mas541 homework

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Problem (1.1)**.**

$$1-\left|\frac{z-w}{1-z\overline{w}}\right|^2 = 1-\frac{(z-w)\left(\overline{z}-\overline{w}\right)}{\left(1-z\overline{w}\right)\left(1-\overline{z}w\right)}$$
$$= \frac{1-\overline{z}w-z\overline{w}+|z|^2|w|^2-|z|^2-|w|^2+z\overline{w}+\overline{z}w}{|1-\overline{z}w|^2}$$
$$= \frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\overline{z}w|^2}$$

Problem (1.2)**.**

Let $f=u+iv$*.* $\partial f=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)$*. Then* $\overline{\partial f}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(u-iv)=\bar{\partial}\bar{f}$*.*

Problem (1.3)**.**

If f *is constant, then* $|f|$ *is also constant. On the other hand, assume* $f=u+iv$ *and* $|f|^2=u^2+v^2$ *is positive real number. (if it is zero, then* f *must be zero)*

$$u^2+v^2=R>0$$

Differentiate both sides of the equation above with x *and* y *respectively, we can get* $uu_x+vv_x=0$*,* $uu_y+vv_y=0$*,* $u_x=v_y$ *and* $u_y=-v_x$*. By simple calculation we can get* $u_x=u_y=v_x=v_y=0$*. Therefore* u,v *are constant.*

Problem (1.4)**.**

Note that $\int_0^{2\pi}e^{ik\theta}d\theta=\int_0^{2\pi}(\cos k\theta+i\sin k\theta)\,d\theta=0$ *for positive integer* k*. Therefore* $\frac{1}{2\pi}\int_0^{2\pi}(z_0+re^{i\theta})^jd\theta=\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{j}\binom{j}{k}z_0^k\left(re^{i\theta}\right)^{j-k}d\theta=z_0^j$*. Similarly, we can get* $\frac{1}{2\pi}\int_0^{2\pi}\overline{(z_0+re^{i\theta})^j}d\theta=\bar{z}_0{}^j$*.*

Since u *is polynomial, we can write it as* $\sum_{l,k}a_{l,k}z^l\bar{z}^k$*. By direct computation, we can get* $\frac{1}{2\pi}\int_0^{2\pi}u\left(z_0+re^{i\theta}\right)d\theta=\sum_{l,k}a_{l,k}z_0^l\bar{z}_0{}^k=u(z_0)$*.*

Problem (1.5)**.**

Let $f=u+iv$*.* $(g\circ f)_x=g_uu_x+g_vv_x$*. Then*

$$(g\circ f)_{xx}=\left(g_{uu}u_x+g_{uv}v_x\right)u_x+g_uu_{xx}+\left(g_{vu}u_x+g_{vv}v_x\right)v_x+g_vv_{xx}$$
$$(g\circ f)_{yy}=\left(g_{uu}u_y+g_{uv}v_y\right)u_y+g_uu_{yy}+\left(g_{vu}u_y+g_{vv}v_y\right)v_y+g_vv_{yy}$$

But we have Cauchy-Riemann equation and $g_{uu}+g_{vv}=0$ *and* $g_{vu}=g_{uv}$*. Also, since* f *is* C^2 *function,* f *is harmonic,* $u_{xy}=u_{yx}$*, and* $v_{xy}=v_{yx}$*. Using* *these equations, we can check that* $(g \circ f)_{xx} + (g \circ f)_{yy} = 0$. *Hence* $(g \circ f)$ *is a harmonic function.*

Problem (2.1)**.**

Let $f = u + iv$*. Then* $\bar{f}f' = ff' - 2ivf'$*, where* ff' *is holomorphic. So,* $\int_\gamma \bar{f}f'dz = \int_\gamma -2ivf'dz = \int_\gamma -2iv(u_x + iv_x)dz = \int_\gamma -2iv(v_y + iv_x)dz = -i\int_a^b (2vv_y + 2ivv_y)(\gamma_1' + i\gamma_2')dt = \alpha$ *where* $\gamma = \gamma_1 + i\gamma_2$*.*

Therefore, real part of $\int_\gamma \bar{f}f'dz$ *is equal to real part of* α*. And it is also equal to* $-\int_a^b Im\left[(2vv_y + i2vv_x)(\gamma_1' + i\gamma_2')\right]dt = -\int_a^b (2vv_x\gamma_1' + 2vv_y\gamma_2')dt = -\int_a^b \frac{d}{dt}(v^2 \circ \gamma)dt = 0$ *since* γ *is closed curve.*

