mas651 exercises

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Problem (5.1.1)**.**

Let $(S, \mathcal{S})$ *be a state space of* X_n *where* $S = \{1, 2, \cdots N\}$ *and* $\mathcal{S} = 2^S$*. Note that* N *is an absorbing state. And* $X_1 = 1$ *with probability* 1*. For fixed* k *such that* $1 \leq k < N$, $k \leq n$,

$$P(X_{n+1} = k + 1 | X_n = k) = \frac{N - k}{N}$$

and

$$P(X_{n+1} = k | X_n = k) = \frac{k}{N}.$$

If $k > n$*, then the above are all* 0*. So it is a temporally inhomogeneous. The Markov property is trivial since the very next state only depends on the current state.*

□

Problem (5.1.2)**.**

$$P(X_4 = 2 | X_3 = 1, X_2 = 1, X_1 = 1, X_0 = 0) = (1/16)/(1/4) = 1/4$$

but

$$P(X_4 = 2 | X_3 = 1, X_2 = 0, X_1 = 0, X_0 = 0) = (1/16)/(1/8) = 1/2.$$

Thus X_n *is not a Markov chain.*

□

Problem (5.1.5)**.**

$$P(X_{n+1} = k + 1 | X_n = k) = \frac{m - k}{m} \frac{b - k}{m}$$

because we must choose a white ball in the left urn and a black ball in the right urn.

$$P(X_{n+1} = k | X_n = k) = \frac{k}{m} \frac{b - k}{m} + \frac{m - k}{m} \frac{m + k - b}{m}$$

since there are two cases, choosing both black or both white.

$$P(X_{n+1} = k - 1 | X_n = k) = \frac{k}{m} \frac{m + k - b}{m}$$

since we must choose a black ball in the left urn and a white ball in the right urn. Note that the sum of the above is 1, so there is no other transition probability.

□

Problem (5.1.6)**.**

$$P(S_{n+1} = k + 1 | S_n = k) = \frac{P(X_{n+1} = 1, S_n = k)}{P(S_n = k)}$$

where the denominator is

$$\int_{\theta\in(0,1)} P(S_n = k|\theta)dP = \binom{n}{x}\frac{x!y!}{(n+1)!} = \frac{1}{n+1}$$

for $x =$*the number of* i *such that* $U_i \le \theta$ *and* $y = n-x$*. Note that* $x = (n+k)/2$ *and* $y = (n-k)/2$ *since* $x+y = n$ *and* $x-y = k$*. The numerator is*

$$\int_{\theta\in(0,1)} P(X_{n+1} = 1, S_n = k|\theta)dP = \binom{n}{x}\frac{(x+1)!y!}{(n+2)!}$$

These are because $P(S_n = k|\theta) = \theta^x(1-\theta)^y\binom{n}{x}$ *and* $P(X_{n+1} = 1, S_n = k|\theta) = P(X_{n+1} = 1|\theta)P(S_n = k|\theta) = \binom{n}{x}\theta^{x+1}(1-\theta)^y$ *and using the kernel of beta distribution.*

Thus, the probability what we want is $(n+k+2)/(2n+4)$ *which depends on* n*. So* X_n *is temporally inhomogeneous.*

$$P(S_{n+1} = k+1|S_1 = t_1, \cdots, S_n = k) = P(X_{n+1} = 1|X_1 = t_1, \cdots, X_n = t_n)$$

where $\sum_{i=1}^n t_i = k$*. We can show the above is equal to* $P(S_{n+1} = k+1|S_n = k) = (n+k+2)/(2n+4)$ *similarly, by omitting the* $\binom{n}{x}$ *term of both denominator and numerator.*

$\square$

Problem (5.2.1)**.**

By the given hint,

$$E(1_A 1_B|\mathcal{F}_n) = E(1_A E(1_B|\mathcal{F}_n)|X_n)$$

so it suffices to show that $E(1_B|\mathcal{F}_n) = E(1_B|X_n)$*.*

Let $Y = 1_{B_n}(\omega_0)\cdots 1_{B_{n+k}}(\omega_k)$*. Then* $Y \circ \theta_n =$ *the indicator function of* $\{X_n \in B_n, \cdots, X_{n+k} \in B_{n+k}\} = B$*. By the markov property,*

