FRANK JONES INTEGRATION THEORY SOLUTIONS

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CHAPTER 6

section A.

Problem 1. "the vanishing property"

First, assume that $\int f d\mu = 0$. Since f is measurable, there exists an increasing seqeunce of nonnegative simple functions which converges to f. Let denote them as s_n. Then $f^{-1}((0,\infty]) = \bigcup_{n=1}^{\infty} s_n^{-1}((0,\infty])$, union of measure zero set. Therefore, we get $\mu(f^{-1}((0,\infty])) = 0$.

Conversely, assume that $\int f d\mu > 0$. Then there exists nonnegative simple function $s \leq f$ such that $\int s d\mu > 0$. Also we can write s as a linear combination of (measurable) characteristic functions, i.e. $\int s d\mu = \sum_{i=1}^{N} \alpha_i \mu(A_i) > 0$. So, $\alpha_i \mu(A_i) > 0$ for at least one integer $1 \leq i \leq N$. By Observing the fact that $A_i \subset f^{-1}((0,\infty])$, we can conclude that $f^{-1}((0,\infty])$ has measure zero implies $\int f d\mu = 0$ by contrapositive.

Problem 2. "the finiteness property"

$E = \{x : f(x) = \infty\}$ is a measurable set since f is measurable. Suppose $\mu(E) > 0$. For any $M \in \mathbb{N}$, choose nonnegative simple measurable function $s_M \leq f$ such that $s_M(x) \geq M$ for $x \in E$. Then $\int f d\mu \geq \int s_M d\mu \geq M\mu(E)$. Therefore, $\int f d\mu \geq M\mu(E)$ for all positive integer M. It means that $\int f d\mu = \infty$. By taking contrapositive, we get what we want.

Problem 3. "compatibility to not finite nonnegative simple functions"

We are only interested in the case when $\alpha_k = \infty, \mu(A_k) > 0$. Since f is measurable, we can consider seqeunce of nonnegative finite simple function $\{t_n\}$ such that $t_n(x) \geq n$ for $x \in A_k$. Therefore, $\int f d\mu \geq \int t_n d\mu \geq \mu(A_k)n$ for all positive integer n, which means $\int f d\mu = \infty$. For other cases, it is easy to check the compatibility.

Problem 4. "scalar multiplication is still valid for $c = \infty$"

First, assume $\int f d\mu = 0$. Then $\mu(f^{-1}((0,\infty])) = 0$ by Problem 1. It is obvious that $g^{-1}((0,\infty]) = f^{-1}((0,\infty])$ for $g = \infty f$. So, $\int g d\mu = 0$. Therefore we get $\int \infty f d\mu = \int f d\mu$.

Second, assume $\int f d\mu > 0$. Then measure of $g^{-1}((0,\infty])$ is positive. So, $\int \infty f d\mu = 0\mu(g^{-1}(0)) + \infty\mu(g^{-1}((0,\infty]))$ which is ∞.

Problem 5. "strict inequality for Fatou's lemma'

Consider this nonnegative finite simple function defined on real line:

$$s_n(x) = \begin{cases} n^2 & \text{if } x \in (0, \frac{1}{n}) \\ 0 & \text{otherwise} \end{cases}$$

$\lim s_n = 0$ and $\lim \int s_n d\mu = \infty$.

Problem 6.

Let $f_k = 1_{A_k}$ where 1_{A_k} is an indicator(characteristic) function. By Fatou's lemma (ILLLI), $\int \liminf_{k\to\infty} 1_{A_k} d\mu \le \liminf_{k\to\infty} \int 1_{A_k} d\mu = \liminf_{k\to\infty} \mu(A_k)$. Now, showing $1_{\liminf A_k} \le \liminf_{k\to\infty} 1_{A_k}$ is left to us. If $x \in \liminf A_k$ then $x \in \bigcap_{i\ge n} A_i$ for some positive integer n. Then $1_{A_i}(x) = 1$ for all positive integer i greater than n. So, $1_{\liminf A_k} \le \inf_{i\ge n} 1_{A_i}$. By letting $n \to \infty$, we get $1_{\liminf A_k} \le \liminf_{k\to\infty} 1_{A_k}$. Therefore, $\mu(\liminf A_k) = \int 1_{\liminf A_k} d\mu \le \int \liminf_{k\to\infty} 1_{A_k} d\mu \le \liminf_{k\to\infty} \int 1_{A_k} d\mu = \liminf_{k\to\infty} \mu(A_k)$.

section B.

