mas541 homework

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Problem (1.1)**.**

$$
\begin{aligned}
1-\left|\frac{z-w}{1-z\overline{w}}\right|^2 &= 1-\frac{(z-w)\left(\overline{z}-\overline{w}\right)}{(1-z\overline{w})\left(1-\overline{z}w\right)} \\
&= \frac{1-\overline{z}w-z\overline{w}+|z|^2|w|^2-|z|^2-|w|^2+z\overline{w}+\overline{z}w}{|1-\overline{z}w|^2} \\
&= \frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\overline{z}w|^2}
\end{aligned}
$$

Problem (1.2)**.**

Let $f=u+iv$. $\partial f=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)$. *Then* $\overline{\partial f}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(u-iv)=\bar{\partial}\bar{f}$.

Problem (1.3)**.**

If f *is constant, then* $|f|$ *is also constant. On the other hand, assume* $f=u+iv$ *and* $|f|^2=u^2+v^2$ *is positive real number. (if it is zero, then* f *must be zero)*

$$u^2+v^2=R>0$$

Differentiate both sides of the equation above with x *and* y *respectively, we can get* $uu_x+vv_x=0$, $uu_y+vv_y=0$, $u_x=v_y$ *and* $u_y=-v_x$. *By simple calculation we can get* $u_x=u_y=v_x=v_y=0$. *Therefore* u,v *are constant.*

Problem (1.4)**.**

Note that $\int_0^{2\pi}e^{ik\theta}d\theta=\int_0^{2\pi}(\cos k\theta+i\sin k\theta)\,d\theta=0$ *for positive integer* k. *Therefore* $\frac{1}{2\pi}\int_0^{2\pi}(z_0+re^{i\theta})^jd\theta=\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{j}\binom{j}{k}z_0^k\left(re^{i\theta}\right)^{j-k}d\theta=z_0^j$. *Similarly, we can get* $\frac{1}{2\pi}\int_0^{2\pi}\overline{(z_0+re^{i\theta})^j}d\theta=\bar{z}_0{}^j$.

Since u *is polynomial, we can write it as* $\sum_{l,k}a_{l,k}z^l\bar{z}^k$. *By direct computation, we can get* $\frac{1}{2\pi}\int_0^{2\pi}u\left(z_0+re^{i\theta}\right)d\theta=\sum_{l,k}a_{l,k}z_0^l\bar{z}_0{}^k=u(z_0)$.

Problem (1.5)**.**

Let $f=u+iv$. $(g\circ f)_x=g_uu_x+g_vv_x$. *Then*

$$
\begin{aligned}
(g\circ f)_{xx} &= \left(g_{uu}u_x+g_{uv}v_x\right)u_x+g_uu_{xx}+\left(g_{vu}u_x+g_{vv}v_x\right)v_x+g_vv_{xx} \\
(g\circ f)_{yy} &= \left(g_{uu}u_y+g_{uv}v_y\right)u_y+g_uu_{yy}+\left(g_{vu}u_y+g_{vv}v_y\right)v_y+g_vv_{yy}
\end{aligned}
$$

But we have Cauchy-Riemann equation and $g_{uu}+g_{vv}=0$ *and* $g_{vu}=g_{uv}$. *Also, since* f *is* C^2 *function,* f *is harmonic,* $u_{xy}=u_{yx}$, *and* $v_{xy}=v_{yx}$. *Using* *these equations, we can check that* $(g \circ f)_{xx} + (g \circ f)_{yy} = 0$*. Hence* $(g \circ f)$ *is a harmonic function.*

Problem (2.1)**.**

Let $f = u + iv$. Then $\bar{f}f' = ff' - 2ivf'$, where ff' is holomorphic. So, $\int_\gamma \bar{f}f'dz = \int_\gamma -2ivf'dz = \int_\gamma -2iv(u_x + iv_x)dz = \int_\gamma -2iv(v_y + iv_x)dz = -i\int_a^b (2vv_y + 2ivv_y)(\gamma_1' + i\gamma_2')dt = \alpha$ where $\gamma = \gamma_1 + i\gamma_2$.

Therefore, real part of $\int_\gamma \bar{f}f'dz$ is equal to real part of α. And it is also equal to $-\int_a^b Im\left[(2vv_y + i2vv_x)(\gamma_1' + i\gamma_2')\right]dt = -\int_a^b (2vv_x\gamma_1' + 2vv_y\gamma_2')dt = -\int_a^b \frac{d}{dt}(v^2 \circ \gamma)dt = 0$ since γ is closed curve.

So, $\int_\gamma \bar{f}f'dz$ is purely imaginary.

Problem (2.2)**.**

Let $f = -u_y$ and $g = u_x$. Then f, g are continuous on U. Since u is harmonic, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ on $U \setminus \{0\}$. So there is $v : U \to \mathbb{R}$ which is C^1 function and $v_x = f$, $v_y = g$ by lemma 2.5.3.

