mas541 homework

20208209 오재민

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Problem (1.1)**.**

$$
\begin{aligned}
1-\left|\frac{z-w}{1-z\overline{w}}\right|^2 &= 1-\frac{(z-w)\left(\overline{z}-\overline{w}\right)}{\left(1-z\overline{w}\right)\left(1-\overline{z}w\right)} \\
&= \frac{1-\overline{z}w-z\overline{w}+|z|^2|w|^2-|z|^2-|w|^2+z\overline{w}+\overline{z}w}{|1-\overline{z}w|^2} \\
&= \frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\overline{z}w|^2}
\end{aligned}
$$

Problem (1.2)**.**

Let $f=u+iv$*.* $\partial f=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)$*. Then* $\overline{\partial f}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(u-iv)=\bar{\partial}\bar{f}$*.*

Problem (1.3)**.**

If f *is constant, then* $|f|$ *is also constant. On the other hand, assume* $f=u+iv$ *and* $|f|^2=u^2+v^2$ *is positive real number. (if it is zero, then* f *must be zero)*

$$u^2+v^2=R>0$$

Differentiate both sides of the equation above with x *and* y *respectively, we can get* $uu_x+vv_x=0$*,* $uu_y+vv_y=0$*,* $u_x=v_y$ *and* $u_y=-v_x$*. By simple calculation we can get* $u_x=u_y=v_x=v_y=0$*. Therefore* u,v *are constant.*

Problem (1.4)**.**

Note that $\int_0^{2\pi}e^{ik\theta}d\theta=\int_0^{2\pi}(\cos k\theta+i\sin k\theta)\,d\theta=0$ *for positive integer* k*. Therefore* $\frac{1}{2\pi}\int_0^{2\pi}(z_0+re^{i\theta})^jd\theta=\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{j}\binom{j}{k}z_0^k\left(re^{i\theta}\right)^{j-k}d\theta=z_0^j$*. Similarly, we can get* $\frac{1}{2\pi}\int_0^{2\pi}\overline{(z_0+re^{i\theta})^j}d\theta=\bar{z}_0{}^j$*.*

Since u *is polynomial, we can write it as* $\sum_{l,k}a_{l,k}z^l\bar{z}^k$*. By direct computation, we can get* $\frac{1}{2\pi}\int_0^{2\pi}u\left(z_0+re^{i\theta}\right)d\theta=\sum_{l,k}a_{l,k}z_0^l\bar{z}_0{}^k=u(z_0)$*.*

Problem (1.5)**.**

Let $f=u+iv$*.* $(g\circ f)_x=g_uu_x+g_vv_x$*. Then*

$$
\begin{aligned}
(g\circ f)_{xx} &= \left(g_{uu}u_x+g_{uv}v_x\right)u_x+g_uu_{xx}+\left(g_{vu}u_x+g_{vv}v_x\right)v_x+g_vv_{xx} \\
(g\circ f)_{yy} &= \left(g_{uu}u_y+g_{uv}v_y\right)u_y+g_uu_{yy}+\left(g_{vu}u_y+g_{vv}v_y\right)v_y+g_vv_{yy}
\end{aligned}
$$

But we have Cauchy-Riemann equation and $g_{uu}+g_{vv}=0$ *and* $g_{vu}=g_{uv}$*. Also, since* f *is* C^2 *function,* f *is harmonic,* $u_{xy}=u_{yx}$*, and* $v_{xy}=v_{yx}$*. Using* *these equations, we can check that* $(g \circ f)_{xx} + (g \circ f)_{yy} = 0$*. Hence* $(g \circ f)$ *is a harmonic function.*

Problem (2.1)**.**

Let $f = u + iv$*. Then* $\bar{f}f' = ff' - 2ivf'$*, where* ff' *is holomorphic. So,* $\int_\gamma \bar{f}f'dz = \int_\gamma -2ivf'dz = \int_\gamma -2iv(u_x + iv_x)dz = \int_\gamma -2iv(v_y + iv_x)dz = -i\int_a^b (2vv_y + 2ivv_y)(\gamma_1' + i\gamma_2')dt = \alpha$ *where* $\gamma = \gamma_1 + i\gamma_2$*.*

