Math 525: Assignment 8

To answer the first question, you will need a theorem from linear algebra. Recall that a matrix/vector is said to be nonnegative/positive if all of its entries are nonnegative/positive.

Definition. Let A be a nonnegative square matrix. We say A is *primitive* if A^m is a positive matrix for some positive integer m.

Theorem (Perron-Frobenius for primitive matrices). Let A be a primitive matrix. Then,

- $\lambda_1 = \rho(A)$ is an eigenvalue of A.
- All other eigenvalues λ_i are smaller in magnitude than λ_1 (i.e., $|\lambda_i| < \lambda_1$).
- $\{x: Ax = \lambda_1 x\} = span(v)$ for some positive vector v.
- 1. A Markov chain whose transition matrix P is primitive is called regular. Let
 - $\mu = (\mu_1, \dots, \mu_m)^{\mathsf{T}}$ be a column vector with $\mu^{\mathsf{T}} e = 1$ where $e = (1, \dots, 1)^{\mathsf{T}}$,
 - P be the transition matrix of a regular Markov chain, and
 - suppose that P^{\dagger} has a linearly independent set of eigenvectors $\{v_1, \ldots, v_m\}$ with corresponding eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$ in descending order of magnitude:

$$|\lambda_1| > |\lambda_2| \ge \cdots \ge |\lambda_m|.$$

- (a) Show that if $\mu^{\dagger}e = 1$, then $\mu^{\dagger}Pe = 1$.
- (b) Let $\mu = c_1 v_1 + \cdots + c_m v_m$ be an eigendecomposition of μ . By the Perron-Frobenius theorem, we can, without loss of generality, pick v_1 to be a positive vector. Show that for any positive integer k,

$$\mu^{\mathsf{T}} P^k = c_1 \lambda_1^k v_1^{\mathsf{T}} + \dots + c_m \lambda_m^k v_m^{\mathsf{T}}.$$

- (c) Show that $\lim_{n\to\infty} \mu^{\mathsf{T}} P^n = c_1 v_1^{\mathsf{T}}$ (**Hint**: Proposition 1.13 of Lecture 16).
- (d) Show that $(c_1v_1^{\mathsf{T}})P = c_1v_1^{\mathsf{T}}$.
- (e) Show that $(c_1v_1^{\dagger})e = 1$ (**Hint**: part (a)) and $c_1 > 0$.
- (f) Part (e) shows that $c_1v_1^{\mathsf{T}}$ is a distribution vector (i.e., it is nonnegative and its entries add up to one). With this in mind, what is the significance of
 - i. Part (c)? (**Hint**: if $\mathbb{P}(X_0 = i) = \mu_i$, then $(\mu^{\mathsf{T}} P^n)_i = \mathbb{P}(X_n = i)$).
 - ii. Part (d)?

2. A coin with probability p of heads is flipped repeatedly. X_n is the result of the n-th coin flip (n = 1 is the first coin flip). Let

$$\tau = \inf \{ n > 1: (X_{n-1}, X_n) = (H, T) \},$$

corresponding to the first time at which we see a heads (H) followed by a tails (T).

- (a) Give an expression for $\mathbb{P}(\tau \leq n)$ (**Hint**: try to think of a transition matrix P such that $\mathbb{P}(\tau \leq n) = \mu^{\mathsf{T}} P^n \nu$ for appropriately chosen column vectors μ and ν).
- (b) Give an expression for $\mathbb{P}(\tau = n)$ (**Hint**: use your results from part (a)).
- (c) (Optional) Let p=1/3 and compute $\mathbb{P}(\tau \leq 10)$ with numerical software such as MATLAB.