MAS550 EXERCISES

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Date: January 6, 2021.

Problem (1.1.2)**.**
Let $A = \Pi_{i=1}^{d}(a_i, b_i]$*. Then*

$$A = \left(\Pi_{i=1}^{d}[a_i - 1, b_i]\right) \cap \left(\Pi_{I=1}^{d}(a_i, b_i + 1)\right)$$

which is intersection of open set and closed set. So, $A \in \mathcal{R}^d$ *therefore* $\sigma(S_d) \subset \mathcal{R}^d$*.*

On the other hand, let $B = \Pi_{i=1}^{d}(a_i, b_i)$ *where* $-\infty < a_i < b_i < \infty$*. We can choose sequences* $\{a_{i,j}\}_{j=1}^{\infty}$ *and* $\{b_{i,j}\}_{j=1}^{\infty}$ *for each* $1 \le i \le d$ *such that* $a_{i,j} \downarrow a_i$ *and* $b_{i,j} \uparrow b_i$*. Then* $B_n = \Pi_{i=1}^{d}(a_{i,n}, b_{i,n}] \uparrow B$*. So* B *is a countable union of open rectangles, hence* $B \in \sigma(S_d)$*. Since such* B *forms basis of topology on* $\mathbb{R}^d$*, we can conclude that* $\mathcal{R}^d \subset \sigma(S_d)$*.*

Problem (1.2.3)**.**
Let F *be a distribution function. It is nonnegative, nondecreasing. So* $\lim_{y\downarrow x} F(y)$ *and* $\lim_{y\uparrow x} F(y)$ *always exist. Let* x *be a point where* F *is discontinuous. Since* F *is discontinuous at* x*, we can assume without loss of generality* $\lim_{y\downarrow x} F(y) > F(x)$*. Choose a rational number* $q_x \in (F(x), \lim_{y\downarrow x} F(y))$*. Then function* $x \mapsto q_x$ *is injective since* F *is nondecreasing. So there is injection from set of discontinuities to rational numbers. Now we can conclude that set of discontinuities is at most countable.*

Problem (1.3.4)**.**

(a) *Let* $f : \mathbb{R}^d \to \mathbb{R}$ *be a continuous function. Consider* $\mathcal{B} = \left\{U \subset \mathbb{R} : f^{-1}(U) \in \mathcal{R}^d\right\}$*. It is well known that* $\mathcal{B}$ *is a* σ*-field. By continuity of* f*,* $\mathcal{B}$ *contains every open set of* $\mathbb{R}$*, hence* $\mathcal{R} \subset \mathcal{B}$*. Therefore* f *is a measurable function.*

(b) *Let* $\mathcal{F}$ *be a* σ*-field that makes all the continuous functions measurable. Let* $\pi_i : \mathbb{R}^d \to \mathbb{R}$ *be the projection on i-th factor, which is continuous. Then* $\cap_{i=1}^{d}\pi_i^{-1}((a_i, b_i)) = \Pi_{i=1}^{d}(a_i, b_i) \in \mathcal{F}$*. Since* $\mathcal{F}$ *contains every open rectangles in* $\mathbb{R}^d$*, we can conclude that* $\mathcal{R}^d \subset \mathcal{F}$*. This means* $\mathcal{R}^d$ *is the smallest such* σ*-field. The fact that* $\mathcal{R}^d$ *makes all the continuous functions measurable is written in (a).*

Problem (1.3.1)**.**

Since $\sigma(X)$ *is the smallest* σ*-field which makes* X *measurable, it sufficient to show that* X *is measurable with respect to* $\sigma\left(X^{-1}(\mathcal{A})\right)$.

Let $X : \Omega \to S$. *It is clear that* $\{X \in A\} \in \sigma\left(X^{-1}(\mathcal{A})\right)$ *for all* $A \in \mathcal{A}$. *But by theorem 1.3.1, since* $\mathcal{A}$ *generates* $\mathcal{S}$, X *is measurable with respect to* $\sigma\left(X^{-1}(\mathcal{A})\right)$.

Therefore we can conlude that $\sigma\left(X^{-1}(\mathcal{A})\right) \subset \sigma(X)$, *and reverse inclusion is canonical since* $X^{-1}(\mathcal{A}) \subset \sigma(X)$.

Problem (1.4.1)**.**

Let $E_n = \left\{x : f(x) > \frac{1}{n}\right\}$. *Then* $\int f d\mu \geq \int_{E_n} f d\mu \geq \int_{E_n} \frac{1}{n} d\mu = \frac{1}{n}\mu(E_n)$. *Therefore* $\mu(E_n) = 0$ *for every positive integer* n. *So,* $\mu(\{f > 0\}) = \sum_{n=1}^{\infty} \mu(E_n) = 0$. *This says* $f = 0$ *a.e.*

Problem (1.4.2)**.** *Since* $E_{n+1,2m} \cup E_{n+1,2m+1} = E_{n,m}$ *and* $\frac{2m+1}{2^{n+1}} \geq \frac{m}{2^n}$, *we can easily see that* $\sum_{m\geq 1} \frac{m}{2^n}\mu(E_{n,m})$ *is monotonically increasing as n grows.*

For every positive integer M, $\sum_{m=1}^{M} \frac{m}{2^n}\mu(E_{n,m}) \leq \int f d\mu$. *So* $\sum_{m\geq 1} \frac{m}{2^n}\mu(E_{n,m}) \leq \int f d\mu$.

Let $s_n = \sum_{m=1}^{n2^n} \frac{m}{2^n} 1_{E_{n,m}}$. *Then* $\int s_n d\mu \leq \sum_{m\geq 1} \frac{m}{2^n}\mu(E_{n,m}) \leq \int f d\mu$. *But* $s_n \uparrow f$ *monotonically. By monotone convergence theorem,* $\lim_{n\to\infty} \int s_n d\mu = \int f d\mu$. *Hence by sandwich lemma, the desired result follows.*

Problem (1.5.1).

First, we will show that $|g| \leq \|g\|_\infty$ *a.e.*

It is true because

$$\begin{aligned}\mu\left(|g| > \|g\|_\infty\right) &= \mu\left(\bigcup_{n=1}^{\infty}\left\{|g| \geq \|g\|_\infty + \frac{1}{n}\right\}\right) \\ &\leq \sum_{n=1}^{\infty}\mu\left(\left\{|g| > \|g\|_\infty + \frac{1}{n}\right\}\right) \\ &= 0\end{aligned}$$

by definition of $\|g\|_\infty$.

