mas540 exercises

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Exercise (1.4)**.**

(a) *Let* $I = [0, 1]$*. Then* $I \backslash \hat{C} = \bigcup_{n=1}^{\infty} \hat{C}_n^c$ *where* $\hat{C}_n$ *is* n*-th stage of constructing Fat Cantor set. Thus,*

$$m(I \setminus \hat{C}) = m(I) - m(\hat{C}) = 1 - m(\hat{C}) = \lim_{n \to \infty} m(\hat{C}_n^c) = \sum_{n=1}^{\infty} 2^{n-1} l_n$$

because $\hat{C}_n^c \uparrow \bigcup_{n=1}^{\infty} \hat{C}_n^c$ *and* $\hat{C}$ *is closed hence measurable. Therfore* $m(\hat{C}) = 1 - \sum_{n=1}^{\infty} 2^{n-1} l_n > 0$*.*

(b) $\hat{C}_k$ *consists of* 2^k *closed intervals whose length are* $(1 - \sum_{n=1}^{k} 2^{n-1} l_n)/2^k$*. Let* $x \in \hat{C}$*. Then* $x \in \hat{C}_k$*. So we can find* $x_k \in I_k$ *such that*

$$|x - x_k| \leq \left(1 - \sum_{n=1}^{k} 2^{n-1} l_n\right) / 2^k + \varepsilon_k l_k$$

for some $0 < \varepsilon_k < 1$*. As* $k \to \infty$*,* $|x - x_k| \to 0$ *since* $l_k \to 0$*.*

(c) *The result of b tells us that every point of* $\hat{C}$ *is a limit point of* I*. And we also know that* $\hat{C}$ *is closed. Hence* $\hat{C}$ *is a perfect set.*

Let $(a, b) \subset \hat{C}$ *and* $a < c < d < b$*. For large* k*,* $l_k < d - c$ *since* $l_k \to 0$*. Then, for* $\hat{C}_k$*,* c *and* d *must lie in different intervals of* $\hat{C}_k$*. So there is* $e \notin \hat{C}_k$ *such that* $c < e < d$*. Then* $[c, d]$ *does not belong to* $\hat{C}_k$ *which is a contradiction. So* $\hat{C}$ *is totally disconnected.*

(d) *It is well known fact that a nonempty perfect set is uncountable. We had learned it in an introductory analysis course and topology course.*

□

Exercise (1.7)**.**

First, we will show that if O *is open, then* δO *is also open. Let* $\delta x \in \delta O$*. Then* $x \in O$*. By openness, there is* $r > 0$ *such that* $Q_r(x) \subset O$ *where* $Q_r(x)$ *is a cube whose side length is* r *and centered at* x*. Thus* $\delta Q_r(x) \subset \delta O$ *and* $\delta Q_r(x)$ *contains* δx*. But a collection of all open rectangles forms a basis of Euclidean space. So* δO *is an open set.*

Next, let a set E *and a positive number* ε *be given. Choose* $O \supset E$ *such that* $m_*(O \setminus E) < \varepsilon / (\delta_1 \cdots \delta_d)$*. Then, there is an union of cube* $\bigcup_{j=1}^{\infty} Q_j \supset O \setminus E$ *such that* $\sum_{j=1}^{\infty} m(Q_j) < \varepsilon / (\delta_1 \cdots \delta_d)$*. Then,*

$$m_*(\delta O \setminus \delta E) = m_*(\delta(O \setminus E)) \leq m_*(\bigcup_{j=1}^{\infty} \delta Q_j) \leq \sum_{j=1}^{\infty} m(\delta Q_j) < \varepsilon.$$

Thus δE *is measurable.*

Now let $E \subset \bigcup_{j=1}^{\infty} Q_j$. Then $\delta E \subset \bigcup \delta Q_j$, so $m(\delta E) \leq \delta_1 \cdots \delta_d \sum_{j=1}^{\infty} m(Q_j)$. Since $\bigcup_{j=1}^{\infty}$ is arbitrary, we get

$$m(\delta E) \leq \delta_1 \cdots \delta_d m(E).$$

Now let $\delta E \subset \bigcup_{j=1}^{\infty} Q'_j$. Then $E \subset \bigcup_{j=1}^{\infty} 1/\delta Q'_j$. So $m(E) \leq \sum_{j=1}^{\infty} m(Q'_j)/(\delta_1 \cdots \delta_d)$. Since $\bigcup_{j=1}^{\infty} Q'_j$ is arbitary, we get

$$m(E) \leq \frac{m(\delta E)}{\delta_1 \cdots \delta_d}$$

and this finishes the proof.

