

Math 525: Assignment 3

1. (Borel measurable) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous.

(a) Let

$$\mathcal{G} = \{(a, b): -\infty < a < b < \infty\}$$

be the set of all open intervals. Show $\sigma(\mathcal{G}) = \mathcal{B}(\mathbb{R})$.¹

(b) Define

$$\mathcal{M} = \{B \subset \mathbb{R}: f^{-1}(B) \in \mathcal{B}(\mathbb{R})\}.$$

Show that \mathcal{M} is a σ -algebra.² This establishes $\sigma(\mathcal{M}) = \mathcal{M}$.

(c) (Optional) Show that for any $G \in \mathcal{G}$, $f^{-1}(G)$ is a countable union of open intervals. This establishes $\mathcal{G} \subset \mathcal{M}$.

(d) Use (a), (b), and (c) to conclude that $\mathcal{B}(\mathbb{R}) \subset \mathcal{M}$ and hence f is Borel measurable.

2. (Distribution function) Let X be a discrete random variable with distribution function F . Show that $\sum_n F(x_n) - F(x_n-) = 1$.

3. (Uniform random variable) Define the function $f: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and the function F by $F(x) = \int_{-\infty}^x f(y)dy$. Let $Y \sim U[0, 1]$. Define $X = F^{-1}(Y)$.

(a) Simplify the expression for X as much as possible.

(b) What is the distribution function of X ?

4. (Expectation) Let X_1, X_2, \dots be nonnegative integer-valued random variables. Suppose they are independent and have the same distribution function F (in this case, we say X_1, X_2, \dots are *independent and identically distributed*, or i.i.d. for short). Show that³

$$\mathbb{E} \min\{X_1, \dots, X_m\} = \sum_{n=1}^{\infty} \mathbb{P}\{X_1 \geq n\}^m.$$

5. (Integrability) Let X be a discrete random variable. Suppose X^2 is integrable. Show that X is integrable.⁴

¹**Hint:** use the fact that $(x, \infty) = (x, x+2) \cup (x+1, x+3) \cup \dots$ and $(-\infty, x] = \mathbb{R} \setminus (x, \infty)$

²**Hint:** use the properties of f^{-1} from the previous assignment

³**Hint:** use the fact that $\mathbb{E}Y = \sum_{n \geq 1} \mathbb{P}(\{Y \geq n\})$ for a nonnegative integer-valued random variable Y

⁴**Hint:** use the inequality $|a| \leq \max\{1, |a|^2\}$