mas541 homework

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2020년 10월 11일

Problem (1.1)**.**

$$
\begin{aligned}
1-\left|\frac{z-w}{1-z\overline{w}}\right|^2 &= 1-\frac{(z-w)\left(\overline{z}-\overline{w}\right)}{(1-z\overline{w})\,(1-\overline{z}w)} \\
&= \frac{1-\overline{z}w-z\overline{w}+|z|^2|w|^2-|z|^2-|w|^2+z\overline{w}+\overline{z}w}{|1-\overline{z}w|^2} \\
&= \frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\overline{z}w|^2}
\end{aligned}
$$

Problem (1.2)**.**

Let $f=u+iv$. $\partial f=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)$. *Then* $\overline{\partial f}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(u-iv)=\bar{\partial}\bar{f}$.

Problem (1.3)**.**

If f *is constant, then* $|f|$ *is also constant. On the other hand, assume* $f=u+iv$ *and* $|f|^2=u^2+v^2$ *is positive real number. (if it is zero, then* f *must be zero)*

$$u^2+v^2=R>0$$

Differentiate both sides of the equation above with x *and* y *respectively, we can get* $uu_x+vv_x=0$, $uu_y+vv_y=0$, $u_x=v_y$ *and* $u_y=-v_x$. *By simple calculation we can get* $u_x=u_y=v_x=v_y=0$. *Therefore* u,v *are constant.*

Problem (1.4)**.**

Note that $\int_0^{2\pi}e^{ik\theta}d\theta=\int_0^{2\pi}\left(\cos k\theta+i\sin k\theta\right)d\theta=0$ *for positive integer* k. *Therefore* $\frac{1}{2\pi}\int_0^{2\pi}(z_0+re^{i\theta})^j d\theta=\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{j}\binom{j}{k}z_0^k\left(re^{i\theta}\right)^{j-k}d\theta=z_0^j$. *Similarly, we can get* $\frac{1}{2\pi}\int_0^{2\pi}\overline{(z_0+re^{i\theta})^j}d\theta=\bar{z}_0{}^j$.

Since u *is polynomial, we can write it as* $\sum_{l,k}a_{l,k}z^l\bar{z}^k$. *By direct computation, we can get* $\frac{1}{2\pi}\int_0^{2\pi}u\left(z_0+re^{i\theta}\right)d\theta=\sum_{l,k}a_{l,k}z_0^l\bar{z}_0{}^k=u(z_0)$.

Problem (1.5)**.**

Let $f=u+iv$. $(g\circ f)_x=g_uu_x+g_vv_x$. *Then*

$$
\begin{aligned}
(g\circ f)_{xx} &= \left(g_{uu}u_x+g_{uv}v_x\right)u_x+g_uu_{xx}+\left(g_{vu}u_x+g_{vv}v_x\right)v_x+g_vv_{xx} \\
(g\circ f)_{yy} &= \left(g_{uu}u_y+g_{uv}v_y\right)u_y+g_uu_{yy}+\left(g_{vu}u_y+g_{vv}v_y\right)v_y+g_vv_{yy}
\end{aligned}
$$

But we have Cauchy-Riemann equation and $g_{uu}+g_{vv}=0$ *and* $g_{vu}=g_{uv}$. *Also, since* f *is* C^2 *function,* f *is harmonic,* $u_{xy}=u_{yx}$, *and* $v_{xy}=v_{yx}$. *Using* *these equations, we can check that* $(g \circ f)_{xx} + (g \circ f)_{yy} = 0$*. Hence* $(g \circ f)$ *is a harmonic function.*

Problem (2.1)**.**

Let $f = u + iv$. *Then* $\bar{f}f' = ff' - 2ivf'$, *where* ff' *is holomorphic. So,* $\int_\gamma \bar{f}f'dz = \int_\gamma -2ivf'dz = \int_\gamma -2iv(u_x + iv_x)dz = \int_\gamma -2iv(v_y + iv_x)dz = -i\int_a^b (2vv_y + 2ivv_y)(\gamma_1' + i\gamma_2')dt = \alpha$ *where* $\gamma = \gamma_1 + i\gamma_2$.

