mas550 homework

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Problem (1.1.2)**.**

Let $A = \Pi_{i=1}^{d} (a_i, b_i]$*. Then*

$$A = \left(\Pi_{i=1}^{d}[a_i - 1, b_i]\right) \cap \left(\Pi_{I=1}^{d}(a_i, b_i + 1)\right)$$

which is intersection of open set and closed set. So, $A \in \mathcal{R}^d$ *therefore* $\sigma(S_d) \subset \mathcal{R}^d$*.*

On the other hand, let $B = \Pi_{i=1}^{d}(a_i, b_i)$ *where* $-\infty < a_i < b_i < \infty$*. We can choose sequences* $\{a_{i,j}\}_{j=1}^{\infty}$ *and* $\{b_{i,j}\}_{j=1}^{\infty}$ *for each* $1 \leq i \leq d$ *such that* $a_{i,j} \downarrow a_i$ *and* $b_{i,j} \uparrow b_i$*. Then* $B_n = \Pi_{i=1}^{d}(a_{i,n}, b_{i,n}] \uparrow B$*. So* B *is a countable union of open rectangles, hence* $B \in \sigma(S_d)$*. Since such* B *forms basis of topology on* $\mathbb{R}^d$*, we can conclude that* $\mathcal{R}^d \subset \sigma(S_d)$*.*

Problem (1.2.3)**.**

Let F *be a distribution function. It is nonnegative, nondecreasing. So* $\lim_{y\downarrow x} F(y)$ *and* $\lim_{y\uparrow x} F(y)$ *always exist. Let* x *be a point where* F *is discontinuous. Since* F *is discontinuous at* x*, we can assume without loss of generality* $\lim_{y\downarrow x} F(y) > F(x)$*. Choose a rational number* $q_x \in (F(x), \lim_{y\downarrow x} F(y))$*. Then function* $x \mapsto q_x$ *is injective since* F *is nondecreasing. So there is injection from set of discontinuities to rational numbers. Now we can conclude that set of discontinuities is at most countable.*

Problem (1.3.4)**.**

(a) *Let* $f : \mathbb{R}^d \to \mathbb{R}$ *be a continuous function. Consider* $\mathcal{B} = \left\{U \subset \mathbb{R} : f^{-1}(U) \in \mathcal{R}^d\right\}$*. It is well known that* $\mathcal{B}$ *is a* σ*-field. By continuity of* f*,* $\mathcal{B}$ *contains every open set of* $\mathbb{R}$*, hence* $\mathcal{R} \subset \mathcal{B}$*. Therefore* f *is a measurable function.*

(b) *Let* $\mathcal{F}$ *be a* σ*-field that makes all the continuous functions measurable. Let* $\pi_i : \mathbb{R}^d \to \mathbb{R}$ *be the projection on* i*-th factor, which is continuous. Then* $\cap_{i=1}^{d}\pi_i^{-1}((a_i, b_i)) = \Pi_{i=1}^{d}(a_i, b_i) \in \mathcal{F}$*. Since* $\mathcal{F}$ *contains every open rectangles in* $\mathbb{R}^d$*, we can conclude that* $\mathcal{R}^d \subset \mathcal{F}$*. This means* $\mathcal{R}^d$ *is the smallest such* σ*-field. The fact that* $\mathcal{R}^d$ *makes all the continuous functions measurable is written in (a).*

Problem (1.3.1)**.**

Since $\sigma(X)$ is the smallest σ-field which makes X measurable, it sufficient to show that X is measurable with respect to $\sigma\left(X^{-1}(\mathcal{A})\right)$.

Let $X : \Omega \to S$. It is clear that $\{X \in A\} \in \sigma\left(X^{-1}(\mathcal{A})\right)$ for all $A \in \mathcal{A}$. But by theorem 1.3.1, since $\mathcal{A}$ generates $\mathcal{S}$, X is measurable with respect to $\sigma\left(X^{-1}(\mathcal{A})\right)$.

Therefore we can conlude that $\sigma\left(X^{-1}(\mathcal{A})\right) \subset \sigma(X)$, and reverse inclusion is canonical since $X^{-1}(\mathcal{A}) \subset \sigma(X)$.

Problem (1.4.1)**.**

Let $E_n = \left\{x : f(x) > \frac{1}{n}\right\}$. Then $\int f d\mu \geq \int_{E_n} f d\mu \geq \int_{E_n} \frac{1}{n} d\mu = \frac{1}{n}\mu(E_n)$. Therefore $\mu(E_n) = 0$ for every positive integer n. So, $\mu(\{f > 0\}) = \sum_{n=1}^{\infty} \mu(E_n) = 0$. This says $f = 0$ a.e.

Problem (1.4.2)**.** *Since $E_{n+1,2m} \cup E_{n+1,2m+1} = E_{n,m}$ and $\frac{2m+1}{2^{n+1}} \geq \frac{m}{2^n}$, we can easily see that $\sum_{m\geq 1} \frac{m}{2^n}\mu(E_{n,m})$ is monotonically increasing as n grows.*

For every positive integer M, $\sum_{m=1}^{M} \frac{m}{2^n}\mu(E_{n,m}) \leq \int f d\mu$. So $\sum_{m\geq 1} \frac{m}{2^n}\mu(E_{n,m}) \leq \int f d\mu$.

