Σ : Covariance Matrix — $d \times d$, symmetric, PSD

PSD: Positive semidefinite

eigen-decomposition

$$U = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_d \\ | & | & & | \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_d \end{bmatrix}$$

$$\Sigma = U \Lambda \underline{U}^T = \boxed{\sum_{i=1}^{d} \lambda_i \underbrace{\vec{u}_i \, \vec{u}_i^T}_{\text{rank}-1 \text{ matrix}}}$$

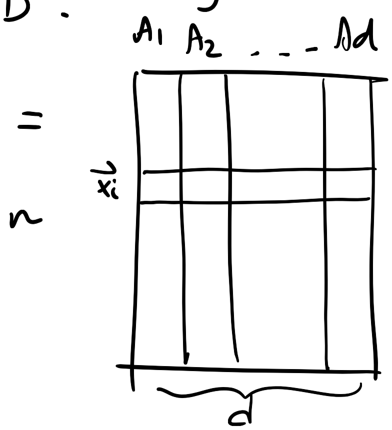
rank : # of linearly independent rows or cols in a matrix

$\vec{u}_i = d \times 1$

$u_i^T = 1 \times d$

SVD : Singular Value Decomposition

$D =$



$D \in \mathbb{R}^{n \times d}$

$$\boxed{D = L \Delta R^T} = V \Delta U^T$$

left singular vectors

right singular vectors

"diagonal" matrix of singular values

$$D \in \mathbb{R}^{n \times d}$$

$$L = \underbrace{\begin{bmatrix} | & | & & | \\ \vec{l}_1 & \vec{l}_2 & \cdots & \vec{l}_n \\ | & | & & | \end{bmatrix}}_{n \times n}$$

$$\Delta = \begin{array}{c} \\ n \\ \\ \end{array} \overset{d}{\begin{bmatrix} \delta_1 & & & & 0 \\ 0 & \delta_2 & & & \\ & & \ddots & & \\ & & & \delta_k & \\ & & & & 0 \; 0 \; 0 \\ \hline & & 0 & & \end{bmatrix}}$$

$d < n$

$$R = \underbrace{\begin{bmatrix} | & | & & | \\ \vec{r}_1 & \vec{r}_2 & \cdots & \vec{r}_d \\ | & | & & | \end{bmatrix}}_{d \times d}$$

$$\underset{n \times d}{D} = \underbrace{\underset{n \times n}{L} \; \Delta}_{n \times d} \; \underset{d \times d}{R^T}$$

$$\underset{\uparrow \atop \text{rank of } D}{k} = \text{rank}(D) \leq \min(n, d) \leq d$$

$$\boxed{D = L \Delta R^T = \sum_{i=1}^{k} \underset{\uparrow \atop \text{scalar}}{\delta_i} \underbrace{\underset{\uparrow \atop n \times 1}{\vec{l}_i} \; \underset{\uparrow \atop 1 \times d}{\vec{r}_i^{\,T}}}_{n \times d} \qquad \delta_1 \geq \delta_2 \geq \delta_3 \cdots \geq \delta_k \geq 0 \cdots 0}$$

$\vec{l}_i$: mixture of cols

$\vec{r}_i$: mixture of rows

$\downarrow$

linear combination

$\downarrow$

weighted sum

$$D = \begin{array}{c|c} & \text{movie} \\ \hline \text{users} & \\ \end{array}$$

$\vec{r}_i$: prototypical user

$\vec{l}_i$: " movie

$D \longrightarrow$ center $\longrightarrow \bar{D}$

Covariance $\boxed{\underline{\underline{\Sigma}} = \frac{1}{n} \bar{D}^T \bar{D}}$

"Covariance" between columns

R : right singular vectors $\underline{\underline{\text{are}}}$ the eigenvectors of $\underbrace{\underline{\bar{D}^T \bar{D}}}_{d \times d}$

