# mas550 homework

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**Problem** (1.1.2)**.**

*Let* $A = \Pi_{i=1}^{d}(a_i, b_i]$*. Then*

$$A = \left(\Pi_{i=1}^{d}[a_i - 1, b_i]\right) \cap \left(\Pi_{I=1}^{d}(a_i, b_i + 1)\right)$$

*which is intersection of open set and closed set. So,* $A \in \mathcal{R}^d$ *therefore* $\sigma(S_d) \subset \mathcal{R}^d$*.*

*On the other hand, let* $B = \Pi_{i=1}^{d}(a_i, b_i)$ *where* $-\infty < a_i < b_i < \infty$*. We can choose sequences* $\{a_{i,j}\}_{j=1}^{\infty}$ *and* $\{b_{i,j}\}_{j=1}^{\infty}$ *for each* $1 \leq i \leq d$ *such that* $a_{i,j} \downarrow a_i$ *and* $b_{i,j} \uparrow b_i$*. Then* $B_n = \Pi_{i=1}^{d}(a_{i,n}, b_{i,n}] \uparrow B$*. So* $B$ *is a countable union of open rectangles, hence* $B \in \sigma(S_d)$*. Since such* $B$ *forms basis of topology on* $\mathbb{R}^d$*, we can conclude that* $\mathcal{R}^d \subset \sigma(S_d)$*.*

**Problem** (1.2.3)**.**

*Let* $F$ *be a distribution function. It is nonnegative, nondecreasing. So* $\lim_{y\downarrow x} F(y)$ *and* $\lim_{y\uparrow x} F(y)$ *always exist. Let* $x$ *be a point where* $F$ *is discontinuous. Since* $F$ *is discontinuous at* $x$*, we can assume without loss of generality* $\lim_{y\downarrow x} F(y) > F(x)$*. Choose a rational number* $q_x \in (F(x), \lim_{y\downarrow x} F(y))$*. Then function* $x \mapsto q_x$ *is injective since* $F$ *is nondecreasing. So there is injection from set of discontinuities to rational numbers. Now we can conclude that set of discontinuities is at most countable.*

**Problem** (1.3.4)**.**

*(a) Let* $f : \mathbb{R}^d \to \mathbb{R}$ *be a continuous function. Consider* $\mathcal{B} = \left\{U \subset \mathbb{R} : f^{-1}(U) \in \mathcal{R}^d\right\}$*. It is well known that* $\mathcal{B}$ *is a* $\sigma$*-field. By continuity of* $f$*,* $\mathcal{B}$ *contains every open set of* $\mathbb{R}$*, hence* $\mathcal{R} \subset \mathcal{B}$*. Therefore* $f$ *is a measurable function.*

*(b) Let* $\mathcal{F}$ *be a* $\sigma$*-field that makes all the continuous functions measurable. Let* $\pi_i : \mathbb{R}^d \to \mathbb{R}$ *be the projection on* $i$*-th factor, which is continuous. Then* $\cap_{i=1}^{d}\pi_i^{-1}((a_i, b_i)) = \Pi_{i=1}^{d}(a_i, b_i) \in \mathcal{F}$*. Since* $\mathcal{F}$ *contains every open rectangles in* $\mathbb{R}^d$*, we can conclude that* $\mathcal{R}^d \subset \mathcal{F}$*. This means* $\mathcal{R}^d$ *is the smallest such* $\sigma$*-field. The fact that* $\mathcal{R}^d$ *makes all the continuous functions measurable is written in (a).*

**Problem** (1.3.1)**.**

*Since $\sigma(X)$ is the smallest $\sigma$-field which makes $X$ measurable, it sufficient to show that $X$ is measurable with respect to $\sigma\left(X^{-1}(\mathcal{A})\right)$.*

*Let $X : \Omega \to S$. It is clear that $\{X \in A\} \in \sigma\left(X^{-1}(\mathcal{A})\right)$ for all $A \in \mathcal{A}$. But by theorem 1.3.1, since $\mathcal{A}$ generates $\mathcal{S}$, $X$ is measurable with respect to $\sigma\left(X^{-1}(\mathcal{A})\right)$.*

