#4.2

$$P_0(R \le 1+t) = 1 - \int_{-\infty}^{\infty} p_1(0,y)\, P_y(T_0 > t)\, dy = \int_{-\infty}^{\infty} p_1(0,y)\, P_y(T_0 \le t)\, dy$$

$$= 2\int_0^{\infty} p_1(0,y)\, P_y(T_0 \le t)\, dy = 2\int_0^{\infty} p_1(0,y) \int_0^t P_y(T_0 = s)\, ds\, dy$$

∴ Fubini, $\int_0^{\infty} 2\, p_1(0,y)\, P_y(T_0 = s)\, dy$ = goal.

* $P_y(T_0 = s) = P_0(T_y = s)$,

dist. of T_y is $\int_0^t \frac{1}{\sqrt{2\pi s^3}}\, y e^{-\frac{y^2}{2s}}\, ds$

∴ $P_0(T_y = s) = \frac{1}{\sqrt{2\pi t^3}}\, y e^{-\frac{y^2}{2t}}$

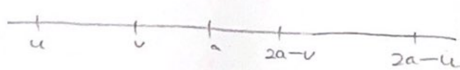
So, $\int_0^{\infty} 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \cdot \frac{1}{\sqrt{2\pi t^3}}\, y e^{-\frac{y^2}{2t}}\, dy$.

$$= \frac{1}{\pi}\int_0^{\infty} e^{-\frac{t+1}{2t} y^2}\, y \cdot \frac{1}{\sqrt{t^3}}\, dy = \frac{t}{t+1} \cdot \frac{1}{\pi} \cdot t^{\frac{3}{2}} = \frac{1}{\pi\sqrt{t}\,(t+1)} \quad \blacksquare$$

#4.3

a) $Y_s(\omega) = \begin{cases} 1 & s<t,\ u < \omega(t-s) < v \\ 0 & \text{o.w.} \end{cases}$ | $\bar{Y}_s(\omega) = \begin{cases} 1 & s<t,\ 2a-v < \omega(t-s) < 2a-u \\ 0 & \text{o.w.} \end{cases}$

Note that $2a-u$: reflection of u to $x=a$



∴ By symmetry of BM with $B_0 = a$, $E_a Y_s = E_a \bar{Y}_s$.

Let $S = \inf\{s : s<t,\ B_s = a\}$. Then $\{S<\infty\} = \{T_a < t\}$

∴ $P_0(T_a < t,\ u < B_t < v) = E_0(Y_S \circ \theta_S\ ;\ S < \infty)$

$$= E_0\big(E_0(Y_S \circ \theta_S \mid \mathcal{F}_S)\ ;\ S<\infty\big) \overset{\text{S.Markovian}}{=} E_0(E_a Y_S\ ;\ S<\infty) = E_0(E_a \bar{Y}_S\ ;\ S<\infty)$$

$$\overset{\text{S.Markovian}}{=} E_0\big(E_0(\bar{Y}_S \circ \theta_S \mid \mathcal{F}_S)\ ;\ S<\infty\big) = E_0(\bar{Y}_S \circ \theta_S\ ;\ S<\infty) = \cancel{P_0(T_a<t,\ 2a-v<B_t}$$

$$= P_0(T_a < t,\ 2a-v < B_t < 2a-u)$$

where first "=" is due to the def of $Y_s(\omega)$

and last "=" is due to the def. of $\bar{Y}_s(\omega)$ ■

#5.3.

i) $T_a < T_b \Rightarrow \sigma = T_a$.

$$\therefore E_x e^{-\lambda T_a} = \underbrace{E_x(e^{-\lambda T_a} ; T_a < T_b)}_{=u} + E_x(e^{-\lambda T_a} ; T_b < T_a)$$

So, consider the last term.

