mas441 homework

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Problem (1.5)**.**

(a) *Let $\varepsilon > 0$ be given. There is open set O containing E such that $m(O \setminus E) < \varepsilon$. Since E is compact set contained in open set O, there is $r > 0$ such that r neighborhood of E is contained in O. For $nr > 1$, $O_n \subset O$. Therefore $m(O_n \setminus E) \le m(O \setminus E) < \varepsilon$. Therefore $\lim_{n\to\infty} m(O_n) = m(E)$.*

(b) *For closed and unbounded set which does not satisfy above, consider $E = \left\{\sum_{k=1}^{n} \frac{1}{k} : n \in \mathbb{N}\right\}$. $m(E) = 0$ because of countability and $m(O_n) = \infty$ since each O_n contains (x, ∞) for some $x > 0$.*

For open and bounded set which does not satisfy (a), consider $E = \bigcup_{i=1}^{\infty} \left(q_i - \frac{\varepsilon}{2^{i+1}}, q_i + \frac{\varepsilon}{2^{i+1}}\right)$ where q_i is enueration of rational numbers between 0 and 1. Then by countable additivity, $m(E) \le \varepsilon$ and $O_n \supset [0, 1]$. Since ε is arbitrary positive number, we can see that E does not satisfy (a).

Problem (1.7)**.**

It will be shown in problem #8 that δE is measurable when E is measurable since δE is image of E under n by n matrix whose i-th diagonal entry is δ_i.

Consider $R = \Pi_{i=1}^{d} [a_i, b_i]$. Then $\delta R = \Pi_{i=1}^{d} [\delta_i a_i, \delta_i b_i]$. It is rectangle, so $|\delta R| = \Pi_{i=1}^{d} |R|$ for all rectangle R.

Now suppose $\delta E \subset \bigcup_{j=1}^{\infty} Q_j$ where Q_j is a cube. Then $E \subset \bigcup_{j=1}^{\infty} \frac{1}{\delta} Q_j$. It leads $m_*(E) \le \sum_{j=1}^{\infty} \Pi_{i=1}^{d} \frac{1}{\delta_i} |Q_j|$. Therefore $\Pi_{i=1}^{d} \delta_i m_*(E) \le \sum_{j=1}^{\infty} |Q_j|$. Since $\bigcup_{j=1}^{\infty} Q_j$ is arbitrary, $\Pi_{i=1}^{d} m_*(E) \le m_*(\delta E)$.

On the contrary, suppose $E \subset \bigcup_{j=1}^{\infty} Q'_j$. Then $\delta E \subset \bigcup_{j=1}^{\infty} \delta Q'_j$. It leads $m_*(\delta E) \le \sum_{j=1}^{\infty} \Pi_{i=1}^{d} |Q'_j| = \Pi_{i=1}^{d} \delta_i \sum_{j=1}^{\infty} |Q'_j|$. Since $\bigcup_{j=1}^{\infty} Q'_j$ is arbitrary, $m_*(\delta E) \le \Pi_{i=1}^{d} \delta_i m_*(E)$.

Problem (1.8)**.**

(a) *Note that $|Lx - Lx'| \le \|L\| |x - x'|$ where $\|L\| = \sup_{|x|=1} |Lx|$. It is well known that $\|L\| < \infty$ for linear operator on d Euclidean space. Therefore L is continuous, which leads compactness of $L(E)$ when E is compact. Also, $\bigcup_{\alpha} L(A_\alpha) = L(\bigcup_{\alpha} A_\alpha)$. It means L preserves F_σ. Because we can represent any F_σ set as countable union of compact set by considering k-disc centered at origin. (k is positive integer)*

(b) *Assume E is measurable. Let $\varepsilon > 0$ be given. There is $F_\sigma \subset E$ such that $m(E \setminus F_\sigma) < \varepsilon$. By definition of Lebesgue measure, there is covering of $E \setminus F_\sigma$ by cubes, $\sum |Q_j| < \varepsilon$.*

Then $m(L(E) - L(F_\sigma)) \le m(L(E \setminus F_\sigma)) \le \sum m_*(L(Q_j)) \le (2\sqrt{d}M)^d \sum m_*(Q_j)$.

