FRANK JONES INTEGRATION THEORY SOLUTIONS

JAEMIN OH

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CHAPTER 2: LEBESGUE MEASURE ON $\mathbb{R}^n$

section A: Construction.

In this section, we will construct Lebesgue measure from set of special rectangles. Although there are many other methods to consturct measure on Euclidean space, method in our textbook looks like intuitive and good to study for beginners. Other textbooks use Caratéodory extension or Urysohn lemma to construct Lebesgue measure on Euclidean space.

Problem 3.

Consider the open set $G \setminus P$ and x in that set. There is ε neighborhood of x contained in $G \setminus P$. But ε neighborhood contains special rectangle I whose measure is positive. Now, take $P' = P \cup I$.

Problem 5. 'at most countably many disjoint open sets'

Let G_i be nonempty set. Pick $x \in G_i$ and consider ε neighborhood contained in G_i. That neighborhood contains point q_i whose components are all rational. Consider the injection $G_i \mapsto q_i$ from $\mathcal{I}$ to countable set. (It is clearly injection because each G_i is disjoint.)

Problem 6. 'the structure of open sets in real line'

Consider an equivalence relation $x \sim y \leftrightarrow x, y$ belongs to some open interval contained in G. Equivalence class of x is a largest open interval containing x.

Problem 8. 'open disk cannot be expressed as disjoint union of open rectangles'

It is trivial since open disk is connected.

Note that regular space with countable basis, metrizable space, compact Hausdorff space are normal space. So Euclidean space is normal space. This fact is useful to prove property 4 of Lebesgue measure on compact sets.

Problem 17.

Let $x = \sum_{k=1}^{\infty} \frac{\alpha_k}{3^k}$ where $\alpha_j \in \{0, 2\}$. Then $1 - x = \sum_{k=1}^{\infty} \frac{2-\alpha_k}{3^k}$ and $2 - \alpha_k \in \{0, 2\}$. Therefore, $1 - x \in C \leftrightarrow x \in C$.

B. Properties of Lebesgue Measure.

(1) $\mathcal{L}$ is complement closed.
(2) $\mathcal{L}$ is $\sigma - \cap\cup$ closed.
(3) $\mathcal{L}$ is set minus closed.
(4) Countable additivity of λ.
(5) Continuity from below.
(6) Continuity from above.
(7) $\mathcal{L}$ contains Borel algebra on $\mathbb{R}^n$.
(8) λ is complete.
(9) Theorem on approximation of $\mathcal{L}$.
(10) On $\mathcal{L}$, $\lambda^* = \lambda_* = \lambda$.
(11) $\lambda(B) = \lambda^*(A) + \lambda_*(B \setminus A)$ if $A \subset B \in \mathcal{L}$.
(12) $A \in \mathcal{L}$ if and only if $\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \cap A^c)$ for all $E \subset \mathbb{R}^n$.

Problem 27.

If $A \subset \bigcup_{k=1}^{\infty} I_k$, then $\lambda^*(A) \leq \lambda^*(\bigcup_{k=1}^{\infty} I_k) \leq \sum_{k=1}^{\infty} \lambda^*(I_k) \leq \sum_{k=1}^{\infty} \lambda(I_k)$. Therefore $\lambda^*(A) \leq \inf\{\}$.

On the contrary, assume $\lambda^*(A) < \inf\{\}$. There is open set G containing A and $\lambda(G) < \inf\{\}$. From problem 9, G can be expressed as a countable union of nonoverlapping special rectangles. That is, $\lambda(G) = \sum_{k=1}^{\infty} \lambda(I_k)$ which is contradiction.

Problem 29.

Use $\lambda(A \cup B) = \lambda(A \setminus B) + \lambda(B \setminus A) + \lambda(A \cap B)$.

Problem 30.

Let G_A, G_B be open sets containing A, B respectively. $\lambda(G_A) + \lambda(G_B) = \lambda(G_A \cup G_B) + \lambda(G_A \cap G_B) \geq \lambda^*(A \cup B) + \lambda^*(A \cap B)$. Since G_A, G_B are arbitrary, $\lambda^*(A) + \lambda^*(B) \geq \lambda^*(A \cup B) + \lambda^*(A \cap B)$.

Problem 31.

Clearly one point set is measure zero and closed. Let $\varepsilon > 0$ be given. There exists open set G_i containing a_i such that $\lambda(G_i \setminus a_i) < \frac{\varepsilon}{2^i}$. Therefore $\lambda(A) < \varepsilon$ which means $\lambda(A) = 0$.

Problem 32.

Let I_k be special 'cube' centered at origin, length of side $= k$. Then $a \times I_k \subset a \times \mathbb{R}^n$. Therefore we get $\lambda(a \times \mathbb{R}^n) = \lim_{k\to\infty} \lambda(a \times I_k) = 0$ by continuity from below.

Problem 34.

There is closed set of positive measure, no interior.

$$[0,1] \setminus \bigcup_{k=1}^{\infty} \left(q_k - \frac{\varepsilon}{2^{k+1}}, q_k + \frac{\varepsilon}{2^{k+1}} \right) \tag{1}$$

Problem 37.

For each positive integer k, choose open set $G_k supsetE$ such that $\lambda(G_k) < \lambda^*(E) + \frac{1}{k}$. Put $A = \bigcap_{k=1}^{\infty} G_k$ which is measurable. Then $\lambda(A) < \lambda^*(E) + \frac{1}{k}$ for all positive integer k. Therefore $\lambda(A) \leq \lambda^*(E)$. Reverse inequality is trivial because $E \subset A$. We can conclude that there exists measurable hull of E which has finite outer measure.

