mas540 exercises

Jaemin Oh

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Exercise (1.4)**.**

(a) *Let $I = [0,1]$. Then $I \backslash \hat{C} = \bigcup_{n=1}^{\infty} \hat{C}_n^c$ where $\hat{C}_n$ is n-th stage of constructing Fat Cantor set. Thus,*

$$m(I \setminus \hat{C}) = m(I) - m(\hat{C}) = 1 - m(\hat{C}) = \lim_{n\to\infty} m(\hat{C}_n^c) = \sum_{n=1}^{\infty} 2^{n-1} l_n$$

because $\hat{C}_n^c \uparrow \bigcup_{n=1}^{\infty} \hat{C}_n^c$ and $\hat{C}$ is closed hence measurable. Therfore $m(\hat{C}) = 1 - \sum_{n=1}^{\infty} 2^{n-1} l_n > 0$.

(b) *$\hat{C}_k$ consists of 2^k closed intervals whose length are $(1 - \sum_{n=1}^{k} 2^{n-1} l_n)/2^k$. Let $x \in \hat{C}$. Then $x \in \hat{C}_k$. So we can find $x_k \in I_k$ such that*

$$|x - x_k| \leq \left(1 - \sum_{n=1}^{k} 2^{n-1} l_n\right) / 2^k + \varepsilon_k l_k$$

for some $0 < \varepsilon_k < 1$. As $k \to \infty$, $|x - x_k| \to 0$ since $l_k \to 0$.

(c) *The result of b tells us that every point of $\hat{C}$ is a limit point of I. And we also know that $\hat{C}$ is closed. Hence $\hat{C}$ is a perfect set.*

Let $(a, b) \subset \hat{C}$ and $a < c < d < b$. For large k, $l_k < d - c$ since $l_k \to 0$. Then, for $\hat{C}_k$, c and d must lie in different intervals of $\hat{C}_k$. So there is $e \notin \hat{C}_k$ such that $c < e < d$. Then $[c, d]$ does not belong to $\hat{C}_k$ which is a contradiction. So $\hat{C}$ is totally disconnected.

(d) *It is well known fact that a nonempty perfect set is uncountable. We had learned it in an introductory analysis course and topology course.*

$\square$

Exercise (1.7)**.**

First, we will show that if O is open, then δO is also open. Let $\delta x \in \delta O$. Then $x \in O$. By openness, there is $r > 0$ such that $Q_r(x) \subset O$ where $Q_r(x)$ is a cube whose side length is r and centered at x. Thus $\delta Q_r(x) \subset \delta O$ and $\delta Q_r(x)$ contains δx. But a collection of all open rectangles forms a basis of Euclidean space. So δO is an open set.

Next, let a set E and a positive number ε be given. Choose $O \supset E$ such that $m_*(O \setminus E) < \varepsilon/(\delta_1 \cdots \delta_d)$. Then, there is an union of cube $\bigcup_{j=1}^{\infty} Q_j \supset O \setminus E$ such that $\sum_{j=1}^{\infty} m(Q_j) < \varepsilon/(\delta_1 \cdots \delta_d)$. Then,

$$m_*(\delta O \setminus \delta E) = m_*(\delta(O \setminus E)) \leq m_*(\bigcup_{j=1}^{\infty} \delta Q_j) \leq \sum_{j=1}^{\infty} m(\delta Q_j) < \varepsilon.$$

Thus δE is measurable.

Now let $E \subset \bigcup_{j=1}^{\infty} Q_j$. Then $\delta E \subset \bigcup \delta Q_j$, so $m(\delta E) \leq \delta_1 \cdots \delta_d \sum_{j=1}^{\infty} m(Q_j)$. Since $\bigcup_{j=1}^{\infty}$ is arbitrary, we get

$$m(\delta E) \leq \delta_1 \cdots \delta_d m(E).$$

Now let $\delta E \subset \bigcup_{j=1}^{\infty} Q'_j$. Then $E \subset \bigcup_{j=1}^{\infty} 1/\delta Q'_j$. So $m(E) \leq \sum_{j=1}^{\infty} m(Q'_j)/(\delta_1 \cdots \delta_d)$. Since $\bigcup_{j=1}^{\infty} Q'_j$ is arbitary, we get

$$m(E) \leq \frac{m(\delta E)}{\delta_1 \cdots \delta_d}$$

and this finishes the proof.

□

Exercise (1.24)**.**

Let s_n be enumeration of $\mathbb{Q} \cap [-1,1]$ and t_n be enumeration of $\mathbb{Q} \cap [-1,1]^c$. When $n = m^2$, put $r_n = t_m$. When $n \in \big(m^2, (m+1)^2\big)$, put $r_n = s_{n-m}$. Then r_n is an enumeration of $\mathbb{Q}$. Also, we get

$$\begin{aligned} m\left(\bigcup_{n=1}^{\infty} (r_n - 1/n, r_n + 1/n)\right) &\leq \sum_{m=1}^{\infty} 2/m^2 + m\left(\bigcup_{n \neq m^2} (r_n - 1/n, r_n + 1/n)\right) \\ &\leq \sum_{m=1}^{\infty} 2/m^2 + 2 + 1 < \infty. \end{aligned}$$

Therefore, finiteness implies nonemptyness of the complement, since the Lebesgue measure of complement is positive.