Notice that last term can be arbitrarily small and $L(F_\sigma)$ is countable union of closed sets. By corollary 3.5, $L(E)$ is measurable.

Problem (1.13)**.**

(a) *Every open set is countable union of almost disjoint cubes. Therefore open set is F_σ. By considering complement, every closed set is countable intersection of open sets.*

(b) *$\mathbb{Q}$ is F_σ set because $\mathbb{Q} = \bigcup_{i=1}^{\infty} \{q_i\}$, where one-point set is closed.*

Assume $\mathbb{Q} = \bigcap_{i=1}^{\infty} G_i$ where G_i is an open set. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, each G_i is open dense subset of $\mathbb{R}$. Consider $G_i \setminus \{q_i\} = G_i'$. It is also dense in $\mathbb{R}$ and open. By Baire's theorem, $\bigcap_{i=1}^{\infty} G_i'$ must be nonempty. But actually $\bigcap_{I=1}^{\infty} G_i'$ is empty. It is contradiction. Therefore $\mathbb{Q}$ is not G_δ set.

(c) *Consider $\mathbb{Q}_{>0} \cup \mathbb{I}_{\le 0}$ where $\mathbb{I}$ is set of irrational number. It is disjoint union of F_σ set and G_δ set. If that set is G_δ set, by intersection(-ing) with positive real numbers, we get $\mathbb{Q}_{>0} = G_\delta$ which is contradiction. If that set is F_σ, its complement is G_δ, and it leads $\mathbb{Q}_{\le 0}$ is G_δ set by intersection with nonpositive real numbers. It also contradicts with (b).*

 # positive rationals and nonpositive rationals are not G_δ set by same reasoning in (b).

Problem (1.14)**.**

(a) *$J_*(E) \le J_*(\bar{E})$ is trivial. Let $E \subset \bigcup_{j=1}^{N} I_j$. Then $\bar{E} \subset \bigcup_{j=1}^{N} \bar{I}_j = \overline{\bigcup_{j=1}^{N} I_j}$. But $\sum |I_j| = \sum |\bar{I}_j|$. Therefore $J_*(\bar{E}) \le \sum_{j=1}^{N} |\bar{I}_j| = \sum_{j=1}^{N} |I_j|$. By taking infimum over all $\bigcup_{j=1}^{N} \supset E$, $J_*\left(\bar{E}\right) \le J_*(E)$.*

(b) *$E = \mathbb{Q} \cap [0,1]$. Then $m(E) = 0$ but covering of E by finitely many intervals must contain $[0,1]$. So $J_*(E) = 1$.*

Problem (1.15)**.**

$m_*^{\mathcal{R}}(E) \leq m_*(E)$ since class of rectangles contains class of cubes.

Assume $m_*^{\mathcal{R}}(E) < m_*(E)$. Then there is $\bigcup_{j=1}^{\infty} R_j$ containing E such that $m_*(E) > \sum |R_j|$ by definition of $m_*^{\mathcal{R}}$. This is impossible since $m_*(E) \leq m_*(\bigcup_{i=1}^{\infty} R_j) \leq \sum m_*(R_j) = \sum |R_j|$ by countable additivity of m_*.

Therefore $m_*^{\mathcal{R}}(E) = m_*(E)$.

Problem (1.16)**.**

(a) $x \in E$ iff for any n, there is $k \geq n$ such that $x \in E_k$ iff $x \in \bigcup_{k \geq n} E_k$ for any n iff $x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$.

Therefore, E is measurable.

(b) $m(E) \leq m\left(\bigcup_{k \geq n} E_k\right) \leq \sum_{k=n}^{\infty} m(E_k)$ for any positive integer n.