So, $\int_\gamma \bar{f}f'dz$ *is purely imaginary.*

Problem (2.2)**.**

Let $f = -u_y$ *and* $g = u_x$*. Then* f, g *are continuous on* U*. Since* u *is harmonic,* $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ *on* $U \setminus \{0\}$*. So there is* $v : U \to \mathbb{R}$ *which is* C^1 *function and* $v_x = f$*,* $v_y = g$ *by lemma 2.5.3.*

Let $F = u + iv$*. Then* F *is* C^1 *function since* u, v *are* C^1*. Since* $v_x = f = -u_y$ *and* $v_y = g = u_x$*,* F *satisfies Cauchy-Riemann equation on* U*. Thus* F *is holomorphic on* U *and real part of* F *is* u*.*

Problem (2.3)**.**

(a) *For* $z \notin [0,1]$*, the map* $w \mapsto \frac{1}{w-z}$ *is holomorphic on* $\mathbb{C}\setminus[0,1]$*. Let* $\gamma(t) = t$ *for* $t \in [0,1]$*. Then* $F(z) = \int_\gamma \frac{dw}{w-z} = \int_0^1 \frac{1}{t-z}dt$ *is well defined.*

For $z \notin [0,1]$*, let* $d > 0$ *be distance between* z *and* $[0,1]$*. For* $|h| < \frac{d}{2}$*, consider* $\frac{F(z+h)-F(z)}{h} = \int_0^1 \frac{1}{(t-z-h)(t-z)}dt$*. Then* $\left|\frac{1}{(t-z-h)(t-z)} - \frac{1}{(t-z)^2}\right| = \left|\frac{h}{(t-z)^2(t-z-h)}\right| \leq |h|\frac{2}{d^3}$ *since* $|t-z| \geq d$ *and* $|t-z-h| \geq \frac{d}{2}$*. Therefore, as* $|h| \to 0$*, integrand converges to* $\frac{1}{(t-z)^2}$ *uniformly on* $t \in [0,1]$*. So* $\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = \int_0^1 \lim_{h\to 0}\frac{1}{(t-z-h)(t-z)}dt = \int_0^1 \frac{1}{(t-z)^2}dt = F'(z)$*.*

By same reasoning, we get $F''(z) = \int_0^1 \frac{1}{(t-z)^3}dt$*. From existence of* F''*,* F' *is continuous. Therefore* F *is* C^1 *function. Existence of complex derivative and* C^1 *implies* F *is holomorphic on* $\mathbb{C}\setminus[0,1]$*.*

(b) *For* $s \in (0,1)$*,* $F(s+i\varepsilon) = \int_0^1 \frac{1}{t-s-i\varepsilon}dt = \int_0^1 \frac{t-s+i\varepsilon}{(t-s)^2+\varepsilon^2}dt = \int_0^1 \frac{t-s}{(t-s)^2+\varepsilon^2}dt + i\int_0^1\int_0^1 \frac{\varepsilon}{(t-s)^2+\varepsilon^2}dt$*. Let* $t-s = \varepsilon\tan\theta$*.* $\varepsilon\tan\theta_0 + s = 0$ *and* $\varepsilon\tan\theta_1 + s = 1$ *for* $-\frac{\pi}{2} < \theta_0, \theta_1 < \frac{\pi}{2}$*. Then* $\sec^2\theta_0 = \frac{s^2}{\varepsilon^2} + 1$*,* $\sec^2\theta_1 = \frac{(1-s)^2}{\varepsilon^2} + 1$*,* $\theta_0 = \tan^{-1}\left(\frac{-s}{\varepsilon}\right)$*, and* $\theta_1 = \tan^{-1}\left(\frac{1-s}{\varepsilon}\right)$*.*

Then $F(s+i\varepsilon) = \int_{\theta_0}^{\theta_1}\tan\theta d\theta + i\int_{\theta_0}^{\theta_1}d\theta = \log\left|\frac{\sec\theta_1}{\sec\theta_0}\right| + i(\theta_1 - \theta_0)$*. As* $\varepsilon \downarrow 0$*,* $F(s+i\varepsilon)$ *goes to* $\frac{1-s}{s} + i\pi$ *by simple calculation.*

Similarly, $F(s-i\varepsilon)$ *goes to* $\frac{1-s}{s}-i\pi$ *as* $\varepsilon \downarrow 0$.

