DQ - COMPLEX FUNCTION THEORY

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1. 2012.01

Problem 1.1. *Let Ω be a simply connected domain. Let f be a meromorphic function on Ω which has finitely many poles. If γ is a piecewise C^1 curve which does not cross any poles of f, then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} Res_f(a_k) Ind_\gamma(a_k)$$

where $\{a_k\}_{k=1}^n$ are poles of f lying inside of γ.

Use this formula and contour $\Gamma = \gamma_1 + \gamma_2$ where $\gamma_1(t) = t$ for $t \in [-R, R]$ and $\gamma_2(t) = Re^{it}$ for $t \in [0, \pi]$.

□

Problem 1.2. *Let f be a such map. Since f is bounded near 0, the Riemann removable singularity theorem says that f extends to the entire function. Then f is bounded entire function, so f is constant. But any constant function cannot be conformal map of A onto B.*

□

Problem 1.3. *Consider this Blaschke factor:*

$$B_{1/2}(z) = \frac{z - 1/2}{1 - z/2}$$

This is an automorphism of $\mathbb{D}$ but has no fixed point.

□

Problem 1.4. *Let $g(z) = f(z)/z$. Since $f(0) = 0$, g is bounded near the origin. So we can regard g as a holomorphic function on the unit disk.*

Now fix $0 \le r < 1$. Then

$$\max_{z \in \overline{D}(0,r)} |g(z)| = \max_{z \in \partial D(0,r)} |g(z)|$$

by the maximum modulus theorem. But the last term is bounded by $1/r$ since $|f| \le 1$. Thus by $r \uparrow 1$, we can get $|g(z)| \le 1$ for $z \in \mathbb{D}$.

□

Problem 1.5.

(a) *omitted. see 2019.02.*

(b) *Consider $g(z) = f(1/z)$. If g has a removable singularity at the origin, then f is bounded entire function, which is a contradiction.*

If g has an essential singularity at the origin, then $g(0 < |z| < 1)$ is dense in $\mathbb{C}$. But, $g(|z| > 1)$ is an open set since g is holomorphic hence open mapping. So, $q \in g(|z| > 1)$ implies the existence of $\varepsilon > 0$ such that

$$D(q, \varepsilon) \subset g(|z| > 1).$$

But we always find $0 < |z'| < 1$ such that $g(z') \in D(q, \varepsilon)$ by the denseness. Therefore

$$g(|z| > 1) \cap g(0 < |z| < 1) \ne \emptyset$$

which contradicts to the injectivity.

So g must have a pole at the origin, and that implies f must be a polynomial. Now the injectivity implies linearity of f.

$\square$

Problem 1.6.

(a) *Choose $r > 0$ such that the zero set of f in $D(0,r)$ consists of the origin only. Note that $F(z) = f(z)/z^m$ is nonvanishing on $D(0,r)$. Since $D(0,r)$ is simply connected and F is nonvanishing, there is $h \in H(D(0,r))$ such that $F = e^h$. By taking $g(z) = z\exp(h(z)/m)$, we get the desired result.*

(b) *It suffices to show that $f(\Omega)$ is open. Let $q \in f(\Omega)$ and choose $\delta > 0$ such that $f(p) = q$ and $\overline{D}(p,\delta) \subset \Omega$. Note that we can choose δ so that $f(\cdot) - q$ is nonvanishing in $\overline{D}'(p,\delta)$.*

Since $\partial D(p,\delta)$ is compact and $f(\cdot) - q$ is nonvanishing, we can choose $\varepsilon > 0$ so that

$$|f(\zeta) - q| > 2\varepsilon$$

for all $\zeta \in \partial D(p,\delta)$.

Define $N : D(q,\varepsilon) \to \mathbb{Z}$ by

$$N(w) = \frac{1}{2\pi i}\int_{\partial D(p,\delta)} \frac{f'(\zeta)}{f(\zeta) - w} d\zeta.$$

This is integer valued by the argument principle and continuous by $\varepsilon > 0$. Thus N is constant and $N(p) = m \geq 1$. Thus $N(w) = m$ for all $w \in D(q,\varepsilon)$. This implies that every $w \in D(q,\varepsilon)$ has a preimage z in $D(p,\delta)$, so $D(q,\varepsilon) \subset f(D(p,\delta)) \subset f(\Omega)$.

