REAL ANALYSIS HW6

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section 6

Problem 1. "the vanishing property"

First, assume that $\int f d\lambda = 0$. Since f is measurable, there exists an increasing seqeunce of nonnegative simple functions which converges to f. Let denote them as s_n. Then $f^{-1}((0,\infty]) = \bigcup_{n=1}^{\infty} s_n^{-1}((0,\infty])$, union of measure zero set. Therefore, we get $\lambda(f^{-1}((0,\infty])) = 0$.

Conversely, assume that $\int f d\lambda > 0$. Then there exists nonnegative simple function $s \leq f$ such that $\int s d\lambda > 0$. Also we can write s as a linear combination of (measurable) characteristic functions, i.e. $\int s d\lambda = \sum_{i=1}^{N} \alpha_i \lambda(A_i) > 0$. So, $\alpha_i \lambda(A_i) > 0$ for at least one integer $1 \leq i \leq N$. By Observing the fact that $A_i \subset f^{-1}((0,\infty])$, we can conclude that $f^{-1}((0,\infty])$ has measure zero implies $\int f d\lambda = 0$ by contrapositive.