mas550 homework

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Problem (4.1.7)**.**

By definition of $\text{Var}(X|\mathcal{F})$*, we get the following:*

$$E\left(\text{Var}(X|\mathcal{F})\right) = EX^2 - E\left(E(X|\mathcal{F})^2\right)$$

And clearly,

$$\text{Var}\left(E(X|\mathcal{F})\right) = E\left(E(X|\mathcal{F})^2\right) - \left(E(E(X|\mathcal{F}))\right)^2$$

Therefore, by summing them vertically, we can get

$$\text{Var}\left(E(X|\mathcal{F})\right) + E\left(\text{Var}(X|\mathcal{F}\right) = EX^2 - \left(E(E(X|\mathcal{F}))\right)^2$$

which is equal to $\text{Var}(X)$ *since the last term is equal to square of* EX*.* □

Problem (4.1.9)**.**

$$\begin{aligned}
\int |X-Y|^2 dP &= \int X^2 - 2XY + Y^2 dP \\
&= \int X^2 - 2E(XY|\mathcal{G}) + Y^2 dP \\
&= \int X^2 - 2XE(Y|\mathcal{G}) + Y^2 dP \\
&= \int X^2 - 2X^2 + Y^2 dP \\
&= EY^2 - EX^2 \\
&= 0
\end{aligned}$$

Therefore, $|X-Y|^2 = 0$ *a.s. which implies* $X = Y$ *a.s. Note that* XY *is integrable by Holder's inequality for* $p = q = 2$ *and finite second moment of* X, Y*.* □

Problem (4.2.3)**.**

Clearly $\mathcal{F}_m \subset \mathcal{F}_{m+1}$ *for all positive integer* m*. Let* $Z_n = X_n \vee Y_n$*, then* Z_n *is clearly* $\mathcal{F}_n$ *measurable.*

Now, let $A \in \mathcal{F}_{n-1}$. Then,

$$
\begin{aligned}
\int_A E(Z_n|\mathcal{F}_{n-1})dP &= \int_A Z_n dP \\
&\geq \int_A X_n dP \vee \int_A Y_n dP \\
&= \int_A E(X_n|\mathcal{F}_{n-1})dP \vee \int_A E(Y_n|\mathcal{F}_{n-1})dP \\
&\geq \int_A X_{n-1} dP \vee \int_A Y_{n-1} dP
\end{aligned}
$$

Therefore $\int_A E(Z_n|\mathcal{F}_{n-1})dP \geq \int_A X_{n-1}, Y_{n-1} dP$ for all $A \in \mathcal{F}_{n-1}$. Since $E(Z_n|\mathcal{F}_{n-1})$ is $\mathcal{F}_{n-1}$ measurable, we can conclude that conditional expectation of Z_n with respect to $\mathcal{F}_{n-1}$ is equal or greater than X_{n-1} and Y_{n-1} a.s.

So, Z_n is a submartingale. □

Problem (4.2.9)**.**

Note that $\{N > n\} = \{N \leq n\}^c \in \mathcal{F}_n$ and $\{N < n\} = \{N \leq n-1\} \in \mathcal{F}_{n-1}$ since N is integer valued. Now, consider the following:

$$
\begin{aligned}
E(Z_n|\mathcal{F}_{n-1}) &= 1_{N \geq n} E(X_n^1|\mathcal{F}_{n-1}) + 1_{N<n} E(X_n^2|\mathcal{F}_{n-1}) \\
&\leq 1_{N \geq n} X_{n-1}^1 + 1_{N<n} X_{n-1}^2 \\
&= 1_{N > n-1} X_{n-1}^1 + 1_{N \leq n-1} X_{n-1}^2 \\
&\leq 1_{N \geq n-1} X_{n-1}^1 + 1_{N < n-1} X_{n-1}^2 \\
&= Z_{n-1}
\end{aligned}
$$

So, Z_n is supermartingale.

Now, consider the Y_n:

First, $Y_n = X_n^1 1_{N>n} + X_N^2 1_{N=n} + X_n^2 1_{N<n} \leq X_n^1 1_{N \geq n} + X_n^2 1_{N<n}$.

$$
\begin{aligned}
E(Y_n|\mathcal{F}_{n-1}) &\leq 1_{N \geq n} E\left(X_n^1|\mathcal{F}_{n-1}\right) +!_{N<n} E\left(X_n^2|\mathcal{F}_{n-1}\right) \\
&\leq 1_{N \geq n} X_{n-1}^1 + 1_{N<n} X_{n-1}^2 \\
&= 1_{N > n-1} X_{n-1}^1 + 1_{N \leq n-1} X_{n-1}^2 \\
&= Y_{n-1}
\end{aligned}
$$

So, Y_n is also a supermartingale. □