mas541 homework

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Problem (1.1)**.**

$$
\begin{aligned}
1-\left|\frac{z-w}{1-z\overline{w}}\right|^2 &= 1-\frac{(z-w)\left(\overline{z}-\overline{w}\right)}{\left(1-z\overline{w}\right)\left(1-\overline{z}w\right)} \\
&= \frac{1-\overline{z}w-z\overline{w}+|z|^2|w|^2-|z|^2-|w|^2+z\overline{w}+\overline{z}w}{|1-\overline{z}w|^2} \\
&= \frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\overline{z}w|^2}
\end{aligned}
$$

Problem (1.2)**.**

Let $f=u+iv$. $\partial f=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)$. Then $\overline{\partial f}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(u-iv)=\bar{\partial}\bar{f}$.

Problem (1.3)**.**

If f is constant, then $|f|$ is also constant. On the other hand, assume $f=u+iv$ and $|f|^2=u^2+v^2$ is positive real number. (if it is zero, then f must be zero)

$$u^2+v^2=R>0$$

Differentiate both sides of the equation above with x and y respectively, we can get $uu_x+vv_x=0$, $uu_y+vv_y=0$, $u_x=v_y$ and $u_y=-v_x$. By simple calculation we can get $u_x=u_y=v_x=v_y=0$. Therefore u,v are constant.

Problem (1.4)**.**

Note that $\int_0^{2\pi}e^{ik\theta}d\theta=\int_0^{2\pi}(\cos k\theta+i\sin k\theta)\,d\theta=0$ for positive integer k. Therefore $\frac{1}{2\pi}\int_0^{2\pi}(z_0+re^{i\theta})^j d\theta=\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{j}\binom{j}{k}z_0^k\left(re^{i\theta}\right)^{j-k}d\theta=z_0^j$. Similarly, we can get $\frac{1}{2\pi}\int_0^{2\pi}\overline{(z_0+re^{i\theta})^j}d\theta=\bar{z_0}^j$.

Since u is polynomial, we can write it as $\sum_{l,k}a_{l,k}z^l\bar{z}^k$. By direct computation, we can get $\frac{1}{2\pi}\int_0^{2\pi}u\left(z_0+re^{i\theta}\right)d\theta=\sum_{l,k}a_{l,k}z_0^l\bar{z_0}^k=u(z_0)$.

Problem (1.5)**.**

Let $f=u+iv$. $(g\circ f)_x=g_uu_x+g_vv_x$. Then

$$
\begin{aligned}
(g\circ f)_{xx} &= \left(g_{uu}u_x+g_{uv}v_x\right)u_x+g_uu_{xx}+\left(g_{vu}u_x+g_{vv}v_x\right)v_x+g_vv_{xx} \\
(g\circ f)_{yy} &= \left(g_{uu}u_y+g_{uv}v_y\right)u_y+g_uu_{yy}+\left(g_{vu}u_y+g_{vv}v_y\right)v_y+g_vv_{yy}
\end{aligned}
$$

But we have Cauchy-Riemann equation and $g_{uu}+g_{vv}=0$ and $g_{vu}=g_{uv}$. Also, since f is C^2 function, f is harmonic, $u_{xy}=u_{yx}$, and $v_{xy}=v_{yx}$. Using *these equations, we can check that* $(g \circ f)_{xx} + (g \circ f)_{yy} = 0$. *Hence* $(g \circ f)$ *is a harmonic function.*

Problem (2.1)**.**

Let $f = u + iv$. *Then* $\bar{f}f' = ff' - 2ivf'$, *where* ff' *is holomorphic. So,* $\int_\gamma \bar{f}f'dz = \int_\gamma -2ivf'dz = \int_\gamma -2iv(u_x + iv_x)dz = \int_\gamma -2iv(v_y + iv_x)dz = -i\int_a^b (2vv_y + 2ivv_y)(\gamma_1' + i\gamma_2')dt = \alpha$ *where* $\gamma = \gamma_1 + i\gamma_2$.

Therefore, real part of $\int_\gamma \bar{f}f'dz$ *is equal to real part of* α. *And it is also equal to* $-\int_a^b Im\left[(2vv_y + i2vv_x)(\gamma_1' + i\gamma_2')\right]dt = -\int_a^b (2vv_x\gamma_1' + 2vv_y\gamma_2')dt = -\int_a^b \frac{d}{dt}(v^2 \circ \gamma)dt = 0$ *since* γ *is closed curve.*

So, $\int_\gamma \bar{f}f'dz$ *is purely imaginary.*

Problem (2.2)**.**

Let $f = -u_y$ *and* $g = u_x$. *Then* f, g *are continuous on* U. *Since* u *is harmonic,* $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ *on* $U \setminus \{0\}$. *So there is* $v : U \to \mathbb{R}$ *which is* C^1 *function and* $v_x = f$, $v_y = g$ *by lemma 2.5.3.*

Let $F = u + iv$. *Then* F *is* C^1 *function since* u, v *are* C^1. *Since* $v_x = f = -u_y$ *and* $v_y = g = u_x$, F *satisfies Cauchy-Riemann equation on* U. *Thus* F *is holomorphic on* U *and real part of* F *is* u.

Problem (2.3)**.**

(a) For $z \notin [0,1]$, *the map* $w \mapsto \frac{1}{w-z}$ *is holomorphic on* $\mathbb{C}\setminus[0,1]$. *Let* $\gamma(t) = t$ *for* $t \in [0,1]$. *Then* $F(z) = \int_\gamma \frac{dw}{w-z} = \int_0^1 \frac{1}{t-z}dt$ *is well defined.*

For $z \notin [0,1]$, *let* $d > 0$ *be distance between* z *and* $[0,1]$. *For* $|h| < \frac{d}{2}$, *consider* $\frac{F(z+h)-F(z)}{h} = \int_0^1 \frac{1}{(t-z-h)(t-z)}dt$. *Then* $\left|\frac{1}{(t-z-h)(t-z)} - \frac{1}{(t-z)^2}\right| = \left|\frac{h}{(t-z)^2(t-z-h)}\right| \le |h|\frac{2}{d^3}$ *since* $|t-z| \ge d$ *and* $|t-z-h| \ge \frac{d}{2}$. *Therefore, as* $|h| \to 0$, *integrand converges to* $\frac{1}{(t-z)^2}$ *uniformly on* $t \in [0,1]$. *So* $\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = \int_0^1 \lim_{h\to 0}\frac{1}{(t-z-h)(t-z)}dt = \int_0^1 \frac{1}{(t-z)^2}dt = F'(z)$.

By same reasoning, we get $F''(z) = \int_0^1 \frac{1}{(t-z)^3}dt$. *From existence of* F'', F' *is continuous. Therefore* F *is* C^1 *function. Existence of complex derivative and* C^1 *implies* F *is holomorphic on* $\mathbb{C}\setminus[0,1]$.

(b) For $s \in (0,1)$, $F(s+i\varepsilon) = \int_0^1 \frac{1}{t-s-i\varepsilon}dt = \int_0^1 \frac{t-s+i\varepsilon}{(t-s)^2+\varepsilon^2}dt = \int_0^1 \frac{t-s}{(t-s)^2+\varepsilon^2}dt + i\int_0^1\int_0^1 \frac{\varepsilon}{(t-s)^2+\varepsilon^2}dt$. *Let* $t - s = \varepsilon\tan\theta$. $\varepsilon\tan\theta_0 + s = 0$ *and* $\varepsilon\tan\theta_1 + s = 1$ *for* $-\frac{\pi}{2} < \theta_0, \theta_1 < \frac{\pi}{2}$. *Then* $\sec^2\theta_0 = \frac{s^2}{\varepsilon^2} + 1$, $\sec^2\theta_1 = \frac{(1-s)^2}{\varepsilon^2} + 1$, $\theta_0 = \tan^{-1}\left(\frac{-s}{\varepsilon}\right)$, *and* $\theta_1 = \tan^{-1}\left(\frac{1-s}{\varepsilon}\right)$.

