mas541 homework

20208209 오재민

2020년 9월 26일

Problem (1).

$$\begin{aligned}1-\left|\frac{z-w}{1-z\overline{w}}\right|^2 &= 1-\frac{(z-w)\left(\overline{z}-\overline{w}\right)}{\left(1-z\overline{w}\right)\left(1-\overline{z}w\right)}\\ &= \frac{1-\overline{z}w-z\overline{w}+|z|^2|w|^2-|z|^2-|w|^2+z\overline{w}+\overline{z}w}{|1-\overline{z}w|^2}\\ &= \frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\overline{z}w|^2}\end{aligned}$$

Problem (2).

Let $f=u+iv$. $\partial f=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)$. *Then* $\overline{\partial f}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(u-iv)=\bar{\partial}\bar{f}$.

Problem (3).

If f *is constant, then* $|f|$ *is also constant. On the other hand, assume* $f=u+iv$ *and* $|f|^2=u^2+v^2$ *is positive real number. (if it is zero, then* f *must be zero)*

$$u^2+v^2=R>0$$

Differentiate both sides of the equation above with x *and* y *respectively, we can get* $uu_x+vv_x=0$, $uu_y+vv_y=0$, $u_x=v_y$ *and* $u_y=-v_x$. *By simple calculation we can get* $u_x=u_y=v_x=v_y=0$. *Therefore* u,v *are constant.*

Problem (4).

Note that $\int_0^{2\pi}e^{ik\theta}d\theta=\int_0^{2\pi}(\cos k\theta+i\sin k\theta)\,d\theta=0$ *for positive integer* k. *Therefore* $\frac{1}{2\pi}\int_0^{2\pi}(z_0+re^{i\theta})^jd\theta=\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{j}\binom{j}{k}z_0^k\left(re^{i\theta}\right)^{j-k}d\theta=z_0^j$. *Similarly, we can get* $\frac{1}{2\pi}\int_0^{2\pi}\overline{(z_0+re^{i\theta})^j}d\theta=\bar{z}_0{}^j$.

Since u *is polynomial, we can write it as* $\sum_{l,k}a_{l,k}z^l\bar{z}^k$. *By direct computation, we can get* $\frac{1}{2\pi}\int_0^{2\pi}u\left(z_0+re^{i\theta}\right)d\theta=\sum_{l,k}a_{l,k}z_0^l\bar{z}_0{}^k=u(z_0)$.

Problem (5).

Let $f=u+iv$. $(g\circ f)_x=g_uu_x+g_vv_x$. *Then*

$$\begin{aligned}(g\circ f)_{xx}&=\left(g_{uu}u_x+g_{uv}v_x\right)u_x+g_uu_{xx}+\left(g_{vu}u_x+g_{vv}v_x\right)v_x+g_vv_{xx}\\(g\circ f)_{yy}&=\left(g_{uu}u_y+g_{uv}v_y\right)u_y+g_uu_{yy}+\left(g_{vu}u_y+g_{vv}v_y\right)v_y+g_vv_{yy}\end{aligned}$$

But we have Cauchy-Riemann equation and $g_{uu}+g_{vv}=0$ *and* $g_{vu}=g_{uv}$. *Also, since* f *is* C^2 *function,* f *is harmonic,* $u_{xy}=u_{yx}$, *and* $v_{xy}=v_{yx}$. *Using* *these equations, we can check that* $(g \circ f)_{xx} + (g \circ f)_{yy} = 0$. *Hence* $(g \circ f)$ *is a harmonic function.*

Problem (1).

Let $f = u + iv$. *Then* $\bar{f}f' = ff' - 2ivf'$, *where* ff' *is holomorphic. So,* $\int_\gamma \bar{f}f'dz = \int_\gamma -2ivf'dz = \int_\gamma -2iv(u_x + iv_x)dz = \int_\gamma -2iv(v_y + iv_x)dz = -i\int_a^b (2vv_y + 2ivv_y)(\gamma_1' + i\gamma_2')dt = \alpha$ *where* $\gamma = \gamma_1 + i\gamma_2$.

Therefore, real part of $\int_\gamma \bar{f}f'dz$ *is equal to real part of* α. *And it is also equal to* $-\int_a^b Im\left[(2vv_y + i2vv_x)(\gamma_1' + i\gamma_2')\right]dt = -\int_a^b (2vv_x\gamma_1' + 2vv_y\gamma_2')dt = -\int_a^b \frac{d}{dt}(v^2 \circ \gamma)dt = 0$ *since* γ *is closed curve.*

So, $\int_\gamma \bar{f}f'dz$ *is purely imaginary.*

Problem (2).