Problem 38.

First assume A is measurable hull of E. $\lambda(A) = \lambda^*(E) < \infty$. But we already know that $\lambda(A) = \lambda^*(E) + \lambda_*(A \setminus E)$. Therefore $\lambda_*(A \setminus E) = 0$.
Conversely assume $\lambda_*(A \setminus E) = 0$. From 11st property of Lebesgue measure, we can get A is measurable hull of E.

Problem 39.

Let $E_k = B(0,k) \setminus B(0,k-1)$ which is measurable partition of $\mathbb{R}^n$. Then $E \cap E_k$ has finite outer measure. By problem 37, there exists measurable hull A_k of $E \cap E_k$. Put $A = \bigcup_{k=1}^{\infty} A_k$ and consider the compact set $K \subset A \setminus E$. But $\lambda(K \cap E_k) \leq \lambda_*(A_k \setminus E \cap E_k) = 0$. By continuity from below, $\lambda(K) = 0$. So $\lambda_*(A \setminus E) = 0$.

Problem 40.

Let A_k be measurable hull of E_k. $B_j = \bigcap_{i=j}^{\infty} A_i$ is also measurable hull of E_j and $B_i \subset B_{i+1}$. Then $\lambda\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \lambda\left(B_k\right) = \lim_{k\to\infty} \lambda^*\left(E_k\right)$.
Also we know that $\lambda^*\left(\cup_{k=1}^{\infty} E_k\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} B_k\right)$. On the contrary, assume $\lambda^*\left(\bigcup_{k=1}^{\infty} E_k\right) < \lambda\left(\bigcup_{k=1}^{\infty} B_k\right)$. There is open set G containing $\bigcup_{k=1}^{\infty} E_k$ whose measure is strictly less than $\bigcup_{k=1}^{\infty} B_k$. But outer measure of G is greater than E_k for each positive integer k. Which is contradiction because $\lim_{k\to\infty} \lambda^*\left(E_k\right) = \lambda\left(\bigcup_{k=1}^{\infty} B_k\right) > \lambda\left(G\right) = \lambda^*\left(G\right)$.

Problem 41.

Put $B_k = \bigcup_{j=k}^{\infty} A_j$. Then $B_k \supset B_{k+1}$. $\lambda\left(\limsup_{k\to\infty} A_k\right) = \lambda\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \lambda\left(B_k\right)$. But $\lambda\left(B_k\right) \leq \sum_{j=k}^{\infty} \lambda\left(A_k\right)$ which goes to 0 as $k \to \infty$. Therefore $\lambda\left(B_k\right) \to 0$.

Problem 42.

Use hint. Or, let $h = \sum_{i=1}^{\infty} 1_{A_i}$, then h is measurable function. So inverse image of $[0,d]$ under h is measurable. Then we get

$$\sum_{i=1}^{\infty} \lambda\left(A_i\right) = \int_{h^{-1}([0,d])} h d\lambda \leq \int_{h^{-1}([0,d])} d d\lambda = d\lambda\left(\bigcup_{k=1}^{\infty} A_k\right)$$

Problem 43.

$B_k = A_k \setminus \bigcup_{j=1}^{k-1} A_j$ when $k \geq 2$. Obviously $B_1 = A_1$. Then $\lambda\left(B_k\right) = \lambda\left(A_k \setminus A_1\right) = \lambda\left(A_k\right)$. Therefore we get what we want.

Problem 44.

$\sum_{k=1}^{\infty} \left(\lambda\left(A_k\right) - \lambda\left(B_k\right)\right) = 0$. It leads $\lambda\left(A_k\right) = \lambda\left(B_k\right)$. So $\lambda\left(A_k\right) = \lambda\left(A_k\right) - \lambda\left(A_k \cap \left(\bigcup_{j=1}^{k-1} A_j\right)\right)$ which leads conclusion.

Problem 45.

For each positive integer k, there are at most countable A_i's such that $\lambda\left(B(0,k) \cap A_i\right) > 0$. Collect such A_i's. Obviously such collection is countable. Then for some positive integer k, there are at most countable A_i's in collection such that $\lambda\left(B(0,k) \cap A_i\right) > 0$. Actually, our collection is all of $\{A_i : i \in \mathcal{I}\}$. Because if $A_i \cap B(0,k)$ has measure zero for all positive integer k, then $\lambda\left(A_i\right) = 0$ by continuity from below. This means $i \notin \mathcal{I}$. Therefore $\mathcal{I}$ is countable.

Problem 46.

Let $B_k = \bigcup_{j=k}^{\infty} A_j$. Then $B_k \supset B_{k+1}$. If $\lambda\left(B_1\right) < \infty$, continuity from above implies

$$\lambda\left(\limsup_{k\to\infty} A_k\right) = \lim_{k\to\infty} \lambda\left(B_k\right) = \limsup_{k\to\infty} \lambda\left(B_k\right) \geq \limsup_{k\to\infty} \lambda\left(A_k\right)$$

$\liminf$ case is similar to above.

Problem 47.

$\varepsilon \leq \limsup \lambda\left(A_k\right) \leq \lambda\left(\limsup A_k\right)$. Therefore $\limsup A_k$ is not empty set because it has nonzero measure.

Problem 48.

