Delta

Delta

- Orthodox Definition:
    - sensitivity of a derivative price to the movement in the underlying asset.
    - It is expressed as the first mathematical derivative of the product with respect to the underlying asset.

$$\text{Delta} = \frac{\partial F}{\partial U}$$

F is the derivative F(U,t)

U is the underlying security

Continuous time models should be used for pricing and getting a benchmark fair value, not to hedge

- Hedging in continuous time will never be possible
- Nothing can be schematized in dynamic markets
    - Attempting to give the delta meaning in terms of risk management is denying the dynamic interaction of parameters

Modified Delta

In the real-world trading perspective, orthodox definition of delta offers no significance

- There is no such thing as an infinitely small move in the market
- Even if there is one, such microscopic move would be nobody's concern

Instead we may use the Modified Delta

$$\text{Modified delta} = \frac{\Delta F}{\Delta U}$$

$$\boxed{\text{Delta} = \frac{1}{2}\left(\frac{\Delta F}{\Delta U^-} + \frac{\Delta F}{\Delta U^+}\right)}$$

- Delta depends on the magnitude of the changes in the underlying security
- It could be a function of either his utility curve or his estimation of future volatility

Example: Misleading Delta

The option trader has the following position:

- Short : $1 million of the 96 calls
  - (delta .824, total continuous delta $824,000 short).
- Long: $1 million of the 104 calls
  - (delta .198, total continuous delta $198,000 long).
- **Total Delta: - $626,000.**
- **Hedge: Buy $626,000 of forward.**

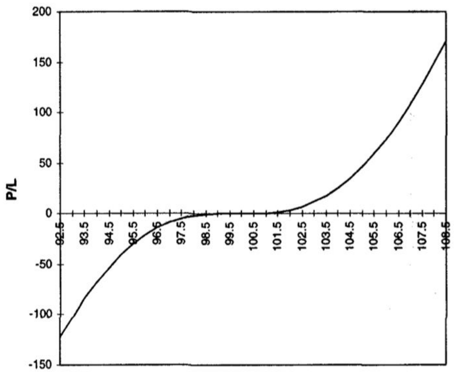


Table 7.1 Inapplicability of the Continuous-Time Delta

Asset Price	P/L 000	Delta 000
92.5	−122	420
93.5	−84	349
95.5	−30	194
97	−9	90
98	−2	39
99	0	8
100	0	0
101	1	16
102	4	53
103	12	106
104	26	171
105	46	241
106	74	311
107	108	377
108.5	171	461

What if the trader buys $550,000 cash instead?

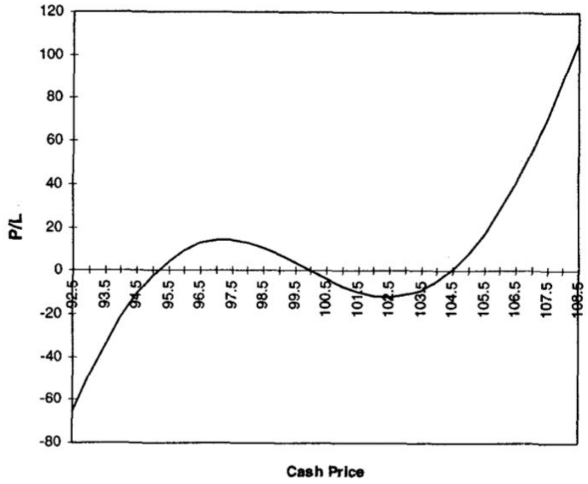


Figure 7.3 Modified delta hedge.