Scatter matrix

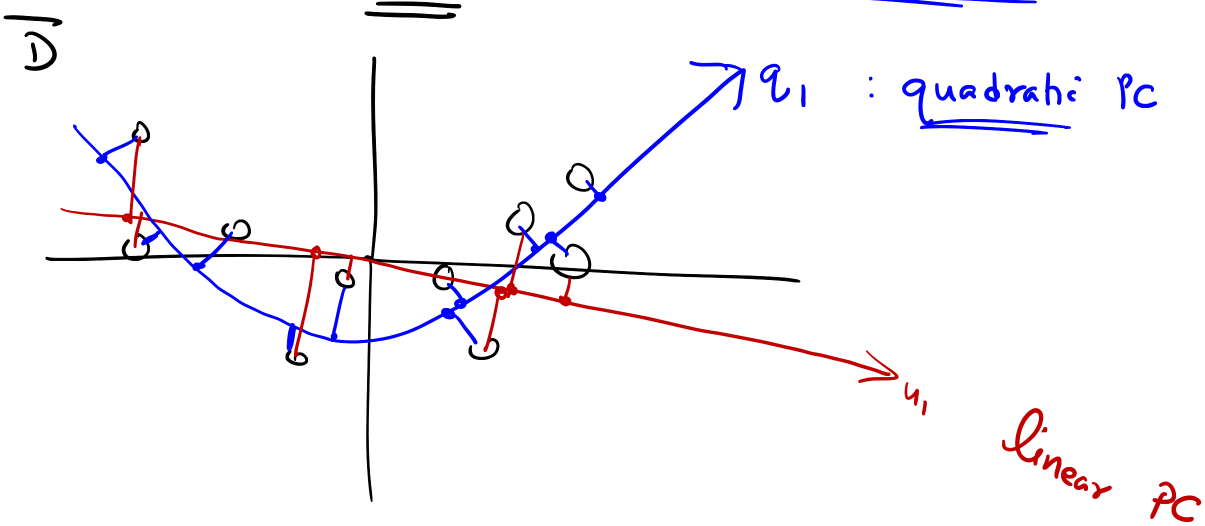
$\longrightarrow$ eigenvectors of Σ

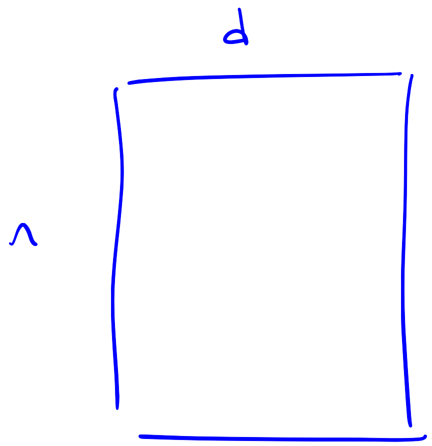
L: left singular vectors are eigenvectors of $\underbrace{\underset{n \times d}{\bar{D}} \; \underset{d \times n}{\bar{D}^T}}_{n \times n}$

"Covariance" between rows

PCA: $\underline{\underline{\text{linear}}}$ method $\longrightarrow$ $\underline{\text{non-linear pc?}}$ $\longleftarrow$ "kernels" $\Downarrow$ graphs

$\bar{D}$





d

high-dimensional data

hypercube : $\underline{d}$

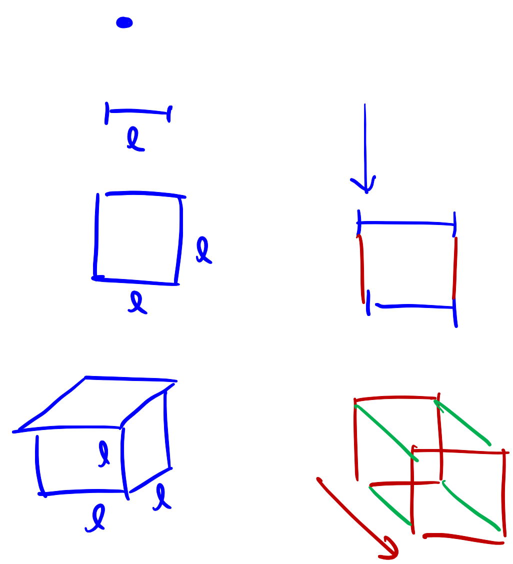
$d=0$

$d=1$

$d=2$

$d=3$

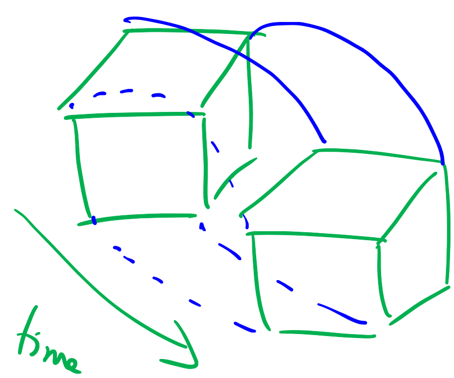
⋮



hypercube

$d=4$

MBR:
Minimum
Bounding
Rectangle/Cube



"Connect all corners"

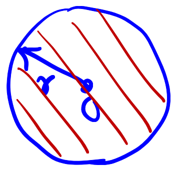
$$H_d(\ell) = \left\{ \vec{x} \;\middle|\; 0 \le x_i \le \ell \quad \forall i=1,\dots,d \right\}$$

