DQ - PROBABILITY

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1. 2012.02

Problem 1.1.

(a) $(\tau = n) = \{S_n > 0, S_k \leq 0 \ \forall k < n\} \in \mathcal{B}$. *And* $\tau < \infty$. *Thus the result follows.*

(b) *Note that* $S_\tau > 0$ *and* $\tau < \infty$ *for all* $\omega \in \Omega$. *Thus, to* S_τ *be random variable, it is enough to check that* $(S_\tau < a)$ *is measurable for* $a > 0$.

$$(S_\tau < a) = \bigcup_{n=1}^{\infty} \{\tau = n\} \cap \{S_n < a\}$$

Therefore S_τ *is a random variable.*

$\square$

Problem 1.2.

Since $P(N < \infty) = 1$, $X_{n \wedge N} \to X_N$ *almost surely. Since* $X_{n \wedge N}$ *is uniformly bounded, bounded convergence theorem implies* $X_{n \wedge N} \to X_N$ *in* L_1. *Thus*

$$EX_{n \wedge N} \to EX_N.$$

But $EX_0 = EX_{n \wedge N}$ *since* $X_{n \wedge N}$ *is a martingale. Therefore the result follows.*

$\square$

2. 2020.08

Problem 2.1. *Use the following formula:*

$$\int_{(a,b]} GdF + FdG = F(b)G(b) - F(a)G(a) + \sum_{x\in(a,b]} \big(F(x) - F(x^-)\big).$$

Note that the last term is the sum of atoms.

In our case, $G(x) = x^3$.

□

Problem 2.2.

$$\frac{\sum X_k}{\sqrt{\sum X_k^2}} = \frac{\sum X_k/n}{\sigma/\sqrt{n}} \frac{\sigma}{\sqrt{\sum X_k^2/n}}$$

Now apply the Central limit theorem, law of large numbers, and Slutsky's theorem.

□

Problem 2.3. *Let X_i's are iid random variables which follow exponential distribution of rate 2. Then given integral is equal to*

$$\mathbb{E}\cos\left(\frac{X_1 + \cdots X_n}{n}\right).$$

By law of large numbers,

$$\frac{X_1 + \cdots X_n}{n} \to_p \mathbb{E}X_1 = \frac{1}{2}.$$

Since cos *is bounded continuous function, by applying bounded convergence theorem, given limit is equal to* $\cos 1/2$.

□

Problem 2.4. *Observe that S_n is a martingale. Thus $|S_n|^2$ is a submartingale, and we will apply Doob's inequality.*

Let $A = \big\{\max_{1\le k\le n} |S_k|^2 \ge \lambda^2\big\}$. Then by Doob's inequality,

$$\lambda^2 P(A) \le \mathbb{E}S_n^2 1_A \le \mathbb{E}|S_n|^2 = \operatorname{Var} S_n.$$

But $A = \{\max_{1\le k\le n} |S_k| \ge \lambda\}$. Therefore we get the result.

□

Problem 2.5. *Consider the corresponding quadratic martingale $M_n = S_n^2 - n\sigma^2 = S_n^2 - n$. Then $M_{n\wedge T}$ is also a martingale. Thus, $\mathbb{E}M_{n\wedge T} = \mathbb{E}M_0 = 0$ implies*

$$\mathbb{E}S_{n\wedge T}^2 = \mathbb{E}n \wedge T.$$

But the right hand side goes to $\mathbb{E}T$ by monotone convergence theorem. And note that

$$|S_{n\wedge T}| \le \max(|a|, |b|).$$

Thus bounded convergence theorem implies $\mathbb{E}S_{n\wedge T}^2 \to \mathbb{E}S_T^2$.

$$\mathbb{E}S_T^2 = P(S_T = a)a^2 + P(S_T = b)b^2 = (x-a)(b-x)$$

To complete the above, we must show that $T < \infty$ a.s. This can be done by considering

$$P\left(T > m(b-a)\right) \le \left(1 - (1/2)^{b-a}\right)^m.$$

□