Therefore, real part of $\int_\gamma \bar{f}f'dz$ *is equal to real part of* α. *And it is also equal to* $-\int_a^b Im\left[(2vv_y + i2vv_x)(\gamma_1' + i\gamma_2')\right]dt = -\int_a^b (2vv_x\gamma_1' + 2vv_y\gamma_2')dt = -\int_a^b \frac{d}{dt}(v^2 \circ \gamma)dt = 0$ *since* γ *is closed curve.*

So, $\int_\gamma \bar{f}f'dz$ *is purely imaginary.*

Problem (2.2)**.**

Let $f = -u_y$ *and* $g = u_x$. *Then* f, g *are continuous on* U. *Since* u *is harmonic,* $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ *on* $U \setminus \{0\}$. *So there is* $v : U \to \mathbb{R}$ *which is* C^1 *function and* $v_x = f$, $v_y = g$ *by lemma 2.5.3.*

Let $F = u + iv$. *Then* F *is* C^1 *function since* u, v *are* C^1. *Since* $v_x = f = -u_y$ *and* $v_y = g = u_x$, F *satisfies Cauchy-Riemann equation on* U. *Thus* F *is holomorphic on* U *and real part of* F *is* u.

Problem (2.3)**.**

(a) For $z \notin [0,1]$, *the map* $w \mapsto \frac{1}{w-z}$ *is holomorphic on* $\mathbb{C}\setminus[0,1]$. *Let* $\gamma(t) = t$ *for* $t \in [0,1]$. *Then* $F(z) = \int_\gamma \frac{dw}{w-z} = \int_0^1 \frac{1}{t-z}dt$ *is well defined.*

For $z \notin [0,1]$, *let* $d > 0$ *be distance between* z *and* $[0,1]$. *For* $|h| < \frac{d}{2}$, *consider* $\frac{F(z+h)-F(z)}{h} = \int_0^1 \frac{1}{(t-z-h)(t-z)}dt$. *Then* $\left|\frac{1}{(t-z-h)(t-z)} - \frac{1}{(t-z)^2}\right| = \left|\frac{h}{(t-z)^2(t-z-h)}\right| \leq |h|\frac{2}{d^3}$ *since* $|t-z| \geq d$ *and* $|t-z-h| \geq \frac{d}{2}$. *Therefore, as* $|h| \to 0$, *integrand converges to* $\frac{1}{(t-z)^2}$ *uniformly on* $t \in [0,1]$. *So* $\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = \int_0^1 \lim_{h\to 0}\frac{1}{(t-z-h)(t-z)}dt = \int_0^1 \frac{1}{(t-z)^2}dt = F'(z)$.

By same reasoning, we get $F''(z) = \int_0^1 \frac{1}{(t-z)^3}dt$. *From existence of* F'', F' *is continuous. Therefore* F *is* C^1 *function. Existence of complex derivative and* C^1 *implies* F *is holomorphic on* $\mathbb{C}\setminus[0,1]$.

(b) For $s \in (0,1)$, $F(s+i\varepsilon) = \int_0^1 \frac{1}{t-s-i\varepsilon}dt = \int_0^1 \frac{t-s+i\varepsilon}{(t-s)^2+\varepsilon^2}dt = \int_0^1 \frac{t-s}{(t-s)^2+\varepsilon^2}dt + i\int_0^1\int_0^1 \frac{\varepsilon}{(t-s)^2+\varepsilon^2}dt$. *Let* $t - s = \varepsilon\tan\theta$. $\varepsilon\tan\theta_0 + s = 0$ *and* $\varepsilon\tan\theta_1 + s = 1$ *for* $-\frac{\pi}{2} < \theta_0, \theta_1 < \frac{\pi}{2}$. *Then* $\sec^2\theta_0 = \frac{s^2}{\varepsilon^2} + 1$, $\sec^2\theta_1 = \frac{(1-s)^2}{\varepsilon^2} + 1$, $\theta_0 = \tan^{-1}\left(\frac{-s}{\varepsilon}\right)$, *and* $\theta_1 = \tan^{-1}\left(\frac{1-s}{\varepsilon}\right)$.

