FRANK JONES INTEGRATION THEORY SOLUTIONS

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section F. Some Calculations.

Problem 31.

$\overline{\mu}(A) = 0$ implies existence of $B, C \in \mathcal{M}$ such that $B \subset A \subset C$ and $\mu(B) = \mu(C) = 0$. So $\emptyset \subset E \subset C$, $\emptyset, C \in \mathcal{M}$ and $\mu(C) = 0$. By definition of $\overline{\mathcal{M}}$, $E \in \overline{\mathcal{M}}$ and $\overline{\mu}(E) = \mu(C) = 0$.

Problem 32.

Let $B \in \overline{\mathcal{M}}$. Then there are $A, C \in \mathcal{M}$ such that $A \subset B \subset C$, $\mu(C \setminus A) = 0$. Let $N = B \setminus A \subset C \setminus A$ which is $\mu-$null set. Then $B = A \cup N$. So every elements in $\overline{\mathcal{M}}$ can be expressed as the form of $A \cup N$ such that $A \in \mathcal{M}$ and N is subset of μ-null set.

On the contrary, consider $A \cup N$, $A \in \mathcal{M}$ and N is subset of N' which is μ-null set. Then $A \subset A \cup N \subset A \cup N'$ with $\mu(A \cup N' \setminus A) = 0$, $A, A \cup N' \in \mathcal{M}$. So $A \cup N \in \overline{\mathcal{M}}$.

Problem 33.

Let $\mathcal{A}$ be an union of $\mathcal{M}$ and collection of subsets of μ-null sets. By problem 32, $\overline{\mathcal{M}} \subset \sigma(\mathcal{A})$. But $\mathcal{A} \subset \overline{\mathcal{M}}$ because $\overline{\mathcal{M}}$ is complete and containing $\mathcal{M}$.

Therefore $\sigma(\mathcal{A}) = \overline{\mathcal{M}}$.

Problem 36.

$\{f > \frac{1}{k}\}$ must be finite since $\sum_{x \in X} f(x) < \infty$. Therefore $\bigcup \{f > \frac{1}{k}\} = \{f > 0\}$ is countable.

Problem 37.

Let F be finite subset of $\mathbb{N}$. Then $\sum_{x \in F} f(x) \leq \sum_{k=1}^{F_{\max}} f(k)$. Therefore $\sum_{x \in F} f(x) \leq \lim_{n \to \infty} \sum k = 1^n f(k)$.

Conversely, $\sum_{k=1}^{n} f(k) \leq \sum_{x \in F} f(x) \leq \sum_{x \in \mathbb{N}} f(x)$ for each n.

Proposition says when counting measure and nonnegative measurable function is given, $\int_X f d\mu = \sum_{x \in X} f(x)$.

Problem 38.

First assume $f \in L^1$. Then $|f|$ is nonnegative function. So $\int_X |f| \, d\mu = \sum_{x \in X} |f(x)| < \infty$ by proposition above.

Conversely, $\sum_{x \in X} |f(x)| = \int_X |f| \, d\mu < \infty$. Therefore $\int_X f^{\pm} d\mu \leq \int_X |f| \, d\mu < \infty$. So $f \in L^1$.

section G. Miscellany.

Problem 43.

Define $\varphi(E) = \int_E f d\mu$. Then $\varphi(\emptyset) = 0$ and φ is countably additive. Let $P_n = \left\{x \in X : f(x) \geq \frac{1}{n}\right\}$. Then P_n is measurable hence $\varphi(P_n) = 0$ for all n. $\varphi(P_n) = \int_{P_n} f d\mu \geq \frac{1}{n}\int_{P_n} f d\mu = \frac{1}{n}\mu(P_n)$. Hence $\mu(P_n) = 0$ for all n. Therefore $\mu(\{x \in X : f(x) > 0\}) = \mu\left(\bigcup_{n=1}^{\infty} P_n\right) = 0$.
Similarly, we can deduce that $\mu(\{x \in X : f(x) < 0\}) = 0$. Therefore $f = 0$ a.e.

Problem 44.

Define $\varphi(E) = \int_E f d\lambda$ for $E \in \mathcal{L}$. Such φ has same property as in problem 43. Note that $\varphi(\{x\}) = 0$ for each $x \in \mathbb{R}$ since one-point set is null set.
Every open set $G \subset \mathbb{R}$ can be expressed as nonoverlapping union of special rectangles. So $\varphi(G) = \sum_{k=1}^{\infty} \varphi([a_k, b_k])$ where $G = \cup_{k=1}^{\infty}[a_k, b_k]$.
Also, $\mathbb{R}$ is open. Therefore $\varphi(F) = \varphi(\mathbb{R} - G)$ for all closed set $F \subset \mathbb{R}$. Then $\varphi(F_\sigma) = 0$. All Lebesgue measurable set E can be expressed as $F_\sigma \cup N$ where N is a null set.
Therefore $\varphi(E) = 0$ for all $E \in \mathcal{L}$. By previous problem, we get $f = 0$ λ-a.e.

Problem 45.

Define $\nu(A) = \lambda(A \cap [-1, 1])$. Then $\int_{[a,b]} 1_E - \frac{1}{2} d\nu = 0$ for all $-\infty < a < b < \infty$.
Note that $\int_{\mathbb{R}} \left|1_E - \frac{1}{2}\right| d\nu = \frac{1}{2}\lambda([-1, 1]) < \infty$ So $1_E - \frac{1}{2} \in L^1(\nu)$.
Every open set $G \subset \mathbb{R}$ can be expressed as countably many nonoverlapping special rectangle $[a_k, b_k]$. Therefore $\int_G \left(1_E - \frac{1}{2}\right) d\nu = 0$. Therefore $\int_F \left(1_E - \frac{1}{2}\right) d\nu = 0$ for all closed $F \subset \mathbb{R}$. And it implies $\int_{F_\sigma} \left(1_E - \frac{1}{2}\right) d\nu = 0$.
Every $A \in \mathcal{L}$ can be expressed as $F_\sigma \cup N$ where N is μ-null set. Therefore $\int_A \left(1_E - \frac{1}{2}\right) d\nu = 0$ for all $A \in \mathcal{L}$. By problem 43, $1_E = \frac{1}{2}$ ν-a.e.
But $x : 1_E \neq \frac{1}{2} = \mathbb{R}$ and $\nu(\mathbb{R}) = \mu([-1, 1]) > 0$ which is contradiction.
Therefore, there is no such E.