Let $s_n = \sum_{m=1}^{n2^n} \frac{m}{2^n} 1_{E_{n,m}}$. Then $\int s_n d\mu \leq \sum_{m\geq 1} \frac{m}{2^n}\mu(E_{n,m}) \leq \int f d\mu$. But $s_n \uparrow f$ monotonically. By monotone convergence theorem, $\lim_{n\to\infty} \int s_n d\mu = \int f d\mu$. Hence by sandwich lemma, the desired result follows.

Problem (1.5.1)**.**

First, we will show that $|g| \le \|g\|_\infty$ *a.e.*

It is true because

$$\begin{aligned}\mu\left(|g| > \|g\|_\infty\right) &= \mu\left(\bigcup_{n=1}^{\infty}\left\{|g| \ge \|g\|_\infty + \frac{1}{n}\right\}\right)\\ &\le \sum_{n=1}^{\infty}\mu\left(\left\{|g| > \|g\|_\infty + \frac{1}{n}\right\}\right)\\ &= 0\end{aligned}$$

by definition of $\|g\|_\infty$.

Hence $|g| \le \|g\|_\infty$ *a.e.*

Then, $\int |fg| d\mu \le \|g\|_\infty \int |f| d\mu = \|g\|_\infty \|f\|_1$.

Problem (1.5.3)**.**

(a) *Since* $p > 1$, $x \mapsto |x|^p$ *is convex function.* $|f+g|^p \le 2^{p-1}(|f|^p + |g|^p)$ *follows from convexity of* $|x|^p$.

$\int |f+g|^p d\mu \le \int 2^p |f|^p d\mu + \int 2^p |g|^p d\mu$. *Therefore finiteness of* $\|f\|_p$ *and* $\|g\|_p$ *leads* $\|f+g\|_p < \infty$.

Now, consider $\int |f+g|^p d\mu = \int |f+g||f+g|^{p-1} d\mu \le \int |f||f+g|^{p-1} d\mu + \int |g||f+g|^{p-1} d\mu$. *Let* q *be Holder conjugate of* p. *Then by applying Holder inequality, we get* $\|f+g\|_p^p \le \|f+g\|_p^{p/q}(\|f\|_p + \|g\|_p)$. *Simple calculating leads Minkowski's inequality.*

(b) *First consider* $p = 1$. *By using triangle inequaltiy, the result follows directly. Next consider* $p = \infty$. $|f+g| \le |f| + |g| \le \|f\|_\infty + \|g\|_\infty$ *a.e. Therefore* $\|f+g\|_\infty \le \|f\|)\infty + \|g\|_\infty$.

Problem (1.6.8)**.**

First assume $g = 1_A$. *Then* $\int g d\mu = \mu(A) = \int_A f(x) dx = \int 1_A f dm$ *where* m *is Lebesgue measure.*

Next, assume $g = \sum_i a_i 1_{A_i}$, *simple function. Then* $\int g d\mu = \sum_i a_i \mu(A_i) = \sum_i a_i \int 1_{A_i} f dm$.

Next, assume g *is nonnegative measurable. Let* $\{s_n\}_{n=1}^{\infty}$ *be increasing sequence of simple function converges to* g *pointwisely. Then* $\int g d\mu = \lim_{n\to\infty} \int s_n d\mu =$

$\lim_{n\to\infty}\int s_n f dm$. But $s_n f \uparrow gf$ since f is nonnegative. By monotone convergence theorem, we can get $\int g d\mu = \int gf dm$.

Last, assume g is integrable function. We can decompose g by $g = g^+ - g^-$. Applying 3rd step for g^+, g^- each, we can get $\int g d\mu = \int g^+ f dm - \int g^- f dm = \int gf dm$ since f is nonnegative.

Problem (1.6.13)**.**

Since $X_n \uparrow X$, $X_n^+ \uparrow X^+$ and $X_n^- \downarrow X^-$. And note that $X_n^- \leq X_1^-$ which is integrable. Apply monotone convergence theorem to X_n^+ and apply dominated convergence theorem to X_n^- to get $\lim EX_n = \lim EX_n^+ - \lim EX_n^- = EX^+ - EX^- = EX$.

Problem (1.7.1)**.**

We need to show that $\int_{X\times Y}|f|d(\mu_1\times\mu_2)<\infty$.

Since $|f|^{\pm}$ *is nonnegative, by Fubini's theorem,* $\int_X\int_Y|f|^{\pm}\mu_2(dy)\mu_1(dx)<\infty$*. Then, their sum is also finite, and the sum is* $\int_{X\times Y}|f|d(\mu_1\times\mu_2)$ *by Fubini's theorem. This leads the conclusion of the exercise.*

Corollary is immediate if we take $\mu_1=c$ *and* $\mu_2=\mu$.

Problem (1.7.3)**.**

1.
$$\begin{aligned}\int_{(a,b]}\{F(y)-F(a)\}\,dG(y)&=\int_{(a,b]}\int_{(a,y]}1\mu(dx)\nu(dy)\\&=\int_{a<x\leq y\leq b}1d(\mu\times\nu)\\&=\mu\times\nu(1<X\leq Y\leq b)\end{aligned}$$
by Fubini's theorem on nonnegative function 1.

