# H&K LINEAR ALGEBRA SOLUTIONS

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CHAPTER 6

## 4. Invariant Subspaces.

Problem 8.

Let $\mathcal{B} = \{\alpha_1, \cdots, \alpha_n\}$ be an ordered basis for $V$. Let $\mathcal{B}_i = span(\alpha_i)$. Since every subspace of $V$ is invariant under $T$, $T\alpha_i = c_i\alpha_i$ for some scalar $c_i$. Let $W_i = span(\alpha_1 + \alpha_i)$, $\beta_i = \alpha_1 + \alpha_i$. $T\beta_i = c_1\alpha_1 + c_i\alpha_i = k\beta_i$. So $c_1 = c_i = k$. Therefore, $T\alpha_i = c_1\alpha_i$ for all $i$. So $T$ is a scalar multiple of the identity operator.

Problem 10.

Let $p$ be a minimal polynomial for $A$. From non-triangulability of $A$, we can say that $p$ has degree 2 irreducible factor since every degree 3 real coefficient polynomial has at least one real zero. Then irreducible factor of $p$ splits into distinct linear factors over $\mathbb{C}$. So, minimal polynomial for $A$ splits into distinct linear factors over $\mathbb{C}$ which is equivalent to $A$ is diagonalizable.

Problem 12.

First assume $t$ is an eigenvalue of $T$. Then there exists $\alpha \in V \setminus 0$ such that $T\alpha = t\alpha$. It is easy to observe that $f(T)\alpha = f(t)\alpha$. So $f(t)$ is an eigenvalue of linear operator $f(T)$.

Conversely, assume $c$ is an eigenvalue of $f(T)$. So there exists nonzero vector $\alpha \in V$. Consider the equation $f(x) = c$. There is $t \in F$ which satisfies that equatio since $F$ is algebraically closed. From $f(t) = c$, $f(x) - c = (x - t)q(x)$. So $(T - tI)q(T)\alpha = 0$. If $q(T)\alpha \neq 0$, we are done. If $q(T)\alpha = 0$, $q(T)$ has 0 as eigenvalue. So we can find $s \in F$ such that $q(s) = 0$. Then $q(x) = (x - s)r(x)$, $q(T)\alpha = (T - sI)r(T)\alpha = 0$. If $r(T)\alpha \neq 0$, we are done. If not $\cdots$. By repeating (such process is finite since f has finite degree), we can conclude that $t$ is an eigenvaue for $T$. When $f$ is degree 1, it is trivial.

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