# HW

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**Problem 1.** *Obviously* $\left|\frac{1}{z}\right| \leq \frac{1}{\varepsilon}$. *Let* $\frac{1}{z} = re^{i\theta}$ *where* $0 < r \leq \frac{1}{\varepsilon}$. *Then real part of* $\frac{1}{z} = r\cos\theta$. *Therefore* $\left|e^{\frac{1}{z}}\right| = e^{r\cos\theta} \leq e^{\frac{1}{\varepsilon}}$. *Thus, given function is bounded in the region* $|z| \geq \varepsilon$.

**Problem 2.** *Let* $z = \log(1+i)^{\pi i}$. *Then* $e^z = (1+i)^{\pi i} = e^{\pi i \log(1+i)}$. *Therefore* $z = 2k\pi i + \pi i \log(1+i) = 2k\pi i + \pi i\left(\ln\sqrt{2} + i\left(\frac{\pi}{4} + 2m\pi\right)\right)$. *So* $z = \left(2k + \frac{1}{2}\ln 2\right)\pi i - \pi^2\left(\frac{1}{4} + 2m\right)$ *where* $k, m$ *are integers.*

**Problem 3.**

$$\begin{aligned}(iy)^{(iy)} &= e^{iy\log(iy)} = e^{iy\left(ln|y| + i\left(\frac{\pi}{2} + k\pi\right)\right)} \\ &= e^{-y\left(\frac{\pi}{2+k\pi} + iy\ln|y|\right)} = e^{-y\left(\frac{\pi}{2} + k\pi\right)}\left(\cos(y\ln|y|) + i\sin(y\ln|y|)\right)\end{aligned}$$

*where* $k$ *is even(odd) when* $y > 0$ *(*$y < 0$*).*

**Problem 4.** $z = re^{i\theta}$. *Then* $z^{\frac{2}{3}} = e^{\frac{2}{3}\log z} = e^{\frac{2}{3}(\ln r + i\theta)} = r^{\frac{2}{3}} \cdot e^{i\left(\frac{2}{3}\theta\right)} = w$. *Therefore* $|w| = r^{\frac{2}{3}}$.

**Problem 5.** *Let* $a = \frac{1}{2i}\log\frac{z+i}{z-i}$. *Then* $\sin a = \frac{e^{ai} - e^{-ai}}{2i}$ *and* $\cos a = \frac{e^{ai} + e^{-ai}}{2}$.

$$\begin{aligned}\cot a &= i\frac{e^{ai} + e^{-ai}}{e^{ai} - e^{-ai}} = i\frac{e^{2ai} + e^{-2ai} + 2}{e^{2ai} - e^{-2ai}} \\ &= i\frac{\frac{z+i}{z-i} + \frac{z-i}{z+i} + 2}{\frac{z+i}{z-i} - \frac{z-i}{z+i}} = i\frac{4z^2}{4zi} = z\end{aligned}$$

*Therefore* $\cot a = z$ *which implies* $\cot^{-1} z = a$.

**Problem 6.**

(1) $\lim_{z\to 0}\frac{\log(1+z)}{z} = 1$ *by theorem 5.26. By continuity of* $e^z$, $\lim_{z\to 0}(1+z)^{\frac{1}{z}} = \lim_{z\to 0} e^{\frac{\log(1+z)}{z}} = e$.
(2) $z^{\frac{1}{2}} = e^{\frac{1}{2}\ln|z| + i\theta}$ *where* $\theta$ *is argument of* $z$. *So,* $z^{\frac{1}{2}} \to 0$ *as* $z \to 0$. *Therefore,* $\lim_{z\to 0}\frac{\sqrt{z}-2}{z-2} = \frac{-2}{-2} = 1$.

**Problem 7.** $u_x = \sqrt{x^2+y^2} + \frac{x^2}{\sqrt{x^2+y^2}}$, $u_y = \frac{xy}{\sqrt{x^2+y^2}}$, $v_x = \frac{xy}{\sqrt{x^2+y^2}}$, *and* $v_y = \sqrt{x^2+y^2} + \frac{y^2}{\sqrt{x^2+y^2}}$. *To satisfy Cauchy-Riemann equation,* $x^2 = y^2$ *and* $xy = 0$ *which is impossible except for* $x = y = 0$.

*Let's compute values of partials.* $u_x(0,0) = \lim_{h\to 0}\frac{h\sqrt{h^2}}{h} = \lim_{h\to 0}|h| = 0$. *Similarly all other partials have* $0$ *at the origin. So given function satisfies Cauchy-Riemann equation at origin. So* $f$ *is differentiable at origin and* $f'(0) = 0$.

**Problem 8.** $f(z) = z^2 + z^2\bar{z}$. *So* $\frac{\partial f}{\partial \bar{z}} = z^2$ *which is zero if and only if* $z = 0$. *So* $f$ *is nowhere differentible except at* $z = 0$. $\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}(z + z\bar{z}) = 0 = f'(0)$. *Also, we cannot say about derivative of* $f$ *on* $\mathbb{C} - 0$. *So we cannot say about second(or higher) order derivative of* $f$ *at origin.*

**Problem 9.** $u_x = e^x\cos(ay)$, $u_y = -ae^x\sin(ay)$, $v_x = e^x\sin(y+b)$, *and* $v_y = e^x\cos(y+b)$. *To satisfy Cauchy-Riemann equation,* $a = \pm 1$ *and* $b = 2k\pi$ *where* $k \in \mathbb{Z}$.

**Problem 10.** *Since real part of $f' = f_x$ is zero, $u_x = 0 = v_y$ by Cauchy-Riemann equation. So $u = \phi_1(y)$ and $v = \phi_2(x)$. Then $\phi_1'(y) = -\phi_2'(x)$ for all $x, y \in \mathbb{R}$ because $u_y = -v_x$. Therefore $\phi_1' = -\phi_2' = c \in \mathbb{R}$. So, put $\phi_1 = cy + d_1$ and $\phi_2 = -cx + d_2$ where $d_1, d_2 \in \mathbb{R}$. Therefore $f(x, y) = (cy + d_1) + i(-cx + d_2)$ which is entire function.*

**Problem 11.** *Automatically, $u_y = -v_x = 0$. $u'(x) = v'(y)$ for all $x, y \in \mathbb{R}$. So $u' = v' = c \in \mathbb{R}$. So $u(x) = cx + d_1$ and $v(y) = cy + d_2$ where $d_1, d_2 \in \mathbb{R}$. Therefore $f(z) = cz + d_1 + id_2$.*

**Problem 12.** *Given function is real function. But analytic real valued function must be constant in the domain. Unfortunately, our function is not constant. So it is not analytic in the domain.*

**Problem 13.** *$f(z) = \ln|z| + i\theta$ where $4\pi < \theta \leq 6\pi$ is argument of $z$. (that is, cutting $\theta = 4\pi$)*

*Clearly, $f$ is analytic on $\mathbb{C} \setminus [0, \infty)$ and $f(-1) = 0 + i\theta$. $\theta = \pi + 2k\pi \in (4\pi, 6\pi]$. So $k = 2$ and $f(-1) = 5\pi i$.*