2.
$$\begin{aligned}\int_{(a,b]}F(y)dG(y)&=\int_{(a,b]}\int_{-\infty}^{y}1\mu(dx)\nu(dy)\\&=\int_{(-\infty,a]}\int_{(a,b]}1\nu(dy)\mu(dx)+\int_{(a,b]}\int_{[x,b]}\nu(dy)\mu(dx)\\&=F(a)\{G(b)-G(a)\}+G(b)\{F(b)-F(a)\}\\&-\int_{(a,b]}G(x)\mu(dx)+\int_{(a,b]}G(x)-G(x^-)\mu(dx)\end{aligned}$$
We can get similar result for $\int_{(a,b]}G(y)dF(y)$*. By simple calculation, we get the conclusion of (2).*

3. *If* $F=G$ *continuous, Then* $\mu(\{x\})=\nu(\{x\})=F(x)-F(x^-)=G(x)-G(x^-)=0)$*. Therefore, by using (2), we can get the conclusion.*

Problem (2.1.3)**.**

1. *If* $h(\alpha)=0$ *for some* $\alpha>0$*, by mean value theorem,* $h'(\beta)=0$ *for some* $\beta\in(0,\alpha)$*. It contradicts to* $h'(x)>0$ *for positive* x*. Therefore* $h>0$ *for positive* x*.*

$x = y$ *iff* $\rho(x, y) = 0$ *iff* $h(\rho(x, y)) = 0$. *And* $h(\rho(x, y)) = h(\rho(y, x))$ *since* $\rho(x, y) = \rho(y, x)$.

Now consider $x \geq y > 0$ *and* $\frac{h(x+y)-h(x)}{y} = h'(x+\theta)$ *and* $\frac{h(y)}{y} = h'(y-\delta)$. *Since* h' *is decreasing,* $h(x + y) - h(x) \leq h(y)$. *Using this, we can prove triangle inequality of* $h \circ \rho$.

2. $h(x) = 1 - \frac{1}{1+x}$ *so* $h'(x) = \frac{1}{(1+x)^2}$ *and* $h''(x) = \frac{-2}{(1+x)^3}$. *Given* h *satisfies all of (1).*

Problem (2.1.9)**.**

Let $\mathcal{A}_1 = \{\{1, 2\}, \{1, 3\}\}, \mathcal{A}_2 = \{\{1, 4\}\}$. *For* $A_1 \in \mathcal{A}_1$ *and* $A_2 \in \mathcal{A}_2$, $P(A_1 \cap A_2) = P(A_1)P(A_2) = 1/4$. *But,* $\sigma(\mathcal{A}_1) = 2^\Omega$ *and* $\sigma(\mathcal{A}_2) = \{\Omega, \{1, 4\}, \{2, 3\}, \emptyset\}$. *They are not independent by considering* $A_1 = \{2, 3, 4\}$ *and* $A_2 = \{2, 3\}$.

Problem (2.2.3)**.**

(a) $f(U_i)$*'s are iid because* $P\left(\bigcap_i (f\circ U_i)\in B_i\right) = P\left(\bigcap_i \left\{U_i \in f^{-1}(B_i)\right\}\right) = \Pi_i P\left(U_i\in f^{-1}(B_i)\right) = \Pi_i P\left(f(U_i)\in B_i\right)$*. Also, for borel set* B*,* $P\left(f(U_i)\in B\right) = P\left(U_i\in f^{-1}(B_i)\right)$ *are all same for* i*.*

$Ef(U_i) = \int_0^1 f(x)dx$*,* $E|f(U_i)| = \int_0^1 |f(x)|dx < \infty$*.*

Now, by WLLN, $\frac{\sum f(U_i)}{n}$ *converges to* $\int_0^1 f(x)dx$ *in probability.*

(b) $P\left(|I_n - I| > a/n^{0.5}\right) \le \frac{n}{a^2}E|I_n - I|^2 = \frac{n}{a^2}Var(I_n) = Var(\sum f(U_i))/na^2 = Var(f(U_i))/a^2 = \left[\int_0^1 f(x)^2dx - \left(\int_0^1 f(x)dx\right)^2\right]/a^2$*.*

Problem (2.2.5)**.**

Note that $P(X_i \le a) = 0$ *for all* $a < e$*.*

$xP(X_i > x) = \frac{e}{\log x} \to 0$ *as* $x \to \infty$*.*

$E|X_i| = EX_i = \int_e^\infty P(X_i > x)dx = \int_e^\infty \frac{e}{x\log x}dx = \infty$ *since* $X_i \ge 0$ *almost surely.*

But $\mu_n = \int_{|X_i|\le n} X_i dP \uparrow EX_i = \infty$ *by monotone convergence theorem.*

Now, theorem 2.2.12 says $\frac{s_n}{n} - \mu_n$ *converges to* 0 *in probability.*

Problem (2.3.5)**.**

(a) *Let* $F_N = \{Y \le N\}$ *and* $Y_n = Y1_{F_n}$. *Then* $EY_n \uparrow EY$ *by MCT. So choose* N *so that* $EY - EY_N < \varepsilon$. *Now consider* $|EX_n - EX| \le E|X_n - X| \le \int_{|X_n - X|>\varepsilon} 2Y dP + \int_{|X_n-X|\le\varepsilon} |X_n - X| dP \le \varepsilon + \int_{|X_n-X|>\varepsilon} 2Y dP$.

Let $E_n = \{|X_n - X| > \varepsilon\}$. *Then* $\int_{E_n} 2Y dP = \int_{E_n} 2Y - 2Y_N + 2Y_N dP \le E(2Y - 2Y_N) + 2NP(E_n)$, *where the last term goes to* 0 *as* $n \to \infty$.

(b) *Let* h, g *be continuous functions,* $h(0) = 0$, $g > 0$ *for large* x, $|h|/g \to 0$ *as* $|x| \to \infty$, *and* $Eg(X_n) \le C < \infty$.

