Final Exam

Math 525: Probability

Last name, first name: ______________________________

Section number: ______________________________

User ID: ______________________________

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

If you are unsure about a question, leave it blank (or cross out any work you have done on it), to be awarded 25% of its points.

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Question 1 (5 points)
Let $\mathbb{P}$ be a probability measure and A and B be events. Show that

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

Question 2 (5 points)
Let $\mathcal{A} = \{(-\infty, x] \colon x \in \mathbb{R}\}$ and $\mathcal{B} = \{[x, \infty) \colon x \in \mathbb{R}\}$. Show that $\mathcal{A} \subset \sigma(\mathcal{B})$.

Question 3 (5 points)
Flip 4 fair coins. Conditional on there being an even number of heads, what is the probability that there are at least 2 heads?

Question 4 (5 points)
Let X be an r.v. and F be its distribution function. Show that $\mathbb{P}\{X < x\} = \lim_{y \uparrow x} F(y)$.

Question 5 (5 points)

$f : \mathbb{R} \to \mathbb{R}$ is *upper semicontinuous* if it can be written as the limit of a sequence $(f_n)_{n\geq 0}$ of nonincreasing (i.e., $f_n \geq f_{n+1}$) continuous functions. Show that f is Borel measurable. **Hint**: it is sufficient to show that $f^{-1}((-\infty, a])$ is a Borel set, which you can do by using the fact that $f_n^{-1}((-\infty, a])$ is a Borel set for each n.

Question 6 (5 points)

Let X be a positive integer-valued r.v. Show that

$$\mathbb{P}\{X = n\} = \frac{1}{n(n+1)} \qquad \text{for } n \geq 1$$

defines a probability distribution. **Hint**: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

Question 7 (5 points)

Let X be an r.v. Give an example of a function f such that $f \circ X$ is an r.v. that is independent of X. **Hint**: f can be as simple as you want.

Question 8 (5 points)

Let X and Y be nonnegative integer-valued r.v.s. Show that

$$\mathbb{P}\{X \leq Y\} = \sum_{n=0}^{\infty} p_n \sum_{k=0}^{n} q_k$$

where $p_n = \mathbb{P}\{Y = n\}$ and $q_n = \mathbb{P}\{X = n\}$.

Question 9 (5 points)
Let X be an r.v. with MGF $M(\theta) = \frac{\lambda}{\lambda - \theta}$. Then $\mathbb{E}[X]$ is equal to...

A. $\frac{\lambda}{(\lambda-\theta)^2}$

B. $\frac{1}{\lambda}$

C. $\frac{2\lambda}{(\lambda-\theta)^3}$

D. $\frac{2}{\lambda^2}$

Question 10 (5 points)
Show that

$$(\mathbb{E}[X])^2 \leq \mathbb{E}[X^2] \qquad \text{for any r.v. } X.$$

Hint: use the *Cauchy-Schwarz inequality* ($\mathbb{E}[XY] \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$).

Question 11 (5 points)
Let $(X_n)_n$ be a sequence of r.v.s such that $X_n \to \infty$ a.s. Show that the probability that $X_n \geq 1$ for all but finitely many n is one. That is, show that

$$\mathbb{P}(\{\omega \mid \exists N\colon \forall n \geq N\colon X_n(\omega) \geq 1\}) = 1.$$

Hint: $X_n \to X$ a.s. means $\mathbb{P}(\{\omega\colon \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1$.

Question 12 (5 points)
Let $(\Lambda_n)_n$ be a sequence of events with $\mathbb{P}(\Lambda_n) \to 0$. Let X be an integrable random variable. Show that $\mathbb{E}[X I_{\Lambda_n}] \to 0$. **Hint**: X is integrable means $\mathbb{E}[|X|] < \infty$.

Question 13 (5 points)

For square-integrable r.v.s X and Y, show that $\sqrt{\mathbb{E}[(X+Y)^2]} \leq \sqrt{\mathbb{E}[X^2]} + \sqrt{\mathbb{E}[Y^2]}$.
Hint: expand $\mathbb{E}[(X+Y)^2]$ and use the Cauchy-Schwarz inequality.

