mas540 exercises

Jaemin Oh

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Exercise (1.4)**.**

(a) *Let $I = [0,1]$. Then $I \backslash \hat{C} = \bigcup_{n=1}^{\infty} \hat{C}_n^c$ where $\hat{C}_n$ is n-th stage of constructing Fat Cantor set. Thus,*

$$m(I \setminus \hat{C}) = m(I) - m(\hat{C}) = 1 - m(\hat{C}) = \lim_{n\to\infty} m(\hat{C}_n^c) = \sum_{n=1}^{\infty} 2^{n-1} l_n$$

because $\hat{C}_n^c \uparrow \bigcup_{n=1}^{\infty} \hat{C}_n^c$ and $\hat{C}$ is closed hence measurable. Therfore $m(\hat{C}) = 1 - \sum_{n=1}^{\infty} 2^{n-1} l_n > 0$.

(b) *$\hat{C}_k$ consists of 2^k closed intervals whose length are $(1 - \sum_{n=1}^{k} 2^{n-1} l_n)/2^k$. Let $x \in \hat{C}$. Then $x \in \hat{C}_k$. So we can find $x_k \in I_k$ such that*

$$|x - x_k| \le \left(1 - \sum_{n=1}^{k} 2^{n-1} l_n\right)/2^k + \varepsilon_k l_k$$

for some $0 < \varepsilon_k < 1$. As $k \to \infty$, $|x - x_k| \to 0$ since $l_k \to 0$.

(c) *The result of b tells us that every point of $\hat{C}$ is a limit point of I. And we also know that $\hat{C}$ is closed. Hence $\hat{C}$ is a perfect set.*

Let $(a,b) \subset \hat{C}$ and $a < c < d < b$. For large k, $l_k < d - c$ since $l_k \to 0$. Then, for $\hat{C}_k$, c and d must lie in different intervals of $\hat{C}_k$. So there is $e \notin \hat{C}_k$ such that $c < e < d$. Then $[c,d]$ does not belong to $\hat{C}_k$ which is a contradiction. So $\hat{C}$ is totally disconnected.

(d) *It is well known fact that a nonempty perfect set is uncountable. We had learned it in an introductory analysis course and topology course.*

$\square$

Exercise (1.7)**.**

First, we will show that if O is open, then δO is also open. Let $\delta x \in \delta O$. Then $x \in O$. By openness, there is $r > 0$ such that $Q_r(x) \subset O$ where $Q_r(x)$ is a cube whose side length is r and centered at x. Thus $\delta Q_r(x) \subset \delta O$ and $\delta Q_r(x)$ contains δx. But a collection of all open rectangles forms a basis of Euclidean space. So δO is an open set.

Next, let a set E and a positive number ε be given. Choose $O \supset E$ such that $m_*(O \setminus E) < \varepsilon/(\delta_1 \cdots \delta_d)$. Then, there is an union of cube $\bigcup_{j=1}^{\infty} Q_j \supset O \setminus E$ such that $\sum_{j=1}^{\infty} m(Q_j) < \varepsilon/(\delta_1 \cdots \delta_d)$. Then,

$$m_*(\delta O \setminus \delta E) = m_*(\delta(O \setminus E)) \le m_*(\bigcup_{j=1}^{\infty} \delta Q_j) \le \sum_{j=1}^{\infty} m(\delta Q_j) < \varepsilon.$$

Thus δE is measurable.

Now let $E \subset \bigcup_{j=1}^{\infty} Q_j$. Then $\delta E \subset \bigcup \delta Q_j$, so $m(\delta E) \leq \delta_1 \cdots \delta_d \sum_{j=1}^{\infty} m(Q_j)$. Since $\bigcup_{j=1}^{\infty}$ is arbitrary, we get

$$m(\delta E) \leq \delta_1 \cdots \delta_d m(E).$$

Now let $\delta E \subset \bigcup_{j=1}^{\infty} Q'_j$. Then $E \subset \bigcup_{j=1}^{\infty} 1/\delta Q'_j$. So $m(E) \leq \sum_{j=1}^{\infty} m(Q'_j)/(\delta_1 \cdots \delta_d)$. Since $\bigcup_{j=1}^{\infty} Q'_j$ is arbitary, we get

$$m(E) \leq \frac{m(\delta E)}{\delta_1 \cdots \delta_d}$$

and this finishes the proof.