*Therefore we can conlude that $\sigma\left(X^{-1}(\mathcal{A})\right) \subset \sigma(X)$, and reverse inclusion is canonical since $X^{-1}(\mathcal{A}) \subset \sigma(X)$.*

**Problem** (1.4.1)**.**

*Let $E_n = \left\{x : f(x) > \frac{1}{n}\right\}$. Then $\int f d\mu \geq \int_{E_n} f d\mu \geq \int_{E_n} \frac{1}{n} d\mu = \frac{1}{n}\mu(E_n)$. Therefore $\mu(E_n) = 0$ for every positive integer $n$. So, $\mu(\{f > 0\}) = \sum_{n=1}^{\infty} \mu(E_n) = 0$. This says $f = 0$ a.e.*

**Problem** (1.4.2)**.** *Since $E_{n+1,2m} \cup E_{n+1,2m+1} = E_{n,m}$ and $\frac{2m+1}{2^{n+1}} \geq \frac{m}{2^n}$, we can easily see that $\sum_{m\geq 1} \frac{m}{2^n}\mu(E_{n,m})$ is monotonically increasing as $n$ grows.*

*For every positive integer $M$, $\sum_{m=1}^{M} \frac{m}{2^n}\mu(E_{n,m}) \leq \int f d\mu$. So $\sum_{m\geq 1} \frac{m}{2^n}\mu(E_{n,m}) \leq \int f d\mu$.*

*Let $s_n = \sum_{m=1}^{n2^n} \frac{m}{2^n} 1_{E_{n,m}}$. Then $\int s_n d\mu \leq \sum_{m\geq 1} \frac{m}{2^n}\mu(E_{n,m}) \leq \int f d\mu$. But $s_n \uparrow f$ monotonically. By monotone convergence theorem, $\lim_{n\to\infty} \int s_n d\mu = \int f d\mu$. Hence by sandwich lemma, the desired result follows.*

**Problem** (1.5.1).

*First, we will show that* $|g| \leq \|g\|_\infty$ *a.e.*

*It is true because*

$$\begin{aligned}\mu\left(|g| > \|g\|_\infty\right) &= \mu\left(\bigcup_{n=1}^{\infty}\left\{|g| \geq \|g\|_\infty + \frac{1}{n}\right\}\right) \\ &\leq \sum_{n=1}^{\infty}\mu\left(\left\{|g| > \|g\|_\infty + \frac{1}{n}\right\}\right) \\ &= 0\end{aligned}$$

*by definition of* $\|g\|_\infty$.

*Hence* $|g| \leq \|g\|_\infty$ *a.e.*

*Then,* $\int |fg| d\mu \leq \|g\|_\infty \int |f| d\mu = \|g\|_\infty \|f\|_1$.

**Problem** (1.5.3).

(a) *Since* $p > 1$, $x \mapsto |x|^p$ *is convex function.* $|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p)$ *follows from convexity of* $|x|^p$.

$\int |f+g|^p d\mu \leq \int 2^p |f|^p d\mu + \int 2^p |g|^p d\mu$. *Therefore finiteness of* $\|f\|_p$ *and* $\|g\|_p$ *leads* $\|f+g\|_p < \infty$.

*Now, consider* $\int |f+g|^p d\mu = \int |f+g||f+g|^{p-1} d\mu \leq \int |f||f+g|^{p-1} d\mu + \int |g||f+g|^{p-1} d\mu$. *Let* $q$ *be Holder conjugate of* $p$. *Then by applying Holder inequality, we get* $\|f+g\|_p^p \leq \|f+g\|_p^{p/q}(\|f\|_p + \|g\|_p)$. *Simple calculating leads Minkowski's inequality.*

(b) *First consider* $p = 1$. *By using triangle inequaltiy, the result follows directly. Next consider* $p = \infty$. $|f+g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty$ *a.e. Therefore* $\|f+g\|_\infty \leq \|f\|)\infty + \|g\|_\infty$.

