# mas651 exercises

Jaemin Oh

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**Problem** (5.1.1)**.**

*Let* $(S, \mathcal{S})$ *be a state space of* $X_n$ *where* $S = \{1, 2, \cdots N\}$ *and* $\mathcal{S} = 2^S$*. Note that* $N$ *is an absorbing state. And* $X_1 = 1$ *with probability* $1$*. For fixed* $k$ *such that* $1 \le k < N$*,* $k \le n$*,*

$$P(X_{n+1} = k+1 | X_n = k) = \frac{N-k}{N}$$

*and*

$$P(X_{n+1} = k | X_n = k) = \frac{k}{N}.$$

*If* $k > n$*, then the above are all* $0$*. So it is a temporally inhomogeneous. The Markov property is trivial since the very next state only depends on the current state.*

□

**Problem** (5.1.2)**.**

$$P(X_4 = 2 | X_3 = 1, X_2 = 1, X_1 = 1, X_0 = 0) = (1/16)/(1/4) = 1/4$$

*but*

$$P(X_4 = 2 | X_3 = 1, X_2 = 0, X_1 = 0, X_0 = 0) = (1/16)/(1/8) = 1/2.$$

*Thus* $X_n$ *is not a Markov chain.*

□

**Problem** (5.1.5)**.**

$$P(X_{n+1} = k+1 | X_n = k) = \frac{m-k}{m}\frac{b-k}{m}$$

*because we must choose a white ball in the left urn and a black ball in the right urn.*

$$P(X_{n+1} = k | X_n = k) = \frac{k}{m}\frac{b-k}{m} + \frac{m-k}{m}\frac{m+k-b}{m}$$

*since there are two cases, choosing both black or both white.*

$$P(X_{n+1} = k-1 | X_n = k) = \frac{k}{m}\frac{m+k-b}{m}$$

*since we must choose a black ball in the left urn and a white ball in the right urn. Note that the sum of the above is 1, so there is no other transition probability.*

□

**Problem** (5.1.6)**.**

$$P(S_{n+1} = k+1 | S_n = k) = \frac{P(X_{n+1} = 1, S_n = k)}{P(S_n = k)}$$

*where the denominator is*

$$\int_{\theta\in(0,1)} P(S_n = k|\theta)dP = \binom{n}{x}\frac{x!y!}{(n+1)!} = \frac{1}{n+1}$$

*for* $x =$*the number of* $i$ *such that* $U_i \leq \theta$ *and* $y = n-x$. *Note that* $x = (n+k)/2$ *and* $y = (n-k)/2$ *since* $x+y = n$ *and* $x - y = k$. *The numerator is*

$$\int_{\theta\in(0,1)} P(X_{n+1} = 1, S_n = k|\theta)dP = \binom{n}{x}\frac{(x+1)!y!}{(n+2)!}$$

*These are because* $P(S_n = k|\theta) = \theta^x(1-\theta)^y\binom{n}{x}$ *and* $P(X_{n+1} = 1, S_n = k|\theta) = P(X_{n+1} = 1|\theta)P(S_n = k|\theta) = \binom{n}{x}\theta^{x+1}(1-\theta)^y$ *and using the kernel of beta distribution.*

*Thus, the probability what we want is* $(n+k+2)/(2n+4)$ *which depends on* $n$. *So* $X_n$ *is temporally inhomogeneous.*

$$P(S_{n+1} = k+1|S_1 = t_1, \cdots, S_n = k) = P(X_{n+1} = 1|X_1 = t_1, \cdots, X_n = t_n)$$

*where* $\sum_{i=1}^n t_i = k$. *We can show the above is equal to* $P(S_{n+1} = k+1|S_n = k) = (n+k+2)/(2n+4)$ *similarly, by omitting the* $\binom{n}{x}$ *term of both denominator and numerator.*

$\square$

**Problem** (5.2.1)**.**

*By the given hint,*

$$E(1_A 1_B|\mathcal{F}_n) = E(1_A E(1_B|\mathcal{F}_n)|X_n)$$

*so it suffices to show that* $E(1_B|\mathcal{F}_n) = E(1_B|X_n)$.

*Let* $Y = 1_{B_n}(\omega_0)\cdots 1_{B_{n+k}}(\omega_k)$. *Then* $Y \circ \theta_n =$ *the indicator function of* $\{X_n \in B_n, \cdots, X_{n+k} \in B_{n+k}\} = B$. *By the markov property,*

$$P(B|\mathcal{F}_n) = E_{X_n} Y.$$

*Let* $\varphi(x) = E_x Y$ *then* $\varphi(X_n)$ *is* $\sigma(X_n)$*-measurable mapping. Thus, when* $B$ *has a form of* $\{X_n \in B_n, \cdots, X_{n+k} \in B_{n+k}\}$ *for some nonnegative integer* $k$,

$$P(B|\mathcal{F}_n) = P(B|X_n).$$

*Note that a collection of such* $B$ *generates* $\sigma(X_n, X_{n+1}, \cdots)$.