Let B_k be open ball of radius k and centered at origin. Then $A \cap B_k \in \mathcal{L}_0$. There exist compact set C_k such that $\lambda\left(A \cap B_k \setminus C_k\right) < \frac{1}{k}$. Set $K_k = \bigcup_{j=1}^{k} C_j \subset A \cap B_k$. Then $K_k \subset K_{k+1}$ and each K_k is compact. Then $\lambda\left(A \cap B_k \cap \bigcap_{j=1}^{\infty} K_j^c\right) \leq \lambda\left(A \cap B_j \cap K_j^c\right) < \frac{1}{j}$ for all $j > k$. So $\lambda\left(A \cap B_k \cap \bigcap_{j=1}^{\infty} K_j^c\right) = 0$. By continuity from below, we get what we want.

CHAPTER 5. ALGEBRAS OF SETS AND MEASURABLE FUNCTIONS

section A. algebras and σ algebras.

Let X be given set. Algebra $\mathcal{A}$ on X denotes subset of 2^X satisfying 3 conditions.

(1) $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$
(2) $\cup$ closed.
(3) complement closed.

Similarly, σ algebra $\mathcal{F}$ on X denotes subset of power set of X, which contains empty set, entire set and is $\sigma - \cup$ closed and complement closed.

Problem 9.

Clearly, $\mathcal{M}_0 \subset \mathcal{M}_1$. So $\mathcal{M}_1 \supset \sigma(\mathcal{M}_0)$ which is the smallest σ algebra containing $\mathcal{M}_0$. Let $A \in \mathcal{M}_1$. Without loss of generality, we can assume A is countable set. Then $A = \{a_i : i \in \mathbb{N}\}$ and each $\{a_i\} \in \mathcal{M}_0$. Therefore A is a countable union of sets in $\mathcal{M}_0$. Which says $\mathcal{M}_1 \subset \sigma(\mathcal{M}_0)$.

Problem 11.

Let $y \in A_x$. Then $A_y \subset A_x$. If $x \notin A_y$, $x \in A_x \setminus A_y \in \mathcal{B}$. So $A_x \subset A_x \setminus A_y$ which is contradiction. All other procedure is easy to prove. Just follow the hint.

section B. Borel Sets.

Borel algebra on X is the smallest σ algebra containing topology τ on X. Borel set denotes member of Borel algebra on X. It is not easy to construct exact Borel algebra on $\mathbb{R}^n$. It is useful to understand Borel algebra on $\mathbb{R}^n$ by smallest σ algebra containing all the special rectangels.

Problem 12.

- $G \in \tau \Rightarrow G = \bigcup_{k=1}^{\infty} I_k$ where I_k is special rectangle. So $G \in \mathcal{M} \Rightarrow \tau \subset mathcalM$.
- $\tau \subset \mathcal{M} \Rightarrow \mathcal{B} \subset \mathcal{M}$.
- $I_k \in \mathcal{N}$. Since I_k is closed, $I_k \in \mathbb{B}$. Therefore $\mathcal{N} \subset \mathcal{B}$.
- c implies $\mathcal{M} \subset \mathcal{B}$.

Problem 14.

By using approximation property of Lebesgue measure, $\Rightarrow$ direction is clear.

On the contrary, assume there is F_σ, G_δ satisfying written properties. Then $\lambda^* (A \setminus F_\sigma) \leq \lambda (G_\delta \setminus F_\sigma) = 0$. Therefore $A \setminus F_\sigma \in \mathcal{L}$ because of completeness. Then $A = A \setminus F_\sigma \cup F_\sigma \in \mathcal{L}$.

section D. Measurable Functions.

Problem 16.

- a only constant functions
- b every functions
- c mapping which maps each E_i to constants.

Problem 17.

$$\phi(x) = \begin{cases} \frac{1}{t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Then,

$$\phi^{-1}\left([-\infty, d]\right) = \begin{cases} t \geq \frac{1}{a} & \text{if } a < 0 \\ 0 \text{ or } t < 0 & \text{if } a = 0 \\ t < 0 \text{ or } 0 \text{ or } t \geq \frac{1}{a} & \text{if } a > 0 \end{cases}$$

Therefore $\phi : \mathbb{R} \to [-\infty, \infty]$ is Borel measurable.

Problem 18.

Observe that $f_n \to f$. Then measurability of f follows immediately.

Problem 19.

- $\{f > 0\} = E \notin \mathcal{L}$ so f is not $\mathcal{L}$ measurable.
- $t > 0 \Rightarrow f^{-1}(t) = \{lnt\} \in \mathcal{L}$. Similarly, we can chect the other cases.

Problem 20.

$$f'(x) = \lim_{n\to\infty} \frac{f\left(x + \frac{1}{n}\right) - f(x)}{\frac{1}{n}}$$

Right hand side of above is Borel measurable because differentibility of f implies continuity of f.

section E. Simple Functions.

Problem 21.

Consider $s^{-1}([t,\infty])$ where $s = \sum_{k=1}^{m} \alpha_k 1_{A_k}$. Rearrange α_k as increasing order, i.e. $\alpha_1 < \alpha_2 < \cdots < \alpha_m$. Then inverse image of $[t,\infty]$ is disjoint union of A_k's. Therefore s is measurable if and only if each A_k is measurable.

Problem 23.

Actually, $f^+ = \sup\{f, 0\}$ and $f^- = -\inf\{f, 0\}$. By MF6, $f^\pm$ are all measurable.

Problem 24.

