

mas541 homework

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Problem (1).

$$\begin{aligned}
1 - \left| \frac{z-w}{1-z\bar{w}} \right|^2 &= 1 - \frac{(z-w)(\bar{z}-\bar{w})}{(1-z\bar{w})(1-\bar{z}w)} \\
&= \frac{1 - \bar{z}w - z\bar{w} + |z|^2|w|^2 - |z|^2 - |w|^2 + z\bar{w} + \bar{z}w}{|1 - \bar{z}w|^2} \\
&= \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2}
\end{aligned}$$

Problem (2).

Let $f = u+iv$. $\partial f = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (u+iv)$. Then $\bar{\partial} f = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (u-iv) = \bar{\partial} \bar{f}$.

Problem (3).

If f is constant, then $|f|$ is also constant. On the other hand, assume $f = u+iv$ and $|f|^2 = u^2 + v^2$ is positive real number. (if it is zero, then f must be zero)

$$u^2 + v^2 = R > 0$$

Differentiate both sides of the equation above with x and y respectively, we can get $uu_x + vv_x = 0$, $uu_y + vv_y = 0$, $u_x = v_y$ and $u_y = -v_x$. By simple calculation we can get $u_x = u_y = v_x = v_y = 0$. Therefore u, v are constant.

Problem (4).

Note that $\int_0^{2\pi} e^{ik\theta} d\theta = \int_0^{2\pi} (\cos k\theta + i \sin k\theta) d\theta = 0$ for positive integer k . Therefore $\frac{1}{2\pi} \int_0^{2\pi} (z_0 + re^{i\theta})^j d\theta = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^j \binom{j}{k} z_0^k (re^{i\theta})^{j-k} d\theta = z_0^j$. Similarly, we can get $\frac{1}{2\pi} \int_0^{2\pi} (z_0 + re^{i\theta})^j d\theta = \bar{z}_0^j$.

Since u is polynomial, we can write it as $\sum_{l,k} a_{l,k} z^l \bar{z}^k$. By direct computation, we can get $\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta = \sum_{l,k} a_{l,k} z_0^l \bar{z}_0^k = u(z_0)$.

Problem (5).

Let $f = u+iv$. $(g \circ f)_x = g_u u_x + g_v v_x$. Then

$$\begin{aligned}
(g \circ f)_{xx} &= (g_{uu} u_x + g_{uv} v_x) u_x + g_u u_{xx} + (g_{vu} u_x + g_{vv} v_x) v_x + g_v v_{xx} \\
(g \circ f)_{yy} &= (g_{uu} u_y + g_{uv} v_y) u_y + g_u u_{yy} + (g_{vu} u_y + g_{vv} v_y) v_y + g_v v_{yy}
\end{aligned}$$

But we have Cauchy-Riemann equation and $g_{uu} + g_{vv} = 0$ and $g_{vu} = g_{uv}$. Also, since f is C^2 function, f is harmonic, $u_{xy} = u_{yx}$, and $v_{xy} = v_{yx}$. Using

these equations, we can check that $(g \circ f)_{xx} + (g \circ f)_{yy} = 0$. Hence $(g \circ f)$ is a harmonic function.

Problem (1).

Let $f = u + iv$. Then $\bar{f}f' = ff' - 2ivf'$, where ff' is holomorphic. So, $\int_{\gamma} \bar{f}f'dz = \int_{\gamma} -2ivf'dz = \int_{\gamma} -2iv(u_x + iv_x)dz = \int_{\gamma} -2iv(v_y + iv_x)dz = -i \int_a^b (2vv_y + 2ivv_x)(\gamma'_1 + i\gamma'_2)dt = \alpha$ where $\gamma = \gamma_1 + i\gamma_2$.

Therefore, real part of $\int_{\gamma} \bar{f}f'dz$ is equal to real part of α . And it is also equal to $-\int_a^b \text{Im}[(2vv_y + i2vv_x)(\gamma'_1 + i\gamma'_2)]dt = -\int_a^b (2vv_x\gamma'_1 + 2vv_y\gamma'_2)dt = -\int_a^b \frac{d}{dt}(v^2 \circ \gamma)dt = 0$ since γ is closed curve.