$\square$

Exercise (1.24).

Let s_n be enumeration of $\mathbb{Q} \cap [-1,1]$ and t_n be enumeration of $\mathbb{Q} \cap [-1,1]^c$. When $n = m^2$, put $r_n = t_m$. When $n \in \big(m^2, (m+1)^2\big)$, put $r_n = s_{n-m}$. Then r_n is an enumeration of $\mathbb{Q}$. Also, we get

$$\begin{aligned} m\left(\bigcup_{n=1}^{\infty} (r_n - 1/n, r_n + 1/n)\right) &\leq \sum_{m=1}^{\infty} 2/m^2 + m\left(\bigcup_{n \neq m^2} (r_n - 1/n, r_n + 1/n)\right) \\ &\leq \sum_{m=1}^{\infty} 2/m^2 + 2 + 1 < \infty. \end{aligned}$$

Therefore, finiteness implies nonemptyness of the complement, since the Lebesgue measure of complement is positive.

$\square$

Exercise (1.35).

First, let's briefly check the idea of constructing φ. Construction can be done by defining a sequence of functions, say φ_n. Put $\varphi_n(0) = 0$ and $\varphi_n(1) = 1$. Let C_{ji} be the i-th stage of constructing C_j. Then φ_i maps the discarded set of stage i to the discarded set of stage i, sequentially, and linearly(positive). We can extend φ_i by assigning value on C_{1i} using linearity and monotonicity. This sequence of functions converges uniformly, thus φ is continuous. The other properties of φ can be checked by this construction.

Let $\mathcal{N} \subset C_1$ be a non-measurable set. Then $\varphi(\mathcal{N}) \subset C_2$ so $\varphi(\mathcal{N})$ is measurable by completeness. If $\varphi(\mathcal{N})$ is a Borel set, then by continuity, $\varphi^{-1}(\varphi(\mathcal{N})) = \mathcal{N}$ must be a Borel set, which is a contradiction. So there is a Lebesgue measurable set which is not Borel measurable.

Since $\varphi(\mathcal{N})$ is measurable, $f = 1_{\varphi(\mathcal{N})}$ is a measurable map. Then $f \circ \varphi(x) = 1_{\mathcal{N}}(x)$ is non-measurable map.

$\square$

Problem (1.4).

(a) A_ε *is clearly bounded, so it is enough to show that the complement is open. Let* $c \notin A_\varepsilon$. *Then* $osc(f,c) < \varepsilon$, *so for some* $r > 0$, $osc(f,c,r) < \varepsilon$. *Choose any* $d \in I(c,r)$. *We can choose* $r^* > 0$ *so that* $I(d,r^*) \subset I(c,r)$. *Then*

$$osc(f,d,r^*) \le osc(f,c,r) < \varepsilon$$

so $osc(f,d) < \varepsilon$, *which says* $I(c,r) \subset J \setminus A_\varepsilon$. *Therefore* $J \setminus A_\varepsilon$ *is open in* J, *hence* A_ε *is compact.*

(b) *Let* D_f *be a set of all discontinuities of* f. *Then for any* $\varepsilon > 0$, $A_\varepsilon \subset D_f$. *So* $m(A_\varepsilon) \le m(D_f) = 0$. *By the definition of Lebesgue measure, there is countably many open intervals which cover* A_ε *and have sum of length* $\le \varepsilon$. *Using compactness, we can choose finite subcover, call them by* $(a_i,b_i)_{i=1}^k$ *where* $a_i < a_{i+1}$. *After discarding all of subcovers from* J, *we get compact subset of* J, *say* J'. *For each* $c \in J'$, *we can choose* r_c *such that* $osc(f,c,2r_c) < \varepsilon$. *Again, using compactness, we can choose finitely many* c*'s. Then finitely many closed intervals* $[c-r_c, c+r_c]$ *have finite intersections. By taking these endpoints(contain* a_i, b_i*'s) as endpoints of our partition(if necessary, consider a refinement), we get*

$$U(f,P) - L(f,P) \le 2M\varepsilon + m(J)\varepsilon$$

where M *is bound of* f. *The first term of estimate comes from* (a_i,b_i)*'s and the second term comes from* J'.

