mas541 homework

20208209 오재민

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Problem (1).

$$
\begin{aligned}
1-\left|\frac{z-w}{1-z\overline{w}}\right|^2 &= 1-\frac{(z-w)(\overline{z}-\overline{w})}{(1-z\overline{w})(1-\overline{z}w)} \\
&= \frac{1-\overline{z}w-z\overline{w}+|z|^2|w|^2-|z|^2-|w|^2+z\overline{w}+\overline{z}w}{|1-\overline{z}w|^2} \\
&= \frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\overline{z}w|^2}
\end{aligned}
$$

Problem (2).

Let $f=u+iv$*.* $\partial f=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)$*. Then* $\overline{\partial f}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(u-iv)=\bar{\partial}\bar{f}$*.*

Problem (3).

If f *is constant, then* $|f|$ *is also constant. On the other hand, assume* $f=u+iv$ *and* $|f|^2=u^2+v^2$ *is positive real number. (if it is zero, then* f *must be zero)*

$$u^2+v^2=R>0$$

Differentiate both sides of the equation above with x *and* y *respectively, we can get* $uu_x+vv_x=0$*,* $uu_y+vv_y=0$*,* $u_x=v_y$ *and* $u_y=-v_x$*. By simple calculation we can get* $u_x=u_y=v_x=v_y=0$*. Therefore* u,v *are constant.*

Problem (4).

Note that $\int_0^{2\pi}e^{ik\theta}d\theta=\int_0^{2\pi}(\cos k\theta+i\sin k\theta)\,d\theta=0$ *for positive integer* k*. Therefore* $\frac{1}{2\pi}\int_0^{2\pi}(z_0+re^{i\theta})^jd\theta=\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{j}\binom{j}{k}z_0^k\left(re^{i\theta}\right)^{j-k}d\theta=z_0^j$*. Similarly, we can get* $\frac{1}{2\pi}\int_0^{2\pi}\overline{(z_0+re^{i\theta})^j}d\theta=\bar{z}_0{}^j$*.*

Since u *is polynomial, we can write it as* $\sum_{l,k}a_{l,k}z^l\bar{z}^k$*. By direct computation, we can get* $\frac{1}{2\pi}\int_0^{2\pi}u\left(z_0+re^{i\theta}\right)d\theta=\sum_{l,k}a_{l,k}z_0^l\bar{z}_0{}^k=u(z_0)$*.*

Problem (5).

Let $f=u+iv$*.* $(g\circ f)_x=g_uu_x+g_vv_x$*. Then*

$$
\begin{aligned}
(g\circ f)_{xx} &= (g_{uu}u_x+g_{uv}v_x)\,u_x+g_uu_{xx}+(g_{vu}u_x+g_{vv}v_x)\,v_x+g_vv_{xx} \\
(g\circ f)_{yy} &= (g_{uu}u_y+g_{uv}v_y)\,u_y+g_uu_{yy}+(g_{vu}u_y+g_{vv}v_y)\,v_y+g_vv_{yy}
\end{aligned}
$$

But we have Cauchy-Riemann equation and $g_{uu}+g_{vv}=0$ *and* $g_{vu}=g_{uv}$*. Using these equations, we can check that* $(g\circ f)_{xx}+(g\circ f)_{yy}=0$*. Hence*

$(g \circ f)$ *is a harmonic function.*