mas550 homework

20208209 오재민

2020년 9월 15일

Problem (1.3.1).

Since $\sigma(X)$ is the smallest σ-field which makes X measurable, it sufficient to show that X is measurable with respect to $\sigma\left(X^{-1}(\mathcal{A})\right)$.

Let $X : \Omega \to S$. It is clear that $\{X \in A\} \in \sigma\left(X^{-1}(\mathcal{A})\right)$ for all $A \in \mathcal{A}$. But by theorem 1.3.1, since $\mathcal{A}$ generates $\mathcal{S}$, X is measurable with respect to $\sigma\left(X^{-1}(\mathcal{A})\right)$.

Therefore we can conlude that $\sigma\left(X^{-1}(\mathcal{A})\right) \subset \sigma(X)$, and reverse inclusion is canonical since $X^{-1}(\mathcal{A}) \subset \sigma(X)$.

Problem (1.4.1).

Let $E_n = \left\{x : f(x) > \frac{1}{n}\right\}$. Then $\int f d\mu \geq \int_{E_n} f d\mu \geq \int_{E_n} \frac{1}{n} d\mu = \frac{1}{n}\mu(E_n)$. Therefore $\mu(E_n) = 0$ for every positive integer n. So, $\mu(\{f > 0\}) = \sum_{n=1}^{\infty} \mu(E_n) = 0$. This says $f = 0$ a.e.

Problem (1.4.2). *Since $E_{n+1,2m} \cup E_{n+1,2m+1} = E_{n,m}$ and $\frac{2m+1}{2^{n+1}} \geq \frac{m}{2^n}$, we can easily see that $\sum_{m\geq 1} \frac{m}{2^n}\mu(E_{n,m})$ is monotonically increasing as n grows.*

For every positive integer M, $\sum_{m=1}^{M} \frac{m}{2^n}\mu(E_{n,m}) \leq \int f d\mu$. So $\sum_{m\geq 1} \frac{m}{2^n}\mu(E_{n,m}) \leq \int f d\mu$.

Let $s_n = \sum_{m=1}^{n2^n} \frac{m}{2^n} 1_{E_{n,m}}$. Then $\int s_n d\mu \leq \sum_{m\geq 1} \frac{m}{2^n}\mu(E_{n,m}) \leq \int f d\mu$. But $s_n \uparrow f$ monotonically. By monotone convergence theorem, $\lim_{n\to\infty} \int s_n d\mu = \int f d\mu$. Hence by sandwich lemma, the desired result follows.