Therefore, real part of $\int_\gamma \bar{f}f'dz$ *is equal to real part of* α*. And it is also equal to* $-\int_a^b Im\left[(2vv_y + i2vv_x)(\gamma_1' + i\gamma_2')\right]dt = -\int_a^b (2vv_x\gamma_1' + 2vv_y\gamma_2')dt = -\int_a^b \frac{d}{dt}(v^2 \circ \gamma)dt = 0$ *since* γ *is closed curve.*

So, $\int_\gamma \bar{f}f'dz$ *is purely imaginary.*

Problem (2.2)**.**

Let $f = -u_y$ *and* $g = u_x$*. Then* f, g *are continuous on* U*. Since* u *is harmonic,* $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ *on* $U \setminus \{0\}$*. So there is* $v : U \to \mathbb{R}$ *which is* C^1 *function and* $v_x = f$*,* $v_y = g$ *by lemma 2.5.3.*

Let $F = u + iv$*. Then* F *is* C^1 *function since* u, v *are* C^1*. Since* $v_x = f = -u_y$ *and* $v_y = g = u_x$*,* F *satisfies Cauchy-Riemann equation on* U*. Thus* F *is holomorphic on* U *and real part of* F *is* u*.*

Problem (2.3)**.**

(a) For $z \notin [0,1]$*, the map* $w \mapsto \frac{1}{w-z}$ *is holomorphic on* $\mathbb{C}\setminus[0,1]$*. Let* $\gamma(t) = t$ *for* $t \in [0,1]$*. Then* $F(z) = \int_\gamma \frac{dw}{w-z} = \int_0^1 \frac{1}{t-z}dt$ *is well defined.*

For $z \notin [0,1]$*, let* $d > 0$ *be distance between* z *and* $[0,1]$*. For* $|h| < \frac{d}{2}$*, consider* $\frac{F(z+h)-F(z)}{h} = \int_0^1 \frac{1}{(t-z-h)(t-z)}dt$*. Then* $\left|\frac{1}{(t-z-h)(t-z)} - \frac{1}{(t-z)^2}\right| = \left|\frac{h}{(t-z)^2(t-z-h)}\right| \le |h|\frac{2}{d^3}$ *since* $|t-z| \ge d$ *and* $|t-z-h| \ge \frac{d}{2}$*. Therefore, as* $|h| \to 0$*, integrand converges to* $\frac{1}{(t-z)^2}$ *uniformly on* $t \in [0,1]$*. So* $\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = \int_0^1 \lim_{h\to 0}\frac{1}{(t-z-h)(t-z)}dt = \int_0^1 \frac{1}{(t-z)^2}dt = F'(z)$*.*

By same reasoning, we get $F''(z) = \int_0^1 \frac{1}{(t-z)^3}dt$*. From existence of* F''*,* F' *is continuous. Therefore* F *is* C^1 *function. Existence of complex derivative and* C^1 *implies* F *is holomorphic on* $\mathbb{C}\setminus[0,1]$*.*

(b) For $s \in (0,1)$*,* $F(s+i\varepsilon) = \int_0^1 \frac{1}{t-s-i\varepsilon}dt = \int_0^1 \frac{t-s+i\varepsilon}{(t-s)^2+\varepsilon^2}dt = \int_0^1 \frac{t-s}{(t-s)^2+\varepsilon^2}dt + i\int_0^1\int_0^1 \frac{\varepsilon}{(t-s)^2+\varepsilon^2}dt$*. Let* $t - s = \varepsilon\tan\theta$*.* $\varepsilon\tan\theta_0 + s = 0$ *and* $\varepsilon\tan\theta_1 + s = 1$ *for* $-\frac{\pi}{2} < \theta_0, \theta_1 < \frac{\pi}{2}$*. Then* $\sec^2\theta_0 = \frac{s^2}{\varepsilon^2} + 1$*,* $\sec^2\theta_1 = \frac{(1-s)^2}{\varepsilon^2} + 1$*,* $\theta_0 = \tan^{-1}\left(\frac{-s}{\varepsilon}\right)$*, and* $\theta_1 = \tan^{-1}\left(\frac{1-s}{\varepsilon}\right)$*.*

Then $F(s+i\varepsilon) = \int_{\theta_0}^{\theta_1}\tan\theta d\theta + i\int_{\theta_0}^{\theta_1}d\theta = \log\left|\frac{\sec\theta_1}{\sec\theta_0}\right| + i(\theta_1 - \theta_0)$*. As* $\varepsilon \downarrow 0$*,* $F(s+i\varepsilon)$ *goes to* $\frac{1-s}{s} + i\pi$ *by simple calculation.*

Similarly, $F(s-i\varepsilon)$ *goes to* $\frac{1-s}{s} - i\pi$ *as* $\varepsilon \downarrow 0$.

