

Math 525: Assignment 1

1. Sometimes it is useful to start with a small set of events \mathcal{G} that is not necessarily a σ -algebra and “generate” a σ -algebra from it. The σ -algebra generated from \mathcal{G} is

$$\sigma(\mathcal{G}) = \bigcap_{\substack{\mathcal{F} \text{ is a } \sigma\text{-algebra on } \Omega \\ \mathcal{G} \subset \mathcal{F}}} \mathcal{F}.$$

That is, $\sigma(\mathcal{G})$ is the intersection of all σ -algebras containing \mathcal{G} . Show that...

- (a) $\sigma(\mathcal{G})$ is a σ -algebra.
 - (b) $\sigma(\mathcal{G}) \subset \sigma(\mathcal{G}')$ whenever $\mathcal{G} \subset \mathcal{G}'$.
 - (c) If \mathcal{F} is a σ -algebra, then $\sigma(\mathcal{F}) = \mathcal{F}$.
 - (d) If \mathcal{F} is a σ -algebra and $\mathcal{G} \subset \mathcal{F}$, then $\sigma(\mathcal{G}) \subset \mathcal{F}$.
2. Show that the following two are algebras but not σ -algebras:
 - (a) All finite subsets of \mathbb{R} together with their complements.
 - (b) All finite unions of intervals in \mathbb{R} of the form $(a, b]$, $(-\infty, a]$, and (b, ∞) .
 3. Prove the principle of inclusion-exclusion. That is, show that if $A_1, \dots, A_n \in \mathcal{F}$, then

$$\begin{aligned} & \mathbb{P}(A_1 \cup \dots \cup A_n) \\ &= \sum_i \mathbb{P}(A_i) - \sum_{i < j} \mathbb{P}(A_i \cap A_j) + \sum_{i < j < k} \mathbb{P}(A_i \cap A_j \cap A_k) - \dots + (-1)^{n+1} \mathbb{P}(A_1 \cap \dots \cap A_n). \end{aligned}$$

Note that when we write $\sum_{i < j}$, we mean $\sum_{i,j \in \{(i,j): i < j\}}$ (and similarly for the sums involving more indices).

Solutions

1. To simplify notation, let

$$\mathcal{M}(\mathcal{G}) = \{\mathcal{F}: \mathcal{F} \text{ is a } \sigma\text{-algebra on } \Omega \text{ and } \mathcal{G} \subset \mathcal{F}\}$$

so that

$$\sigma(\mathcal{G}) = \bigcap_{\mathcal{F} \in \mathcal{M}(\mathcal{G})} \mathcal{F}.$$

Note that as a trivial consequence of the definition, $\sigma(\mathcal{G}) \supset \mathcal{G}$.