Then $F(s+i\varepsilon) = \int_{\theta_0}^{\theta_1}\tan\theta d\theta + i\int_{\theta_0}^{\theta_1}d\theta = \log\left|\frac{\sec\theta_1}{\sec\theta_0}\right| + i(\theta_1 - \theta_0)$. *As* $\varepsilon \downarrow 0$, $F(s+i\varepsilon)$ *goes to* $\frac{1-s}{s} + i\pi$ *by simple calculation.*

Similarly, $F(s-i\varepsilon)$ *goes to* $\frac{1-s}{s}-i\pi$ *as* $\varepsilon\downarrow 0$.

(c) *Consider* $F(-\varepsilon)=\int_0^1 \frac{1}{t+\varepsilon}dt=\log\frac{1+\varepsilon}{\varepsilon}$. *It goes to* ∞ *as* $\varepsilon\downarrow 0$.

Consider $F(1+\varepsilon)=\int_0^1 \frac{1}{t-1-\varepsilon}dt=\log\frac{\varepsilon}{1+\varepsilon}$. *It goes to* $-\infty$ *as* $\varepsilon\downarrow 0$. *Therefore, for* $s=0,1$, $\lim_{z\notin[0,1]\to s}F(z)$ *does not exists.*

Problem (2.4)**.**

First consider $p\equiv 0$. *We can easily see that* $\sup_{z\in C}|z^{-n}|=1$ *so desired value* ≤ 1.

Note that $|p(z)-z^{-n}|=|z^np(z)-1|$. *Thus,* $1=\frac{1}{2\pi i}\int_C\frac{z^np(z)-1}{z}dz\leq \sup_{z\in C}|z^np(z)-1|$.

Those leads the conclusion.

Problem (2.5)**.**

It is enough to show γ *and* μ *are path homotopic. Definte* $H(t,s)=(1-s)\gamma(t)+\frac{\gamma(t)}{|\gamma(t)|}s$. *Then* $H(t,1)=\mu(t)$ *and* $H(t,0)=\gamma(t)$ *by reparametrization. And* H *is continuous because* $\gamma(t)\neq 0$. *Therefore* H *is path homotopy between* γ *and* μ. *Since line integration is invariant under path homotopy, we get* $\int_\gamma F(\zeta)d\zeta=\int_\mu F(\zeta)d\zeta$.

Problem (3.1).

It suffices to show that $\int_\gamma f(z)dz = 0$ for rectangle γ whose edges are parallel to coordinate axes by Morera's theorem.

First, assume that γ intersects with $[0,1]$ only finitely many points. Let p be such point. Then p must be on (wlog) left edge of γ. Let $a+ib$, $a+ic$ be two vertices incident with left edge. ($b > c$) Let $\rho(t) = a + i(tc + (1-t)b)$. Consider $f \circ \rho$. It is continuous and equals to $\frac{\partial}{\partial t}F(\rho(t))$ except for $\gamma^{-1}(p)$ where F is antiderivative of f on $\mathbb{C} \setminus [0,1]$. Then lemma 2.3.1 says $f(\rho(t)) = \frac{\partial}{\partial t}F(\rho(t))$ even for $\gamma^{-1}(p)$. Therefore $\int_\rho f(z)dz = F(a+ic) - F(a+ib)$. By using this result, we can easily calculate $\int_\gamma f(z)dz = 0$.

Now, assume that (wlog) upper edge of γ intersects with $[0,1]$. Let $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ which are upper edge, left edge, bottom edge, and right edge respectively, parametrized like ρ of above, positive oriented. Consider φ made by shrinking side edges of γ so that distance between of upper edges of φ and γ less than δ, while bottom edge is fixed. Also note that δ is chosen so that $d(z_0, z_1) < \delta$ implies $d(f(z_0), f(z_1)) < \varepsilon$.

$$\begin{aligned}\left|\int_\gamma f(z)dz - \int_\varphi f(z)dz\right| \le & \left|\int_{\gamma_2 - \varphi_2} f(z)dz + \int_{\gamma_4 - \varphi_4} f(z)dz\right| \\ & + (\text{ length of } \gamma_1)\,\varepsilon\end{aligned}$$

And, second term of above goes to 0 as distance between φ_1 and γ_1 goes to 0 by continuity and result of first case. Actually $\int_\varphi f(z)dz = 0$ because φ does not intersect with $[0,1]$. Thus we have shown that $\int_\gamma f(z)dz = 0$.

