Math 525: Lecture 6

January 23, 2018

1 Moments

Definition 1.1. Let X be a discrete random variable and k be a positive integer. Suppose X^k is integrable. Then we call $\mathbb{E}[|X|^k]$ the k-th *absolute moment* of X, $\mathbb{E}[X^k]$ the k-th *raw moment* of X, and $\mathbb{E}[(X - \mathbb{E}[X])^k]$ the k-th *central moment* of X.

Note that the first moment is the expectation and the second central moment is the variance. The k-th raw moment is also sometimes simply called the k-th moment.

Example 1.2. Let X be a positive integer-valued random variable satisfying

$$\mathbb{P}(\{X = n\}) = c\frac{1}{n^3}$$

where c is a "normalizing constant" chosen such that

$$\sum_{n\geq 1} \mathbb{P}(\{X = n\}) = c \sum_{n \geq 1} \frac{1}{n^3} = 1.$$

This random variable has a finite expectation:

$$\mathbb{E}\left[X\right] = \sum_{n \geq 1} n \left(c\frac{1}{n^3} \right) = c \sum_{n\geq 1} \frac{1}{n^2} < \infty.$$

However, its variance is infinite:

$$\mathbb{E}\left[X^2\right] = \sum_{n\geq 1} n^2 \left(c\frac{1}{n^3} \right) = c \sum_{n \geq 1} \frac{1}{n} = \infty.$$

The same technique can be used to make a random variable whose first k moments are finite but all of its subsequent moments are infinite.

Proposition 1.3. *Let X and Y be discrete random variables and k be a positive integer. If X^k and Y^k are integrable, so too is $(X + Y)^k$.*

Proof. For any real numbers x and y,

$$|x+y|^k \le (2\max\{|x|,|y|\})^k = 2^k \max\{|x|^k,|y|^k\} \le 2^k|x|^k + 2^k|y|^k.$$

Therefore,

$$|X+Y|^k \le 2^k|X|^k + 2^k|Y|^k,$$

from which the desired result follows by taking expectations of both sides. □

Proposition 1.4. *Let X be a discrete random variable and k be a positive integer. If X^k is integrable, so too is X^j for each $0 \le j \le k$.*

Proof. For any real number $x \ge 0$,

$$x^j \le \max\left\{x^k, 1\right\} \le x^k + 1.$$

Therefore,

$$|X|^j \le |X|^k + 1,$$

from which the desired result follows by taking expectations of both sides. □

Corollary 1.5. *Let X be a discrete random variable and k be a positive integer. If X^k is integrable, so too is $(X - \mathbb{E}X)^k$ (and vice versa).*

It is understood that the statement $(X-\mathbb{E}X)^k$ is integrable requires also the integrability of X (otherwise we would not even be able to talk about $\mathbb{E}X$, let alone $(X-\mathbb{E}X)^k$).

Proof. Suppose X^k is integrable. Let $Y = -\mathbb{E}X$ and apply Proposition 1.3 to see that $(X-\mathbb{E}X)^k$ is integrable.

Suppose $(X-\mathbb{E}X)^k$ is integrable. Then,

$$|X|^k = |(X-\mathbb{E}X)+\mathbb{E}X|^k \le (|X-\mathbb{E}X| + |\mathbb{E}X|)^k \le \sum_{j=0}^{k}\binom{k}{j}|X-\mathbb{E}X|^j\,|\mathbb{E}X|^{k-j}.$$

Now take expectations of both sides and apply Proposition 1.4. □

2 Moment generating functions

Last lecture, we looked at the probability generating function G of a discrete **nonnegative integer-valued** random variable X,

$$G(t) = \mathbb{E}\left[t^X\right].$$

In this lecture, we start by letting X be **any** discrete random variable and examining the moment generating function M of X,

$$M(\theta) = \mathbb{E}\left[e^{\theta X}\right].$$

As usual, we have been a bit cavalier in defining M, which is only well-defined at values of $\theta \in \mathbb{R}$ for which the random variable $e^{\theta X}$ is integrable. Remember the Taylor series for e^x is

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots = \sum_{n \geq 0} \frac{1}{n!} x^n.$$