Let $F = u + iv$. Then F is C^1 function since u, v are C^1. Since $v_x = f = -u_y$ and $v_y = g = u_x$, F satisfies Cauchy-Riemann equation on U. Thus F is holomorphic on U and real part of F is u.

Problem (2.3)**.**

(a) *For $z \notin [0,1]$, the map $w \mapsto \frac{1}{w-z}$ is holomorphic on $\mathbb{C}\setminus[0,1]$. Let $\gamma(t) = t$ for $t \in [0,1]$. Then $F(z) = \int_\gamma \frac{dw}{w-z} = \int_0^1 \frac{1}{t-z}dt$ is well defined.*

For $z \notin [0,1]$, let $d > 0$ be distance between z and $[0,1]$. For $|h| < \frac{d}{2}$, consider $\frac{F(z+h)-F(z)}{h} = \int_0^1 \frac{1}{(t-z-h)(t-z)}dt$. Then $\left|\frac{1}{(t-z-h)(t-z)} - \frac{1}{(t-z)^2}\right| = \left|\frac{h}{(t-z)^2(t-z-h)}\right| \leq |h|\frac{2}{d^3}$ since $|t-z| \geq d$ and $|t-z-h| \geq \frac{d}{2}$. Therefore, as $|h| \to 0$, integrand converges to $\frac{1}{(t-z)^2}$ uniformly on $t \in [0,1]$. So $\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = \int_0^1 \lim_{h\to 0} \frac{1}{(t-z-h)(t-z)}dt = \int_0^1 \frac{1}{(t-z)^2}dt = F'(z)$.

By same reasoning, we get $F''(z) = \int_0^1 \frac{1}{(t-z)^3}dt$. From existence of F'', F' is continuous. Therefore F is C^1 function. Existence of complex derivative and C^1 implies F is holomorphic on $\mathbb{C}\setminus[0,1]$.

(b) *For $s \in (0,1)$, $F(s+i\varepsilon) = \int_0^1 \frac{1}{t-s-i\varepsilon}dt = \int_0^1 \frac{t-s+i\varepsilon}{(t-s)^2+\varepsilon^2}dt = \int_0^1 \frac{t-s}{(t-s)^2+\varepsilon^2}dt + i\int_0^1 \int_0^1 \frac{\varepsilon}{(t-s)^2+\varepsilon^2}dt$. Let $t - s = \varepsilon\tan\theta$. $\varepsilon\tan\theta_0 + s = 0$ and $\varepsilon\tan\theta_1 + s = 1$ for $-\frac{\pi}{2} < \theta_0, \theta_1 < \frac{\pi}{2}$. Then $\sec^2\theta_0 = \frac{s^2}{\varepsilon^2} + 1$, $\sec^2\theta_1 = \frac{(1-s)^2}{\varepsilon^2} + 1$, $\theta_0 = \tan^{-1}\left(\frac{-s}{\varepsilon}\right)$, and $\theta_1 = \tan^{-1}\left(\frac{1-s}{\varepsilon}\right)$.*

Then $F(s+i\varepsilon) = \int_{\theta_0}^{\theta_1} \tan\theta d\theta + i\int_{\theta_0}^{\theta_1} d\theta = \log\left|\frac{\sec\theta_1}{\sec\theta_0}\right| + i(\theta_1 - \theta_0)$. As $\varepsilon \downarrow 0$, $F(s+i\varepsilon)$ goes to $\frac{1-s}{s} + i\pi$ by simple calculation.

Similarly, $F(s-i\varepsilon)$ goes to $\frac{1-s}{s} - i\pi$ as $\varepsilon \downarrow 0$.

(c) *Consider $F(-\varepsilon) = \int_0^1 \frac{1}{t+\varepsilon} dt = \log \frac{1+\varepsilon}{\varepsilon}$. It goes to ∞ as $\varepsilon \downarrow 0$.*

Consider $F(1+\varepsilon) = \int_0^1 \frac{1}{t-1-\varepsilon} dt = \log \frac{\varepsilon}{1+\varepsilon}$. It goes to $-\infty$ as $\varepsilon \downarrow 0$. Therefore, for $s = 0, 1$, $\lim_{z \notin [0,1] \to s} F(z)$ does not exists.

Problem (2.4)**.**

First consider $p \equiv 0$. We can easily see that $\sup_{z \in C} |z^{-n}| = 1$ so desired value ≤ 1.

Note that $|p(z) - z^{-n}| = |z^n p(z) - 1|$. Thus, $1 = \frac{1}{2\pi i} \int_C \frac{z^n p(z) - 1}{z} dz \leq \sup_{z \in C} |z^n p(z) - 1|$.

Those leads the conclusion.

