REAL ANALYSIS HW6

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section 6

Problem 1. "the vanishing property"

First, assume that $\int f d\mu = 0$. Since f is measurable, there exists an increasing seqeunce of nonnegative simple functions which converges to f. Let denote them as s_n. Then $f^{-1}((0,\infty]) = \bigcup_{n=1}^{\infty} s_n^{-1}((0,\infty])$, union of measure zero set. Therefore, we get $\mu(f^{-1}((0,\infty])) = 0$.

Conversely, assume that $\int f d\mu > 0$. Then there exists nonnegative simple function $s \leq f$ such that $\int s d\mu > 0$. Also we can write s as a linear combination of (measurable) characteristic functions, i.e. $\int s d\mu = \sum_{i=1}^{N} \alpha_i \mu(A_i) > 0$. So, $\alpha_i \mu(A_i) > 0$ for at least one integer $1 \leq i \leq N$. By Observing the fact that $A_i \subset f^{-1}((0,\infty])$, we can conclude that $f^{-1}((0,\infty])$ has measure zero implies $\int f d\mu = 0$ by contrapositive.

Problem 2. "the finiteness property"

$E = \{x : f(x) = \infty\}$ is a measurable set since f is measurable. Suppose $\mu(E) > 0$. For any $M \in \mathbb{N}$, choose nonnegative simple measurable function $s_M \leq f$ such that $s_M(x) \geq M$ for $x \in E$. Then $\int f d\mu \geq \int s_M d\mu \geq M\mu(E)$. Therefore, $\int f d\mu \geq M\mu(E)$ for all positive integer M. It means that $\int f d\mu = \infty$. By taking contrapositive, we get what we want.

Problem 6.

Let $f_k = 1_{A_k}$ where 1_{A_k} is an indicator(characteristic) function. By Fatou's lemma (ILLLI), $\int \liminf_{k\to\infty} 1_{A_k} d\mu \leq \liminf_{k\to\infty} \int 1_{A_k} d\mu = \liminf_{k\to\infty} \mu(A_k)$. Now, showing $1_{\liminf A_k} \leq \liminf_{k\to\infty} 1_{A_k}$ is left to us. If $x \in \liminf A_k$ then $x \in \bigcap_{i\geq n} A_i$ for some positive integer n. Then $1_{A_i}(x) = 1$ for all positive integer i greater than n. So, $1_{\liminf A_k} \leq \inf_{i\geq n} 1_{A_i}$. By letting $n \to \infty$, we get $1_{\liminf A_k} \leq \liminf_{k\to\infty} 1_{A_k}$. So, $\mu(\liminf A_k) = \int 1_{\liminf A_k} d\mu \leq \int \liminf_{k\to\infty} 1_{A_k} d\mu \leq \liminf_{k\to\infty} \int 1_{A_k} d\mu = \liminf_{k\to\infty} \mu(A_k)$.