By first, second case and Morera's thm, f is actually entire function.

Problem (3.2).

For $0 < r < 1$, $|f^{(n)}(0)| \le \frac{n!}{r^n}\frac{1}{1-r}$ by using Cauchy estimate. $r^n(1-r)$ is maximized when $r = \frac{n}{n+1}$. So, when $r = \frac{n}{n+1}$, we get best estimate of $|f^{(n)}(0)|$.

Problem (3.3).

(a) *Since K is compact subset of open set U, there is $r > 0$ such that for all $x \in K$, closure of $D(x,r)$ is in U. Then, $|f(z)|^2 \le \frac{1}{2\pi}\left|\int_{\partial D(z,r} \frac{f^2(w)}{w-z}dw\right| \le \frac{1}{2\pi}\int_0^{2\pi} |f^2(z+re^{i\theta})d\theta|$. By multiplying ρ both sides and integrating from 0 to r, we can get the following:*

$$
\begin{aligned}
\frac{r^2}{2}|f(z)|^2 &\leq \frac{1}{2\pi}\int_0^r\int_0^{2\pi}\rho|f^2(z+re^{i\theta})|d\theta d\rho \\
&= \frac{1}{2\pi}\int_{\overline{D}(z,r)}|f|^2dm \\
&= \frac{1}{2\pi}\int_U|f|^2dm
\end{aligned}
$$

for all $z \in K$, where m is lebesgue measure, using Holder's inequality and polar coordinate integration.

Therefore $C = \frac{1}{r\sqrt{\pi}}$

(b) *If f is identically zero, possible.*

Else if f is constant, then $\int_{\mathbb{C}}|f|dm = \infty$ since measure of complex plane is ∞.

Else, that is f is nonconstant entire function, then f must be unbounded. So, there is $\delta > 0$ such that $|f| \geq 1$ for all $|z| > \delta$. Then $\int_{\mathbb{C}}|f|dm \geq m(\{z : |z| > \delta\}) = \infty$.

Problem (3.4). (a) *Since $\frac{z}{e^z-1}$ is bounded near 0, it has removable singularity at 0. So we can regard it as holomorphic function. Note that $e^z - 1 = 0$ when z is integer multiple of $2\pi i$. So, given power series converges on unit disc. Now, multiply $e^z - 1$ both sides. Since $e^z - 1$ is entire and given power serires converges absolutely on $\bar{D}(0,r)$ where $0 < r < 1$, we can write $z = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n\sum_{n=1}^{\infty}\frac{1}{n!}z^n$. Since z is entire, coefficient of power series is unique. By comparing coefficients of both sides, we can get given recursion formula.*

$\lim_{z\to 0}\frac{z}{e^z-1} = 1 = B_0$. *From this, by simple calculation, $B_1 = \frac{-1}{2}$, $B_2 = \frac{1}{6}$, and $B_3 = 0$.*

Consider $-z = f(z) - f(-z) = \sum_{n=0}^{\infty}2\frac{B_{2n+1}}{(2n+1)!}z^{2n+1}$. This makes sense because f is holomorphic on unit disc. By comparing coefficient of this series, we can get $B_{2m+1} = 0$ for $m \geq 1$.

