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#2.

a) $X \in \overline{\mathcal{M}}$ obviously. $(\because \mathcal{M} \subset \overline{\mathcal{M}})$

$A \in \overline{\mathcal{M}} \Rightarrow A = E \cup Z$, $Z \subset F$, $\mu(F) = 0$, $E, F \in \mathcal{M}$.

$\Rightarrow A^c = (E \cup F)^c \cup (F \setminus Z)$

Note that $(E \cup F)^c \in \mathcal{M}$ and $F \setminus Z \subset F$.

$\therefore A^c \in \overline{\mathcal{M}}$.

$A_i \in \overline{\mathcal{M}} \Rightarrow A_i = E_i \cup Z_i$, $Z_i \subset F_i$.

$\Rightarrow \bigcup A_i = \left(\bigcup_i E_i\right) \cup \left(\bigcup_i Z_i\right)$, $\quad \bigcup_i Z_i \subset \bigcup_i F_i$

But $\mu\left(\bigcup_i F_i\right) \le \sum_i \mu(F_i) = 0$.

so $\bigcup_i A_i \in \overline{\mathcal{M}}$. $\quad \therefore \overline{\mathcal{M}}$: σ-algebra

$\mathcal{F}$: another σ-algebra, contains $\mathcal{M}$, all subsets of μ-null sets.

$A \in \overline{\mathcal{M}} \Rightarrow A = E \cup Z$. But $E \in \mathcal{F}$ $(\because \mathcal{M} \subset \mathcal{F})$ and $Z \in \mathcal{F}$ by def.

$\therefore A \in \mathcal{F}$. $\quad \therefore \overline{\mathcal{M}} \subset \mathcal{F}$. $\quad \therefore \overline{\mathcal{M}}$ is the smallest.

b) ① w-defined.

$A \in \overline{\mathcal{M}}$ $(= E_1 \cup Z_1 = E_2 \cup Z_2)$ $\quad (Z_i \subset F_i,\ \mu(F_i) = 0)$.

$\mu(E_1) \le \mu(E_2 \cup F_2) \le \mu(E_2) + \mu(F_2) = \mu(E_2)$.

Similarly, $\mu(E_2) \le \mu(E_1)$ $\quad \therefore \mu(E_2) = \mu(E_1)$.

$\therefore \overline{\mu}(E_1 \cup Z_1) = \mu(E_1) = \mu(E_2) = \overline{\mu}(E_2 \cup Z_2)$. $\quad \therefore$ w-def.

② A_i : disjoint, $\in \overline{\mathcal{M}}$. $\quad A_i = E_i \cup Z_i$

Since A_i's are disjoint, E_i : disjoint, Z_i : disjoint.

$\Rightarrow \overline{\mu}\left(\dot{\bigcup_i} A_i\right) = \overline{\mu}\left(\left(\dot{\bigcup_i} E_i\right) \cup \left(\dot{\bigcup_i} Z_i\right)\right) = \mu\left(\dot{\bigcup_i} E_i\right) = \sum_i \mu(E_i) = \sum_i \overline{\mu}(A_i)$.

Since $\bigcup_i Z_i \subset \bigcup_i F_i$: μ-null set.

$\blacksquare$

#3.

① Let E be a L-msb set. A : any set

Choose $G_\delta \supset A$ s.t. $m_*(G_\delta) = m_*(A)$. ($* \; G_\delta$: L-msb)

Then $m_*(A) = \underbrace{m_*(G_\delta)}_{m_*=m} = \underbrace{m_*(G_\delta \cap E) + m_*(G_\delta \setminus E)}_{m_*=m.} \underset{\text{monotonicity of } m_*.}{\geq} m_*(A \cap E) + m_*(A \setminus E)$

$\therefore$ E: Caratheodory msb

② E : Carathéodory msb, $m_*(E) < \infty$.

