mas651 exercises

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Problem (5.1.1).

Let $(S, \mathcal{S})$ be a state space of X_n where $S = \{1, 2, \cdots N\}$ and $\mathcal{S} = 2^S$. Note that N is an absorbing state. And $X_1 = 1$ with probability 1. For fixed k such that $1 \le k < N$, $k \le n$,

$$P(X_{n+1} = k + 1 | X_n = k) = \frac{N - k}{N}$$

and

$$P(X_{n+1} = k | X_n = k) = \frac{k}{N}.$$

If $k > n$, then the above are all 0. So it is a temporally inhomogeneous. The Markov property is trivial since the very next state only depends on the current state.

□

Problem (5.1.2).

$$P(X_4 = 2 | X_3 = 1, X_2 = 1, X_1 = 1, X_0 = 0) = (1/16)/(1/4) = 1/4$$

but

$$P(X_4 = 2 | X_3 = 1, X_2 = 0, X_1 = 0, X_0 = 0) = (1/16)/(1/8) = 1/2.$$

Thus X_n is not a Markov chain.

□

Problem (5.1.5).

$$P(X_{n+1} = k + 1 | X_n = k) = \frac{m - k}{m} \frac{b - k}{m}$$

because we must choose a white ball in the left urn and a black ball in the right urn.

$$P(X_{n+1} = k | X_n = k) = \frac{k}{m} \frac{b - k}{m} + \frac{m - k}{m} \frac{m + k - b}{m}$$

since there are two cases, choosing both black or both white.

$$P(X_{n+1} = k - 1 | X_n = k) = \frac{k}{m} \frac{m + k - b}{m}$$

since we must choose a black ball in the left urn and a white ball in the right urn. Note that the sum of the above is 1, so there is no other transition probability.

□

Problem (5.1.6).

$$P(S_{n+1} = k + 1 | S_n = k) = \frac{P(X_{n+1} = 1, S_n = k)}{P(S_n = k)}$$

2

where the denominator is

$$\int_{\theta \in (0,1)} P(S_n = k|\theta) dP = \binom{n}{x} \frac{x!y!}{(n+1)!} = \frac{1}{n+1}$$

for $x =$ the number of i such that $U_i \leq \theta$ and $y = n - x$. Note that $x = (n+k)/2$ and $y = (n-k)/2$ since $x + y = n$ and $x - y = k$. The numerator is

$$\int_{\theta \in (0,1)} P(X_{n+1} = 1, S_n = k|\theta) dP = \binom{n}{x} \frac{(x+1)!y!}{(n+2)!}$$

These are because $P(S_n = k|\theta) = \theta^x (1-\theta)^y \binom{n}{x}$ and $P(X_{n+1} = 1, S_n = k|\theta) = P(X_{n+1} = 1|\theta) P(S_n = k|\theta) = \binom{n}{x} \theta^{x+1}(1-\theta)^y$ and using the kernel of beta distribution.

Thus, the probability what we want is $(n+k+2)/(2n+4)$ which depends on n. So X_n is temporally inhomogeneous.

$$P(S_{n+1} = k+1|S_1 = t_1, \cdots, S_n = k) = P(X_{n+1} = 1|X_1 = t_1, \cdots, X_n = t_n)$$

where $\sum_{i=1}^n t_i = k$. We can show the above is equal to $P(S_{n+1} = k+1|S_n = k) = (n+k+2)/(2n+4)$ similarly, by omitting the $\binom{n}{x}$ term of both denominator and numerator.

$\square$

Problem (5.2.1).

By the given hint,

$$E(1_A 1_B | \mathcal{F}_n) = E(1_A E(1_B | \mathcal{F}_n) | X_n)$$

so it suffices to show that $E(1_B | \mathcal{F}_n) = E(1_B | X_n)$.

Let $Y = 1_{B_n}(\omega_0) \cdots 1_{B_{n+k}}(\omega_k)$. Then $Y \circ \theta_n = $ the indicator function of $\{X_n \in B_n, \cdots, X_{n+k} \in B_{n+k}\} = B$. By the markov property,

$$P(B | \mathcal{F}_n) = E_{X_n} Y.$$

Let $\varphi(x) = E_x Y$ then $\varphi(X_n)$ is $\sigma(X_n)$-measurable mapping. Thus, when B has a form of $\{X_n \in B_n, \cdots, X_{n+k} \in B_{n+k}\}$ for some nonnegative integer k,

$$P(B | \mathcal{F}_n) = P(B | X_n).$$

Note that a collection of such B generates $\sigma(X_n, X_{n+1}, \cdots)$.