*Now let* $\mathcal{G} = \{C : P(C|\mathcal{F}_n) = P(C|X_n)\}$. *By putting* $B_{n+i} = S$ *for* $0 \leq i \leq k$, *we earn* $\Omega_0 \in \mathcal{G}$. *If* $C, D \in \mathcal{G}$ *and* $C \subset D$, *then by properties of conditional expectation,* $D \setminus C \in \mathcal{G}$. *If* $C_i \in \mathcal{G}$ *and* $C_i \uparrow C$ *then by monotone convergence theorem for conditional expectation,* $C \in \mathcal{G}$. *Thus* $\mathcal{G}$ *is a lambda system containing a collection of* $B$*'s which generates* $\sigma(X_n, \cdots)$. *Therefore, by Dynkin's theorem, the third equation is satisfied by any* $B \in \sigma(X_n, \cdots)$. *By the first equation, we can derive the conclusion.*

$\square$

**Problem** (5.2.4)**.**

*First, claim that*

$$P_x(X_n = y|T_y = m) = P_y(X_{n-m} = y).$$

*This is because*

$$\begin{aligned}
P_x(X_n = y|T_y = m) &= \frac{P_x(X_n = y, T_y = m)}{P_x(T_y = m)} \\
&= \frac{\int_{T_y = m} 1_{(X_n = y)} dP_x}{P_x(T_y = m)} \\
&= \frac{\int_{T_y = m} E\left(1_{(X_n = y)}|\mathcal{F}_m\right) dP_x}{P_x(T_y = m)} \\
&= \frac{\int_{T_y = m} E_{X_m} 1_{(X_{n-m} = y)} dP_x}{P_x(T_y = m)} \\
&= \frac{P_x(T_y = m) P_y(X_{n-m} = y)}{P_x(T_y = m)}.
\end{aligned}$$

*Now, note that* $P_x(X_n = y) = \sum_{m=1}^{n} P_x(X_n = y, T_y = m)$. *From this and* *the above discussion,*

$$\begin{aligned} p^n(x,y) = P_x\left(X_n = y\right) &= \sum_{m=1}^{n} P_x\left(X_n = y | T_y = m\right) P_x\left(T_y = m\right) \\ &= \sum_{m=1}^{n} P_y\left(X_{n-m} = y\right) P_x\left(T_y = m\right) = \sum_{m=1}^{n} P_x\left(T_y = m\right) p^{n-m}(y,y). \end{aligned}$$

□

**Problem** (5.2.6).

*Fix $x \in S \setminus C$. Since $P_x(T_C = \infty) = \lim_{M\to\infty} P_x(T_C > M) < 1$, we can choose $N_x$ and $\varepsilon$ so that*

$$P_x(T_C > M) \le 1 - \varepsilon$$

*whenever $M \ge N_x$. Note that we can choose $N_x$ as an integer. Put $N = \max_{x\in S\setminus C} N_x$. Now we get*

$$\begin{aligned} P_y(T_C > 2N) &= \sum_{x\in S\setminus C} P_y(T_C > 2N, T_C > N, X_N = x) \\ &= \sum_{x\in S\setminus C} P_y\left(T_C > 2N | X_N = x, T_C > N\right) P_y(X_n = x, T_C > N) \\ &\le \sum_{x\in S\setminus C} P_x(T_C > N) P_y(X_N = x, T_C > N) \\ &\le (1-\varepsilon) \sum_{x\in S\setminus C} P_y(X_N = x, T_C > N) \\ &\le (1-\varepsilon)^2. \end{aligned}$$

*By induction, the result follows.*

*Remark* 1. By $k \to \infty$, we can say that $P_y\left(T_C = \infty\right) = 0$. That is, $P_y\left(T_C < \infty\right) = 1$.

□

**Problem** (5.2.7).

1. *It is similar to the manipulation of problem 5.2.4:*

$$\begin{aligned} P_x\left(V_A < V_B\right) &= \sum_y P_x\left(V_A < V_B, X_1 = y\right) = \sum_y P_x\left(V_A < V_B | X_1 = y\right) P_x\left(X_1 = y\right) \\ &= \sum_y p(x,y) P_x\left(V_A < V_B | X_1 = y\right) = \sum_y p(x,y) P_y\left(V_A < V_B\right) \end{aligned}$$

*where the first term is $h(x)$ and the last term is $\sum_y p(x,y)h(y)$.*

2. *I think we must further assume that* $h$ *is bounded and measurable. For convenience, let* $\tau = V_A \wedge V_B = V_{A\cup B}$. *By the equation (5.2.2) of our textbook, we get*

$$\begin{aligned} E_x\left(h(X_{n+1})|\mathcal{F}_n\right) &= \sum_y p(X_n, y)h(y) \\ &= h(X_n) \end{aligned}$$

*for* $X_n \notin A \cup B$.