If we substitute this into $M(\theta)$, we obtain

$$M(\theta) = \mathbb{E}\left[\sum_{n \geq 0} \frac{\theta^n}{n!} X^n\right].$$

Now, we would like to distribute the expectation over the sum to conclude

$$M(\theta) = \sum_{n \geq 0} \frac{\theta^n}{n!} \mathbb{E}\left[X^n\right]. \tag{1}$$

However, while we know from last lecture that we can distribute the expectation over a **finite** sum (from the property $\mathbb{E}[aX + bY] = a\mathbb{E}X + b\mathbb{E}Y$), we cannot argue about infinite sums yet, so the conclusion (1) is just heuristic! We will defer a rigorous proof of this claim to a future lecture. For the time being, let's proceed assuming (1) is true. If we take derivatives with respect to θ,

$$\begin{aligned}
M'(\theta) &= \sum_{n \geq 1} \frac{\theta^{n-1}}{(n-1)!} \mathbb{E}\left[X^n\right] \\
M''(\theta) &= \sum_{n \geq 2} \frac{\theta^{n-2}}{(n-2)!} \mathbb{E}\left[X^n\right] \\
&\vdots \\
M^{(k)}(\theta) &= \sum_{n \geq k} \frac{\theta^{n-k}}{(n-k)!} \mathbb{E}\left[X^n\right]
\end{aligned}$$

and we can conclude

$$M^{(k)}(0) = \mathbb{E}\left[X^k\right], \qquad k = 1, 2, \ldots \tag{2}$$

Note that we have also ignored the fact that to evaluate the k-th derivative at θ_0, we require M to be defined in a neighborhood of θ_0. Regardless, if we proceed ignoring this issue, we deduce from (2) that the moment generating function generates the moments (perhaps unsurprisingly, given its name).

Remark 2.1. Note that $M(0) = 1$ since $M(0) = \mathbb{E}[X^0] = \mathbb{E}[1]$. This is true for any random variable, since 1 is integrable.

3 Special discrete distributions

There are a handful of discrete distributions which come up frequently in applications. Our last topic today is to study some of these special distributions and compute their moments.

3.1 Bernoulli

A random variable X has a *Bernoulli distribution* if

$$\mathbb{P}(\{X = 1\}) = p \qquad \text{and} \qquad \mathbb{P}(\{X = 0\}) = 1 - p$$

for some $0 \leq p \leq 1$. We will often simply write $X \sim \text{Bernoulli}(p)$ to indicate such a random variable.

Example 3.1. Toss a coin once, corresponding to the sample space $\Omega = \{H, T\}$. Define X by $X(H) = 1$ and $X(T) = 0$. Then, X has a Bernoulli distribution.

The moment generating function of $X \sim \text{Bernoulli}(p)$ is

$$M(\theta) = \mathbb{E}\left[e^{\theta X}\right] = e^{\theta \cdot 0}\mathbb{P}(\{X = 0\}) + e^{\theta \cdot 1}\mathbb{P}(\{X = 1\}) = (1 - p) + e^{\theta} p.$$

Note that $M^{(k)}(\theta) = e^{\theta} p$. Therefore, $\mathbb{E}X^k = M^{(k)}(0) = p$ for all $k = 1, 2, \ldots$ From this, it follows that

$$\text{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2 = p - p^2 = p\,(1 - p)\,.$$

3.2 Binomial

A random variable X has a *binomial distribution* with parameters $n \in \{1, 2, \ldots\}$ and $0 \leq p \leq 1$ if

$$\mathbb{P}(\{X = k\}) = \binom{n}{k} p^k\,(1 - p)^{n-k}\,, \qquad k = 0, 1, 2, \ldots, n.$$

We will often simply write $X \sim B(n, p)$ to indicate such a random variable. Note that the above implies that X only takes values in $\{0, 1, \ldots, n\}$ with positive probability:

Proposition 3.2. *Let $X \sim B(n, p)$. Then,*

$$\sum_{k=0}^{n} \mathbb{P}(\{X = k\}) = 1.$$

Proof. By the binomial theorem,

$$\sum_{k=0}^{n} \mathbb{P}(\{X = k\}) = \sum_{k=0}^{n} \binom{n}{k} p^k\,(1 - p)^{n-k} = (p + 1 - p)^n = 1^n = 1. \qquad \square$$

Example 3.3. Toss the same coin n times. Let X be the number of heads witnessed in all n coin tosses. Assume that the probability of getting heads on each toss is $0 \leq p \leq 1$. Then, $X \sim B(n, p)$.