Now let $\mathcal{G} = \{C : P(C | \mathcal{F}_n) = P(C | X_n)\}$. By putting $B_{n+i} = S$ for $0 \leq i \leq k$, we earn $\Omega_0 \in \mathcal{G}$. If $C, D \in \mathcal{G}$ and $C \subset D$, then by properties of conditional expectation, $D \setminus C \in \mathcal{G}$. If $C_i \in \mathcal{G}$ and $C_i \uparrow C$ then by monotone convergence theorem for conditional expectation, $C \in \mathcal{G}$. Thus $\mathcal{G}$ is a lambda system containing a collection of B's which generates $\sigma(X_n, \cdots)$. Therefore, by Dynkin's theorem, the third equation is satisfied by any $B \in \sigma(X_n, \cdots)$. By the first equation, we can derive the conclusion.

$\square$

Problem (5.2.4).

First, claim that

$$P_x(X_n = y | T_y = m) = P_y(X_{n-m} = y).$$

This is because

$$P_x(X_n = y | T_y = m) = \frac{P_x(X_n = y, T_y = m)}{P_x(T_y = m)}$$

$$= \frac{\int_{T_y = m} 1_{(X_n = y)} dP_x}{P_x(T_y = m)}$$

$$= \frac{\int_{T_y = m} E(1_{(X_n = y)} | \mathcal{F}_m) dP_x}{P_x(T_y = m)}$$

$$= \frac{\int_{T_y = m} E_{X_m} 1_{(X_{n-m} = y)} dP_x}{P_x(T_y = m)}$$

$$= \frac{P_x(T_y = m) P_y(X_{n-m} = y)}{P_x(T_y = m)}.$$

Now, note that $P_x(X_n = y) = \sum_{m=1}^{n} P_x(X_n = y, T_y = m)$. From this and

4

the above discussion,

$$p^n(x, y) = P_x(X_n = y) = \sum_{m=1}^{n} P_x(X_n = y | T_y = m) P_x(T_y = m)$$

$$= \sum_{m=1}^{n} P_y(X_{n-m} = y) P_x(T_y = m) = \sum_{m=1}^{n} P_x(T_y = m) p^{n-m}(y, y).$$

$\square$

Problem (5.2.6).

Fix $x \in S \setminus C$. Since $P_x(T_C = \infty) = \lim_{M \to \infty} P_x(T_C > M) < 1$, we can choose N_x and ε so that

$$P_x(T_C > M) \leq 1 - \varepsilon$$

whenever $M \geq N_x$. Note that we can choose N_x as an integer. Put $N = \max_{x \in S \setminus C} N_x$. Now we get

$$P_y(T_C > 2N) = \sum_{x \in S \setminus C} P_y(T_C > 2N, T_C > N, X_N = x)$$

$$= \sum_{x \in S \setminus C} P_y(T_C > 2N | X_N = x, T_C > N) P_y(X_n = x, T_C > N)$$

$$\leq \sum_{x \in S \setminus C} P_x(T_C > N) P_y(X_N = x, T_C > N)$$

$$\leq (1 - \varepsilon) \sum_{x \in S \setminus C} P_y(X_N = x, T_C > N)$$

$$\leq (1 - \varepsilon)^2.$$

By induction, the result follows.

Remark 1. By $k \to \infty$, we can say that $P_y(T_C = \infty) = 0$. That is, $P_y(T_C < \infty) = 1$.

$\square$

Problem (5.2.7).

1. It is similar to the manipulation of problem 5.2.4:

$$P_x(V_A < V_B) = \sum_{y} P_x(V_A < V_B, X_1 = y) = \sum_{y} P_x(V_A < V_B | X_1 = y) P_x(X_1 = y)$$

$$= \sum_{y} p(x, y) P_x(V_A < V_B | X_1 = y) = \sum_{y} p(x, y) P_y(V_A < V_B)$$

where the first term is $h(x)$ and the last term is $\sum_{y} p(x, y) h(y)$.

2. *I think we must further assume that h is bounded and measurable. For convenience, let $\tau = V_A \wedge V_B = V_{A \cup B}$. By the equation (5.2.2) of our textbook, we get*

$$E_x\left(h(X_{n+1})|\mathcal{F}_n\right) = \sum_y p(X_n, y)h(y)$$
$$= h(X_n)$$

for $X_n \notin A \cup B$.