(c) *Since* $D_f \subset \bigcup_{n=1}^\infty A_{1/n}$, *so* $m(A_{1/n}) = 0$ *leads the conclusion. Assume not, i.e.* $m(A_{1/n}) > \varepsilon$. *Take partition* P *such that* $U(f,P) - L(f,P) < \varepsilon/n$. *Let* $[a,b]$ *be interval of* P *whose interior intersects to* $A_{1/n}$. *Then*

$$\sup_{x,y\in[a,b]} |f(x) - f(y)| \ge \frac{1}{n}.$$

But $m(A_{1/n}) > \varepsilon$. *So*

$$\begin{aligned}
&\sum_{[a,b]\cap A_{1/n}\neq\emptyset} \left[\sup_{x\in[a,b]} f(x) - \inf_{y\in[a,b]} f(y)\right] m\left(A_{1/n}\cap[a,b]\right) \\
&= \sum_{[a,b]\cap A_{1/n}\neq\emptyset} \sup_{x,y\in[a,b]} |f(x)-f(y)|\, m\left(A_{1/n}\cap[a,b]\right) \\
&\ge \frac{\varepsilon}{n} \\
&> U(f,P) - L(f,P)
\end{aligned}$$

which is a contradiction.

□

Exercise (2.2).

Let $\varepsilon > 0$. *Choose* $g \in C_c(\mathbb{R}^d)$ *such that* $\|f - g\|_1 < \varepsilon$. *Let the domain of* g *is contained in* $B_r(0)$. *For* $x \in B_r(0)$,

$$|x - \delta x| = |1 - \delta||x| \le r|1 - \delta| < \xi$$

if $|1 - \delta|$ *is small. Let* $\xi > 0$ *be a number which satisfies* $|x - y| < \xi \Rightarrow |g(x) - g(y)| < \varepsilon$. *Then, for enoughly small* $|1-\delta|$, *we get* $|x - \delta x| < \xi \Rightarrow |g(\delta x) - g(x)| < \varepsilon$. *Thus we get* $\|g_\delta - g\| \le \varepsilon m(B_r(0))$, $\|f - g\| < \varepsilon$, $\|f_\delta - g_\delta\| < K\varepsilon$. *Therefore*

$$\|f - f_\delta\| \le \|f - g\| + \|g - g_\delta\| + \|g_\delta - f_\delta\| \le (m(B_r(0)) + 1 + K)\,\varepsilon.$$

This says as $\delta \to 1$, $\|f_\delta - f\| \to 0$.

□

Exercise (2.6).

(a) *Let* $n \in \mathbb{N}$. *On* $[n, n+1]$, *define*

$$f(x) = \begin{cases} n & \text{if } n \le x \le n + 1/n^3 \\ 1/n^3 & \text{if } n + 2/n^3 \le x \le n + 1 - 1/n^3 \\ \text{linear} & \text{otherwise.} \end{cases}$$

Then

$$\int_{[n,n+1]} f(x)dx \le \frac{1}{n^2} + \frac{1}{n^3}n\frac{1}{2} + \left(1 - \frac{3}{n^3}\right)\frac{1}{n^3} + \frac{1}{n^3}(n+1)\frac{1}{2} = \frac{2n+3}{2n^3} + \frac{1}{n^2} - \frac{3}{n^6}.$$

Now, reflect f *to the y-axis. Define* f *on* $(-1, 1)$ *by* 1. *Then*

$$\int_{\mathbb{R}} f dm \le 2 + 2\left(\sum_{n\ge1}\left(\frac{4n+2}{2n^3} - \frac{3}{n^6}\right)\right) < \infty.$$

But clearly $\limsup_{x\to\infty} f(x) = \infty$.

(b) *By same manipulation used in #2.24.b, the result follows. See after If* φ *does not vanish* $\sim$.

□

Exercise (2.19).

Let $g(x, \alpha) = 1_{E_\alpha}(x)1_{(0,\infty)}(\alpha)$. *Since* g *is nonnegative, Tonelli's theorem can be applied.*

$$\begin{aligned} \int_{\mathbb{R}^d\times\mathbb{R}} g dm &= \int_{\mathbb{R}^d}\int_{\mathbb{R}} g_x d\alpha dx = \int_{\mathbb{R}^d} |f(x)|dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^d} g^\alpha dx d\alpha = \int_{(0,\infty)} m(E_\alpha)d\alpha. \end{aligned}$$

Because $g_x(\alpha) = 1_{(0<\alpha<|f(x)|)}(\alpha)$ *and* $g^\alpha(x) = 1_{(0<\alpha<|f(x)|)}(x)$. □

Exercise (2.24).

