mas550 homework

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Problem (1.1.2)**.**

Let $A = \Pi_{i=1}^{d}(a_i, b_i]$*. Then*

$$A = \left(\Pi_{i=1}^{d}[a_i - 1, b_i]\right) \cap \left(\Pi_{I=1}^{d}(a_i, b_i + 1)\right)$$

which is intersection of open set and closed set. So, $A \in \mathcal{R}^d$ *therefore* $\sigma(S_d) \subset \mathcal{R}^d$*.*

On the other hand, let $B = \Pi_{i=1}^{d}(a_i, b_i)$ *where* $-\infty < a_i < b_i < \infty$*. We can choose sequences* $\{a_{i,j}\}_{j=1}^{\infty}$ *and* $\{b_{i,j}\}_{j=1}^{\infty}$ *for each* $1 \le i \le d$ *such that* $a_{i,j} \downarrow a_i$ *and* $b_{i,j} \uparrow b_i$*. Then* $B_n = \Pi_{i=1}^{d}(a_{i,n}, b_{i,n}] \uparrow B$*. So* B *is a countable union of open rectangles, hence* $B \in \sigma(S_d)$*. Since such* B *forms basis of topology on* $\mathbb{R}^d$*, we can conclude that* $\mathcal{R}^d \subset \sigma(S_d)$*.*

Problem (1.2.3)**.**

Let F *be a distribution function. It is nonnegative, nondecreasing. So* $\lim_{y\downarrow x} F(y)$ *and* $\lim_{y\uparrow x} F(y)$ *always exist. Let* x *be a point where* F *is discontinuous. Since* F *is discontinuous at* x*, we can assume without loss of generality* $\lim_{y\downarrow x} F(y) > F(x)$*. Choose a rational number* $q_x \in (F(x), \lim_{y\downarrow x} F(y))$*. Then function* $x \mapsto q_x$ *is injective since* F *is nondecreasing. So there is injection from set of discontinuities to rational numbers. Now we can conclude that set of discontinuities is at most countable.*

Problem (1.3.4)**.**

(a) *Let* $f : \mathbb{R}^d \to \mathbb{R}$ *be a continuous function. Consider* $\mathcal{B} = \left\{U \subset \mathbb{R} : f^{-1}(U) \in \mathcal{R}^d\right\}$*. It is well known that* $\mathcal{B}$ *is a* σ*-field. By continuity of* f*,* $\mathcal{B}$ *contains every open set of* $\mathbb{R}$*, hence* $\mathcal{R} \subset \mathcal{B}$*. Therefore* f *is a measurable function.*

(b) *Let* $\mathcal{F}$ *be a* σ*-field that makes all the continuous functions measurable. Let* $\pi_i : \mathbb{R}^d \to \mathbb{R}$ *be the projection on* i*-th factor, which is continuous. Then* $\cap_{i=1}^{d}\pi_i^{-1}((a_i, b_i)) = \Pi_{i=1}^{d}(a_i, b_i) \in \mathcal{F}$*. Since* $\mathcal{F}$ *contains every open rectangles in* $\mathbb{R}^d$*, we can conclude that* $\mathcal{R}^d \subset \mathcal{F}$*. This means* $\mathcal{R}^d$ *is the smallest such* σ*-field. The fact that* $\mathcal{R}^d$ *makes all the continuous functions measurable is written in (a).*

Problem (1.3.1)**.**

Since $\sigma(X)$ *is the smallest* σ*-field which makes* X *measurable, it sufficient to show that* X *is measurable with respect to* $\sigma\left(X^{-1}(\mathcal{A})\right)$.

Let $X : \Omega \to S$. *It is clear that* $\{X \in A\} \in \sigma\left(X^{-1}(\mathcal{A})\right)$ *for all* $A \in \mathcal{A}$. *But by theorem 1.3.1, since* $\mathcal{A}$ *generates* $\mathcal{S}$, X *is measurable with respect to* $\sigma\left(X^{-1}(\mathcal{A})\right)$.

Therefore we can conlude that $\sigma\left(X^{-1}(\mathcal{A})\right) \subset \sigma(X)$, *and reverse inclusion is canonical since* $X^{-1}(\mathcal{A}) \subset \sigma(X)$.

Problem (1.4.1)**.**

Let $E_n = \left\{x : f(x) > \frac{1}{n}\right\}$. *Then* $\int f d\mu \geq \int_{E_n} f d\mu \geq \int_{E_n} \frac{1}{n} d\mu = \frac{1}{n}\mu(E_n)$. *Therefore* $\mu(E_n) = 0$ *for every positive integer* n. *So,* $\mu(\{f > 0\}) = \sum_{n=1}^{\infty} \mu(E_n) = 0$. *This says* $f = 0$ *a.e.*

Problem (1.4.2)**.** *Since* $E_{n+1,2m} \cup E_{n+1,2m+1} = E_{n,m}$ *and* $\frac{2m+1}{2^{n+1}} \geq \frac{m}{2^n}$, *we can easily see that* $\sum_{m\geq 1} \frac{m}{2^n}\mu(E_{n,m})$ *is monotonically increasing as* n *grows.*

For every positive integer M, $\sum_{m=1}^{M} \frac{m}{2^n}\mu(E_{n,m}) \leq \int f d\mu$. *So* $\sum_{m\geq 1} \frac{m}{2^n}\mu(E_{n,m}) \leq \int f d\mu$.