□

Exercise (1.35)**.**

First, let's briefly check the idea of constructing φ. Construction can be done by defining a sequence of functions, say φ_n. Put $\varphi_n(0) = 0$ and $\varphi_n(1) = 1$. Let C_{ji} be the i-th stage of constructing C_j. Then φ_i maps the discarded set of stage i to the discarded set of stage i, sequentially, and linearly(positive). We can extend φ_i by assigning value on C_{1i} using linearity and monotonicity. This sequence of functions converges uniformly, thus φ is continuous. The other properties of φ can be checked by this construction.

Let $\mathcal{N} \subset C_1$ be a non-measurable set. Then $\varphi(\mathcal{N}) \subset C_2$ so $\varphi(\mathcal{N})$ is measurable by completeness. If $\varphi(\mathcal{N})$ is a Borel set, then by continuity, $\varphi^{-1}(\varphi(\mathcal{N})) = \mathcal{N}$ must be a Borel set, which is a contradiction. So there is a Lebesgue measurable set which is not Borel measurable.

Since $\varphi(\mathcal{N})$ is measurable, $f = 1_{\varphi(\mathcal{N})}$ is a measurable map. Then $f \circ \varphi(x) = 1_{\mathcal{N}}(x)$ is non-measurable map.

□

Problem (1.4)**.**

(a) *A_ε is clearly bounded, so it is enough to show that the complement is open. Let $c \notin A_\varepsilon$. Then $osc(f,c) < \varepsilon$, so for some $r > 0$, $osc(f,c,r) < \varepsilon$. Choose any $d \in I(c,r)$. We can choose $r^* > 0$ so that $I(d,r^*) \subset I(c,r)$. Then*

$$osc(f,d,r^*) \leq osc(f,c,r) < \varepsilon$$

so $osc(f,d) < \varepsilon$, which says $I(c,r) \subset J \setminus A_\varepsilon$. Therefore $J \setminus A_\varepsilon$ is open in J, hence A_ε is compact.

(b) *Let D_f be a set of all discontinuities of f. Then for any $\varepsilon > 0$, $A_\varepsilon \subset D_f$. So $m(A_\varepsilon) \leq m(D_f) = 0$. By the definition of Lebesgue measure, there is countably many open intervals which cover A_ε and have sum of length $\leq \varepsilon$. Using compactness, we can choose finite subcover, call them by $(a_i, b_i)_{i=1}^k$ where $a_i < a_{i+1}$. After discarding all of subcovers from J, we get compact subset of J, say J'. For each $c \in J'$, we can choose r_c such that $osc(f,c,2r_c) < \varepsilon$. Again, using compactness, we can choose finitely many c's. Then finitely many closed intervals $[c - r_c, c + r_c]$ have finite intersections. By taking these endpoints(contain a_i, b_i's) as endpoints of our partition(if necessary, consider a refinement), we get*

$$U(f,P) - L(f,P) \leq 2M\varepsilon + m(J)\varepsilon$$

where M is bound of f. The first term of estimate comes from (a_i, b_i)'s and the second term comes from J'.

(c) *Since $D_f \subset \bigcup_{n=1}^\infty A_{1/n}$, so $m(A_{1/n}) = 0$ leads the conclusion. Assume not, i.e. $m(A_{1/n}) > \varepsilon$. Take partition P such that $U(f,P) - L(f,P) < \varepsilon/n$. Let $[a,b]$ be interval of P whose interior intersects to $A_{1/n}$. Then*

$$\sup_{x,y \in [a,b]} |f(x) - f(y)| \geq \frac{1}{n}.$$

But $m(A_{1/n}) > \varepsilon$. So

$$\begin{aligned}
&\sum_{[a,b]\cap A_{1/n} \neq \emptyset} \left[\sup_{x \in [a,b]} f(x) - \inf_{y \in [a,b]} f(y) \right] m\left(A_{1/n} \cap [a,b]\right) \\
&= \sum_{[a,b]\cap A_{1/n} \neq \emptyset} \sup_{x,y \in [a,b]} |f(x) - f(y)|\, m\left(A_{1/n} \cap [a,b]\right) \\
&\geq \frac{\varepsilon}{n} \\
&> U(f,P) - L(f,P)
\end{aligned}$$

which is a contradiction.

□

Exercise (2.2).

