$\Pr(j=k|x) = \frac{\exp(u_k(x))}{\sum_j \exp(u_j(x))}$ and $V(x) = \log\left(\sum_j \exp(u_j(x))\right) + \gamma$ in a Static Discrete Choice Problem with T1EV Disturbance

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Suppose an agent has J choices, indexed by $j \in \mathcal{J}$. Conditional on the observable state of the agent denoted as x, suppose we know the functional form of the deterministic utility $u_j(x), \forall j \in \mathcal{J}$. The structural disturbances $\varepsilon_j,\ j \in \mathcal{J}$ are additively separable to each decision and follow i.i.d. Type 1 Extreme Value (T1EV) distribution.[1] In other words, the agent solves

$$\max_{j\in\mathcal{J}} u_j(x) + \varepsilon_j$$

WLOG, let's calculate the probability of choosing choice a conditional on x. For simplicity, let's use the following notation: $u_j(x) \equiv u_j$, $(u_i - u_j) \equiv m_{ij}$, and $\varepsilon_a \equiv \varepsilon$ when appropriate. The conditional choice probability of choosing $j \in \mathcal{J}$, $\Pr(j = a|x)$, is:

$$\begin{aligned}
&\Pr(j=a|x) = \Pr\Big(\big[u_a + \varepsilon_a \geq u_j + \varepsilon_j\big],\ \forall j \in \mathcal{J}\backslash\{a\}|x\Big) \text{ by definition}\\
&= \mathbb{E}_{\varepsilon_a}\left[\Pr\Big(\big[u_a + \varepsilon_a \geq u_j + \varepsilon_j\big],\ \forall j \in \mathcal{J}\backslash\{a\}\Big| x, \varepsilon_a\Big)\right] \text{ by the law of iterated expectations}\\
&= \mathbb{E}_{\varepsilon_a}\left[\prod_{j\in\mathcal{J}\backslash\{a\}} \Pr\Big(\big[u_a + \varepsilon_a \geq u_j + \varepsilon_j\big]\Big| x, \varepsilon_a\Big)\right] \quad \because \forall j,\ \varepsilon_j \text{ are i.i.d.}\\
&= \int_{-\infty}^{\infty} e^{-\sum_{j\in\mathcal{J}\backslash\{a\}} e^{-(m_{aj}+\varepsilon)}} \cdot g(\varepsilon) d\varepsilon \quad \because \text{Apply cdf for } \varepsilon_j,\ \forall\ j \in \mathcal{J}\backslash\{a\} \text{ and integrate } \varepsilon \text{ over the real line}\\
&= \int_{-\infty}^{\infty} e^{-\sum_{j\in\mathcal{J}\backslash\{a\}} e^{-(m_{aj}+\varepsilon)}} \cdot e^{-e^{-\varepsilon}} e^{-\varepsilon} d\varepsilon = \int_{-\infty}^{\infty} e^{(-e^{-\varepsilon})(1+\sum_{j\in\mathcal{J}\backslash\{a\}} e^{-m_{aj}})} e^{-\varepsilon} d\varepsilon
\end{aligned}$$

Change of variable: $t = e^{-\varepsilon} \to dt = -e^{-\varepsilon} d\varepsilon$. Then, $\varepsilon \to -\infty \Rightarrow t \to \infty$ and $\varepsilon \to \infty \Rightarrow t \to 0$.

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{(-e^{-\varepsilon})(1+\sum_{j\in\mathcal{J}\backslash\{a\}} e^{-m_{aj}})} e^{-\varepsilon} d\varepsilon = \int_{\infty}^{0} -e^{-t(1+\sum_{j\in\mathcal{J}\backslash\{a\}} e^{-m_{aj}})} dt = \int_{0}^{\infty} e^{-t(1+\sum_{j\in\mathcal{J}\backslash\{a\}} e^{-m_{aj}})} dt\\
&= -\frac{e^{-t(1+\sum_{j\in\mathcal{J}\backslash\{a\}} e^{-m_{aj}})}}{1+\sum_{j\in\mathcal{J}\backslash\{a\}} e^{-m_{aj}}}\Bigg|_{t=0}^{t=\infty} = \frac{1}{1+\sum_{j\in\mathcal{J}\backslash\{a\}} e^{-m_{aj}}} = \frac{1}{1+\sum_{j\in\mathcal{J}\backslash\{a\}} e^{-(u_a-u_j)}} = \frac{\exp(u_a(x))}{\sum_{j\in\mathcal{J}} \exp(u_j(x))}
\end{aligned}$$

So, we can finally say that $\Pr(j=k|x) = \frac{\exp(u_k(x))}{\sum_{j\in\mathcal{J}}\exp(u_j(x))}$.

