2021s mas651 lecture note

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March 14, 2021

Contents

Chapter 1

Markov Chain

1.1 Construction, Markov Properties

Let $(S, \mathcal{S})$ be a measurable space. This will be a state space of our Markov Chain.

Definition 1.1.1 (Transition Probability)**.** A function $p : S \times \mathcal{S} \to \mathbb{R}$ is a transition probability if it satisfies two condition:

1. For each $B \in \mathcal{S}$, $p(\cdot, B)$ is a measurable mapping.
2. For each $x \in S$, $p(x, \cdot)$ is a probability measure.

Remark. By the second condition, $p(\cdot, B)$ is a bounded measurable mapping.

If any transition probability is given, then we can define a Markov chain with a natural filtration $\mathcal{F}_n$.

Definition 1.1.2 (Markov Chain)**.** $(X_n, \mathcal{F}_n)$ is a markov chain with transition probability p if

$$P\left(X_{n+1} \in B | \mathcal{F}_n\right) = p\left(X_n, B\right).$$

After defining of Markov chain, we must have a curiosity about the existence. The existence of X_n can be shown by the Kolmogorov extension theorem. First, assume an initial distribution μ on S. Second, define

$$\nu_{0,\ldots,n}\left(B_0, \ldots B_n\right) = \int_{B_0} \mu(dx_0) \cdots \int_{B_n} p(x_{n-1}, dx_n)$$

Then $\{\nu_{0,\ldots n}\}_{n=0}^{\infty}$ is a collection of finite dimensinal distributions which is consistent. Thus, by the Kolmogorov extension theorem, there is a measure P_μ on $(\Omega_0, \mathcal{F}_\infty) = (S^\omega, \mathcal{S}^\omega)$ which satisfies

$$P_\mu\left(X_0 \in B_0, \ldots X_n \in B_n\right) = \int_{B_0} \mu(dx_0) \cdots \int_{B_n} p(x_{n-1}, dx_n)$$

where X_n is a coordinate map of $\omega \in \Omega_0$.

Remark. $\omega = (\omega_0, \omega_1, \dots)$ so we can interpret ω_n as a n-th state of our markov chain. $\theta(\omega) = (\omega_n, \omega_{n+1}, \dots)$ is a shift operator, which makes the original n-th state be an initial.

We have constructed P_μ and corresponding X_n. But we don't know whether X_n is a markov chain or not. To show that X_n is actually a markov chain, we should use 1.1.2.

Theorem 1.1.3. X_n above satisfies

$$P_\mu\left(X_{n+1} \in B | \mathcal{F}_n\right) = p\left(X_n, B\right)$$

Proof. To prove this, we must show

$$\int_A 1_{(X_{n+1} \in B)} dP_\mu = \int_A p\left(X_n, B\right) dP_\mu$$

for all $A \in \mathcal{F}_n$. Thanks to $\pi - \lambda$ theorem, we can restrict ourselves to $A = \{X_0 \in B_0, \dots X_n \in B_n\}$. To simplicity, put $B = B_{n+1}$. Then the target equation is equal to

$$P_\mu\left(X_0 \in B_0, \dots, X_{n+1} \in B_{n+1}\right) = \int_{B_0} \mu(dx_0) \cdots \int_{B_n} p(x_{n-1}, dx_n) p(x_n, B_{n+1}).$$

And we want that the last expression is equal to

$$\int_A p(X_n, B) dP_\mu.$$

To do this, first replace $p(x_n, B_{n+1})$ to $1_C(x_n)$. We can easily check that the equality is true for any indicator function. Thus the equality is true for any simple function, and true for any bounded measurable function by BCT. Note that $p(\cdot, B_{n+1})$ is a bounded measurable mapping.

□

To go further, we need a very useful theorem which is called the Monotone Class Theorem.

Theorem 1.1.4 (Monotone Class Theorem)**.** Let $\mathcal{A}$ be a π-system that contains Ω and let $\mathcal{H}$ be a collection of real valued functions which satisfies:

1. If $A \in \mathcal{A}$, then the corresponding indicator function belongs to $\mathcal{H}$.
2. If $f, g \in \mathcal{H}$, then for any real linear combination of them belongs to $\mathcal{H}$.
3. If $0 \le f_n \uparrow f \le M$ and $f_n \in \mathcal{H}$, then $f \in \mathcal{H}$.

Then $\mathcal{H}$ contains all $\sigma(\mathcal{A})$ measurable functions which are bounded.

Proof. $\mathcal{G} = \{A : 1_A \in \mathcal{H}\}$ is a λ-system containing $\mathcal{A}$. Thus $\sigma(\mathcal{A}) \subset \mathcal{G}$ which means $1_A \in \mathcal{H}$ for all $A \in \sigma(\mathcal{A})$. Then $\mathcal{H}$ contains all simple functions which are $\sigma(\mathcal{A})$ measurable, hence the conclusion follows from the third condition above. □

Remark. A class of bounded measurable function $f : S \to \mathbb{R}$ which satisfies

$$E\left(f(X_{n+1})|\mathcal{F}_n\right) = \int p(X_n, dy)f(y)$$

is actually a monotone class. Because 1.1.2 says the first condition for $\mathcal{A} = \mathcal{S}$, and the others are satisfied by the elementary properties of expectation and integration.

By induction, we can extend this result by

$$E\left[\prod_{m=0}^{n} f_m(X_m)\right] = \int \mu(dx_0)f_0(x_0)\cdots\int p_{n-1}(x_{n-1}dx_n)f_n(x_n)$$

where the subscript for p stands for temporally inhomogeneous transition probability.