Then $F(s+i\varepsilon) = \int_{\theta_0}^{\theta_1}\tan\theta d\theta + i\int_{\theta_0}^{\theta_1}d\theta = \log\left|\frac{\sec\theta_1}{\sec\theta_0}\right| + i(\theta_1 - \theta_0)$. *As* $\varepsilon \downarrow 0$, $F(s+i\varepsilon)$ *goes to* $\frac{1-s}{s} + i\pi$ *by simple calculation.*

Similarly, $F(s - i\varepsilon)$ goes to $\frac{1-s}{s} - i\pi$ as $\varepsilon \downarrow 0$.

(c) *Consider $F(-\varepsilon) = \int_0^1 \frac{1}{t+\varepsilon} dt = \log \frac{1+\varepsilon}{\varepsilon}$. It goes to ∞ as $\varepsilon \downarrow 0$.*

Consider $F(1+\varepsilon) = \int_0^1 \frac{1}{t-1-\varepsilon} dt = \log \frac{\varepsilon}{1+\varepsilon}$. It goes to $-\infty$ as $\varepsilon \downarrow 0$. Therefore, for $s = 0, 1$, $\lim_{z \notin [0,1] \to s} F(z)$ does not exists.

Problem (2.4)**.**

First consider $p \equiv 0$. We can easily see that $\sup_{z \in C} |z^{-n}| = 1$ so desired value ≤ 1.

Note that $|p(z) - z^{-n}| = |z^n p(z) - 1|$. Thus, $1 = \frac{1}{2\pi i} \int_C \frac{z^n p(z) - 1}{z} dz \leq \sup_{z \in C} |z^n p(z) - 1|$.

Those leads the conclusion.

Problem (2.5)**.**

It is enough to show γ and μ are path homotopic. Definte $H(t, s) = (1-s)\gamma(t) + \frac{\gamma(t)}{|\gamma(t)|} s$. Then $H(t, 1) = \mu(t)$ and $H(t, 0) = \gamma(t)$ by reparametrization. And H is continuous because $\gamma(t) \neq 0$. Therefore H is path homotopy between γ and μ. Since line integration is invariant under path homotopy, we get $\int_\gamma F(\zeta) d\zeta = \int_\mu F(\zeta) d\zeta$.

Problem (3.1).

It suffices to show that $\int_\gamma f(z)dz = 0$ for rectangle γ whose edges are parallel to coordinate axes by Morera's theorem.

First, assume that γ intersects with $[0,1]$ only finitely many points. Let p be such point. Then p must be on (wlog) left edge of γ. Let $a+ib$, $a+ic$ be two vertices incident with left edge. ($b > c$) Let $\rho(t) = a+i(tc+(1-t)b)$. Consider $f \circ \rho$. It is continuous and equals to $\frac{\partial}{\partial t}F(\rho(t))$ except for $\gamma^{-1}(p)$ where F is antiderivative of f on $\mathbb{C} \setminus [0,1]$. Then lemma 2.3.1 says $f(\rho(t)) = \frac{\partial}{\partial t}F(\rho(t))$ even for $\gamma^{-1}(p)$. Therefore $\int_\rho f(z)dz = F(a+ic) - F(a+ib)$. By using this result, we can easily calculate $\int_\gamma f(z)dz = 0$.

Now, assume that (wlog) upper edge of γ intersects with $[0,1]$. Let $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ which are upper edge, left edge, bottom edge, and right edge respectively, parametrized like ρ of above, positive oriented. Consider φ made by shrinking side edges of γ so that distance between of upper edges of φ and γ less than δ, while bottom edge is fixed. Also note that δ is chosen so that $d(z_0,z_1) < \delta$ implies $d(f(z_0), f(z_1)) < \varepsilon$.

$$\left|\int_\gamma f(z)dz - \int_\varphi f(z)dz\right| \leq \left|\int_{\gamma_2-\varphi_2} f(z)dz + \int_{\gamma_4-\varphi_4} f(z)dz\right| + (\text{ length of } \gamma_1)\,\varepsilon$$

And, second term of above goes to 0 as distance between φ_1 and γ_1 goes to 0 by continuity and result of first case. Actually $\int_\varphi f(z)dz = 0$ because φ does not intersect with $[0,1]$. Thus we have shown that $\int_\gamma f(z)dz = 0$.

By first, second case and Morera's thm, f is actually entire function.

Problem (3.2).

For $0 < r < 1$, $|f^{(n)}(0)| \leq \frac{n!}{r^n}\frac{1}{1-r}$ by using Cauchy estimate. $r^n(1-r)$ is maximized when $r = \frac{n}{n+1}$. So, when $r = \frac{n}{n+1}$, we get best estimate of $|f^{(n)}(0)|$.

Problem (3.3).

(a) *Since K is compact subset of open set U, there is $r > 0$ such that for all $x \in K$, closure of $D(x,r)$ is in U. Then, $|f(z)|^2 \leq \frac{1}{2\pi}\left|\int_{\partial D(z,r} \frac{f^2(w)}{w-z}dw\right| \leq \frac{1}{2\pi}\int_0^{2\pi}|f^2(z+re^{i\theta})d\theta|$. By multiplying ρ both sides and integrating from 0 to r, we can get the following:*

$$\begin{aligned}\frac{r^2}{2}|f(z)|^2 &\leq \frac{1}{2\pi}\int_0^r\int_0^{2\pi}\rho|f^2(z+re^{i\theta})|d\theta d\rho\\ &= \frac{1}{2\pi}\int_{\overline{D}(z,r)}|f|^2dm\\ &= \frac{1}{2\pi}\int_U|f|^2dm\end{aligned}$$

for all $z \in K$, where m is lebesgue measure, using Holder's inequality and polar coordinate integration.

Therefore $C = \frac{1}{r\sqrt{\pi}}$

(b) *If f is identically zero, possible.*

Else if f is constant, then $\int_{\mathbb{C}}|f|dm = \infty$ since measure of complex plane is ∞.

Else, that is f is nonconstant entire function, then f must be unbounded. So, there is $\delta > 0$ such that $|f| \geq 1$ for all $|z| > \delta$. Then $\int_{\mathbb{C}}|f|dm \geq m(\{z : |z| > \delta\}) = \infty$.

Problem (3.4). (a) *Since $\frac{z}{e^z-1}$ is bounded near 0, it has removable singularity at 0. So we can regard it as holomorphic function. Note that $e^z - 1 = 0$ when z is integer multiple of $2\pi i$. So, given power series converges on unit disc. Now, multiply $e^z - 1$ both sides. Since $e^z - 1$ is entire and given power serires converges absolutely on $\bar{D}(0, r)$ where $0 < r < 1$, we can write $z = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n\sum_{n=1}^{\infty}\frac{1}{n!}z^n$. Since z is entire, coefficient of power series is unique. By comparing coefficients of both sides, we can get given recursion formula.*

$\lim_{z\to 0}\frac{z}{e^z-1} = 1 = B_0$. *From this, by simple calculation, $B_1 = \frac{-1}{2}$, $B_2 = \frac{1}{6}$, and $B_3 = 0$.*

Consider $-z = f(z) - f(-z) = \sum_{n=0}^{\infty}2\frac{B_{2n+1}}{(2n+1)!}z^{2n+1}$. This makes sense because f is holomorphic on unit disc. By comparing coefficient of this series, we can get $B_{2m+1} = 0$ for $m \geq 1$.