Now, put $Y_n = h(X_n)$. Then

$$Y_{n \wedge \tau} - Y_0 = h\left(X_{n \wedge \tau}\right) - h(X_0) = \sum_{k=1}^n 1_{(\tau \geq k)}\left(Y_k - Y_{k-1}\right).$$

By using the above,

$$E_x\left(Y_{n+1 \wedge \tau} - Y_0|\mathcal{F}_n\right) = \sum_{k=1}^{n+1} 1_{(\tau \geq k)} E_x\left(Y_k - Y_{k-1}|\mathcal{F}_n\right)$$
$$= 1_{(\tau \geq n+1)}\left(Y_n - Y_0\right) + 1_{(\tau < n+1)}\left(Y_\tau - Y_0\right)$$
$$= 1_{(\tau > n)}(Y_n - Y_0) + 1_{(\tau \leq n)}(Y_\tau - Y_0)$$
$$= Y_{n \wedge \tau} - Y_0.$$

So $h(X_{n \wedge \tau})$ is a martingale. Note that the first equality is due to (5.2.2), and the last is due to optional stopping.

3. *We assumed that h is bounded. Thus, our martingale is uniformly bounded, so the optional stopping theorem can be applied:*

$$x = E_x h(X_0) = E_x h(X_\tau) = E_x\left[E_x\left(h(X_\tau)|\mathcal{F}_\tau\right)\right]$$

where the last term is equal to

$$E_x\left[E_{X_\tau} h(X_0)\right] = E_x\left[1_{(X_\tau \in A)} + 0 \cdot 1_{(X_\tau \in B)}\right] = E_x\left[1_{(X_\tau \in A)}\right].$$

The above is because $P_x(\tau < \infty) = 1$ and h is 1 on A and 0 on B. Note that this implies the result, since $X_\tau \in A$ is equivalent to $V_A < V_B$.

$\square$

Problem (5.2.8)**.**

Let $\tau = V_0 \wedge V_N$. Then $X_{n \wedge \tau}$ is an uniformly bounded martingale since the state space of X_n is finite. By the optional stopping theorem, we get

$$E_x X_0 = E_x X_\tau$$

where the LHS is equal to x. Note that, by the remark 1, we can say that $P_x(\tau < \infty) = 1$. Then the RHS of the above eqn is equal to $0 P_x(V_0 < V_N) + N P_x(V_N < V_0)$. Thus,

$$x = N P_x(V_N < V_0).$$

$\square$

Problem (5.2.11)**.**

1. It is similar to the manipulation of problem 5.2.7:

$$E_x V_A = \sum_{k \geq 1} P_x \left(V_A \geq k \right)$$

$$= P_x \left(V_A \geq 1 \right) + \sum_{k \geq 2} P_x \left(V_A \geq k \right)$$

$$= 1 + \sum_{k \geq 2} P_x \left(V_A \geq k \right)$$

$$= 1 + \sum_{k \geq 2} \sum_y P_x \left(V_A \geq k | X_1 = y \right) P_x \left(X_1 = y \right)$$

$$= 1 + \sum_y p(x, y) \sum_{k \geq 2} P_y \left(V_A \geq k - 1 \right) = 1 + \sum_y p(x, y) E_y V_A$$

where $P_x \left(V_A \geq 1 \right) = 1$ since x lies outside of A.

Also, $E_x V_A < \infty$ because

$$E_x \frac{V_A}{N} = \sum_{k \geq 1} P_x \left(V_A \geq kN \right)$$

$$\leq \sum_{k \geq 1} (1 - \varepsilon)^k < \infty.$$

2. I think we should assume the measurability, and boundedness of g. By the manipulation used in problem 5.2.7, we get:

$$E_x \left(g(X_{n+1}) + n + 1 | \mathcal{F}_n \right) = n + 1 + \sum_y p(X_n, y) g(y)$$

$$= n + g(X_n)$$

for $X_n \notin A$.

Now put $Y_n = g(X_n) + n$ and $\tau = V_A$ for convenience. Then

$$E_x \left(Y_{n+1 \wedge \tau} - Y_0 | \mathcal{F}_n \right) = \sum_{k=1}^{n+1} 1_{(\tau \geq k)} E_x \left(Y_k - Y_{k-1} | \mathcal{F}_n \right)$$

$$= 1_{(\tau \geq n+1)}(Y_n - Y_0) + 1_{(\tau < n+1)}(Y_\tau - Y_0)$$

$$= Y_{n \wedge \tau} - Y_0.$$

So $X_{n \wedge V_A} + n \wedge V_A$ is a martingale.

3. From the boundedness of g and the fact that V_A is L^1 function, our martingale is uniformly integrable. Thus we can apply optional stopping theorem:

$$E_x g(X_0) = E_x \left[V_A + g(X_{V_A}) \right]$$

7

where the first term is $g(x)$ and the second term is $E_x V_A + E_x g(X_{V_A})$. But X_{V_A} lies in A and g is 0 on A. Thus the second term of the equation is $E_x V_A$.

□

Problem (5.3.1).

Abbreviation of notation: $P(X_1 \leq x_1, \cdots, X_n \leq x_n)$ as $P(X \leq x)$, which is a distribution function of a random vector.