**Problem** (1.6.8).

*First assume* $g = 1_A$. *Then* $\int g d\mu = \mu(A) = \int_A f(x)dx = \int 1_A f dm$ *where* $m$ *is Lebesgue measure.*

*Next, assume* $g = \sum_i a_i 1_{A_i}$, *simple function. Then* $\int g d\mu = \sum_i a_i \mu(A_i) = \sum_i a_i \int 1_{A_i} f dm$.

*Next, assume* $g$ *is nonnegative measurable. Let* $\{s_n\}_{n=1}^{\infty}$ *be increasing sequence of simple function converges to g pointwisely. Then* $\int g d\mu = \lim_{n\to\infty} \int s_n d\mu =$

$\lim_{n\to\infty} \int s_n f dm$. *But* $s_n f \uparrow gf$ *since* $f$ *is nonnegative. By monotone convergence theorem, we can get* $\int g d\mu = \int gf dm$.

*Last, assume* $g$ *is integrable function. We can decompose* $g$ *by* $g = g^+ - g^-$. *Applying 3rd step for* $g^+, g^-$ *each, we can get* $\int g d\mu = \int g^+ f dm - \int g^- f dm = \int gf dm$ *since* $f$ *is nonnegative.*

**Problem** (1.6.13)**.**

*Since* $X_n \uparrow X$, $X_n^+ \uparrow X^+$ *and* $X_n^- \downarrow X^-$. *And note that* $X_n^- \leq X_1^-$ *which is integrable. Apply monotone convergence theorem to* $X_n^+$ *and apply dominated convergence theorem to* $X_n^-$ *to get* $\lim EX_n = \lim EX_n^+ - \lim EX_n^- = EX^+ - EX^- = EX$.

**Problem** (1.7.1)**.**

*We need to show that* $\int_{X\times Y} |f| d(\mu_1 \times \mu_2) < \infty$.

*Since* $|f|^{\pm}$ *is nonnegative, by Fubini's theorem,* $\int_X \int_Y |f|^{\pm} \mu_2(dy)\mu_1(dx) < \infty$. *Then, their sum is also finite, and the sum is* $\int_{X\times Y} |f| d(\mu_1 \times \mu_2)$ *by Fubini's theorem. This leads the conclusion of the exercise.*

*Corollary is immediate if we take* $\mu_1 = c$ *and* $\mu_2 = \mu$.

**Problem** (1.7.3)**.**

1.

$$\begin{aligned}\int_{(a,b]} \{F(y) - F(a)\}\, dG(y) &= \int_{(a,b]} \int_{(a,y]} 1\mu(dx)\nu(dy)\\ &= \int_{a<x\le y\le b} 1 d(\mu \times \nu)\\ &= \mu \times \nu(1 < X \le Y \le b)\end{aligned}$$

*by Fubini's theorem on nonnegative function* 1.

2.

$$\begin{aligned}\int_{(a,b]} F(y)dG(y) &= \int_{(a,b]} \int_{-\infty}^{y} 1\mu(dx)\nu(dy)\\ &= \int_{(-\infty,a]} \int_{(a,b]} 1\nu(dy)\mu(dx) + \int_{(a,b]} \int_{[x,b]} \nu(dy)\mu(dx)\\ &= F(a)\{G(b) - G(a)\} + G(b)\{F(b) - F(a)\}\\ &- \int_{(a,b]} G(x)\mu(dx) + \int_{(a,b]} G(x) - G(x^-)\mu(dx)\end{aligned}$$

*We can get similar result for* $\int_{(a,b]} G(y)dF(y)$. *By simple calculation, we get the conclusion of (2).*

3. *If* $F = G$ *continuous, Then* $\mu(\{x\}) = \nu(\{x\}) = F(x) - F(x^-) = G(x) - G(x^-) = 0)$. *Therefore, by using (2), we can get the conclusion.*

**Problem** (2.1.3)**.**

1. *If* $h(\alpha) = 0$ *for some* $\alpha > 0$, *by mean value theorem,* $h'(\beta) = 0$ *for some* $\beta \in (0, \alpha)$. *It contradicts to* $h'(x) > 0$ *for positive* $x$. *Therefore* $h > 0$ *for positive* $x$.

