Math 525: Assignment 8

To answer the first question, you will need a theorem from linear algebra. Recall that a matrix/vector is said to be nonnegative/positive if all of its entries are nonnegative/positive.

Definition. Let A be a nonnegative matrix. We say A is *primitive* if A^m is a positive matrix for some positive integer m.

Theorem (Perron-Frobenius for primitive matrices)**.** *Let A be a primitive matrix. Then,*

- $\lambda_1 = \rho(A)$ *is an eigenvalue of A.*
- *All other eigenvalues λ_j are smaller in magnitude than λ_1 (i.e., $|\lambda_j| < \lambda_1$).*
- $\{x \colon Ax = \lambda_1 x\} = \mathit{span}(v)$ *for some positive vector v.*

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1. A Markov chain whose transition matrix P is primitive is called *regular*. Let
   - $\mu = (\mu_1, \ldots, \mu_m)^\intercal$ be a column vector with $\mu^\intercal e = 1$ where $e = (1, \ldots, 1)^\intercal$,
   - P be the transition matrix of a regular Markov chain, and
   - suppose that $P^\intercal$ has a linearly independent set of eigenvectors $\{v_1, \ldots, v_m\}$ with corresponding eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$ in descending order of magnitude:

   $$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_m|.$$

   (a) Show that if $\mu^\intercal e = 1$, then $\mu^\intercal P e = 1$.

   (b) Let $\mu = c_1 v_1 + \cdots + c_m v_m$ be an eigendecomposition of μ. By the Perron-Frobenius theorem, we can, without loss of generality, pick v_1 to be a positive vector. Show that for any positive integer k,

   $$\mu^\intercal P^k = c_1 \lambda_1^k v_1^\intercal + \cdots + c_m \lambda_m^k v_m^\intercal.$$

   (c) Show that $\lim_{n\to\infty} \mu^\intercal P^n = c_1 v_1^\intercal$ (**Hint**: Proposition 1.13 of Lecture 16).

   (d) Show that $(c_1 v_1^\intercal) P = c_1 v_1^\intercal$.

   (e) Show that $(c_1 v_1^\intercal) e = 1$ (**Hint**: part (a)) and $c_1 > 0$.

   (f) Part (e) shows that $c_1 v_1^\intercal$ is a distribution vector (i.e., it is nonnegative and its entries add up to one). With this in mind, what is the significance of

      i. Part (c)? (**Hint**: if $\mathbb{P}(X_0 = i) = \mu_i$, then $(\mu^\intercal P^n)_i = \mathbb{P}(X_n = i)$).

      ii. Part (d)?

2. A coin with probability p of heads is flipped repeatedly. X_n is the result of the n-th coin flip ($n = 1$ is the first coin flip). Let

$$\tau = \inf \{n > 1 \colon (X_{n-1}, X_n) = (H, T)\},$$

corresponding to the first time at which we see a heads (H) followed by a tails (T).

(a) Give an expression for $\mathbb{P}(\tau \leq n)$ (**Hint**: try to think of a transition matrix P such that $\mathbb{P}(\tau \leq n) = \mu^\intercal P^n \nu$ for appropriately chosen column vectors μ and ν).

(b) Give an expression for $\mathbb{P}(\tau = n)$ (**Hint**: use your results from part (b)).

(c) (Optional) Let $p = 1/3$ and compute $\mathbb{P}(\tau \leq 10)$ with numerical software such as MATLAB.