RAGS

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Problem (13.1)**.**

I think the given condition should be modified: A is $\nu-$null whenever $\mu(A) = 0$. With this, $\nu = \nu^+ - \nu^-$ and they are all absolutely continuous positive measure with respect to μ. Therefore we can apply the Radon-Nikodym theorem, so $d\nu^+ = f_1 d\mu$ and $d\nu^- = f_2 d\mu$. Then $f = f_1 - f_2$ is what we desired.

□

Problem (13.2)**.**

Statement: Let ν be a finite signed measure and let μ be σ-finite positive measure. Then $\nu = \nu_a + \nu_s$ where $\nu_a \ll \mu$ and $\nu_s \perp \mu$. Note that $\nu_a \ll \mu$ means $|\nu_a| \ll \mu$.

This can be proved by decomposing ν into $\nu^+ - \nu^-$.

□

Problem (13.3)**.**

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Problem (13.4)**.**

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Problem (13.5)**.**

First assume that $\nu \ll \mu$ and $\mu \ll \nu$. Then there are f_1, f_2 such that

$$d\nu = f_1 d\mu, \;\; d\mu = f_2 d\nu.$$

By change of variable formula,

$$\mu(A) = \int_A f_2 d\nu = \int_A f_2 f_1 d\mu.$$

Thus, $\mu(f_1 f_2 = 0) > 0$ gives the contradiction.

For the other direction, assume that $d\nu = f d\mu$ for $f > 0$ μ-a.e. Clearly $\nu \ll \mu$. Now let $E_n = \{f > 1/n\}$. When $\nu(A) = 0$,

$$\frac{1}{n}\mu(A \cap E_n) \le \int_A f d\mu = \nu(A) = 0.$$

Thus $\mu(A \cap E_n) = 0$ for all n, and by letting $n \to \infty$, we can get $\mu(A) = 0$.

□

Problem (13.6)**.**

First,

$$\mu(f = 0) \le \mu\left(f \le \frac{1}{n}\right) \le \frac{1}{n}\rho\left(f \le \frac{1}{n}\right) \le \frac{1}{n}\rho(X)$$

thus by letting $n \to \infty$ we can get $\mu(f = 0) = 0$, which means $f > 0$ μ-a.e.

Now,

$$\rho(A) = \mu(A) + \nu(A) = \int_A f + g d\rho$$

for all A. Thus

$$\int_A f + g - 1 d\rho = 0$$

which means $f + g = 1$ ρ-a.e.

Since $\nu \ll \mu$, there is h such that $d\nu = h d\mu$. But,

$$\int_A g d\rho = \nu(A) = \int_A h d\mu = \int_A h f d\rho$$

by change of variable formula. Thus $g = hf$ ρ*-a.e. Therefore* $h = d\nu/d\mu = g/f$.

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