mas541 homework

20208209 오재민

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Problem (1).

Let $f = u + iv$. *Then* $\bar{f}f' = ff' - 2ivf'$, *where* ff' *is holomorphic. So,* $\int_\gamma \bar{f}f'dz = \int_\gamma -2ivf'dz = \int_\gamma -2iv(u_x + iv_x)dz = \int_\gamma -2iv(v_y + iv_x)dz = -i\int_a^b (2vv_y + 2ivv_y)(\gamma_1' + i\gamma_2')dt = \alpha$ *where* $\gamma = \gamma_1 + i\gamma_2$.

Therefore, real part of $\int_\gamma \bar{f}f'dz$ *is equal to real part of* α. *And it is also equal to* $-\int_a^b Im\left[(2vv_y + i2vv_x)(\gamma_1' + i\gamma_2')\right]dt = -\int_a^b (2vv_x\gamma_1' + 2vv_y\gamma_2')dt = -\int_a^b \frac{d}{dt}(v^2 \circ \gamma)dt = 0$ *since* γ *is closed curve.*

So, $\int_\gamma \bar{f}f'dz$ *is purely imaginary.*

Problem (2).

Let $f = -u_y$ *and* $g = u_x$. *Then* f, g *are continuous on* U. *Since* u *is harmonic,* $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ *on* $U \setminus \{0\}$. *So there is* $v : U \to \mathbb{R}$ *which is* C^1 *function and* $v_x = f$, $v_y = g$ *by lemma 2.5.3.*

Let $F = u + iv$. *Then* F *is* C^1 *function since* u, v *are* C^1. *Since* $v_x = f = -u_y$ *and* $v_y = g = u_x$, F *satisfies Cauchy-Riemann equation on* U. *Thus* F *is holomorphic on* U *and real part of* F *is* u.

Problem (3).

(a) For $z \notin [0,1]$, *the map* $w \mapsto \frac{1}{w-z}$ *is holomorphic on* $\mathbb{C}\setminus[0,1]$. *Let* $\gamma(t) = t$ *for* $t \in [0,1]$. *Then* $F(z) = \int_\gamma \frac{dw}{w-z} = \int_0^1 \frac{1}{t-z}dt$ *is well defined.*

For $z \notin [0,1]$, *let* $d > 0$ *be distance between* z *and* $[0,1]$. *For* $|h| < \frac{d}{2}$, *consider* $\frac{F(z+h)-F(z)}{h} = \int_0^1 \frac{1}{(t-z-h)(t-z)}dt$. *Then* $\left|\frac{1}{(t-z-h)(t-z)} - \frac{1}{(t-z)^2}\right| = \left|\frac{h}{(t-z)^2(t-z-h)}\right| \le |h|\frac{2}{d^3}$ *since* $|t-z| \ge d$ *and* $|t-z-h| \ge \frac{d}{2}$. *Therefore, as* $|h| \to 0$, *integrand converges to* $\frac{1}{(t-z)^2}$ *uniformly on* $t \in [0,1]$. *So* $\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = \int_0^1 \lim_{h\to 0} \frac{1}{(t-z-h)(t-z)}dt = \int_0^1 \frac{1}{(t-z)^2}dt = F'(z)$.

By same reasoning, we get $F''(z) = \int_0^1 \frac{1}{(t-z)^3}dt$. *From existence of* F'', F' *is continuous. Therefore* F *is* C^1 *function. Existence of complex derivative and* C^1 *implies* F *is holomorphic on* $\mathbb{C}\setminus[0,1]$.

(b) For $s \in (0,1)$, $F(s+i\varepsilon) = \int_0^1 \frac{1}{t-s-i\varepsilon}dt = \int_0^1 \frac{t-s+i\varepsilon}{(t-s)^2+\varepsilon^2}dt = \int_0^1 \frac{t-s}{(t-s)^2+\varepsilon^2}dt + i\int_0^1 \int_0^1 \frac{\varepsilon}{(t-s)^2+\varepsilon^2}dt$. *Let* $t - s = \varepsilon\tan\theta$. $\varepsilon\tan\theta_0 + s = 0$ *and* $\varepsilon\tan\theta_1 + s = 1$ *for* $-\frac{\pi}{2} < \theta_0, \theta_1 < \frac{\pi}{2}$. *Then* $\sec^2\theta_0 = \frac{s^2}{\varepsilon^2} + 1$, $\sec^2\theta_1 = \frac{(1-s)^2}{\varepsilon^2} + 1$, $\theta_0 = \tan^{-1}\left(\frac{-s}{\varepsilon}\right)$, *and* $\theta_1 = \tan^{-1}\left(\frac{1-s}{\varepsilon}\right)$.

Then $F(s+i\varepsilon) = \int_{\theta_0}^{\theta_1} \tan\theta d\theta + i\int_{\theta_0}^{\theta_1} d\theta = \log\left|\frac{\sec\theta_1}{\sec\theta_0}\right| + i(\theta_1 - \theta_0)$. *As* $\varepsilon \downarrow 0$, $F(s+i\varepsilon)$ *goes to* $\frac{1-s}{s} + i\pi$ *by simple calculation.*

Similarly, $F(s-i\varepsilon)$ goes to $\frac{1-s}{s} - i\pi$ as $\varepsilon \downarrow 0$.

(c) *Consider $F(-\varepsilon) = \int_0^1 \frac{1}{t+\varepsilon} dt = \log \frac{1+\varepsilon}{\varepsilon}$. It goes to ∞ as $\varepsilon \downarrow 0$.*

Consider $F(1+\varepsilon) = \int_0^1 \frac{1}{t-1-\varepsilon} dt = \log \frac{\varepsilon}{1+\varepsilon}$. It goes to $-\infty$ as $\varepsilon \downarrow 0$. Therefore, for $s = 0, 1$, $\lim_{z \notin [0,1] \to s} F(z)$ does not exists.

Problem (4)**.**

IDK where to start...

Problem (5)**.**

It is enough to show γ and μ are path homotopic. Definte $H(t,s) = (1-s)\gamma(t) + \frac{\gamma(t)}{|\gamma(t)|}s$. Then $H(t,1) = \mu(t)$ and $H(t,0) = \gamma(t)$ by reparametrization. And H is continuous because $\gamma(t) \neq 0$. Therefore H is path homotopy between γ and μ. Since line integration is invariant under path homotopy, we get $\int_\gamma F(\zeta) d\zeta = \int_\mu F(\zeta) d\zeta$.