# mas550 homework

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**Problem** (1.1.2).

*Let* $A = \Pi_{i=1}^{d}(a_i, b_i]$*. Then*

$$A = \left(\Pi_{i=1}^{d}[a_i - 1, b_i]\right) \cap \left(\Pi_{I=1}^{d}(a_i, b_i + 1)\right)$$

*which is intersection of open set and closed set. So,* $A \in \mathcal{R}^d$ *therefore* $\sigma(S_d) \subset \mathcal{R}^d$.

*On the other hand, let* $B = \Pi_{i=1}^{d}(a_i, b_i)$ *where* $-\infty < a_i < b_i < \infty$*. We can choose sequences* $\{a_{i,j}\}_{j=1}^{\infty}$ *and* $\{b_{i,j}\}_{j=1}^{\infty}$ *for each* $1 \leq i \leq d$ *such that* $a_{i,j} \downarrow a_i$ *and* $b_{i,j} \uparrow b_i$*. Then* $B_n = \Pi_{i=1}^{d}(a_{i,n}, b_{i,n}] \uparrow B$*. So* $B$ *is a countable union of open rectangles, hence* $B \in \sigma(S_d)$*. Since such* $B$ *forms basis of topology on* $\mathbb{R}^d$*, we can conclude that* $\mathcal{R}^d \subset \sigma(S_d)$.

**Problem** (1.2.3).

*Let* $F$ *be a distribution function. It is nonnegative, nondecreasing. So* $\lim_{y \downarrow x} F(y)$ *and* $\lim_{y \uparrow x} F(y)$ *always exist. Let* $x$ *be a point where* $F$ *is discontinuous. Since* $F$ *is discontinuous at* $x$*, we can assume without loss of generality* $\lim_{y \downarrow x} F(y) > F(x)$*. Choose a rational number* $q_x \in (F(x), \lim_{y \downarrow x} F(y))$*. Then function* $x \mapsto q_x$ *is injective since* $F$ *is nondecreasing. So there is injection from set of discontinuities to rational numbers. Now we can conclude that set of discontinuities is at most countable.*

**Problem** (1.3.4).

(a) *Let* $f : \mathbb{R}^d \to \mathbb{R}$ *be a continuous function. Consider* $\mathcal{B} = \left\{U \subset \mathbb{R} : f^{-1}(U) \in \mathcal{R}^d\right\}$*. It is well known that* $\mathcal{B}$ *is a* $\sigma$*-field. By continuity of* $f$*,* $\mathcal{B}$ *contains every open set of* $\mathbb{R}$*, hence* $\mathcal{R} \subset \mathcal{B}$*. Therefore* $f$ *is a measurable function.*

(b) *Let* $\mathcal{F}$ *be a* $\sigma$*-field that makes all the continuous functions measurable. Let* $\pi_i : \mathbb{R}^d \to \mathbb{R}$ *be the projection on* $i$*-th factor, which is continuous. Then* $\cap_{i=1}^{d}\pi_i^{-1}((a_i, b_i)) = \Pi_{i=1}^{d}(a_i, b_i) \in \mathcal{F}$*. Since* $\mathcal{F}$ *contains every open rectangles in* $\mathbb{R}^d$*, we can conclude that* $\mathcal{R}^d \subset \mathcal{F}$*. This means* $\mathcal{R}^d$ *is the smallest such* $\sigma$*-field. The fact that* $\mathcal{R}^d$ *makes all the continuous functions measurable is written in (a).*