Choose M *so large that* $g > 0$ *on* $|x| > M$. $\varepsilon_M = \sup_{|x|\ge M} |h|/g$ *and* $\bar{Y} = Y1_{|Y|\le M}$.

Then $|Eh(X_n) - Eh(X)| \le E|h(X_n) - Eh(\bar{X}_n)| + E|h(\bar{X}_n - h(\bar{X})| + E|h(\bar{X}) - h(X)|$. *First term and third term are bounded by* $\varepsilon_M C$ *which goes to* 0 *as* $M \to \infty$. *And the second term goes to* 0 *as* $n \to \infty$ *by bounded convergence thm.*

Therefore the conclusions hold.

Problem (2.3.6.)**.**

(a) *We already show that* $\rho(x, y) = \frac{|x-y|}{1+|x-y|}$ *is a metric in problem 2.1.3.*

First consider $d(X, Y) = 0$ *iff* $E\frac{|X-Y|}{1+|X-Y|} = 0$ *iff* $\frac{|X-Y|}{1+|X-Y|} = 0$ *a.s. iff* $X = Y$ *a.s.*

Next, it is trivial to check $d(X, Y) = d(Y, X)$.

Lastly, $d(X, Z) = E\rho(X, Z) \le E(\rho(X, Y) + \rho(Y, Z)) = E\rho(X, Y) + E\rho(Y, Z) = d(X, Y) + d(Y, Z)$.

Therefore given function is a metric of class of random variables.

(b) *First assume* $X_n \to X$ *in probability. Then* $\frac{|X_n-X|}{1+|X_n-X|} \le 1$ *and it goes to* 0 *in probability. So bounded convergence thm implies* $d(X_n, X) \to 0$.

Next assume $d(X_n, X) \to 0$ *as* $n \to 0$.

$$
\begin{aligned}
P(|X_n - X| > \varepsilon) &= P\left(\frac{|X_n - X|}{1+|X_n - X|} > \frac{\varepsilon}{1+\varepsilon}\right) \\
&\le E\frac{|X_n - X|}{1+|X_n - X|}\frac{1+\varepsilon}{\varepsilon} \\
&= d(X_n, X)\frac{1+\varepsilon}{\varepsilon} \to 0
\end{aligned}
$$

by Markov's inequality.

Problem (2.3.8)**.**

Independence of A_n implies independence of A_n^c. Let $B_n = \cap_{k=1}^n A_k^c$. Then $0 = P\left(\bigcap_{n=1}^\infty A_n^c\right) = \lim_{n\to\infty} P(B_n)$.

So, for arbitrary $\varepsilon > 0$, there is a positive integer N_ε such that $n \ge N_\varepsilon$ implies $P(B_n) = P\left(\cap_{k=1}^n A_k^c\right) = \Pi_{k=1}^n \left(1 - P(A_k)\right) = e^{\sum_{k=1}^n \log(1-P(A_k))} < \varepsilon$. But as $n \to \infty$

$$\lim_{n\to\infty} e^{\sum_{k=1}^n \log(1-P(A_k))} = 0$$

This means that $\sum_{k=1}^\infty \log\left(1 - P(A_k)\right) = -\infty$, therefore $\log\left(1 - P(A_k)\right)$ does not converge to 0, which is equivalent to that $P(A_k)$ does not converge to 0. Therefore $\sum_{n=1}^\infty P(A_n) = \infty$.

Problem (2.3.12)**.**

Let $\Omega = \{\omega_i : i \in \mathbb{N}\}$. Without loss of generality, we can assume $P(\{\omega_i\}) > 0$ for all $i \in \mathbb{N}$.

If there is ω_i such that $X_n(\omega_i)$ does not converge to $X(\omega_i)$, then for some $\varepsilon > 0$, and for all $N \in \mathbb{N}$, there is $n_N \ge N$ but $|X_{n_N}(\omega_i) - X(\omega_i)| > \varepsilon$.

This means $\{|X_{n_N} - X| > \varepsilon\}$ contains ω_i for all N. So $0 < P(\{\omega_i\}) \le P\left(|X_{n_N} - X| > \varepsilon\right)$.

But $X_n \to X$ in probability implies $X_{n_N} \to X$ in probability. This contradicts to above. Therefore there is no such w_i hence X_n converges to X almost surely.

Problem (2.5.2)**.**

If $E|X_1|^p = \infty$*, then for each positive integer* k*,* $E|X_1|^p \leq \sum_n P(|X_1|^p > nk) = \infty$*. But* $P(|X_1|^p > nk) = P(|X_n| > (nk)^{1/p})$*. Then by Borel Cantelli lemma* $P(|X_n| > (nk)^{1/p} i.o.) = 1$*. That is,* $\limsup_n |X_n|/n^{1/p} \geq k^{1/p}$ *for infinitely many* k*. Therefore* $\limsup_n |X_n|/n^{1/p} = \infty$*.*

But $|X_n| \leq |S_n| + |S_{n-1}|$*. That leads* $\limsup_n |S_n|/n^{1/p} = \infty$*. By taking contrapositive, we get the conclusion.*

Problem (2.5.5)**.**

The first one leads the second one directly because Kolmogorov's three series lemma with $A = 1$ *tells it.*

The second one implies the third one because $\frac{X_n}{1+X_n} \leq 1_{X_n>1} + X_n 1_{X_n \leq 1}$ *and monotone convergence theorem.*

