mas441 homework

20208209 Jaemin Oh

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Problem (6.5)**.**

(a) *Let $\varepsilon > 0$ be given. There is open set O containing E such that $m(O \setminus E) < \varepsilon$. Since E is compact set contained in open set O, there is $r > 0$ such that that r neighborhood of E is contained in O. For $nr > 1$, $O_n \subset O$. Therefore $m(O_n \setminus E) \leq m(O \setminus E) < \varepsilon$. Therefore $\lim_{n\to\infty} m(O_n) = m(E)$.*

(b) *For closed and unbounded set which does not satisfy above, consider $E = \{\sum_{k=1}^{n} : n \in \mathbb{N}\}$. $m(E) = 0$ because of countability and $m(O_n) = \infty$ since each O_n contains (x, ∞) for some $x > 0$.*

For open and bounded set which does not satisfy (a), consider $E = \bigcup_{i=1}^{\infty} \left(q_i - \frac{\varepsilon}{2^{i+1}}, q_i + \frac{\varepsilon}{2^{i+1}}\right)$ where q_i is enueration of rational numbers between 0 and 1. Then by countable additivity, $m(E) \leq \varepsilon$ and $O_n \supset [0, 1]$. Since ε is arbitrary positive number, we can see that E does not satisfy (a).

Problem (6.7)**.**

It will be shown in problem #8 that δE is measurable when E is measurable since δE is image of E under n by n matrix whose i-th diagonal entry is δ_i.

Consider $R = \Pi_{i=1}^{d} [a_i, b_i]$. Then $\delta R = \Pi_{i=1}^{d} [\delta_i a_i, \delta_i b_i]$. It is rectangle, so $|\delta R| = \Pi_{i=1}^{d} |R|$ for all rectangle R.

Now suppose $\delta E \subset \bigcup_{j=1}^{\infty} Q_j$ where Q_j is a cube. Then $E \subset \bigcup_{j=1}^{\infty} \frac{1}{\delta} Q_j$. It leads $m_*(E) \leq \sum_{j=1}^{\infty} \Pi_{i=1}^{d} \frac{1}{\delta_i} |Q_j|$. Therefore $\Pi_{i=1}^{d} \delta_i m_*(E) \leq \sum_{j=1}^{\infty} |Q_j|$. Since $\bigcup_{j=1}^{\infty} Q_j$ is arbitrary, $\Pi_{i=1}^{d} m_*(E) \leq m_*(\delta E)$.

On the contrary, suppose $E \subset \bigcup_{j=1}^{\infty} Q'_j$. Then $\delta E \subset \bigcup_{j=1}^{\infty} \delta Q'_j$. It leads $m_*(\delta E) \leq \sum_{j=1}^{\infty} \Pi_{i=1}^{d} |Q'_j| = \Pi_{i=1}^{d} \delta_i \sum_{j=1}^{\infty} |Q'_j|$. Since $\bigcup_{j=1}^{\infty} Q'_j$ is arbitrary, $m_*(\delta E) \leq \Pi_{i=1}^{d} \delta_i m_*(E)$.

Problem (6.8)**.**

(a) *Note that $|Lx - Lx'| \leq \|L\| |x - x'|$ where $\|L\| = \sup_{|x|=1} |Lx|$. It is well known that $\|L\| < \infty$ for linear operator on d Euclidean space. Therefore L is continuous, which leads compactness of $L(E)$ when E is compact. Also, $\bigcup_\alpha L(A_\alpha) = L(\bigcup_\alpha A_\alpha)$. It means L preserves F_σ. Because we can represent any F_σ set as countable union of compact set by considering k-disc centered at origin. (k is positive integer)*

(b) *Assume E is measurable. Let $\varepsilon > 0$ be given. There is $F_\sigma \subset E$ such that $m(E \setminus F_\sigma) < \varepsilon$. By definition of Lebesgue measure, there is covering of $E \setminus F_\sigma$ by cubes, $\sum |Q_j| < \varepsilon$.*

Then $m(L(E) - L(F_\sigma)) \leq m(L(E \setminus F_\sigma)) \leq \sum m_*(L(Q_j)) \leq (2\sqrt{d}M)^d \sum m_*(Q_j)$.