- (a) We claim that the intersection $\mathcal{F} = \bigcap_{\alpha} \mathcal{F}_{\alpha}$ of σ -algebras \mathcal{F}_{α} is itself a σ -algebra. Since $\sigma(\mathcal{G})$ is, by definition, the intersection of σ -algebras, the desired result follows from this claim. The claim is established by three points: (i) $\emptyset \in \mathcal{F}$ because $\emptyset \in \mathcal{F}_{\alpha}$ for each α . (ii) Suppose $A \in \mathcal{F}$. Then, $A \in \mathcal{F}_{\alpha}$ for each α and hence $A^c \in \mathcal{F}_{\alpha}$ for each α , from which it follows that $A^c \in \mathcal{F}$. (iii) Suppose $A_1, A_2, \dots \in \mathcal{F}$. Then, $A_1, A_2, \dots \in \mathcal{F}_{\alpha}$ for each α and hence $\bigcup_{n \geq 1} A_n \in \mathcal{F}_{\alpha}$ for each α , from which it follows that $\bigcup_{n \geq 1} A_n \in \mathcal{F}$.
 - (b) Suppose $\mathcal{G} \subset \mathcal{G}'$. Since any σ -algebra which contains \mathcal{G}' must also contain \mathcal{G} , we have $\mathcal{M}(\mathcal{G}) \supset \mathcal{M}(\mathcal{G}')$, from which the desired result follows.
 - (c) Suppose \mathcal{F} is a σ -algebra. Since $\mathcal{F} \in \mathcal{M}(\mathcal{F})$, it follows that $\sigma(\mathcal{F}) \subset \mathcal{F}$. Since we already know $\sigma(\mathcal{F}) \supset \mathcal{F}$, the desired result follows.
 - (d) If $\mathcal{G} \subset \mathcal{F}$, part (b) tells us $\sigma(\mathcal{G}) \subset \sigma(\mathcal{F})$. If \mathcal{F} is a σ -algebra, part (c) tells us $\sigma(\mathcal{F}) = \mathcal{F}$. Combining these two points, the desired result follows.
2. Checking that these are algebras is straightforward, so I will just point out why they are not σ -algebras:
 - (a) Consider $\mathbb{N} = \{1, 2, \dots\}$. Neither \mathbb{N} nor $\mathbb{N}^c = \mathbb{R} \setminus \mathbb{N}$ are finite subsets of \mathbb{R} , and hence \mathbb{N} is not in the algebra. However, we can write \mathbb{N} as a countable union of elements of the algebra: $\mathbb{N} = \bigcup_{n \geq 1} \{n\}$.
 - (b) The set $(-\infty, 0)$ can be written as a countable union of elements of the algebra: $(-\infty, 0) = \bigcup_{n \geq 1} (-\infty, -1/n]$.
 3. Suppose the claim holds for n . We attempt to establish it for $n + 1$. Note that

$$\begin{aligned} \mathbb{P}(A_1 \cup \dots \cup A_{n+1}) &= \mathbb{P}((A_1 \cup \dots \cup A_n) \cup A_{n+1}) \\ &= \mathbb{P}(A_1 \cup \dots \cup A_n) + \mathbb{P}(A_{n+1}) - \mathbb{P}((A_1 \cup \dots \cup A_n) \cap A_{n+1}) \end{aligned}$$

Let's handle each term in the sum separately. First, note that by our induction hypothesis,

$$\begin{aligned} &\mathbb{P}(A_1 \cup \dots \cup A_n) \\ &= \sum_{i \leq n} \mathbb{P}(A_i) - \sum_{i < j \leq n} \mathbb{P}(A_i \cap A_j) + \sum_{i < j < k \leq n} \mathbb{P}(A_i \cap A_j \cap A_k) - \dots + (-1)^{n+1} \mathbb{P}(A_1 \cap \dots \cap A_n). \end{aligned}$$

Similarly, applying our induction hypothesis,

$$\begin{aligned}
\mathbb{P}((A_1 \cup \dots \cup A_n) \cap A_{n+1}) &= \mathbb{P}((A_1 \cap A_{n+1}) \cup \dots \cup (A_n \cap A_{n+1})) \\
&= \sum_{i \leq n} \mathbb{P}(A_i \cap A_{n+1}) - \sum_{i < j \leq n} \mathbb{P}(A_i \cap A_j \cap A_{n+1}) + \sum_{i < j < k \leq n} \mathbb{P}(A_i \cap A_j \cap A_k \cap A_{n+1}) \\
&\quad - \dots + (-1)^{n+1} \mathbb{P}(A_1 \cap \dots \cap A_n \cap A_{n+1}).
\end{aligned}$$

Now, we can simplify our expression for $\mathbb{P}(A_1 \cup \dots \cup A_{n+1})$:

$$\begin{aligned}
\mathbb{P}(A_1 \cup \dots \cup A_{n+1}) &= \sum_{i \leq n+1} \mathbb{P}(A_i) - \sum_{i < j \leq n+1} \mathbb{P}(A_i \cap A_j) + \sum_{i < j < k \leq n+1} \mathbb{P}(A_i \cap A_j \cap A_k) \\
&\quad - \dots + (-1)^{n+2} \mathbb{P}(A_1 \cap \dots \cap A_{n+1}).
\end{aligned}$$