Problem (2.5)**.**

It is enough to show γ and μ are path homotopic. Definte $H(t,s) = (1-s)\gamma(t) + \frac{\gamma(t)}{|\gamma(t)|} s$. Then $H(t,1) = \mu(t)$ and $H(t,0) = \gamma(t)$ by reparametrization. And H is continuous because $\gamma(t) \neq 0$. Therefore H is path homotopy between γ and μ. Since line integration is invariant under path homotopy, we get $\int_\gamma F(\zeta) d\zeta = \int_\mu F(\zeta) d\zeta$.

Problem (3.1).

It suffices to show that $\int_\gamma f(z)dz = 0$ for rectangle γ whose edges are parallel to coordinate axes by Morera's theorem.

First, assume that γ intersects with $[0,1]$ only finitely many points. Let p be such point. Then p must be on (wlog) left edge of γ. Let $a+ib$, $a+ic$ be two vertices incident with left edge. ($b > c$) Let $\rho(t) = a+i(tc+(1-t)b)$. Consider $f \circ \rho$. It is continuous and equals to $\frac{\partial}{\partial t}F(\rho(t))$ except for $\gamma^{-1}(p)$ where F is antiderivative of f on $\mathbb{C} \setminus [0,1]$. Then lemma 2.3.1 says $f(\rho(t)) = \frac{\partial}{\partial t}F(\rho(t))$ even for $\gamma^{-1}(p)$. Therefore $\int_\rho f(z)dz = F(a+ic) - F(a+ib)$. By using this result, we can easily calculate $\int_\gamma f(z)dz = 0$.

Now, assume that (wlog) upper edge of γ intersects with $[0,1]$. Let $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ which are upper edge, left edge, bottom edge, and right edge respectively, parametrized like ρ of above, positive oriented. Consider φ made by shrinking side edges of γ so that distance between of upper edges of φ and γ less than δ, while bottom edge is fixed. Also note that δ is chosen so that $d(z_0, z_1) < \delta$ implies $d(f(z_0), f(z_1)) < \varepsilon$.

$$\left| \int_\gamma f(z)dz - \int_\varphi f(z)dz \right| \leq \left| \int_{\gamma_2 - \varphi_2} f(z)dz + \int_{\gamma_4 - \varphi_4} f(z)dz \right| + (\text{ length of } \gamma_1)\,\varepsilon$$

And, second term of above goes to 0 as distance between φ_1 and γ_1 goes to 0 by continuity and result of first case. Actually $\int_\varphi f(z)dz = 0$ because φ does not intersect with $[0,1]$. Thus we have shown that $\int_\gamma f(z)dz = 0$.

By first, second case and Morera's thm, f is actually entire function.

Problem (3.2).

For $0 < r < 1$, $|f^{(n)}(0)| \leq \frac{n!}{r^n}\frac{1}{1-r}$ by using Cauchy estimate. $r^n(1-r)$ is maximized when $r = \frac{n}{n+1}$. So, when $r = \frac{n}{n+1}$, we get best estimate of $|f^{(n)}(0)|$.

Problem (3.3).

(a) *Since K is compact subset of open set U, there is $r > 0$ such that for all $x \in K$, closure of $D(x,r)$ is in U. Then, $|f(z)|^2 \leq \frac{1}{2\pi}\left|\int_{\partial D(z,r} \frac{f^2(w)}{w-z}dw\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f^2(z+re^{i\theta})d\theta|$. By multiplying ρ both sides and integrating from 0 to r, we can get the following:*

$$\begin{aligned}\frac{r^2}{2}|f(z)|^2 &\leq \frac{1}{2\pi}\int_0^r\int_0^{2\pi}\rho|f^2(z+re^{i\theta})|d\theta d\rho\\ &= \frac{1}{2\pi}\int_{\overline{D}(z,r)}|f|^2dm\\ &= \frac{1}{2\pi}\int_U|f|^2dm\end{aligned}$$

for all $z \in K$, where m is lebesgue measure, using Holder's inequality and polar coordinate integration.

Therefore $C = \frac{1}{r\sqrt{\pi}}$

(b) *If f is identically zero, possible.*

Else if f is constant, then $\int_{\mathbb{C}}|f|dm = \infty$ since measure of complex plane is ∞.

Else, that is f is nonconstant entire function, then f must be unbounded. So, there is $\delta > 0$ such that $|f| \geq 1$ for all $|z| > \delta$. Then $\int_{\mathbb{C}}|f|dm \geq m(\{z : |z| > \delta\}) = \infty$.

Problem (3.4)**.** (a) *Since $\frac{z}{e^z-1}$ is bounded near 0, it has removable singularity at 0. So we can regard it as holomorphic function. Note that $e^z - 1 = 0$ when z is integer multiple of $2\pi i$. So, given power series converges on unit disc. Now, multiply $e^z - 1$ both sides. Since $e^z - 1$ is entire and given power serires converges absolutely on $\bar{D}(0, r)$ where $0 < r < 1$, we can write $z = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n\sum_{n=1}^{\infty}\frac{1}{n!}z^n$. Since z is entire, coefficient of power series is unique. By comparing coefficients of both sides, we can get given recursion formula.*

$\lim_{z\to 0}\frac{z}{e^z-1} = 1 = B_0$. *From this, by simple calculation, $B_1 = \frac{-1}{2}$, $B_2 = \frac{1}{6}$, and $B_3 = 0$.*

Consider $-z = f(z) - f(-z) = \sum_{n=0}^{\infty}2\frac{B_{2n+1}}{(2n+1)!}z^{2n+1}$. This makes sense because f is holomorphic on unit disc. By comparing coefficient of this series, we can get $B_{2m+1} = 0$ for $m \geq 1$.