Let $f = -u_y$ *and* $g = u_x$. *Then* f, g *are continuous on* U. *Since* u *is harmonic,* $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ *on* $U \setminus \{0\}$. *So there is* $v : U \to \mathbb{R}$ *which is* C^1 *function and* $v_x = f$, $v_y = g$ *by lemma 2.5.3.*

Let $F = u + iv$. *Then* F *is* C^1 *function since* u, v *are* C^1. *Since* $v_x = f = -u_y$ *and* $v_y = g = u_x$, F *satisfies Cauchy-Riemann equation on* U. *Thus* F *is holomorphic on* U *and real part of* F *is* u.

Problem (3).

(a) *For* $z \notin [0,1]$, *the map* $w \mapsto \frac{1}{w-z}$ *is holomorphic on* $\mathbb{C}\setminus[0,1]$. *Let* $\gamma(t) = t$ *for* $t \in [0,1]$. *Then* $F(z) = \int_\gamma \frac{dw}{w-z} = \int_0^1 \frac{1}{t-z}dt$ *is well defined.*

For $z \notin [0,1]$, *let* $d > 0$ *be distance between* z *and* $[0,1]$. *For* $|h| < \frac{d}{2}$, *consider* $\frac{F(z+h)-F(z)}{h} = \int_0^1 \frac{1}{(t-z-h)(t-z)}dt$. *Then* $\left|\frac{1}{(t-z-h)(t-z)} - \frac{1}{(t-z)^2}\right| = \left|\frac{h}{(t-z)^2(t-z-h)}\right| \leq |h|\frac{2}{d^3}$ *since* $|t-z| \geq d$ *and* $|t-z-h| \geq \frac{d}{2}$. *Therefore, as* $|h| \to 0$, *integrand converges to* $\frac{1}{(t-z)^2}$ *uniformly on* $t \in [0,1]$. *So* $\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = \int_0^1 \lim_{h\to 0}\frac{1}{(t-z-h)(t-z)}dt = \int_0^1 \frac{1}{(t-z)^2}dt = F'(z)$.

By same reasoning, we get $F''(z) = \int_0^1 \frac{1}{(t-z)^3}dt$. *From existence of* F'', F' *is continuous. Therefore* F *is* C^1 *function. Existence of complex derivative and* C^1 *implies* F *is holomorphic on* $\mathbb{C}\setminus[0,1]$.

(b) *For* $s \in (0,1)$, $F(s+i\varepsilon) = \int_0^1 \frac{1}{t-s-i\varepsilon}dt = \int_0^1 \frac{t-s+i\varepsilon}{(t-s)^2+\varepsilon^2}dt = \int_0^1 \frac{t-s}{(t-s)^2+\varepsilon^2}dt + i\int_0^1 \int_0^1 \frac{\varepsilon}{(t-s)^2+\varepsilon^2}dt$. *Let* $t - s = \varepsilon\tan\theta$. $\varepsilon\tan\theta_0 + s = 0$ *and* $\varepsilon\tan\theta_1 + s = 1$ *for* $-\frac{\pi}{2} < \theta_0, \theta_1 < \frac{\pi}{2}$. *Then* $\sec^2\theta_0 = \frac{s^2}{\varepsilon^2} + 1$, $\sec^2\theta_1 = \frac{(1-s)^2}{\varepsilon^2} + 1$, $\theta_0 = \tan^{-1}\left(\frac{-s}{\varepsilon}\right)$, *and* $\theta_1 = \tan^{-1}\left(\frac{1-s}{\varepsilon}\right)$.

Then $F(s+i\varepsilon) = \int_{\theta_0}^{\theta_1}\tan\theta d\theta + i\int_{\theta_0}^{\theta_1}d\theta = \log\left|\frac{\sec\theta_1}{\sec\theta_0}\right| + i(\theta_1 - \theta_0)$. *As* $\varepsilon \downarrow 0$, $F(s+i\varepsilon)$ *goes to* $\frac{1-s}{s} + i\pi$ *by simple calculation.*

Similarly, $F(s - i\varepsilon)$ goes to $\frac{1-s}{s} - i\pi$ as $\varepsilon \downarrow 0$.

(c) *Consider $F(-\varepsilon) = \int_0^1 \frac{1}{t+\varepsilon} dt = \log \frac{1+\varepsilon}{\varepsilon}$. It goes to ∞ as $\varepsilon \downarrow 0$.*

Consider $F(1+\varepsilon) = \int_0^1 \frac{1}{t-1-\varepsilon} dt = \log \frac{\varepsilon}{1+\varepsilon}$. It goes to $-\infty$ as $\varepsilon \downarrow 0$. Therefore, for $s = 0, 1$, $\lim_{z \notin [0,1] \to s} F(z)$ does not exists.

Problem (4)**.**

IDK where to start...