Notice that last term can be arbitrarily small and $L(F_\sigma)$ is countable union of closed sets. By corollary 3.5, $L(E)$ is measurable.

Problem (6.13)**.**

(a) *Every open set is countable union of almost disjoint cubes. Therefore open set is F_σ. By considering complement, every closed set is countable intersection of open sets.*

(b) *$\mathbb{Q}$ is F_σ set because $\mathbb{Q} = \bigcup_{i=1}^{\infty} \{q_i\}$, where one-point set is closed.*

Assume $\mathbb{Q} = \bigcap_{i=1}^{\infty} G_i$ where G_i is an open set. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, each G_i is open dense subset of $\mathbb{R}$. Consider $G_i \setminus \{q_i\} = G_i'$. It is also dense in $\mathbb{R}$ and open. By Baire's theorem, $\bigcap_{i=1}^{\infty} G_i'$ must be nonempty. But actually $\bigcap_{I=1}^{\infty} G_i'$ is empty. It is contradiction. Therefore $\mathbb{Q}$ is not G_δ set.

(c) *Consider $\mathbb{Q}_{>0} \cup \mathbb{I}_{\leq 0}$ where $\mathbb{I}$ is set of irrational number. It is disjoint union of F_σ set and G_δ set. If that set is G_δ set, by intersection(-ing) with positive real numbers, we get $\mathbb{Q}_{>0} = G_\delta$ which is contradiction. If that set is F_σ, its complement is G_δ, and it leads $\mathbb{Q}_{\leq 0}$ is G_δ set by intersection with nonpositive real numbers. It also contradicts with (b).*

 # positive rationals and nonpositive rationals are not G_δ set by same reasoning in (b).

Problem (6.14)**.**

(a) *$J_*(E) \leq J_*(\bar{E})$ is trivial. Let $E \subset \bigcup_{j=1}^{N} I_j$. Then $\bar{E} \subset \bigcup_{j=1}^{N} \bar{I}_j = \overline{\bigcup_{j=1}^{N} I_j}$. But $\sum |I_j| = \sum |\bar{I}_j|$. Therefore $J_*(\bar{E}) \leq \sum_{j=1}^{N} |\bar{I}_j| = \sum_{j=1}^{N} |I_j|$. By taking infimum over all $\bigcup_{j=1}^{N} \supset E$, $J_*\left(\bar{E}\right) \leq J_*(E)$.*

(b) *$E = \mathbb{Q} \cap [0,1]$. Then $m(E) = 0$ but covering of E by finitely many intervals must contain $[0,1]$. So $J_*(E) = 1$.*

Problem (6.15)**.**

$m_*^{\mathcal{R}}(E) \le m_*(E)$ since class of rectangles contains class of cubes.

Assume $m_*^{\mathcal{R}}(E) < m_*(E)$. Then there is $\bigcup_{j=1}^{\infty} R_j$ containing E such that $m_*(E) > \sum |R_j|$ by definition of $m_*^{\mathcal{R}}$. This is impossible since $m_*(E) \le m_*(\bigcup_{i=1}^{\infty} R_j) \le \sum m_*(R_j) = \sum |R_j|$ by countable additivity of m_*.

Therefore $m_*^{\mathcal{R}}(E) = m_*(E)$.

Problem (6.16)**.**

(a) $x \in E$ iff for any n, there is $k \ge n$ such that $x \in E_k$ iff $x \in \bigcup_{k\ge n} E_k$ for any n iff $x \in \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k$.

Therefore, E is measurable.

(b) $m(E) \le m\left(\bigcup_{k\ge n} E_k\right) \le \sum_{k=n}^{\infty} m(E_k)$ for any positive integer n.