(b) *We alreadly notice that $e^z - 1$ is zero when z is integer multiple of $2\pi i$. But $\lim_{z\to 2k\pi i}\frac{z}{e^z-1}$ is not bounded when $k \neq 0$. Therefore, $\frac{z}{e^z-1}$ is holomorphic on $D(0, 2\pi)$ and is not holomorphic outside of that disc. Since* *power seriese representation of holomorphic function at P has radius of convergence at least $d(P,U)$, we can say radius of convergence of the series is 2π.*

Problem (3.5)**.**

f' is holomorphic on unit disc. Let $r = \sup_{z\in K}|z|$. Since K is compact, $|f'| \leq M$ on K and r is positive but less than 1. Let $\gamma(t) = tz^n$ which connects origin and z^n. $|f(z^n) - f(0)| = \left|\int_\gamma f'dz\right| \leq M\sup_{z\in K}|z|^n = Mr^n$. Therefore, $|\sum_{n=1}^{\infty} f(z^n)| \leq \sum_{n=1}^{\infty}|f(z^n)| \leq \sum_{n=1}^{\infty} Mr^n < \infty$ because r is positive but less than 1.

Problem (4.1)**.**

Notice that f does not vanish on $\mathbb{C} \setminus \{0\}$. Therefore $g(z) = \frac{1}{f(z)}$ is holomorphic on $\mathbb{C} \setminus \{0\}$. Near 0, g is bounded since $\sqrt{|z|}$ goes to 0 as z goes to 0. This means g has removable singularity at 0 and therefore entire. But $g(z) \le \sqrt{|z|}$, so g must be constant by Cauchy integral formula.

Then f must be constant also, and this is contradiction. Therefore there is no such holomorphic function.

Problem (4.2)**.**

Let $g(z) = f\left(\frac{1}{z}\right)$. Then $g \to 0$ as $z \to 0$. Therefore g is entire. Also, $g(z)/z$ is entire since $\lim_{z\to 0} g(z)/z = g'(0)$ hence bounded near 0.

Now, consider given integral. Let $\zeta = e^{it}$ and $t = 2\pi - s$. Then given integral is $\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(e^{-is})}{e^{-is}-z} ie^{-is} ds = \frac{1}{2\pi i}\int_0^{2\pi} \frac{g(e^{is})}{e^{is}-e^{2is}z} ie^{is} ds = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{g(\zeta)}{\zeta z\left(\frac{1}{z}-w\right)} d\zeta$

Therefore given integral is equal to $\frac{1}{2\pi i}\int_{|\zeta|=1} \frac{h(\zeta)}{\frac{1}{z}-\zeta} d\zeta$ where $h(\zeta) = \frac{g(\zeta)}{\zeta z}$. Thus, it is equal to $-g(1/z) = -f(z)$.

Problem (4.3)**.**

f maps $re^{i\theta}$ to $\sqrt{r}e^{i\left(\frac{\theta}{2}+k(z)\pi\right)}$ where $k(z) \in \mathbb{Z}$. To f be continuous, $k(z)$ must be all even or all odd.

First assume that $k(z)$ is all even. Then $f'(0) = \lim_{\mathbb{R}\ni h\to 0} \frac{f(h)}{h} = \lim_{\mathbb{R}\ni h\to 0} \frac{\sqrt{h}}{h} = \infty$, which is contradiction.

Similarly, if $k(z)$ is all odd, $f'(0)$ does not exist.

Therefore existence of such f leads $0 \notin U$.

Let ι be identity function of U. Since $z \notin U$, ι does not vanish on U, hence $1/\iota$ is holomorphic on U. Since U is hsc, $1/\iota$ has holomorphic antiderivative φ.

Now consider the derivative of $\iota(z)e^{-\varphi(z)}$. Simple calculation leads that it is equal to 0. Hence $\iota(z) = ce^{\varphi(z)}$ for some constant c. Therefore $\iota(z) = e^{\psi(z)}$ for some holomorphic ψ on U.

Take $f = e^{\frac{1}{2}\psi}$. Then f satisfies what we want.

