Conformal Self Mappings

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Contents

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- f is a function of U into V
- f is conformal if f is bijective and holomorphic.
- conformal $=$ biholomorphic
- If h is holomorphic function of U, and U is somewhat complicated, by considering $h \circ f$, we can change the domain of h.

characterizing conformal self mapping of $\mathbb{C}$

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- In fact, above form is all of them.
- Note that we are considering not just entire function. Conformal self mappings of $\mathbb{C}$ has more condition than entire function.

Lemma 6.1.2.
If $f : \mathbb{C} \to \mathbb{C}$ is a conformal then $\lim_{|z|\to\infty} |f(z)| = \infty$.

proof of lemma 6.1.2.

- Fix M. We want to show existence of N such that $f(\{z : |z| > N\}) \subset \{w : |w| > M\}$.

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- Existence of N is clear since RHS of above is compact($\Rightarrow$ closed and bounded).

characterizing conformal self mapping of $\mathbb{C}$

lemma 6.1.3

f is a conformal self mapping of $\mathbb{C}$. Then there are $B, D > 0$ such that $|z| > D$ implies $|f(z)| < B|z|$.

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- There is C such that $|z| > C$ implies $|f(z)| > 1$.

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- Near 0, $\left|f(\frac{1}{z})\right| < \frac{B}{|z|}$.
- $z \mapsto \frac{1}{z}$ leads the conclusion.

characterizing

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- But f must be nonconstant. So $f(z) = az + b$ for $a \neq 0$.
- We are characterized conformal self mappings of $\mathbb{C}$.

remark

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- Why n? Because we cannot say $g'(0) \neq 0$. But, $g^{(n)}(0) \neq 0$ for some n since g is nonconstant since h is nonconstant.
- Note that entire function φ which satisfies $\lim_{|z|\to\infty} |\varphi(z)| = \infty$ must be polynomial.

characterizing conformal self mapping of unit disc

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lemma 6.2.1.
$f : D \to D$ is biholomorphic which fixes origin iff $f(z) = wz$ for $|w| = 1$

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- Uniqueness of Schwarz lemma tells us that $f(z) = f'(0)z$.

Mobius transformation

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theorem 6.2.3.

f is conformal self mapping of unit disc. Then $f(z) = w\varphi_a(z)$ for some $|a| < 1$ and $|w| = 1$.

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- Take $a = -bw^{-1}$.

preliminaries of linear fractional transformation

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- Note that image of north-pole in S^2 under projection is ∞.
- Also, above definition of limit in extended plane is congruent to definition using metric.

linear fractional transformation

- Let $ad - bc \neq 0$, $a, b, c, d \in \mathbb{C}$. $f(z) = \frac{az+b}{cz+d}$ is called linear fractional transformation if it satisfies two more conditions.

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- Note that $f(p_i) \to f(p_0)$ when $p_i \to p_0$ for all $p_0 \in \mathbb{C} \cup \infty$.
- Above says continuity of f on extended plane.