Consider the function below:

$$\phi_n(x) = \begin{cases} \frac{i-1}{2^n} & \text{if } \frac{i-1}{2^n} \le x \le \frac{i}{2^n} \\ n & \text{if } x \ge n \end{cases}$$

Since f is bounded, there exists $M \ge 0$ such that $|f(x)| \le M$ for all $x \in X$. Also note that ϕ_n converges to $i : x \mapsto x$ where $|x| \le M$. So $\phi_n \circ f^\pm$ uniformly converges to $f^\pm$ repectively. This means $\phi_n \circ f^+ - \phi_n \circ f^-$ uniformly converges to f.

CHAPTER 6

section A. Nonnegative Functions.

Problem 1. "the vanishing property"

First, assume that $\int f d\mu = 0$. Since f is measurable, there exists an increasing seqeunce of nonnegative simple functions which converges to f. Let denote them as s_n. Then $f^{-1}((0,\infty]) = \bigcup_{n=1}^{\infty} s_n^{-1}((0,\infty])$, union of measure zero set. Therefore, we get $\mu(f^{-1}((0,\infty])) = 0$.

Conversely, assume that $\int f d\mu > 0$. Then there exists nonnegative simple function $s \le f$ such that $\int s d\mu > 0$. Also we can write s as a linear combination of (measurable) characteristic functions, i.e. $\int s d\mu = \sum_{i=1}^{N} \alpha_i \mu(A_i) > 0$. So, $\alpha_i \mu(A_i) > 0$ for at least one integer $1 \le i \le N$. By Observing the fact that $A_i \subset f^{-1}((0,\infty])$, we can conclude that $f^{-1}((0,\infty])$ has measure zero implies $\int f d\mu = 0$ by contrapositive.

Problem 2. "the finiteness property"

$E = \{x : f(x) = \infty\}$ is a measurable set since f is measurable. Suppose $\mu(E) > 0$. For any $M \in \mathbb{N}$, choose nonnegative simple measurable function $s_M \le f$ such that $s_M(x) \ge M$ for $x \in E$. Then $\int f d\mu \ge \int s_M d\mu \ge M\mu(E)$. Therefore, $\int f d\mu \ge M\mu(E)$ for all positive integer M. It means that $\int f d\mu = \infty$. By taking contrapositive, we get what we want.

Problem 3. "compatibility to not finite nonnegative simple functions"

We are only interested in the case when $\alpha_k = \infty, \mu(A_k) > 0$. Since f is measurable, we can consider seqeunce of nonnegative finite simple function $\{t_n\}$ such that $t_n(x) \ge n$ for $x \in A_k$. Therefore, $\int f d\mu \ge \int t_n d\mu \ge \mu(A_k)n$ for all positive integer n, which means $\int f d\mu = \infty$. For other cases, it is easy to check the compatibility.

Problem 4. "scalar multiplication is still valid for $c = \infty$"

First, assume $\int f d\mu = 0$. Then $\mu(f^{-1}((0,\infty])) = 0$ by Problem 1. It is obvious that $g^{-1}((0,\infty]) = f^{-1}((0,\infty])$ for $g = \infty f$. So, $\int g d\mu = 0$. Therefore we get $\int \infty f d\mu = \int f d\mu$.

Second, assume $\int f d\mu > 0$. Then measure of $g^{-1}((0,\infty])$ is positive. So, $\int \infty f d\mu = 0\mu(g^{-1}(0)) + \infty\mu(g^{-1}((0,\infty]))$ which is ∞.

Problem 5. "strict inequality for Fatou's lemma'

Consider this nonnegative finite simple function defined on real line:

$$s_n(x) = \begin{cases} n^2 & \text{if } x \in (0, \frac{1}{n}) \\ 0 & \text{otherwise} \end{cases}$$

$\lim s_n = 0$ and $\lim \int s_n d\mu = \infty$.

Problem 6.

Let $f_k = 1_{A_k}$ where 1_{A_k} is an indicator(characteristic) function. By Fatou's lemma (ILLLI), $\int \liminf_{k\to\infty} 1_{A_k} d\mu \le \liminf_{k\to\infty} \int 1_{A_k} d\mu = \liminf_{k\to\infty} \mu(A_k)$. Now, showing $1_{\liminf A_k} \le \liminf_{k\to\infty} 1_{A_k}$ is left to us. If $x \in \liminf A_k$ then $x \in \bigcap_{i\ge n} A_i$ for some positive integer n. Then $1_{A_i}(x) = 1$ for all positive integer i greater than n. So, $1_{\liminf A_k} \le \inf_{i\ge n} 1_{A_i}$. By letting $n \to \infty$, we get $1_{\liminf A_k} \le \liminf_{k\to\infty} 1_{A_k}$. Therefore, $\mu(\liminf A_k) = \int 1_{\liminf A_k} d\mu \le \int \liminf_{k\to\infty} 1_{A_k} d\mu \le \liminf_{k\to\infty} \int 1_{A_k} d\mu = \liminf_{k\to\infty} \mu(A_k)$.

section B. General Measurable Functions.

Problem 9. "iff condition for (finite) integrability"

Let $f \in \mathcal{L}^1(\mu)$. Then $\int f_\pm d\mu < \infty$, so there sum is also finite ($= \int |f| d\mu$).
Conversely, assume $|f| \in \mathcal{L}^1(\mu)$. Then $\int f_\pm d\mu \leq \int |f| d\mu < \infty$. Therefore $f \in \mathcal{L}^1(\mu)$.

Problem 10. "dominated integrability"

If f is measurable, $f_\pm$ is also measurable. So $f_+ + f_- = |f|$ is measurable. $\int |f| d\mu \leq \int |g| d\mu < \infty$. Therefore, $|f| \in \mathcal{L}^1(\mu)$ by Problem 9.

