RCA SOLUTIONS

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1. Abstract Integration

Problem (10).

Since each f_n is bounded and $\mu(X) < \infty$, $f_n \in L^1$. We can conclude that f is also bounded hence $f \in L^1$ because f_n is uniformly convergent to f and each f_n is bounded.

$$\begin{aligned}\left|\int_X f_n d\mu - \int_X f d\mu\right| &\leq \int_X |f_n - f| \, d\mu \\ &\leq \varepsilon\mu(X)\end{aligned}$$

if n is large, because of uniform convergence. So we can see that $\lim \int_X f_n d\mu = \int_X f d\mu$.

4. Elementary Hilbert Space Theory

Problem (1).

Let $m \in M$. Then $(m, x) = 0$ for all $x \in M^{\perp}$. So $m \in \left(M^{\perp}\right)^{\perp} \Rightarrow M \leq \left(M^{\perp}\right)^{\perp}$.

On the contrary, let $m \in \left(M^{\perp}\right)^{\perp}$. Decompose m by $Pm + Qm$ where $Pm \in M$ and $Qm \in M^{\perp}$. For $x \in M^{\perp}$, $(m, x) = (Qm, x) = 0$. Take $x = Qm$. Then $Qm = 0$, which leads $m = Pm \in M$. Hence $\left(M^{\perp}\right)^{\perp} \leq M$.

Problem (5).

Since L is continuous linear functional on H, there is nonzero unique $y \in H$ such that $Lx = (x, y)$ for all $x \in H$. (When L is nontrivial linear functional.)

For all $x \in M$, $(x, y) = Lx = 0$. So $y \in M^{\perp}$.

If dimension of $M^{\perp}$ is bigger than 1, we can take $z \in M^{\perp}$ such that $z \neq 0$ and $\{z, y\}$ is linearly independent.

Now, consider the following:

$$u = z - \frac{(z, y)}{(y, y)} y$$

Then $(u, y) = 0$ and $u \in M^{\perp}$, so $u = 0$. This contradicts to independency of $\{z, y\}$. So dimension of $M^{\perp}$ is smaller than 1.

Problem (7).

Let $N_0 = 0$, choose N_1 so that $\sum_{n=N_0+1}^{N_1} a_n^2 > 1$. For chosen $N_0, \cdots, N_k$, choose N_{k+1} so that $\sum_{n=N_k+1}^{N_{k+1}} a_n^2 > 1$. And put $E_k = \{N_{k-1}, \cdots, N_k\}$.

Let $s_k = \sum_{n \in E_k} a_n^2 > 1$, and $c_k = \frac{1}{k s_k}$. Then, done.

Problem (8).

Let $\{v_\beta : \beta \in B\}$ be an orthonormal basis of H_2. Then $H_2 \cong l^2(B)$. Additionally, assume $H_1 \cong l^2(A)$ and without loss of generality, there exists an injection $\varphi : A \hookrightarrow B$. Let Q be a class of all finite linear combination of $\{v_\beta : \beta \in \varphi(A)\}$. Then $\overline{Q}$ is Hilbert space, being closed subspace of H_2.

Then $\overline{Q} \cong l^2(\varphi(A)) \cong l^2(A) \cong H_1$. So H_1 is isomorphic to $\overline{Q}$, which is closed subspace of H_2.