$\square$

Exercise (1.24)**.**

Let s_n be enumeration of $\mathbb{Q} \cap [-1, 1]$ and t_n be enumeration of $\mathbb{Q} \cap [-1, 1]^c$. When $n = m^2$, put $r_n = t_m$. When $n \in \left(m^2, (m+1)^2\right)$, put $r_n = s_{n-m}$. Then r_n is an enumeration of $\mathbb{Q}$. Also, we get

$$\begin{aligned} m\left(\bigcup_{n=1}^{\infty} (r_n - 1/n, r_n + 1/n)\right) &\leq \sum_{m=1}^{\infty} 2/m^2 + m\left(\bigcup_{n \neq m^2} (r_n - 1/n, r_n + 1/n)\right) \\ &\leq \sum_{m=1}^{\infty} 2/m^2 + 2 + 1 < \infty. \end{aligned}$$

Therefore, finiteness implies nonemptyness of the complement, since the Lebesgue measure of complement is positive.

$\square$

Exercise (1.35)**.**

First, let's briefly check the idea of constructing φ. Construction can be done by defining a sequence of functions, say φ_n. Put $\varphi_n(0) = 0$ and $\varphi_n(1) = 1$. Let C_{ji} be the i-th stage of constructing C_j. Then φ_i maps the discarded set of stage i to the discarded set of stage i, sequentially, and linearly(positive). We can extend φ_i by assigning value on C_{1i} using linearity and monotonicity. This sequence of functions converges uniformly, thus φ is continuous. The other properties of φ can be checked by this construction.

Let $\mathcal{N} \subset C_1$ be a non-measurable set. Then $\varphi(\mathcal{N}) \subset C_2$ so $\varphi(\mathcal{N})$ is measurable by completeness. If $\varphi(\mathcal{N})$ is a Borel set, then by continuity, $\varphi^{-1}(\varphi(\mathcal{N})) = \mathcal{N}$ must be a Borel set, which is a contradiction. So there is a Lebesgue measurable set which is not Borel measurable.

Since $\varphi(\mathcal{N})$ is measurable, $f = 1_{\varphi(\mathcal{N})}$ is a measurable map. Then $f \circ \varphi(x) = 1_{\mathcal{N}}(x)$ is non-measurable map.

$\square$

Problem (1.4)**.**

(a) A_ε *is clearly bounded, so it is enough to show that the complement is open. Let* $c \notin A_\varepsilon$. *Then* $osc(f,c) < \varepsilon$, *so for some* $r > 0$, $osc(f,c,r) < \varepsilon$. *Choose any* $d \in I(c,r)$. *We can choose* $r^* > 0$ *so that* $I(d,r^*) \subset I(c,r)$. *Then*

$$osc(f,d,r^*) \le osc(f,c,r) < \varepsilon$$

so $osc(f,d) < \varepsilon$, *which says* $I(c,r) \subset J \setminus A_\varepsilon$. *Therefore* $J \setminus A_\varepsilon$ *is open in* J, *hence* A_ε *is compact.*

(b) *Let* D_f *be a set of all discontinuities of* f. *Then for any* $\varepsilon > 0$, $A_\varepsilon \subset D_f$. *So* $m(A_\varepsilon) \le m(D_f) = 0$. *By the definition of Lebesgue measure, there is countably many open intervals which cover* A_ε *and have sum of length* $\le \varepsilon$. *Using compactness, we can choose finite subcover, call them by* $(a_i,b_i)_{i=1}^k$ *where* $a_i < a_{i+1}$. *After discarding all of subcovers from* J, *we get compact subset of* J, *say* J'. *For each* $c \in J'$, *we can choose* r_c *such that* $osc(f,c,2r_c) < \varepsilon$. *Again, using compactness, we can choose finitely many* c*'s. Then finitely many closed intervals* $[c - r_c, c + r_c]$ *have finite intersections. By taking these endpoints(contain* a_i, b_i*'s) as endpoints of our partition(if necessary, consider a refinement), we get*

$$U(f,P) - L(f,P) \le 2M\varepsilon + m(J)\varepsilon$$

where M *is bound of* f. *The first term of estimate comes from* (a_i,b_i)*'s and the second term comes from* J'.

