SOLUTIONS FOR TOPOLOGY WRITTEN BY J.MUNKRES

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22. The Quotient Topology

22.1. **Quotient Topology.**

22.2. **Topological Group.**

Problem (5).

(a) *Let $\varphi_\alpha : xH \mapsto \alpha \cdot xH$ be the map induced by f_α. Then φ_α is clearly bijective. Let V be an open set in G/H. We want to show that $\varphi_\alpha^{-1}(V)$ is open in G. Since G/H is quotient space, it is good to consider $p^{-1}\left(\varphi_\alpha^{-1}(V)\right)$ where p corresponds each $x \in G$ to xH.*

$$\begin{aligned} p^{-1}\left(\varphi_\alpha^{-1}(V)\right) &= \{x \in G : \alpha \cdot xH \in V\} \\ &= \left\{\alpha^{-1} \cdot x : xH \in V\right\} \\ &= f_{\alpha^{-1}}\left(p^{-1}(V)\right) \end{aligned}$$

But the last one is open since $f_g : x \mapsto g \cdot x, g \in G$ is homeomorphism, and p is a quotient map. Note that $\varphi_\alpha^{-1} = \varphi_{\alpha^{-1}}$. So φ_α is a homeomorphism.

(b) *Note that G/H is a homogeneous space by a). So, closedness of $\{eH\}$ implies closedness of every other singleton sets. Now Consider*

$$\begin{aligned} p^{-1}\left(G/H \setminus \{eH\}\right) &= \{x \in G : xH \in G/H \setminus \{eH\}\} \\ &= H^c \end{aligned}$$

Therefore $G/H \setminus \{eH\}$ is open, hence $\{eH\}$ is closed.

(c) *Let V be an open set in G. We want to check whether $p(V)$ is open or not. So, it is good to consider*

$$\begin{aligned} p^{-1}\left(p(V)\right) &= \{x \in G : xH \in p(V)\} \\ &= \bigcup_{x \in V} xH \\ &= \{x \cdot h : x \in V, h \in H\} \\ &= \bigcup_{h \in H} V \cdot h \end{aligned}$$

But $V \cdot h = \{x \cdot h : x \in V\} = g_h(V)$ is open since g_h is homeomorphism. (see problem 4) So, last one of above equation is union of open sets, therefore $p(V)$ is open. So p is open mapping.

(d) *By b), G/H is T_1 space. Let $f : xH \mapsto x^{-1}H$, V be an open set in G/H. We want to show that $f^{-1}(V)$ is open in G/H. So, consider*

$$\begin{aligned} p^{-1}\left(f^{-1}(V)\right) &= \left\{g \in G : g^{-1}H \in V\right\} \\ &= \left\{g^{-1} \in G : gH \in V\right\} \end{aligned}$$

But note that $F : x \mapsto x^{-1}$ is a homeomorphism, and last one of above equation is $F\left(p^{-1}(V)\right)$, therefore open.

Now consider $g : (xH, yH) \mapsto x \cdot yH$. Let $(xH, yH) \in g^{-1}(V)$. Then $x \cdot yH \in V \leftrightarrow p(x \cdot y) \in V$. Therefore $x \cdot y \in p^{-1}(V)$ which is open. Let $G : (x, y) \mapsto x \cdot y$. Then $(x, y) \in G^{-1}\left(p^{-1}(V)\right)$ which is open in $G \times G$. So there *exists basis element of* $G \times G$ *such that* $(x, y) \in B_1 \times B_2 \subset G^{-1}\left(p^{-1}(V)\right)$. *Then,*

$$\begin{aligned}(p \times p)(x, y) = (xH, yH) &\subset p(B_1) \times p(B_2) \\ &\subset (p \times p)\left(G^{-1}\left(p^{-1}(V)\right)\right) \subset g^{-1}(V)\end{aligned}$$

So $g^{-1}(V)$ *is open since* p *is open mapping therefore* $p(B_1) \times p(B_2)$ *is open.*

35. The Tietze Extension Theorem

Problem (1).

Let A, B be disjoint closed subspace of X. Define f as following:

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$

Then $f : A \cup B \to [0,1]$ is continuous (by pasting lemma).

By Tietze extension theorem, we can extend f to $\overline{f} : X \to [0,1]$. This $\overline{f}$ is what we can get from Urysohn lemma.

Problem (5).

(a) Let (X, A, f) be given. Since $f : A \to \mathbb{R}^J$, $f_\alpha : A \to \mathbb{R}$. Apply the Tietze extension theorem to f_α. Then we get continuous function $\overline{f} : X \to \mathbb{R}^J$ which satisfies $\left(\overline{f}\right)_\alpha = \overline{f_\alpha}$. This $\overline{f}$ is what we want.

(b) Without loss of generality, we can assume that Y is a retract of $\mathbb{R}^J$. Let $f : A \to Y$ be a continuous function. By expanding codomain, we can get $f' : A \to \mathbb{R}^J$ which is continuous. Let $\overline{f}$ be the extension of f'. Then $r \circ \overline{f}$ is continuous extension of f where r is a retraction of $\mathbb{R}^J$ into Y.

Problem (6).

(a) Let h be a homeomorphism of Y_o onto Y. Since Y has universal extension property, we can extend h to $\overline{h} : X \to Y$ which is continuous. Then $h^{-1} \circ \overline{h}$ is the retraction of X into Y_o.

(b) Note that (b) of problem 5 still holds if we replace $\mathbb{R}^J$ to $[0,1]^J$. Fix $y \in Y$ and choose neighborhood V_y of y. By Urysohn lemma, there is a continuous function f_y such that $f_y(y) = 1$ and vanishes outside of V_y. Let $F(x) = (f_y(x))_{y \in Y}$. Then F is imbedding of Y into $[0,1]^Y$ by theorem 34.2.

Since F is imbedding of Y into $[0,1]^J$, $F(Y) \cong Y$. Now, it is sufficient to show that $F(Y)$ is retract of $[0,1]^J$. But it follows directly by (b) of problem 5 since $F(Y)$ is compact hence closed, $[0,1]^J$ is compact Hausdorff hence normal.