Choose $G_\delta \supset E$, $m_*(G_\delta) = m_*(E)$

$\Rightarrow$ $m_*(G_\delta) \geq m_*(G_\delta \cap E) + m_*(G_\delta \setminus E) = m_*(E) + m_*(G_\delta \setminus E)$.

So $m_*(G_\delta \setminus E) = 0$ ($\because m_*(G_\delta) = m_*(E) < \infty$).

$\therefore$ $G_\delta \setminus E$: L-msb by completeness. $\therefore$ $E = G_\delta \setminus (G_\delta \setminus E)$: L-msb

Now, E : any Caratheodory msb set.

$E_n = E \cap [-n,n]^d$, then $[-n,n]$: Caratheodory msb by ①.

$\Rightarrow$ E_n : Caratheodory msb, $^{(\because ②)}$ so $E = \bigcup_n E_n$ is also Carathéodory msb.

∎

#10.

a) Support of ν_1, ν_2, μ = A_1, A_2, B

$\nu_1 \perp \mu$, $\nu_2 \perp \mu \Rightarrow A_1 \cap B = \phi$, $A_2 \cap B = \phi$.

$\therefore (A_1 \cup A_2) \cap B = \phi \Rightarrow \nu_1 + \nu_2 \perp \mu$

$*\ (\nu_1+\nu_2)(E) = \nu_1(E) + \nu_2(E) = \nu_1(E \cap A_1) + \nu_2(E \cap A_2)$

$= \nu_1(E \cap (A_1 \cup A_2)) + \nu_2(E \cap (A_1 \cup A_2))$.

$*\ \mu(E) = \mu(E \cap B_1) = \mu(E \cap B_1 \cap B_2)$.

b) $\mu(E) = 0 \Rightarrow \nu_1(E) = \nu_2(E) = 0 \Rightarrow (\nu_1+\nu_2)(E) = 0$. $\therefore \nu_1 + \nu_2 \ll \mu$.

c) $|\nu_1|(E) = \sup \sum_{i=1}^{\infty} |\nu_1(E_i)| = \sup \sum |\nu_1(E_i \cap A_1)| = |\nu|(E \cap A_1)$

where $\sum E_i = E$

$\therefore |\nu_1|$ is supported on A_1. (Similarly, $|\nu_2|$ is supported on A_2).

Since $A_1 \cap A_2 = \phi$, we have $|\nu_1| \perp |\nu_2|$.

d) $|\nu|(E) = 0 \Rightarrow \sup \sum_{i=1}^{\infty} |\nu(E_i)| = 0 \Rightarrow |\nu(E_i)| = 0 \Rightarrow \nu(E_i) = 0$.

$\Rightarrow \nu(E) = \sum_{i=1}^{\infty} \nu(E_i) = 0$. $\therefore \nu \ll |\nu|$

e) ν, μ : supported on A, B.

$\nu(E) = \nu(A \cap E)$ But $\mu(A \cap E) = \mu(A \cap E \cap B) = 0$. $(\because A \cap B = \phi)$.

$\Rightarrow \mu(A \cap E) = 0$. Since $\nu \ll \mu$, $\nu(A \cap E) = 0$. $\Rightarrow \nu(E) = 0$.

11.

i) F_A : abs. continuous $\Rightarrow$ $F_A' \in L^1$, $\int_a^b F_A' dx = F_A(b) - F_A(a)$

Since F_A' is L^1, $\nu(E) = \int_E F_A' dx$ is a positive measure. ($F_A' \geq 0$).

Also, $\mu_A(a,b] = F_A(b) - F_A(a) = \int_{(a,b]} F_A' dx = \nu(a,b]$.

Since $\mathbb{R}$ is σ-finite, by Caratheodory ext'n thm,

$\nu_0(a,b] = \int_{(a,b]} F_A' dx$ uniquely extends to ν.

