FRANK JONES INTEGRATION THEORY SOLUTIONS

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CHAPTER 2: LEBESGUE MEASURE ON $\mathbb{R}^n$

section A: Construction.

In this section, we will construct Lebesgue measure from set of special rectangles. Although there are many other methods to consturct measure on Euclidean space, method in our textbook looks like intuitive and good to study for beginners. Other textbooks use Caratéodory extension or Urysohn lemma to construct Lebesgue measure on Euclidean space.

Problem 3.

Consider the open set $G \setminus P$ and x in that set. There is ε neighborhood of x contained in $G \setminus P$. But ε neighborhood contains special rectangle I whose measure is positive. Now, take $P' = P \cup I$.

Problem 5. 'at most countably many disjoint open sets'

Let G_i be nonempty set. Pick $x \in G_i$ and consider ε neighborhood contained in G_i. That neighborhood contains point q_i whose components are all rational. Consider the injection $G_i \mapsto q_i$ from $\mathcal{I}$ to countable set. (It is clearly injection because each G_i is disjoint.)

Problem 6. 'the structure of open sets in real line'

Consider an equivalence relation $x \sim y \leftrightarrow x, y$ belongs to some open interval contained in G. Equivalence class of x is a largest open interval containing x.

Problem 8. 'open disk cannot be expressed as disjoint union of open rectangles'

It is trivial since open disk is connected.

Note that regular space with countable basis, metrizable space, compact Hausdorff space are normal space. So Euclidean space is normal space. This fact is useful to prove property 4 of Lebesgue measure on compact sets.

Problem 17.

Let $x = \sum_{k=1}^{\infty} \frac{\alpha_k}{3^k}$ where $\alpha_j \in \{0, 2\}$. Then $1 - x = \sum_{k=1}^{\infty} \frac{2-\alpha_k}{3^k}$ and $2 - \alpha_k \in \{0, 2\}$. Therefore, $1 - x \in C \leftrightarrow x \in C$.

B. Properties of Lebesgue Measure

(1) $\mathcal{L}$ is complement closed.
(2) $\mathcal{L}$ is $\sigma - \cap\cup$ closed.
(3) $\mathcal{L}$ is set minus closed.
(4) Countable additivity of λ.
(5) Continuity from below.
(6) Continuity from above.
(7) $\mathcal{L}$ contains Borel algebra on $\mathbb{R}^n$.
(8) λ is complete.
(9) Theorem on approximation of $\mathcal{L}$.
(10) On $\mathcal{L}$, $\lambda^* = \lambda_* = \lambda$.
(11) $\lambda(B) = \lambda^*(A) + \lambda_*(B \setminus A)$ if $A \subset B \in \mathcal{L}$.
(12) $A \in \mathcal{L}$ if and only if $\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \cap A^c)$ for all $E \subset \mathbb{R}^n$.

Problem 27.

If $A \subset \bigcup_{k=1}^{\infty} I_k$, then $\lambda^*(A) \leq \lambda^*(\bigcup_{k=1}^{\infty} I_k) \leq \sum_{k=1}^{\infty} \lambda^*(I_k) \leq \sum_{k=1}^{\infty} \lambda(I_k)$. Therefore $\lambda^*(A) \leq \inf\{\}$.

On the contrary, assume $\lambda^*(A) < \inf\{\}$. There is open set G containing A and $\lambda(G) < \inf\{\}$. From problem 9, G can be expressed as a countable union of nonoverlapping special rectangles. That is, $\lambda(G) = \sum_{k=1}^{\infty} \lambda(I_k)$ which is contradiction.

Problem 29.

Use $\lambda(A \cup B) = \lambda(A \setminus B) + \lambda(B \setminus A) + \lambda(A \cap B)$.

Problem 30.