Problem 11.

It is obvious that $|f_k| \leq |f|$ and $f \in \mathcal{L}^1(\mu)$. So, if each f_ks are measurable, by dominated convergence thm, done. Actually, $f_k = f \cdot 1_{A_k \cap E_k}$ where $A_k = [-k, k]$ and $E_k = f^{-1}([-k, k])$. There exists sequence of nonnegative finite simple function $\{s_i\}$ which converges to f since f is measurable. So, $\lim s_i 1_{a_k \cap E_k} = f_k$ is measurable.

Problem 12.

Likewise, it is enough to show that $f(x)e^{-\frac{|x|^2}{k}} = f_k(x)$ is measurable. $e^{-\frac{|x|^2}{k}}$ is continuous, hence Borel measurable, hence Lebesgue measurable. There exists sequence of nonnegative finite simple function $\{s_k\}$ which converges to f since f is measurable. So, $\lim_{i \to \infty} s_i e^{-\frac{|x|^2}{k}} = f_k$ is measurable.

Problem 13. 'alternative proof for problem 2.42'

For each $x \in X$, there are at most $d \in \mathbb{N}$ distinct A_k containing x. Fix positive integer N. Let $I_x = \{k \in \mathbb{N} : k \leq N \text{ and } x \in A_k\}$. Clearly $\sum_{k=1}^N 1_{A_k} \leq d 1_A$ where $A = \bigcup_{i=1}^\infty A_i$. By integrating both sides, $\sum_{k=1}^N \mu(A_k) \leq d\mu(A)$. By Letting $N \to \infty$, we got the result in Problem 2.42.
(similar, another) $\sum_{k=1}^N 1_{A_k} \leq |I_x| \leq d = d 1_A$ for all N. So $\sum_{k=1}^\infty 1_{A_k} \leq d 1_A$. By integrating both sides and monotone convergence thm, we got it.

Problem 14.

Let $\mathcal{I}$ be set of all sequences which are strictly increasing positive integers and length m. Such set is countable. Now, $\bigcup_{i \in \mathcal{I}} \bigcap_{j=1}^m A_{i_j} = E_m$ and it is measurable.
Similar to Problem 13, $m 1_{E_m} \leq \sum_{i=1}^\infty 1_{A_i}$. So, $m\mu(E_m) = \int m 1_{E_m} d\mu \leq \int \sum_{i=1}^\infty 1_{A_i} d\mu = \sum_{i=1}^\infty \int 1_{A_i} d\mu = \sum_{i=1}^\infty \mu(A_i)$ by monotone convergence thm.

section C. Almost Everywhere.

Problem 16.

Clearly, $|f| = 0$ almost everywhere, and $|f|$ is measurable. Consider nonnegative finite simple function $s \leq |f|$. Then $s = 0$ almost everywhere and $s = \sum_{i=1}^{N} \alpha_i 1_{A_i}$, where A_i is null set if $\alpha_i \neq 0$. Therefore, $\int s d\mu = 0$, which implies $\int |f| d\mu = 0$. So, $f \in \mathcal{L}^1(\mu)$, $\left|\int f d\mu\right| = 0 \Rightarrow \int f d\mu = 0$.

Problem 17.

Note that $f \sim g \Leftrightarrow f = g$ a.e. is an equivalence relation. So, $g = h$ a.e. Let g be measurable and $E_t = [-\infty, t]$, $N = \{x : g(x) \neq h(x)\}$. Then $h^{-1}(E_t) \backslash g^{-1}(E_t) \subset N$ and $g^{-1}(E_t) \setminus h^{-1}(E_t) \subset N$. Since μ is complete, they are all null sets. So $h^{-1}(E_t) \cup g^{-1}(E_t)$ is measurable. Therefore, $h^{-1}(E_t)$ is also measurable, which implies measurability of h.

Comment: In problem 19 and 21, we treat the functions in L^1. So $f(x) - g(x)$ is well defined except on null set N which is $\{x : f(x) = \pm\infty \text{ or } g(x) = \pm\infty\}$.

Problem 19.

It is trivial to check whether f is measurable or not. (Surely measurable.)
There are countably many corresponding null sets. Let N be a union of such null sets. Then all assumptions of problem are valid except on N. Note that $\lim_{k\to\infty} |f_k(x)| \leq \lim_{k\to\infty} g_k(x) = g(x)$ almost everywhere and $g \in L^1$ so $f \in L^1$. Now consider $g_k + g - |f - f_k| = h_k$ which is nonnegative measurable function except on N. By applying Fatou's lemma, we can get :

$$\int \liminf_{k\to\infty} h_k d\lambda \leq \liminf_{k\to\infty} \left(\int g d\lambda + \int (g_k - |f - f_k|) \, d\lambda \right) =$$

$$\int g d\lambda - \limsup_{k\to\infty} \left(\int (|f - f_k| - g_k) \, d\lambda \right) \leq 0$$

Therefore $\int g d\lambda + \limsup_{k\to\infty} \left(\int (|f - f_k| - g_k) \, d\lambda \right) \leq 0$. But we know that $\int g d\lambda = \limsup \int g_k d\lambda$ and $\limsup (a_k + b_k) \leq \limsup (a_k) + \limsup (b_k)$. Thus,

$$\limsup_{k\to\infty} \left(\int (g_k + |f - f_k| - g_k) \, d\lambda \right)$$

which means $\lim_{k\to\infty} \int |f - f_k| \, d\lambda = 0$ because limsup of nonnegative sequence goes to positive (or infinity) when it does not go to 0.