But, since $\sum_{k=1}^{\infty} m(E_k) < \infty$, for given $\varepsilon > 0$, there is positive integer N such that $n \geq N$ implies $\sum_{k=n}^{\infty} m(E_k) < \varepsilon$. Therefore $m(E) < \varepsilon$ for every positive ε. This means $m(E) = 0$.

Problem (1.17)**.**

Fix k. $m(|f_k| = \infty) = \lim_{n\to\infty} m(|f_k| > n) = 0$. So we can choose positive integer N_k such that $N_k \leq N_{k+1}$ and $m(|f_k| > N_k) < 2^{-k}$. Let $N_k = \frac{c_k}{k}$.

Then $\sum_{k=1}^{\infty} m\left(\frac{|f_k|}{c_k} > \frac{1}{k}\right) \leq 1 < \infty$. By Borel-Cantelli lemma, $m(\limsup E_k) = 0$. So, if $x \notin \limsup E_k$, then $x \in \bigcap_{k \geq n} E_k^c$ for some positive integer n, and it means $\frac{|f_k(x)|}{c_k} \leq \frac{1}{k}$ for all $k \geq n$. Therefore $\lim_{k\to\infty} \frac{|f_k|}{c_k} = 0$ for almost every x.

Problem (1.18)**.**

First consider characteristic function 1_E of finite measure set E. There are $F_n \subset E \subset G_n$ where F_n, G_n are closed, open repectively and $m(G_n \setminus F_n) < 2^{-n}$.

We can assume G_n is decreasing by considering $G_n' = \bigcap_{k=1}^{n} G_n$. Similarly, we can regard F_n as increasing sequence of closed sets.

Now, for each n, there is Urysohn function f_n which is continuous, vanishes outside of G_n and equal to 1 on F_n. Then clearly $f_n \to 1_E$ as $n \to \infty$ except on $\bigcap_{n\geq 1} (G_n \setminus F_n)$. But $m(\bigcap_{n\geq 1} (G_n \setminus F_n) = 0$. This says there is sequence of continuous function whose a.e. limit is 1_E.

From now on, using the above, consider measurable function f. There is sequence of simple function s_n whose pointwise limit is f. By above, for each n, there is $\{f_{n,k}\}_{k=1}^{\infty}$ whose a.e. limit is s_n.

Choose $k_n \leq k_{n+1}$ so that $m(\{f_{n,k_n} \neq s_n\}) < 2^{-n}$. By Borel Cantelli lemma, $m(\limsup A_n) = 0$ where $A_n = \{f_{n,k_n} \neq s_n\}$. If $x \in (\limsup A_n)^c$, then $x \in \bigcap_{n \geq N} A_n^c$ for some N, then $f_{n,k_n} = s_n$ for $n \geq N$. Therefore $\lim_{n\to\infty} f_{n,k_n} = \lim_{n\to\infty} s_n = f$ for almost every x.

Problem (1.22)**.**

Assume $f = 1_{[0,1]}$ a.e. where 1_A denotes characteristic function of A. If $f \neq 1$ for some $x \in (0,1)$, there is $\delta > 0$ such that $(x-\delta, x+\delta) \subset (0,1)$ and $f \neq 1$ on $(x-\delta, x+\delta)$ by continuity. It contradicts with $f = 1_{[0,1]}$ a.e. Therefore $f = 1$ for $x \in (0,1)$. Similarly, $f = 0$ for $|x| > 1$. Then f must be discontinuous at $x = 0, 1$. It leads the fact that there is no such f.

Problem (1.23)**.**

Fix n. Then $\mathbb{R} = \bigcup_{k\in\mathbb{Z}} \left(\frac{k}{n}, \frac{k+1}{n}\right]$. So for each $x \in \mathbb{R}$, there exists unique k such that $x \in \left(\frac{k}{n}, \frac{k+1}{n}\right]$. Now, fix y. For $x \in \left(\frac{k}{n}, \frac{k+1}{n}\right]$, define $f_n(x,y)$ as follows:

$$f_n(x,y) = n\left[f\left(\frac{k}{n}, y\right)\left(\frac{k+1}{n} - x\right) + f\left(\frac{k+1}{n}, y\right)\left(x - \frac{k}{n}\right)\right]$$

It is line segemnt connecting $\left(\frac{k}{n}, f\left(\frac{k}{n}, y\right)\right)$ and $\left(\frac{k+1}{n}, f\left(\frac{k+1}{n}, y\right)\right)$. Note that it is sum of product of two continuous functions. Hence f_n is measurable.