*Now, put* $Y_n = h(X_n)$. *Then*

$$Y_{n\wedge\tau} - Y_0 = h\left(X_{n\wedge\tau}\right) - h(X_0) = \sum_{k=1}^{n} 1_{(\tau\geq k)}\left(Y_k - Y_{k-1}\right).$$

*By using the above,*

$$\begin{aligned} E_x\left(Y_{n+1\wedge\tau} - Y_0|\mathcal{F}_n\right) &= \sum_{k=1}^{n+1} 1_{(\tau\geq k)} E_x\left(Y_k - Y_{k-1}|\mathcal{F}_n\right) \\ &= 1_{(\tau\geq n+1)}\left(Y_n - Y_0\right) + 1_{(\tau<n+1)}\left(Y_\tau - Y_0\right) \\ &= 1_{(\tau>n)}(Y_n - Y_0) + 1_{(\tau\leq n)}(Y_\tau - Y_0) \\ &= Y_{n\wedge\tau} - Y_0. \end{aligned}$$

*So* $h(X_{n\wedge\tau})$ *is a martingale. Note that the first equality is due to (5.2.2), and the last is due to optional stopping.*

3. *We assumed that h is bounded. Thus, our martingale is uniformly bounded, so the optional stopping theorem can be applied:*

$$x = E_x h(X_0) = E_x h(X_\tau) = E_x\left[E_x\left(h(X_\tau)|\mathcal{F}_\tau\right)\right]$$

*where the last term is equal to*

$$E_x\left[E_{X_\tau} h(X_0)\right] = E_x\left[1_{(X_\tau\in A)} + 0\cdot 1_{(X_\tau\in B)}\right] = E_x\left[1_{(X_\tau\in A)}\right].$$

*The above is because* $P_x(\tau < \infty) = 1$ *and* $h$ *is* 1 *on* $A$ *and* 0 *on* $B$. *Note that this implies the result, since* $X_\tau \in A$ *is equivalent to* $V_A < V_B$.

□

**Problem** (5.2.8)**.**

*Let* $\tau = V_0 \wedge V_N$. *Then* $X_{n\wedge\tau}$ *is an uniformly bounded martingale since the state space of* $X_n$ *is finite. By the optional stopping theorem, we get*

$$E_x X_0 = E_x X_\tau$$

*where the LHS is equal to* $x$. *Note that, by the remark 1, we can say that* $P_x\left(\tau < \infty\right) = 1$. *Then the RHS of the above eqn is equal to* $0P_x\left(V_0 < V_N\right) + NP_x\left(V_N < V_0\right)$. *Thus,*

$$x = NP_x\left(V_N < V_0\right).$$

□

**Problem** (5.2.11)**.**

1. *It is similar to the manipulation of problem 5.2.7:*

$$\begin{aligned}
E_x V_A &= \sum_{k\geq 1} P_x\left(V_A \geq k\right) \\
&= P_x\left(V_A \geq 1\right) + \sum_{k\geq 2} P_x\left(V_A \geq k\right) \\
&= 1 + \sum_{k\geq 2} P_x\left(V_A \geq k\right) \\
&= 1 + \sum_{k\geq 2}\sum_{y} P_x\left(V_A \geq k | X_1 = y\right) P_x\left(X_1 = y\right) \\
&= 1 + \sum_{y} p(x,y) \sum_{k\geq 2} P_y\left(V_A \geq k-1\right) = 1 + \sum_{y} p(x,y) E_y V_A
\end{aligned}$$

   *where $P_x\left(V_A \geq 1\right) = 1$ since $x$ lies outside of $A$.*

   *Also, $E_x V_A < \infty$ because*

$$\begin{aligned}
E_x \frac{V_A}{N} &= \sum_{k\geq 1} P_x\left(V_A \geq kN\right) \\
&\leq \sum_{k\geq 1} (1-\varepsilon)^k < \infty.
\end{aligned}$$

2. *I think we should assume the measurability, and boundedness of $g$. By the manipulation used in problem 5.2.7, we get:*

$$\begin{aligned}
E_x\left(g(X_{n+1}) + n + 1 | \mathcal{F}_n\right) &= n + 1 + \sum_{y} p(X_n, y) g(y) \\
&= n + g(X_n)
\end{aligned}$$

   *for $X_n \notin A$.*

   *Now put $Y_n = g(X_n) + n$ and $\tau = V_A$ for convenience. Then*

$$\begin{aligned}
E_x\left(Y_{n+1\wedge\tau} - Y_0 | \mathcal{F}_n\right) &= \sum_{k=1}^{n+1} 1_{(\tau\geq k)} E_x\left(Y_k - Y_{k-1} | \mathcal{F}_n\right) \\
&= 1_{(\tau\geq n+1)}(Y_n - Y_0) + 1_{(\tau< n+1)}(Y_\tau - Y_0) \\
&= Y_{n\wedge\tau} - Y_0.
\end{aligned}$$

   *So $X_{n\wedge V_A} + n\wedge V_A$ is a martingale.*

3. *From the boundedness of $g$ and the fact that $V_A$ is $L^1$ function, our martingale is uniformly integrable. Thus we can apply optional stopping theorem:*

$$E_x g(X_0) = E_x\left[V_A + g(X_{V_A})\right]$$

*where the first term is $g(x)$ and the second term is $E_x V_A + E_x g(X_{V_A})$. But $X_{V_A}$ lies in $A$ and $g$ is 0 on $A$. Thus the second term of the equation is $E_x V_A$.*

$\square$