mas540 exercises

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Exercise (1.4)**.**

(a) *Let* $I = [0,1]$*. Then* $I \backslash \hat{C} = \bigcup_{n=1}^{\infty} \hat{C}_n^c$ *where* $\hat{C}_n$ *is n-th stage of constructing Fat Cantor set. Thus,*

$$m(I \setminus \hat{C}) = m(I) - m(\hat{C}) = 1 - m(\hat{C}) = \lim_{n\to\infty} m(\hat{C}_n^c) = \sum_{n=1}^{\infty} 2^{n-1} l_n$$

because $\hat{C}_n^c \uparrow \bigcup_{n=1}^{\infty} \hat{C}_n^c$ *and* $\hat{C}$ *is closed hence measurable. Therfore* $m(\hat{C}) = 1 - \sum_{n=1}^{\infty} 2^{n-1} l_n > 0$*.*

(b) $\hat{C}_k$ *consists of* 2^k *closed intervals whose length are* $(1 - \sum_{n=1}^{k} 2^{n-1} l_n)/2^k$*. Let* $x \in \hat{C}$*. Then* $x \in \hat{C}_k$*. So we can find* $x_k \in I_k$ *such that*

$$|x - x_k| \le \left(1 - \sum_{n=1}^{k} 2^{n-1} l_n\right)/2^k + \varepsilon_k l_k$$

for some $0 < \varepsilon_k < 1$*. As* $k \to \infty$*,* $|x - x_k| \to 0$ *since* $l_k \to 0$*.*

(c) *The result of b tells us that every point of* $\hat{C}$ *is a limit point of* I*. And we also know that* $\hat{C}$ *is closed. Hence* $\hat{C}$ *is a perfect set.*

Let $(a,b) \subset \hat{C}$ *and* $a < c < d < b$*. For large* k*,* $l_k < d - c$ *since* $l_k \to 0$*. Then, for* $\hat{C}_k$*,* c *and* d *must lie in different intervals of* $\hat{C}_k$*. So there is* $e \notin \hat{C}_k$ *such that* $c < e < d$*. Then* $[c,d]$ *does not belong to* $\hat{C}_k$ *which is a contradiction. So* $\hat{C}$ *is totally disconnected.*

(d) *It is well known fact that a nonempty perfect set is uncountable. We had learned it in an introductory analysis course and topology course.*

□

Exercise (1.7)**.**

First, we will show that if O *is open, then* δO *is also open. Let* $\delta x \in \delta O$*. Then* $x \in O$*. By openness, there is* $r > 0$ *such that* $Q_r(x) \subset O$ *where* $Q_r(x)$ *is a cube whose side length is* r *and centered at* x*. Thus* $\delta Q_r(x) \subset \delta O$ *and* $\delta Q_r(x)$ *contains* δx*. But a collection of all open rectangles forms a basis of Euclidean space. So* δO *is an open set.*

Next, let a set E *and a positive number* ε *be given. Choose* $O \supset E$ *such that* $m_*(O \setminus E) < \varepsilon/(\delta_1 \cdots \delta_d)$*. Then, there is an union of cube* $\bigcup_{j=1}^{\infty} Q_j \supset O \setminus E$ *such that* $\sum_{j=1}^{\infty} m(Q_j) < \varepsilon/(\delta_1 \cdots \delta_d)$*. Then,*

$$m_*(\delta O \setminus \delta E) = m_*(\delta(O \setminus E)) \le m_*(\bigcup_{j=1}^{\infty} \delta Q_j) \le \sum_{j=1}^{\infty} m(\delta Q_j) < \varepsilon.$$

Thus δE *is measurable.*

Now let $E \subset \bigcup_{j=1}^{\infty} Q_j$. Then $\delta E \subset \bigcup \delta Q_j$, so $m(\delta E) \leq \delta_1 \cdots \delta_d \sum_{j=1}^{\infty} m(Q_j)$. Since $\bigcup_{j=1}^{\infty}$ is arbitrary, we get

$$m(\delta E) \leq \delta_1 \cdots \delta_d m(E).$$

Now let $\delta E \subset \bigcup_{j=1}^{\infty} Q'_j$. Then $E \subset \bigcup_{j=1}^{\infty} 1/\delta Q'_j$. So $m(E) \leq \sum_{j=1}^{\infty} m(Q'_j)/(\delta_1 \cdots \delta_d)$. Since $\bigcup_{j=1}^{\infty} Q'_j$ is arbitary, we get

$$m(E) \leq \frac{m(\delta E)}{\delta_1 \cdots \delta_d}$$

and this finishes the proof.

□

Exercise (1.24).

