MAS651 EXERCISES

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Problem (5.1.1)**.**

Let $(S, \mathcal{S})$ *be a state space of* X_n *where* $S = \{1, 2, \cdots N\}$ *and* $\mathcal{S} = 2^S$*. Note that* N *is an absorbing state. And* $X_1 = 1$ *with probability* 1*. For fixed* k *such that* $1 \leq k < N$*,* $k \leq n$*,*

$$P(X_{n+1} = k+1 | X_n = k) = \frac{N-k}{N}$$

and

$$P(X_{n+1} = k | X_n = k) = \frac{k}{N}.$$

If $k > n$*, then the above are all* 0*. So it is a temporally inhomogeneous. The Markov property is trivial since the very next state only depends on the current state.*

□

Problem (5.1.2)**.**

$$P(X_4 = 2 | X_3 = 1, X_2 = 1, X_1 = 1, X_0 = 0) = (1/16)/(1/4) = 1/4$$

but

$$P(X_4 = 2 | X_3 = 1, X_2 = 0, X_1 = 0, X_0 = 0) = (1/16)/(1/8) = 1/2.$$

Thus X_n *is not a Markov chain.*

□

Problem (5.1.5)**.**

$$P(X_{n+1} = k+1 | X_n = k) = \frac{m-k}{m}\frac{b-k}{m}$$

because we must choose a white ball in the left urn and a black ball in the right urn.

$$P(X_{n+1} = k | X_n = k) = \frac{k}{m}\frac{b-k}{m} + \frac{m-k}{m}\frac{m+k-b}{m}$$

since there are two cases, choosing both black or both white.

$$P(X_{n+1} = k-1 | X_n = k) = \frac{k}{m}\frac{m+k-b}{m}$$

since we must choose a black ball in the left urn and a white ball in the right urn. Note that the sum of the above is 1, so there is no other transition probability.

□

Problem (5.1.6)**.**

$$P(S_{n+1} = k+1 | S_n = k) = \frac{P(X_{n+1} = 1, S_n = k)}{P(S_n = k)}$$

where the denominator is

$$\int_{\theta \in (0,1)} P(S_n = k | \theta) dP = \binom{n}{x} \frac{x! y!}{(n+1)!} = \frac{1}{n+1}$$

for $x =$*the number of* i *such that* $U_i \leq \theta$ *and* $y = n - x$*. Note that* $x = (n+k)/2$ *and* $y = (n-k)/2$ *since* $x + y = n$ *and* $x - y = k$*. The numerator is*

$$\int_{\theta \in (0,1)} P(X_{n+1} = 1, S_n = k | \theta) dP = \binom{n}{x} \frac{(x+1)! y!}{(n+2)!}$$

These are because $P(S_n = k|\theta) = \theta^x (1-\theta)^y \binom{n}{x}$ *and* $P(X_{n+1} = 1, S_n = k | \theta) = P(X_{n+1} = 1|\theta) P(S_n = k|\theta) = \binom{n}{x} \theta^{x+1} (1-\theta)^y$ *and using the kernel of beta distribution.*

Thus, the probability what we want is $(n+k+2)/(2n+4)$ *which depends on* n. *So* X_n *is temporally inhomogeneous.*

$$P(S_{n+1} = k+1 | S_1 = t_1, \cdots, S_n = k) = P(X_{n+1} = 1 | X_1 = t_1, \cdots, X_n = t_n)$$

where $\sum_{i=1}^{n} t_i = k$. *We can show the above is equal to* $P(S_{n+1} = k+1 | S_n = k) = (n+k+2)/(2n+4)$ *similarly, by omitting the* $\binom{n}{x}$ *term of both denominator and numerator.*

$\square$