complex - hw6

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Problem 1

$$\begin{aligned}\int_C \overline{z}dz &= \int_0^\pi \left(5e^{-it}+3\right)\left(5ie^{it}\right)dt \\ &= \int_0^\pi 25i + 15ie^{it}dt \\ &= 25i\pi - 30\end{aligned} \tag{1}$$

Problem 2 *For* $|z| = 1$, $\left|\frac{2z+1}{5+z^2}\right| \leq \frac{3}{4}$. *Therefore*

$$\left|\int_{|z|=1} \frac{2z+1}{5+z^2}\right| \leq \int_{|z|=1} \left|\frac{2z+1}{5+z^2}\right| |dz| \leq \frac{3}{2}\pi \tag{2}$$

Problem 3 *(a)* $|z| = \sqrt{\frac{t^4}{9}+t^2}$. *So*

$$\begin{aligned}\int_C |z|^2 dz &= \int_0^1 \left(\frac{t^4}{9}+t^2\right)\left(\frac{2t}{3}+i\right)dt \\ &= \int_0^1 \frac{2}{27}t^5 + \frac{2}{3}t^3 dt + i\int_0^1 \frac{1}{9}t^4 + t^2 dt \\ &= \frac{29}{162} + i\frac{16}{45}\end{aligned} \tag{3}$$

(b)

$$
\begin{aligned}
\int_C Re(z)|dz| &= \int_0^1 \frac{t^2}{3}\sqrt{\frac{4}{9}t^2+1}dt \\
&= \int_0^{tan^{-1}\left(\frac{2}{3}\right)} \frac{9}{8}\tan^2\theta\sec^3\theta d\theta \\
&= \frac{1}{4}\sec^3 x\tan x - \frac{1}{8}\sec x\tan x + \frac{1}{8}\ln|\sec x+\tan x|\,|_0^{\tan^{-1}\left(\frac{2}{3}\right)} \\
&= \frac{17\sqrt{13}}{288} - \frac{9}{64}\ln\left(\frac{2}{3}+\frac{\sqrt{13}}{3}\right)
\end{aligned} \tag{4}
$$

Problem 4

$$
\int_C z|z|dz = \int_0^{\pi} R^2e^{it}\left(Rie^{it}\right)dt + \int_{-R}^{R} t|t|dt = 0 \tag{5}
$$

because each of them is 0.

Problem 5 *(a) Given integral is*

$$
\begin{aligned}
\int_1^2 -x^2dx + \int_2^1 0dx + \int_0^{-1} 3y^2dy + \int_{-1}^0 5y^2dy \\
= -\frac{1}{3}7 - 1 + \frac{5}{3} = -\frac{5}{3}
\end{aligned} \tag{6}
$$

(b) Let $x = r\cos\theta$ *and* $y = r\sin\theta$*. Then given integral is*

$$
\begin{aligned}
&= \int_{-\pi}^{\pi} r^2\cos^2\theta\sin^2\theta d\theta \\
&= \frac{r^4}{4}\int_{\pi}^{\pi}\frac{1-\cos 4\theta}{2}d\theta = \frac{r^2}{4}\pi
\end{aligned} \tag{7}
$$

Problem 6 *Since* e^z *is entire,* $\int_{|z|=1} e^z dz = 0$*. So its real and imaginary part* *are zero. Put $z = e^{it}$. Then*

$$\begin{aligned}
\int_{|z|=1} e^z dz &= \int_{-\pi}^{\pi} e^{e^{it}} ie^{it} dt = 0 \\
&= \int_{-\pi}^{\pi} e^{\cos t + i(\sin t + t)} idt \\
&= i\int_{-\pi}^{\pi} e^{\cos t} \cos(\sin t + t)\, dt - \int_{-\pi}^{\pi} e^{\cos t} \sin(\sin t + t)\, dt
\end{aligned} \tag{8}$$

Problem 7 *(a)*

$$\begin{aligned}
\int_{|z-i|=1} \frac{dz}{1+z^2} &= \frac{1}{2i} \int_{|z-i|=1} \left(\frac{1}{z-i} - \frac{1}{z+i} \right) dz \\
&= \pi
\end{aligned} \tag{9}$$

since $\frac{1}{z+i}$ is analytic except for $z = -i$.

(b) Consider $C_1 = |z - i| = \varepsilon_1$ and $C_2 = |z + i| = \varepsilon_2$ where C_1, C_2 are lying inside of contour $|z| = 2$. (it is possible since $i, -i$ are interior point of $|z| \leq 2$) Then

$$\begin{aligned}
\int_{|z|=2} \frac{dz}{z^2+1} &= \int_{C_1} \frac{dz}{z^2+1} + \int_{C_2} \frac{dz}{z^2+1} \\
&= \frac{1}{2i} \int_{C_1} \left(\frac{1}{z-i} - \frac{1}{z+i} \right) dz + \frac{1}{2i} \int_{C_2} \left(\frac{1}{z-i} - \frac{1}{z+i} \right) dz \\
&= \pi + (-\pi) = 0
\end{aligned} \tag{10}$$

Problem 8 *Let $F(z) = Logz$. Then $F'(z) = \frac{1}{z}$ and $\frac{1}{z}$ is continuous on right half plane. By Fundamental Theorem of Integration,*

$$\int_{z_0}^{z_1} \frac{1}{z} dz = Log\,(z_1) - Log\,(z_0) \tag{11}$$

Problem 9 *Note that $P(z)$ is entire. So given integral is $P^{(n+1)}(1) = 0$.*

Problem 10 *By joining terminal point and initial point of γ with straight line, we can get simple closed contour C. Consider $C_1 = |z - \pi i| = \varepsilon_1$ and $C_2 = |z + \pi i| = \varepsilon_2$ such that each circle lies inside of contour C. Then $\int_C \frac{dz}{z^2\pi^2} = \int_{C_1} \frac{dz}{z^2+\pi^2} + \int_{C_2} \frac{dz}{z^2+\pi^2} = \int_\gamma \frac{dz}{z^2+\pi^2} = \int_\delta \frac{dz}{z^2+\pi^2}$ where $\delta(t) = -t$ for $-2\pi \leq t \leq 0$.*

But, $\int_{C_1} \frac{dz}{z^2+\pi^2} = \frac{1}{2\pi i} \int_{C_1} \left(\frac{1}{z-\pi i} - \frac{1}{z+\pi i} \right) dz = 1$ *and similarly,* $\int_{C_2} \frac{dz}{z^2+\pi^2} = -1$*. So* $\int_C \frac{dz}{z^2+\pi^2} = 0$*, therefore* $\int_\gamma \frac{dz}{z^2+\pi^2} = -\int_\delta \frac{dz}{z^2+\pi^2}$.

So given integral is

$$
\begin{aligned}
-\int_\delta \frac{dz}{z^2+\pi^2} &= \int_{-2\pi}^{0} \frac{dt}{t^2+\pi^2} \\
&= \int_{\tan^{-1}(-2)}^{0} \frac{dt}{\pi} \\
&= \frac{\tan^{-1}(2)}{\pi}
\end{aligned} \tag{12}
$$