So, $\int_{\gamma} \bar{f}f'dz$ is purely imaginary.

Problem (2).

Let $f = -u_y$ and $g = u_x$. Then f, g are continuous on U . Since u is harmonic, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ on $U \setminus \{0\}$. So there is $v : U \rightarrow \mathbb{R}$ which is C^1 function and $v_x = f$, $v_y = g$ by lemma 2.5.3.

Let $F = u + iv$. Then F is C^1 function since u, v are C^1 . Since $v_x = f = -u_y$ and $v_y = g = u_x$, F satisfies Cauchy-Riemann equation on U . Thus F is holomorphic on U and real part of F is u .

Problem (3).

- (a) For $z \notin [0, 1]$, the map $w \mapsto \frac{1}{w-z}$ is holomorphic on $\mathbb{C} \setminus [0, 1]$. Let $\gamma(t) = t$ for $t \in [0, 1]$. Then $F(z) = \int_{\gamma} \frac{dw}{w-z} = \int_0^1 \frac{1}{t-z}dt$ is well defined.

For $z \notin [0, 1]$, let $d > 0$ be distance between z and $[0, 1]$. For $|h| < \frac{d}{2}$, consider $\frac{F(z+h)-F(z)}{h} = \int_0^1 \frac{1}{(t-z-h)(t-z)}dt$. Then $\left| \frac{1}{(t-z-h)(t-z)} - \frac{1}{(t-z)^2} \right| = \left| \frac{h}{(t-z)^2(t-z-h)} \right| \leq |h| \frac{2}{d^3}$ since $|t-z| \geq d$ and $|t-z-h| \geq \frac{d}{2}$. Therefore, as $|h| \rightarrow 0$, integrand converges to $\frac{1}{(t-z)^2}$ uniformly on $t \in [0, 1]$. So $\lim_{h \rightarrow 0} \frac{F(z+h)-F(z)}{h} = \int_0^1 \lim_{h \rightarrow 0} \frac{1}{(t-z-h)(t-z)}dt = \int_0^1 \frac{1}{(t-z)^2}dt = F'(z)$.

By same reasoning, we get $F''(z) = \int_0^1 \frac{1}{(t-z)^3}dt$. From existence of F'' , F' is continuous. Therefore F is C^1 function. Existence of complex derivative and C^1 implies F is holomorphic on $\mathbb{C} \setminus [0, 1]$.

- (b) For $s \in (0, 1)$, $F(s+i\varepsilon) = \int_0^1 \frac{1}{t-s-i\varepsilon}dt = \int_0^1 \frac{t-s+i\varepsilon}{(t-s)^2+\varepsilon^2}dt = \int_0^1 \frac{t-s}{(t-s)^2+\varepsilon^2}dt + i \int_0^1 \frac{\varepsilon}{(t-s)^2+\varepsilon^2}dt$. Let $t-s = \varepsilon \tan \theta$. $\varepsilon \tan \theta_0 + s = 0$ and $\varepsilon \tan \theta_1 + s = 1$ for $-\frac{\pi}{2} < \theta_0, \theta_1 < \frac{\pi}{2}$. Then $\sec^2 \theta_0 = \frac{s^2}{\varepsilon^2} + 1$, $\sec^2 \theta_1 = \frac{(1-s)^2}{\varepsilon^2} + 1$, $\theta_0 = \tan^{-1}(\frac{-s}{\varepsilon})$, and $\theta_1 = \tan^{-1}(\frac{1-s}{\varepsilon})$.

Then $F(s+i\varepsilon) = \int_{\theta_0}^{\theta_1} \tan \theta d\theta + i \int_{\theta_0}^{\theta_1} d\theta = \log \left| \frac{\sec \theta_1}{\sec \theta_0} \right| + i(\theta_1 - \theta_0)$. As $\varepsilon \downarrow 0$, $F(s+i\varepsilon)$ goes to $\frac{1-s}{s} + i\pi$ by simple calculation.

