Conformal Self Mappings

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October 12, 2020

Contents

conformal map

- U, V : open subsets of $\mathbb{C}$
- f is a function of U into V
- f is conformal if f is bijective and holomorphic.
- conformal $=$ biholomorphic
- If h is holomorphic function of U, and U is somewhat complicated, by considering $h \circ f$, we can change the domain of h.
- why conformal? it preserves the angle.

characterizing conformal self mapping of $\mathbb{C}$

- natural example: $az + b$ for $a \neq 0$
- In fact, above form is all of them.
- Note that we are considering not just entire function. Conformal self mappings of $\mathbb{C}$ has more condition than entire function.

Lemma 6.1.2.

If $f : \mathbb{C} \to \mathbb{C}$ is a conformal then $\lim_{|z| \to \infty} |f(z)| = \infty$.

proof of lemma 6.1.2.

- Fix M. We want to show existence of N such that $f(\{z : |z| > N\}) \subset \{w : |w| > M\}$.
- Above is equivalent to $\{z : |z| > N\} \subset f^{-1}(\{w : |w| > N\})$ since f is bijective.
- Above is equivalent to $\{z : |z| \leq N\} \supset f^{-1}(\{w : |w| \leq M\})$ by taking complement.
- Existence of N is clear since RHS of above is compact($\Rightarrow$ closed and bounded).

characterizing conformal self mapping of $\mathbb{C}$

We already know that f must be polynomial when f is entire and $f \to \infty$ as $|z| \to \infty$. Using this and fundamental thm of algebra, we can characterize conformal self mapping of complex plane.

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lemma 6.1.3

f is a conformal self mapping of $\mathbb{C}$. Then there are $B, D > 0$ such that $|z| > D$ implies $|f(z)| < B|z|$.

proof of lemma 6.1.3.

- There is C such that $|z| > C$ implies $|f(z)| > 1$.
- Define $g(z) = 1/f\left(\frac{1}{z}\right)$ for $z \in D'(0, \frac{1}{C})$.
- As $z \to 0$, $g \to 0$. So g has removable singularity at 0.
- $g'(0) \neq 0$ since g is injection since f is injection.
- Near 0, $\left|\frac{g(z)}{z}\right| > \frac{1}{B}$.
- Near 0, $\left|f(\frac{1}{z})\right| < \frac{B}{|z|}$.
- $z \mapsto \frac{1}{z}$ leads the conclusion.

characterizing

- Consider $|f^{(n)}(0)| \leq \frac{n!}{r^n} \sup_{w \in \partial D(0,r)} |f(w)|$.
- For $r > D$, supremum above $\leq Br$ by lemma 6.1.3.
- If $n > 1$, by letting $r \to \infty$, n-th derivative of f at 0 must be zero.
- Therefore f must be polynomial of degree at most 1.
- But f must be nonconstant. So $f(z) = az + b$ for $a \neq 0$.
- We are characterized conformal self mappings of $\mathbb{C}$.

remark

- h is holomorphic on $\{z : |z| > \alpha\}$ and $\lim_{|z|\to\infty} |h(z)| = \infty$.
- By same procedure in proof of lemma 6.1.3, we can conclude that there are $B, D > 0$ such that $|z| > D \Rightarrow |h(z)| < B|z|^n$ for some n.
- Why n? Because we cannot say $g'(0) \neq 0$. But, $g^{(n)}(0) \neq 0$ for some n since g is nonconstant since h is nonconstant. *$g(z) = 1/h(1/z)$
- Note that entire function φ which satisfies $\lim_{|z|\to\infty} |\varphi(z)| = \infty$ must be polynomial.

characterizing conformal self mapping of unit disc

- natural example : rotation ($f(z) = wz$ for $|w| = 1$)
- In fact, above form is all of them which fixes origin.

lemma 6.2.1.
$f : D \to D$ is biholomorphic which fixes origin iff $f(z) = wz$ for $|w| = 1$

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proof of lemma 6.2.1.

- $g = f^{-1}$. Then both of f, g are fixing origin.
- Schwarz lemma says $|f'(0)|$ and $|g'(0)|$ are ≤ 1.
- Chain rule says $f'(0)g'(0) = 1$. This leads $|f'(0)| = |g'(0)| = 1$.
- Uniqueness of Schwarz lemma tells us that $f(z) = f'(0)z$.

Mobius transformation

- $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ for $|a| < 1$ is called Mobius transformation.
- Theorem 5.5.2 says φ_a is conformal self mapping of unit disc. So we take it for granted.

theorem 6.2.3.

f is conformal self mapping of unit disc. Then $f(z) = w\varphi_a(z)$ for some $|a| < 1$ and $|w| = 1$.

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proof of theorem 6.2.3.

