CSE2215 - Computer Graphics

Introduction to CG

Painting by Numbers

Elmar Eisemann

Delft University of Technology

Welcome!

Elmar Eisemann

Ricardo Marroquim

Mathijs Molenaar

Martin Skrodzki
(Linear Algebra Recap)

Michael Weinmann
(helps with project)

Many TAs in the background who you will encounter during the practical sessions!

Introduction

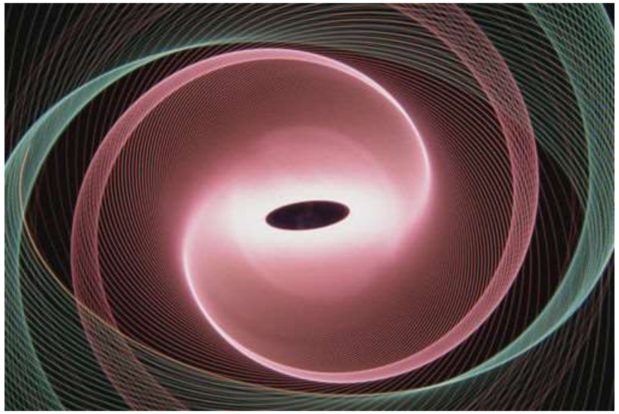
- Computer Graphics
  - part of computer science

Manipulation, creation, and display of visual and geometric information with a computer

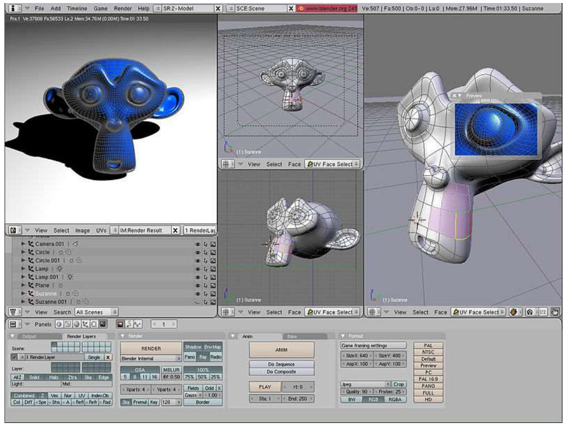
Not: using Photoshop

Instead: making Photoshop

First hit on Google in 2012... don't ask! **;)**

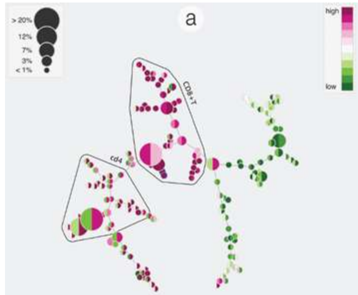


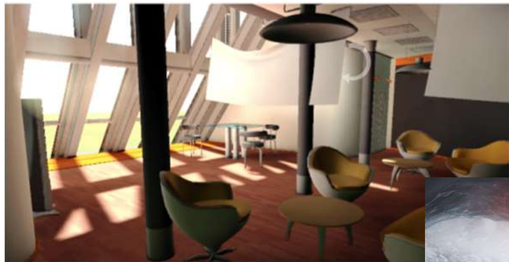
Hit on Google in 2023... voila! ;)

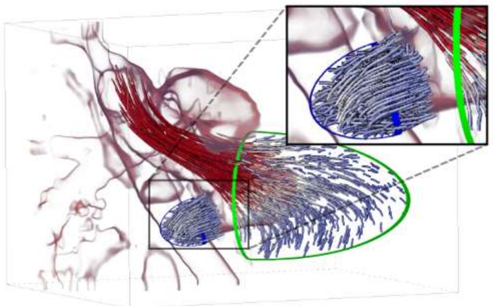


Computer Graphics and Visualization

- CG impacts many domains
  - Big Data Analysis
  - Scientific Visualization
  - Architecture/Design
  - Education
  - Entertainment
  - …







Computer Graphics and Visualization

Delft University of Technology

Computer Graphics and Visualization

TU Delft
Delft University of Technology

Computer Graphics and Visualization

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What do we cover in this course?

- Key Concepts and basics of CG
- Images
- Geometry
- Transformations/Animation
- Materials and Light
- Textures
- Shadows

CSE2215 Computer Graphics

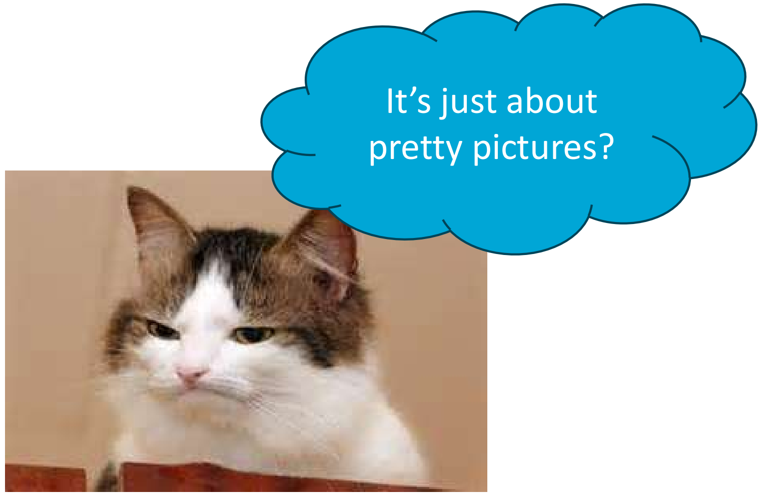
- Key Concepts

relevant for midterm

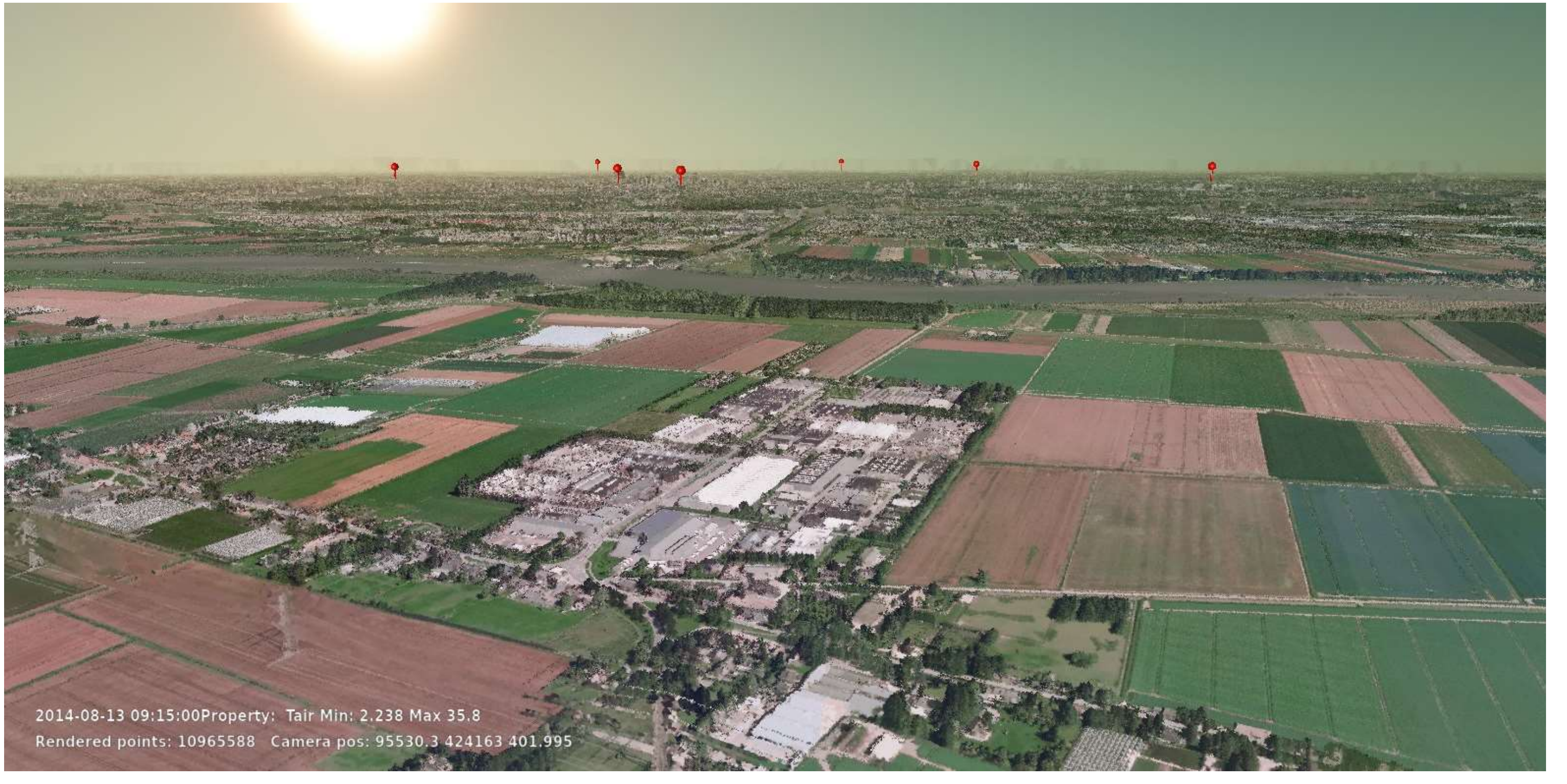
- Images
- Geometry
- Transformations
- Materials and Light
- Textures
- Shadows
- Advanced Ray Tracing

relevant for project

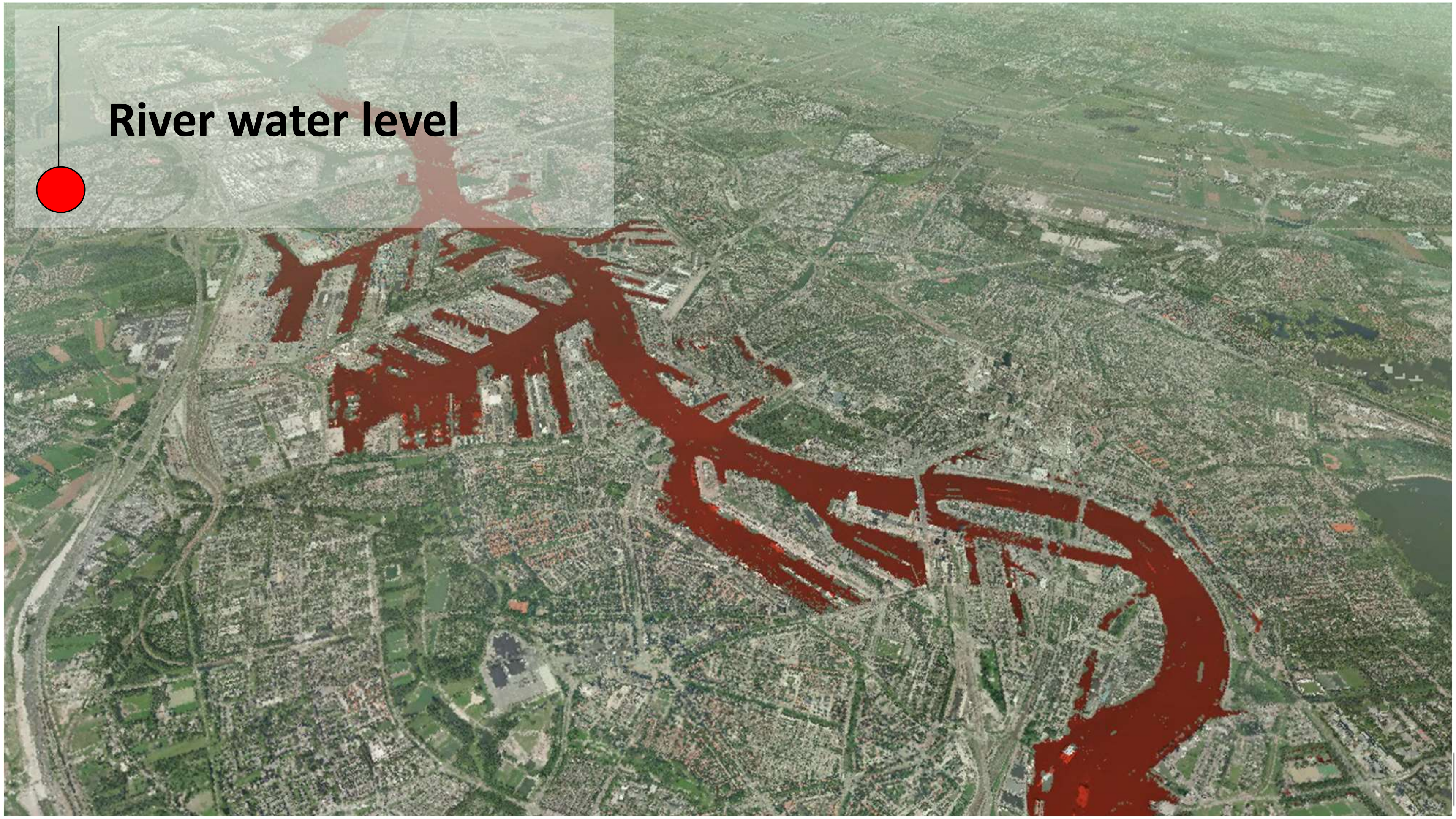
- Data Structures



Large-Scale Visualization



Models



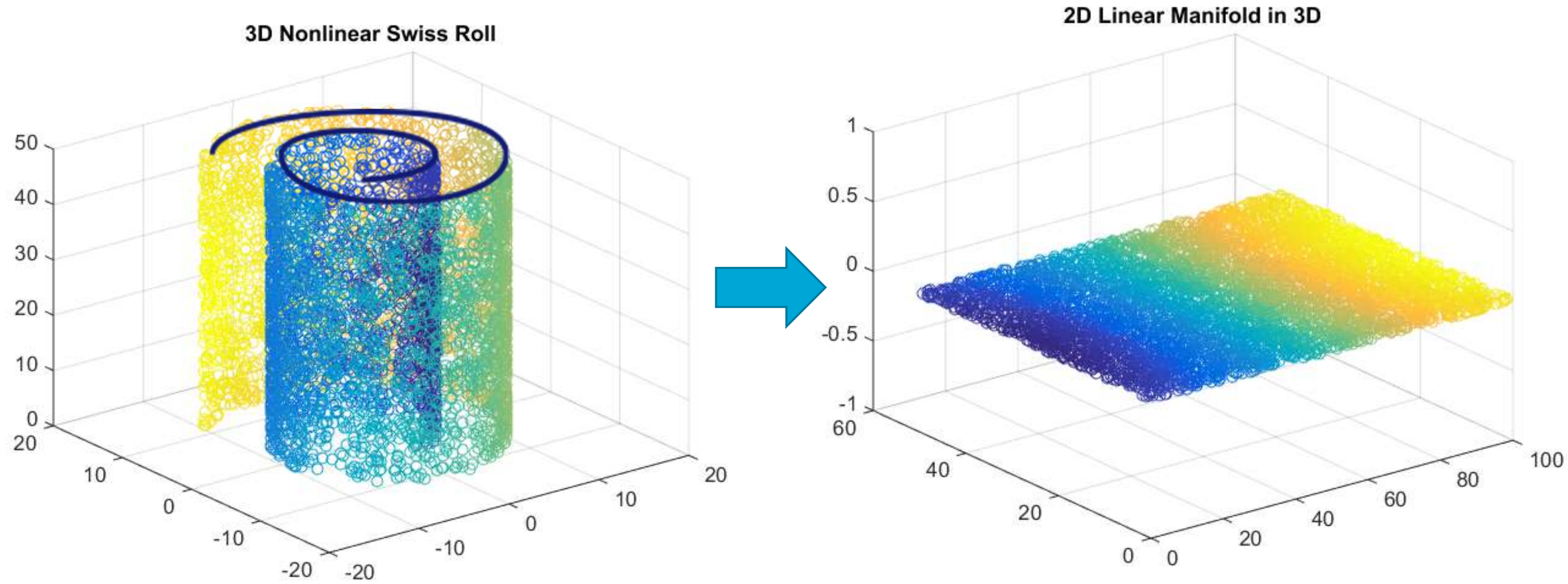
Creating the “Omniverse” for Machine Learning

- NVIDIAs vision for future machine learning


D & T FOODS INT'L

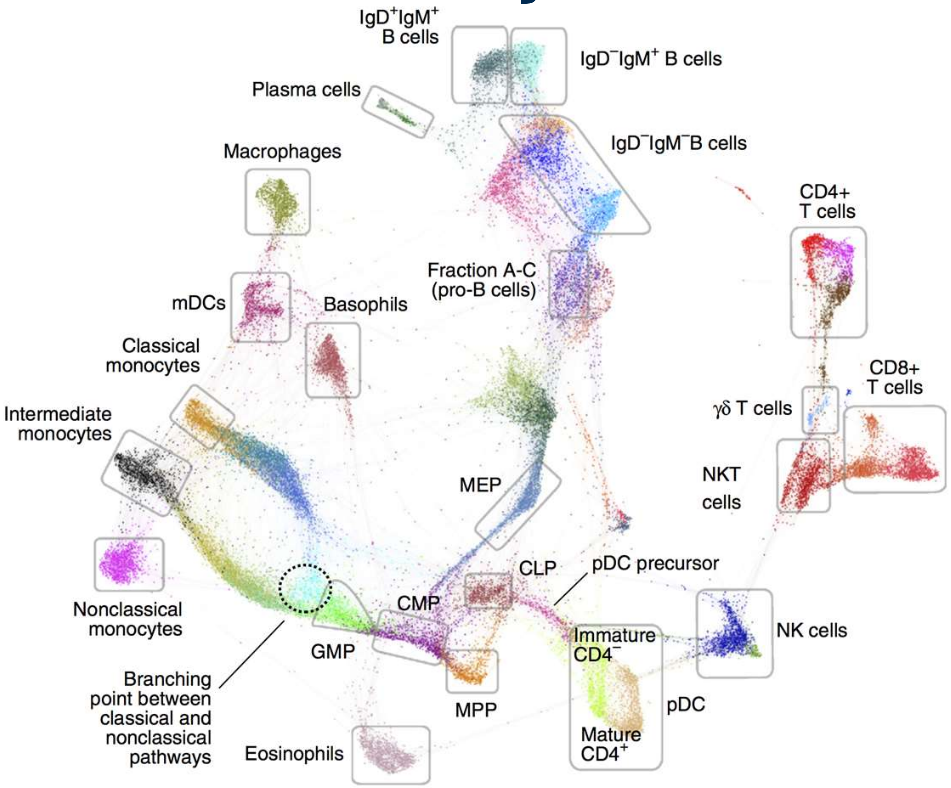

D & T FOODS INT'L

Acceleration Structures for Dimensionality Reduction



- Original t-SNE: $O(n^2)$ 2 days(!)
- Barnes-Hut SNE: O(n log(n)) 8 min
- Linear t-SNE: O(n) 10 sec.
- Dual Hiearchy t-SNE: O(n) < 2 sec.

Dimensionality Reduction in the Medical Domain



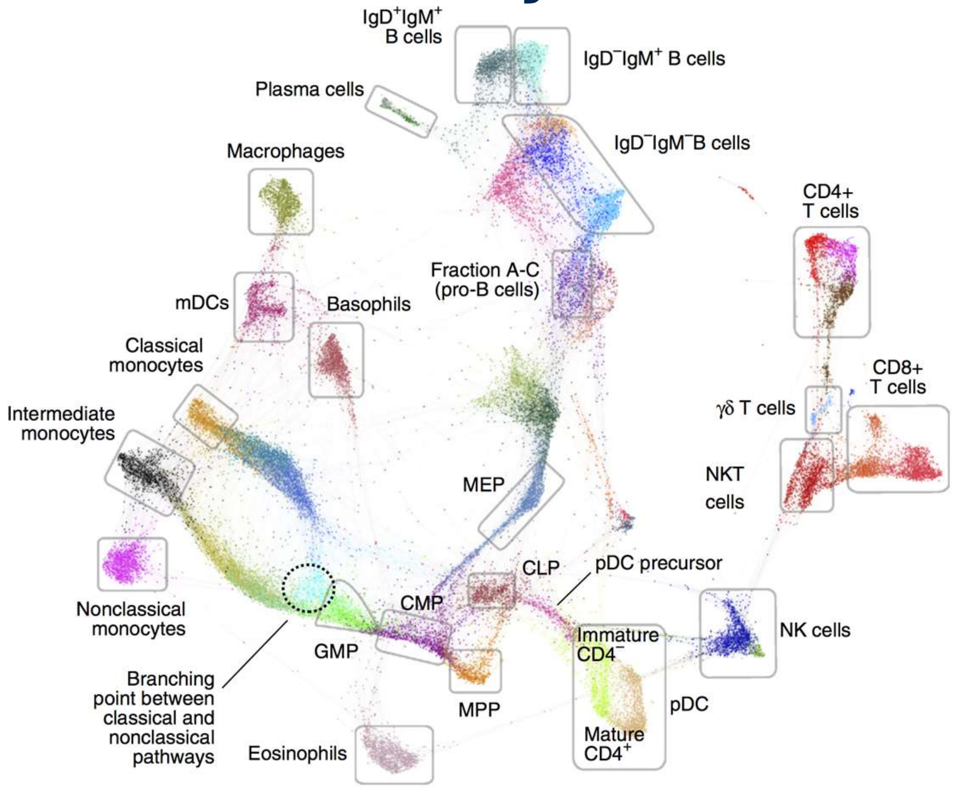
Samusik et al. Nature Methods, 2016

22 hours

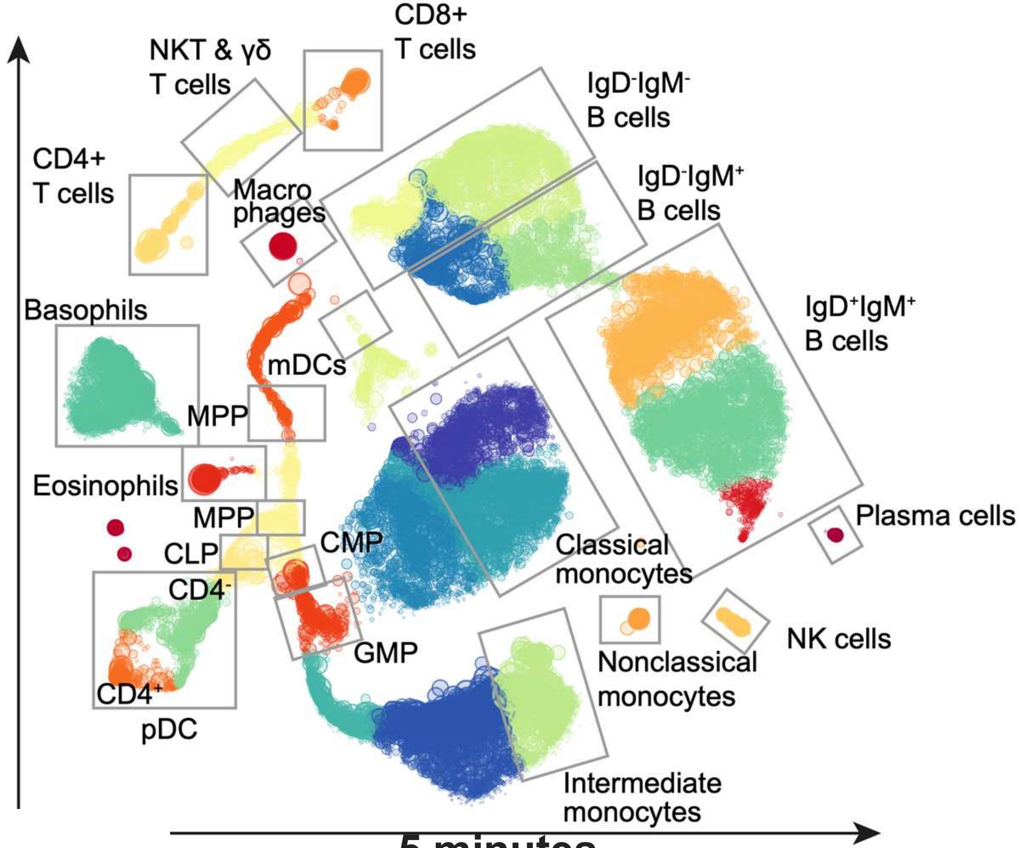
30k cells visualized

[van Unen, Höllt, Pezzotti, Li, Reinders, Eisemann, Koning, Vilanova, Lelieveldt: Nature Communications 2017]

Dimensionality Reduction in the Medical Domain



Samusik et al. Nature Methods, 2016



22 hours
30k cells visualized

5 minutes
Zoom in for full 1M

[van Unen, Höllt, Pezzotti, Li, Reinders, Eisemann, Koning, Vilanova, Lelieveldt: Nature Communications 2017]

CSE2215

- The course provides basic (!) knowledge to tackle such challenges.

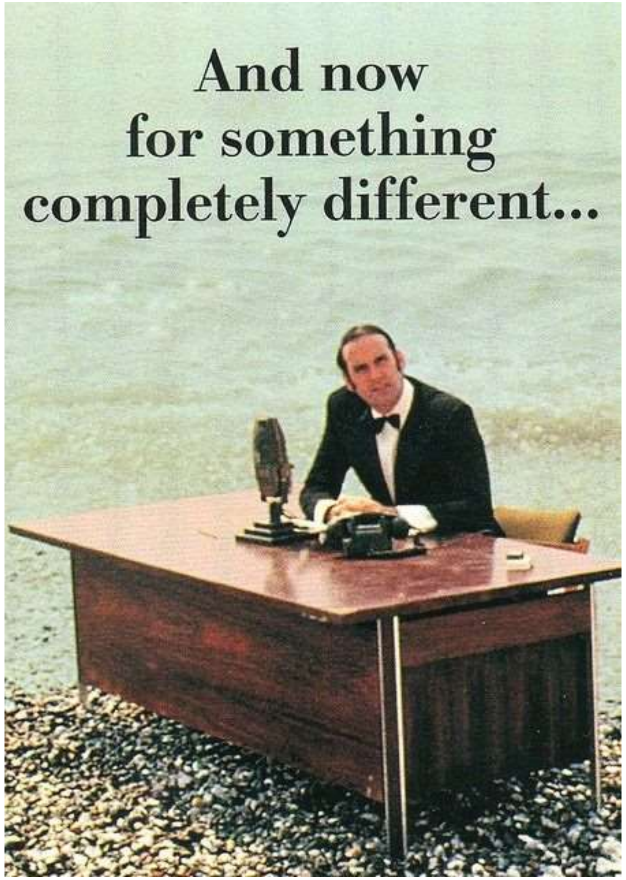
- An image says more than a thousand words…

CSE2215

CSE2215 Study Goals

- S1- Explain and compare the structure and properties of standard algorithms and data structures linked to Computer Graphics.
- S2- Execute and visualize standard algorithms and data structures.
- S3- Use mathematical methods to analyze, create, apply algorithms and data structures, as well as understanding time and space complexity of image-generation algorithms
- S4- Apply mathematical modeling and theory of geometric computations and transformations, object representations, simulation, and encoding.
- S5- Implement algorithms and data structures using the C++ programming language and OpenGL.
- S6- Apply the knowledge obtained in this course to problems of other fields

At the moment of suspense...



Administration...

CS2215 Computer Graphics

- How to test my knowledge?
- Theory:
  - Exam
  - Use in projects/assignment
- Practical Skills:
  - Tutorials
  - C++ and OpenGL
  - Assignments
  - Projects

CS2215 Computer Graphics

- **Expected prior knowledge**
  - Object-oriented programming;
  - Algorithms and Data Structures;
  - Linear Algebra;
  - Calculus

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Assessment

- **Practical Grade** (at least 5.0)
  - 5 Assignments (40% of practical grade)
    – referred to as **Assignment Score**

    (individual, first 3 determine project groups)
  - Final Project (60% of practical grade)
    – referred to as **Project Score**

    (group project with individual grades)
- **Exam Grade** (at least 5.0)
- **Final Grade** (at least 6.0)
  - (Practical Grade + Exam Grade) /2

Student Example

- Student A:
  - Scores all 5 Assignments with a 2.5
  - Scores final project with a 10
  - Scores exam with a 4

  - Practical Grade: (0.4 * 2.5+ 0.6 * 10) = 1+6 = 7
    - 5 Assignments (40% of practical grade)
    - Final Project (60% of practical grade)

  - Final Grade: (Practical + Exam)/2, thus (7+6)/2=6.5

Attention!

Assessment

- **Practical Grade** (at least 5.0)
  - 5 Assignments (40% of practical grade) – referred to as **Assignment Score**

    (individual, first 3 determine project groups)
  - Final Project (60% of practical grade) – referred to as **Project Score**

    (group project with individual grades)
- **Exam Grade** (at least 5.0)
- **Final Grade** (at least 6.0)
  - (Practical Grade + Exam Grade) /2

Student Examples

- Student A:
  - Scores all 5 Assignments with a 2.5
  - Scores final project with a 10
  - Scores exam with a 6 >5?

  - Practical Grade: (0.4 * 2.5+ 0.6 * 10) = 1+6 = 7 >5?
    - 5 Assignments (40% of practical grade)
    - Final Project (60% of practical grade)
  - Final Grade: (Practical + Exam)/2, thus 7+6/2=6.5 >6?

What to do, if you do not pass?

- **Exam resit:** Open to everyone
- **The repair of the Assignment score** takes form of a multi-component assignment and can involve a computer assessment.
- **The repair of the final project score** implies

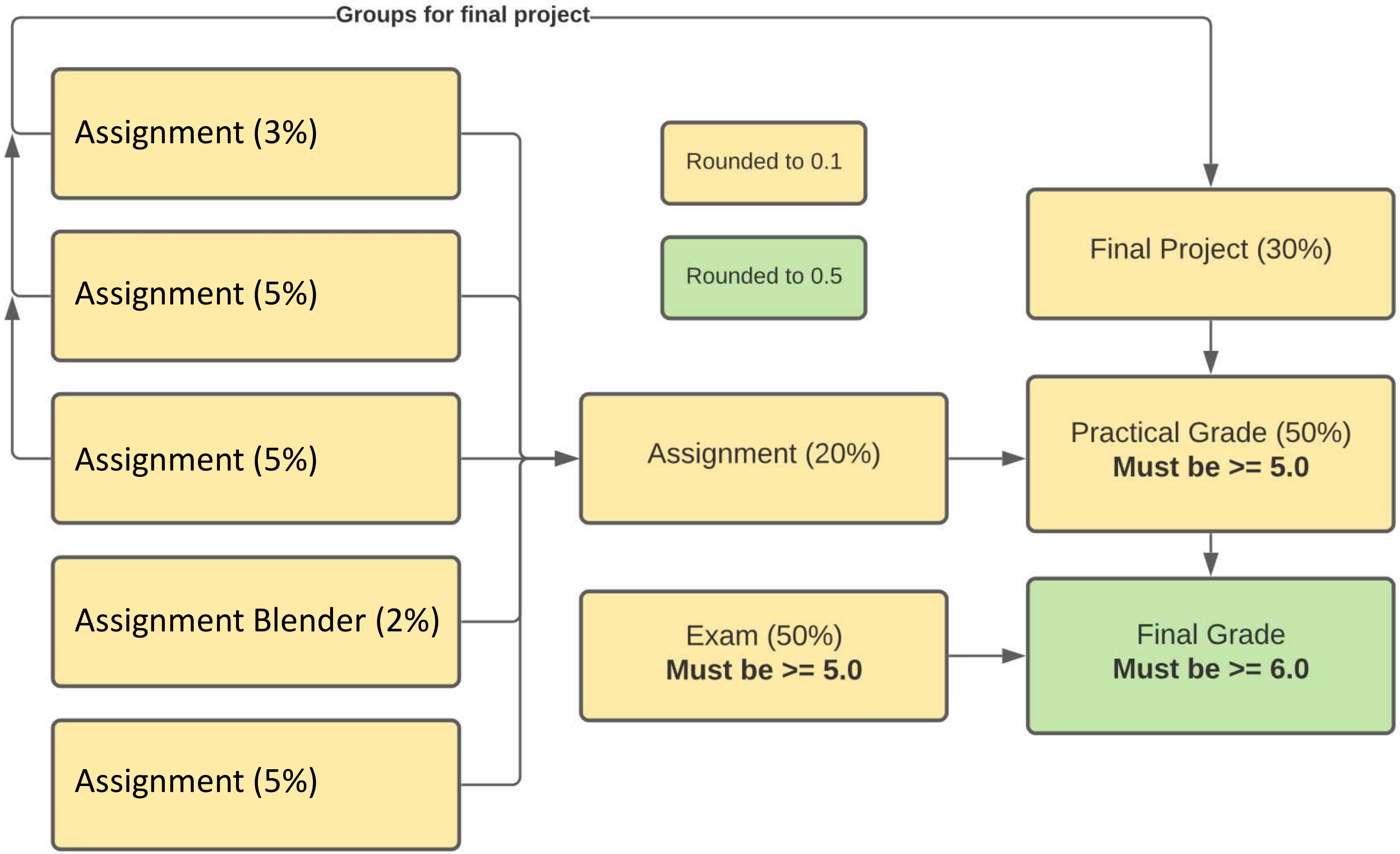
  at least 6.0 on the midterm exam (not resit)

  at least 6.0 in the assignments (before any repairs)

  it will take the form of an individual assignment.

Repairs/Resits are expected to take place in the following quarter.
Following TU Delft regulations, a repair can maximally lead to a 6.0

Assessment Overview – also available on Brightspace



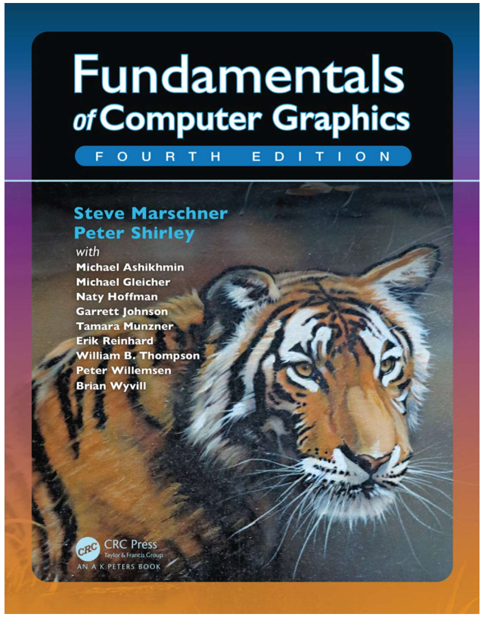
How is your work evaluated?

- Assignments
  - Automatic Correction and Feedback

- Exam
  - Mix of multiple choice (MC) and open questions
  - Grading of MC questions: wrong answer=0, skip=1, correct=2
    - If you cannot find your answer, skip to get partial points for your attempt
    - Do not guess, as this is likely to be unsuccessful
  - Passing of the MC part guaranteed if 70% of the points are reached.

Books

You can get me as
an e-Book at the
library!



- Fundamentals of Computer Graphics
  by Marschner, Shirley

- Computer Graphics. Principles and Practice
  by James D. Foley, Andries VanDam, Steven K. Feiner

- Real-time Rendering
  by Tomas Akenine-Möller, Eric Haines, Naty Hoffman - Peters, Wellesley

Questions

- Answers.ewi.tudeflt.nl / FAQs on Brightspace
- TAs in practicals
- Teachers in class/break

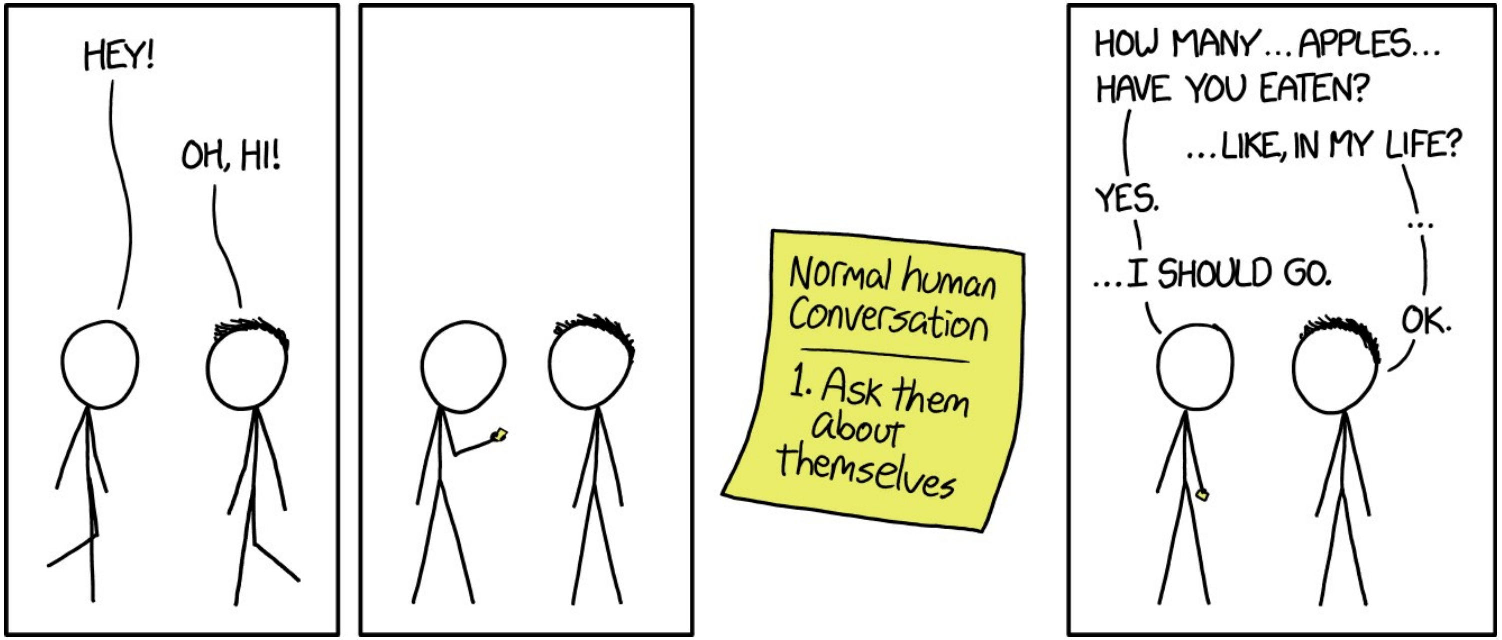
- **Last resort:** Course email cg-cs-ewi@tudelft.nl
  - Processing time: ~6 workdays
  - **Use TU Delft email and add your student number**

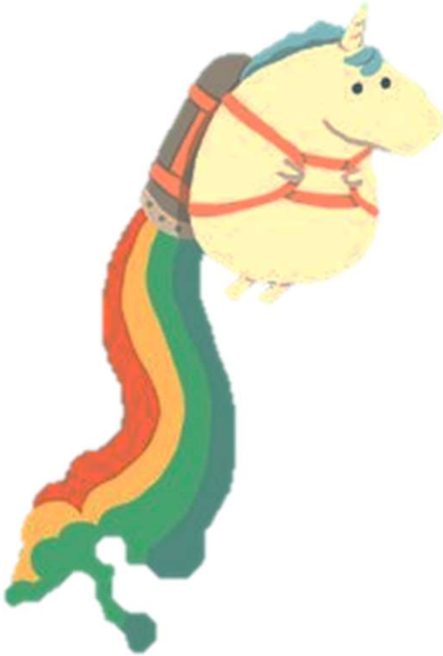
Information overkill...

Where do I find all the info?

- On Brightspace
  - Tasks pop up every week
    - (e.g., Lecture 4.5h, readup/revise 3h, exercises 1h, Algebra test 2h, assignments 2h, setup 0.5h “=“ 13h)
  - Content is available after the lectures
  - Assignment download and hand in
  - Assessment explanations
  - Additional exercises
  - Reading instructions
  - FAQs
  - and much more…

Questions?


HEY!
OH, HI!
Normal human Conversation
1. Ask them about themselves
HOW MANY... APPLES... HAVE YOU EATEN?
YES.
...LIKE, IN MY LIFE?
...
...I SHOULD GO.
OK.



Let's get started!

Today's goals

- S1- Explain and compare the structure and properties of standard algorithms and data structures linked to Computer Graphics.
- [S6- Apply the knowledge obtained in this course to problems of other fields]

- Reading material:
  - Introduction (Chapter 1)
  - Color (Pages 493-495)
  - Basic Ray tracing (Pages 69 – 71 up to Section 4.2)

What is Computer Graphics about?

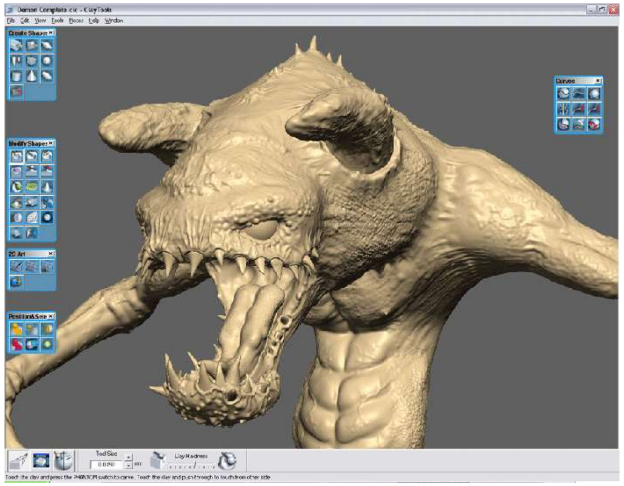
- Modeling - Making content
- Animation - Making movement
- Rendering - Making images

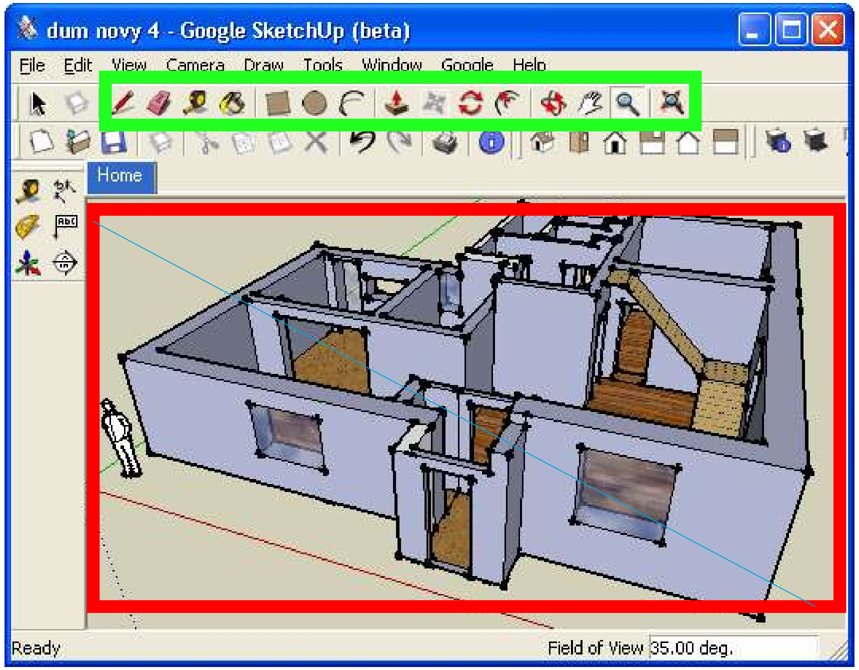
Modeling

- Create 3D Objects

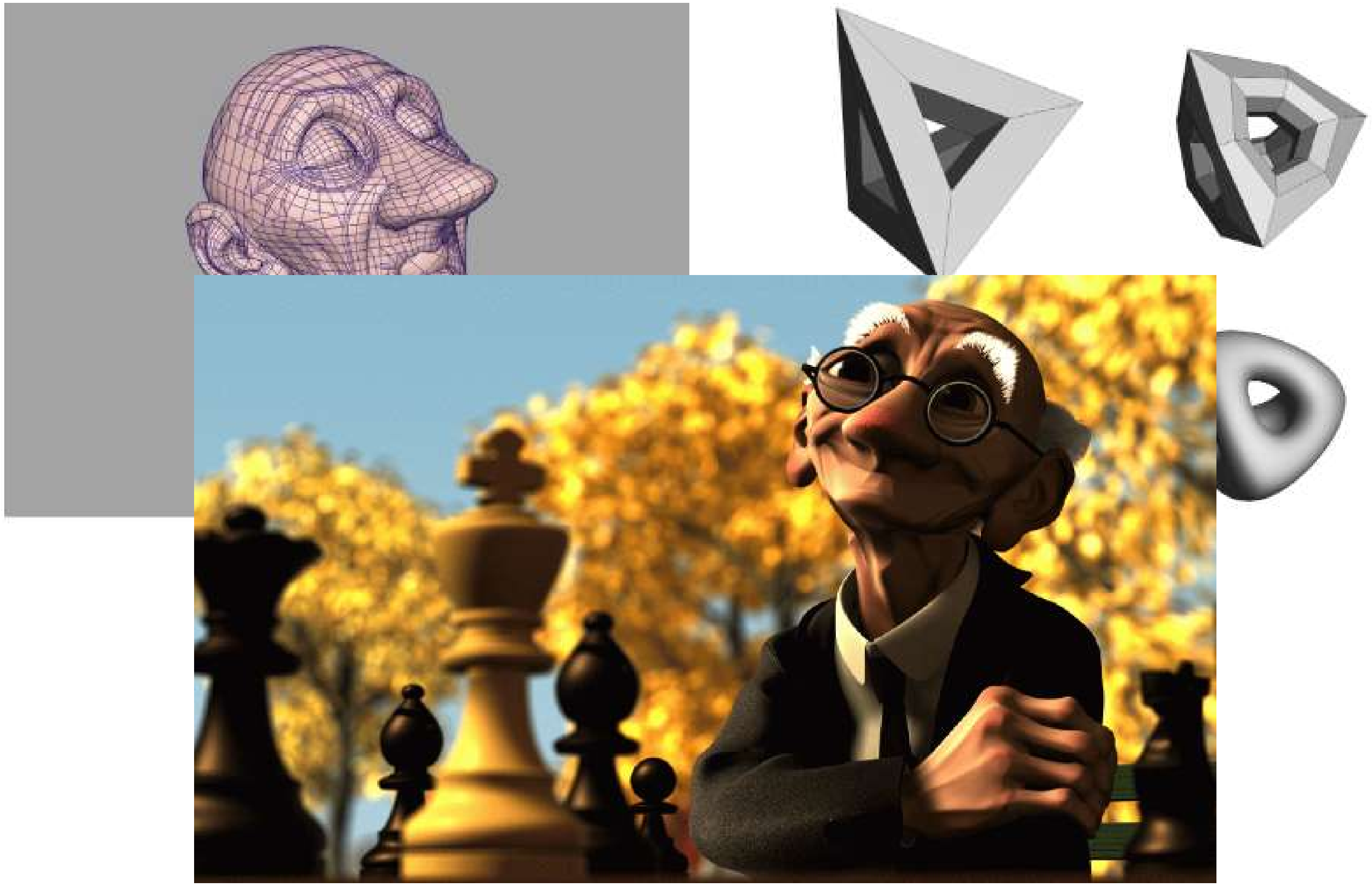
  e.g.,

  - Geometry
  - Analysis
  - Representations





Subdivision Surfaces



What is Computer Graphics about?

- Modeling - Making content
- Animation - Making movement
- Rendering - Making images

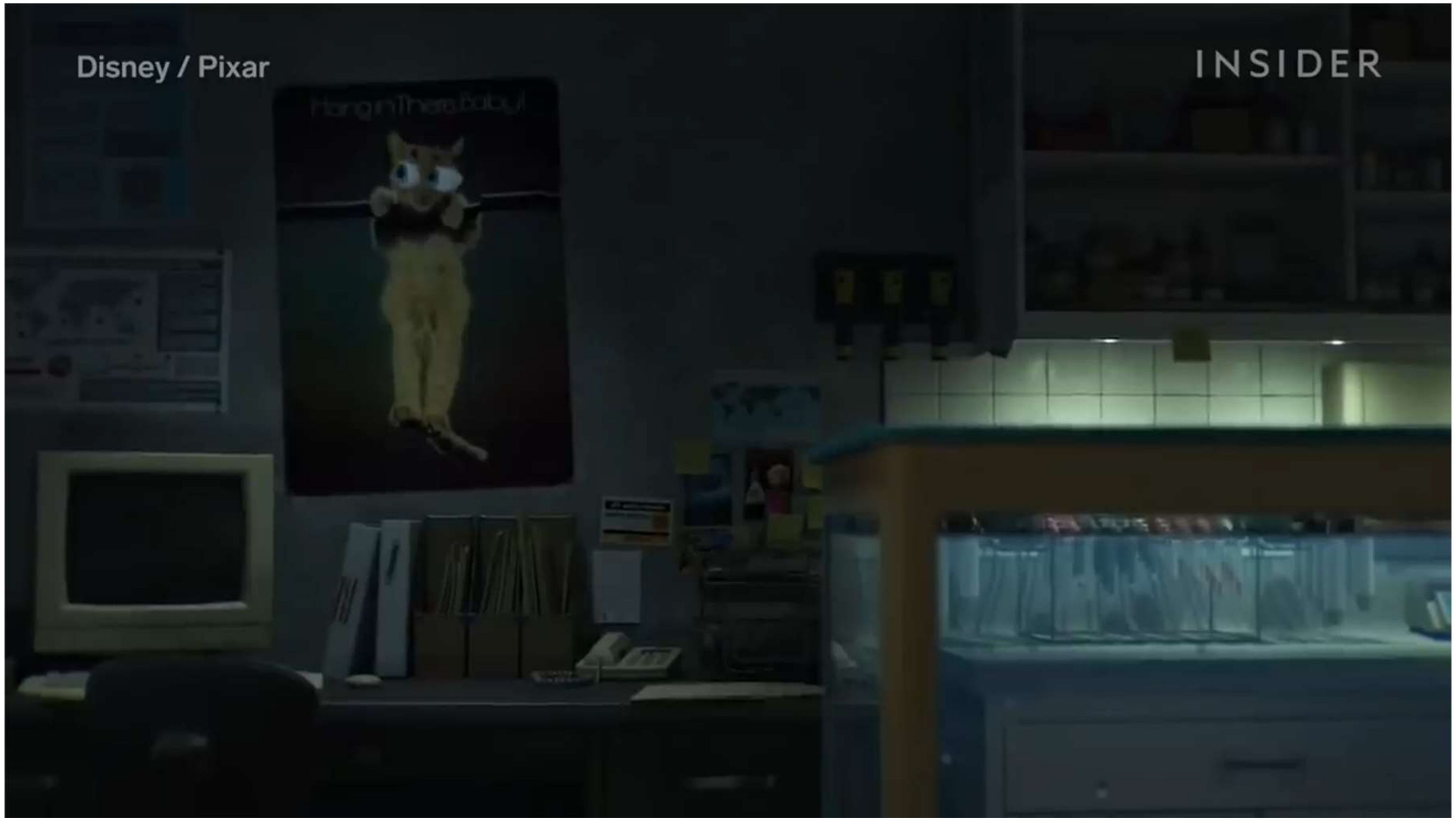
Animation

- Synthesize Movement

  e.g.,

  - Data analysis (e.g., PCA)
  - Data Interpolation
  - Differential analysis
  - Physics

Extreme Example



Source: Insider
https://www.youtube.com/watch?v=qTPKGVrFtQU

What is Computer Graphics about?

- Modeling - Making content
- Animation - Making movement
- Rendering - Making images

Rendering

- Making images

  e.g.,

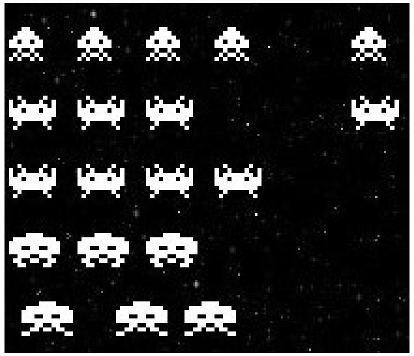
  - Physics
  - Math
  - Perception
  - User interfaces
  - Electrical Engineering
  - ...

Rendering

Source: Insider
https://www.youtube.com/watch?v=qTPKGVrFtQU

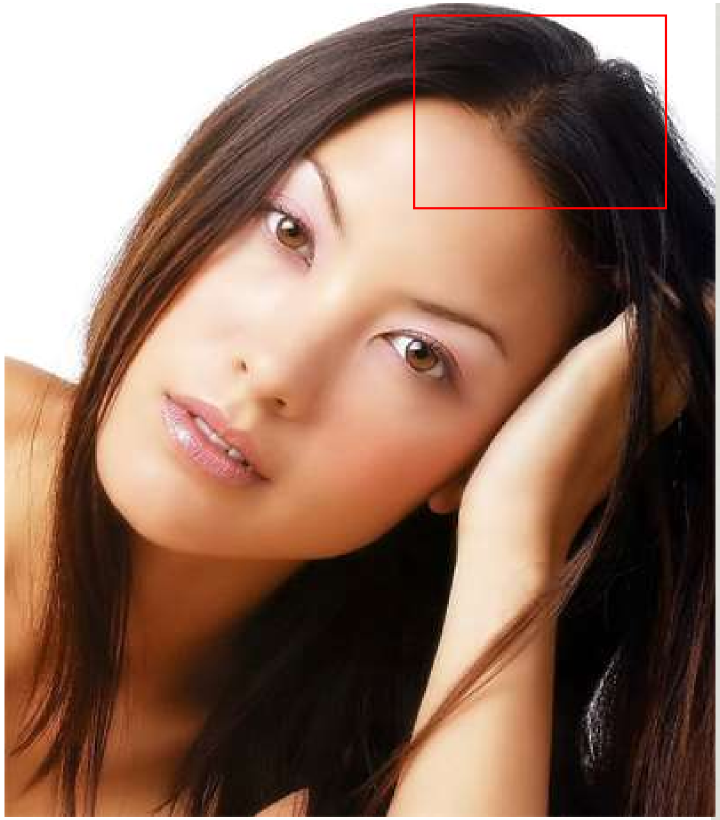
Introduction

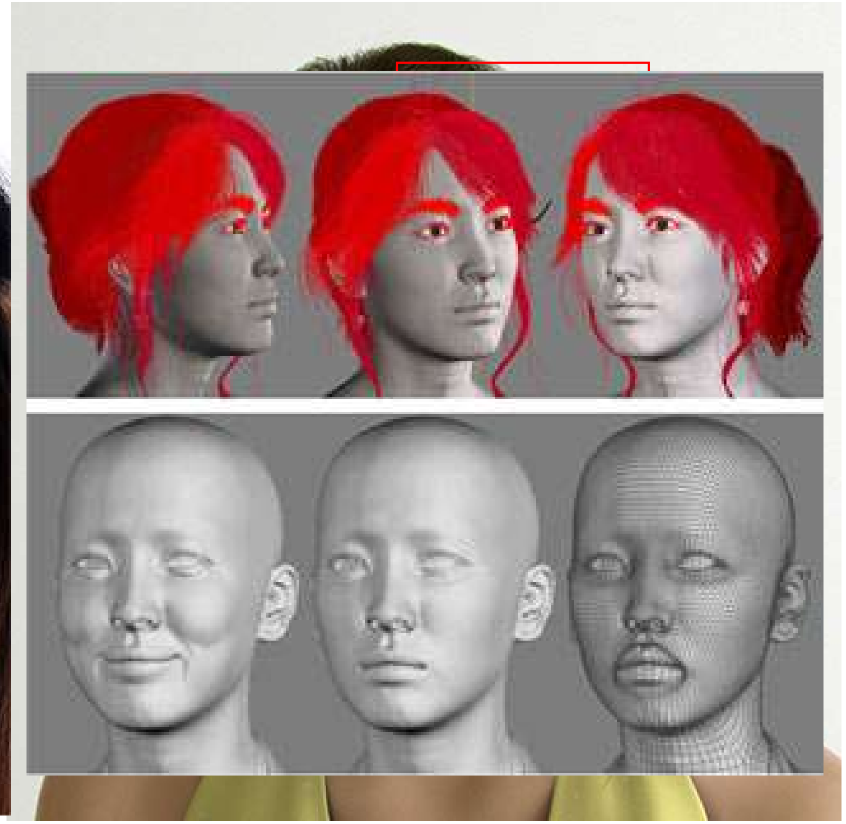
- Graphics advances at an incredible pace



1978 – Space Invaders | 1990 - Loom | 2007 - Unreal | 2023 - Starfield

Photo or Computation?





Can we do better?

- Yes, we can!

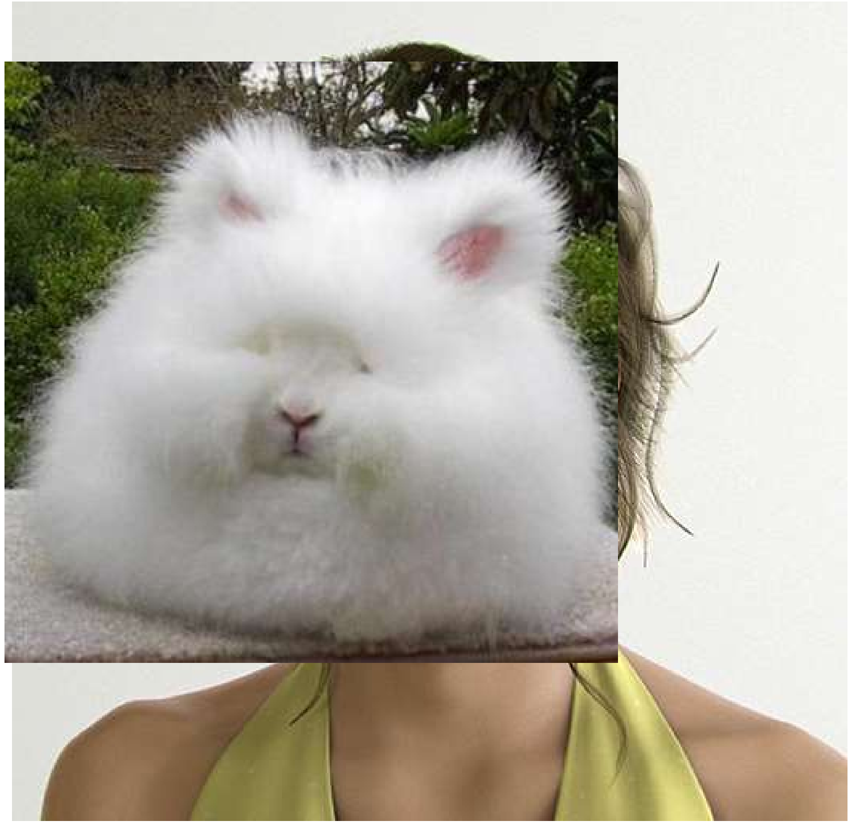
It should not...

...take months of work

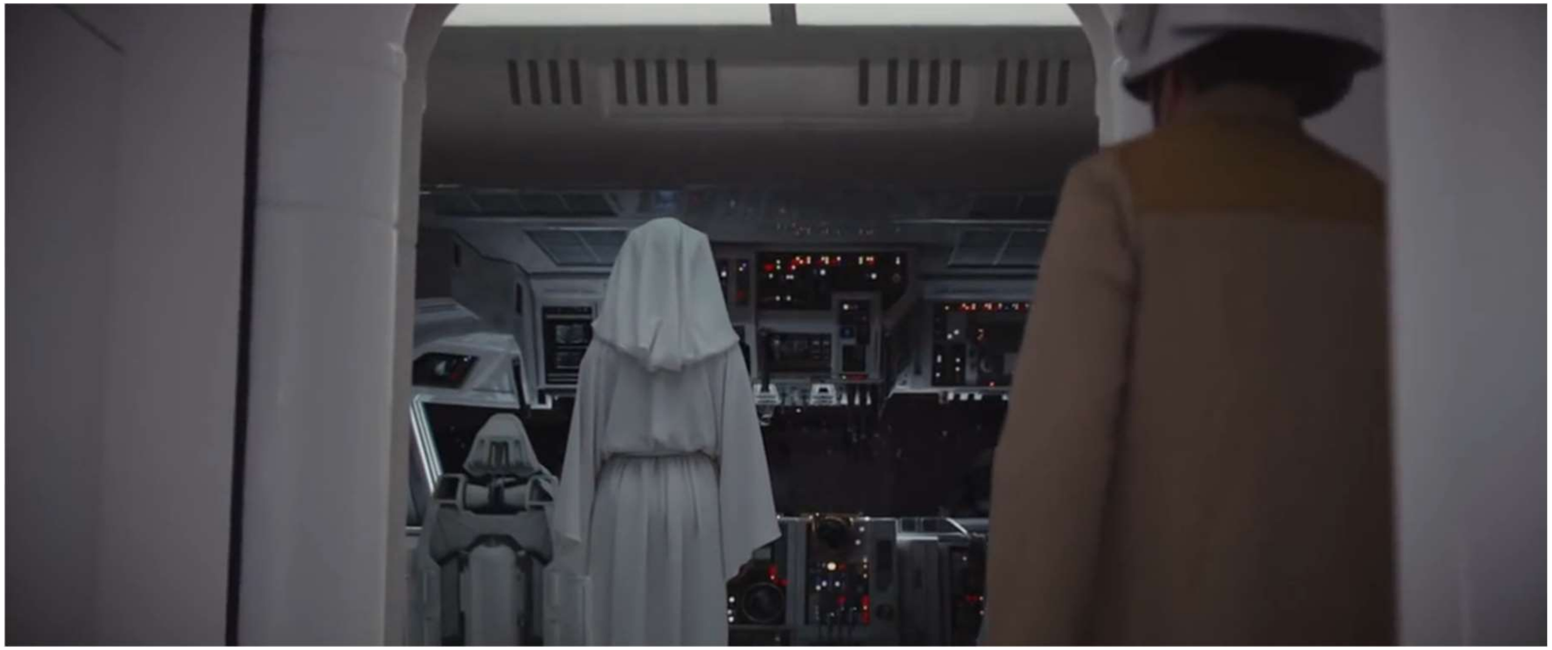
...take hours of computation

...only result in

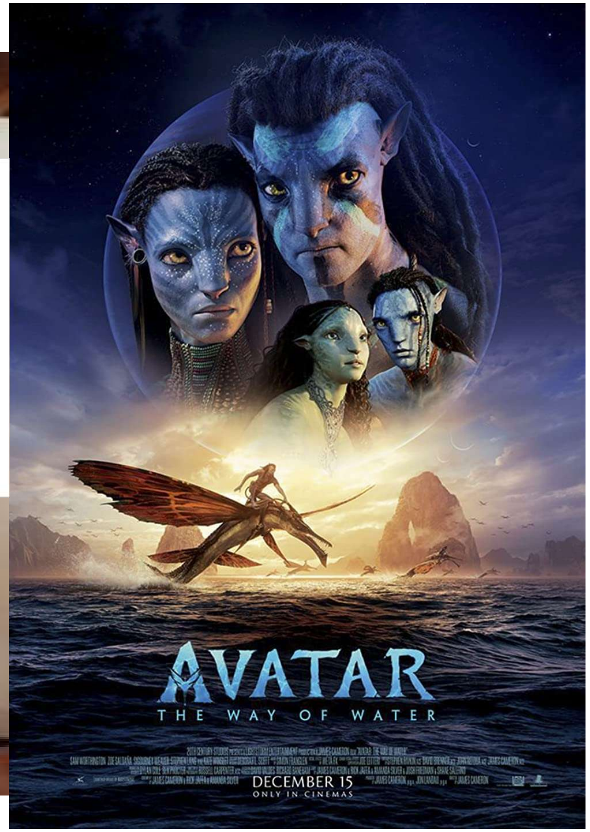
one view, pose, and light !



Intermezzo



Extreme Computation Times

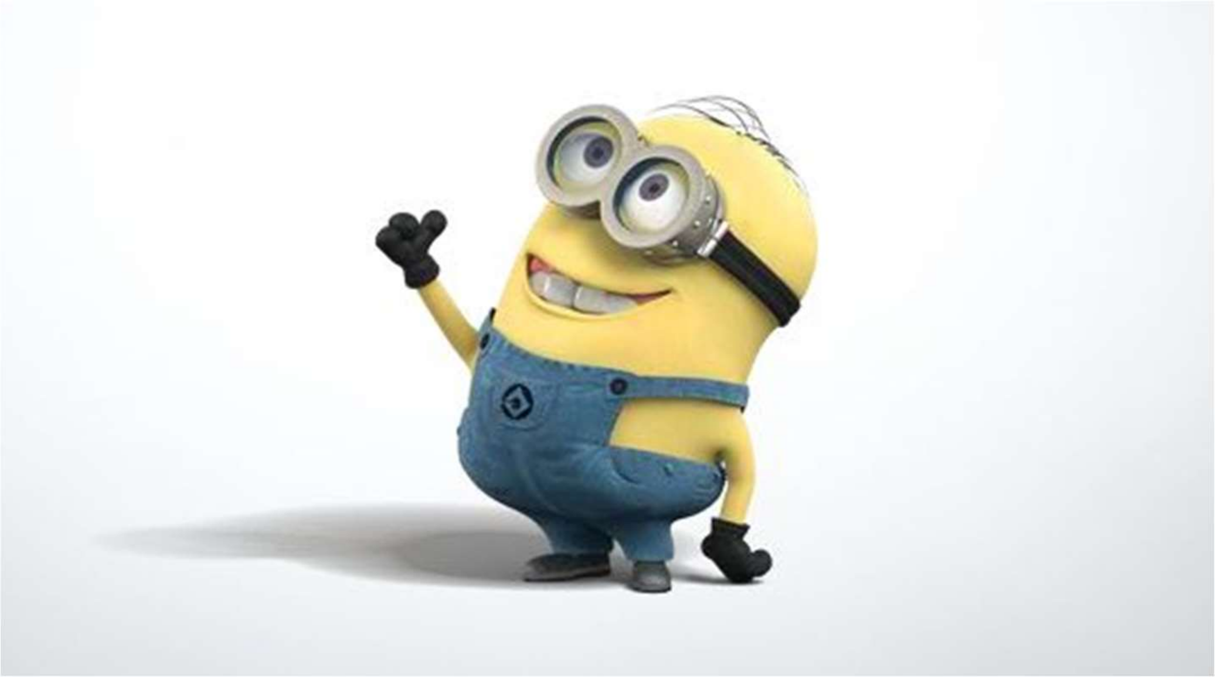


- Big Hero Six – copyright Disney

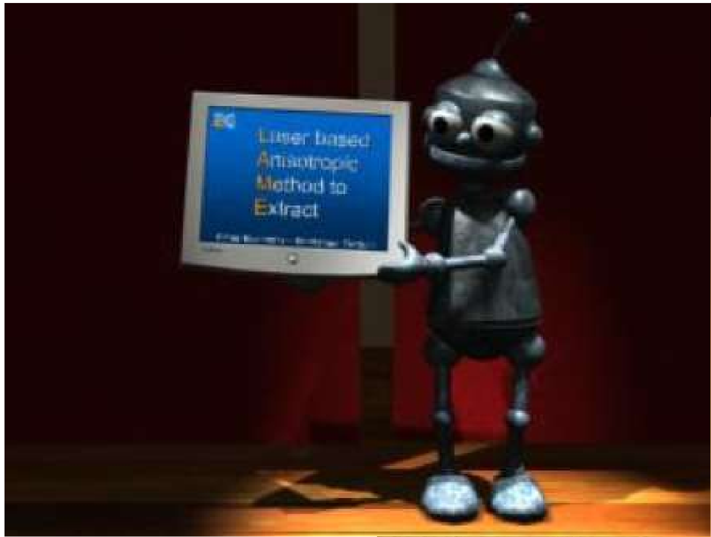
Financial Facts - Rendering

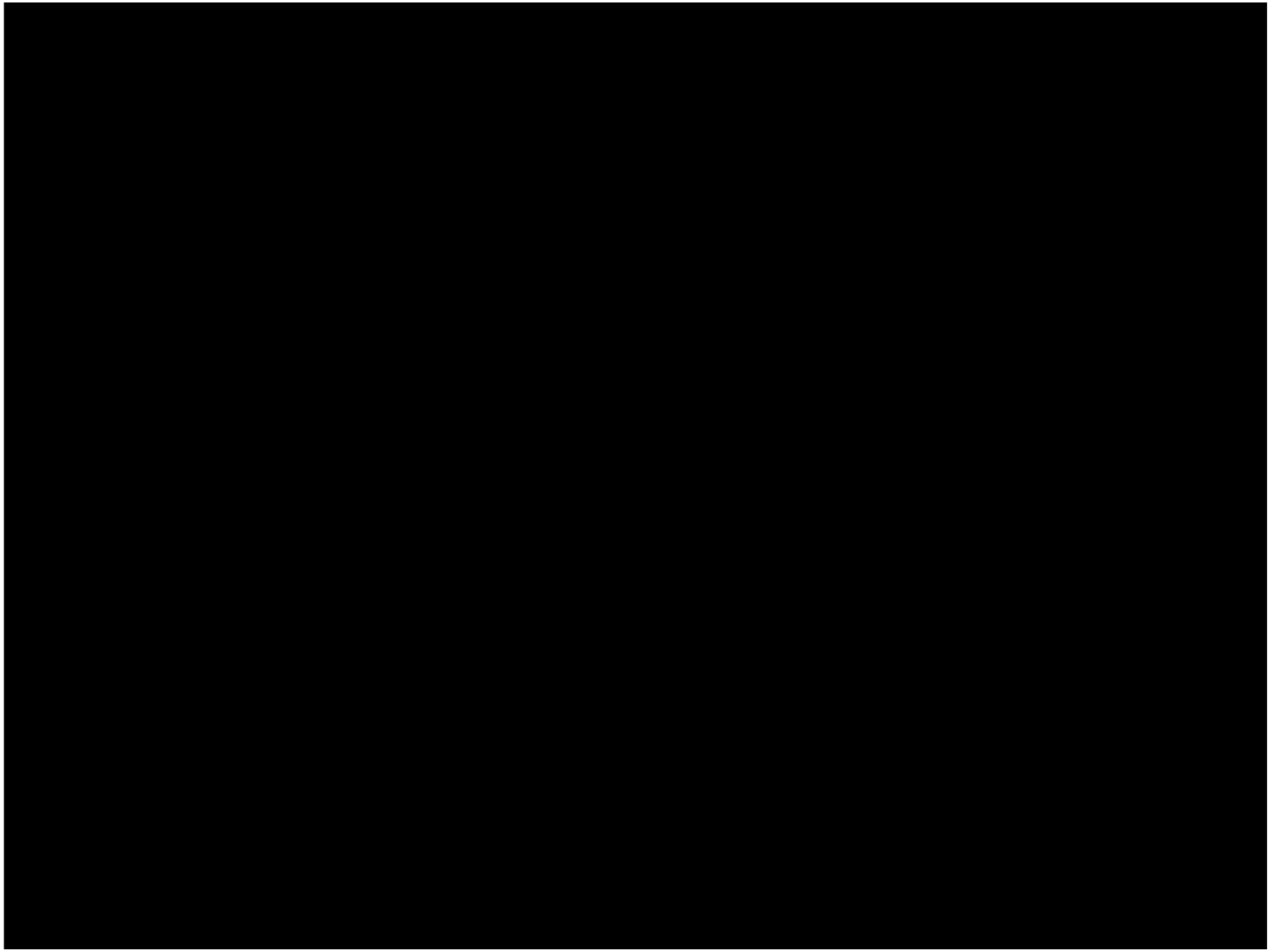
- Despicable me:

500.000 € for electricity



What do we get for it ?





During the practicals: blender

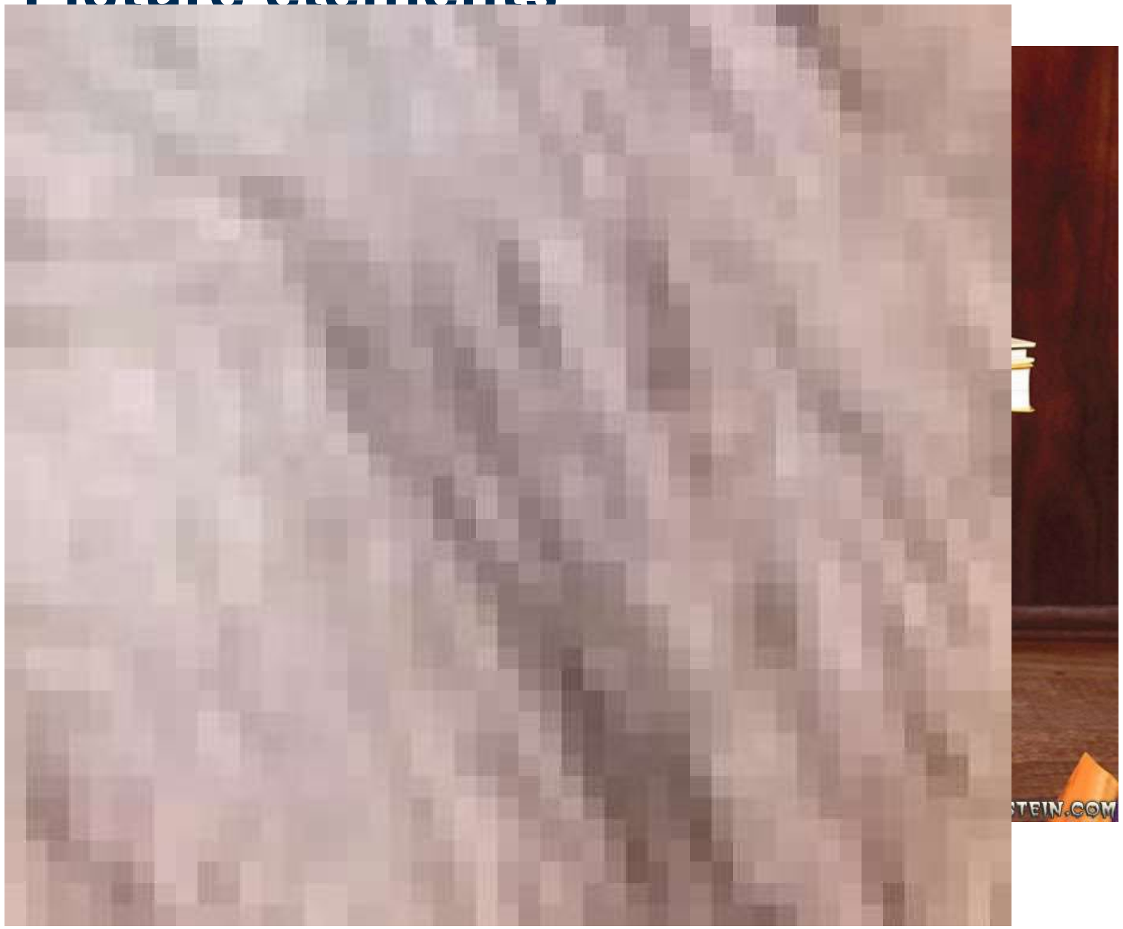
- Learn to use such tools yourselves!

How to produce an image?