Let $\varphi = f * g$.

(a) *Choose* $h > 0$ *small so that* $\|f_h - f\|_1 < \varepsilon$. *Then*

$$|\varphi(x+h)-\varphi(x)| \le \int |f(x+h-y)-f(x-y)||g(y)|dy \le B\|f_h-f\|_1 < B\varepsilon.$$

Thus φ *is uniformly continuous.*

(b) *By Tonelli's theorem,*

$$\|\varphi\|_1 \le \iint |f(x-y)||g(y)|dydx \le \|f\|_1 \int |g(y)|dy = \|f\|_1\|g\|_1 < \infty.$$

So $\varphi \in L^1$. *Note that* φ *is uniformly continuous by (a).*

If φ *does not vanish at infinity, then there exists* $\varepsilon > 0$ *such that for all* $M > 0$, *there is* $|x_M| \ge M : \ |\varphi(x_M)| > 2\varepsilon$. *By uniform continuity, there is* $\delta > 0$ *such that* $|x-y| < \delta \Rightarrow |\varphi(x) - \varphi(y)| < \varepsilon$. *We can get strictly increasing sequence* $y_i \in \{x_M : M > 0\}$ *such that* $B_\delta(y_i) \cap B_\delta(y_j) = \emptyset$ *whenever* $i \neq j$.

Note that for $x \in B_\delta(y_i)$, $|\varphi(x)| > \varepsilon$. *Thus*

$$\int |\varphi|dx \ge \sum_{i=1}^{\infty} \varepsilon m(B_\delta(y_i)) = \infty.$$

But the above contradicts to $\varphi \in L^1$.

□

Problem (2.3).

Let $E_k = \{|f_k - f| > \varepsilon\}$. *By the Markov inequality,*

$$m(E_k) \le \frac{1}{\varepsilon}\int |f_k - f|dm.$$

Since $f_k \to f$ *in* L^1, *we get* $m(E_k) \to 0$. *Thus* L^1 *convergence implies the convergence in measure.*

For counterexample, consider $f_k = k1_{(0,1/k)}$. *Then* $\int f_k dm = 1$. *But* $m(|f_k| > \varepsilon) \le 1/k$ *so* $f_k \to 0$ *in measure. But, as we seen,* f_k *does not converge to* 0 *in* L^1. *Thus the converse of the previous result is not true.* □

Exercise (3.2).

Let $\{L_\delta\}$ *be any approximation to the identity. Then, by triangle inequality,* $\{K_\delta + L_\delta\}$ *is also approximation to the identity because of the third condition. Therefore*

$$f * (K_\delta + L_\delta)(x) \to f(x) \text{ a.e. } x$$

as $\delta \to 0$ *by theorem 2.1.*

But,

$$\begin{aligned} f * (K_\delta + L_\delta)(x) &= \int f(x-y)(K_\delta(y) + L_\delta(y))dy \\ &= f * K_\delta(x) + f * L_\delta(x). \end{aligned}$$

Since $f * L_\delta(x) \to f(x)$ *for a.e.* x*,* $f * K_\delta(x) \to 0$ *for a.e.* x *necessarily.*

□

Exercise (3.5).

(a) *By the change of variable formula*($\log x = t$)*,*

$$\begin{aligned} \int_{\mathbb{R}} |f(x)|dx &= \int_{-1/2}^{1/2} f(x)dx \\ &= \int_{-\infty}^{-\log 2} \frac{1}{t^2}dt = \frac{1}{\log 2} < \infty. \end{aligned}$$

(b) *Let* $\varepsilon > 0$*. Then*

$$\begin{aligned} f^*(x) &\geq \frac{1}{2|x| + 2\varepsilon} \int_{-|x|-\varepsilon}^{|x|+\varepsilon} \frac{dt}{t(\log t)^2} \\ &= \frac{1}{|x| + \varepsilon} \int_0^{|x|+\varepsilon} \frac{dt}{t(\log t)^2} \\ &= \frac{1}{-\log(|x| + \varepsilon)(|x| + \varepsilon)}. \end{aligned}$$

Since $\varepsilon > 0$ *is arbitrary, by taking* $\varepsilon \downarrow 0$*, we obtain*

$$f^*(x) \geq \frac{1}{|x| \log \frac{1}{|x|}}.$$

But $1/(-|x| \log |x|)$ *is clearly non-locally integrable function. This is by integrating on the interval containing* 0 *and the change of variable formula, used above.*

□

Exercise (3.12).