(c) *If $f'(p) = 0$, then p is not simple, say $f(p) = q$ of order $m \geq 2$. Since f' is holomorphic, we can choose δ_1 so that p is isolated in $D(p,\delta_1)$ in the sense of simple points. Now choose $\delta(< \delta_1), \varepsilon > 0$ such that $\overline{D}(p,\delta) \subset \Omega$ and $D(q,2\varepsilon) \subset f(D(p,\delta)) \setminus f(\partial D(p,\delta))$.*

For $w \in D(q,\varepsilon)$, we can define

$$N(w) = \frac{1}{2\pi i}\int_{\partial D(p,\delta)} \frac{f'(\zeta)}{f(\zeta) - w} d\zeta.$$

Then by $\varepsilon > 0$ and the argument principle, N is constant. Therefore each $w \in D(q,\varepsilon)$ has m preimages in $D(p,\delta)$ counting multiplicities. But every points in $D(p,\delta)$ is simple except for p. Thus we can say that w has m distinct preimages in $D(p,\delta)$ if $w \neq q$. And this contradicts to the injectivity.

$\square$

2. 2019.02

Problem 2.1 (Casorati-Weierstrass)**.** *If the image of f is not dense in $\mathbb{C}$, then there are $\varepsilon > 0$ and $w \in \mathbb{C}$ such that*

$$|f(z) - w| > \varepsilon$$

for all $z \in D'(z_0, r)$. Now consider $g(z) = 1/(f(z) - w)$. Then the modulus of g is bounded by $1/\varepsilon$. So the Riemann removable singularity theorem implies that $g \in H(D(z_0, r))$.

If $g(z_0) = 0$, then f has a pole at z_0, which is contradiction. If $g(z_0) \neq 0$, then f must be bounded near z_0, which contradicts to the essential singularity.

□

Problem 2.2. *Observe that the given polynomial is a partial sum of* $\exp(z)$*. Since the radius of convergence of the power series of* $\exp(z)$ *is ∞, the given polynomial converges locally uniformly.*

Note that $|\exp(z)| \geq \exp(-R)$ *on $z \in \partial D(0, R)$. Thus, if we take n so large that*

$$|P_n(z) - \exp(z)| < \exp(-R)$$

for all $z \in \partial D(0, R)$, then Rouche's theorem implies the result because $\exp(z)$ *is nonvanishing.*

□

Problem 2.3. *Since the modulus of f is 1 on the boundary of the unit disk, the modulus of f is bounded by 1 on the entire unit disk. Thus, by the maximum modulus principle, f is a self mapping of the unit disk.*

Problem 2.4. *Fix $r > 0$ and choose N such that $|a_n| > 2r$ whenever $n \geq N$. Then*

$$\sum_{n \geq N} \left| \frac{r}{a_n} \right|^n \leq \sum_{n \geq N} \left(\frac{1}{2} \right)^n < \infty.$$

Thus, for each $r > 0$,

$$\sum_{n \in \mathbb{N}} \left| \frac{r}{a_n} \right|^n < \infty.$$

This implies

$$\prod_{n \in \mathbb{N}} E_{n-1} \left(\frac{z}{a_n} \right)$$

is an entire function.

(explanation about the zeros are needed)

□

Problem 2.5. *If $f(0) = 0$, then the result follows trivially. So assume that $f(0) \neq 0$. Consider*

$$g(z) = \frac{f(z)}{\prod_{k=1}^{n(R)} B_{a_k/R}(z/R)}$$

where $n(R)$ denotes the number of zeros of f in $D(0, R)$. Then g is nonvanishing. So $\log |g|$ *is harmonic, and the mean value property implies*

$$\log |g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |g(Re^{i\theta})| d\theta$$

which is equivalent to

$$\log|f(0)| - \sum_{k=1}^{n(R)} \log\left|\frac{a_k}{R}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta.$$

Thus, the result follows immediately if we observe that $|a_k/R| \leq |r/R| \leq 1$ *and* $n(R) \geq n(r) \geq n$.

$\square$

Problem 2.6. *Let* K *be a compact subset of the unit disk. Then we can find* $0 \leq r < 1$ *such that* $K \subset \overline{D}(0,r)$. *Note that*

$$|f(z)| \leq \sum_{n\geq 1} |a_n||z|^n \leq \sum_{n\geq 1} nr^n < \infty.$$

Thus, $\mathcal{F}$ *is locally uniformly bounded. Then second Montel's theorem implies the result.*

$\square$