*$x = y$ iff $\rho(x, y) = 0$ iff $h(\rho(x, y)) = 0$. And $h(\rho(x, y)) = h(\rho(y, x))$ since $\rho(x, y) = \rho(y, x)$.*

*Now consider $x \geq y > 0$ and $\frac{h(x+y)-h(x)}{y} = h'(x+\theta)$ and $\frac{h(y)}{y} = h'(y-\delta)$. Since $h'$ is decreasing, $h(x + y) - h(x) \leq h(y)$. Using this, we can prove triangle inequality of $h \circ \rho$.*

2. *$h(x) = 1 - \frac{1}{1+x}$ so $h'(x) = \frac{1}{(1+x)^2}$ and $h''(x) = \frac{-2}{(1+x)^3}$. Given $h$ satisfies all of (1).*

**Problem** (2.1.9)**.**

*Let $\mathcal{A}_1 = \{\{1, 2\}, \{1, 3\}\}, \mathcal{A}_2 = \{\{1, 4\}\}$. For $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, $P(A_1 \cap A_2) = P(A_1)P(A_2) = 1/4$. But, $\sigma(\mathcal{A}_1) = 2^\Omega$ and $\sigma(\mathcal{A}_2) = \{\Omega, \{1, 4\}, \{2, 3\}, \emptyset\}$. They are not independent by considering $A_1 = \{2, 3, 4\}$ and $A_2 = \{2, 3\}$.*

**Problem** (2.2.3)**.**

(a) *$f(U_i)$'s are iid because* $P\left(\bigcap_i (f \circ U_i) \in B_i\right) = P\left(\bigcap_i \left\{U_i \in f^{-1}(B_i)\right\}\right) = \Pi_i P\left(U_i \in f^{-1}(B_i)\right) = \Pi_i P\left(f(U_i) \in B_i\right)$. *Also, for borel set* $B$, $P\left(f(U_i) \in B\right) = P\left(U_i \in f^{-1}(B_i)\right)$ *are all same for* $i$.

$Ef(U_i) = \int_0^1 f(x)dx$, $E|f(U_i)| = \int_0^1 |f(x)|dx < \infty$.

*Now, by WLLN,* $\frac{\sum f(U_i)}{n}$ *converges to* $\int_0^1 f(x)dx$ *in probability.*

(b) $P\left(|I_n - I| > a/n^{0.5}\right) \leq \frac{n}{a^2} E|I_n - I|^2 = \frac{n}{a^2} Var(I_n) = Var(\sum f(U_i))/na^2 = Var(f(U_i))/a^2 = \left[\int_0^1 f(x)^2 dx - \left(\int_0^1 f(x)dx\right)^2\right]/a^2$.

**Problem** (2.2.5)**.**

*Note that* $P(X_i \leq a) = 0$ *for all* $a < e$.

$xP(X_i > x) = \frac{e}{\log x} \to 0$ *as* $x \to \infty$.

$E|X_i| = EX_i = \int_e^\infty P(X_i > x)dx = \int_e^\infty \frac{e}{x \log x} dx = \infty$ *since* $X_i \geq 0$ *almost surely.*

*But* $\mu_n = \int_{|X_i| \leq n} X_i dP \uparrow EX_i = \infty$ *by monotone convergence theorem.*

*Now, theorem 2.2.12 says* $\frac{s_n}{n} - \mu_n$ *converges to* $0$ *in probability.*

**Problem** (2.3.5)**.**

(a) *Let* $F_N = \{Y \le N\}$ *and* $Y_n = Y 1_{F_n}$. *Then* $EY_n \uparrow EY$ *by MCT. So choose* $N$ *so that* $EY - EY_N < \varepsilon$. *Now consider* $|EX_n - EX| \le E|X_n - X| \le \int_{|X_n - X|>\varepsilon} 2Y dP + \int_{|X_n-X|\le\varepsilon} |X_n - X| dP \le \varepsilon + \int_{|X_n-X|>\varepsilon} 2Y dP$.