Similarly, $F(s - i\varepsilon)$ goes to $\frac{1-s}{s} - i\pi$ as $\varepsilon \downarrow 0$.

(c) Consider $F(-\varepsilon) = \int_0^1 \frac{1}{t+\varepsilon} dt = \log \frac{1+\varepsilon}{\varepsilon}$. It goes to ∞ as $\varepsilon \downarrow 0$.

Consider $F(1 + \varepsilon) = \int_0^1 \frac{1}{t-1-\varepsilon} dt = \log \frac{\varepsilon}{1+\varepsilon}$. It goes to $-\infty$ as $\varepsilon \downarrow 0$.

Therefore, for $s = 0, 1$, $\lim_{z \notin [0,1] \rightarrow s} F(z)$ does not exist.

Problem (4).

First consider $p \equiv 0$. We can easily see that $\sup_{z \in C} |z^{-n}| = 1$ so desired value ≤ 1 .

Note that $|p(z) - z^{-n}| = |z^n p(z) - 1|$. Thus, $1 = \frac{1}{2\pi i} \int_C \frac{z^n p(z) - 1}{z} dz \leq \sup_{z \in C} |z^n p(z) - 1|$.

Those leads the conclusion.

Problem (5).

It is enough to show γ and μ are path homotopic. Define $H(t, s) = (1-s)\gamma(t) + \frac{\gamma(t)}{|\gamma(t)|}s$. Then $H(t, 1) = \mu(t)$ and $H(t, 0) = \gamma(t)$ by reparametrization. And H is continuous because $\gamma(t) \neq 0$. Therefore H is path homotopy between γ and μ . Since line integration is invariant under path homotopy, we get $\int_\gamma F(\zeta) d\zeta = \int_\mu F(\zeta) d\zeta$.

Problem (1).

It suffices to show that $\int_{\gamma} f(z)dz = 0$ for rectangle γ whose edges are parallel to coordinate axes by Morera's theorem.

First, assume that γ intersects with $[0, 1]$ only finitely many points. Let p be such point. Then p must be on (wlog) left edge of γ . Let $a + ib, a + ic$ be two vertices incident with left edge. ($b > c$) Let $\rho(t) = a + i(tc + (1-t)b)$. Consider $f \circ \rho$. It is continuous and equals to $\frac{\partial}{\partial t} F(\rho(t))$ except for $\gamma^{-1}(p)$ where F is antiderivative of f on $\mathbb{C} \setminus [0, 1]$. Then lemma 2.3.1 says $f(\rho(t)) = \frac{\partial}{\partial t} F(\rho(t))$ even for $\gamma^{-1}(p)$. Therefore $\int_{\rho} f(z)dz = F(a + ic) - F(a + ib)$. By using this result, we can easily calculate $\int_{\gamma} f(z)dz = 0$.

Now, assume that (wlog) upper edge of γ intersects with $[0, 1]$. Let $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ which are upper edge, left edge, bottom edge, and right edge respectively, parametrized like ρ of above, positive oriented. Consider φ made by shrinking side edges of γ so that distance between of upper edges of φ and γ less than δ , while bottom edge is fixed. Also note that δ is chosen so that $d(z_0, z_1) < \delta$ implies $d(f(z_0), f(z_1)) < \varepsilon$.

$$\left| \int_{\gamma} f(z)dz - \int_{\varphi} f(z)dz \right| \leq \left| \int_{\gamma_2 - \varphi_2} f(z)dz + \int_{\gamma_4 - \varphi_4} f(z)dz \right| + (\text{length of } \gamma_1) \varepsilon$$

And, second term of above goes to 0 as distance between φ_1 and γ_1 goes to 0 by continuity and result of first case. Actually $\int_{\varphi} f(z)dz = 0$ because φ does not intersect with $[0, 1]$. Thus we have shown that $\int_{\gamma} f(z)dz = 0$.

By first, second case and Morera's thm, f is actually entire function.

Problem (2).