Hypercube in d-dim with edge length ℓ

Hypersphere

$S_d(r)$: hypersphere centered at $\vec{O}$ in d-dim, with radius r

2D:



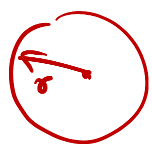
$$S_d(r) = \left\{ \vec{x} \;\middle|\; \|\vec{x}\|^2 = r^2 \right\}$$

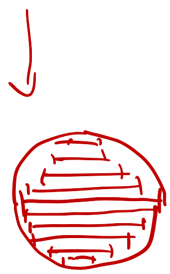
Hyperball $B_d(r) = \left\{ \vec{x} \;\middle|\; \|\vec{x}\|^2 \leq r^2 \right\}$

$d = 0 \longrightarrow$ •

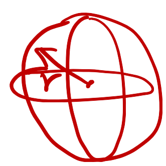
$d = 1 \longrightarrow$

$d = 2 \longrightarrow$





$d = 3 \longrightarrow$



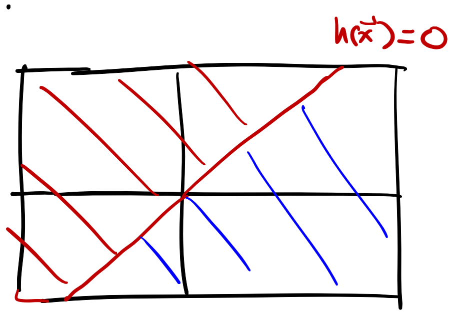
⋮

hypersphere

$d = 4$?

Hyper plane :

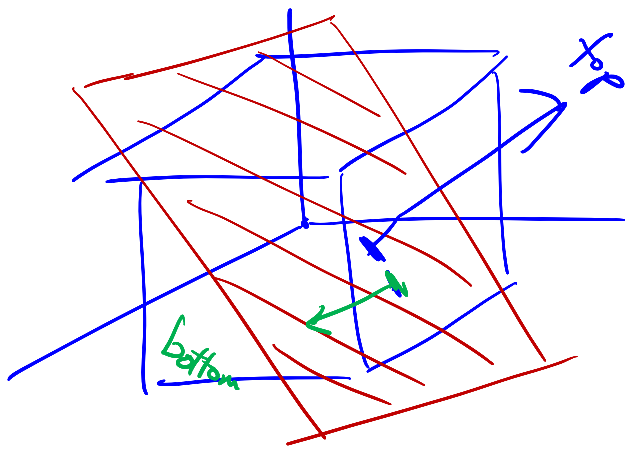
2 d Space



hyperplane / line (lives in 2 d space)

Is a 1 d object

3d



hyperplane is 2d plane

↳ linear object or flat

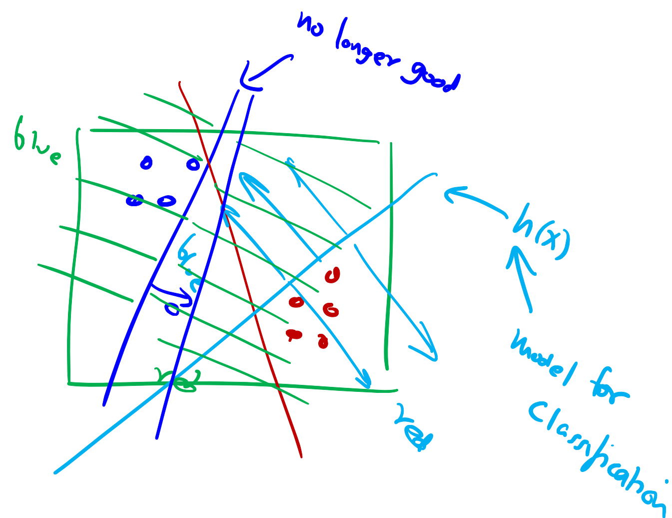
set of all vectors $\vec{x}$ (or points)

such that $\boxed{h(\vec{x}) = 0}$

$h(\vec{x}) = 0$

$h(\vec{x})$ divides d-dim into

half-space



One half-space

$h(\vec{x}) < 0$

$h(\vec{x}) = 0$

lie on the hyperplane

2^{nd}

$h(\vec{x}) > 0$

$$h(\vec{x}) = w_1 x_1 + w_2 x_2 + \cdots + w_d x_d + b$$

weight w_i per dimension

bias

h as a function has $d+1$ parameters $w_1, w_2 \ldots, w_d, b$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$$

$$= \vec{w}^T \vec{x} + b$$

scalar

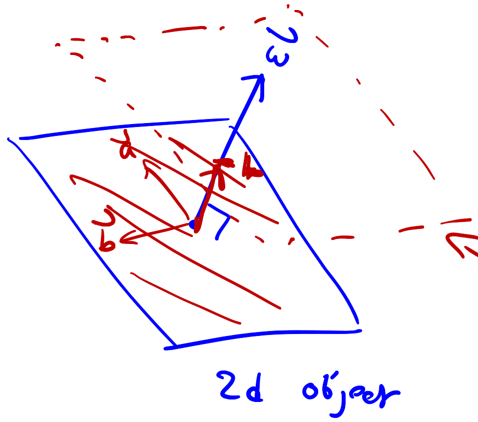
$$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix}$$

