FRANK JONES INTEGRATION THEORY SOLUTIONS

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CHATER 7: LEBESGUE INTEGRAL ON $\mathbb{R}^n$

section A: Riemann Integral.

Problem 1.

Suppose 1_A is LSC. Let $x \in A$ and $0 < 1_A(x) = 1$. By definition of LSC at x, there exists $\delta > 0$ such that $1_A(y) > 0$ for all $y \in B(x;\delta)$. So $1_A(y) = 1$ and therefore $B(x;\delta) \subset A$. Thus A is open.

On the contrary, suppose A is open. For $x \in A$, consider $t < 1 = 1_A(x)$. Since A is open, there exists $\delta > 0$ such that $B(x;\delta) \subset A$. Then we get $y \in B(x;\delta) \Rightarrow t < 1_A(y) = 1$. For $x \notin A$, consider $t < 0 = 1_A(x)$. Take any $\delta > 0$. Then $y \in B(x;\delta) \Rightarrow t < 0 \leq 1_A(y)$. Therefore 1_A is LSC if A is open.

Note that f is LSC if and only if $\forall t \in \bar{\mathbb{R}}$, $\{f > t\}$ is open. And f is LSC at x if and only if x is interior point of every $\{f > t\}$ for $t < f(x)$.

Problem 2.

For $x \in A^\circ$, there exists $\delta > 0$ such that $B(x;\delta) \subset A$. Then $\inf_{y \in B(x;\delta)} 1_A(y) = 1$ so lower envelope of 1_A at x is same as $1_{A^\circ}(x) = 1$.

Now assume $x \notin A^\circ$. Then, for every $\delta > 0$ $B(x;\delta) \cap A \neq \emptyset$. Then $\inf_{y \in B(x;\delta)} 1_A(y) = 0$ for some small δ. Then lower envelope of 1_A is zero at x, which is same as $1_{A^\circ}(x)$.

If $x \in \bar{A}$, for all $\delta > 0$ $B(x;\delta) \cap A \neq \emptyset$. Therefore $\sup_{y \in B(x;\delta)} 1_A(y) = 1$, so upper envelope of 1_A is same as $1_{\bar{A}}$.

If $x \notin \bar{A}$, there exists $\delta > 0$ such that $B(x;\delta) \cap A = \emptyset$. Then $\sup_{y \in B(x;\delta)} 1_A(y) = 0$. So upper envelope of 1_A is same as $1_{\bar{A}}$.

Problem 3.

For each $x \in \mathbb{R}^n$, let $t < \min_i f_i(x)$. Then $t < f_i(x)$ for all $i \in \mathcal{I}$. For each i, there exists δ_i such that $y \in B(x;\delta_i)$ implies $t < f_i(y)$. Take $\delta = \max_i \delta_i$. Then $y \in B(x;\delta)$ implies $t < \min_i f_i(y)$. So $\min_i f_i$ is LSC.

Now consider $A_i = \left(-\frac{1}{i}, \frac{1}{i}\right) \subset \mathbb{R}$. Since A_i is open, by problem 1, 1_{A_i} is LSC. Let $A = \bigcap_{i=1}^{\infty} A_i$ then $1_A = \inf_i 1_{A_i}$. It is not semicontinuous by considering the set $\left\{1_A > \frac{1}{2}\right\} = \{0\}$.

Problem 4.

Let $\tau_f \geq f$ and $\tau_g \geq g$ where τ_f and τ_g are step functions. Note that $\tau_f + \tau_g$ is also step function greater than $f + g$. Then all others follow directly.

Problem 5.

Let $\varepsilon > 0$ be given. We can choose positive integer N such that

(1) $|f(x) - f_n(x)| < \varepsilon$ if $n \geq N$ and for all $x \in \mathbb{R}^n = X$.
(2) $\int_I (\tau_N - \sigma_N)\, d\lambda < \varepsilon$

where τ_N, σ_N is simple function bigger, smaller than f_N respectively.
For every $x \in X$,

$$\sigma_N(x) - \varepsilon \leq f_N(x) - \varepsilon < f < f_N(x) + \varepsilon \leq \tau_N(x) + \varepsilon$$

Then, $\int_I \sigma_N d\lambda - \varepsilon\lambda(I) \leq r\underline{\int_I} f d\lambda \leq r\overline{\int_I} f d\lambda \leq \int_I \tau_N d\lambda + \varepsilon\lambda(I)$ Because $\sigma_N - \varepsilon$ is step function smaller than f and $\tau_N + \varepsilon$ is also step function bigger than f.