Also, consider below:

$$\begin{aligned} f_n(x,y) - f(x,y) = &\left[f\left(\frac{k}{n}, y\right) - f(x,y)\right](k+1-nx) \\ &+ \left[f\left(\frac{k+1}{n}, y\right) - f(x,y)\right](nx-k) \end{aligned}$$

Note that $k < nx \leq k+1$ hence $0 \leq k+1-nx \leq 1$ and $0 \leq nx - k \leq 1$. By continuity of $f(\cdot, y)$, as $n \to \infty$, $f_n(x,y) - f(x,y) \to 0$ since $\frac{k}{n}, \frac{k+1}{n} \to x$.

Therefore $f(x,y)$ is pointwise limit of measurable function hence measurable.

Problem (1.25)**.**

Let E be measurable. Then E^c is also measurable. By definition of measurability, there is open set O containing E^c such that $m_*(O \backslash E^c) = m_*(E \backslash O^c) < \varepsilon$. Therefore E is measurable in new sense.

Assume that E is measurable in new sense. For each $\varepsilon > 0$, there is closed $F \subset E$ such that $m_*(E \setminus F) = m_*(F^c \setminus E^c) < \varepsilon$. It leads measurability of E^c and therefore E is measurable in old sense because class of measurable sets is closed under complement set operation.

Problem (1.26)**.**

$m_*(E \setminus A) \le m_*(B \setminus A) = m(B) - m(A) = 0$ since measure of B is finite. Therefore $E \setminus A$ is zero measure set, therefore measurable. $E = E \setminus A \cup A$ which is union of two measurable set. Therefore E is measurable.

Problem (1.27)**.**

Let $Q_t = [-\frac{1}{2}t, \frac{1}{2}t]^d$ and $K_t = E_1 \cup (E_2 \cap Q_t)$. Clearly K_t is compact for each $t \ge 0$. It is straightforward from definition that $E_1 \subset K_t \subset E_2$.

Note that $K_0 = E_1$ and $K_M = E_2$ for large M such that $E_2 \subset Q_M$. Now define the function $\varphi(t) = m(K_t)$. Then, for s, t, $|\varphi(s) - \varphi(t)| \le |m(Q_s) - m(Q_t)| = |s^d - t^d|$. For every $\varepsilon > 0$, there is $\delta > 0$ such that $|s-t| < \delta$ implies $|s^d - t^d| < \varepsilon$. This leads continuity of $\varphi(t)$.

Since domain of φ is connected and codomain of φ is ordered, we can use intermediate value theorem. So there is $p \in [0, M]$ so that $m(E_1) < m(K_p) < m(E_2)$. And clearly $E_1 \subset K_p \subset E_2$.

Problem (1.28)**.**

Let $\alpha \in (0, 1)$. $\frac{1}{\alpha} m_*(E) > m_*(E)$ so there is open set O containing E such that $m_*(E) = m_*(E \cap \bigcup_{j \ge 1} I_j) = m_*(\bigcup_{j \ge 1} E \cap I_j) > \alpha m_*(O) = \alpha \sum_{j \ge 1} m_*(I_j)$ where I_j's are disjoint interval whose union is O.

If $m_*(E \cap I_j) < \alpha m_*(I_j)$ for all positive integer j, then $m_*(E) \le \sum_{j \ge 1} m_*(E \cap I_j) \le \alpha \sum_{j \ge 1} m_*(I_j)$ which contradicts to above.

Therefore there is I_j such that $m_*(E \cap I_j) \ge \alpha m_*(I_j)$.