Let $\varepsilon > 0$. Choose $g \in C_c(\mathbb{R}^d)$ such that $\|f-g\|_1 < \varepsilon$. Let the domain of g is contained in $B_r(0)$. For $x \in B_r(0)$,

$$|x - \delta x| = |1-\delta||x| \le r|1-\delta| < \xi$$

if $|1-\delta|$ is small. Let $\xi > 0$ be a number which satisfies $|x-y| < \xi \Rightarrow |g(x) - g(y)| < \varepsilon$. Then, for enoughly small $|1-\delta|$, we get $|x-\delta x| < \xi \Rightarrow |g(\delta x) - g(x)| < \varepsilon$. Thus we get $\|g_\delta - g\| \le \varepsilon m(B_r(0))$, $\|f-g\| < \varepsilon$, $\|f_\delta - g_\delta\| < K\varepsilon$. Therefore

$$\|f - f_\delta\| \le \|f-g\| + \|g - g_\delta\| + \|g_\delta - f_\delta\| \le (m(B_r(0)) + 1 + K)\,\varepsilon.$$

This says as $\delta \to 1$, $\|f_\delta - f\| \to 0$.

□

Exercise (2.6).

(a) *Let $n \in \mathbb{N}$. On $[n, n+1]$, define*

$$f(x) = \begin{cases} n & \text{if } n \le x \le n + 1/n^3 \\ 1/n^3 & \text{if } n + 2/n^3 \le x \le n+1-1/n^3 \\ \text{linear} & \text{otherwise.} \end{cases}$$

Then

$$\int_{[n,n+1]} f(x)dx \le \frac{1}{n^2} + \frac{1}{n^3}n\frac{1}{2} + \left(1 - \frac{3}{n^3}\right)\frac{1}{n^3} + \frac{1}{n^3}(n+1)\frac{1}{2} = \frac{2n+3}{2n^3} + \frac{1}{n^2} - \frac{3}{n^6}.$$

Now, reflect f to the y-axis. Define f on $(-1,1)$ by 1. Then

$$\int_{\mathbb{R}} f dm \le 2 + 2\left(\sum_{n\ge 1}\left(\frac{4n+2}{2n^3} - \frac{3}{n^6}\right)\right) < \infty.$$

But clearly $\limsup_{x\to\infty} f(x) = \infty$.

(b) *By same manipulation used in #2.24.b, the result follows. See after If φ does not vanish $\sim$.*

□

Exercise (2.19).

Let $g(x, \alpha) = 1_{E_\alpha}(x)1_{(0,\infty)}(\alpha)$. Since g is nonnegative, Tonelli's theorem can be applied.

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}} g dm &= \int_{\mathbb{R}^d}\int_{\mathbb{R}} g_x d\alpha dx = \int_{\mathbb{R}^d} |f(x)| dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^d} g^\alpha dx d\alpha = \int_{(0,\infty)} m(E_\alpha) d\alpha.\end{aligned}$$

Because $g_x(\alpha) = 1_{(0<\alpha<|f(x)|)}(\alpha)$ and $g^\alpha(x) = 1_{(0<\alpha<|f(x)|)}(x)$. □

Exercise (2.24)**.**
Let $\varphi = f * g$.

(a) *Choose* $h > 0$ *small so that* $\|f_h - f\|_1 < \varepsilon$. *Then*

$$|\varphi(x+h) - \varphi(x)| \leq \int |f(x+h-y) - f(x-y)||g(y)|dy \leq B\|f_h - f\|_1 < B\varepsilon.$$

Thus φ *is uniformly continuous.*

(b) *By Tonelli's theorem,*

$$\|\varphi\|_1 \leq \iint |f(x-y)||g(y)|dydx \leq \|f\|_1 \int |g(y)|dy = \|f\|_1\|g\|_1 < \infty.$$

So $\varphi \in L^1$. *Note that* φ *is uniformly continuous by (a).*

If φ *does not vanish at infinity, then there exists* $\varepsilon > 0$ *such that for all* $M > 0$, *there is* $|x_M| \geq M :\ |\varphi(x_M)| > 2\varepsilon$. *By uniform continuity, there is* $\delta > 0$ *such that* $|x - y| < \delta \Rightarrow |\varphi(x) - \varphi(y)| < \varepsilon$. *We can get strictly increasing sequence* $y_i \in \{x_M : M > 0\}$ *such that* $B_\delta(y_i) \cap B_\delta(y_j) = \emptyset$ *whenever* $i \neq j$.

Note that for $x \in B_\delta(y_i)$, $|\varphi(x)| > \varepsilon$. *Thus*

$$\int |\varphi|dx \geq \sum_{i=1}^{\infty} \varepsilon m(B_\delta(y_i)) = \infty.$$

But the above contradicts to $\varphi \in L^1$.

□

Problem (2.3)**.**
Let $E_k = \{|f_k - f| > \varepsilon\}$. *By the Markov inequality,*

$$m(E_k) \leq \frac{1}{\varepsilon}\int |f_k - f|dm.$$

Since $f_k \to f$ *in* L^1, *we get* $m(E_k) \to 0$. *Thus* L^1 *convergence implies the convergence in measure.*

For counterexample, consider $f_k = k1_{(0,1/k)}$. *Then* $\int f_k dm = 1$. *But* $m(|f_k| > \varepsilon) \leq 1/k$ *so* $f_k \to 0$ *in measure. But, as we seen,* f_k *does not converge to* 0 *in* L^1. *Thus the converse of the previous result is not true.* □

Exercise (3.2).