(b) *We alreadly notice that $e^z - 1$ is zero when z is integer multiple of $2\pi i$. But $\lim_{z\to 2k\pi i}\frac{z}{e^z-1}$ is not bounded when $k \neq 0$. Therefore, $\frac{z}{e^z-1}$ is holomorphic on $D(0, 2\pi)$ and is not holomorphic outside of that disc. Since* *power seriese representation of holomorphic function at P has radius of convergence at least $d(P, U)$, we can say radius of convergence of the series is 2π.*

Problem (3.5)**.**

f' is holomorphic on unit disc. Let $r = \sup_{z \in K} |z|$. Since K is compact, $|f'| \leq M$ on K and r is positive but less than 1. Let $\gamma(t) = tz^n$ which connects origin and z^n. $|f(z^n) - f(0)| = \left|\int_\gamma f' dz\right| \leq M \sup_{z \in K} |z|^n = Mr^n$. Therefore, $|\sum_{n=1}^{\infty} f(z^n)| \leq \sum_{n=1}^{\infty} |f(z^n)| \leq \sum_{n=1}^{\infty} Mr^n < \infty$ because r is positive but less than 1.

Problem (4.1)**.**

Notice that f does not vanish on $\mathbb{C}\setminus\{0\}$. Therefore $g(z)=\frac{1}{f(z)}$ is holomorphic on $\mathbb{C}\setminus\{0\}$. Near 0, g is bounded since $\sqrt{|z|}$ goes to 0 as z goes to 0. This means g has removable singularity at 0 and therefore entire. But $g(z)\le\sqrt{|z|}$, so g must be constant by Cauchy integral formula.

Then f must be constant also, and this is contradiction. Therefore there is no such holomorphic function.

Problem (4.2)**.**

Let $g(z)=f\left(\frac{1}{z}\right)$. Then $g\to 0$ as $z\to 0$. Therefore g is entire. Also, $g(z)/z$ is entire since $\lim_{z\to 0} g(z)/z = g'(0)$ hence bounded near 0.

Now, consider given integral. Let $\zeta=e^{it}$ and $t=2\pi-s$. Then given integral is $\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(e^{-is})}{e^{-is}-z}ie^{-is}ds=\frac{1}{2\pi i}\int_0^{2\pi}\frac{g(e^{is})}{e^{is}-e^{2is}z}ie^{is}ds=\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{g(\zeta)}{\zeta z\left(\frac{1}{z}-w\right)}d\zeta$

Therefore given integral is equal to $\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{h(\zeta)}{\frac{1}{z}-\zeta}d\zeta$ where $h(\zeta)=\frac{g(\zeta)}{\zeta z}$. Thus, it is equal to $-g(1/z)=-f(z)$.

Problem (4.3)**.**

f maps $re^{i\theta}$ to $\sqrt{r}e^{i\left(\frac{\theta}{2}+k(z)\pi\right)}$ where $k(z)\in\mathbb{Z}$. To f be continuous, $k(z)$ must be all even or all odd.

First assume that $k(z)$ is all even. Then $f'(0)=\lim_{\mathbb{R}\ni h\to 0}\frac{f(h)}{h}=\lim_{\mathbb{R}\ni h\to 0}\frac{\sqrt{h}}{h}=\infty$, which is contradiction.

Similarly, if $k(z)$ is all odd, $f'(0)$ does not exist.

Therefore existence of such f leads $0\notin U$.

Let ι be identity function of U. Since $z\notin U$, ι does not vanish on U, hence $1/\iota$ is holomorphic on U. Since U is hsc, $1/\iota$ has holomorphic antiderivative φ.

Now consider the derivative of $\iota(z)e^{-\varphi(z)}$. Simple calculation leads that it is equal to 0. Hence $\iota(z)=ce^{\varphi(z)}$ for some constant c. Therefore $\iota(z)=e^{\psi(z)}$ for some holomorphic ψ on U.

Take $f=e^{\frac{1}{2}\psi}$. Then f satisfies what we want.

Problem (4.4)**.**

(a) *Let γ_R be the contour used in example 4.6.5.*

First, consider $\int_0^\infty \frac{1}{x^a(x+1)}dx$*. To calculate this, take* $f(z) = z^{-a}/(1+z)$ *where* $0 < arg(z) < 2\pi$*. By residue thm,* $2\pi i e^{-a\pi i} = \int_0^\infty \frac{1}{r^a(r+1)}dr\left(1-e^{-2a\pi i}\right)$*. Therefore* $\int_0^\infty \frac{1}{x^a(x+1)}dx = \pi\csc(\pi a)$*.*

Now, $\int_{\gamma_R} \frac{\log z}{z^a(1+z)}dz = 2\pi i e^{-a\pi i}\pi i$ *by residue thm. But as* $R \to \infty$*, that integral goes to* $(1-e^{-2a\pi i})\int_0^\infty \frac{\log r}{r^a(r+1)}dr - e^{-2a\pi i}\int_0^\infty \frac{2\pi i \log r}{r^a(r+1)}dr$*.*

By simple calculation, the value we want is equal to $\frac{i\pi^2}{\sin(\pi a)} + \frac{\pi^2 e^{-a\pi i}}{\sin^2(\pi a)} = \frac{\pi^2\cos(\pi a)}{\sin^2(\pi a)}$*.*

(b) *Consider* $f(z) = \frac{\pi\cot(\pi z)}{(z+\alpha)^2}$ *and* Γ_n *= square centered at origin, each edges is parallel to real or imaginary axis, length of edge is* $2n+1$*.*

Then $\int_{\Gamma_n} f(z)dz$ *goes to 0 as* $n \to \infty$ *by considering modulus of* $f(z)$*, and index of* Γ_n *at each singularites is 1, and residues are* $\frac{1}{(k+\alpha)^2}$ *at* $z = k$ *and* $-\frac{\pi^2}{\sin^2(\pi\alpha)^2}$ *at* $z = -\alpha$*.*

Above calculation leads the conclusion.

Problem (4.5)**.**

Note that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *is holomorphic iff* f *is meromorphic on* $\hat{\mathbb{C}}$*.*

(a) *First consider 'if' part. Let* f *be rational function. We already knows that rational function is meromorphic on entire complex plane. So, we need to show that rational function is meromorphic at* ∞*.*

Let $f(z) = \frac{(z-Q_1)^{m_1}\cdots(z-Q_l)^{m_l}}{(z-P_1)^{n_1}\cdots(z-P_k)^{n_k}}$*. Since* f *has finitely many pole in complex plane, we can choose* $M > 0$ *so that* f *has no pole on* $\{z : |z| > M\}$*. For* $0 < |w| < \frac{1}{M}$*, consider* $g(w) = f(1/w)$*. Then* g *is holomorphic.*

Let $\sum_i n_i = N$ *and* $\sum_j m_j = M$*. If* $M = N$*,* $g \to 1$ *as* $z \to 0$*. If* $M > N$*,* $g \to 0$ *as* $z \to 0$*. If* $M < N$*,* $g \to \infty$ *if* $z \to 0$*. Hence* g *is meromorphic near* 0*, which means that* f *is meromorphic at* ∞*.*

Second, consider 'only if' part. Either f *has a pole or removable singularity at* ∞*,* f *has finitely many poles in complex plane. So* $f(z)(z-P_1)^{n_1}\cdots(z-P_k)^{n_k} = F(z)$ *is entire where* n_i *is order of pole* P_i*.*

Consider $F(1/z) = g(z)$ *for* $z \neq 0$*. As* $z \to 0$*,* $g \to \infty$ *or* α *for some* $\alpha \in \mathbb{C}$ *by simple calculation. Therefore* F *has a pole or removable singularity at* ∞*.*

If F has removable singularity at ∞, F must be bounded, hence constant by Liouville's thm.

If F has a pole at ∞, F must be polynomial since its modulus diverges.

In both cases, F must be rational function.