Problem (1.37)**.**

Consider $f 1_{[-n,n]}$. It is uniformly continuous on $[-n, n]$. Let $\varepsilon > 0$ be arbitrary. choose $\delta > 0$ less than n such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$ for all $x, y \in [-n, n]$.

For each $x \in [-n, n]$*, consider* $\left(x - \frac{\delta}{2}, x + \frac{\delta}{2}\right)$*. Such interval forms open cover of* $[-n, n]$*. We can cover* $[-n, n]$ *by at most* $\frac{2n+1}{\delta}$ *number of such intervals. Let* Γ_n *be graph of* $f 1_{[-n,n]}$*. Then* $m_*(\Gamma_n) \leq \frac{2n+1}{\delta} \delta 2\varepsilon = 2(2n+1)\varepsilon$ *which can be arbitrarily small. Therefore* $m_*(\Gamma_n) = 0$ *for all* n *and* $m(\Gamma) = \sum_{n=1}^{\infty} m(\Gamma_n) = 0$ *where* Γ *is graph of* f*.*

Problem (3.5)**.**

(a) It is enough to show that $\int_0^{1/2} \frac{dx}{x(\log x)^2} < \infty$.

$$\begin{aligned}
\int_0^{1/2} \frac{dx}{x(\log x)^2} &= \lim_{n\to\infty} \int_{1/n}^{1/2} \frac{dx}{x(\log x)^2} \\
&= \lim_{n\to\infty} \int_{-\log n}^{-\log 2} \frac{dt}{t^2} \\
&= \lim\left(1/\log 2 - 1/\log n\right) = 1/\log 2 < \infty
\end{aligned}$$

By MCT and the change of variable formula. Thus f *is integrable.*

(b) Fix $x \in [-1/2, 1/2]$*. Let* $\varepsilon > 0$ *be small. Consider* $B = (-|x| - \varepsilon, |x| + \varepsilon)$.

$$\begin{aligned}
f^*(x) &\geq \frac{1}{2|x| + 2\varepsilon} \int_{-|x|-\varepsilon}^{|x|+\varepsilon} \frac{dt}{|t|(\log|t|)^2} \\
&= \frac{1}{|x| + \varepsilon} \int_0^{|x|+\varepsilon} \frac{dt}{t(\log t)^2} \\
&\geq \frac{1}{|x| + \varepsilon} \int_0^{|x|} \frac{dt}{t(\log t)^2} \\
&= \frac{1}{|x| + \varepsilon} \frac{1}{-\log|x|}
\end{aligned}$$

by similar calculation to above. The above inequality for maximal function of f *holds for all small* $\varepsilon > 0$*, thus we can say that* $f^*(x) \geq 1/\left(-|x|\log|x|\right)$.

Now, it remains to show that f^* *is not locally integrable. This can be done by considering the following:*

$$\begin{aligned}
\int_{-1/2}^{1/2} f^*(x)dx &\geq 2\int_0^{1/2} \frac{dx}{-x\log x} \\
&= 2\lim_{n\to\infty} \int_{1/n}^{1/2} \frac{dx}{-x\log x} \\
&= 2\lim_{n\to\infty} \left(\log\left(1/\log 2\right) - \log\left(1/\log n\right)\right) = \infty
\end{aligned}$$

Thus f^* *is not integrable on* $(-1/2, 1/2)$*, this implies the result.*

$\square$

Problem (3.12).

By chain rule, F' exists for all $x \neq 0$. But,

$$\lim_{h\to 0} \frac{F(h)}{h} = \lim_{h\to 0} h \sin(1/h^2) = 0$$

Thus F' exists for all $x \in \mathbb{R}$.

For $1/\sqrt{2n\pi + \pi/6} \leq x \leq 1/\sqrt{2n\pi - \pi/6}$, $2n\pi - \pi/6 \leq 1/x^2 \leq 2n\pi + \pi/6$, thus $\cos 1/x^2 \geq sqrt3/2$ and $\left|\sin 1/x^2\right| \leq 1/2$. So $|F'| \geq 2/x \cos 1/x^2 - 2x \left|\sin 1/x^2\right| \geq \sqrt{3}\sqrt{2n\pi - \pi/6} - 1/\sqrt{2n\pi - \pi/6}$.