$\parallel$

$\mu_0(a,b] = \mu_A(a,b]$ " to μ_A

$\therefore \nu = \mu_A$

$\therefore \mu_A(E) = \int_E F_A' dx = \int_E F' dx$ ($\because$ $F_J' = 0$, $F_C' = 0$ a.e.)

so $F_A' = F'$ a.e.

ii) ① $f = 1_E$. $\mu(E) = \int f d\mu = \int_E F' dx = \int f F' dx$.

② f = simple ftn. $\Rightarrow$ by ① and linearity, clear

③ $f \geq 0$ $\Rightarrow$ $s_n \uparrow f$, s_n : simple.

$$\Rightarrow \int f d\mu \underset{\text{MCT}}{=} \lim_{n\to\infty} \int s_n d\mu \underset{\text{by ②}}{=} \lim_{n\to\infty} \int s_n F' dx \underset{\text{MCT, } F' \geq 0 \ (\because \text{increasing})}{=} \int f F' dx.$$

④ Any f. $= f^+ - f^-$ so apply ③ to $f^\pm$ respectively.

(To avoid $\infty - \infty$, consider $f \in L^1(\mu)$.)

iii) By ~~pag~~ b) of page 285, μ_J has point mass at discontinuities of F.

And they are the only ones. (μ_J : supported on discontinuities of F).

$$\mu_C(E) = \inf \sum_{j=1}^{\infty} \left[F_C(b_j) - F_C(a_j)\right] \underset{(\text{cor. 3.7.})}{\leq} \inf \sum_{j=1}^{\infty} \int_{a_j}^{b_j} F_C' dx = 0.$$

$\Rightarrow \mu_C \equiv 0$.

$\therefore \mu_J + \mu_C$: supported on $D(F)$: discontinuities of F.

But, since F is increasing, $D(F)$: at most countable

$\therefore m(D(F)) = 0$. so m is supported on $D(F)^c$. $\blacksquare$

Problem #2

WANT TO SHOW : $\mu(f^{-1}(E)) = \mu(E)$ $\forall f \in SL^2(\mathbb{R})$

$$\int_{f^{-1}(E)} \frac{dxdy}{y^2} \xrightarrow{(x,y) \xrightarrow{f} (u,v)} \int_E \frac{dudv}{v^2} = \mu(E)$$

Because

① $1_{f^{-1}(E)}(x,y) = \begin{cases} 1 & f(x,y) \in E \Leftrightarrow (u,v) \in E \\ 0 & \text{o.w.} \end{cases}$

② $dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \left|\frac{1}{f'(z)}\right|^2 dudv$ by Problem 1 & C-R eqn.

$\left(\frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}, \Rightarrow |J| = |f'(z)|^2 \right)$

③ $(u,v) = w. \Rightarrow f(z) = w, \quad f^{-1}(w) = x+iy.$

$$f^{-1}(w) = \frac{dw - b}{-cw + a} = \frac{(dw-b)(-c\bar{w}+a)}{|-cw+a|^2} \quad (\because a,b,c,d \in \mathbb{R}).$$

$$= \frac{-dc\bar{w}w + adw + bc\bar{w} - ab}{|-cw+a|^2}, \text{ imaginary part } = \frac{v}{|-cw+a|^2} = y$$

$$\therefore \frac{1}{y^2} = \frac{|-cw+a|^2}{v^2}.$$

But, $\left|\frac{1}{f'(z)}\right|^2 = |(f^{-1})'(w)|^2 = \left|\frac{d(-c\cancel{w}+a) + c(d\cancel{w}-b)}{(-cw+a)^2}\right|^2 = \left|\frac{1}{(-cw+a)^2}\right|^2.$

$(\because$ ~~a,b,c,d~~ $ad-bc=1).$

Therefore, by ①.②.③,

$$\int_{f^{-1}(E)} \frac{dxdy}{y^2} = \int_E \frac{|-cw+a|^4}{v^2} \cdot \frac{1}{|-cw+a|^4} dudv = \int_E \frac{dudv}{v^2} = \mu(E)$$

$\parallel$

$\mu(f^{-1}(E)).$