HW

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Comment: In problem 19 and 21, we treat the functions in L^1. So $f(x) - g(x)$ is well defined except on null set N which is $\{x : f(x) = \pm\infty$ or $g(x) = \pm\infty\}$.

Problem 19.

It is trivial to check whether f is measurable or not. (Surely measurable.)
There are countably many corresponding null sets. Let N be a union of such null sets. Then all assumptions of problem are valid except on N. Note that $\lim_{k\to\infty} |f_k(x)| \le \lim_{k\to\infty} g_k(x) = g(x)$ almost everywhere and $g \in L^1$ so $f \in L^1$. Now consider $g_k + g - |f - f_k| = h_k$ which is nonnegative measurable function except on N. By applying Fatou's lemma, we can get :

$$\int \liminf_{k\to\infty} h_k d\lambda \le \liminf_{k\to\infty} \left(\int g d\lambda + \int (g_k - |f - f_k|)\, d\lambda \right) = \int g d\lambda - \limsup_{k\to\infty} \left(\int (|f - f_k| - g_k)\, d\lambda \right) \le 0$$

Therefore $\int g d\lambda + \limsup_{k\to\infty} \left(\int (|f - f_k| - g_k)\, d\lambda \right) \le 0$. But we know that $\int g d\lambda = \limsup \int g_k d\lambda$ and $\limsup (a_k + b_k) \le \limsup (a_k) + \limsup (b_k)$. Thus,

$$\limsup_{k\to\infty} \left(\int (g_k + |f - f_k| - g_k)\, d\lambda \right) \le 0$$

which means $\lim_{k\to\infty} \int |f - f_k|\, d\lambda = 0$ because limsup of nonnegative sequence goes to positive (or infinity) when it does not go to 0.

Then $\lim \left| \int f_k d\lambda - \int f d\lambda \right| = 0$, which implies conclusion of our problem.

section D. Integration over subsets of $\mathbb{R}^n$.

Problem 20.

It is obvious that $\{x \in E : -\infty \le f(x) \le t\} \subset E$ for every $t \in [-\infty, \infty]$. By completeness of Lebesgue measure λ, every subset of null set is measurable. Hence f is measurable.

Now consider $0 \le s \le f_+ 1_E$ where s is nonnegative simple function. s can have positive value on subset of E. Therefore $\int s d\lambda = 0$. So $\int f_+ 1_E d\lambda = 0$. Similarly, we can show that $\int f_- 1_E d\lambda = 0$. Thus we get $\int_E f d\lambda = 0$ for all measurable function defined on E.

Problem 21.

There are countably many corresponding null sets. Let N be a union of such null sets. Then all assumptions of problem are valid except on N. Now consider $|f - f_k| \le f + f_k = g_k \in L^1$ a.e. and $\lim g_k = 2f$ exists a.e. and $\lim \int g_k d\lambda = \int 2 f d\lambda$. So all the assumptions of problem 19 are satisfied.

Therefore $\lim_{k\to\infty} \int |f_k - f|\, d\lambda = \int \lim_{k\to\infty} |f_k - f|\, d\lambda = 0$.

From above and $\int_E f d\lambda \leq \int f d\lambda$ for nonnegative measurable function f, we can get

$$\lim \left| \int_E f d\lambda - \int_E f_k d\lambda \right| \leq \lim \int_E |f - f_k| \, d\lambda = 0$$

So $\lim_{k\to\infty} \int_E f_k d\lambda = \lim_{k\to\infty} \int_E f d\lambda$.

section E. Generalization of Measure Space.

Problem 22.

Let $A_1 = A$, $A_2 = B \setminus A$, $A_k = \emptyset$ for $k \geq 3$. Then by countable additivity of μ, $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu(B) = \mu(A) + \mu(B \setminus A)$. Therefore $\mu(A) \leq \mu(B)$ because $B \setminus A \in \mathcal{M}$ and $\mu(B \setminus A) \geq 0$.

Problem 24.

Let $B_1 = A_1$, $B_k = A_k \setminus \bigcup_{j=1}^{k-1} A_j \subset A_k$ for $k \geq 2$. Then B_k's are pairwise disjoint and in $\mathcal{M}$. Also $\bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k$. Therefore $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(B_k) \leq \sum_{k=1}^{\infty} \mu(A_k)$.

Problem 25.

Let $B_1 = A_1$ and $B_k = A_k \setminus A_{k-1}$ for $k \geq 2$. Then B_k's are pairwise disjoint and union from index 1 to index $N \in \mathbb{N} \cup \infty$ is same for that of A_k's. Therefore $\mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(B_k) = \lim \sum_{k=1}^{n} \mu(B_k) = \lim \mu(A_n)$.