By using the above,

$$\begin{aligned}
\int_0^1 |F'| dm &\geq \sum_{n=1}^{\infty} \left(1/\sqrt{2n\pi - \pi/6} - 1/\sqrt{2n\pi + \pi/6}\right)\left(\sqrt{3}\sqrt{2n\pi - \pi/6} - 1/\sqrt{2n\pi - \pi/6}\right) \\
&= \sum_{n=1}^{\infty} \frac{\pi/\sqrt{3}}{\sqrt{2n\pi + \pi/6}\left(\sqrt{2n\pi + \pi/6} + \sqrt{2n\pi - \pi/6}\right)} \\
&\quad - \sum_{n=1}^{\infty} \frac{\pi/3}{(2n\pi - \pi/6)\sqrt{2n\pi + \pi/6}\left(\sqrt{2n\pi + \pi/6} + \sqrt{2n\pi - \pi/6}\right)}
\end{aligned}$$

where the last sum converges and previous one diverges (by p-test.) Thus F' is non-integrable.

$\square$

Problem (3.14).

(a) *Given $\varepsilon > 0$, we can always find $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Therefore,*

$$\limsup_{h\downarrow 0} \frac{F(x+h) - F(x)}{h} = \lim_{n\to\infty} \sup_{0<h<1/n} \frac{F(x+h) - F(x)}{h}$$

Let q be a rational number. Clearly, the following is true:

$$\sup_{0<q<1/n} \frac{F(x+q) - F(x)}{q} \leq \sup_{0<h<1/n} \frac{F(x+h) - F(x)}{h} \tag{3.14.2}$$

If the above inequality is strict, then there is $h' \in (0, 1/n)$ such that

$$\sup_{0<q<1/n} \frac{F(x+q) - F(x)}{q} < \frac{F(x+h') - F(x)}{h'} \tag{3.14.3}$$

Let $q_i \in (0, 1/n)$ be a sequence of rational numbers such that $q_i \to h'$. Then by continuity of F and $h' > 0$,

$$\lim_{i\to\infty} \frac{F(x+q_i) - F(x)}{q_i} = \frac{F(x+h') - F(x)}{h'}$$

But the above is impossible due to strict inequality in 3.14.3. Thus, only equality can be possible in 3.14.2.

Therefore,

$$\limsup_{h\downarrow 0} \frac{F(x+h) - F(x)}{h} = \inf_n \sup_{0<q<1/n} \frac{F(x+q) - F(x)}{q}$$

which is countable $\limsup$. *And* $\frac{F(x+q)-F(x)}{q}$ *is measurable due to the continuity of F. Thus the result follows.*

□

Problem (3.15)**.**

Let F be function of bounded variation on $[a, b]$. We can write $F(x) = F(a) + P_F(a, x) - N_F(a, x)$ for $x \in [a, b]$. Let $F_1(x) = F(a) + P_F(a, x)$ and F_2 be the other. Clearly F_i's are bounded, increasing function. Thus it remains to show that P_F, N_F are continuous provided by the continuity of F.

Let $\varepsilon > 0$ be given, choose $\delta > 0$ such that $|F(x+h) - F(x)| < \varepsilon$ if $|h| < \delta$. Without loss of generality, assume $h > 0$. By the definition of P_F, there are partitions of $[a, x], [a, x+h]$ such that $P_F(a, x) - \sum_+ (F(t_j) - F(t_{j-1})) < \varepsilon$ and $P_F(a, x+h) - \sum_+ (F(s_j) - F(s_{j-1})) < \varepsilon$. Now consider the common refinement of those two partition. Then:

$$\begin{aligned} P_F(a, x+h) - P_F(a, x) &\le 2\varepsilon + \sum_+ (F(s_j) - F(s_{j-1})) - \sum_+ (F(t_j) - F(t_{j-1})) \\ &= 2\varepsilon + \sum_+ \left(F(s'_j) - F(s'_{j-1})\right) \\ &\le 2\varepsilon + |F(x+h) - F(x)| \le 3\varepsilon \end{aligned}$$

where s'_j is a positive part of partition of $[x, x+h]$. When $h < 0$, by similar manipulation, we can get the same result. Thus P_F is continuous.