Let G_A, G_B be open sets containing A, B respectively. $\lambda(G_A) + \lambda(G_B) = \lambda(G_A \cup G_B) + \lambda(G_A \cap G_B) \geq \lambda^*(A \cup B) + \lambda^*(A \cap B)$. Since G_A, G_B are arbitrary, $\lambda^*(A) + \lambda^*(B) \geq \lambda^*(A \cup B) + \lambda^*(A \cap B)$.

Problem 31.

Clearly one point set is measure zero and closed. Let $\varepsilon > 0$ be given. There exists open set G_i containing a_i such that $\lambda(G_i \setminus a_i) < \frac{\varepsilon}{2^i}$. Therefore $\lambda(A) < \varepsilon$ which means $\lambda(A) = 0$.

Problem 32.

Let I_k be special 'cube' centered at origin, length of side $= k$. Then $a \times I_k \subset a \times \mathbb{R}^n$. Therefore we get $\lambda(a \times \mathbb{R}^n) = \lim_{k\to\infty} \lambda(a \times I_k) = 0$ by continuity from below.

Problem 34.

There is closed set of positive measure, no interior.

$$[0,1] \setminus \bigcup_{k=1}^{\infty} \left(q_k - \frac{\varepsilon}{2^{k+1}}, q_k + \frac{\varepsilon}{2^{k+1}} \right) \tag{1}$$

Problem 37.

For each positive integer k, choose open set $G_k supsetE$ such that $\lambda(G_k) < \lambda^*(E) + \frac{1}{k}$. Put $A = \bigcap_{k=1}^{\infty} G_k$ which is measurable. Then $\lambda(A) < \lambda^*(E) + \frac{1}{k}$ for all positive integer k. Therefore $\lambda(A) \leq \lambda^*(E)$. Reverse inequality is trivial because $E \subset A$. We can conclude that there exists measurable hull of E which has finite outer measure.

Problem 38.

First assume A is measurable hull of E. $\lambda(A) = \lambda^*(E) < \infty$. But we already know that $\lambda(A) = \lambda^*(E) + \lambda_*(A \setminus E)$. Therefore $\lambda_*(A \setminus E) = 0$.

Conversely assume $\lambda_*(A \setminus E) = 0$. From 11st property of Lebesgue measure, we can get A is measurable hull of E.

Problem 39.

Let $E_k = B(0,k) \setminus B(0,k-1)$ which is measurable partition of $\mathbb{R}^n$. Then $E \cap E_k$ has finite outer measure. By problem 37, there exists measurable hull A_k of $E \cap E_k$. Put $A = \bigcup_{k=1}^{\infty} A_k$ and consider the compact set $K \subset A \setminus E$. But $\lambda(K \cap E_k) \leq \lambda_*(A_k \setminus E \cap E_k) = 0$. By continuity from below, $\lambda(K) = 0$. So $\lambda_*(A \setminus E) = 0$.

Problem 40.

CHAPTER 6

section A.

Problem 1. ”the vanishing property”

First, assume that $\int f d\mu = 0$. Since f is measurable, there exists an increasing seqeunce of nonnegative simple functions which converges to f. Let denote them as s_n. Then $f^{-1}((0,\infty]) = \bigcup_{n=1}^{\infty} s_n^{-1}((0,\infty])$, union of measure zero set. Therefore, we get $\mu(f^{-1}((0,\infty])) = 0$.

Conversely, assume that $\int f d\mu > 0$. Then there exists nonnegative simple function $s \leq f$ such that $\int s d\mu > 0$. Also we can write s as a linear combination of (measurable) characteristic functions, i.e. $\int s d\mu = \sum_{i=1}^{N} \alpha_i \mu(A_i) > 0$. So, $\alpha_i \mu(A_i) > 0$ for at least one integer $1 \leq i \leq N$. By Observing the fact that $A_i \subset f^{-1}((0,\infty])$, we can conclude that $f^{-1}((0,\infty])$ has measure zero implies $\int f d\mu = 0$ by contrapositive.

