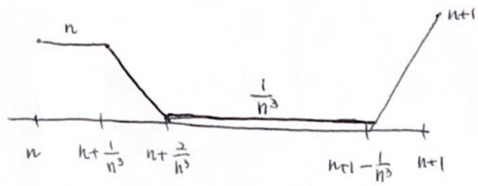
MAS 441 HW 2.

#6. a)



$= f|_{[n,n+1]}$ $(n \geq 1)$

$$\to \int_n^{n+1} f(x)\,dx \leq \frac{1}{n^2} + \frac{1}{n^3} \times n \times \frac{1}{2} + \left(1 - \frac{3}{n^3}\right) \times \frac{1}{n^3} + \frac{1}{n^3} \times (n+1) \times \frac{1}{2}$$

$$= \frac{2n+1}{2n^3} + \frac{1}{n^2} + \frac{1}{n^3} - \frac{3}{n^6}.$$

reflect f to y-axis.

Define $f|_{(-1,1)} = 1$.

$$\Rightarrow \int_{\mathbb{R}} f \leq 2 + 2\left(\sum_{n=1}^{\infty} \left(\frac{4n+2}{2n^3} - \frac{3}{n^6}\right)\right) < \infty.$$

But clearly $\limsup_{x \to \infty} f(x) = \infty$. ■

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b) Assume $|f| \not\to 0$. as $|x| \to \infty$.

For some $\varepsilon > 0$, $\forall M > 0$ $\exists |x_M| \geq M$: $|f(x_M)| > \varepsilon$.

Since f is uniformly continuous, $|f|$ is also unif. continuous.

$\exists \delta > 0$ s.t. $||\cancel{f}(x)| - |\cancel{f}(y)|| < \delta \Rightarrow ||f(x)| - |f(y)|| < \frac{\varepsilon}{2}$.

$\Rightarrow$ on δ-ball of x_M, $|f| > \frac{\varepsilon}{2}$.

$$\therefore \int |f| \geq \sum_{k=1}^{\infty} \int_{b(x_{M_k}, \delta)} |f| = \infty.$$

(* $M_1 = 1$, $M_k = x_{M_{k-1}} + 2\delta$.

then δ-ball of x_{M_k} : disjoint, $x_{M_k} \to \infty$).

■

#8. ~~Choose~~ Let $\varepsilon > 0$ be given. Choose $\delta > 0$ s.t.

$m(E) < \delta \Rightarrow \int_E |f| < \varepsilon$. (possible by integrability of f).

$$|x-y| < \delta \Rightarrow |F(x) - F(y)| \le \int_{[\min(x,y), \max(x,y)]} |f| < \varepsilon$$

because $m([\min(x,y), \max(x,y)]) = |x-y| < \delta$. ∎

#9. Consider $\alpha \cdot 1_{E_\alpha} \le f \cdot 1_{E_\alpha} \le f$.

$$\Rightarrow \int \alpha 1_{E_\alpha} \le \int f$$

$$\int \alpha 1_{E_\alpha} = \alpha\, m(E_\alpha) \qquad \therefore\ m(E_\alpha) \le \frac{1}{\alpha} \int f.$$ ∎

#11. Let $E_n = \{ f < -\frac{1}{n} \}$. Then $E_n \uparrow \{ f < 0 \}$.

$$\therefore 0 \le \int_{E_n} f \le \int_{E_n} \left(-\frac{1}{n}\right) = -\frac{1}{n} m(E_n) \Rightarrow -\frac{1}{n} m(E_n) \ge 0 \Rightarrow m(E_n) = 0.$$

$$\therefore m(f<0) = \lim_{n\to\infty} m(E_n) = 0.$$

$\therefore \boxed{f \ge 0 \text{ a.e.}}$ ∎

#18. Integrability of $|f(x) - f(y)|$ implies integrability of $y \mapsto \int_0^1 |f(x)-f(y)|\,dx$

$\Rightarrow \int_0^1 |f(x) - f(y)|\,dx$ is finite for a.e. $y \in [0,1]$. Choose y.

$$\Rightarrow \int_0^1 |f(x)|\,dx \le \int_0^1 |f(x) - f(y)| + |f(y)|\,dx = \underbrace{|f(y)|}_{\text{finite}} + \underbrace{\int_0^1 |f(x) - f(y)|\,dx}_{\text{finite}}.$$

$\therefore$ f : integrable on $[0,1]$. ∎

#19. Define $g: \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x,\alpha) = 1_{E_\alpha}(x) \cdot 1_{(0,\infty)}(\alpha).$$

g: nonnegative $\Rightarrow$ Tonelli's thm is applicable.

$$\Rightarrow \int_{\mathbb{R}^d \times \mathbb{R}} g = \int_{\mathbb{R}^d}\int_{\mathbb{R}} g_x \, d\alpha \, dx = \underline{\int_{\mathbb{R}^d} |f(x)| \, dx}.$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}^d} g^\alpha \, dx \, d\alpha = \underline{\int_{(0,\infty)} m(E_\alpha) \, d\alpha} \qquad (=).$$

Because $g_x(\alpha) = 1_{E_\alpha}(x)\, 1_{(0,\infty)}(\alpha) = 1_{\{0<\alpha<|f(x)|\}}(\alpha)$.

$$\Rightarrow \int_{\mathbb{R}} g_x(\alpha)\, d\alpha = |f(x)|.$$

$$g^\alpha_\alpha(x) = 1_{E_\alpha}(x)\, 1_{(0,\infty)}(\alpha) = 1_{\{0<\alpha<|f(x)|\}}(x)$$

$$\Rightarrow \int_{\mathbb{R}^d} g^\alpha(x)\, dx = m(|f| > \alpha).$$

Section 6 Problem #3. $(\mu = m)$

$E_k = \{ |f_k - f| > \varepsilon \}$.

By Chebyshev's Inequality, $\mu(E_k) \le \frac{1}{\varepsilon} \int |f_k - f| \, d\mu$.

$\therefore \mu(E_k) \to 0$ as $k \to \infty$ $(\because f_k \to f$ in $L_1)$

$\therefore f_k \to f$ in L^1 $\Rightarrow$ $f_k \xrightarrow{\mu} f$

Consider $f_k = k \cdot 1_{(0,\frac{1}{k})}$. $\int f_k = 1$

But $\mu(|f_k| > \varepsilon) \le \frac{1}{k} \to 0$ as $k \to \infty$.

$\therefore f_k \to 0$ in measure.

but $f_k \not\to 0$ in L^1

$(\because \int f_k = 1$ but $\int 0 = 0)$

$\therefore$ Converse is not true.