Let $\{L_\delta\}$ *be any approximation to the identity. Then, by triangle inequality,* $\{K_\delta + L_\delta\}$ *is also approximation to the identity because of the third condition. Therefore*

$$f * (K_\delta + L_\delta)(x) \to f(x) \text{ a.e. } x$$

as $\delta \to 0$ *by theorem 2.1.*

But,

$$\begin{aligned} f * (K_\delta + L_\delta)(x) &= \int f(x-y)(K_\delta(y) + L_\delta(y))dy \\ &= f * K_\delta(x) + f * L_\delta(x). \end{aligned}$$

Since $f * L_\delta(x) \to f(x)$ *for a.e.* x*,* $f * K_\delta(x) \to 0$ *for a.e.* x *necessarily.*

□

Exercise (3.5).

(a) *By the change of variable formula(*$\log x = t$*),*

$$\begin{aligned} \int_{\mathbb{R}} |f(x)|dx &= \int_{-1/2}^{1/2} f(x)dx \\ &= \int_{-\infty}^{-\log 2} \frac{1}{t^2} dt = \frac{1}{\log 2} < \infty. \end{aligned}$$

(b) *Let* $\varepsilon > 0$*. Then*

$$\begin{aligned} f^*(x) &\geq \frac{1}{2|x| + 2\varepsilon} \int_{-|x|-\varepsilon}^{|x|+\varepsilon} \frac{dt}{t(\log t)^2} \\ &= \frac{1}{|x| + \varepsilon} \int_0^{|x|+\varepsilon} \frac{dt}{t(\log t)^2} \\ &= \frac{1}{-\log(|x| + \varepsilon)(|x| + \varepsilon)}. \end{aligned}$$

Since $\varepsilon > 0$ *is arbitrary, by taking* $\varepsilon \downarrow 0$*, we obtain*

$$f^*(x) \geq \frac{1}{|x| \log \frac{1}{|x|}}.$$

But $1/(-|x| \log |x|)$ *is clearly non-locally integrable function. This is by integrating on the interval containing* 0 *and the change of variable formula, used above.*

□

Exercise (3.12).

By chain rule, F' exists for all $x \neq 0$. But,

$$\lim_{h\to 0} \frac{F(h)}{h} = \lim_{h\to 0} h\sin(1/h^2) = 0$$

Thus F' exists for all $x \in \mathbb{R}$.

For $1/\sqrt{2n\pi + \pi/6} \leq x \leq 1/\sqrt{2n\pi - \pi/6}$, $2n\pi - \pi/6 \leq 1/x^2 \leq 2n\pi + \pi/6$, thus $\cos 1/x^2 \geq \sqrt{3}/2$ and $\left|\sin 1/x^2\right| \leq 1/2$. So $|F'| \geq 2/x\cos 1/x^2 - 2x\left|\sin 1/x^2\right| \geq \sqrt{3}\sqrt{2n\pi - \pi/6} - 1/\sqrt{2n\pi - \pi/6}$.

By using the above,

$$\begin{aligned}
\int_0^1 |F'|dm &\geq \sum_{n=1}^{\infty} \left(1/\sqrt{2n\pi - \pi/6} - 1/\sqrt{2n\pi + \pi/6}\right)\left(\sqrt{3}\sqrt{2n\pi - \pi/6} - 1/\sqrt{2n\pi - \pi/6}\right) \\
&= \sum_{n=1}^{\infty} \frac{\pi/\sqrt{3}}{\sqrt{2n\pi + \pi/6}\left(\sqrt{2n\pi + \pi/6} + \sqrt{2n\pi - \pi/6}\right)} \\
&- \sum_{n=1}^{\infty} \frac{\pi/3}{(2n\pi - \pi/6)\sqrt{2n\pi + \pi/6}\left(\sqrt{2n\pi + \pi/6} + \sqrt{2n\pi - \pi/6}\right)}
\end{aligned}$$

where the last sum converges and previous one diverges (by p-test.) Thus F' is non-integrable.

□

Exercise (3.23)**.**

(a) Follow the hint.

$$(D^+G_\varepsilon)(x_0) = (D^+F)(x_0) + \varepsilon > 0.$$

This means, for sufficiently small $h > 0$,

$$G_\varepsilon(x_0 + h) > G_\varepsilon(x_0) \geq 0.$$

This contradicts to our choice of x_0.

(b) Use the Mean value theorem.

□

Exercise (3.25)**.**

(a) Let f be the function given in the hint. Note that all of points in any open set O is a point of Lebesgue density. This is because, we can only consider small ball B_x contained in O. Thus

$$\liminf \frac{m(O_n \cap B)}{m(B)} = 1$$

for all $x \in E$. Therefore

$$\begin{aligned}\liminf \frac{1}{m(B)} \int_B f dm &= \liminf \sum_{n \geq 1} \frac{m(O_n \cap B)}{m(B)} \\ &\geq \sum_{n \geq 1} \liminf \frac{m(O_n \cap B)}{m(B)} = \sum_{n \geq 1} 1 = \infty.\end{aligned}$$

(b) *Let $F(x) = \int_{-\infty}^{x} f(t)dt$ where f is the function found in a. Then F satisfies the given condition.*

□

Exercise (3.32).