Problem 2. ”the finiteness property”

$E = \{x : f(x) = \infty\}$ is a measurable set since f is measurable. Suppose $\mu(E) > 0$. For any $M \in \mathbb{N}$, choose nonnegative simple measurable function $s_M \leq f$ such that $s_M(x) \geq M$ for $x \in E$. Then $\int f d\mu \geq \int s_M d\mu \geq M\mu(E)$. Therefore, $\int f d\mu \geq M\mu(E)$ for all positive integer M. It means that $\int f d\mu = \infty$. By taking contrapositive, we get what we want.

Problem 3. ”compatibility to not finite nonnegative simple functions”

We are only interested in the case when $\alpha_k = \infty, \mu(A_k) > 0$. Since f is measurable, we can consider seqeunce of nonnegative finite simple function $\{t_n\}$ such that $t_n(x) \geq n$ for $x \in A_k$. Therefore, $\int f d\mu \geq \int t_n d\mu \geq \mu(A_k)n$ for all positive integer n, which means $\int f d\mu = \infty$. For other cases, it is easy to check the compatibility.

Problem 4. ”scalar multiplication is still valid for $c = \infty$”

First, assume $\int f d\mu = 0$. Then $\mu(f^{-1}((0,\infty])) = 0$ by Problem 1. It is obvious that $g^{-1}((0,\infty]) = f^{-1}((0,\infty])$ for $g = \infty f$. So, $\int g d\mu = 0$. Therefore we get $\int \infty f d\mu = \int f d\mu$.

Second, assume $\int f d\mu > 0$. Then measure of $g^{-1}((0,\infty])$ is positive. So, $\int \infty f d\mu = 0\mu(g^{-1}(0)) + \infty\mu(g^{-1}((0,\infty]))$ which is ∞.

Problem 5. ”strict inequality for Fatou’s lemma’

Consider this nonnegative finite simple function defined on real line:

$$s_n(x) = \begin{cases} n^2 & \text{if } x \in (0, \frac{1}{n}) \\ 0 & \text{otherwise} \end{cases}$$

$\lim s_n = 0$ and $\lim \int s_n d\mu = \infty$.

Problem 6.

Let $f_k = 1_{A_k}$ where 1_{A_k} is an indicator(characteristic) function. By Fatou's lemma (ILLLI), $\int \liminf_{k\to\infty} 1_{A_k} d\mu \leq \liminf_{k\to\infty} \int 1_{A_k} d\mu = \liminf_{k\to\infty} \mu(A_k)$. Now, showing $1_{\liminf A_k} \leq \liminf_{k\to\infty} 1_{A_k}$ is left to us. If $x \in \liminf A_k$ then $x \in \bigcap_{i\geq n} A_i$ for some positive integer n. Then $1_{A_i}(x) = 1$ for all positive integer i greater than n. So, $1_{\liminf A_k} \leq \inf_{i\geq n} 1_{A_i}$. By letting $n \to \infty$, we get $1_{\liminf A_k} \leq \liminf_{k\to\infty} 1_{A_k}$. Therefore, $\mu(\liminf A_k) = \int 1_{\liminf A_k} d\mu \leq \int \liminf_{k\to\infty} 1_{A_k} d\mu \leq \liminf_{k\to\infty} \int 1_{A_k} d\mu = \liminf_{k\to\infty} \mu(A_k)$.

section B.

Problem 9. "iff condition for (finite) integrability"

Let $f \in \mathcal{L}^1(\mu)$. Then $\int f_\pm d\mu < \infty$, so there sum is also finite ($= \int |f| d\mu$).

Conversely, assume $|f| \in \mathcal{L}^1(\mu)$. Then $\int f_\pm d\mu \leq \int |f| d\mu < \infty$. Therefore $f \in \mathcal{L}^1(\mu)$.

Problem 10. "dominated integrability"

If f is measurable, $f_\pm$ is also measurable. So $f_+ + f_- = |f|$ is measurable. $\int |f| d\mu \leq \int |g| d\mu < \infty$. Therefore, $|f| \in \mathcal{L}^1(\mu)$ by Problem 9.

