## Math 525: Assignment 2

- 1. (Independence) Let A and B be independent events. Show that  $A^c = \Omega \setminus A$  and  $B^c = \Omega \setminus B$  are independent. Also show that A and  $B^c$  are independent (and so too, by symmetry, are  $A^c$  and B).
- 2. (Conditional probability) Roll two (fair) dice. What is the probability that at least one of the two dice is four given that their sum is seven?
- 3. (Conditional probability) Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Let  $B \in \mathcal{F}$  be an event with  $\mathbb{P}(B) > 0$ . Define the function  $\mathbb{T}: \mathcal{F} \to [0, 1]$  by  $\mathbb{T}(A) = \mathbb{P}(A \mid B)$ .
  - (a) Show that  $(\Omega, \mathcal{F}, \mathbb{T})$  is a probability space. **Hint**: since we already know that  $\mathcal{F}$  is a  $\sigma$ -algebra on  $\Omega$ , we need only check that  $\mathbb{T}$  is a probability measure.
  - (b) Why did we require  $\mathbb{P}(B) > 0$ ?
- 4. (Counting) Consider a circular table with  $N \geq 4$  seats. Barbie and Ken are two of N dinner guests to be seated at this table. In how many ways can the dinner guests be seated such that...
  - (a) Barbie and Ken are not adjacent.
  - (b) Barbie and Ken are adjacent.
- 5. (Counting) Prove the binomial theorem. That is, prove that for a positive integer n and real numbers a and b,

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}.$$

**Hint**: an easy way to do this is with induction.

- 6. (Inverse images) Let  $f: A \to B$ . Show that for any  $H, J \subset B$ ,
  - (a)  $f^{-1}(H \cup J) = f^{-1}(H) \cup f^{-1}(J)$ .
  - (b)  $f^{-1}(H^c) = (f^{-1}(H))^c$ .
  - (c)  $f^{-1}(H \cap J) = f^{-1}(H) \cap f^{-1}(J)$ . **Hint**: use parts (a) and (b).
- 7. (Random variables) Let  $(X_n)_{n\geq 1}$  be a sequence of random variables. Prove that  $\sup_n X_n$  is also a random variable.