Hence $|g| \leq \|g\|_\infty$ *a.e.*

Then, $\int |fg| d\mu \leq \|g\|_\infty \int |f| d\mu = \|g\|_\infty \|f\|_1$.

Problem (1.5.3).

(a) *Since* $p > 1$, $x \mapsto |x|^p$ *is convex function.* $|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p)$ *follows from convexity of* $|x|^p$.

$\int |f+g|^p d\mu \leq \int 2^p |f|^p d\mu + \int 2^p |g|^p d\mu$. *Therefore finiteness of* $\|f\|_p$ *and* $\|g\|_p$ *leads* $\|f+g\|_p < \infty$.

Now, consider $\int |f+g|^p d\mu = \int |f+g||f+g|^{p-1} d\mu \leq \int |f||f+g|^{p-1} d\mu + \int |g||f+g|^{p-1} d\mu$. *Let* q *be Holder conjugate of* p. *Then by applying Holder inequality, we get* $\|f+g\|_p^p \leq \|f+g\|_p^{p/q}(\|f\|_p + \|g\|_p)$. *Simple calculating leads Minkowski's inequality.*

(b) *First consider* $p = 1$. *By using triangle inequaltiy, the result follows directly. Next consider* $p = \infty$. $|f+g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty$ *a.e. Therefore* $\|f+g\|_\infty \leq \|f\|)\infty + \|g\|_\infty$.

Problem (1.6.8).

First assume $g = 1_A$. *Then* $\int g d\mu = \mu(A) = \int_A f(x) dx = \int 1_A f dm$ *where* m *is Lebesgue measure.*

Next, assume $g = \sum_i a_i 1_{A_i}$, *simple function. Then* $\int g d\mu = \sum_i a_i \mu(A_i) = \sum_i a_i \int 1_{A_i} f dm$.

Next, assume g *is nonnegative measurable. Let* $\{s_n\}_{n=1}^{\infty}$ *be increasing sequence of simple function converges to* g *pointwisely. Then* $\int g d\mu = \lim_{n\to\infty} \int s_n d\mu = \lim_{n\to\infty} \int s_n f dm$. *But* $s_n f \uparrow gf$ *since* f *is nonnegative. By monotone convergence theorem, we can get* $\int g d\mu = \int g f dm$.

Last, assume g *is integrable function. We can decompose* g *by* $g = g^+ - g^-$. *Applying 3rd step for* g^+, g^- *each, we can get* $\int g d\mu = \int g^+ f dm - \int g^- f dm = \int g f dm$ *since* f *is nonnegative.*

Problem (1.6.13).

Since $X_n \uparrow X$, $X_n^+ \uparrow X^+$ *and* $X_n^- \downarrow X^-$. *And note that* $X_n^- \leq X_1^-$ *which is integrable. Apply monotone convergence theorem to* X_n^+ *and apply dominated convergence theorem to* X_n^- *to get* $\lim EX_n = \lim EX_n^+ - \lim EX_n^- = EX^+ - EX^- = EX$.

Problem (1.7.1)**.**

We need to show that $\int_{X\times Y}|f|d(\mu_1\times\mu_2)<\infty$.

Since $|f|^{\pm}$ *is nonnegative, by Fubini's theorem,* $\int_X\int_Y |f|^{\pm}\mu_2(dy)\mu_1(dx)<\infty$. *Then, their sum is also finite, and the sum is* $\int_{X\times Y}|f|d(\mu_1\times\mu_2)$ *by Fubini's theorem. This leads the conclusion of the exercise.*

Corollary is immediate if we take $\mu_1=c$ *and* $\mu_2=\mu$.

Problem (1.7.3)**.**

(1)

$$\begin{aligned}\int_{(a,b]}\{F(y)-F(a)\}\,dG(y)&=\int_{(a,b]}\int_{(a,y]}1\mu(dx)\nu(dy)\\&=\int_{a<x\le y\le b}1d(\mu\times\nu)\\&=\mu\times\nu(1<X\le Y\le b)\end{aligned}$$

by Fubini's theorem on nonnegative function 1.

(2)

$$\begin{aligned}\int_{(a,b]}F(y)dG(y)&=\int_{(a,b]}\int_{-\infty}^{y}1\mu(dx)\nu(dy)\\&=\int_{(-\infty,a]}\int_{(a,b]}1\nu(dy)\mu(dx)+\int_{(a,b]}\int_{[x,b]}\nu(dy)\mu(dx)\\&=F(a)\{G(b)-G(a)\}+G(b)\{F(b)-F(a)\}\\&-\int_{(a,b]}G(x)\mu(dx)+\int_{(a,b]}G(x)-G(x^-)\mu(dx)\end{aligned}$$

We can get similar result for $\int_{(a,b]}G(y)dF(y)$. *By simple calculation, we get the conclusion of (2).*

(3) If $F=G$ *continuous, Then* $\mu(\{x\})=\nu(\{x\})=F(x)-F(x^-)=G(x)-G(x^-)=0)$. *Therefore, by using (2), we can get the conclusion.*

Problem (2.1.3)**.**

(1) If $h(\alpha)=0$ *for some* $\alpha>0$, *by mean value theorem,* $h'(\beta)=0$ *for some* $\beta\in(0,\alpha)$. *It contradicts to* $h'(x)>0$ *for positive* x. *Therefore* $h>0$ *for positive* x.

$x=y$ *iff* $\rho(x,y)=0$ *iff* $h(\rho(x,y))=0$. *And* $h(\rho(x,y))=h(\rho(y,x))$ *since* $\rho(x,y)=\rho(y,x)$.

Now consider $x\ge y>0$ *and* $\frac{h(x+y)-h(x)}{y}=h'(x+\theta)$ *and* $\frac{h(y)}{y}=h'(y-\delta)$. *Since* h' *is decreasing,* $h(x+y)-h(x)\le h(y)$. *Using this, we can prove triangle inequality of* $h\circ\rho$.

(2) $h(x)=1-\frac{1}{1+x}$ *so* $h'(x)=\frac{1}{(1+x)^2}$ *and* $h''(x)=\frac{-2}{(1+x)^3}$. *Given* h *satisfies all of (1).*

Problem (2.1.9)**.**

Let $\mathcal{A}_1=\{\{1,2\},\{1,3\}\},\mathcal{A}_2=\{\{1,4\}\}$. *For* $A_1\in\mathcal{A}_1$ *and* $A_2\in\mathcal{A}_2$, $P(A_1\cap A_2)=P(A_1)P(A_2)=1/4$. *But,* $\sigma(\mathcal{A}_1)=2^{\Omega}$*and* $\sigma(\mathcal{A}_2)=\{\Omega,\{1,4\},\{2,3\},\emptyset\}$. *They are not independent by considering* $A_1=\{2,3,4\}$ *and* $A_2=\{2,3\}$.