(b) *We alrealdy notice that $e^z - 1$ is zero when z is integer multiple of $2\pi i$. But $\lim_{z\to 2k\pi i}\frac{z}{e^z-1}$ is not bounded when $k \neq 0$. Therefore, $\frac{z}{e^z-1}$ is holomorphic on $D(0, 2\pi)$ and is not holomorphic outside of that disc. Since* *power seriese representation of holomorphic function at P has radius of convergence at least $d(P,U)$, we can say radius of convergence of the series is 2π.*

Problem (3.5)**.**

f' is holomorphic on unit disc. Let $r = \sup_{z\in K}|z|$. Since K is compact, $|f'| \leq M$ on K and r is positive but less than 1. Let $\gamma(t) = tz^n$ which connects origin and z^n. $|f(z^n) - f(0)| = \left|\int_\gamma f' dz\right| \leq M \sup_{z\in K}|z|^n = Mr^n$. Therefore, $|\sum_{n=1}^{\infty} f(z^n)| \leq \sum_{n=1}^{\infty} |f(z^n)| \leq \sum_{n=1}^{\infty} Mr^n < \infty$ because r is positive but less than 1.

Problem (4.1)**.**

Notice that f does not vanish on $\mathbb{C}\setminus\{0\}$. Therefore $g(z)=\frac{1}{f(z)}$ is holomorphic on $\mathbb{C}\setminus\{0\}$. Near 0, g is bounded since $\sqrt{|z|}$ goes to 0 as z goes to 0. This means g has removable singularity at 0 and therefore entire. But $g(z)\le\sqrt{|z|}$, so g must be constant by Cauchy integral formula.

Then f must be constant also, and this is contradiction. Therefore there is no such holomorphic function.

Problem (4.2)**.**

Let $g(z)=f\left(\frac{1}{z}\right)$. Then $g\to 0$ as $z\to 0$. Therefore g is entire. Also, $g(z)/z$ is entire since $\lim_{z\to 0} g(z)/z=g'(0)$ hence bounded near 0.

Now, consider given integral. Let $\zeta=e^{it}$ and $t=2\pi-s$. Then given integral is $\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(e^{-is})}{e^{-is}-z}ie^{-is}ds=\frac{1}{2\pi i}\int_0^{2\pi}\frac{g(e^{is})}{e^{is}-e^{2is}z}ie^{is}ds=\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{g(\zeta)}{\zeta z\left(\frac{1}{z}-w\right)}d\zeta$

Therefore given integral is equal to $\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{h(\zeta)}{\frac{1}{z}-\zeta}d\zeta$ where $h(\zeta)=\frac{g(\zeta)}{\zeta z}$. Thus, it is equal to $-g(1/z)=-f(z)$.

Problem (4.3)**.**

f maps $re^{i\theta}$ to $\sqrt{r}e^{i\left(\frac{\theta}{2}+k(z)\pi\right)}$ where $k(z)\in\mathbb{Z}$. To f be continuous, $k(z)$ must be all even or all odd.

First assume that $k(z)$ is all even. Then $f'(0)=\lim_{\mathbb{R}\ni h\to 0}\frac{f(h)}{h}=\lim_{\mathbb{R}\ni h\to 0}\frac{\sqrt{h}}{h}=\infty$, which is contradiction.

Similarly, if $k(z)$ is all odd, $f'(0)$ does not exist.

Therefore existence of such f leads $0\notin U$.

Let ι be identity function of U. Since $z\notin U$, ι does not vanish on U, hence $1/\iota$ is holomorphic on U. Since U is hsc, $1/\iota$ has holomorphic antiderivative φ.

Now consider the derivative of $\iota(z)e^{-\varphi(z)}$. Simple calculation leads that it is equal to 0. Hence $\iota(z)=ce^{\varphi(z)}$ for some constant c. Therefore $\iota(z)=e^{\psi(z)}$ for some holomorphic ψ on U.

Take $f=e^{\frac{1}{2}\psi}$. Then f satisfies what we want.

Problem (4.4)**.**

(a) *Let γ_R be the contour used in example 4.6.5.*

First, consider $\int_0^\infty \frac{1}{x^a(x+1)}dx$. *To calculate this, take* $f(z) = z^{-a}/(1+z)$ *where* $0 < arg(z) < 2\pi$. *By residue thm,* $2\pi i e^{-a\pi i} = \int_0^\infty \frac{1}{r^a(r+1)}dr\left(1 - e^{-2a\pi i}\right)$. *Therefore* $\int_0^\infty \frac{1}{x^a(x+1)}dx = \pi\csc(\pi a)$.

