

# mas540 exercises

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**Exercise (1.4).**

- (a) Let  $I = [0, 1]$ . Then  $I \setminus \hat{C} = \bigcup_{n=1}^{\infty} \hat{C}_n^c$  where  $\hat{C}_n$  is  $n$ -th stage of constructing Fat Cantor set. Thus,

$$m(I \setminus \hat{C}) = m(I) - m(\hat{C}) = 1 - m(\hat{C}) = \lim_{n \rightarrow \infty} m(\hat{C}_n^c) = \sum_{n=1}^{\infty} 2^{n-1} l_n$$

because  $\hat{C}_n^c \uparrow \bigcup_{n=1}^{\infty} \hat{C}_n^c$  and  $\hat{C}$  is closed hence measurable. Therefore  $m(\hat{C}) = 1 - \sum_{n=1}^{\infty} 2^{n-1} l_n > 0$ .

- (b)  $\hat{C}_k$  consists of  $2^k$  closed intervals whose length are  $(1 - \sum_{n=1}^k 2^{n-1} l_n)/2^k$ . Let  $x \in \hat{C}$ . Then  $x \in \hat{C}_k$ . So we can find  $x_k \in I_k$  such that

$$|x - x_k| \leq \left(1 - \sum_{n=1}^k 2^{n-1} l_n\right) / 2^k + \varepsilon_k l_k$$

for some  $0 < \varepsilon_k < 1$ . As  $k \rightarrow \infty$ ,  $|x - x_k| \rightarrow 0$  since  $l_k \rightarrow 0$ .

- (c) The result of b tells us that every point of  $\hat{C}$  is a limit point of  $I$ . And we also know that  $\hat{C}$  is closed. Hence  $\hat{C}$  is a perfect set.

Let  $(a, b) \subset \hat{C}$  and  $a < c < d < b$ . For large  $k$ ,  $l_k < d - c$  since  $l_k \rightarrow 0$ . Then, for  $\hat{C}_k$ ,  $c$  and  $d$  must lie in different intervals of  $\hat{C}_k$ . So there is  $e \notin \hat{C}_k$  such that  $c < e < d$ . Then  $[c, d]$  does not belong to  $\hat{C}_k$  which is a contradiction. So  $\hat{C}$  is totally disconnected.

- (d) It is well known fact that a nonempty perfect set is uncountable. We had learned it in an introductory analysis course and topology course.

□

**Exercise (1.7).**

First, we will show that if  $O$  is open, then  $\delta O$  is also open. Let  $\delta x \in \delta O$ . Then  $x \in O$ . By openness, there is  $r > 0$  such that  $Q_r(x) \subset O$  where  $Q_r(x)$  is a cube whose side length is  $r$  and centered at  $x$ . Thus  $\delta Q_r(x) \subset \delta O$  and  $\delta Q_r(x)$  contains  $\delta x$ . But a collection of all open rectangles forms a basis of Euclidean space. So  $\delta O$  is an open set.

Next, let a set  $E$  and a positive number  $\varepsilon$  be given. Choose  $O \supset E$  such that  $m_*(O \setminus E) < \varepsilon/(\delta_1 \cdots \delta_d)$ . Then, there is a union of cube  $\bigcup_{j=1}^{\infty} Q_j \supset O \setminus E$  such that  $\sum_{j=1}^{\infty} m(Q_j) < \varepsilon/(\delta_1 \cdots \delta_d)$ . Then,

$$m_*(\delta O \setminus \delta E) = m_*(\delta(O \setminus E)) \leq m_*\left(\bigcup_{j=1}^{\infty} \delta Q_j\right) \leq \sum_{j=1}^{\infty} m(\delta Q_j) < \varepsilon.$$

Thus  $\delta E$  is measurable.