Problem (2.2.3)**.**

(a) *$f(U_i)$'s are iid because* $P\left(\bigcap_i (f \circ U_i) \in B_i\right) = P\left(\bigcap_i \left\{U_i \in f^{-1}(B_i)\right\}\right) = \Pi_i P\left(U_i \in f^{-1}(B_i)\right) = \Pi_i P\left(f(U_i) \in B_i\right)$. *Also, for borel set* B, $P\left(f(U_i) \in B\right) = P\left(U_i \in f^{-1}(B_i)\right)$ *are all same for* i.

$Ef(U_i) = \int_0^1 f(x)dx$, $E|f(U_i)| = \int_0^1 |f(x)|dx < \infty$.

Now, by WLLN, $\frac{\sum f(U_i)}{n}$ *converges to* $\int_0^1 f(x)dx$ *in probability.*

(b) $P\left(|I_n - I| > a/n^{0.5}\right) \leq \frac{n}{a^2} E|I_n - I|^2 = \frac{n}{a^2} Var(I_n) = Var(\sum f(U_i))/na^2 = Var(f(U_i))/a^2 = \left[\int_0^1 f(x)^2 dx - \left(\int_0^1 f(x)dx\right)^2\right]/a^2$.

Problem (2.2.5)**.**

Note that $P(X_i \leq a) = 0$ *for all* $a < e$.

$xP(X_i > x) = \frac{e}{\log x} \to 0$ *as* $x \to \infty$.

$E|X_i| = EX_i = \int_e^\infty P(X_i > x)dx = \int_e^\infty \frac{e}{x \log x} dx = \infty$ *since* $X_i \geq 0$ *almost surely.*

But $\mu_n = \int_{|X_i| \leq n} X_i dP \uparrow EX_i = \infty$ *by monotone convergence theorem.*

Now, theorem 2.2.12 says $\frac{s_n}{n} - \mu_n$ *converges to* 0 *in probability.*

Problem (2.3.5)**.**

(a) *Let $F_N = \{Y \le N\}$ and $Y_n = Y1_{F_n}$. Then $EY_n \uparrow EY$ by MCT. So choose N so that $EY - EY_N < \varepsilon$. Now consider $|EX_n - EX| \le E|X_n - X| \le \int_{|X_n - X| > \varepsilon} 2Y dP + \int_{|X_n - X| \le \varepsilon} |X_n - X| dP \le \varepsilon + \int_{|X_n - X| > \varepsilon} 2Y dP$.*

Let $E_n = \{|X_n - X| > \varepsilon\}$. Then $\int_{E_n} 2Y dP = \int_{E_n} 2Y - 2Y_N + 2Y_N dP \le E(2Y - 2Y_N) + 2NP(E_n)$, where the last term goes to 0 as $n \to \infty$.

(b) *Let h, g be continuous functions, $h(0) = 0$, $g > 0$ for large x, $|h|/g \to 0$ as $|x| \to \infty$, and $Eg(X_n) \le C < \infty$.*

Choose M so large that $g > 0$ on $|x| > M$. $\varepsilon_M = \sup_{|x| \ge M} |h|/g$ and $\bar{Y} = Y1_{|Y| \le M}$.

Then $|Eh(X_n) - Eh(X)| \le E|h(X_n) - Eh(\bar{X}_n)| + E|h(\bar{X}_n - h(\bar{X})| + E|h(\bar{X}) - h(X)|$. First term and third term are bounded by $\varepsilon_M C$ which goes to 0 as $M \to \infty$. And the second term goes to 0 as $n \to \infty$ by bounded convergence thm.

Therefore the conclusions hold.

Problem (2.3.6.)**.**

(a) *We already show that $\rho(x, y) = \frac{|x-y|}{1+|x-y|}$ is a metric in problem 2.1.3.*

First consider $d(X, Y) = 0$ iff $E\frac{|X-Y|}{1+|X-Y|} = 0$ iff $\frac{|X-Y|}{1+|X-Y|} = 0$ a.s. iff $X = Y$ a.s.

Next, it is trivial to check $d(X, Y) = d(Y, X)$.

Lastly, $d(X, Z) = E\rho(X, Z) \le E(\rho(X, Y) + \rho(Y, Z)) = E\rho(X, Y) + E\rho(Y, Z) = d(X, Y) + d(Y, Z)$.

Therefore given function is a metric of class of random variables.

(b) *First assume $X_n \to X$ in probability. Then $\frac{|X_n - X|}{1+|X_n - X|} \le 1$ and it goes to 0 in probability. So bounded convergence thm implies $d(X_n, X) \to 0$.*

Next assume $d(X_n, X) \to 0$ as $n \to 0$.

$$
\begin{aligned}
P(|X_n - X| > \varepsilon) &= P\left(\frac{|X_n - X|}{1 + |X_n - X|} > \frac{\varepsilon}{1 + \varepsilon}\right) \\
&\le E\frac{|X_n - X|}{1 + |X_n - X|}\frac{1 + \varepsilon}{\varepsilon} \\
&= d(X_n, X)\frac{1 + \varepsilon}{\varepsilon} \to 0
\end{aligned}
$$

by Markov's inequality.

Problem (2.3.8)**.**

Independence of A_n implies independence of A_n^c. Let $B_n = \cap_{k=1}^n A_k^c$. Then $0 = P\left(\bigcap_{n=1}^\infty A_n^c\right) = \lim_{n\to\infty} P(B_n)$.

So, for arbitrary $\varepsilon > 0$, there is a positive integer N_ε such that $n \ge N_\varepsilon$ implies $P(B_n) = P\left(\cap_{k=1}^n A_k^c\right) = \Pi_{k=1}^n (1 - P(A_k)) = e^{\sum_{k=1}^n \log(1-P(A_k))} < \varepsilon$. But as $n \to \infty$

$$
\lim_{n\to\infty} e^{\sum_{k=1}^n \log(1-P(A_k))} = 0
$$

This means that $\sum_{k=1}^\infty \log(1 - P(A_k)) = -\infty$, therefore $\log(1 - P(A_k))$ does not converge to 0, which is equivalent to that $P(A_k)$ does not converge to 0. Therefore $\sum_{n=1}^\infty P(A_n) = \infty$.