Problem (4.4)**.**

(a) *Let γ_R be the contour used in example 4.6.5.*

First, consider $\int_0^\infty \frac{1}{x^a(x+1)}dx$*. To calculate this, take* $f(z) = z^{-a}/(1+z)$ *where* $0 < arg(z) < 2\pi$*. By residue thm,* $2\pi i e^{-a\pi i} = \int_0^\infty \frac{1}{r^a(r+1)}dr\left(1 - e^{-2a\pi i}\right)$*. Therefore* $\int_0^\infty \frac{1}{x^a(x+1)}dx = \pi\csc(\pi a)$*.*

Now, $\int_{\gamma_R} \frac{\log z}{z^a(1+z)}dz = 2\pi i e^{-a\pi i}\pi i$ *by residue thm. But as* $R \to \infty$*, that integral goes to* $(1 - e^{-2a\pi i})\int_0^\infty \frac{\log r}{r^a(r+1)}dr - e^{-2a\pi i}\int_0^\infty \frac{2\pi i \log r}{r^a(r+1)}dr$*.*

By simple calculation, the value we want is equal to $\frac{i\pi^2}{\sin(\pi a)} + \frac{\pi^2 e^{-a\pi i}}{\sin^2(\pi a)} = \frac{\pi^2\cos(\pi a)}{\sin^2(\pi a)}$*.*

(b) *Consider* $f(z) = \frac{\pi\cot(\pi z)}{(z+\alpha)^2}$ *and* Γ_n *= square centered at origin, each edges is parallel to real or imaginary axis, length of edge is* $2n+1$*.*

Then $\int_{\Gamma_n} f(z)dz$ *goes to 0 as* $n \to \infty$ *by considering modulus of* $f(z)$*, and index of* Γ_n *at each singularites is 1, and residues are* $\frac{1}{(k+\alpha)^2}$ *at* $z = k$ *and* $-\frac{\pi^2}{\sin^2(\pi\alpha)^2}$ *at* $z = -\alpha$*.*

Above calculation leads the conclusion.

Problem (4.5)**.**

Note that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *is holomorphic iff* f *is meromorphic on* $\hat{\mathbb{C}}$*.*

(a) *First consider 'if' part. Let* f *be rational function. We already knows that rational function is meromorphic on entire complex plane. So, we need to show that rational function is meromorphic at* ∞*.*

Let $f(z) = \frac{(z-Q_1)^{m_1}\cdots(z-Q_l)^{m_l}}{(z-P_1)^{n_1}\cdots(z-P_k)^{n_k}}$*. Since* f *has finitely many pole in complex plane, we can choose* $M > 0$ *so that* f *has no pole on* $\{z : |z| > M\}$*. For* $0 < |w| < \frac{1}{M}$*, consider* $g(w) = f(1/w)$*. Then* g *is holomorphic.*

Let $\sum_i n_i = N$ *and* $\sum_j m_j = M$*. If* $M = N$*,* $g \to 1$ *as* $z \to 0$*. If* $M > N$*,* $g \to 0$ *as* $z \to 0$*. If* $M < N$*,* $g \to \infty$ *if* $z \to 0$*. Hence* g *is meromorphic near* 0*, which means that* f *is meromorphic at* ∞*.*

Second, consider 'only if' part. Either f *has a pole or removable singularity at* ∞*,* f *has finitely many poles in complex plane. So* $f(z)(z-P_1)^{n_1}\cdots(z-P_k)^{n_k} = F(z)$ *is entire where* n_i *is order of pole* P_i*.*

Consider $F(1/z) = g(z)$ *for* $z \neq 0$*. As* $z \to 0$*,* $g \to \infty$ *or* α *for some* $\alpha \in \mathbb{C}$ *by simple calculation. Therefore* F *has a pole or removable singularity at* ∞*.*

If F has removable singularity at ∞, F must be bounded, hence constant by Liouville's thm.

If F has a pole at ∞, F must be polynomial since its modulus diverges.

In both cases, F must be rational function.

(b) *Note that $z \mapsto \frac{az+b}{cz+d}$ for $ad-bc \neq 0$ is biholomorphic function of Riemann sphere. Also note that biholomorphic function of $\mathbb{C}$ must have a form of $\alpha z + \beta$ for $\alpha \neq 0$ by fundamental thm of algebra.*

Now consider biholomorphic f on Riemann sphere. Let $f(\infty) = b$ and $\varphi_b(z) = \frac{-\bar{b}-1}{z-b}$. Then $\varphi_b \circ f$ is biholomorphic function of Riemann sphere, which maps $\infty \to \infty$. Therefore $\varphi_b \circ f$ is biholomophic function of complex plane hence $\varphi_b(f(z)) = \alpha z + \beta$. Then $f(z) = \frac{-b\alpha z - b\beta + 1}{-\alpha z - \beta - \bar{b}}$, which is linear frational transformation.