Now, $\int_{\gamma_R} \frac{\log z}{z^a(1+z)}dz = 2\pi i e^{-a\pi i}\pi i$ *by residue thm. But as* $R \to \infty$, *that integral goes to* $(1 - e^{-2a\pi i})\int_0^\infty \frac{\log r}{r^a(r+1)}dr - e^{-2a\pi i}\int_0^\infty \frac{2\pi i \log r}{r^a(r+1)}dr$.

By simple calculation, the value we want is equal to $\frac{i\pi^2}{\sin(\pi a)} + \frac{\pi^2 e^{-a\pi i}}{\sin^2(\pi a)} = \frac{\pi^2\cos(\pi a)}{\sin^2(\pi a)}$.

(b) *Consider* $f(z) = \frac{\pi\cot(\pi z)}{(z+\alpha)^2}$ *and* Γ_n = *square centered at origin, each edges is parallel to real or imaginary axis, length of edge is* $2n+1$.

Then $\int_{\Gamma_n} f(z)dz$ *goes to 0 as* $n \to \infty$ *by considering modulus of* $f(z)$, *and index of* Γ_n *at each singularites is 1, and residues are* $\frac{1}{(k+\alpha)^2}$ *at* $z = k$ *and* $-\frac{\pi^2}{\sin^2(\pi\alpha)^2}$ *at* $z = -\alpha$.

Above calculation leads the conclusion.

Problem (4.5)**.**

Note that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *is holomorphic iff* f *is meromorphic on* $\hat{\mathbb{C}}$.

(a) *First consider 'if' part. Let* f *be rational function. We already knows that rational function is meromorphic on entire complex plane. So, we need to show that rational function is meromorphic at* ∞.

Let $f(z) = \frac{(z-Q_1)^{m_1}\cdots(z-Q_l)^{m_l}}{(z-P_1)^{n_1}\cdots(z-P_k)^{n_k}}$. *Since* f *has finitely many pole in complex plane, we can choose* $M > 0$ *so that* f *has no pole on* $\{z : |z| > M\}$. *For* $0 < |w| < \frac{1}{M}$, *consider* $g(w) = f(1/w)$. *Then* g *is holomorphic.*

Let $\sum_i n_i = N$ *and* $\sum_j m_j = M$. *If* $M = N$, $g \to 1$ *as* $z \to 0$. *If* $M > N$, $g \to 0$ *as* $z \to 0$. *If* $M < N$, $g \to \infty$ *if* $z \to 0$. *Hence* g *is meromorphic near* 0, *which means that* f *is meromorphic at* ∞.

Second, consider 'only if' part. Either f *has a pole or removable singularity at* ∞, f *has finitely many poles in complex plane. So* $f(z)(z-P_1)^{n_1}\cdots(z-P_k)^{n_k} = F(z)$ *is entire where* n_i *is order of pole* P_i.

Consider $F(1/z) = g(z)$ *for* $z \neq 0$. *As* $z \to 0$, $g \to \infty$ *or* α *for some* $\alpha \in \mathbb{C}$ *by simple calculation. Therefore* F *has a pole or removable singularity at* ∞.

If F has removable singularity at ∞, F must be bounded, hence constant by Liouville's thm.

If F has a pole at ∞, F must be polynomial since its modulus diverges.

In both cases, F must be rational function.

(b) *Note that $z \mapsto \frac{az+b}{cz+d}$ for $ad-bc \neq 0$ is biholomorphic function of Riemann sphere. Also note that biholomorphic function of $\mathbb{C}$ must have a form of $\alpha z + \beta$ for $\alpha \neq 0$ by fundamental thm of algebra.*

Now consider biholomorphic f on Riemann sphere. Let $f(\infty) = b$ and $\varphi_b(z) = \frac{-\bar{b}-1}{z-b}$. Then $\varphi_b \circ f$ is biholomorphic function of Riemann sphere, which maps $\infty \to \infty$. Therefore $\varphi_b \circ f$ is biholomophic function of complex plane hence $\varphi_b(f(z)) = \alpha z + \beta$. Then $f(z) = \frac{-b\alpha z - b\beta + 1}{-\alpha z - \beta - \bar{b}}$, which is linear frational transformation.