Problem (2.3.12)**.**

Let $\Omega = \{\omega_i : i \in \mathbb{N}\}$*. Without loss of generality, we can assume* $P(\{\omega_i\}) > 0$ *for all* $i \in \mathbb{N}$*.*

If there is ω_i *such that* $X_n(\omega_i)$ *does not converge to* $X(\omega_i)$*, then for some* $\varepsilon > 0$*, and for all* $N \in \mathbb{N}$*, there is* $n_N \geq N$ *but* $|X_{n_N}(\omega_i) - X(\omega_i)| > \varepsilon$*.*

This means $\{|X_{n_N} - X| > \varepsilon\}$ *contains* ω_i *for all* N*. So* $0 < P(\{\omega_i\}) \leq P\left(|X_{n_N} - X| > \varepsilon\right)$*.*

But $X_n \to X$ *in probability implies* $X_{n_N} \to X$ *in probability. This contradicts to above. Therefore there is no such* w_i *hence* X_n *converges to* X *almost surely.*

Problem (2.5.2)**.**

If $E|X_1|^p = \infty$*, then for each positive integer* k*,* $E|X_1|^p \le \sum_n P(|X_1|^p > nk) = \infty$*. But* $P(|X_1|^p > nk) = P(|X_n| > (nk)^{1/p})$*. Then by Borel Cantelli lemma* $P(|X_n| > (nk)^{1/p} i.o.) = 1$*. That is,* $\limsup_n |X_n|/n^{1/p} \ge k^{1/p}$ *for infinitely many* k*. Therefore* $\limsup_n |X_n|/n^{1/p} = \infty$*.*

But $|X_n| \le |S_n| + |S_{n-1}|$*. That leads* $\limsup_n |S_n|/n^{1/p} = \infty$*. By taking contrapositive, we get the conclusion.*

Problem (2.5.5)**.**

The first one leads the second one directly because Kolmogorov's three series lemma with $A = 1$ *tells it.*

The second one implies the third one because $\frac{X_n}{1+X_n} \le 1_{X_n > 1} + X_n 1_{X_n \le 1}$ *and monotone convergence theorem.*

The third one implies $\sum_n \frac{X_n}{1+X_n} < \infty$ *a.s. And convergence of* $\sum_n \frac{a_n}{1+a_n}$ *for* $a_n \ge 0$ *gives the convergence of* $\sum_n a_n$*. It is because* $\lim a_n = 0$ *and* $|a_N + \cdots a_{N+n}| \le (1+\varepsilon)\left|\frac{a_N}{1+a_N} + \cdots + \frac{a_{N+n}}{1+a_{N+n}}\right|$ *for large* N*. Therefore* $\sum_{k=1}^n a_k$ *is cauchy hence converges. Therefore* $\sum_n X_n$ *converges a.s.*

Problem (3.2.4)**.**

Since $X_n \to X_\infty$ *in distribution, there are* $Y_n =_d X_n$ *and* $Y_\infty =_d X_\infty$ *such that* $Y_n \to Y_\infty$ *a.s.*

Then $g(Y_n) \ge 0$ *and* $g(Y_n) \to g(Y_\infty)$ *a.s. Therefore by Fatou's lemma,* $\liminf Eg(Y_n) \ge Eg(Y_\infty)$ *which is equivalent to* $\liminf Eg(X_n) \ge Eg(X_\infty)$ *since* $X_n =_d Y_n$ *for all* $n \in \mathbb{N} \cup \infty$*.*

Problem (3.2.5)**.**

There are $Y_n \to Y_\infty$ *a.s. and distribution function of* Y_n *is equal to* F_n*.* $F_\infty = F$*.*

Then by theorem 1.6.8, $Eh(Y_n) \to Eh(Y_\infty)$ *which is equivalent to* $\int h(x)dF_n(x) \to \int h(x)dF(x)$ *because distribution function of* Y_n *is* F_n*.*

Problem (3.2.12)**.**

First assume that $X_n \to X$ in probability. Let f be any continuous, bounded function. Then, by bounded convergence theorem, $f(X_n) \to f(X)$ in L_1. Thus $X_n \to X$ in distribution.

Now assume that $X_n \to c$ in distribution. Let F be distribution function of c. Since c is constant, $F(c+\varepsilon) = 1$, $F(c) = 1$, and $F(c-\varepsilon) = 0$ for any $\varepsilon > 0$. Note that the discountinuity point of F is only c. Then

$$P(|X_n - c| \leq \varepsilon) \geq P(X_n \leq c + \varepsilon) - P(X_n \leq c - \varepsilon) \to 1$$

Therefore, $X_n \to c$ in probability.

□

Problem (3.2.13)**.** *[Slutsky's theorem]*

First note that the portmanteau theorem holds not only for bounded continuous function but also for bounded uniformly continuous function. Because, when we look the proof, the trapezoid function which we used is bounded, uniformly continuous(actually, Lipschitz).

Now without loss of generality, assume that $c = 0$. Let f be bounded, uniformly continuous function. Let $w_\delta(f) = \sup_{|x-y|\leq\delta} |f(x) - f(y)|$. Then $w_\delta(f) \to 0$ as $\delta \to 0$ because of uniform continuity. Now, consider the following:

$$\begin{aligned}
&|Ef(X_n + Y_n) - Ef(X)| \\
&\leq |Ef(X_n + Y_n) - Ef(X_n)| + |Ef(X_n) - Ef(X)| \\
&\leq w_\delta(f) + 2\sup|f| P(|Y_n| > \delta) + o(1)
\end{aligned}$$

By taking $\limsup_{n\to 0}$, the last term $\leq w_\delta(f)$. Therefore, by $\delta \to 0$, we get the result.

□

Problem (3.2.14)**.**

¡++¿

Problem (4.1.1)**.**

$$P(G|A) = P(A \cap G)/P(A) = \int_G 1_A dP / \int_\Omega 1_A dP = \int_g P(A|\mathcal{G})dP / \int_\Omega P(A|\mathcal{G})dP$$

□

Problem (4.1.2)**.**
Let $A \in \mathcal{F}$*. Then*

$$a^2 \int_A 1_{(|X| \geq a)} dP \leq \int_A X^2 1_{(|X| \geq a)} dP \leq \int_A X^2 dP$$

which leads

$$a^2 P(|X| \geq a|\mathcal{F}) \leq E(X^2|\mathcal{F}).$$

□

Problem (4.1.3)**.**
Let θ *be any real number. Then*

$$E\left((X + \theta Y)^2|\mathcal{G}\right) \geq 0.$$

By linearity of conditional expectation, we get

$$E(X^2|\mathcal{G}) + 2\theta E(XY|\mathcal{G}) + \theta^2 E(Y^2|\mathcal{G}) \geq 0.$$

Thus, the conclusion follows.

