

mas541 homework

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Problem (1).

$$\begin{aligned}
1 - \left| \frac{z-w}{1-z\bar{w}} \right|^2 &= 1 - \frac{(z-w)(\bar{z}-\bar{w})}{(1-z\bar{w})(1-\bar{z}w)} \\
&= \frac{1 - \bar{z}w - z\bar{w} + |z|^2|w|^2 - |z|^2 - |w|^2 + z\bar{w} + \bar{z}w}{|1 - \bar{z}w|^2} \\
&= \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2}
\end{aligned}$$

Problem (2).

Let $f = u+iv$. $\partial f = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (u+iv)$. Then $\bar{\partial} f = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (u-iv) = \bar{\partial} \bar{f}$.

Problem (3).

If f is constant, then $|f|$ is also constant. On the other hand, assume $f = u+iv$ and $|f|^2 = u^2 + v^2$ is positive real number. (if it is zero, then f must be zero)

$$u^2 + v^2 = R > 0$$

Differentiate both sides of the equation above with x and y respectively, we can get $uu_x + vv_x = 0$, $uu_y + vv_y = 0$, $u_x = v_y$ and $u_y = -v_x$. By simple calculation we can get $u_x = u_y = v_x = v_y = 0$. Therefore u, v are constant.

Problem (4).

Note that $\int_0^{2\pi} e^{ik\theta} d\theta = \int_0^{2\pi} (\cos k\theta + i \sin k\theta) d\theta = 0$ for positive integer k . Therefore $\frac{1}{2\pi} \int_0^{2\pi} (z_0 + re^{i\theta})^j d\theta = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^j \binom{j}{k} z_0^k (re^{i\theta})^{j-k} d\theta = z_0^j$. Similarly, we can get $\frac{1}{2\pi} \int_0^{2\pi} (z_0 + re^{i\theta})^j d\theta = \bar{z}_0^j$.

Since u is polynomial, we can write it as $\sum_{l,k} a_{l,k} z^l \bar{z}^k$. By direct computation, we can get $\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta = \sum_{l,k} a_{l,k} z_0^l \bar{z}_0^k = u(z_0)$.

Problem (5).

Let $f = u+iv$. $(g \circ f)_x = g_u u_x + g_v v_x$. Then

$$\begin{aligned}
(g \circ f)_{xx} &= (g_{uu} u_x + g_{uv} v_x) u_x + g_u u_{xx} + (g_{vu} u_x + g_{vv} v_x) v_x + g_v v_{xx} \\
(g \circ f)_{yy} &= (g_{uu} u_y + g_{uv} v_y) u_y + g_u u_{yy} + (g_{vu} u_y + g_{vv} v_y) v_y + g_v v_{yy}
\end{aligned}$$

But we have Cauchy-Riemann equation and $g_{uu} + g_{vv} = 0$ and $g_{vu} = g_{uv}$. Also, since f is C^2 function, f is harmonic, $u_{xy} = u_{yx}$, and $v_{xy} = v_{yx}$. Using

these equations, we can check that $(g \circ f)_{xx} + (g \circ f)_{yy} = 0$. Hence $(g \circ f)$ is a harmonic function.

Problem (1).

Let $f = u + iv$. Then $\bar{f}f' = ff' - 2ivf'$, where ff' is holomorphic. So, $\int_{\gamma} \bar{f}f'dz = \int_{\gamma} -2ivf'dz = \int_{\gamma} -2iv(u_x + iv_x)dz = \int_{\gamma} -2iv(v_y + iv_x)dz = -i \int_a^b (2vv_y + 2ivv_x)(\gamma'_1 + i\gamma'_2)dt = \alpha$ where $\gamma = \gamma_1 + i\gamma_2$.

Therefore, real part of $\int_{\gamma} \bar{f}f'dz$ is equal to real part of α . And it is also equal to $-\int_a^b \text{Im}[(2vv_y + i2vv_x)(\gamma'_1 + i\gamma'_2)]dt = -\int_a^b (2vv_x\gamma'_1 + 2vv_y\gamma'_2)dt = -\int_a^b \frac{d}{dt}(v^2 \circ \gamma)dt = 0$ since γ is closed curve.

So, $\int_{\gamma} \bar{f}f'dz$ is purely imaginary.

Problem (2).

Let $f = -u_y$ and $g = u_x$. Then f, g are continuous on U . Since u is harmonic, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ on $U \setminus \{0\}$. So there is $v : U \rightarrow \mathbb{R}$ which is C^1 function and $v_x = f$, $v_y = g$ by lemma 2.5.3.

Let $F = u + iv$. Then F is C^1 function since u, v are C^1 . Since $v_x = f = -u_y$ and $v_y = g = u_x$, F satisfies Cauchy-Riemann equation on U . Thus F is holomorphic on U and real part of F is u .

