails}\}}Y + I_{\{\text{heads}\}}.$$
Its distribution function is
$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{2} & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1. \end{cases}$$

Definition 5.3. If X has a discrete distribution, we define its probability mass function (PMF) $p\colon \mathbb{R} \to [0,1]$
$$p(x) = \mathbb{P}(\{X = x\}).$$

Definition 5.4. A discrete random variable X is said to be Bernoulli(q) if its probability mass function is only nonzero at the points $x = 0, 1$ (i.e., $p(1) = q$ and $p(0) = 1 - q$).

Example 5.5. I_A is a Bernoulli random variable whenever $A \in \mathcal{F}$.

Exercise 5.6. Let X and Y be independent Bernoulli(1/2) random variables. Let $Z = (Y - X)^2$. Note that Z is also Bernoulli and
$$p_Z(1) = p_X(1)p_Y(0) + p_X(0)p_Y(1) = 1/2.$$
Prove that X, Z are independent (and hence so too are Y, Z) but X, Y, Z are not independent.