mas651 exercises

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Problem (5.1.1)**.**

Let $(S, \mathcal{S})$ *be a state space of* X_n *where* $S = \{1, 2, \cdots N\}$ *and* $\mathcal{S} = 2^S$*. Note that* N *is an absorbing state. And* $X_1 = 1$ *with probability* 1*. For fixed* k *such that* $1 \leq k < N$*,* $k \leq n$*,*

$$P(X_{n+1} = k + 1 | X_n = k) = \frac{N - k}{N}$$

and

$$P(X_{n+1} = k | X_n = k) = \frac{k}{N}.$$

If $k > n$*, then the above are all* 0*. So it is a temporally inhomogeneous. The Markov property is trivial since the very next state only depends on the current state.*

□

Problem (5.1.2)**.**

$$P(X_4 = 2 | X_3 = 1, X_2 = 1, X_1 = 1, X_0 = 0) = (1/16)/(1/4) = 1/4$$

but

$$P(X_4 = 2 | X_3 = 1, X_2 = 0, X_1 = 0, X_0 = 0) = (1/16)/(1/8) = 1/2.$$

Thus X_n *is not a Markov chain.*

□

Problem (5.1.5)**.**

$$P(X_{n+1} = k + 1 | X_n = k) = \frac{m - k}{m} \frac{b - k}{m}$$

because we must choose a white ball in the left urn and a black ball in the right urn.

$$P(X_{n+1} = k | X_n = k) = \frac{k}{m} \frac{b - k}{m} + \frac{m - k}{m} \frac{m + k - b}{m}$$

since there are two cases, choosing both black or both white.

$$P(X_{n+1} = k - 1 | X_n = k) = \frac{k}{m} \frac{m + k - b}{m}$$

since we must choose a black ball in the left urn and a white ball in the right urn. Note that the sum of the above is 1, so there is no other transition probability.

□

Problem (5.1.6)**.**

$$P(S_{n+1} = k + 1 | S_n = k) = \frac{P(X_{n+1} = 1, S_n = k)}{P(S_n = k)}$$

where the denominator is

$$\int_{\theta \in (0,1)} P(S_n = k|\theta) dP = \binom{n}{x} \frac{x!y!}{(n+1)!} = \frac{1}{n+1}$$

for $x =$*the number of* i *such that* $U_i \leq \theta$ *and* $y = n - x$*. Note that* $x = (n+k)/2$ *and* $y = (n-k)/2$ *since* $x + y = n$ *and* $x - y = k$*. The numerator is*

$$\int_{\theta \in (0,1)} P(X_{n+1} = 1, S_n = k|\theta) dP = \binom{n}{x} \frac{(x+1)!y!}{(n+2)!}$$

These are because $P(S_n = k|\theta) = \theta^x (1-\theta)^y \binom{n}{x}$ *and* $P(X_{n+1} = 1, S_n = k|\theta) = P(X_{n+1} = 1|\theta) P(S_n = k|\theta) = \binom{n}{x} \theta^{x+1} (1-\theta)^y$ *and using the kernel of beta distribution.*

Thus, the probability what we want is $(n+k+2)/(2n+4)$ *which depends on* n*. So* X_n *is temporally inhomogeneous.*

$$P(S_{n+1} = k+1|S_1 = t_1, \cdots, S_n = k) = P(X_{n+1} = 1|X_1 = t_1, \cdots, X_n = t_n)$$

where $\sum_{i=1}^n t_i = k$*. We can show the above is equal to* $P(S_{n+1} = k+1|S_n = k) = (n+k+2)/(2n+4)$ *similarly, by omitting the* $\binom{n}{x}$ *term of both denominator and numerator.*

$\square$

Problem (5.2.1).

By the given hint,

$$E(1_A 1_B|\mathcal{F}_n) = E(1_A E(1_B|\mathcal{F}_n)|X_n)$$

so it suffices to show that $E(1_B|\mathcal{F}_n) = E(1_B|X_n)$.

Let $Y = 1_{B_n}(\omega_0)\cdots 1_{B_{n+k}}(\omega_k)$. *Then* $Y \circ \theta_n =$ *the indicator function of* $\{X_n \in B_n, \cdots, X_{n+k} \in B_{n+k}\} = B$. *By the markov property,*

$$P(B|\mathcal{F}_n) = E_{X_n}Y.$$

Let $\varphi(x) = E_x Y$ *then* $\varphi(X_n)$ *is* $\sigma(X_n)$*-measurable mapping. Thus, when* B *has a form of* $\{X_n \in B_n, \cdots, X_{n+k} \in B_{n+k}\}$ *for some nonnegative integer* k,

$$P(B|\mathcal{F}_n) = P(B|X_n).$$

Note that a collection of such B *generates* $\sigma(X_n, X_{n+1}, \cdots)$.

Now let $\mathcal{G} = \{C : P(C|\mathcal{F}_n) = P(C|X_n)\}$. *By putting* $B_{n+i} = S$ *for* $0 \leq i \leq k$, *we earn* $\Omega_0 \in \mathcal{G}$. *If* $C, D \in \mathcal{G}$ *and* $C \subset D$, *then by properties of conditional expectation,* $D \setminus C \in \mathcal{G}$. *If* $C_i \in \mathcal{G}$ *and* $C_i \uparrow C$ *then by monotone convergence theorem for conditional expectation,* $C \in \mathcal{G}$. *Thus* $\mathcal{G}$ *is a lambda system containing a collection of* B*'s which generates* $\sigma(X_n, \cdots)$. *Therefore, by Dynkin's theorem, the third equation is satisfied by any* $B \in \sigma(X_n, \cdots)$. *By the first equation, we can derive the conclusion.*

□

Problem (5.2.4).

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Problem (5.2.6).

Fix $x \in S \setminus C$. *Since* $P_x(T_C = \infty) = \lim_{M\to\infty} P_x(T_C > M) < 1$, *we can choose* N_x *and* ε *so that*

$$P_x(T_C > M) \leq 1 - \varepsilon$$

whenever $M \geq N_x$. *Note that we can choose* N_x *as an integer. Put* $N = \max_{x \in S\setminus C} N_x$. *Now we get*

$$\begin{aligned}
P_y(T_C > 2N) &= \sum_{x \in S\setminus C} P_y(T_C > 2N, T_C > N, X_N = x) \\
&= \sum_{x \in S\setminus C} P_y\left(T_C > 2N | X_N = x, T_C > N\right) P_y(X_n = x, T_C > N) \\
&\leq \sum_{x \in S\setminus C} P_x(T_C > N) P_y(X_N = x, T_C > N) \\
&\leq (1-\varepsilon) \sum_{x \in S\setminus C} P_y(X_N = x, T_C > N) \\
&\leq (1-\varepsilon)^2.
\end{aligned}$$

By induction, the result follows.

□

Problem (5.2.7).

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Problem (5.2.8).

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Problem (5.2.11).

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