(c) *Consider* $F(-\varepsilon) = \int_0^1 \frac{1}{t+\varepsilon} dt = \log \frac{1+\varepsilon}{\varepsilon}$. *It goes to* ∞ *as* $\varepsilon \downarrow 0$.

Consider $F(1+\varepsilon) = \int_0^1 \frac{1}{t-1-\varepsilon} dt = \log \frac{\varepsilon}{1+\varepsilon}$. *It goes to* $-\infty$ *as* $\varepsilon \downarrow 0$. *Therefore, for* $s = 0, 1$, $\lim_{z \notin [0,1] \to s} F(z)$ *does not exists.*

Problem (2.4)**.**

First consider $p \equiv 0$. *We can easily see that* $\sup_{z \in C} |z^{-n}| = 1$ *so desired value* ≤ 1.

Note that $|p(z) - z^{-n}| = |z^n p(z) - 1|$. *Thus,* $1 = \frac{1}{2\pi i} \int_C \frac{z^n p(z) - 1}{z} dz \leq \sup_{z \in C} |z^n p(z) - 1|$.

Those leads the conclusion.

Problem (2.5)**.**

It is enough to show γ *and* μ *are path homotopic. Definte* $H(t,s) = (1-s)\gamma(t) + \frac{\gamma(t)}{|\gamma(t)|} s$. *Then* $H(t,1) = \mu(t)$ *and* $H(t,0) = \gamma(t)$ *by reparametrization. And* H *is continuous because* $\gamma(t) \neq 0$. *Therefore* H *is path homotopy between* γ *and* μ. *Since line integration is invariant under path homotopy, we get* $\int_\gamma F(\zeta) d\zeta = \int_\mu F(\zeta) d\zeta$.

Problem (3.1)**.**

It suffices to show that $\int_\gamma f(z)dz = 0$ *for rectangle* γ *whose edges are parallel to coordinate axes by Morera's theorem.*

First, assume that γ *intersects with* $[0,1]$ *only finitely many points. Let* p *be such point. Then* p *must be on (wlog) left edge of* γ*. Let* $a+ib$, $a+ic$ *be two vertices incident with left edge. (*$b > c$*) Let* $\rho(t) = a + i(tc + (1-t)b)$*. Consider* $f \circ \rho$*. It is continuous and equals to* $\frac{\partial}{\partial t}F(\rho(t))$ *except for* $\gamma^{-1}(p)$ *where* F *is antiderivative of* f *on* $\mathbb{C} \setminus [0,1]$*. Then lemma 2.3.1 says* $f(\rho(t)) = \frac{\partial}{\partial t}F(\rho(t))$ *even for* $\gamma^{-1}(p)$*. Therefore* $\int_\rho f(z)dz = F(a+ic) - F(a+ib)$*. By using this result, we can easily calculate* $\int_\gamma f(z)dz = 0$*.*

Now, assume that (wlog) upper edge of γ *intersects with* $[0,1]$*. Let* $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ *which are upper edge, left edge, bottom edge, and right edge respectively, parametrized like* ρ *of above, positive oriented. Consider* φ *made by shrinking side edges of* γ *so that distance between of upper edges of* φ *and* γ *less than* δ*, while bottom edge is fixed. Also note that* δ *is chosen so that* $d(z_0, z_1) < \delta$ *implies* $d(f(z_0), f(z_1)) < \varepsilon$*.*

$$\left|\int_\gamma f(z)dz - \int_\varphi f(z)dz\right| \le \left|\int_{\gamma_2 - \varphi_2} f(z)dz + \int_{\gamma_4 - \varphi_4} f(z)dz\right| \\ + (\text{ length of } \gamma_1)\,\varepsilon$$