*Let* $E_n = \{|X_n - X| > \varepsilon\}$. *Then* $\int_{E_n} 2Y dP = \int_{E_n} 2Y - 2Y_N + 2Y_N dP \le E(2Y - 2Y_N) + 2NP(E_n)$, *where the last term goes to* $0$ *as* $n \to \infty$.

(b) *Let* $h, g$ *be continuous functions,* $h(0) = 0$, $g > 0$ *for large* $x$, $|h|/g \to 0$ *as* $|x| \to \infty$, *and* $Eg(X_n) \le C < \infty$.

*Choose* $M$ *so large that* $g > 0$ *on* $|x| > M$. $\varepsilon_M = \sup_{|x|\ge M} |h|/g$ *and* $\bar{Y} = Y 1_{|Y|\le M}$.

*Then* $|Eh(X_n) - Eh(X)| \le E|h(X_n) - Eh(\bar{X}_n)| + E|h(\bar{X}_n - h(\bar{X})| + E|h(\bar{X}) - h(X)|$. *First term and third term are bounded by* $\varepsilon_M C$ *which goes to* $0$ *as* $M \to \infty$. *And the second term goes to* $0$ *as* $n \to \infty$ *by bounded convergence thm.*

*Therefore the conclusions hold.*

**Problem** (2.3.6.)**.**

(a) *We already show that* $\rho(x, y) = \frac{|x-y|}{1+|x-y|}$ *is a metric in problem 2.1.3.*

*First consider* $d(X, Y) = 0$ *iff* $E\frac{|X-Y|}{1+|X-Y|} = 0$ *iff* $\frac{|X-Y|}{1+|X-Y|} = 0$ *a.s. iff* $X = Y$ *a.s.*

*Next, it is trivial to check* $d(X, Y) = d(Y, X)$.

*Lastly,* $d(X, Z) = E\rho(X, Z) \le E\left(\rho(X, Y) + \rho(Y, Z)\right) = E\rho(X, Y) + E\rho(Y, Z) = d(X, Y) + d(Y, Z)$.

*Therefore given function is a metric of class of random variables.*

(b) *First assume* $X_n \to X$ *in probability. Then* $\frac{|X_n-X|}{1+|X_n-X|} \le 1$ *and it goes to* $0$ *in probability. So bounded convergence thm implies* $d(X_n, X) \to 0$.

*Next assume* $d(X_n, X) \to 0$ *as* $n \to 0$.

$$
\begin{aligned}
P(|X_n - X| > \varepsilon) &= P\left(\frac{|X_n - X|}{1+|X_n - X|} > \frac{\varepsilon}{1+\varepsilon}\right) \\
&\le E\frac{|X_n - X|}{1+|X_n - X|}\frac{1+\varepsilon}{\varepsilon} \\
&= d(X_n, X)\frac{1+\varepsilon}{\varepsilon} \to 0
\end{aligned}
$$

*by Markov's inequality.*

**Problem** (2.3.8)**.**

*Independence of* $A_n$ *implies independence of* $A_n^c$*. Let* $B_n = \cap_{k=1}^n A_k^c$*. Then* $0 = P\left(\bigcap_{n=1}^\infty A_n^c\right) = \lim_{n\to\infty} P(B_n)$.

*So, for arbitrary* $\varepsilon > 0$*, there is a positive integer* $N_\varepsilon$ *such that* $n \ge N_\varepsilon$ *implies* $P(B_n) = P\left(\cap_{k=1}^n A_k^c\right) = \Pi_{k=1}^n \left(1 - P(A_k)\right) = e^{\sum_{k=1}^n \log(1-P(A_k))} < \varepsilon$*. But as* $n \to \infty$

$$
\lim_{n\to\infty} e^{\sum_{k=1}^n \log(1-P(A_k))} = 0
$$

*This means that* $\sum_{k=1}^\infty \log\left(1 - P(A_k)\right) = -\infty$*, therefore* $\log\left(1 - P(A_k)\right)$ *does not converge to* $0$*, which is equivalent to that* $P(A_k)$ *does not converge to* $0$*. Therefore* $\sum_{n=1}^\infty P(A_n) = \infty$.