Wider Variations:

93 or 107: “felt long”

Between 99 and 101: “felt short”

95 or 103: “felt squarish”

Table 7.2 A Modified Delta Hedge

	P/L 000	Delta 000
92.5	$\boxed{-65}$	344
94	−22	235
95	−2	157
96	10	81
97	14	14
98	13	−37
99	7	−68
100	0	−76
101	−7	−60
102	−11	−23
103	−11	30
104	−5	95
105	8	165
106	28	235
108.5	$\boxed{107}$	385

1. Maximum loss has dropped from -121 to -65
2. Maximum profit has dropped from 171 to 107.
3. Provides more delta neutral result and has widened the neutrality increment

Delta as a Measure for Risk

Delta shortcoming:

It provides the same measure to an extremely risky position and an extremely safe one.

Example:

Case1: A trader is long 1000 calls

Case 2: A trader is short 1000 puts of the same delta

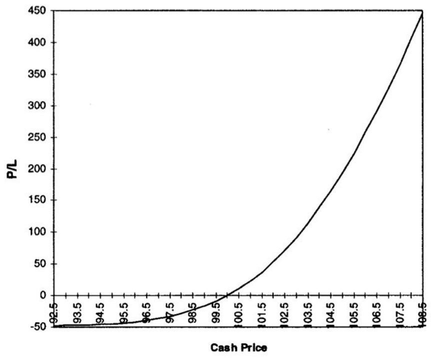


Figure 7.4 P/L for a long/low delta call.

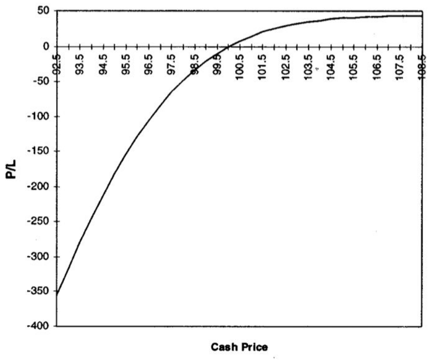


Figure 7.5 P/L for a short/low delta put.

Both have the same delta: $200,000 approximately

Delta for a Forward

- A Non-Interest Bearing Asset

$$F = e^{rt} S$$

- Foreign Currency Forwards

$$F = e^{(r - rf)t} S$$

Because Forward does not immediately deliver the profits and losses, we need to discount it back to the cash using e^{-rt}

$$\text{P/L of Forward} = e^{-rt} e^{(r - rf)t} \Delta S = e^{-rf\,t} \Delta S$$

$$\text{Delta of a Forward} = e^{-rf\,t}$$

➡ *Red Flag: Delta's sole dependence on the foreign rate*
Same forward yet two different hedge ratios based on their home currency

- Foreign Cross Currency Pairs
    - Spot expressed in units of currency 2 per currency 1
    - Forward formula:

$$F = e^{(r1 - r2)t}\ S$$

    - Need to discount it back to the cash, but at which rate? Currency 1 rate or Currency 2 rate

$$\text{Delta of a Forward} = e^{-r(r1 - r2)t}$$

Red Flag: Cannot translate unrealized profits and losses into home currency without incurring a foreign exchange exposure in one of the currencies

- Stock of Stock index

$$\text{Delta of Forward} = e^{-rt}\, e^{(r-d)t} = e^{-dt}$$

Delta for a Forward-Forward

- Delta for a Forward-Forward
    - Arithmetic sum of one long, one short position in the same instrument at two different dates
    - The delta of F(t1,t2) will be equal to

Delta F(t2) - Delta F(t1)

Red Flag: The profits to t1 need to be discounted at a different rate than the profits to t2

Delta for a Future

- Difference between Futures & Forward:
  - Unlike Forward, Futures daily settlement feature eliminates the discounting need.

1. Foreign Currency

$$\text{Future} = F\, e^{(r - rf)t} S,$$

$$\text{Delta of Future} = e^{(r - rf)t}$$

2. Stock or Stock index

$$\text{Delta of Future} = e^{(r - d)t}, \text{ with } d \text{ the dividend rate.}$$

3. Foreign Cross Currency

$$\text{Delta of a Future} = e^{(r1 - r2)t}$$

Delta, Volatility, and Extreme Volatility

$$S_t = S_0 \, exp\{(\mu - .5\sigma^2)\, t + \sigma \sqrt{t}\, Z\}$$

Z follows a reduced centered normal distribution such as $p(z = x) = exp(-x^2/2) / \sqrt{2\pi}$.