Therefore we can get

$$r\overline{\int_I} f d\lambda - r\underline{\int_I} f\lambda < \varepsilon + 2\varepsilon\lambda(I)$$

which implies Riemann integrability of f.

By definition of uniform convergence, $r\int |f - f_N|\, d\lambda \leq \varepsilon\lambda(I)$.
So, $\lim_{N\to\infty} r\int |f - f_N|\, d\lambda = 0$, which implies $\lim_{N\to\infty} \left| r\int f d\lambda - r\int f_N d\lambda \right| = 0$ then conclusion follows.

Problem 6.

(a) $g(x) = 0$ for all $x \in I = [0,1]$. $f(x) = 1_{\mathbb{Q}\cap I}(x)$. f is nowhere continuous but $f = g$ almost everywhere.
(b) Consider the following funnction f :

$$f(x) = \begin{cases} 0 & \text{if } x \in I \cap C \\ \frac{1}{x} & \text{if } x \in I \cap C^c \end{cases}$$

where C is Cantor ternary set and $I = [0,1]$.

For $x \in I \cap C^c$, $x \in I_{j,k} = \left(\frac{2k}{3^j}, \frac{2k+1}{3^j}\right)$ for some k, j. On $I_{j,k}$, function $i : x \mapsto x$ is continuous and $i(x) \neq 0$. So, the map $\varphi : x \mapsto \frac{1}{x}$ is also continuous on $I_{j,k}$. Therefore, φ is continuous at x. Therefore, φ is continuous a.e. on I.

Note that $f(x) = \frac{1}{x}$ a.e. on I. Let $g = f$ a.e. on I. Then $g(x) = \frac{1}{x}$ a.e. on I. Then g is discontinuous at $x = 0$. Therefore, f is continuous a.e. on I and there is no continuous function such that $f = g$ a.e. on I.

Problem 7.

For $x \in [a,b]$, $na \leq nx \leq nb$, so $nx \in \frac{1}{2} + \mathbb{Z}$ for finitely many x. Therefore, (nx) is discontinuous at most finitely many points, which implies (nx) is Riemann integrable. Then $\frac{(nx)}{n^2}$ also Riemann integrable, and their finite summation $f_k(x) = \sum_{n=1}^k \frac{(nx)}{n^2}$ is also Riemann integrable.

Now, let $\varepsilon > 0$ be given. Since $\sum_{n=1}^\infty$ converges, there exists positive integer N such that $\sum_{k=N}^\infty < \varepsilon$. Consider $m > n \geq N$ and following:

$$|f_m(x) - f_n(x)| \leq \left| \sum_{k=n+1}^m \frac{(kx)}{k^2} \right| \leq \sum_{k=n+1}^m \frac{|(kx)|}{k^2} \leq \sum_{k=n+1}^m \frac{1}{k^2} < \varepsilon$$

for all $x \in [a,b]$ since $-1 \leq (nx) \leq 1$. Therefore, f_n is uniformly Cauchy, which implies uniform convergence of f_n to f. By problem 5, f is Riemann integrable since each f_n is Riemann integrable and $f_n \rightrightarrows f$.

Problem 9.

With out loss of generality, assume that f is monotonically increasing.

Let $\varepsilon > 0$ be given. Consider $a = x_0 < x_1 < \cdots < x_n = b$ where $\max_{1\le k\le n} \lambda\left([x_{k-1}, x_k]\right) < \frac{\varepsilon}{f(b)-f(a)}$. (If $f(a) = f(b)$, conclusion follows trivially so let us assume that $f(a) < f(b)$.