The third one implies $\sum_n \frac{X_n}{1+X_n} < \infty$ *a.s. And convergence of* $\sum_n \frac{a_n}{1+a_n}$ *for* $a_n \geq 0$ *gives the convergence of* $\sum_n a_n$*. It is because* $lim a_n = 0$ *and* $|a_N + \cdots a_{N+n}| \leq (1+\varepsilon) \left| \frac{a_N}{1+a_N} + \cdots + \frac{a_{N+n}}{1+a_{N+n}} \right|$ *for large* N*. Therefore* $\sum_{k=1}^n a_k$ *is cauchy hence converges. Therefore* $\sum_n X_n$ *converges a.s.*

Problem (3.2.4)**.**

Since $X_n \to X_\infty$ *in distribution, there are* $Y_n =_d X_n$ *and* $Y_\infty =_d X_\infty$ *such that* $Y_n \to Y_\infty$ *a.s.*

Then $g(Y_n) \geq 0$ *and* $g(Y_n) \to g(Y_\infty)$ *a.s. Therefore by Fatou's lemma,* $\liminf Eg(Y_n) \geq Eg(Y_\infty)$ *which is equivalent to* $\liminf Eg(X_n) \geq Eg(X_\infty)$ *since* $X_n =_d Y_n$ *for all* $n \in \mathbb{N} \cup \infty$*.*

Problem (3.2.5)**.**

There are $Y_n \to Y_\infty$ *a.s. and distribution function of* Y_n *is equal to* F_n*.* $F_\infty = F$*.*

Then by theorem 1.6.8, $Eh(Y_n) \to Eh(Y_\infty)$ *which is equivalent to* $\int h(x)dF_n(x) \to \int h(x)dF(x)$ *because distribution function of* Y_n *is* F_n*.*

Problem (4.1.7)**.**

By definition of $\mathrm{Var}(X|\mathcal{F})$*, we get the following:*

$$E\left(\mathrm{Var}(X|\mathcal{F})\right) = EX^2 - E\left(E(X|\mathcal{F})^2\right)$$

And clearly,

$$\mathrm{Var}\left(E(X|\mathcal{F})\right) = E\left(E(X|\mathcal{F})^2\right) - \left(E(E(X|\mathcal{F}))\right)^2$$

Therefore, by summing them vertically, we can get

$$\mathrm{Var}\left(E(X|\mathcal{F})\right) + E\left(\mathrm{Var}(X|\mathcal{F}\right) = EX^2 - \left(E(E(X|\mathcal{F}))\right)^2$$

which is equal to $\mathrm{Var}(X)$ *since the last term is equal to square of* EX*.* □

Problem (4.1.9)**.**

$$\begin{aligned}\int |X-Y|^2 dP &= \int X^2 - 2XY + Y^2 dP \\ &= \int X^2 - 2E(XY|\mathcal{G}) + Y^2 dP \\ &= \int X^2 - 2XE(Y|\mathcal{G}) + Y^2 dP \\ &= \int X^2 - 2X^2 + Y^2 dP \\ &= EY^2 - EX^2 \\ &= 0\end{aligned}$$

Therefore, $|X-Y|^2 = 0$ *a.s. which implies* $X = Y$ *a.s. Note that* XY *is integrable by Holder's inequality for* $p = q = 2$ *and finite second moment of* X, Y*.* □

Problem (4.2.3)**.**

Clearly $\mathcal{F}_m \subset \mathcal{F}_{m+1}$ *for all positive integer* m*. Let* $Z_n = X_n \vee Y_n$*, then* Z_n *is clearly* $\mathcal{F}_n$ *measurable.*

Now, let $A \in \mathcal{F}_{n-1}$. Then,

$$\begin{aligned}
\int_A E(Z_n|\mathcal{F}_{n-1})dP &= \int_A Z_n dP \\
&\geq \int_A X_n dP \vee \int_A Y_n dP \\
&= \int_A E(X_n|\mathcal{F}_{n-1})dP \vee \int_A E(Y_n|\mathcal{F}_{n-1})dP \\
&\geq \int_A X_{n-1} dP \vee \int_A Y_{n-1} dP
\end{aligned}$$

Therefore $\int_A E(Z_n|\mathcal{F}_{n-1})dP \geq \int_A X_{n-1}, Y_{n-1} dP$ for all $A \in \mathcal{F}_{n-1}$. Since $E(Z_n|\mathcal{F}_{n-1})$ is $\mathcal{F}_{n-1}$ measurable, we can conclude that conditional expectation of Z_n with respect to $\mathcal{F}_{n-1}$ is equal or greater than X_{n-1} and Y_{n-1} a.s.

So, Z_n is a submartingale. $\square$

Problem (4.2.9)**.**

Note that $\{N > n\} = \{N \leq n\}^c \in \mathcal{F}_n$ and $\{N < n\} = \{N \leq n-1\} \in \mathcal{F}_{n-1}$ since N is integer valued. Now, consider the following:

$$\begin{aligned}
E(Z_n|\mathcal{F}_{n-1}) &= 1_{N \geq n} E(X_n^1|\mathcal{F}_{n-1}) + 1_{N<n} E(X_n^2|\mathcal{F}_{n-1}) \\
&\leq 1_{N \geq n} X_{n-1}^1 + 1_{N<n} X_{n-1}^2 \\
&= 1_{N > n-1} X_{n-1}^1 + 1_{N \leq n-1} X_{n-1}^2 \\
&\leq 1_{N \geq n-1} X_{n-1}^1 + 1_{N < n-1} X_{n-1}^2 \\
&= Z_{n-1}
\end{aligned}$$

So, Z_n is supermartingale.

