

Math 525: Assignment 1

1. Sometimes it is useful to start with a small set of events \mathcal{G} that is not necessarily a σ -algebra and “generate” a σ -algebra from it. The σ -algebra generated from \mathcal{G} is

$$\sigma(\mathcal{G}) = \bigcap_{\substack{\mathcal{F} \text{ is a } \sigma\text{-algebra on } \Omega \\ \mathcal{G} \subset \mathcal{F}}} \mathcal{F}.$$

That is, $\sigma(\mathcal{G})$ is the intersection of all σ -algebras containing \mathcal{G} . Show that...

- (a) $\sigma(\mathcal{G})$ is a σ -algebra.
 - (b) $\sigma(\mathcal{G}) \subset \sigma(\mathcal{G}')$ whenever $\mathcal{G} \subset \mathcal{G}'$.
 - (c) If \mathcal{F} is a σ -algebra, then $\sigma(\mathcal{F}) = \mathcal{F}$.
 - (d) If \mathcal{F} is a σ -algebra and $\mathcal{G} \subset \mathcal{F}$, then $\sigma(\mathcal{G}) \subset \mathcal{F}$.
2. Show that the following two are algebras but not σ -algebras:
 - (a) All finite subsets of \mathbb{R} together with their complements.
 - (b) All finite unions of intervals in \mathbb{R} of the form $(a, b]$, $(-\infty, a]$, and (b, ∞) .
 3. Prove the principle of inclusion-exclusion. That is, show that if $A_1, \dots, A_n \in \mathcal{F}$, then

$$\begin{aligned} & \mathbb{P}(A_1 \cup \dots \cup A_n) \\ &= \sum_i \mathbb{P}(A_i) - \sum_{i < j} \mathbb{P}(A_i \cap A_j) + \sum_{i < j < k} \mathbb{P}(A_i \cap A_j \cap A_k) - \dots + (-1)^{n+1} \mathbb{P}(A_1 \cap \dots \cap A_n). \end{aligned}$$

Note that when we write $\sum_{i < j}$, we mean $\sum_{i, j \in \{(i, j) : i < j\}}$ (and similarly for the sums involving more indices).