(c) *Since* $D_f \subset \bigcup_{n=1}^{\infty} A_{1/n}$, *so* $m(A_{1/n}) = 0$ *leads the conclusion. Assume not, i.e.* $m(A_{1/n}) > \varepsilon$. *Take partition* P *such that* $U(f,P) - L(f,P) < \varepsilon/n$. *Let* $[a,b]$ *be interval of* P *whose interior intersects to* $A_{1/n}$. *Then*

$$\sup_{x,y\in[a,b]} |f(x) - f(y)| \ge \frac{1}{n}.$$

But $m(A_{1/n}) > \varepsilon$. *So*

$$\begin{aligned}
&\sum_{[a,b]\cap A_{1/n}\neq\emptyset} \left[\sup_{x\in[a,b]} f(x) - \inf_{y\in[a,b]} f(y)\right] m\left(A_{1/n} \cap [a,b]\right) \\
&= \sum_{[a,b]\cap A_{1/n}\neq\emptyset} \sup_{x,y\in[a,b]} |f(x) - f(y)|\, m\left(A_{1/n} \cap [a,b]\right) \\
&\ge \frac{\varepsilon}{n} \\
&> U(f,P) - L(f,P)
\end{aligned}$$

which is a contradiction.

□

Exercise (2.2).

Let $\varepsilon > 0$. Choose $g \in C_c(\mathbb{R}^d)$ such that $\|f - g\|_1 < \varepsilon$. Let the domain of g is contained in $B_r(0)$. For $x \in B_r(0)$,

$$|x - \delta x| = |1 - \delta||x| \leq r|1 - \delta| < \xi$$

if $|1 - \delta|$ is small. Let $\xi > 0$ be a number which satisfies $|x - y| < \xi \Rightarrow |g(x) - g(y)| < \varepsilon$. Then, for enoughly small $|1-\delta|$, we get $|x-\delta x| < \xi \Rightarrow |g(\delta x)-g(x)| < \varepsilon$. Thus we get $\|g_\delta - g\| \leq \varepsilon m(B_r(0))$, $\|f - g\| < \varepsilon$, $\|f_\delta - g_\delta\| < K\varepsilon$. Therefore

$$\|f - f_\delta\| \leq \|f - g\| + \|g - g_\delta\| + \|g_\delta - f_\delta\| \leq (m(B_r(0)) + 1 + K)\,\varepsilon.$$

This says as $\delta \to 1$, $\|f_\delta - f\| \to 0$.

$\square$

Exercise (2.6).

(a) *Let $n \in \mathbb{N}$. On $[n, n+1]$, define*

$$f(x) = \begin{cases} n & \text{if } n \leq x \leq n + 1/n^3 \\ 1/n^3 & \text{if } n + 2/n^3 \leq x \leq n + 1 - 1/n^3 \\ \text{linear} & \text{otherwise.} \end{cases}$$

Then

$$\int_{[n,n+1]} f(x)dx \leq \frac{1}{n^2} + \frac{1}{n^3} n \frac{1}{2} + \left(1 - \frac{3}{n^3}\right) \frac{1}{n^3} + \frac{1}{n^3}(n+1)\frac{1}{2} = \frac{2n+3}{2n^3} + \frac{1}{n^2} - \frac{3}{n^6}.$$

Now, reflect f to the y-axis. Define f on $(-1, 1)$ by 1. Then

$$\int_{\mathbb{R}} f dm \leq 2 + 2\left(\sum_{n \geq 1}\left(\frac{4n+2}{2n^3} - \frac{3}{n^6}\right)\right) < \infty.$$

But clearly $\limsup_{x \to \infty} f(x) = \infty$.

(b) *By same manipulation used in #2.24.b, the result follows. See after If φ does not vanish $\sim$.*

$\square$

Exercise (2.19).

Let $g(x, \alpha) = 1_{E_\alpha}(x) 1_{(0,\infty)}(\alpha)$. Since g is nonnegative, Tonelli's theorem can be applied.

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}} g dm &= \int_{\mathbb{R}^d} \int_{\mathbb{R}} g_x d\alpha dx = \int_{\mathbb{R}^d} |f(x)| dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} g^\alpha dx d\alpha = \int_{(0,\infty)} m(E_\alpha) d\alpha. \end{aligned}$$

Because $g_x(\alpha) = 1_{(0<\alpha<|f(x)|)}(\alpha)$ and $g^\alpha(x) = 1_{(0<\alpha<|f(x)|)}(x)$. $\square$

Exercise (2.24).

Let $\varphi = f * g$.