Let $I = [a, b]$ and $\sigma : I \to \mathbb{R}$ such that $\sigma\left((x_{k-1}, x_k)\right) = \{f(x_{k-1})\}$, $\sigma(x_k) = f(a)$. Similarly, let $\tau : I \to \mathbb{R}$ such that $\tau\left((x_{k-1}, x_k)\right) = \{f(x_k)\}$ and $\tau(x_k) = f(b)$. Then σ, τ are step functions satisfying $\sigma \le f \le \tau$. So,

$$\begin{aligned}\int_I (\tau - \sigma)\, d\lambda &= \sum_{k=1}^{n} \left(f(x_k) - f(x_{k-1})\right) \lambda\left([x_{k-1}, x_k]\right)\\ &< \sum_{k=1}^{n} \left(f(x_k) - f(x_{k-1})\right) \frac{\varepsilon}{f(b) - f(a)} = \varepsilon\end{aligned}$$

which implies Riemann integrability of f on I.

Let $x < x'$ be points where f is discontinuous. Since f is monotonic, $f(x-) = \lim_{y\uparrow x} f(y)$ and $f(x+) = \lim_{y\downarrow x} f(y)$ exist. By monotonicity of f, we can easily deduce that $f(x-) \le f(x) \le f(x+) \le f(x'-) \le f(x') \le f(x'+)$. Since f is discontinuous at x, x', $f(t-) < f(t+)$ for $t = x, x'$. Choose $q_t \in (f(t-), f(t+))$ for $t = x, x'$. Then $q_x < q_{x'}$. The map $x \mapsto q_x$ is hence injective. So, there are at most countably many discontinuous points of f on I.

Therefore f is continuous a.e. on I.

Problem 10.

Consider 1_C where C is Cantor ternary set. If $x \in C$, $x \notin C^\circ$ since C has empty interior. So, for any $\delta > 0$, there exists $y \in B(x; \delta)$ such that $y \notin C$. Then $1_C(x) - 1_C(y) = 1$. So 1_C is discontinuous at $x \in C$.

On the contrary if $x \in C^c$, $x \in I_{j,k}$ for some j, k (we'll use notation of problem 6.) Then there is $\delta > 0$ such that $B(x; \delta) \subset I_{j,k} \subset C^c$, so $d(x, y) < \delta$ implies $1_C(x) - 1_C(y) = 0 < \varepsilon$ for any $\varepsilon > 0$. Therefore 1_C is continuous on C^c.

1_C has an uncountable set of discontinuities(C) and continuous a.e. on $[0, 1]$. Therefore 1_C is Riemann integrable.

Problem 11.

If f is Riemann integrable and $f = 1_A$ a.e. on $I = [0, 1]$, $r\int f d\lambda = \int_I f d\lambda = \int_I 1_A d\lambda = \lambda(A) > 0$.

Let σ be a step function, $\sigma \le f$. Then $\sigma \le 1_A$ a.e. on I and let N be corresponding null set.

Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be endpoints of special rectangles corresponding to σ. If $(x_{k-1}, x_k) \cap J_i = \emptyset$ for all positive $i = \frac{2m-1}{2^t} < 1$ ($A = I \setminus \bigcup J_i$, we'll use notation of section 4.B), $(x_{k-1}, x_k) \subset A$ which contradicts the fact that A has empty interior. So, $(x_{k-1}, x_k) \cap J_i \neq \emptyset$ for some i.

If $(x_{k-1}, x_k) \setminus N \subset A$, $(x_{k-1}, x_k) \setminus N \cap J_i = \emptyset$, then $(x_{k-1}, x_k) \cap J_i \subset N$. But $J_i \cap (x_{k-1}, x_k)$ is open and nonempty, so it has positive measure which contradicts $\lambda(N) = 0$.

Therefore for each k, $(x_{k-1}, x_k) \setminus N \cap A^c \neq \emptyset$. So $\sigma\left((x_{k-1}, x_k)\right) = \xi$ where $\xi \leq 0$. It means $\sigma \leq 0$.

Then $r \int f d\lambda = \sup_{\sigma \leq f} \int_I \sigma d\lambda \leq 0$ which contradicts to $r \int f d\lambda = \lambda\left(A\right) > 0$.