Problem (5.1)**.**

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ and assume that $P(z) = 0$ has no solution. Then by the argument principle, $\frac{1}{2\pi i}\int_{\partial D(Q,R)} \frac{P'(\zeta)}{P(\zeta)} d\zeta = 0$ for all $R > 0$. That integral is equal to $\frac{1}{2\pi i}\int_0^{2\pi} \frac{P'(Q+Re^{i\theta})}{P(Q+Re^{i\theta})} Rie^{i\theta} d\theta$. But, as $R \to \infty$, integrand of above goes to in uniformly on $0 \le \theta \le 2\pi$. Therefore, the integral above goes to $n > 0$ which is the degree of P. It is contradiction. Thus $P(z) = 0$ has at least one solution in complex plane.

Problem (5.2)**.**

Assume the existence of such f. Since f is bounded near 0, Riemann removable singularity theorem says that f can be extended to the function which is holomorphic on entire unit disc.

If modulus of $f(0)$ is equal to 1 or 2, then image of the unit disc under f is not open which contradicts to the open mapping theorem. So $f(0) \in \{w : 1 < |w| < 2\}$.

Since f is surjective function of the punctured unit disc onto the annulus, we can find $w \neq 0$ such that $f(0) = f(w)$. Choose two disjoint neighborhood U_w, U_0 of $w, 0$ respectively. Then by the open mapping theorem, $f(U_w)$ and $f(U_0)$ are open and $f(0) \in f(U_w) \cap f(U_0)$. Since $f(U_w) \cap f(U_0)$ is open, we can choose small neighborhood of $f(0)$ contained in the previous set. And therefore we can choose $f(0) \neq \alpha \in f(U_w) \cap f(U_0)$. This cannot be happen since f is injective.

Thus there is no such f.

Problem (5.3)**.**

(a) *Choose $R > \lambda$, and choose n so large that $\lambda - 1 \ge 1/n$. Then $\bar{D}(R, R - \frac{1}{n}) \subset$ Right half plane.*

Then for $\zeta \in \partial D(R, R - 1/n)$, $|e^{-\zeta}| < 1 \le \lambda - 1/n \le |\zeta - \lambda|$. Put $f(z) = e^{-z} + z - \lambda$ and $g(z) = z - \lambda$. Then by above and Rouche's theorem, f and g has same zero on $D(R, R - 1/n)$. But any $z \in$ Right half plane must be inside of $D(R, R - 1/n)$ for some large R and n. This means f and g have same zero on the right half plane.

But $g(z) = 0$ has unique solution. Therefore $e^{-z} + z - \lambda = 0$ has unique solution on the right half plane.

(b) *Fix $z' \in U$. Note that $U \setminus \{z'\}$ is still a domain. Let $g_k(z) = f_k(z) - f_k(z')$ for $z \in U \setminus \{z'\}$. Since f_j is an injective holomorphic function on U, g_k does not vanish on $U \setminus \{z'\}$. Uniform convergence of f_j on compact subsets of U implies uniform convergence of g_k on compact subsets of $U \setminus \{z'\}$. Since g_k is nonvanishing function, by Hurwitz's theorem, $\lim_{k\to\infty} g_k(z) = f(z) - f(z')$ does not vanish or identically zero.*

If it is identically zero on $U \setminus \{z'\}$, then f must be constant function on U. If it is nonvanishing on $U \setminus \{z'\}$, then $f(z'') = f(z')$ implies $z'' = z'$. Thus f must be injective.

Problem (5.4)**.**

It seems to be solved by the maximum modulus principle (or theorem), but I don't know where to start.

Problem (5.5)**.**

For $z \in S$, $|\varphi(z)| = \left|\frac{e^{2\pi z i}-1}{e^{2\pi z i}+1}\right|$, and the real part of $e^{2\pi z i} > 0$ because $z \in S$. Then it is clear that $|\varphi(z)| < 1$. Also $\varphi(0) = 0$.

Therefore $\varphi \circ f : D \to D$ is holomorphic and it fixes the origin. Then Schwarz's lemma says $|\varphi'(0) f'(0)| \leq 1$. But $\varphi'(0) = \pi$. Therefore $|f'(0)| \leq 1/\pi$. The equality holds only if $\varphi(f(z)) = wz$ for some $|w| = 1$.

Problem (9.1)**.**

First, by considering the Maclaurine series of $\cos z$, $\cos\sqrt{z}$ *is an entire function. Now, note that* $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, *so modulus of* $\cos\sqrt{z}$ *is bounded by* $e^{|z|/2}$. *Therefore* $\lambda(\cos\sqrt{z}) \leq 1/2$. *Since genus is nonnegative integer bounded by order, the genus of* $\cos\sqrt{z}$ *must be 0.*

Now consider $f(z) = \sin^2 z$. *Its zero set is* $\{k\pi\}$ *where* k *is an integer. Note that the smallest nonnegative integer* p *satisfying* $\sum_{k\neq 0} |k\pi|^{-p-1}$ *is 1. Therefore the rank of* f *is 1. Since* $f(z) = \frac{e^{2iz}+e^{-2iz}-2}{-4}$, *its modulus is bounded by* $e^{2|z|}$. *Thus* $\lambda(f) \leq 1$. *But we know the relation :* $1 =$ *rank* $\leq$ *genus* $\leq$ *order* ≤ 1. *Therefore the genus of* $\sin^2 z$ *is one.*