Problem (3).

- (a) For $z \notin [0, 1]$, the map $w \mapsto \frac{1}{w-z}$ is holomorphic on $\mathbb{C} \setminus [0, 1]$. Let $\gamma(t) = t$ for $t \in [0, 1]$. Then $F(z) = \int_{\gamma} \frac{dw}{w-z} = \int_0^1 \frac{1}{t-z}dt$ is well defined.

For $z \notin [0, 1]$, let $d > 0$ be distance between z and $[0, 1]$. For $|h| < \frac{d}{2}$, consider $\frac{F(z+h)-F(z)}{h} = \int_0^1 \frac{1}{(t-z-h)(t-z)}dt$. Then $\left| \frac{1}{(t-z-h)(t-z)} - \frac{1}{(t-z)^2} \right| = \left| \frac{h}{(t-z)^2(t-z-h)} \right| \leq |h| \frac{2}{d^3}$ since $|t-z| \geq d$ and $|t-z-h| \geq \frac{d}{2}$. Therefore, as $|h| \rightarrow 0$, integrand converges to $\frac{1}{(t-z)^2}$ uniformly on $t \in [0, 1]$. So $\lim_{h \rightarrow 0} \frac{F(z+h)-F(z)}{h} = \int_0^1 \lim_{h \rightarrow 0} \frac{1}{(t-z-h)(t-z)}dt = \int_0^1 \frac{1}{(t-z)^2}dt = F'(z)$.

By same reasoning, we get $F''(z) = \int_0^1 \frac{1}{(t-z)^3}dt$. From existence of F'' , F' is continuous. Therefore F is C^1 function. Existence of complex derivative and C^1 implies F is holomorphic on $\mathbb{C} \setminus [0, 1]$.

- (b) For $s \in (0, 1)$, $F(s+i\varepsilon) = \int_0^1 \frac{1}{t-s-i\varepsilon}dt = \int_0^1 \frac{t-s+i\varepsilon}{(t-s)^2+\varepsilon^2}dt = \int_0^1 \frac{t-s}{(t-s)^2+\varepsilon^2}dt + i \int_0^1 \frac{\varepsilon}{(t-s)^2+\varepsilon^2}dt$. Let $t-s = \varepsilon \tan \theta$. $\varepsilon \tan \theta_0 + s = 0$ and $\varepsilon \tan \theta_1 + s = 1$ for $-\frac{\pi}{2} < \theta_0, \theta_1 < \frac{\pi}{2}$. Then $\sec^2 \theta_0 = \frac{s^2}{\varepsilon^2} + 1$, $\sec^2 \theta_1 = \frac{(1-s)^2}{\varepsilon^2} + 1$, $\theta_0 = \tan^{-1}(\frac{-s}{\varepsilon})$, and $\theta_1 = \tan^{-1}(\frac{1-s}{\varepsilon})$.

Then $F(s+i\varepsilon) = \int_{\theta_0}^{\theta_1} \tan \theta d\theta + i \int_{\theta_0}^{\theta_1} d\theta = \log \left| \frac{\sec \theta_1}{\sec \theta_0} \right| + i(\theta_1 - \theta_0)$. As $\varepsilon \downarrow 0$, $F(s+i\varepsilon)$ goes to $\frac{1-s}{s} + i\pi$ by simple calculation.

Similarly, $F(s - i\varepsilon)$ goes to $\frac{1-s}{s} - i\pi$ as $\varepsilon \downarrow 0$.

(c) Consider $F(-\varepsilon) = \int_0^1 \frac{1}{t+\varepsilon} dt = \log \frac{1+\varepsilon}{\varepsilon}$. It goes to ∞ as $\varepsilon \downarrow 0$.

Consider $F(1 + \varepsilon) = \int_0^1 \frac{1}{t-1-\varepsilon} dt = \log \frac{\varepsilon}{1+\varepsilon}$. It goes to $-\infty$ as $\varepsilon \downarrow 0$.

Therefore, for $s = 0, 1$, $\lim_{z \notin [0,1] \rightarrow s} F(z)$ does not exist.

Problem (4).

IDK where to start...

Problem (5).

It is enough to show γ and μ are path homotopic. Define $H(t, s) = (1-s)\gamma(t) + \frac{\gamma(t)}{|\gamma(t)|}s$. Then $H(t, 1) = \mu(t)$ and $H(t, 0) = \gamma(t)$ by reparametrization. And H is continuous because $\gamma(t) \neq 0$. Therefore H is path homotopy between γ and μ . Since line integration is invariant under path homotopy, we get $\int_\gamma F(\zeta) d\zeta = \int_\mu F(\zeta) d\zeta$.