Problem 11.

It is obvious that $|f_k| \leq |f|$ and $f \in \mathcal{L}^1(\mu)$. So, if each f_ks are measurable, by dominated convergence thm, done. Actually, $f_k = f \cdot 1_{A_k \cap E_k}$ where $A_k = [-k, k]$ and $E_k = f^{-1}([-k, k])$. There exists sequence of nonnegative finite simple function $\{s_i\}$ which converges to f since f is measurable. So, $\lim s_i 1_{a_k \cap E_k} = f_k$ is measurable.

Problem 12.

Likewise, it is enough to show that $f(x)e^{-\frac{|x|^2}{k}} = f_k(x)$ is measurable. $e^{-\frac{|x|^2}{k}}$ is continuous, hence Borel measurable, hence Lebesgue measurable. There exists sequence of nonnegative finite simple function $\{s_k\}$ which converges to f since f is measurable. So, $\lim_{i\to\infty} s_i e^{-\frac{|x|^2}{k}} = f_k$ is measurable.

Problem 13. 'alternative proof for problem 2.42'

For each $x \in X$, there are at most $d \in \mathbb{N}$ distinct A_k containing x. Fix positive integer N. Let $I_x = \{k \in \mathbb{N} : k \leq N \text{ and } x \in A_k\}$. Clearly $\sum_{k=1}^N 1_{A_k} \leq d 1_A$ where $A = \bigcup_{i=1}^\infty A_i$. By integrating both sides, $\sum_{k=1}^N \mu(A_k) \leq d\mu(A)$. By Letting $N \to \infty$, we got the result in Problem 2.42.

(similar, another) $\sum_{k=1}^N 1_{A_k} \leq |I_x| \leq d = d1_A$ for all N. So $\sum_{k=1}^\infty 1_{A_k} \leq d1_A$. By integrating both sides and monotone convergence thm, we got it.

Problem 14.

Let $\mathcal{I}$ be set of all sequences which are strictly increasing positive integers and length m. Such set is countable. Now, $\bigcup_{i\in\mathcal{I}}\bigcap_{j=1}^{m} A_{i_j} = E_m$ and it is measurable.

Similar to Problem 13, $m1_{E_m} \leq \sum_{i=1}^{\infty} 1_{A_i}$. So, $m\mu(E_m) = \int m1_{E_m} d\mu \leq \int \sum_{i=1}^{\infty} 1_{A_i} d\mu = \sum_{i=1}^{\infty} \int 1_{A_i} d\mu = \sum_{i=1}^{\infty} \mu(A_i)$ by monotone convergence thm.

section C.

Problem 16.

Clearly, $|f| = 0$ almost everywhere, and $|f|$ is measurable. Consider nonnegative finite simple function $s \leq |f|$. Then $s = 0$ almost everywhere and $s = \sum_{i=1}^{N} \alpha_i 1_{A_i}$, where A_i is null set if $\alpha_i \neq 0$. Therefore, $\int s d\mu = 0$, which implies $\int |f| d\mu = 0$. So, $f \in \mathcal{L}^1(\mu)$, $\left|\int f d\mu\right| = 0 \Rightarrow \int f d\mu = 0$.

Problem 17.

Note that $f \sim g \Leftrightarrow f = g$ a.e. is an equivalence relation. So, $g = h$ a.e. Let g be measurable and $E_t = [-\infty, t]$, $N = \{x : g(x) \neq h(x)\}$. Then $h^{-1}(E_t) \setminus g^{-1}(E_t) \subset N$ and $g^{-1}(E_t) \setminus h^{-1}(E_t) \subset N$. Since μ is complete, they are all null sets. So $h^{-1}(E_t) \cup g^{-1}(E_t)$ is measurable. Therefore, $h^{-1}(E_t)$ is also measurable, which implies measurability of h.