DQ - REAL ANALYSIS

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Problem 1. *Consider*

$$\mathcal{F} = \left\{ A \subset \mathbb{R}^n : f^{-1}(A) \in \mathcal{B}(\mathbb{R}^m) \right\}.$$

Then $\mathcal{F}$ is a $\sigma-$algebra containing every open set of $\mathbb{R}^n$. Thus $\mathcal{F}$ contains Borel $\sigma-$algebra of $\mathbb{R}^n$.

$\square$

Problem 2. *omitted.*

$\square$

Problem 3. *Clearly, $1/q < 1/r < 1/p$. Let $\theta \in (0,1)$ such that*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$

Also,

$$|f|^r = |f|^{r\theta}|f|^{r(1-\theta)}$$

Apply Holder's inequality.

$\square$

Problem 4. *First, consider $f_n = |f| \wedge n$. Let $\varepsilon > 0$ be given. By using MCT, we can find $\delta > 0$ such that*

$$m(E) < \delta \Rightarrow \int_E |f|dm < \varepsilon$$

But, the above implies the absolute continuity of F.

$\square$

Problem 5. *Let*

$$M = \{f \in \mathcal{H} : L(f) = 0\}$$

Since L is bounded linear functional, it is continuous, hence M is a closed subspace of $\mathcal{H}$.

If $M = \mathcal{H}$, then put $g = 0$. If $M \neq \mathcal{H}$, then there is nonzero $h \in M^{\perp}$. Consider

$$w = hL(f) - fL(h)$$

Then $L(w) = 0$, so $w \in M$, thus $w \perp h$. So,

$$0 = (w|h) = L(f)(h|h) - L(h)(f|h)$$

By taking $g = \overline{L(h)}h/(h|h)$, we have proved the existence of such element.

Let g_1, g_2 be such elements. Then $L(f) = (f|g_1) = (f|g_2)$ for all $f \in \mathcal{H}$. Put $f = g_1 - g_2$. Then

$$(g_1 - g_2|g_1) = (g_1 - g_2|g_2)$$

which leads $\|g_1 - g_2\| = 0$. Therefore the uniqueness follows.

$\square$

Problem 6.

(a) *Let E be a set of exterior measure zero. Let A be any set. Then*

$$m_*(A) \geq m_*(A \cap E^c) = m_*(A \cap E^c) + m_*(A \cap E)$$

where the last term is zero by monotonicity of exterior measure.

(b) *omitted.*

$\square$