mas550 homework

20208209 오재민

2020년 10월 21일

Problem (2.5.2).

If $E|X_1|^p = \infty$*, then for each positive integer* k*,* $E|X_1|^p \le \sum_n P(|X_1|^p > nk) = \infty$*. But* $P(|X_1|^p > nk) = P(|X_n| > (nk)^{1/p})$*. Then by Borel Cantelli lemma* $P(|X_n| > (nk)^{1/p} i.o.) = 1$*. That is,* $\limsup_n |X_n|/n^{1/p} \ge k^{1/p}$ *for infinitely many* k*. Therefore* $\limsup_n |X_n|/n^{1/p} = \infty$*.*

But $|X_n| \le |S_n| + |S_{n-1}|$*. That leads* $\limsup_n |S_n|/n^{1/p} = \infty$*. By taking contrapositive, we get the conclusion.*

Problem (2.5.5).

The first one leads the second one directly because Kolmogorov's three series lemma with $A = 1$ *tells it.*

The second one implies the third one because $\frac{X_n}{1+X_n} \le 1_{X_n > 1} + X_n 1_{X_n \le 1}$ *and monotone convergence theorem.*

The third one implies $\sum_n \frac{X_n}{1+X_n} < \infty$ *a.s. And convergence of* $\sum_n \frac{a_n}{1+a_n}$ *for* $a_n \ge 0$ *gives the convergence of* $\sum_n a_n$*. It is because* $lim a_n = 0$ *and* $|a_N + \cdots a_{N+n}| \le (1+\varepsilon)\left|\frac{a_N}{1+a_N} + \cdots + \frac{a_{N+n}}{1+a_{N+n}}\right|$ *for large* N*. Therefore* $\sum_{k=1}^n a_k$ *is cauchy hence converges. Therefore* $\sum_n X_n$ *converges a.s.*

Problem (3.2.4).

Since $X_n \to X_\infty$ *in distribution, there are* $Y_n =_d X_n$ *and* $Y_\infty =_d X_\infty$ *such that* $Y_n \to Y_\infty$ *a.s.*

Then $g(Y_n) \ge 0$ *and* $g(Y_n) \to g(Y_\infty)$ *a.s. Therefore by Fatou's lemma,* $\liminf Eg(Y_n) \ge Eg(Y_\infty)$ *which is equivalent to* $\liminf Eg(X_n) \ge Eg(X_\infty)$ *since* $X_n =_d Y_n$ *for all* $n \in \mathbb{N} \cup \infty$*.*

Problem (3.2.5).

There are $Y_n \to Y_\infty$ *a.s. and distribution function of* Y_n *is equal to* F_n*.* $F_\infty = F$*.*

Then by theorem 1.6.8, $Eh(Y_n) \to Eh(Y_\infty)$ *which is equivalent to* $\int h(x)dF_n(x) \to \int h(x)dF(x)$ *because distribution function of* Y_n *is* F_n*.*