H&K LINEAR ALGEBRA SOLUTIONS

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CHAPTER 6

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4. Invariant Subspaces.

Problem 8.

Let $\mathcal{B} = \{\alpha_1, \cdots, \alpha_n\}$ be an ordered basis for V. Let $\mathcal{B}_i = span(\alpha_i)$. Since every subspace of V is invariant under T, $T\alpha_i = c_i\alpha_i$ for some scalar c_i. Let $W_i = span(\alpha_1 + \alpha_i)$, $\beta_i = \alpha_1 + \alpha_i$. $T\beta_i = c_1\alpha_1 + c_i\alpha_i = k\beta_i$. So $c_1 = c_i = k$. Therefore, $T\alpha_i = c_1\alpha_i$ for all i. So T is a scalar multiple of the identity operator.

Problem 10.

Let p be a minimal polynomial for A. From non-triangulability of A, we can say that p has degree 2 irreducible factor since every degree 3 real coefficient polynomial has at least one real zero. Then irreducible factor of p splits into distinct linear factors over $\mathbb{C}$. So, minimal polynomial for A splits into distinct linear factors over $\mathbb{C}$ which is equivalent to A is diagonalizable.

Problem 12.

First assume t is an eigenvalue of T. Then there exists $\alpha \in V \setminus 0$ such that $T\alpha = t\alpha$. It is easy to observe that $f(T)\alpha = f(t)\alpha$. So $f(t)$ is an eigenvalue of linear operator $f(T)$.

Conversely, assume c is an eigenvalue of $f(T)$. So there exists nonzero vector $\alpha \in V$. Consider the equation $f(x) = c$. There is $t \in F$ which satisfies that equatio since F is algebraically closed. From $f(t) = c$, $f(x) - c = (x - t)q(x)$. So $(T - tI)q(T)\alpha = 0$. If $q(T)\alpha \neq 0$, we are done. If $q(T)\alpha = 0$, $q(T)$ has 0 as eigenvalue. So we can find $s \in F$ such that $q(s) = 0$. Then $q(x) = (x - s)r(x)$, $f(s) = f(t) = c$, $q(T)\alpha = (T - sI)r(T)\alpha = 0$. If $r(T)\alpha \neq 0$, we are done. If not $\cdots$. By repeating (such process is finite since f has finite degree), we can conclude that $c = f(a)$ for some eigenvalue a of T. When f is degree 1, it is trivial.

5. Simultaneous Triangulization.

In 6-4, we studied about conditions for triangulability of only one linear operator on finite dimension vector space V. But in this section, we are curious about conditions for triangulability of *collection of linear operators* on V. That is, we want find ordered basis for V which represents each linear operator in given collection as triangular matrix.

In fact, if commuting collection is given and each linear operator is triangulable, then we can find ordered basis for V which triangulize every linear operator in given collection simultaneously.

Above sentence is still valid if we replace 'triangul-' to 'diagonal-'.