$\vec{w}$: weight vector : "normal" vector to the hyperplane

Orthogonal vector " " "

$\mathbb{R}^3$

$d=3$



new hyperplane @ b

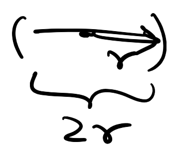
bias gives the offset

Volume of the d-dim unit hypersphere $(r=1)$

$$\mathrm{Vol}\big(S_d(r)\big) \equiv \mathrm{Vol}\big(S_d(1)\big)$$

radius

$d=1$ $\mathrm{vol}\big(S_1(r)\big) = 2r$



$d=2$ $\mathrm{vol}\big(S_2(r)\big) = \pi r^2$

$d=3$ $\mathrm{Vol}\big(S_3(r)\big) = \frac{4}{3}\pi r^3$

$$\mathrm{Vol}\big(S_d(r)\big) = K_d\, r^d$$

$$= \left(\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}\right) r^d$$

Gamma function

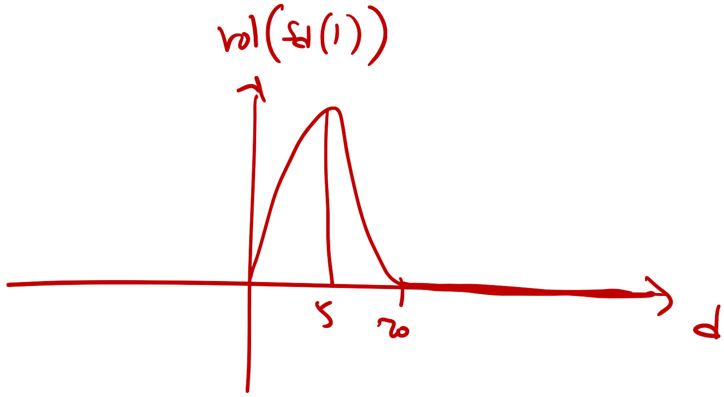
kind of like factorial

$\approx \left(\frac{d}{2}\right)!$ when d is even

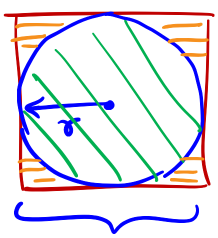
$r = 1$

$$\lim_{d\to\infty} \mathrm{vol}(S_d(1)) = \lim_{d\to\infty} \boxed{\frac{\pi^{d/2}}{(d/2)!}} \to 0$$

approx



volume goes to zero!



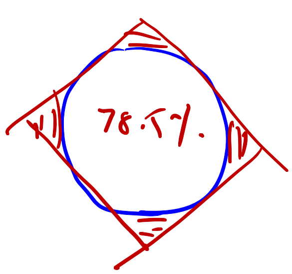
inscribed hypersphere in a hypercube

Ratio: $$\frac{\mathrm{vol}(S_d(r))}{\mathrm{vol}(H_d(2r))} = \frac{\frac{\pi^{d/2}}{(d/2)!}\cancel{r^d}}{2^d\,\cancel{r^d}}$$

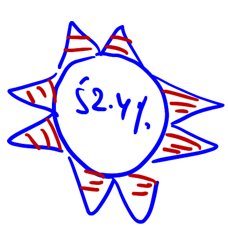
$$\lim_{d\to\infty} \text{ratio} \longrightarrow 0$$

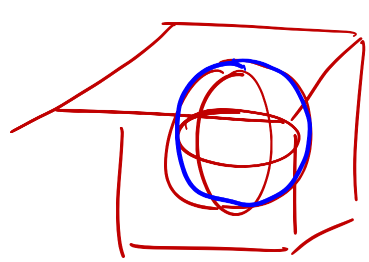
Volume inside the hypersphere is 'zero' negligible compared to the corners

2d



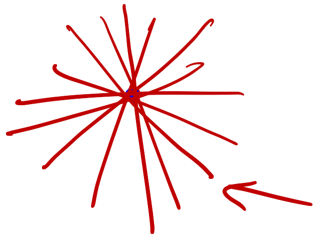
3d





8 corners 2^d corners

d-dim



Corner

2^d corners