- Computers can calculate…

Making images with a Computer

Pixels – Picture elements

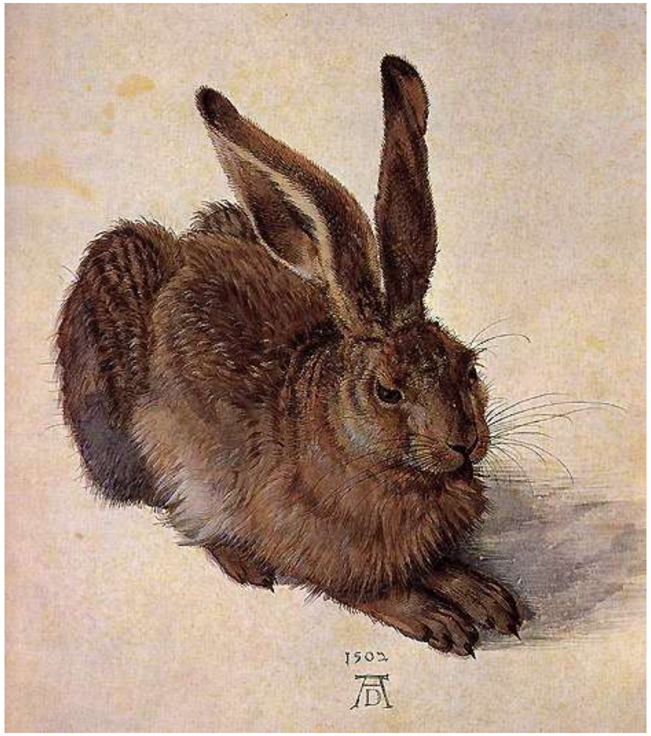


Today

- Give a glimpse on how images are computed
- Explain some of the most-basic principles
- Short outlook on things to come

Producing Images in the Real World

- Albrecht Dürer, 16th century



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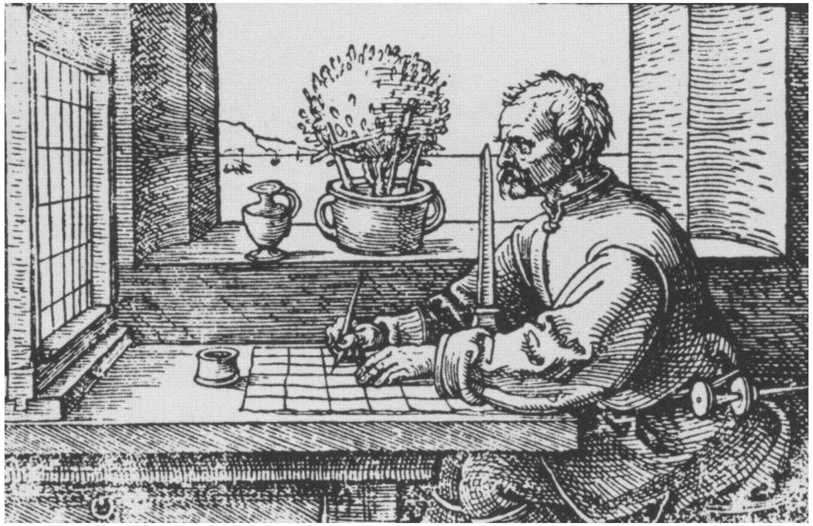
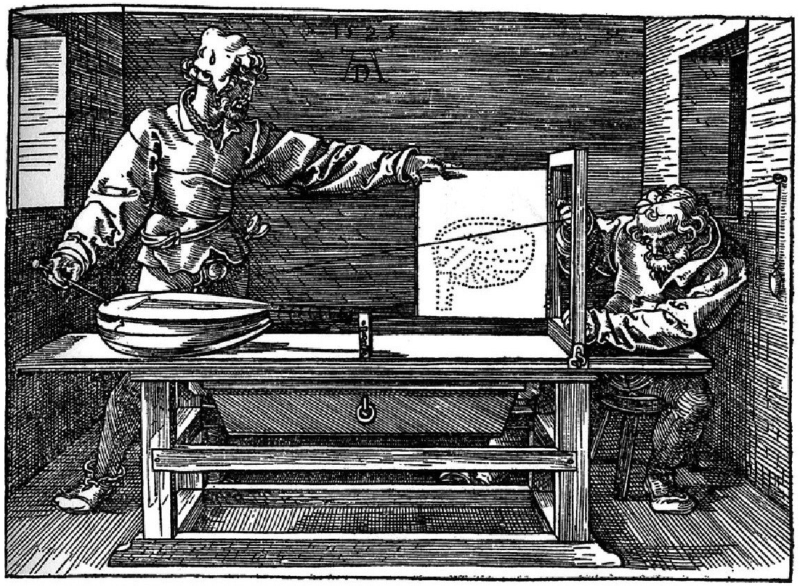
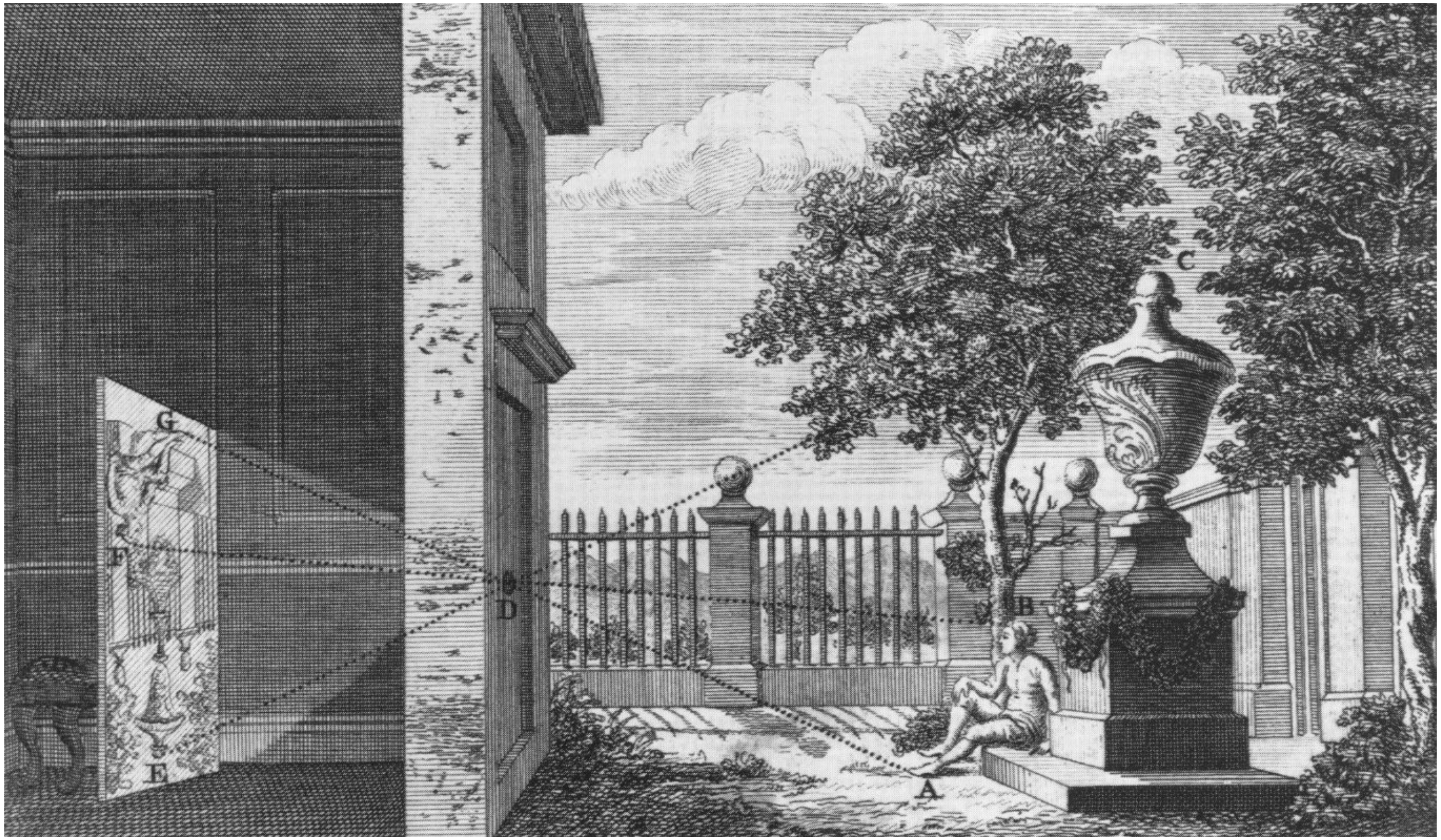




Producing Images in the Real World

Producing Images in the Real World



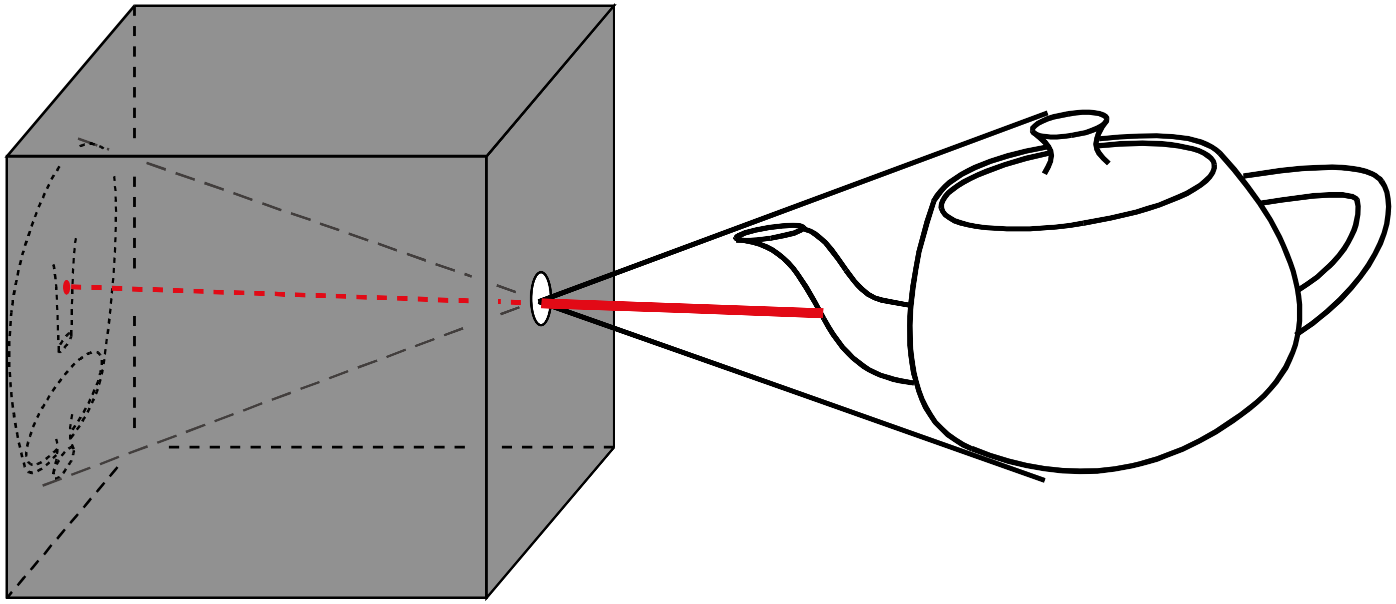
Camera obscura

Producing Images in the Real World

- A photo of such a camera [Abellardo]

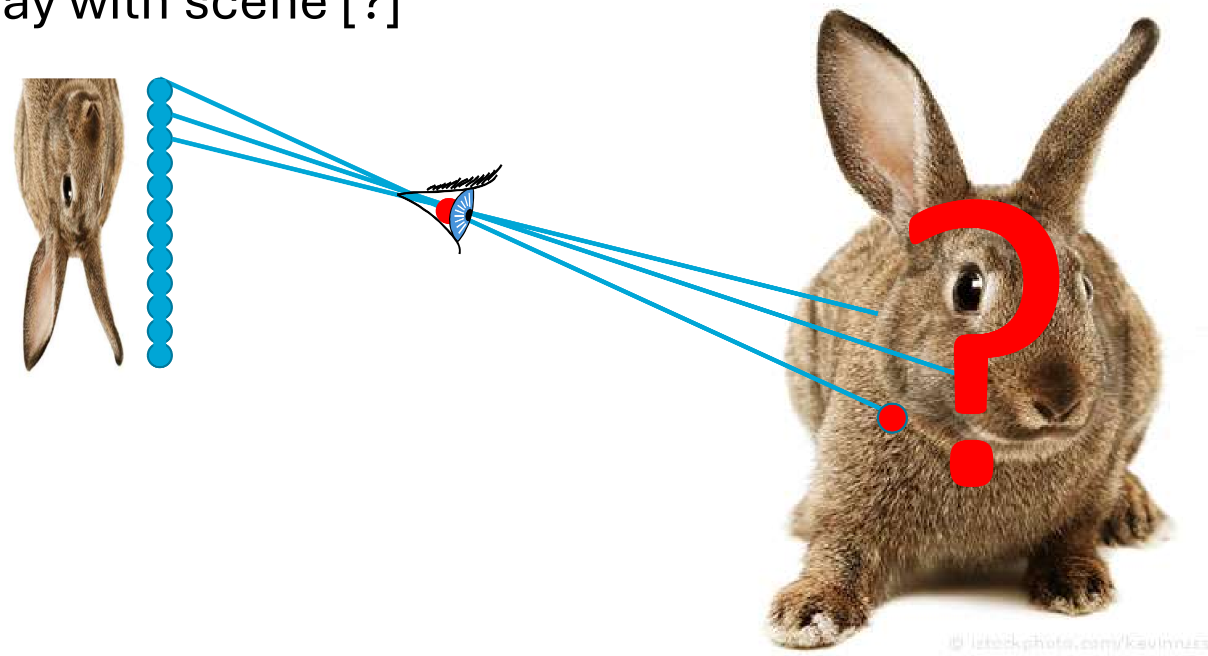
Pinhole camera

- Box with hole
- Perfect image for “point-sized” hole



Virtual Camera

- Take a pixel on the image in the virtual world
- Compute ray through pixel and camera center
- Intersect ray with scene [?]



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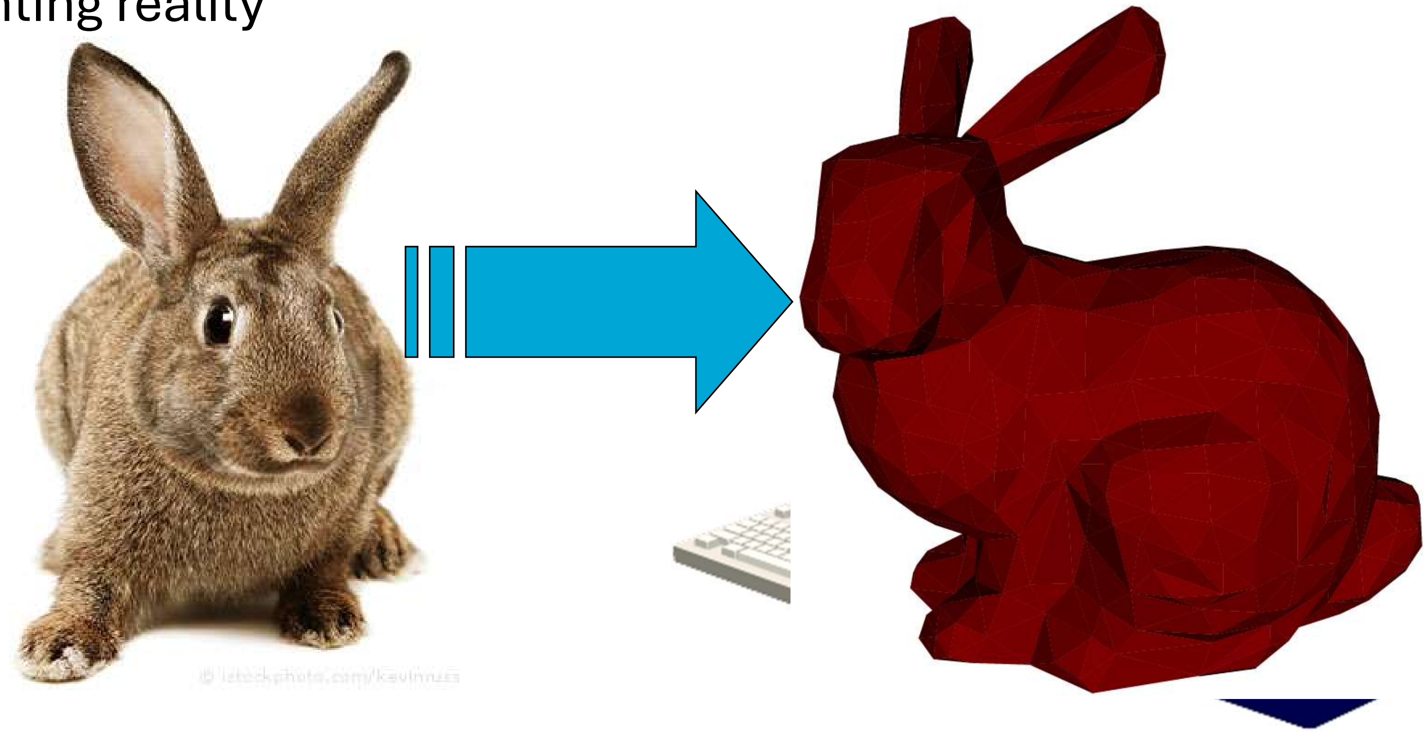
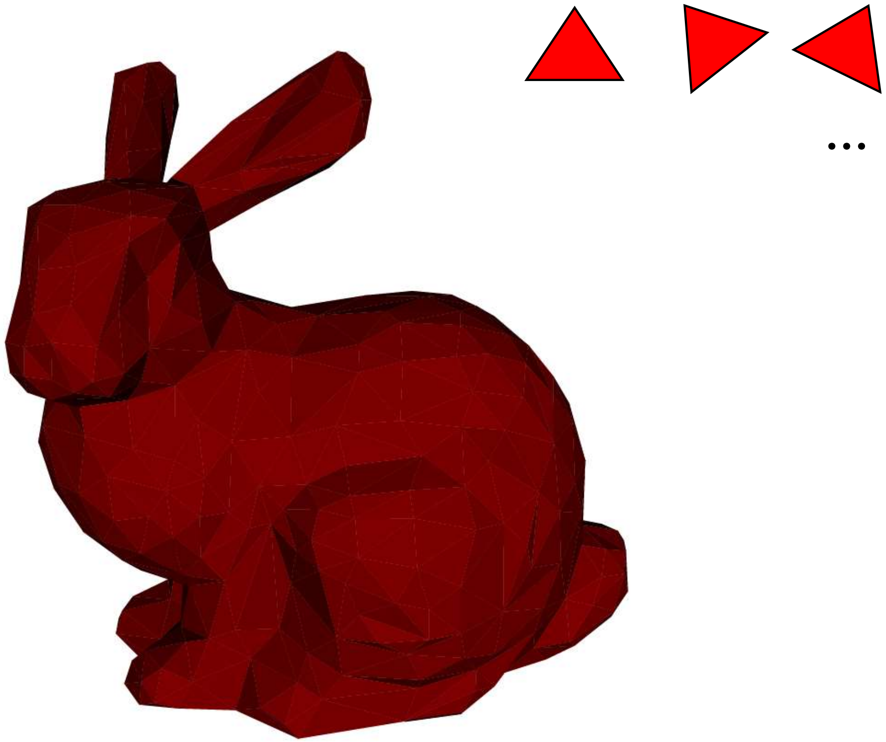
Models

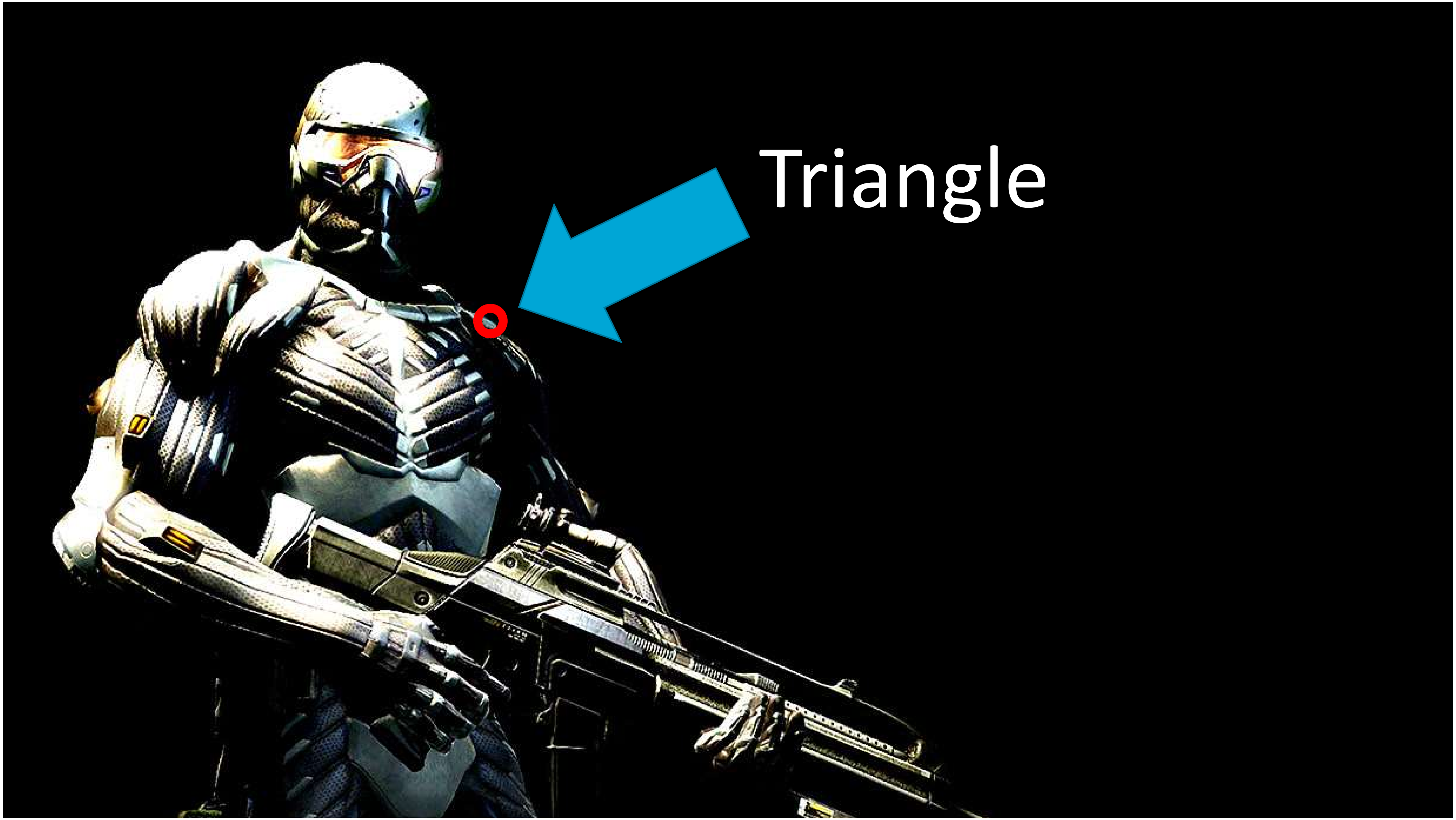
- Representing reality



Models

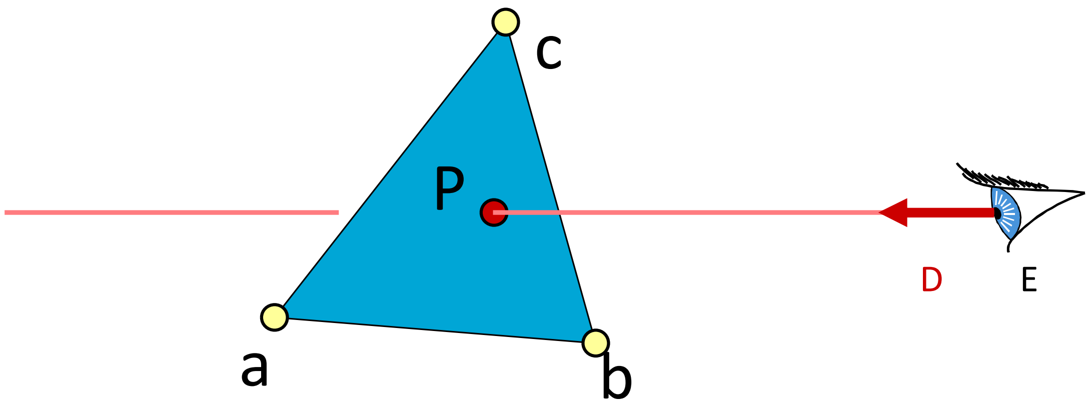
- Models are typically lists of triangles




Triangle

Solve Equations

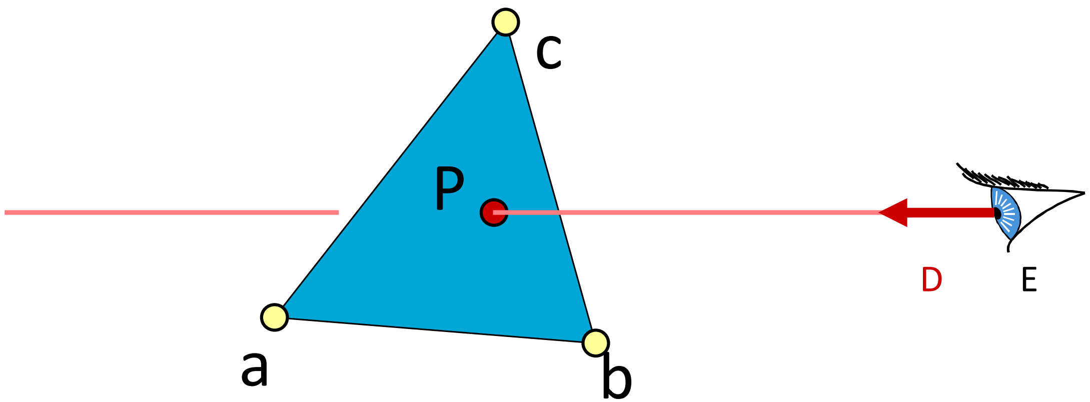
- E+tD= a+β (b-a)+γ (c-a)



Solve Equations

- $E_x+tD_x= a_x+\beta\ (b_x-a_x)+\gamma\ (c_x-a_x)$
- $E_y+tD_y= a_y+\beta\ (b_y-a_y)+\gamma\ (c_y-a_y)$
- $E_z+tD_z= a_z+\beta\ (b_z-a_z)+\gamma\ (c_z-a_z)$

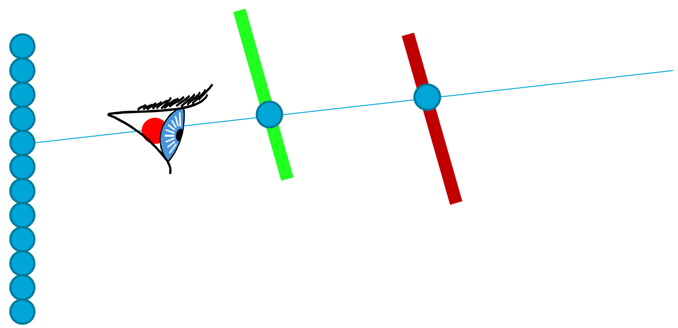
$$\begin{bmatrix} a_x - b_x & a_x - c_x & D_x \\ a_y - b_y & a_y - c_y & D_y \\ a_z - b_z & a_z - c_z & D_z \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} a_x - E_x \\ a_y - E_y \\ a_z - E_z \end{bmatrix}$$



Finally,
test if P is on the triangle or outside the triangle.
Try to do this at home!

Produce Final Image

- Keep the closest intersection point

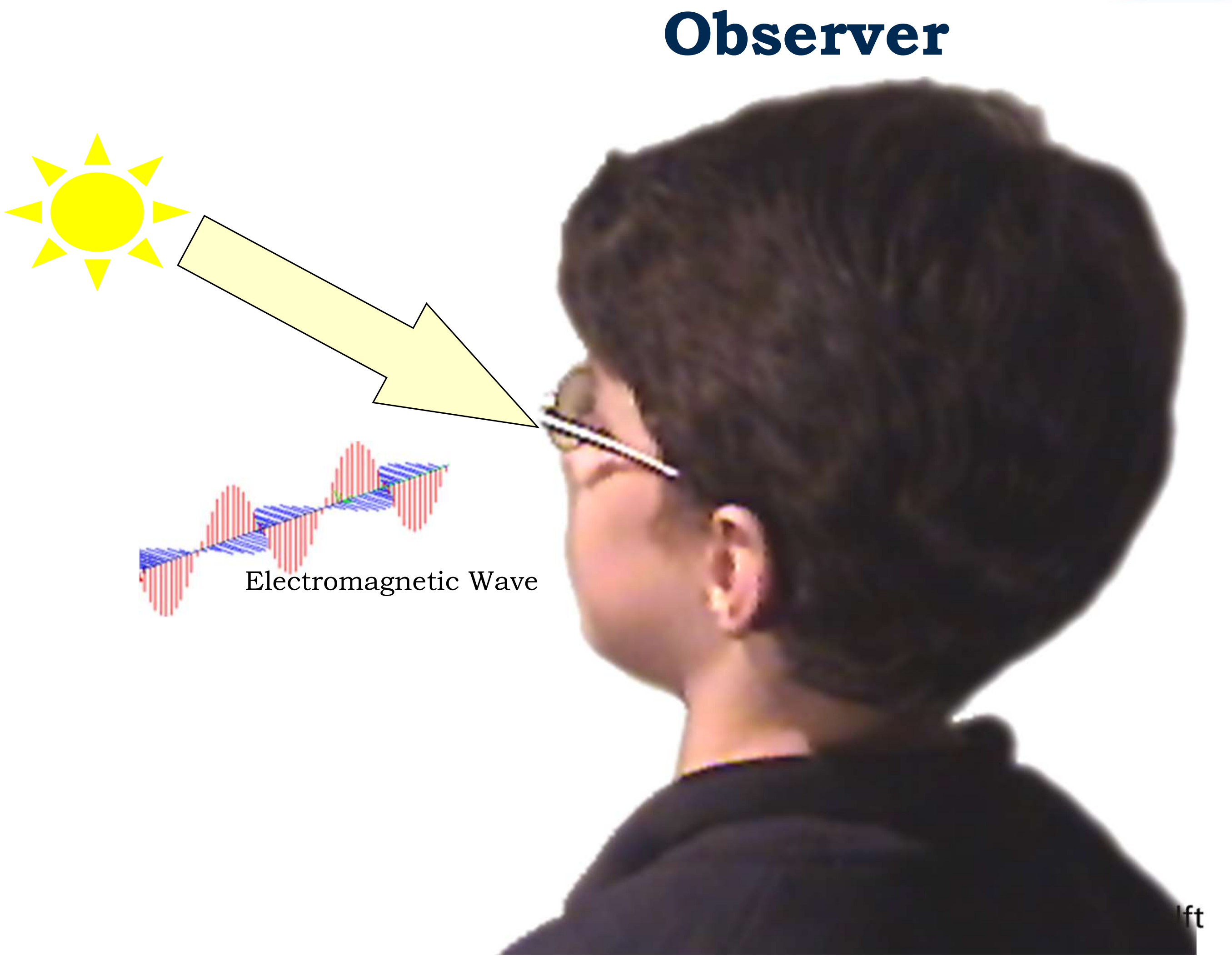


Ray Tracing - Recap

```
For each pixel
    Distance=MAX
    Color=0
    Ray=computeRay(pixel)
    For each triangle
      (CurrColor,CurrDistance)=computeIntersection(Ray)
      If (CurrDistance<Distance)
                 Distance=CurrDistance
                 Color=CurrColor
```

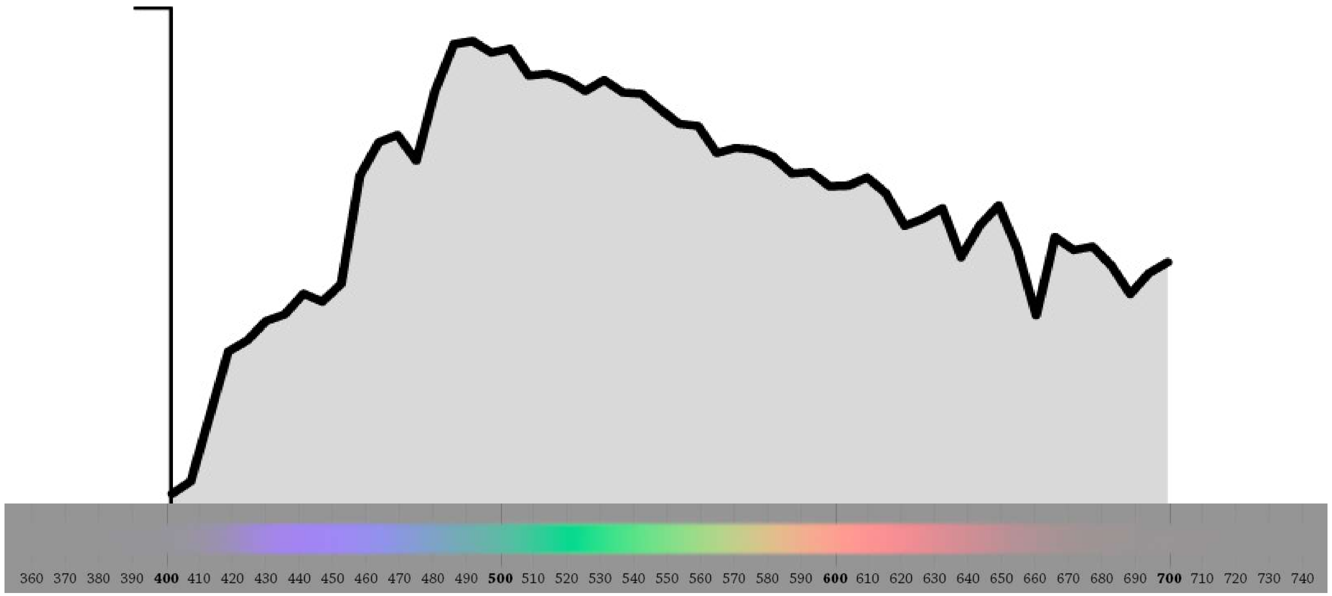
What do we see ?

- Light is registered
  by our eyes...
  and perceived as color



Physical Definition

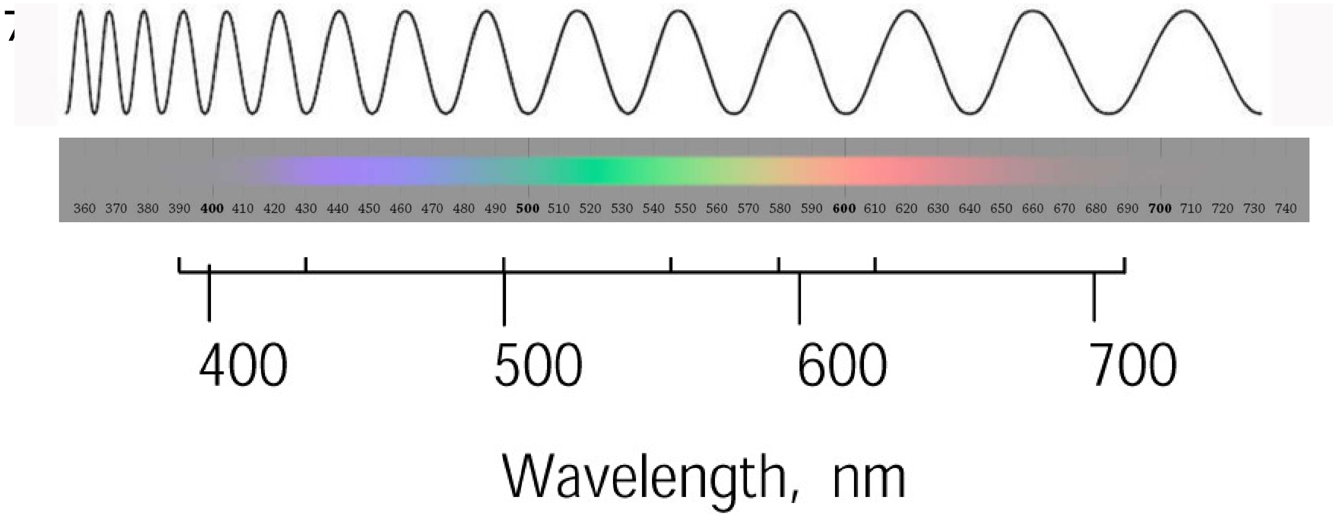
- Light = Distribution of power over a spectrum



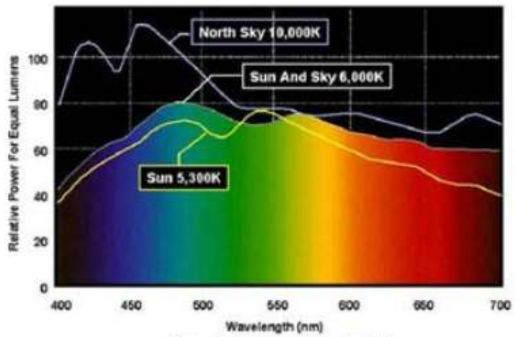
TU Delft

Light Spectrum

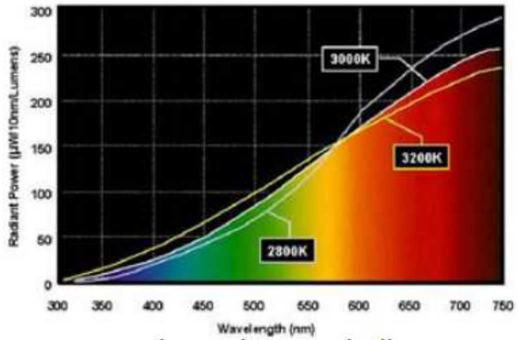
- Visible colors between
  380 nm (violette) and 720 nm (red)
- Outside visible range
  - Below 380 nm : ultra-violet
  - Above 7



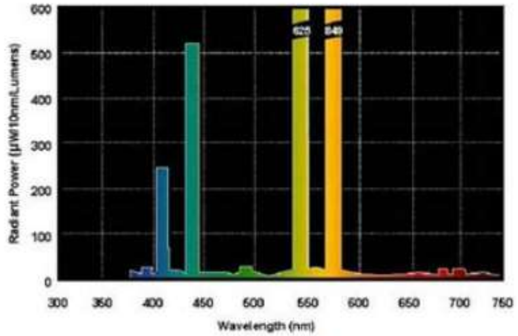
Examples of Spectra



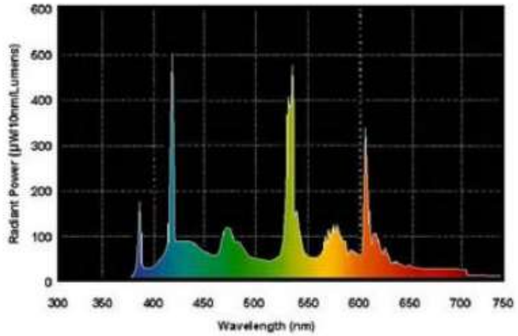
Outdoor daylight



Incandescent bulb

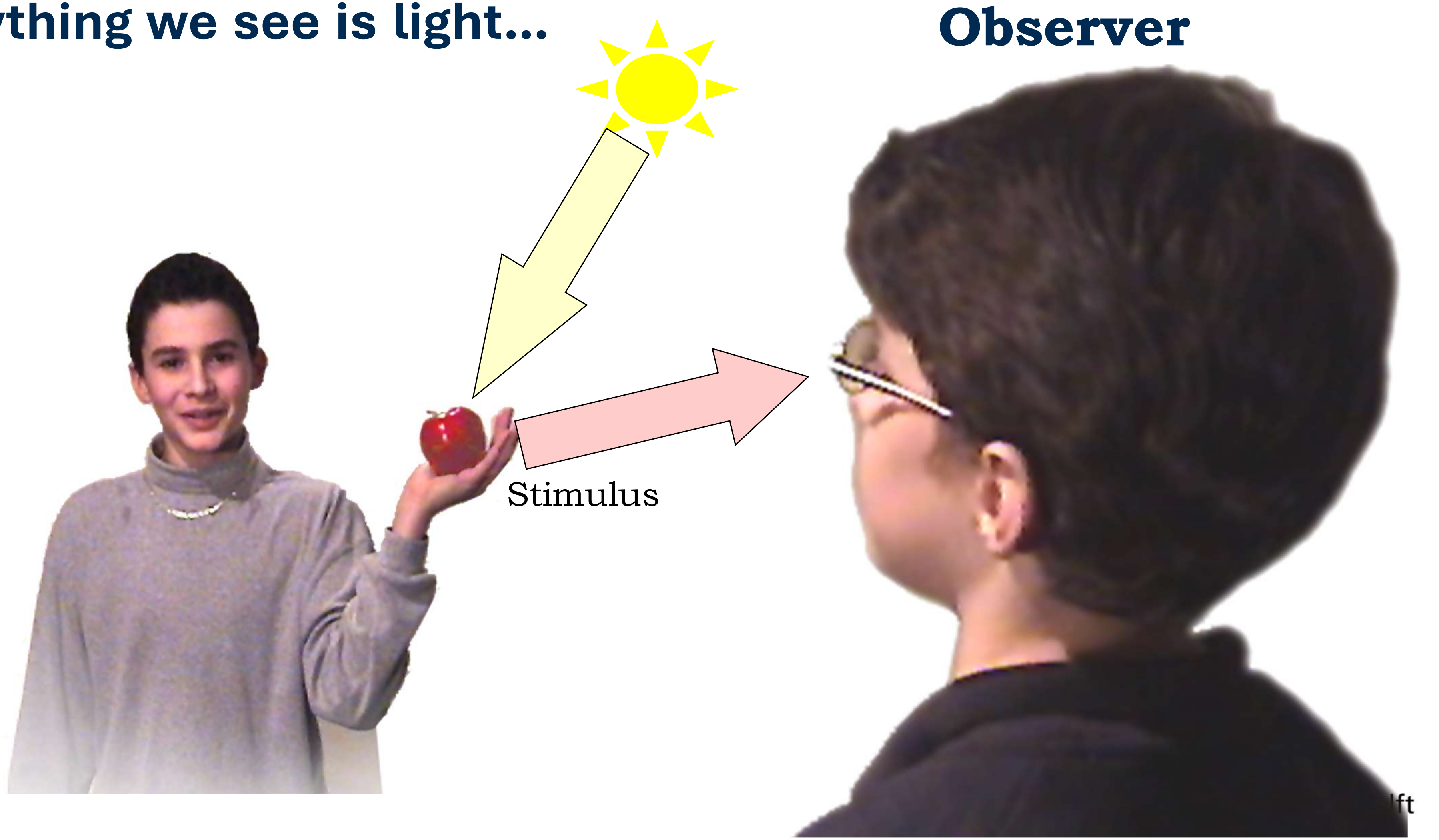


Mercury lamp

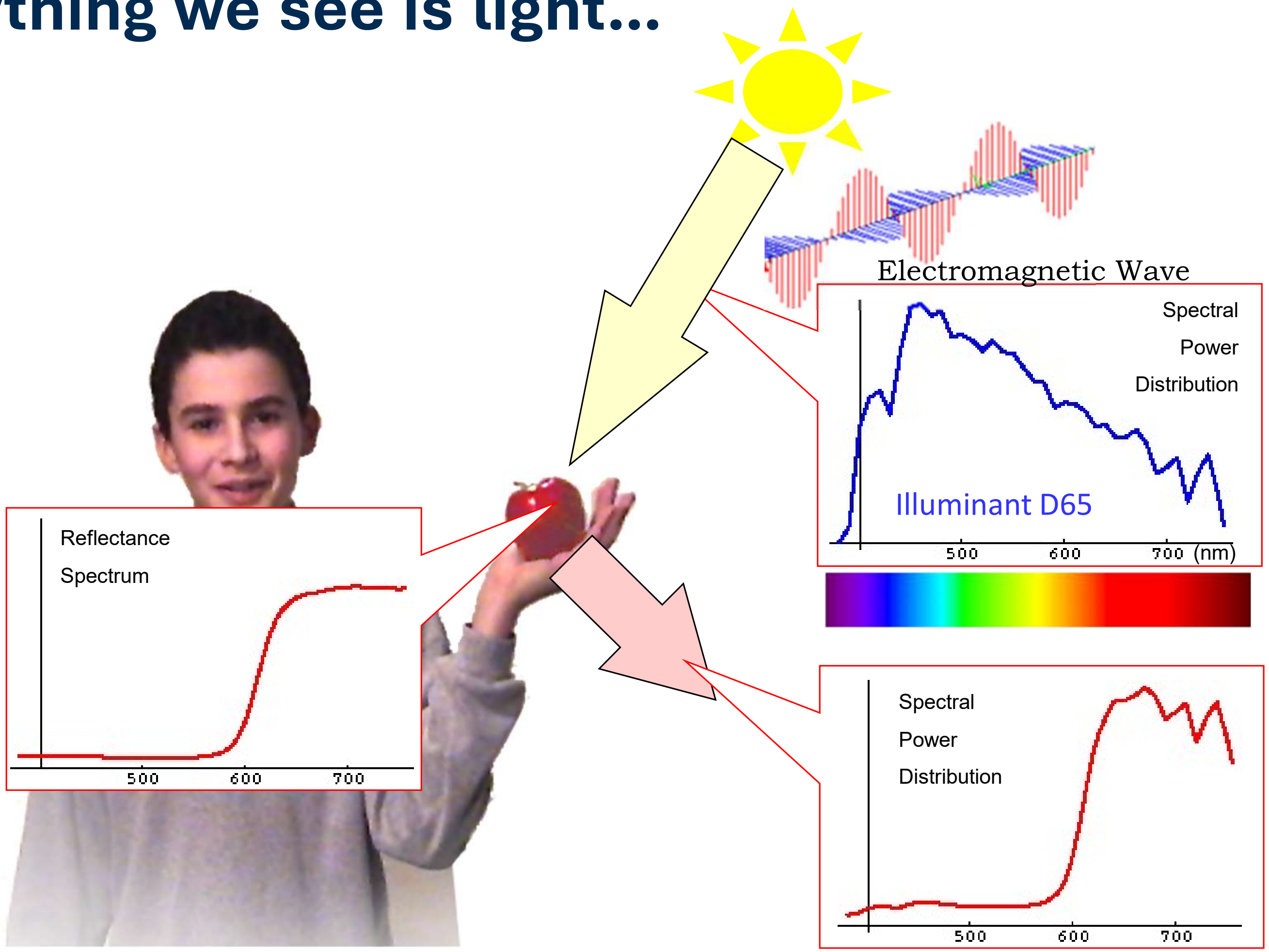


SP65 triphosphor fluorescent

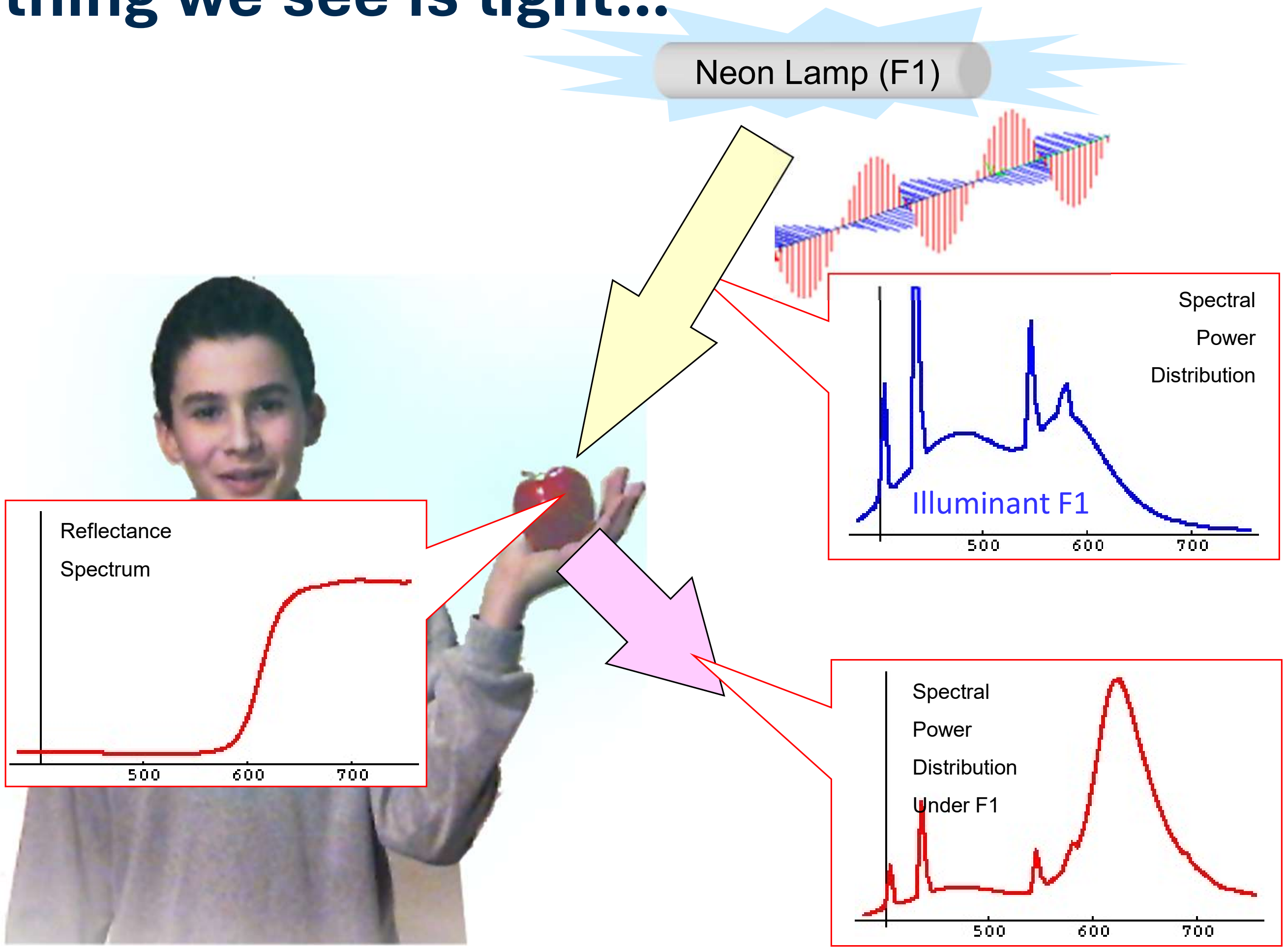
Everything we see is light...



Everything we see is light…

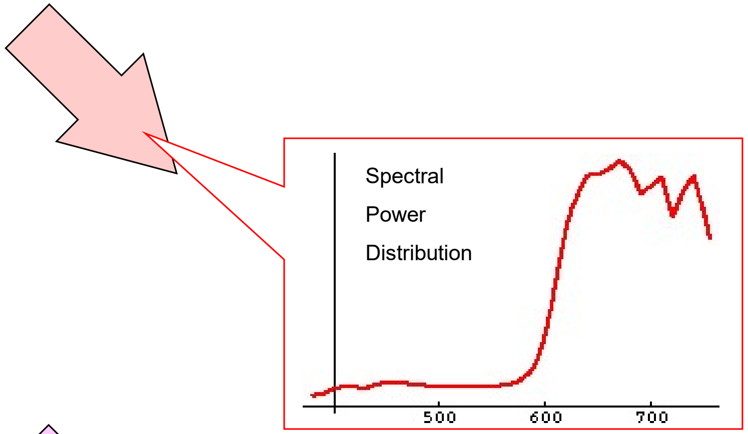


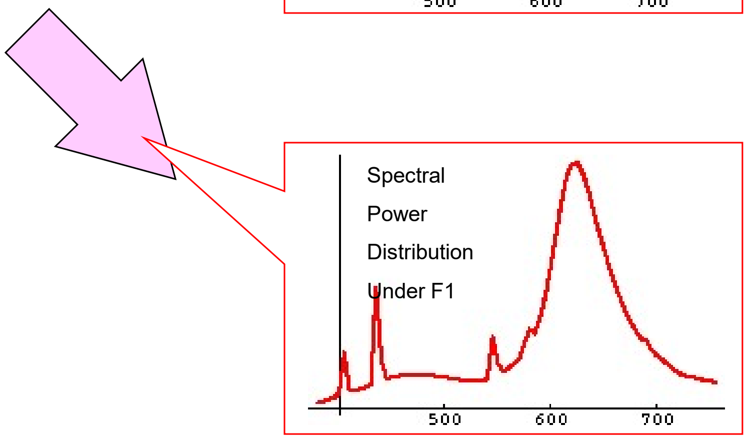
Everything we see is light…



Everything we see is light...

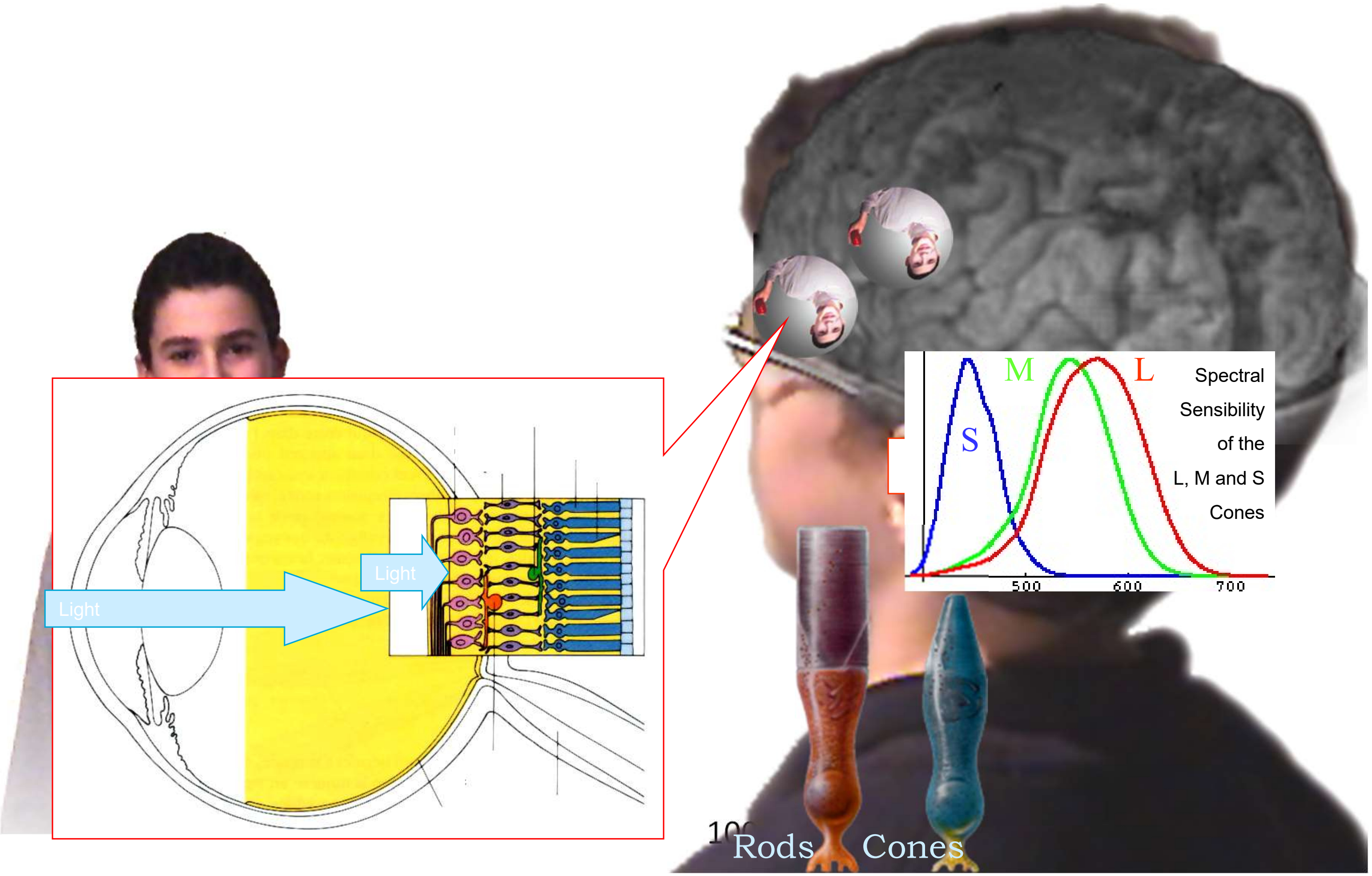
...but we cannot always distinguish it, as our eyes are not perfect.





Observer

Eye Biology



TU Delft

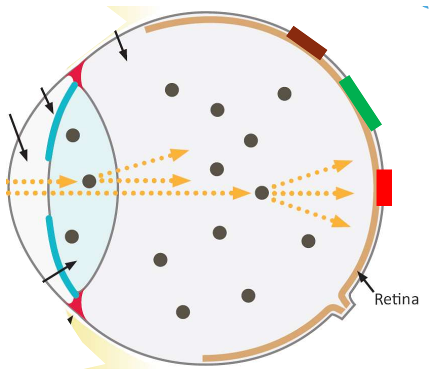
Eye Biology

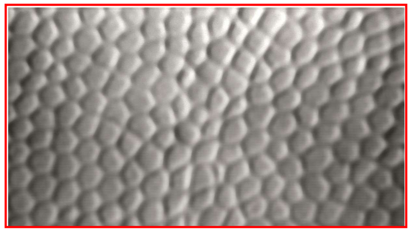
- Cones :
  - Chromatic perception (3 types-LMS)
  - Concentrated in center of retina
  - 6 to 7 million in retinal center
    - 3 times full HD

- Rods :
  - Achromatic perception
  - Low-light vision

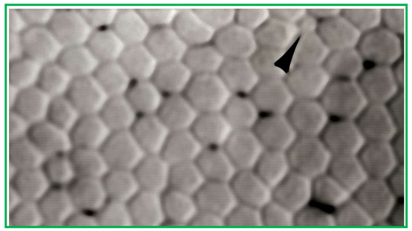
Eye Biology

- Retina consists of Cones & Rods

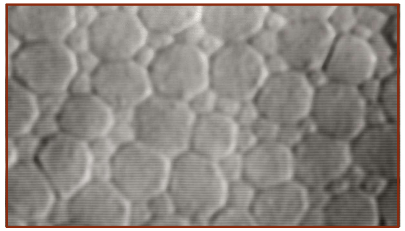




Center – fovea
only cones



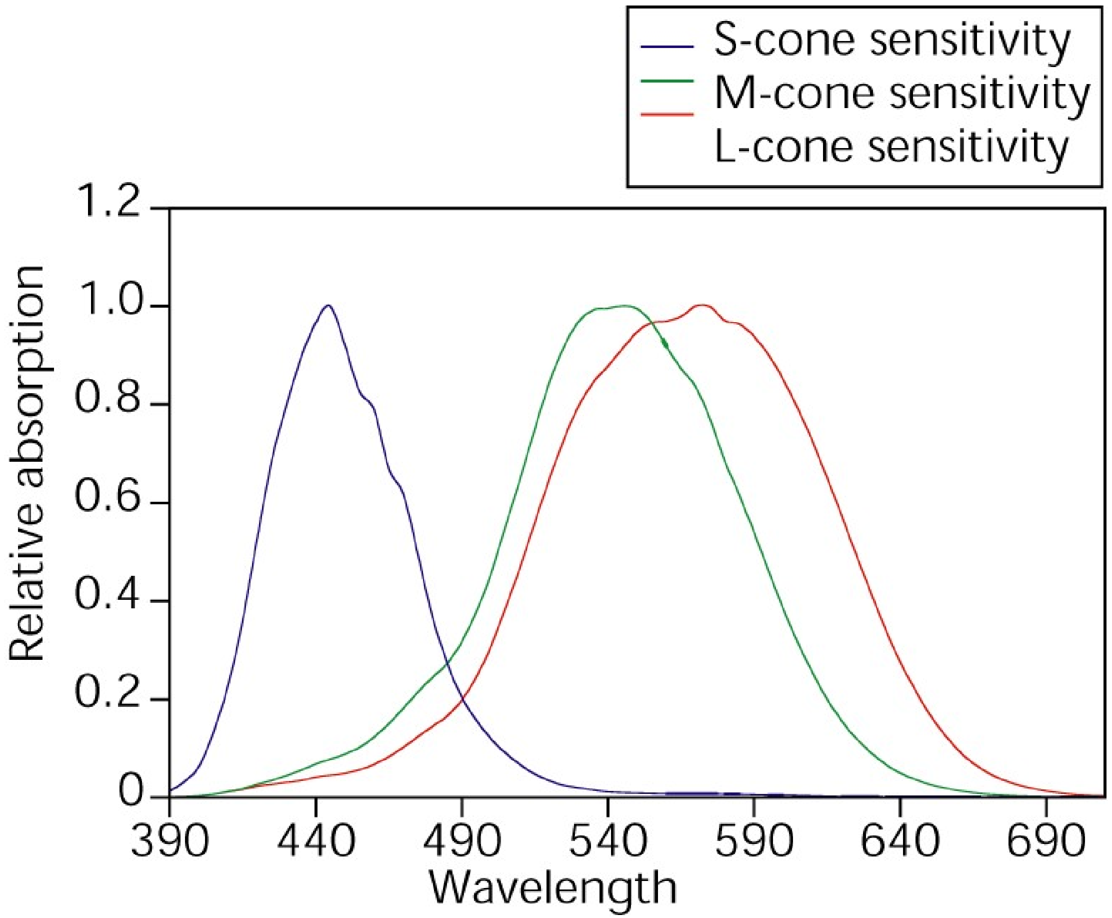
Boundary region
Mix of both



Periphery
Mix of both (more rods)

Curcio, C. A., Sloan, K. R., Kalina, R. E., Hendrickson, A. E., 1990. Human photoreceptor topography. J Comp Neurol 292, 497-523

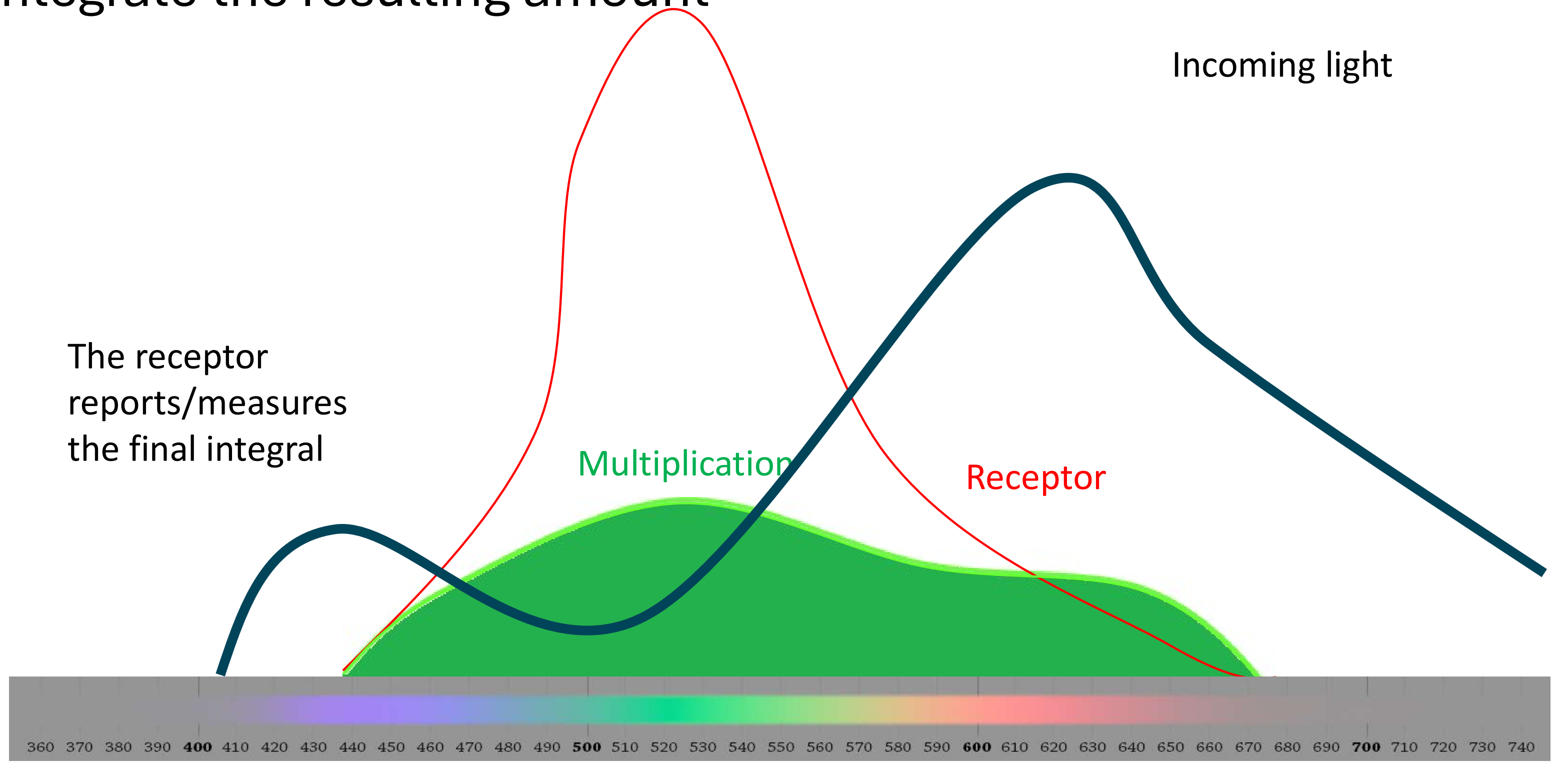
3 Cone types



S-cone sensitivity
M-cone sensitivity
L-cone sensitivity
Relative absorption
1.2
1.0
0.8
0.6
0.4
0.2
0
390
440
490
540
590
640
690
Wavelength

Simplified Receptor/Light Interaction

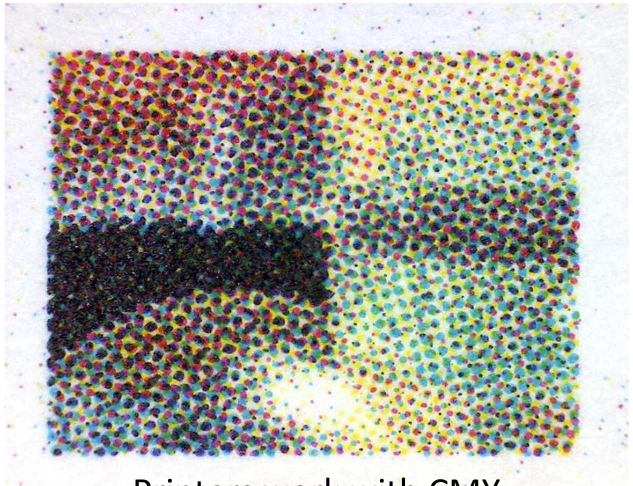
- Multiply incoming light and receptor response curve and integrate the resulting amount



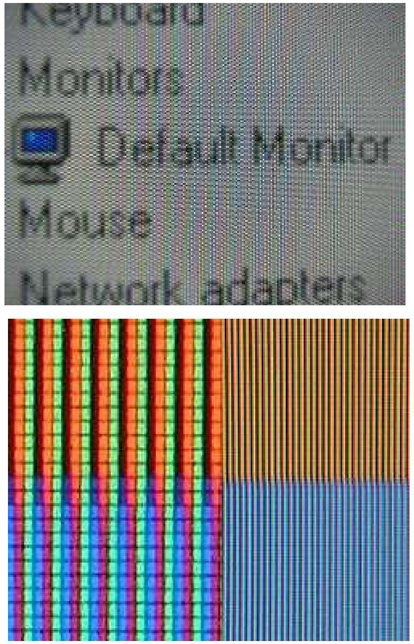
Receptor and Incoming Light

- How many dimensions?

3 !

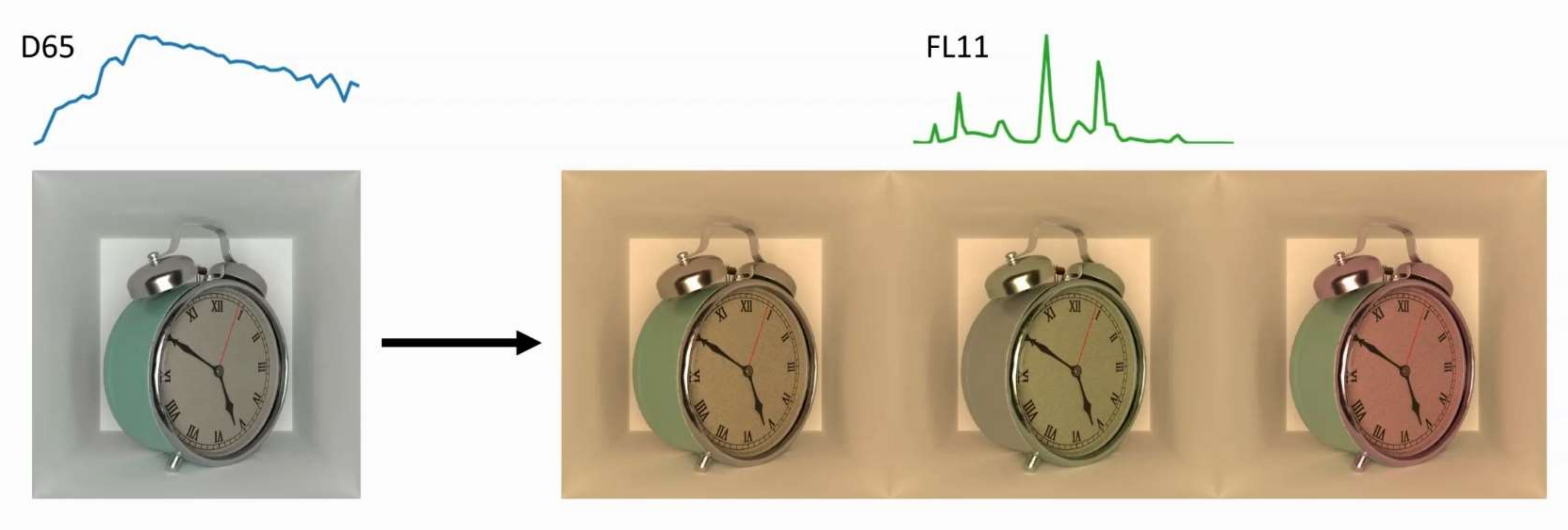


Printers work with CMY ink



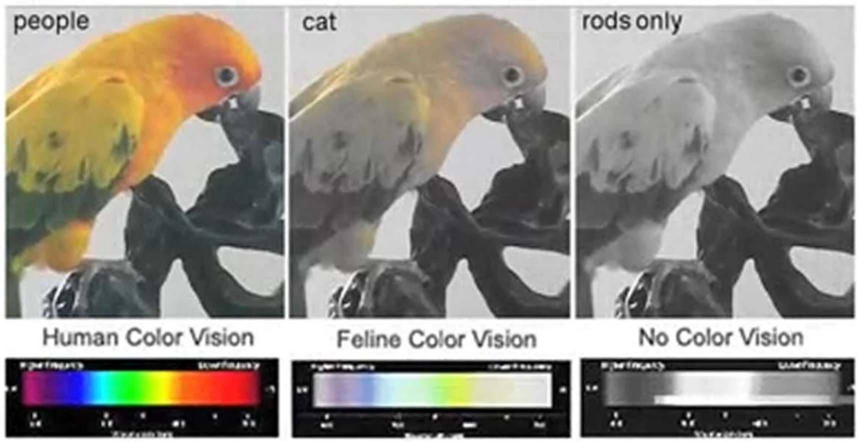
Screens work with RGB mask

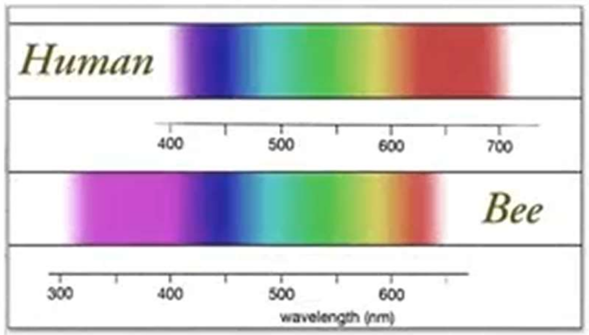
Metamerisms



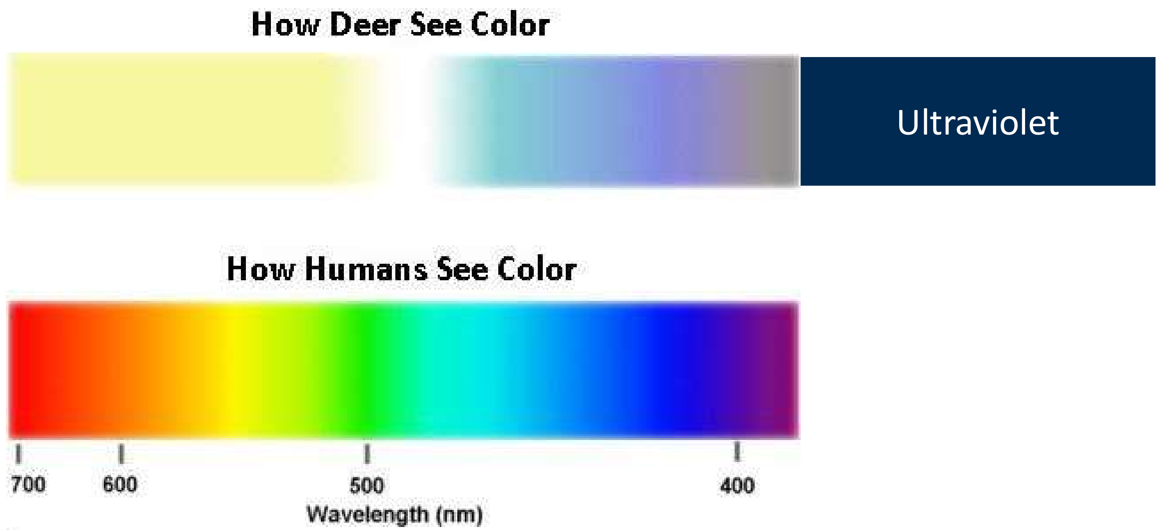
[Van de Ruit & Eisemann SIGGRAPH 23]

Different Species = Different View of the world





Different Species = Different View of the world



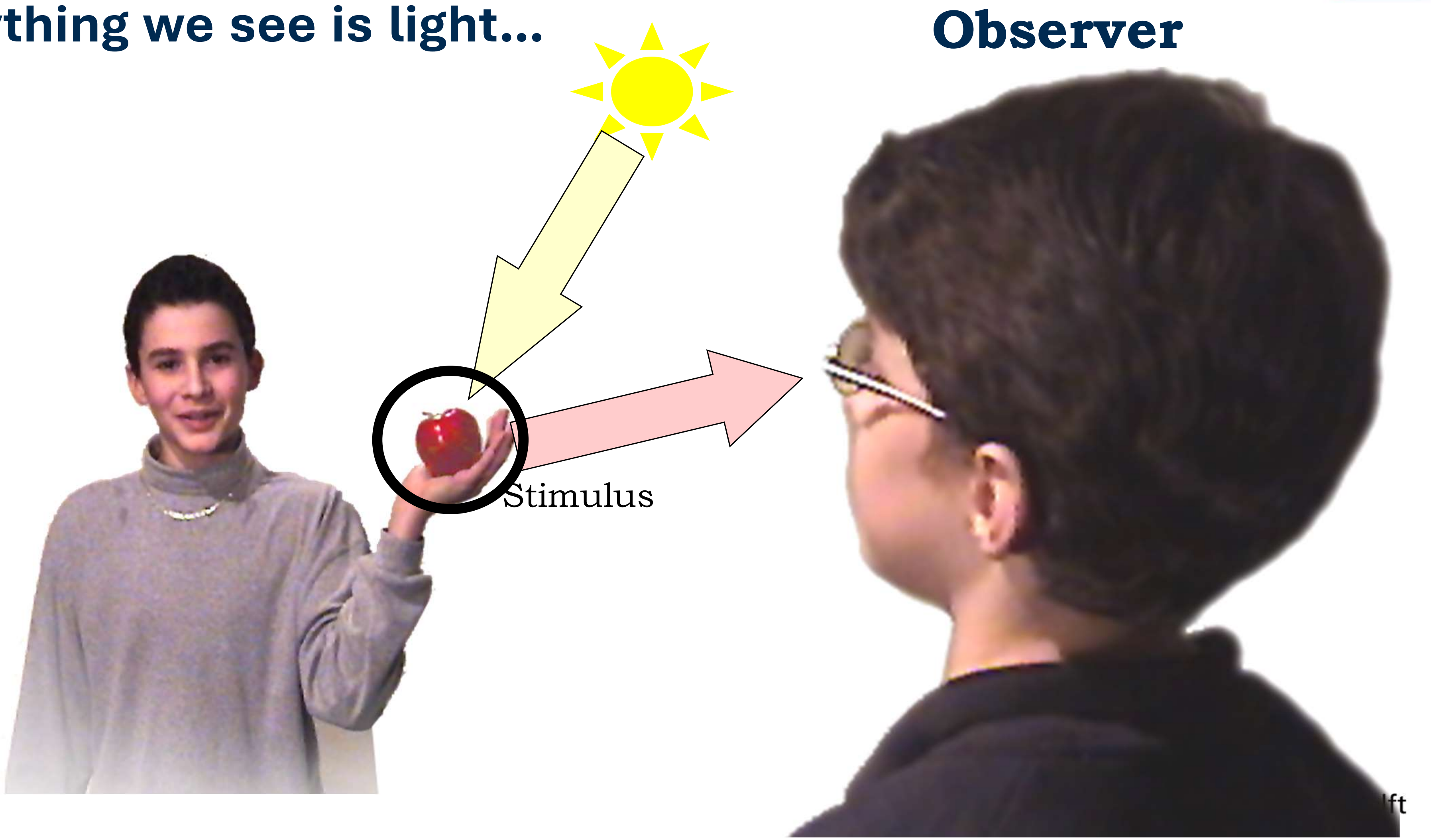
Different Species = Different View of the world

Confirmed:

Deer See Ultraviolet, What Does This Mean To Hunters?

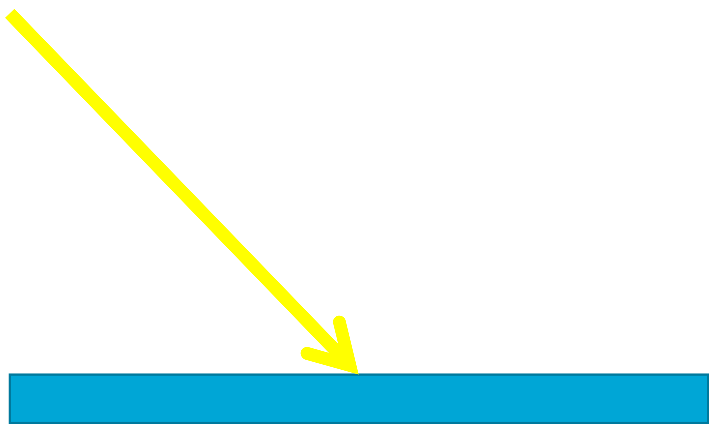
https://bowhunting.net/2019/02/confirmed-deer-see-ultraviolet-what-does-this-mean-to-hunters/

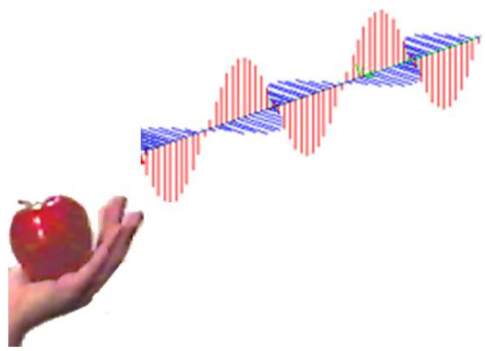
Everything we see is light...



Reflection

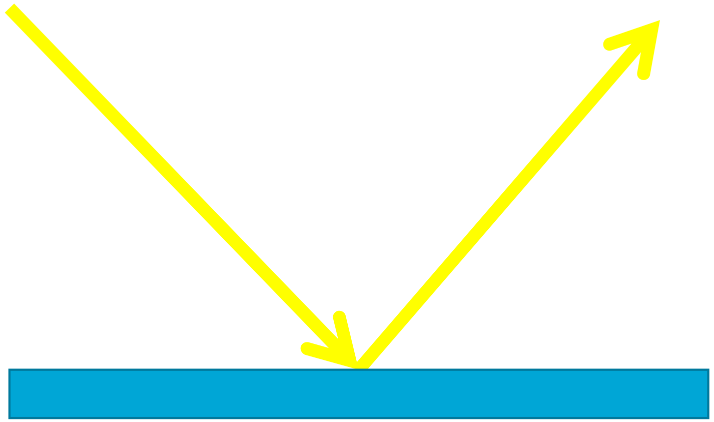
- What happens when the light hits a surface?