And, second term of above goes to 0 *as distance between* φ_1 *and* γ_1 *goes to 0 by continuity and result of first case. Actually* $\int_\varphi f(z)dz = 0$ *because* φ *does not intersect with* $[0,1]$*. Thus we have shown that* $\int_\gamma f(z)dz = 0$*.*

By first, second case and Morera's thm, f *is actually entire function.*

Problem (3.2)**.**

For $0 < r < 1$, $|f^{(n)}(0)| \le \frac{n!}{r^n}\frac{1}{1-r}$ *by using Cauchy estimate.* $r^n(1-r)$ *is maximized when* $r = \frac{n}{n+1}$*. So, when* $r = \frac{n}{n+1}$*, we get best estimate of* $|f^{(n)}(0)|$*.*

Problem (3.3)**.**

(a) *Since* K *is compact subset of open set* U*, there is* $r > 0$ *such that for all* $x \in K$*, closure of* $D(x,r)$ *is in* U*. Then,* $|f(z)|^2 \le \frac{1}{2\pi}\left|\int_{\partial D(z,r}\frac{f^2(w)}{w-z}dw\right| \le \frac{1}{2\pi}\int_0^{2\pi}|f^2(z+re^{i\theta})d\theta|$*. By multiplying* ρ *both sides and integrating from* 0 *to* r*, we can get the following:*

$$\begin{aligned}\frac{r^2}{2}|f(z)|^2 &\leq \frac{1}{2\pi}\int_0^r \int_0^{2\pi} \rho|f^2(z+re^{i\theta})|d\theta d\rho \\ &= \frac{1}{2\pi}\int_{\overline{D}(z,r)} |f|^2 dm \\ &= \frac{1}{2\pi}\int_U |f|^2 dm\end{aligned}$$

for all $z \in K$, where m is lebesgue measure, using Holder's inequality and polar coordinate integration.

Therefore $C = \frac{1}{r\sqrt{\pi}}$

(b) *If f is identically zero, possible.*

Else if f is constant, then $\int_{\mathbb{C}} |f|dm = \infty$ since measure of complex plane is ∞.

Else, that is f is nonconstant entire function, then f must be unbounded. So, there is $\delta > 0$ such that $|f| \geq 1$ for all $|z| > \delta$. Then $\int_{\mathbb{C}} |f|dm \geq m(\{z : |z| > \delta\}) = \infty$.

Problem (3.4). (a) *Since $\frac{z}{e^z-1}$ is bounded near 0, it has removable singularity at 0. So we can regard it as holomorphic function. Note that $e^z - 1 = 0$ when z is integer multiple of $2\pi i$. So, given power series converges on unit disc. Now, multiply $e^z - 1$ both sides. Since $e^z - 1$ is entire and given power serires converges absolutely on $\bar{D}(0, r)$ where $0 < r < 1$, we can write $z = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \sum_{n=1}^{\infty} \frac{1}{n!} z^n$. Since z is entire, coefficient of power series is unique. By comparing coefficients of both sides, we can get given recursion formula.*

$\lim_{z\to 0} \frac{z}{e^z-1} = 1 = B_0$. *From this, by simple calculation, $B_1 = \frac{-1}{2}$, $B_2 = \frac{1}{6}$, and $B_3 = 0$.*

Consider $-z = f(z) - f(-z) = \sum_{n=0}^{\infty} 2\frac{B_{2n+1}}{(2n+1)!} z^{2n+1}$. This makes sense because f is holomorphic on unit disc. By comparing coefficient of this series, we can get $B_{2m+1} = 0$ for $m \geq 1$.

