

# FRANK JONES INTEGRATION THEORY SOLUTIONS

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## CHAPTER 6

**section A.**

Problem 1. "the vanishing property"

First, assume that  $\int f d\mu = 0$ . Since  $f$  is measurable, there exists an increasing sequence of nonnegative simple functions which converges to  $f$ . Let denote them as  $s_n$ . Then  $f^{-1}((0, \infty]) = \bigcup_{n=1}^{\infty} s_n^{-1}((0, \infty])$ , union of measure zero set. Therefore, we get  $\mu(f^{-1}((0, \infty])) = 0$ .

Conversely, assume that  $\int f d\mu > 0$ . Then there exists nonnegative simple function  $s \leq f$  such that  $\int s d\mu > 0$ . Also we can write  $s$  as a linear combination of (measurable) characteristic functions, i.e.  $\int s d\mu = \sum_{i=1}^N \alpha_i \mu(A_i) > 0$ . So,  $\alpha_i \mu(A_i) > 0$  for at least one integer  $1 \leq i \leq N$ . By Observing the fact that  $A_i \subset f^{-1}((0, \infty])$ , we can conclude that  $f^{-1}((0, \infty])$  has measure zero implies  $\int f d\mu = 0$  by contrapositive.

Problem 2. "the finiteness property"

$E = \{x : f(x) = \infty\}$  is a measurable set since  $f$  is measurable. Suppose  $\mu(E) > 0$ . For any  $M \in \mathbb{N}$ , choose nonnegative simple measurable function  $s_M \leq f$  such that  $s_M(x) \geq M$  for  $x \in E$ . Then  $\int f d\mu \geq \int s_M d\mu \geq M\mu(E)$ . Therefore,  $\int f d\mu \geq M\mu(E)$  for all positive integer  $M$ . It means that  $\int f d\mu = \infty$ . By taking contrapositive, we get what we want.

Problem 3. "compatibility to not finite nonnegative simple functions"

We are only interested in the case when  $\alpha_k = \infty, \mu(A_k) > 0$ . Since  $f$  is measurable, we can consider sequence of nonnegative finite simple function  $\{t_n\}$  such that  $t_n(x) \geq n$  for  $x \in A_k$ . Therefore,  $\int f d\mu \geq \int t_n d\mu \geq \mu(A_k)n$  for all positive integer  $n$ , which means  $\int f d\mu = \infty$ . For other cases, it is easy to check the compatibility.

Problem 4. "scalar multiplication is still valid for  $c = \infty$ "

First, assume  $\int f d\mu = 0$ . Then  $\mu(f^{-1}((0, \infty])) = 0$  by Problem 1. It is obvious that  $g^{-1}((0, \infty]) = f^{-1}((0, \infty])$  for  $g = \infty f$ . So,  $\int g d\mu = 0$ . Therefore we get  $\int \infty f d\mu = \int f d\mu$ .

Second, assume  $\int f d\mu > 0$ . Then measure of  $g^{-1}((0, \infty])$  is positive. So,  $\int \infty f d\mu = 0\mu(g^{-1}(0)) + \infty\mu(g^{-1}((0, \infty]))$  which is  $\infty$ .

Problem 5. "strict inequality for Fatou's lemma"

Consider this nonnegative finite simple function defined on real line:

$$s_n(x) = \begin{cases} n^2 & \text{if } x \in (0, \frac{1}{n}) \\ 0 & \text{otherwise} \end{cases}$$

$\lim s_n = 0$  and  $\lim \int s_n d\mu = \infty$ .

Problem 6.

Let  $f_k = 1_{A_k}$  where  $1_{A_k}$  is an indicator(characteristic) function. By Fatou's lemma (ILLLI),  $\int \liminf_{k \rightarrow \infty} 1_{A_k} d\mu \leq \liminf_{k \rightarrow \infty} \int 1_{A_k} d\mu = \liminf_{k \rightarrow \infty} \mu(A_k)$ . Now, showing  $1_{\liminf_{k \rightarrow \infty} A_k} \leq \liminf_{k \rightarrow \infty} 1_{A_k}$  is left to us. If  $x \in \liminf_{k \rightarrow \infty} A_k$  then  $x \in \bigcap_{i \geq n} A_i$  for some positive integer  $n$ . Then  $1_{A_i}(x) = 1$  for all positive integer  $i$  greater than  $n$ . So,  $1_{\liminf_{k \rightarrow \infty} A_k} \leq \inf_{i \geq n} 1_{A_i}$ . By letting  $n \rightarrow \infty$ , we get  $1_{\liminf_{k \rightarrow \infty} A_k} \leq \liminf_{k \rightarrow \infty} 1_{A_k}$ . Therefore,  $\mu(\liminf_{k \rightarrow \infty} A_k) = \int 1_{\liminf_{k \rightarrow \infty} A_k} d\mu \leq \int \liminf_{k \rightarrow \infty} 1_{A_k} d\mu \leq \liminf_{k \rightarrow \infty} \int 1_{A_k} d\mu = \liminf_{k \rightarrow \infty} \mu(A_k)$ .