(c) *Consider* $F(-\varepsilon)=\int_0^1 \frac{1}{t+\varepsilon}dt=\log\frac{1+\varepsilon}{\varepsilon}$. *It goes to* ∞ *as* $\varepsilon\downarrow 0$.

Consider $F(1+\varepsilon)=\int_0^1 \frac{1}{t-1-\varepsilon}dt=\log\frac{\varepsilon}{1+\varepsilon}$. *It goes to* $-\infty$ *as* $\varepsilon\downarrow 0$. *Therefore, for* $s=0,1$, $\lim_{z\notin[0,1]\to s}F(z)$ *does not exists.*

Problem (2.4)**.**

First consider $p\equiv 0$. *We can easily see that* $\sup_{z\in C}|z^{-n}|=1$ *so desired value* ≤ 1.

Note that $|p(z)-z^{-n}|=|z^n p(z)-1|$. *Thus,* $1=\frac{1}{2\pi i}\int_C \frac{z^n p(z)-1}{z}dz \leq \sup_{z\in C}|z^n p(z)-1|$.

Those leads the conclusion.

Problem (2.5)**.**

It is enough to show γ *and* μ *are path homotopic. Definte* $H(t,s)=(1-s)\gamma(t)+\frac{\gamma(t)}{|\gamma(t)|}s$. *Then* $H(t,1)=\mu(t)$ *and* $H(t,0)=\gamma(t)$ *by reparametrization. And* H *is continuous because* $\gamma(t)\neq 0$. *Therefore* H *is path homotopy between* γ *and* μ. *Since line integration is invariant under path homotopy, we get* $\int_\gamma F(\zeta)d\zeta=\int_\mu F(\zeta)d\zeta$.

Problem (3.1).

It suffices to show that $\int_\gamma f(z)dz = 0$ for rectangle γ whose edges are parallel to coordinate axes by Morera's theorem.

First, assume that γ intersects with $[0,1]$ only finitely many points. Let p be such point. Then p must be on (wlog) left edge of γ. Let $a+ib$, $a+ic$ be two vertices incident with left edge. ($b > c$) Let $\rho(t) = a + i(tc + (1-t)b)$. Consider $f \circ \rho$. It is continuous and equals to $\frac{\partial}{\partial t}F(\rho(t))$ except for $\gamma^{-1}(p)$ where F is antiderivative of f on $\mathbb{C} \setminus [0,1]$. Then lemma 2.3.1 says $f(\rho(t)) = \frac{\partial}{\partial t}F(\rho(t))$ even for $\gamma^{-1}(p)$. Therefore $\int_\rho f(z)dz = F(a+ic) - F(a+ib)$. By using this result, we can easily calculate $\int_\gamma f(z)dz = 0$.

Now, assume that (wlog) upper edge of γ intersects with $[0,1]$. Let $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ which are upper edge, left edge, bottom edge, and right edge respectively, parametrized like ρ of above, positive oriented. Consider φ made by shrinking side edges of γ so that distance between of upper edges of φ and γ less than δ, while bottom edge is fixed. Also note that δ is chosen so that $d(z_0, z_1) < \delta$ implies $d(f(z_0), f(z_1)) < \varepsilon$.

$$\left| \int_\gamma f(z)dz - \int_\varphi f(z)dz \right| \leq \left| \int_{\gamma_2 - \varphi_2} f(z)dz + \int_{\gamma_4 - \varphi_4} f(z)dz \right| \\ + (\text{ length of } \gamma_1)\, \varepsilon$$

And, second term of above goes to 0 as distance between φ_1 and γ_1 goes to 0 by continuity and result of first case. Actually $\int_\varphi f(z)dz = 0$ because φ does not intersect with $[0,1]$. Thus we have shown that $\int_\gamma f(z)dz = 0$.

By first, second case and Morera's thm, f is actually entire function.

Problem (3.2).

For $0 < r < 1$, $|f^{(n)}(0)| \leq \frac{n!}{r^n}\frac{1}{1-r}$ by using Cauchy estimate. $r^n(1-r)$ is maximized when $r = \frac{n}{n+1}$. So, when $r = \frac{n}{n+1}$, we get best estimate of $|f^{(n)}(0)|$.

Problem (3.3).