**Problem** (2.3.12)**.**

*Let* $\Omega = \{\omega_i : i \in \mathbb{N}\}$*. Without loss of generality, we can assume* $P(\{\omega_i\}) > 0$ *for all* $i \in \mathbb{N}$.

*If there is* $\omega_i$ *such that* $X_n(\omega_i)$ *does not converge to* $X(\omega_i)$*, then for some* $\varepsilon > 0$*, and for all* $N \in \mathbb{N}$*, there is* $n_N \ge N$ *but* $|X_{n_N}(\omega_i) - X(\omega_i)| > \varepsilon$.

*This means* $\{|X_{n_N} - X| > \varepsilon\}$ *contains* $\omega_i$ *for all* $N$*. So* $0 < P(\{\omega_i\}) \le P\left(|X_{n_N} - X| > \varepsilon\right)$.

*But* $X_n \to X$ *in probability implies* $X_{n_N} \to X$ *in probability. This contradicts to above. Therefore there is no such* $w_i$ *hence* $X_n$ *converges to* $X$ *almost surely.*

**Problem** (2.5.2).

*If* $E|X_1|^p = \infty$*, then for each positive integer* $k$*,* $E|X_1|^p \leq \sum_n P(|X_1|^p > nk) = \infty$*. But* $P(|X_1|^p > nk) = P(|X_n| > (nk)^{1/p})$*. Then by Borel Cantelli lemma* $P(|X_n| > (nk)^{1/p} i.o.) = 1$*. That is,* $\limsup_n |X_n|/n^{1/p} \geq k^{1/p}$ *for infinitely many* $k$*. Therefore* $\limsup_n |X_n|/n^{1/p} = \infty$*.*

*But* $|X_n| \leq |S_n| + |S_{n-1}|$*. That leads* $\limsup_n |S_n|/n^{1/p} = \infty$*. By taking contrapositive, we get the conclusion.*

**Problem** (2.5.5).

*The first one leads the second one directly because Kolmogorov's three series lemma with* $A = 1$ *tells it.*

*The second one implies the third one because* $\frac{X_n}{1+X_n} \leq 1_{X_n > 1} + X_n 1_{X_n \leq 1}$ *and monotone convergence theorem.*

*The third one implies* $\sum_n \frac{X_n}{1+X_n} < \infty$ *a.s. And convergence of* $\sum_n \frac{a_n}{1+a_n}$ *for* $a_n \geq 0$ *gives the convergence of* $\sum_n a_n$*. It is because* $lim a_n = 0$ *and* $|a_N + \cdots a_{N+n}| \leq (1+\varepsilon)\left|\frac{a_N}{1+a_N} + \cdots + \frac{a_{N+n}}{1+a_{N+n}}\right|$ *for large* $N$*. Therefore* $\sum_{k=1}^n a_k$ *is cauchy hence converges. Therefore* $\sum_n X_n$ *converges a.s.*

**Problem** (3.2.4).

*Since* $X_n \to X_\infty$ *in distribution, there are* $Y_n =_d X_n$ *and* $Y_\infty =_d X_\infty$ *such that* $Y_n \to Y_\infty$ *a.s.*

*Then* $g(Y_n) \geq 0$ *and* $g(Y_n) \to g(Y_\infty)$ *a.s. Therefore by Fatou's lemma,* $\liminf Eg(Y_n) \geq Eg(Y_\infty)$ *which is equivalent to* $\liminf Eg(X_n) \geq Eg(X_\infty)$ *since* $X_n =_d Y_n$ *for all* $n \in \mathbb{N} \cup \infty$*.*

**Problem** (3.2.5).

*There are* $Y_n \to Y_\infty$ *a.s. and distribution function of* $Y_n$ *is equal to* $F_n$*.* $F_\infty = F$*.*

*Then by theorem 1.6.8,* $Eh(Y_n) \to Eh(Y_\infty)$ *which is equivalent to* $\int h(x)dF_n(x) \to \int h(x)dF(x)$ *because distribution function of* $Y_n$ *is* $F_n$*.*