By chain rule, F' exists for all $x \neq 0$. But,

$$\lim_{h\to 0} \frac{F(h)}{h} = \lim_{h\to 0} h\sin(1/h^2) = 0$$

Thus F' exists for all $x \in \mathbb{R}$.

For $1/\sqrt{2n\pi + \pi/6} \leq x \leq 1/\sqrt{2n\pi - \pi/6}$, $2n\pi - \pi/6 \leq 1/x^2 \leq 2n\pi + \pi/6$, thus $\cos 1/x^2 \geq \sqrt{3}/2$ and $\left|\sin 1/x^2\right| \leq 1/2$. So $|F'| \geq 2/x\cos 1/x^2 - 2x\left|\sin 1/x^2\right| \geq \sqrt{3}\sqrt{2n\pi - \pi/6} - 1/\sqrt{2n\pi - \pi/6}$.

By using the above,

$$\begin{aligned}\int_0^1 |F'|dm &\geq \sum_{n=1}^{\infty}\left(1/\sqrt{2n\pi - \pi/6} - 1/\sqrt{2n\pi + \pi/6}\right)\left(\sqrt{3}\sqrt{2n\pi - \pi/6} - 1/\sqrt{2n\pi - \pi/6}\right) \\ &= \sum_{n=1}^{\infty} \frac{\pi/\sqrt{3}}{\sqrt{2n\pi + \pi/6}\left(\sqrt{2n\pi + \pi/6} + \sqrt{2n\pi - \pi/6}\right)} \\ &- \sum_{n=1}^{\infty} \frac{\pi/3}{(2n\pi - \pi/6)\sqrt{2n\pi + \pi/6}\left(\sqrt{2n\pi + \pi/6} + \sqrt{2n\pi - \pi/6}\right)}\end{aligned}$$

where the last sum converges and previous one diverges (by p-test.) Thus F' is non-integrable.

□

Exercise (3.23)**.**

(a) Follow the hint.

$$(D^+G_\varepsilon)(x_0) = (D^+F)(x_0) + \varepsilon > 0.$$

This means, for sufficiently small $h > 0$,

$$G_\varepsilon(x_0 + h) > G_\varepsilon(x_0) \geq 0.$$

This contradicts to our choice of x_0.

(b) Use the Mean value theorem.

□

Exercise (3.25)**.**

(a) Let f be the function given in the hint. Note that all of points in any open set O is a point of Lebesgue density. This is because, we can only consider small ball B_x contained in O. Thus

$$\liminf \frac{m(O_n \cap B)}{m(B)} = 1$$

for all $x \in E$. Therefore

$$\begin{aligned}\liminf \frac{1}{m(B)} \int_B f dm &= \liminf \sum_{n \geq 1} \frac{m(O_n \cap B)}{m(B)} \\ &\geq \sum_{n \geq 1} \liminf \frac{m(O_n \cap B)}{m(B)} = \sum_{n \geq 1} 1 = \infty.\end{aligned}$$

(b) *Let $F(x) = \int_{-\infty}^{x} f(t)dt$ where f is the function found in a. Then F satisfies the given condition.*

□

Exercise (3.32)**.**

Assume the Lipschitz condition. Take $\delta = \varepsilon/M$ when $\varepsilon > 0$ is given. For (a_i, b_i) such that $\sum_i (b_i - a_i) < \delta$, then $\sum_i |f(b_i) - f(a_i)| \leq M \sum_i (b_i - a_i) < M\delta = \varepsilon$. Thus f is absolutely continuous. So f' exists a.e. Now consider the following:

$$|f'(x)| = \lim_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|} \leq M$$

Thus $|f'| \leq M$ a.e. x.

For the other direction, without loss of generality, assume $x \leq y$. Since f is absolutely continuous, f' exists a.e, and $\int_x^y f' dm = f(y) - f(x)$. Thus, $|f(x) - f(y)| = \left|\int_x^y f' dm\right| \leq \int_x^y |f'| dm \leq (y - x)M = |x - y|M$.