(a) *Since K is compact subset of open set U, there is $r > 0$ such that for all $x \in K$, closure of $D(x, r)$ is in U. Then, $|f(z)|^2 \leq \frac{1}{2\pi}\left|\int_{\partial D(z,r} \frac{f^2(w)}{w-z}dw\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f^2(z + re^{i\theta})d\theta|$. By multiplying ρ both sides and integrating from 0 to r, we can get the following:*

$$\begin{aligned}\frac{r^2}{2}|f(z)|^2 &\le \frac{1}{2\pi}\int_0^r\int_0^{2\pi}\rho|f^2(z+re^{i\theta})|d\theta d\rho\\ &= \frac{1}{2\pi}\int_{\overline{D}(z,r)}|f|^2dm\\ &= \frac{1}{2\pi}\int_U|f|^2dm\end{aligned}$$

for all $z \in K$, where m is lebesgue measure, using Holder's inequality and polar coordinate integration.

Therefore $C = \frac{1}{r\sqrt{\pi}}$

(b) *If f is identically zero, possible.*

Else if f is constant, then $\int_{\mathbb{C}}|f|dm = \infty$ since measure of complex plane is ∞.

Else, that is f is nonconstant entire function, then f must be unbounded. So, there is $\delta > 0$ such that $|f| \ge 1$ for all $|z| > \delta$. Then $\int_{\mathbb{C}}|f|dm \ge m(\{z : |z| > \delta\}) = \infty$.

Problem (3.4). (a) *Since $\frac{z}{e^z-1}$ is bounded near 0, it has removable singularity at 0. So we can regard it as holomorphic function. Note that $e^z-1=0$ when z is integer multiple of $2\pi i$. So, given power series converges on unit disc. Now, multiply e^z-1 both sides. Since e^z-1 is entire and given power serires converges absolutely on $\bar{D}(0,r)$ where $0<r<1$, we can write $z = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n\sum_{n=1}^{\infty}\frac{1}{n!}z^n$. Since z is entire, coefficient of power series is unique. By comparing coefficients of both sides, we can get given recursion formula.*

$\lim_{z\to 0}\frac{z}{e^z-1} = 1 = B_0$. *From this, by simple calculation, $B_1 = \frac{-1}{2}$, $B_2 = \frac{1}{6}$, and $B_3 = 0$.*

Consider $-z = f(z) - f(-z) = \sum_{n=0}^{\infty}2\frac{B_{2n+1}}{(2n+1)!}z^{2n+1}$. This makes sense because f is holomorphic on unit disc. By comparing coefficient of this series, we can get $B_{2m+1} = 0$ for $m \ge 1$.

(b) *We alrealdy notice that e^z-1 is zero when z is integer multiple of $2\pi i$. But $\lim_{z\to 2k\pi i}\frac{z}{e^z-1}$ is not bounded when $k \ne 0$. Therefore, $\frac{z}{e^z-1}$ is holomorphic on $D(0,2\pi)$ and is not holomorphic outside of that disc. Since* *power seriese representation of holomorphic function at P has radius of convergence at least $d(P,U)$, we can say radius of convergence of the series is 2π.*

Problem (3.5)**.**

f' is holomorphic on unit disc. Let $r = \sup_{z\in K}|z|$. Since K is compact, $|f'| \leq M$ on K and r is positive but less than 1. Let $\gamma(t) = tz^n$ which connects origin and z^n. $|f(z^n) - f(0)| = \left|\int_\gamma f' dz\right| \leq M \sup_{z\in K}|z|^n = Mr^n$. Therefore, $|\sum_{n=1}^{\infty} f(z^n)| \leq \sum_{n=1}^{\infty}|f(z^n)| \leq \sum_{n=1}^{\infty} Mr^n < \infty$ because r is positive but less than 1.

Problem (4.1)**.**

Notice that f does not vanish on $\mathbb{C}\setminus\{0\}$. Therefore $g(z)=\frac{1}{f(z)}$ is holomorphic on $\mathbb{C}\setminus\{0\}$. Near 0, g is bounded since $\sqrt{|z|}$ goes to 0 as z goes to 0. This means g has removable singularity at 0 and therefore entire. But $g(z)\leq\sqrt{|z|}$, so g must be constant by Cauchy integral formula.

Then f must be constant also, and this is contradiction. Therefore there is no such holomorphic function.

Problem (4.2)**.**

Let $g(z)=f\left(\frac{1}{z}\right)$. Then $g\to 0$ as $z\to 0$. Therefore g is entire. Also, $g(z)/z$ is entire since $\lim_{z\to 0} g(z)/z = g'(0)$ hence bounded near 0.