□

Problem (4.1.5)**.**
Let $P(a) = P(b) = P(c) = 1/3$*,* $X = 1_{(a)}$*,* $\mathcal{F}_1 = \sigma(\{a\})$*, and* $\mathcal{F}_2 = \sigma(\{b\})$*. Then*

$$E(X|\mathcal{F}_2) = \frac{1}{2} 1_{\{a,c\}}$$

but

$$E\left(E(X|\mathcal{F}_2)|\mathcal{F}_1\right) = \frac{1}{3} 1_{(a)} + \frac{1}{4} 1_{\{b,c\}}.$$

□

Problem (4.1.6)**.**
Note that

$$\int E(X|\mathcal{F})^2 dP = \int E(XE(X|\mathcal{F})|\mathcal{F})dP = \int XE(X|\mathcal{F})dP.$$

Use the above.

□

Problem (4.1.7)**.**
By definition of $\mathrm{Var}(X|\mathcal{F})$*, we get the following:*

$$E\left(\mathrm{Var}(X|\mathcal{F})\right) = EX^2 - E\left(E(X|\mathcal{F})^2\right)$$

And clearly,

$$\mathrm{Var}\left(E(X|\mathcal{F})\right) = E\left(E(X|\mathcal{F})^2\right) - \left(E(E(X|\mathcal{F}))\right)^2$$

Therefore, by summing them vertically, we can get

$$\mathrm{Var}\left(E(X|\mathcal{F})\right) + E\left(\mathrm{Var}(X|\mathcal{F}) = EX^2 - \left(E(E(X|\mathcal{F}))\right)^2\right.$$

which is equal to $\mathrm{Var}(X)$ *since the last term is equal to square of* EX*.* □

Problem (4.1.9)**.**

$$\begin{aligned}\int |X-Y|^2 dP &= \int X^2 - 2XY + Y^2 dP \\ &= \int X^2 - 2E(XY|\mathcal{G}) + Y^2 dP \\ &= \int X^2 - 2XE(Y|\mathcal{G}) + Y^2 dP \\ &= \int X^2 - 2X^2 + Y^2 dP \\ &= EY^2 - EX^2 \\ &= 0\end{aligned}$$

Therefore, $|X-Y|^2 = 0$ a.s. which implies $X = Y$ a.s. Note that XY is integrable by Holder's inequality for $p = q = 2$ and finite second moment of X, Y. □

Problem (4.1.10)**.**

Let $Z = E(X|\mathcal{G})$.

$$\begin{aligned}EX^+ - EX^- = EX = EZ &= EZ^+ - EZ^- \\ &= \int_{Z>0} ZdP + \int_{Z<0} ZdP \\ &= \int_{Z>0} XdP + \int_{Z<0} XdP\end{aligned}$$

also,

$$\begin{aligned}EX^+ + EX^- = E|X| = E|Z| &= EZ^+ + EZ^- \\ &= \int_{Z>0} ZdP - \int_{Z<0} ZdP \\ &= \int_{Z>0} XdP - \int_{Z<0} XdP\end{aligned}$$

thus

$$\int X^+ - 1_{Z>0} X dP = 0.$$

But, from $X^+ \geq 1_{Z>0}X$, we can catch $X^+ = 1_{Z>0}X$ almost surely. Similarly, $X^- = -1_{Z<0}X$ almost surely. Therefore each sign functions of X and $E(X|\mathcal{G})$ are equal almost surely.

Now consider

$$\{Y < E(Y|\mathcal{G})\} = \bigcup_{i\in\mathbb{N}} \{Y < q_i < E(Y|\mathcal{G})\} = \bigcup_{i\in\mathbb{N}} B_i$$

where q_i's are enumeration of the rationals.

Note that $B_i = \{Y - q_i < 0\} \cap \{E(Y|\mathcal{G}) - q_i > 0\}$. But, by using the fact that each sign functions of $Y - q_i$ and $E(Y|\mathcal{G}) - q_i$ are equal almost surely, we get

$$P(B_i) = 0$$

for each $i \in \mathbb{N}$. Thus $P(Y < E(Y|\mathcal{G})) = 0$.

By similar assertion, $Y = E(Y|\mathcal{G})$ almost surely.

□

Problem (4.2.3)**.**

Clearly $\mathcal{F}_m \subset \mathcal{F}_{m+1}$ for all positive integer m. Let $Z_n = X_n \vee Y_n$, then Z_n is clearly $\mathcal{F}_n$ measurable.

Now, let $A \in \mathcal{F}_{n-1}$. Then,

$$\begin{aligned}
\int_A E(Z_n|\mathcal{F}_{n-1})dP &= \int_A Z_n dP \\
&\geq \int_A X_n dP \vee \int_A Y_n dP \\
&= \int_A E(X_n|\mathcal{F}_{n-1})dP \vee \int_A E(Y_n|\mathcal{F}_{n-1})dP \\
&\geq \int_A X_{n-1} dP \vee \int_A Y_{n-1} dP
\end{aligned}$$

Therefore $\int_A E(Z_n|\mathcal{F}_{n-1})dP \geq \int_A X_{n-1}, Y_{n-1}dP$ for all $A \in \mathcal{F}_{n-1}$. Since $E(Z_n|\mathcal{F}_{n-1})$ is $\mathcal{F}_{n-1}$ measurable, we can conclude that conditional expectation of Z_n with respect to $\mathcal{F}_{n-1}$ is equal or greater than X_{n-1} and Y_{n-1} a.s.

So, Z_n is a submartingale. □

Problem (4.2.9)**.**

Note that $\{N > n\} = \{N \leq n\}^c \in \mathcal{F}_n$ and $\{N < n\} = \{N \leq n-1\} \in \mathcal{F}_{n-1}$ since N is integer valued. Now, consider the following:

$$\begin{aligned}
E(Z_n|\mathcal{F}_{n-1}) &= 1_{N \geq n} E(X_n^1|\mathcal{F}_{n-1}) + 1_{N<n} E(X_n^2|\mathcal{F}_{n-1}) \\
&\leq 1_{N \geq n} X_{n-1}^1 + 1_{N<n} X_{n-1}^2 \\
&= 1_{N > n-1} X_{n-1}^1 + 1_{N \leq n-1} X_{n-1}^2 \\
&\leq 1_{N \geq n-1} X_{n-1}^1 + 1_{N < n-1} X_{n-1}^2 \\
&= Z_{n-1}
\end{aligned}$$

So, Z_n is supermartingale.