Assume the Lipschitz condition. Take $\delta = \varepsilon/M$ when $\varepsilon > 0$ is given. For (a_i, b_i) such that $\sum_i (b_i - a_i) < \delta$, then $\sum_i |f(b_i) - f(a_i)| \leq M \sum_i (b_i - a_i) < M\delta = \varepsilon$. Thus f is absolutely continuous. So f' exists a.e. Now consider the following:

$$|f'(x)| = \lim_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|} \leq M$$

Thus $|f'| \leq M$ a.e. x.

For the other direction, without loss of generality, assume $x \leq y$. Since f is absolutely continuous, f' exists a.e, and $\int_x^y f' dm = f(y) - f(x)$. Thus, $|f(x) - f(y)| = \left|\int_x^y f' dm\right| \leq \int_x^y |f'| dm \leq (y - x)M = |x - y|M$.

□

Problem (3.5).

First, assume that $F' \geq 0$ a.e. Let E be the set, $F'(x) < 0$. According to exercise 25, we can find Φ which is increasing, absolutely continuous, and $D_{\pm}\Phi(x) = \infty$ for all $x \in E$. Note that $\infty = D_{+}\Phi(x) \leq D^{+}\Phi(x)$. Now, for $\delta > 0$, consider $F + \delta\Phi$. On E, $D^{+}(F + \delta\Phi) = \infty > 0$. On E^c, $D^{+}(F + \delta\Phi) = F' + \delta\Phi' \geq 0$. Therefore, by exercise 23, $F + \delta\Phi$ is an increasing function. So

$$F(x) - F(a) + \delta(\Phi(x) - \Phi(a)) \geq 0.$$

Since $\delta > 0$ is arbitrary, we can assert $F(x) \geq F(a)$ whenever $x \geq a$.

Now we'll solve the problem using the above. Let $G(x) = \int_a^x F' dm$. Then $G'(x) = F'(x)$ a.e. by Lebesgue differentiation theorem. Thus $G'(x) - F'(x) \geq 0$ a.e. Then, the above implies $G(x) - G(a) - F(x) + F(a) \geq 0$. Since we can say that $G'(x) - F'(x) \leq 0$ a.e. also, we obtain $G(x) - G(a) - F(x) + F(a) \leq 0$. But $G(a) = 0$. Therefore $F(x) - F(a) = G(x) = \int_a^x F' dm$. Since F' is integrable, $\nu(B) = \int_B F' dm$ is absolutely continuous with respect to m so F is absolutely continuous.

□

Exercise (4.4)**.**

First, let's show the completeness. Let $\{f_n\} \subset l^2(\mathbb{Z})$ *be a Cauchy sequence. Choose* n_k *such that* $\|f_{n_{k+1}} - f_{n_k}\| < 2^{-k+1}$*. Define* $f = f_{n_1} + \sum_{k=1}^{\infty}(f_{n_{k+1}} - f_{n_k})$ *and* $g = |f_{n_1}| + \sum_{k=1}^{\infty}|f_{n_{k+1}} - f_{n_k}|$*. Note that* $\|g\| \leq \|f_{n_1}\| + \sum_{k=1}^{\infty}\|f_{n_{k+1}} - f_{n_k}\| \leq \|f_{n_1}\| + 2 < \infty$*. Because, for each* $i \in \mathbb{Z}$*,* $|f(i)| \leq g(i) \leq \|g\| < \infty$*, we can observe that* $f(i)$ *is absolutely converges. Thus* f *is well defined function, also in* l^2 *(*$\because \|f\| \leq \|g\| < \infty$*). Now let's show that* $f_{n_k} \to f$*.*

$$\|f - f_{n_k}\| \leq \sum_{m=k}^{\infty} \|f_{n_{m+1}} - f_{n_m}\| \leq 2^{-k}.$$

So $f_{n_k} \to f$ *as* $k \to \infty$ *in* l^2*. Therefore* $l^2(\mathbb{Z})$ *is complete.*

Now let's show the separability. Let $\mathcal{B}$ *be the set of all rational sequence in* $l^2(\mathbb{Z})$*. Clearly, it is nonempty since the zero sequence is in* $\mathcal{B}$*. Let* $f \in l^2$*. Fix* $\varepsilon > 0$*. For each* $i \in \mathbb{Z}$*, choose* q_i *such that*

$$|f(i) - q_i|^2 < \frac{\varepsilon^2}{2^{|i|}}.$$

Let $q : i \mapsto q_i$*. Then* $\|q\| \leq \|q - f\| + \|f\|$*, where*

$$\begin{aligned} \|q - f\| &= \left(\sum_{-\infty}^{\infty} |q_i - f(i)|^2 \right)^{1/2} \\ &\leq \left(\sum_{-\infty}^{\infty} \frac{\varepsilon^2}{2^{|i|}} \right)^{1/2} \\ &= \sqrt{3}\varepsilon. \end{aligned}$$