(b) *We alrealdy notice that $e^z - 1$ is zero when z is integer multiple of $2\pi i$. But $\lim_{z\to 2k\pi i} \frac{z}{e^z-1}$ is not bounded when $k \neq 0$. Therefore, $\frac{z}{e^z-1}$ is holomorphic on $D(0, 2\pi)$ and is not holomorphic outside of that disc. Since* *power seriese representation of holomorphic function at P has radius of convergence at least $d(P,U)$, we can say radius of convergence of the series is 2π.*

Problem (3.5)**.**

f' is holomorphic on unit disc. Let $r = \sup_{z\in K}|z|$. Since K is compact, $|f'| \leq M$ on K and r is positive but less than 1. Let $\gamma(t) = tz^n$ which connects origin and z^n. $|f(z^n) - f(0)| = \left|\int_\gamma f'dz\right| \leq M\sup_{z\in K}|z|^n = Mr^n$. Therefore, $|\sum_{n=1}^{\infty} f(z^n)| \leq \sum_{n=1}^{\infty}|f(z^n)| \leq \sum_{n=1}^{\infty} Mr^n < \infty$ because r is positive but less than 1.

Problem (4.1)**.**

Notice that f does not vanish on $\mathbb{C}\setminus\{0\}$. Therefore $g(z)=\frac{1}{f(z)}$ is holomorphic on $\mathbb{C}\setminus\{0\}$. Near 0, g is bounded since $\sqrt{|z|}$ goes to 0 as z goes to 0. This means g has removable singularity at 0 and therefore entire. But $g(z)\leq\sqrt{|z|}$, so g must be constant by Cauchy integral formula.

Then f must be constant also, and this is contradiction. Therefore there is no such holomorphic function.

Problem (4.2)**.**

Let $g(z)=f\left(\frac{1}{z}\right)$. Then $g\to 0$ as $z\to 0$. Therefore g is entire. Also, $g(z)/z$ is entire since $\lim_{z\to 0}g(z)/z=g'(0)$ hence bounded near 0.

Now, consider given integral. Let $\zeta=e^{it}$ and $t=2\pi-s$. Then given integral is $\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(e^{-is})}{e^{-is}-z}ie^{-is}ds=\frac{1}{2\pi i}\int_0^{2\pi}\frac{g(e^{is})}{e^{is}-e^{2is}z}ie^{is}ds=\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{g(\zeta)}{\zeta z\left(\frac{1}{z}-w\right)}d\zeta$

Therefore given integral is equal to $\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{h(\zeta)}{\frac{1}{z}-\zeta}d\zeta$ where $h(\zeta)=\frac{g(\zeta)}{\zeta z}$. Thus, it is equal to $-g(1/z)=-f(z)$.

Problem (4.3)**.**

f maps $re^{i\theta}$ to $\sqrt{r}e^{i\left(\frac{\theta}{2}+k(z)\pi\right)}$ where $k(z)\in\mathbb{Z}$. To f be continuous, $k(z)$ must be all even or all odd.

First assume that $k(z)$ is all even. Then $f'(0)=\lim_{\mathbb{R}\ni h\to 0}\frac{f(h)}{h}=\lim_{\mathbb{R}\ni h\to 0}\frac{\sqrt{h}}{h}=\infty$, which is contradiction.

Similarly, if $k(z)$ is all odd, $f'(0)$ does not exist.

Therefore existence of such f leads $0\notin U$.

Let ι be identity function of U. Since $z\notin U$, ι does not vanish on U, hence $1/\iota$ is holomorphic on U. Since U is hsc, $1/\iota$ has holomorphic antiderivative φ.

Now consider the derivative of $\iota(z)e^{-\varphi(z)}$. Simple calculation leads that it is equal to 0. Hence $\iota(z)=ce^{\varphi(z)}$ for some constant c. Therefore $\iota(z)=e^{\psi(z)}$ for some holomorphic ψ on U.

Take $f=e^{\frac{1}{2}\psi}$. Then f satisfies what we want.

Problem (4.4)**.**

(a) *Let γ_R be the contour used in example 4.6.5.*

First, consider $\int_0^\infty \frac{1}{x^a(x+1)}dx$. To calculate this, take $f(z) = z^{-a}/(1+z)$ where $0 < arg(z) < 2\pi$. By residue thm, $2\pi i e^{-a\pi i} = \int_0^\infty \frac{1}{r^a(r+1)}dr\left(1 - e^{-2a\pi i}\right)$. Therefore $\int_0^\infty \frac{1}{x^a(x+1)}dx = \pi\csc(\pi a)$.