The rise in volatility has the tendency to increase the expected final price of the stock through compounding, while the drift would make it decrease by $.5\sigma^2$)

Table 7.3 Asset Values One Period Ahead

	Z						
σ (%)	−3	−2	−1	0	1	2	3
10.00	90.11	93.28	96.55	99.94	103.45	107.08	110.84
15.70	84.88	89.60	94.59	99.85	105.41	111.28	117.47
25.00	76.91	83.84	91.40	99.85	108.60	118.39	129.05
50.00	58.72	69.77	82.91	98.52	117.07	139.12	165.31
100.00	33.47	47.26	66.73	94.22	133.04	187.86	265.27

Extreme Volatility - Price Distribution Skew

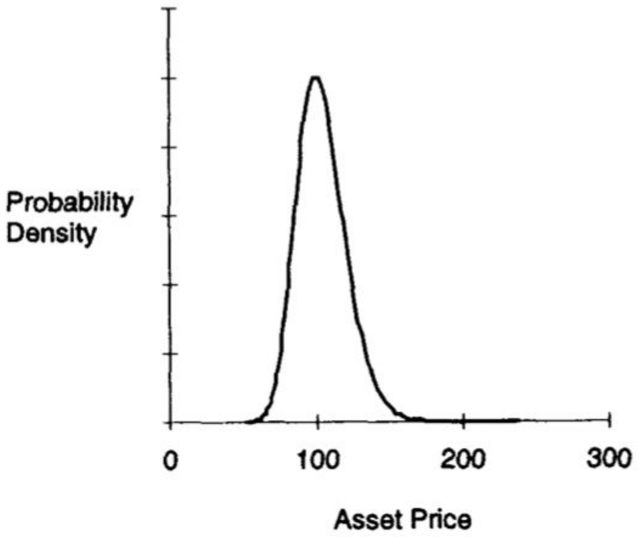


Figure 7.6 Asset distribution at medium volatility.

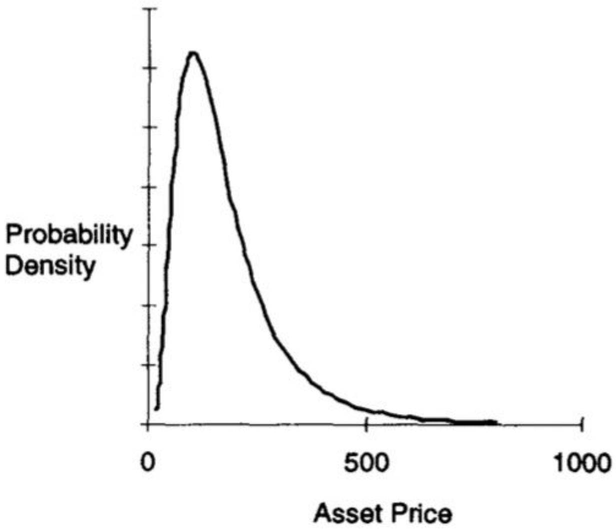


Figure 7.7 Asset distribution at high volatility.

At higher volatility, the distribution develops an increasing skew to the right.