$$P(B|\mathcal{F}_n) = E_{X_n} Y.$$

Let $\varphi(x) = E_x Y$ *then* $\varphi(X_n)$ *is* $\sigma(X_n)$*-measurable mapping. Thus, when* B *has a form of* $\{X_n \in B_n, \cdots, X_{n+k} \in B_{n+k}\}$ *for some nonnegative integer* k*,*

$$P(B|\mathcal{F}_n) = P(B|X_n).$$

Note that a collection of such B *generates* $\sigma(X_n, X_{n+1}, \cdots)$*.*

Now let $\mathcal{G} = \{C : P(C|\mathcal{F}_n) = P(C|X_n)\}$*. By putting* $B_{n+i} = S$ *for* $0 \leq i \leq k$*, we earn* $\Omega_0 \in \mathcal{G}$*. If* $C, D \in \mathcal{G}$ *and* $C \subset D$*, then by properties of conditional expectation,* $D \setminus C \in \mathcal{G}$*. If* $C_i \in \mathcal{G}$ *and* $C_i \uparrow C$ *then by monotone convergence theorem for conditional expectation,* $C \in \mathcal{G}$*. Thus* $\mathcal{G}$ *is a lambda system containing a collection of* B*'s which generates* $\sigma(X_n, \cdots)$*. Therefore, by Dynkin's theorem, the third equation is satisfied by any* $B \in \sigma(X_n, \cdots)$*. By the first equation, we can derive the conclusion.*

$\square$

Problem (5.2.4)**.**

First, claim that

$$P_x(X_n = y|T_y = m) = P_y(X_{n-m} = y).$$

This is because

$$\begin{aligned} P_x(X_n = y|T_y = m) &= \frac{P_x(X_n = y, T_y = m)}{P_x(T_y = m)} \\ &= \frac{\int_{T_y=m} 1_{(X_n=y)} dP_x}{P_x(T_y = m)} \\ &= \frac{\int_{T_y=m} E\left(1_{(X_n=y)}|\mathcal{F}_m\right) dP_x}{P_x(T_y = m)} \\ &= \frac{\int_{T_y=m} E_{X_m} 1_{(X_{n-m}=y)} dP_x}{P_x(T_y = m)} \\ &= \frac{P_x(T_y = m) P_y(X_{n-m} = y)}{P_x(T_y = m)}. \end{aligned}$$

Now, note that $P_x(X_n = y) = \sum_{m=1}^{n} P_x(X_n = y, T_y = m)$*. From this and* *the above discussion,*

$$\begin{aligned}
p^n(x,y) = P_x\left(X_n = y\right) &= \sum_{m=1}^{n} P_x\left(X_n = y | T_y = m\right) P_x\left(T_y = m\right) \\
&= \sum_{m=1}^{n} P_y\left(X_{n-m} = y\right) P_x\left(T_y = m\right) = \sum_{m=1}^{n} P_x\left(T_y = m\right) p^{n-m}(y,y).
\end{aligned}$$

□

Problem (5.2.6).

Fix $x \in S \setminus C$. Since $P_x(T_C = \infty) = \lim_{M\to\infty} P_x(T_C > M) < 1$, we can choose N_x and ε so that

$$P_x(T_C > M) \leq 1 - \varepsilon$$

whenever $M \geq N_x$. Note that we can choose N_x as an integer. Put $N = \max_{x \in S\setminus C} N_x$. Now we get

$$\begin{aligned}
P_y(T_C > 2N) &= \sum_{x \in S\setminus C} P_y(T_C > 2N, T_C > N, X_N = x) \\
&= \sum_{x \in S\setminus C} P_y\left(T_C > 2N | X_N = x, T_C > N\right) P_y(X_n = x, T_C > N) \\
&\leq \sum_{x \in S\setminus C} P_x(T_C > N) P_y(X_N = x, T_C > N) \\
&\leq (1-\varepsilon) \sum_{x \in S\setminus C} P_y(X_N = x, T_C > N) \\
&\leq (1-\varepsilon)^2.
\end{aligned}$$

By induction, the result follows.

Remark 1. By $k \to \infty$, we can say that $P_y\left(T_C = \infty\right) = 0$. That is, $P_y\left(T_C < \infty\right) = 1$.

□

Problem (5.2.7).

