mas540 exercises

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Exercise (4.4)**.**

First, let's show the completeness. Let $\{f_n\} \subset l^2(\mathbb{Z})$ *be a Cauchy sequence. Choose* n_k *such that* $\|f_{n_{k+1}} - f_{n_k}\| < 2^{-k+1}$. *Define* $f = f_{n_1} + \sum_{k=1}^{\infty}(f_{n_{k+1}} - f_{n_k})$ *and* $g = |f_{n_1}| + \sum_{k=1}^{\infty}|f_{n_{k+1}} - f_{n_k}|$. *Note that* $\|g\| \leq \|f_{n_1}\| + \sum_{k=1}^{\infty}\|f_{n_{k+1}} - f_{n_k}\| \leq \|f_{n_1}\| + 2 < \infty$. *Because, for each* $i \in \mathbb{Z}$, $|f(i)| \leq g(i) \leq \|g\| < \infty$, *we can observe that* $f(i)$ *is absolutely converges. Thus* f *is well defined function, also in* l^2 $(\because \|f\| \leq \|g\| < \infty)$. *Now let's show that* $f_{n_k} \to f$.

$$\|f - f_{n_k}\| \leq \sum_{m=k}^{\infty} \|f_{n_{m+1}} - f_{n_m}\| \leq 2^{-k}.$$

So $f_{n_k} \to f$ *as* $k \to \infty$ *in* l^2. *Therefore* $l^2(\mathbb{Z})$ *is complete.*

Now let's show the separability. Let $\mathcal{B}$ *be the set of all rational sequence in* $l^2(\mathbb{Z})$. *Clearly, it is nonempty since the zero sequence is in* $\mathcal{B}$. *Let* $f \in l^2$. *Fix* $\varepsilon > 0$. *For each* $i \in \mathbb{Z}$, *choose* q_i *such that*

$$|f(i) - q_i|^2 < \frac{\varepsilon^2}{2^{|i|}}.$$

Let $q : i \mapsto q_i$. *Then* $\|q\| \leq \|q - f\| + \|f\|$, *where*

$$\begin{aligned}
\|q - f\| &= \left(\sum_{-\infty}^{\infty} |q_i - f(i)|^2\right)^{1/2} \\
&\leq \left(\sum_{-\infty}^{\infty} \frac{\varepsilon^2}{2^{|i|}}\right)^{1/2} \\
&= \sqrt{3}\varepsilon.
\end{aligned}$$

Since $\|f\| < \infty$, *we can see that* $q \in l^2$ *and* $\|f - q\| \leq \sqrt{3}\varepsilon$. *Note that* $q \in l^2$ *implies* $q \in \mathcal{B}$. *So* $\mathcal{B}$ *is dense in* l^2, *and clearly* $\mathcal{B}$ *is countable set.*

□

Exercise (4.15)**.**

Let $\{e_1, e_2, \cdots, e_n\}$ *be an orthonormal basis of* $\mathcal{H}_1$. *Let* $f \in \mathcal{H}_1$, $\|f\| = 1$. *Then* $f = \sum_{i=1}^{n} c_i e_i$, *where* $\sqrt{\sum |c_i|^2} = 1$. *Then*

$$\begin{aligned}
\|Tf\| &= \|c_1 T e_1 + \cdots c_n T e_n\| \\
&\leq |c_1|\|Te_1\| + \cdots + |c_n|\|Te_n\| \\
&\leq \sum_{i=1}^{n} |c_i| M \\
&\leq M \left(\sum_{i=1}^{n} |c_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} 1\right)^{1/2} \\
&= \sqrt{n} M < \infty
\end{aligned}$$

where $M = \max_{1\le i\le n} \|Te_i\|$. *Since* n *is fixed, the above says that* T *is bounded operator.*

$\square$

Exercise (4.22)**.**

(a) *Polarization identity:*

$$(f,g) = \frac{1}{4}\left[\|f+g\|^2 - \|f-g\|^2 + i\|f+ig\|^2 - i\|f-ig\|^2\right].$$

This can be shown by using the hint. (Actually, we have seen it in the lecture.)