Since $\|f\| < \infty$*, we can see that* $q \in l^2$ *and* $\|f - q\| \leq \sqrt{3}\varepsilon$*. Note that* $q \in l^2$ *implies* $q \in \mathcal{B}$*. So* $\mathcal{B}$ *is dense in* l^2*, and clearly* $\mathcal{B}$ *is countable set.*

□

Exercise (4.15)**.**

Let $\{e_1, e_2, \cdots, e_n\}$ *be an orthonormal basis of* $\mathcal{H}_1$*. Let* $f \in \mathcal{H}_1$*,* $\|f\| = 1$*. Then* $f = \sum_{i=1}^{n} c_i e_i$*, where* $\sqrt{\sum |c_i|^2} = 1$*. Then*

$$\begin{aligned} \|Tf\| &= \|c_1 Te_1 + \cdots c_n Te_n\| \\ &\leq |c_1|\|Te_1\| + \cdots + |c_n|\|Te_n\| \\ &\leq \sum_{i=1}^{n} |c_i| M \\ &\leq M \left(\sum_{i=1}^{n} |c_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} 1 \right)^{1/2} \\ &= \sqrt{n} M < \infty \end{aligned}$$

where $M = \max_{1\le i\le n} \|Te_i\|$. *Since* n *is fixed, the above says that* T *is bounded operator.*

$\square$

Exercise (4.22)**.**

(a) *Polarization identity:*

$$(f,g) = \frac{1}{4}\left[\|f+g\|^2 - \|f-g\|^2 + i\|f+ig\|^2 - i\|f-ig\|^2\right].$$

This can be shown by using the hint. (Actually, we have seen it in the lecture.)

Put Tf, Tg *in the place of* f, g *respectively. Since* T *is linear and* $\|Tf\| = \|f\|$, *we can easily see that* $(f,g) = (Tf, Tg)$.

Now fix $g \in \mathcal{H}$. *Then* $(f, T^*Tg) = (Tf, Tg)$ *by the definition of adjoint, and* $(Tf, Tg) = (f,g)$ *by isometric property of* T. *Thus*

$$(f, T^*Tg - g) = 0$$

for all $f \in \mathcal{H}$. *Therefore* $T^*T = I$ *by taking* $f = T^*Tg - g$.

(b) *Let's show the injectivity. Let* $Tf = Tg$. *Then*

$$0 = \|Tf - Tg\| = \|f-g\| \Rightarrow f = g.$$

Thus T *is bijective isometry. Therefore it is an unitary operator.*

Now fix $g \in \mathcal{H}$. *For each* $f \in \mathcal{H}$, *there is* h *such that* $f = Th$ *because of the surjectivity. Then*

$$\begin{aligned}(f, TT^*g) &= (Th, TT^*g)\\ &= (h, T^*TT^*g)\\ &= (h, T^*g)\\ &= (Th, g)\\ &= (f,g)\end{aligned}$$

by the definition of the adjoint and $T^*T = I$ *because* T *is an isometry. Therefore*

$$(f, TT^*g - g) = 0$$

for all $f \in \mathcal{H}$. *By taking* $f = TT^*g - g$, *we can conclude that* $TT^* = I$.

(c) *Let* $\mathcal{H} = l^2(\mathbb{N})$. *Let* $f = (f(1), f(2), \cdots) \in \mathcal{H}$. *Define* $T : (f(1), f(2), \cdots) \mapsto (0, f(1), f(2), \cdots)$. *Clearly* T *is a linear operator, but non-surjective. If we show that* T *is isometry, then we are done.*

$$\|Tf\|^2 = 0 + \sum_{i=1}^{\infty} |f(i)|^2 = \|f\|^2.$$

SoT *is an isometry, which is not unitary.*

(d) *Note that unitary operator is isometry. So, by (a) and Cauchy Schwartz inequality,*

$$(Tf, Tf) = (f, T^*Tf) \le \|f\|\|T^*Tf\| = \|f\|^2.$$

Thus $\|Tf\| \le \|f\|$.

For the other direction,

$$\begin{aligned}(f, f) &= (T^*Tf, T^*Tf)\\ &= (Tf, TT^*Tf)\\ &\le \|Tf\|\|TT^*Tf\|.\end{aligned}$$

But $\|TT^*Tf\|^2 = (TT^*Tf, TT^*Tf) = (Tf, TT^*TT^*Tf) = (Tf, Tf) = \|Tf\|^2$ *since* $(T^*T)^*(T^*T) = T^*TT^*T = I$ *by (a). Therefore* $(f, f) \le (Tf, Tf)$, *which completes the proof.*

□

Exercise (4.32)**.**

(a) $T(cf + dg)(t) = t(cf + dg)(t) = ctf(t) + dtg(t) = cT(f)(t) + dT(g)(t)$ *so* T *is linear. Note that* $t^2 \le 1$ *on* $[0, 1]$. *So*

$$\|Tf\|^2 = \int_0^1 t^2|f(t)|^2 dt \le \int_0^1 |f(t)|^2 dt = \|f\|^2$$

which says that $\|T\| \le 1$.