- The mean remains the same
- The median needs to slide left in an amount commensurate to the volatility level

Extreme Volatility – Delta

Spot = 100

Table 7.4 Deltas with Extreme Shifts in Volatility

VOL	90 Put Delta	110 Call Delta	90 Call Delta
10	−0.06	0.09	0.94
15	−0.15	0.20	0.85
20	−0.21	0.27	0.79
30	−0.27	0.36	0.73
40	−0.30	0.42	0.70
50	−0.32	0.46	0.68
60	−0.32	0.49	0.68
80	−0.32	0.54	0.68
100	−0.31	0.59	0.69
120	−0.29	0.62	0.71
140	−0.27	0.65	0.73
160	−0.26	0.68	0.74
180	−0.24	0.71	0.76

Gamma

Contents

Simple Gamma

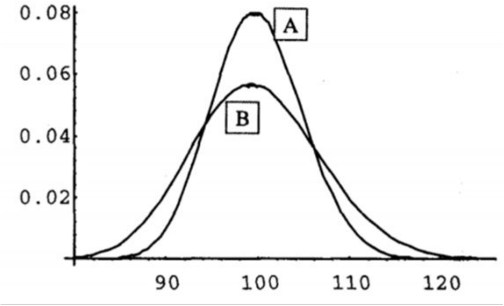
-Second derivative of the derivative with respect to the asset price.

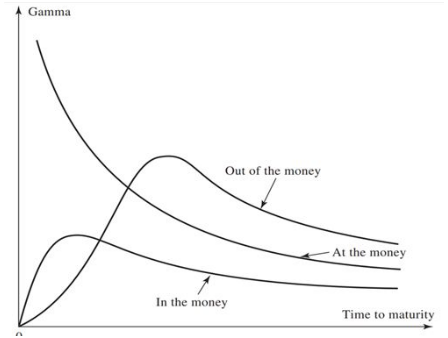
$$\frac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$$

-Same for put and call. It only depends on your position.

-Maximum at ATM.

-Short-term option gamma > long-term option gamma





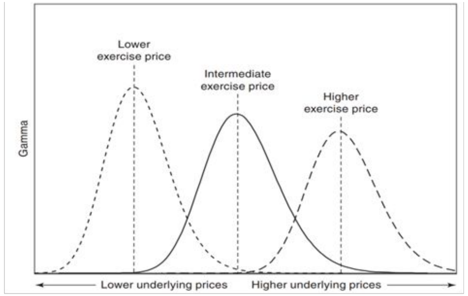
Simple Gamma

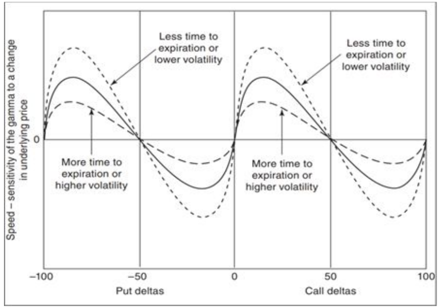
-Gamma is larger with smaller strike price

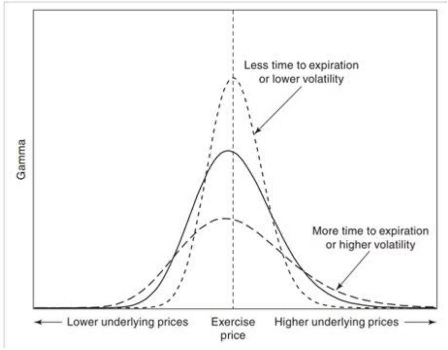
one point price change with underlying with 50 is greater than one point price change with the underlying 100

-Gamma is smaller with higher volatility (the effect of volatility and time are similar).

$$\frac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$$







Imperfections of Gamma

<u>Problem with using gamma as an indicator</u>

-does not provide long-term stability. Only an instantaneous hedge.

<u>Solution:</u>

-Vary the underlying price and calculate the actual change in hedge ratio (delta).