For $0 < r < 1$, $|f^{(n)}(0)| \leq \frac{n!}{r^n} \frac{1}{1-r}$ by using Cauchy estimate. $r^n(1-r)$ is maximized when $r = \frac{n}{n+1}$. So, when $r = \frac{n}{n+1}$, we get best estimate of $|f^{(n)}(0)|$.

Problem (3).

- (a) Since K is compact subset of open set U , there is $r > 0$ such that for all $x \in K$, closure of $D(x, r)$ is in U . Then, $|f(z)|^2 \leq \frac{1}{2\pi} \left| \int_{\partial D(z, r)} \frac{f^2(w)}{w-z} dw \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f^2(z + re^{i\theta})| d\theta$. By multiplying ρ both sides and integrating from 0 to r , we can get the following:

$$\begin{aligned}
\frac{r^2}{2} |f(z)|^2 &\leq \frac{1}{2\pi} \int_0^r \int_0^{2\pi} \rho |f^2(z + re^{i\theta})| d\theta d\rho \\
&= \frac{1}{2\pi} \int_{\bar{D}(z,r)} |f|^2 dm \\
&= \frac{1}{2\pi} \int_U |f|^2 dm
\end{aligned}$$

for all $z \in K$, where m is lebesgue measure, using Holder's inequality and polar coordinate integration.

Therefore $C = \frac{1}{r\sqrt{\pi}}$

(b) If f is identically zero, possible.

Else if f is constant, then $\int_{\mathbb{C}} |f| dm = \infty$ since measure of complex plane is ∞ .

Else, that is f is nonconstant entire function, then f must be unbounded. So, there is $\delta > 0$ such that $|f| \geq 1$ for all $|z| > \delta$. Then $\int_{\mathbb{C}} |f| dm \geq m(\{z : |z| > \delta\}) = \infty$.

Problem (4). (a) Since $\frac{z}{e^z - 1}$ is bounded near 0, it has removable singularity at 0. So we can regard it as holomorphic function. Note that $e^z - 1 = 0$ when z is integer multiple of $2\pi i$. So, given power series converges on unit disc. Now, multiply $e^z - 1$ both sides. Since $e^z - 1$ is entire and given power series converges absolutely on $\bar{D}(0, r)$ where $0 < r < 1$, we can write $z = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \sum_{n=1}^{\infty} \frac{1}{n!} z^n$. Since z is entire, coefficient of power series is unique. By comparing coefficients of both sides, we can get given recursion formula.

$\lim_{z \rightarrow 0} \frac{z}{e^z - 1} = 1 = B_0$. From this, by simple calculation, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, and $B_3 = 0$.

Consider $-z = f(z) - f(-z) = \sum_{n=0}^{\infty} 2 \frac{B_{2n+1}}{(2n+1)!} z^{2n+1}$. This makes sense because f is holomorphic on unit disc. By comparing coefficient of this series, we can get $B_{2m+1} = 0$ for $m \geq 1$.

(b) We already notice that $e^z - 1$ is zero when z is integer multiple of $2\pi i$. But $\lim_{z \rightarrow 2k\pi i} \frac{z}{e^z - 1}$ is not bounded when $k \neq 0$. Therefore, $\frac{z}{e^z - 1}$ is holomorphic on $D(0, 2\pi)$ and is not holomorphic outside of that disc. Since

power series representation of holomorphic function at P has radius of convergence at least $d(P, U)$, we can say radius of convergence of the series is 2π .

Problem (5).

f' is holomorphic on unit disc. Let $r = \sup_{z \in K} |z|$. Since K is compact, $|f'| \leq M$ on K and r is positive but less than 1. Let $\gamma(t) = tz^n$ which connects origin and z^n . $|f(z^n) - f(0)| = \left| \int_{\gamma} f' dz \right| \leq M \sup_{z \in K} |z|^n = Mr^n$. Therefore, $|\sum_{n=1}^{\infty} f(z^n)| \leq \sum_{n=1}^{\infty} |f(z^n)| \leq \sum_{n=1}^{\infty} Mr^n < \infty$ because r is positive but less than 1.