(b) *Note that $z \mapsto \frac{az+b}{cz+d}$ for $ad-bc \neq 0$ is biholomorphic function of Riemann sphere. Also note that biholomorphic function of $\mathbb{C}$ must have a form of $\alpha z + \beta$ for $\alpha \neq 0$ by fundamental thm of algebra.*

Now consider biholomorphic f on Riemann sphere. Let $f(\infty) = b$ and $\varphi_b(z) = \frac{-\bar{b}-1}{z-b}$. Then $\varphi_b \circ f$ is biholomorphic function of Riemann sphere, which maps $\infty \to \infty$. Therefore $\varphi_b \circ f$ is biholomophic function of complex plane hence $\varphi_b(f(z)) = \alpha z + \beta$. Then $f(z) = \frac{-b\alpha z - b\beta + 1}{-\alpha z - \beta - \bar{b}}$, which is linear frational transformation.

Problem (5.1)**.**

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ *and assume that* $P(z) = 0$ *has no solution. Then by the argument principle,* $\frac{1}{2\pi i}\int_{\partial D(Q,R)} \frac{P'(\zeta)}{P(\zeta)}d\zeta = 0$ *for all* $R > 0$*. That integral is equal to* $\frac{1}{2\pi i}\int_0^{2\pi} \frac{P'(Q+Re^{i\theta})}{P(Q+Re^{i\theta})}Rie^{i\theta}d\theta$*. But, as* $R \to \infty$*, integrand of above goes to in uniformly on* $0 \leq \theta \leq 2\pi$*. Therefore, the integral above goes to* $n > 0$ *which is the degree of* P*. It is contradiction. Thus* $P(z) = 0$ *has at least one solution in complex plane.*

Problem (5.2)**.**

Assume the existence of such f*. Since* f *is bounded near* 0*, Riemann removable singularity theorem says that* f *can be extended to the function which is holomorphic on entire unit disc.*

If modulus of $f(0)$ *is equal to 1 or 2, then image of the unit disc under* f *is not open which contradicts to the open mapping theorem. So* $f(0) \in \{w : 1 < |w| < 2\}$*.*

Since f *is surjective function of the punctured unit disc onto the annulus, we can find* $w \neq 0$ *such that* $f(0) = f(w)$*. Choose two disjoint neighborhood* U_w, U_0 *of* $w, 0$ *respectively. Then by the open mapping theorem,* $f(U_w)$ *and* $f(U_0)$ *are open and* $f(0) \in f(U_w) \cap f(U_0)$*. Since* $f(U_w) \cap f(U_0)$ *is open, we can choose small neighborhood of* $f(0)$ *contained in the previous set. And therefore we can choose* $f(0) \neq \alpha \in f(U_w) \cap f(U_0)$*. This cannot be happen since* f *is injective.*

Thus there is no such f*.*

Problem (5.3)**.**

(a) *Choose* $R > \lambda$*, and choose* n *so large that* $\lambda - 1 \geq 1/n$*. Then* $\bar{D}(R, R - \frac{1}{n}) \subset$ *Right half plane.*

   Then for $\zeta \in \partial D(R, R - 1/n)$*,* $|e^{-\zeta}| < 1 \leq \lambda - 1/n \leq |\zeta - \lambda|$*. Put* $f(z) = e^{-z} + z - \lambda$ *and* $g(z) = z - \lambda$*. Then by above and Rouche's theorem,* f *and* g *has same zero on* $D(R, R - 1/n)$*. But any* $z \in$ *Right half plane must be inside of* $D(R, R - 1/n)$ *for some large* R *and* n*. This means* f *and* g *have same zero on the right half plane.*

   But $g(z) = 0$ *has unique solution. Therefore* $e^{-z} + z - \lambda = 0$ *has unique solution on the right half plane.*

(b) *Fix $z' \in U$. Note that $U \setminus \{z'\}$ is still a domain. Let $g_k(z) = f_k(z) - f_k(z')$ for $z \in U \setminus \{z'\}$. Since f_j is an injective holomorphic function on U, g_k does not vanish on $U \setminus \{z'\}$. Uniform convergence of f_j on compact subsets of U implies uniform convergence of g_k on compact subsets of $U \setminus \{z'\}$. Since g_k is nonvanishing function, by Hurwitz's theorem, $\lim_{k\to\infty} g_k(z) = f(z) - f(z')$ does not vanish or identically zero.*

If it is identically zero on $U \setminus \{z'\}$, then f must be constant function on U. If it is nonvanishing on $U \setminus \{z'\}$, then $f(z'') = f(z')$ implies $z'' = z'$. Thus f must be injective.

Problem (5.4)**.**

It seems to be solved by the maximum modulus principle (or theorem), but I don't know where to start.

Problem (5.5)**.**

For $z \in S$, $|\varphi(z)| = \left|\frac{e^{2\pi zi}-1}{e^{2\pi zi}+1}\right|$, and the real part of $e^{2\pi zi} > 0$ because $z \in S$. Then it is clear that $|\varphi(z)| < 1$. Also $\varphi(0) = 0$.

Therefore $\varphi \circ f : D \to D$ is holomorphic and it fixes the origin. Then Schwarz's lemma says $|\varphi'(0)f'(0)| \le 1$. But $\varphi'(0) = \pi$. Therefore $|f'(0)| \le 1/\pi$. The equality holds only if $\varphi(f(z)) = wz$ for some $|w| = 1$.

Problem (9.1)**.**

First, by considering the Maclaurine series of $\cos z$, $\cos\sqrt{z}$ *is an entire function. Now, note that* $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, *so modulus of* $\cos\sqrt{z}$ *is bounded by* $e^{|z|/2}$. *Therefore* $\lambda(\cos\sqrt{z}) \le 1/2$. *Since genus is nonnegative integer bounded by order, the genus of* $\cos\sqrt{z}$ *must be 0.*

Now consider $f(z) = \sin^2 z$. *Its zero set is* $\{k\pi\}$ *where* k *is an integer. Note that the smallest nonnegative integer* p *satisfying* $\sum_{k\neq 0} |k\pi|^{-p-1}$ *is 1. Therefore the rank of* f *is 1. Since* $f(z) = \frac{e^{2iz}+e^{-2iz}-2}{-4}$, *its modulus is bounded by* $e^{2|z|}$. *Thus* $\lambda(f) \le 1$. *But we know the relation :* $1 = rank \le genus \le order \le 1$. *Therefore the genus of* $\sin^2 z$ *is one.*

Now consider $g(z) = \sin z^2$. *The zero set of* g *is* $\{\sqrt{k\pi}\}$ *where* k *is an integer. Note that the smallest nonnegative integer* p *satisfying* $\sum 2|\sqrt{k\pi}|^{-p-1}$ *is 2. Therefore the rank of* g *is 2. Since* $g(z) = \frac{e^{iz^2}-e^{-iz^2}}{2i}$, *its modulus is bounded by* $e^{|z|^2}$. *Thus* $\lambda(g) \le 2$. *So,* $2 = rank \le genus \le order \le 2$.

$\square$

Problem (9.2)**.**

It is well known fact that $\left\{e^{in\sqrt{2}\pi} : n \in \mathbb{N}\right\}$ *is dense in* S^1. *Let* $a_n = \frac{2^n-1}{2^n} e^{in\sqrt{2}\pi}$. *Then every point on* S^1 *is accumulation point of* $\{a_n\}_{n=1}^{\infty}$. *Note that* $\sum 1 - |a_n| = \sum 2^{-n} < \infty$. *Therefore the corresponding Blaschke product* $B(z) = \Pi_n - \frac{\bar{a_n}}{|a_n|} B_{a_n}(z)$ *is holomorphic on the unit disc* D *and vanishes on* $\{a_n\}_{n=1}^{\infty}$ *exactly. But, if* $w \in \partial D$, *then* w *is accumulation point of the zero set of* B. *Thus if* w *is regular, then extension of* B *on small neighborhood of* w *is identically zero, which is contradiction. So* B *is the desired one.*

$\square$

Let f be an entire function. Let $M(r) = \sup_{|z|=r} |f(z)|$. Before #3 and #4, we need the followings:

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \lambda$$

$$\limsup_{n\to\infty} \frac{n\log n}{-\log|a_n|} = \lambda$$

where a_n is the n-th Maclaurine coefficient of f.