Now let  $E \subset \bigcup_{j=1}^{\infty} Q_j$ . Then  $\delta E \subset \bigcup \delta Q_j$ , so  $m(\delta E) \leq \delta_1 \cdots \delta_d \sum_{j=1}^{\infty} m(Q_j)$ . Since  $\bigcup_{j=1}^{\infty} Q_j$  is arbitrary, we get

$$m(\delta E) \leq \delta_1 \cdots \delta_d m(E).$$

Now let  $\delta E \subset \bigcup_{j=1}^{\infty} Q'_j$ . Then  $E \subset \bigcup_{j=1}^{\infty} 1/\delta Q'_j$ . So  $m(E) \leq \sum_{j=1}^{\infty} m(Q'_j)/(\delta_1 \cdots \delta_d)$ . Since  $\bigcup_{j=1}^{\infty} Q'_j$  is arbitrary, we get

$$m(E) \leq \frac{m(\delta E)}{\delta_1 \cdots \delta_d}$$

and this finishes the proof. □

**Exercise (1.24).**

Let  $s_n$  be enumeration of  $\mathbb{Q} \cap [-1, 1]$  and  $t_n$  be enumeration of  $\mathbb{Q} \cap [-1, 1]^c$ . When  $n = m^2$ , put  $r_n = t_m$ . When  $n \in (m^2, (m+1)^2)$ , put  $r_n = s_{n-m}$ . Then  $r_n$  is an enumeration of  $\mathbb{Q}$ . Also, we get

$$\begin{aligned} m\left(\bigcup_{n=1}^{\infty} (r_n - 1/n, r_n + 1/n)\right) &\leq \sum_{m=1}^{\infty} 2/m^2 + m\left(\bigcup_{n \neq m^2} (r_n - 1/n, r_n + 1/n)\right) \\ &\leq \sum_{m=1}^{\infty} 2/m^2 + 2 + 1 < \infty. \end{aligned}$$

Therefore, finiteness implies nonemptiness of the complement, since the Lebesgue measure of complement is positive. □

**Exercise (1.35).**

First, let's briefly check the idea of constructing  $\varphi$ . Construction can be done by defining a sequence of functions, say  $\varphi_n$ . Put  $\varphi_n(0) = 0$  and  $\varphi_n(1) = 1$ . Let  $C_{ji}$  be the  $i$ -th stage of constructing  $C_j$ . Then  $\varphi_i$  maps the discarded set of stage  $i$  to the discarded set of stage  $i$ , sequentially, and linearly(positive). We can extend  $\varphi_i$  by assigning value on  $C_{1i}$  using linearity and monotonicity. This sequence of functions converges uniformly, thus  $\varphi$  is continuous. The other properties of  $\varphi$  can be checked by this construction.

Let  $\mathcal{N} \subset C_1$  be a non-measurable set. Then  $\varphi(\mathcal{N}) \subset C_2$  so  $\varphi(\mathcal{N})$  is measurable by completeness. If  $\varphi(\mathcal{N})$  is a Borel set, then by continuity,  $\varphi^{-1}(\varphi(\mathcal{N})) = \mathcal{N}$  must be a Borel set, which is a contradiction. So there is a Lebesgue measurable set which is not Borel measurable.

Since  $\varphi(\mathcal{N})$  is measurable,  $f = 1_{\varphi(\mathcal{N})}$  is a measurable map. Then  $f \circ \varphi(x) = 1_{\mathcal{N}}(x)$  is non-measurable map. □

**Problem (1.4).**

- (a)  $A_\varepsilon$  is clearly bounded, so it is enough to show that the complement is open. Let  $c \notin A_\varepsilon$ . Then  $\text{osc}(f, c) < \varepsilon$ , so for some  $r > 0$ ,  $\text{osc}(f, c, r) < \varepsilon$ . Choose any  $d \in I(c, r)$ . We can choose  $r^* > 0$  so that  $I(d, r^*) \subset I(c, r)$ . Then

$$\text{osc}(f, d, r^*) \leq \text{osc}(f, c, r) < \varepsilon$$

so  $\text{osc}(f, d) < \varepsilon$ , which says  $I(c, r) \subset J \setminus A_\varepsilon$ . Therefore  $J \setminus A_\varepsilon$  is open in  $J$ , hence  $A_\varepsilon$  is compact.

