mas541 homework

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Problem (5.1)**.**

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ *and assume that* $P(z) = 0$ *has no solution. Then by the argument principle,* $\frac{1}{2\pi i}\int_{\partial D(Q,R)} \frac{P'(\zeta)}{P(\zeta)}d\zeta = 0$ *for all* $R > 0$*. That integral is equal to* $\frac{1}{2\pi i}\int_0^{2\pi} \frac{P'(Q+Re^{i\theta})}{P(Q+Re^{i\theta})}Rie^{i\theta}d\theta$*. But, as* $R \to \infty$*, integrand of above goes to in uniformly on* $0 \leq \theta \leq 2\pi$*. Therefore, the integral above goes to* $n > 0$ *which is the degree of* P*. It is contradiction. Thus* $P(z) = 0$ *has at least one solution in complex plane.*

Problem (5.2)**.**

Assume the existence of such f*. Since* f *is bounded near* 0*, Riemann removable singularity theorem says that* f *can be extended to the function which is holomorphic on entire unit disc.*

If modulus of $f(0)$ *is equal to 1 or 2, then image of the unit disc under* f *is not open which contradicts to the open mapping theorem. So* $f(0) \in \{w : 1 < |w| < 2\}$*.*

Since f *is surjective function of the punctured unit disc onto the annulus, we can find* $w \neq 0$ *such that* $f(0) = f(w)$*. Choose two disjoint neighborhood* U_w, U_0 *of* $w, 0$ *respectively. Then by the open mapping theorem,* $f(U_w)$ *and* $f(U_0)$ *are open and* $f(0) \in f(U_w) \cap f(U_0)$*. Since* $f(U_w) \cap f(U_0)$ *is open, we can choose small neighborhood of* $f(0)$ *contained in the previous set. And therefore we can choose* $f(0) \neq \alpha \in f(U_w) \cap f(U_0)$*. This cannot be happen since* f *is injective.*

Thus there is no such f*.*

Problem (5.3)**.**

(a) Choose $R > \lambda$*, and choose* n *so large that* $\lambda - 1 \geq 1/n$*. Then* $\bar{D}(R, R - \frac{1}{n}) \subset$ *Right half plane.*

Then for $\zeta \in \partial D(R, R - 1/n)$*,* $|e^{-\zeta}| < 1 \leq \lambda - 1/n \leq |\zeta - \lambda|$*. Put* $f(z) = e^{-z} + z - \lambda$ *and* $g(z) = z - \lambda$*. Then by above and Rouche's theorem,* f *and* g *has same zero on* $D(R, R - 1/n)$*. But any* $z \in$ *Right half plane must be inside of* $D(R, R - 1/n)$ *for some large* R *and* n*. This means* f *and* g *have same zero on the right half plane.*

But $g(z) = 0$ *has unique solution. Therefore* $e^{-z} + z - \lambda = 0$ *has unique solution on the right half plane.*

(b) *Fix $z' \in U$. Note that $U \setminus \{z'\}$ is still a domain. Let $g_k(z) = f_k(z) - f_k(z')$ for $z \in U \setminus \{z'\}$. Since f_j is an injective holomorphic function on U, g_k does not vanish on $U \setminus \{z'\}$. Uniform convergence of f_j on compact subsets of U implies uniform convergence of g_k on compact subsets of $U \setminus \{z'\}$. Since g_k is nonvanishing function, by Hurwitz's theorem, $\lim_{k\to\infty} g_k(z) = f(z) - f(z')$ does not vanish or identically zero.*

If it is identically zero on $U \setminus \{z'\}$, then f must be constant function on U. If it is nonvanishing on $U \setminus \{z'\}$, then $f(z'') = f(z')$ implies $z'' = z'$. Thus f must be injective.

Problem (5.4)**.**

It seems to be solved by the maximum modulus principle (or theorem), but I don't know where to start.

Problem (5.5)**.**

For $z \in S$, $|\varphi(z)| = \left|\frac{e^{2\pi zi}-1}{e^{2\pi zi}+1}\right|$, and the real part of $e^{2\pi zi} > 0$ because $z \in S$. Then it is clear that $|\varphi(z)| < 1$. Also $\varphi(0) = 0$.

Therefore $\varphi \circ f : D \to D$ is holomorphic and it fixes the origin. Then Schwarz's lemma says $|\varphi'(0)f'(0)| \leq 1$. But $\varphi'(0) = \pi$. Therefore $|f'(0)| \leq 1/\pi$. The equality holds only if $\varphi(f(z)) = wz$ for some $|w| = 1$.