Estimation of discrete choice models often relies on getting a closed-form relationship between the ex-ante expected utility and the choice probability conditional on the state variable. Let's compute the ex-ante expected utility from the discrete choice problem. In other words,

$$\begin{aligned}
V(x) &= \int \max_{j\in\mathcal{J}}\{u_j(x) + \varepsilon_j\} g(\vec{\varepsilon}) d\vec{\varepsilon} = \sum_{j\in\mathcal{J}} \int (u_j + \varepsilon_j) \mathbb{1}\{u_j + \varepsilon_j \geq u_k + \varepsilon_k,\ \forall k \in \mathcal{J}\backslash\{j\}\} g(\vec{\varepsilon}) d\vec{\varepsilon}\\
&= \sum_{j\in\mathcal{J}} \int_{-\infty}^{\infty} (u_j + \varepsilon_j) e^{-e^{-\varepsilon_j}(1+\sum_{k\in\mathcal{J}\in\{j\}} e^{-m_{jk}})} e^{-\varepsilon_j} d\varepsilon_j
\end{aligned}$$

[1] If ε follows Type 1 Extreme Value distribution, the pdf is $f(\varepsilon) = e^{-\varepsilon} e^{e^{-\varepsilon}}$ and the cdf is $F(\varepsilon) = e^{-e^{-\varepsilon}}$.

Where the second line is exploiting the cdf form of iid T1EV structural errors across the decisions. Again WLOG, I calculate the first case, $j = a$, as the representative term of the summation. Denoting $\varepsilon_a \coloneqq \varepsilon$,

$$\begin{aligned}
&\int_{-\infty}^{\infty} (u_a + \varepsilon_a) e^{-e^{-\varepsilon_a}(1+\sum_{k\in\mathcal{J}\in\{a\}} e^{-m_{ak}})} e^{-\varepsilon_a} d\varepsilon_a \\
&= \underbrace{\int_{-\infty}^{\infty} u_a e^{-e^{-\varepsilon}(1+\sum_{k\in\mathcal{J}\setminus\{a\}} e^{-m_{ak}})} e^{-\varepsilon} d\varepsilon}_{(1)} + \underbrace{\int_{-\infty}^{\infty} \varepsilon e^{-e^{-\varepsilon}(1+\sum_{k\in\mathcal{J}\setminus\{a\}} e^{-m_{ak}})} e^{-\varepsilon} d\varepsilon}_{(2)}
\end{aligned}$$

Using the same change of variable, $t = e^{-\varepsilon}$, term (1) becomes

$$\begin{aligned}
&\int_{-\infty}^{\infty} u_a e^{(1+\sum_{k\in\mathcal{J}\setminus\{a\}} e^{-m_{ak}})} e^{-\varepsilon} d\varepsilon = u_a \int_0^{\infty} e^{-t(1+\sum_{k\in\mathcal{J}\setminus\{a\}} e^{-m_{ak}})} dt = u_a \frac{-e^{-t(1+\sum_{k\in\mathcal{J}\setminus\{a\}} e^{-m_{ak}})}}{1+\sum_{k\in\mathcal{J}\setminus\{a\}} e^{-m_{ak}}}\Bigg|_{t=0}^{t=\infty} \\
&= \frac{u_a}{1+\sum_{k\in\mathcal{J}\setminus\{a\}} e^{-m_{ak}}} = \frac{u_a}{1+\sum_{k\in\mathcal{J}\setminus\{a\}} e^{-(u_a-u_k)}} = \frac{u_a \exp(u_a)}{\sum_{j\in\mathcal{J}} \exp(u_j)}
\end{aligned}$$

Of course, noticing that we are taking expectation, we can simply say that (1) is

$$u_a \Pr\Big(\big[u_a + \varepsilon_a \geq u_j + \varepsilon_j\big],\ \forall j \in \mathcal{J}\setminus\{j\}\Big) = \frac{u_a \exp(u_a)}{\sum_j \exp(u_j)}$$