All the above calculation can be applied to the process showing the continuity of N_F.

$\square$

Problem (3.16).

(a) *Let $F(x) = F(a) + P_F(a,x) - N_F(a,x) = F_1(x) + F_2(x)$. Since F, F_1, F_2 are of bounded variation(because F_i's are increasing), their derivative exists a.e. Further, $F' = F_1' - F_2'$. Since F_i's are increasing, we can say $F_i' \geq 0$ a.e. So $F' = F_1' - F_2' \leq F_1' + F_2'$. Similarly, $-F' \leq F_1' + F_2'$. So $|F'| \leq F_1' + F_2'$. By integrating the previous inequality from a to b, we get $\int_a^b |F'| dm \leq \int_a^b F_1' + F_2' dm$.*

But the last one is $\leq F_1(b) - F_1(a) + F_2(b) - F_2(a) = P_F(a,b) + N_F(a,b) = T_F(a,b)$.

$\square$

Problem (3.24).

(a) *$F(x) \leq F(b)$ so F is bdd increasing function. Let F_J be corresponding jump function of F. Then $F - F_J$ is continuous and increasing, so bounded. By theorem 3.11, derivative of $F - F_J$ exists a.e., ≥ 0, and integrable. So, $F_A(x) = \int_a^x (F - F_J)' dm \leq (F - F_J)(x) - (F - F_J)(a)$. Then clearly F_A is absolutely continuous. Let $F_C(x) = F(x) - F_J(x) - F_A(x)$. Then F_C is sum of continuous functions, so continuous. And since F_A is absolutely continuous, its derivative is equal to $(F - F_J)'$ so $F_C' = 0$ a.e. Note that $(F - F_J)' \geq 0$ implies F_A is increasing. Also note that for $x \leq y$,*

$$\begin{aligned} F_C(x) - F_C(y) &= F(x) - F(y) - F_J(x) + F_J(y) + \int_x^y (F - F_J)' dm \\ &\leq F(x) - F(y) - F_J(x) + F_J(y) + (F - F_J)(y) - (F - F_J)(x) \\ &= 0 \end{aligned}$$

so F_C is increasing function.

(b) *Let $F = F_A + F_C + F_J$. Note that F_J can vary up to additive constant. Since $F_C' = 0$ a.e., derivative of $F - F_J$ and derivative of F_A are equal* *almost everywhere. So, due to absolute continuity of* F_A, $F_A(x)-F_A(a) = \int_a^x F'dm = \int_a^x (F-F_J)'dm$. *This means that* F_J *determines* F_A *up to additive constant. Then* F_C *is determined automatically.*

$\square$

Problem (3.32)**.**

Assume the Lipschitz condition. Take $\delta = \varepsilon/M$ *when* $\varepsilon > 0$ *is given. For* (a_i, b_i) *such that* $\sum_i (b_i - a_i) < \delta$, *then* $\sum_i |f(b_i) - f(a_i)| \leq M \sum_i (b_i - a_i) < M\delta = \varepsilon$. *Thus* f *is absolutely continuous. So* f' *exists a.e. Now consider the following:*

$$|f'(x)| = \lim_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|} \leq M$$

Thus $|f'| \leq M$ *a.e.* x.

For the other direction, without loss of generality, assume $x \leq y$. *Since* f *is absolutely continuous,* f' *exists a.e, and* $\int_x^y f'dm = f(y) - f(x)$. *Thus,* $|f(x) - f(y)| = \left|\int_x^y f'dm\right| \leq \int_x^y |f'|dm \leq (y-x)M = |x-y|M$.

$\square$