□

Problem (3.5)**.**

First, assume that $F' \geq 0$ a.e. Let E be the set, $F'(x) < 0$. According to exercise 25, we can find Φ which is increasing, absolutely continuous, and $D_{\pm}\Phi(x) = \infty$ for all $x \in E$. Note that $\infty = D_+\Phi(x) \leq D^+\Phi(x)$. Now, for $\delta > 0$, consider $F + \delta\Phi$. On E, $D^+(F + \delta\Phi) = \infty > 0$. On E^c, $D^+(F + \delta\Phi) = F' + \delta\Phi' \geq 0$. Therefore, by exercise 23, $F + \delta\Phi$ is an increasing function. So

$$F(x) - F(a) + \delta(\Phi(x) - \Phi(a)) \geq 0.$$

Since $\delta > 0$ is arbitrary, we can assert $F(x) \geq F(a)$ whenever $x \geq a$.

Now we'll solve the problem using the above. Let $G(x) = \int_a^x F' dm$. Then $G'(x) = F'(x)$ a.e. by Lebesgue differentiation theorem. Thus $G'(x) - F'(x) \geq 0$ a.e. Then, the above implies $G(x) - G(a) - F(x) + F(a) \geq 0$. Since we can say that $G'(x) - F'(x) \leq 0$ a.e. also, we obtain $G(x) - G(a) - F(x) + F(a) \leq 0$. But $G(a) = 0$. Therefore $F(x) - F(a) = G(x) = \int_a^x F' dm$. Since F' is integrable, $\nu(B) = \int_B F' dm$ is absolutely continuous with respect to m so F is absolutely continuous.

□

Exercise (4.4)**.**

First, let's show the completeness. Let $\{f_n\} \subset l^2(\mathbb{Z})$ *be a Cauchy sequence. Choose* n_k *such that* $\|f_{n_{k+1}} - f_{n_k}\| < 2^{-k+1}$*. Define* $f = f_{n_1} + \sum_{k=1}^{\infty}(f_{n_{k+1}} - f_{n_k})$ *and* $g = |f_{n_1}| + \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$*. Note that* $\|g\| \leq \|f_{n_1}\| + \sum_{k=1}^{\infty} \|f_{n_{k+1}} - f_{n_k}\| \leq \|f_{n_1}\| + 2 < \infty$*. Because, for each* $i \in \mathbb{Z}$*,* $|f(i)| \leq g(i) \leq \|g\| < \infty$*, we can observe that* $f(i)$ *is absolutely converges. Thus* f *is well defined function, also in* l^2 *(*$\because \|f\| \leq \|g\| < \infty$*). Now let's show that* $f_{n_k} \to f$*.*

$$\|f - f_{n_k}\| \leq \sum_{m=k}^{\infty} \|f_{n_{m+1}} - f_{n_m}\| \leq 2^{-k}.$$

So $f_{n_k} \to f$ *as* $k \to \infty$ *in* l^2*. Therefore* $l^2(\mathbb{Z})$ *is complete.*

Now let's show the separability. Let $\mathcal{B}$ *be the set of all rational sequence in* $l^2(\mathbb{Z})$*. Clearly, it is nonempty since the zero sequence is in* $\mathcal{B}$*. Let* $f \in l^2$*. Fix* $\varepsilon > 0$*. For each* $i \in \mathbb{Z}$*, choose* q_i *such that*

$$|f(i) - q_i|^2 < \frac{\varepsilon^2}{2^{|i|}}.$$

Let $q : i \mapsto q_i$*. Then* $\|q\| \leq \|q - f\| + \|f\|$*, where*

$$\begin{aligned}
\|q - f\| &= \left(\sum_{-\infty}^{\infty} |q_i - f(i)|^2 \right)^{1/2} \\
&\leq \left(\sum_{-\infty}^{\infty} \frac{\varepsilon^2}{2^{|i|}} \right)^{1/2} \\
&= \sqrt{3}\varepsilon.
\end{aligned}$$

Since $\|f\| < \infty$*, we can see that* $q \in l^2$ *and* $\|f - q\| \leq \sqrt{3}\varepsilon$*. Note that* $q \in l^2$ *implies* $q \in \mathcal{B}$*. So* $\mathcal{B}$ *is dense in* l^2*, and clearly* $\mathcal{B}$ *is countable set.*