Now, consider given integral. Let $\zeta=e^{it}$ and $t=2\pi-s$. Then given integral is $\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(e^{-is})}{e^{-is}-z}ie^{-is}ds=\frac{1}{2\pi i}\int_0^{2\pi}\frac{g(e^{is})}{e^{is}-e^{2is}z}ie^{is}ds=\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{g(\zeta)}{\zeta z\left(\frac{1}{z}-w\right)}d\zeta$

Therefore given integral is equal to $\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{h(\zeta)}{\frac{1}{z}-\zeta}d\zeta$ where $h(\zeta)=\frac{g(\zeta)}{\zeta z}$. Thus, it is equal to $-g(1/z)=-f(z)$.

Problem (4.3)**.**

f maps $re^{i\theta}$ to $\sqrt{r}e^{i\left(\frac{\theta}{2}+k(z)\pi\right)}$ where $k(z)\in\mathbb{Z}$. To f be continuous, $k(z)$ must be all even or all odd.

First assume that $k(z)$ is all even. Then $f'(0)=\lim_{\mathbb{R}\ni h\to 0}\frac{f(h)}{h}=\lim_{\mathbb{R}\ni h\to 0}\frac{\sqrt{h}}{h}=\infty$, which is contradiction.

Similarly, if $k(z)$ is all odd, $f'(0)$ does not exist.

Therefore existence of such f leads $0\notin U$.

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Problem (4.4)**.**

(a)

(b) *Consider $f(z)=\frac{\pi\cot(\pi z)}{(z+\alpha)^2}$ and Γ_n = square centered at origin, each edges is parallel to real or imaginary axis, length of edge is $2n+1$.*

Then $\int_{\Gamma_n} f(z)dz$ goes to 0 as $n\to\infty$ by considering modulus of $f(z)$, and index of Γ_n at each singularites is 1, and residues are $\frac{1}{(k+\alpha)^2}$ at $z=k$ and $-\frac{\pi^2}{\sin^2(\pi\alpha)^2}$ at $z=-\alpha$.

Above calculation leads the conclusion.

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Problem (4.5).

Note that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic iff f is meromorphic on $\hat{\mathbb{C}}$.

(a) *First consider 'if' part. Let f be rational function. We already knows that rational function is meromorphic on entire complex plane. So, we need to show that rational function is meromorphic at ∞.*

Let $f(z) = \frac{(z-Q_1)^{m_1}\cdots(z-Q_l)^{m_l}}{(z-P_1)^{n_1}\cdots(z-P_k)^{n_k}}$. Since f has finitely many pole in complex plane, we can choose $M > 0$ so that f has no pole on $\{z : |z| > M\}$. For $0 < |w| < \frac{1}{M}$, consider $g(w) = f(1/w)$. Then g is holomorphic.

Let $\sum_i n_i = N$ and $\sum_j m_j = M$. If $M = N$, $g \to 1$ as $z \to 0$. If $M > N$, $g \to 0$ as $z \to 0$. If $M < N$, $g \to \infty$ if $z \to 0$. Hence g is meromorphic near 0, which means that f is meromorphic at ∞.

Second, consider 'only if' part. Either f has a pole or removable singularity at ∞, f has finitely many poles in complex plane. So $f(z)(z - P_1)^{n_1} \cdots (z - P_k)^{n_k} = F(z)$ is entire where n_i is order of pole P_i.

Consider $F(1/z) = g(z)$ for $z \neq 0$. As $z \to 0$, $g \to \infty$ or α for some $\alpha \in \mathbb{C}$ by simple calculation. Therefore F has a pole or removable singularity at ∞.

If F has removable singularity at ∞, F must be bounded, hence constant by Liouville's thm.

If F has a pole at ∞, F must be polynomial since its modulus diverges.

In both cases, F must be rational function.

(b) *Note that $z \mapsto \frac{az+b}{cz+d}$ for $ad-bc \neq 0$ is biholomorphic function of Riemann sphere. Also note that biholomorphic function of $\mathbb{C}$ must have a form of $\alpha z + \beta$ for $\alpha \neq 0$ by fundamental thm of algebra.*

Now consider biholomorphic f on Riemann sphere. Let $f(\infty) = b$ and $\varphi_b(z) = \frac{-\bar{b}-1}{z-b}$. Then $\varphi_b \circ f$ is biholomorphic function of Riemann sphere, which maps $\infty \to \infty$. Therefore $\varphi_b \circ f$ is biholomophic function of complex plane hence $\varphi_b(f(z)) = \alpha z + \beta$. Then $f(z) = \frac{-b\alpha z - b\beta + 1}{-\alpha z - \beta - b}$, which is linear frational transformation.

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