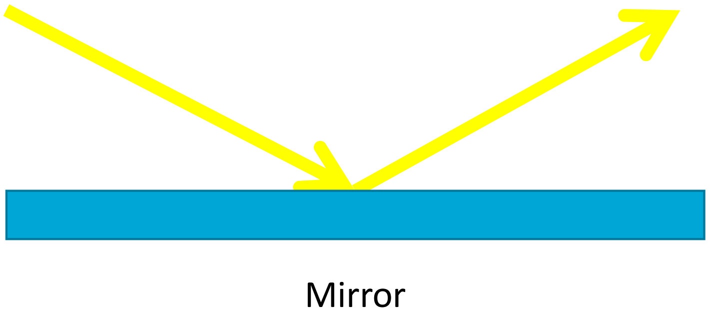
Reflection

- What happens when the light hits a surface?



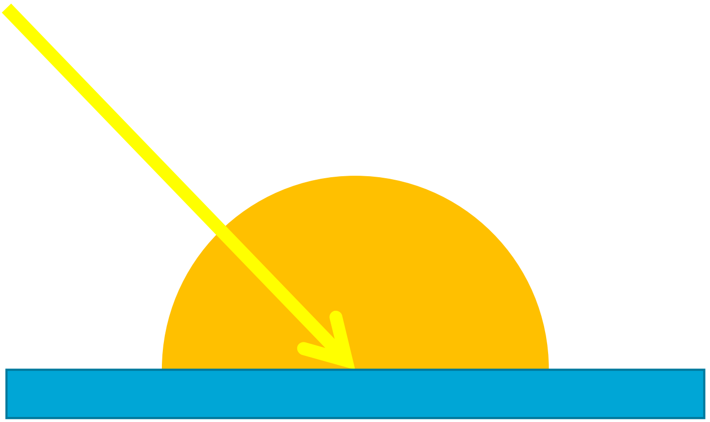
Reflection

- What happens when the light hits a surface?



Reflection

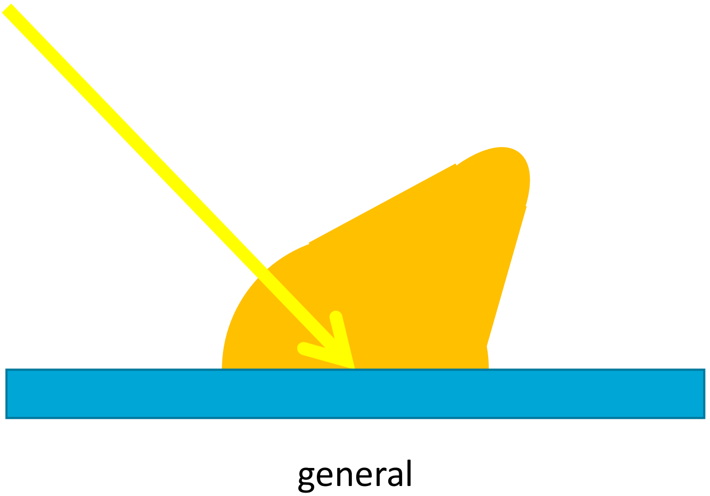
- What happens when the light hits a surface?



perfectly diffuse

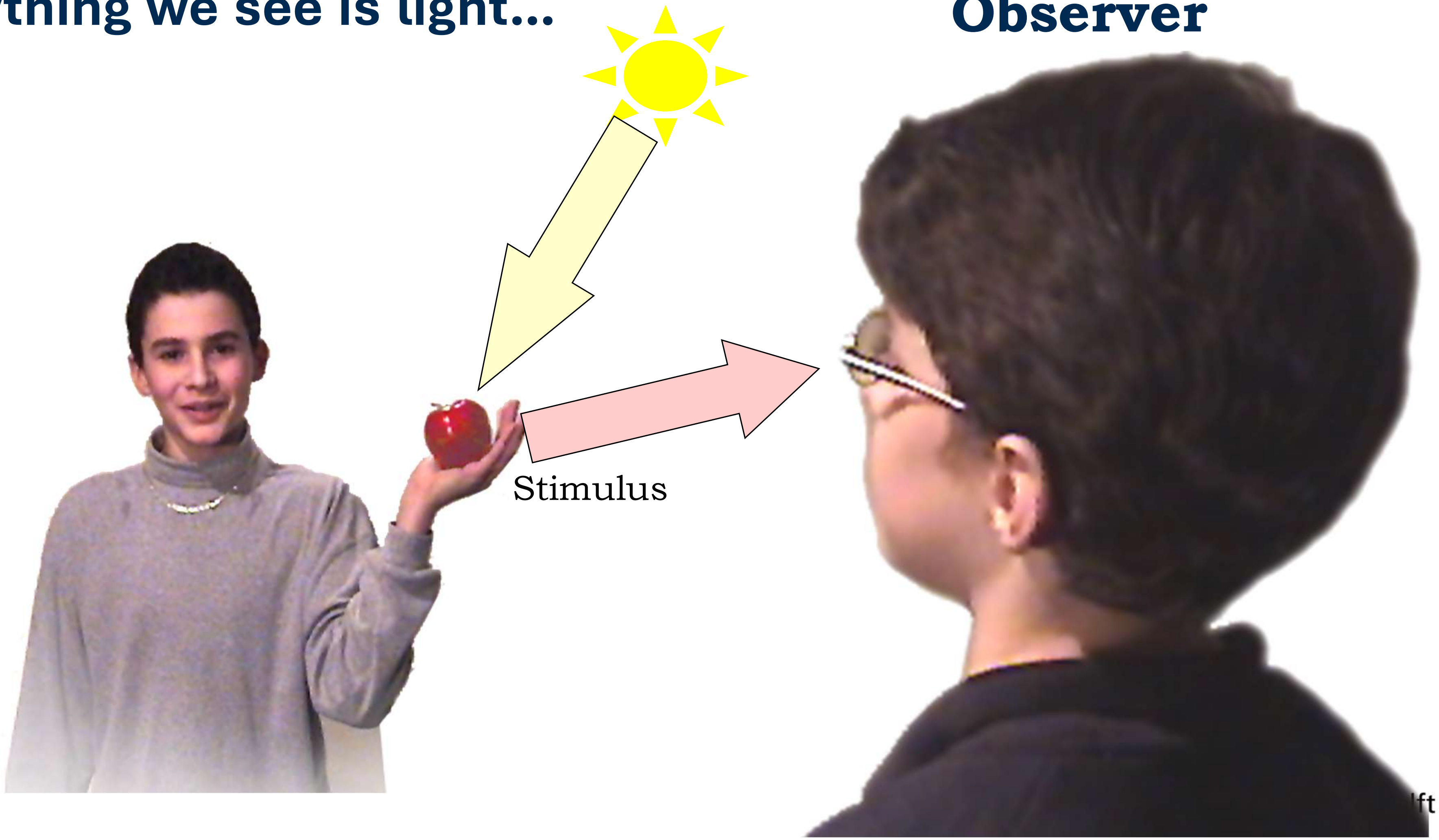
Reflection

- What happens when the light hits a surface?

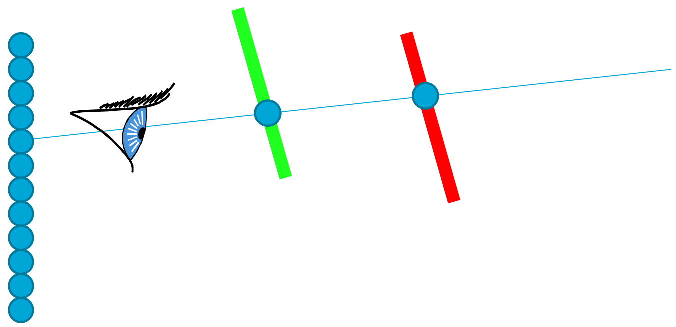


Everything we see is light...

Observer



Ray Tracing - Cost



For each pixel

    Distance=MAX

    Color=0

    R=computeRay(pixel)

    For each triangle

        (CurrColor,CurrDistance)=testIntersection(R)

        If (CurrDistance<Distance)

            Distance=CurrDistance

            Color=CurrColor

Performance Analysis

- Stupid implementation:
- Ray Tracing:

  Cost = Pixels * Triangles

e.g., 100.000 triangles and a 1000^2 screen:

Raytracing: 100.000 * 1.000.000 = 10^11

Performance Analysis

- Smart implementation:
- Ray Tracing:

Cost = Pixels * log(Triangles)

+ building a structure

e.g., 100.000 triangles and a 1000^2 screen:

Raytracing: 1.000.000 * 5 + X = 5 * 10^6

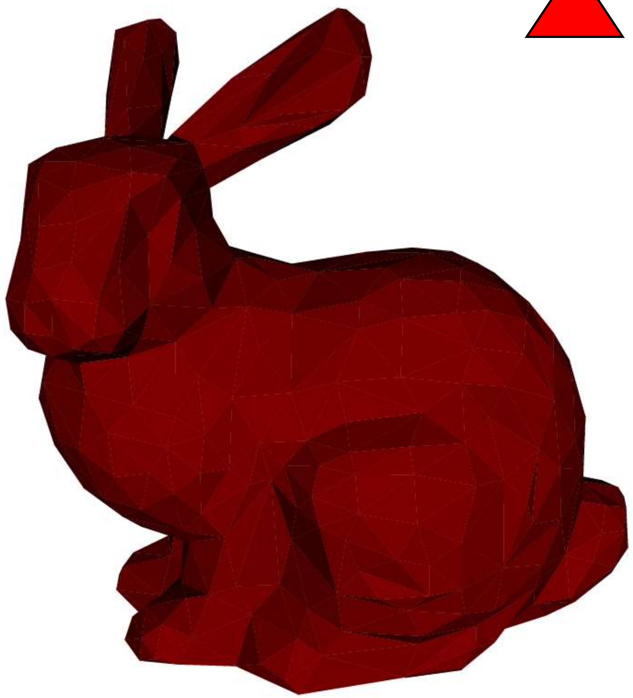
What is currently used? (30 Images/Sec)

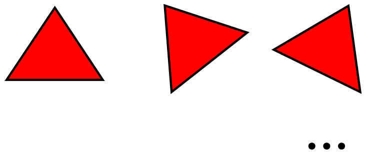
Alternative approach:

Rasterization via the Graphics Pipeline

Simplified Graphics Pipeline

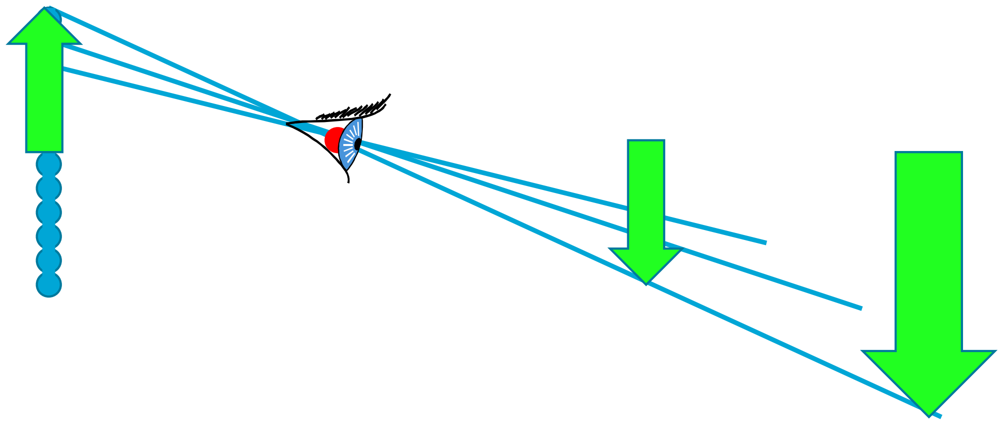
- Models are typically lists of triangles





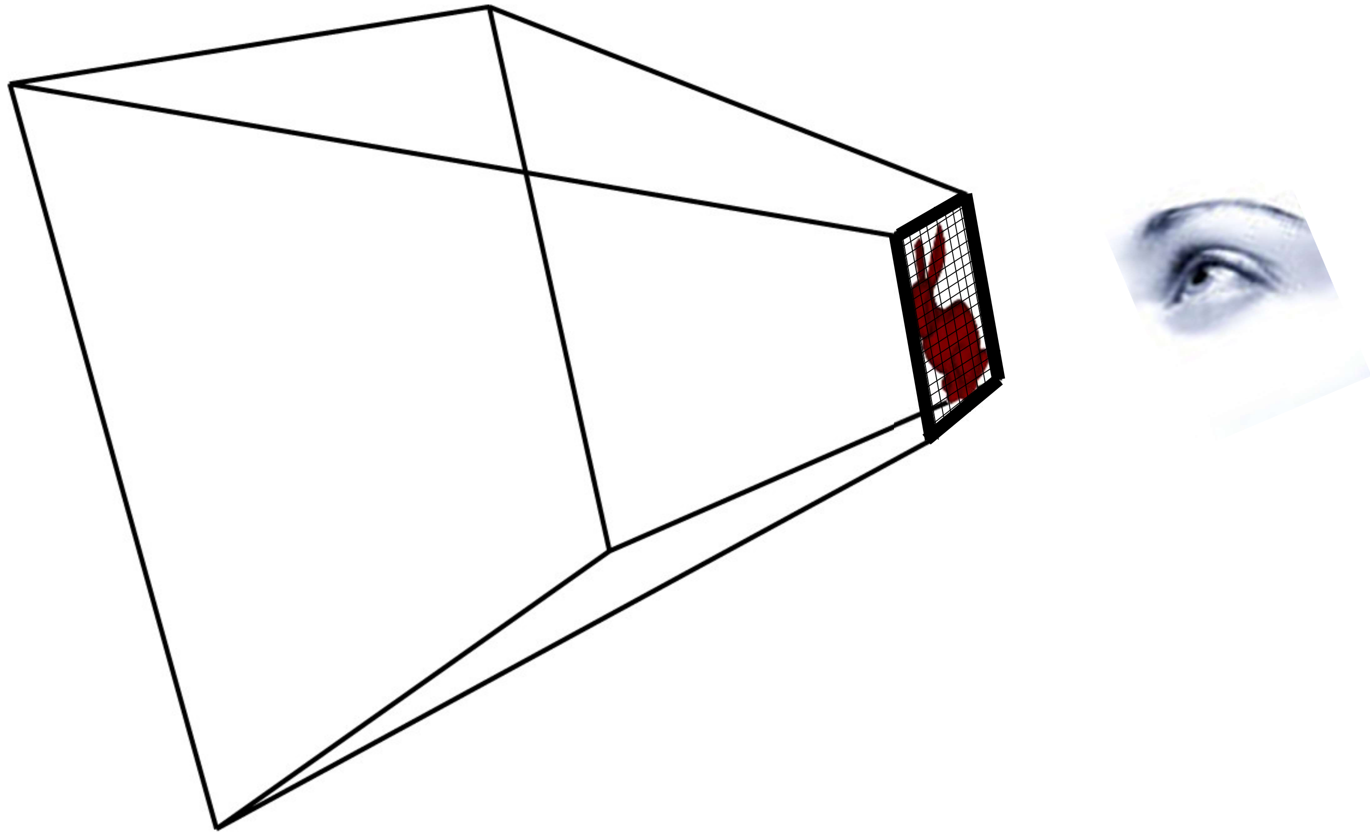
Virtual Camera

- Camera Plane in front of the eye



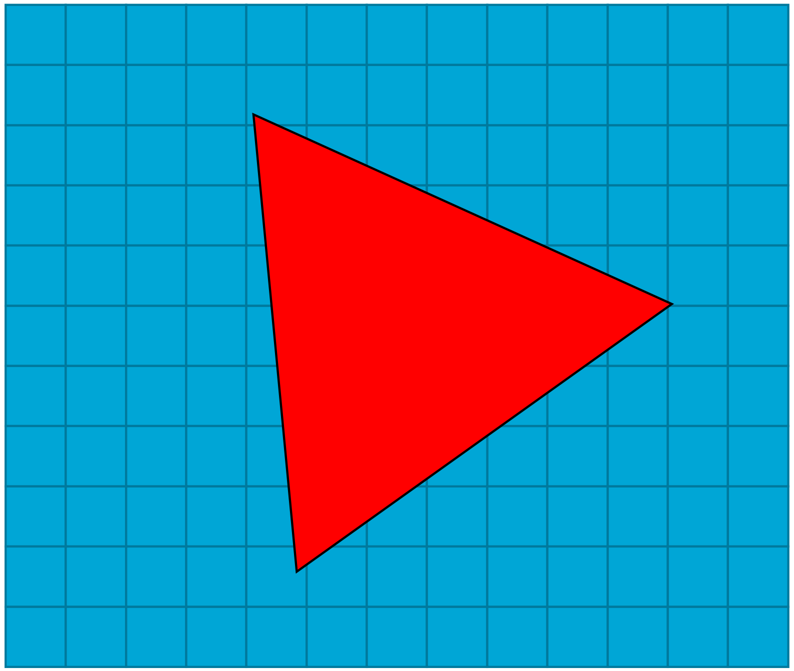
Simplified Graphics Pipeline

- Projection: Transform coordinates to screen



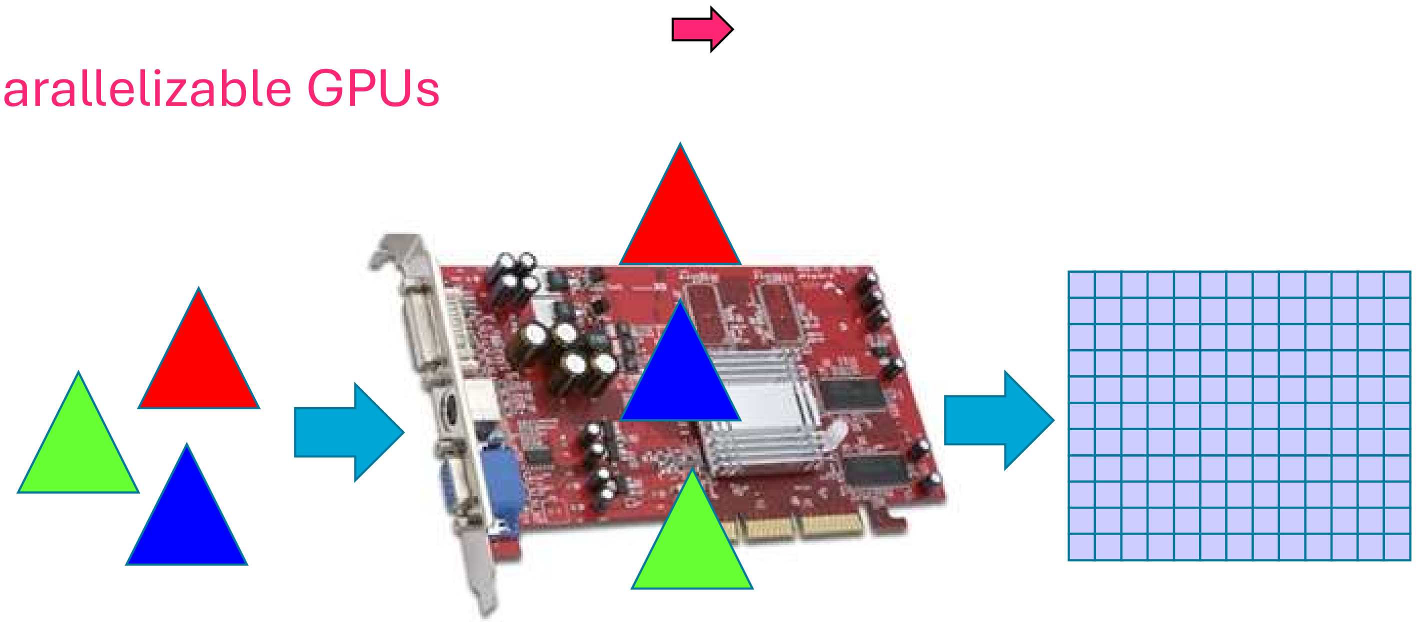
Simplified Graphics Pipeline

- Rasterization: Fill screen pixels



Simplified Graphics Pipeline

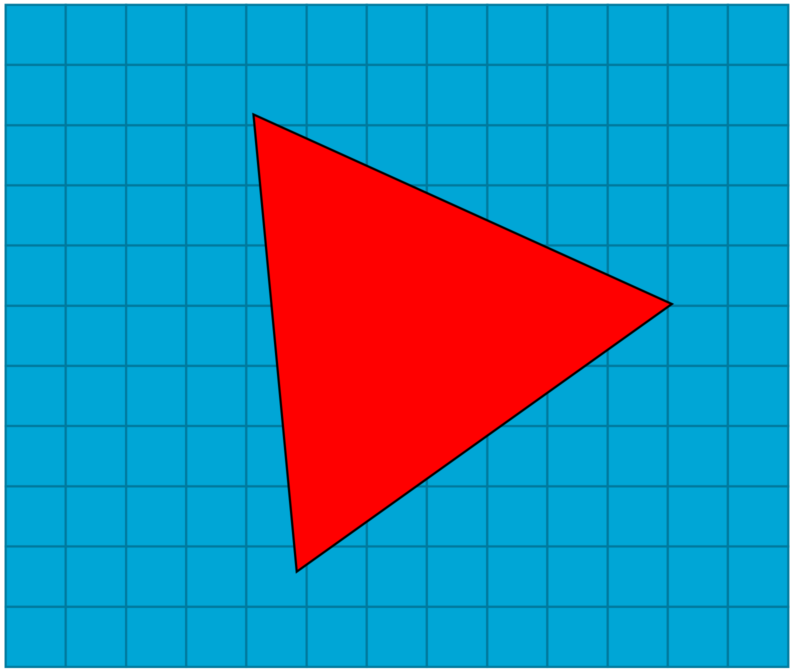
- Highly parallelizable GPUs



…Nvidia 1080 listed with 3584 cores…

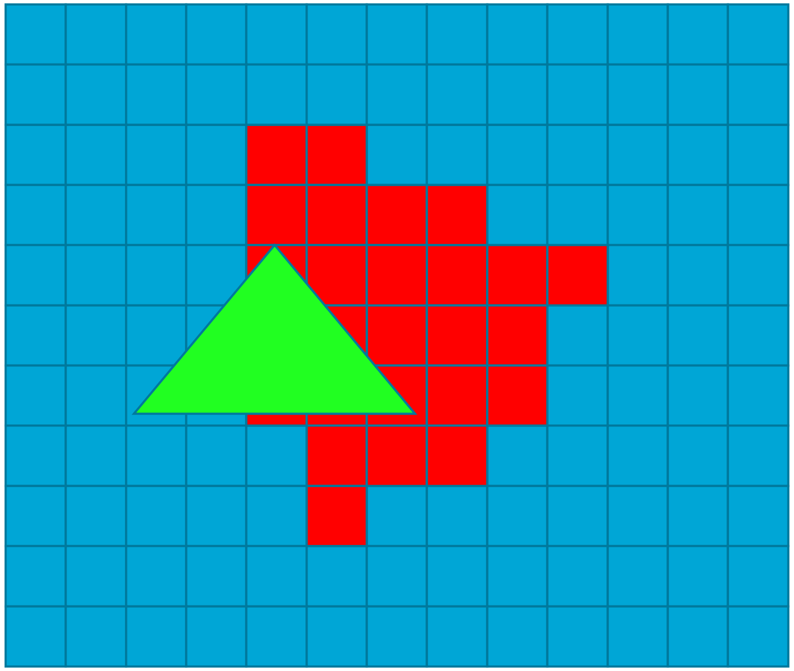
Simplified Graphics Pipeline

- Catch: Let's look at a second triangle...



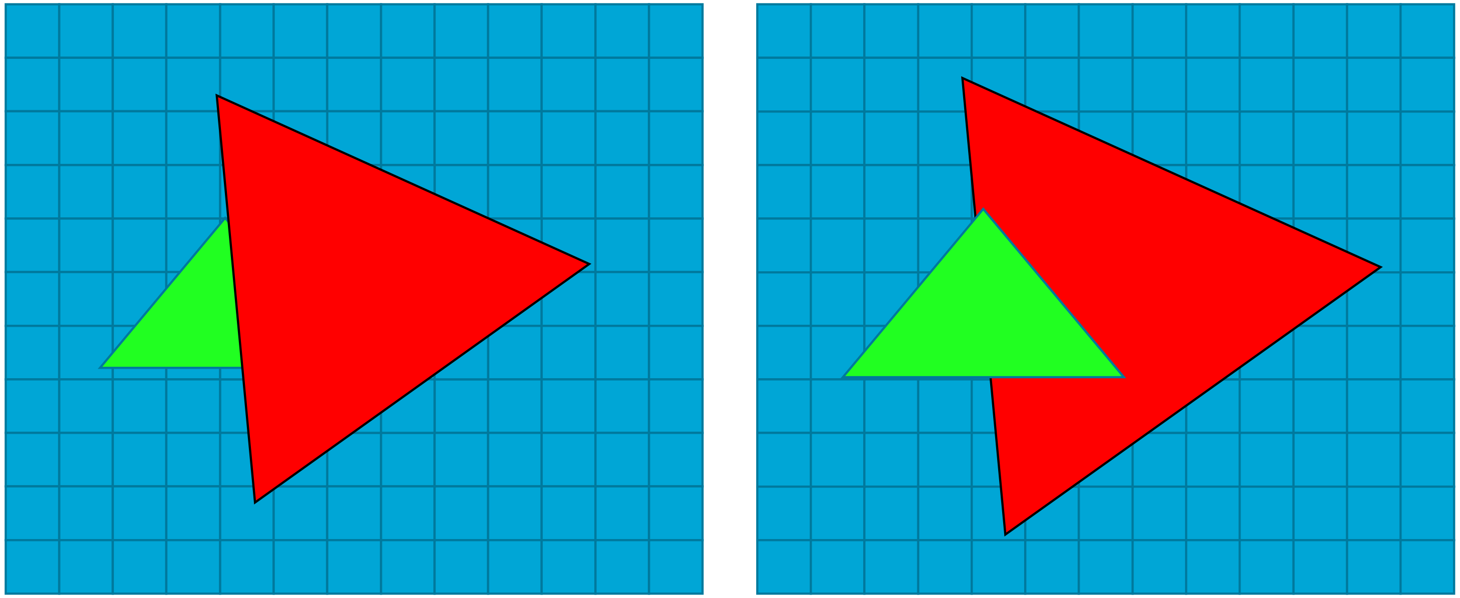
Simplified Graphics Pipeline

- Catch: Let's look at a second triangle...



Simplified Graphics Pipeline

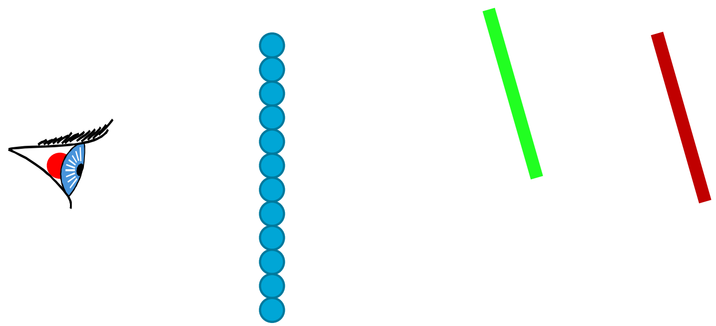
- Catch: Triangle drawing order changes result



As for ray tracing: we need to know the closest triangle in a pixel

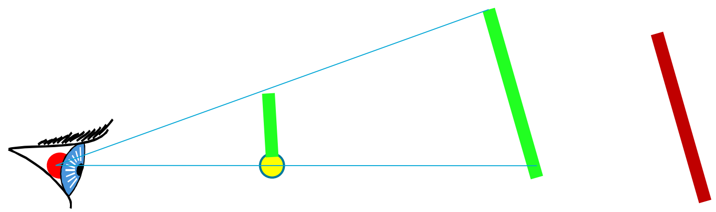
Simplified Graphics Pipeline

- Depth Test: Avoid sorting!
- Store a depth in each pixel



Simplified Graphics Pipeline

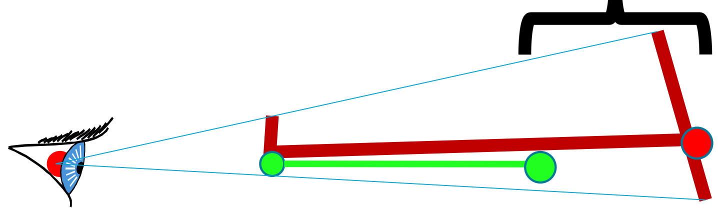
- Depth Test: Avoid sorting!
- Store a depth in each pixel



Simplified Graphics Pipeline

- Depth Test: Avoid sorting!
- Store a depth in each pixel

Compare new distance to stored distance
Update pixel only if new distance is nearer



Simplified Graphics Pipeline

- Depth Test: Avoid sorting!
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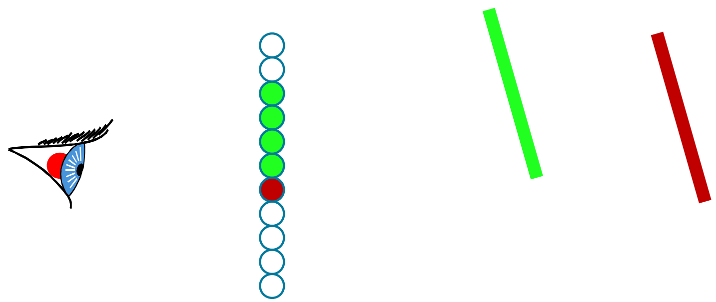


Cost of Rasterization

Algorithm:

For each triangle

```
projTri=projectTriangle(triangle)
fillPixels(projTri)
```

Cost = Triangles + “drawn pixels”

Performance Analysis

Ray Tracing:

Cost = Pixels * log(Triangles) + structure

vs.

Rasterization:

Cost = Triangles + "drawn pixels"

e.g., 100.000 triangles and a 1000^2 screen:

Raytracing: X+5 * 1.000.000

Rasterization: 100.000 + "drawn pixels"

Raytracing/Rasterization : ~50

Performance Analysis – More Geometry…

Ray Tracing:

Cost = Pixels * log(Triangles) + structure

vs.

Rasterization:

Cost = Triangles + “drawn pixels”

e.g., 100.000.000 triangles and a 1000^2 screen:

Raytracing: X+8 * 1.000.000

Rasterization: 100.000.000 + “drawn pixels”

Raytracing/Rasterization : ~0.1

But Rasterization can be made smarter
with acceleration structures too…

What complexity do we work with?

- Today's Games:
  - Often around 30K (Gears of War 3 and later)
  - Up to 500.000 triangles (Lamborghini Reventon - 562,786 Forza4)
- Today's Movies
  - Many Billions...

TU Delft

How many triangles in this shot of Avatar 2 ?

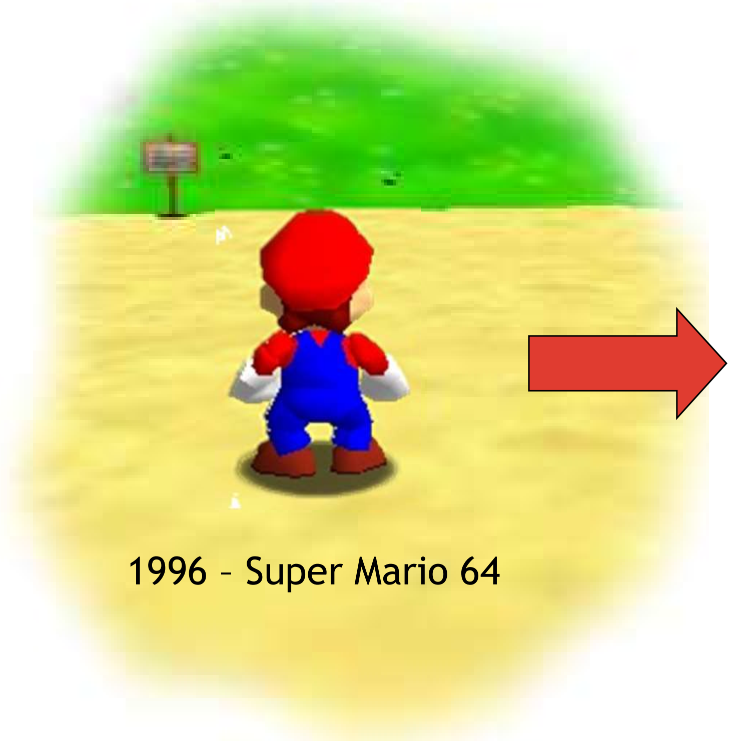
Local Computations

- Processor only knows its *current triangle (project or intersect)*

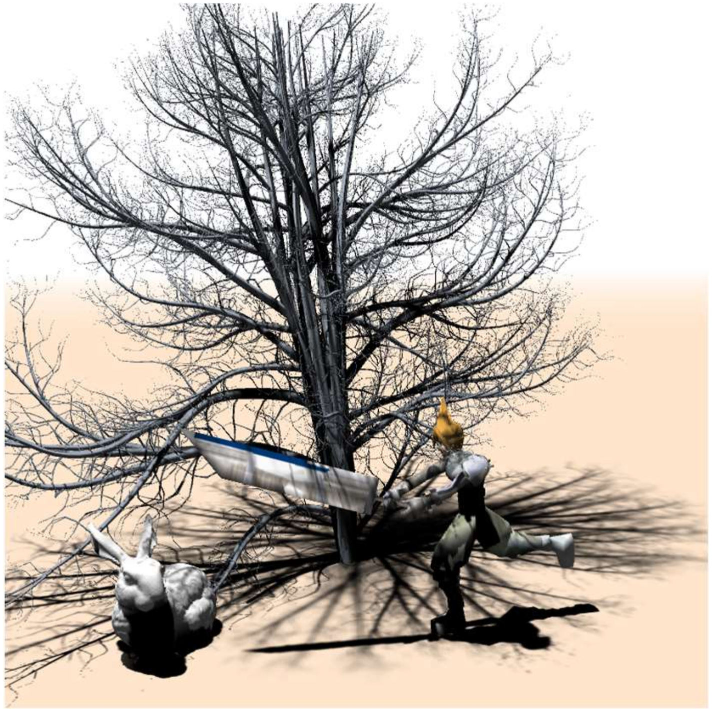
  generally NOT enough

Non-Local Problems

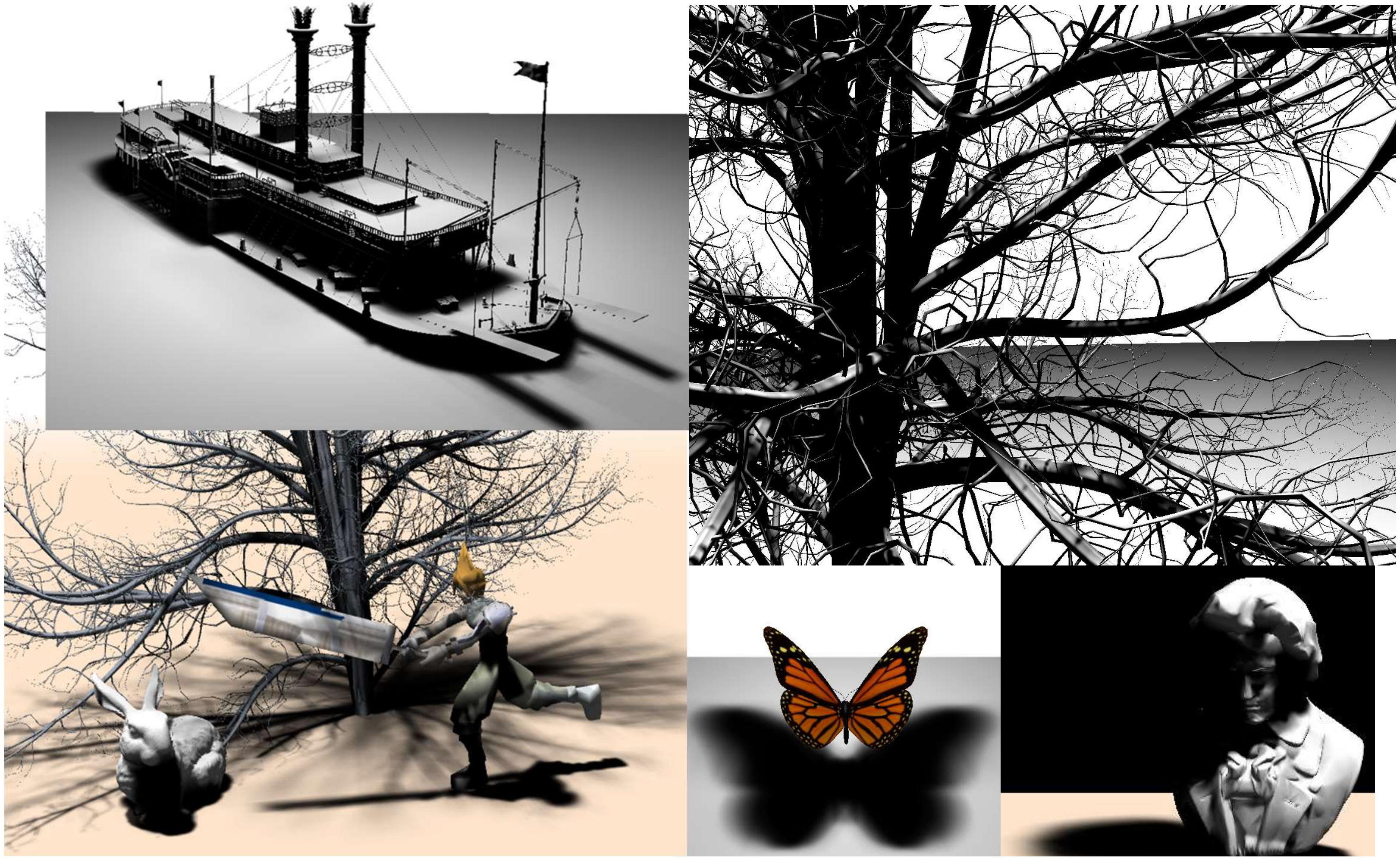
- Challenges beyond local computations
  - Shadows



1996 – Super Mario 64

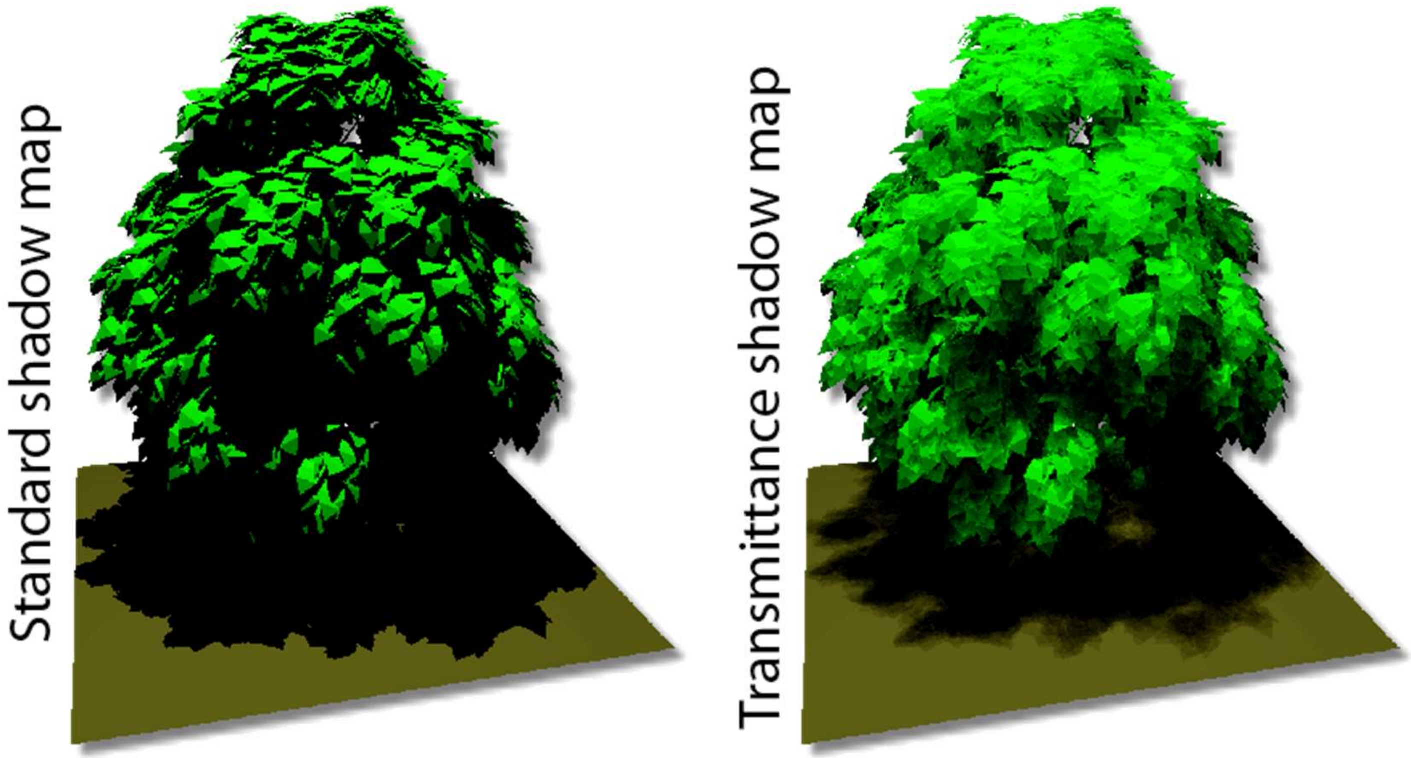


Occlusion Textures for Plausible Soft Shadows



Non-Local Problems

- Challenges beyond local computations
  - Transmittance

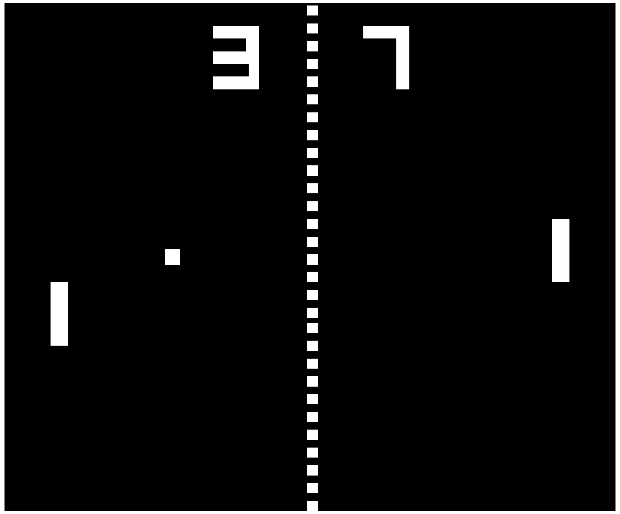


Non-Local Problems

- Challenges beyond local computations
  - Refraction/Translucency

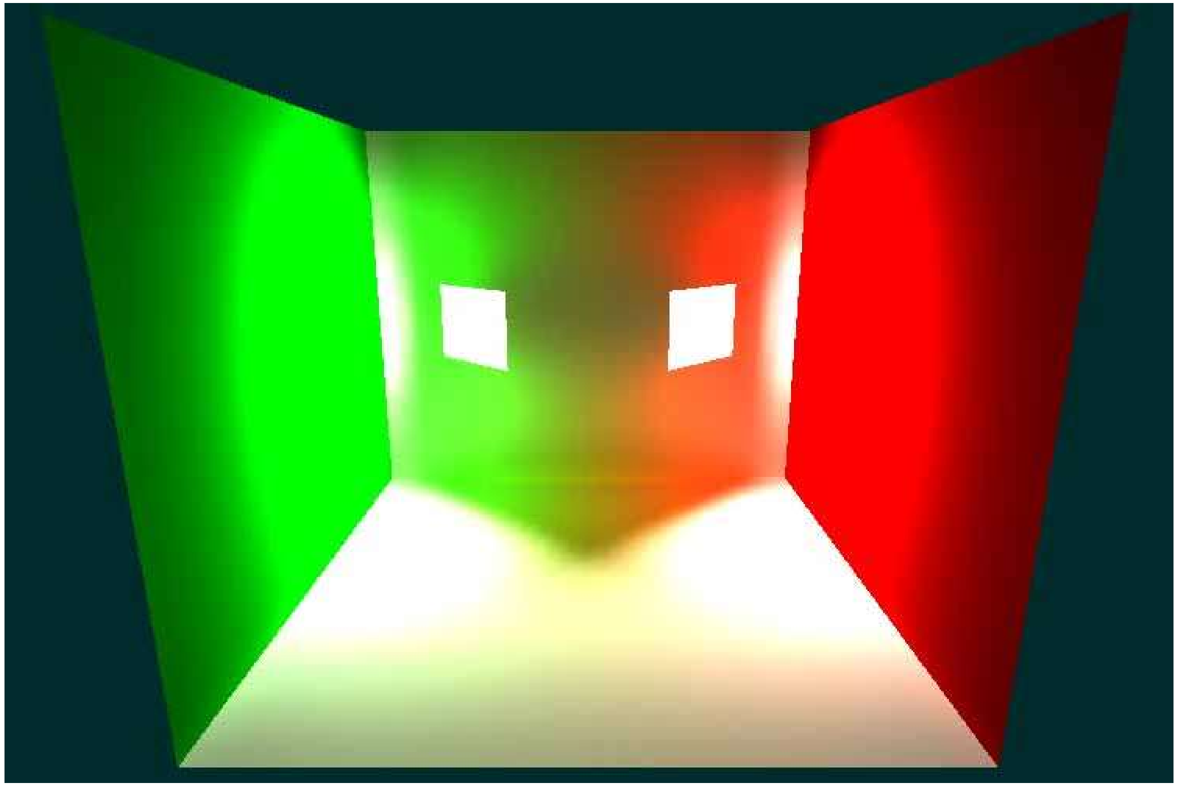
Non-Local Problems

- Challenges beyond local computations
  - Collision Detection



1972 - Pong

Global Illumination




In all examples
SM resolution: 1024x1024
scattering: < 3 ms
Scene complexity and extent do only marginally
We support dynamic lights and viewpoints.

Resume

- Introduction
  - Definition of Computer Graphics
  - Basics of how we perceive images/colors
  - Creating images on a computer:
    - Raytracing
    - Rasterization (including Depth Buffer)

Thank you very much

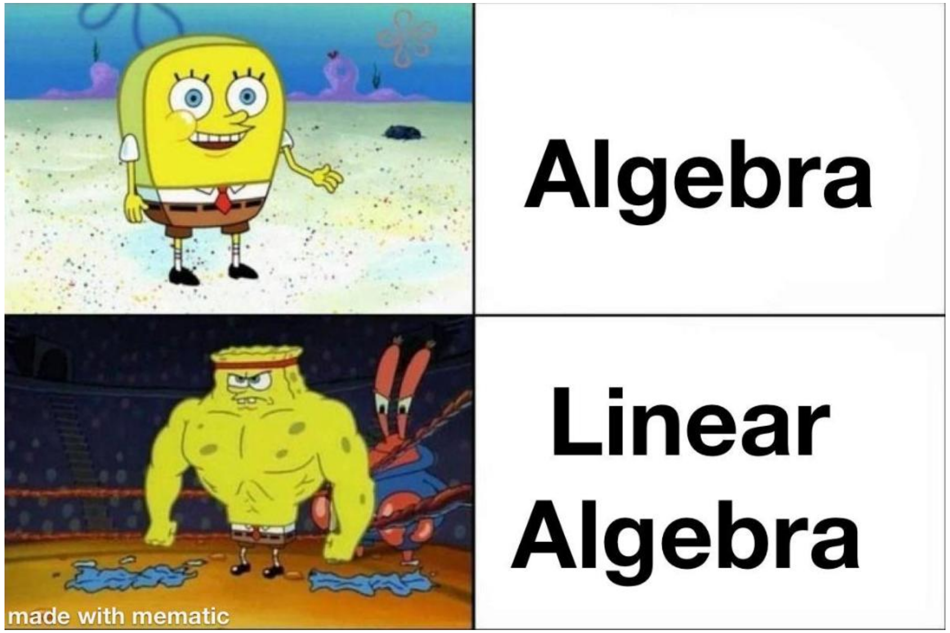
for your attention!

Computer Graphics – Linear Algebra Recap

Martin Skrodzki

Computer Graphics and Visualization Group
Delft University of Technology, The Netherlands

November 13th, 2024



Computer Graphics – Why linear algebra?

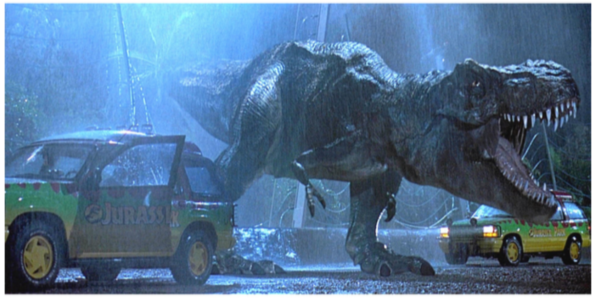
`https://vevox.app/#/m/106717265`

Computer Graphics – Why linear algebra?

`https://vevox.app/#/m/106717265`

According to Google:

Movies



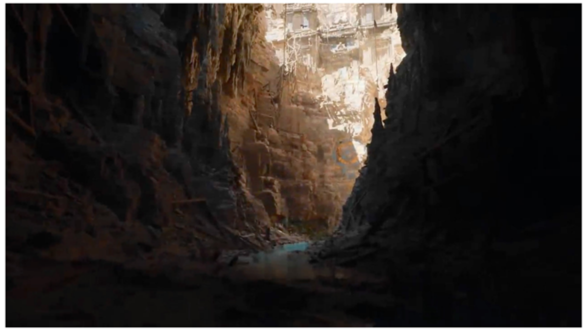
Jurassic Park (1993)

The Matrix (1999)

Motion Capture

Andy Serkis in The Two Towers

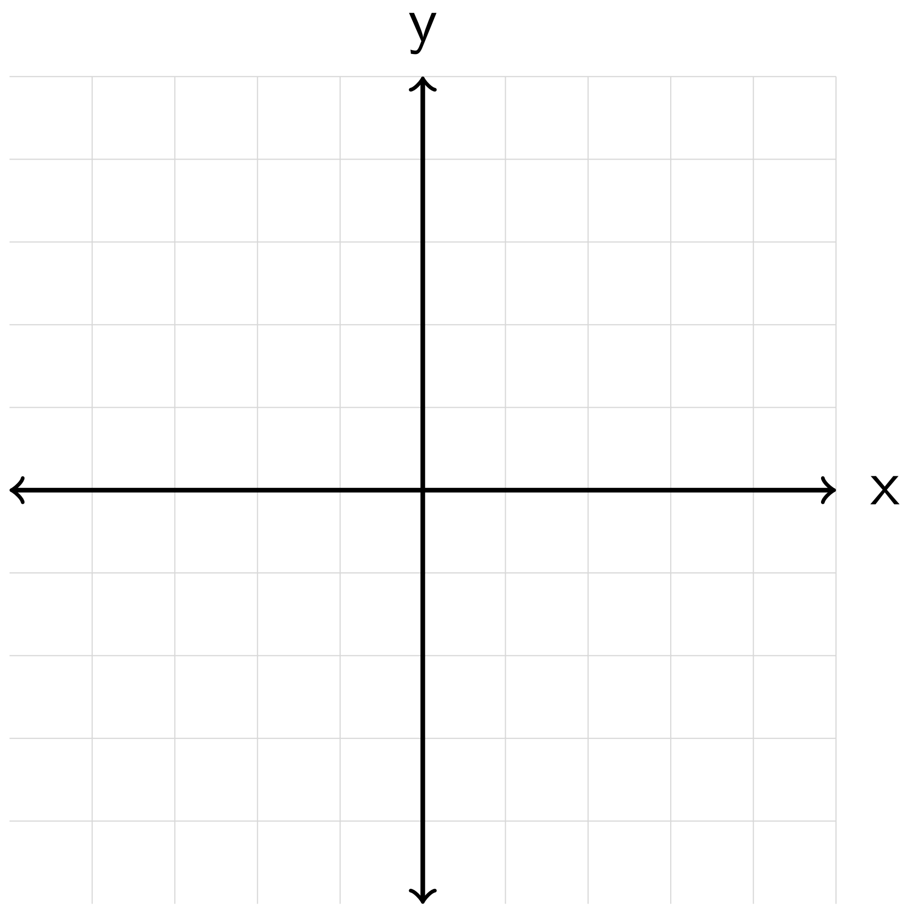
Games



Unreal Engine 5 Demo Realtime in PS5 (2020)

Addition and Subtraction of Vectors

`https://vevox.app/#/m/106717265`

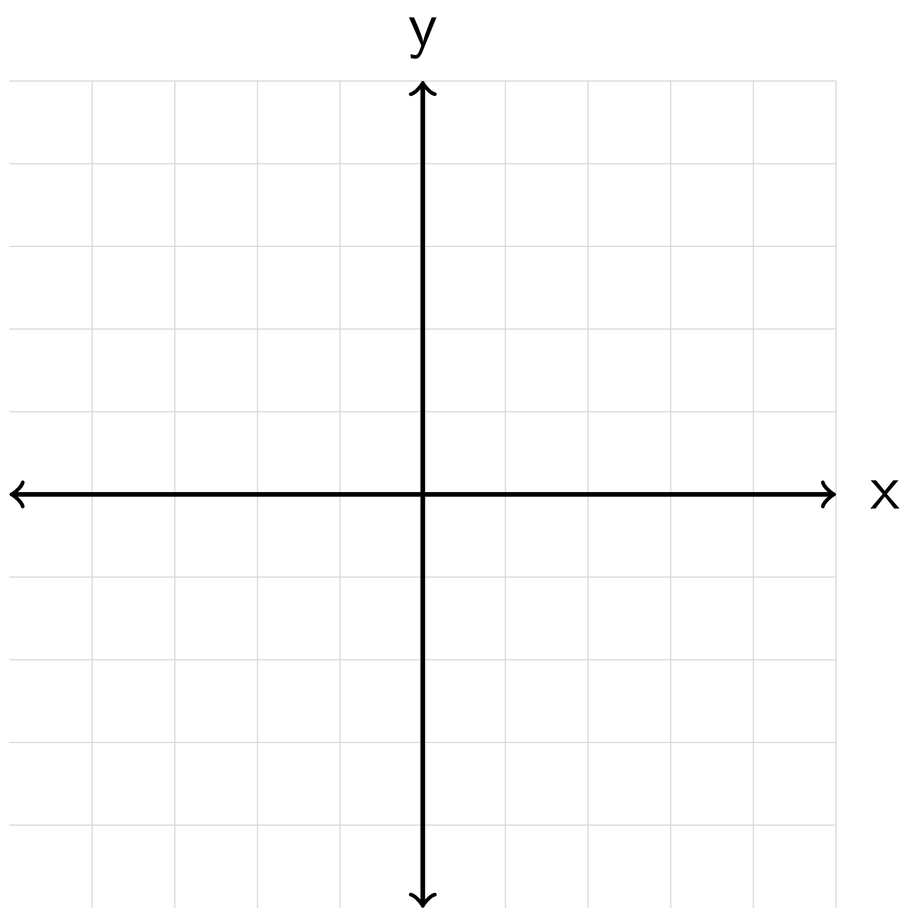


Addition and Subtraction of Vectors

https://vevox.app/#/m/106717265

For two vectors

$$\mathbf{v} = (v_1, v_2, v_3)^t = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$



Addition and Subtraction of Vectors

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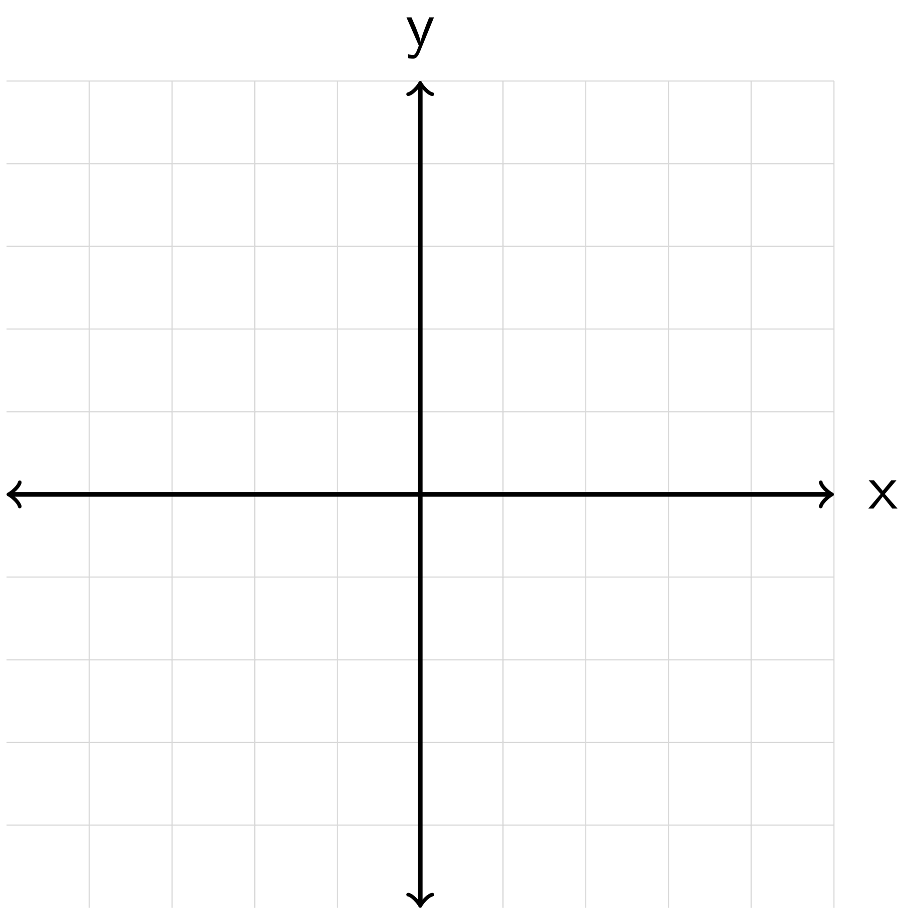

Addition and Subtraction of Vectors

https://vevox.app/#/m/106717265

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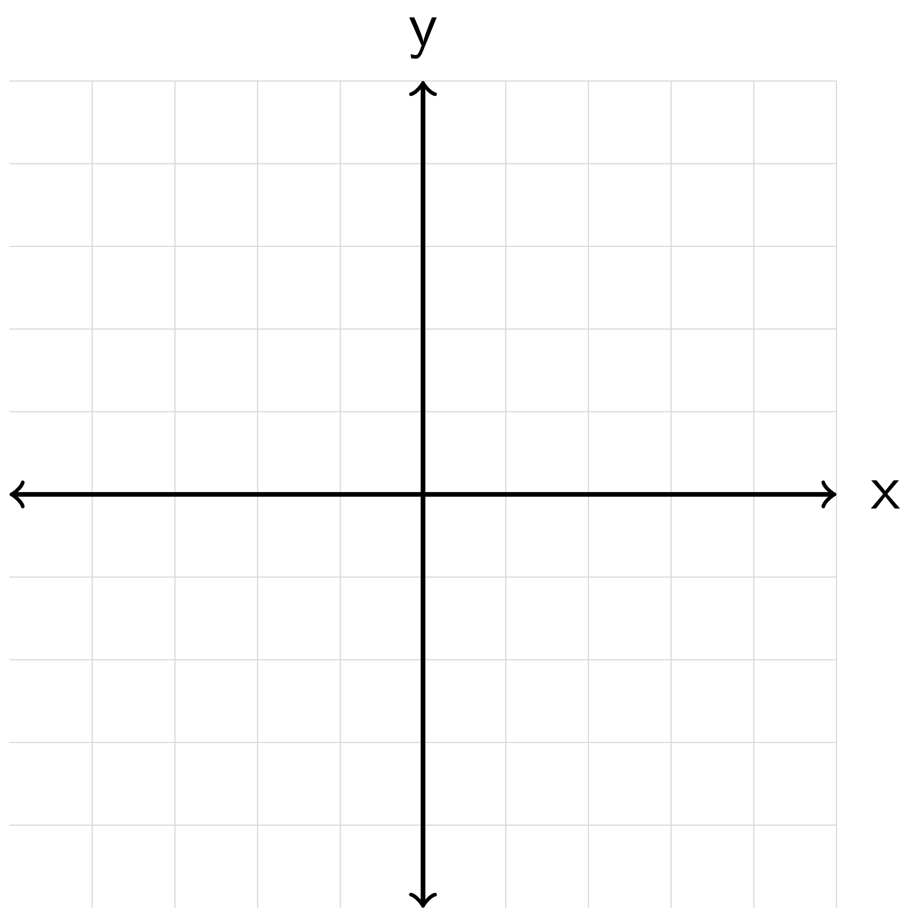

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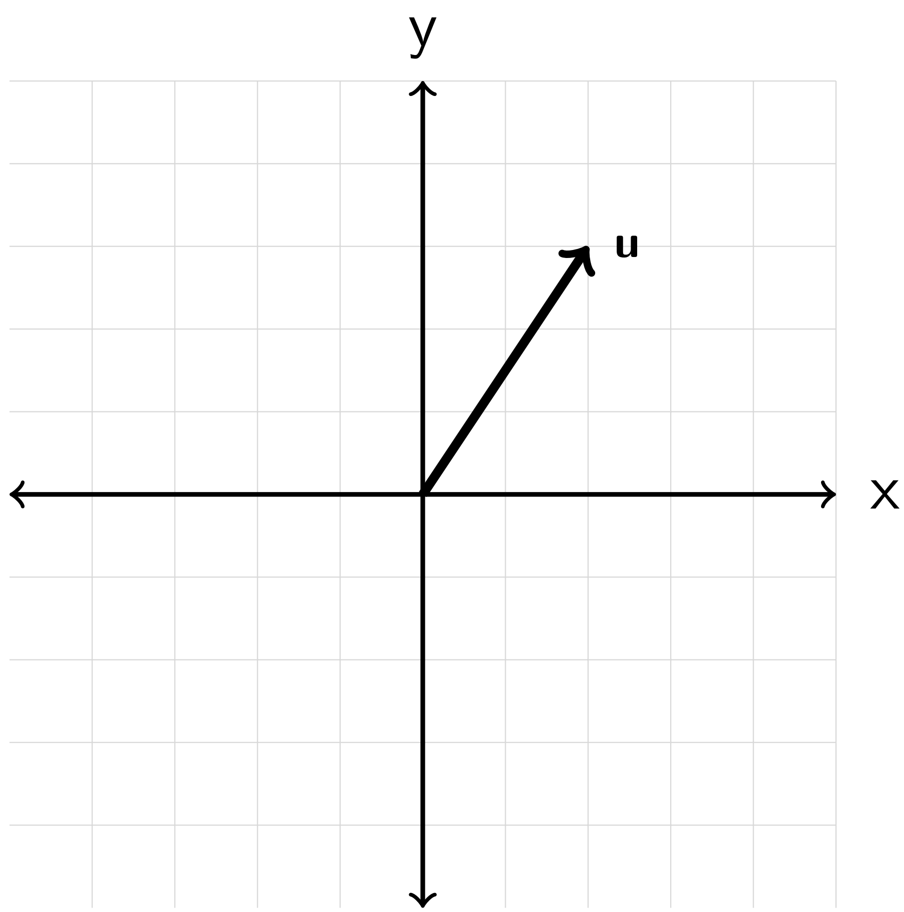

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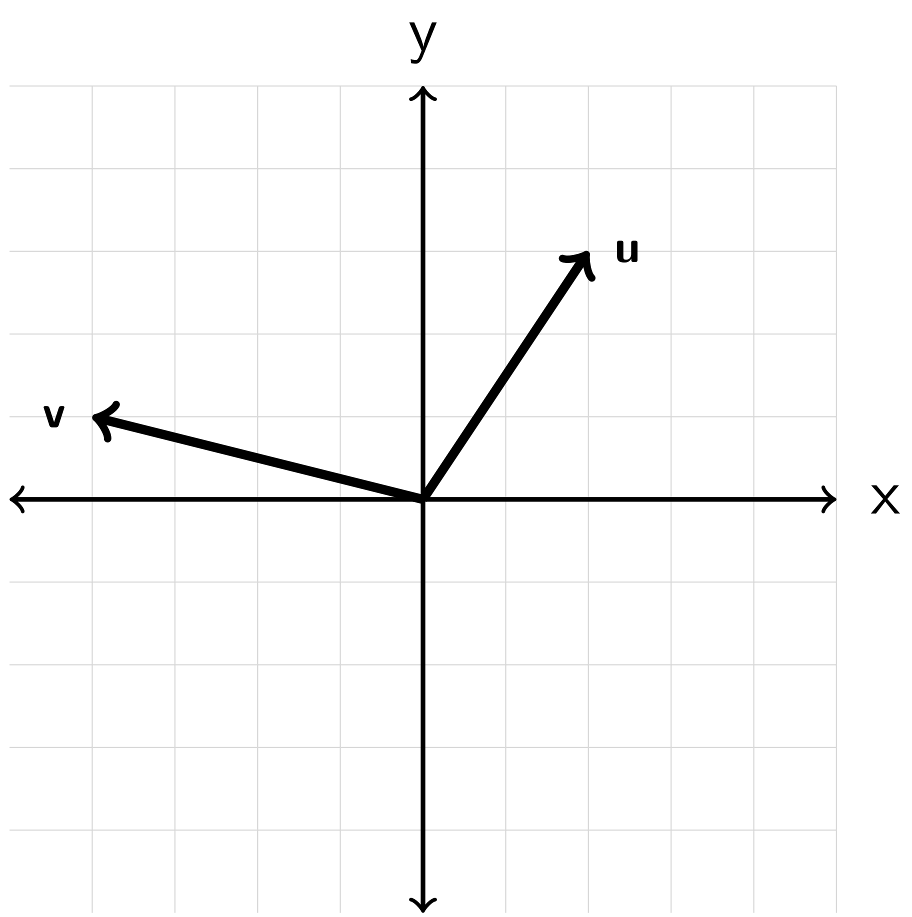

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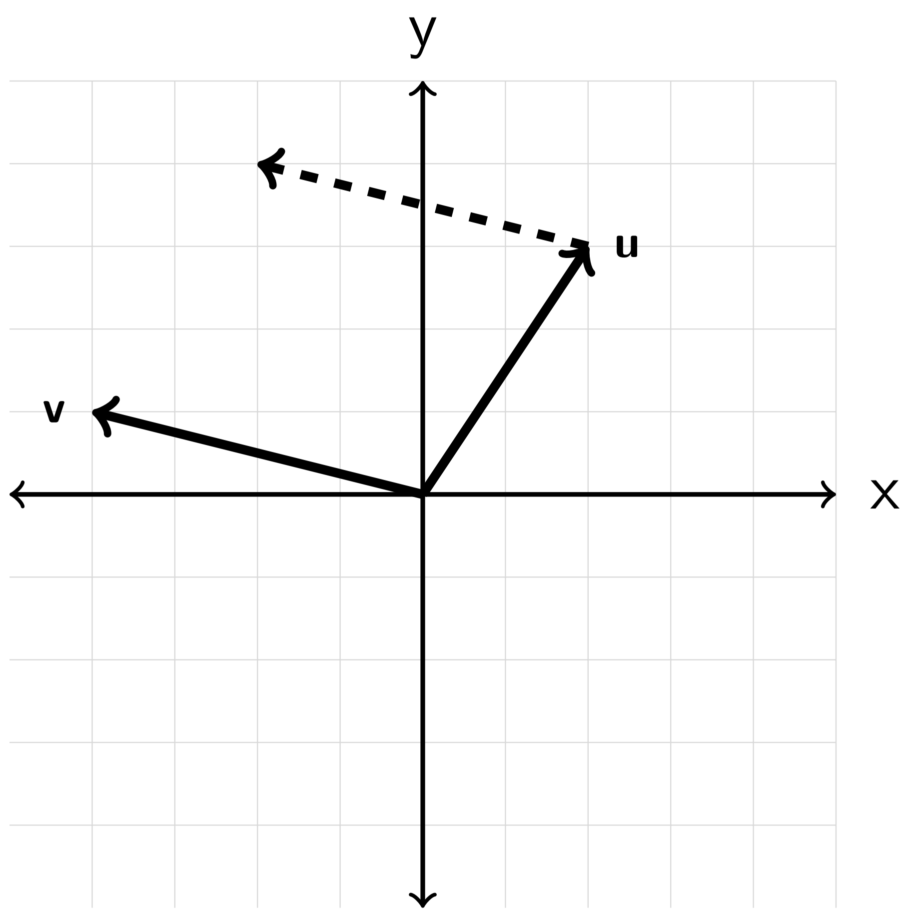

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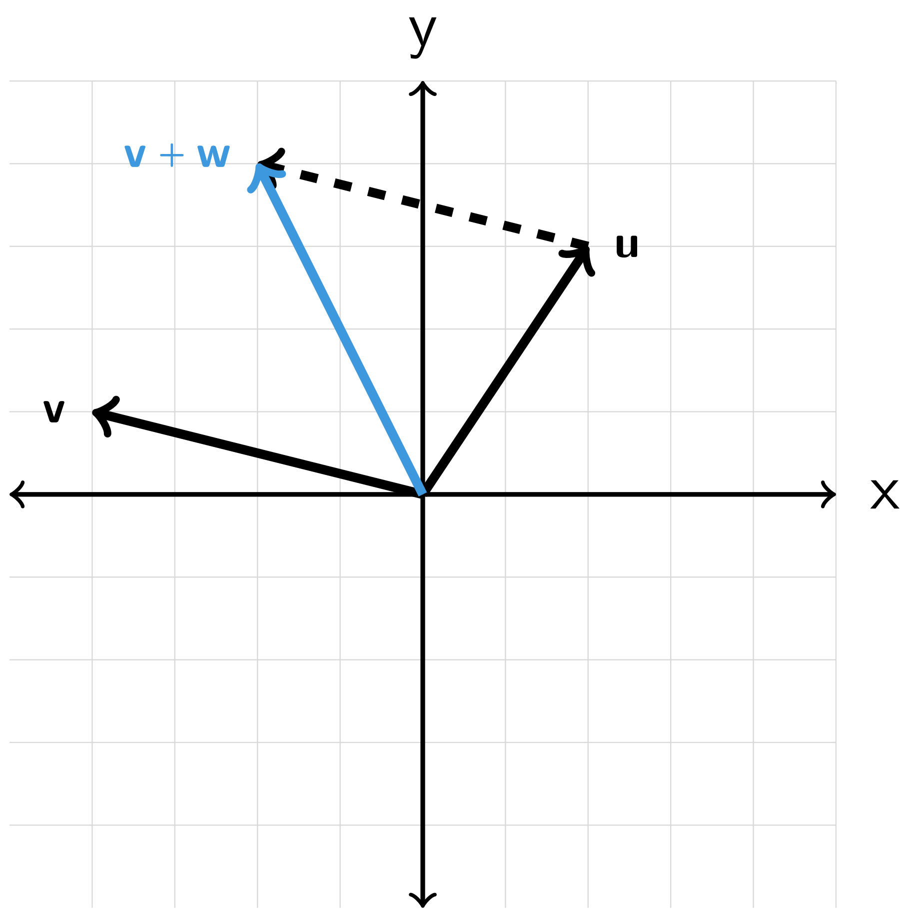

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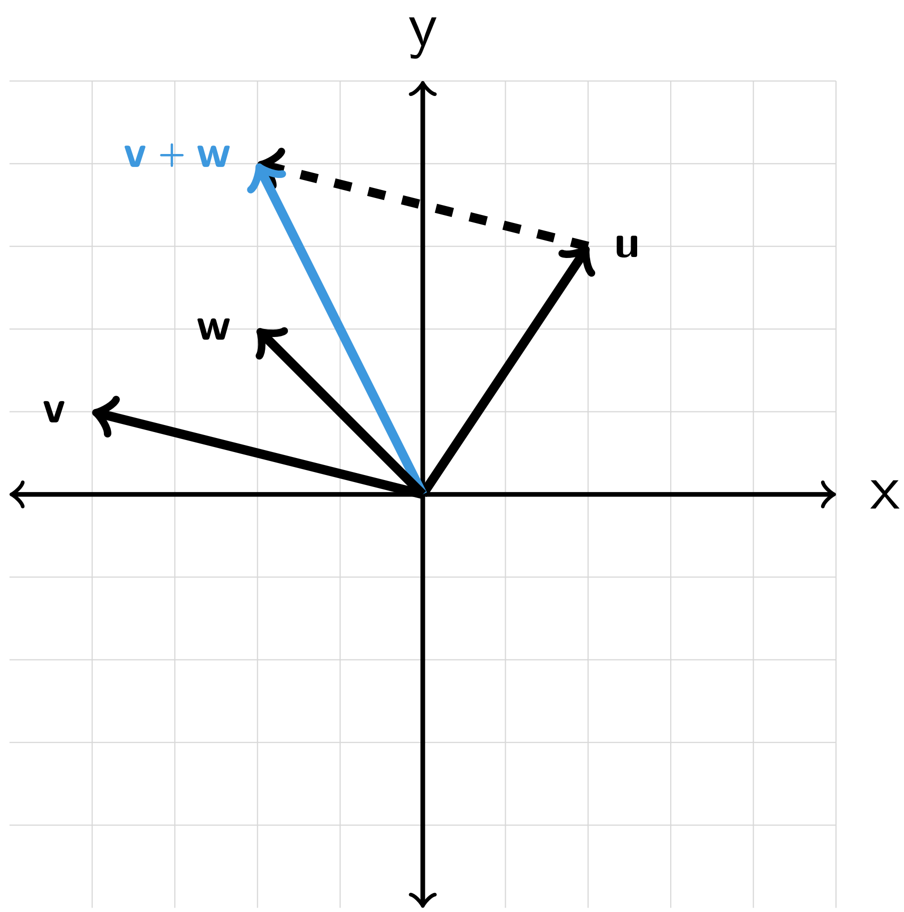

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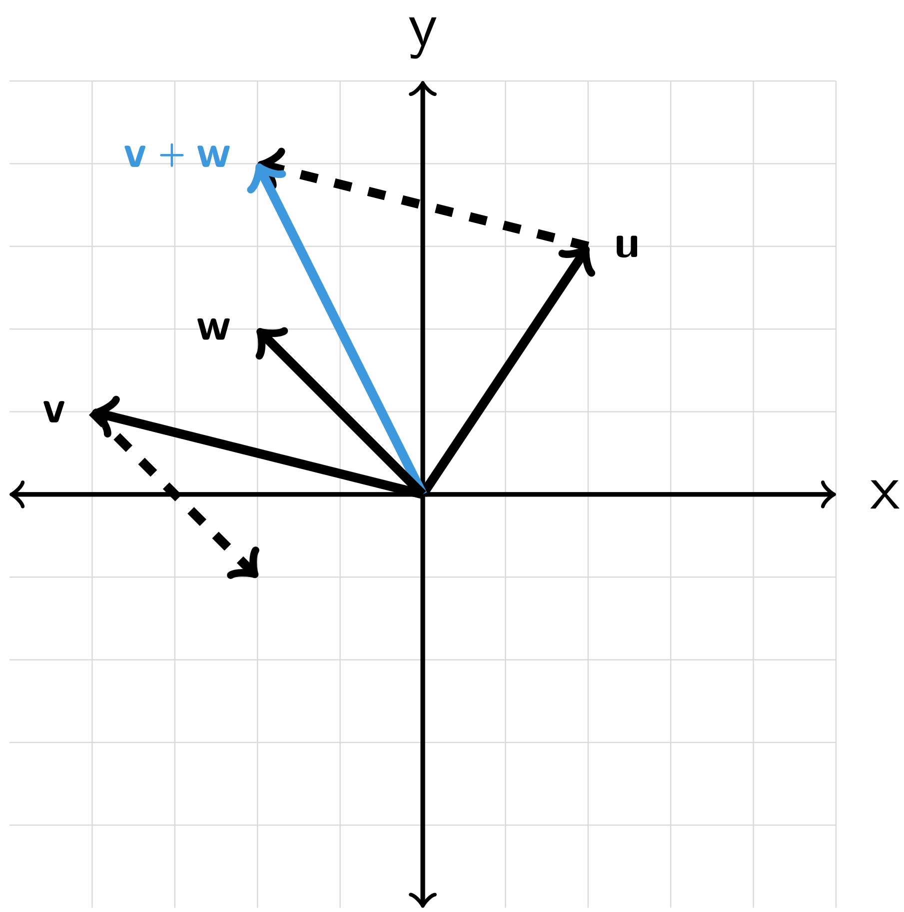

