HW

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Problem 1. *$u_{xx} = (2y+6xy)_x = 6y$ and $u_{yy} = \left(2x+3x^2-3y^2\right)_y = -6y$, so given function u is harmonic. Let v be a harmonic conjugate of u. Then $f = u+iv$ must satisfy Cauchy - Riemann equation. Therefore $v_y = u_x = 2y+6xy$, $v = y^2+3xy^2+\varphi(x)$. $v_x = 3y^2+\varphi'(x) = 3y^2-3x^2-2x$. So $\varphi(x) = -x^3-x^2+C$ and $v = y^2+3xy^2-x^3-x^2+C$ where $C \in \mathbb{R}$.*

Problem 2.

$$\begin{aligned} u_x &= e^x\left(x\sin y + y\cos y + \sin y\right) = v_y \\ u_y &= e^x\left(x\cos y + \cos y - y\sin y\right) = -v_x \end{aligned} \tag{1}$$

Because $\int xe^x dx = (x-1)e^x + c$ for some $c \in \mathbb{R}$, we get

$$v = -\cos y\,(x-1)\,e^x - e^x\left(\cos y - y\sin y\right) + \varphi(y) \tag{2}$$

by integrating v_x with respect to x.

We can get $\varphi'(y) = 0$ by differntiating (2) and comparing with v_x in (1). Therefore,

$$v = -e^x(x-1)\cos y - e^x\left(\cos y - y\sin y\right) + c \tag{3}$$

for some $c \in \mathbb{R}$.

Problem 3.

$$\begin{aligned} u_x + v_x &= -2e^{-x}\left(\cos y - \sin y\right) = u_x - u_y \\ u_y + v_y &= 2e^{-x}\left(-\sin y - \cos y\right) = u_x + u_y \end{aligned} \tag{4}$$

By CR equation, $u_x = v_y$ and $u_y = -v_x$. By (4) and CR equation, we can get

$$\begin{aligned} u_x &= 2e^{-x}\left(-\cos y\right) = v_y \\ u_y &= 2e^{-x}\left(-\sin y\right) = -v_x \end{aligned} \tag{5}$$

So,

$$\begin{aligned} u &= 2e^{-x}\cos y + c_1 \\ v &= -2e^{-x}\sin y + c_2 \end{aligned} \tag{6}$$

Therefore, $f(x,y) = 2e^{-x}\left(\cos y + i\sin(-y)\right) + C = 2e^{-x}e^{-iy} + C = 2e^{-z} + C$ where $C \in \mathbb{C}$.

Problem 4. *Let $f = u+iv$. $zf(z) = (x+iy)(u+iv) = xu - yv + i(uy+vx)$.*

$$\begin{aligned} (xu-yv)_{xx} &= \left(u + xu_x - yv_x\right)_x = \left(2u_x + xu_{xx} - yv_{xx}\right) \\ (xu-yv)_{yy} &= \left(xu_y - v - yv_y\right)_y = \left(xu_{yy} - 2v_y - yv_{yy}\right) \end{aligned} \tag{7}$$

Since f and $zf(z)$ are both harmonic, by (7), we can get $u_x = v_y$. Similar computation on imaginary part of $zf(z)$ yields $u_y = -v_x$. Also note that each partials of f is continuous on D. Therefore, f is analytic on D.

Problem 5. *We can choose $\varepsilon > 0$ such that $L+\varepsilon < 1$. Then, there exists positive integer N such that $\sup_{k\geq n}\left|\frac{a_{k+1}}{a_k}\right| < L+\varepsilon$ for all $n \geq N$. So $\left|\frac{a_{n+1}}{a_n}\right| < L+\varepsilon$ for all $n \geq N$. We can deduce $|a_{k+N}| < (L+\varepsilon)^k\,|a_N|$.*

Therefore,

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{N-1}|a_n| + \sum_{n=N}^{\infty}|a_N| \le \sum_{n=1}^{N-1}|a_n| + \sum_{k=0}^{\infty}|a_N|(L+\varepsilon)^k < \infty \tag{8}$$

Problem 6.

(a) For $E = \{z : z \neq \pm ni, n \in \mathbb{N}\}$, given sequence converges pointwisely to zero. For $|z| \le R \in \mathbb{R}_{\ge 0}$ and $n^2 \ge N \ge R^2 + \frac{R}{\varepsilon}$ for some small positive ε,

$$\left|\frac{z}{z^2+n^2}\right| \le \frac{|z|}{|n^2-|z|^2|} \le \frac{R}{n^2-R^2} < \varepsilon \tag{9}$$

Therefore, for $E \bigcap \{|z| \le R\}$, given sequence converges uniformly for any $R \in \mathbb{R}_{\ge 0}$.