Put Tf, Tg *in the place of* f, g *respectively. Since* T *is linear and* $\|Tf\| = \|f\|$*, we can easily see that* $(f,g) = (Tf, Tg)$.

Now fix $g \in \mathcal{H}$. *Then* $(f, T^*Tg) = (Tf, Tg)$ *by the definition of adjoint, and* $(Tf, Tg) = (f, g)$ *by isometric property of* T. *Thus*

$$(f, T^*Tg - g) = 0$$

for all $f \in \mathcal{H}$. *Therefore* $T^*T = I$ *by taking* $f = T^*Tg - g$.

(b) *Let's show the injectivity. Let* $Tf = Tg$. *Then*

$$0 = \|Tf - Tg\| = \|f - g\| \Rightarrow f = g.$$

Thus T *is bijective isometry. Therefore it is an unitary operator.*

Now fix $g \in \mathcal{H}$. *For each* $f \in \mathcal{H}$*, there is* h *such that* $f = Th$ *because of the surjectivity. Then*

$$\begin{aligned}(f, TT^*g) &= (Th, TT^*g)\\ &= (h, T^*TT^*g)\\ &= (h, T^*g)\\ &= (Th, g)\\ &= (f, g)\end{aligned}$$

by the definition of the adjoint and $T^*T = I$ *because* T *is an isometry. Therefore*

$$(f, TT^*g - g) = 0$$

for all $f \in \mathcal{H}$. *By taking* $f = TT^*g - g$*, we can conclude that* $TT^* = I$.

(c) *Let* $\mathcal{H} = l^2(\mathbb{N})$. *Let* $f = (f(1), f(2), \cdots) \in \mathcal{H}$. *Define* $T : (f(1), f(2), \cdots) \mapsto (0, f(1), f(2), \cdots)$. *Clearly* T *is a linear operator, but non-surjective. If we show that* T *is isometry, then we are done.*

$$\|Tf\|^2 = 0 + \sum_{i=1}^{\infty} |f(i)|^2 = \|f\|^2.$$

SoT *is an isometry, which is not unitary.*

(d) *Note that unitary operator is isometry. So, by (a) and Cauchy Schwartz inequality,*

$$(Tf, Tf) = (f, T^*Tf) \leq \|f\|\|T^*Tf\| = \|f\|^2.$$

Thus $\|Tf\| \leq \|f\|$.

For the other direction,

$$\begin{aligned}(f, f) &= (T^*Tf, T^*Tf)\\ &= (Tf, TT^*Tf)\\ &\leq \|Tf\|\|TT^*Tf\|.\end{aligned}$$

But $\|TT^*Tf\|^2 = (TT^*Tf, TT^*Tf) = (Tf, TT^*TT^*Tf) = (Tf, Tf) = \|Tf\|^2$ *since* $(T^*T)^*(T^*T) = T^*TT^*T = I$ *by (a). Therefore* $(f, f) \leq (Tf, Tf)$*, which completes the proof.*

□

Exercise (4.32)**.**

(a) $T(cf + dg)(t) = t(cf + dg)(t) = ctf(t) + dtg(t) = cT(f)(t) + dT(g)(t)$ *so* T *is linear. Note that* $t^2 \leq 1$ *on* $[0, 1]$*. So*

$$\|Tf\|^2 = \int_0^1 t^2|f(t)|^2 dt \leq \int_0^1 |f(t)|^2 dt = \|f\|^2$$

which says that $\|T\| \leq 1$.

Also,

$$\begin{aligned}(Tf, g) &= \int_0^1 tf(t)\overline{g(t)}dt\\ &= \int_0^1 f(t)\overline{tg(t)}dt = (f, Tg)\end{aligned}$$

hence $Tg = T^*g$ *for all* $g \in L^2[0, 1]$ *by same argument used in exercise 22. Thus* T *is a bounded linear operator with* $T = T^*$.