## section B.

Problem 9. "iff condition for (finite) integrability"

Let  $f \in \mathcal{L}^1(\mu)$ . Then  $\int f_{\pm} d\mu < \infty$ , so there sum is also finite ( $= \int |f| d\mu$ ).

Conversely, assume  $|f| \in \mathcal{L}^1(\mu)$ . Then  $\int f_{\pm} d\mu \leq \int |f| d\mu < \infty$ . Therefore  $f \in \mathcal{L}^1(\mu)$ .

Problem 10. "dominated integrability"

If  $f$  is measurable,  $f_{\pm}$  is also measurable. So  $f_+ + f_- = |f|$  is measurable.  $\int |f| d\mu \leq \int |g| d\mu < \infty$ . Therefore,  $|f| \in \mathcal{L}^1(\mu)$  by Problem 9.

Problem 11.

It is obvious that  $|f_k| \leq |f|$  and  $f \in \mathcal{L}^1(\mu)$ . So, if each  $f_k$ s are measurable, by dominated convergence thm, done. Actually,  $f_k = f \cdot 1_{A_k \cap E_k}$  where  $A_k = [-k, k]$  and  $E_k = f^{-1}([-k, k])$ . There exists sequence of nonnegative finite simple function  $\{s_i\}$  which converges to  $f$  since  $f$  is measurable. So,  $\lim s_i 1_{A_k \cap E_k} = f_k$  is measurable.

Problem 12.

Likewise, it is enough to show that  $f(x)e^{-\frac{|x|^2}{k}} = f_k(x)$  is measurable.  $e^{-\frac{|x|^2}{k}}$  is continuous, hence Borel measurable, hence Lebesgue measurable. There exists sequence of nonnegative finite simple function  $\{s_k\}$  which converges to  $f$  since  $f$  is measurable. So,  $\lim_{i \rightarrow \infty} s_i e^{-\frac{|x|^2}{k}} = f_k$  is measurable.

Problem 13. 'alternative proof for problem 2.42'

For each  $x \in X$ , there are at most  $d \in \mathbb{N}$  distinct  $A_k$  containing  $x$ . Fix positive integer  $N$ . Let  $I_x = \{k \in \mathbb{N} : k \leq N \text{ and } x \in A_k\}$ . Clearly  $\sum_{k=1}^N 1_{A_k} \leq d 1_A$  where  $A = \bigcup_{i=1}^{\infty} A_i$ . By integrating both sides,  $\sum_{k=1}^N \mu(A_k) \leq d\mu(A)$ . By Letting  $N \rightarrow \infty$ , we got the result in Problem 2.42.

(similar, another)  $\sum_{k=1}^N 1_{A_k} \leq |I_x| \leq d = d 1_A$  for all  $N$ . So  $\sum_{k=1}^{\infty} 1_{A_k} \leq d 1_A$ . By integrating both sides and monotone convergence thm, we got it.

Problem 14.

Let  $\mathcal{I}$  be set of all sequences which are strictly increasing positive integers and length  $m$ . Such set is countable. Now,  $\bigcup_{i \in \mathcal{I}} \bigcap_{j=1}^m A_{i_j} = E_m$  and it is measurable.

Similar to Problem 13,  $m1_{E_m} \leq \sum_{i=1}^{\infty} 1_{A_i}$ . So,  $m\mu(E_m) = \int m1_{E_m} d\mu \leq \int \sum_{i=1}^{\infty} 1_{A_i} d\mu = \sum_{i=1}^{\infty} \int 1_{A_i} d\mu = \sum_{i=1}^{\infty} \mu(A_i)$  by monotone convergence thm.

**section C.**

Problem 16.

Clearly,  $|f| = 0$  almost everywhere, and  $|f|$  is measurable. Consider nonnegative finite simple function  $s \leq |f|$ . Then  $s = 0$  almost everywhere and  $s = \sum_{i=1}^N \alpha_i 1_{A_i}$ , where  $A_i$  is null set if  $\alpha_i \neq 0$ . Therefore,  $\int s d\mu = 0$ , which implies  $\int |f| d\mu = 0$ . So,  $f \in \mathcal{L}^1(\mu)$ ,  $|\int f d\mu| = 0 \Rightarrow \int f d\mu = 0$ .

Problem 17.

Note that  $f \sim g \Leftrightarrow f = g$  a.e. is an equivalence relation. So,  $g = h$  a.e. Let  $g$  be measurable and  $E_t = [-\infty, t]$ ,  $N = \{x : g(x) \neq h(x)\}$ . Then  $h^{-1}(E_t) \setminus g^{-1}(E_t) \subset N$  and  $g^{-1}(E_t) \setminus h^{-1}(E_t) \subset N$ . Since  $\mu$  is complete, they are all null sets. So  $h^{-1}(E_t) \cup g^{-1}(E_t)$  is measurable. Therefore,  $h^{-1}(E_t)$  is also measurable, which implies measurability of  $h$ .