Then $\lim \left| \int f_k d\lambda - \int f d\lambda \right| = 0$, which implies conclusion of our problem.

section D. Integration over subsets of $\mathbb{R}^n$.

Problem 20.

It is obvious that $\{x \in E : -\infty \leq f(x) \leq t\} \subset E$ for every $t \in [-\infty, \infty]$. By completeness of Lebesgue measure λ, every subset of null set is measurable. Hence f is measurable.

Now consider $0 \leq s \leq f_+ 1_E$ where s is nonnegative simple function. s can have positive value on subset of E. Therefore $\int s d\lambda = 0$. So $\int f_+ 1_E d\lambda = 0$. Similarly, we can show that $\int f_- 1_E d\lambda = 0$. Thus we get $\int_E f d\lambda = 0$ for all measurable function defined on E.

Problem 21.

There are countably many corresponding null sets. Let N be a union of such null sets. Then all assumptions of problem are valid except on N. Now consider $|f - f_k| \leq f + f_k = g_k \in L^1$ a.e. and $\lim g_k = 2f$ exists a.e. and $\lim \int g_k d\lambda = \int 2f d\lambda$. So all the assumptions of problem 19 is satisfied.

Therefore $\lim_{k\to\infty} \int |f_k - f| \, d\lambda = \int \lim_{k\to\infty} |f_k - f| \, d\lambda = 0$.

From above and $\int_E f d\lambda \leq \int f d\lambda$ for nonnegative measurable function f, we can get

$$\lim \left| \int_E f d\lambda - \int_E f_k d\lambda \right| \leq \lim \int_E |f - f_k| \, d\lambda = 0$$

So $\lim_{k\to\infty} \int_E f_k d\lambda = \lim_{k\to\infty} \int_E f d\lambda$.

section E. Generalization of Measure Space.

Problem 22.

Let $A_1 = A$, $A_2 = B \setminus A$, $A_k = \emptyset$ for $k \geq 3$. Then by countable additivity of μ, $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu(B) = \mu(A) + \mu(B \setminus A)$. Therefore $\mu(A) \leq \mu(B)$ because $B \setminus A \in \mathcal{M}$ and $\mu(B \setminus A) \geq 0$.

Problem 24.

Let $B_1 = A_1$, $B_k = A_k \setminus \bigcup_{j=1}^{k-1} A_j \subset A_k$ for $k \geq 2$. Then B_k's are pairwise disjoint and in $\mathcal{M}$. Also $\bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k$. Therefore $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(B_k) \leq \sum_{k=1}^{\infty} \mu(A_k)$.

Problem 25.

Let $B_1 = A_1$ and $B_k = A_k \setminus A_{k-1}$ for $k \geq 2$. Then B_k's are pairwise disjoint and union from index 1 to index $N \in \mathbb{N} \cup \infty$ is same for that of A_k's. Therefore $\mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(B_k) = \lim \sum_{k=1}^{n} \mu(B_k) = \lim \mu(A_n)$.

section F. Some Calculations.

Problem 36.

$\{f > \frac{1}{k}\}$ must be finite since $\sum_{x\in X} f(x) < \infty$. Therefore $\bigcup\{f > \frac{1}{k}\} = \{f > 0\}$ is countable.

Problem 37.

Let F be finite subset of $\mathbb{N}$. Then $\sum_{x\in F} f(x) \leq \sum_{k=1}^{F_{\max}} f(k)$. Therefore $\sum_{x\in F} f(x) \leq \lim_{n\to\infty} \sum k = 1^n f(k)$.

Conversely, $\sum_{k=1}^{n} f(k) \leq \sum_{x\in F} f(x) \leq \sum_{x\in\mathbb{N}} f(x)$ for each n.

Proposition says when counting measure and nonnegative measurable function is given, $\int_X f d\mu = \sum_{x\in X} f(x)$.

Problem 38.

First assume $f \in L^1$. Then $|f|$ is nonnegative function. So $\int_X |f|\, d\mu = \sum_{x\in X} |f(x)| < \infty$ by proposition above.

Conversely, $\sum_{x\in X} |f(x)| = \int_X |f|\, d\mu < \infty$. Therefore $\int_X f^{\pm} d\mu \leq \int_X |f|\, d\mu < \infty$. So $f \in L^1$.

CHATER 7: Lebesgue Integral on $\mathbb{R}^n$

section A: Riemann Integral.

Problem 1.

Suppose 1_A is LSC. Let $x \in A$ and $0 < 1_A(x) = 1$. By definition of LSC at x, there exists $\delta > 0$ such that $1_A(y) > 0$ for all $y \in B(x;\delta)$. So $1_A(y) = 1$ and therefore $B(x;\delta) \subset A$. Thus A is open.

On the contrary, suppose A is open. For $x \in A$, consider $t < 1 = 1_A(x)$. Since A is open, there exists $\delta > 0$ such that $B(x;\delta) \subset A$. Then we get $y \in B(x;\delta) \Rightarrow t < 1_A(y) = 1$. For $x \notin A$, consider $t < 0 = 1_A(x)$. Take any $\delta > 0$. Then $y \in B(x;\delta) \Rightarrow t < 0 \le 1_A(y)$. Therefore 1_A is LSC if A is open.

Note that f is LSC if and only if $\forall t \in \bar{\mathbb{R}}$, $\{f > t\}$ is open. And f is LSC at x if and only if x is interior point of every $\{f > t\}$ for $t < f(x)$.

Problem 2.

