MAS541 HW6.

#1.

a) $f: z \mapsto (z-i)^2$ on upper half plane is onto function.
$\mathbb{H} \to \mathbb{C}$

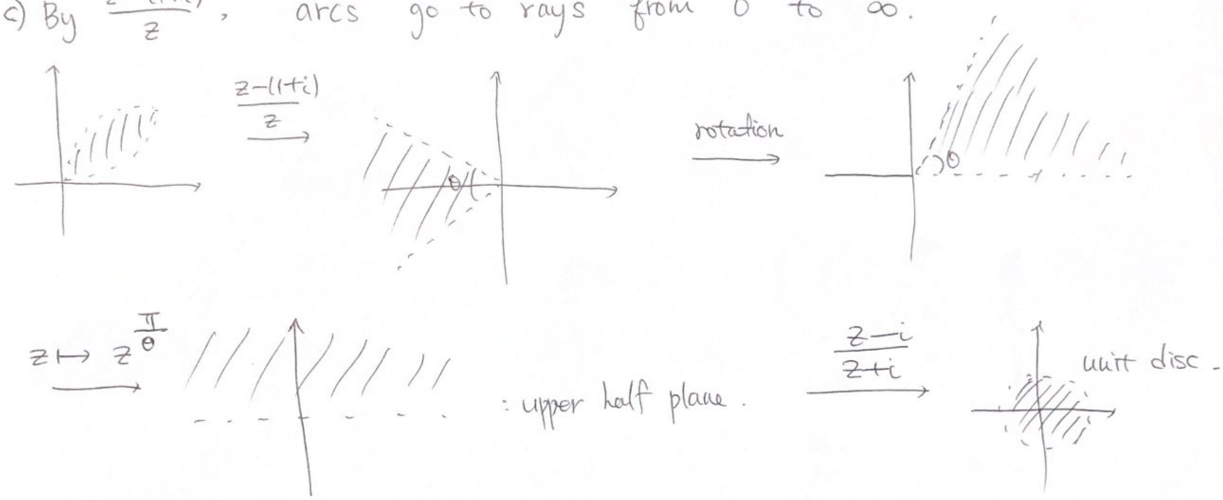
$g: z \mapsto \frac{z-i}{z+i}$ on upper half plane is biholomorphic map onto the unit disk.

$\therefore$ ~~$f: z \mapsto (z-i)^2$~~ $\therefore f \circ g^{-1}$ is hlmrph map of the unit disc onto $\mathbb{C}$. ∎

b). $f: z \mapsto z^2$ is biholomorphic function of upper half plane onto $\mathbb{C}\setminus[0,\infty)$
$\mathbb{H} \to \mathbb{C}\setminus[0,\infty)$

$\therefore$ $f \circ g^{-1}$ is bihlmrph ftn of the unit disc onto $\mathbb{C}\setminus[0,\infty)$. ∎

c) By $\frac{z-(1+i)}{z}$, arcs go to rays from 0 to ∞.



$\therefore$ composition of above : $\Omega_3 \to$ unit disc, biholomorphic. ∎.

(calculating θ and rotation angle is always possible.)

#2. $f \in Aut(\mathbb{C} \setminus \{0\})$.

f has isolated singularity at 0.

① f has essential singularity at 0.

$f(D(1,\frac{1}{3}))$: open. $f(1) \in f(D(1,\frac{1}{3}))$

so $\exists\ \varepsilon > 0$ s.t. $D(f(1),\varepsilon) \subset f(D(1,\frac{1}{3}))$.

Choose $|\alpha| < \frac{1}{3}$ so that.

$f(\alpha) \in D(f(1),\varepsilon)$. (by Weierstrass, Casorati thm).

$\Rightarrow f(\alpha) \in f(D(1,\frac{1}{3}))$ but $\alpha \notin D(1,\frac{1}{3})$.

this contradicts to injectiveness of f.

② f has removable singularity at 0.

Then f extends to entire ftn f.

If $f(0) \neq 0$, $f(0) = \alpha$,

then for $w \in \mathbb{C} \setminus \{0\}$: $f(w) = \alpha$,

Consider $U_w \cap U_0 = \phi$.

$\rightarrow \underbrace{f(U_w) \cap f(U_0)}_{\text{open}} \ni \alpha$

($\Rightarrow\!\Leftarrow$).

$\therefore f(0) = 0$. $\Rightarrow f : \mathbb{C} \to \mathbb{C}$, biholomorphic,

$f = az + b$, $f : 0 \mapsto 0$. $\therefore \underline{az}$ $(a \neq 0)$

③ pole at 0.

$\rightarrow \frac{1}{f} : 0 \longmapsto 0$

$\therefore \frac{1}{f} = az$. $\therefore f = \frac{1}{az}$ $(a \neq 0)$

∎

$f \in Aut(\mathbb{C} \setminus \{P_1, \cdots, P_k\})$

By same reasoning, f cannot have ess. singularity at each P_i and ∞. $\Rightarrow f$: meromorphic on $\hat{\mathbb{C}}$.

$\therefore f$ can be uniquely extended to hlmrph ftn of $\hat{\mathbb{C}}$ into $\hat{\mathbb{C}}$.

