Math 525: Assignment 9 Solutions

1. Let $U = \min(X, Y)$ and $V = \max(X, Y)$ for brevity. Note that $U \leq u$ if and only if $X \leq u$ <u>or</u> $Y \leq u$. Similarly, $V \leq v$ if and only if $X \leq v$ <u>and</u> $Y \leq v$. Therefore,

$$F_{UV}(u, v) \equiv \mathbb{P}(U \leq u, V \leq v) = \begin{cases} v^2 & \text{if } 0 \leq v \leq u \leq 1 \\ 2uv - u^2 & \text{if } 0 \leq u \leq v \leq 1 \end{cases}$$

(it might help to draw a diagram to understand the above). Differentiating away from the discontinuity,

$$f_{UV}(u, v) = \frac{\partial^2 F_{UV}}{\partial u \partial v}(u, v) = \begin{cases} 2 & \text{if } 0 \leq v < u \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

You can verify that this is the density by integrating.

2. We first show that $X \sim \mathcal{N}(0, 1)$ ($Y \sim \mathcal{N}(0, 1)$ is obtained similarly). The marginal density is

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) = \begin{cases} \int_0^\infty \frac{1}{\pi} e^{-(x^2+y^2)/2} dy & \text{if } x \geq 0 \\ \int_{-\infty}^0 \frac{1}{\pi} e^{-(x^2+y^2)/2} dy & \text{if } x < 0. \end{cases}$$

By symmetry,

$$f_X(x) = \int_0^\infty \frac{1}{\pi} e^{-(x^2+y^2)/2} dy = \frac{1}{\pi} e^{-x^2/2} \int_0^\infty e^{-y^2/2} dy = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

To see that X and Y are not independent, note that by symmetry, the quantity

$$\mathbb{E}[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x, y) dx dy = \frac{2}{\pi} \int_0^\infty \int_0^\infty xy e^{-(x^2+y^2)/2} dx dy$$

is positive (and hence not equal to $\mathbb{E}X\mathbb{E}Y = 0$).

3.

(a) To see that $X \sim \mathcal{N}(0, 1)$, compare its characteristic function to that of a $\mathcal{N}(0, 1)$ random variable:

$$\begin{aligned} \varphi_X(t) = \varphi_{aU+bV}(t) = \varphi_{aU}(t)\varphi_{bV}(t) &= \exp\left(-\frac{a^2t^2}{2}\right) \exp\left(-\frac{b^2t^2}{2}\right) \\ &= \exp\left(-\frac{(a^2+b^2)t^2}{2}\right) = \exp\left(-\frac{t^2}{2}\right). \end{aligned}$$

(b) Note that

$$\mathbb{E}[XY] = \mathbb{E}[(aU + bV)(cU + dV)] = \mathbb{E}\left[acU^2 + 2(ad + bc)UV + bdV^2\right]$$
$$= ac\mathbb{E}\left[U^2\right] + 2(ad + bc)\mathbb{E}U\mathbb{E}V + bd\mathbb{E}\left[V^2\right] = ac + bd = \rho.$$

(c) The joint characteristic function of X and Y is

$$\varphi_{XY}(t,s) = \mathbb{E}[\exp(itX + isY)] = \mathbb{E}[\exp(i(ta + sc)U + i(tb + sd)V)]$$
$$= \varphi_U(ta + sc)\varphi_V(tb + sd) = \exp\left(-\frac{(ta + sc)^2}{2}\right)\exp\left(-\frac{(tb + sd)^2}{2}\right)$$
$$= \exp\left(-\frac{(a^2 + b^2)t^2 + 2(ac + bd)st + (c^2 + d^2)s^2}{2}\right) = \exp\left(-\frac{t^2 + 2\rho st + s^2}{2}\right).$$

(d) The inverse Fourier transform of the joint characteristic function is

$$f_{XY}(x,y) = \mathcal{F}^{-1}[\varphi_{XY}](x,y)$$
$$= \frac{1}{(2\pi)^2}\int\int_{\mathbb{R}^2} \varphi_{XY}(t,s)\exp(-it(tx + sy))\,dtds$$
$$= \frac{1}{(2\pi)^2}\int\int_{\mathbb{R}^2} \exp\left(-\frac{t^2 + 2\rho st + s^2}{2}\right)\exp(-i(tx + sy))\,dtds.$$

Note that the term $t^2 + 2\rho st + s^2$ is nothing other than the quadratic form

$$\begin{pmatrix} t & s \end{pmatrix}\Sigma\begin{pmatrix} t \\ s \end{pmatrix} \qquad \text{where} \qquad \Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Therefore, letting $\mathbf{t} = (t,s)^\intercal$ and $\mathbf{x} = (x,y)^\intercal$, we can rewrite the integral as

$$\mathcal{I}(\mathbf{x}) \equiv \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \exp\left(-\frac{1}{2}\mathbf{t}^\intercal\Sigma\mathbf{t}\right)\exp\left(-i\mathbf{t}^\intercal\mathbf{x}\right)d\mathbf{t}.$$

Moreover, we have the following factorization of Σ:

$$\Sigma = U^\intercal U \qquad \text{where} \qquad U = \begin{pmatrix} 1 & \frac{\rho}{\sqrt{1-\rho^2}} \end{pmatrix}.$$

so that $\mathbf{t}^\intercal\Sigma\mathbf{t} = (U\mathbf{t})^\intercal(U\mathbf{t})$. This should inspire the change of variables $\mathbf{u} = U\mathbf{t}$. Note that the substitution has Jacobian

$$\det(U^{-1}) = \frac{1}{\det(U)} = \frac{1}{\sqrt{1-\rho^2}}.$$

Performing the substitution,

$$\mathcal{I}(\mathbf{x}) = \frac{1}{\sqrt{1-\rho^2}}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \exp\left(-\frac{1}{2}\mathbf{u}^\intercal\mathbf{u}\right)\exp\left(-i\mathbf{u}^\intercal U^{-\intercal}\mathbf{x}\right)d\mathbf{u}.$$

It is straightforward to derive the following inverse Fourier transform:

$$\mathcal{F}^{-1}\left[\mathbf{u} \mapsto \exp\left(-\frac{1}{2}\mathbf{u}^\intercal\mathbf{u}\right)\right] = \mathbf{y} \mapsto \frac{1}{2\pi}\exp\left(-\frac{1}{2}\mathbf{y}^\intercal\mathbf{y}\right). \tag{1}$$

Now, note that $\mathcal{I}(\mathbf{x})$ is nothing other (1) evaluated at $\mathbf{y} = U^{-1}\mathbf{x}$, with a multiplicative factor of $1/\sqrt{1-\rho^2}$. That is,

$$\mathcal{I}(\mathbf{x}) = \frac{1}{\sqrt{1-\rho^2}}\frac{1}{2\pi}\exp\left(-\frac{1}{2}(U^{-1}\mathbf{x})^\intercal(U^{-1}\mathbf{x})\right).$$

Now, a straightforward computation reveals that

$$(U^{-1}\mathbf{x})^\intercal(U^{-1}\mathbf{x}) = \frac{x^2 - 2\rho xy + y^2}{1-\rho^2}.$$

(e) Part (b) tell us

$$\rho \neq 0 \implies X, Y \text{ dependent},$$

which is equivalent to saying

$$X, Y \text{ independent} \implies \rho = 0$$

by modus tollens. The converse of this claim is obtained by noting that if $\rho = 0$, then the joint density becomes a product of standard normal densities:

$$f_{XY}(x,y) = \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right) = \underbrace{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)}_{f(x)}\underbrace{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)}_{f(y)}.$$

Therefore, X and Y are independent.