(a) *Choose* $h > 0$ *small so that* $\|f_h - f\|_1 < \varepsilon$. *Then*

$$|\varphi(x+h) - \varphi(x)| \le \int |f(x+h-y) - f(x-y)||g(y)|dy \le B\|f_h - f\|_1 < B\varepsilon.$$

Thus φ *is uniformly continuous.*

(b) *By Tonelli's theorem,*

$$\|\varphi\|_1 \le \iint |f(x-y)||g(y)|dydx \le \|f\|_1 \int |g(y)|dy = \|f\|_1\|g\|_1 < \infty.$$

So $\varphi \in L^1$. *Note that* φ *is uniformly continuous by (a).*

If φ *does not vanish at infinity, then there exists* $\varepsilon > 0$ *such that for all* $M > 0$, *there is* $|x_M| \ge M :\ |\varphi(x_M)| > 2\varepsilon$. *By uniform continuity, there is* $\delta > 0$ *such that* $|x - y| < \delta \Rightarrow |\varphi(x) - \varphi(y)| < \varepsilon$. *We can get strictly increasing sequence* $y_i \in \{x_M : M > 0\}$ *such that* $B_\delta(y_i) \cap B_\delta(y_j) = \emptyset$ *whenever* $i \ne j$.

Note that for $x \in B_\delta(y_i)$, $|\varphi(x)| > \varepsilon$. *Thus*

$$\int |\varphi|dx \ge \sum_{i=1}^{\infty} \varepsilon m(B_\delta(y_i)) = \infty.$$

But the above contradicts to $\varphi \in L^1$.

□

Problem (2.3).

Let $E_k = \{|f_k - f| > \varepsilon\}$. *By the Markov inequality,*

$$m(E_k) \le \frac{1}{\varepsilon}\int |f_k - f|dm.$$

Since $f_k \to f$ *in* L^1, *we get* $m(E_k) \to 0$. *Thus* L^1 *convergence implies the convergence in measure.*

For counterexample, consider $f_k = k1_{(0,1/k)}$. *Then* $\int f_k dm = 1$. *But* $m(|f_k| > \varepsilon) \le 1/k$ *so* $f_k \to 0$ *in measure. But, as we seen,* f_k *does not converge to* 0 *in* L^1. *Thus the converse of the previous result is not true.* □

Exercise (3.2).

Let $\{L_\delta\}$ be any approximation to the identity. Then, by triangle inequality, $\{K_\delta + L_\delta\}$ is also approximation to the identity because of the third condition. Therefore

$$f * (K_\delta + L_\delta)(x) \to f(x) \text{ a.e. } x$$

as $\delta \to 0$ by theorem 2.1.

But,

$$\begin{aligned} f * (K_\delta + L_\delta)(x) &= \int f(x-y)(K_\delta(y) + L_\delta(y))dy \\ &= f * K_\delta(x) + f * L_\delta(x). \end{aligned}$$

Since $f * L_\delta(x) \to f(x)$ for a.e. x, $f * K_\delta(x) \to 0$ for a.e. x necessarily.

□

Exercise (3.5).

(a) *By the change of variable formula($\log x = t$),*

$$\begin{aligned} \int_{\mathbb{R}} |f(x)|dx &= \int_{-1/2}^{1/2} f(x)dx \\ &= \int_{-\infty}^{-\log 2} \frac{1}{t^2} dt = \frac{1}{\log 2} < \infty. \end{aligned}$$

(b) *Let $\varepsilon > 0$. Then*

$$\begin{aligned} f^*(x) &\geq \frac{1}{2|x| + 2\varepsilon} \int_{-|x|-\varepsilon}^{|x|+\varepsilon} \frac{dt}{t(\log t)^2} \\ &= \frac{1}{|x| + \varepsilon} \int_0^{|x|+\varepsilon} \frac{dt}{t(\log t)^2} \\ &= \frac{1}{-\log(|x|+\varepsilon)(|x|+\varepsilon)}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, by taking $\varepsilon \downarrow 0$, we obtain

$$f^*(x) \geq \frac{1}{|x| \log \frac{1}{|x|}}.$$

But $1/(-|x|\log|x|)$ is clearly non-locally integrable function. This is by integrating on the interval containing 0 and the change of variable formula, used above.

□

Exercise (3.12).

By chain rule, F' exists for all $x \neq 0$. But,

$$\lim_{h\to 0}\frac{F(h)}{h} = \lim_{h\to 0} h\sin(1/h^2) = 0$$

Thus F' exists for all $x \in \mathbb{R}$.

For $1/\sqrt{2n\pi + \pi/6} \leq x \leq 1/\sqrt{2n\pi - \pi/6}$, $2n\pi - \pi/6 \leq 1/x^2 \leq 2n\pi + \pi/6$, thus $\cos 1/x^2 \geq \sqrt{3}/2$ and $\left|\sin 1/x^2\right| \leq 1/2$. So $|F'| \geq 2/x\cos 1/x^2 - 2x\left|\sin 1/x^2\right| \geq \sqrt{3}\sqrt{2n\pi - \pi/6} - 1/\sqrt{2n\pi - \pi/6}$.