If f has poles at P_1, P_2, then $\exists\ \delta_1, \delta_2 > 0$ s.t.

$|f| > 1$ on $D'(P_1,\delta_1) \cup D'(P_2,\delta_2) = U$.

$\therefore \frac{1}{f}$ is well defined on U, removable singularity at P_1.

Since $\frac{1}{f} \to 0$ as $z \to P_1$, $\frac{1}{f}(P_1) = 0$.

Then $\frac{1}{f}(P_2) \neq 0$ by ② of left-side. ($\Rightarrow\!\Leftarrow$)

$\therefore f$ has at most 1 pole.

Thus f must permute $\{P_1, \cdots, P_k, \infty\}$.

and f : linear fractional transformation.

If $k=2 \Rightarrow f(z) = \frac{P_1(z-P_1) - P_2(P_1-P_2)}{z-P_2}$ $(P_1 \mapsto P_2 \mapsto \infty \mapsto P_1)$

$k > 2 \Rightarrow$ rotation or identity.

(if rotation can permute $\{P_1, \cdots, P_k\}$).

∎

#3. a) $f_1 = f$. $\quad f_{k+1} = f \circ f_k$. Since Ω is bdd,

$\mathcal{F} = \{f_n : n \geq 1\}$ is a normal family. (By 1st Montel Thm).

$\therefore \exists \{n_k\}$ s.t. $f_{n_k} \to F$ normally.

$\therefore \quad f'_{n_k}(P) \to F'(P)$, but $f'_{n_k}(P) = [f'(P)]^{n_k}$.

If $|f'(P)| > 1$, $\quad |f'(P)|^{n_k} \to \infty$. $\quad (\Rightarrow\Leftarrow)$.

$\therefore \quad |f'(P)| \leq 1$. $\blacksquare$

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b) By a), $\quad |\phi'(P)| \leq 1$.

But $P \in \Omega$, $\quad \phi'(P) = 1$.

$\therefore$ ϕ' attains its maximum modulus at P. (interior)

$\therefore$ ϕ' must be constant by max. modulus principle.

$\therefore \phi'(z) = 1$, $\quad \therefore \phi(z) = z + C$.

But $\phi(P) = P$. $\quad \therefore C = 0$.

$\therefore \quad \phi(z) = z$, identity. $\blacksquare$

#4. Fix cpt subset K of the unit disc.

Let $r = \sup_{z \in K} |z|$, $\underline{0 \le r < 1}$.

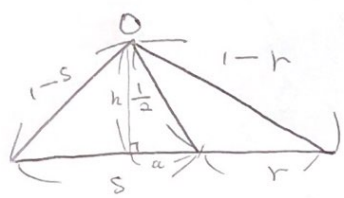
$\Rightarrow \forall f \in \mathcal{F}, \quad |f(z)| \le \sum a_n |z^n| \le \sum n \cdot r^n < \infty.$

$\therefore$ $\mathcal{F}$ is uniformly bounded by $\sum n \cdot r^n$ on K.

$\therefore$ $\mathcal{F}$ is bounded on cpt subsets.

$\Rightarrow$ 2nd Montel thm says that $\mathcal{F}$ is a normal family. ▨

#5. O : the origin.



wlog, assume $s \ge r$. $(s = r \Rightarrow a = 0)$.

$\Rightarrow$ ① $(1-s)^2 - (s-a)^2 = h^2$

② $\frac{1}{4} - a^2 = h^2$

③ $(1-r)^2 - (r+a)^2 = h^2$.

① + ② $\Rightarrow 1 - 2s + 2as = \frac{1}{4}. \Rightarrow a = \frac{1}{2s}\left(2s - \frac{3}{4}\right) = 1 - \frac{3}{8s}$

② + ③ $\Rightarrow 1 - 2r - 2ar = \frac{1}{4}. \Rightarrow a = \frac{1}{2r}\left(\frac{3}{4} - 2r\right) = \frac{3}{8r} - 1.$

$\Rightarrow 1 - \frac{3}{8s} = \frac{3}{8r} - 1$

$\Rightarrow 2 = \frac{3}{8}\left(\frac{1}{s} + \frac{1}{r}\right) \Rightarrow \frac{1}{s} + \frac{1}{r} = \frac{16}{3}$ ▨

Consider $\phi_{\frac{1}{2}}(z) = \dfrac{z - \frac{1}{2}}{1 - \frac{1}{2}z}$. This maps given circles to  $\cdots (*)$

circles externally tangent at the origin and internally tangent with the unit circle resp. (i.e. 1−1 correspondence btw'n given situation and 'new situation' described above)

~~Circle of~~ ~~moved by~~

Center of circle(*) moved by $\phi_{-\frac{1}{2}}(z)$ is $\dfrac{4 + e^{-i\theta} + 4e^{i\theta}}{8 + 2e^{-i\theta} + 2e^{i\theta}} = a, \ \bar{a}$.

~~dist~~ $\Rightarrow s = |a - \frac{1}{2}|, \ r = |\bar{a} - \frac{1}{2}| \Rightarrow$ Simple Calcul. says $\frac{1}{r} + \frac{1}{s} = \frac{16}{3}$.