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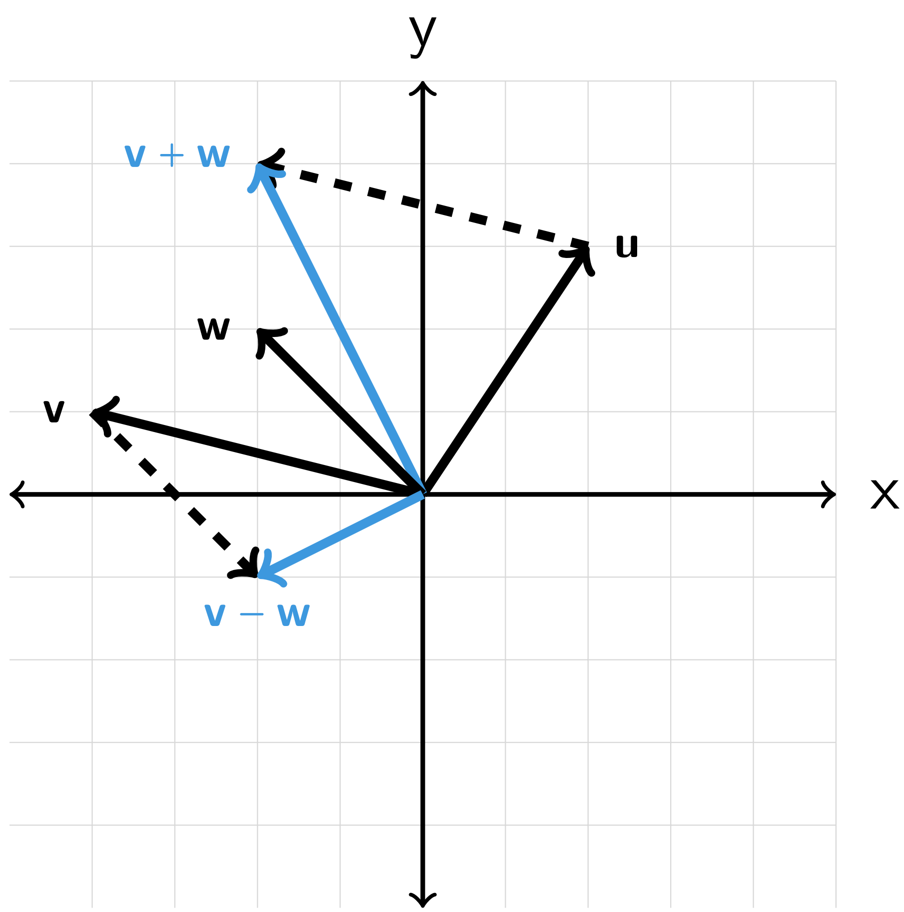

Length of a vector

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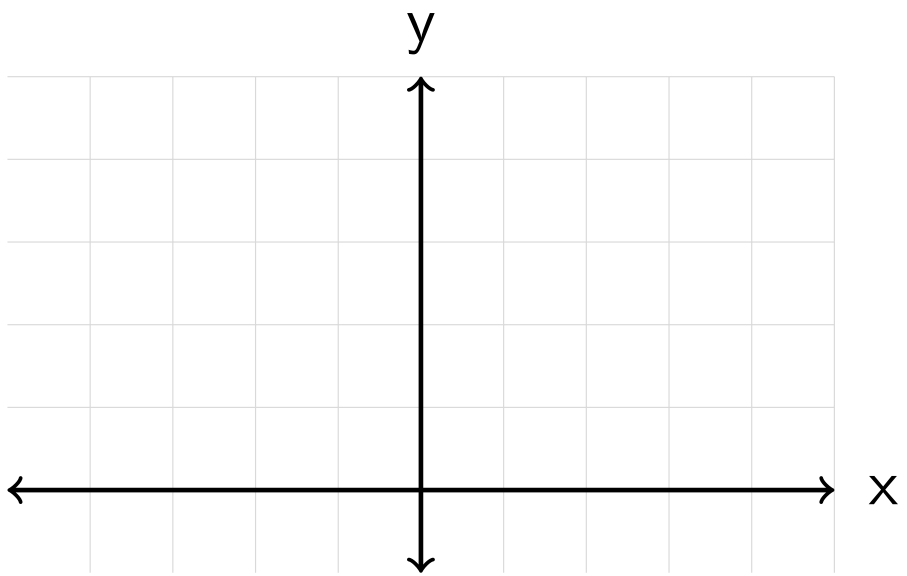

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aka Pythagoras' Theorem both in 2D

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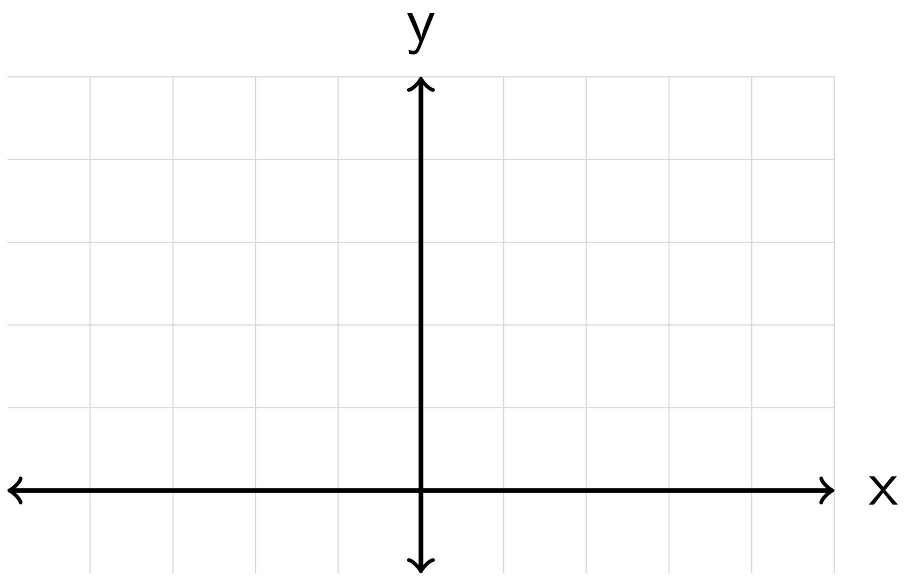

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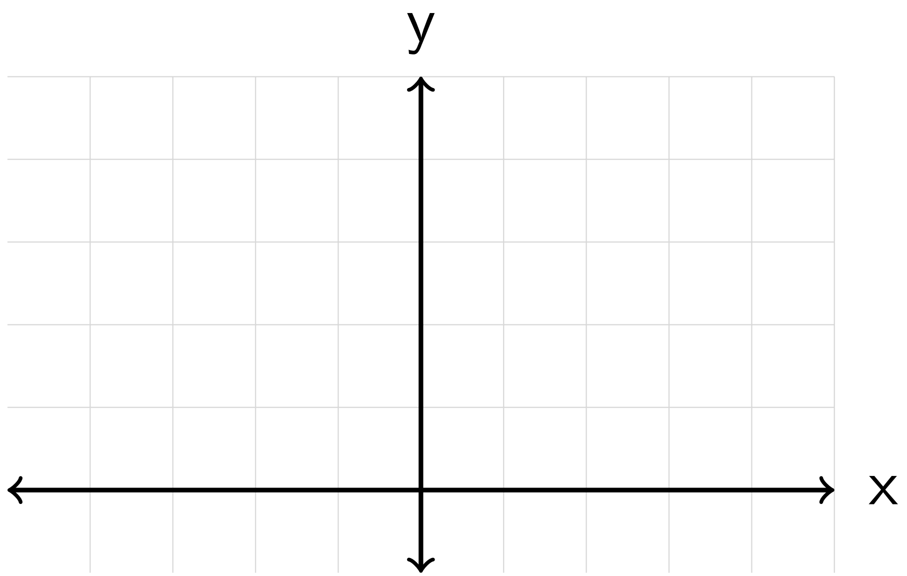

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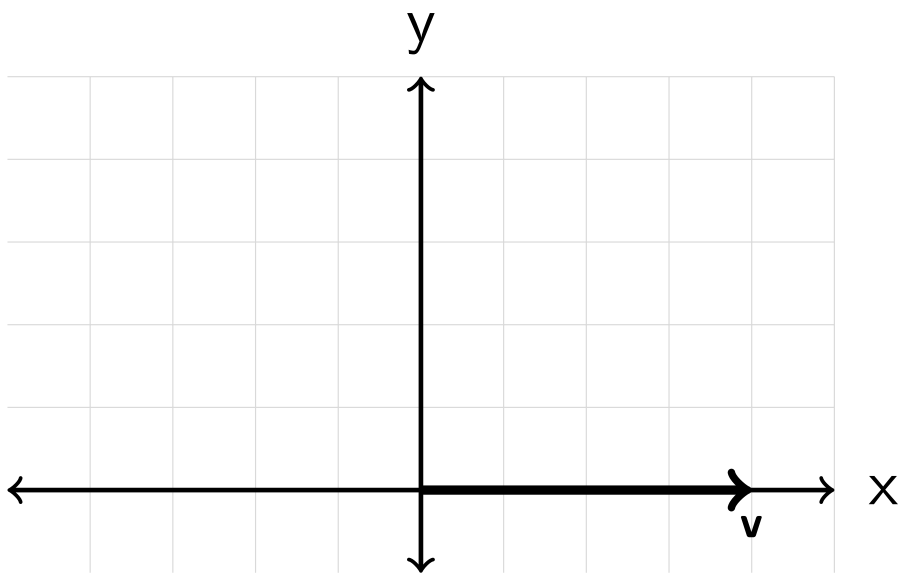

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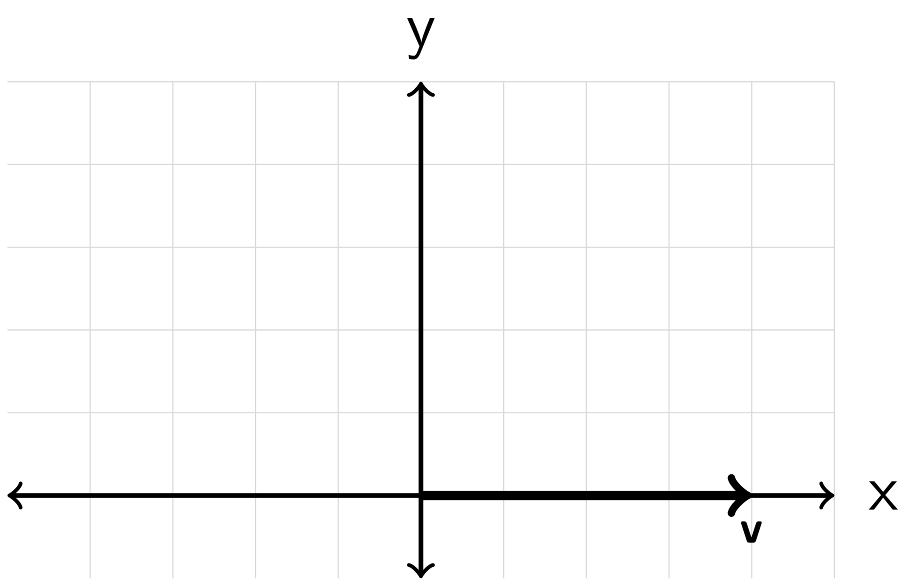

$$\|\mathbf{v}\|_2 = \left\| \binom{2}{0} \right\|_2 = \sqrt{2^2 + 0^2} = \sqrt{4} = 2$$

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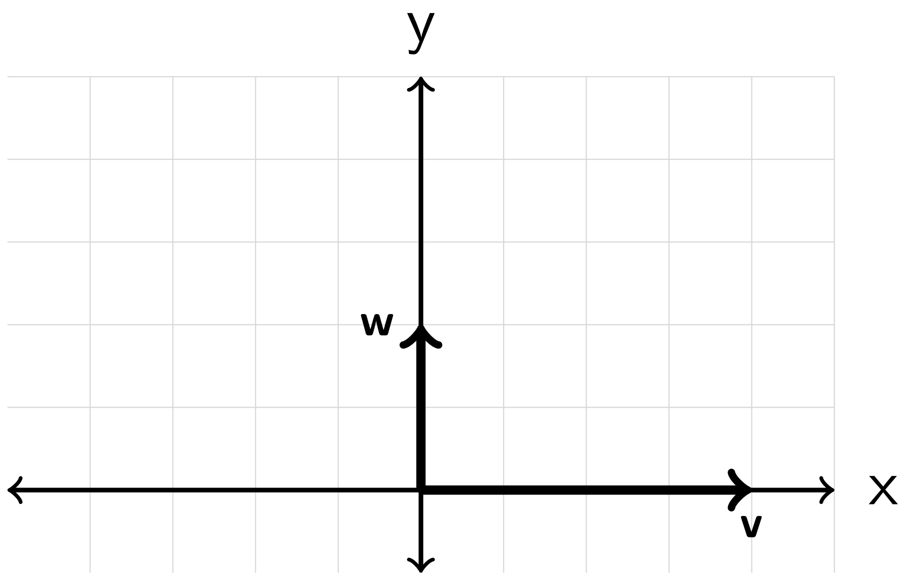

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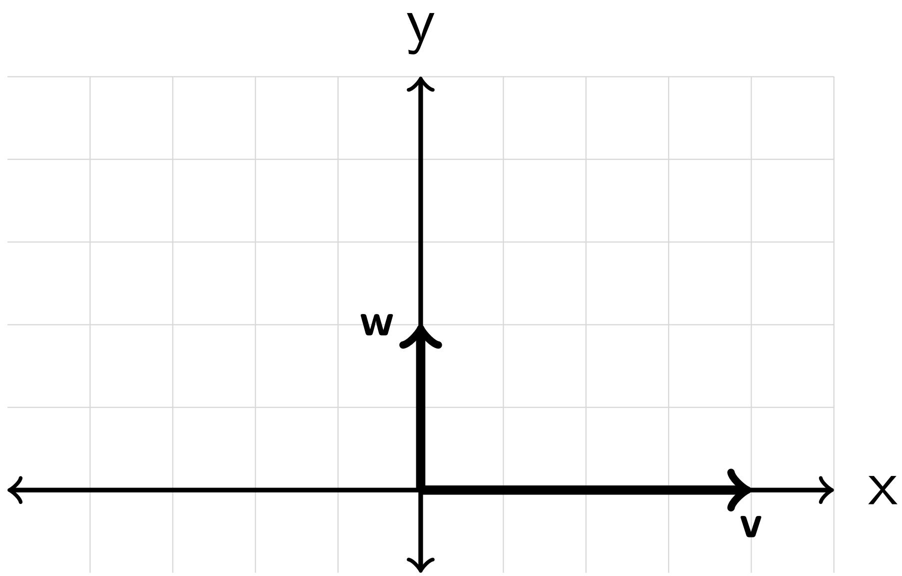

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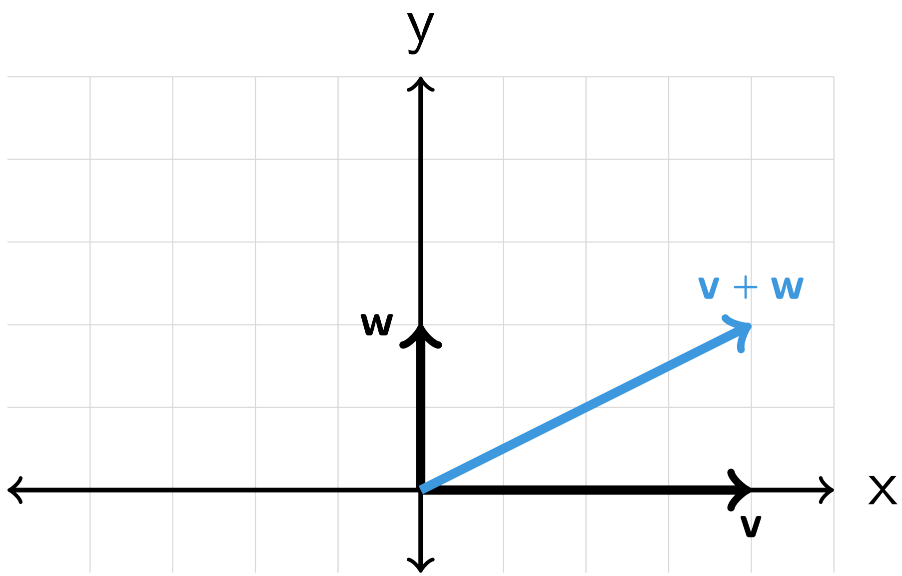

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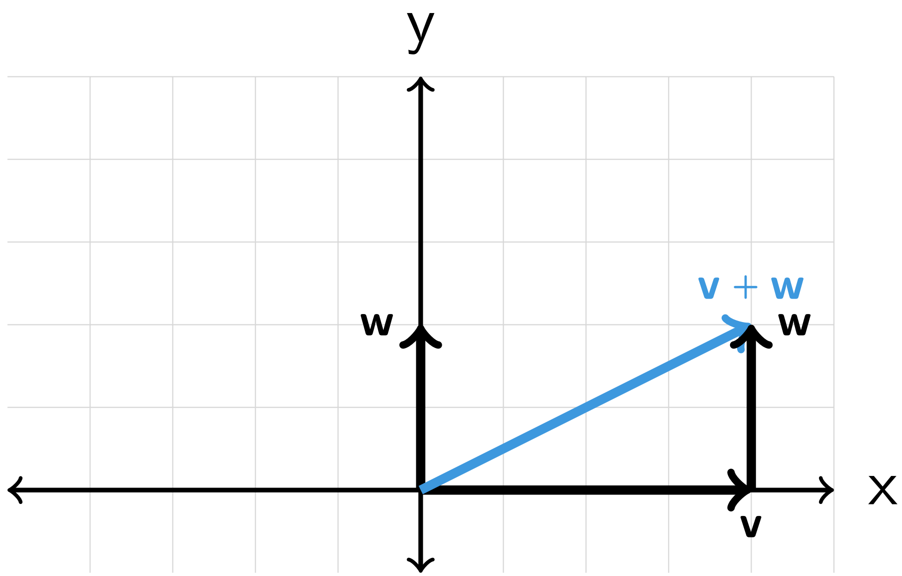

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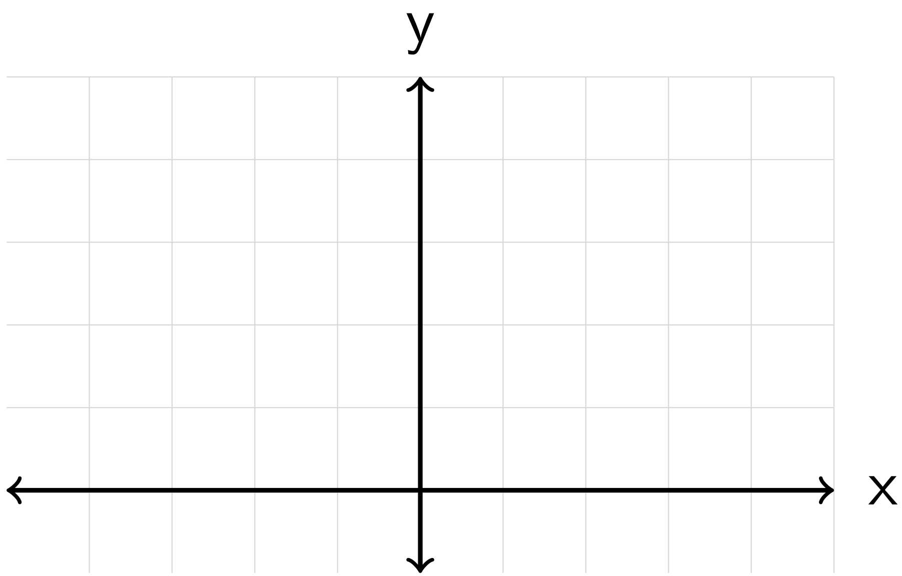
$$\|\mathbf{v} + \mathbf{w}\|_2 = \left\|\binom{2}{1}\right\|_2 = \sqrt{2^2 + 1^2} = \sqrt{5}$$

Combination: Distance between points

`https://vevox.app/#/m/106717265`



Combination: Distance between points

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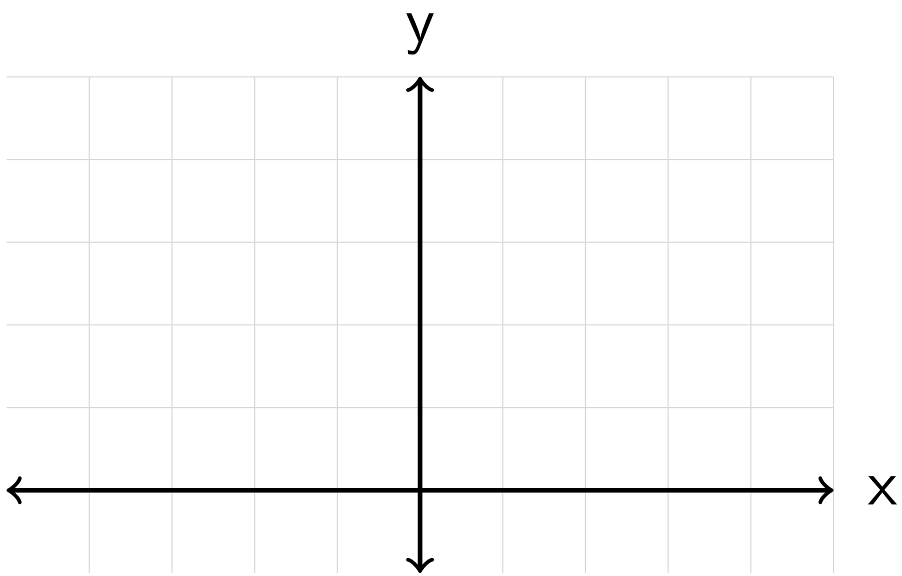
Having subtraction and length, can compute the distance between two points $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ as

$$\left\|\mathbf{v} - \mathbf{w}\right\|_2 = \left\|\mathbf{w} - \mathbf{v}\right\|_2 .$$



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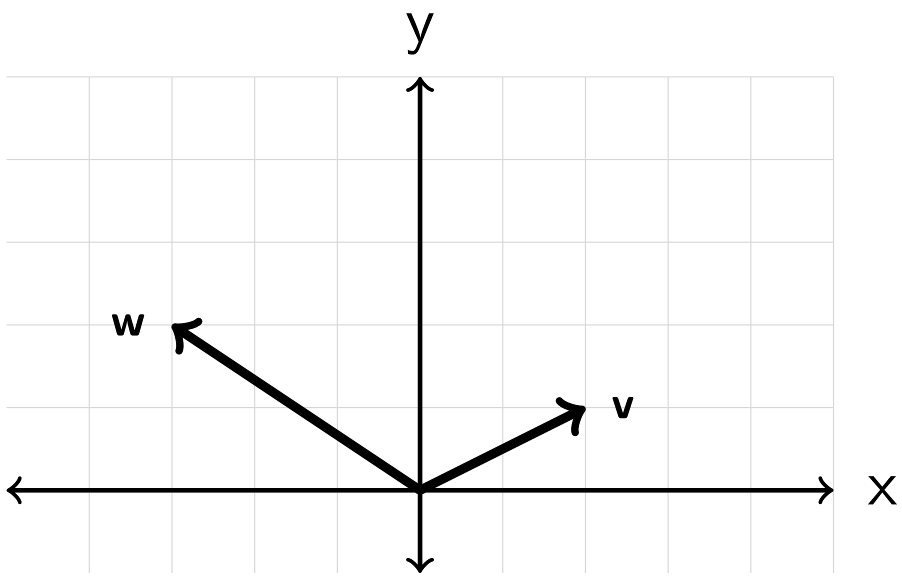

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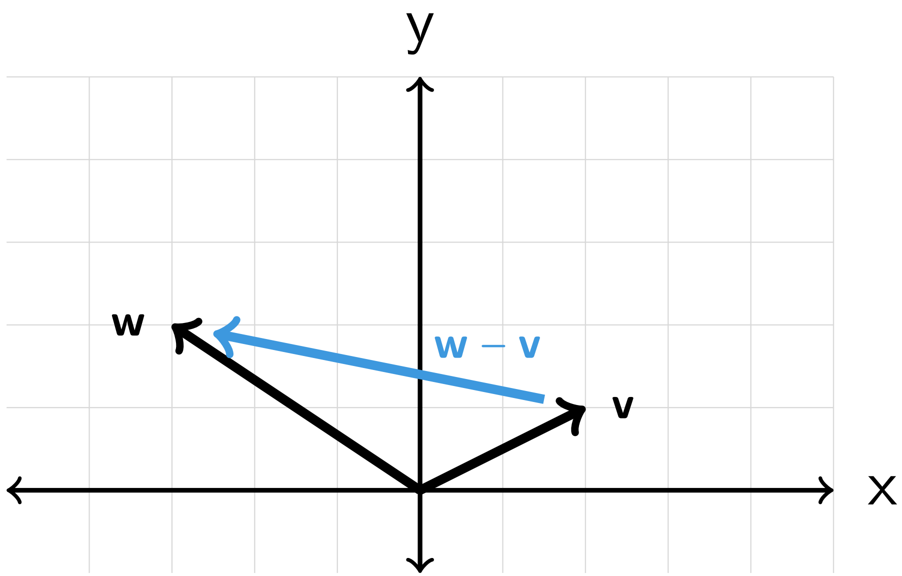

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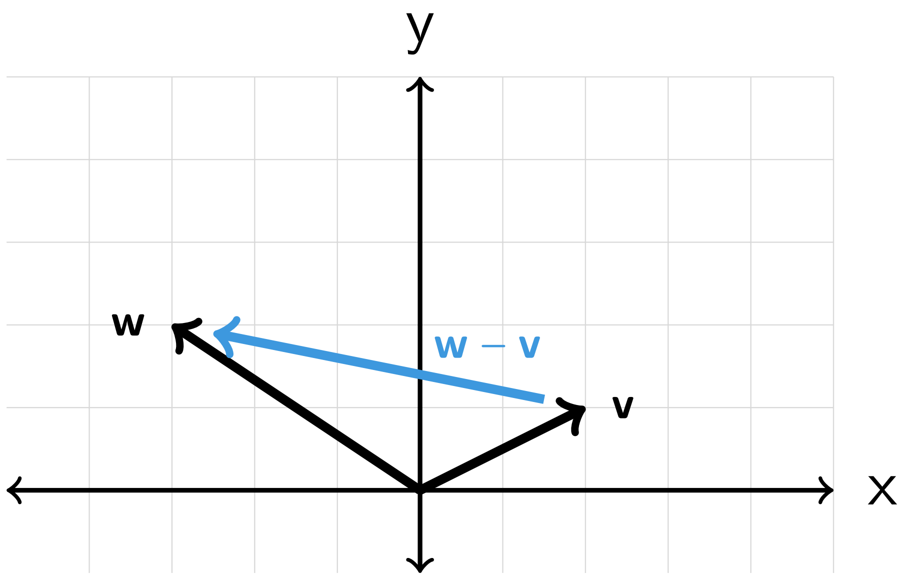

$$\|\mathbf{w} - \mathbf{v}\|_2 = \left\| \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} -\frac{5}{2} \\ \frac{1}{2} \end{pmatrix} \right\|_2$$

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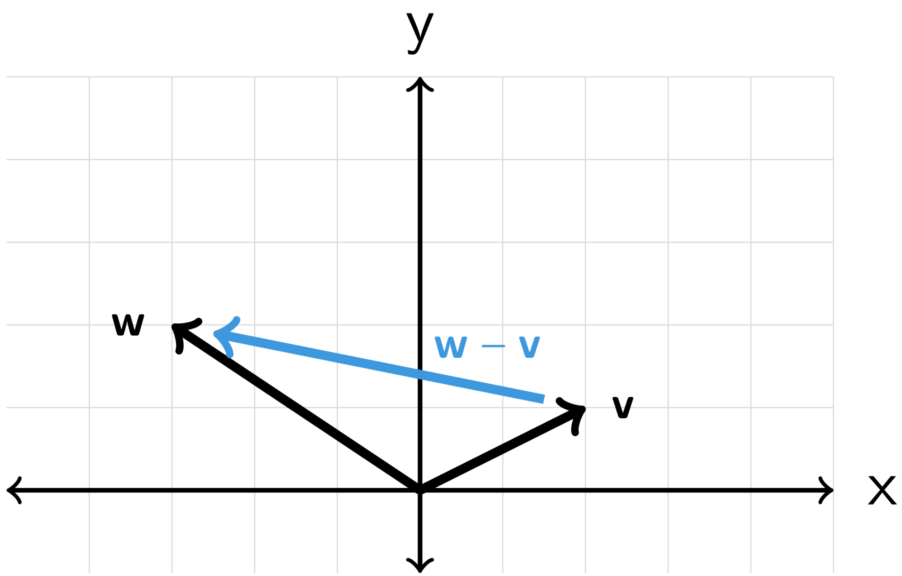

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$$= \sqrt{\left(-\frac{5}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{26}{4}}$$

$$\approx 2.54951$$

Dot Product

`https://vevox.app/#/m/106717265`

For vectors $\mathbf{v} = (v1, v_2, v_3)^t$ and $w = (w_1, w_2, w_3)^t$, the *dot product* is

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

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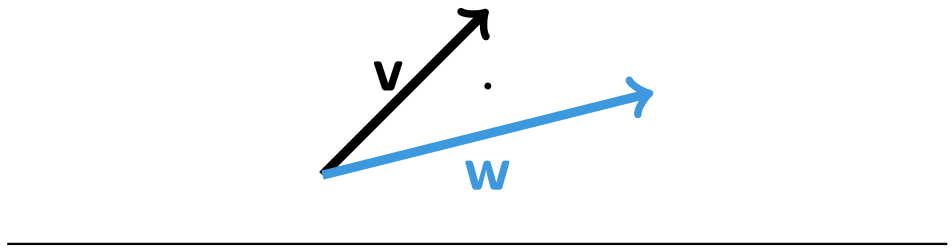

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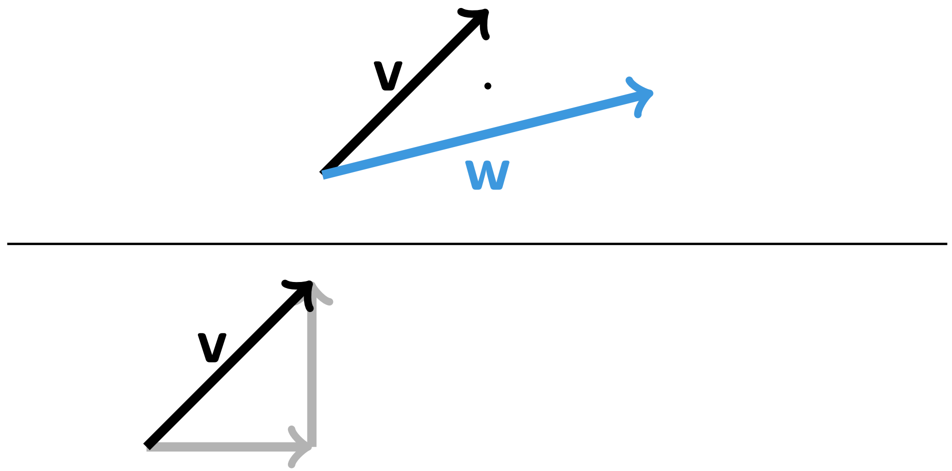

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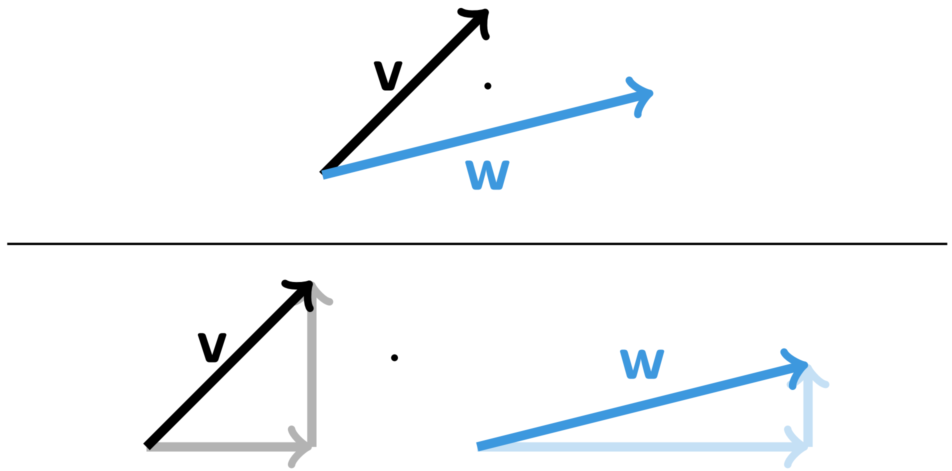

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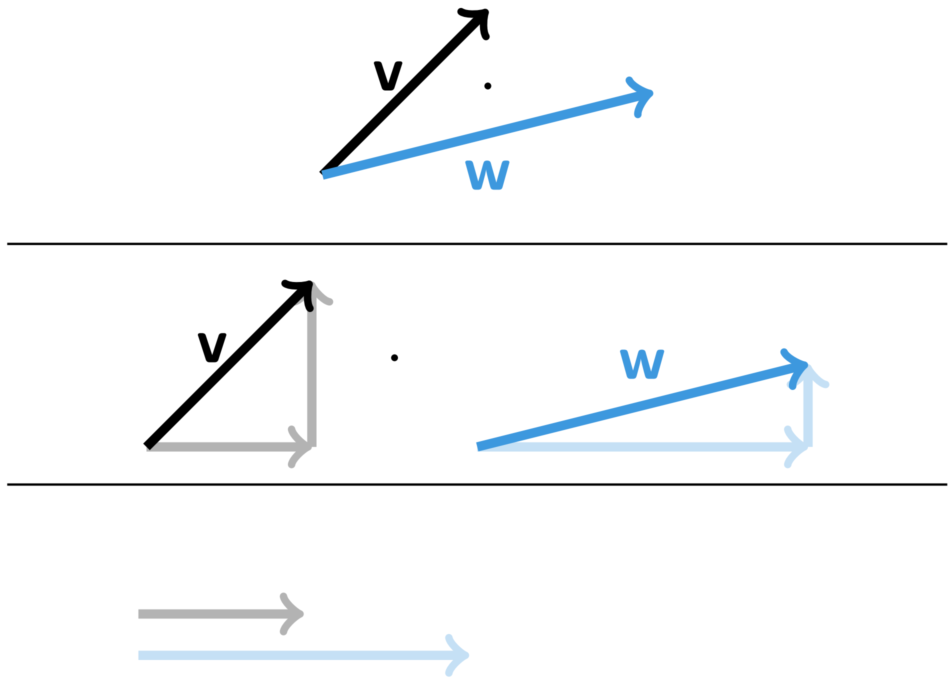

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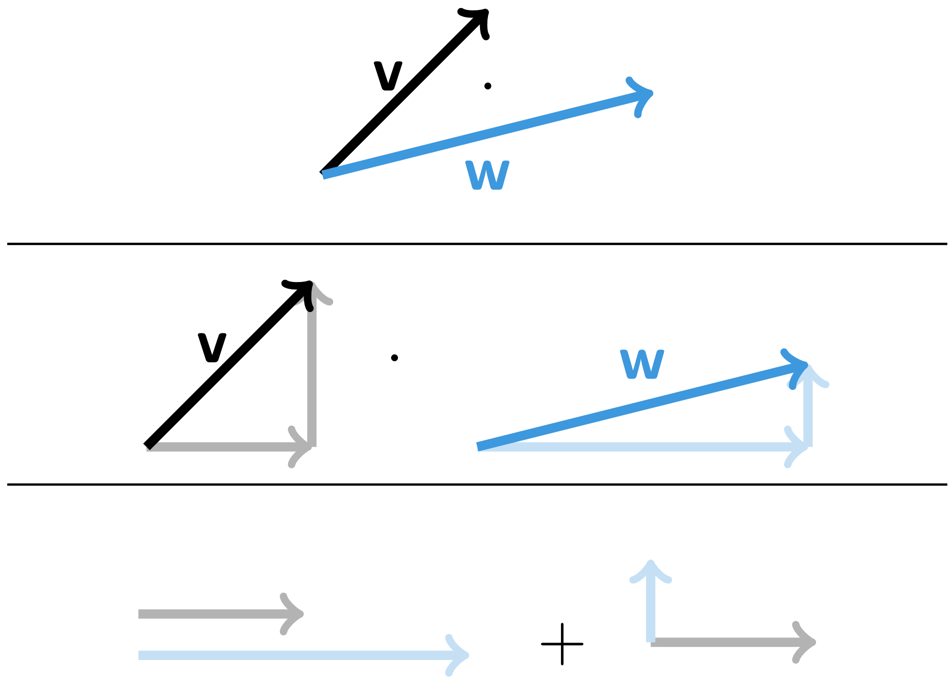

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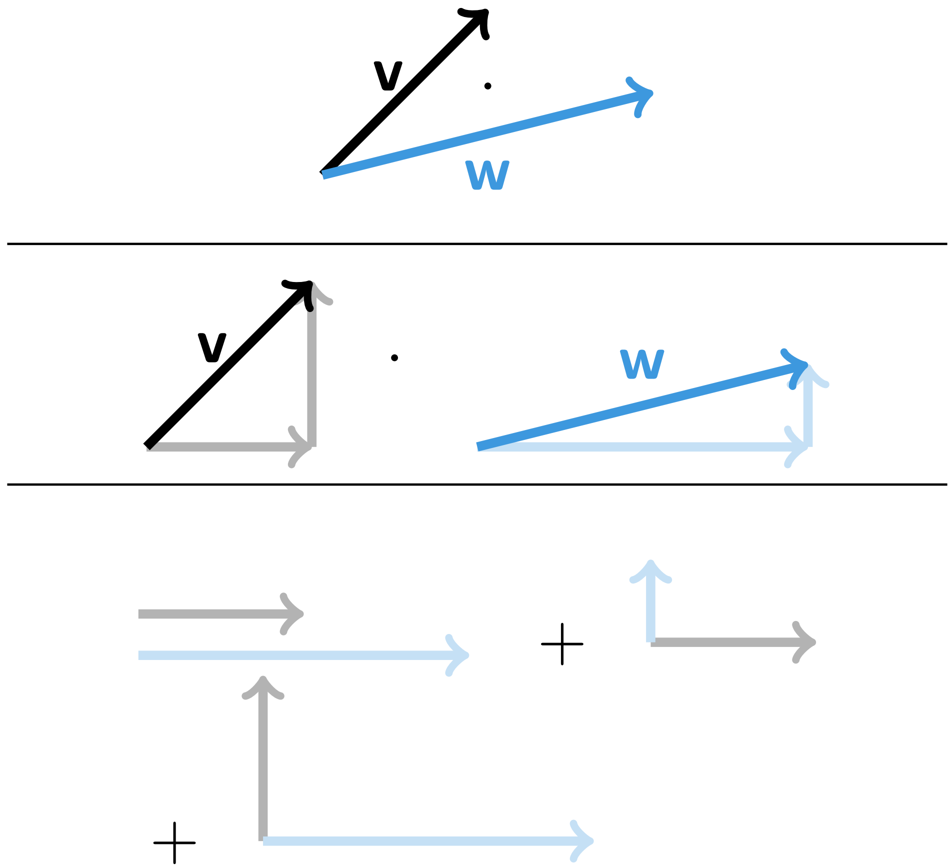

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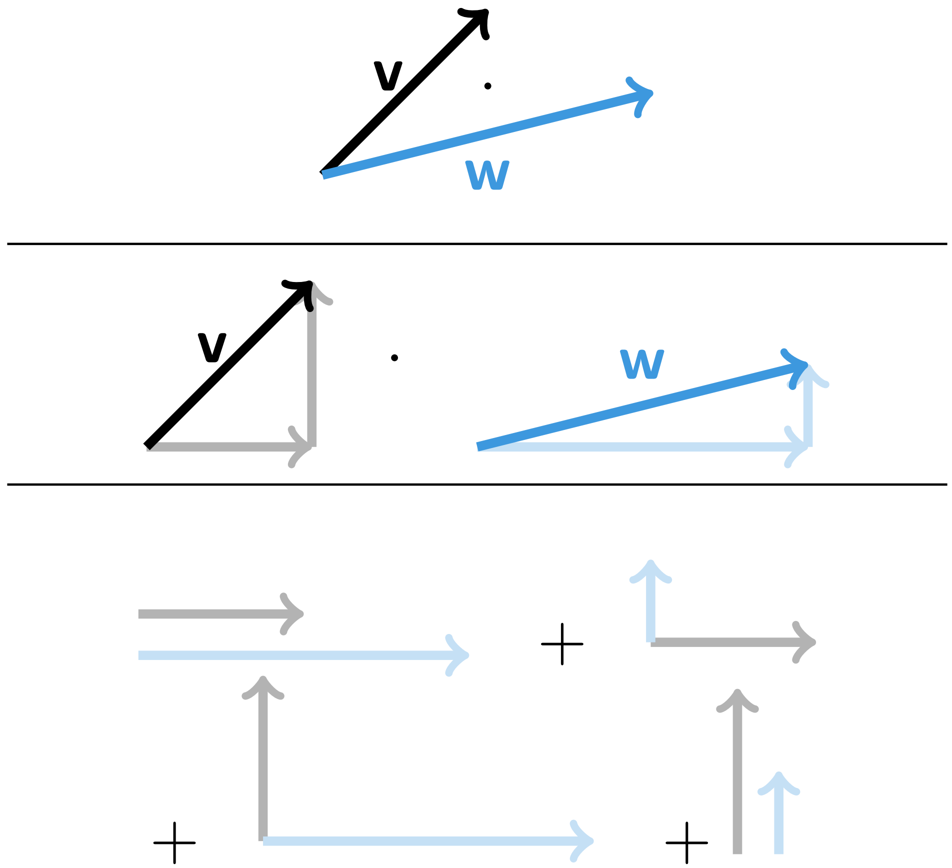

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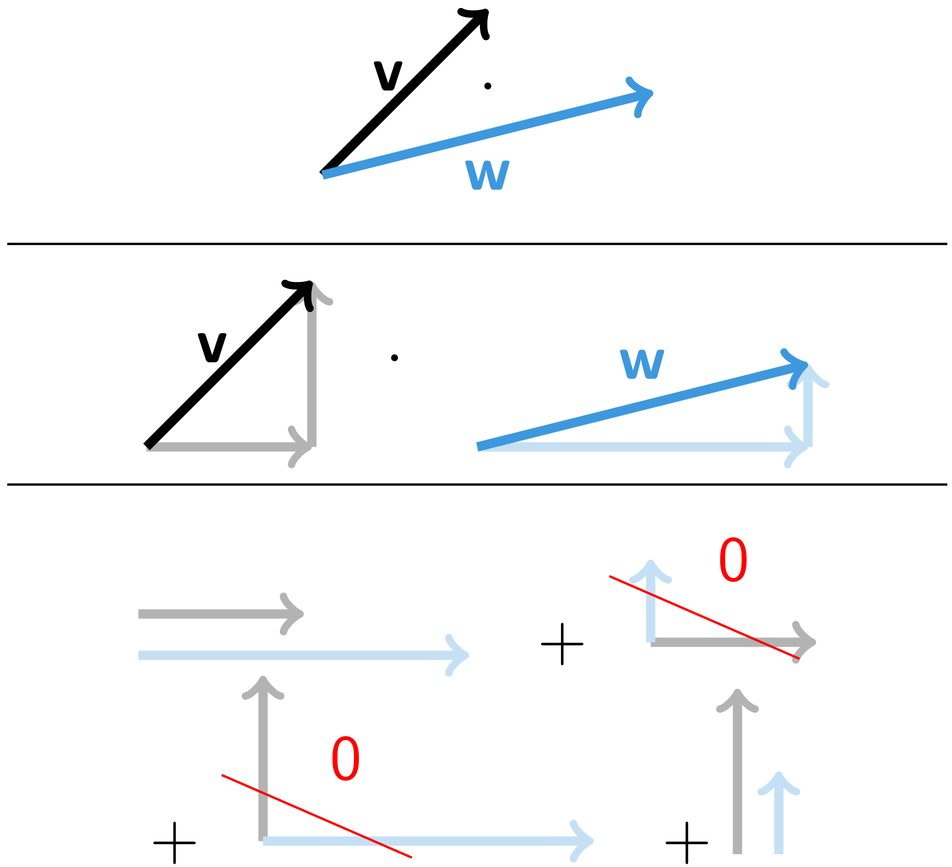

Angles between Vectors

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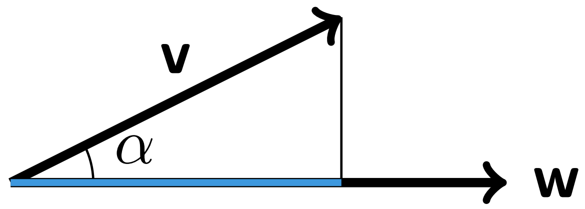

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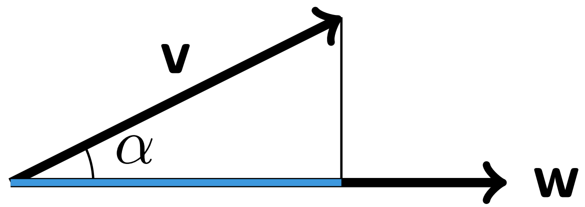

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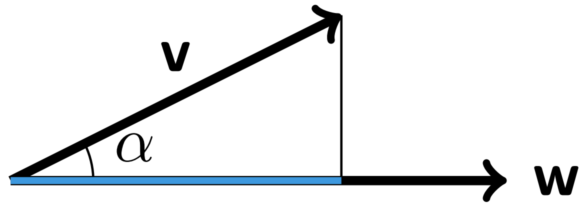

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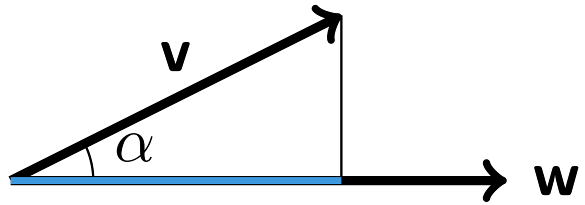

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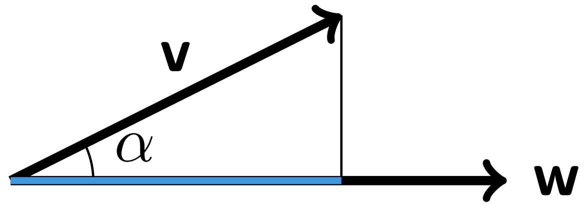
It's 0, providing a test for orthogonality.



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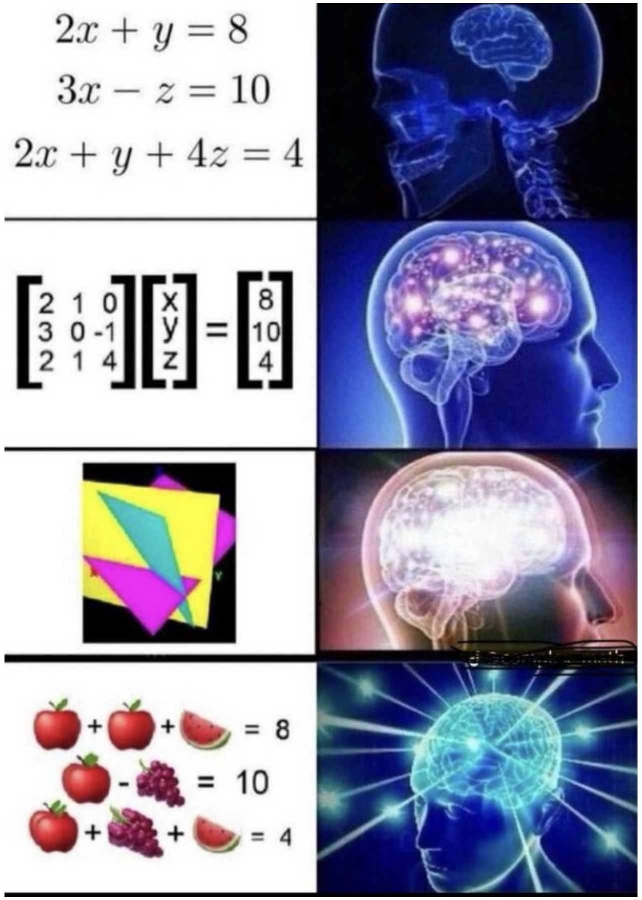

Cross product

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Given two vectors $\mathbf{a} = (a_1, a_2, a_3)^t$ and $\mathbf{b} = (b_1, b_2, b_3)^t$, their cross product is

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.$$

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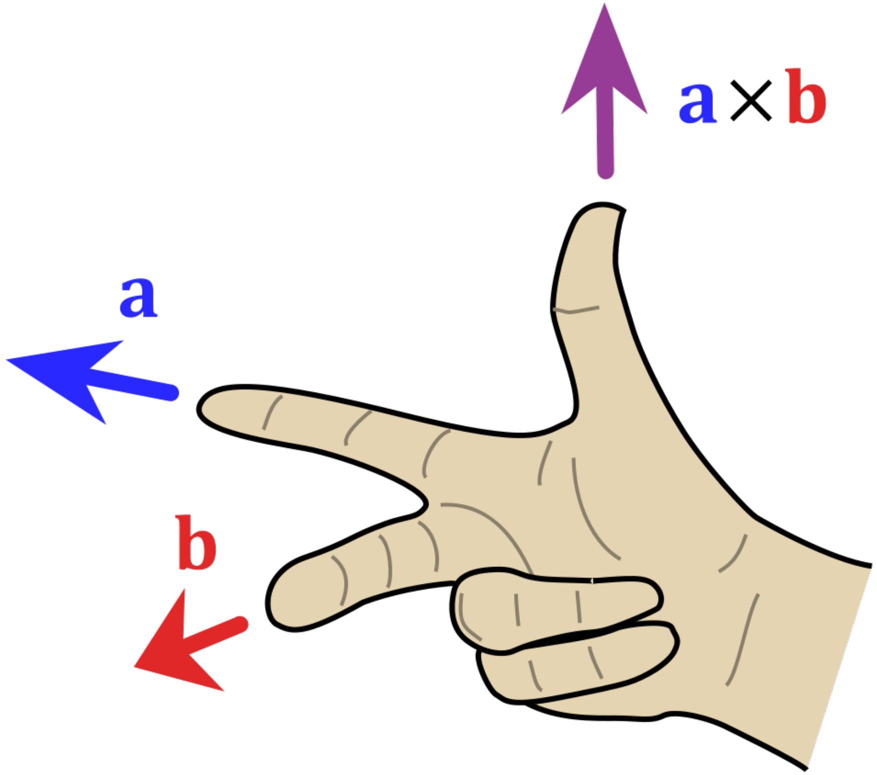

Figure: Right hand rule, Acdx on Wikipedia, (CC BY-SA 3.0).

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2	
3	
-1	
2	
3	

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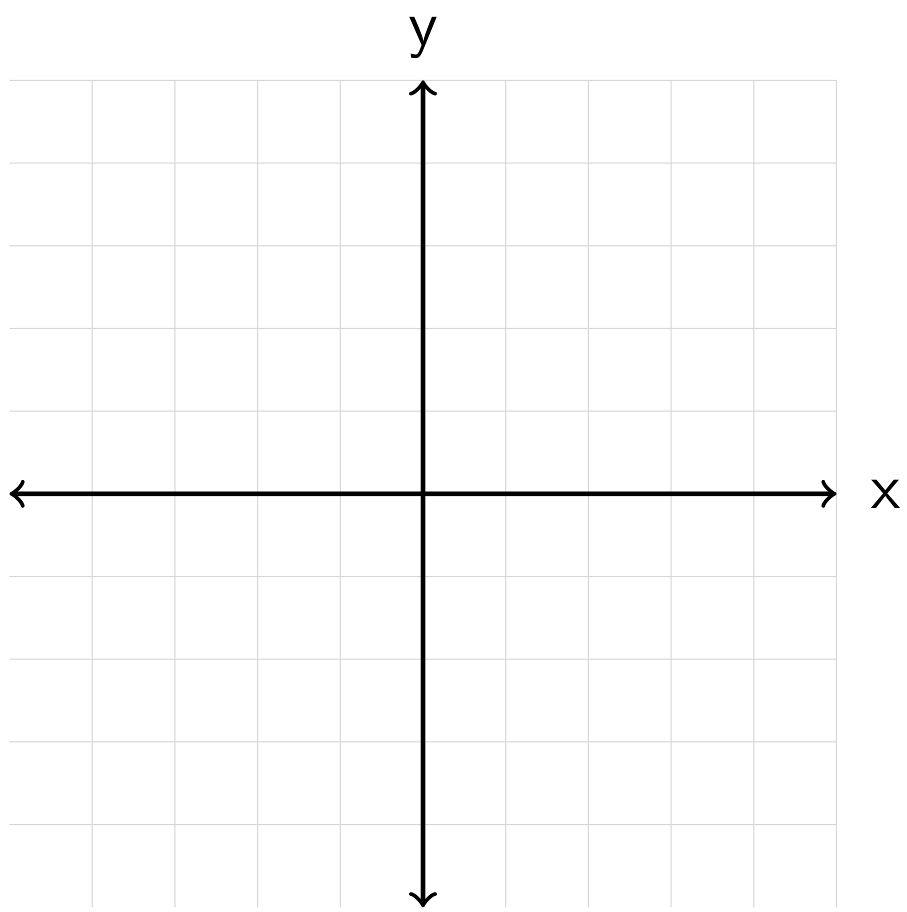

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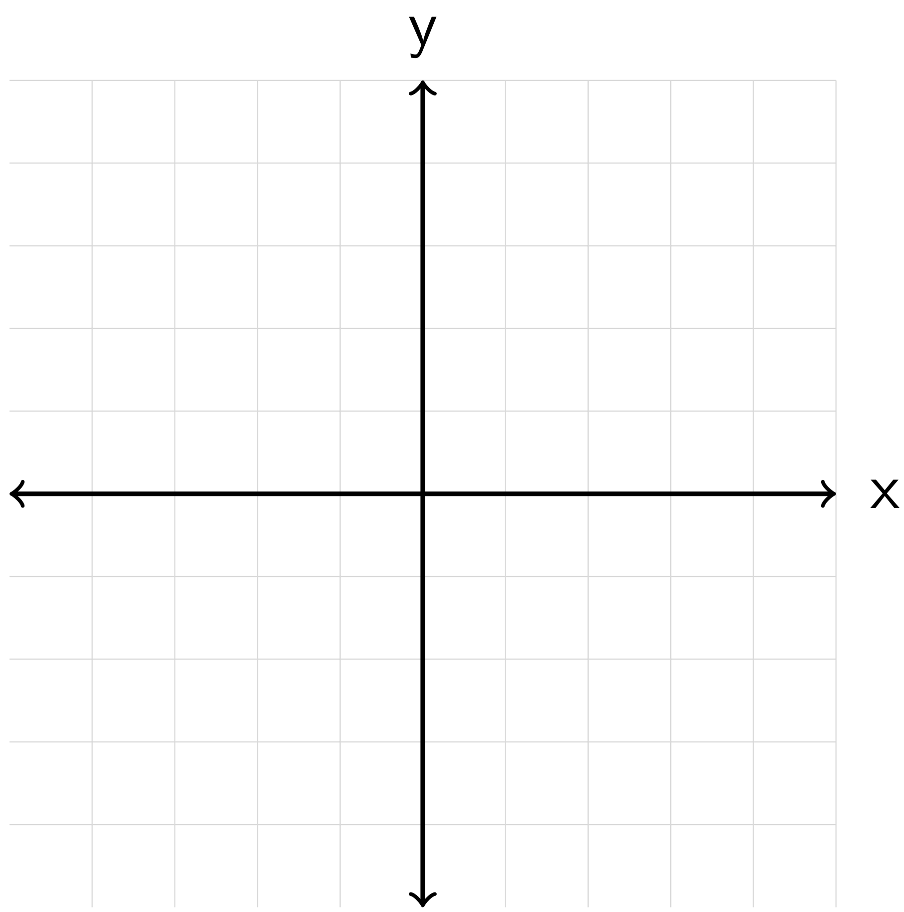

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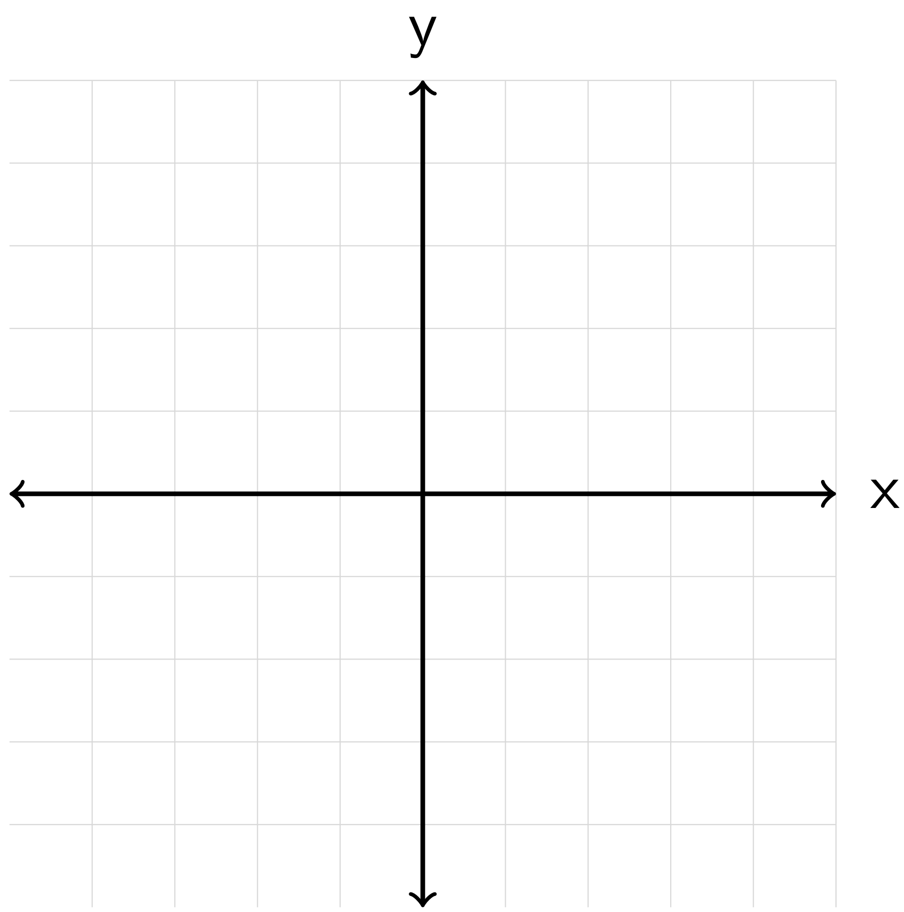

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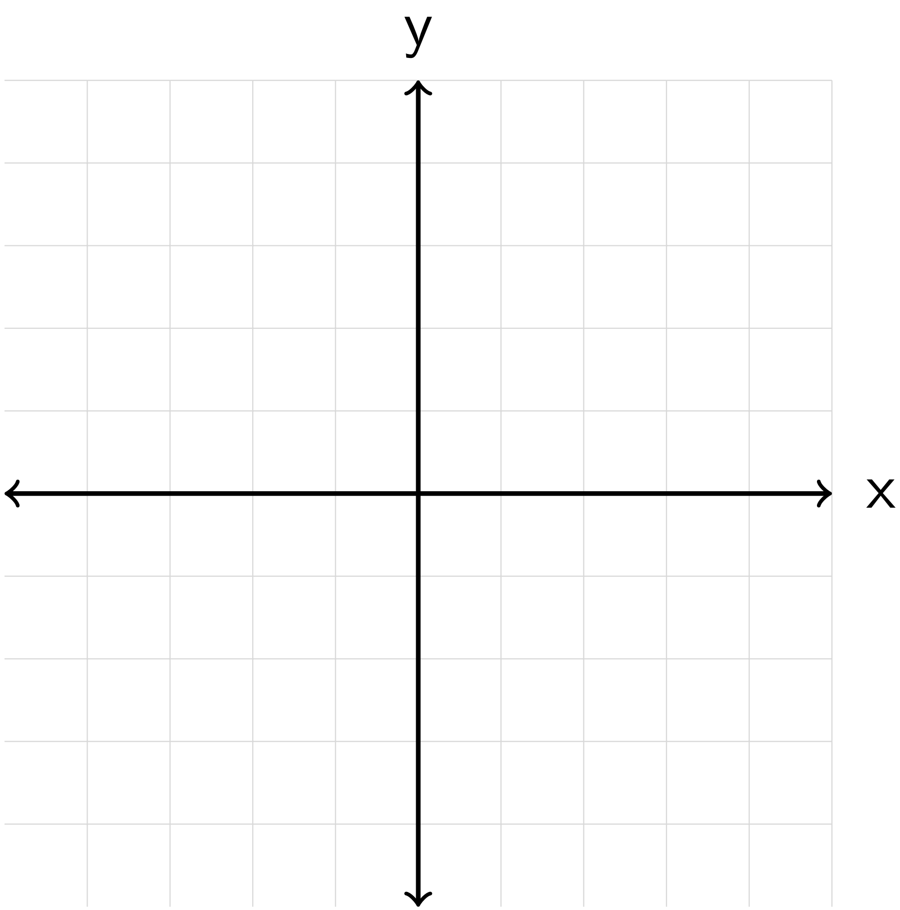

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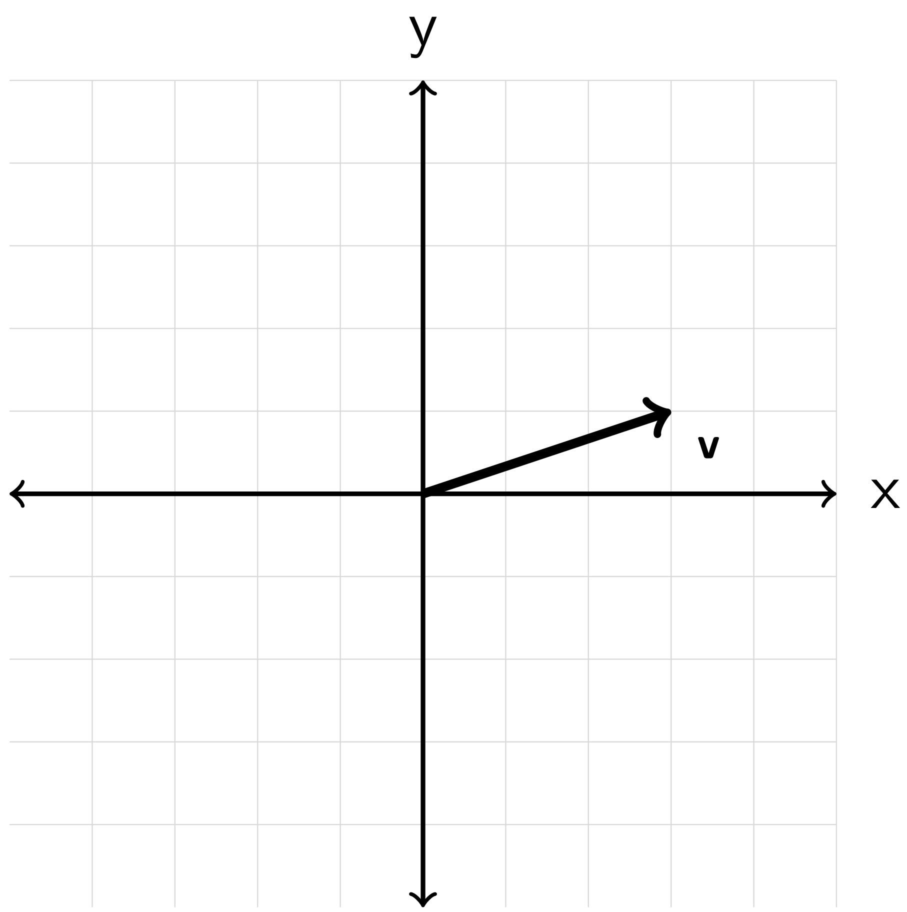

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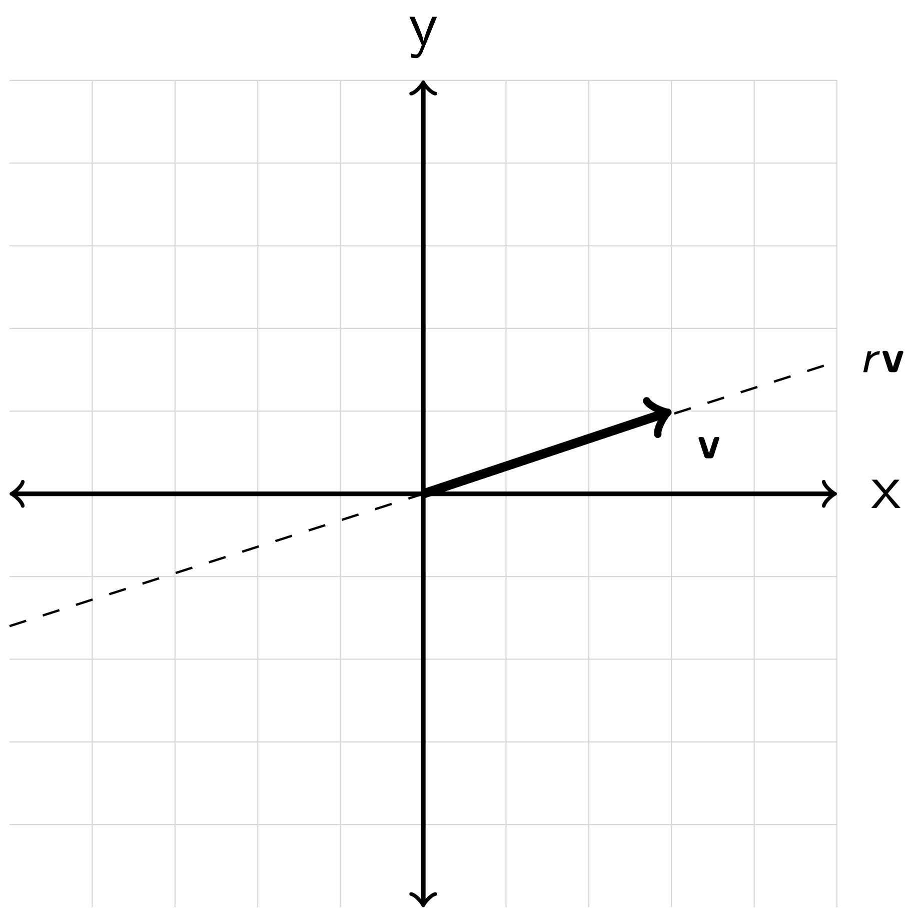

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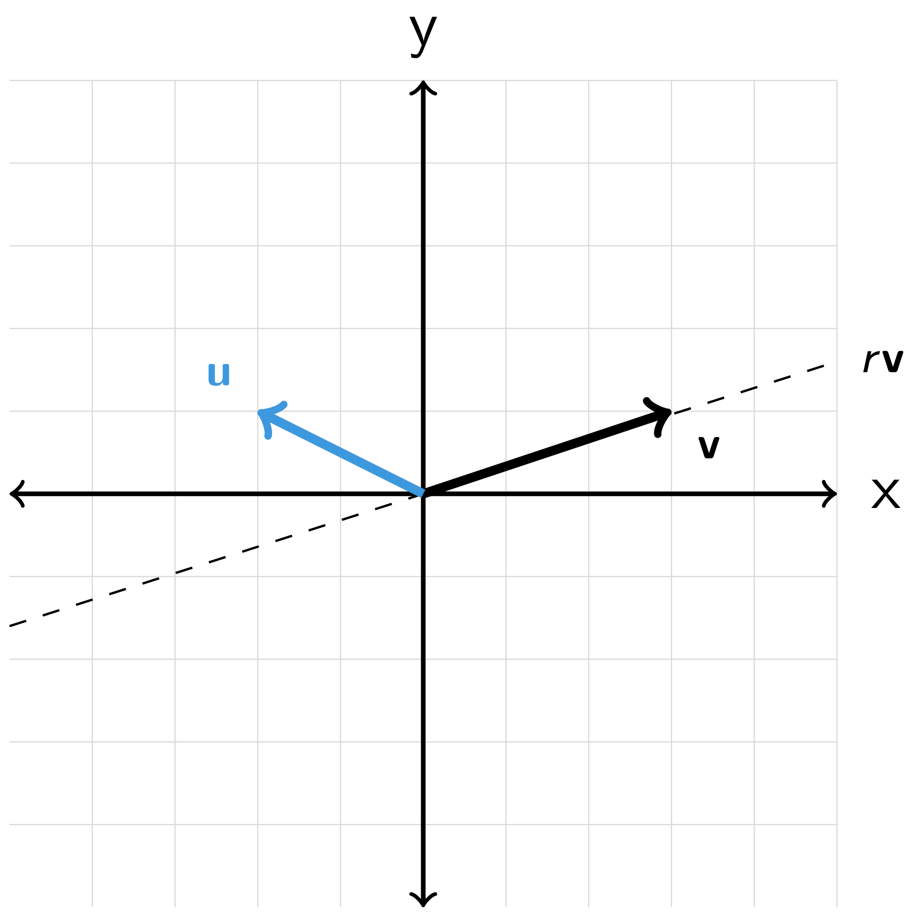
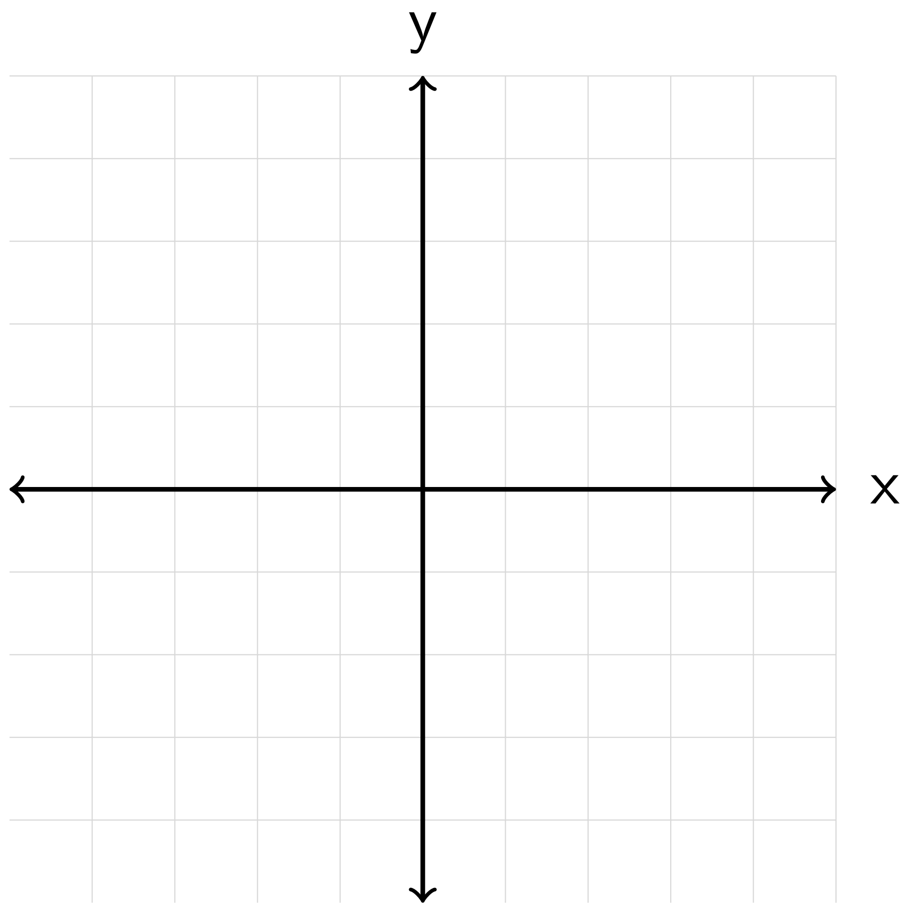


Finding orthogonal vectors



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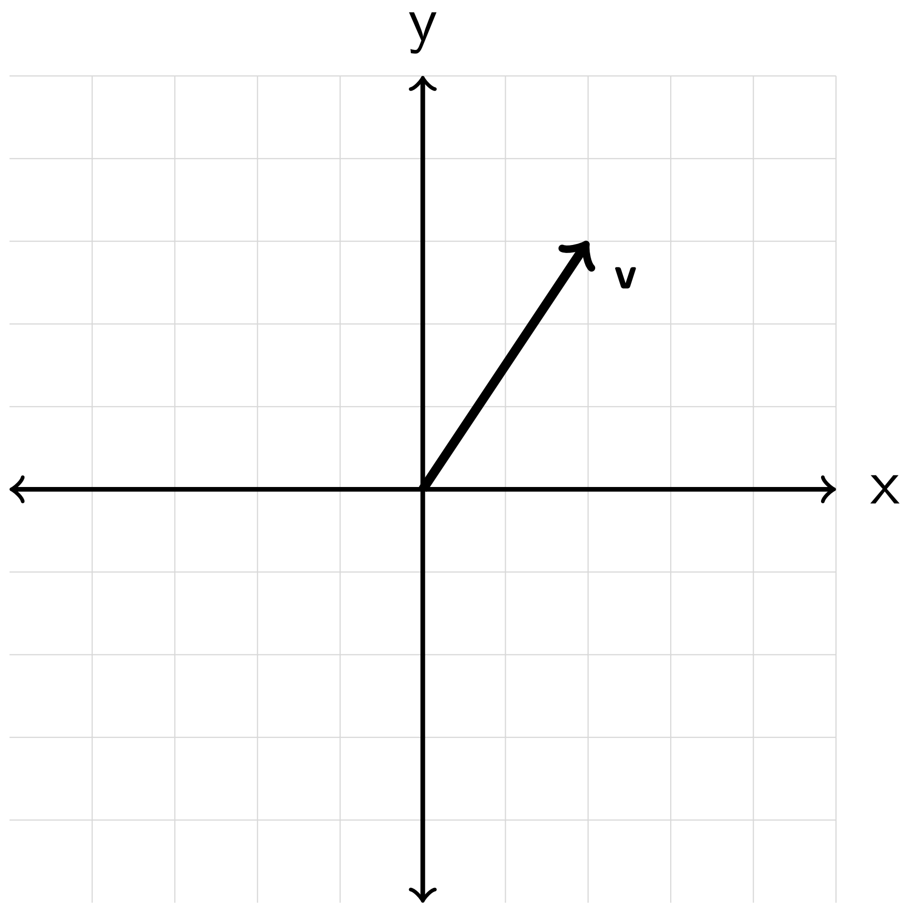

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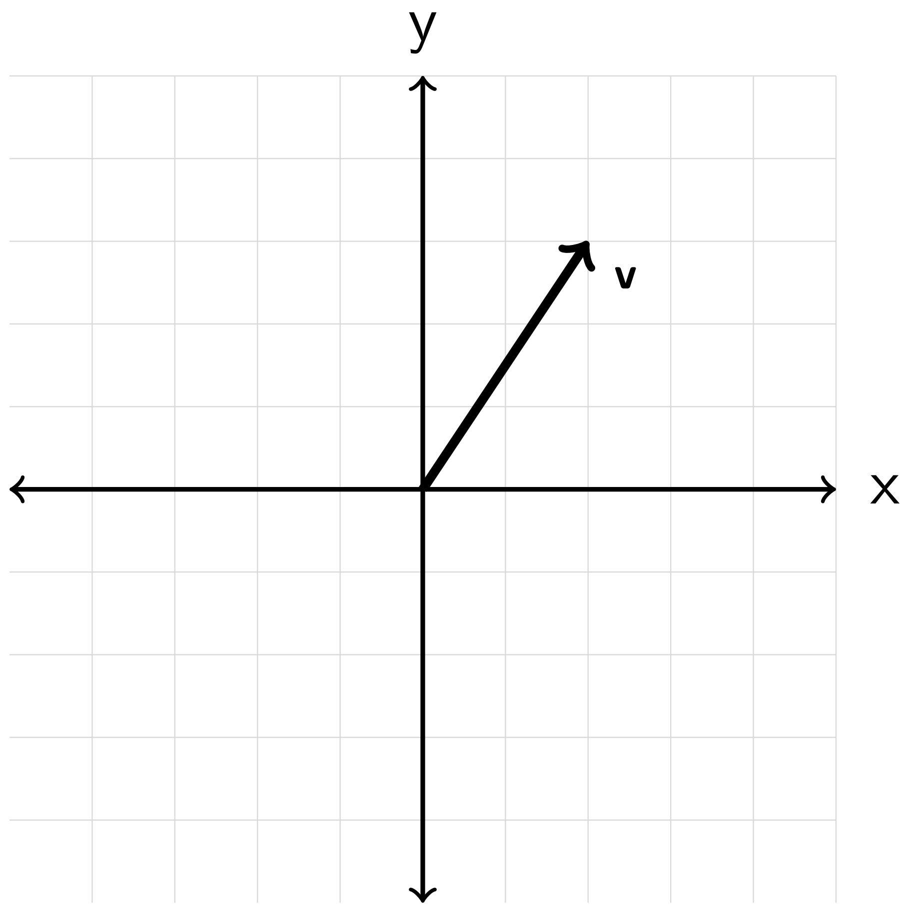

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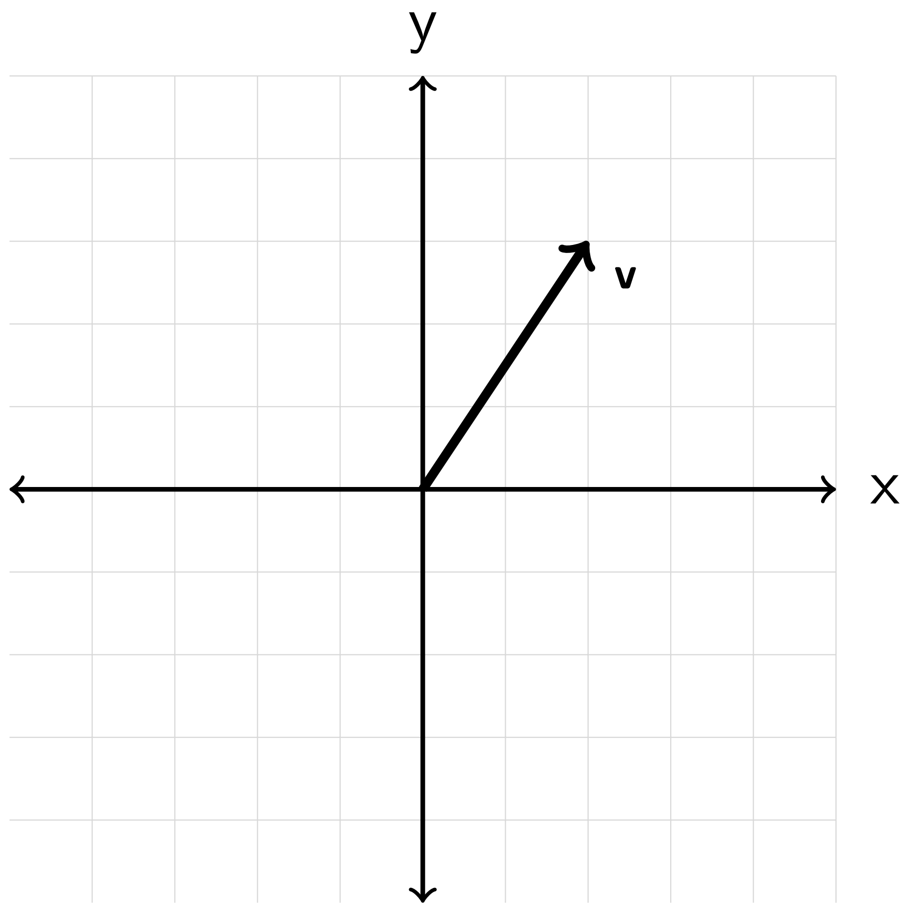

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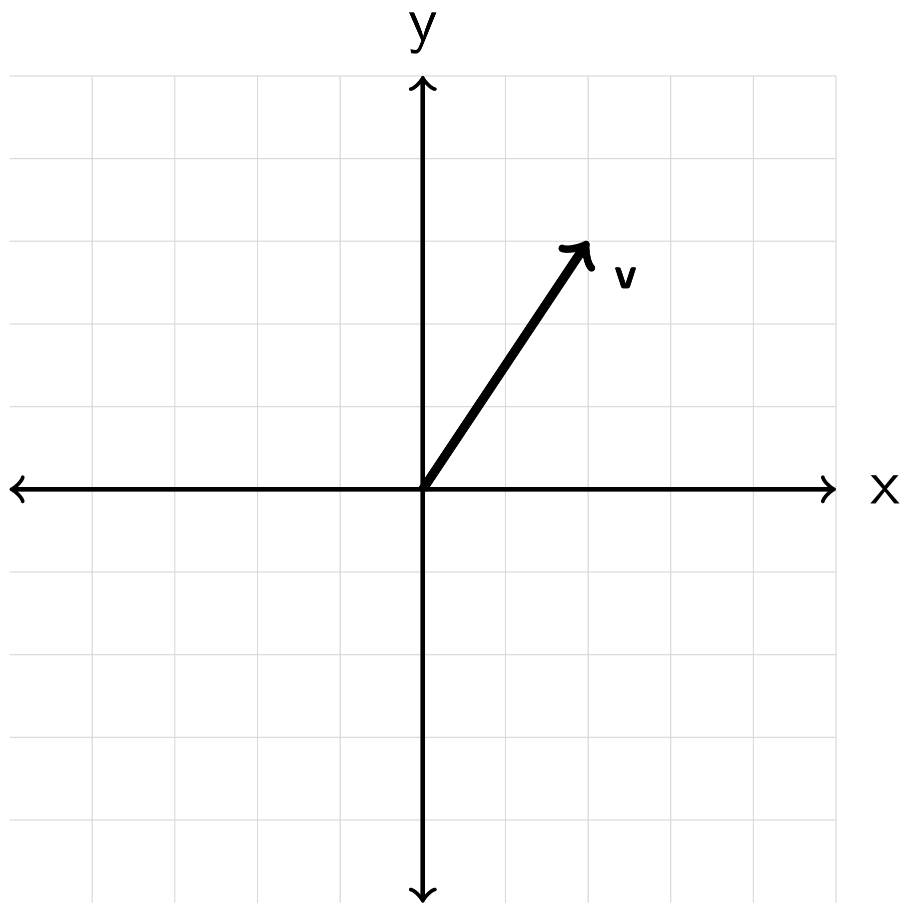
That is, all vectors of the form $\mathbf{w} = r\begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$ are orthogonal to $\mathbf{v}$.