Now, consider the Y_n:

First, $Y_n = X_n^1 1_{N>n} + X_N^2 1_{N=n} + X_n^2 1_{N<n} \leq X_n^1 1_{N \geq n} + X_n^2 1_{N<n}$.

$$\begin{aligned}
E(Y_n|\mathcal{F}_{n-1}) &\leq 1_{N \geq n} E\left(X_n^1|\mathcal{F}_{n-1}\right) + !_{N<n} E\left(X_n^2|\mathcal{F}_{n-1}\right) \\
&\leq 1_{N \geq n} X_{n-1}^1 + 1_{N<n} X_{n-1}^2 \\
&= 1_{N > n-1} X_{n-1}^1 + 1_{N \leq n-1} X_{n-1}^2 \\
&= Y_{n-1}
\end{aligned}$$

So, Y_n is also a supermartingale. $\square$

Problem (4.2.8).

Let $\nu = \inf\left\{k : \Pi_{m=1}^{k}(1+Y_m) > M\right\}$ *for* $M > 0$*. Let* $U_n = MX_n\Pi_{m=1}^{n-1}(1+Y_m)^{-1}$*. Clealry* ν *is a stopping time. Now we claim that* $U_{n\wedge\nu}$ *is positive supermartingale.*

$$
\begin{aligned}
E\left(U_{n+1\wedge\nu}|\mathcal{F}_n\right) &= E\left(U_\nu 1_{\{n+1>\nu\}} + U_{n+1}1_{\{n+1\le\nu\}}|\mathcal{F}_n\right)\\
&\le U_\nu 1_{\{\nu\le n\}} + 1_{\{n+1\le\nu\}}M\Pi_{m=1}^{n}(1+Y_m)^{-1}X_n(1+Y_n)\\
&= U_\nu 1_{\{\nu\le n\}} + 1_{\{n<\nu\}}M\Pi_{m=1}^{n-1}(1+Y_m)^{-1}X_n\\
&= U_{\nu\wedge n}
\end{aligned}
$$

Above manipulation is possible since $\{n+1\le\nu\} = \{\nu\le n\}^c$*. Thus* $U_{n\wedge\nu}$ *is a positive supermartingale, so it converges almost surely.*

Note that $\sum Y_n < \infty$ *implies* $\Pi(1+Y_n) < \infty$ *by considering* $1+x \le \exp(x)$ *and its partial product. Now fix* w *so that* $U_{\nu\wedge n}(w)$ *and* $\Pi(1+Y_n(w))$ *are convergent. Choose* $M > \Pi(1+Y_n)$*. Then* $\nu = \infty$*, so* $U_{\nu\wedge n} = U_n$*. But we know that* $U_{\nu\wedge n}(w)$ *converges, say to* K*. Then for that* w*,* $X_n(w) \to K(w)\Pi(1+Y_n(w))/M$*. Thus we can say that* X_n *converges almost surely.*

□

Problem (4.3.3).

It is very similar to #4.2.8.

Let $\nu = \inf\left\{k : \sum_{m=1}^{k} Y_m > M\right\}$ *for* $M > 0$*. Clearly,* ν *is a stopping time. Let* $U_n = X_n - \sum_{m<n} Y_m + M$*. Then clearly,* $U_{n\wedge\nu}$ *is nonnegative random variables. Now we claim that* $U_{n\wedge\nu}$ *is a supermartingale.*

$$
\begin{aligned}
E\left(U_{n+1\wedge\nu}|\mathcal{F}_n\right) &= E\left(U_\nu 1_{\{\nu<n+1\}} + U_{n+1}1_{\{n+1\le\nu\}}|\mathcal{F}_n\right)\\
&\le U_\nu 1_{\{\nu<n+1\}} + 1_{\{n+1\le\nu\}}\left(X_n + Y_n - \sum_{m<n+1} Y_m + M\right)\\
&= U_\nu 1_{\{\nu\le n\}} + U_n 1_{\{n<\nu\}}\\
&= U_{\nu\wedge n}
\end{aligned}
$$

Above is possible since $\{\nu\ge n+1\} = \{\nu\le n\}^c \in \mathcal{F}_n$*. Thus* $U_{n\wedge\nu}$ *is a positive supermartingale, so it converges almost surely.*

Now, fix w so that $U_{n\wedge\nu}(w), \sum Y_n(w)$ both are convergent. Choose $M > \sum Y_n(w)$. Then $\nu = \infty$ so $U_{n\wedge\nu} = U_n$. Then we can say that $U_n(w) \to K(w)$, so $X_n(w) \to K(w) - M + \sum Y_n(w)$. Thus X_n converges almost surely.

$\square$

Problem (4.3.4)**.**

Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of independent random variables such that $P(Y_n = 1) = p_n$. Also let $P(Y_n = 0) = 1 - p_n$. Since Y_n are indepdendent, by Borel Canteli lemma (1st and 2nd both) implies that

$$\sum_{n\geq 1} p_n = \sum_{n\geq 1} P(Y_n = 1) = \infty \Leftrightarrow P(Y_n = 1 i.o.) = 1$$

Note that $\cap_{n=N}^{N+k} \{Y_n = 0\} \downarrow \cap_{n=N}^{\infty} \{Y_n = 0\}$. So $\Pi_{n=N}^{N+k}(1-p_n) \to \Pi_{n=N}^{\infty}(1-p_n)$ as $k \to \infty$.