=>**Measure gamma change due to underlying price movement**

> ***Up-gamma and Down-gamma***
>
> *Up-gamma: change in delta should the asset price move higher by some defined increment*
>
> *Down–gamma: Same as up-gamma with the asset moving lower*

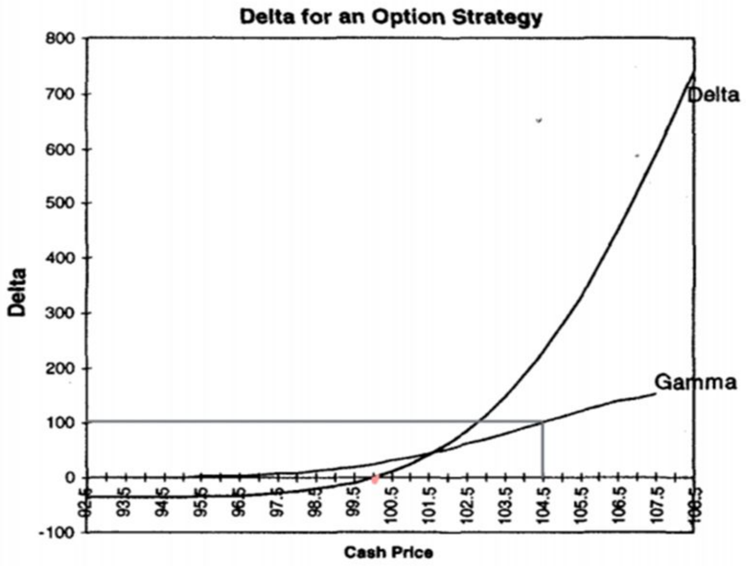


Figure 8.2 An unstable delta for a single option.

Calculation of Up and Down Gamma

-Formula

Up Gamma = Delta(x+d) - Delta(x)

Down Gamma = Delta(x) – Delta(x-d)

-Using its simple average has a problem due to *risk reversal point

=> Gamma needs to be measured with respect to the origin

Table 8.2 Up-Gamma and Down-Gamma for a Risk Reversal

	Delta	Up-Gamma	Down-Gamma
93.5	449	−140	−165
94	375	−126	−153
94.5	309	−112	−140
95	249	−98	−126
95.5	197	−84	−112
96	151	−71	−98
96.5	113	−59	−84
97	80	−47	−71
97.5	54	−36	−59
98	33	−26	−47
98.5	18	−16	−36
99	7	−7	−26
99.5	2	2	−16
100.0	0	11	−7
100.5	3	20	2
101	11	29	11
101.5	23	38	20
102	40	48	29
102.5	61	58	38
103	87	68	48
103.5	119	79	58
104	156	90	68
104.5	198	100	79
105	245	111	90
105.5	298	121	100
106	356	130	111
106.5	419	139	121
107	486	146	130
107.5	558	152	139

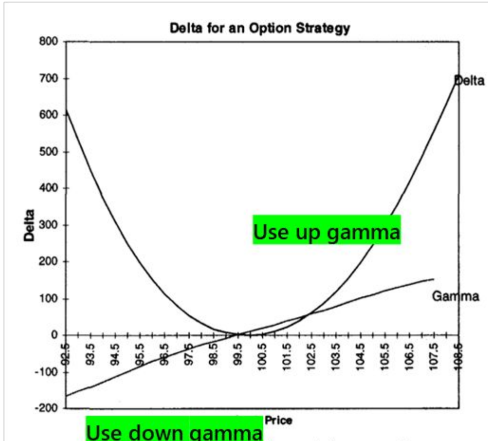


Figure 8.3 Gamma for a risk reversal.

*Risk Reversal: Any position that has an up-gamma over some increment of a different sign than the down-gamma

Gamma correction for different maturities

[Example]

Calendar spread - use two different maturities.

They might have different variance because futures can be positively correlated with the cash (unequal variance).

	Position	Delta	Gamma
Sep (90 days)	Long 2000 contracts	1030	100
March (270 days)	Short 2000 contracts	(1054)	(58)

Gamma correction for different maturities

-Two adjustments

1)The back month could have a lower or higher volatility exposure than the front owing to the **present valuing.**

However, this effect is week.

2)Back month can have different gamma owning to the **stability of the basis** (the cash-future relationship).