We apply the same change of variable for (2) while realizing that $t = e^{-\varepsilon} \Rightarrow -\log t = \varepsilon$:

$$\begin{aligned}
\int_{-\infty}^{\infty} \varepsilon e^{(-e^{\varepsilon})(1+\sum_{k\in\mathcal{J}\setminus\{a\}} e^{-m_{ak}})} e^{-\varepsilon} d\varepsilon &= \int_{\infty}^{0} -(\log t)\, e^{-t(1+\sum_{k\in\mathcal{J}\setminus\{a\}} e^{-m_{ak}})} (-1) dt \\
&= \int_{\infty}^{0} (\log t)\, e^{-t(1+\sum_{k\in\mathcal{J}\setminus\{a\}} e^{-m_{ak}})} dt
\end{aligned}$$

Let's denote $M \equiv (1 + \sum_{k\in\mathcal{J}\setminus\{a\}} e^{-m_{ak}})$:

$$\begin{aligned}
&\int_{\infty}^{0} (\log t)\, e^{-t(1+\sum_{k\in\mathcal{J}\setminus\{a\}} e^{-m_{ak}})} dt = \int_{\infty}^{0} \big(\log tM \; - \log M\big) e^{-tM} dt \\
&= \int_0^{\infty} -\log(tM) e^{-tM} dt + \int_0^{\infty} \log M e^{-tM} dt \\
&= \int_0^{\infty} -\frac{1}{M} (\log y) e^{-y} dy - \frac{(\log M) e^{-tM}}{M}\Bigg|_0^{\infty} = \gamma \frac{1}{M} + \frac{\log M}{M} \\
&= \frac{\gamma}{1+\sum_{k\in\mathcal{J}\setminus\{a\}} e^{-m_{ak}}} + \frac{\log\left(1+\sum_{k\in\mathcal{J}\setminus\{a\}} e^{-m_{ak}}\right)}{1+\sum_{k\in\mathcal{J}\setminus\{a\}} e^{-m_{ak}}} \\
&= \frac{\gamma \exp(u_a)}{\sum_{j\in\mathcal{J}} \exp(u_j)} + \frac{\exp(u_a)}{\sum_{j\in\mathcal{J}} \exp(u_j)} \log\left(\frac{\sum_{j\in\mathcal{J}} \exp(u_j)}{\exp(u_a)}\right) \\
&= \frac{\gamma \exp(u_a)}{\sum_{j\in\mathcal{J}} \exp(u_j)} + \frac{\exp(u_a)}{\sum_{j\in\mathcal{J}} \exp(u_j)} \log\left(\sum_{j\in\mathcal{J}} \exp(u_j)\right) - \frac{u_a \exp(u_a)}{\sum_{j\in\mathcal{J}} \exp(u_j)}
\end{aligned}$$

We applied change of variable again in the second line: $y = tM \Rightarrow dy = Mdt$ and γ is Euler's constant. Note that the derivation of $-\int_0^{\infty} (\log y) e^{-y} dy = \gamma$ involves $\Gamma'(1) = \gamma$. Knitting (1) and (2) together, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} (u_a + \varepsilon_a) e^{-e^{-\varepsilon_a}(1+\sum_{k\in\mathcal{J}\in\{a\}} e^{-m_{ak}})} e^{-\varepsilon_a} d\varepsilon_a \\
&= \underbrace{\frac{u_a \exp(u_a)}{\sum_{k\in\mathcal{J}} \exp(u_k)}}_{(1)} + \underbrace{\frac{\gamma \exp(u_a)}{\sum_{k\in\mathcal{J}} \exp(u_k)} + \frac{\exp(u_a)}{\sum_{k\in\mathcal{J}} \exp(u_k)} \log\left(\sum_{k\in\mathcal{J}} \exp(u_k)\right) - \frac{u_a \exp(u_a)}{\sum_{k\in\mathcal{J}} \exp(u_k)}}_{(2)} \\
&= \frac{\gamma \exp(u_a)}{\sum_{k\in\mathcal{J}} \exp(u_k)} + \frac{\exp(u_a)}{\sum_{k\in\mathcal{J}} \exp(u_k)} \log\left(\sum_{k\in\mathcal{J}} \exp(u_k)\right)
\end{aligned}$$

Broadcasting this result across all $j \in \mathcal{J}$, we finally get $V(x)$:

$$V(x) = \sum_{j \in \mathcal{J}} \left(\frac{\exp(u_j)}{\sum_{k \in \mathcal{J}} \exp(u_k)} \log \left(\sum_{k \in \mathcal{J}} \exp(u_k) \right) + \frac{\gamma \exp(u_j)}{\sum_{k \in \mathcal{J}} \exp(u_k)} \right) = \log \Big(\sum_{j \in \mathcal{J}} \exp(u_j(x)) \Big) + \gamma$$