Now consider $g(z) = \sin z^2$. *The zero set of* g *is* $\{\sqrt{k\pi}\}$ *where* k *is an integer. Note that the smallest nonnegative integer* p *satisfying* $\sum 2|\sqrt{k\pi}|^{-p-1}$ *is 2. Therefore the rank of* g *is 2. Since* $g(z) = \frac{e^{iz^2}-e^{-iz^2}}{2i}$, *its modulus is bounded by* $e^{|z|^2}$. *Thus* $\lambda(g) \leq 2$. *So,* $2 =$ *rank* $\leq$ *genus* $\leq$ *order* ≤ 2.

□

Problem (9.2)**.**

It is well known fact that $\left\{e^{in\sqrt{2}\pi} : n \in \mathbb{N}\right\}$ *is dense in* S^1. *Let* $a_n = \frac{2^n-1}{2^n}e^{in\sqrt{2}\pi}$. *Then every point on* S^1 *is accumulation point of* $\{a_n\}_{n=1}^{\infty}$. *Note that* $\sum 1 - |a_n| = \sum 2^{-n} < \infty$. *Therefore the corresponding Blaschke product* $B(z) = \Pi_n - \frac{\bar{a_n}}{|a_n|}B_{a_n}(z)$ *is holomorphic on the unit disc* D *and vanishes on* $\{a_n\}_{n=1}^{\infty}$ *exactly. But, if* $w \in \partial D$, *then* w *is accumulation point of the zero set of* B. *Thus if* w *is regular, then extension of* B *on small neighborhood of* w *is identically zero, which is contradiction. So* B *is the desired one.*

□

Let f be an entire function. Let $M(r) = \sup_{|z|=r} |f(z)|$. Before #3 and #4, we need the followings:

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \lambda$$

$$\limsup_{n\to\infty} \frac{n\log n}{-\log|a_n|} = \lambda$$

where a_n is the n-th Maclaurine coefficient of f.

For the first formula, let $\rho < a = \limsup \frac{\log\log M(r)}{\log r}$. Then there is $r_n \uparrow \infty$ such that $\rho < \frac{\log\log M(r_n)}{\log r_n}$. Then $M(r_n) > \exp(r_n^\rho)$ which says $\lambda \geq \rho$. Since ρ is arbitrary, we can deduce that $\lambda \geq a$.

For the other direction, let $\rho < \lambda$. Then there is increasing sequnce $r_n \uparrow \infty$ such that $M(r_n) > \exp(r_n^\rho)$. Thus $\log\log M(r_n)/\log r_n \geq \rho$ which leads $a \geq \rho$. Since $\rho \leq \lambda$ is arbitrary, $a \geq \lambda$.

For the second formula, let $\mu = \limsup_n \frac{n\log n}{-\log|a_n|}$. If $\mu = \infty$, then $\lambda \leq \mu$ directly. So assume $\mu < \infty$ and $\varepsilon > 0$. Then $0 \leq \frac{n\log n}{-\log|a_n|} \leq \mu + \varepsilon$ for $n \geq N$. By simple calculation, $|a_n| \leq n^{-n/(\mu+\varepsilon)}$. Thus $M(r) \leq \sum |a_n| r^n \leq \sum n^{-n/(\mu+\varepsilon)} r^n = \sum_{n<(2r)^{\mu+\varepsilon}} () + \sum_{n \geq (2r)^{\mu+\varepsilon}} () = S_1 + S_2$.

$$\begin{aligned}
S_1 &\leq r^{(2r)^{\mu+\varepsilon}} \sum_n n^{-n/(\mu+\varepsilon)} \\
&= O(r^{(2r)^{\mu+\varepsilon}} = O(\exp((2r)^{\mu+\varepsilon}\log r)) \\
&= O(\exp(r^{\mu+2\varepsilon}))
\end{aligned}$$

And $n^{-1/(\mu+\varepsilon)} r \leq 1/2$ yields $S_2 \leq 1$. Thus $M(r) = O(\exp(r^{\mu+2\varepsilon}))$, which implies $\lambda \leq \mu + 2\varepsilon$. By letting $\varepsilon \downarrow 0$, we get $\lambda \leq \mu$.

For the other direction, let $0 < \tau < \mu$. Then $\tau \leq \frac{n\log n}{-\log|a_n|}$ for infinitely many n which goes to ∞. For those n, $\log|a_n| \geq \frac{-n\log n}{\tau}$. By cauchy's thm, we know that $|a_n| \leq M(r) r^{-n}$. So,

$$\begin{aligned}
\log M(r) &\geq \log|a_n| + n\log r \\
&\geq n\left(\log r - \frac{\log n}{\tau}\right)
\end{aligned}$$

By taking $r_n = (en)^{1/\tau}$, $\log M(r_n) \geq n/\tau = r_n^\tau/(e\tau)$. So

$$\frac{\log\log M(r_n)}{\log r_n} \geq \frac{\tau\log r_n - \log e\tau}{\log r_n}$$

thus $\limsup \geq \tau$. Since τ is arbitrary, we get $\lambda \geq \mu$ by the first formula.