For the first formula, let $\rho < a = \limsup \frac{\log\log M(r)}{\log r}$. Then there is $r_n \uparrow \infty$ such that $\rho < \frac{\log\log M(r_n)}{\log r_n}$. Then $M(r_n) > \exp(r_n^\rho)$ which says $\lambda \geq \rho$. Since ρ is arbitrary, we can deduce that $\lambda \geq a$.

For the other direction, let $\rho < \lambda$. Then there is increasing sequnce $r_n \uparrow \infty$ such that $M(r_n) > \exp(r_n^\rho)$. Thus $\log\log M(r_n)/\log r_n \geq \rho$ which leads $a \geq \rho$. Since $\rho \leq \lambda$ is arbitrary, $a \geq \lambda$.

For the second formula, let $\mu = \limsup_n \frac{n\log n}{-\log|a_n|}$. If $\mu = \infty$, then $\lambda \leq \mu$ directly. So assume $\mu < \infty$ and $\varepsilon > 0$. Then $0 \leq \frac{n\log n}{-\log|a_n|} \leq \mu + \varepsilon$ for $n \geq N$. By simple calculation, $|a_n| \leq n^{-n/(\mu+\varepsilon)}$. Thus $M(r) \leq \sum |a_n| r^n \leq \sum n^{-n/(\mu+\varepsilon)} r^n = \sum_{n<(2r)^{\mu+\varepsilon}}() + \sum_{n\geq(2r)^{\mu+\varepsilon}}() = S_1 + S_2$.

$$
\begin{aligned}
S_1 &\leq r^{(2r)^{\mu+\varepsilon}} \sum_n n^{-n/(\mu+\varepsilon)} \\
&= O(r^{(2r)^{\mu+\varepsilon}} = O(\exp((2r)^{\mu+\varepsilon}\log r)) \\
&= O(\exp(r^{\mu+2\varepsilon}))
\end{aligned}
$$

And $n^{-1/(\mu+\varepsilon)} r \leq 1/2$ yields $S_2 \leq 1$. Thus $M(r) = O(\exp(r^{\mu+2\varepsilon}))$, which implies $\lambda \leq \mu + 2\varepsilon$. By letting $\varepsilon \downarrow 0$, we get $\lambda \leq \mu$.

For the other direction, let $0 < \tau < \mu$. Then $\tau \leq \frac{n\log n}{-\log|a_n|}$ for infinitely many n which goes to ∞. For those n, $\log|a_n| \geq \frac{-n\log n}{\tau}$. By cauchy's thm, we know that $|a_n| \leq M(r) r^{-n}$. So,

$$
\begin{aligned}
\log M(r) &\geq \log|a_n| + n\log r \\
&\geq n\left(\log r - \frac{\log n}{\tau}\right)
\end{aligned}
$$

By taking $r_n = (en)^{1/\tau}$, $\log M(r_n) \geq n/\tau = r_n^\tau/(e\tau)$. So

$$
\frac{\log\log M(r_n)}{\log r_n} \geq \frac{\tau\log r_n - \log e\tau}{\log r_n}
$$

thus $\limsup \geq \tau$. Since τ is arbitrary, we get $\lambda \geq \mu$ by the first formula.

□

Problem (9.3)**.**

If $\sum a_n z^n$ is an entire function, then its order is determined by $\limsup_{n\to\infty} \frac{n\log n}{-\log|a_n|}$.

(a) *First represent f as the Maclaurine series. Let a_n be its n-th coefficient.*

But $\limsup_n \frac{n \log n}{-\log n - \log |a_n|} = \limsup_n \frac{n\log n}{-\log |a_n|}$*. So the order of* f *and* f' *are same.*

(b) *Note that* $\log E_n(z) = z^{n+1}/(n+1) + z^{n+2}/(n+2) + \cdots$ *by power series. Also,* $\log |z| \le |\log z| = |\log |z| + iarg(z)|$*. So* $\log |E_n(z)| \le |z|^{n+1}/(1 - |z|)$ *for* $|z| < 1$*.*

By definition of E_n*, it is also clear that* $\log |E_n| \le \log |E_{n-1}| + |z|^n$*. Now we claim that* $\log |E_n| \le (2n+1)|z|^{n+1}$*. This can be done by the following:*

$$\begin{aligned}\log |E_n| &\le |z| \log |E_n| + |z|^{n+1} \\ &\le |z|(\log |E_{n-1}| + |z|^n) + |z|^{n+1} \\ &\le |z|(2n|z|^n) + |z|^{n+1}\end{aligned}$$

for $|z| < 1$ *and induction. The case when* $|z| \ge 1$ *can be done by using the part of above.*

Now put $n = \mu =$*genus. Let* P *be the canonical product of given entire function with rate* μ*. Then* $\log |P| \le (2\mu+1)|z|^{\mu+1} \sum_n |a_n|^{-\mu-1}$*. Since* $f = cz^m e^g P$ *where the degree of* g *is less or equal to* μ*, the order of* f *is thus determined by* P*. The above inequality implies* $\lambda(f) \le \mu + 1$*.*

(c) *Let* a_n *be a sequence of zeros of* f*. Since we know that the order of* f *and* f' *are same,* $\lambda(f) \le 1$*. Thus* $\sum_n |a_n|^{-1-1} < \infty$*. But* $\sum_n (\sqrt{n})^{-1-1} \le \sum_n |a_n|^{-1-1} < \infty$ *which is contradiction. Therefore* f *must be constant, so* $f(z) = 0$ *for every* z*.*

□

Problem (9.4)**.**

Let a_n *be* n*-th coefficient of* g*. Then* $\limsup_{n\to\infty} |a_n|^{1/n} = 0$ *so the radius of convergence is* ∞*, thus* g *is an entire function.*

By Stirling's formula, $\log(n!) = n\log n - n + O(\log n)$*. Therefore* $\frac{n\log n}{\log(n!)} \to 1$ *as* $n \to \infty$*.*

$\frac{n\log n}{-\log a_n} = \frac{n \log n}{\alpha \log(n!)} \to 1/\alpha$ *as* $n \to \infty$*. Therefore the order of* g *is* $1/\alpha$*.*

□

Problem (9.5)**.**

By considering the Maclaurine series of $\sin z$*,* $\sin\sqrt{z}/\sqrt{z}$ *is holomorphic by the Riemann removable singularity theorem. And by simple calculation, its order is bigger than* 0 *and smaller or equal to* $1/2$*.*

Now, consider $f(z) = \sin z/z$*. Since the order of* f *is finite and* f *is entire, it can omit at most one complex number. If* f *omit the value* c*,then* $f(z) - c$ *is nonvanishing, so* $f(z) - c = \exp(g(z))$*. But the degree of* g *must be* 0 *or* 1 *since the order of* f *is less or equal to* 1*. If the degree of* g *is zero, then* $f(z)-c$ *is constant which is contradiction. So we can say that* $f(z) - c = \exp(az + b)$*. But, as* $|z| \to \infty$*,* $\left|\frac{f(z)-c}{\exp(az+b)}\right| \to 0$ *which is contradiction because it must be equal to* 1*. Therefore, we can conclude that* $f(z)$ *assumes every complex value.*