- (b) Let  $D_f$  be a set of all discontinuities of  $f$ . Then for any  $\varepsilon > 0$ ,  $A_\varepsilon \subset D_f$ . So  $m(A_\varepsilon) \leq m(D_f) = 0$ . By the definition of Lebesgue measure, there is countably many open intervals which cover  $A_\varepsilon$  and have sum of length  $\leq \varepsilon$ . Using compactness, we can choose finite subcover, call them by  $(a_i, b_i)_{i=1}^k$  where  $a_i < a_{i+1}$ . After discarding all of subcovers from  $J$ , we get compact subset of  $J$ , say  $J'$ . For each  $c \in J'$ , we can choose  $r_c$  such that  $\text{osc}(f, c, 2r_c) < \varepsilon$ . Again, using compactness, we can choose finitely many  $c$ 's. Then finitely many closed intervals  $[c - r_c, c + r_c]$  have finite intersections. By taking these endpoints (contain  $a_i, b_i$ 's) as endpoints of our partition (if necessary, consider a refinement), we get

$$U(f, P) - L(f, P) \leq 2M\varepsilon + m(J)\varepsilon$$

where  $M$  is bound of  $f$ . The first term of estimate comes from  $(a_i, b_i)$ 's and the second term comes from  $J'$ .

- (c) Since  $D_f \subset \bigcup_{n=1}^{\infty} A_{1/n}$ , so  $m(A_{1/n}) = 0$  leads the conclusion. Assume not, i.e.  $m(A_{1/n}) > \varepsilon$ . Take partition  $P$  such that  $U(f, P) - L(f, P) < \varepsilon/n$ . Let  $[a, b]$  be interval of  $P$  whose interior intersects to  $A_{1/n}$ . Then

$$\sup_{x, y \in [a, b]} |f(x) - f(y)| \geq \frac{1}{n}.$$

But  $m(A_{1/n}) > \varepsilon$ . So

$$\begin{aligned} & \sum_{[a, b] \cap A_{1/n} \neq \emptyset} \left[ \sup_{x \in [a, b]} f(x) - \inf_{y \in [a, b]} f(y) \right] m(A_{1/n} \cap [a, b]) \\ &= \sum_{[a, b] \cap A_{1/n} \neq \emptyset} \sup_{x, y \in [a, b]} |f(x) - f(y)| m(A_{1/n} \cap [a, b]) \\ &\geq \frac{\varepsilon}{n} \\ &> U(f, P) - L(f, P) \end{aligned}$$

which is a contradiction.

□

**Exercise (2.2).**

Let  $\varepsilon > 0$ . Choose  $g \in C_c(\mathbb{R}^d)$  such that  $\|f - g\|_1 < \varepsilon$ . Let the domain of  $g$  is contained in  $B_r(0)$ . For  $x \in B_r(0)$ ,

$$|x - \delta x| = |1 - \delta||x| \leq r|1 - \delta| < \xi$$

if  $|1 - \delta|$  is small. Let  $\xi > 0$  be a number which satisfies  $|x - y| < \xi \Rightarrow |g(x) - g(y)| < \varepsilon$ . Then, for enough small  $|1 - \delta|$ , we get  $|x - \delta x| < \xi \Rightarrow |g(\delta x) - g(x)| < \varepsilon$ . Thus we get  $\|g_\delta - g\| \leq \varepsilon m(B_r(0))$ ,  $\|f - g\| < \varepsilon$ ,  $\|f_\delta - g_\delta\| < K\varepsilon$ . Therefore

$$\|f - f_\delta\| \leq \|f - g\| + \|g - g_\delta\| + \|g_\delta - f_\delta\| \leq (m(B_r(0)) + 1 + K)\varepsilon.$$

This says as  $\delta \rightarrow 1$ ,  $\|f_\delta - f\| \rightarrow 0$ .