To check the volatility of the back month,

Use single factor model: use relative volatilities of every month.

Use covariance matrix of forward months (deriving it by construction).

Gamma correction for different maturities

Assume March had 12% more volatility than September

=>September moved one point? March moves 1.12 points

Multiply 1.12 to the March Gamma

	Position	Delta	Gamma	True Gamma (in front contract equivalent)
Sep (90 days)	Long 2000 contracts	1030	100	100
March (270 days)	Short 2000 contracts	(1054)	(58)	(65)

> ***Risk Management Rule:*** Gammas of different maturities cannot be compared without proper adjustment.

Shadow Gamma

-Formula

At point x_0 with $\nu' < x_0 < \nu$

Shadow up-gamma (x_0) = $(\text{Delta}(x, V + \text{Sig}(x)) - \text{Delta}(x_0, V))/(x - \nu 0)$

Shadow down-gamma (x_0) = $(\text{Delta}(x_0, V) - \text{Delta}(x', V + \text{Sig}(x')))/(x_0 - \nu')$

-Computation of the forecast changes in delta taking into account the changes in volatility and its impact on the position

=>**Measure gamma change due to underlying price movement and volatility movement**

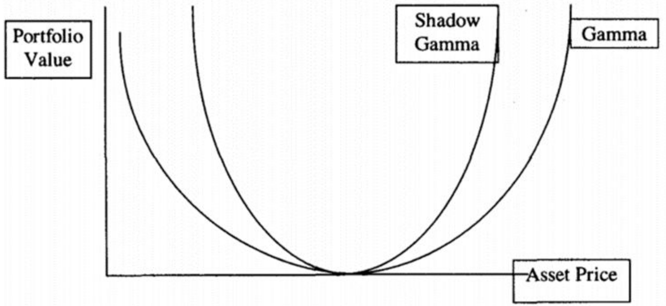


Figure 8.4 Comparing P/L from gamma and shadow gamma.

Shadow Gamma

-Formula (when A<B<C)

Shadow up –gamma (B) = (Delta (C, revised vol) – Delta (B, original vol))/(C-B), where underlying asset price is assumed to go up to C from B

Shadow down-gamma (B) = (Delta (B, original vol) – Delta (A, revised vol))/(B-A), where underlying asset price is assumed to go down to A from B

-Table 8.3 interpretation (when asset price is 100=B)

1.Up gamma =292, Down gamma = 300

2.Shadow up gamma = 337

(337 - 0) / (101-100)

Shadow down gamma = 346

(0-(-346)) / (100-99)

Table 8.3 Shadow Gamma

Asset	P/L Unchanged Volatility	P/L Higher Volatility (100 b.p)	Delta Unchanged Volatility	Delta Higher Volatility (100 b.p)	Delta Difference
96	279	352	−1,605	−1,730	−126
96.5	206	273	−1,323	−1,445	−122
97	146	207	−1,071	−1,185	−114
97.5	99	154	−846	−947	−102
98	61	112	−645	−731	−86
98.5	34	81	−464	−532	−67
99	15	59	−300	−346	−47
99.5	4	46	−146	−171	−25
100	0	39	0	−1	−2
100.5	4	46	145	167	22
101	15	58	292	337	44
101.5	33	80	447	513	66
102	59	110	612	698	86
102.5	94	150	793	896	103
103	139	200	991	1,109	118
103.5	194	261	1,209	1,339	130
104	260	334	1,450	1,588	137

Reacting to Volatility movements

-Large movements are generally accompanied by price jumps. Some moves might take place gently. Recompute P&L and cover delta gamma positions to individual needs.

=>**Construct a volatility map** (forecast made by experience)

Threshold Gamma

-A variation of the shadow gamma method. **Volatility is not a linear junction of the move** and requires the use of schedules of moves (scenario analysis).