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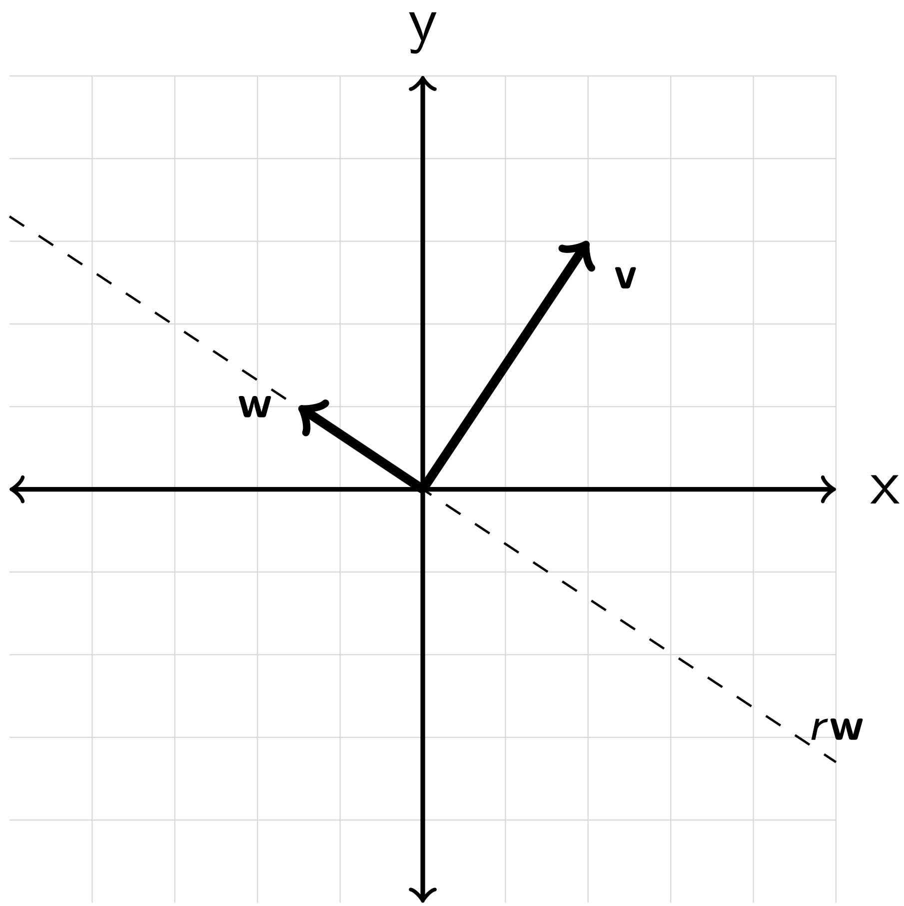

Basis

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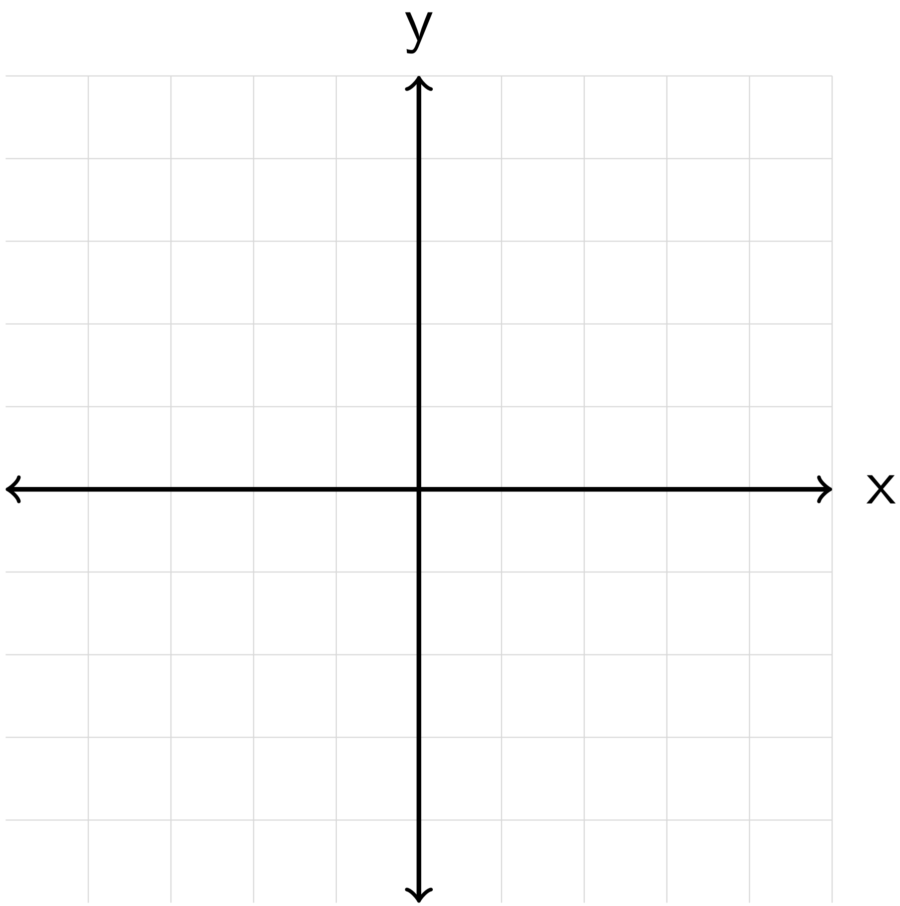

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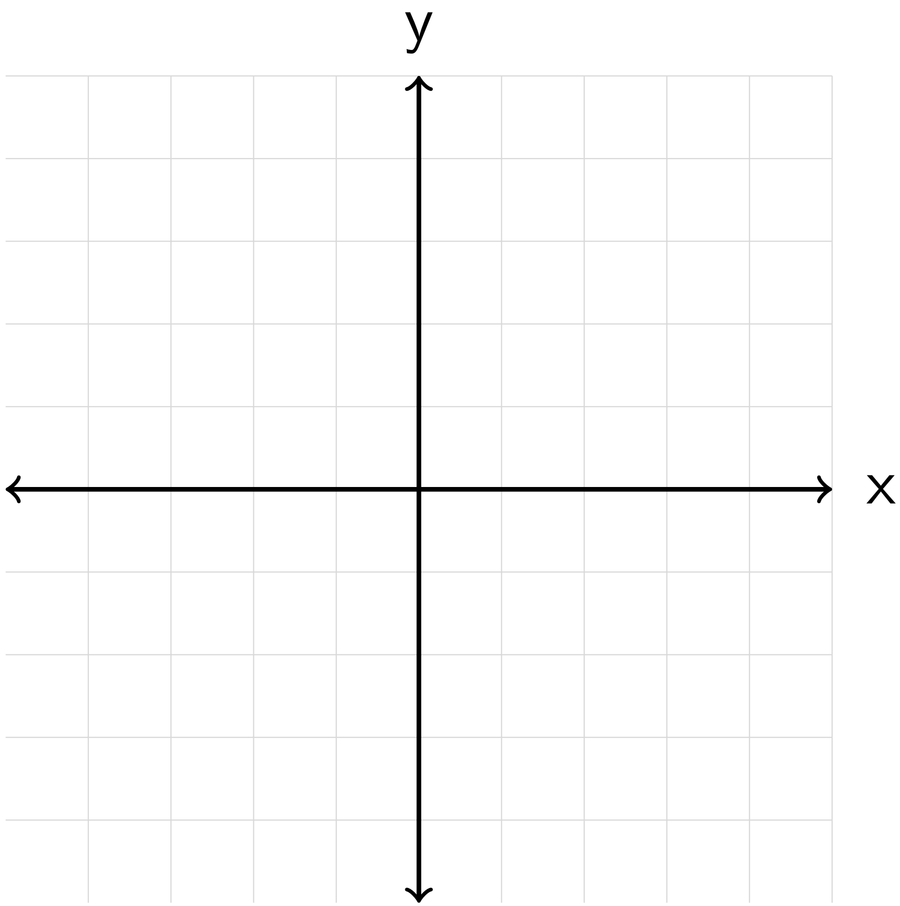

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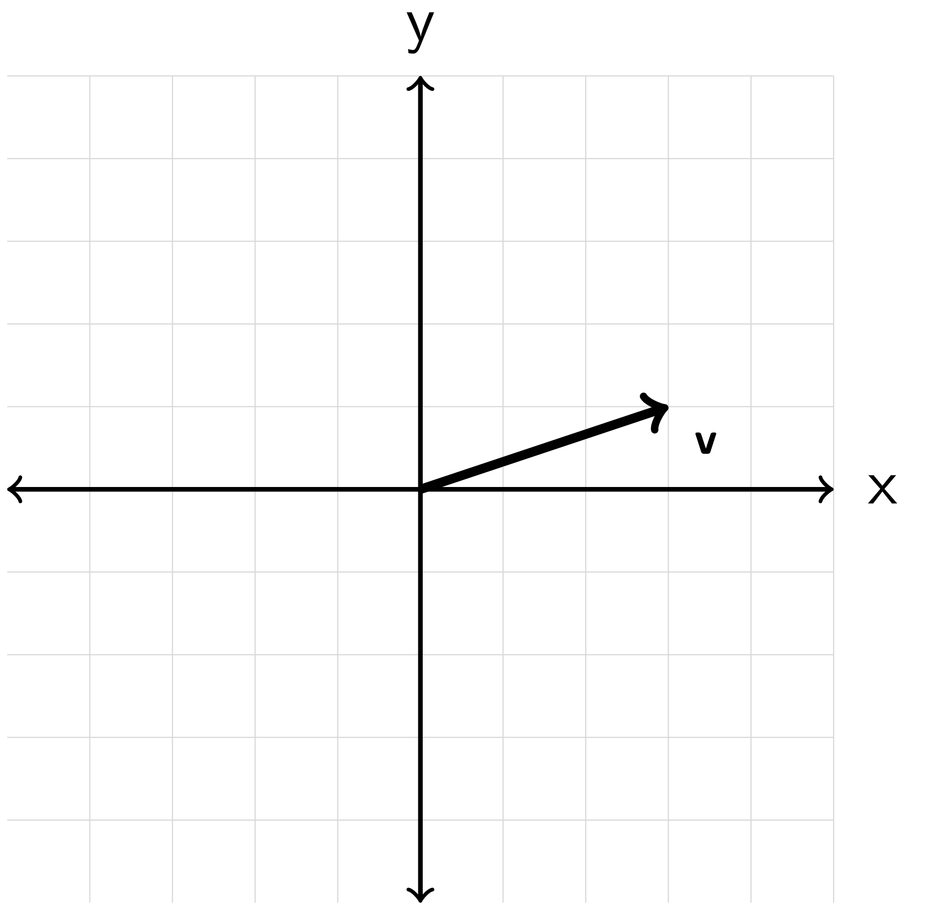

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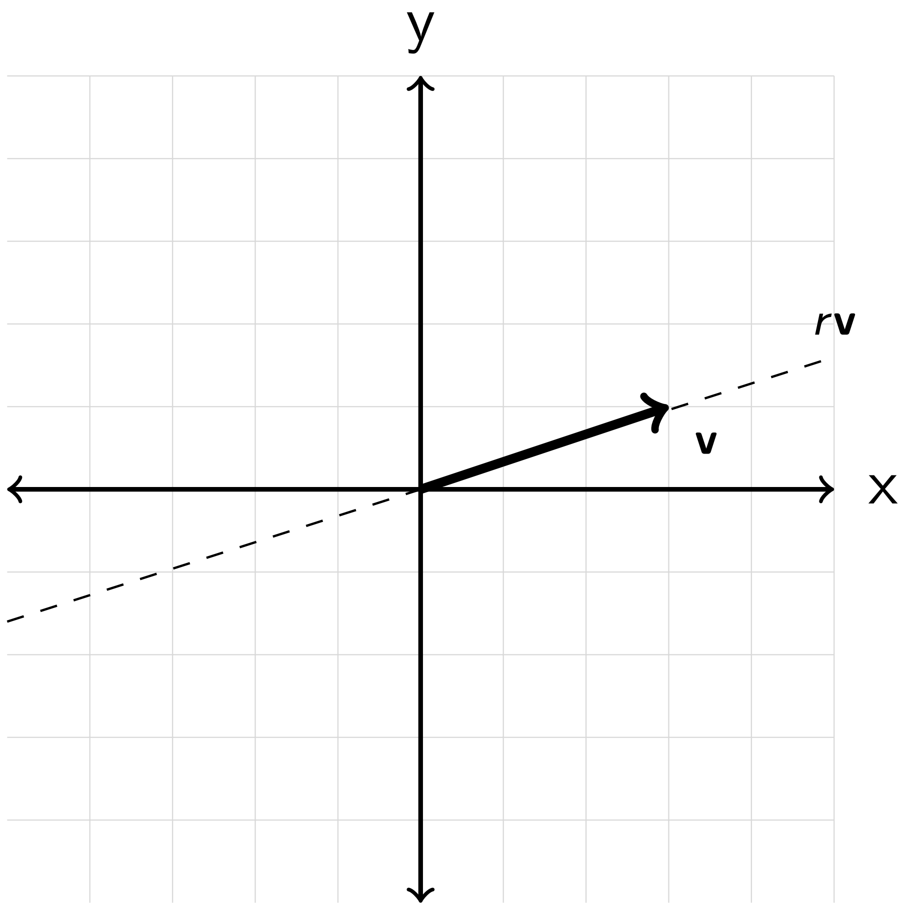

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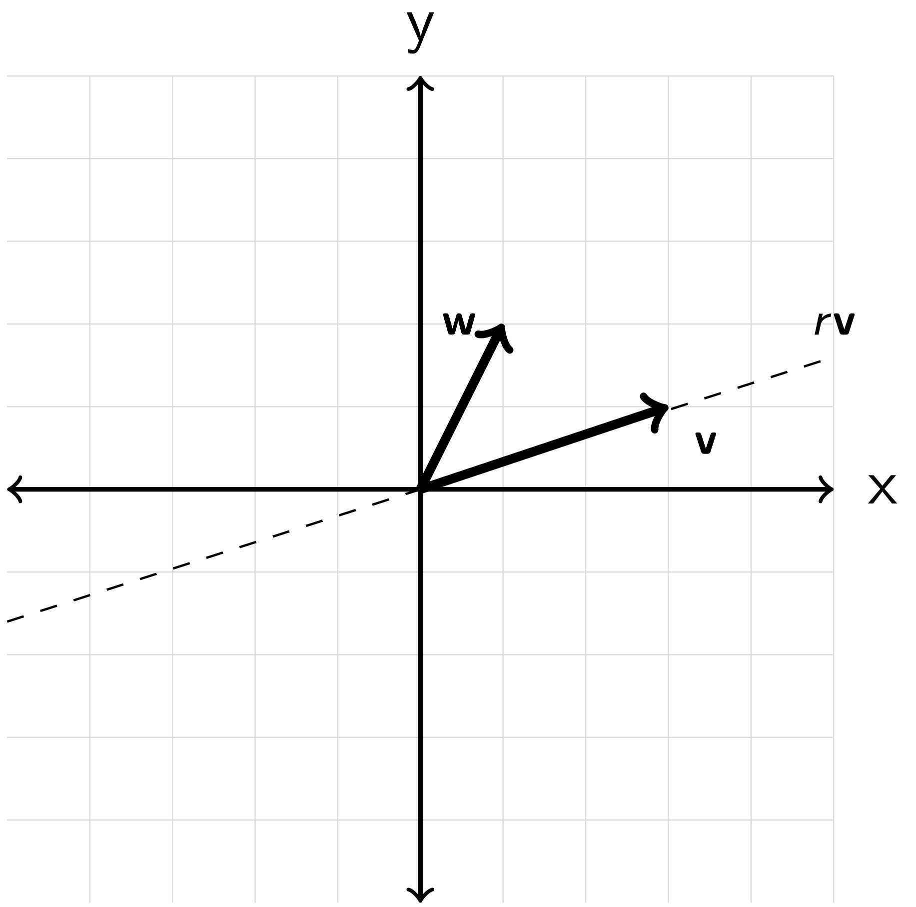

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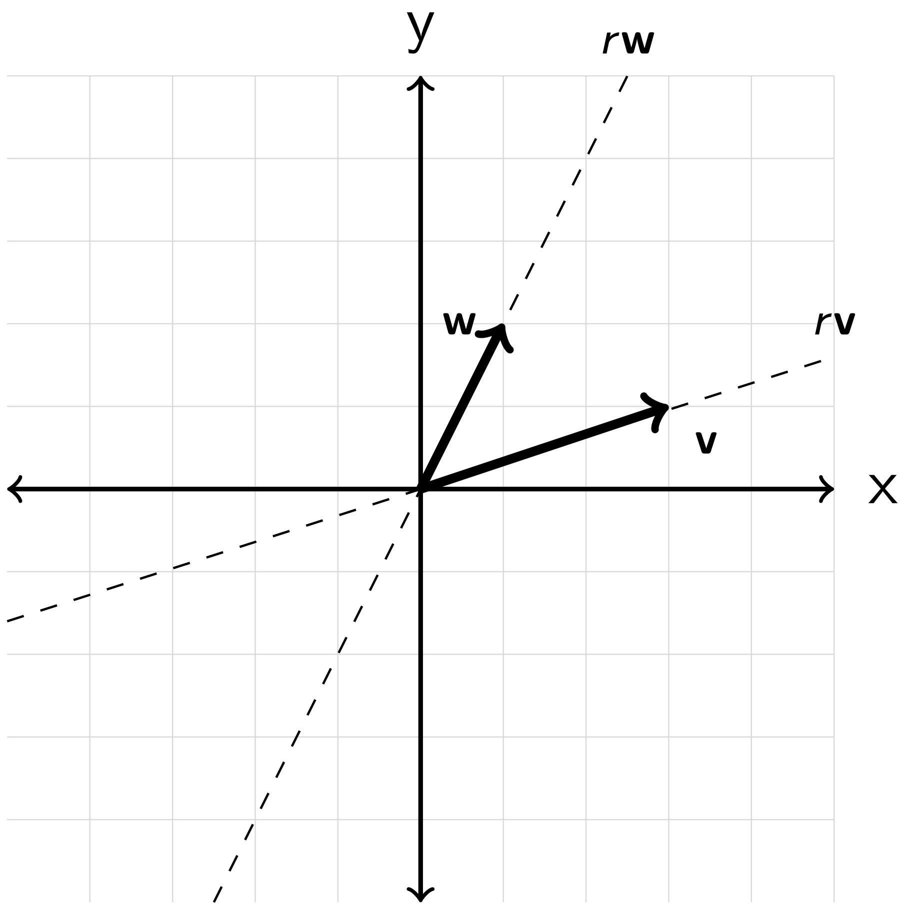

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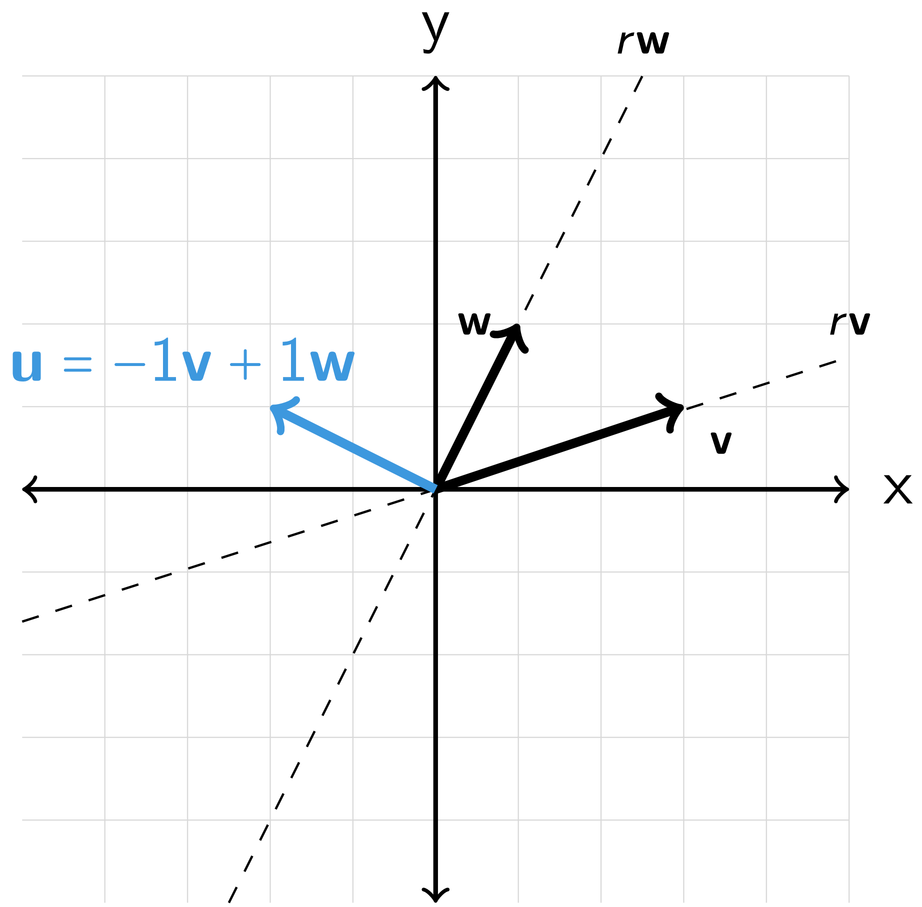

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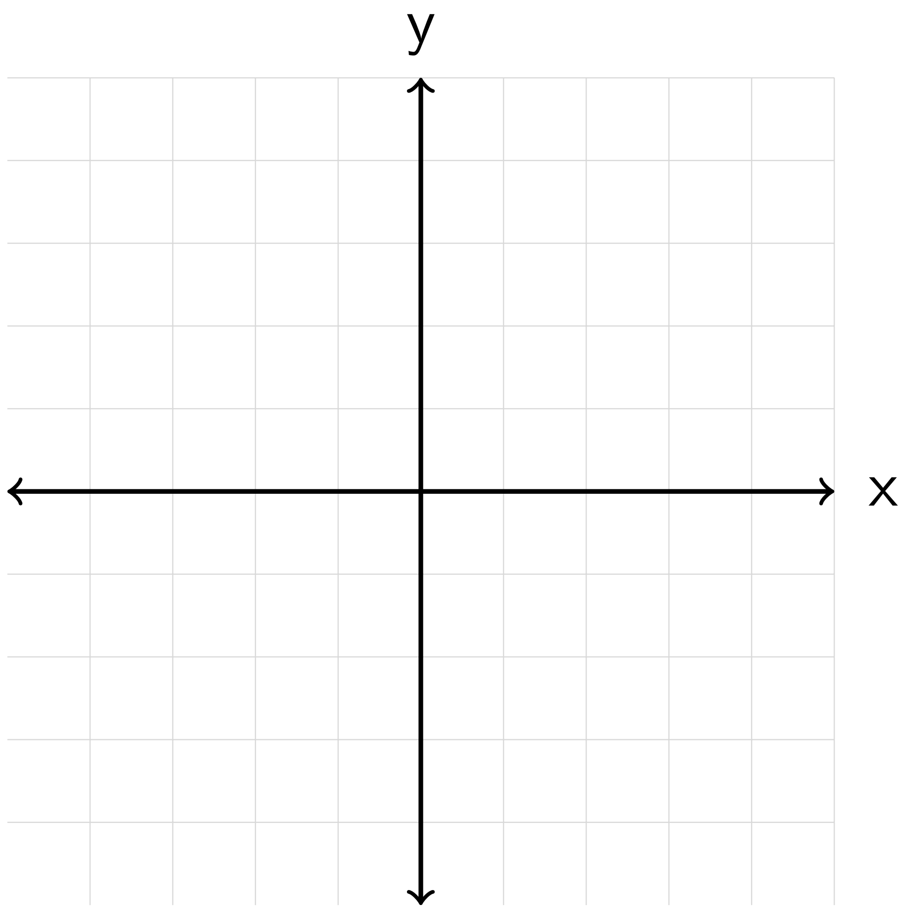

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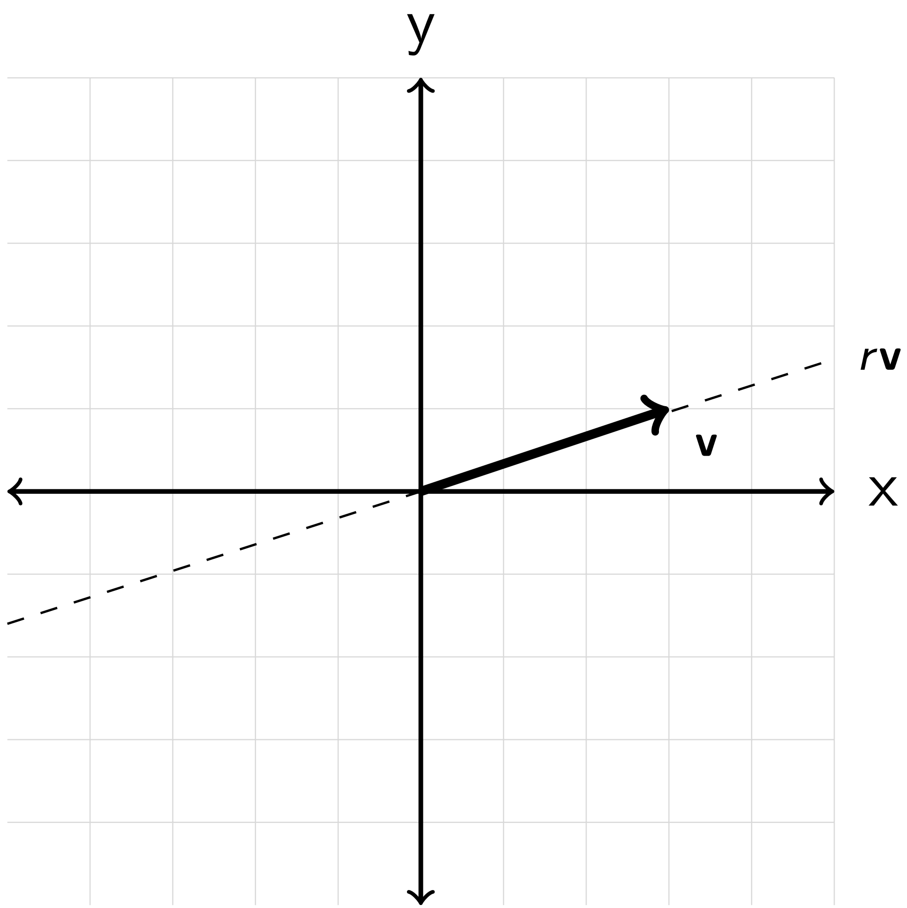

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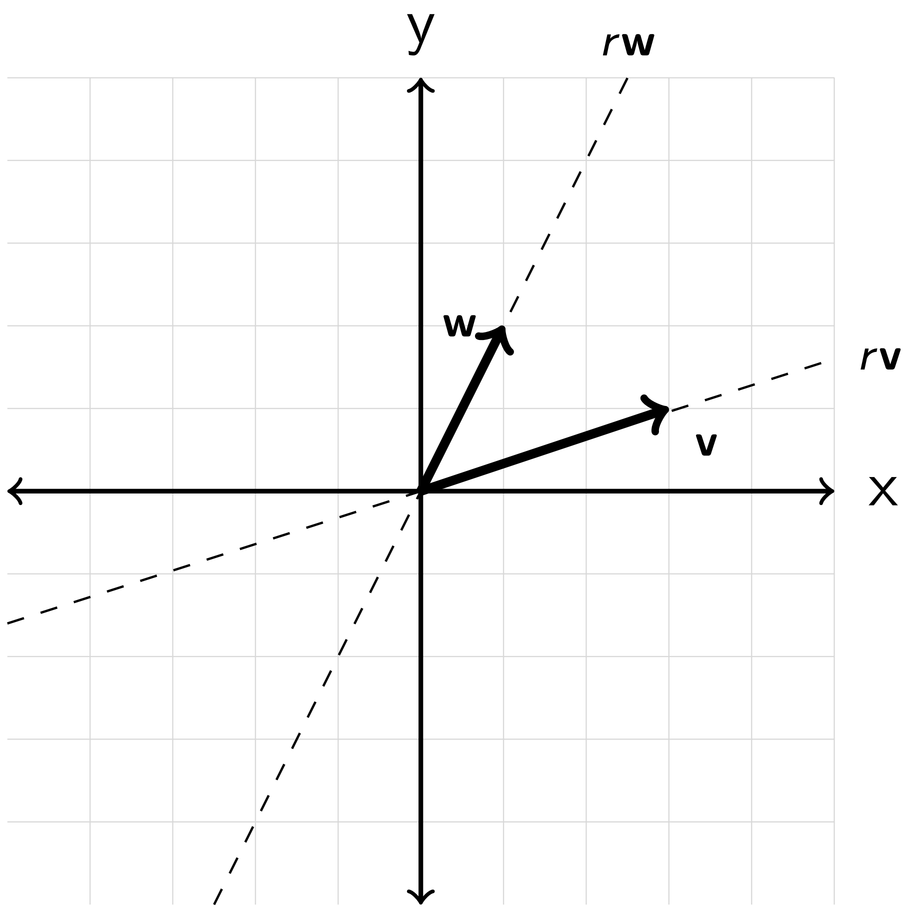

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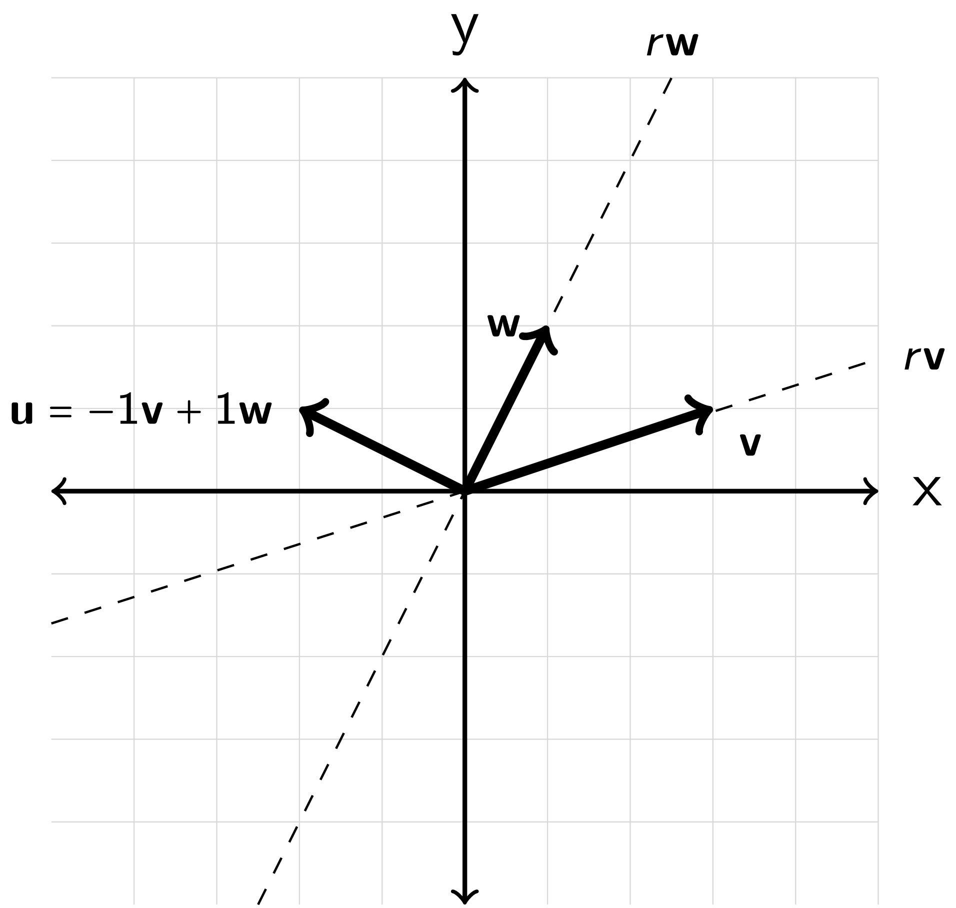

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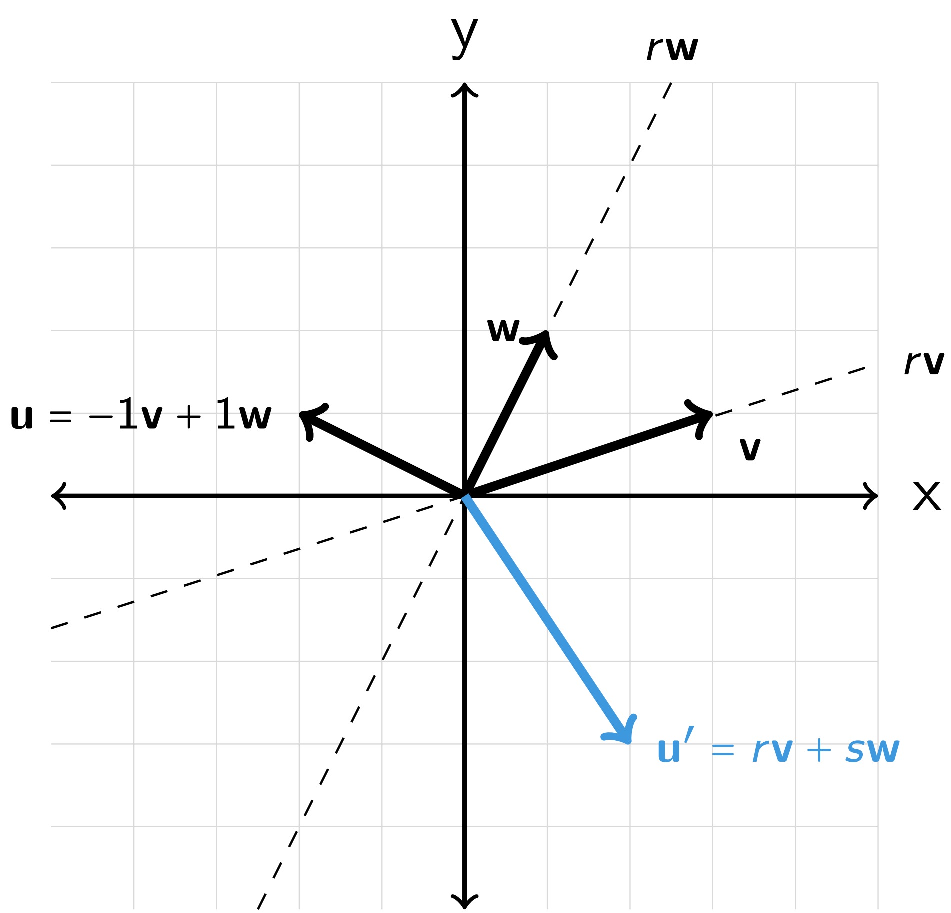

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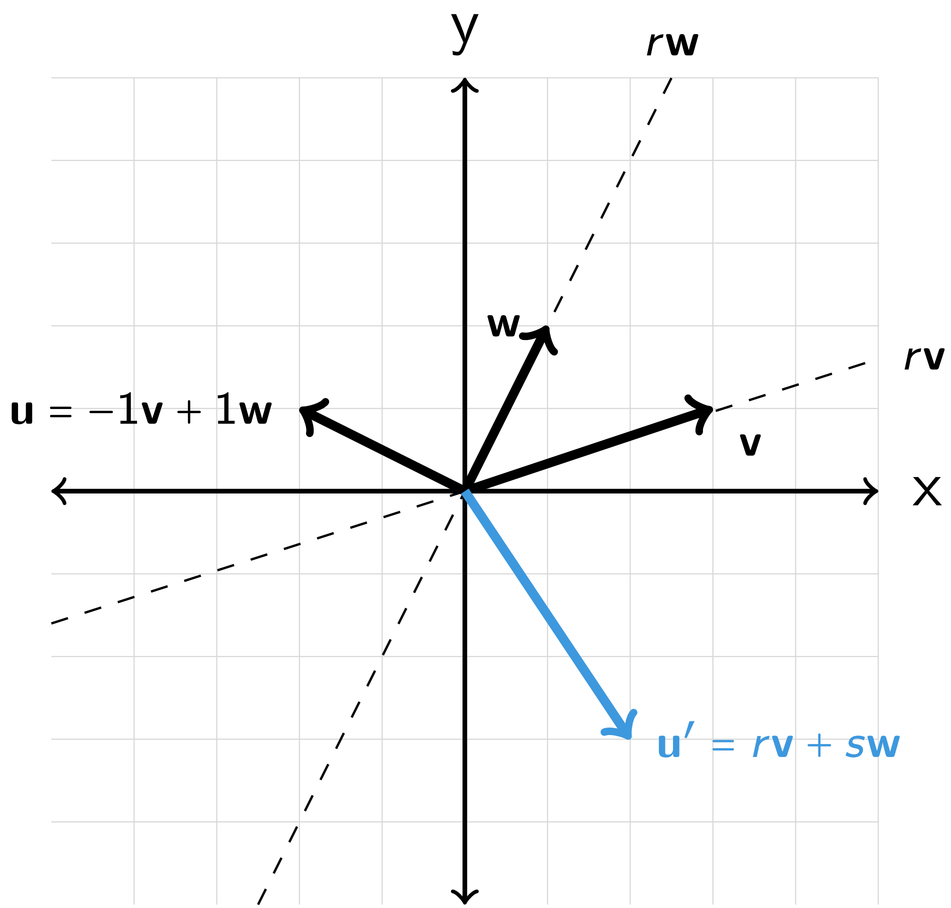

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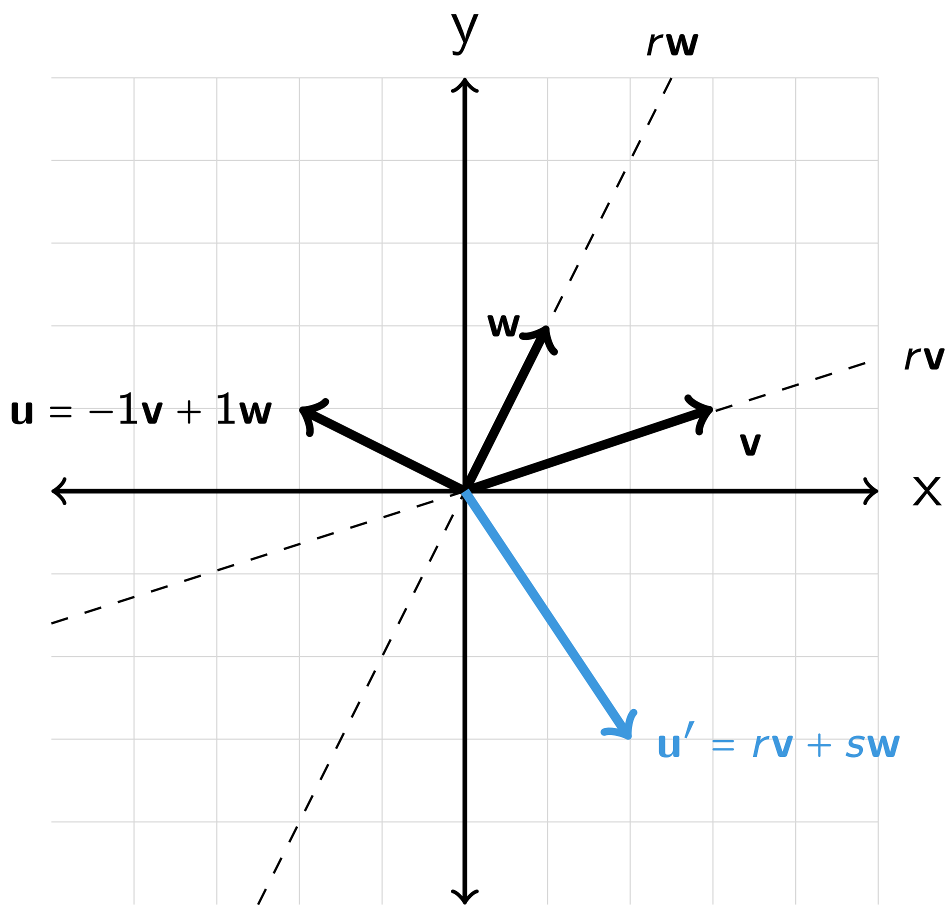

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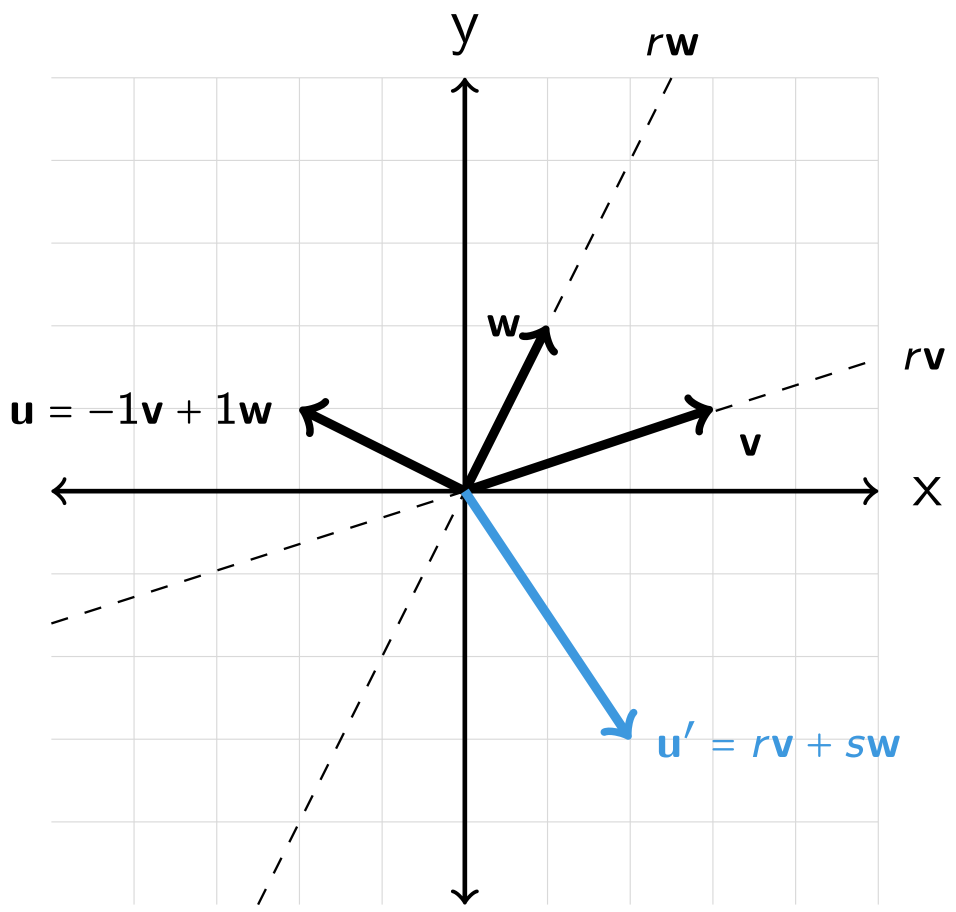

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$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 4 & 8 \end{pmatrix}, \text{ but } \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 2 & 10 \end{pmatrix}.$$

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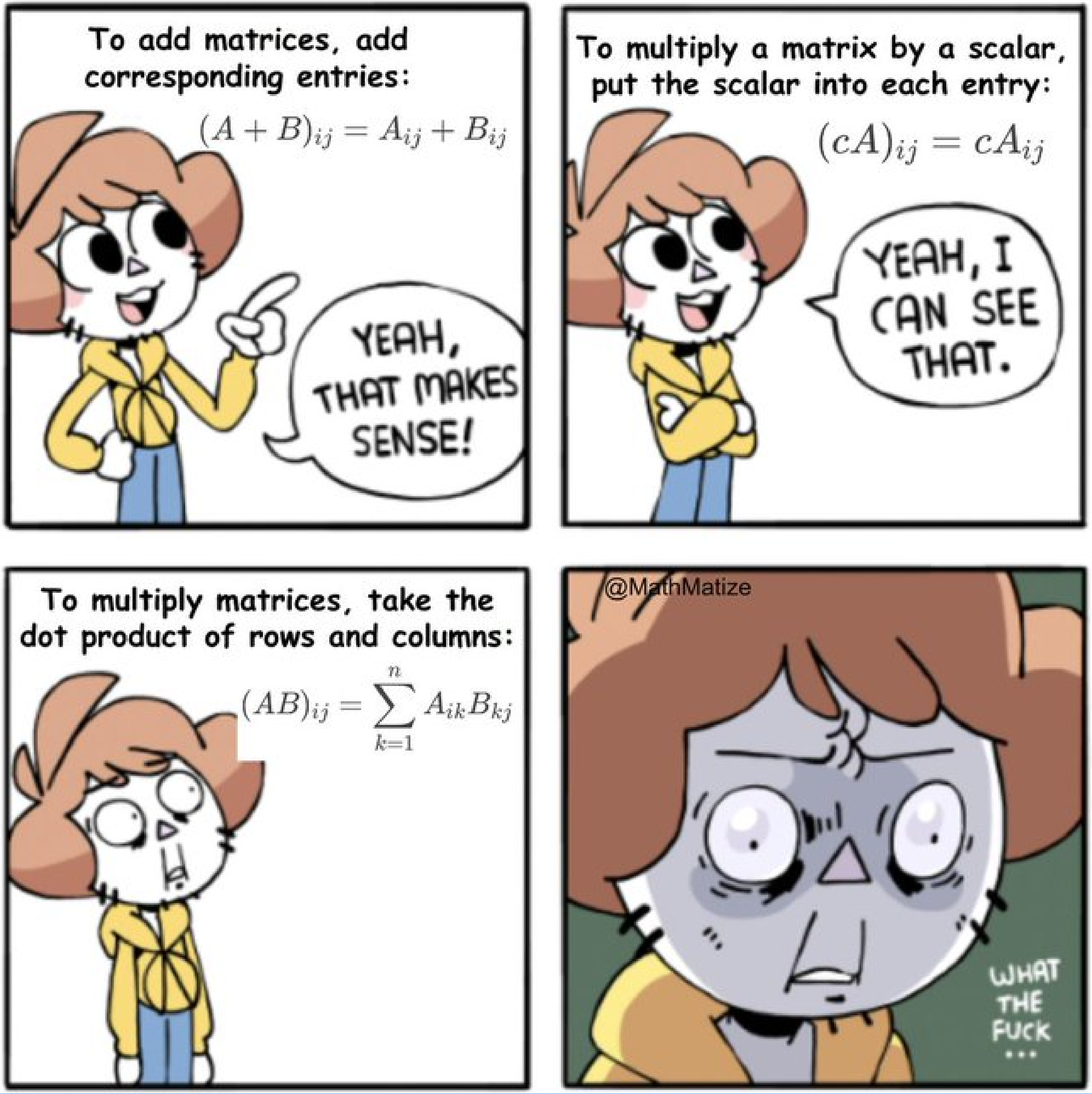
Matrix Multiplication



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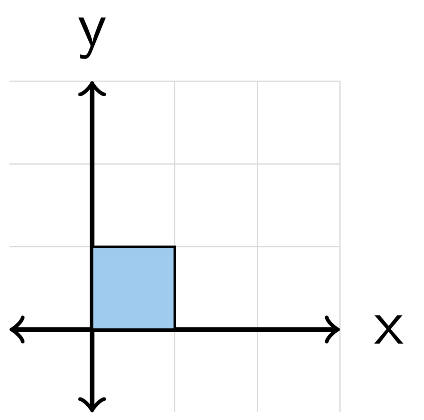

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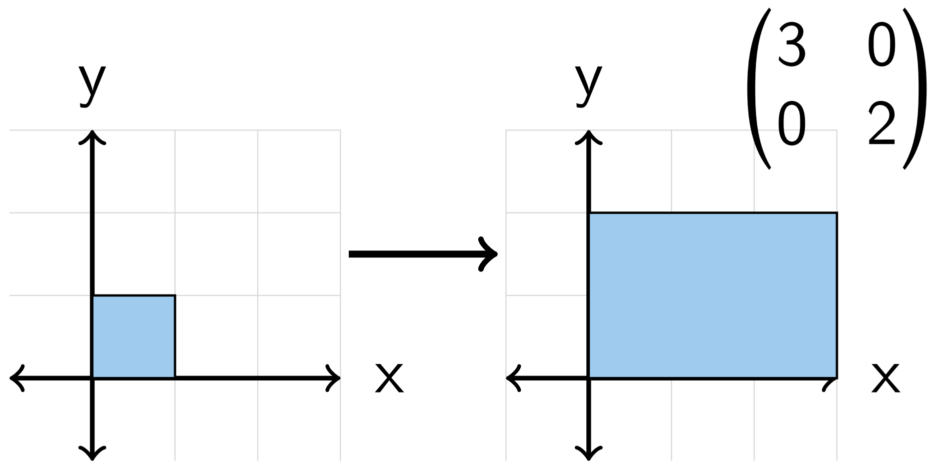

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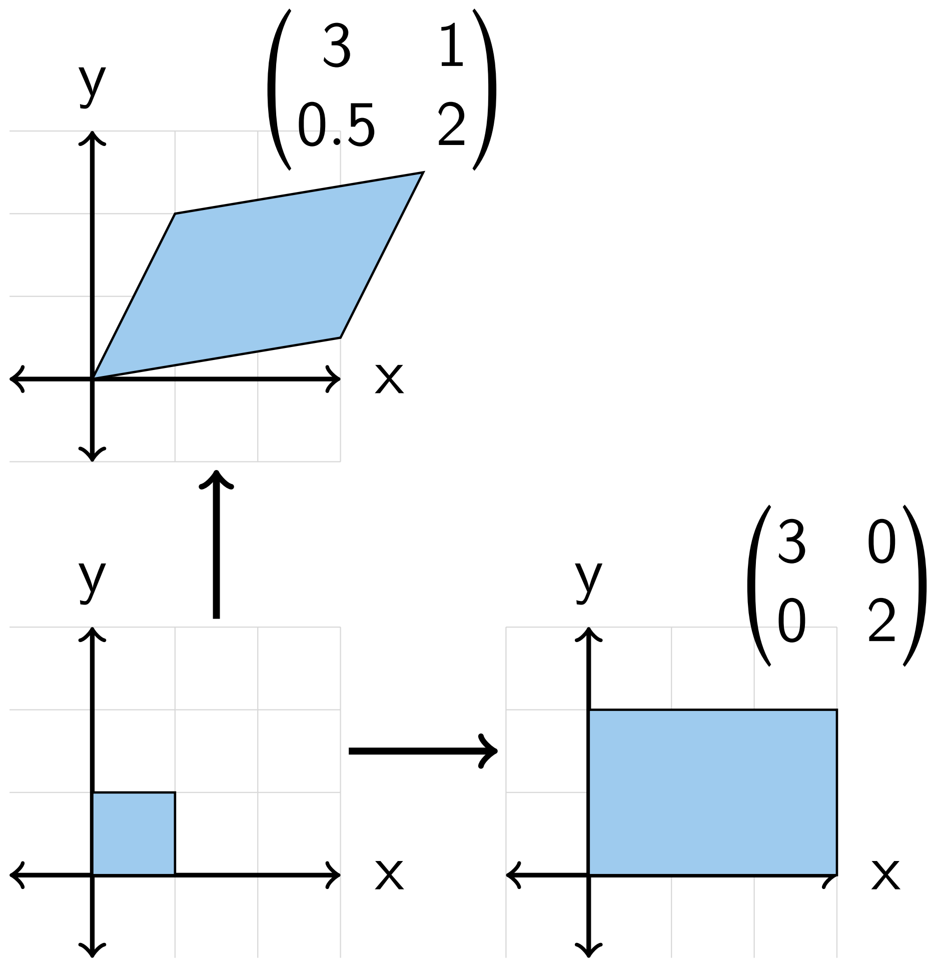

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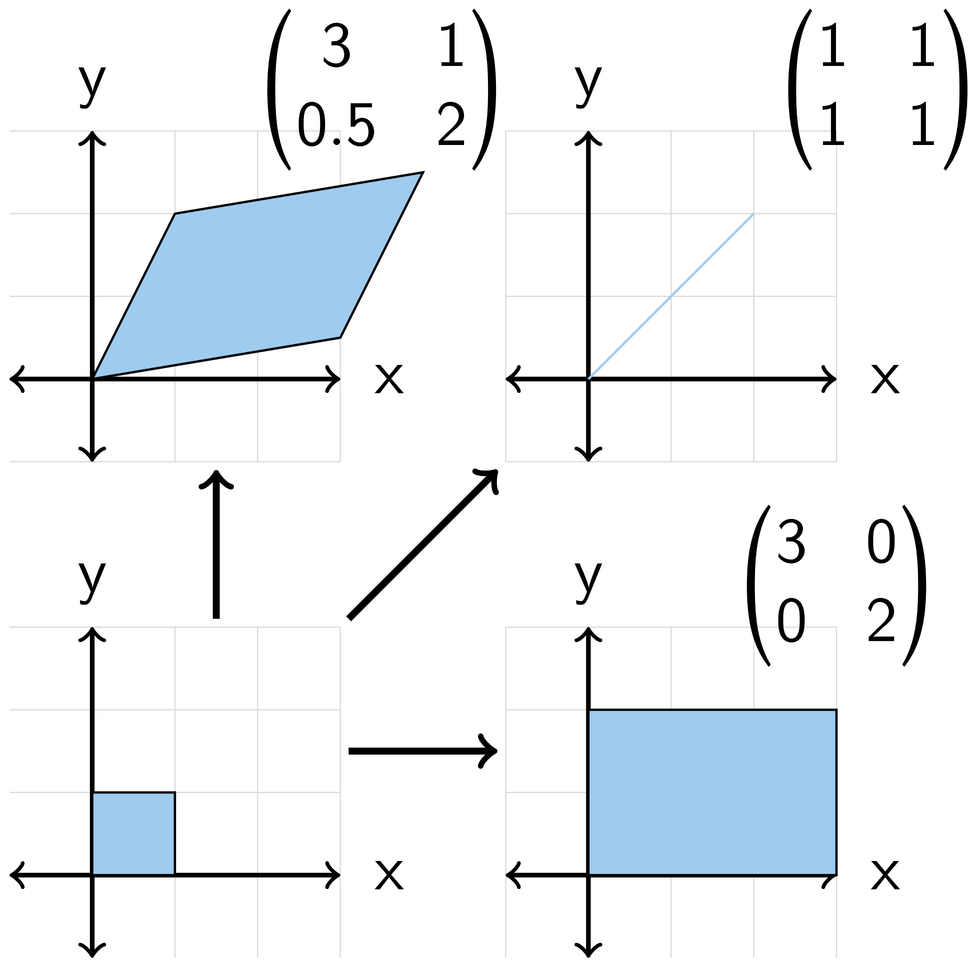
Question: What happens in the degenerate case?



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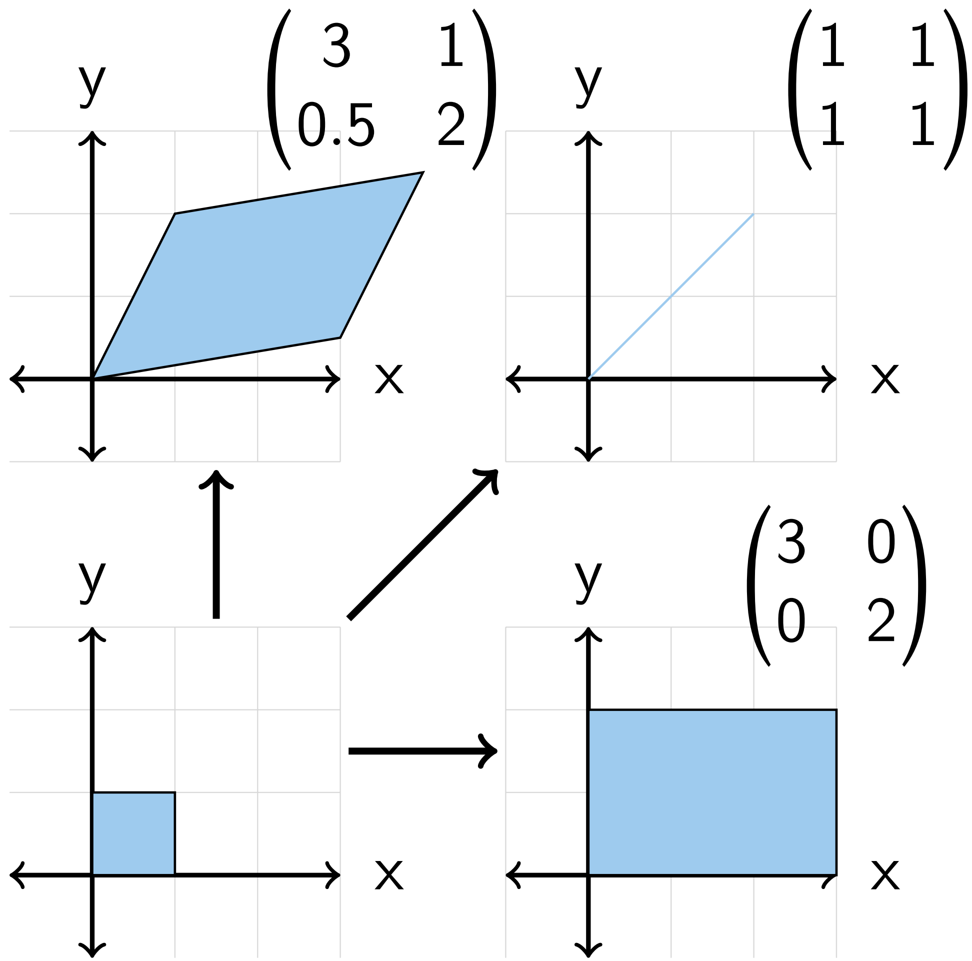
For a matrix $A' = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ it is:

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$



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a_{11}	a_{12}	a_{13}
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Determinant

$$a_{11}a_{22}a_{33}$$

$$\begin{array}{|c|c|c|c:c:}
\hline
a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\
\hline
a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\
\hline
a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\
\hline
\end{array}$$

Determinant

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31}$$

a_{11}	a_{12}	a_{13}	a_{11}	a_{12}
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\hline
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a_{11}	a_{12}	a_{13}	a_{11}	a_{12}
a_{21}	a_{22}	a_{23}	a_{21}	a_{22}
a_{31}	a_{32}	a_{33}	a_{31}	a_{32}

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a_{31}	a_{32}	a_{33}	a_{31}	a_{32}

Determinant

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

$$\begin{array}{|c|c|c|c:c:}
\hline
a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\
\hline
a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\
\hline
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\hline
\end{array}$$

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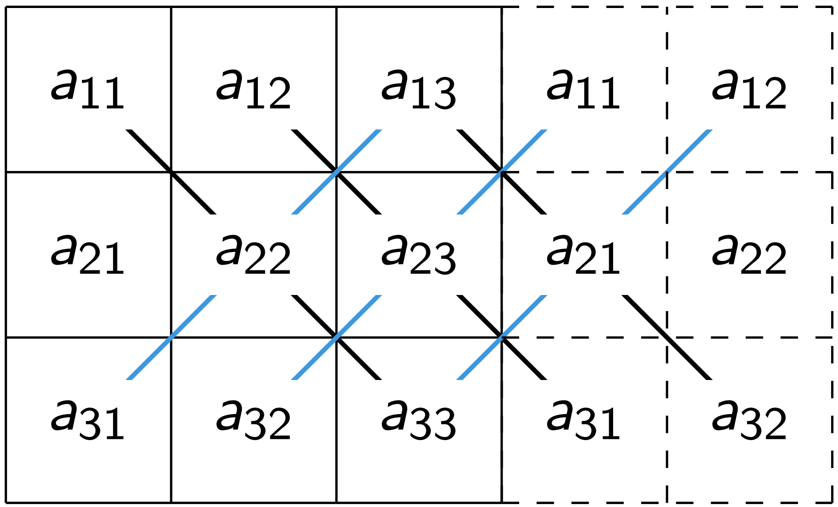

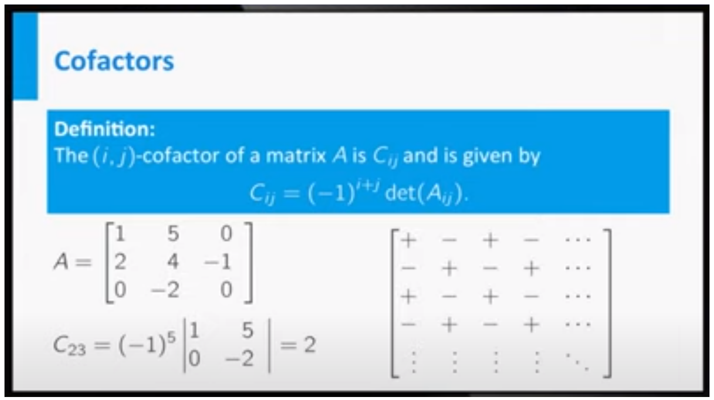

Figure: `https://ocw.tudelft.nl/course-lectures/determinants/`

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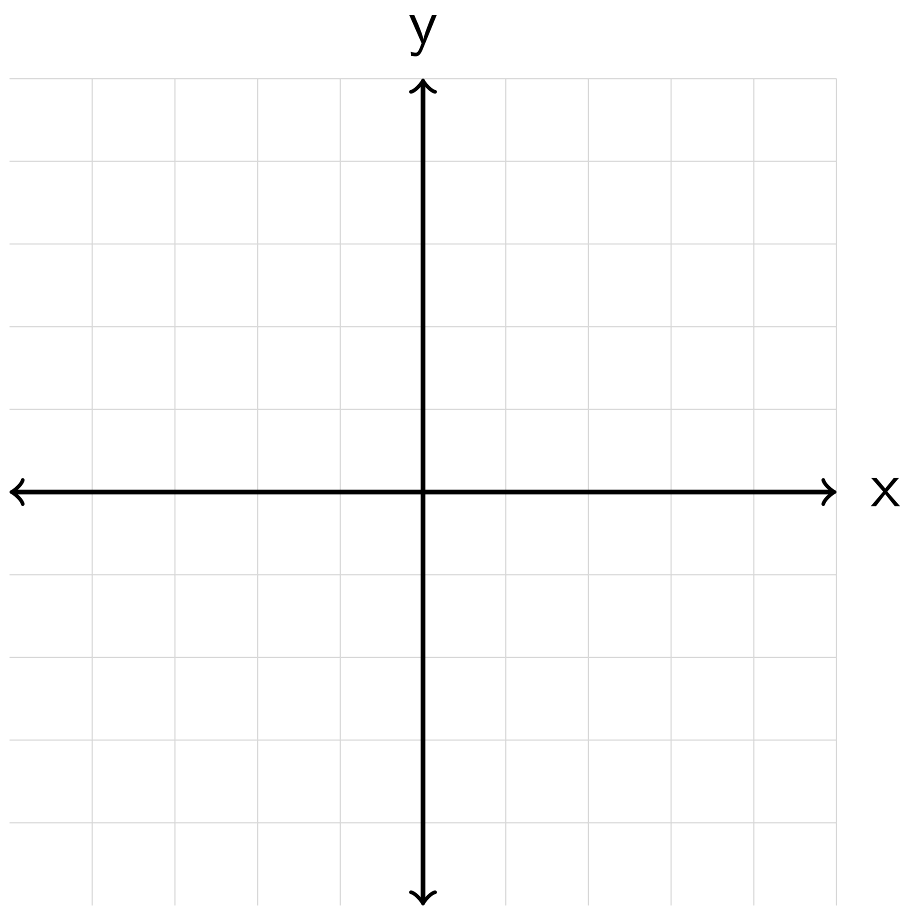
Different option in the exercise!

Definition Line

`https://vevox.app/#/m/106717265`



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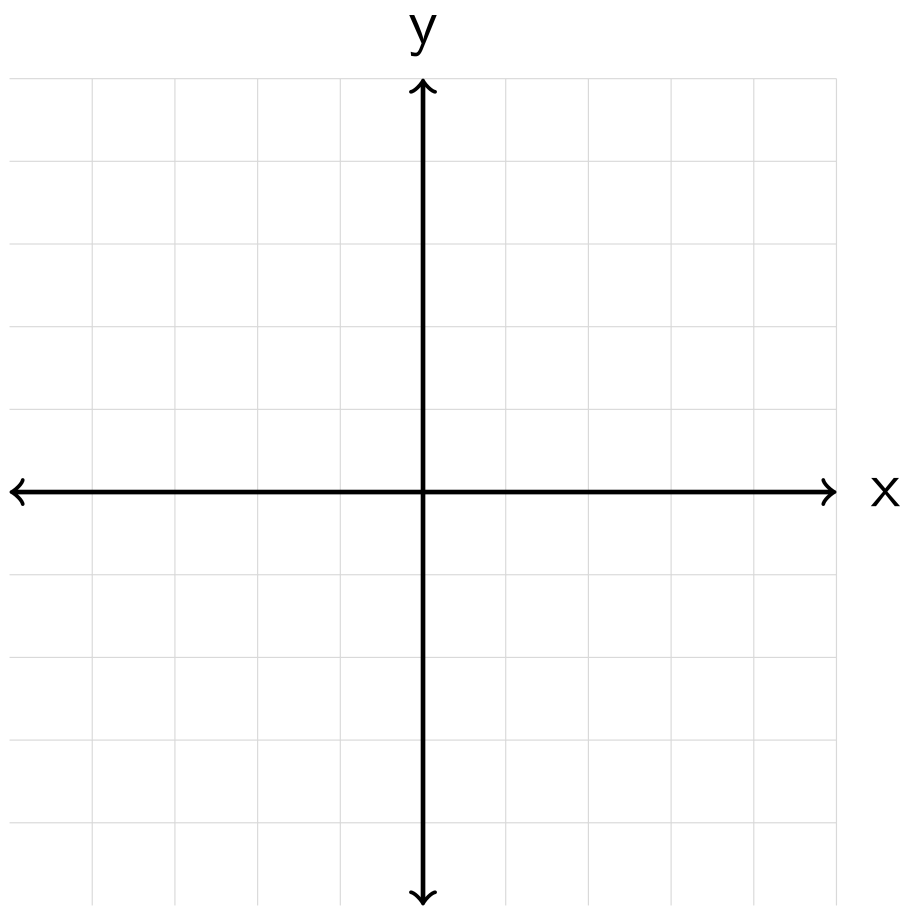
A *line* in $\mathbb{R}^2$ an be represented via the formula $y = mx + b$ for some scalars $m, b \in \mathbb{R}$.



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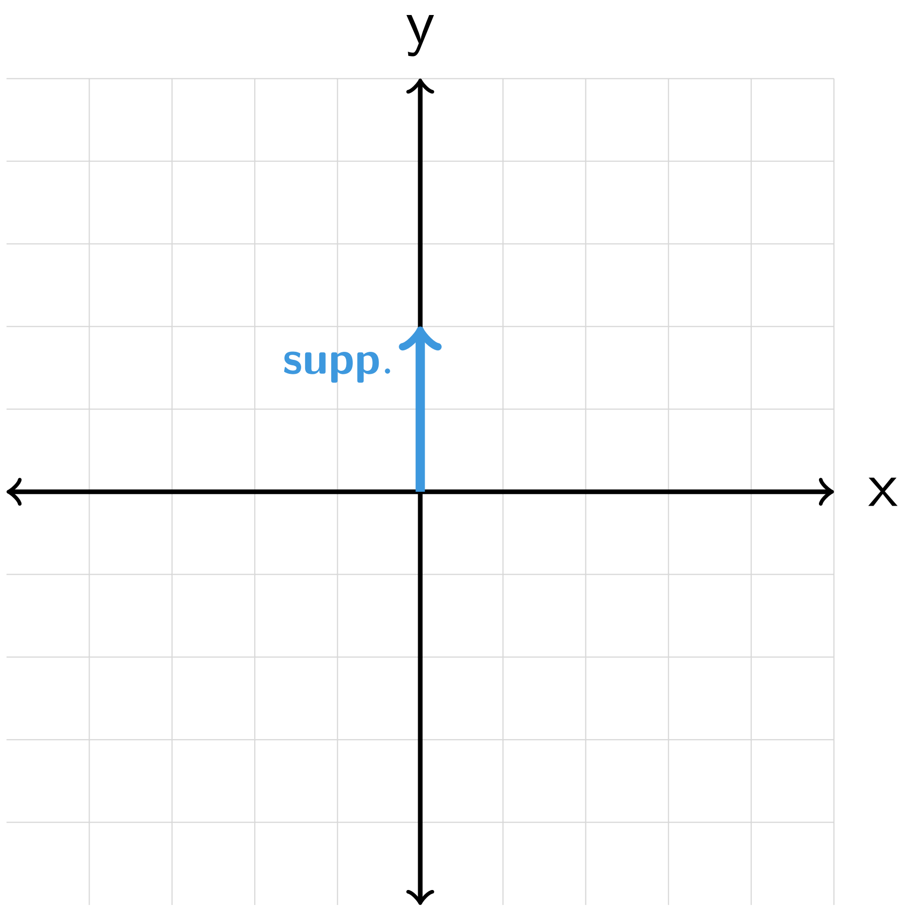

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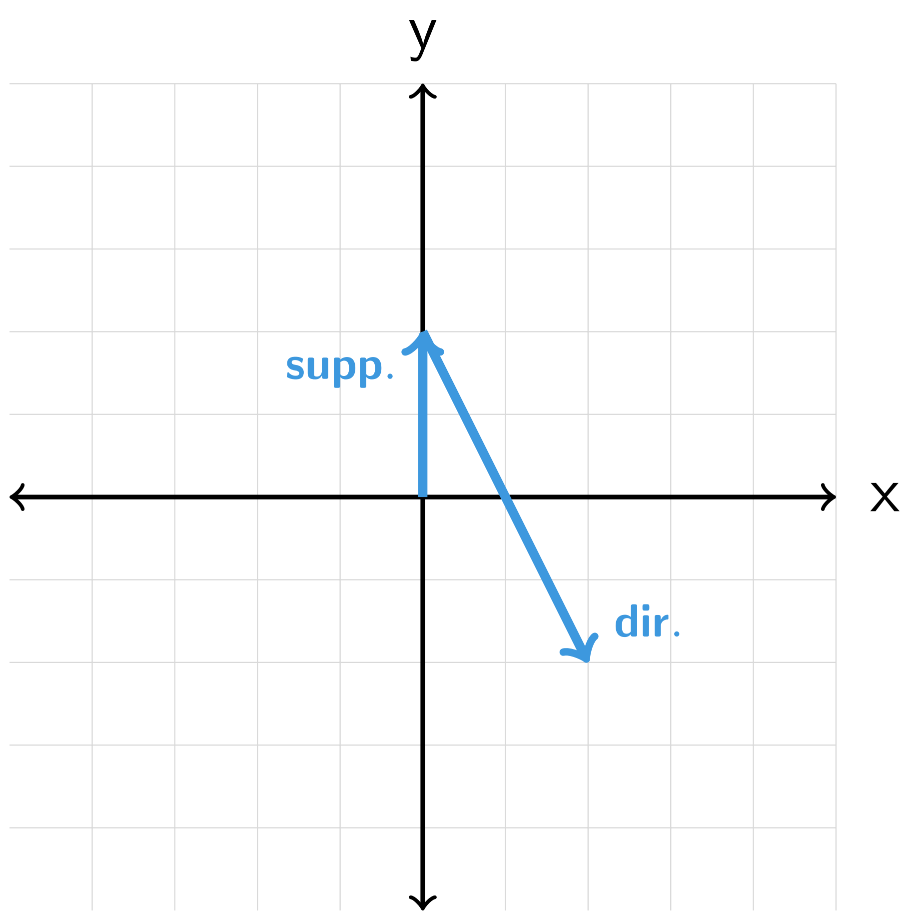

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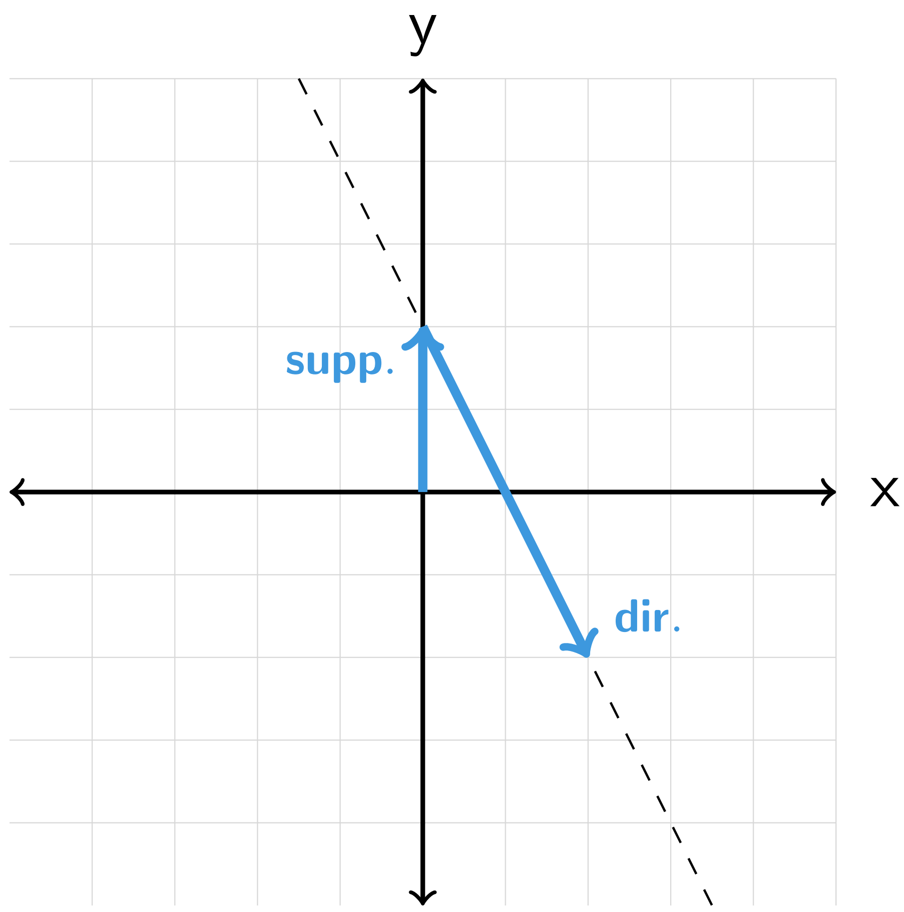
$$y = -2x + 1$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} + r \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$



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for r

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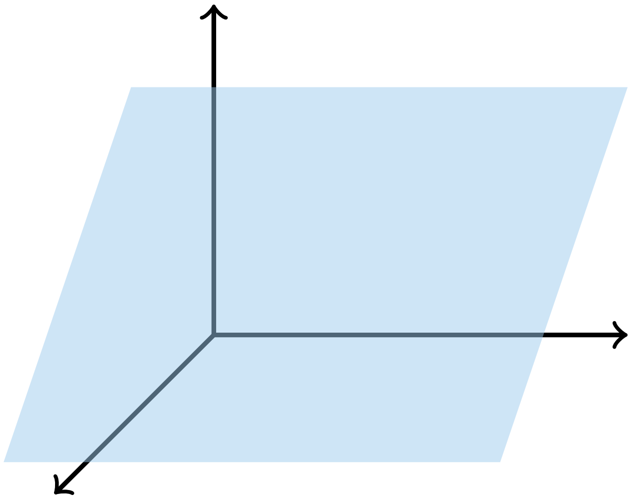
for r and see that $r = 1$ is a solution, while for the point $\mathbf{w} = (0,1,1)$ is not on the line as the linear system

$$(1,0,0)^t + r(-1,2,1)^t = (0,1,1)$$

does not have any solution.