Problem 9. "iff condition for (finite) integrability"

Let $f \in \mathcal{L}^1(\mu)$. Then $\int f_\pm d\mu < \infty$, so there sum is also finite ($= \int |f| d\mu$).

Conversely, assume $|f| \in \mathcal{L}^1(\mu)$. Then $\int f_\pm d\mu \le \int |f| d\mu < \infty$. Therefore $f \in \mathcal{L}^1(\mu)$.

Problem 10. "dominated integrability"

If f is measurable, $f_\pm$ is also measurable. So $f_+ + f_- = |f|$ is measurable. $\int |f| d\mu \le \int |g| d\mu < \infty$. Therefore, $|f| \in \mathcal{L}^1(\mu)$ by Problem 9.

Problem 11.

It is obvious that $|f_k| \le |f|$ and $f \in \mathcal{L}^1(\mu)$. So, if each f_ks are measurable, by dominated convergence thm, done. Actually, $f_k = f \cdot 1_{A_k \cap E_k}$ where $A_k = [-k, k]$ and $E_k = f^{-1}([-k, k])$. There exists sequence of nonnegative finite simple function $\{s_i\}$ which converges to f since f is measurable. So, $\lim s_i 1_{a_k \cap E_k} = f_k$ is measurable.

Problem 12.

Likewise, it is enough to show that $f(x)e^{-\frac{|x|^2}{k}} = f_k(x)$ is measurable. $e^{-\frac{|x|^2}{k}}$ is continuous, hence Borel measurable, hence Lebesgue measurable. There exists sequence of nonnegative finite simple function $\{s_k\}$ which converges to f since f is measurable. So, $\lim_{i\to\infty} s_i e^{-\frac{|x|^2}{k}} = f_k$ is measurable.

Problem 13. 'alternative proof for problem 2.42'

For each $x \in X$, there are at most $d \in \mathbb{N}$ distinct A_k containing x. Fix positive integer N. Let $I_x = \{k \in \mathbb{N} : k \leq N \text{ and } x \in A_k\}$. Clearly $\sum_{k=1}^{N} 1_{A_k} \leq d1_A$ where $A = \bigcup_{i=1}^{\infty} A_i$. By integrating both sides, $\sum_{k=1}^{N} \mu(A_k) \leq d\mu(A)$. By Letting $N \to \infty$, we got the result in Problem 2.42.

(similar, another) $\sum_{k=1}^{N} 1_{A_k} \leq |I_x| \leq d = d1_A$ for all N. So $\sum_{k=1}^{\infty} 1_{A_k} \leq d1_A$. By integrating both sides and monotone convergence thm, we got it.

Problem 14.

Let $\mathcal{I}$ be set of all sequences which are strictly increasing positive integers and length m. Such set is countable. Now, $\bigcup_{i \in \mathcal{I}} \bigcap_{j=1}^{m} A_{i_j} = E_m$ and it is measurable.

Similar to Problem 13, $m1_{E_m} \leq \sum_{i=1}^{\infty} 1_{A_i}$. So, $m\mu(E_m) = \int m1_{E_m} d\mu \leq \int \sum_{i=1}^{\infty} 1_{A_i} d\mu = \sum_{i=1}^{\infty} \int 1_{A_i} d\mu = \sum_{i=1}^{\infty} \mu(A_i)$ by monotone convergence thm.

section C.

Problem 16.

Clearly, $|f| = 0$ almost everywhere, and $|f|$ is measurable. Consider nonnegative finite simple function $s \leq |f|$. Then $s = 0$ almost everywhere and $s = \sum_{i=1}^{N} \alpha_i 1_{A_i}$, where A_i is null set if $\alpha_i \neq 0$. Therefore, $\int s d\mu = 0$, which implies $\int |f| d\mu = 0$. So, $f \in \mathcal{L}^1(\mu)$, $\left|\int f d\mu\right| = 0 \Rightarrow \int f d\mu = 0$.

Problem 17.

Note that $f \sim g \Leftrightarrow f = g$ a.e. is an equivalence relation. So, $g = h$ a.e. Let g be measurable and $E_t = [-\infty, t]$, $N = \{x : g(x) \neq h(x)\}$. Then $h^{-1}(E_t) \setminus g^{-1}(E_t) \subset N$ and $g^{-1}(E_t) \setminus h^{-1}(E_t) \subset N$. Since μ is complete, they are all null sets. So $h^{-1}(E_t) \cup g^{-1}(E_t)$ is measurable. Therefore, $h^{-1}(E_t)$ is also measurable, which implies measurability of h.