□

**Exercise (2.6).**

(a) Let  $n \in \mathbb{N}$ . On  $[n, n+1]$ , define

$$f(x) = \begin{cases} n & \text{if } n \leq x \leq n + 1/n^3 \\ 1/n^3 & \text{if } n + 2/n^3 \leq x \leq n + 1 - 1/n^3 \\ \text{linear} & \text{otherwise.} \end{cases}$$

Then

$$\int_{[n, n+1]} f(x) dx \leq \frac{1}{n^2} + \frac{1}{n^3} n \frac{1}{2} + \left(1 - \frac{3}{n^3}\right) \frac{1}{n^3} + \frac{1}{n^3} (n+1) \frac{1}{2} = \frac{2n+3}{2n^3} + \frac{1}{n^2} - \frac{3}{n^6}.$$

Now, reflect  $f$  to the  $y$ -axis. Define  $f$  on  $(-1, 1)$  by 1. Then

$$\int_{\mathbb{R}} f dm \leq 2 + 2 \left( \sum_{n \geq 1} \left( \frac{4n+2}{2n^3} - \frac{3}{n^6} \right) \right) < \infty.$$

But clearly  $\limsup_{x \rightarrow \infty} f(x) = \infty$ .

(b) By same manipulation used in #2.24.b, the result follows. See after If  $\varphi$  does not vanish  $\sim$ .

□

**Exercise (2.19).**

Let  $g(x, \alpha) = 1_{E_\alpha}(x) 1_{(0, \infty)}(\alpha)$ . Since  $g$  is nonnegative, Tonelli's theorem can be applied.

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}} g dm &= \int_{\mathbb{R}^d} \int_{\mathbb{R}} g_x d\alpha dx = \int_{\mathbb{R}^d} |f(x)| dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} g^\alpha dx d\alpha = \int_{(0, \infty)} m(E_\alpha) d\alpha. \end{aligned}$$

Because  $g_x(\alpha) = 1_{(0 < \alpha < |f(x)|)}(\alpha)$  and  $g^\alpha(x) = 1_{(0 < \alpha < |f(x)|)}(x)$ .

□

**Exercise (2.24).**

Let  $\varphi = f * g$ .

(a) Choose  $h > 0$  small so that  $\|f_h - f\|_1 < \varepsilon$ . Then

$$|\varphi(x+h) - \varphi(x)| \leq \int |f(x+h-y) - f(x-y)| |g(y)| dy \leq B \|f_h - f\|_1 < B\varepsilon.$$

Thus  $\varphi$  is uniformly continuous.

(b) By Tonelli's theorem,

$$\|\varphi\|_1 \leq \iint |f(x-y)| |g(y)| dy dx \leq \|f\|_1 \int |g(y)| dy = \|f\|_1 \|g\|_1 < \infty.$$

So  $\varphi \in L^1$ . Note that  $\varphi$  is uniformly continuous by (a).

If  $\varphi$  does not vanish at infinity, then there exists  $\varepsilon > 0$  such that for all  $M > 0$ , there is  $|x_M| \geq M : |\varphi(x_M)| > 2\varepsilon$ . By uniform continuity, there is  $\delta > 0$  such that  $|x - y| < \delta \Rightarrow |\varphi(x) - \varphi(y)| < \varepsilon$ . We can get strictly increasing sequence  $y_i \in \{x_M : M > 0\}$  such that  $B_\delta(y_i) \cap B_\delta(y_j) = \emptyset$  whenever  $i \neq j$ .

Note that for  $x \in B_\delta(y_i)$ ,  $|\varphi(x)| > \varepsilon$ . Thus

$$\int |\varphi| dx \geq \sum_{i=1}^{\infty} \varepsilon m(B_\delta(y_i)) = \infty.$$

But the above contradicts to  $\varphi \in L^1$ .

□

**Problem (2.3).**

Let  $E_k = \{|f_k - f| > \varepsilon\}$ . By the Markov inequality,

$$m(E_k) \leq \frac{1}{\varepsilon} \int |f_k - f| dm.$$

Since  $f_k \rightarrow f$  in  $L^1$ , we get  $m(E_k) \rightarrow 0$ . Thus  $L^1$  convergence implies the convergence in measure.

For counterexample, consider  $f_k = k 1_{(0, 1/k)}$ . Then  $\int f_k dm = 1$ . But  $m(|f_k| > \varepsilon) \leq 1/k$  so  $f_k \rightarrow 0$  in measure. But, as we seen,  $f_k$  does not converge to 0 in  $L^1$ . Thus the converse of the previous result is not true. □