For $x \in A^\circ$, there exists $\delta > 0$ such that $B(x;\delta) \subset A$. Then $\inf_{y \in B(x;\delta)} 1_A(y) = 1$ so lower envelope of 1_A at x is same as $1_{A^\circ}(x) = 1$.

Now assume $x \notin A^\circ$. Then, for every $\delta > 0$ $B(x;\delta) \cap A \neq \emptyset$. Then $\inf_{y \in B(x;\delta)} 1_A(y) = 0$ for some small δ. Then lower envelope of 1_A is zero at x, which is same as $1_{A^\circ}(x)$.

If $x \in \bar{A}$, for all $\delta > 0$ $B(x;\delta) \cap A \neq \emptyset$. Therefore $\sup_{y \in B(x;\delta)} 1_A(y) = 1$, so upper envelope of 1_A is same as $1_{\bar{A}}$.

If $x \notin \bar{A}$, there exists $\delta > 0$ such that $B(x;\delta) \cap A = \emptyset$. Then $\sup_{y \in B(x;\delta)} 1_A(y) = 0$. So upper envelope of 1_A is same as $1_{\bar{A}}$.

Problem 3.

For each $x \in \mathbb{R}^n$, let $t < \min_i f_i(x)$. Then $t < f_i(x)$ for all $i \in \mathcal{I}$. For each i, there exists δ_i such that $y \in B(x;\delta_i)$ implies $t < f_i(y)$. Take $\delta = \max_i \delta_i$. Then $y \in B(x;\delta)$ implies $t < \min_i f_i(y)$. So $\min_i f_i$ is LSC.

Now consider $A_i = \left(-\frac{1}{i}, \frac{1}{i}\right) \subset \mathbb{R}$. Since A_i is open, by problem 1, 1_{A_i} is LSC. Let $A = \bigcap_{i=1}^{\infty} A_i$ then $1_A = \inf_i 1_{A_i}$. It is not semicontinuous by considering the set $\left\{1_A > \frac{1}{2}\right\} = \{0\}$.

Problem 4.

Let $\tau_f \ge f$ and $\tau_g \ge g$ where τ_f and τ_g are step functions. Note that $\tau_f + \tau_g$ is also step function greater than $f + g$. Then all others follow directly.

Problem 5.

Let $\varepsilon > 0$ be given. We can choose positive integer N such that

(1) $|f(x) - f_n(x)| < \varepsilon$ if $n \ge N$ and for all $x \in \mathbb{R}^n = X$.
(2) $\int_I (\tau_N - \sigma_N)\, d\lambda < \varepsilon$

where τ_N, σ_N is simple function bigger, smaller than f_N respectively.
For every $x \in X$,

$$\sigma_N(x) - \varepsilon \leq f_N(x) - \varepsilon < f < f_N(x) + \varepsilon \leq \tau_N(x) + \varepsilon$$

Then, $\int_I \sigma_N d\lambda - \varepsilon\lambda(I) \leq r\underline{\int_I} f d\lambda \leq r\overline{\int_I} f d\lambda \leq \int_I \tau_N d\lambda + \varepsilon\lambda(I)$ Because $\sigma_N - \varepsilon$ is step function smaller than f and $\tau_N + \varepsilon$ is also step function bigger than f.

Therefore we can get

$$r\overline{\int_I} f d\lambda - r\underline{\int_I} f\lambda < \varepsilon + 2\varepsilon\lambda(I)$$

which implies Riemann integrability of f.

By definition of uniform convergence, $r\int |f - f_N| \, d\lambda \leq \varepsilon\lambda(I)$.
So, $\lim_{N\to\infty} r\int |f - f_N| \, d\lambda = 0$, which implies $\lim_{N\to\infty} \left| r\int f d\lambda - r\int f_N d\lambda \right| = 0$ then conclusion follows.

Problem 6.

(a) $g(x) = 0$ for all $x \in I = [0, 1]$. $f(x) = 1_{\mathbb{Q}\cap I}(x)$. f is nowhere continuous but $f = g$ almost everywhere.
(b) Consider the following funnction f :

$$f(x) = \begin{cases} 0 & \text{if } x \in I \cap C \\ \frac{1}{x} & \text{if } x \in I \cap C^c \end{cases}$$

where C is Cantor ternary set and $I = [0, 1]$.

For $x \in I \cap C^c$, $x \in I_{j,k} = \left(\frac{2k}{3^j}, \frac{2k+1}{3^j}\right)$ for some k, j. On $I_{j,k}$, function $i : x \mapsto x$ is continuous and $i(x) \neq 0$. So, the map $\varphi : x \mapsto \frac{1}{x}$ is also continuous on $I_{j,k}$. Therefore, φ is continuous at x. Therefore, φ is continuous a.e. on I.

Note that $f(x) = \frac{1}{x}$ a.e. on I. Let $g = f$ a.e. on I. Then $g(x) = \frac{1}{x}$ a.e. on I. Then g is discontinuous at $x = 0$. Therefore, f is continuous a.e. on I and there is no continuous function such that $f = g$ a.e. on I.

Problem 7.