Let $f_n(t) = \sqrt{2n+1}t^n$*. Then* $\|f_n\|^2 = \int_0^1 (2n+1)t^{2n}dt = 1$ *for all* n*. Thus* $f_n \in$ *the unit ball of* $L^2[0, 1]$*. For any subsequence* f_{n_k}*,*

$$\begin{aligned}&\|Tf_{n_k} - Tf_{n_l}\|^2\\ &= \int_0^1 (2n_k + 1)t^{2n_k+2} + (2n_l + 1)t^{2n_l+2} - 2\sqrt{(n_k + 1)(n_l + 1)}t^{(n_k+1)(n_l+1)}dt\\ &= \frac{2n_k + 1}{2n_k + 3} + \frac{2n_l + 1}{2n_l + 3} - \frac{2\sqrt{(n_k + 1)(n_l + 1)}}{(n_k + 1)(n_l + 1) + 1}.\end{aligned}$$

As $n_k, n_l \to \infty$*, the first two terms go to* 1 *respectively, but the last term go to* 0*. So the sequence does not converge. Hence* T *is non-compact.*

(b) Suppose $T\varphi = \lambda\varphi$. Then $t\varphi(t) = \lambda\varphi(t)$ for all $t \in [0,1]$. Then $t\varphi(t)1_{\varphi\neq 0}(t) = \lambda\varphi(t)1_{\varphi\neq 0}(t)$, so $1_{\varphi\neq 0}(t) = 0$, which means $\varphi = 0$. But the zero vector cannot be an eigenvector, hence there is no eigenvector.

$\square$

Problem (4.1)**.**

Let X be a collection of linearly independent subsets of $\mathcal{H}$. Impose partial order by the inclusion. Note that X is nonempty since the empty set is in X.

We'll use Zorn's lemma which is equivalent to the AC. Let Y be any totally ordered subset of X. $L_Y = \bigcup_{w\in Y} w$. Then every finite subset of L_Y is in Y, since Y is totally ordered. Hence L_Y is linearly independent, so $L_Y \in X$. But, note that L_Y is an upperbound of Y in X. So Zorn's lemma gives L_m which is maximal element of X.

Now assert that L_m is an algebraic basis of $\mathcal{H}$. Since $L_m \in X$, L_m is linearly independent. If L_m does not span $\mathcal{H}$, then there is $f \in \mathcal{H}$ outside of span L_m. Define $L_f = L_m \cup \{f\}$. Then L_f is strictly larger than L_m. But, L_f is linearly independent, since f is outside of span of L_m. Thus $L_f \in X$, which contradicts to the maximality of L_m. Hence L_m spans $\mathcal{H}$ algebraically, so L_m is an algebraic basis.

Now $L_m = \{a_\alpha : \alpha \in I\}$. Let $B = \left\{e_\alpha = \frac{a_\alpha}{\|a_\alpha\|} : \alpha \in I\right\}$. Then B is an algebraic basis, consists of unit vectors.

Choose $\{e_i\}_{i\in\mathbb{N}}$. For $f \in \mathcal{H}$,

$$f = \sum_{\alpha\in F} c_\alpha e_\alpha = \sum_{\alpha\in F\setminus\mathbb{N}} c_\alpha e_\alpha + \sum_{i=1}^{N} c_i e_i$$

where F is finite set. Define $l(f) = \sum_{i=1}^{N} ic_i$. Note that N depends on f. Clearly, l is linear: $l(cf + dg) = c\sum_{i=1}^{N} ic_i + d\sum_{i=1}^{N} id_i = cl(f) + dl(g)$. Also $l(e_i) = i$. But, $|l(e_i)| = i \to \infty$ as $i \to \infty$, even though $\|e_i\| = 1$. This says l is unbounded linear functional.

$\square$