Let $c \in \mathbb{C}$ *be given. Then the solution of* $f(z) = c$ *exists, say* α*. Then* α^2 *is a solution of* $f(\sqrt{z}) = c$*. Therefore* c *is in the image of* $f(\sqrt{z})$*, which is entire of nonintegral finite order. Thus there are infinitely many solutions of* $f(\sqrt{z}) = c$*, say* $w_1, w_2, \cdots$*. Then* $\sqrt{w_1}, \sqrt{w_2}, \cdots$ *are the infinite solutions of* $f(z) = c$*.*

$\square$

Problem (10.1)**.**

(a) *Let* $\cos z = g(z)$*.* $g'(z) = -\sin z$ *hence* $g'(\pi/2) = -1 \neq 0$*. Note that* $g(\pi/2) = 0$*. So, by theorem 5.2.2, there are* $\delta, \varepsilon > 0$ *such that each* $q \in D(0, \varepsilon)$ *has unique inverse image under* g*, and the inverse image of* q *lies in* $D(\pi/2, \delta)$*. It is well known that* $f : q \mapsto g^{-1}(q)$ *on* $D(0, \varepsilon)$ *is holomorphic.*

Therefore, we have the function element (f, U)*. Uniqueness(up to* ε*) follows from the uniqueness of inverse image of* g *on* $D(0, \varepsilon)$*.*

(b) *Let* α *be any complex number.* $\cos w = \alpha$ *can be rewritten as* $t^2 - 2\alpha t + 1 = 0$ *for* $t = e^{iw}$*. The former equation has order 2 solution when* $\alpha = \pm 1$*, and otherwise, has simple two solutions. Thus, when* $\alpha \neq \pm 1$*, by choosing one of two solutions, we can apply theorem 5.2.2 again. So we can find function element of* arccos (f, U) *where* U *is a disc centered at* α*, whose preimage under* $g(z) = \cos z$ *contains one of two solutions as described before.*

Now, let $\Delta \subset \mathbb{C} \setminus \{-1, 1\}$ *be a disc, where* g^{-1} *is well defined and holomorphic. Then* (f, Δ) *is a function element of* arccos*. Let* $D = g^{-1}(\Delta)$*. Then* g *is a conformal mapping of* D *onto* Δ*.*

When $\Delta \cap \Delta' \neq \emptyset$*, we can find corresponding* D, D' *which intersects. Then, by letting* f *be the inverse of* $g|_D$*, we can get the function element* (f, Δ)*. Since* g *has unique inverse on* $D \cap D'$*,* $f = f'$ *on* $\Delta \cap \Delta'$*. Thus* $(f, \Delta), (f', \Delta')$ *are direct analytic continuation. This process may be continued.*

Let γ *be a path from the origin to* $\alpha \in \mathbb{C} \setminus \{-1, 1\}$*. From the origin, we can apply the above procedure along* γ*. Then, by using compactness of the image of* γ*, we can cover the image by finite chain of* Δ_i *such that* (f_i, Δ_i) *is a direct analytic continuation of* (f_{i-1}, Δ_{i-1})*.*

It says that (f, U) *from (a) admits unrestricted continuation in* $\mathbb{C} \setminus \{-1, 1\}$*.*

(c) *If* $z_0 = \pm 1$*, then* $\sin h(z_0) = 0$*. By chain rule,* $-\sin h(z) h'(z) = 1$*. Putting* $z = z_0$ *leads contradiction. So* $z_0 \neq \pm 1$*.*

Now, let (f, U) *be that of (a) and note that for given* $\Delta \subset \mathbb{C} \setminus \{-1, 1\}$ *which is a disc centered at* α*, there are exactly two function elements by* *solving the equation* $\cos w = \alpha$*. Let* $\gamma_1, \gamma_2, \gamma_3$ *be curves where the index of* $\gamma_1 - \gamma_2$ *at* 1 *is* ± 1 *but* 0 *at* -1*, and the inndex of* $\gamma_1 - \gamma_3$ *at* -1 *is* ∓ 1 *but* 0 *at* 1*.*

Then (h, Δ) *can be achieved by analytic continuation along one of* γ_i*'s. Because, if not,* (f, U) *defines global* arccos *on* $\mathbb{C} \setminus \{-1, 1\}$ *which is impossible.*

Impossibility follows from this observation: Let $\delta(t) = 1 + \varepsilon e^{2\pi i t}$*. Analytic continuation of* (f, U) *along* δ *leads another function element defined on* U *which is a disc centered at the origin. In fact, this observation leads the conclusion: given* (h, Δ) *is a member of equivalence class determined by* (f, U)*.*

$\square$

Problem (10.2)**.**

(a) *Let* $u = s + t, v = t/s$*. Then the integral must be:*

$$\int_0^\infty \int_0^\infty \frac{v^{z-1}}{1+v} e^{-u} du dv = \int_0^\infty \frac{v^{z-1}}{v+1} dv$$

Since $0 < 1 - z < 1$*, by calculating residue (similar to #4 (a) of hw4), we can get* $= \pi / \sin \pi(1-z) = \pi / \sin \pi z$*.*

From holomorphy of $\Gamma(z)\Gamma(1-z), \pi/\sin \pi z$ *on* $\mathbb{C} \setminus \mathbb{Z}$*, they are same by the identity theorem.*

(b) *Note that* $\Gamma(z) = \int_0^1 e^{-t} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt = S_1 + S_2$*. Then* $|S_2| \le \int_1^\infty e^{-t} t^{s-1} dt$ *where* $s = Re(z)$*. When* $s \ge 1/2$*, take* $s \le n \le s+1$*. For such* n*,* $|S_2| \le \int_1^\infty e^{-t} t^n dt = \Gamma(n+1) = n! \le n^n = e^{n \log n} \le e^{(s+1)\log(s+1)}$*. Since* $|\sin \pi z| \le e^{|z|}$*,* $\left|\frac{\sin \pi z}{\pi} \Gamma(z)\right| \le e^{C|z|\log|z|}$*.*

$|S_1| \le |\int_0^1 \sum_{n=0}^\infty t^{n+s-1} (-1)^n / n! dt| = |\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)}|$*. But, the last term is bounded by constant if* $s \ge 1/2$*.*

Thus the result holds for $Re(z) = s \ge 1/2$*.*

(c) *First, (a) says that* $1/\Gamma$ *is entire and has simple zeros at nonnegative integers. Then (b) says that the order of entire function* $1/\Gamma$ *is* 1*. Thus* *the Hadamard factorization theorem implies:*

$$\frac{1}{\Gamma(z)} = e^{Az+B} z \Pi_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{\frac{-z}{n}}$$

Note that $B = 0$ *by considering* $z\Gamma(z) = \Gamma(z+1) \to 1$ *as* $z \to 0$*. Next, by putting* $z = 1$,

$$\begin{aligned}
e^{-A} &= \Pi_{n=1}^{\infty} (1 + 1/n)\, e^{\frac{-1}{n}} \\
&= \exp(\sum_{n=1}^{\infty} \left(\log(1 + \frac{1}{n}) - \frac{1}{n}\right) \\
&= \lim_{N \to \infty} \exp\left(\sum_{n=1}^{N} (\log(1 + 1/n) - 1/n\right) \\
&= \lim_{N \to \infty} \exp\left(-\sum_{n=1}^{N} 1/n + \log N + \log(1 + 1/N)\right) \\
&= e^{-\gamma}
\end{aligned}$$

□

Problem (10.3)**.**

Every element of Γ *can be expressed as finite product of* $\mu, \omega, \mu^{-1}, \omega^{-1}$*. So, we'll use the induction on the length of* $f \in \Gamma$.