For $x \in [a, b]$, $na \leq nx \leq nb$, so $nx \in \frac{1}{2} + \mathbb{Z}$ for finitely many x. Therefore, (nx) is discontinuous at most finitely many points, which implies (nx) is Riemann integrable. Then $\frac{(nx)}{n^2}$ also Riemann integrable, and their finite summation $f_k(x) = \sum_{n=1}^{k} \frac{(nx)}{n^2}$ is also Riemann integrable.
Now, let $\varepsilon > 0$ be given. Since $\sum_{n=1}^{\infty}$ converges, there exists positive integer N such that $\sum_{k=N}^{\infty} < \varepsilon$. Consider $m > n \geq N$ and following:

$$|f_m(x) - f_n(x)| \leq \left| \sum_{k=n+1}^{m} \frac{(kx)}{k^2} \right| \leq \sum_{k=n+1}^{m} \frac{|(kx)|}{k^2} \leq \sum_{k=n+1}^{m} \frac{1}{k^2} < \varepsilon$$

for all $x \in [a, b]$ since $-1 \leq (nx) \leq 1$. Therefore, f_n is uniformly Cauchy, which implies uniform convergence of f_n to f. By problem 5, f is Riemann integrable since each f_n is Riemann integrable and $f_n \rightrightarrows f$.

Problem 9.

With out loss of generality, assume that f is monotonically increasing.

Let $\varepsilon > 0$ be given. Consider $a = x_0 < x_1 < \cdots < x_n = b$ where $\max_{1 \le k \le n} \lambda\left([x_{k-1}, x_k]\right) < \frac{\varepsilon}{f(b)-f(a)}$. (If $f(a) = f(b)$, conclusion follows trivially so let us assume that $f(a) < f(b)$.

Let $I = [a, b]$ and $\sigma : I \to \mathbb{R}$ such that $\sigma\left((x_{k-1}, x_k)\right) = \{f(x_{k-1})\}$, $\sigma(x_k) = f(a)$. Similarly, let $\tau : I \to \mathbb{R}$ such that $\tau\left((x_{k-1}, x_k)\right) = \{f\left(x_k\right)\}$ and $\tau(x_k) = f(b)$. Then σ, τ are step functions satisfying $\sigma \le f \le \tau$. So,

$$\begin{aligned}\int_I (\tau - \sigma)\, d\lambda &= \sum_{k=1}^{n} \left(f(x_k) - f(x_{k-1})\right) \lambda\left([x_{k-1}, x_k]\right) \\ &< \sum_{k=1}^{n} \left(f(x_k) - f(x_{k-1})\right) \frac{\varepsilon}{f(b) - f(a)} = \varepsilon\end{aligned}$$

which implies Riemann integrability of f on I.

Let $x < x'$ be points where f is discontinuous. Since f is monotonic, $f(x-) = \lim_{y \uparrow x} f(y)$ and $f(x+) = \lim_{y \downarrow x} f(y)$ exist. By monotonicity of f, we can easily deduce that $f(x-) \le f(x) \le f(x+) \le f(x'-) \le f(x') \le f(x'+)$. Since f is discontinuous at x, x', $f(t-) < f(t+)$ for $t = x, x'$. Choose $q_t \in (f(t-), f(t+))$ for $t = x, x'$. Then $q_x < q_{x'}$. The map $x \mapsto q_x$ is hence injective. So, there are at most countably many discontinuous points of f on I.

Therefore f is continuous a.e. on I.

Problem 10.

Consider 1_C where C is Cantor ternary set. If $x \in C$, $x \notin C^\circ$ since C has empty interior. So, for any $\delta > 0$, there exists $y \in B(x; \delta)$ such that $y \notin C$. Then $1_C(x) - 1_C(y) = 1$. So 1_C is discontinuous at $x \in C$.

On the contrary if $x \in C^c$, $x \in I_{j,k}$ for some j, k (we'll use notation of problem 6.) Then there is $\delta > 0$ such that $B(x; \delta) \subset I_{j,k} \subset C^c$, so $d(x, y) < \delta$ implies $1_C(x) - 1_C(y) = 0 < \varepsilon$ for any $\varepsilon > 0$. Therefore 1_C is continuous on C^c.

1_C has an uncountable set of discontinuities(C) and continuous a.e. on $[0, 1]$. Therefore 1_C is Riemann integrable.

Problem 11.

If f is Riemann integrable and $f = 1_A$ a.e. on $I = [0, 1]$, $r\int f d\lambda = \int_I f d\lambda = \int_I 1_A d\lambda = \lambda\left(A\right) > 0$.

Let σ be a step function, $\sigma \le f$. Then $\sigma \le 1_A$ a.e. on I and let N be corresponding null set.

Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be endpoints of special rectangles corresponding to σ. If $(x_{k-1}, x_k) \cap J_i = \emptyset$ for all positive $i = \frac{2m-1}{2^t} < 1$ ($A = I \setminus \bigcup J_i$, we'll use notation of section 4.B), $(x_{k-1}, x_k) \subset A$ which contradicts the fact that A has empty interior. So, $(x_{k-1}, x_k) \cap J_i \neq \emptyset$ for some i.

If $(x_{k-1}, x_k) \setminus N \subset A$, $(x_{k-1}, x_k) \setminus N \cap J_i = \emptyset$, then $(x_{k-1}, x_k) \cap J_i \subset N$. But $J_i \cap (x_{k-1}, x_k)$ is open and nonempty, so it has positive measure which contradicts $\lambda(N) = 0$.

Therefore for each k, $(x_{k-1}, x_k) \setminus N \cap A^c \neq \emptyset$. So $\sigma\left((x_{k-1}, x_k)\right) = \xi$ where $\xi \leq 0$. It means $\sigma \leq 0$.

Then $r \int f d\lambda = \sup_{\sigma \leq f} \int_I \sigma d\lambda \leq 0$ which contradicts to $r \int f d\lambda = \lambda\left(A\right) > 0$.