1. *It is similar to the manipulation of problem 5.2.4:*

$$\begin{aligned}
P_x\left(V_A < V_B\right) &= \sum_y P_x\left(V_A < V_B, X_1 = y\right) = \sum_y P_x\left(V_A < V_B | X_1 = y\right) P_x\left(X_1 = y\right) \\
&= \sum_y p(x,y) P_x\left(V_A < V_B | X_1 = y\right) = \sum_y p(x,y) P_y\left(V_A < V_B\right)
\end{aligned}$$

where the first term is $h(x)$ and the last term is $\sum_y p(x,y)h(y)$.

2. *I think we must further assume that h is bounded and measurable. For convenience, let $\tau = V_A \wedge V_B = V_{A\cup B}$. By the equation (5.2.2) of our textbook, we get*

$$\begin{aligned} E_x \left(h(X_{n+1})|\mathcal{F}_n\right) &= \sum_y p(X_n, y)h(y) \\ &= h(X_n) \end{aligned}$$

for $X_n \notin A \cup B$. So $h(X_{n\wedge\tau})$ is a martingale. Note that the first equality is due to (5.2.2), and the last is due to optional stopping.

3. *We assumed that h is bounded. Thus, our martingale is uniformly bounded, so the optional stopping theorem can be applied:*

$$x = E_x h(X_0) = E_x h(X_\tau) = E_x \left[E_x \left(h(X_\tau)|\mathcal{F}_\tau\right)\right]$$

where the last term is equal to

$$E_x \left[E_{X_\tau} h(X_0)\right] = E_x \left[1_{(X_\tau \in A)} + 0 \cdot 1_{(X_\tau \in B)}\right] = E_x \left[1_{(X_\tau \in A)}\right].$$

The above is because $P_x(\tau < \infty) = 1$ and h is 1 on A and 0 on B. Note that this implies the result, since $X_\tau \in A$ is equivalent to $V_A < V_B$.

□

Problem (5.2.8)**.**

Let $\tau = V_0 \wedge V_N$. Then $X_{n\wedge\tau}$ is an uniformly bounded martingale since the state space of X_n is finite. By the optional stopping theorem, we get

$$E_x X_0 = E_x X_\tau$$

where the LHS is equal to x. Note that, by the remark 1, we can say that $P_x(\tau < \infty) = 1$. Then the RHS of the above eqn is equal to $0 P_x(V_0 < V_N) + N P_x(V_N < V_0)$. Thus,

$$x = N P_x (V_N < V_0).$$

□

Problem (5.2.11)**.**

1. *It is similar to the manipulation of problem 5.2.7:*

$$\begin{aligned} E_x V_A &= \sum_{k\ge 1} P_x(V_A \ge k) \\ &= P_x(V_A \ge 1) + \sum_{k\ge 2} P_x(V_A \ge k) \\ &= 1 + \sum_{k\ge 2} P_x(V_A \ge k) \\ &= 1 + \sum_{k\ge 2}\sum_y P_x(V_A \ge k|X_1 = y) P_x(X_1 = y) \\ &= 1 + \sum_y p(x,y) \sum_{k\ge 2} P_y(V_A \ge k-1) = 1 + \sum_y p(x,y) E_y V_A \end{aligned}$$

where $P_x(V_A \geq 1) = 1$ *since* x *lies outside of* A.

Also, $E_x V_A < \infty$ *because*

$$\begin{aligned} E_x \frac{V_A}{N} &= \sum_{k \geq 1} P_x(V_A \geq kN) \\ &\leq \sum_{k \geq 1} (1-\varepsilon)^k < \infty. \end{aligned}$$

2. *I think we should assume the measurability, and boundedness of* g. *By the manipulation used in problem 5.2.7, we get:*

$$\begin{aligned} E_x(g(X_{n+1}) + n + 1 | \mathcal{F}_n) &= n + 1 + \sum_y p(X_n, y) g(y) \\ &= n + g(X_n) \end{aligned}$$

   for $X_n \notin A$. *So* $X_{n \wedge V_A} + n \wedge V_A$ *is a martingale.*

3. *From the boundedness of* g *and the fact that* V_A *is* L^1 *function, our martingale is uniformly integrable. Thus we can apply optional stopping theorem:*

$$E_x g(X_0) = E_x [V_A + g(X_{V_A})]$$

   where the first term is $g(x)$ *and the second term is* $E_x V_A + E_x g(X_{V_A})$. *But* X_{V_A} *lies in* A *and* g *is 0 on* A. *Thus the second term of the equation is* $E_x V_A$.

$\square$