$$E_x(e^{-\lambda T_a} ; T_b < T_a) = E_x(e^{-\lambda(T_a - \sigma + \sigma)} ; T_b < T_a)$$

$$= E_x\left(E_x(e^{-\lambda(T_a - \sigma)} e^{-\lambda\sigma} \mid \mathcal{F}_\sigma) ; T_b < T_a \right).$$

$$= E_x\left(e^{-\lambda\sigma} E_x(e^{-\lambda(T_a - \sigma)} \mid \mathcal{F}_\sigma) ; T_b < T_a \right)$$

$$= E_x(e^{-\lambda\sigma}(E_b e^{-\lambda T_a}) ; T_b < T_a) = E_b e^{-\lambda T_a} \cdot \underbrace{E_x(e^{-\lambda\sigma} ; T_b < T_a)}_{=v}$$

ii) $a < x < b$

So by 7.5.5, $E_x e^{-\lambda T_b} = e^{\sqrt{2\lambda}(x-b)}$

$$E_x e^{-\lambda T_a} \underset{\substack{\uparrow \\ \text{symmetry} \\ \text{of BM.}}}{=} E_{(-x)} e^{-\lambda T_{(-a)}} = e^{\sqrt{2\lambda}(a-x)}.$$

$-x < -a$.

So, by i), we have $e^{\sqrt{2\lambda}(a-x)} = u + v e^{\sqrt{2\lambda}(a-b)}$ ... ①

$(\because a<b \Rightarrow E_b e^{-\lambda T_a} = E_{-b} e^{-\lambda T_{-a}} = e^{\sqrt{2\lambda}(-b+a)})$

Also, by interchanging a, b of i),

we have $E_x e^{-\lambda T_b} = v + u E_a e^{-\lambda T_b}$.

$$\Leftrightarrow \quad e^{\sqrt{2\lambda}(x-b)} = v + u e^{\sqrt{2\lambda}(a-b)} \quad \cdots ②.$$

$e^{\sqrt{2\lambda}(b-a)} \times ① - ②$ gives

$$e^{\sqrt{2\lambda}(b-x)} - e^{\sqrt{2\lambda}(x-b)} = u\left(e^{\sqrt{2\lambda}(b-a)} - e^{\sqrt{2\lambda}(a-b)} \right)$$

~~By~~ Divide both sides by 2

$$\Rightarrow \sinh(\sqrt{2\lambda}(b-x)) = u \cdot \sinh(\sqrt{2\lambda}(b-a)).$$

v can be computed in similar way. ∎

5.4.

~~Above~~ Refer to the paragraph above thm 7.5.9,

$B_t^4 - 6B_t^2 \cdot t + 3t^2$ is a mg.

So, by thm 7.5.1,

$$E\left(B_{T\wedge t}^4 - 6B_{T\wedge t}^2 \cdot (T\wedge t) + 3(T\wedge t)^2\right) = EB_0^4 \cancel{- 6B_0^2}$$

$\boxed{\text{Assume } B_0 = 0.}$

Let $(a,b) \subset (-c,c)$. $U = \inf\{t : B_t \notin (-c,c)\}$

By thm 7.5.5, $ET \underset{\text{by def.}}{\le} EU = c^2 < \infty$.

Thus T is L^1 ftn

$$\therefore \quad EB_{T\wedge t}^4 + 3E(T\wedge t)^2 = E\left[6B_{T\wedge t}^2 \cdot (T\wedge t)\right]$$

① $B_{T\wedge t}^4 \le c^4 \leftarrow$ BCT

② $T\wedge t \uparrow T \leftarrow$ MCT

③. $B_{T\wedge t}^2 \cdot (T\wedge t) \le c^2 \cdot U \leftarrow$ DCT

$\therefore$ as $t \uparrow \infty$, the above becomes

$$EB_T^4 + 3ET^2 = 6E[B_T^2 T]$$

ⓘ

$$\therefore \quad 3ET^2 \le 6EB_T^2 \cdot T \Rightarrow ET^2 \le 2EB_T^2 \cdot T$$

$$\le 2 \cdot (EB_T^4)^{\frac{1}{2}} (ET^2)^{\frac{1}{2}} \quad \text{(Cauchy-Schwarz)}$$

$$\Rightarrow \underline{ET^2 \le 4EB_T^4}$$

Also, ⓘⓘ $EB_T^4 \le 6EB_T^2 \cdot T \le 6 \cdot (EB_T^4)^{\frac{1}{2}} (ET^2)^{\frac{1}{2}}$ (C-S ineq.)

$$\Rightarrow \underline{EB_T^4 \le 36ET^2}$$

▨