convex - hw5

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Problem 1 *P can be orthogonally diagonalized. So* $P = Q\Lambda Q^{-1}$ *where* Λ *is diagonal matrix whose entries are eigenvalues of* P *and columns of* Q *are orthonormal eigenbasis. Let* $\{\alpha_i\}_{i=1}^{\infty}$ *be the orthonormal eigenbasis. Then* $x = \sum_{i=1}^{\infty} x_i\alpha_i$, $x^tPx = \sum_{i=1}^{\infty} \lambda_i x_i^2$. *Since* P *is not positive semidefinite, there is at least one negative eigenvalue. Let* $\lambda_i < 0$. *Then by letting* $x_j = 0$ *for* $j \neq i$ *and* $x_i \to \infty$, *we can get* $x^tPx \to -\infty$. *So problem is unbounded if* P *is not positive semidefinite.*

Problem 2 *Note that given objective function is convex since* $f(x, y) = x^2/y$ *is convex. By* $\mathrm{dom} f$, *Slator's condition is satisfied. Let* $c^tx + d = y > 0$. $L(x, y, \lambda, \nu) = \frac{1}{y}\left(x^tA^tAx - 2b^tAx + b^tb\right) - \lambda y + \nu\left(c^tx + d - y\right)$. *To minimize* L, *consider* $\nabla_x L = \frac{1}{y}\left(2A^tAx - 2A^tb\right) + \nu c = 0$. *Then* $x^* = \left(A^tA\right)^{-1}\left(A^tb - \frac{1}{2}y\nu c\right)$. *But we know that the duality gap is tight. So* $g(\lambda^*, \nu^*) = L(x^*, y^*, \lambda^*, \nu^*) = f(x^*)$. *So minimizer of given problem is* x^* *which can be represented as* $x_1 + tx_2$.

Problem 3 $x_1 = \gamma\left(\frac{\gamma-1}{\gamma+1}\right)^k$ *and* $x_2 = \left(\frac{1-\gamma}{1+\gamma}\right)^k$. *We will using induction.*

$$\begin{aligned}(x_1', x_2') &= (x_1, x_2) + \mathit{argmin}_t f\left((x_1, x_2) + t\nabla f(x_1, x_2)\right)\nabla f(x_1, x_2) \\ &= (x_1, x_2) + t^*(x_1, \gamma x_2) = (x_1(1 + t^*), x_2(1 + t^*\gamma)\end{aligned} \tag{1}$$

where $t^* = \frac{-x_1^2 - \gamma^2 x_2^2}{x_1^2 + \gamma^3 x_2^2}$. *By 'hard' calculation, we can get* $(x_1', x_2') = \left(\gamma\left(\frac{\gamma-1}{\gamma+1}\right)^{k+1}, \left(\frac{1-\gamma}{\gamma+1}\right)^{k+1}\right)$

Problem 4 (a) *Let* $f(x) = \frac{e^x}{1+e^x}$. *Then* $f'(x) = \frac{e^x}{(1+e^x)^2} > 0$, *so* f *is increasing function of* x. *Therefore maximizing* p *is equivalent to maximizing* $a^tx + b$ *subject to given constraints.*

(b) *Note that $f(x)$ is log concave. Since log is monotonic function, maximizing given objective is equivalent to maximizing* $\log($*objective*$)$*. By taking log, we obtain new objective* $a^t x + b - \log(1 + e^{a^t x + b}) + \log(c^t x + d)$*. Therefore, it is convex optimization.*

Problem 5 (a) *Let optimal $t > 0$. Then $a^t x_i - b \geq t > 0$ and $a^t y_i - b \leq < -t < 0$ so $\{x_i\}_{i=1}^N$ and $\{y_i\}_{i=1}^N$ are linearly separated by the hyperplane $a^t z - b = 0$.*

Let $\{x_i\}_{i=1}^N$ and $\{y_i\}_{i=1}^N$ be linearly separated. Then there is (a, b) such that $a^t x_i - b > 0$ and $a^t y_i - b < 0$. If $a = 0$, contradiction. By multiplying normalizing constant, we can assume $\|a\| = 1$. Let $t = \min\{a^t x_i - b, -a^t y_i + b\} > 0$*. Then t, a, b satisfies all the conditions of given problem. So optimal $t^* \geq t > 0$.*

If inequality is not tight, by considering all variables divided with $\|a\|$ we can get larger optimal t than older ones. So inequality must be tight.

(b) *From $\|a\| \leq 1$, we can get $\|\tilde{a}\| \leq \frac{1}{t}$. From tightness of inequality, we can get optimal t by maximizing $\frac{1}{\|\tilde{a}\|}$ which is equivalent to minimizing $\|\tilde{a}\|$. So given problem is equivalent to new problem.*