But, since $\sum_{k=1}^{\infty} m(E_k) < \infty$, for given $\varepsilon > 0$, there is positive integer N such that $n \ge N$ implies $\sum_{k=n}^{\infty} m(E_k) < \varepsilon$. Therefore $m(E) < \varepsilon$ for every positive ε. This means $m(E) = 0$.

Problem (6.17)**.**

Fix k. $m\left(|f_k| = \infty\right) = \lim_{n\to\infty} m\left(|f_k| > n\right) = 0$. So we can choose positive integer N_k such that $N_k \le N_{k+1}$ and $m\left(|f_k| > N_k\right) < 2^{-k}$. Let $N_k = \frac{c_k}{k}$.

Then $\sum_{k=1}^{\infty} m\left(\frac{|f_k|}{c_k} > \frac{1}{k}\right) \le 1 < \infty$. By Borel-Cantelli lemma, $m\left(\limsup E_k\right) = 0$. So, if $x \notin \limsup E_k$, then $x \in \bigcap_{k\ge n} E_k^c$ for some positive integer n, and it means $\frac{|f_k(x)|}{c_k} \le \frac{1}{k}$ for all $k \ge n$. Therefore $\lim_{k\to\infty} \frac{|f_k|}{c_k} = 0$ for almost every x.

Problem (6.18)**.**

First consider characteristic function 1_E of finite measure set E. There are $F_n \subset E \subset G_n$ where F_n, G_n are closed, open repectively and $m(G_n \setminus F_n) < 2^{-n}$.

We can assume G_n is decreasing by considering $G_n' = \bigcap_{k=1}^{n} G_n$. Similarly, we can regard F_n as increasing sequence of closed sets.

Now, for each n, there is Urysohn function f_n which is continuous, vanishes outside of G_n and equal to 1 on F_n. Then clearly $f_n \to 1_E$ as $n \to \infty$ except on $\bigcap_{n\ge 1}(G_n \setminus F_n)$. But $m(\bigcap_{n\ge 1}(G_n \setminus F_n) = 0$. This says there is sequence of continuous function whose a.e. limit is 1_E.

From now on, using the above, consider measurable function f. There is sequence of simple function s_n whose pointwise limit is f. By above, for each n, there is $\{f_{n,k}\}_{k=1}^{\infty}$ whose a.e. limit is s_n.

Choose k_n so that $m(\{f_{n,k_n} \neq s_n\}) < 2^{-n}$. By Borel Cantelli lemma, $m(\limsup A_n) = 0$ where $A_n = \{f_{n,k_n} \neq s_n\}$. If $x \in (\limsup A_n)^c$, then $x \in \bigcap_{n \geq N} A_n^c$ for some N, then $f_{n,k_n} = s_n$ for $n \geq N$. Therefore $\lim_{n\to\infty} f_{n,k_n} = \lim_{n\to\infty} s_n = f$ for almost every x.

Problem (6.22)**.**

Assume $f = 1_{[0,1]}$ a.e. where 1_A denotes characteristic function of A. If $f \neq 1$ for some $x \in (0,1)$, there is $\delta > 0$ such that $(x-\delta, x+\delta) \subset (0,1)$ and $f \neq 1$ on $(x-\delta, x+\delta)$ by continuity. It contradicts with $f = 1_{[0,1]}$ a.e. Therefore $f = 1$ for $x \in (0,1)$. Similarly, $f = 0$ for $|x| > 1$. Then f must be discontinuous at $x = 0, 1$. It leads the fact that there is no such f.

Problem (6.23)**.**

Fix n. Then $\mathbb{R} = \bigcup_{k\in\mathbb{Z}} \left(\frac{k}{n}, \frac{k+1}{n}\right]$. So for each $x \in \mathbb{R}$, there exists unique k such that $x \in \left(\frac{k}{n}, \frac{k+1}{n}\right]$. Now, fix y. For $x \in \left(\frac{k}{n}, \frac{k+1}{n}\right]$, define $f_n(x,y)$ as follows:

$$f_n(x,y) = n\left[f\left(\frac{k}{n}, y\right)\left(\frac{k+1}{n} - x\right) + f\left(\frac{k+1}{n}, y\right)\left(x - \frac{k}{n}\right)\right]$$

It is line segemnt connecting $\left(\frac{k}{n}, f\left(\frac{k}{n}, y\right)\right)$ and $\left(\frac{k+1}{n}, f\left(\frac{k+1}{n}, y\right)\right)$. Note that it is sum of product of two continuous functions. Hence f_n is measurable.