Problem (5)**.**

It is enough to show γ and μ are path homotopic. Definte $H(t, s) = (1-s)\gamma(t) + \frac{\gamma(t)}{|\gamma(t)|}s$. Then $H(t, 1) = \mu(t)$ and $H(t, 0) = \gamma(t)$ by reparametrization. And H is continuous because $\gamma(t) \neq 0$. Therefore H is path homotopy between γ and μ. Since line integration is invariant under path homotopy, we get $\int_\gamma F(\zeta) d\zeta = \int_\mu F(\zeta) d\zeta$.

Problem (1).

¡++¿

Problem (2).

For $0 < r < 1$, $|f^{(n)}(0)| \leq \frac{n!}{r^n}\frac{1}{1-r}$ *by using Cauchy estimate.* $r^n(1-r)$ *is maximized when* $r = \frac{n}{n+1}$. *So, when* $r = \frac{n}{n+1}$, *we get best estimate of* $|f^{(n)}(0)|$.

Problem (3).

(a) *Since* K *is compact subset of open set* U, *there is* $r > 0$ *such that for all* $x \in K$, *closure of* $D(x,r)$ *is in* U. *Then,* $|f(z)| \leq \frac{1}{2\pi}\left|\int_{\partial D(z,r} \frac{f(w)}{w-z}dw\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z+re^{i\theta}d\theta|$. *By multiplying* ρ *both sides and integrating from* 0 *to* r, *we can get the following:*

$$\begin{aligned}\frac{r^2}{2}|f(z)| &\leq \frac{1}{2\pi}\int_0^r\int_0^{2\pi} \rho|f(z+re^{i\theta})|d\theta d\rho \\ &= \frac{1}{2\pi}\int_{\overline{D}(z,r)} |f|dm \\ &\leq \frac{1}{2\pi}\left(\int_U |f|dm\right)^{1/2} m\left(\overline{D}(z,r)\right) \\ &\leq \frac{m(U)}{2\pi}\left(\int_U |f|dm\right)^{1/2}\end{aligned}$$

for all $z \in K$, *where* m *is lebesgue measure, using Holder's inequality and polar coordinate integration.*

(b) *If* f *is identically zero, possible.*

Else if f *is constant, then* $\int_{\mathbb{C}} |f|dm = \infty$ *since measure of complex plane is* ∞.

Else, that is f *is nonconstant entire function, then* f *must be unbounded. So, there is* $\delta > 0$ *such that* $|f| \geq 1$ *for all* $|z| > \delta$. *Then* $\int_{\mathbb{C}} |f|dm \geq m(\{z : |z| > \delta\}) = \infty$.

Problem (4). (a) *Since* $\frac{z}{e^z-1}$ *is bounded near 0, it has removable singularity at* 0. *So we can regard it as holomorphic function. Note that* $e^z - 1 = 0$ *when* z *is integer multiple of* $2\pi i$. *So, given power series converges on* *unit disc. Now, multiply* $e^z - 1$ *both sides. Since* $e^z - 1$ *is entire and given power serires converges absolutely on* $\bar{D}(0, r)$ *where* $0 < r < 1$*, we can write* $z = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \sum_{n=1}^{\infty} \frac{1}{n!} z^n$*. Since* z *is entire, coefficient of power series is unique. By comparing coefficients of both sides, we can get given recursion formula.*

$\lim z \to 0 \frac{z}{e^z - 1} = 1 = B_0$*. From this, by simple calculation,* $B_1 = \frac{-1}{2}$*,* $B_2 = \frac{1}{6}$*, and* $B_3 = 0$*.*

(b) *We already notice that* $e^z - 1$ *is zero when* z *is integer multiple of* $2\pi i$*. But* $\lim_{z \to 2k\pi i} \frac{z}{e^z - 1}$ *is not bounded when* $k \neq 0$*. Therefore,* $\frac{z}{e^z - 1}$ *is holomorphic on* $D(0, 2\pi)$ *and is not holomorphic outside of that disc. Since power seriese representation of holomorphic function at* P *has radius of convergence at least* $d(P, U)$*, we can say radius of convergence of the series is* 2π*.*

Problem (5)**.**

f' *is holomorphic on unit disc. Let* $r = \sup_{z \in K} |z|$*. Since* K *is compact,* $|f'| \leq M$ *on* K *and r is positive but less than 1. Let* $\gamma(t) = tz^n$ *which connects origin and* z^n*.* $|f(z^n) - f(0)| = \left| \int_\gamma f' dz \right| \leq M \sup_{z \in K} |z|^n = Mr^n$*. Therefore,* $|\sum_{n=1}^{\infty} f(z^n)| \leq \sum_{n=1}^{\infty} |f(z^n)| \leq \sum_{n=1}^{\infty} Mr^n < \infty$ *because* r *is positive but less than 1.*