Math 525: Assignment 6

1. Let X be an absolutely continuous random variable. Show that if its probability density is even (i.e., $f(x) = f(-x)$), then its characteristic function must be real (i.e., $\phi(t) = \overline{\phi(t)}$).

2. Let $X_n \to X$ a.s. Show that $\phi_n \to \phi$ pointwise (here, ϕ_n and ϕ are the characteristic functions of X_n and X).

3. Let X be a random variable and ϕ be its characteristic function. Show that $|\phi|^2$ is also a characteristic function. You'll need to use the following facts:

   - For any complex number z, $|z|^2 = z\overline{z}$.
   - You can construct a random variable Y that has the same distribution as X but is independent of it[1] so that $\mathbb{E}[f(X)]\mathbb{E}[g(Y)] = \mathbb{E}[f(X)g(Y)]$ for any Borel measurable functions f and g.

4. Suppose $(X_n)_n$ and $(Y_n)_n$ converge in probability to X and Y, respectively. Show that $(X_n + Y_n)_n$ converges in distribution to $X + Y$ (hint: revisiting the proof for "convergence in probability implies convergence in distribution" may help).

[1] Rigorously, let X be a random variable on the space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, we can form the product space $(\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, \mathbb{P} \times \mathbb{P})$. Denote a member of the new sample space $\Omega \times \Omega$ as $\omega \equiv (\omega_1, \omega_2)$. Define $X_1(\omega_1, \omega_2) = X(\omega_1)$ and $X_2(\omega_1, \omega_2) = X(\omega_2)$ as two copies of the random variable X that are independent.