First, for length 1 f*, the assertion trivially holds. Let length of* f *be* n*. Then,*

$$\mu \circ f(z) = \frac{az + b}{(2a + c)z + 2b + d}$$

where a, d *are odds and* b, c *are evens. Thus* $\mu \circ f$ *satisfies the assertion. Also,*

$$\omega \circ f(z) = \frac{(a + c)z + b + d}{cz + d}$$

so $\omega \circ f$ *satisfies the assertion.*

Similarly, for the inverses of μ, ω*, we can check the assertion. Therefore, the assertion holds by induction.*

□

Problem (10.4)**.**

(a) *Since f is doubly periodic, it is sufficient to show that the residue of f at the origin is zero. Let*

$$\begin{aligned}
\gamma_1(t) &= \frac{1}{2}it \\
\gamma_2(t) &= -t + i\frac{1}{2} \\
\gamma_3(t) &= -\frac{1}{2} - it \\
\gamma_4(t) &= t - i\frac{1}{2}
\end{aligned}$$

where $-1/2 \leq t \leq 1/2$. Then by adjoining the above paths, we get the curve γ whose image is the square centered at the origin.

Now, integrate f along γ. Then

$$\int_\gamma f = \sum_{i=1}^{4} \int_{\gamma_i} f$$

But we can easily check that integral of f along γ_i and γ_{i+2} are cancelled by its double periodicity($i = 1, 2$). Thus $\int_\gamma f = 0$. Therefore, by the residue theorem, $Res_0(f) = 0$. This completes the proof.

$\square$

(b) *Let α be any complex number. Let Λ be the integer lattice.*

If $\inf_{z\in\mathbb{C}\setminus\Lambda} |\wp(z) - \alpha| = \varepsilon > 0$, then $1/(\wp(z) - \alpha)$ is bounded by $1/\varepsilon$ on $z \in \mathbb{C} \setminus \Lambda$. For $z \in \Lambda$, $1/(\wp(z) - \alpha) = 0$ so by the Riemann removable singularity theorem, $1/(\wp(z) - \alpha)$ is entire but bounded. So it must be constant which is contradiction. Thus, infimum of $|\wp(z) - \alpha|$ over $\mathbb{C} \setminus \Lambda$ equals to 0 for any complex number α.

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence such that $|\wp(z_n) - \alpha| \to 0$ as $n \to \infty$. Since $\wp(z)$ is doubly periodic, by translating each z_n appropriately, we can regard $\{z_n\}$ as a sequence contained in the (closed) unit square whose vertices are $(0, 0), (0, 1), (1, 0), (1, 1)$.

Then $\{z_n\}$ is contained in compact set, so it has convergent subsequence $\{z_{n_k}\}$. But z_{n_k} cannot be converges to the vertices described above. Because $\wp(z) - \alpha = \infty$ at those vertices. So z_{n_k} must converges to an-

other points of the unit square described above. Then, by continuity, $|\wp(z) - \alpha| = 0$ for some $z \in$ [the unit square except the vertices].

It leads surjectiveness of $\wp$.

$\square$

Problem (10.5)**.**

(a) *Fix t and consider the following equation of z:*

$$\wp(z) = \gamma(t)$$

This equation always has a solution since range of $\wp$ is $\mathbb{C}$. Let α be a solution of the above equation. Then $\wp'(\alpha) \neq 0$. So, (holomorphic) inverse function theorem can be applied. From z_0, we can analytically continue this function to $\gamma(1)$ along γ(detail: same as problem 1). Then $\Gamma : t \mapsto \wp^{-1}(\gamma(t))$ is what we want.

Uniqueness directly follows from construction. Since $\wp(z) = \gamma(t)$ has simple solution thus $\wp$ is locally invertible.

(b) *By definition of line integral,*

$$\begin{aligned}
\int_\gamma \frac{dw}{\sqrt{4w^3 - C_1 w + C_2}} &= \int_0^1 \frac{\gamma'(t)}{\sqrt{4\gamma(t)^3 - C_1\gamma(t) + C_2}} dt \\
&= \int_0^1 \frac{\wp'(\Gamma(t))\Gamma'(t)}{\wp'(\Gamma(t))} dt \\
&= \int_0^1 \Gamma'(t) dt \\
&= \Gamma(1) - \Gamma(0).
\end{aligned}$$

$\square$

Problem (1)**.**

Note that $H_p \subset L_p(D(0,1))$. Thus $\|\cdot\|_p$ is well defined norm of H_p. H_p is vector space by Minkowski's inequality.

Let $K \subset D = D(0,1)$ be compact. Then, there is $r > 0$ such that $D(z,r) \subset D$ for each $z \in K$. Then,

$$\begin{aligned} |f(z)|^p &\le \frac{1}{2\pi}\left|\int_{\partial D(z,r)} \frac{f^p(\zeta)}{\zeta - z} d\zeta\right| \\ &\le \frac{1}{2\pi}\int_0^{2\pi} |f(z+re^{i\theta})|^p \theta. \end{aligned}$$

By multiplying ρ both sides and integrating from 0 to r, we get the following:

$$\begin{aligned} \frac{r^2}{2}|f(z)|^p &\le \frac{1}{2\pi}\int_0^r \int_0^{2\pi} \rho|f(z+re^{i\theta})|^p d\theta d\rho \\ &= \frac{1}{2\pi}\int_{D(z,r)} |f|^p dm \\ &\le \frac{1}{2\pi}\int_{D(0,1)} |f|^p dm \end{aligned}$$

by polar coordinate integration.

Thus, $\sup_{z\in K}|f(z)| \le 1/(\pi r^2)^{1/p}\|f\|_p = C\|f\|_p$, where C depends on K only.

Let $\{f_k\}$ be a Cauchy in H_p. And fix $K \subset D$. Then

$$\sup_{z\in K}|f_n(z) - f_m(z)| \le_K C\|f_m - f_n\|_p$$

Thus f_k is uniformly Cauchy in K, which says normal convergence of f_k. Let the normal limit of f_k be f(holomorphy of f follows from the fact that f is the normal limit of f_k, which are holomorphic).

Let $\varepsilon > 0$ be given. Choose N such that $\|f_n - f_m\|_p \le \varepsilon$ for $m, n \ge N$. Then, by Fatou's lemma,

$$\int_D |f - f_m|^p dm \le \liminf_{n\to\infty}\int_D |f_n - f_m|^p dm \le \pi\varepsilon^p$$

Therefore $f - f_m \in H_p$ and $\|f - f_m\|_p$ goes to 0.

□

Problem (2)**.**

1. *Fix* $r > 0$*. Then, as* $r \to \infty$*,*

$$\left|\frac{f^{(n)}(0)}{n!}\right| \le \frac{|f(z)|}{r^n} \to 0.$$

if $n \ge m$*. Because, by maximum modulus principle, maximum of* $|f(z)|$ *on* $\overline{D}(0,r)$ *must occur on* $\partial D(0,r)$ *and by the given condition.*

Thus, for $n \ge m$*,* n*-th Maclaurine coefficient is equal to* 0*. Therefore* f *must be a polynomial of degree at most* $m-1$*.*

2. *To satisfy* $\lim_{|z|\to\infty} f(z)/|z|^m = \infty$*,* $\lim_{|z|\to\infty} f(z)$ *must be* ∞*. Such* f *must be polynomial by theorem 4.7.5. But any polynomial* f *does not satisfy the given condition for all positive integer* m*.*

Therefore, we can conclude that we cannot find such entire function.