Also, consider below:

$$\begin{aligned} f_n(x,y) - f(x,y) &= \left[f\left(\frac{k}{n}, y\right) - f(x,y)\right](k+1-nx) \\ &\quad + \left[f\left(\frac{k+1}{n}, y\right) - f(x,y)\right](nx-k) \end{aligned}$$

Note that $k < nx \leq k+1$ hence $0 \leq k+1-nx \leq 1$ and $0 \leq nx-k \leq 1$. By continuity of $f(\cdot, y)$, as $n \to \infty$, $f_n(x,y) - f(x,y) \to 0$ since $\frac{k}{n}, \frac{k+1}{n} \to x$.

Therefore $f(x,y)$ is pointwise limit of measurable function hence measurable.

Problem (6.25)**.**

Let E be measurable. Then E^c is also measurable. By definition of measurability, there is open set O containing E^c such that $m_*(O \backslash E^c) = m_*(E \backslash O^c) < \varepsilon$. Therefore E is measurable in new sense.

Assume that E is measurable in new sense. For each $\varepsilon > 0$, there is closed $F \subset E$ such that $m_*(E \setminus F) = m_*(F^c \setminus E^c) < \varepsilon$. It leads measurability of E^c and therefore E is measurable in old sense because class of measurable sets is closed under complement set operation.

Problem (6.26)**.**

$m_*(E \setminus A) \leq m_*(B \setminus A) = m(B) - m(A) = 0$ *since measure of B is finite. Therefore $E \setminus A$ is zero measure set, therefore measurable. $E = E \setminus A \cup A$ which is union of two measurable set. Therefore E is measurable.*

Problem (6.28)**.**

Let $\alpha \in (0, 1)$. $\frac{1}{\alpha} m_*(E) > m_*(E)$ so there is open set O containing E such that $m_*(E) = m_*(E \cap \bigcup_{j \geq 1} I_j) = m_*(\bigcup_{j \geq 1} E \cap I_j) > \alpha m_*(O) = \alpha \sum_{j \geq 1} m_*(I_j)$ where I_j's are disjoint interval whose union is O.

If $m_*(E \cap I_j) < \alpha m_*(I_j)$ for all positive integer j, then $m_*(E) \leq \sum_{j \geq 1} m_*(E \cap I_j) \leq \alpha \sum_{j \geq 1} m_*(I_j)$ which contradicts to above.

Therefore there is I_j such that $m_*(E \cap I_j) \geq \alpha m_*(I_j)$.

Problem (6.37)**.**

Consider $f 1_{[-n,n]}$. It is uniformly continuous on $[-n, n]$. Let $\varepsilon > 0$ be arbitrary. choose $\delta > 0$ less than n such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$ for all $x, y \in [-n, n]$.

For each $x \in [-n, n]$, consider $\left(x - \frac{\delta}{2}, x + \frac{\delta}{2}\right)$. Such interval forms open cover of $[-n, n]$. We can cover $[-n, n]$ by at most $\frac{2n+1}{\delta}$ number of such intervals. Let Γ_n be graph of $f 1_{[-n,n]}$. Then $m_*(\Gamma_n) \leq \frac{2n+1}{\delta} \delta 2\varepsilon = 2(2n + 1)\varepsilon$ which can be arbitrarily small. Therefore $m_*(\Gamma_n) = 0$ for all n and $m(\Gamma) = \sum_{n=1}^{\infty} m(\Gamma_n) = 0$ where Γ is graph of f.