By using the above,

$$\begin{aligned}
\int_0^1 |F'|dm &\geq \sum_{n=1}^{\infty}\left(1/\sqrt{2n\pi - \pi/6} - 1/\sqrt{2n\pi + \pi/6}\right)\left(\sqrt{3}\sqrt{2n\pi - \pi/6} - 1/\sqrt{2n\pi - \pi/6}\right) \\
&= \sum_{n=1}^{\infty}\frac{\pi/\sqrt{3}}{\sqrt{2n\pi + \pi/6}\left(\sqrt{2n\pi + \pi/6} + \sqrt{2n\pi - \pi/6}\right)} \\
&- \sum_{n=1}^{\infty}\frac{\pi/3}{(2n\pi - \pi/6)\sqrt{2n\pi + \pi/6}\left(\sqrt{2n\pi + \pi/6} + \sqrt{2n\pi - \pi/6}\right)}
\end{aligned}$$

where the last sum converges and previous one diverges (by p-test.) Thus F' is non-integrable.

□

Exercise (3.23)**.**

(a) *Follow the hint.*

$$(D^+G_\varepsilon)(x_0) = (D^+F)(x_0) + \varepsilon > 0.$$

This means, for sufficiently small $h > 0$,

$$G_\varepsilon(x_0 + h) > G_\varepsilon(x_0) \geq 0.$$

This contradicts to our choice of x_0.

(b) *Use the Mean value theorem.*

□

Exercise (3.25)**.**

(a) *Let f be the function given in the hint. Note that all of points in any open set O is a point of Lebesgue density. This is because, we can only consider small ball B_x contained in O. Thus*

$$\liminf \frac{m(O_n \cap B)}{m(B)} = 1$$

for all $x \in E$. Therefore

$$\liminf \frac{1}{m(B)} \int_B f dm = \liminf \sum_{n\geq 1} \frac{m(O_n \cap B)}{m(B)}$$
$$\geq \sum_{n\geq 1} \liminf \frac{m(O_n \cap B)}{m(B)} = \sum_{n\geq 1} 1 = \infty.$$

(b) *Let $F(x) = \int_{-\infty}^{x} f(t)dt$ where f is the function found in a. Then F satisfies the given condition.*

□

Exercise (3.32).

Assume the Lipschitz condition. Take $\delta = \varepsilon/M$ when $\varepsilon > 0$ is given. For (a_i, b_i) such that $\sum_i (b_i - a_i) < \delta$, then $\sum_i |f(b_i) - f(a_i)| \leq M \sum_i (b_i - a_i) < M\delta = \varepsilon$. Thus f is absolutely continuous. So f' exists a.e. Now consider the following:

$$|f'(x)| = \lim_{h\to 0} \frac{|f(x+h) - f(x)|}{|h|} \leq M$$

Thus $|f'| \leq M$ a.e. x.

For the other direction, without loss of generality, assume $x \leq y$. Since f is absolutely continuous, f' exists a.e, and $\int_x^y f' dm = f(y) - f(x)$. Thus, $|f(x) - f(y)| = \left|\int_x^y f' dm\right| \leq \int_x^y |f'| dm \leq (y - x)M = |x - y|M$.

Problem (3.5).

First, assume that $F' \geq 0$ a.e. Let E be the set, $F'(x) < 0$. According to exercise 25, we can find Φ which is increasing, absolutely continuous, and $D_\pm \Phi(x) = \infty$ for all $x \in E$. Note that $\infty = D_+\Phi(x) \leq D^+\Phi(x)$. Now, for $\delta > 0$, consider $F + \delta\Phi$. On E, $D^+(F + \delta\Phi) = \infty > 0$. On E^c, $D^+(F + \delta\Phi) = F' + \delta\Phi' \geq 0$. Therefore, by exercise 23, $F + \delta\Phi$ is an increasing function. So

$$F(x) - F(a) + \delta(\Phi(x) - \Phi(a)) \geq 0.$$

Since $\delta > 0$ is arbitrary, we can assert $F(x) \geq F(a)$ whenever $x \geq a$.

Now we'll solve the problem using the above. Let $G(x) = \int_a^x F' dm$. Then $G'(x) = F'(x)$ a.e. by Lebesgue differentiation theorem. Thus $G'(x) - F'(x) \geq 0$ a.e. Then, the above implies $G(x) - G(a) - F(x) + F(a) \geq 0$. Since we can say that $G'(x) - F'(x) \leq 0$ a.e. also, we obtain $G(x) - G(a) - F(x) + F(a) \leq 0$. But $G(a) = 0$. Therefore $F(x) - F(a) = G(x) = \int_a^x F' dm$.

□