Now, consider the Y_n:

First, $Y_n = X_n^1 1_{N>n} + X_N^2 1_{N=n} + X_n^2 1_{N<n} \leq X_n^1 1_{N \geq n} + X_n^2 1_{N<n}$.

$$\begin{aligned}
E(Y_n|\mathcal{F}_{n-1}) &\leq 1_{N \geq n} E\left(X_n^1|\mathcal{F}_{n-1}\right) +!_{N<n} E\left(X_n^2|\mathcal{F}_{n-1}\right) \\
&\leq 1_{N \geq n} X_{n-1}^1 + 1_{N<n} X_{n-1}^2 \\
&= 1_{N > n-1} X_{n-1}^1 + 1_{N \leq n-1} X_{n-1}^2 \\
&= Y_{n-1}
\end{aligned}$$

So, Y_n is also a supermartingale. □

Problem (4.2.8)**.**

Let $\nu = \inf\left\{k : \Pi_{m=1}^{k}(1+Y_m) > M\right\}$ *for* $M > 0$. *Let* $U_n = MX_n\Pi_{m=1}^{n-1}(1+Y_m)^{-1}$. *Clealry* ν *is a stopping time. Now we claim that* $U_{n\wedge\nu}$ *is positive supermartingale.*

$$\begin{aligned}E\left(U_{n+1\wedge\nu}|\mathcal{F}_n\right) &= E\left(U_\nu 1_{\{n+1>\nu\}} + U_{n+1}1_{\{n+1\le\nu\}}|\mathcal{F}_n\right)\\ &\le U_\nu 1_{\{\nu\le n\}} + 1_{\{n+1\le\nu\}}M\Pi_{m=1}^{n}(1+Y_m)^{-1}X_n(1+Y_n)\\ &= U_\nu 1_{\{\nu\le n\}} + 1_{\{n<\nu\}}M\Pi_{m=1}^{n-1}(1+Y_m)^{-1}X_n\\ &= U_{\nu\wedge n}\end{aligned}$$

Above manipulation is possible since $\{n+1\le\nu\} = \{\nu\le n\}^c$. *Thus* $U_{n\wedge\nu}$ *is a positive supermartingale, so it converges almost surely.*

Note that $\sum Y_n < \infty$ *implies* $\Pi(1+Y_n) < \infty$ *by considering* $1+x \le \exp(x)$ *and its partial product. Now fix* w *so that* $U_{\nu\wedge n}(w)$ *and* $\Pi(1+Y_n(w))$ *are convergent. Choose* $M > \Pi(1+Y_n)$. *Then* $\nu = \infty$, *so* $U_{\nu\wedge n} = U_n$. *But we know that* $U_{\nu\wedge n}(w)$ *converges, say to* K. *Then for that* w, $X_n(w) \to K(w)\Pi(1+Y_n(w))/M$. *Thus we can say that* X_n *converges almost surely.*

□

Problem (4.3.3)**.**

It is very similar to #4.2.8.

Let $\nu = \inf\left\{k : \sum_{m=1}^{k} Y_m > M\right\}$ *for* $M > 0$. *Clearly,* ν *is a stopping time. Let* $U_n = X_n - \sum_{m<n} Y_m + M$. *Then clearly,* $U_{n\wedge\nu}$ *is nonnegative random variables. Now we claim that* $U_{n\wedge\nu}$ *is a supermartingale.*

$$\begin{aligned}E\left(U_{n+1\wedge\nu}|\mathcal{F}_n\right) &= E\left(U_\nu 1_{\{\nu<n+1\}} + U_{n+1}1_{\{n+1\le\nu\}}|\mathcal{F}_n\right)\\ &\le U_\nu 1_{\{\nu<n+1\}} + 1_{\{n+1\le\nu\}}\left(X_n + Y_n - \sum_{m<n+1} Y_m + M\right)\\ &= U_\nu 1_{\{\nu\le n\}} + U_n 1_{\{n<\nu\}}\\ &= U_{\nu\wedge n}\end{aligned}$$

Above is possible since $\{\nu \ge n+1\} = \{\nu\le n\}^c \in \mathcal{F}_n$. *Thus* $U_{n\wedge\nu}$ *is a positive supermartingale, so it converges almost surely.*

Now, fix w *so that* $U_{n\wedge\nu}(w), \sum Y_n(w)$ *both are convergent. Choose* $M > \sum Y_n(w)$. *Then* $\nu = \infty$ *so* $U_{n\wedge\nu} = U_n$. *Then we can say that* $U_n(w) \to K(w)$, *so* $X_n(w) \to K(w) - M + \sum Y_n(w)$. *Thus* X_n *converges almost surely.*

□

Problem (4.3.4)**.**

Let $\{Y_n\}_{n=1}^{\infty}$ *be a sequence of independent random variables such that* $P(Y_n = 1) = p_n$. *Also let* $P(Y_n = 0) = 1 - p_n$. *Since* Y_n *are indepdendent, by Borel Canteli lemma (1st and 2nd both) implies that*

$$\sum_{n\ge1} p_n = \sum_{n\ge1} P(Y_n = 1) = \infty \Leftrightarrow P(Y_n = 1 i.o.) = 1$$

Note that $\cap_{n=N}^{N+k}\{Y_n = 0\} \downarrow \cap_{n=N}^{\infty}\{Y_n = 0\}$. *So* $\Pi_{n=N}^{N+k}(1-p_n) \to \Pi_{n=N}^{\infty}(1-p_n)$ *as* $k\to\infty$.

Since $P(Y_n = 1 i.o.) = P(\cap_{N=1}^{\infty}\cup_{n\ge N}\{Y_n = 1\}) = 1$, *we can get the following:*

$$
\begin{aligned}
P(\bigcap_{N\geq 1}\bigcup_{n\geq N}\{Y_n=0\}) = 0 &= \lim_{N\to\infty} P\left(\bigcap_{n\geq N}\{Y_n=0\}\right) \\
&= \lim_{N\to\infty}\lim_{k\to\infty} P\left(\bigcap_{n=N}^{N+k}\{Y_n=0\}\right) \\
&= \lim_{N\to\infty}\lim_{k\to\infty} \Pi_{n=N}^{N+k}(1-p_n) \\
&= \lim_{N\to\infty} \Pi_{n\geq N}(1-p_n)
\end{aligned}
$$

But, $\Pi_{n\geq N}(1-p_n) \leq \Pi_{n\geq M}(1-p_n)$ *where* $M \geq N$ *since* $1-p_n \leq 1$. *Therefore we can see that* $\Pi_{n\geq N}(1-p_n) \leq \lim_{N\to\infty}\Pi_{n\geq N}(1-p_n) = 0$ *by above. So,* $\Pi_{n\geq N}(1-p_n) = 0$ *for all positive integer* N.