First, let's see that they are identically distributed. Since y is recurrent, the strong Markov property always holds when we are considering $\mathcal{F}_{R_k}$.

$$P_y(\nu_k \leq x) = E_y P_y(\nu_k \leq x | \mathcal{F}_{R_{k-1}}) = E_y E_{X_{R_{k-1}}} 1_{(\nu_1 \leq x)} = P_y(\nu_1 \leq x).$$

So ν_k are identically distributed.

Now, let's see that they are independent.

$$\begin{aligned}
P_y(\nu_1 \leq x_1, \cdots, \nu_n \leq x_n) &= E_y P_y(\nu_1 \leq x_1, \cdots, \nu_n \leq x_n | \mathcal{F}_{R_{n-1}}) \\
&= E_y \left[1_{(\nu_1 \leq x_1)} \cdots 1_{(\nu_{n-1} \leq x_{n-1})} P_y(\nu_1 \leq x_n) \right] \\
&= P_y(\nu_1 \leq x_n) P_y(\nu_1 \leq x_1, \cdots, \nu_{n-1} \leq x_{n-1}) \\
&= \cdots \\
&= P_y(\nu_n \leq x_n) \cdots P_y(\nu_1 \leq x_1)
\end{aligned}$$

since they are identically distributed. Note that $\nu_1, \cdots \nu_{n-1}$ are $\mathcal{F}_{R_{n-1}}$ measurable. This is because $\left\{ X_{R_{n-2}+i} \in B \right\} \cap \{ R_{n-1} = k \} \in \mathcal{F}_k$. $\qquad \square$

Problem (5.3.2).

On $\{ T_y < \infty \}$, by the strong Markov property,

$$\rho_{xy} = P_y(T_z < \infty) = E_x \left(1_{(T_y + T_z < \infty)} | \mathcal{F}_{T_y} \right).$$

Thus,

$$\begin{aligned}
\rho_{xy} \rho_{yz} &= E_x 1_{(T_y < \infty)} E_x \left(1_{(T_y + T_z < \infty)} | \mathcal{F}_{T_y} \right) \\
&= E_x 1_{(T_y < \infty)} 1_{(T_y + T_z < \infty)} \\
&= P_x \left(T_y + T_z < \infty \right) \leq P_x \left(T_z < \infty \right) = \rho_{xz}.
\end{aligned}$$

Problem (5.3.5).

As in the proof of theorem 5.3.8, $\varphi(X_{n \wedge \tau})$ is a nonnegative supermartingale. So the supmartingale converges to Y a.s. From the modified condition $\varphi \to 0$, we know that $\{ x : \varphi(x) > M \}$ is a finite set. So X_n visits $\{ x : \varphi(x) > M \}$ only finitely many times for all $M > 0$. Thus $\varphi(X_n) \to 0$ as $n \to \infty$. If $\tau < \infty$ almost surely, then $\varphi(X_{n \wedge \tau}) \to \varphi(X_\tau) = 0$. But we have the other condition: $\varphi > 0$ on F. So $\varphi(X_\tau) = 0$ cannot happen; thus $P_x(\tau = \infty) > 0$.

If the chain is recurrent, then $\tau < \infty$ a.s. But our case is not the case, so the chain must be transient. $\qquad \square$

Problem (5.3.7).

First assume the recurrence. Let f be a superharmonic function, so $f(X_n)$ is a nonnegative supermartingale. By the martingale convergence theorem, $f(X_n) \to$

Y a.s. By the recurrence, $P(X_n = x \text{ i.o.}) = 1$ for all $x \in S$. So $P(f(X_n) = f(x) \text{ i.o.}) = 1$, which says $f(X_n) \to f(x) = Y$ a.s. But $x \in S$ is arbitrary, f must be a constant.

Now, assume the transience. Fix $z \in S$. Let $V = \inf\{n \geq 0 : X_n = z\}$. Let $f(x) = P_x(V < \infty)$. We will show that f is a nonconstant superharmonic function. For $x \neq z$,

$$f(x) = P_x(V < \infty) = \sum_y p(x,y) P_y(V < \infty) = \sum_y p(x,y) f(y).$$

For $x = z$,
$$\sum_y p(z,y) f(y) \leq \sum_y p(z,y) = 1 = f(z)$$

since $f \leq 1$. Thus, f is superharmonic.

Now claim that there is $y \in S$ such that $f(y) < 1$. If no such y exists, then $f(y) = 1$ for all $y \in S$. This says that $P_y(V < \infty) = 1$ for all y which is equivalent to the recurrence of z. But we assumed the recurrence of our chain. So there is $y \in S : f(y) < 1$. And this says f is nonconstant.

$\square$