mas550 homework

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Problem (1.5.1).

First, we will show that $|g| \leq \|g\|_\infty$ *a.e.*

It is true because

$$\begin{aligned}\mu\left(|g| > \|g\|_\infty\right) &= \mu\left(\bigcup_{n=1}^{\infty}\left\{|g| \geq \|g\|_\infty + \frac{1}{n}\right\}\right)\\ &\leq \sum_{n=1}^{\infty}\mu\left(\left\{|g| > \|g\|_\infty + \frac{1}{n}\right\}\right)\\ &= 0\end{aligned}$$

by definition of $\|g\|_\infty$.

Hence $|g| \leq \|g\|_\infty$ *a.e.*

Then, $\int |fg| d\mu \leq \|g\|_\infty \int |f| d\mu = \|g\|_\infty \|f\|_1$.

Problem (1.5.3).

(a) *Since* $p > 1$, $x \mapsto |x|^p$ *is convex function.* $|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p)$ *follows from convexity of* $|x|^p$.

   $\int |f+g|^p d\mu \leq \int 2^p |f|^p d\mu + \int 2^p |g|^p d\mu$. *Therefore finiteness of* $\|f\|_p$ *and* $\|g\|_p$ *leads* $\|f+g\|_p < \infty$.

   Now, consider $\int |f+g|^p d\mu = \int |f+g||f+g|^{p-1} d\mu \leq \int |f||f+g|^{p-1} d\mu + \int |g||f+g|^{p-1} d\mu$. *Let* q *be Holder conjugate of* p. *Then by applying Holder inequality, we get* $\|f+g\|_p^p \leq \|f+g\|_p^{p/q}(\|f\|_p + \|g\|_p)$. *Simple calculating leads Minkowski's inequality.*

(b) *First consider* $p = 1$. *By using triangle inequaltiy, the result follows directly. Next consider* $p = \infty$. $|f+g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty$ *a.e. Therefore* $\|f+g\|_\infty \leq \|f\|)\infty + \|g\|_\infty$.

Problem (1.6.8).

First assume $g = 1_A$. *Then* $\int g d\mu = \mu(A) = \int_A f(x) dx = \int 1_A f dm$ *where* m *is Lebesgue measure.*

Next, assume $g = \sum_i a_i 1_{A_i}$, *simple function. Then* $\int g d\mu = \sum_i a_i \mu(A_i) = \sum_i a_i \int 1_{A_i} f dm$.

Next, assume g *is nonnegative measurable. Let* $\{s_n\}_{n=1}^\infty$ *be increasing sequence of simple function converges to* g *pointwisely. Then* $\int g d\mu = \lim_{n\to\infty} \int s_n d\mu =$

$\lim_{n\to\infty}\int s_n f dm$. *But* $s_n f \uparrow gf$ *since* f *is nonnegative. By monotone convergence theorem, we can get* $\int g d\mu = \int gf dm$.

Last, assume g *is integrable function. We can decompose* g *by* $g = g^+ - g^-$. *Applying 3rd step for* g^+, g^- *each, we can get* $\int g d\mu = \int g^+ f dm - \int g^- f dm = \int gf dm$ *since* f *is nonnegative.*

Problem (1.6.13)**.**

Since $X_n \uparrow X$, $X_n^+ \uparrow X^+$ *and* $X_n^- \downarrow X^-$. *And note that* $X_n^- \leq X_1^-$ *which is integrable. Apply monotone convergence theorem to* X_n^+ *and apply dominated convergence theorem to* X_n^- *to get* $\lim EX_n = \lim EX_n^+ - \lim EX_n^- = EX^+ - EX^- = EX$.