Let $s_n = \sum_{m=1}^{n2^n} \frac{m}{2^n} 1_{E_{n,m}}$. *Then* $\int s_n d\mu \leq \sum_{m\geq 1} \frac{m}{2^n}\mu(E_{n,m}) \leq \int f d\mu$. *But* $s_n \uparrow f$ *monotonically. By monotone convergence theorem,* $\lim_{n\to\infty} \int s_n d\mu = \int f d\mu$. *Hence by sandwich lemma, the desired result follows.*

Problem (1.5.1)**.**

First, we will show that $|g| \le \|g\|_\infty$ *a.e.*

It is true because

$$\begin{aligned}\mu\left(|g| > \|g\|_\infty\right) &= \mu\left(\bigcup_{n=1}^{\infty}\left\{|g| \ge \|g\|_\infty + \frac{1}{n}\right\}\right)\\ &\le \sum_{n=1}^{\infty}\mu\left(\left\{|g| > \|g\|_\infty + \frac{1}{n}\right\}\right)\\ &= 0\end{aligned}$$

by definition of $\|g\|_\infty$.

Hence $|g| \le \|g\|_\infty$ *a.e.*

Then, $\int |fg| d\mu \le \|g\|_\infty \int |f| d\mu = \|g\|_\infty \|f\|_1$.

Problem (1.5.3)**.**

(a) *Since* $p > 1$, $x \mapsto |x|^p$ *is convex function.* $|f+g|^p \le 2^{p-1}(|f|^p + |g|^p)$ *follows from convexity of* $|x|^p$.

$\int |f+g|^p d\mu \le \int 2^p |f|^p d\mu + \int 2^p |g|^p d\mu$. *Therefore finiteness of* $\|f\|_p$ *and* $\|g\|_p$ *leads* $\|f+g\|_p < \infty$.

Now, consider $\int |f+g|^p d\mu = \int |f+g||f+g|^{p-1} d\mu \le \int |f||f+g|^{p-1} d\mu + \int |g||f+g|^{p-1} d\mu$. *Let* q *be Holder conjugate of* p. *Then by applying Holder inequality, we get* $\|f+g\|_p^p \le \|f+g\|_p^{p/q}(\|f\|_p + \|g\|_p)$. *Simple calculating leads Minkowski's inequality.*

(b) *First consider* $p = 1$. *By using triangle inequaltiy, the result follows directly. Next consider* $p = \infty$. $|f+g| \le |f| + |g| \le \|f\|_\infty + \|g\|_\infty$ *a.e. Therefore* $\|f+g\|_\infty \le \|f\|)\infty + \|g\|_\infty$.

Problem (1.6.8)**.**

First assume $g = 1_A$. *Then* $\int g d\mu = \mu(A) = \int_A f(x) dx = \int 1_A f dm$ *where* m *is Lebesgue measure.*

Next, assume $g = \sum_i a_i 1_{A_i}$, *simple function. Then* $\int g d\mu = \sum_i a_i \mu(A_i) = \sum_i a_i \int 1_{A_i} f dm$.

Next, assume g *is nonnegative measurable. Let* $\{s_n\}_{n=1}^{\infty}$ *be increasing sequence of simple function converges to g pointwisely. Then* $\int g d\mu = \lim_{n\to\infty} \int s_n d\mu =$

$\lim_{n\to\infty} \int s_n f dm$. *But* $s_n f \uparrow gf$ *since* f *is nonnegative. By monotone convergence theorem, we can get* $\int g d\mu = \int gf dm$.

Last, assume g *is integrable function. We can decompose* g *by* $g = g^+ - g^-$. *Applying 3rd step for* g^+, g^- *each, we can get* $\int g d\mu = \int g^+ f dm - \int g^- f dm = \int gf dm$ *since* f *is nonnegative.*

Problem (1.6.13)**.**

Since $X_n \uparrow X$, $X_n^+ \uparrow X^+$ *and* $X_n^- \downarrow X^-$. *And note that* $X_n^- \leq X_1^-$ *which is integrable. Apply monotone convergence theorem to* X_n^+ *and apply dominated convergence theorem to* X_n^- *to get* $\lim EX_n = \lim EX_n^+ - \lim EX_n^- = EX^+ - EX^- = EX$.