For the other direction, suppose $\Pi_{n\geq 1}(1-p_n) = 0$. *Then its partial product must converge to zero. It means that* $\Pi_{n\geq N}(1-p_n) = 0$ *for every* N. *Then* $\lim_N P(\cap_{n\geq N}\{Y_n = 0\}) = 0$. *So* $P(Y_n = 1 i.o.) = 1$ *which implies the result.* □

Problem (4.4.1).

Note that $(N = j) \in \mathcal{F}_j$ and $E(X_k|\mathcal{F}_j) \geq X_j$ since X_i is a submartingale. Thus we get the following:

$$\int_{N=j} E(X_k|\mathcal{F}_j) = \int_{N=j} X_k \geq \int_{N=j} X_j$$

Now, by summing the above about j, we get the desired result, which is the second proof of $EX_N \leq EX_k$ for 4.4.1.

Problem (4.4.2).

Let $K_n = 1_{(M<n\leq N}$. Then K_n is predictable, hence $(K \cdot X)_n$ is a submartingale. By simple calculation, $(K \cdot X)_n = X_{n\wedge N} - X_{n\wedge M}$. Thus, $E(K \cdot X)_k \geq 0$ implies $EX_{k\wedge N} \geq EX_{k\wedge M}$ which is the result.

Problem (4.4.7).

$\lambda > 0$. For $c > 0$,

$$\begin{aligned} P(\max_{1\leq m\leq n} X_m \geq \lambda) &\leq P\left(\max_{1\leq m\leq n} (X_m + c)^2 \geq (\lambda + c)^2\right) \\ &\leq \frac{E(X_n + c)^2}{(\lambda + c)^2} \end{aligned}$$

since $(X_m + c)^2$ is a submartingale, and by the Doob's inequality.

Note that $E(X_n + c)^2 = EX_n^2 + c^2$ since $EX_n = EX_0 = 0$. The minimum of the last term(with respect to c) occurs when $c = EX_n^2/\lambda$ by differentiating it. And, its minimum value is $EX_n^2/(EX_n^2 + \lambda^2)$.

□

Problem (4.4.9).

Consider the following:

$$\begin{aligned} E\left(X_m - X_{m-1}\right)\left(Y_m - Y_{m-1}\right) &= EX_mY_m - EX_mY_{m-1} - EX_{m-1}Y_m + EX_{m-1}Y_{m-1} \\ &= EX_mY_m - E\left(E(X_mY_{m-1}|\mathcal{F}_{m-1})\right) \\ &\quad - E\left(E(X_{m-1}Y_m|\mathcal{F}_{m-1})\right) + EX_{m-1}Y_{m-1} \\ &= EX_mY_m - E\left(Y_{m-1}E(X_m|\mathcal{F}_{m-1})\right) \\ &\quad - E\left(X_{m-1}E(Y_m|\mathcal{F}_{m-1})\right) + EX_{m-1}Y_{m-1} \\ &= EX_mY_m - EX_{m-1}Y_{m-1} \end{aligned}$$

So the result follows directly. Note that X_mY_m is integrable due to the Cauchy-Schwartz inequality.

□

Problem (4.4.10).

By the problem 4.4.9, $EX_n^2 = EX_0^2 + \sum_{m=1}^{n} E\xi_m^2 \leq EX_0^2 + \sum_{m=1}^{\infty} E\xi_m^2 < \infty$. So $\sup_n E|X_n|^2 \leq EX_0^2 + \sum_{m=1}^{\infty} E\xi_m^2 < \infty$*. Therefore, the L_2 martingale convergence theorem implies the result.*

□

Problem (4.6.1).

Let $\mathcal{F}_n = \sigma\left(Y_1, \cdots Y_n\right)$. Then $E(\theta|Y_1, \cdots Y_n) = E(\theta|\mathcal{F}_n)$. By theorem 4.6.8, $E(\theta|\mathcal{F}_n) \to E(\theta|\mathcal{F}_\infty)$ a.s. and in L_1. Now, it remains to show that $E(\theta|\mathcal{F}_\infty) = \theta$.

Conditioning on θ, we can get the followings:

$$E(Y_i|\theta) = E(Z_i + \theta|\theta) = \theta + E(Z_i|\theta) = \theta$$
$$E(|Y_i - \theta|^2|\theta) = E(Z_i^2|\theta) = E(Z_i^2) = 0$$

by independence of Z_i and θ.

Define $X_n = \sum_{i=1}^{n} Y_i/n$. Clearly, X_n are $\mathcal{F}_\infty$ measurable. By the above observations, we can easily check that $E(|X_n - \theta|^2|\theta) = 1/n$. So, by integrating both sides, $E|X_n - \theta|^2 = 1/n$. Therefore $X_n \to \theta$ in L_2. By the fact that L_2 convergent sequence has a almost sure convergent subsequence, we can say that $X_{n_k} \to \theta$. But each X_{n_k} is $\mathcal{F}_\infty$ measurable, we can say that θ is $\mathcal{F}_\infty$ measurable.

Thus, $E(\theta|\mathcal{F}_\infty) = \theta$.

□

Problem (4.6.7)**.**

By triangle inequality,

$$|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)| \leq |E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_n)| + |E(Y|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)|$$

Write the above as $S_1 \leq S_2 + S_3$. Then clealry $ES_3 \to 0$ as $n \to \infty$ by theorem 4.6.8. To estimate ES_2,

$$\begin{aligned} ES_2 &\leq E\left(E\left(|Y_n - Y||\mathcal{F}_n\right)\right) \\ &= E|Y_n - Y| \end{aligned}$$

by Jensen's inequality. Therefore, L_1 convergence of Y_n implies $ES_2 \to 0$ as $n \to \infty$.