Since $P(Y_n = 1 i.o.) = P(\cap_{N=1}^{\infty} \cup_{n\geq N} \{Y_n = 1\}) = 1$, we can get the following:

$$\begin{aligned}
P(\bigcap_{N\geq 1} \bigcup_{n\geq N} \{Y_n = 0\}) = 0 &= \lim_{N\to\infty} P\left(\bigcap_{n\geq N} \{Y_n = 0\}\right) \\
&= \lim_{N\to\infty} \lim_{k\to\infty} P\left(\bigcap_{n=N}^{N+k} \{Y_n = 0\}\right) \\
&= \lim_{N\to\infty} \lim_{k\to\infty} \Pi_{n=N}^{N+k}(1-p_n) \\
&= \lim_{N\to\infty} \Pi_{n\geq N}(1-p_n)
\end{aligned}$$

But, $\Pi_{n\geq N}(1-p_n) \leq \Pi_{n\geq M}(1-p_n)$ where $M \geq N$ since $1-p_n \leq 1$. Therefore we can see that $\Pi_{n\geq N}(1-p_n) \leq \lim_{N\to\infty} \Pi_{n\geq N}(1-p_n) = 0$ by above. So, $\Pi_{n\geq N}(1-p_n) = 0$ for all positive integer N.

For the other direction, suppose $\Pi_{n\geq 1}(1-p_n) = 0$. Then its partial product must converge to zero. It means that $\Pi_{n\geq N}(1-p_n) = 0$ for every N. Then $\lim_N P(\cap_{n\geq N} \{Y_n = 0\}) = 0$. So $P(Y_n = 1 i.o.) = 1$ which implies the result.

$\square$

Problem (4.4.7)**.**

$\lambda > 0$. For $c > 0$,

$$\begin{aligned} P(\max_{1\le m\le n} X_m \ge \lambda) &\le P\left(\max_{1\le m\le n}(X_m+c)^2 \ge (\lambda+c)^2\right) \\ &\le \frac{E(X_n+c)^2}{(\lambda+c)^2} \end{aligned}$$

since $(X_m+c)^2$ is a submartingale, and by the Doob's inequality.

Note that $E(X_n+c)^2 = EX_n^2 + c^2$ since $EX_n = EX_0 = 0$. The minimum of the last term(with respect to c) occurs when $c = EX_n^2/\lambda$ by differentiating it. And, its minimum value is $EX_n^2/(EX_n^2+\lambda^2)$.

□

Problem (4.4.9)**.**

Consider the following:

$$\begin{aligned} E\left(X_m - X_{m-1}\right)\left(Y_m - Y_{m-1}\right) &= EX_mY_m - EX_mY_{m-1} - EX_{m-1}Y_m + EX_{m-1}Y_{m-1} \\ &= EX_mY_m - E\left(E(X_mY_{m-1}|\mathcal{F}_{m-1})\right) \\ &\quad - E\left(E(X_{m-1}Y_m|\mathcal{F}_{m-1})\right) + EX_{m-1}Y_{m-1} \\ &= EX_mY_m - E\left(Y_{m-1}E(X_m|\mathcal{F}_{m-1})\right) \\ &\quad - E\left(X_{m-1}E(Y_m|\mathcal{F}_{m-1})\right) + EX_{m-1}Y_{m-1} \\ &= EX_mY_m - EX_{m-1}Y_{m-1} \end{aligned}$$

So the result follows directly. Note that X_mY_m is integrable due to the Cauchy-Schwartz inequality.

□

Problem (4.4.10)**.**

By the problem 4.4.9, $EX_n^2 = EX_0^2 + \sum_{m=1}^n E\xi_m^2 \le EX_0^2 + \sum_{m=1}^\infty E\xi_m^2 < \infty$. So $\sup_n E|X_n|^2 \le EX_0^2 + \sum_{m=1}^\infty E\xi_m^2 < \infty$. Therefore, the L_2 martingale convergence theorem implies the result.

□

Problem (4.6.1)**.**

Let $\mathcal{F}_n = \sigma(Y_1, \cdots Y_n)$*. Then* $E(\theta|Y_1, \cdots Y_n) = E(\theta|\mathcal{F}_n)$*. By theorem 4.6.8,* $E(\theta|\mathcal{F}_n) \to E(\theta|\mathcal{F}_\infty)$ *a.s. and in* L_1*. Now, it remains to show that* $E(\theta|\mathcal{F}_\infty) = \theta$*.*

Conditioning on θ*, we can get the followings:*

$$E(Y_i|\theta) = E(Z_i + \theta|\theta) = \theta + E(Z_i|\theta) = \theta$$

$$E(|Y_i - \theta|^2|\theta) = E(Z_i^2|\theta) = E(Z_i^2) = 0$$

by independence of Z_i *and* θ*.*

Define $X_n = \sum_{i=1}^n Y_i/n$*. Clearly,* X_n *are* $\mathcal{F}_\infty$ *measurable. By the above observations, we can easily check that* $E(|X_n - \theta|^2|\theta) = 1/n$*. So, by integrating both sides,* $E|X_n - \theta|^2 = 1/n$*. Therefore* $X_n \to \theta$ *in* L_2*. By the fact that* L_2 *convergent sequence has a almost sure convergent subsequence, we can say that* $X_{n_k} \to \theta$*. But each* X_{n_k} *is* $\mathcal{F}_\infty$ *measurable, we can say that* θ *is* $\mathcal{F}_\infty$ *measurable.*

Thus, $E(\theta|\mathcal{F}_\infty) = \theta$*.*

□

Problem (4.6.7)**.**

By triangle inequality,

$$|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)| \leq |E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_n)| + |E(Y|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)|$$

Write the above as $S_1 \leq S_2 + S_3$*. Then clealry* $ES_3 \to 0$ *as* $n \to \infty$ *by theorem 4.6.8. To estimate* ES_2*,*

$$\begin{aligned} ES_2 &\leq E\left(E\left(|Y_n - Y||\mathcal{F}_n\right)\right) \\ &= E|Y_n - Y| \end{aligned}$$

by Jensen's inequality. Therefore, L_1 *convergence of* Y_n *implies* $ES_2 \to 0$ *as* $n \to \infty$*.*

□

Problem (4.7.2).