Now, $\int_{\gamma_R} \frac{\log z}{z^a(1+z)}dz = 2\pi i e^{-a\pi i}\pi i$ by residue thm. But as $R \to \infty$, that integral goes to $(1 - e^{-2a\pi i})\int_0^\infty \frac{\log r}{r^a(r+1)}dr - e^{-2a\pi i}\int_0^\infty \frac{2\pi i \log r}{r^a(r+1)}dr$.

By simple calculation, the value we want is equal to $\frac{i\pi^2}{\sin(\pi a)} + \frac{\pi^2 e^{-a\pi i}}{\sin^2(\pi a)} = \frac{\pi^2\cos(\pi a)}{\sin^2(\pi a)}$.

(b) Consider $f(z) = \frac{\pi\cot(\pi z)}{(z+\alpha)^2}$ and Γ_n = square centered at origin, each edges is parallel to real or imaginary axis, length of edge is $2n+1$.

Then $\int_{\Gamma_n} f(z)dz$ goes to 0 as $n \to \infty$ by considering modulus of $f(z)$, and index of Γ_n at each singularites is 1, and residues are $\frac{1}{(k+\alpha)^2}$ at $z = k$ and $-\frac{\pi^2}{\sin^2(\pi\alpha)^2}$ at $z = -\alpha$.

Above calculation leads the conclusion.

Problem (4.5).

Note that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic iff f is meromorphic on $\hat{\mathbb{C}}$.

(a) First consider 'if' part. Let f be rational function. We already knows that rational function is meromorphic on entire complex plane. So, we need to show that rational function is meromorphic at ∞.

Let $f(z) = \frac{(z-Q_1)^{m_1}\cdots(z-Q_l)^{m_l}}{(z-P_1)^{n_1}\cdots(z-P_k)^{n_k}}$. Since f has finitely many pole in complex plane, we can choose $M > 0$ so that f has no pole on $\{z : |z| > M\}$. For $0 < |w| < \frac{1}{M}$, consider $g(w) = f(1/w)$. Then g is holomorphic.

Let $\sum_i n_i = N$ and $\sum_j m_j = M$. If $M = N$, $g \to 1$ as $z \to 0$. If $M > N$, $g \to 0$ as $z \to 0$. If $M < N$, $g \to \infty$ if $z \to 0$. Hence g is meromorphic near 0, which means that f is meromorphic at ∞.

Second, consider 'only if' part. Either f has a pole or removable singularity at ∞, f has finitely many poles in complex plane. So $f(z)(z-P_1)^{n_1}\cdots(z-P_k)^{n_k} = F(z)$ is entire where n_i is order of pole P_i.

Consider $F(1/z) = g(z)$ for $z \neq 0$. As $z \to 0$, $g \to \infty$ or α for some $\alpha \in \mathbb{C}$ by simple calculation. Therefore F has a pole or removable singularity at ∞.

If F has removable singularity at ∞, F must be bounded, hence constant by Liouville's thm.

If F has a pole at ∞, F must be polynomial since its modulus diverges.

In both cases, F must be rational function.

(b) Note that $z \mapsto \frac{az+b}{cz+d}$ for $ad-bc \neq 0$ is biholomorphic function of Riemann sphere. Also note that biholomorphic function of $\mathbb{C}$ must have a form of $\alpha z + \beta$ for $\alpha \neq 0$ by fundamental thm of algebra.

Now consider biholomorphic f on Riemann sphere. Let $f(\infty) = b$ and $\varphi_b(z) = \frac{-\bar{b}-1}{z-b}$. Then $\varphi_b \circ f$ is biholomorphic function of Riemann sphere, which maps $\infty \to \infty$. Therefore $\varphi_b \circ f$ is biholomophic function of complex plane hence $\varphi_b(f(z)) = \alpha z + \beta$. Then $f(z) = \frac{-b\alpha z - b\beta + 1}{-\alpha z - \beta - \bar{b}}$, which is linear frational transformation.