Definition Plane

Question: How to define a plane?



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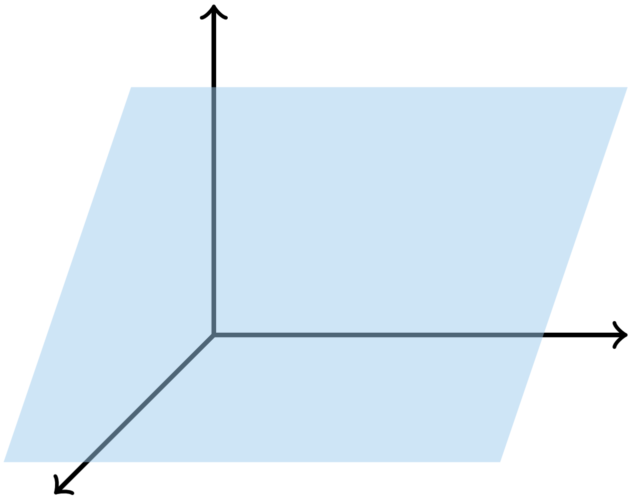
Question: How to define a plane?

Analog to spanning a line, in $\mathbb{R}^3$, we can span a plane by two spanning vectors, i.e., it is of the form

$$\mathbf{u} + r\mathbf{v} + s\mathbf{w}$$

with support vector $\mathbf{u}$ and spanning vectors $\mathbf{v}, \mathbf{w}$.



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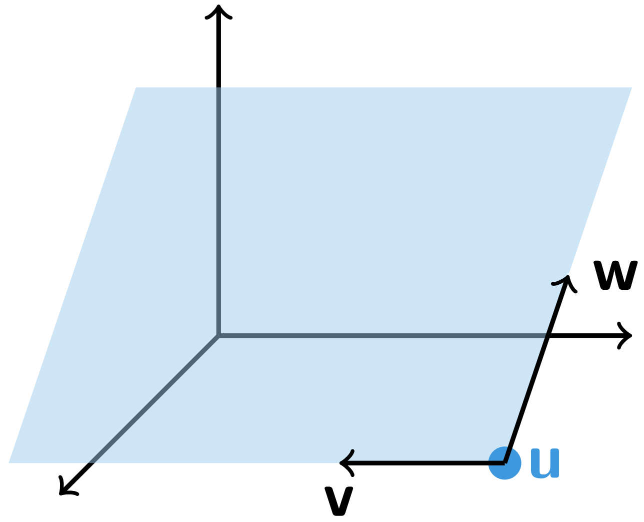

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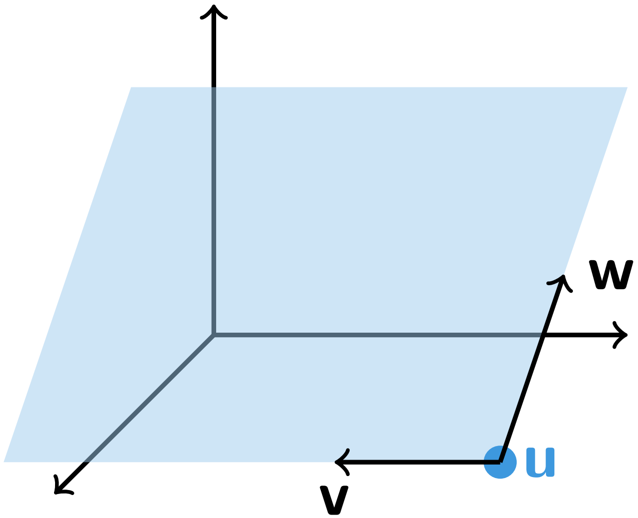

Alternatively, find a *normal* vector $\mathbf{n}$, which is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. Then, any point $\mathbf{p}$ on the plane satisfies

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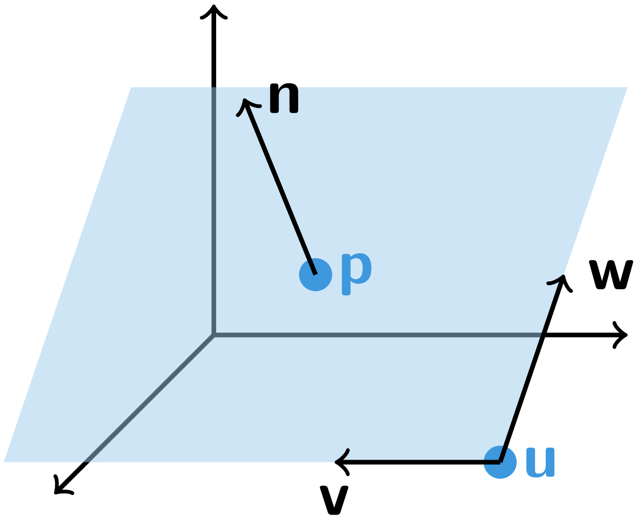

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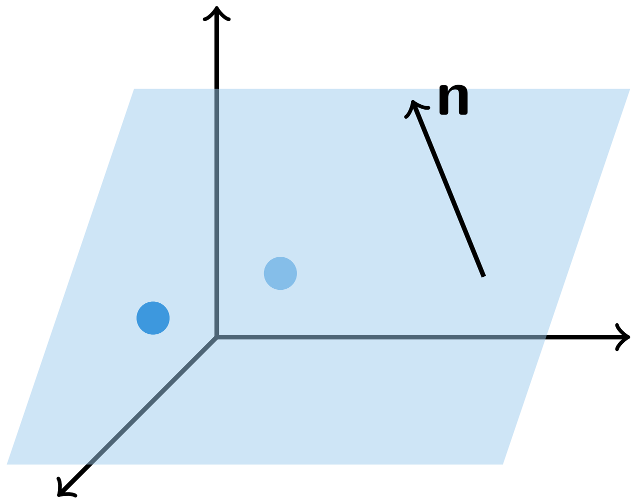
$$(0,1,0)^t + r(0,1,0)^t + s(1,0,-1)^t.$$

Points on sides of a plane

For points, $\mathbf{v}, \mathbf{w}$, you can see how they are ordered along a line spanned by $\mathbf{u}$ by looking at the dot product.
Hence, comparing the dot product of the normal with points on the plane and outside of the plane gives directions.



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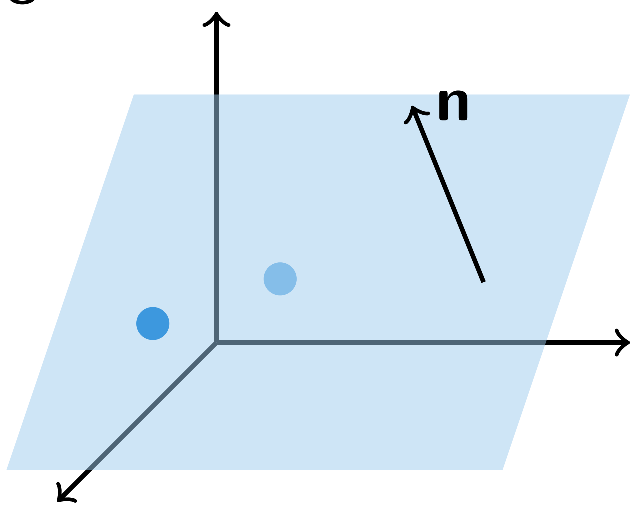

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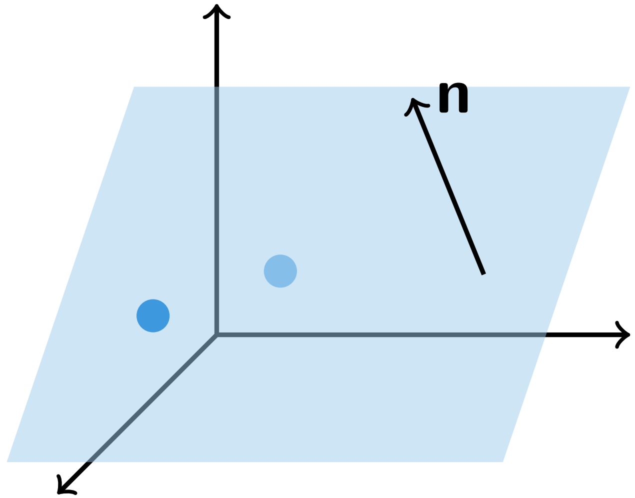

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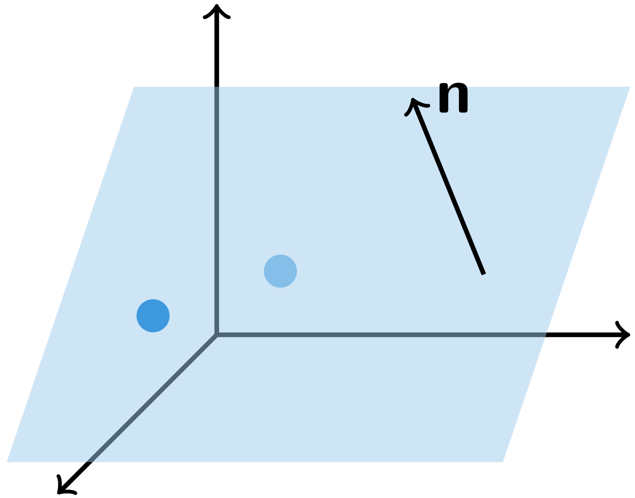

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While the points $(1,0,1)$ and $(0,2,2)$ yield

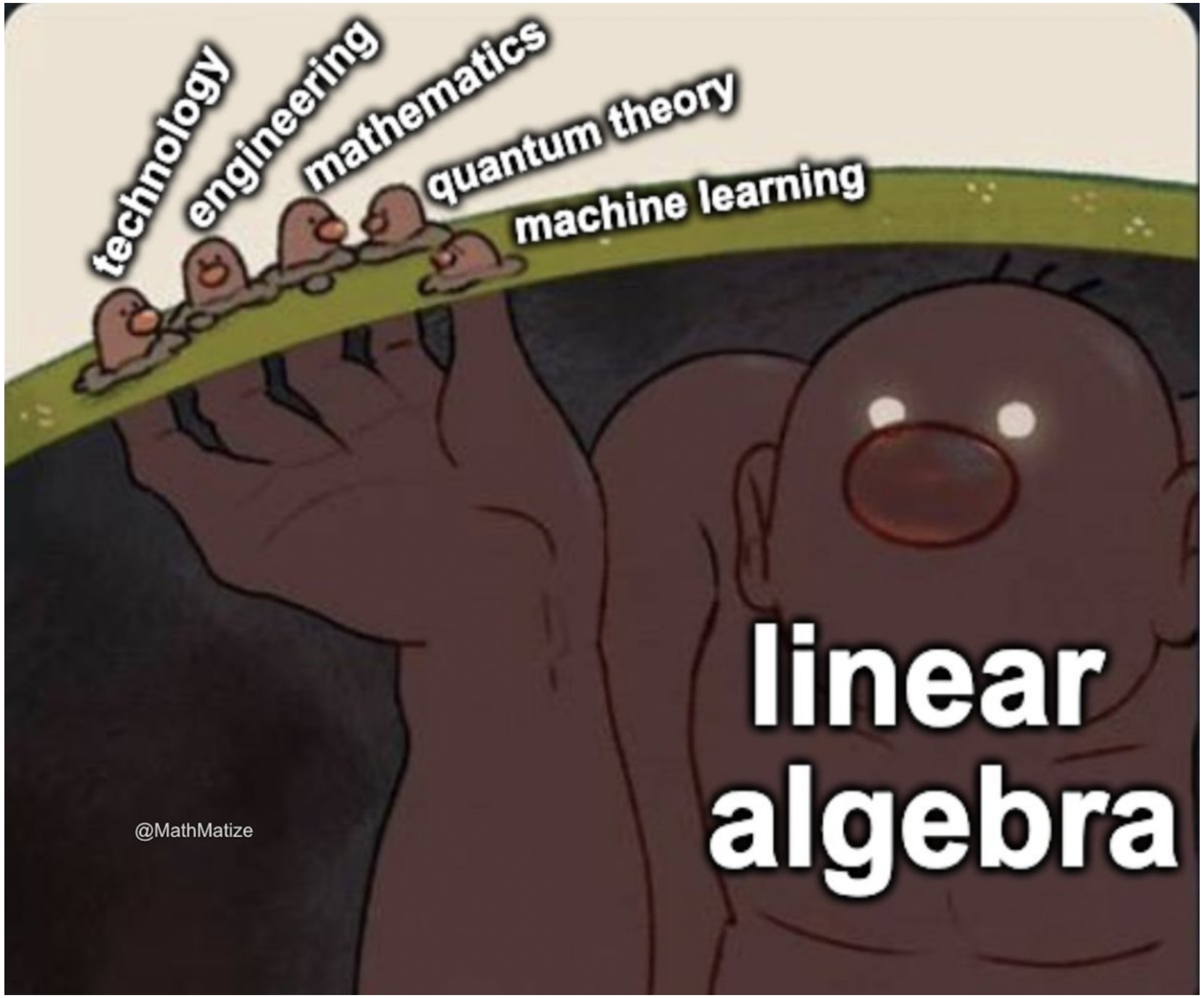
$$(0,1,1)^t \cdot (1,0,1)^t = 1 < 3$$
$$(0,1,1)^t \cdot (0,2,2)^t = 4 > 3$$

hence they lie on opposite sites of the plane.

Summary

How Computer Graphics continues:

- Setup assignment (1a) due tomorrow.
- Do the WebLab assignment (1b).
- Do the Programming assignment (1c).
- Ask questions you have at this point.



Don't worry – it will all be on Brightspace.

CSE2215 - Computer Graphics

Images and Algebra

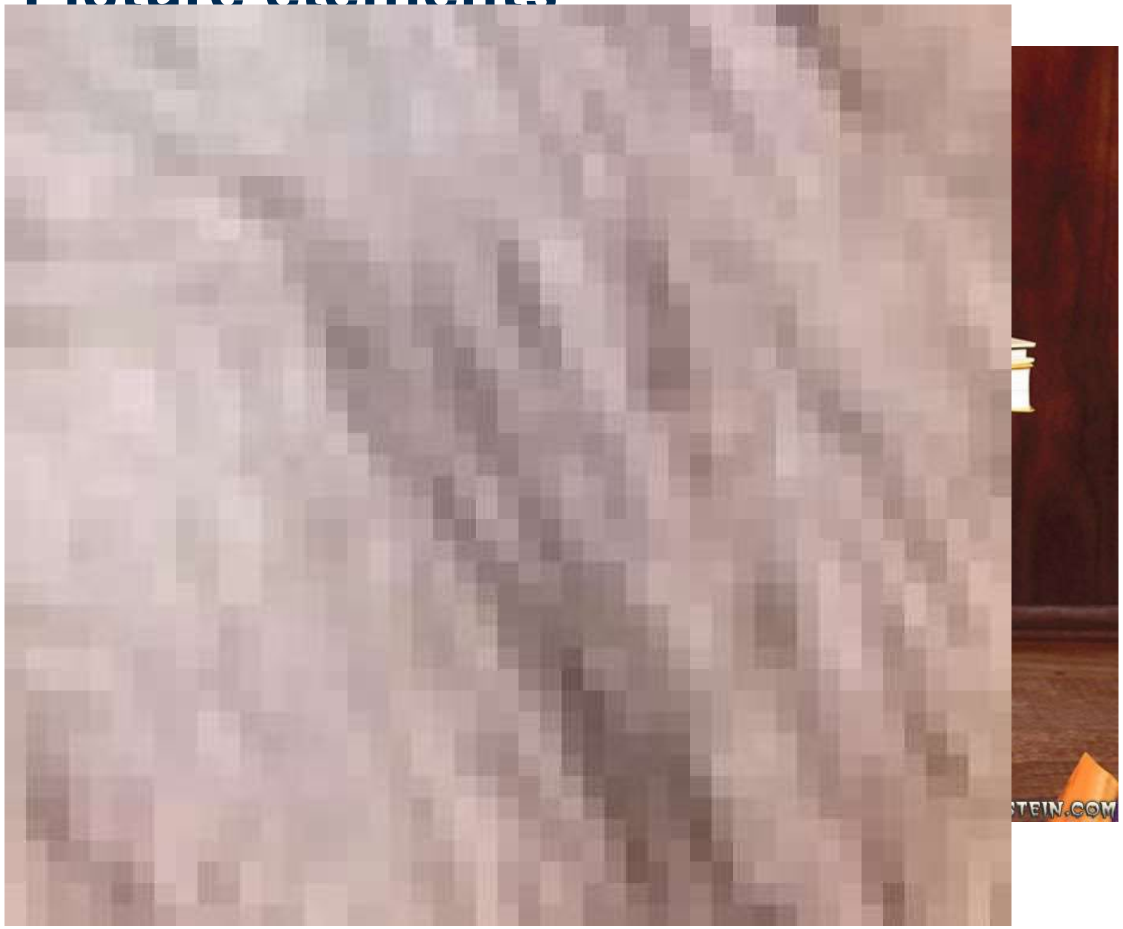
Taking care of your image

Elmar Eisemann

Delft University of Technology

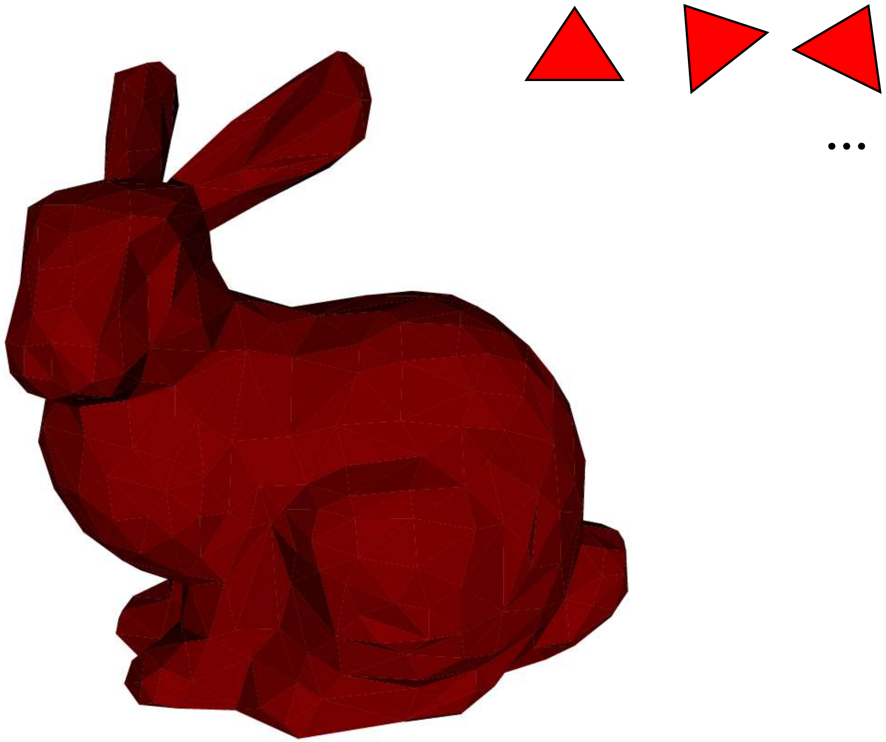
Making images with a Computer

Pixels – Picture elements



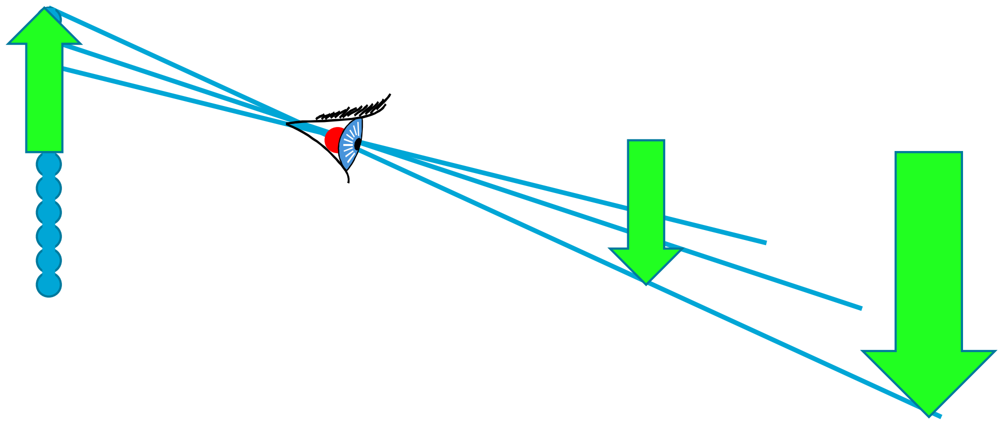
Simplified Graphics Pipeline

- Models are typically lists of triangles



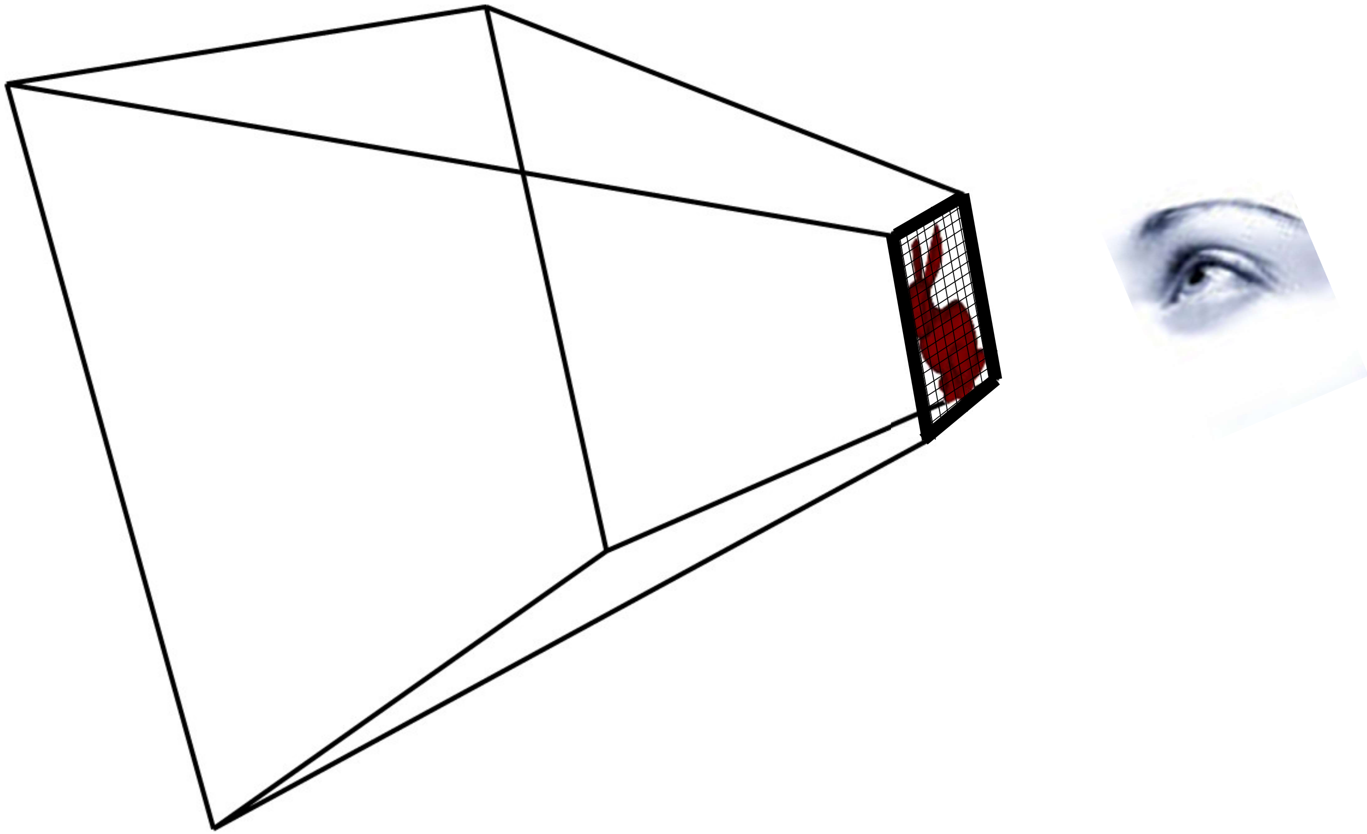
Virtual Camera

- Camera Plane in front of the eye



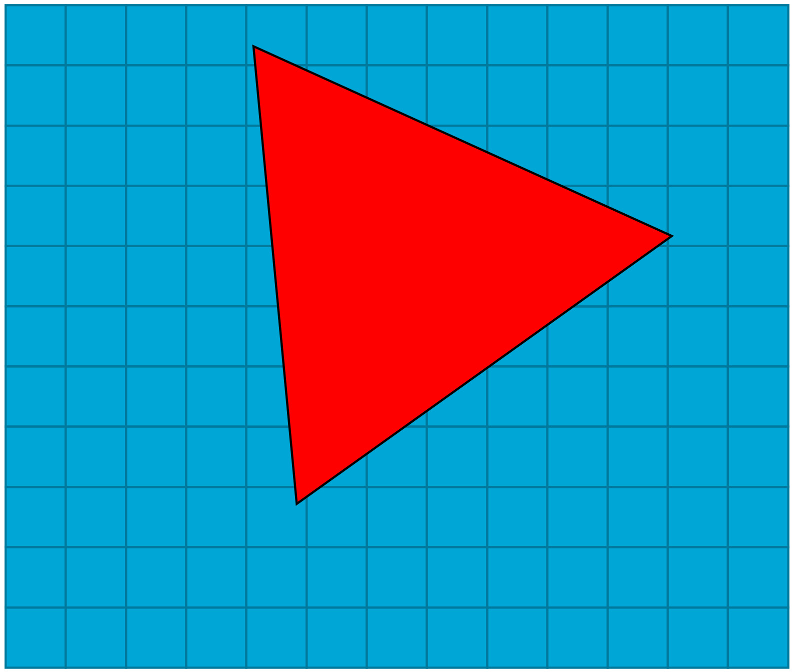
Simplified Graphics Pipeline

- Projection: Transform coordinates to screen



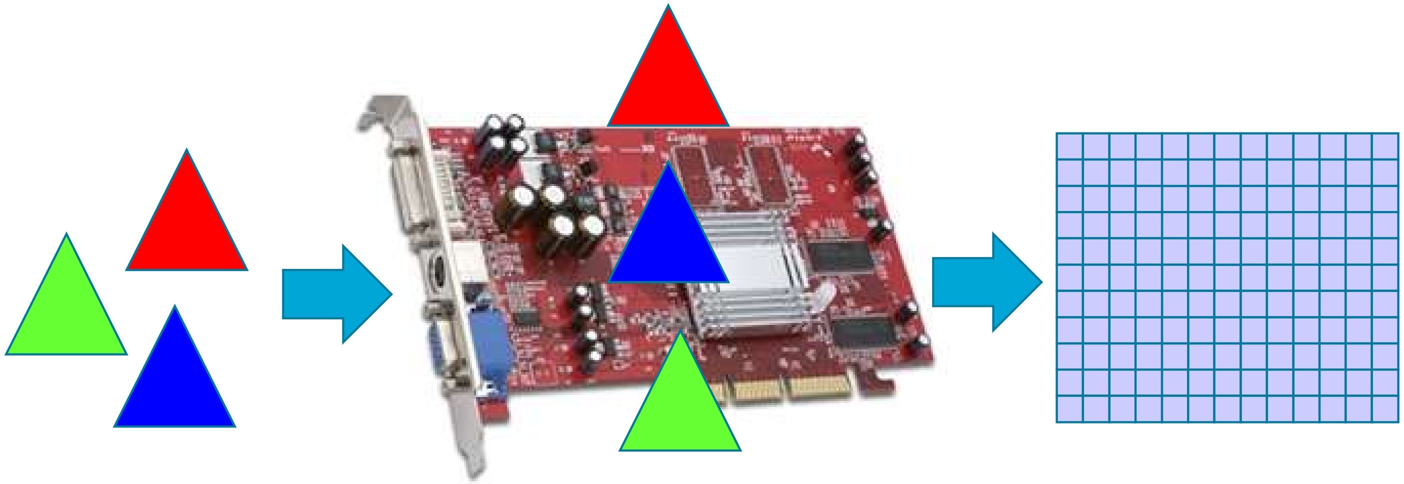
Simplified Graphics Pipeline

- Rasterization: Fill screen pixels



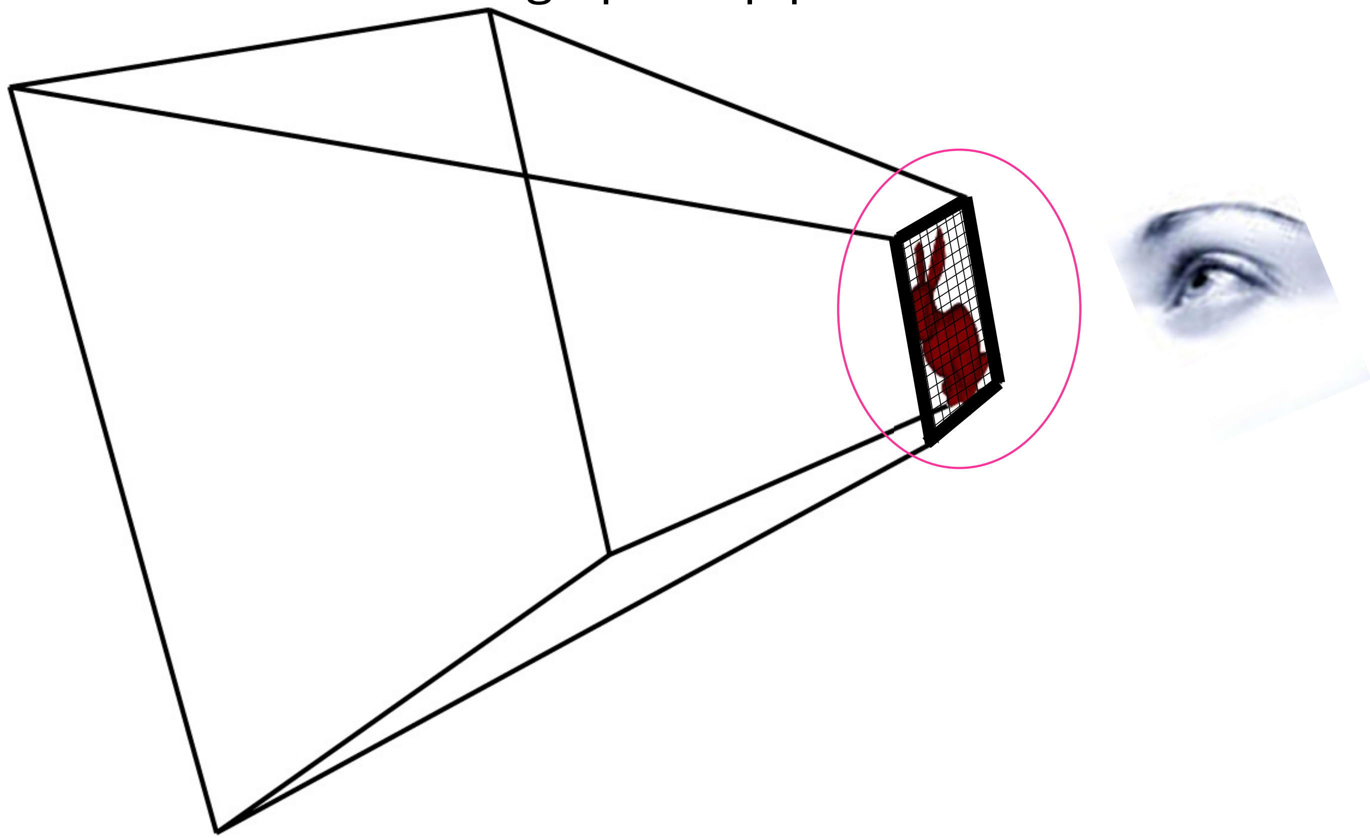
Simplified Graphics Pipeline

- Highly parallelizable ➡ GPUs



Today

- We will work towards the graphics pipeline…



Relevant Study Goals for Today

- S4 Apply mathematical modeling and theory of geometric computations and transformations, object representations, simulation, and encoding.
  - We learn about the mathematical basics for complex object representations
- S5 Implement algorithms and data structures using the C++ programming language
  - We see some rudimentary code to implement basic image operations
- S6 Apply the knowledge obtained in this course to problems of other fields
  - We will see several application examples beyond computer graphics.

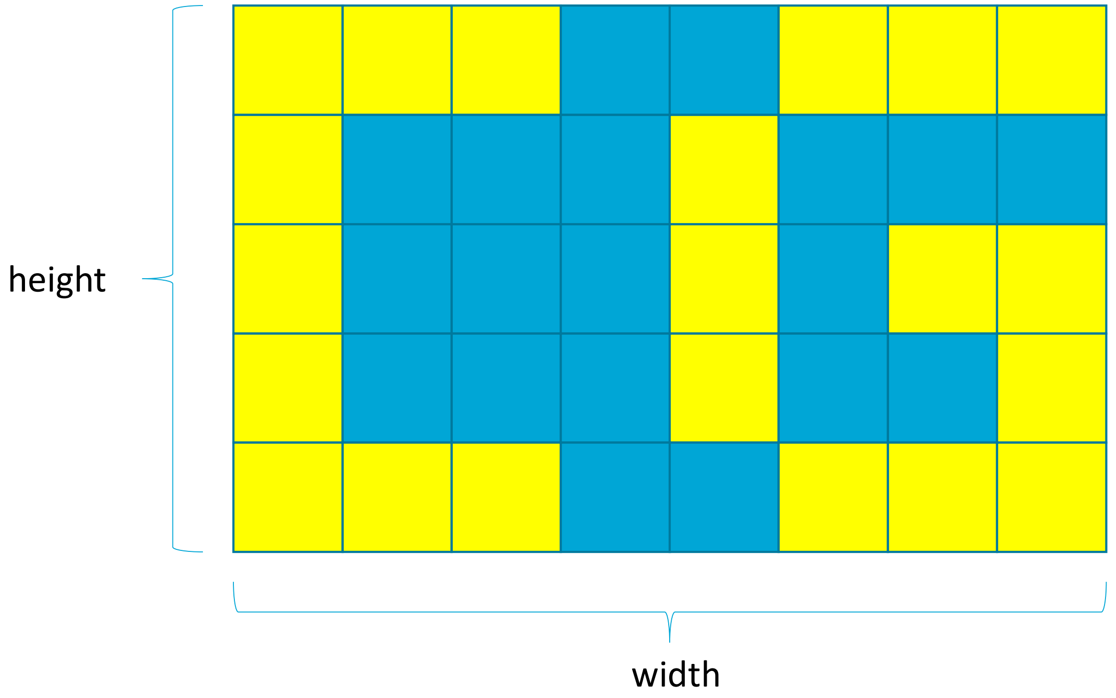
Images

Images

- What is an image?
- How to represent it in memory?
- How to access individual pixels?
- How can we process images?
- How are images stored?

Representation

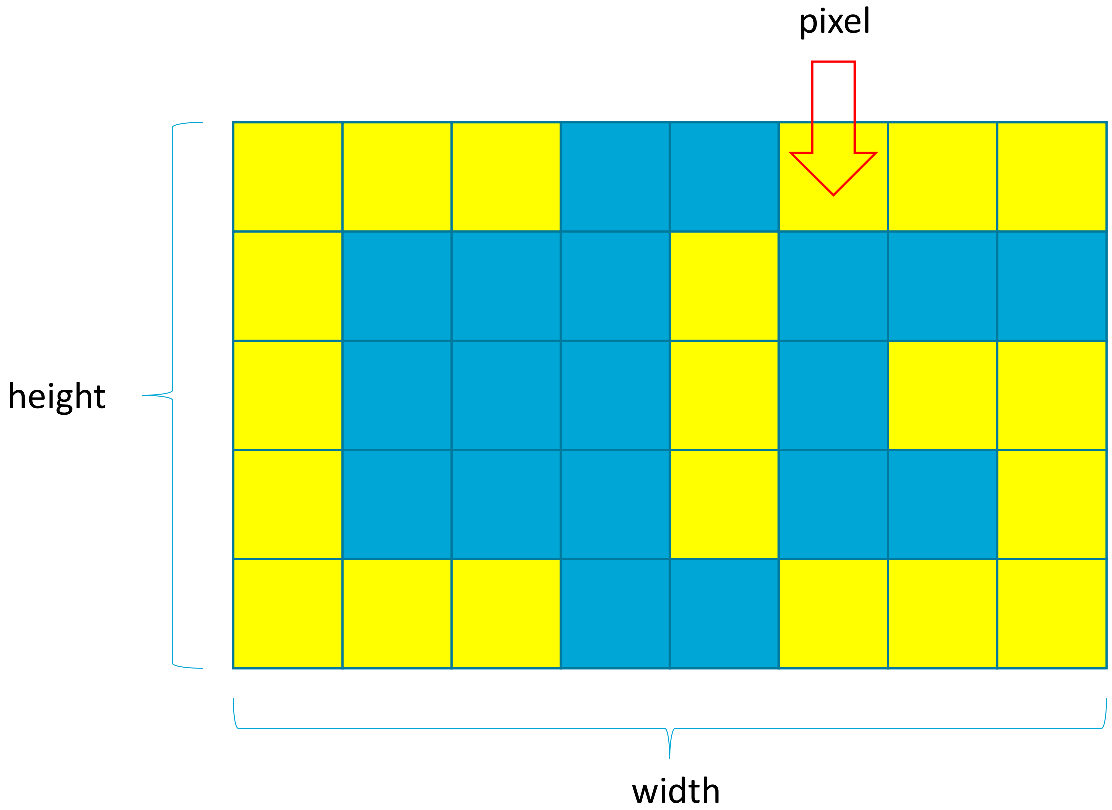
- Image



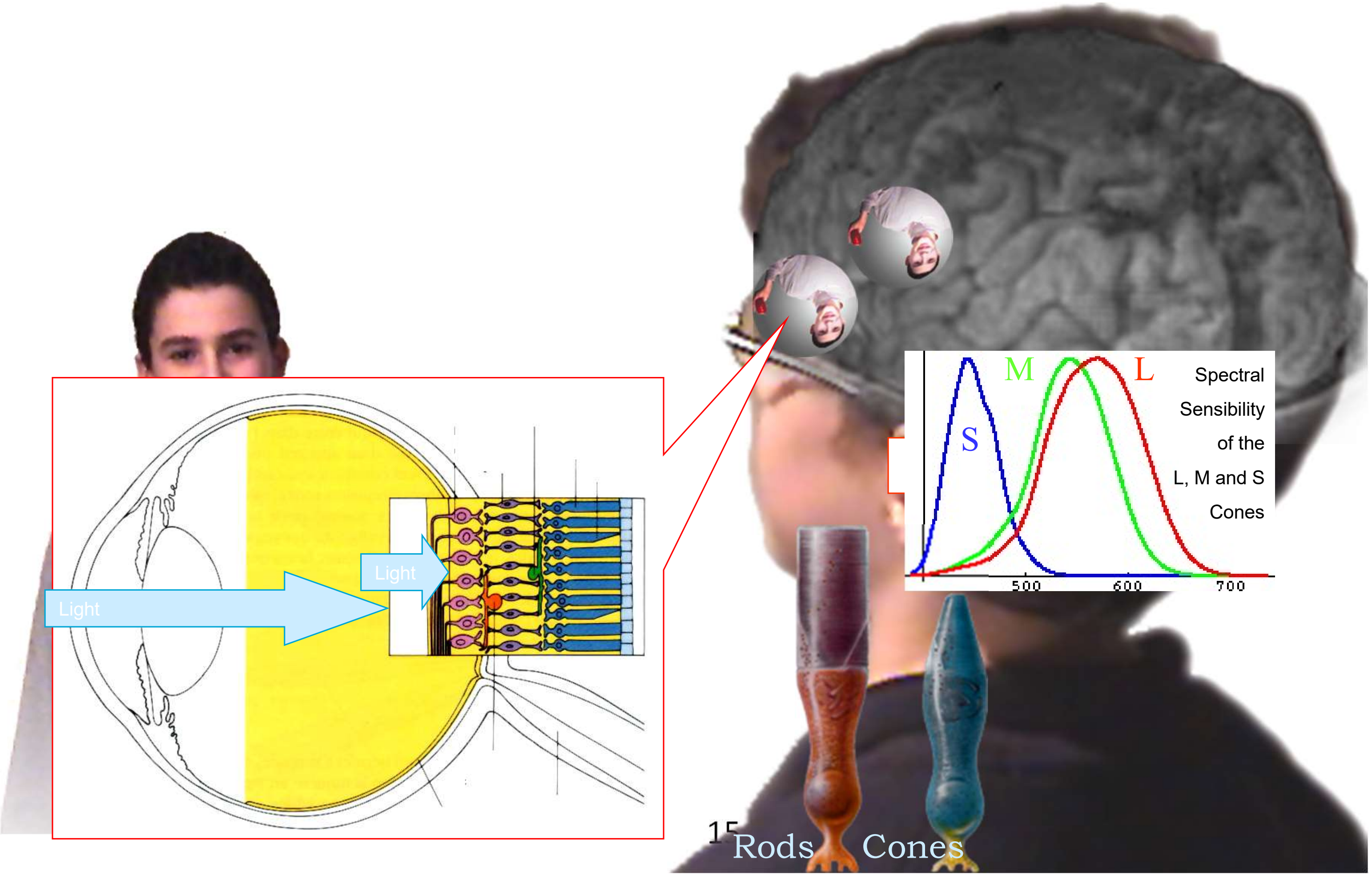
TU Delft

Representation

- Image

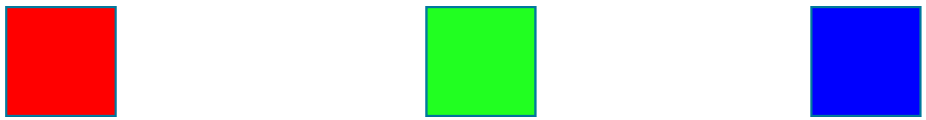


Eye Biology


Light
Light
S
M
L
Spectral
Sensibility
of the
L, M and S
Cones
500
600
700
Rods
Cones

Representation

- Pixel:
- Contains 3 components Red, Green, Blue

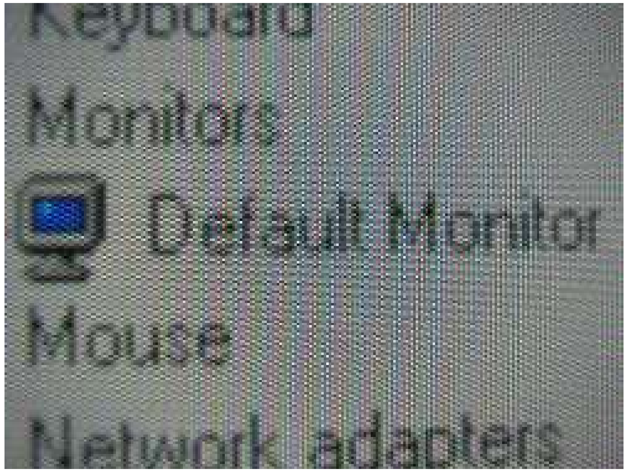


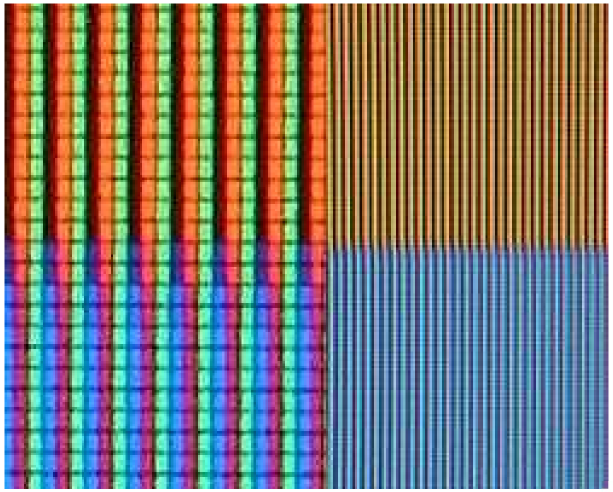
Representation

- Standard Screens work with RGB

- A value of 1 in a color channel activates the channel as strongly as possible

- For example (1,0,0) is the strongest red





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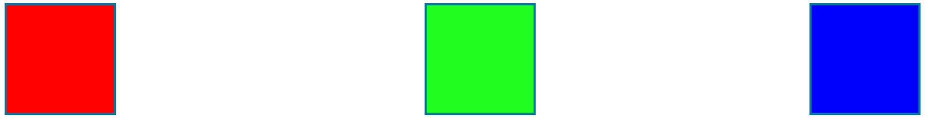
- Pixel:
- Contains 3 components Red, Green, Blue



- Typically, continuous values in [0,1].
- E.g., (1,0.5,0)

Grab a paint program
and test some values for yourself!
Often values between 0,255 instead of 0,1

Images

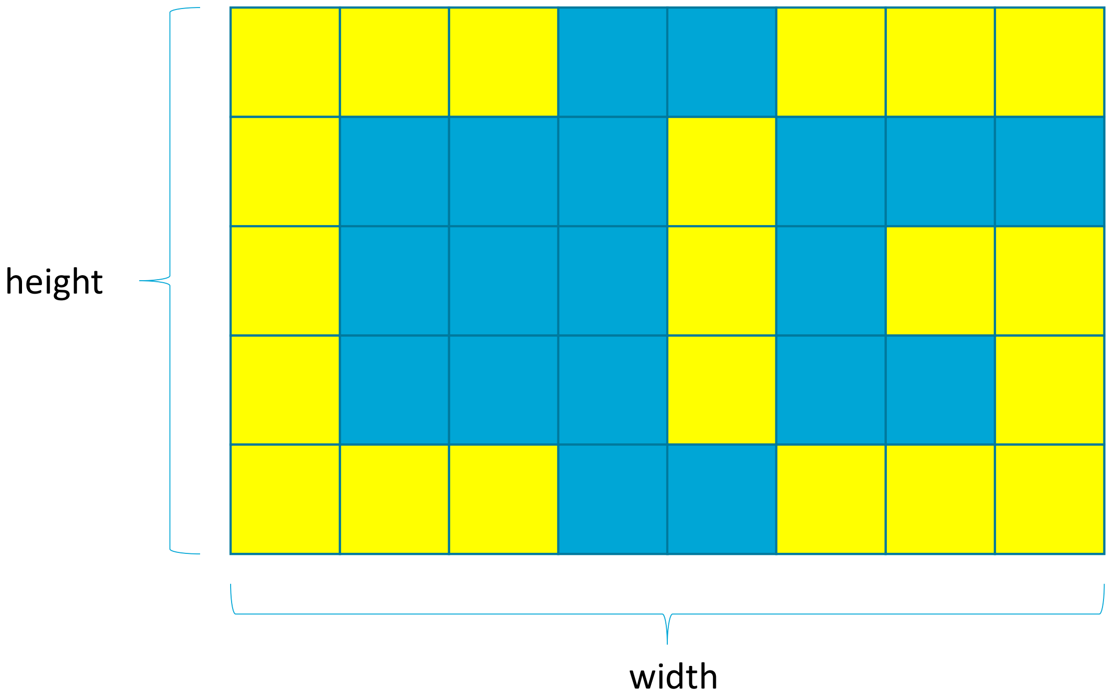
- What is an image?
- How to represent it in memory?
- How to access individual pixels?
- How can we process images?
- How are images stored?

Representation

- Color values are discretized/quantized:
  - most common for image storage and display output:
    8 bit per color channel (256 values)
  - conversion of float v in [0,1] to 8bit: v*255;

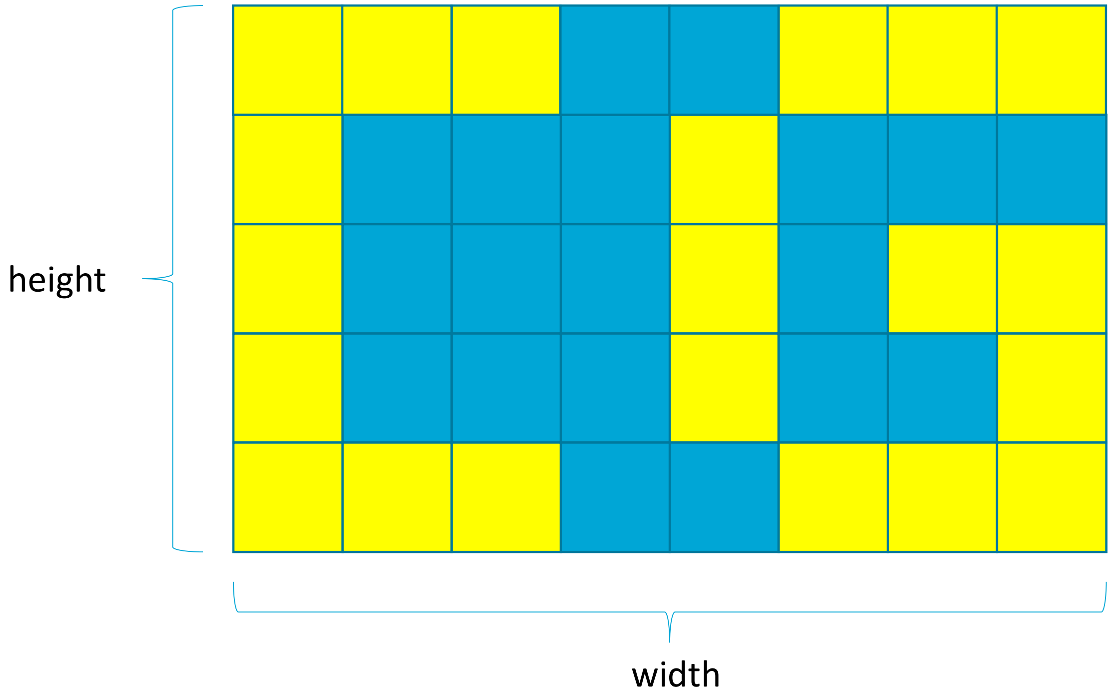
Representation

- Image representation in memory?



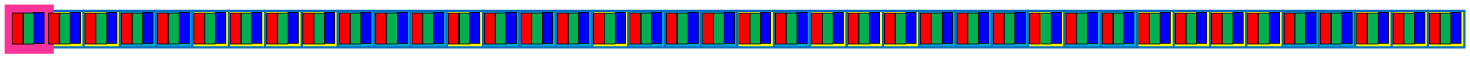
Representation

- Image representation in memory?

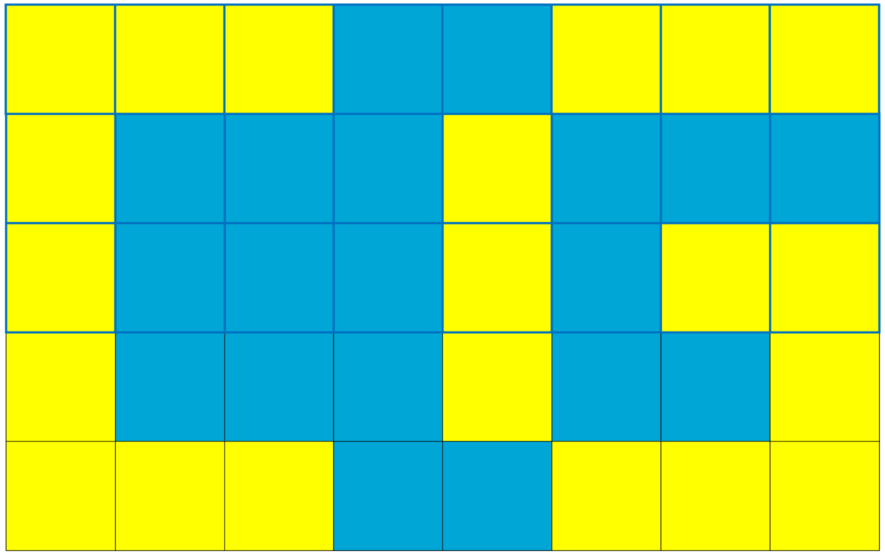


Representation

- Image representation in memory?



3 values per pixel
(Red, Green, Blue)



Representation

- Image representation in memory?

Images

- What is an image?
- How to represent it in memory?
- How to access individual pixels?
- How can we process images?
- How are images stored?

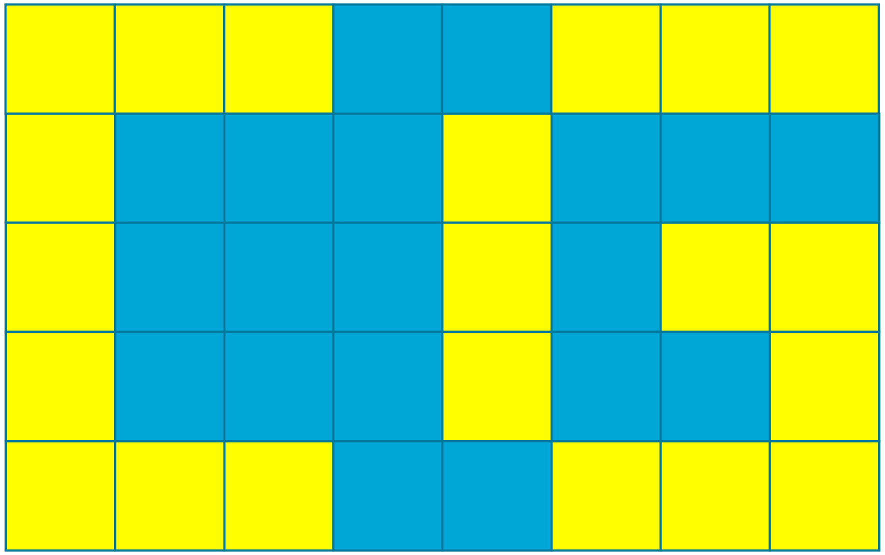
Representation

- How to access pixel (i,j)?
- Find index to access corresponding memory

- Solution: 3*(j*width+i)   3 for the 3 values per pixel (RGB)
  - Need to know the width!
    Typically stored in an image file
  - What about height?
    Usually as well but could be derived from data.

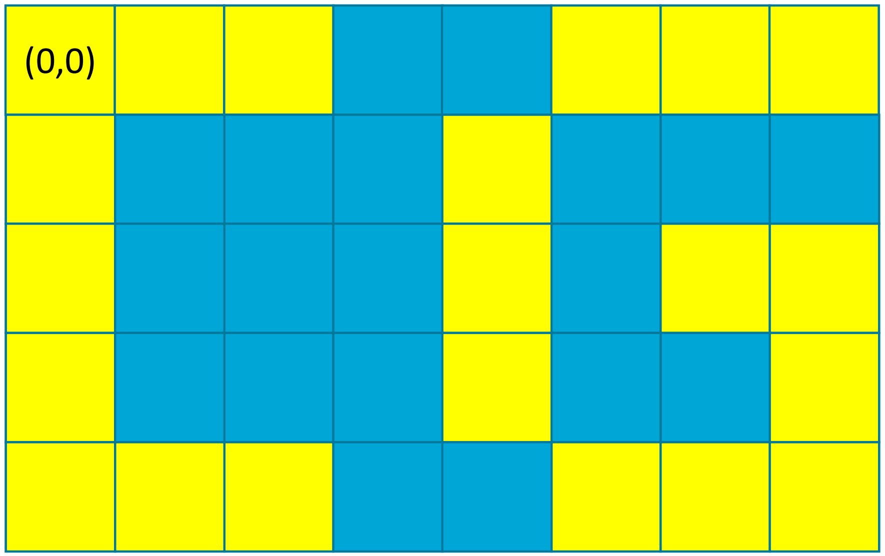
Representation

- Example:
- One channel (grayscale) image (resolution 8x5)
- What index accesses pixel (3,4)?



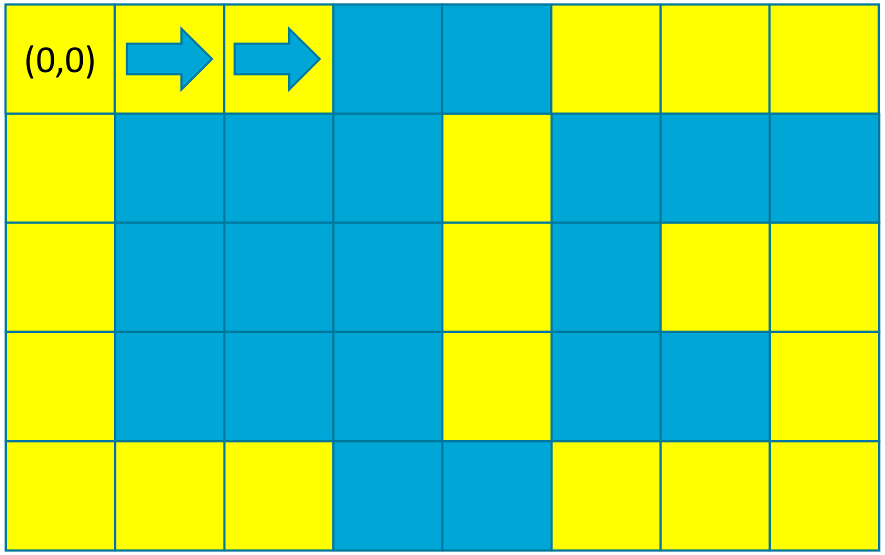
Representation

- Example:
- One channel (grayscale) image (resolution 8x5)
- What index accesses pixel (3,4)?



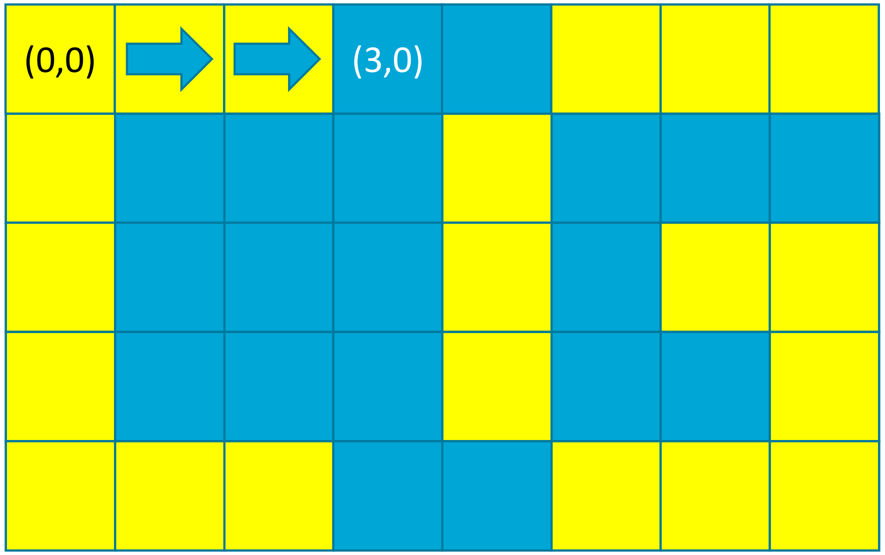
Representation

- Example:
- One channel (grayscale) image (resolution 8x5)
- What index accesses pixel (3,4)?



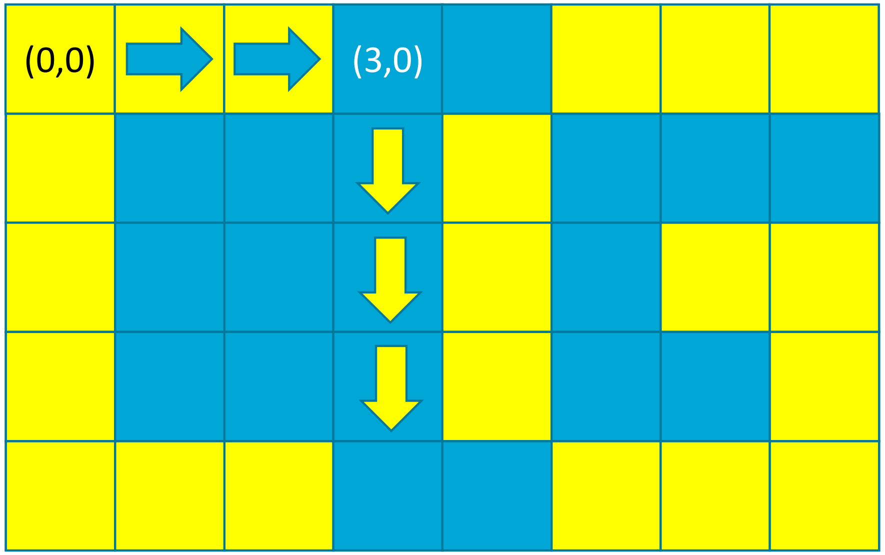
Representation

- Example:
- One channel (grayscale) image (resolution 8x5)
- What index accesses pixel (3,4)?



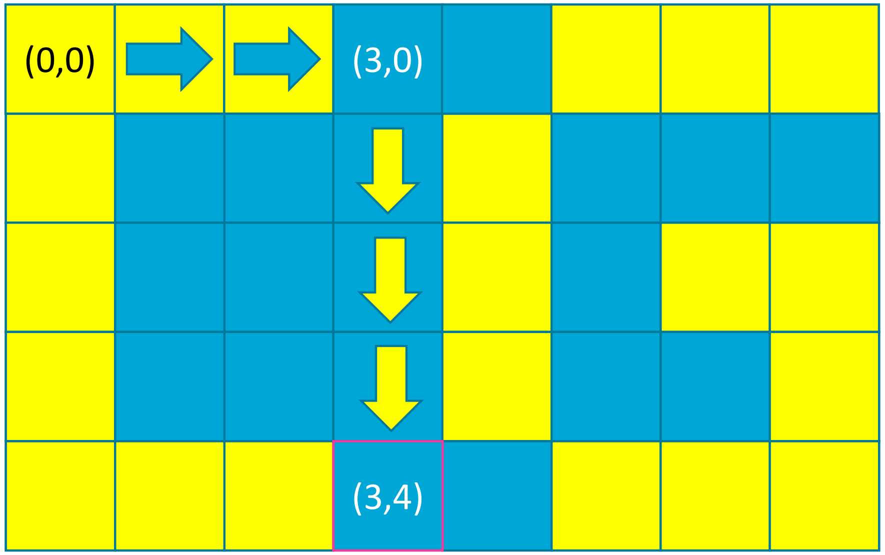
Representation

- Example:
- One channel (grayscale) image (resolution 8x5)
- What index accesses pixel (3,4)?



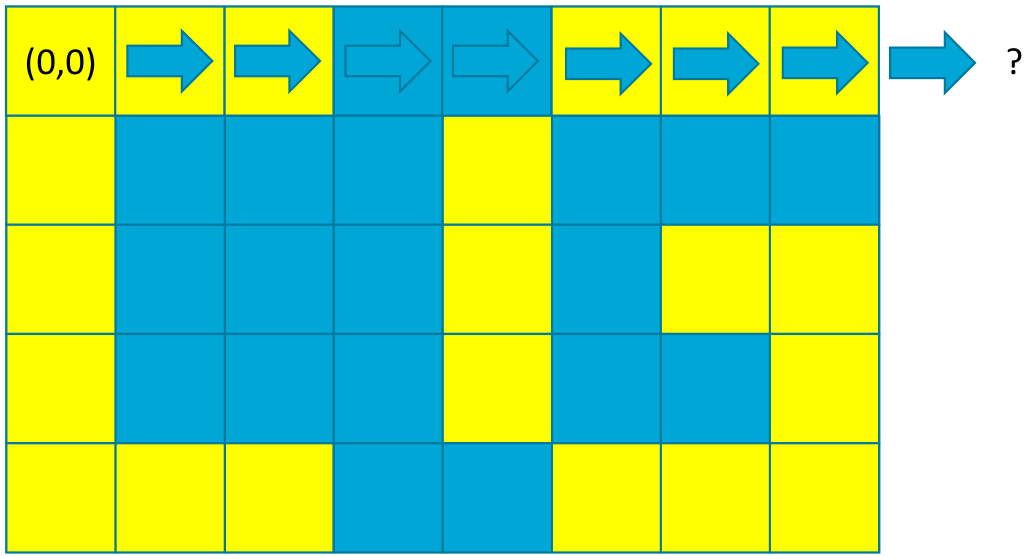
Representation

- Example:
- One channel (grayscale) image (resolution 8x5)
- What index accesses pixel (3,4)?



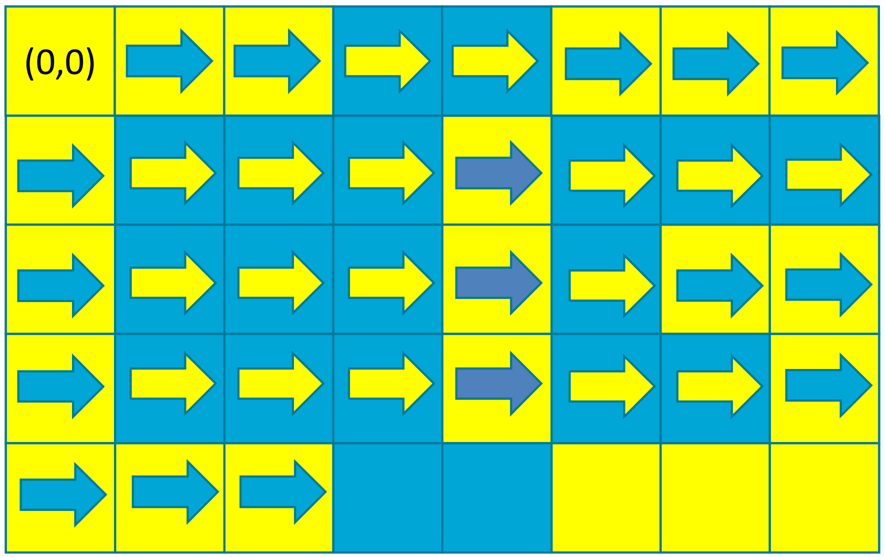
Representation

- Example:
- One channel (grayscale) image (resolution 8x5)
- What index accesses pixel (3,4)? ~~3~~*(j*width+i) $\boxed{=4*8+3=35}$



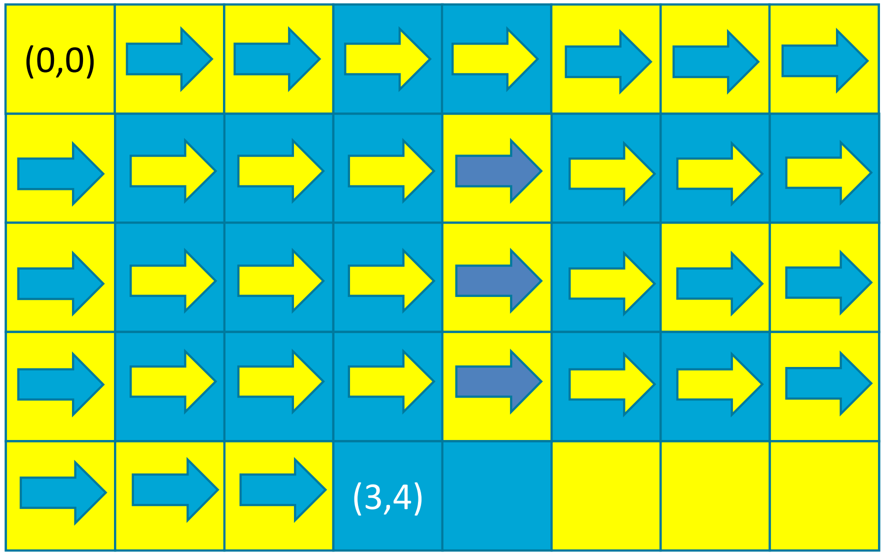
Representation

- Example:
- One channel (grayscale) image (resolution 8x5)
- What index accesses pixel (3,4)?~~3*~~(j*width+i) =4*8+3=35

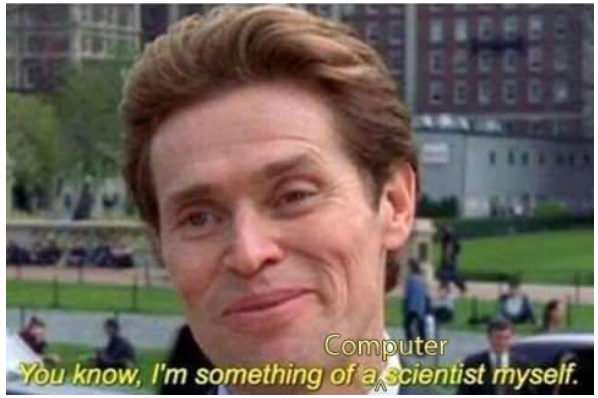


Representation

- Example:
- One channel (grayscale) image (resolution 8x5)
- What index accesses pixel (3,4)? ~~3*~~(j*width+i) =4*8+3=35

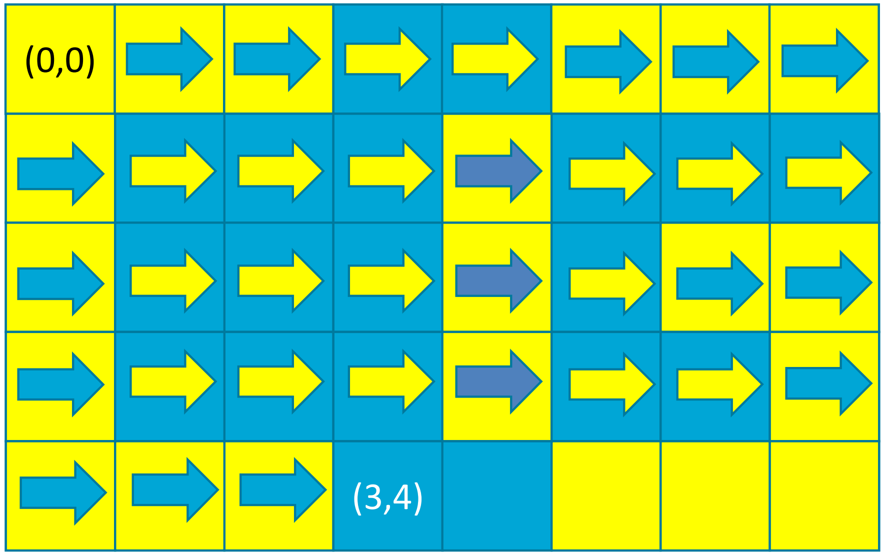


When you start counting from 0 instead of 1



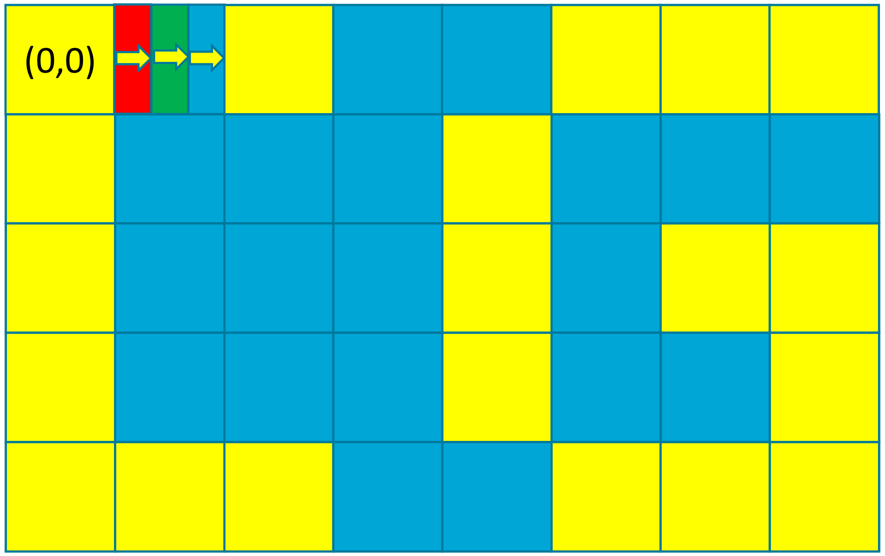
Representation

- Example:
- **Three-channel (RGB)** image (resolution 8x5)
- What index accesses pixel (3,4)?



Representation

- Example:
- **Three-channel (RGB)** image (resolution 8x5)
- What index accesses pixel (3,4)? 3*(j*width+i) =3*(4*8+3)
  =105



Representation

- How to find a pixel (i,j)
  given index I for a grayscale image?

- Solution:

  i= I%width   //modulo = remainder of division

  j= I / width

  (**ATTENTION**: Integer division! E.g.,5/3=1)

Representation

- Let's test it:
- Grayscale Image (resolution 3x3), index I=8...
- Solution:

i= I%width   i= 8%3= 2   // 8=2*3+2=8

j= I / width   j= 8/3 = 2   // , thus pixel (2,2)

(**ATTENTION**: Integer division! E.g.,5/3=1)

Why does it work?
I/width are the full rows that fit into I
I%width is the remainder, thus the offset in the row.

Simple Image Class

Disclaimer: the following code is not a coding paradigm, e.g., you should use **const** qualifiers and private variables. The goal here is to reduce the amount of text…

```
class Image
{public:
  std::vector<float> data;
  int w, h;
};
```

Simple Image Class

```
class Image
{public:
  std::vector<float> data;
  int w, h;
  Image(int wl, int hl){w=wl; h=hl; data.resize(3*w*h);};
  Image(const Image & i){w=i.w; h=i.h, data=i.data;};
};
```

Simple Image Class

```
class Image
{public:
  std::vector<float> data;
  int w, h;
  Image(int wl, int hl){w=wl; h=hl; data.resize(3*w*h);};
  Image(const Image & i){w=i.w; h=i.h, data=i.data;};
  float & pixel(int i, int j, int col){
    return data[3*(i+j*w)+col]; //Problem?
  };
};
```

Simple Image Class

```
class Image
{public:
 std::vector<float> data; float border=0;
 int w, h;
 Image(int wl, int hl){w=wl; h=hl; data.resize(3*w*h);};
 Image(const Image & i){w=i.w; h=i.h, data=i.data;};
 float & pixel(int i, int j, int col){
  if (i<0 || i>=w || j<0 || j>=h || col<0 || col>2)
   return border;
  else return data[3*(i+j*w)+col];
 };
};
```

Images

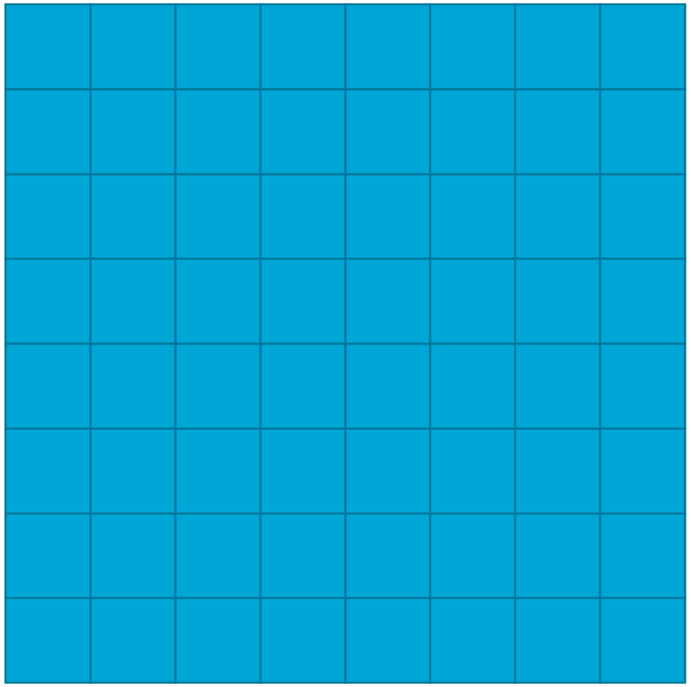
- What is an image?
- How to represent it in memory?
- How to access individual pixels?
- How can we process images?
- How are images stored?