Also,

$$\begin{aligned}(Tf, g) &= \int_0^1 tf(t)\overline{g(t)}dt\\ &= \int_0^1 f(t)\overline{tg(t)}dt = (f, Tg)\end{aligned}$$

hence $Tg = T^*g$ *for all* $g \in L^2[0, 1]$ *by same argument used in exercise 22. Thus* T *is a bounded linear operator with* $T = T^*$.

Let $f_n(t) = \sqrt{2n + 1}t^n$. *Then* $\|f_n\|^2 = \int_0^1 (2n+1)t^{2n}dt = 1$ *for all* n. *Thus* $f_n \in$ *the unit ball of* $L^2[0, 1]$. *For any subsequence* f_{n_k},

$$\begin{aligned}&\|Tf_{n_k} - Tf_{n_l}\|^2\\ &= \int_0^1 (2n_k + 1)t^{2n_k+2} + (2n_l + 1)t^{2n_l+2} - 2\sqrt{(n_k + 1)(n_l + 1)}t^{(n_k+1)(n_l+1)}dt\\ &= \frac{2n_k + 1}{2n_k + 3} + \frac{2n_l + 1}{2n_l + 3} - \frac{2\sqrt{(n_k + 1)(n_l + 1)}}{(n_k + 1)(n_l + 1) + 1}.\end{aligned}$$

As $n_k, n_l \to \infty$, *the first two terms go to* 1 *respectively, but the last term go to* 0. *So the sequence does not converge. Hence* T *is non-compact.*

(b) *Suppose $T\varphi = \lambda\varphi$. Then $t\varphi(t) = \lambda\varphi(t)$ for all $t \in [0,1]$. Then $t\varphi(t)1_{\varphi\neq 0}(t) = \lambda\varphi(t)1_{\varphi\neq 0}(t)$, so $1_{\varphi\neq 0}(t) = 0$, which means $\varphi = 0$. But the zero vector cannot be an eigenvector, hence there is no eigenvector.*

$\square$

Problem (4.1)**.**

Let X be a collection of linearly independent subsets of $\mathcal{H}$. Impose partial order by the inclusion. Note that X is nonempty since the empty set is in X.

We'll use Zorn's lemma which is equivalent to the AC. Let Y be any totally ordered subset of X. $L_Y = \bigcup_{w\in Y} w$. Then every finite subset of L_Y is in Y, since Y is totally ordered. Hence L_Y is linearly independent, so $L_Y \in X$. But, note that L_Y is an upperbound of Y in X. So Zorn's lemma gives L_m which is maximal element of X.

Now assert that L_m is an algebraic basis of $\mathcal{H}$. Since $L_m \in X$, L_m is linearly independent. If L_m does not span $\mathcal{H}$, then there is $f \in \mathcal{H}$ outside of span L_m. Define $L_f = L_m \cup \{f\}$. Then L_f is strictly larger than L_m. But, L_f is linearly independent, since f is outside of span of L_m. Thus $L_f \in X$, which contradicts to the maximality of L_m. Hence L_m spans $\mathcal{H}$ algebraically, so L_m is an algebraic basis.

Now $L_m = \{a_\alpha : \alpha \in I\}$. Let $B = \left\{e_\alpha = \frac{a_\alpha}{\|a_\alpha\|} : \alpha \in I\right\}$. Then B is an algebraic basis, consists of unit vectors.

Choose $\{e_i\}_{i\in\mathbb{N}}$. For $f \in \mathcal{H}$,

$$f = \sum_{\alpha\in F} c_\alpha e_\alpha = \sum_{\alpha\in F\setminus\mathbb{N}} c_\alpha e_\alpha + \sum_{i=1}^{N} c_i e_i$$

where F is finite set. Define $l(f) = \sum_{i=1}^{N} ic_i$. Note that N depends on f. Clearly, l is linear: $l(cf + dg) = c\sum_{i=1}^{N} ic_i + d\sum_{i=1}^{N} id_i = cl(f) + dl(g)$. Also $l(e_i) = i$. But, $|l(e_i)| = i \to \infty$ as $i \to \infty$, even though $\|e_i\| = 1$. This says l is unbounded linear functional.

$\square$

Exercise (7.1).

Assume that m_α is sigma finite. Then there is $\bigcup E_i = \mathbb{R}^d$ such that E_i's are mutually disjoint and have finite Hausdorff measure. Since $\alpha < d$, we have $m_d(E_i) = 0$. Then countable additivity of m_d implies

$$m_d(\mathbb{R}^d) = \sum_i m_d(E_i) = 0$$

which contradicts to $m_d(\mathbb{R}^d) = \infty$.

□

Exercise (7.3).

Since $|x - y| \leq 1$, we have

$$|f(x) - f(y)| \leq M|x - y|^\gamma \leq M|x - y|.$$

Then

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M|x - y|^{\gamma - 1}.$$

By taking $y \to x$, we have $|f'(x)| = 0$ for all $x \in (0, 1)$. Note that f is lipschitz continuous.

For any $[\alpha, \beta] \subset (0, 1)$, $f = c$ on $[\alpha, \beta]$ since its derivative vanishes on $[\alpha, \beta]$. Since $[\alpha, \beta]$ is arbitrary, $f = c$ on $(0, 1)$. Since f is continuous on $[0, 1]$, f is constant on $[0, 1]$.