$\square$

Exercise (4.15)**.**

Let $\{e_1, e_2, \cdots, e_n\}$ *be an orthonormal basis of* $\mathcal{H}_1$*. Let* $f \in \mathcal{H}_1$*,* $\|f\| = 1$*. Then* $f = \sum_{i=1}^{n} c_i e_i$*, where* $\sqrt{\sum |c_i|^2} = 1$*. Then*

$$\begin{aligned}
\|Tf\| &= \|c_1 Te_1 + \cdots c_n Te_n\| \\
&\leq |c_1| \|Te_1\| + \cdots + |c_n| \|Te_n\| \\
&\leq \sum_{i=1}^{n} |c_i| M \\
&\leq M \left(\sum_{i=1}^{n} |c_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} 1 \right)^{1/2} \\
&= \sqrt{n} M < \infty
\end{aligned}$$

where $M = \max_{1\le i\le n} \|Te_i\|$. *Since* n *is fixed, the above says that* T *is bounded operator.*

□

Exercise (4.22)**.**

(a) *Polarization identity:*

$$(f,g) = \frac{1}{4}\left[\|f+g\|^2 - \|f-g\|^2 + i\|f+ig\|^2 - i\|f-ig\|^2\right].$$

This can be shown by using the hint. (Actually, we have seen it in the lecture.)

Put Tf, Tg *in the place of* f, g *respectively. Since* T *is linear and* $\|Tf\| = \|f\|$, *we can easily see that* $(f,g) = (Tf,Tg)$.

Now fix $g \in \mathcal{H}$. *Then* $(f, T^*Tg) = (Tf, Tg)$ *by the definition of adjoint, and* $(Tf,Tg) = (f,g)$ *by isometric property of* T. *Thus*

$$(f, T^*Tg - g) = 0$$

for all $f \in \mathcal{H}$. *Therefore* $T^*T = I$ *by taking* $f = T^*Tg - g$.

(b) *Let's show the injectivity. Let* $Tf = Tg$. *Then*

$$0 = \|Tf - Tg\| = \|f-g\| \Rightarrow f = g.$$

Thus T *is bijective isometry. Therefore it is an unitary operator.*

Now fix $g \in \mathcal{H}$. *For each* $f \in \mathcal{H}$, *there is* h *such that* $f = Th$ *because of the surjectivity. Then*

$$\begin{aligned}(f, TT^*g) &= (Th, TT^*g)\\ &= (h, T^*TT^*g)\\ &= (h, T^*g)\\ &= (Th, g)\\ &= (f,g)\end{aligned}$$

by the definition of the adjoint and $T^*T = I$ *because* T *is an isometry. Therefore*

$$(f, TT^*g - g) = 0$$

for all $f \in \mathcal{H}$. *By taking* $f = TT^*g - g$, *we can conclude that* $TT^* = I$.

(c) *Let* $\mathcal{H} = l^2(\mathbb{N})$. *Let* $f = (f(1), f(2), \cdots) \in \mathcal{H}$. *Define* $T : (f(1), f(2), \cdots) \mapsto (0, f(1), f(2), \cdots)$. *Clearly* T *is a linear operator, but non-surjective. If we show that* T *is isometry, then we are done.*

$$\|Tf\|^2 = 0 + \sum_{i=1}^{\infty} |f(i)|^2 = \|f\|^2.$$

SoT *is an isometry, which is not unitary.*

(d) *Note that unitary operator is isometry. So, by (a) and Cauchy Schwartz inequality,*

$$(Tf, Tf) = (f, T^*Tf) \leq \|f\|\|T^*Tf\| = \|f\|^2.$$

Thus $\|Tf\| \leq \|f\|$.

For the other direction,

$$\begin{aligned}(f, f) &= (T^*Tf, T^*Tf) \\ &= (Tf, TT^*Tf) \\ &\leq \|Tf\|\|TT^*Tf\|.\end{aligned}$$

But $\|TT^*Tf\|^2 = (TT^*Tf, TT^*Tf) = (Tf, TT^*TT^*Tf) = (Tf, Tf) = \|Tf\|^2$ *since* $(T^*T)^*(T^*T) = T^*TT^*T = I$ *by (a). Therefore* $(f, f) \leq (Tf, Tf)$*, which completes the proof.*

□

Exercise (4.32)**.**

(a) $T(cf + dg)(t) = t(cf + dg)(t) = ctf(t) + dtg(t) = cT(f)(t) + dT(g)(t)$ *so* T *is linear. Note that* $t^2 \leq 1$ *on* $[0, 1]$*. So*

$$\|Tf\|^2 = \int_0^1 t^2|f(t)|^2 dt \leq \int_0^1 |f(t)|^2 dt = \|f\|^2$$

which says that $\|T\| \leq 1$.