Question 14 (5 points)

Let $(X_n)_{n\geq 1}$ be a sequence of r.v.s and $f \geq 0$ be a <u>continuous</u> function such that

$$\mathbb{E}\left[f(X_n)\right] \leq \frac{1}{n} \qquad \text{for all } n \geq 1.$$

Suppose $X_n \to x \in \mathbb{R}$ in L^p for some p. Show that $f(x) = 0$.
Hint: use the fact that L^p convergence implies a.s. convergence along a subsequence and apply *Fatou's lemma*, which tells us $\liminf_{n\to\infty} \mathbb{E}[f(X_n)] \geq \mathbb{E}[\liminf_{n\to\infty} f(X_n)]$.

Question 15 (5 points)

Repeatedly flip a coin with probability p of heads. Let X_n be the number of heads seen after the n-th flip ($X_0 = 0$). Let τ be the number of flips until you see $N \geq 1$ heads. Then, $\mathbb{E}[\tau] = \sum_{n\geq 1} n\mu^\intercal \left(P^n - P^{n-1}\right) \nu$ where $\mu = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^\intercal$, $\nu = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^\intercal$, and $P \in \mathbb{R}^{(N+1)\times(N+1)}$. Fill in the nonzero entries of P:

$$P = \begin{pmatrix} \square & \square & & & \\ & \square & \square & & \\ & & \ddots & \ddots & \\ & & & \square & \square \\ & & & & \square \end{pmatrix}$$

Question 16 (5 points)

Let $f : [0,1]^d \to \mathbb{R}$. To reduce the error $|\frac{1}{N}\sum_{n=1}^N f(z^n) - \int_{[0,1]^d} f(\boldsymbol{x})d\boldsymbol{x}|$ in quasi-Monte Carlo, you should pick the sequence $(z^n)_n$ such that...

A. The variation of f is as low as possible.

B. The variation of f is as high as possible.

C. The star discrepancy of $P = \{z^1, \ldots, z^N\}$ (for each N) is as low as possible.

D. The star discrepancy of $P = \{z^1, \ldots, z^N\}$ (for each N) is as high as possible.

Question 17 (5 points)
You have a discrete stochastic process $(X_n)_{n\in T}$ where $T = \{0, 1, \dots, N\}$. You can transform this into a Markov process by defining $(Y_n)_{n\in T}$ by

$$Y_n = (X_0, \dots, X_n).$$

Suppose the state space of the X process is S with $|S| < \infty$. How large is the state space of the Y process?

A. $\sum_{n=0}^{N} |S|^{n+1} = |S|\frac{|S|^{N+1}-1}{|S|-1}$

B. $\prod_{n=0}^{N} |S|^{n+1} = |S|^{\frac{1}{2}(N+1)(N+2)}$

C. $|S|^{N+1}$

D. $(N+1)|S|$

Question 18 (5 points)
Consider the matrix $\begin{pmatrix} 1/2 & 0 & 1/2 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. This matrix is... (check <u>all</u> that apply)

- ◯ irreducible
- ◯ aperiodic
- ◯ primitive
- ◯ a transition matrix

Question 19 (5 points)
Check <u>all</u> true statements.

- ◯ A discrete process is said to satisfy the *Markov (a.k.a. memoryless) property* if the conditional probability distribution of future states (conditional on both past and present states) depend only upon the present state.
- ◯ The *strong Markov property* establishes a strong law of large numbers for a discrete process that satisfies the Markov property.
- ◯ The *strong Markov property* extends the Markov property to stopping times.
- ◯ Any random variable which takes values in $[0, \infty]$ is a *stopping time.*

Question 20 (5 points)
Fill in the blank. The PageRank algorithm represents a random surfer's location in the web as a ______ Markov chain. This guarantees that it has a limiting distribution.

A. irreducible

B. regular

C. stationary

D. nonnegative