

Math 525: Assignment 7 Solutions

1. We know X_n, Y_n converge in distribution to $X, Y \sim \text{Poisson}(\lambda)$. Therefore,

$$\phi_{Z_n}(t) = \mathbb{E} [e^{it(X_n - Y_n)}] = \mathbb{E} [e^{itX_n}] \mathbb{E} [e^{-itY_n}] = \phi_{X_n}(t) \phi_{Y_n}(-t) \rightarrow \phi_X(t) \phi_Y(-t)$$

by one direction of Lévy's continuity theorem. Therefore,

$$\phi_X(t) \phi_Y(-t) = \exp(\lambda(e^{it} - 1)) \exp(\lambda(e^{-it} - 1)) = \exp(\lambda(e^{it} - e^{-it})) = \exp(2\lambda(\cosh t - 1)).$$

By the other direction of Lévy's continuity theorem, Z_n converges in distribution to some random variable Z with the above characteristic function $\phi_X \phi_Y$, as desired.

2. See Exercise 2.2 of Lecture 15.
3. To show that $((X_n, X_{n+1}))_{n \geq 0}$ is a Markov chain, note that

$$\begin{aligned} \mathbb{P}((X_n, X_{n+1}) = (i_n, i_{n+1}) \mid (X_0, X_1) = (i_0, i_1), \dots, (X_{n-1}, X_n) = (i_{n-1}, i_n)) \\ = \mathbb{P}(X_{n+1} = i_{n+1} \mid X_0 = i_0, \dots, X_n = i_n) = \mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n) \\ = \mathbb{P}(X_{n+1} = i_{n+1} \mid X_{n-1} = i_{n-1}, X_n = i_n) \\ = \mathbb{P}((X_n, X_{n+1}) = (i_n, i_{n+1}) \mid (X_{n-1}, X_n) = (i_{n-1}, i_n)). \end{aligned}$$

This follows from

4.

- (a) If P is a transition matrix, then

$$\sum_j (P^2)_{ij} = \sum_j \sum_k P_{ik} P_{kj} = \sum_k \sum_j P_{ik} P_{kj} = \sum_k P_{ik} \sum_j P_{kj} = \sum_k P_{ik} = 1.$$

- (b) If P is a bistochastic matrix, then P and P^\top are transition matrices. Then, by part (a), P^2 and $(P^\top)^2 = (P^2)^\top$ are transition matrices. Therefore, P^2 is bistochastic.

5.

- (a) The multiplicity of the eigenvalue is 2.
- (b) The multiplicity of the eigenvalue is 1.
- (c) How many states there are that correspond to the “end” of the game. These “end” states are called *absorbing states* in the Markov chain literature.