Let $w_N = \sup\{|Y_n - Y_m| : m, n \le N\}$. Then clearly $|Y_n - Y_{-\infty}| \le w_N$ for $n \le N$. Since $Y_n \to Y_{-\infty}$ a.s. as $n \to -\infty$, $w_N \to 0$ as $N \to -\infty$ almost surely. Note that $w_N \le 2Z \in L_1$. Thus $w_N \in L_1$. So, $\mathbb{E}(w_N|\mathcal{F}_{-\infty}) \to 0$ as $N \to -\infty$.

Let

$$\begin{aligned}
&|E(Y_n|\mathcal{F}_n) - E(Y_{-\infty}|\mathcal{F}_{-\infty})| \\
&\le |E(Y_n|\mathcal{F}_n) - E(Y_{-\infty}|\mathcal{F}_n)| + |E(Y_{-\infty}|\mathcal{F}_n) - E(Y_{-\infty}|\mathcal{F}_{-\infty})| \\
&= S_1 + S_2
\end{aligned}$$

Now, consider the following:

$$\begin{aligned}
\limsup_{n\to-\infty} |E(Y_n|\mathcal{F}_n) - E(Y_{-\infty}|\mathcal{F}_n)| &\le \limsup E(|Y_n - Y_{-\infty}|\,|\mathcal{F}_n) \\
&\le \lim_{n\to-\infty} E(w_N|\mathcal{F}_n) \\
&= E(w_N|\mathcal{F}_{-\infty})
\end{aligned}$$

So, by $N \to -\infty$, $\limsup_{n\to-\infty} S_1 = 0$. $\limsup_{n\to-\infty} S_2 = 0$ because $E(Y_{-\infty}|\mathcal{F}_n)$ is a backward martingale so it converges to $E(Y_{-\infty}|\mathcal{F}_{-\infty})$ a.s. and in L_1.

□

Problem (4.8.4).

Let $M_n = S_n^2 - n\sigma^2$. Then M_n is a quadratic martingale. Since $n \wedge T$ is bounded stopping time, we have $EM_{n\wedge T} = EM_0 = 0$. Thus $ES_{n\wedge T}^2 = \sigma^2 E(n \wedge T)$. As $n \to \infty$, $E(n \wedge T) \to ET$ by MCT.

Now consider $E|S_T - S_{n\wedge T}|^2 = E\left(\sum_{m=n+1}^{\infty} 1_{(m\le T)}\xi_m\right)^2$. Note that

$$E(1_{m\le T})(1_{m+k\le T})\xi_m\xi_{m+k} = (E\xi_{m+k})E1_{m\le T}1_{m+k\le T}\xi_m = 0$$

Thus $E|S_T - S_{n\wedge T}|^2 = \sum_{m=n+1}^{\infty} E1_{(m\le T)}\xi_m^2 = \sigma^2 \sum_{m=n+1}^{\infty} P(m \le T)$. But, $\sum_{m=1}^{\infty} P(m \le T) = ET < \infty$. So we can say $E|S_T - S_{n\wedge T}|^2 \to 0$ as $n \to 0$.

Therefore $S_{n\wedge T} \to S_T$ in L_2, so $S_{n\wedge T}^2 \to S_T^2$ in L_1, which leads the conclusion.

□

Problem (4.8.7).

Note that $ET = a^2$ *by theorem 4.8.7. Claim :* $(b, c) = (3, 2)$.

$$\begin{aligned}
&E(Y_{n+1}|\mathcal{F}_n) \\
&= 1 + 6S_n^2 + S_n^4 - 6(n+1)(1+S_n^2) + 3(n+1)^2 + 2(n+1) \\
&= S_n^4 - 6nS_n^2 + 3n^2 + 2n
\end{aligned}$$

Since $n \wedge T$ *is bounded stopping time, we can get* $EY_0 = EY_{n\wedge T}$. *Thus* $3E(n \wedge T)^2 = 6E\left[(n \wedge T)S_{n\wedge T}^2\right] - ES_{n\wedge T}^4 - 2E_{n\wedge T}$.

But, by MCT, $E(n\wedge T)^2 \to ET^2$ *and* $E(n\wedge T) \to ET$. *And* $S_{n\wedge T}$ *is bounded, so BCT implies* $ES_{n\wedge T}^m \to a^m$. *Thus,* $3ET^2 = 6a^4 - a^4 - 2ET = 5a^4 - 2a^2$.

Note that $(n \wedge T)S_{n\wedge T}^2 \leq a^2T \in L_1$. *Thus* $E(n \wedge T)S_{n\wedge T}^2 \to a^4$ *by DCT.*

□

Problem (4.8.9)**.**

¡++¿