□

Exercise (7.7).

Let $\delta > 0$ be given. To show the result, we have to show that for any F_i with $|F_i| \leq \delta$ and $C \subset \bigcup F_i$, we have $\sum |F_i|^\alpha \geq 1$.

We can replace F_i to open set G_i.

Since G_i can be expressed as disjoint union of open intervals, we can replace G_i to I_i.

Since C is compact, by choosing finite open cover, we can only consider finite family of I_i's. Also, we can replace I_i to its closure T_i.

Now we can remove T_i from its family if T_i does not contain any point of C.

If $J, J' \subset T_i$, where J, J' are closed intervals which appear in construction procedure of Cantor ternary set and may be in the different stage, then choose largest such J, J'. By their construction, $J \cup K \cup J' \subset T_i$ where $K \subset C^c$ and $|J|, |J'| \leq |K|$. By concavity,

$$(|J| + |K| + |J'|)^\alpha \geq (3/2(|J| + |J'|))^\alpha = 2\left((|J| + |J'|)/2\right)^\alpha \geq |J|^\alpha + |J'|^\alpha.$$

Thus we can replace T_i to J, J' since this replacement reduces the sum.

Among all J, J', choose the shortest one. Then it has length 3^{-k}.

Now consider C_k where $C = \cap_k C_k$. Then

$$\sum |T_i|^\alpha \geq \sum_{i=1}^{2^k} (3^{-k})^\alpha = 1$$

which leads the conclusion.

$\square$

Exercise (7.10)**.**

Let $S_k = 1 - \sum_{j=1}^{k} 2^{j-1} l_j$. *We want* $S_1 = 2/3$, $S_2 = 2/3 \times 3/4$, $\ldots$, $S_k = 2/(k+1)$. *Then* $S_k \to 0$ *as* k *goes to infinity.*

By the definition of S_k, *we can earn the closed form of* l_k. *Since* $S_k = S_{k-1} - 2^{k-1} l_k$,

$$l_k = \frac{1}{2^{k-1}} \left(S_{k-1} - S_k\right) = \frac{1}{2^{k-2}} \frac{1}{(k+1)(k+2)}.$$

Let $\hat{C}$ *be our fat Cantor set and* $\hat{C} = \cap_k \hat{C}_k$. *Then every* $\hat{C}_k$ *consists of* 2^k *disjoint closed intervals, whose total length is* $2/(k+2) = S_k$. *Thus* $\hat{C}_k$ *is disjoint union of* $1/(2^{k-1}(k+2))$ *length intervals.*

Since $\hat{C} \subset \hat{C}_k$, *given* $\delta > 0$,

$$H_\alpha^\delta(\hat{C}) \le \sum_{i=1}^{2^k} \left(\frac{2}{2^k(k+2)}\right)^\alpha = \frac{2^\alpha}{2^{k(\alpha-1)}} \frac{1}{(k+2)^\alpha}.$$

If $\alpha \ge 1$, *then the last term goes to* 0 *as* k *goes to infinity. Note that if* k *is sufficiently large, then length of each interval in* $\hat{C}_k$ *is shorter than* δ. *So* $H_\alpha^\delta(\hat{C}) = 0$ *for all* δ. *So Hausdorff dimension of* $\hat{C} \le 1$.

Now we will show dim$\hat{C} \ge 1$. *For* $\alpha < 1$, *similar to exercise 7, it is enough to show that* $\sum |T_i|^\alpha \ge 1$ *and there is* $\hat{C}_k \subset \bigcup T_i$. *Thus*

$$\sum |T_i|^\alpha \ge \frac{1}{(k+2)^\alpha} \frac{2^\alpha}{2^{k(\alpha-1)}}.$$

Note that if $n \ge k$, *then* $\hat{C}_n \subset \bigcup T_i$. *So we can let* $k \to \infty$. *Since* $\alpha < 1$, *by letting* $k \to \infty$, *we have*

$$\sum |T_i|^\alpha = \infty.$$

This says $m_\alpha(\hat{C}) = \infty$ *for* $\alpha < 1$. *And this says dim*$\hat{C} \ge 1$.

$\square$

Exercise (7.16)**.**

Continuity can be proven by same reasoning written in page 338 of our textbook.

Assume differentibility and take $u_n = k/4^n$ *and* $v_n = (k+1)/4^n$ *so that* $u_n \le t \le v_n$. *Note that* $K^l(u_n) = K_n^l(u_n)$, *and* $K^l(v_n) = K_n^l(V_n)$. *This says* $|K^l(u_n) - K^l(v_n)| \le l^n$, *because they are directly adjoined vertices of* n*-th stage construction.*

Thus

$$\left|\frac{K^l(u_n) - K^l(v_n)}{u_n - v_n}\right| = \frac{l^n}{1/4^n} = (4l)^n.$$

By letting $n \to \infty$, *the limit goes to* ∞. *Thus modulus of derivative at* t *does not exists, which is contradiction.*

$\square$