HW

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Notation: μ is Lebesgue measure, 1_A is characteristic function, X is Euclidean space.

Problem 11.

It is obvious that $|f_k| \leq |f|$ and $|f| \in \mathcal{L}^1(\mu)$. So, if each f_ks are measurable, by dominated convergence thm, done. Actually, $f_k = f \cdot 1_{A_k \cap E_k}$ where $A_k = [-k, k]$ and $E_k = f^{-1}([-k, k])$. There exists sequence of nonnegative simple function $\{s_i : X \to (-\infty, \infty)\}$ which converges to f since f is measurable. So, $\lim_{i\to\infty} s_i 1_{a_k \cap E_k} = f_k$ is measurable because product of finite measurable function is measurable and limit of measurable function is measurable.

Also note that $1_{A_k \cap E_k} \to 1$ as $k \to \infty$ so $f_k \to f$.

Problem 12.

Likewise, it is enough to show that $f(x)e^{-\frac{|x|^2}{k}} = f_k(x)$ is measurable because $|f_k| \leq |f| \in \mathcal{L}^1$ and $f_k \to f$. $e^{-\frac{|x|^2}{k}}$ is continuous, hence Borel measurable, hence Lebesgue measurable. There exists sequence of nonnegative simple function $\{s_i : X \to (-\infty, \infty)\}$ which converges to f since f is measurable. So, $\lim_{i\to\infty} s_i e^{-\frac{|x|^2}{k}} = f_k$ is measurable because product of finite measurable function is measurable and limit of measurable function is measurable.

Problem 13. 'alternative proof for problem 2.42'

For each $x \in X$, there are at most $d \in \mathbb{N}$ distinct A_k containing x. Fix positive integer N. Clearly $\sum_{k=1}^{N} 1_{A_k} \leq d1_A$ where $A = \bigcup_{i=1}^{\infty} A_i$. By integrating both sides, $\sum_{k=1}^{N} \mu(A_k) \leq d\mu(A)$ by linearity of integral operator. Since N is arbitrary, we got the result in Problem 2.42.

Problem 14.

Let $\mathcal{I}$ be set of all sequences which are strictly increasing positive integers and length m. Such set is countable. Now, $\bigcup_{i\in\mathcal{I}} \bigcap_{j=1}^{m} A_{i_j} = E_m$ and it is measurable.

Similar to Problem 13, $m1_{E_m} \leq \sum_{i=1}^{\infty} 1_{A_i}$. So,

$$m\mu(E_m) = \int m1_{E_m} d\mu \leq \int \sum_{i=1}^{\infty} 1_{A_i} d\mu = \sum_{i=1}^{\infty} \int 1_{A_i} d\mu = \sum_{i=1}^{\infty} \mu(A_i)$$

by monotone convergence thm.

section C.

Problem 16.

Clearly, $|f| = 0$ almost everywhere, and $|f|$ is measurable. Consider nonnegative finite simple function $s \leq |f|$. Then $s = 0$ almost everywhere and $s = \sum_{i=1}^{N} \alpha_i 1_{A_i}$, where A_i is null set if $\alpha_i \neq 0$. Therefore, $\int s d\mu = 0$, which implies $\int |f| d\mu = 0$. So,

$f \in \mathcal{L}^1(\mu)$, $\left|\int f d\mu\right| \leq \int |f| d\mu = 0 \Rightarrow \int f d\mu = 0$.

Problem 17.

Note that $f \sim g \Leftrightarrow f = g$ a.e. is an equivalence relation. (transitivity can be verified from contrapositive of $g = f$ a.e. and $f = h$ a.e. $\Rightarrow g = h$ a.e.) So, $g = h$ a.e. Let g be measurable and $E_t = [-\infty, t]$, $N = \{x : g(x) \neq h(x)\}$. Then $h^{-1}(E_t) \setminus g^{-1}(E_t) \subset N$ and $g^{-1}(E_t) \setminus h^{-1}(E_t) \subset N$. Since μ is complete, they are all null sets. So $h^{-1}(E_t) \cup g^{-1}(E_t)$ is measurable because $g^{-1}(E_t)$ is measurable. Therefore, $h^{-1}(E_t)$ is also measurable, which implies measurability of h. Similarly, measurability of h implies measurability of g.