Also,

$$\begin{aligned}(Tf, g) &= \int_0^1 tf(t)\overline{g(t)}dt \\ &= \int_0^1 f(t)\overline{tg(t)}dt = (f, Tg)\end{aligned}$$

hence $Tg = T^*g$ *for all* $g \in L^2[0, 1]$ *by same argument used in exercise 22. Thus* T *is a bounded linear operator with* $T = T^*$.

Let $f_n(t) = \sqrt{2n + 1}t^n$*. Then* $\|f_n\|^2 = \int_0^1 (2n+1)t^{2n} dt = 1$ *for all* n*. Thus* $f_n \in$ *the unit ball of* $L^2[0, 1]$*. For any subsequence* f_{n_k}*,*

$$\begin{aligned}&\|Tf_{n_k} - Tf_{n_l}\|^2 \\ &= \int_0^1 (2n_k + 1)t^{2n_k+2} + (2n_l + 1)t^{2n_l+2} - 2\sqrt{(n_k + 1)(n_l + 1)}t^{(n_k+1)(n_l+1)}dt \\ &= \frac{2n_k + 1}{2n_k + 3} + \frac{2n_l + 1}{2n_l + 3} - \frac{2\sqrt{(n_k + 1)(n_l + 1)}}{(n_k + 1)(n_l + 1) + 1}.\end{aligned}$$

As $n_k, n_l \to \infty$*, the first two terms go to* 1 *respectively, but the last term go to* 0*. So the sequence does not converge. Hence* T *is non-compact.*

(b) Suppose $T\varphi = \lambda\varphi$. Then $t\varphi(t) = \lambda\varphi(t)$ for all $t \in [0,1]$. Then $t\varphi(t)1_{\varphi\neq 0}(t) = \lambda\varphi(t)1_{\varphi\neq 0}(t)$, so $1_{\varphi\neq 0}(t) = 0$, which means $\varphi = 0$. But the zero vector cannot be an eigenvector, hence there is no eigenvector.

$\square$

Problem (4.1).

Let X be a collection of linearly independent subsets of $\mathcal{H}$. Impose partial order by the inclusion. Note that X is nonempty since the empty set is in X.

We'll use Zorn's lemma which is equivalent to the AC. Let Y be any totally ordered subset of X. $L_Y = \bigcup_{w\in Y} w$. Then every finite subset of L_Y is in Y, since Y is totally ordered. Hence L_Y is linearly independent, so $L_Y \in X$. But, note that L_Y is an upperbound of Y in X. So Zorn's lemma gives L_m which is maximal element of X.

Now assert that L_m is an algebraic basis of $\mathcal{H}$. Since $L_m \in X$, L_m is linearly independent. If L_m does not span $\mathcal{H}$, then there is $f \in \mathcal{H}$ outside of span L_m. Define $L_f = L_m \cup \{f\}$. Then L_f is strictly larger than L_m. But, L_f is linearly independent, since f is outside of span of L_m. Thus $L_f \in X$, which contradicts to the maximality of L_m. Hence L_m spans $\mathcal{H}$ algebraically, so L_m is an algebraic basis.

Now $L_m = \{a_\alpha : \alpha \in I\}$. Let $B = \left\{e_\alpha = \frac{a_\alpha}{\|a_\alpha\|} : \alpha \in I\right\}$. Then B is an algebraic basis, consists of unit vectors.

Choose $\{e_i\}_{i\in\mathbb{N}}$. For $f \in \mathcal{H}$,

$$f = \sum_{\alpha\in F} c_\alpha e_\alpha = \sum_{\alpha\in F\setminus\mathbb{N}} c_\alpha e_\alpha + \sum_{i=1}^{N} c_i e_i$$

where F is finite set. Define $l(f) = \sum_{i=1}^N ic_i$. Note that N depends on f. Clearly, l is linear: $l(cf+dg) = c\sum_{i=1}^N ic_i + d\sum_{i=1}^N id_i = cl(f) + dl(g)$. Also $l(e_i) = i$. But, $|l(e_i)| = i \to \infty$ as $i \to \infty$, even though $\|e_i\| = 1$. This says l is unbounded linear functional.

$\square$