Let s_n be enumeration of $\mathbb{Q} \cap [-1,1]$ and t_n be enumeration of $\mathbb{Q} \cap [-1,1]^c$. When $n = m^2$, put $r_n = t_m$. When $n \in \left(m^2, (m+1)^2\right)$, put $r_n = s_{n-m}$. Then r_n is an enumeration of $\mathbb{Q}$. Also, we get

$$\begin{aligned} m\left(\bigcup_{n=1}^{\infty} (r_n - 1/n, r_n + 1/n)\right) &\leq \sum_{m=1}^{\infty} 2/m^2 + m\left(\bigcup_{n \neq m^2} (r_n - 1/n, r_n + 1/n)\right) \\ &\leq \sum_{m=1}^{\infty} 2/m^2 + 2 + 1 < \infty. \end{aligned}$$

Therefore, finiteness implies nonemptyness of the complement, since the Lebesgue measure of complement is positive.

□

Exercise (1.35).

First, let's briefly check the idea of constructing φ. Construction can be done by defining a sequence of functions, say φ_n. Put $\varphi_n(0) = 0$ and $\varphi_n(1) = 1$. Let C_{ji} be the i-th stage of constructing C_j. Then φ_i maps the discarded set of stage i to the discarded set of stage i, sequentially, and linearly(positive). We can extend φ_i by assigning value on C_{1i} using linearity and monotonicity. This sequence of functions converges uniformly, thus φ is continuous. The other properties of φ can be checked by this construction.

Let $\mathcal{N} \subset C_1$ be a non-measurable set. Then $\varphi(\mathcal{N}) \subset C_2$ so $\varphi(\mathcal{N})$ is measurable by completeness. If $\varphi(\mathcal{N})$ is a Borel set, then by continuity, $\varphi^{-1}(\varphi(\mathcal{N})) = \mathcal{N}$ must be a Borel set, which is a contradiction. So there is a Lebesgue measurable set which is not Borel measurable.

Since $\varphi(\mathcal{N})$ is measurable, $f = 1_{\varphi(\mathcal{N})}$ is a measurable map. Then $f \circ \varphi(x) = 1_{\mathcal{N}}(x)$ is non-measurable map.

□

Problem (1.4).

(a) A_ε *is clearly bounded, so it is enough to show that the complement is open. Let* $c \notin A_\varepsilon$. *Then* $osc(f, c) < \varepsilon$, *so for some* $r > 0$, $osc(f, c, r) < \varepsilon$. *Choose any* $d \in I(c, r)$. *We can choose* $r^* > 0$ *so that* $I(d, r^*) \subset I(c, r)$. *Then*

$$osc(f, d, r^*) \leq osc(f, c, r) < \varepsilon$$

so $osc(f, d) < \varepsilon$, *which says* $I(c, r) \subset J \setminus A_\varepsilon$. *Therefore* $J \setminus A_\varepsilon$ *is open in* J, *hence* A_ε *is compact.*

(b) *Let* D_f *be a set of all discontinuities of* f. *Then for any* $\varepsilon > 0$, $A_\varepsilon \subset D_f$. *So* $m(A_\varepsilon) \leq m(D_f) = 0$. *By the definition of Lebesgue measure, there is countably many open intervals which cover* A_ε *and have sum of length* $\leq \varepsilon$. *Using compactness, we can choose finite subcover, call them by* $(a_i, b_i)_{i=1}^k$ *where* $a_i < a_{i+1}$. *After discarding all of subcovers from* J, *we get compact subset of* J, *say* J'. *For each* $c \in J'$, *we can choose* r_c *such that* $osc(f, c, 2r_c) < \varepsilon$. *Again, using compactness, we can choose finitely many* c*'s. Then finitely many closed intervals* $[c - r_c, c + r_c]$ *have finite intersections. By taking these endpoints(contain* a_i, b_i*'s) as endpoints of our partition(if necessary, consider a refinement), we get*

$$U(f, P) - L(f, P) \leq 2M\varepsilon + m(J)\varepsilon$$

where M *is bound of* f. *The first term of estimate comes from* (a_i, b_i)*'s and the second term comes from* J'.

(c) *Since* $D_f \subset \bigcup_{n=1}^\infty A_{1/n}$, *so* $m(A_{1/n}) = 0$ *leads the conclusion. Assume not, i.e.* $m(A_{1/n}) > \varepsilon$. *Take partition* P *such that* $U(f, P) - L(f, P) < \varepsilon/n$. *Let* $[a, b]$ *be interval of* P *whose interior intersects to* $A_{1/n}$. *Then*

$$\sup_{x,y \in [a,b]} |f(x) - f(y)| \geq \frac{1}{n}.$$

But $m(A_{1/n}) > \varepsilon$. *So*

$$\begin{aligned}
&\sum_{[a,b] \cap A_{1/n} \neq \emptyset} \left[\sup_{x \in [a,b]} f(x) - \inf_{y \in [a,b]} f(y) \right] m\left(A_{1/n} \cap [a, b]\right) \\
&= \sum_{[a,b] \cap A_{1/n} \neq \emptyset} \sup_{x,y \in [a,b]} |f(x) - f(y)| \, m\left(A_{1/n} \cap [a, b]\right) \\
&\geq \frac{\varepsilon}{n} \\
&> U(f, P) - L(f, P)
\end{aligned}$$

which is a contradiction.

$\square$