Table 8.4 Map of Volatility at Different Price Levels

Starting Price	Price One Day Hence	Resulting Volatility Change (3 Month Options)
	105	1
	104	0.5
	103	unch*
	102	−0.5
	101	−0.2
100	100	unch
	99	unch
	98	0.5
	97	1
	96	2
	95	3
	94	4
	93	7 (panic)
	92	10
	91	10
	90	10

*Unchanged.

Reacting to Volatility movements

-Taking volatility skew into account

=>With *biased assets, the gamma needs to take into account the behavior of

volatility and the movement along the “skew curve”.

-Indication of future volatility can be drawn from where the out-of-the-money calls and puts are presently trading.

-In short, **skew gamma(asymmetrical shadow gamma) should be used instead of regular shadow gamma if volatility skew is present.**

*Biased Asset: Asymmetrical assets that cause anxiety during price drops

Other types of Gamma

1) **GARCH Gamma**

-As markets move, so will the future volatility, and this information needs to be incorporated into the future delta (volatility moves in clusters).

=> **the difference between present and future delta is called GARCH Gamma**

-Predicts both future historical and implied volatility, whereas shadow gamma does not (it is simply a function of path taken by the underlying securities).

2) **Advanced Shadow Gamma**

-**takes into account the trader's expected volatility and interest rate moves that accompany the changes in the asset price** (both volatility and interest rate curves are expected to shift in a nonparallel way).

-It could be easily used in a commodity that has both its price change and its volatility correlated to the interest rates.

Table 8.5 Map of Volatility and Interest Rate Differentials at Different Price Levels

Starting Price	Price One Day Hence	Resulting Volatility Change (1 Year Option) (3 Month Options)	Resulting Interest Rate Differential (1 Year Forward) (3 Month Differential)†
	105	1	
	104	0.5	−.75%
	103	unch*	−.5
	102	−0.5	−.5
	101	−0.2	0
100	100	unch	0
	98	unch	0
	96	0.5	2
	94	1	4
	92	2 (panic)	10
	90	3	20
	88	?	?
	86	?	?

*Unchanged.

Example: Assume the existence of the imaginary currency of Syldavia. SYL-USD is the symbol for the currency pair.

US Dollar Rd (one year) = 6%.

Syldavian currency Rf = 20%.

Spot price = 100.

The one-year forward price, satisfying the covered interest rate parity formula (100 × Exp(.06 − .20)), would trade at 86.93.

Case Study: The Syldavian Elections

Assume results of the ballots are due in one hour and there are only two states of P&L decided by the results.

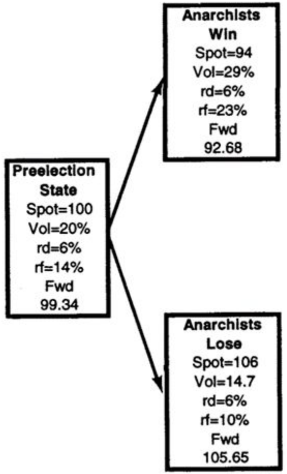


Figure 8.6 Outcomes of Syldavian elections.

Table 8.6 Risk Matrix and the Outcomes of the Syldavian Elections

	P/L			Delta		
Asset	V = 14.7%	V = 20%	V = 29%	V = 14.7%	V = 20%	V = 29%
93	14	−86	−225	334	383	462
94	40	−52	−181 Scenario A	189	282	406
95	52	−29	−143	65	193	357
96	54	−13	−110	−32	118	317
97	47	−5	−80	−104	60	287
98	34	−1	−52	−150	20	267
99	18	0	−26	−173	1	258
100	−8	0 Starting Point	15	−175	0 Starting Point	260
101	−17	1	26	−156	19	272
102	−31	4	55	−118	55	293
103	−40	12	85	−57	108	324
104	−41	26	120	25	174	362
105	−34	48	158	127	252	407
106	−15 Scenario B	77	201	247	339	457
107	16	116	250	379	431	510

Short Shadow Gamma

Vega