SOLUTIONS FOR TOPOLOGY WRITTEN BY J.MUNKRES

JAEMIN OH

Date: July 9, 2020.

35. The Tietze Extension Theorem

Problem (1)**.**

Let A, B be disjoint closed subspace of X. Define f as following:

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$

Then $f : A \cup B \to [0,1]$ is continuous (by pasting lemma).

By Tietze extension theorem, we can extend f to $\overline{f} : X \to [0,1]$. This $\overline{f}$ is what we can get from Urysohn lemma.

Problem (5)**.**

(a) *Let (X, A, f) be given. Since $f : A \to \mathbb{R}^J$, $f_\alpha : A \to \mathbb{R}$. Apply the Tietze extension theorem to f_α. Then we get continuous function $\overline{f} : X \to \mathbb{R}^J$ which satisfies $\left(\overline{f}\right)_\alpha = \overline{f_\alpha}$. This $\overline{f}$ is what we want.*

(b) *Without loss of generality, we can assume that Y is a retract of $\mathbb{R}^J$. Let $f : A \to Y$ be a continuous function. By expanding codomain, we can get $f' : A \to \mathbb{R}^J$ which is continuous. Let $\overline{f}$ be the extension of f'. Then $r \circ \overline{f}$ is continuous extension of f where r is a retraction of $\mathbb{R}^J$ into Y.*

Problem (6)**.**

(a) *Let h be a homeomorphism of Y_o onto Y. Since Y has universal extension property, we can extend h to $\overline{h} : X \to Y$ which is continuous. Then $h^{-1} \circ \overline{h}$ is the retraction of X into Y_o.*

(b) *Note that (b) of problem 5 still holds if we replace $\mathbb{R}^J$ to $[0,1]^J$. Fix $y \in Y$ and choose neighborhood V_y of y. By Urysohn lemma, there is a continuous function f_y such that $f_y(y) = 1$ and vanishes outside of V_y. Let $F(x) = (f_y(x))_{y \in Y}$. Then F is imbedding of Y into $[0,1]^Y$ by theorem 34.2.*

Since F is imbedding of Y into $[0,1]^J$, $F(Y) \cong Y$. Now, it is sufficient to show that $F(Y)$ is retract of $[0,1]^J$. But it follows directly by (b) of problem 5 since $F(Y)$ is compact hence closed, $[0,1]^J$ is compact Hausdorff hence normal.