- $f(0) = b$. Let $g = \varphi_b \circ f$. Then g fixes origin.
- $g(z) = wz$ for some $|w| = 1$. Namely, $f(z) = \varphi_b^{-1}(wz)$. Note that any f must be such form.
- But $\varphi_b^{-1} = \varphi_{-b}$.
- $f(z) = \frac{wz+b}{1+\bar{b}wz}$
- Simple calculation leads $f(z) = w\varphi_{-bw^{-1}}(z)$.
- Take $a = -bw^{-1}$.

automorphism group

- Set of all conformal self mapping of unit disc forms group under composition. It is called automorphism group of unit disc.
- Mobius transformation denotes automorphism of unit disc.
- Further, fix U then $\{$conformal self mapping of $U\}$ forms a group under composition.
- it is called automorphism group of U.

preliminaries of linear fractional transformation

- Riemann sphere is $\mathbb{C} \cup \infty \cong S^2$ by stereographic projection.
- $p_i \to p_0$ in R-sphere is equivalent to $\pi^{-1}(p_i) \to \pi^{-1}(p_0)$ in S^2.
- Note that image of north-pole in S^2 under projection is ∞.
- Also, above definition of limit in extended plane is congruent to definition using metric.

linear fractional transformation

- $g : \mathbb{C} \to \mathbb{C}$ is meromorphic iff $\hat{g} : \mathbb{C} \to \hat{\mathbb{C}}$ is holomoprhic.
- $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$
- Let $ad - bc \neq 0$, $a, b, c, d \in \mathbb{C}$. $f(z) = \frac{az+b}{cz+d}$ is called linear fractional transformation if it satisfies two more conditions.
- If $c = 0$, $f(\infty) = \infty$. In this case, f is a linear map.
- If $c \neq 0$, $f(-\frac{d}{c}) = \infty$, $f(\infty) = \frac{a}{c}$.
- Note that $f(p_i) \to f(p_0)$ when $p_i \to p_0$ for all $p_0 \in \mathbb{C} \cup \infty$.
- Above says continuity of f on the Riemann sphere.

linear fractional transformation

- $[[a, b], [c, d]] = A \in GL_2(\mathbb{C})$
- $A \cdot z = \frac{az+b}{cz+d}$ is a group action (by simple calculation)
- $A^{-1} \cdot (A \cdot z) = I \cdot z = z$, hence $A \cdot z$ has the inverse, hence bijective
- We already know that $A \cdot z$ is continuous on Riemann sphere(by previous frame), hence homeomorphism

l.f. transformation as conformal self mapping of Riemann sphere

- When $c = 0$, $g(z) = 1/f(1/z)$ is holomorphic near 0 (because $f(\infty) = \infty$) hence f is holomorphic at ∞.
- When $c \neq 0$, $h(z) = 1/f(z)$ is holomorphic near $z = -d/c$.
  Also $g(z) = 1/f(1/z)$ is holomoprhic near 0.
  Therefore f is holomorphic at $-d/c$ and ∞.
- In both cases, f is self conformal mapping of the Riemann sphere.

characterizing conformal self mapping of the Riemann sphere

- Let φ be a conformal self mapping of the Riemann sphere.
- If φ maps ∞ to ∞, then φ must be linear.
- If $\varphi(\infty) = a$, then exists ψ: l.f. transformation maps a to ∞
  Then $\psi \circ \varphi$ maps ∞ to ∞.
  By above, $\psi \circ \varphi$ must be linear, and by considering $\varphi(z) = \psi^{-1}(\alpha z + \beta)$, we can conclude that φ must be l.f. transformation.
- Thm 6.3.5 : f is conformal self mapping of the Riemann sphere iff f is l.f. transformation.

geometric property of l.f. transformation

- Any line on Riemann sphere can be regarded as circle by stereographic projection.
- Any l.f. transformation can be represented as composite of translation, dialation, and inversion.
- Translation and dialation maps circle on Riemann sphere to circle.
- In fact, inversion maps circle to circle.

inversion maps circle to circle

- $\alpha(x^2 + y^2) + \beta x + \gamma y + \delta = 0$ represents arbitrary circle on Riemann sphere.
- Let $z = x + iy$ and $w = 1/z = u + iv$. Then simple calculation yields $x = u/(u^2 + v^2)$ and $y = -v/(u^2 + v^2)$.
- $\alpha + \beta u - \gamma v + \delta(u^2 + v^2) = 0$ represents generalized circle also.
- therefore we can see that inversion maps circle to circle on Riemann sphere.
- note that $0 \mapsto \infty$ by inversion. So circle pass through origin goes to line does not pass origin and so forth.

thm 6.3.6.

- Consider $f : z \mapsto \frac{z-i}{z+i}$.
- This maps $(1, 0, \infty, i)$ to $(\frac{1-i}{1+i} = -i, -1, 1, 0)$ respectively.
- Note that $1, 0, \infty$ are points in boundary of upper half plane.
- And $-i, -1, 1$ are points in boundary of unit disc.
- f maps boundary of upper half plane onto boundary of unit disc.
- f maps upper half plane onto unit disc since f is continuous and upper half plane is connected.