Problem (5.1)**.**

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ *and assume that* $P(z) = 0$ *has no solution. Then by the argument principle,* $\frac{1}{2\pi i}\int_{\partial D(Q,R)} \frac{P'(\zeta)}{P(\zeta)}d\zeta = 0$ *for all* $R > 0$*. That integral is equal to* $\frac{1}{2\pi i}\int_0^{2\pi} \frac{P'(Q+Re^{i\theta})}{P(Q+Re^{i\theta})}Rie^{i\theta}d\theta$*. But, as* $R \to \infty$*, integrand of above goes to in uniformly on* $0 \le \theta \le 2\pi$*. Therefore, the integral above goes to* $n > 0$ *which is the degree of* P*. It is contradiction. Thus* $P(z) = 0$ *has at least one solution in complex plane.*

Problem (5.2)**.**

Assume the existence of such f*. Since* f *is bounded near* 0*, Riemann removable singularity theorem says that* f *can be extended to the function which is holomorphic on entire unit disc.*

If modulus of $f(0)$ *is equal to 1 or 2, then image of the unit disc under* f *is not open which contradicts to the open mapping theorem. So* $f(0) \in \{w : 1 < |w| < 2\}$.

Since f *is surjective function of the punctured unit disc onto the annulus, we can find* $w \neq 0$ *such that* $f(0) = f(w)$*. Choose two disjoint neighborhood* U_w, U_0 *of* $w, 0$ *respectively. Then by the open mapping theorem,* $f(U_w)$ *and* $f(U_0)$ *are open and* $f(0) \in f(U_w) \cap f(U_0)$*. Since* $f(U_w) \cap f(U_0)$ *is open, we can choose small neighborhood of* $f(0)$ *contained in the previous set. And therefore we can choose* $f(0) \neq \alpha \in f(U_w) \cap f(U_0)$*. This cannot be happen since* f *is injective.*

Thus there is no such f*.*

Problem (5.3)**.**

(a) *Choose* $R > \lambda$*, and choose* n *so large that* $\lambda - 1 \ge 1/n$*. Then* $\bar{D}(R, R - \frac{1}{n}) \subset$ *Right half plane.*

  Then for $\zeta \in \partial D(R, R - 1/n)$, $|e^{-\zeta}| < 1 \le \lambda - 1/n \le |\zeta - \lambda|$*. Put* $f(z) = e^{-z} + z - \lambda$ *and* $g(z) = z - \lambda$*. Then by above and Rouche's theorem,* f *and* g *has same zero on* $D(R, R - 1/n)$*. But any* $z \in$ *Right half plane must be inside of* $D(R, R - 1/n)$ *for some large* R *and* n*. This means* f *and* g *have same zero on the right half plane.*

  But $g(z) = 0$ *has unique solution. Therefore* $e^{-z} + z - \lambda = 0$ *has unique solution on the right half plane.*

(b) *Fix $z' \in U$. Note that $U \setminus \{z'\}$ is still a domain. Let $g_k(z) = f_k(z) - f_k(z')$ for $z \in U \setminus \{z'\}$. Since f_j is an injective holomorphic function on U, g_k does not vanish on $U \setminus \{z'\}$. Uniform convergence of f_j on compact subsets of U implies uniform convergence of g_k on compact subsets of $U \setminus \{z'\}$. Since g_k is nonvanishing function, by Hurwitz's theorem, $\lim_{k\to\infty} g_k(z) = f(z) - f(z')$ does not vanish or identically zero.*

If it is identically zero on $U \setminus \{z'\}$, then f must be constant function on U. If it is nonvanishing on $U \setminus \{z'\}$, then $f(z'') = f(z')$ implies $z'' = z'$. Thus f must be injective.

Problem (5.4)**.**

It seems to be solved by the maximum modulus principle (or theorem), but I don't know where to start.

Problem (5.5)**.**

For $z \in S$, $|\varphi(z)| = \left|\frac{e^{2\pi z i}-1}{e^{2\pi z i}+1}\right|$, and the real part of $e^{2\pi z i} > 0$ because $z \in S$. Then it is clear that $|\varphi(z)| < 1$. Also $\varphi(0) = 0$.

Therefore $\varphi \circ f : D \to D$ is holomorphic and it fixes the origin. Then Schwarz's lemma says $|\varphi'(0) f'(0)| \leq 1$. But $\varphi'(0) = \pi$. Therefore $|f'(0)| \leq 1/\pi$. The equality holds only if $\varphi(f(z)) = wz$ for some $|w| = 1$.