□

Problem (4.7.2)**.**
Let $w_N = \sup\{|Y_n - Y_m| : m, n \le N\}$. Then clearly $|Y_n - Y_{-\infty}| \le w_N$ for $n \le N$. Since $Y_n \to Y_{-\infty}$ a.s. as $n \to -\infty$, $w_N \to 0$ as $N \to -\infty$ almost surely. Note that $w_N \le 2Z \in L_1$. Thus $w_N \in L_1$. So, $\mathbb{E}(w_N|\mathcal{F}_{-\infty}) \to 0$ as $N \to -\infty$.
Let

$$\begin{aligned} &|E(Y_n|\mathcal{F}_n) - E(Y_{-\infty}|\mathcal{F}_{-\infty})| \\ &\le |E(Y_n|\mathcal{F}_n) - E(Y_{-\infty}|\mathcal{F}_n)| + |E(Y_{-\infty}|\mathcal{F}_n) - E(Y_{-\infty}|\mathcal{F}_{-\infty})| \\ &= S_1 + S_2 \end{aligned}$$

Now, consider the following:

$$\begin{aligned} \limsup_{n\to-\infty} |E(Y_n|\mathcal{F}_n) - E(Y_{-\infty}|\mathcal{F}_n)| &\le \limsup E(|Y_n - Y_{-\infty}|\,|\mathcal{F}_n) \\ &\le \lim_{n\to-\infty} E(w_N|\mathcal{F}_n) \\ &= E(w_N|\mathcal{F}_{-\infty}) \end{aligned}$$

So, by $N \to -\infty$, $\limsup_{n\to-\infty} S_1 = 0$. $\limsup_{n\to-\infty} S_2 = 0$ because $E(Y_{-\infty}|\mathcal{F}_n)$ is a backward martingale so it converges to $E(Y_{-\infty}|\mathcal{F}_{-\infty})$ a.s. and in L_1.

□

Problem (4.8.1)**.**
Let $K_n = 1_{L<n\le M}$. Then $(K\cdot X)_n = X_{n\wedge M} - X_{n\wedge L}$ is a submartingale. Thus $EX_{n\wedge L} \le EX_{n\wedge M}$. Similarly, $EX^+_{n\wedge L} \le EX^+_{n\wedge M}$. Since $X_{n\wedge M}$ is uniformly integrable submartingale, $EX_{n\wedge M} \to EX_M$ as $n \to \infty$. If we can show that $X_{n\wedge L}$ is also a uniformly integrable submartingale, then $EX_L \le EX_M$.
Note that $\sup_n EY^+_{n\wedge L} \le \sup_n EY^+_{n\wedge M} \le \sup_n E|Y_{n\wedge M}| < \infty$ by uniform integrability of $Y_{n\wedge M}$. Thus, by martingale convergence theorem, we get $Y_{n\wedge L} \to Y_L$ almost surely and $Y_L \in L_1$.
Now it remains to show that $Y_n 1_{n<L}$ is uniformly integrable.

$$E(|Y_{n\wedge L}|; |Y_{n\wedge L}| > K, n < L) = E\left(|Y_{n\wedge M}|; |Y_{n\wedge M}| > K, n < L \le M\right)$$

And $\sup$ of the last term goes to 0 as $K \to \infty$ by uniform integrability of $Y_{n\wedge M}$. Therefore, by theorem 4.8.2, $Y_{n\wedge L}$ is uniformly integrable, hence $EY_L \le EY_M$.

¡++¿

Problem (4.8.4)**.**
Let $M_n = S_n^2 - n\sigma^2$. Then M_n is a quadratic martingale. Since $n\wedge T$ is bounded stopping time, we have $EM_{n\wedge T} = EM_0 = 0$. Thus $ES^2_{n\wedge T} = \sigma^2 E(n\wedge T)$. As $n \to \infty$, $E(n\wedge T) \to ET$ by MCT.
Now consider $E|S_T - S_{n\wedge T}|^2 = E\left(\sum_{m=n+1}^{\infty} 1_{(m\le T)}\xi_m\right)^2$. Note that

$$E(1_{m\le T})(1_{m+k\le T})\xi_m\xi_{m+k} = (E\xi_{m+k})E1_{m\le T}1_{m+k\le T}\xi_m = 0$$

Thus $E|S_T - S_{n\wedge T}|^2 = \sum_{m=n+1}^{\infty} E1_{(m\le T)}\xi_m^2 = \sigma^2\sum_{m=n+1}^{\infty} P(m \le T)$. But, $\sum_{m=1}^{\infty} P(m\le T) = ET < \infty$. So we can say $E|S_T - S_{n\wedge T}|^2 \to 0$ as $n \to 0$.
Therefore $S_{n\wedge T} \to S_T$ in L_2, so $S^2_{n\wedge T} \to S^2_T$ in L_1, which leads the conclusion.

□

Problem (4.8.7)**.**

Note that $ET = a^2$ by theorem 4.8.7. Claim : $(b, c) = (3, 2)$.

$$
\begin{aligned}
&E(Y_{n+1}|\mathcal{F}_n)\\
&= 1 + 6S_n^2 + S_n^4 - 6(n+1)(1+S_n^2) + 3(n+1)^2 + 2(n+1)\\
&= S_n^4 - 6nS_n^2 + 3n^2 + 2n
\end{aligned}
$$

Since $n \wedge T$ is bounded stopping time, we can get $EY_0 = EY_{n\wedge T}$. Thus $3E(n\wedge T)^2 = 6E\left[(n \wedge T)S_{n\wedge T}^2\right] - ES_{n\wedge T}^4 - 2E_{n\wedge T}$.

But, by MCT, $E(n \wedge T)^2 \to ET^2$ and $E(n \wedge T) \to ET$. And $S_{n\wedge T}$ is bounded, so BCT implies $ES_{n\wedge T}^m \to a^m$. Thus, $3ET^2 = 6a^4 - a^4 - 2ET = 5a^4 - 2a^2$.

Note that $(n \wedge T)S_{n\wedge T}^2 \le a^2T \in L_1$. Thus $E(n \wedge T)S_{n\wedge T}^2 \to a^4$ by DCT.

□

Problem (4.8.9)**.**

Since $n \wedge T$ is a bounded stopping time, $EX_0 = EX_{n\wedge T} = EX_n = 1$ which is sames as follows:

$$1 = EX_n 1_{(n<T)} + EX_T 1_{(n\ge T)} = EX_n 1_{(n<T)} + \exp(\theta_0 a)P(n \ge T)$$

But, by MCT, the last term of the above $\to \exp(\theta_0 a)P(T < \infty)$.

Since $E\xi_i > 0$, $ES_n \to \infty$ as $n \to \infty$. Then, since $x \mapsto \exp(x)$ is a convex function,

$$E\exp(\theta_0 S_n) \le \exp(\theta_0 ES_n).$$

Thus $EX_n 1_{(n<T)} \le EX_n \le \exp(\theta_0 ES_n) \to 0$ as $n \to \infty$ due to $\theta_0 < 0$.

Therefore $1 = \lim_n EX_n 1_{(n<T)} + \exp(\theta_0 a)P(T < \infty) = \exp(\theta_0 a)P(T < \infty)$, which says the conclusion.

□