To see this, consider the case in which the first k tosses result in heads (H) and the remainder result in tails (T). This is captured by the sample

$$\underbrace{HH \cdots H}_{k \text{ times}}\underbrace{TT \cdots T}_{n-k \text{ times}}.$$

This sample occurs with probability $p^k(1 - p)^{n-k}$. However, there are $\binom{n}{k}$ permutations of the letters above, from which we obtain the expression

$$\mathbb{P}(\{X = k\}) = \binom{n}{k} p^k\,(1 - p)^{n-k}\,.$$

The moment generating function of $X \sim B(n, p)$ is

$$M(\theta) = \mathbb{E}\left[e^{\theta X}\right] = \sum_{k=0}^{n} e^{\theta k}\mathbb{P}(\{X = k\}) = \sum_{k=0}^{n} \binom{n}{k} \left(e^{\theta} p\right)^{k} (1-p)^{n-k} = \left(\left(e^{\theta} - 1\right) p + 1\right)^{n}.$$

Taking derivatives,

$$\begin{aligned} M'(\theta) &= e^{\theta} np M(\theta)^{(n-1)/n} \\ M''(\theta) &= M'(\theta) + e^{2\theta} (n-1) np^2 M(\theta)^{(n-2)/n}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}X &= M'(0) = M(0)^{(n-1)/n} np = np \\ \mathbb{E}\left[X^2\right] &= M''(0) = M'(0) + M(0)^{(n-2)/n} (n-1) np^2 = np(1 + (n-1)p) \end{aligned}$$

and hence

$$\operatorname{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2 = np(1 + (n-1)p) - (np)^2 = np(1-p).$$

3.3 Poisson

A random variable X has a *Poisson distribution* with parameter $\lambda > 0$ if

$$\mathbb{P}(\{X = k\}) = \frac{\lambda^k}{k!} e^{-\lambda}, \qquad k = 0, 1, 2, \ldots$$

We will often simply write $X \sim \text{Poisson}(\lambda)$ to indicate such a random variable. Note that the above implies that X only takes values in $\{0, 1, 2, \ldots\}$ with positive probability:

Proposition 3.4. *Let* $X \sim \text{Poisson}(\lambda)$. *Then,*

$$\sum_{k \geq 0} \mathbb{P}(\{X = k\}) = 1.$$

Proof. By the Taylor expansion of e^x,

$$\sum_{k \geq 0} \mathbb{P}(\{X = k\}) = \sum_{k \geq 0} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k \geq 0} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1. \qquad \square$$

Before we motivate the Poisson distribution, let's blindly compute its moment generating function:

$$M(\theta) = \mathbb{E}\left[e^{\theta X}\right] = \sum_{k \geq 0} e^{\theta k} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k \geq 0} \frac{\left(\lambda e^{\theta}\right)^k}{k!} = e^{-\lambda} e^{\lambda e^{\theta}} = e^{\lambda\left(e^{\theta} - 1\right)}.$$

Taking derivatives,

$$\begin{aligned} M'(\theta) &= \lambda e^{\theta} M(\theta) \\ M''(\theta) &= M'(\theta)\left(\lambda e^{\theta} + 1\right) \end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}X &= M'(0) = \lambda e^0 M(0) = \lambda \\ \mathbb{E}\left[X^2\right] &= M''(0) = M'(0)\left(\lambda e^0 + 1\right) = \lambda(\lambda+1)\end{aligned}$$

and hence

$$\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2 = \lambda(\lambda+1) - \lambda^2 = \lambda.$$

One way to motivate the Poisson distribution is through the following observation:

Proposition 3.5. *Let $\lambda > 0$ and suppose that $np \to \lambda$ as $n \to \infty$. Then,*

$$\lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}, \qquad k = 0, 1, 2, \ldots$$

We recognize the left hand side in the above from $B(n,p)$. The above suggests that Poisson(np) captures the number of successes in n trials, each having probability p, as the number of trials becomes large.

Example 3.6. The number of market crashes per annum could be modelled as a Poisson(λ) random variable with, for example, $\lambda = 0.1$ (one crash every ten years).