$\square$

Problem (4)**.**

Let $K = \{r_1 \le |z| \le r_2\} \subset \{1 < z < 2\}$*, i.e.* $1 < r_1 < r_2 < 2$*. Then* $f^{-1}(K)$ *is compact, by continuity of* f^{-1}*. Since* $f^{-1}(K)$ *is compact, we can find* $1 > R > 0$ *such that* $f^{-1}(K) \subset \overline{D}(0,R)$*. Thus, for* $1 > |z| > R$*,* $f(z)$ *lies in the outside of* K *by bijectiveness of* f*.*

But, $\{1 > |z| > R\}$ *is connected subset of the unit disk. But, the image of* $\{1 > |z| > R\}$ *under* f *lies outside of* K*, which contradicts to connectivity.*

$\square$

Problem (5)**.**

Fix $r > 0$*. We can choose* k *such that* $|a_k| < r < |a_{k+1}|$*. Since* P_n *has simple zero at* a_i*,* $1/P_n$ *has simple pole at* a_i*. Note that* $Res_{1/P_n}(a_i) = \Pi_{j\ne i} 1/(a_i - a_j)$*. Thus, by residue theorem,*

$$\int_{|z|=r} \frac{dz}{P_n(z)} = 2\pi i \sum_{i=1}^{k} \Pi_{j\ne i} \frac{1}{(a_i - a_j)}$$

since index of $|z| = r$ *about each* a_i *is* 1*.*

$\square$

Problem (6).

The degree of f must be even. Otherwise, f must have at least one real root. Since f has real coefficients, if α is a root of $f(z) = 0$, then $\overline{\alpha}$ is also a root. Now let the degree of $f = 2n$. Then, by the above, f has n zeros in the upper half plane and the other n zeros are in the lower half plane.

Let $a_1, \cdots a_k$ be a set of distinct zeros of f which lie in the upper half plane. Since f can have multiple zeros, we cannot say a_i's are simple.

Now consider $\gamma_1^R(t) = t$, $\gamma_2^R(s) = Re^{2\pi is}$ where $-R \leq t \leq R$ and $0 \leq s \leq 1$. Then

$$\int_{\gamma_1^R+\gamma_2^R} \frac{dt}{f(t)} = 2\pi i \sum_{i=1}^{k} Res_{1/f}(a_i)$$

by the residue theorem. But, as $R \to \infty$,

$$\int_{\gamma_2^R} \frac{dt}{|f(t)|} \lesssim \frac{\pi R}{R^{2n}} \to 0$$

where $2n$ is the degree of f.

Also,

$$\int_{\gamma_1^R} \frac{dt}{f(t)} \to \int_{-\infty}^{\infty} \frac{dt}{f(t)}$$

by DCT.

Thus,

$$\int_{-\infty}^{\infty} \frac{dt}{f(t)} = 2\pi i \sum_{i=1}^{k} Res_{1/f}(a_i)$$

□

Problem (7).

1. *Let $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$. Note that the images of f_n are uniformly bounded by Ω. Then by Montel's theorem, $\mathcal{F}$ is a normal family. Note that $f_n'(z_0) = [f'(z_0)]^n$. But there is a subsequence f_{n_k} such that $f_{n_k} \to F$ normally. Then $f_{n_k}' \to F'$ normally. Thus, if $|f'(z_0)| > 1$, then $f_{n_k}'(z_0)$ cannot converge. Therefore $|f'(z_0)| \leq 1$.*

2. *Without loss of generality, assume $z_0 = 0$. If $|f'(0)| < 1$, then $\sum_{n=0}^{\infty} f'(0)^n$ converges, thus $f_n'(0) \to 0$. So, $F'(0) = \lim_{k\to\infty} f_{n_k}'(0) = 0$.*

By hard calculation,

$$f_n''(0) = f''(0) \sum_{k=n-1}^{2n-2} f'(0)^k$$

which goes to 0 *as* $n \to \infty$ *since* $\sum_{n\geq 0} f'(0)^n$ *converges. Thus,* $F''(0) = \lim_{k\to\infty} f_{n_k}''(0) = 0$.

Similar calculation yields the remaining result, i.e. $F^{(n)}(0) = 0$. *Thus* $F(z) = 0$ *for all* $z \in \Omega$.

Problem (8)**.**

Let $\Omega = \{|z| > 1\}$ *and* $D' = \{0 < |z| < 1\}$. *Then* Ω *is conformally equivalent to* D' *by the inversion. Let* $f : D' \to D'$ *be a conformal self mapping. Since* f *is bounded near* 0, $f(0)$ *can be defined by the Riemann removable singularity theorem. So now we can regard* f *as a map from* $D = \{|z| < 1\}$ *to* $\overline{D}$. *If* $|f(0)| = 1$, *then image of* D *under* f *is not open, so it contradicts to the open mapping theorem.*

Else if $0 < |f(0)| < 1$, *then we can choose* $w \in D'$ *such that* $f(0) = f(w)$. *Now, choose* U_0, U_w *which are disjoint open sets containing* $0, w$ *respectively. Then* $f(U_0), f(U_w)$ *are open sets by the open mapping theorem. Since they are intersects* $(f(0), f(w))$, *there is* $r > 0$ *such that* $D(f(0), r) \subset f(U_0) \cap f(U_w)$. *Choosing* $\beta \in D(f(0), r)$ *which is not equal to* $f(0)$ *leads contradiction to the fact that* f *is injective function of* D'. *Thus* $f(0) = 0$.

But, conformal self mapping of the unit disk which fixes origin must be a rotation. Thus $f(z) = \alpha z$ *for some* $|\alpha| = 1$. *So,* $z \mapsto 1/z \mapsto \alpha/z \mapsto z/\alpha$ *is a conformal self mapping of* Ω. *That is, it must be a rotation.*

$\square$

Problem (9)**.**

Let $h = (z-i)/(z+i)$, $f(z) = \log|z|$. *Then* $f\circ h, u\circ h$ *are harmonic function of the upper half plane except* i. *Since* $f, u \to 0$ *as* $|z| \to 1$, $f \circ h, u \circ h \to 0$ *as* $|z| \to 0$. *Thus, by the Schwarz reflection principle, we can regard* $f \circ h, u \circ h$ *be a harmonic function of* $\mathbb{C} \setminus \{-i, i\}$.

Let $f \circ h = F$ *and* $u \circ h = U$. *Since* F, U *are equal to* 0 *on the real axis,*

$$\frac{\partial F}{\partial x} = \frac{\partial U}{\partial x} = 0$$

on the real axis. Thus, by Laplace equation,

$$\frac{\partial^2 F}{\partial y^2} = \frac{\partial^2 U}{\partial y^2} = 0$$

on the real axis. Note that F, U *are nonconstant. So*

$$\frac{\partial F}{\partial y} = C, \frac{\partial U}{\partial y} = D$$

on the real axis($C, D \neq 0$).

By Laplace equation, $F_x - iF_y$ *and* $U_x - iU_y$ *are holomorphic on* $\mathbb{C} \setminus \{-i, i\}$. *Therefore, on the real axis,* $D(F_x - iF_y) = C(U_x - iU_y)$. *And, by the identity theorem, they are equal on all of* $\mathbb{C} \setminus \{-i, i\}$. *Thus*

$$\frac{D}{C} F_x = U_x$$

on all of doubly punctured plane, and by integrating them, we get

$$DF(z) = CU(z)$$

since $F(0) = U(0) = 0$.

Now, by considering the upper half plane and $F \circ h^{-1}, U \circ h^{-1}$, u *must be positive constant multiple of* $f(z) = \log |z|$.

$\square$

Problem (10)**.**

Counterexample: the identity $\iota(z) = z$.

Let $f(z) = \int_\gamma \exp(-\zeta^2) d\zeta$ *where* $\gamma(t) = zt$ *for* $0 \leq t \leq 1$. *Then* $f'(z) = \exp(-z^2) \neq 0$.

Note that $f(0) = 0$. *When* $z \neq 0$, $f(z) = z \int_0^1 \exp(-z^2 t^2) dt$. *Thus* $f(z)^2 = z^2 \int_0^1 \int_0^1 \exp(-z^2(t^2 + s^2)) dt ds$.

By polar coordinate integration, $f(z)^2 = z^2 \int_0^{\pi/2} \int_0^{r(\theta)} r \exp(-z^2 r^2) dr d\theta$. *And this is not equal to zero when* $z \neq 0$. *Thus* f *has simple zero at* 0.

$\square$