□

Problem (9.3)**.**

If $\sum a_n z^n$ is an entire function, then its order is determined by $\limsup_{n\to\infty} \frac{n\log n}{-\log|a_n|}$.

(a) *First represent f as the Maclaurine series. Let a_n be its n-th coefficient.*

But $\limsup_n \frac{n \log n}{-\log n - \log |a_n|} = \limsup_n \frac{n \log n}{-\log |a_n|}$. *So the order of* f *and* f' *are same.*

(b) *Note that* $\log E_n(z) = z^{n+1}/(n+1) + z^{n+2}/(n+2) + \cdots$ *by power series. Also,* $\log |z| \leq |\log z| = |\log |z| + iarg(z)|$. *So* $\log |E_n(z)| \leq |z|^{n+1}/(1 - |z|)$ *for* $|z| < 1$.

By definition of E_n*, it is also clear that* $\log |E_n| \leq \log |E_{n-1}| + |z|^n$. *Now we claim that* $\log |E_n| \leq (2n+1)|z|^{n+1}$. *This can be done by the following:*

$$\begin{aligned}\log |E_n| &\leq |z| \log |E_n| + |z|^{n+1} \\ &\leq |z|(\log |E_{n-1}| + |z|^n) + |z|^{n+1} \\ &\leq |z|(2n|z|^n) + |z|^{n+1}\end{aligned}$$

for $|z| < 1$ *and induction. The case when* $|z| \geq 1$ *can be done by using the part of above.*

Now put $n = \mu =$*genus. Let* P *be the canonical product of given entire function with rate* μ. *Then* $\log |P| \leq (2\mu + 1)|z|^{\mu+1} \sum_n |a_n|^{-\mu-1}$. *Since* $f = cz^m e^g P$ *where the degree of* g *is less or equal to* μ*, the order of* f *is thus determined by* P. *The above inequality implies* $\lambda(f) \leq \mu + 1$.

(c) *Let* a_n *be a sequence of zeros of* f. *Since we know that the order of* f *and* f' *are same,* $\lambda(f) \leq 1$. *Thus* $\sum_n |a_n|^{-1-1} < \infty$. *But* $\sum_n (\sqrt{n})^{-1-1} \leq \sum_n |a_n|^{-1-1} < \infty$ *which is contradiction. Therefore* f *must be constant, so* $f(z) = 0$ *for every* z.

□

Problem (9.4)**.**

Let a_n *be* n*-th coefficient of* g. *Then* $\limsup_{n\to\infty} |a_n|^{1/n} = 0$ *so the radius of convergence is* ∞*, thus* g *is an entire function.*

By Stirling's formula, $\log(n!) = n \log n - n + O(\log n)$. *Therefore* $\frac{n \log n}{\log(n!)} \to 1$ *as* $n \to \infty$.

$\frac{n \log n}{-\log a_n} = \frac{n \log n}{\alpha \log(n!)} \to 1/\alpha$ *as* $n \to \infty$. *Therefore the order of* g *is* $1/\alpha$.

□

Problem (9.5)**.**

By considering the Maclaurine series of $\sin z$, $\sin\sqrt{z}/\sqrt{z}$ *is holomorphic by the Riemann removable singularity theorem. And by simple calculation, its order is bigger than* 0 *and smaller or equal to* $1/2$.

Now, consider $f(z) = \sin z/z$. *Since the order of* f *is finite and* f *is entire, it can omit at most one complex number. If* f *omit the value* c,*then* $f(z) - c$ *is nonvanishing, so* $f(z) - c = \exp(g(z))$. *But the degree of* g *must be* 0 *or* 1 *since the order of* f *is less or equal to* 1. *If the degree of* g *is zero, then* $f(z)-c$ *is constant which is contradiction. So we can say that* $f(z) - c = \exp(az + b)$. *But, as* $|z| \to \infty$, $\left|\frac{f(z)-c}{\exp(az+b)}\right| \to 0$ *which is contradiction because it must be equal to* 1. *Therefore, we can conclude that* $f(z)$ *assumes every complex value.*

Let $c \in \mathbb{C}$ *be given. Then the solution of* $f(z) = c$ *exists, say* α. *Then* α^2 *is a solution of* $f(\sqrt{z}) = c$. *Therefore* c *is in the image of* $f(\sqrt{z})$, *which is entire of nonintegral finite order. Thus there are infinitely many solutions of* $f(\sqrt{z}) = c$, *say* $w_1, w_2, \cdots$. *Then* $\sqrt{w_1}, \sqrt{w_2}, \cdots$ *are the infinite solutions of* $f(z) = c$.

$\square$