(b) Let $z = x + iy$. Then $\frac{e^{nx+iny}}{n}$ converges uniformly when $x \le 0$. If $x > 0$, $\frac{e^{nx}}{n} \uparrow \infty$ as $n \uparrow \infty$ which implies $\frac{e^{n}z}{n}$ does not converge.

Problem 7. *When $z = 1$, $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. So given series does not converge absolutely on E.*

Let $E_1 = \{z : |z| \le r\}$ and $E_2 = \{z : r \le z \le 1, z \in \mathbb{R}\}$. On E_1, given series converges uniformly because its radii of convergence is 1. For $z \in E_2$, let $\varepsilon > 0$ be given. take positive integer N such that $N\varepsilon > 1$. Then, we get $\left|\sum_{k=n}^{n+p}\frac{(-1)^k z^k}{k}\right| \le \left|\frac{(-1)^n z^n}{n}\right| \le \frac{1}{n} < \varepsilon$ if $n \ge N$. So given series converges uniformly on E_2.

Finally, let ε be given. There exist positive integers N_1, N_2 such that $n \ge N_1$ implies $\left|\sum_{k=n}^{n+p}\frac{(-1)^k z^k}{k}\right| < \varepsilon$ for $z \in E_1$ and $n \ge N_2$ implies $\left|\sum_{k=n}^{n+p}\frac{(-1)^k z^k}{k}\right| < \varepsilon$ for $z \in E_2$. If we take $n \ge \max\{N_1, N_2\}$, we can see that given series is uniformly cauchy on E.

Problem 8. *For $|1 + z^2| > 1$, given series converges absolutely. So it converges absolutely on region $|1 + z^2| \ge R > 1$ for any R.*
For $1 < R \le |z^2 + 1| \le R'$, we can see that

$$\left|\frac{z^2}{(1+z^2)^n}\right| \le \frac{R'+1}{R^n} = M_n \tag{10}$$

By M-test, we can conclude that given series uniformly converges on region $\{z : 1 < R \le |z^2 + 1| \le R'\}$.

Problem 9. *On the contrary, assume $\{a_n z_0^n\}$ is bounded. Take r such that $R < r < |z_0|$. Then we get*

$$|a_n r^n| = |a_n z_0^n|\frac{r^n}{|z_0|^n} \le M\left(\frac{r}{|z_0|}\right)^n \tag{11}$$

for some M since $\{a_n z_0^n\}$ is bounded. Then by M-test, $\sum_{n=0}^{\infty} a_n r^n$ converges. It contradicts to the condition that radii of convergence is R.

Problem 10.

(a) $\limsup_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$ so radii of convergence is $\frac{1}{e}$.

(b) $\limsup_{n\to\infty}\left(\frac{2^n+3^n}{4^n+5^n}\right)^{\frac{1}{n}} = \limsup_{n\to\infty}\frac{3}{5}\left(\frac{1+\frac{2^n}{3^n}}{1+\frac{4^n}{5^n}}\right)^{\frac{1}{n}} = \frac{3}{5}$. *So radii of convergence is* $\frac{5}{3}$.

Problem 11. *Let* $w = \frac{-z^2}{8}$. *Then given series is*

$$\sum_{n=0}^{\infty} w^n \tag{12}$$

This series has radii of convergence 1. So, it converges for $|w| < 1$ *which is equivalent to* $|z| < 2\sqrt{2}$. *Also,* $\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}$ *if* $|w| < 1$. *So we get following:*

$$\sum_{n=0}^{\infty} w^n = \frac{1}{1-w} = \frac{1}{1+\frac{z^2}{8}} = \frac{8}{8+z^2} \tag{13}$$

Problem 12. *Note that for* $n > 10$, $n^2 \le n!$ *by induction.* $f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n$. *Let* $z - a = w$ *and* $a_n = \frac{f^{(n)}(a)}{n!}$. *Then*

$$\limsup_{n\to\infty} = \limsup_{n\to\infty}|a_n|^{\frac{1}{n}} \le \limsup_{n\to\infty}\left|\frac{M}{n!}\right|^{\frac{1}{n}} = 0. \tag{14}$$

So radii of convergence is ∞, *so the given function is entire because it can be represented as series. Note that last equality of (14) follows from* $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$ *which is entire.*