Filter

Filter

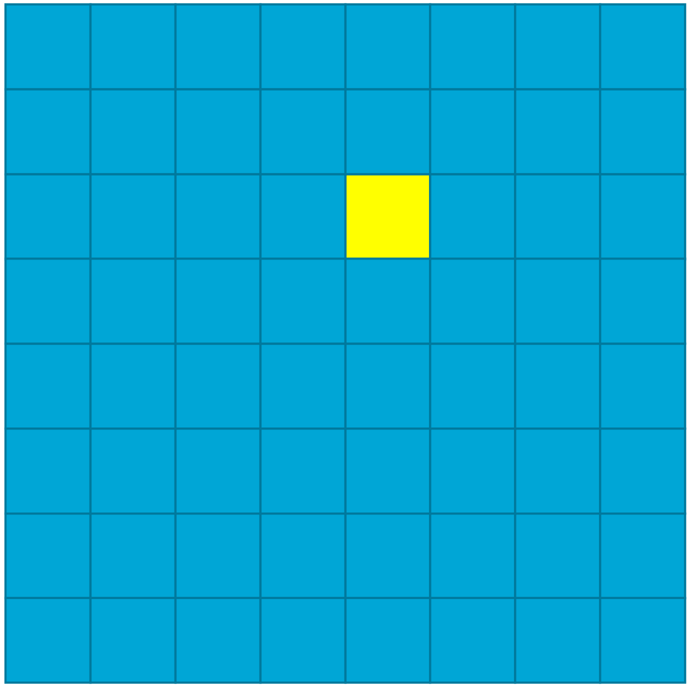
Filter

- Example Box Filter 3x3



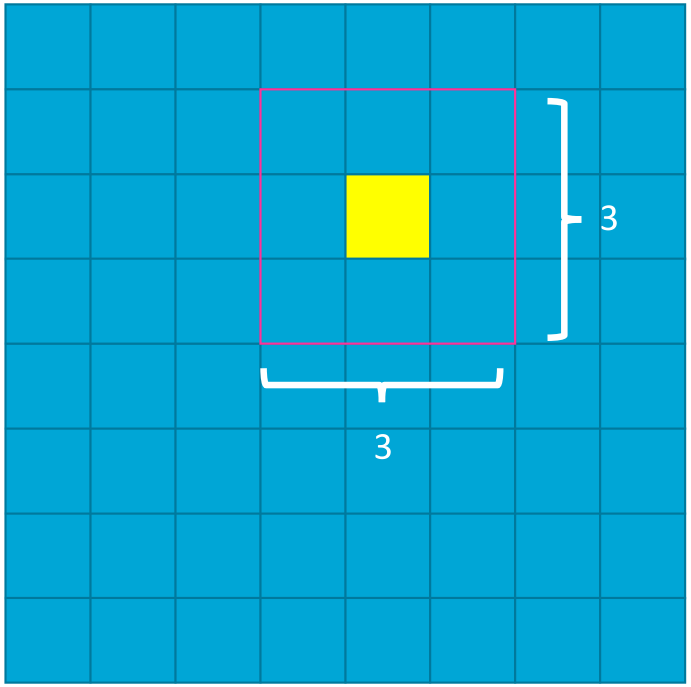
Filter

- Example Box Filter 3x3



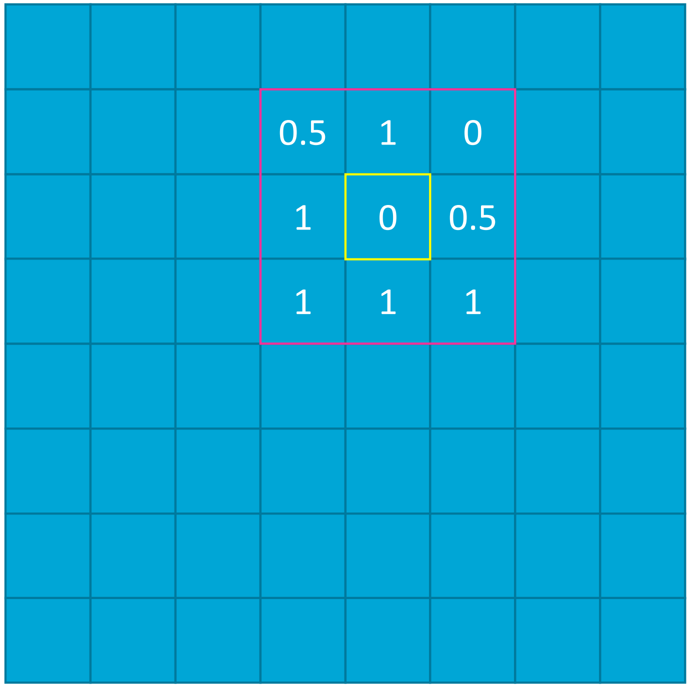
Filter

- Example Box Filter 3x3



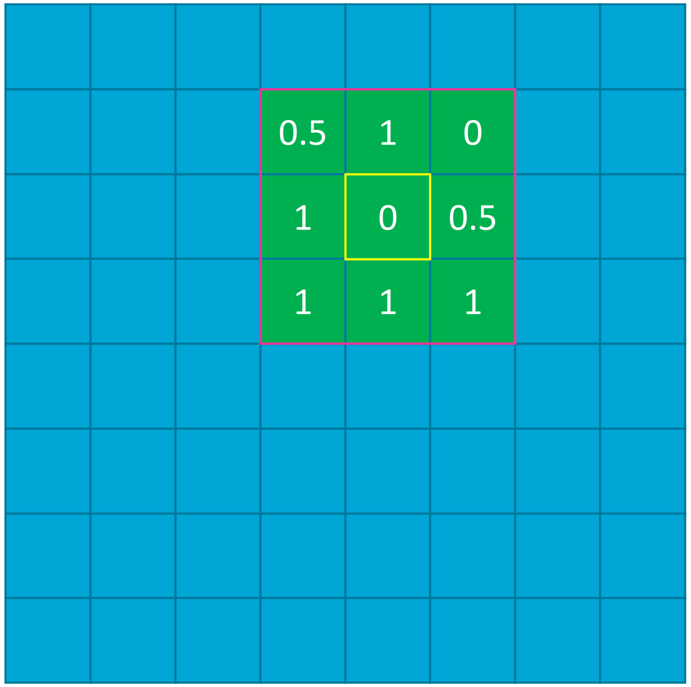
Filter

- Example Box Filter 3x3



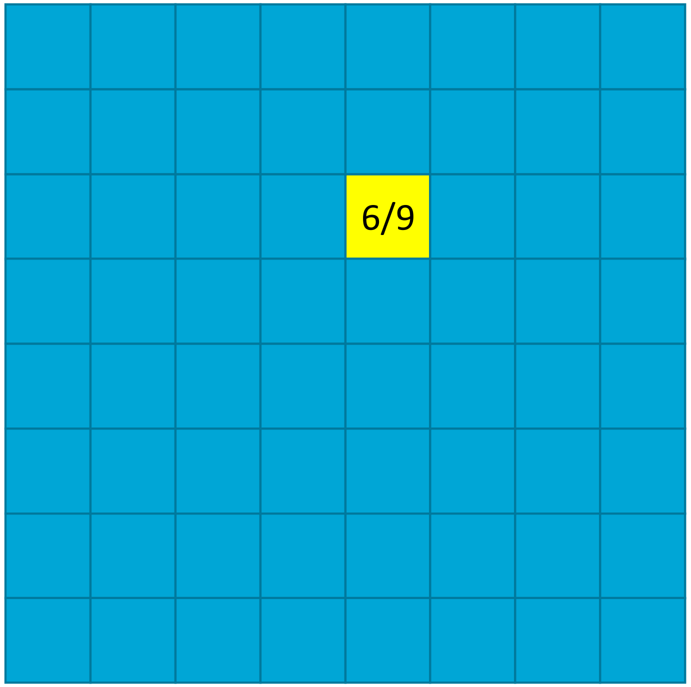
Filter

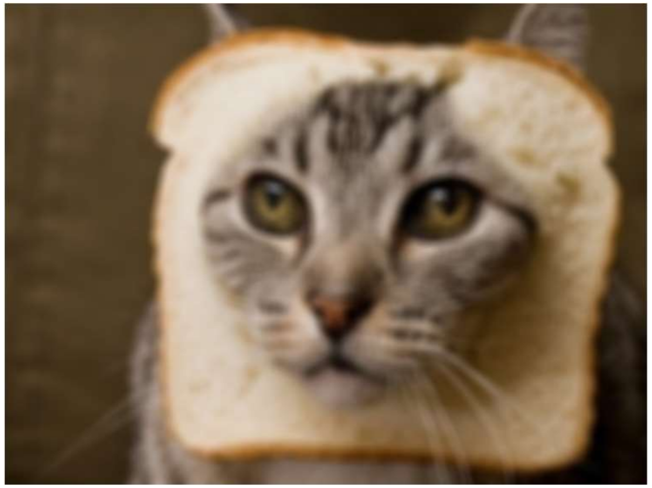
- Example Box Filter 3x3



Filter

- Example Box Filter 3x3





Box Filter Code 1/2

```
Image FilterImage(Image & source, int filterSize)

{

 // we create a result image
 Image result(source);



//and process every channel of every pixel independently

 for (int i=0;i<source.w;++i)

  for (int j=0;j<source.h;++j) // for each pixel

   for (int col=0;col<3;++col) // for each color channel

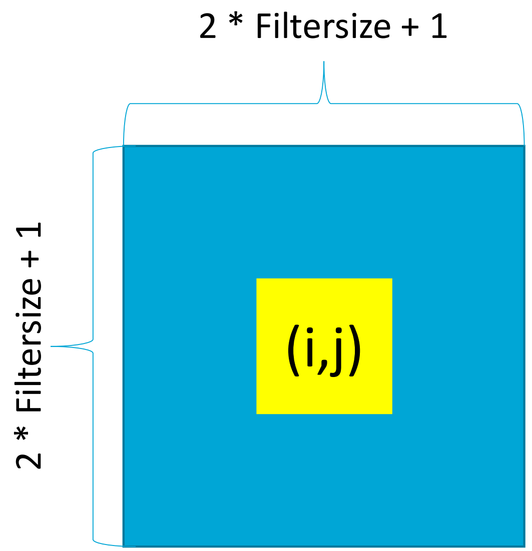
    result.pixel(i,j,col)=boxFilter(source, i, j, col, filterSize);

 return result;

};
```

Box Filter Code 2/2

```
float boxFilter (Image & source, int i, int j, int col, int filterSize)
{
  filterSize=max(1,filterSize);
  float sum=0;
  //Average pixels in the box-filter region
  for (int x=-filterSize;x<filterSize+1;++x)
    for (int y=-filterSize;y<filterSize+1;++y)
      sum+=source.pixel(i+x,j+y,col);
  sum/=(2*filterSize+1)*(2*filterSize+1);
  return sum;
};
```



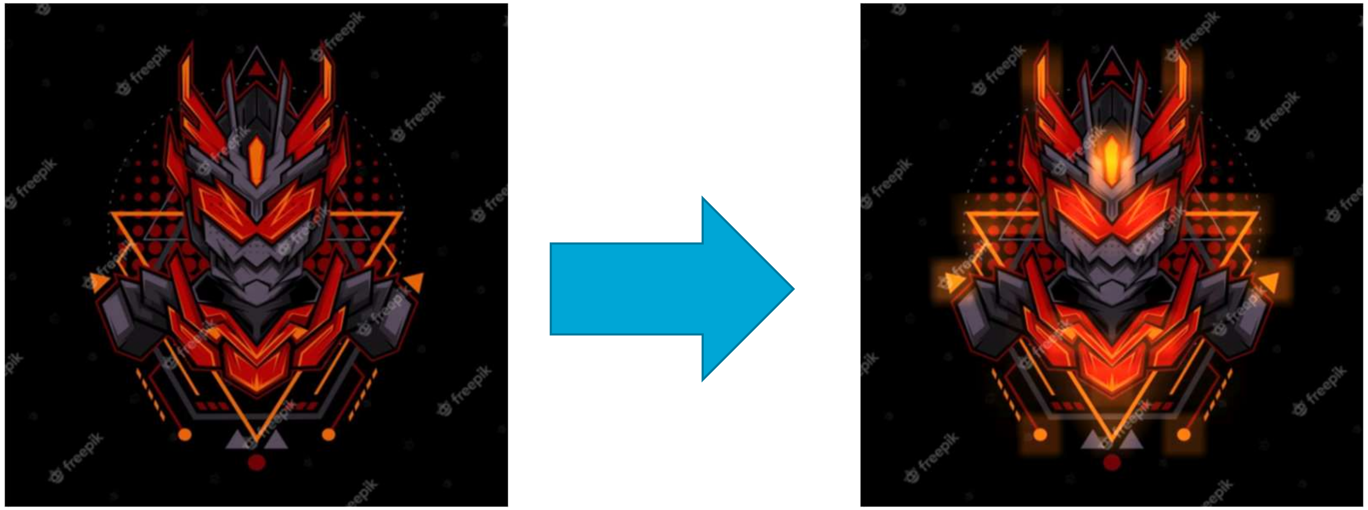
Note: 3x3 Box Filter means Filtersize=1

Example Application

- Bloom effects (here Witcher 3)

Example Application

- Bloom effects



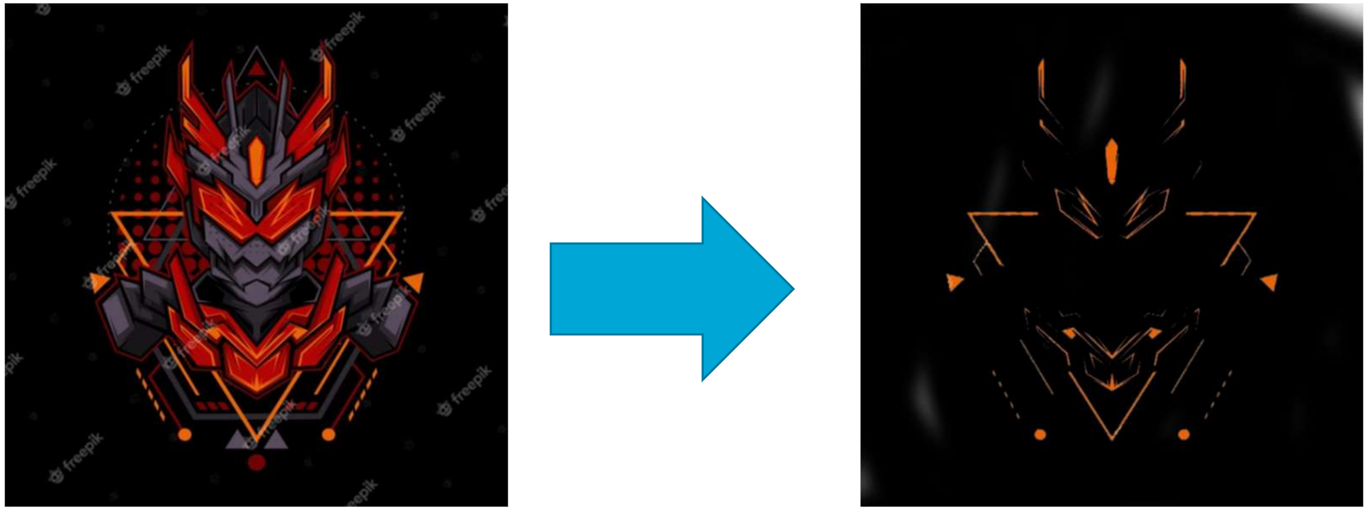
https://nl.freepik.com/premium-vector/dark-fire-ranger-geometry-style_3799192.htm

Example Application

- Bloom effects

Example Application

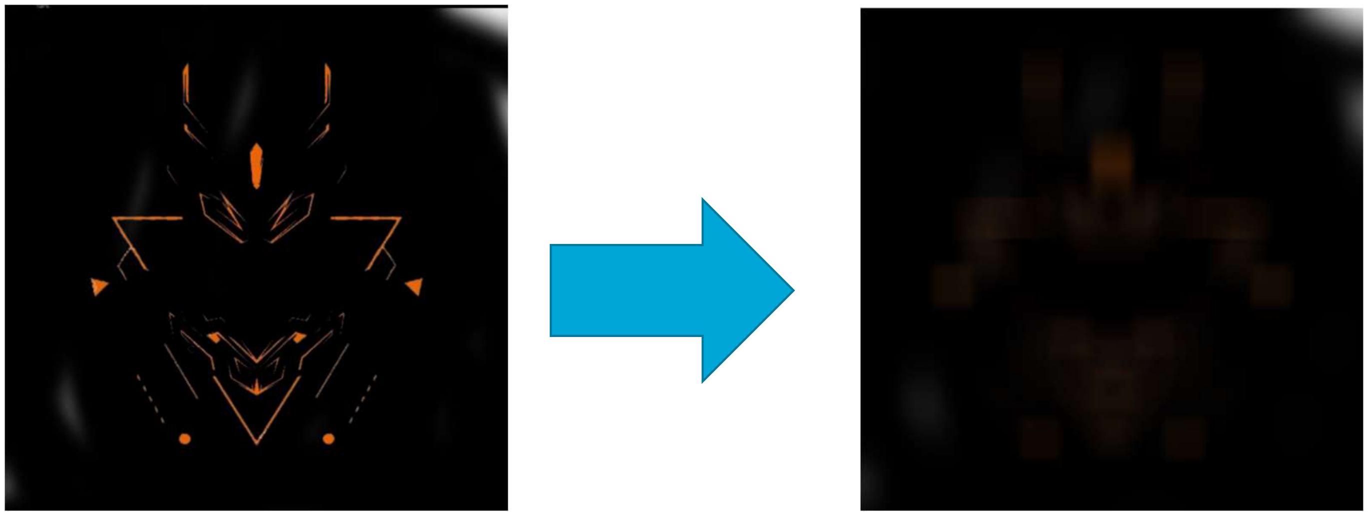
1) Threshold – only keep large values (e.g., >0.9)



Example Application

1) Threshold – only keep large values (e.g., >0.9)

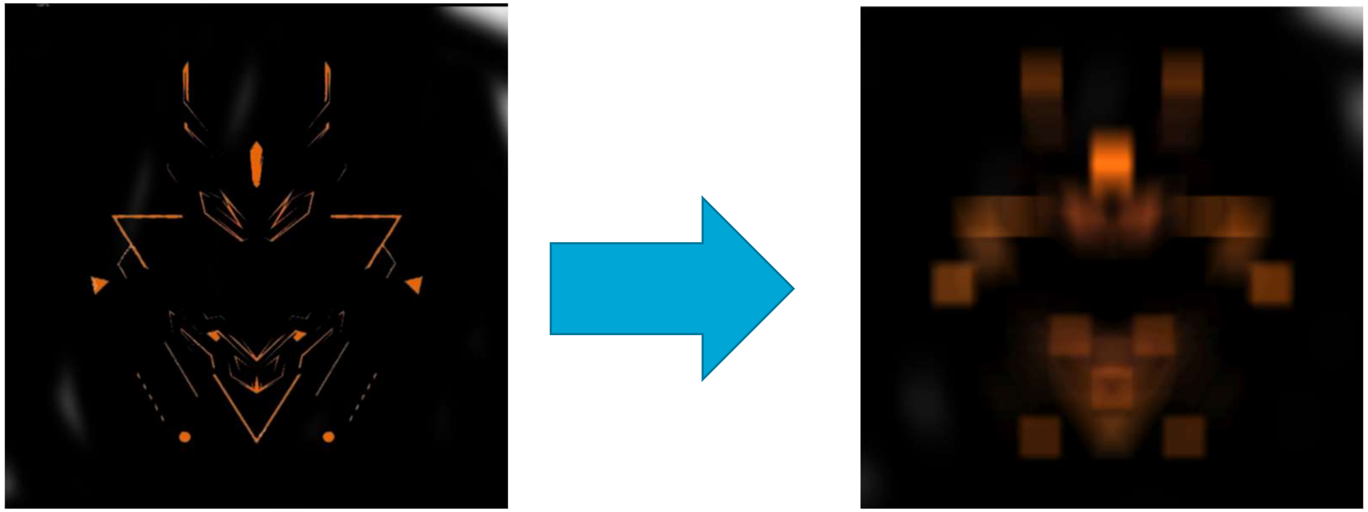
2) Box filter on thresholded image



Example Application

1) Threshold – only keep large values (e.g., >0.9)

2) Box filter on thresholded image and scale



Example Application

1) Threshold – only keep large values (e.g., >0.9)

2) Box filter on thresholded image and scale

3) Add to the original

General Filter Code

```
float filteredPixel(Image & source, Image & filter
                              int i, int j, int col)

{

  float sum=0;

  for (int x=0;x<filter.w;++x)

    for (int y=0;y<filter.h;++y)

      sum+=

              filter.pixel(x,y,col)

              * source.pixel(i+x-filter.w/2,j+y-filter.h/2,col);  //Place center of the filter at (i,j)

  return sum;

};
```

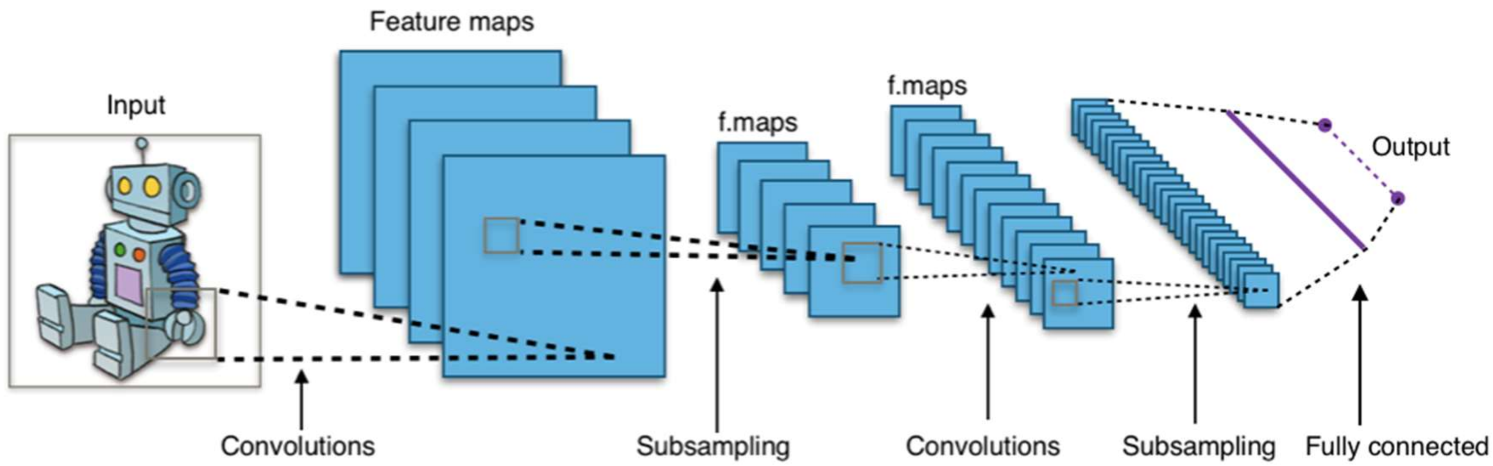
If you want to perform a box filtering, the image “filter” should contain 1.0f/(filter.w*filter.h) in all pixels.
Homework: verify this claim

Example Application 2

Machine Learning:

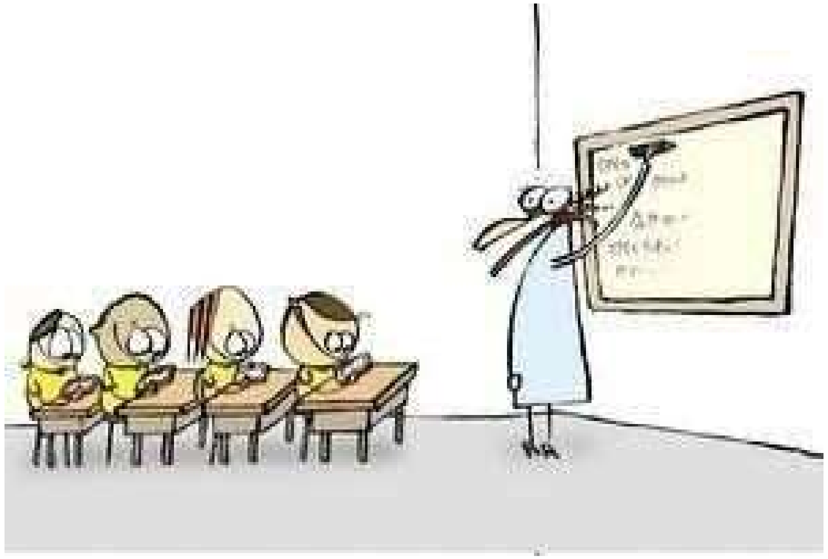
Convolutional Neural Networks Successive filtering and thresholding.

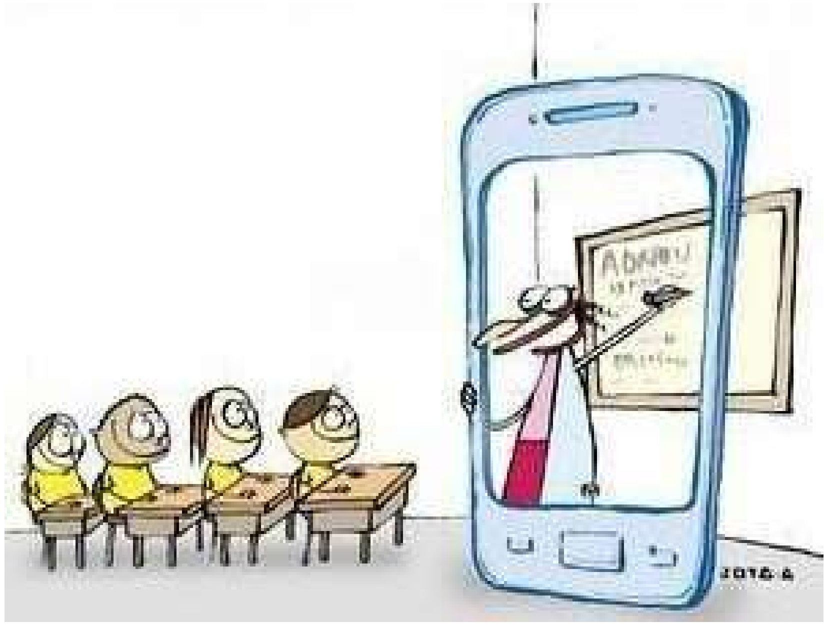
“Filters” are optimized during training.



https://upload.wikimedia.org/wikipedia/commons/6/63/Typical_cnn.png

Questions?





Images

- What is an image?
- How to represent it in memory?
- How to access individual pixels?
- How can we process images?
- How are images stored?

Storing an image

- Several formats for images
  - Some compress data but lose details
  - Others are lossless but require more space

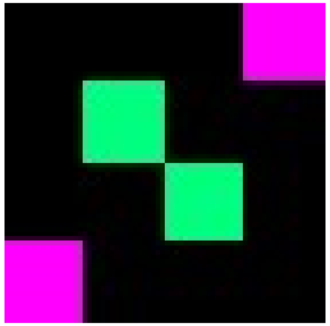
- Many options, we will only cover two:
  - PPM – very simple, big file size, lossless
  - JPEG – complex, small file size, lossy

Simple Image Format: PPM

Each PPM image consists of a header and image data

Example file opened in Text Editor:

```
P3
4 4
15
0  0  0    0  0  0    0  0  0   15  0 15
0  0  0    0 15  7    0  0  0    0  0  0
0  0  0    0  0  0    0 15  7    0  0  0
15  0 15    0  0  0    0  0  0    0  0  0
```



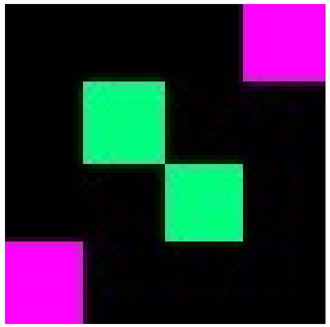
Simple Image Format: PPM

Each PPM image consists of a header and image data. Header contains:

- "magic number“, e.g., “P3” for human-readable pixel values in RGB format
- Image width <Whitespace> Image height <Whitespace>
- Maximum color value between [0,65535] <Whitespace> (usually a newline).

Example file opened in Text Editor:

```
P3
4 4
15
 0  0  0    0  0  0    0  0  0   15  0 15
 0  0  0    0 15  7    0  0  0    0  0  0
 0  0  0    0  0  0    0 15  7    0  0  0
15  0 15    0  0  0    0  0  0    0  0  0
```



TU Delft

Simple Image Format: PPM

Each PPM image consists of a header and image data. Header contains:

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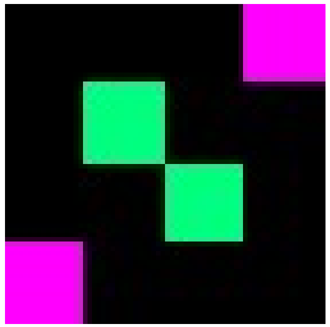
4 4
15
 0  0  0   0  0  0   0  0  0  15  0 15
 0  0  0   0 15  7   0  0  0   0  0  0
 0  0  0   0  0  0   0 15  7   0  0  0
15  0 15   0  0  0   0  0  0   0  0  0
```



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- "magic number“, e.g., “P3” for human-readable pixel values in RGB format
- **Image width <Whitespace> Image height <Whitespace>**
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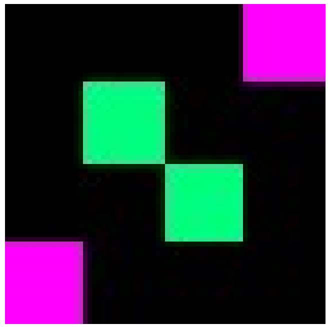
Example file opened in Text Editor:

```
P3
4 4
15
 0  0  0    0  0  0    0  0  0   15  0 15
 0  0  0    0 15  7    0  0  0    0  0  0
 0  0  0    0  0  0    0 15  7    0  0  0
15  0 15    0  0  0    0  0  0    0  0  0
```



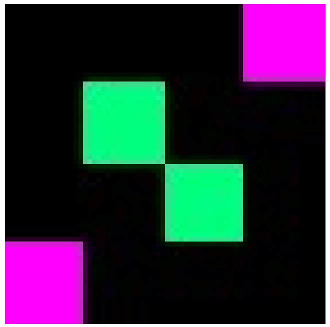
Simple Image Format: PPM

Each PPM image consists of a header and image data. Header contains:

- "magic number“, e.g., “P3” for human-readable pixel values in RGB format
- Image width <Whitespace> Image height <Whitespace>
- **Maximum color value between [0,65535] <Whitespace> (usually a newline).**

Example file opened in Text Editor:

```
P3
4 4
15
 0  0  0    0  0  0    0  0  0   15  0 15
 0  0  0    0 15  7    0  0  0    0  0  0
 0  0  0    0  0  0    0 15  7    0  0  0
15  0 15    0  0  0    0  0  0    0  0  0
```



Simple Image Format: PPM

Each PPM image consists of a header and image data. Header contains:

- "magic number“, e.g., “P3” for human-readable pixel values in RGB format
- Image width <Whitespace> Image height <Whitespace>
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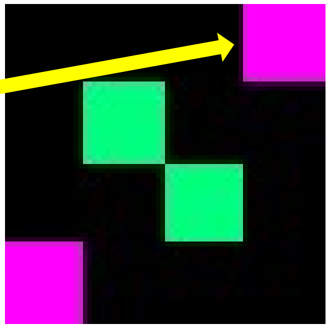
Example file opened in Text Editor:

```
P3
4 4
15
 0  0  0    0  0  0    0  0  0   15  0 15
 0  0  0    0 15  7    0  0  0    0  0  0
 0  0  0    0  0  0    0 15  7    0  0  0
15  0 15    0  0  0    0  0  0    0  0  0
```



Simple Image Format: PPM

- Simple but inefficient in terms of storage

- Filesize directly related to
  - bytes per color channel (if not human-readable, otherwise worse…)
  - resolution

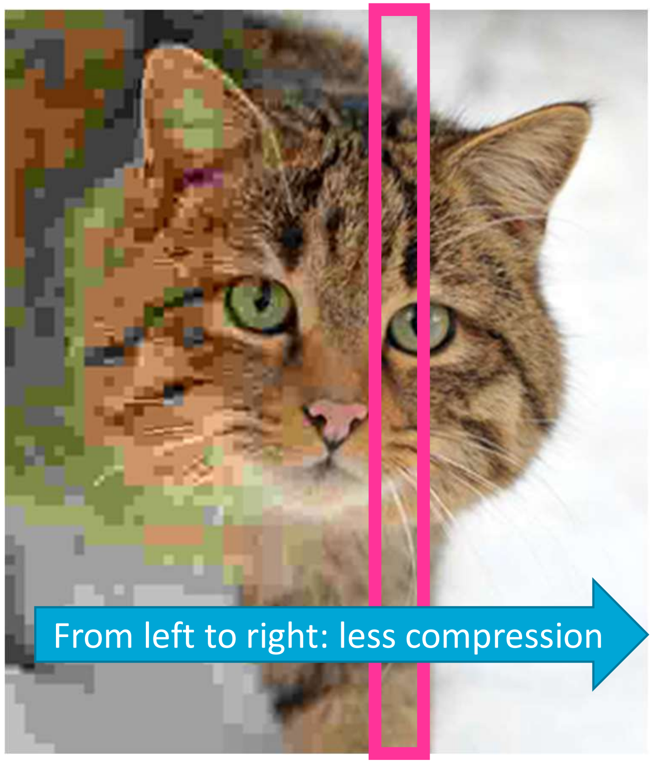
Storing an image

- Many options, we will only cover two:
  - PPM – very simple, big file size, lossless
  - **JPEG – complex, small file size, lossy**

DISCLAIMER:
Method is complicated!
You will NOT be asked to reproduce
or fully understand the details of JPEG compression!
The topic will, for example, come back in **Signal Processing**

Complex Image Format: JPEG

- Compression by reducing quality (lossy)
- 1:10 compression still has high quality



Complex Image Format: JPEG

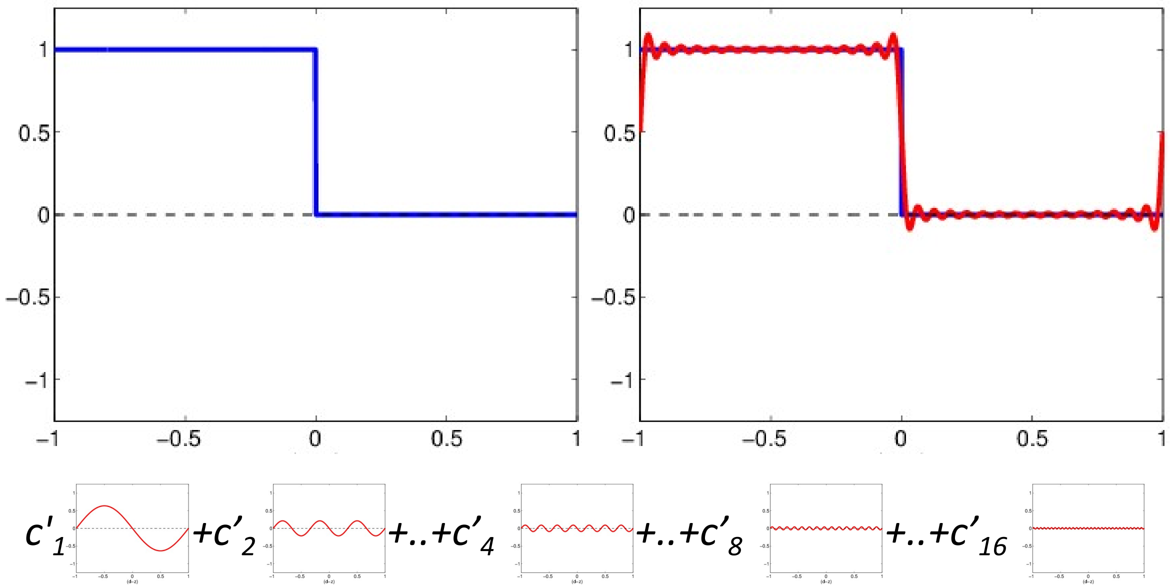
Main idea:

- Use frequency decomposition
- Remove high frequencies first

Complex Image Format: JPEG

Slightly simplified, we can say that:

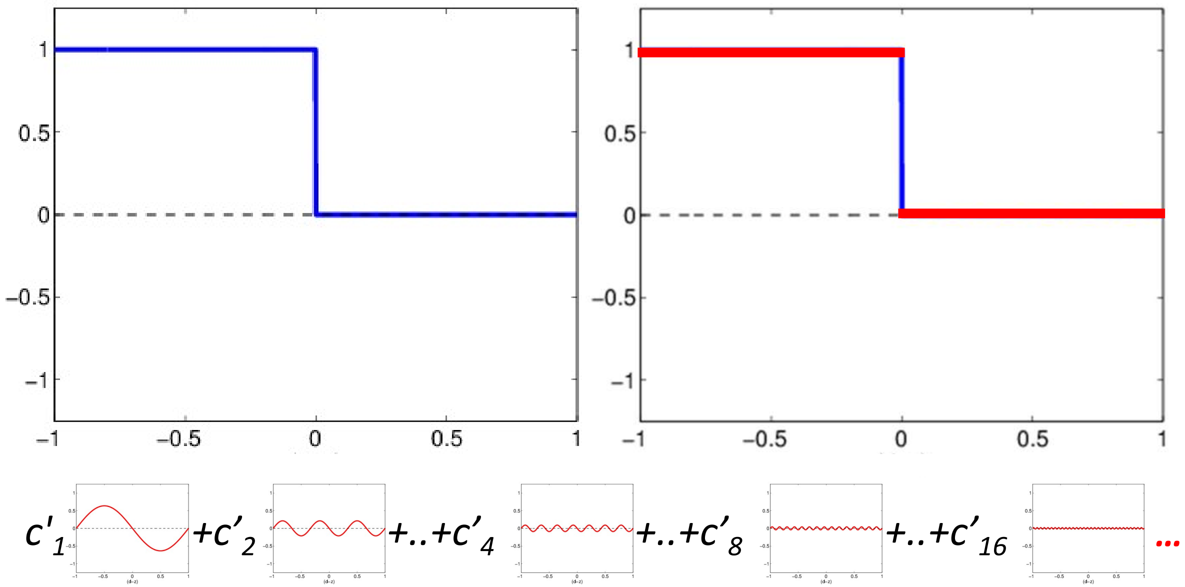
- Any (Riemann) integrable function has a converging Fourier series



Complex Image Format: JPEG

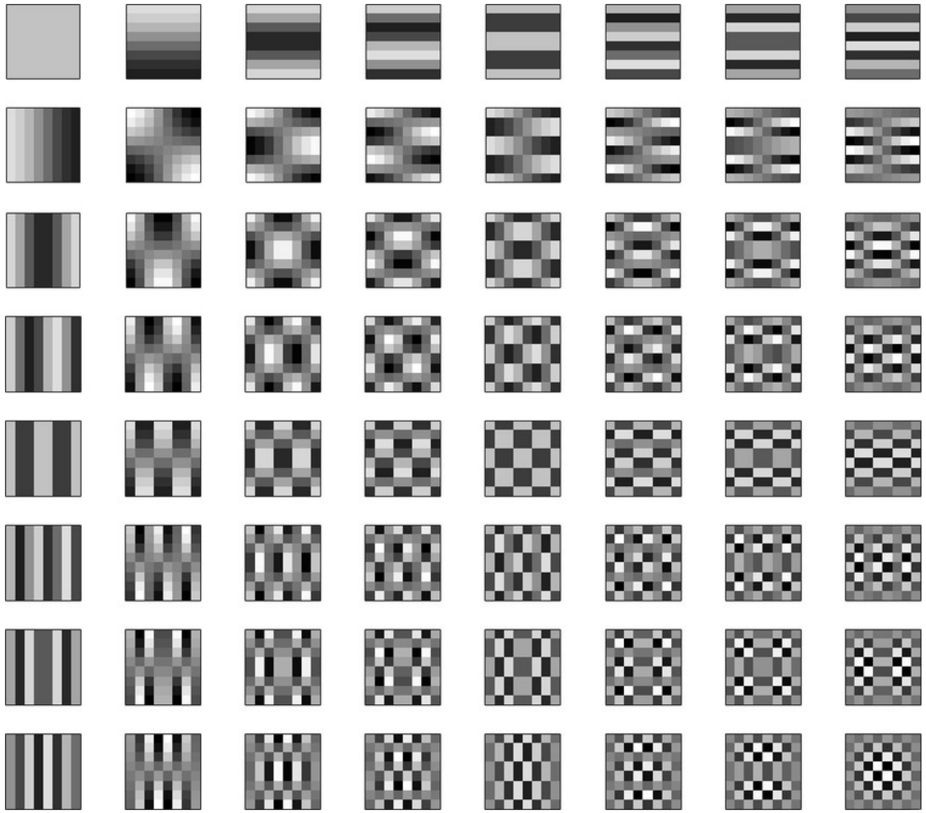
Slightly simplified, we can say that:

- Any (Riemann) integrable function has a converging Fourier series



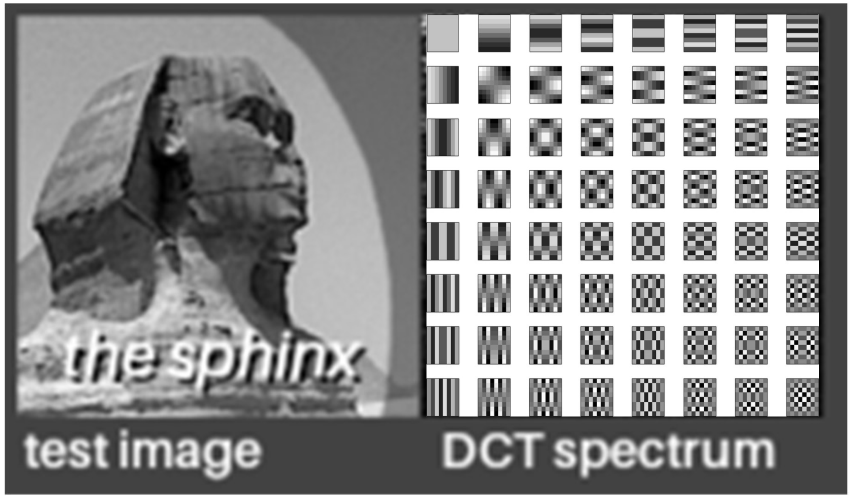
Complex Image Format: JPEG

- Discrete Cosine Transform
- These are the base functions that are linearly combined to reproduce the image



Complex Image Format: JPEG

- Discrete Cosine Transform

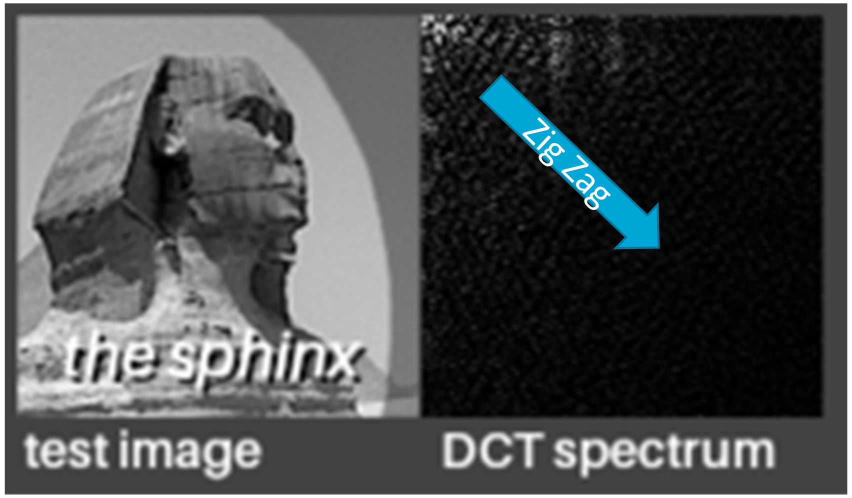


Observation:
Coefficients of high frequencies are low (bottom right is mostly black), this holds generally for natural images.

Wikimedia

Complex Image Format: JPEG

- Discrete Cosine Transform



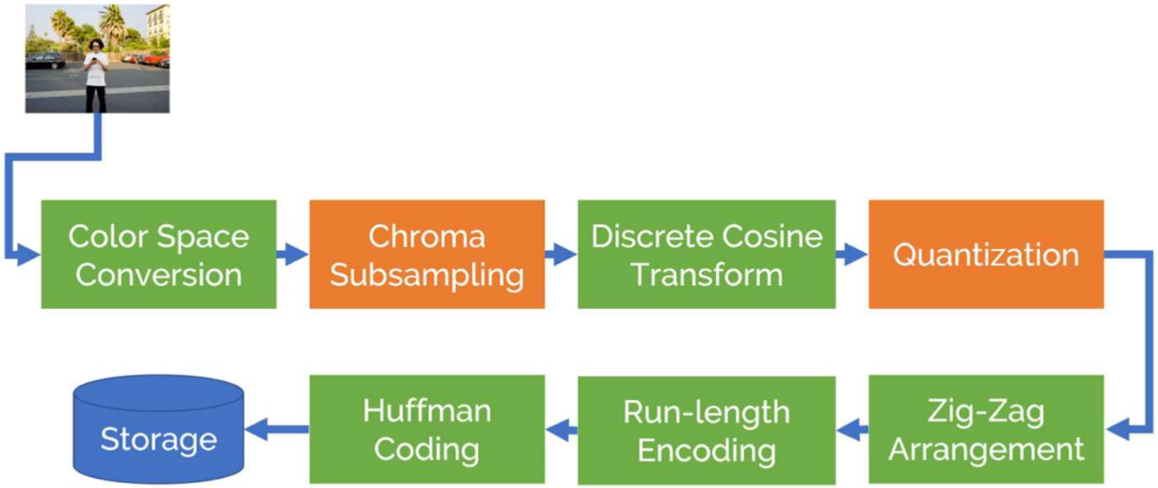
Wikimedia

Observation:
Coefficients of high frequencies are low (bottom right is mostly black), this holds generally for natural images.
Idea:
Quantize values (e.g., transform to bytes)
Move from top left to bottom right (zigzag)
Efficient encoding (e.g., count consecutive 0s)

1	3	6
2	5	8
4	7	9

Want to know more about JPEG?

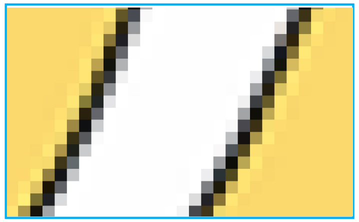
- Principle of JPEG compression
- Leo Isikdogan: https://www.youtube.com/watch?v=Ba89cI9eIg8



Details on Signal Processing are in the book as well Chapter 9 – NOT mandatory to read

JPEG Compression

- Very efficient
- Widespread use (e.g., internet and cameras)
- Quality is slightly reduced



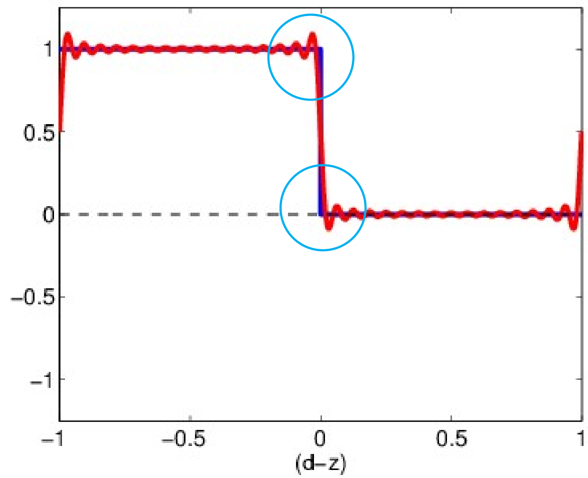


Image Storage != Image in use

- When working with images (modifications, display, conversion...) the image is usually in the decompressed form

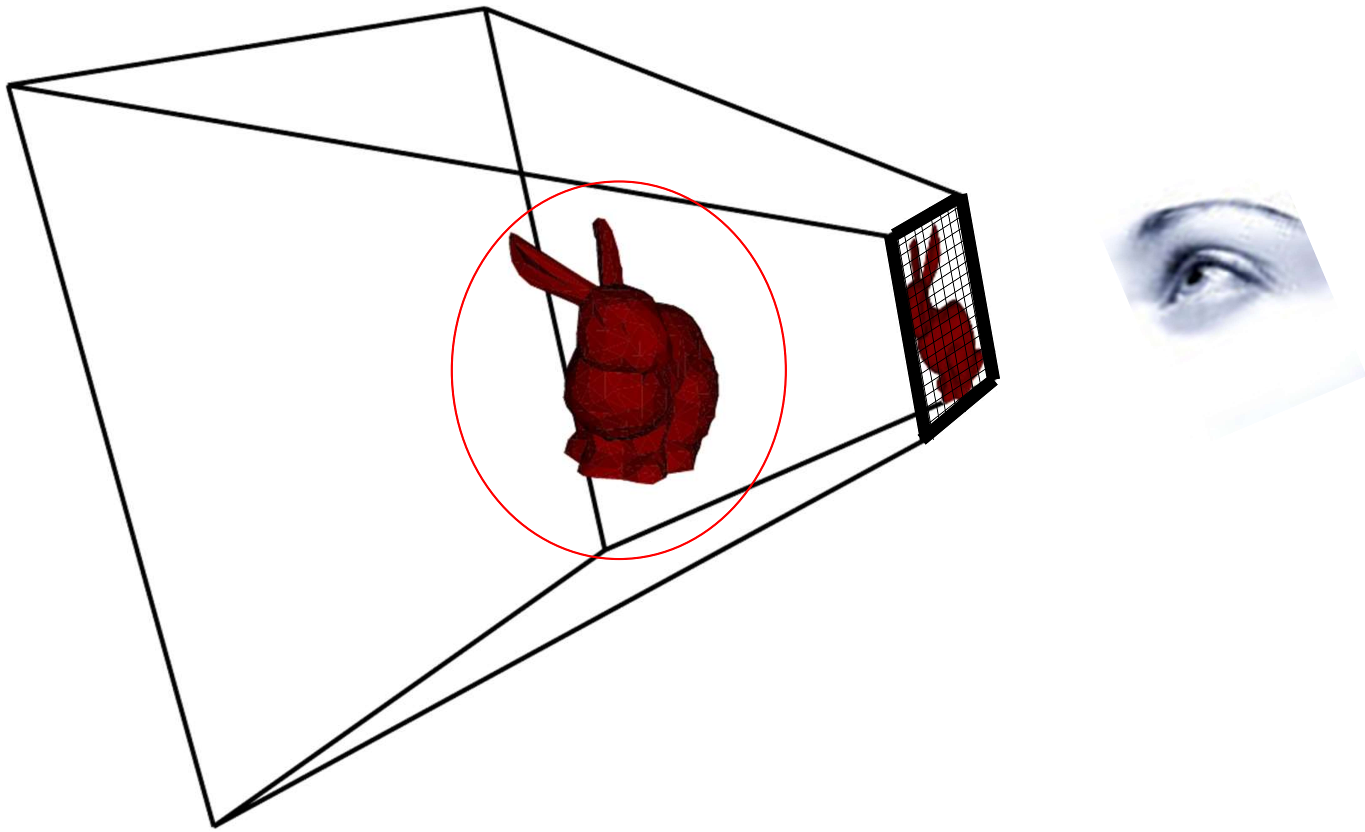
- JPEG requires more memory when the image is loaded in memory compared to its file size

Images

- What is an image?
- How to represent it in memory?
- How to access individual pixels?
- How can we process images?
- How are images stored?

BREAK

Today’s Part 2 - Geometry

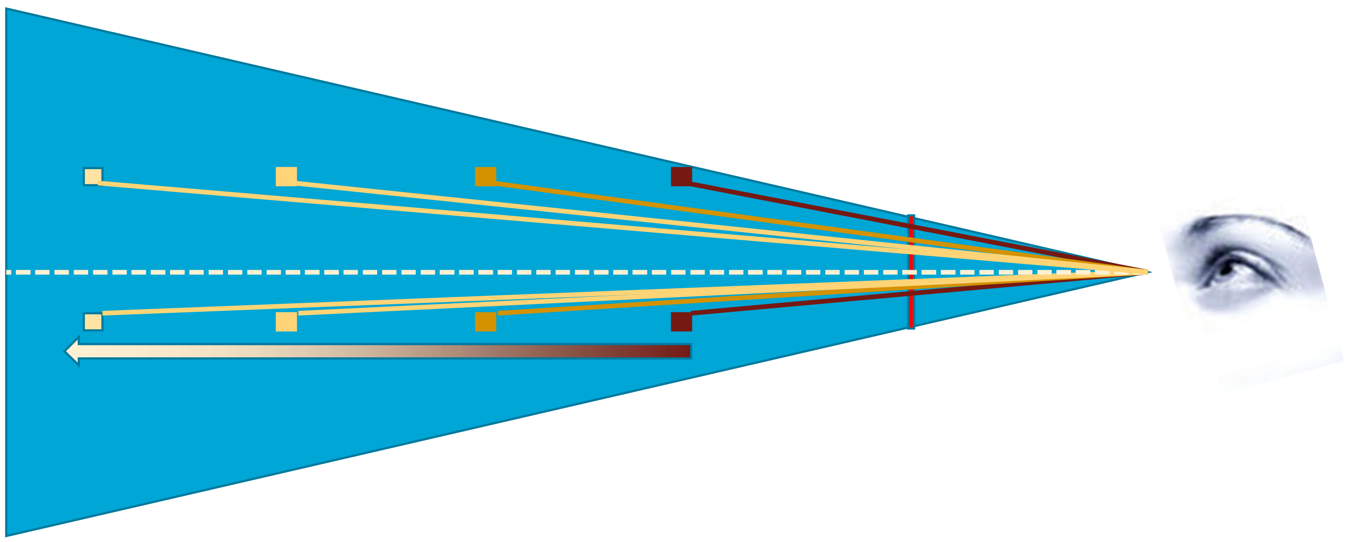


Linking Algebra and GPUs

What has algebra to do with Graphics Cards (GPUs)?

How to draw with accurate perspective?

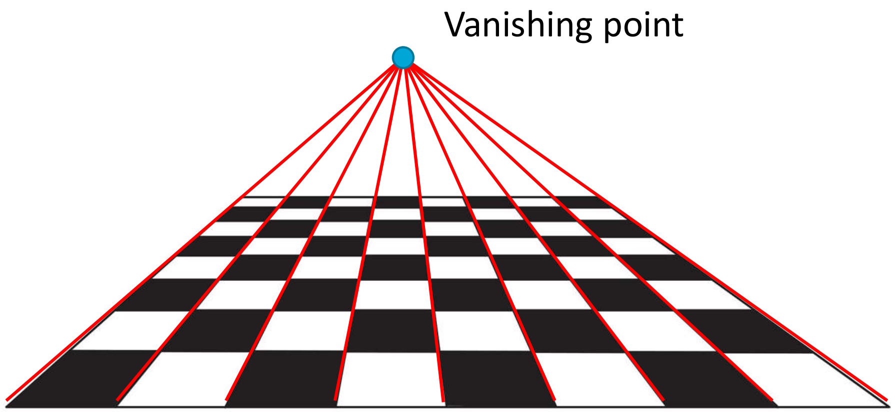
Linear Perspective



Linear Perspective

Linear Perspective

- Central Perspective



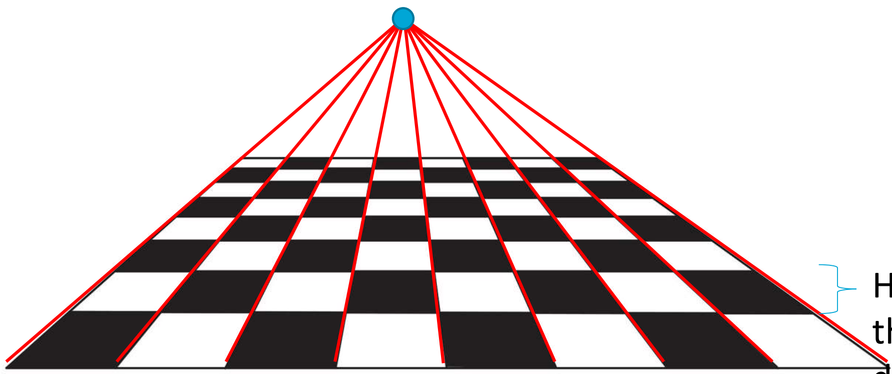
Johannes Vermeer, *The Milkmaid,* Rijksmuseum

Detail of an x-ray of *The Milkmaid*

(courtesy Rijksmuseum)

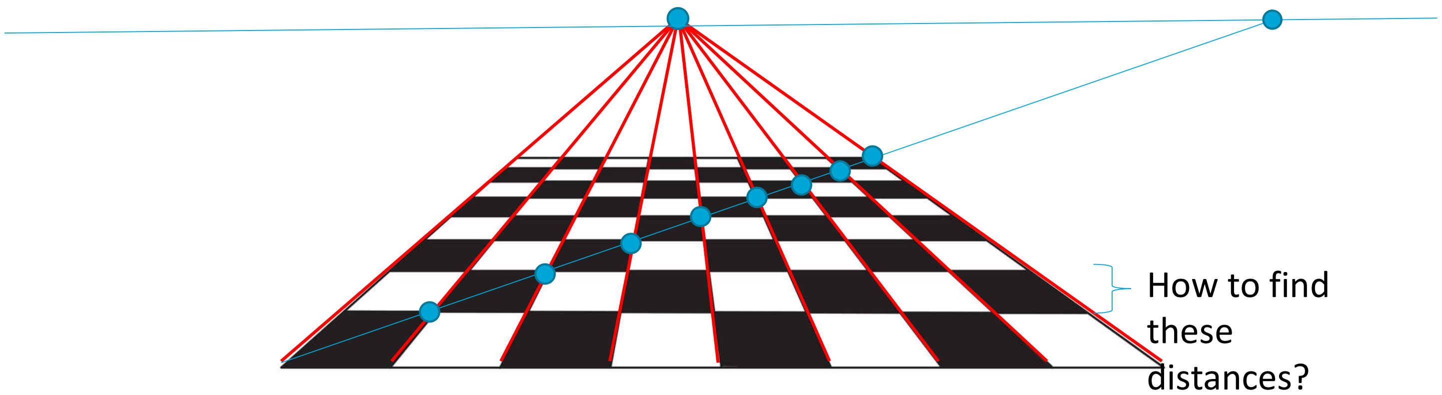
Linear Perspective

- Central Perspective



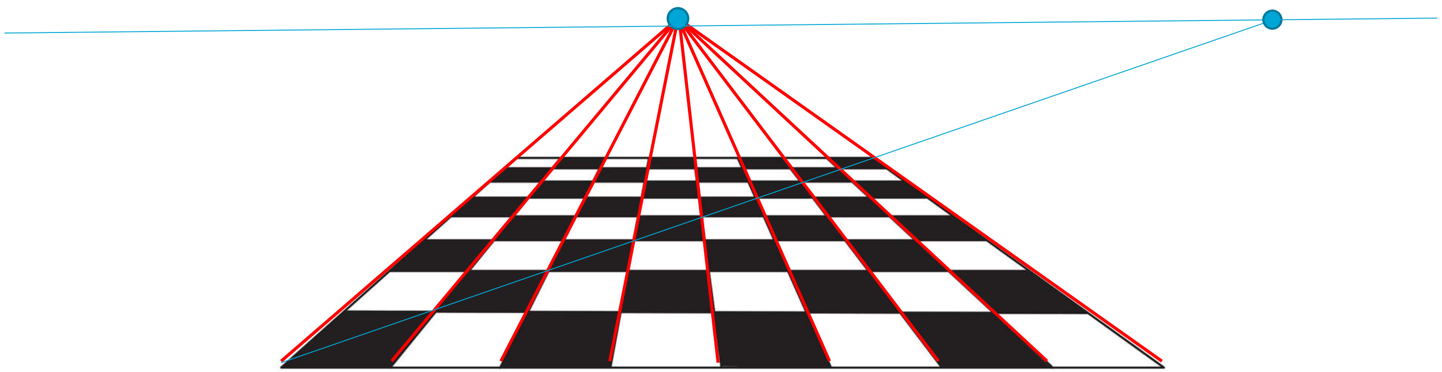
Linear Perspective

- Central Perspective



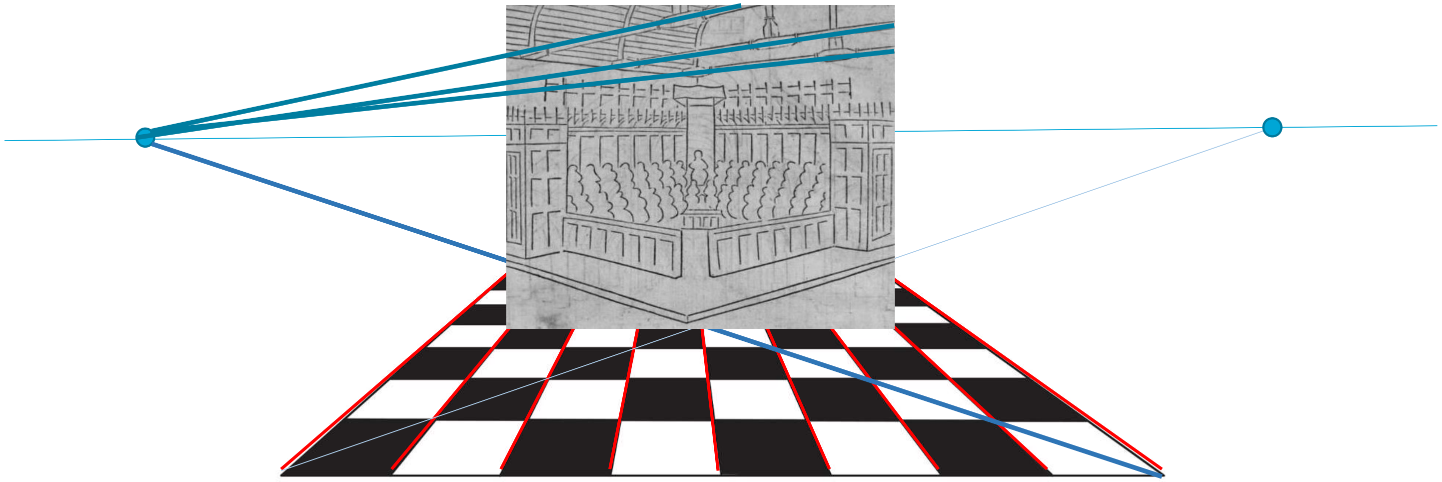
Linear Perspective

- Central Perspective



Linear Perspective

- Two Point Perspective



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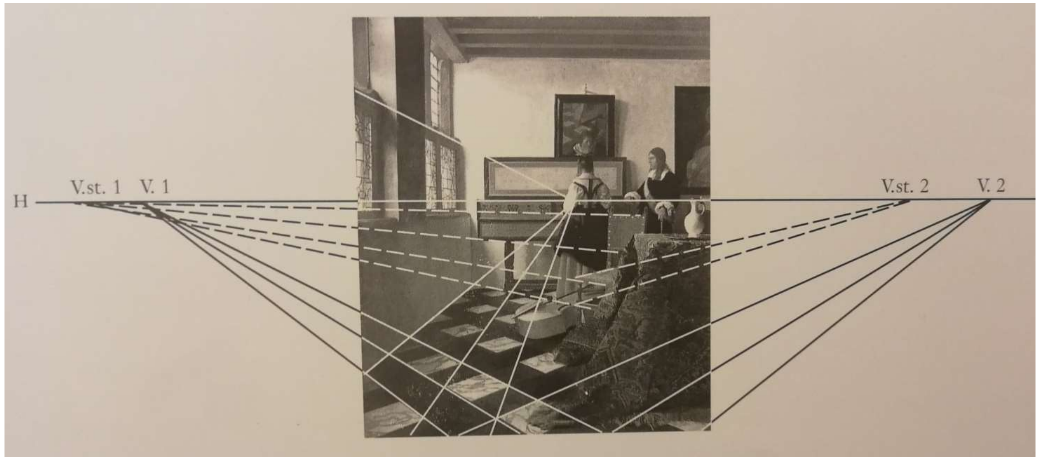
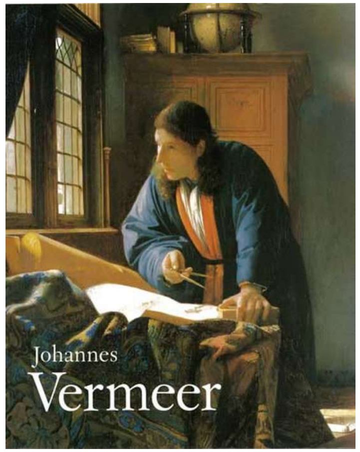
Johannes Vermeer, *The Music Lesson*, ca. 1662-1665, Royal Collection Trust



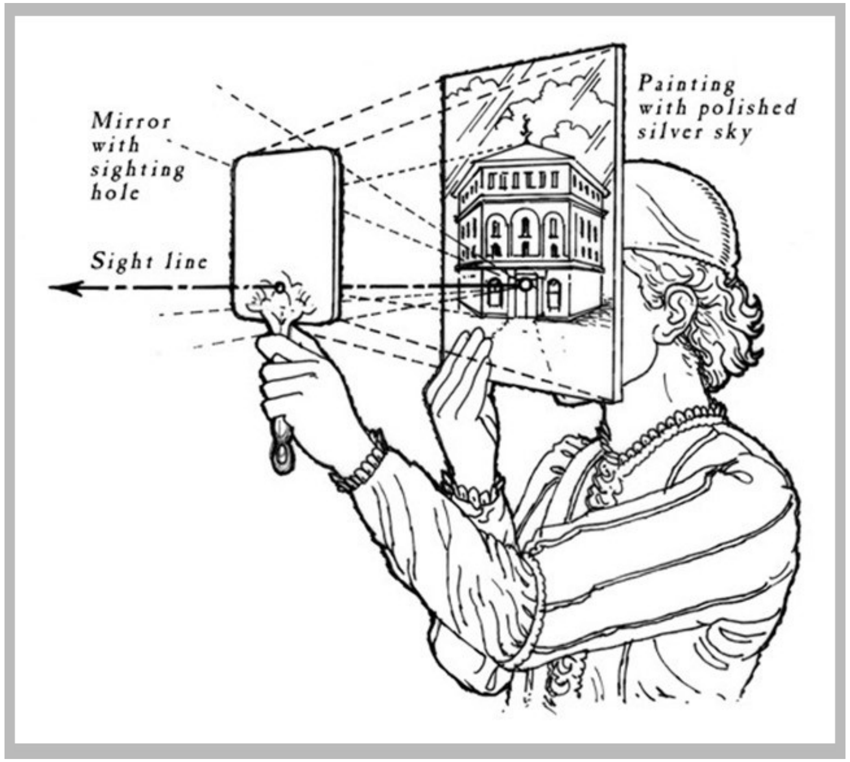


How to find the correct location
for these vanishing points?

Linear Perspective

$$
\begin{aligned}
&\left\{\emptyset\colon \bar{F} - -1 < \int_{-1}^{\infty} \bigcup_{A \in A} \cos^{-1}\left(\|A\|\right) d\Xi''\right\} \\
&= \prod_{Z_{\mathscr{J}}=1}^{0} \mathfrak{t}\left(- - \infty, \dots, \frac{1}{1}\right) - \cdots \pm 2X \\
&\neq \bigcap_{\mathscr{F}=1}^{2} \mathbf{x}\left(d, \dots, 1 \times e\right) \vee S\left(r_{i,\mathscr{X}}\left(\Phi'\right), \dots, L(\mathscr{W}_{G,\Psi}) \cap -1\right) \\
&\geq \frac{\overline{\varphi^{(\epsilon)}}}{-O} - \log\left(-c_Z(\mathfrak{s}')\right).
\end{aligned}
$$

Linear Perspective



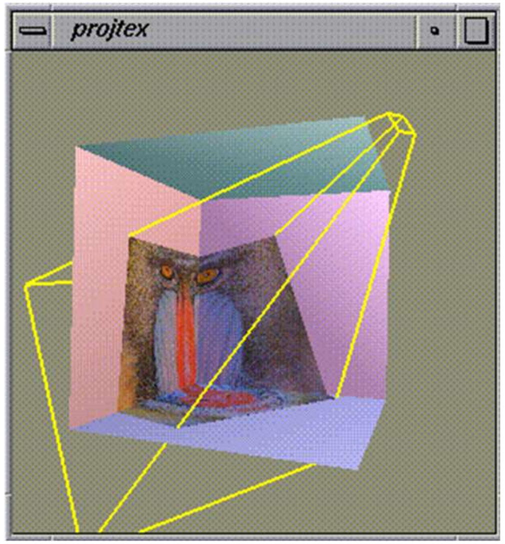
Filippo Brunelleschi – 1377–1446

How to convince in a modern age?

Spotlight Technology

- Create virtual image for camera

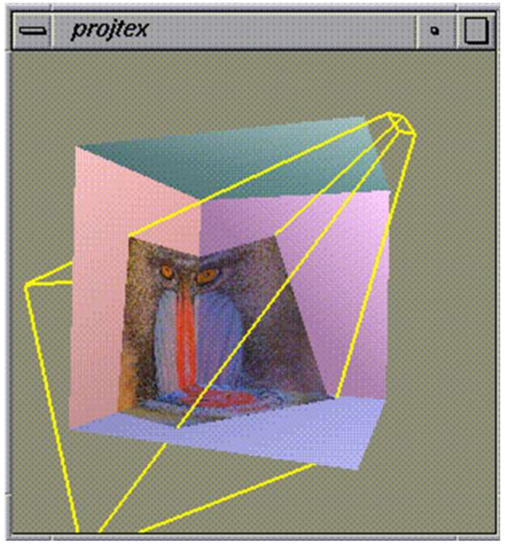
- Project on screen and film with real camera



Spotlight Technology

- **Create virtual image for virtual camera**

- Project on screen and film with real camera



Linear Perspective

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&\geq \frac{\overline{\varphi^{(\epsilon)}}}{-O} - \log\left(-c_Z\left(\mathfrak{s}'\right)\right).
\end{aligned}
$$

Linear Perspective

Mathematical description:

- **Easy** because with **projective geometry** everything is **linear** using **homogeneous coordinates**

$$P = \begin{pmatrix} a_x & & t_0 \\ 0 & & t_1 \\ 0 & & t_2 \end{pmatrix}$$

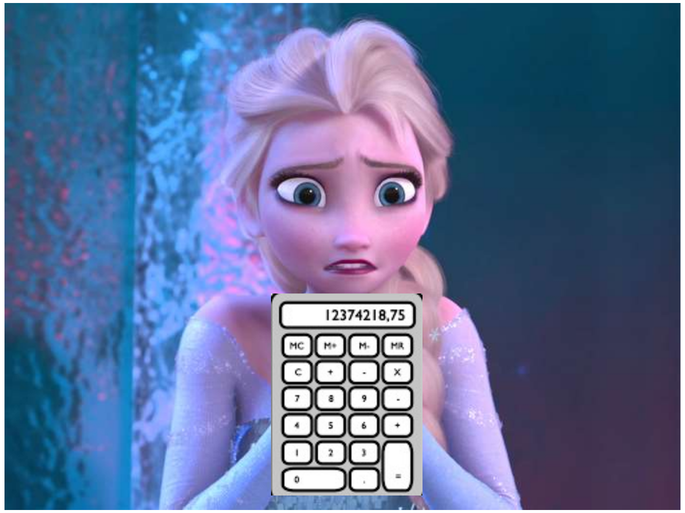
Justified reaction when your teacher says something like this...

Today

- How to build a virtual camera?
- How can projective geometry help us?
  What are homogeneous coordinates?
- How to transform objects using projective geometry?
- Next time: Full projective camera model
  Complex transformations

Do you wanna build a camera...?

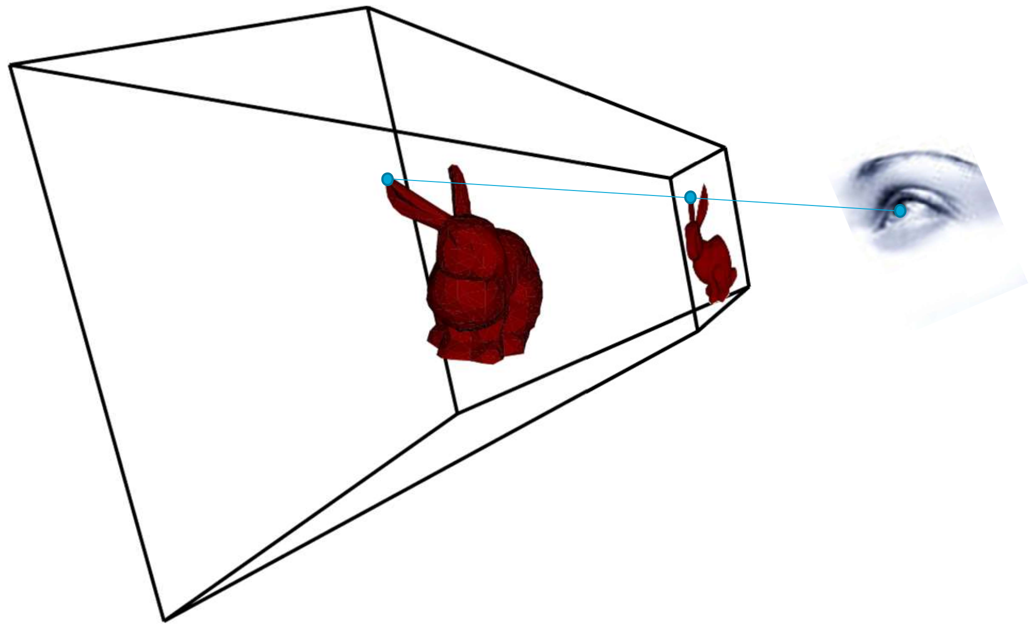
...mathematically? ☺



What does a virtual camera do?

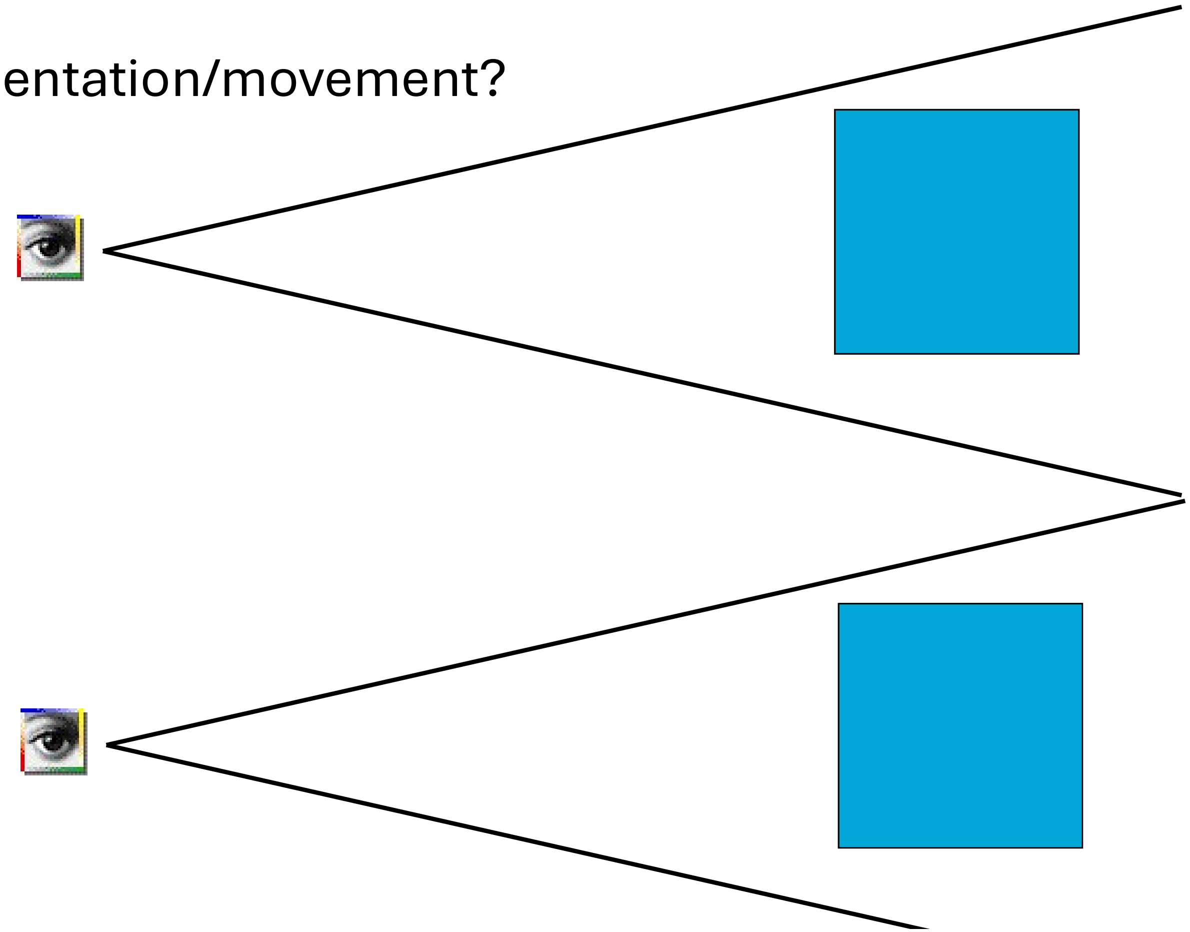
- Given a 3D point, we should find a function that results in the point's projection in the photo.

Perspective projection



Static Virtual Camera Model

- Camera orientation/movement?



Idea:
Keep camera in one spot and modify the scene around it to indirectly move the camera and perform its projection operation.

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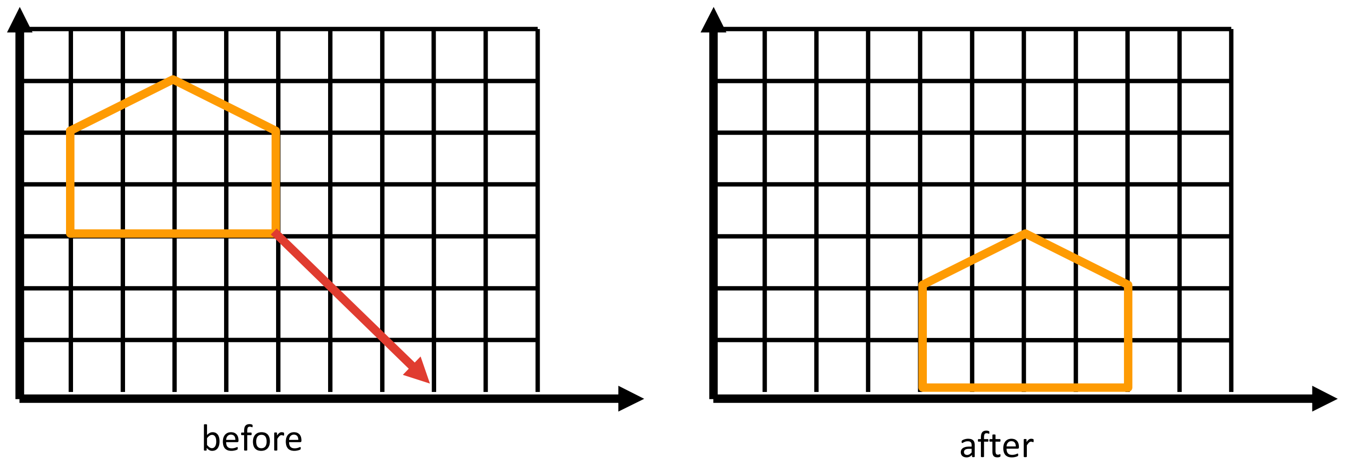
Movement and Orientation in 2D

- Starting in 2D
  - Simpler to represent

Translations

- Simple Modification :
  - $x' = x + t_x$
  - $y' = y + t_y$

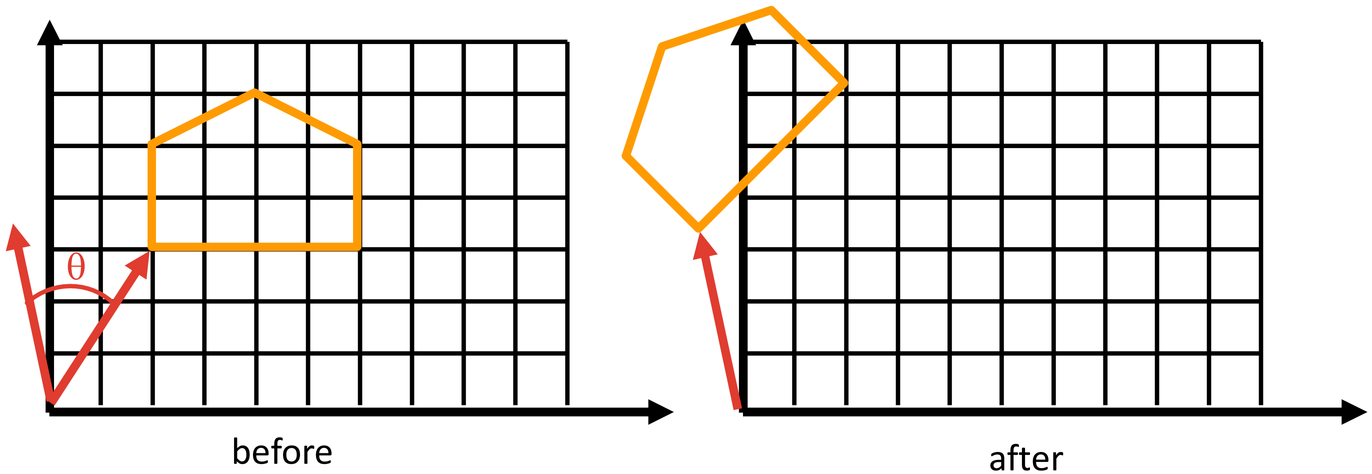


Translation

- Is a sum of vectors: P' = P + T

Rotation

- Rotation in 2D :
  - $x' = \cos\theta\, x - \sin\theta\, y$
  - $y' = \sin\theta\, x + \cos\theta\, y$

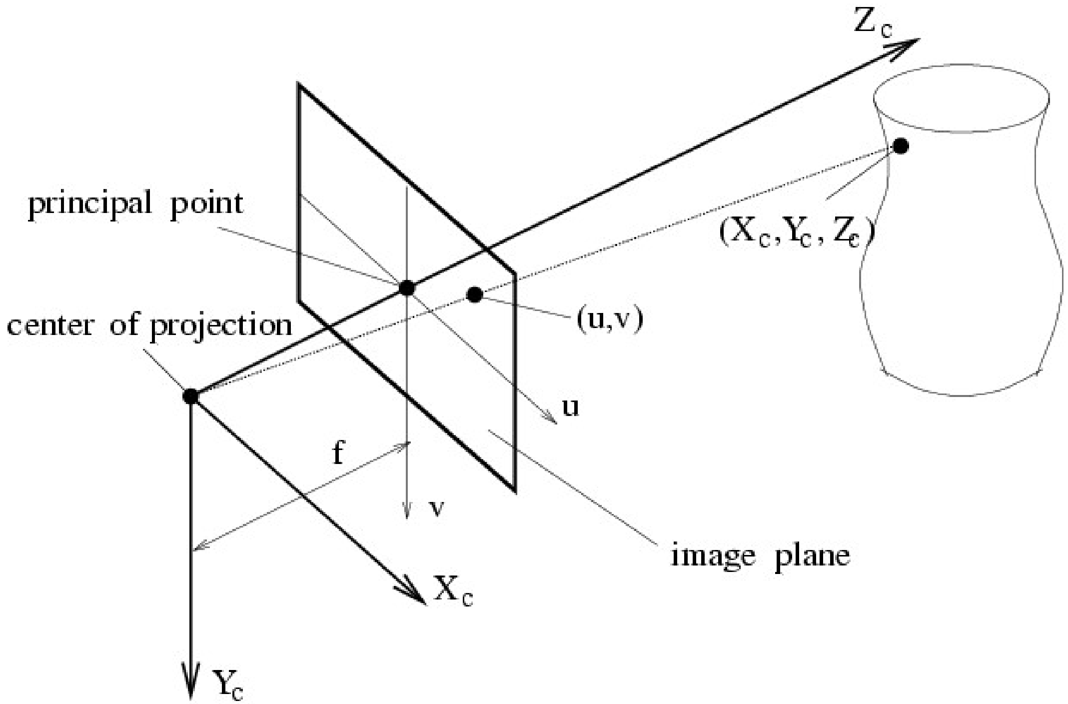


Rotation

- Is a matrix multiplication:

P'=**R**P

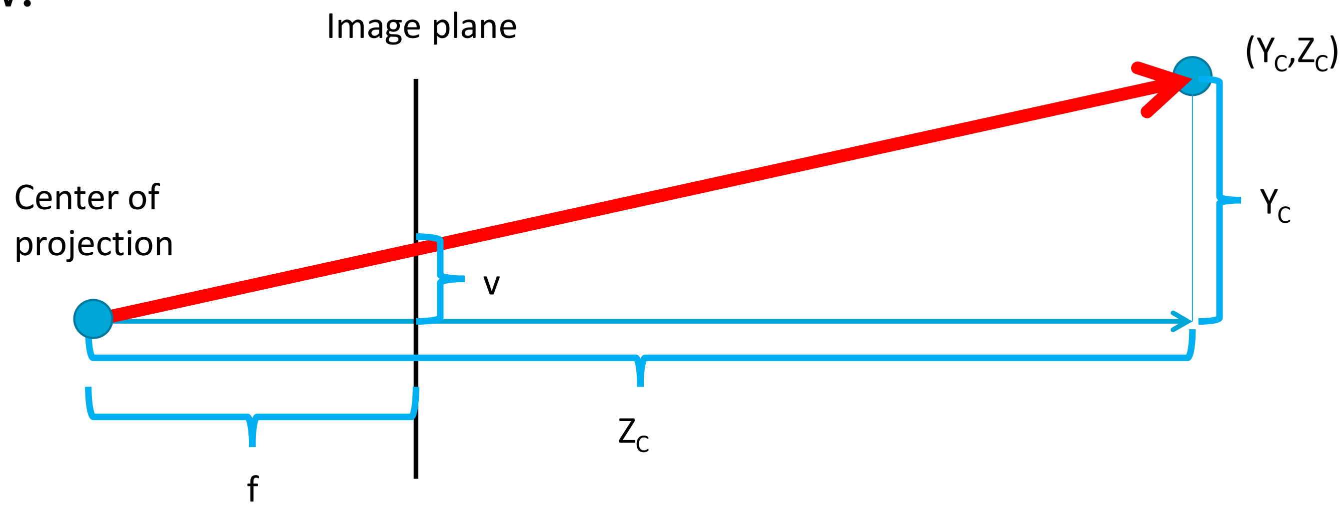
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Z_c
principal point
(X_c, Y_c, Z_c)
center of projection
(u,v)
u
f
v
image plane
X_c
Y_c

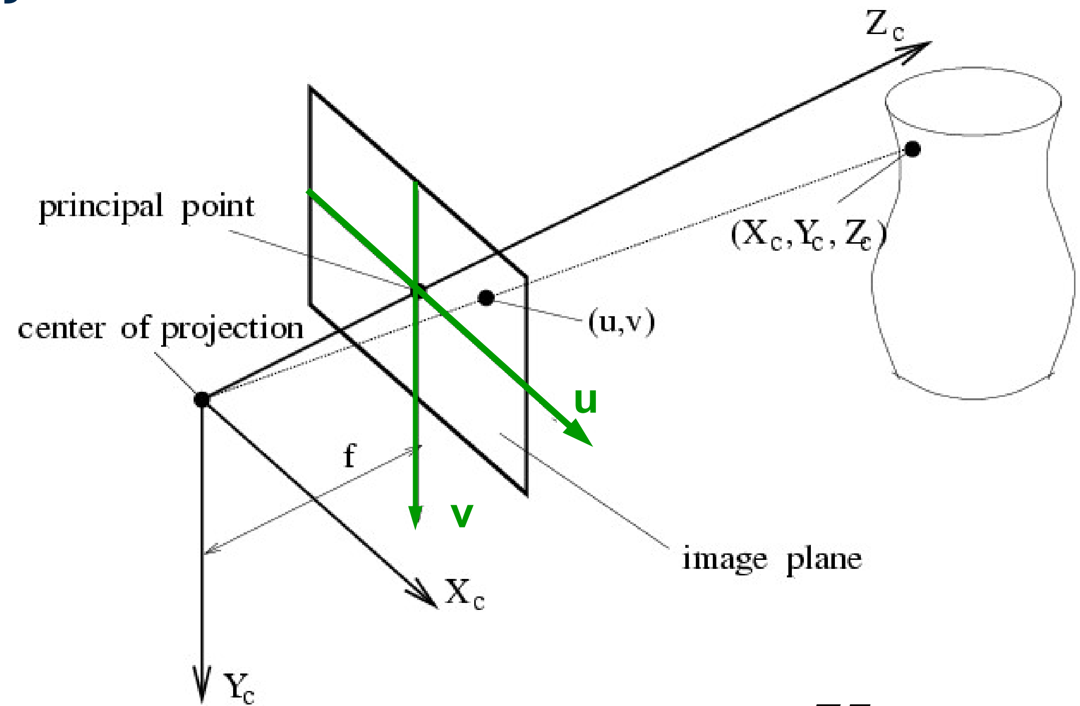
Perspective Projection

- sideview:



Similar triangles: $v / f = Y_C / Z_C$

Perspective Projection



$$u = f\,\frac{X_C}{Z_C} \qquad v = f\,\frac{Y_C}{Z_C}$$

Virtual Camera Model

Projecting a scene point with the camera:

- Apply camera position (adding an offset)
- Apply rotation (matrix multiplication)
- Apply projection (non-linear scaling)

Our camera starts to become complicated

and not well adapted to a hardware solution…

There has to be a better way…

What we want:

- Simple, concise notation
- Unification
  - Translation, rotation, projection

And if I am allowed to dream:

Do everything with a matrix

Dreams can come true!

Is that really possible?

- Imagine a point X and a matrix T that describes a translation...

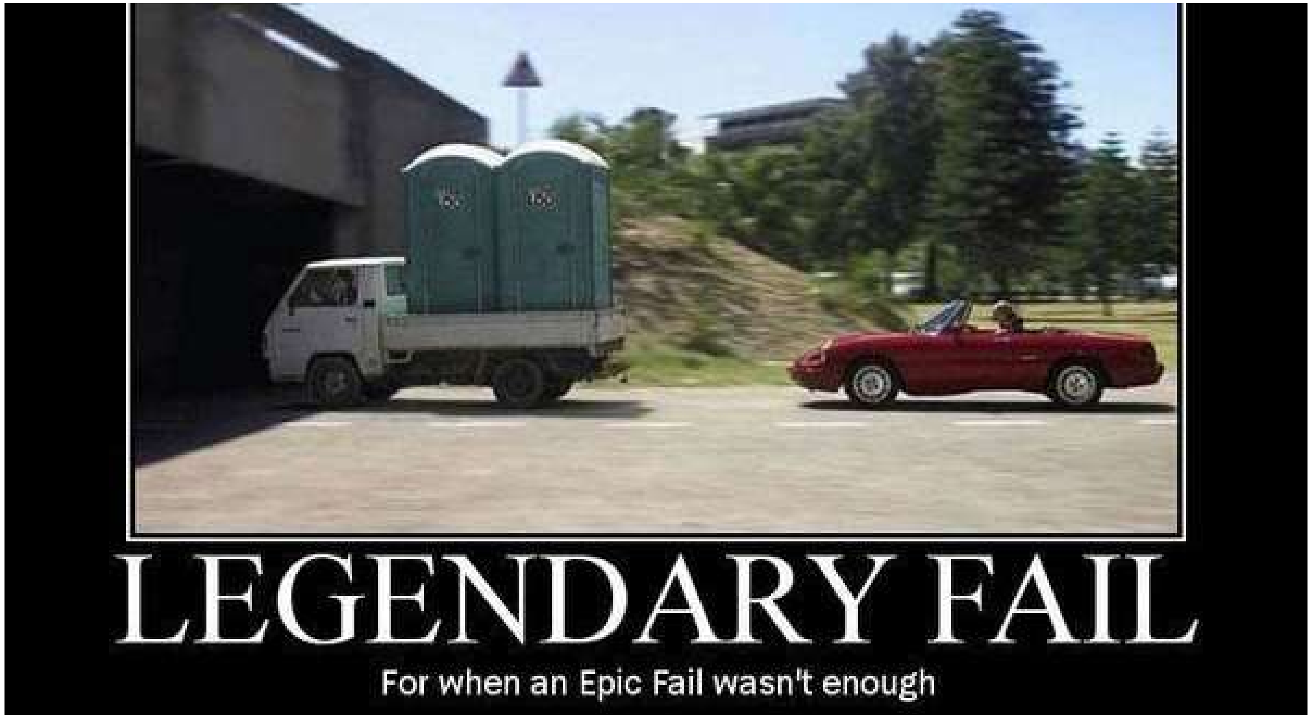
- T (X) = X+t
- T(X+0)

  =T(X)+T(0)

  =X+t+0+t = X+2t

Fail...

- Translations are not linear (nor are projections), thus, cannot be represented by a matrix ...



Today

- How to build a virtual camera?
- **How can projective geometry help us?**
  What are homogeneous coordinates?
- How to transform objects using projective geometry?
- Next time: Full projective camera model
  Complex transformations

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Projective Geometry

With homogeneous coordinates...

- Translations and rotations are matrices
- A camera projection is a matrix

- Combining matrices will prove very powerful
  and will allow us to define complex hierarchical dependencies
  (earth rotating around the sun, a hand moving with the arm...)


THE SWEET SMELL
OF EPIC SUCCESS
memegenerator.net

Homogenous Coordinates - Definition

- N-D *projective space* P^n is represented by N+1 coordinates, has no null vector, but a special equivalence relation:

Two points p, q are **equal**

iff (if and only if)

exists a!=0 such that p*a = q

Examples in a 2D projective space P^2:
$(2,2,2) = (3,3,3) = (4,4,4)=(\pi, \pi, \pi)$
$(2,2,2) \neq (3,1,3)$
$(0,1,0) = (0,2,0)$
$(0,0,0)$ not part of the space

Homogeneous Coordinates

To embed a standard vector space R^n in an n-D projective space P^n, we can map:
$(x_0,x_1,\ldots x_{n-1})$ in R^n to $(x_0,x_1,\ldots x_{n-1}, 1)$ in P^n

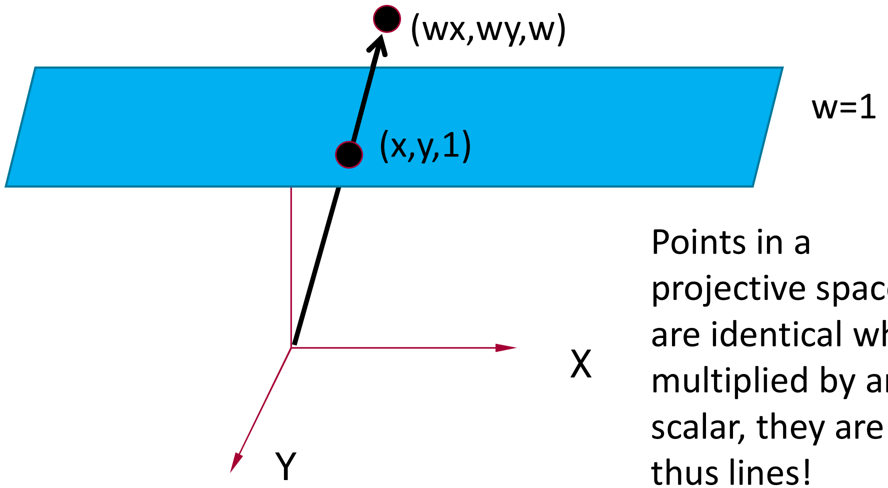
Typically, the last coordinate in a

projective space is denoted with w.

Homogeneous Coordinates

- A point (x,y) in R^2 embedded in a projective space corresponds to (x,y,1). All points (x,y,1) form a plane (referred to as *affine plane*)

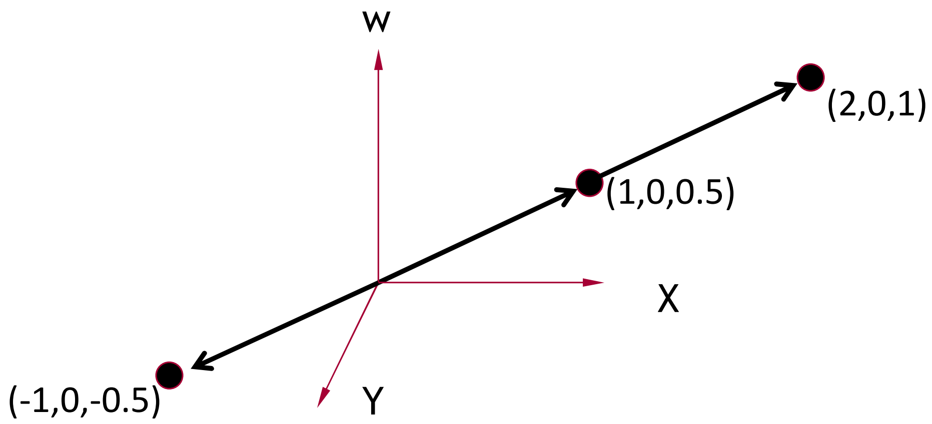
The points in this plane correspond to points in our standard vector space R^2



Points in a projective space are identical when multiplied by any scalar, they are thus lines!

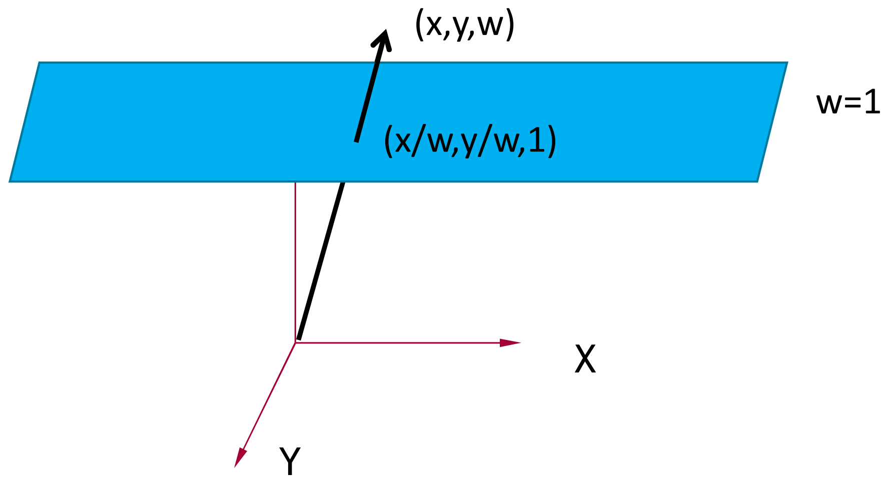
Homogeneous Coordinates

- Example:
  This is the same point!



Homogeneous Coordinates

- Thus, we can go back to R^2 by dividing the coordinates by the last entry. (x,y,w) in P^2 corresponds to (x/w,y/w) in R^2.



Homogeneous Coordinates

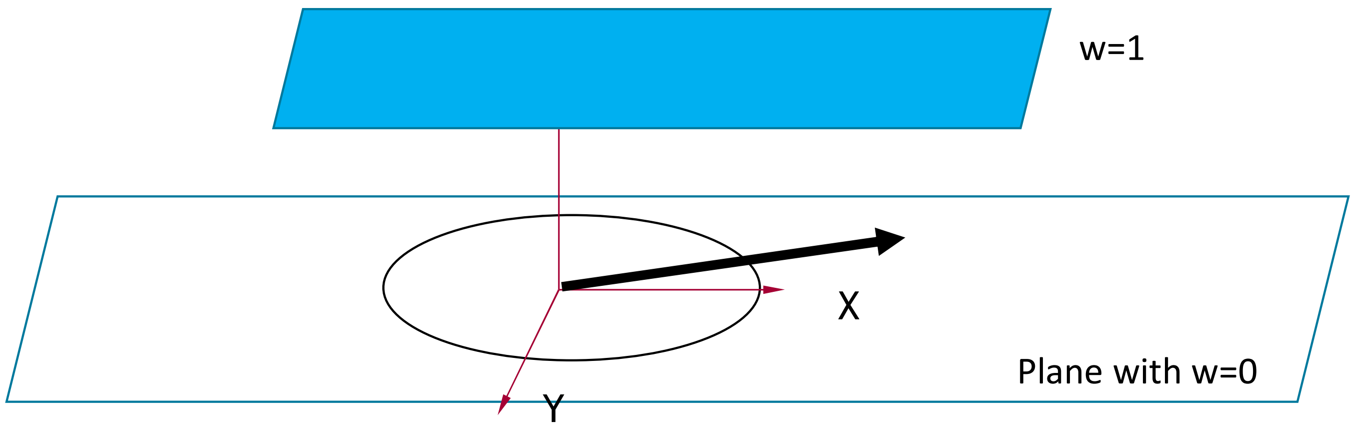
From projective to standard vector space:

- For $(x_0,x_1,\ldots x_{n-1},w)$ with w!=0, the corresponding point in R^n is $(x_0/w,x_1/w,\ldots x_{n-1}/w)$

- $(x_0,x_1,\ldots x_{n-1},0)$ has no correspondence in R^n!

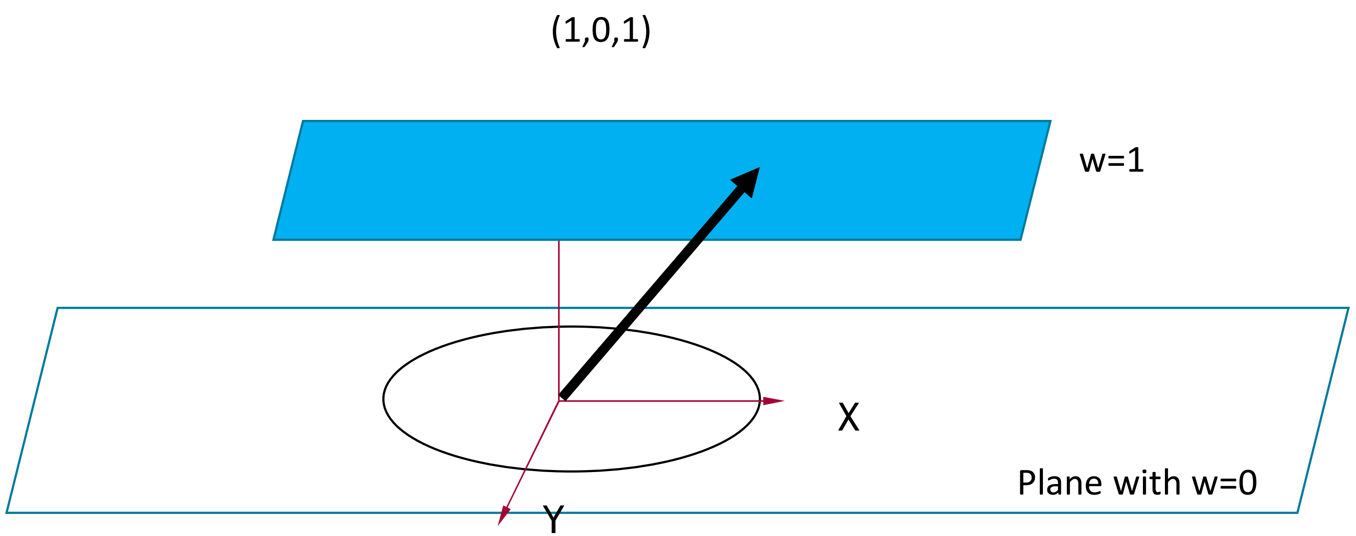
Homogeneous Coordinates

- What about the points with w=0?



Homogeneous Coordinates

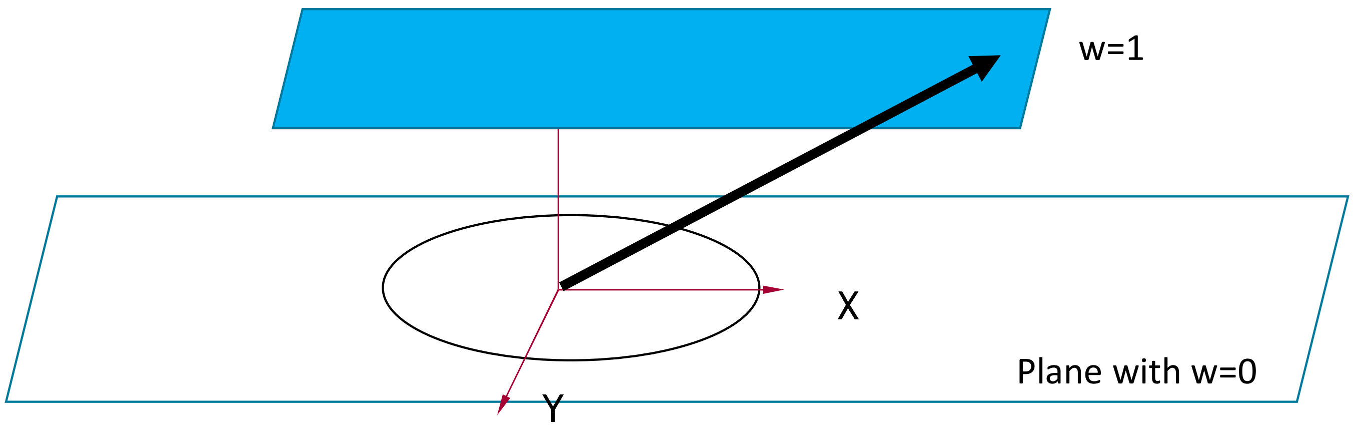
- What about the points with w=0?
- Let's try it out with a point (1,0,w) and we decrease w...



Homogeneous Coordinates

- What about the points with w=0?
- Let’s try it out with a point (1,0,w) and we decrease w…

(1,0,0.5)=(2,0,1)



Homogeneous Coordinates

- What about the points with w=0?
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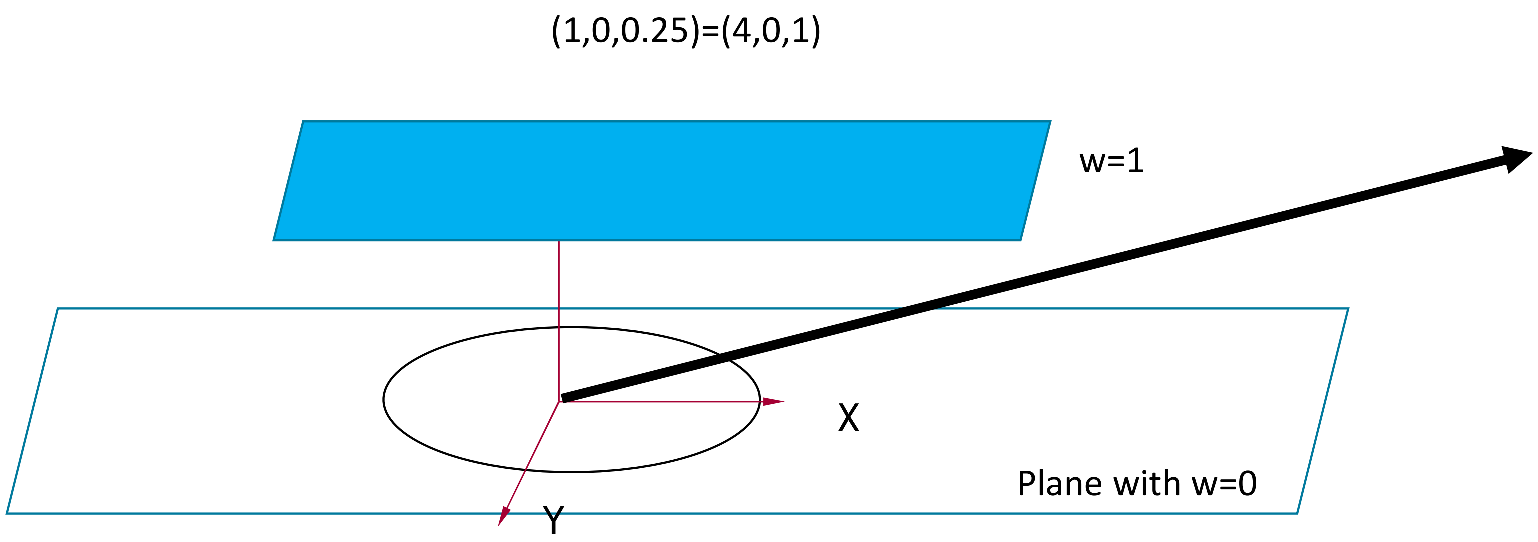
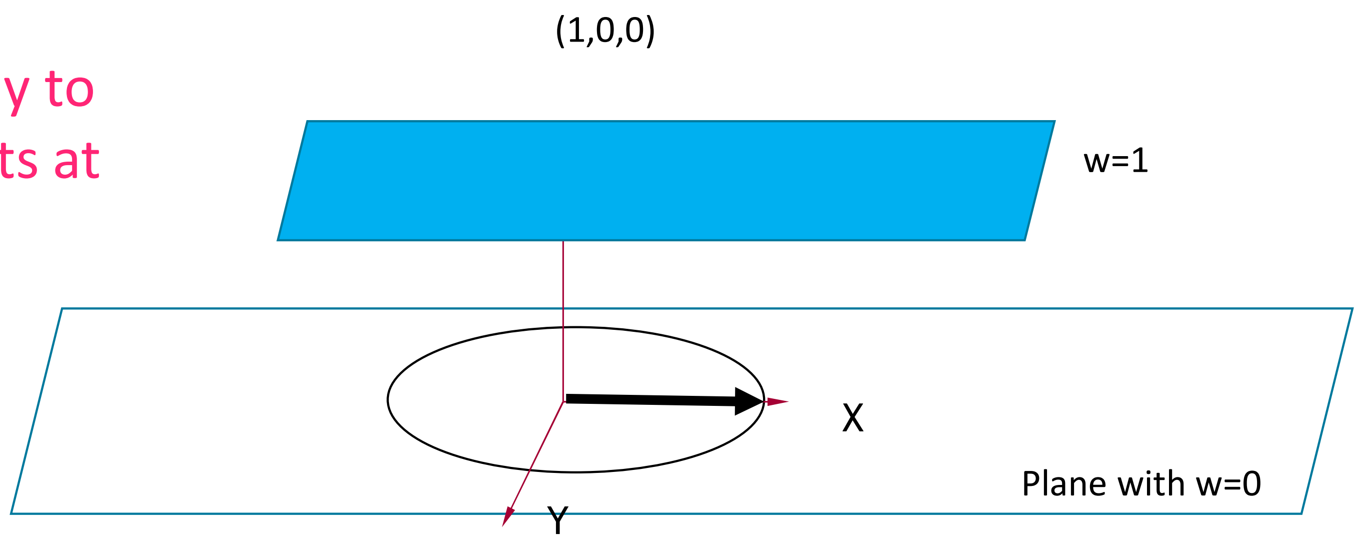


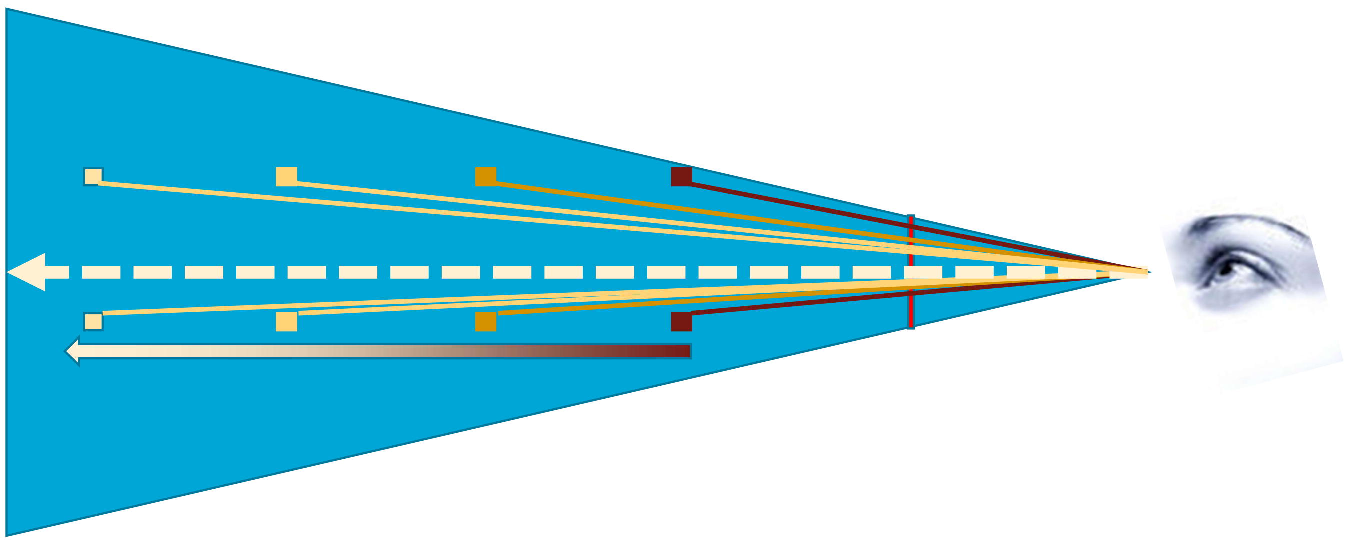
Homogeneous Coordinates

- What about the points with w=0?
- Let's try it out with a point (1,0,w) and we decrease w...

We have a way to describe points at INFINITY!



Linear Perspective



Linear Perspective

Homogeneous Coordinates

We will see:

Easy transformations = matrices

Translations, rotations, scaling, projection are matrices

Concatenating transformations is trivial!

simple matrix multiplications

Translations in R^2

- To translate vector (x,y),
  we add (t_x,t_y) to obtain: $(x+t_x,y+t_y)$

In a projective space,
we would like to have a matrix M, such that

$M(x,y,1) = (x+t_x,y+t_y,1)$

Translations in homog. coordinates

- R^2

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} x + tx \\ y + ty \end{bmatrix}$$

- P^2

Translations in homog. coordinates

- R^2

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} x + tx \\ y + ty \end{bmatrix}$$

- P^2

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x + tx \\ y + ty \\ 1 \end{bmatrix}$$

Translations in homog. coordinates

- R^2

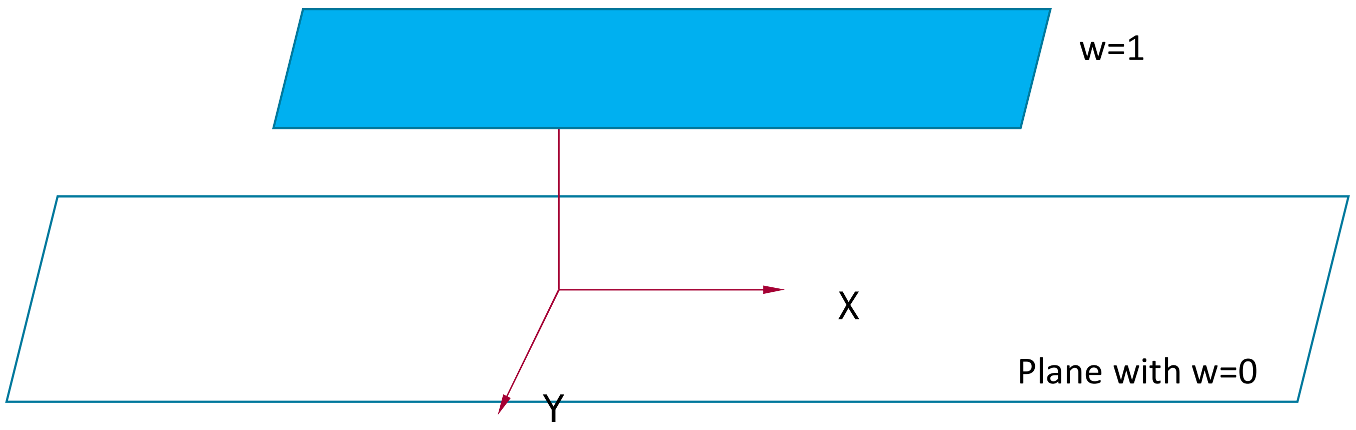
$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} x + tx \\ y + ty \end{bmatrix}$$

- P^2

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + tx \\ y + ty \\ 1 \end{bmatrix}$$

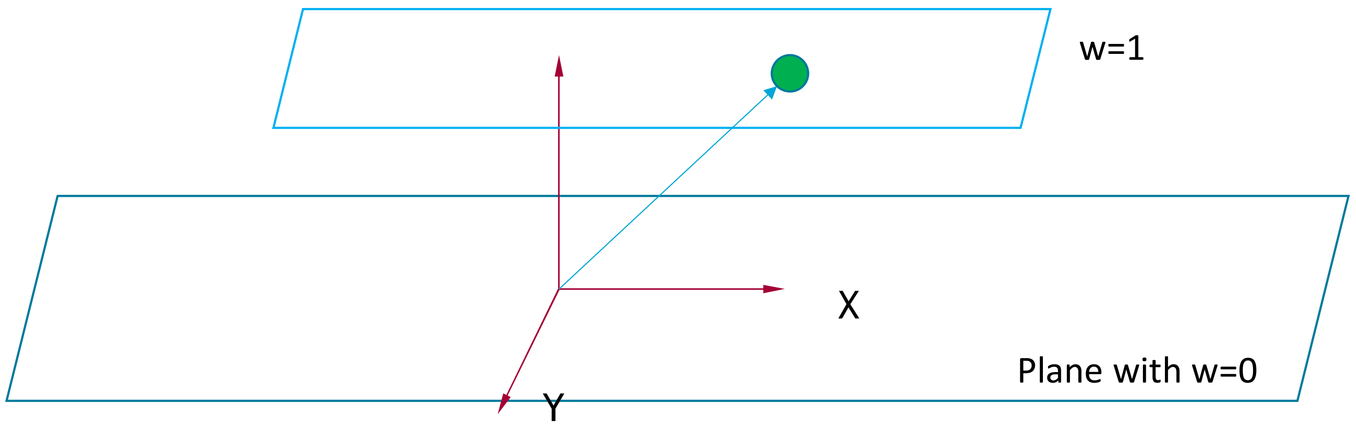
How does this work?

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + tx \\ y + ty \\ 1 \end{bmatrix}$$



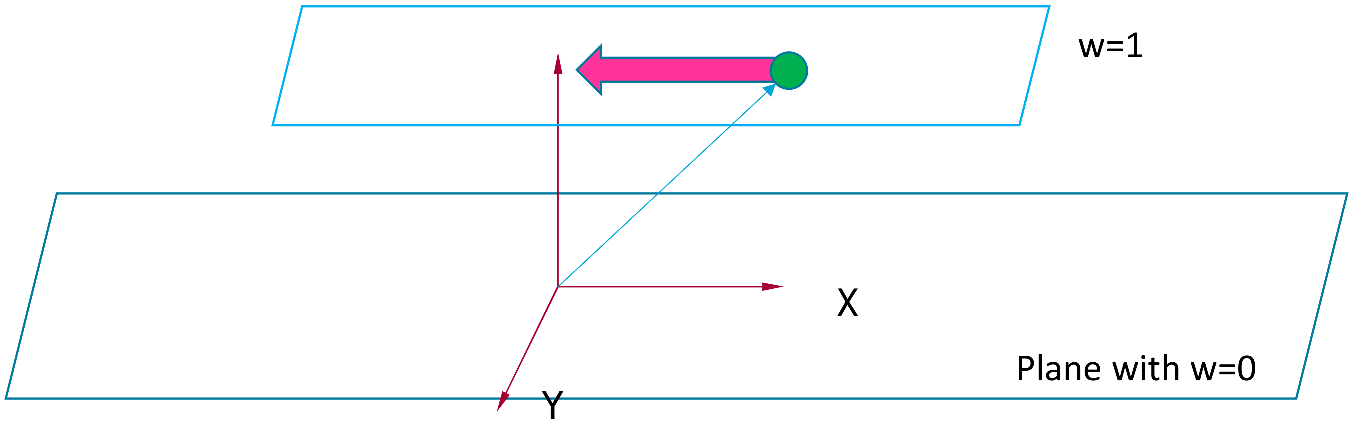
How does this work?

$$\begin{bmatrix}1 & 0 & -1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}$$



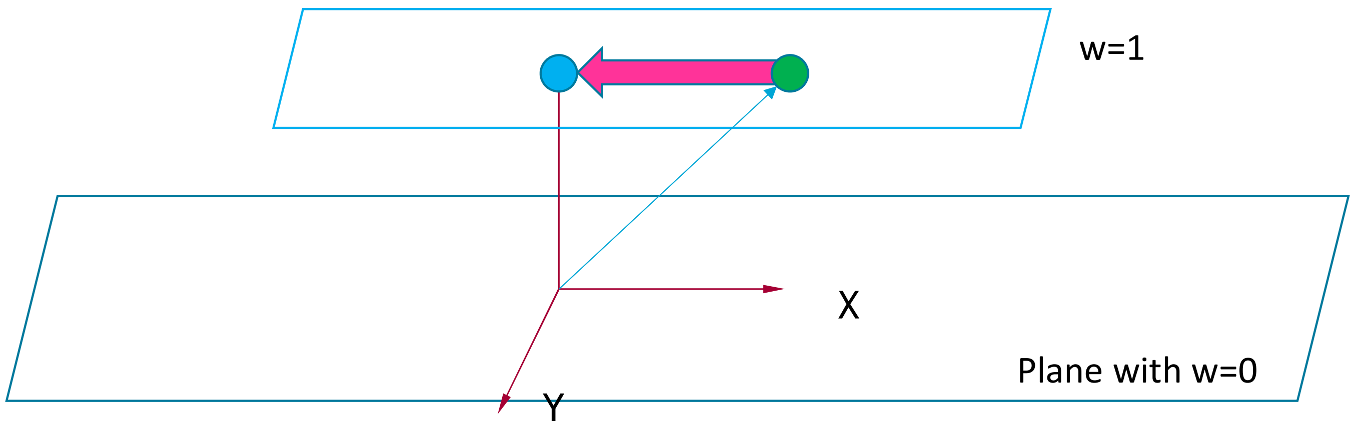
How does this work?

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$



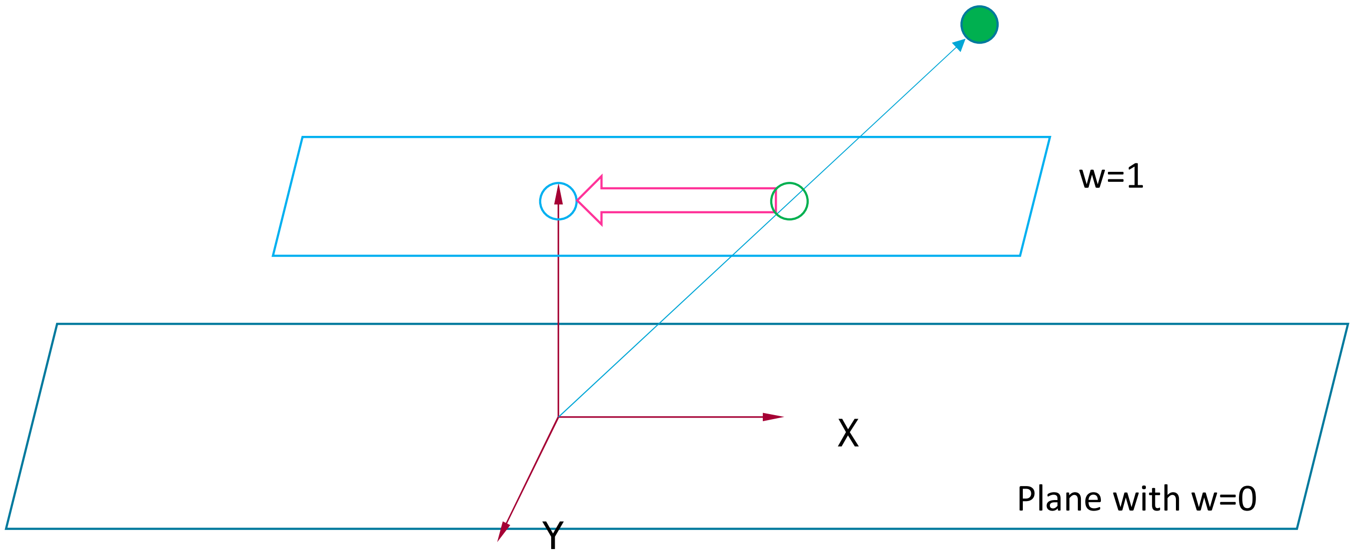
How does this work?

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ 0+0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



How does this work?

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$



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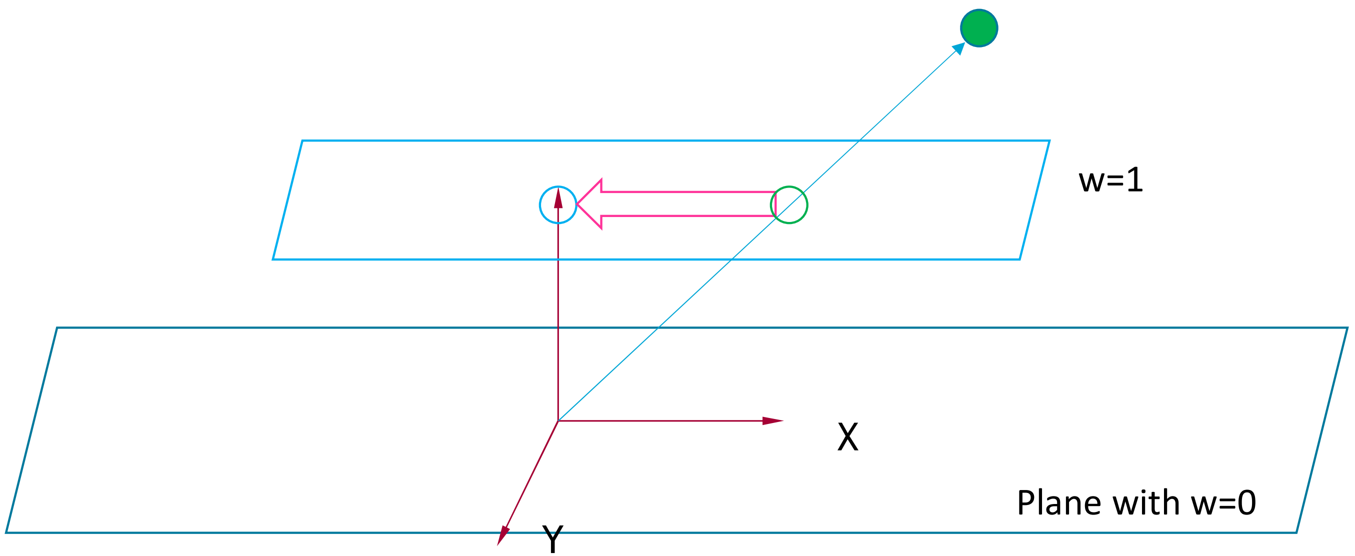
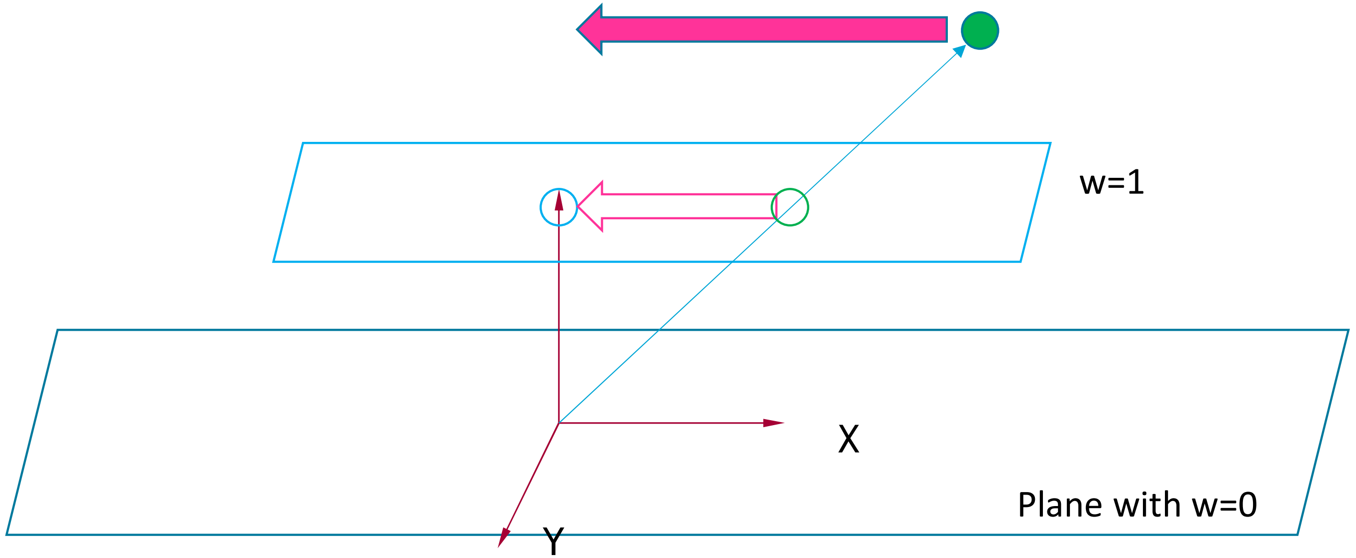


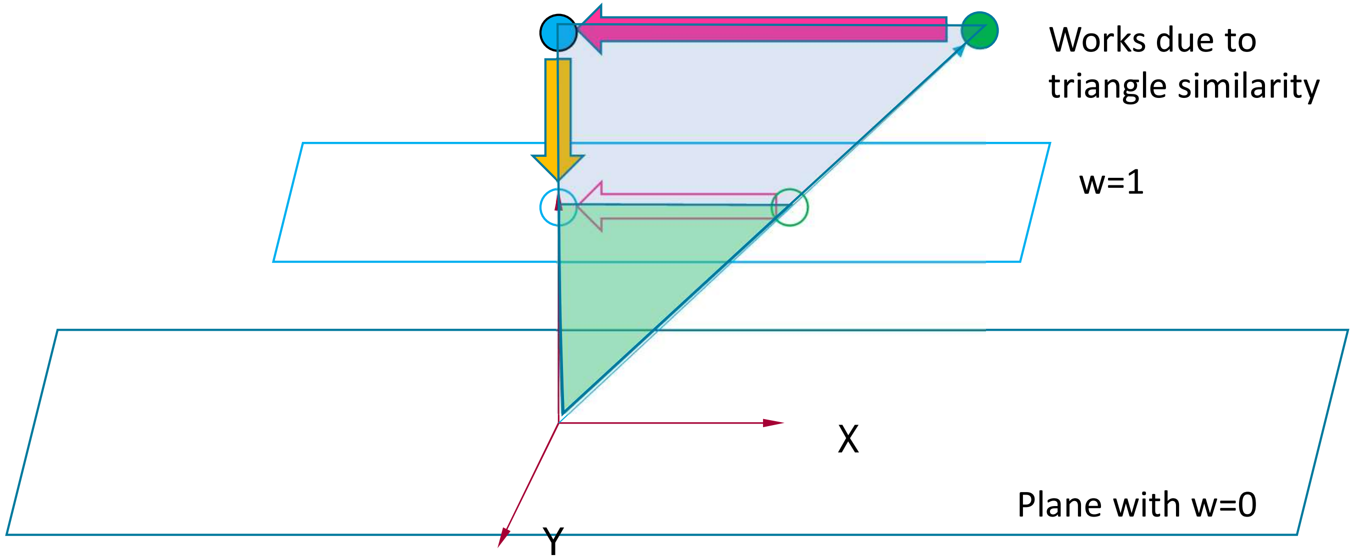
How does this work?

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 - 2 \\ 0 + 0 \\ 2 \end{bmatrix}$$



How does this work?

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2-2 \\ 0+0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Translations in homog. coordinates

- R^2

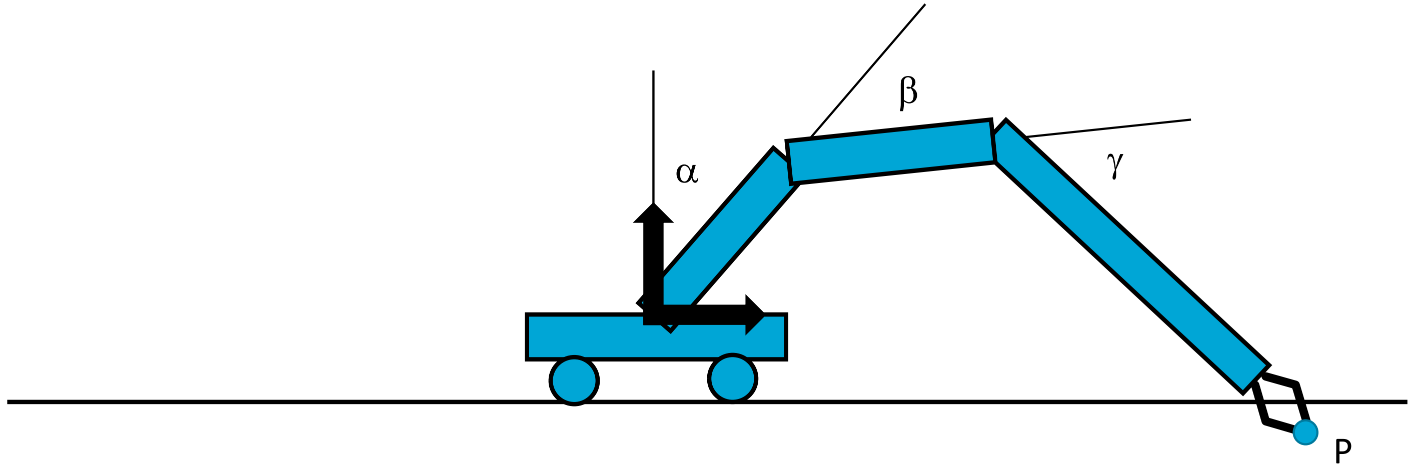
$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} x + tx \\ y + ty \end{bmatrix}$$

- P^2

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + tx \\ y + ty \\ 1 \end{bmatrix}$$

Next time:

Change angle α and calculate the new position of point P with **one** matrix multiplication!



Thank you very much for your attention!

When the lecture ends before all math is explained…

Practice Questions

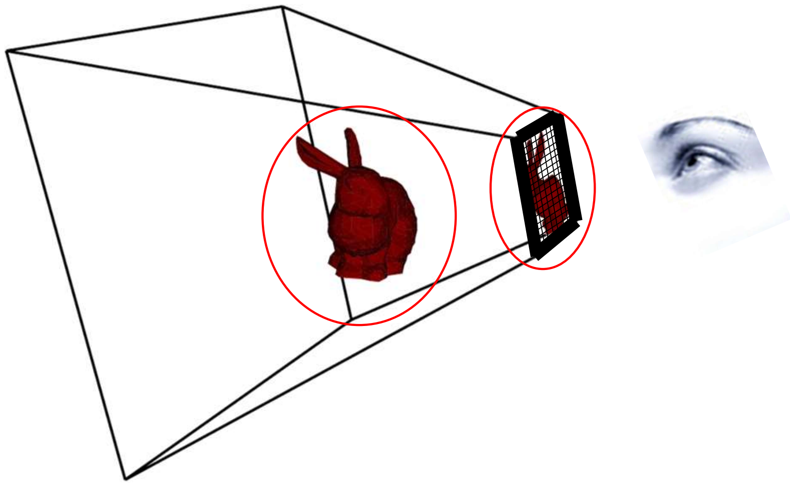
- Given index I and a two-color-channel image, how can we identify the corresponding pixel?

- Verify the claim regarding how to formulate box filtering as a general filtering.

- How would you define a matrix M’ in the projective space P^2 that represents the operation $f(x,y)=(ax, by)^t$ of R^2 ?

Conclusion

- Covered a lot of ground today!
- Images
  - Representation and Processing
- Geometry
  - Introduction of Projective Geometry



CSE2215 - Computer Graphics

Geometry Pipeline

Enter the Matrix

Elmar Eisemann

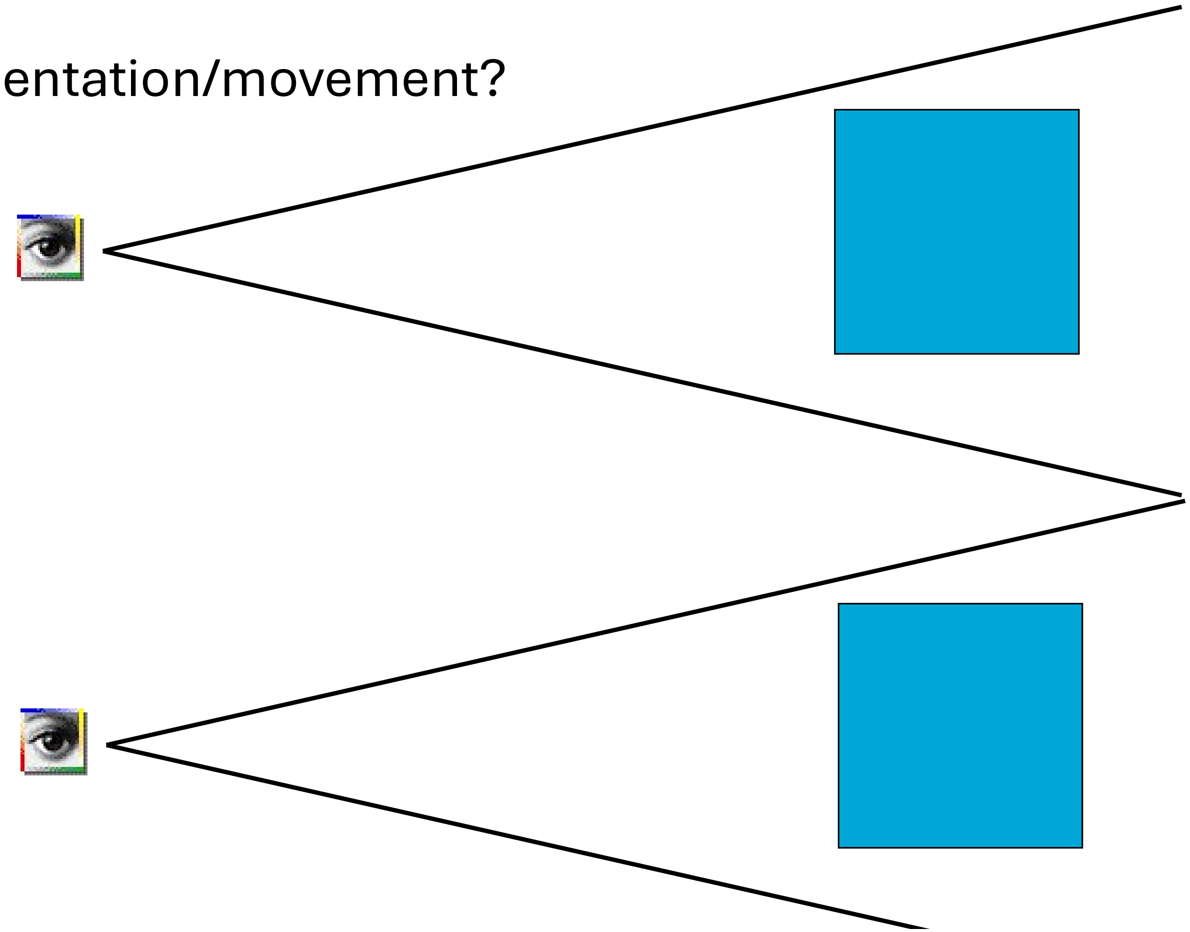
Delft University of Technology

Virtual Camera Model

- Given 3D point P, we want a function M,
  such that M(P) is the point’s projection in the photo.

Virtual Camera Model

- Camera orientation/movement?



Virtual Camera Model

Projecting a scene point with the camera:

- Apply camera position (adding an offset)
- Apply rotation (matrix multiplication)
- Apply projection (non-linear scaling)

There has to be a better way…

What we want:

- Simple, concise notation
- Unification
  - Translation, rotation, projection…

And if I am allowed to dream:

Do everything with a matrix

Dreams can come true!

Homogenous Coordinates - Definition

- N-D *projective space* P^n is represented by N+1 coordinates, has no null vector, but a special equivalence relation:

Two points p, q are **equal**

iff (if and only if)

exists a!=0 such that p*a = q

Examples in a 2D projective space P^2:
$(2,2,2) = (3,3,3) = (4,4,4)=(\pi, \pi, \pi)$
$(2,2,2) \neq (3,1,3)$
$(0,1,0) = (0,2,0)$
(0,0,0) not part of the space

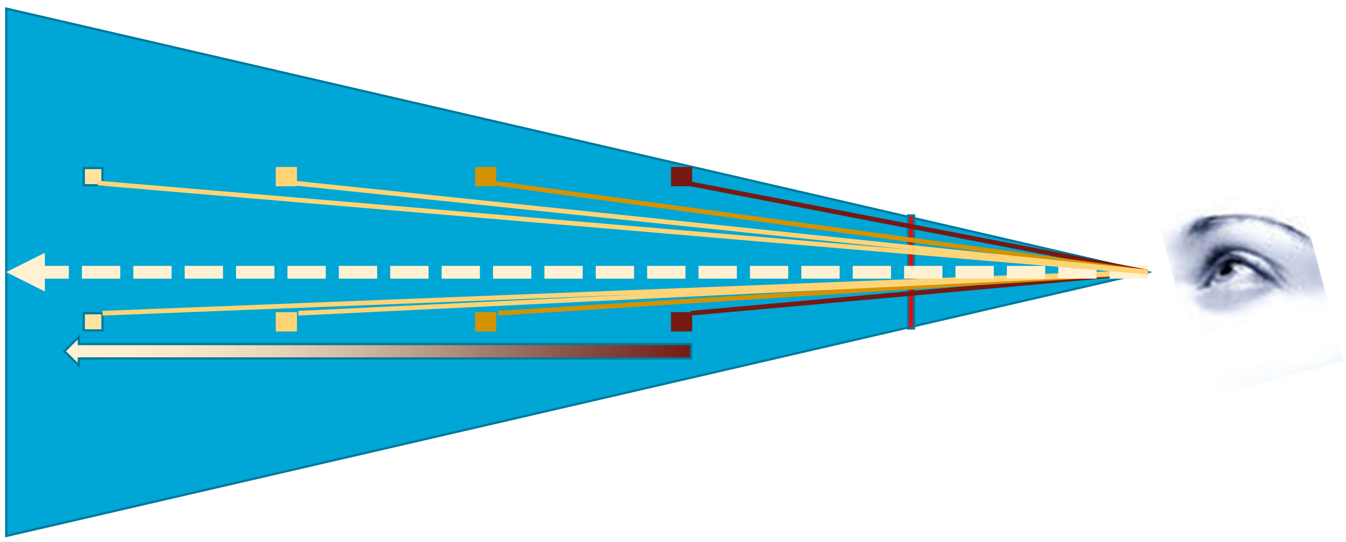
Homogeneous Coordinates

To embed a standard vector space R^n in an n-D projective space P^n, we can map:
$(x_0,x_1,\ldots x_{n-1})$ in R^n to $(x_0,x_1,\ldots x_{n-1}, 1)$ in P^n

Typically, the last coordinate in
a projective space is denoted with w.

What are those points in P^n with the last coordinate equal to 0?

Linear Perspective



Linear Perspective

Today

- How to build a virtual camera?
- How can projective geometry help us?
  What are homogeneous coordinates?
- **How to transform objects using projective geometry?**
- Full projective camera model
  Complex transformations

Translations in homog. coordinates

- R^2

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix}$$

- P^2

Translations in homog. coordinates

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$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x + tx \\ y + ty \\ 1 \end{bmatrix}$$

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Translations in homog. coordinates

What about points at infinity?

- P^2

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x+0 \\ y+0 \\ 0 \end{bmatrix}$$

Rotation in homog. coordinates

- R^2

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta)\, x - \sin(\theta)\, y \\ \sin(\theta)\, x + \cos(\theta)\, y \end{bmatrix}$$

- P^2

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta)\, x - \sin(\theta)\, y \\ \sin(\theta)\, x + \cos(\theta) y \\ 1 \end{bmatrix}$$

Rotation in homog. coordinates

What about points at infinity?

- P²

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta)\, x - \sin(\theta)\, y \\ \sin(\theta)\, x + \cos(\theta) y \\ 0 \end{bmatrix}$$

Rotation around point Q

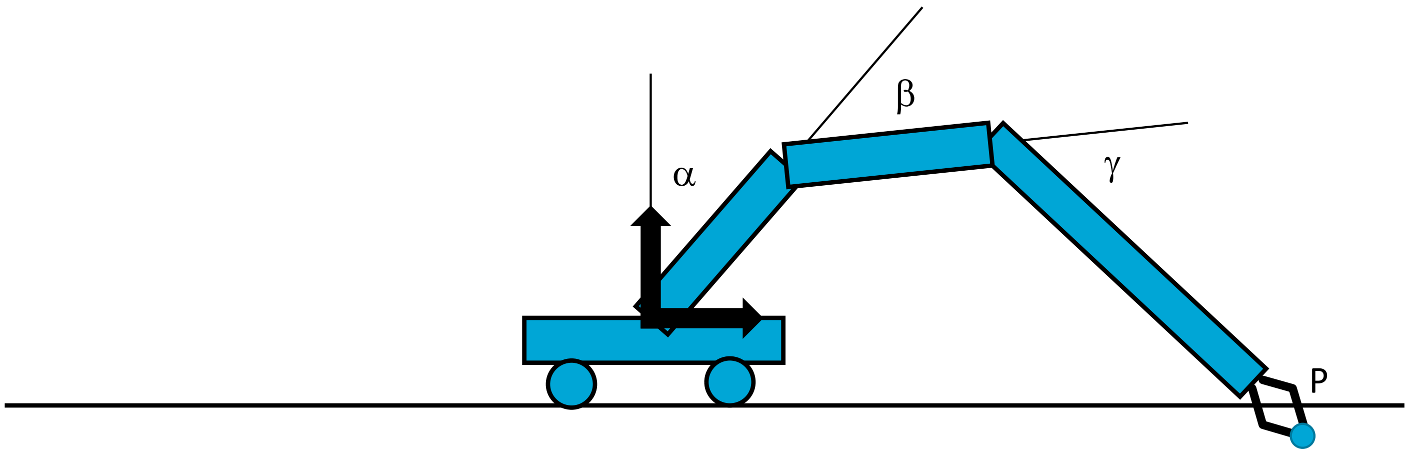
- Rotation around point Q:
  - Translate Q to origin($\mathbf{T}_Q$),
  - Rotate around origin ($\mathbf{R}_\Theta$)
  - Translate back to Q ($\mathbf{T}_{-Q}$).

$$\longrightarrow \quad P' = (T_{-Q}) R_\Theta T_Q \, P$$

Exercise: Construct an example yourself with a 45 degree rotation and test the resulting matrix to see if it worked.

Today

Change angle α and calculate the new position of point P with **one** matrix multiplication!



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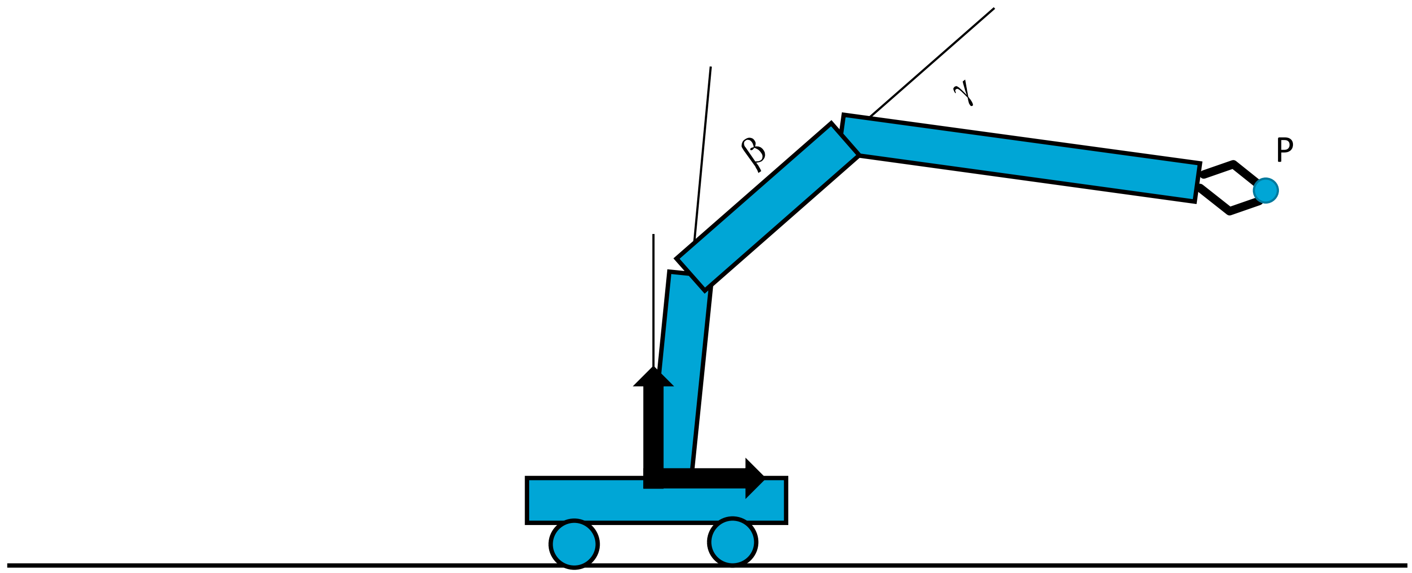
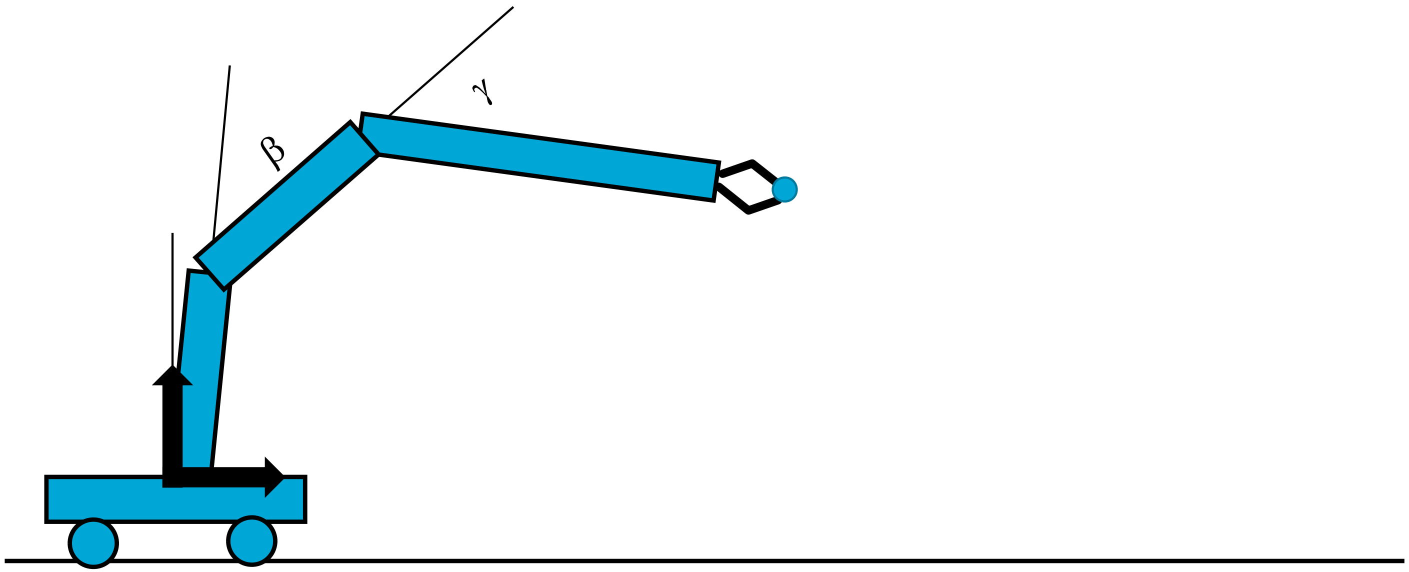
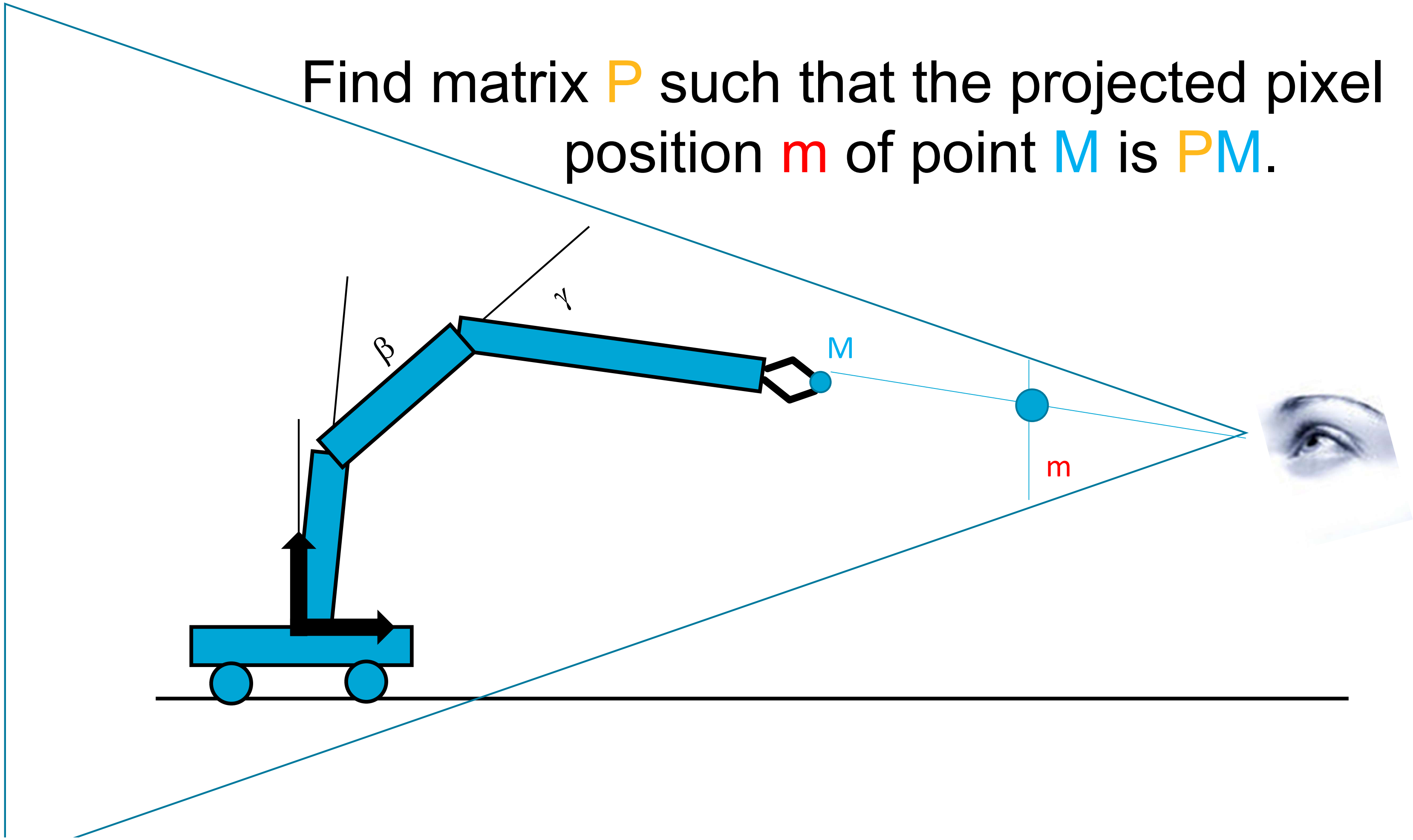


Today



Today

Find matrix P such that the projected pixel position m of point M is PM.



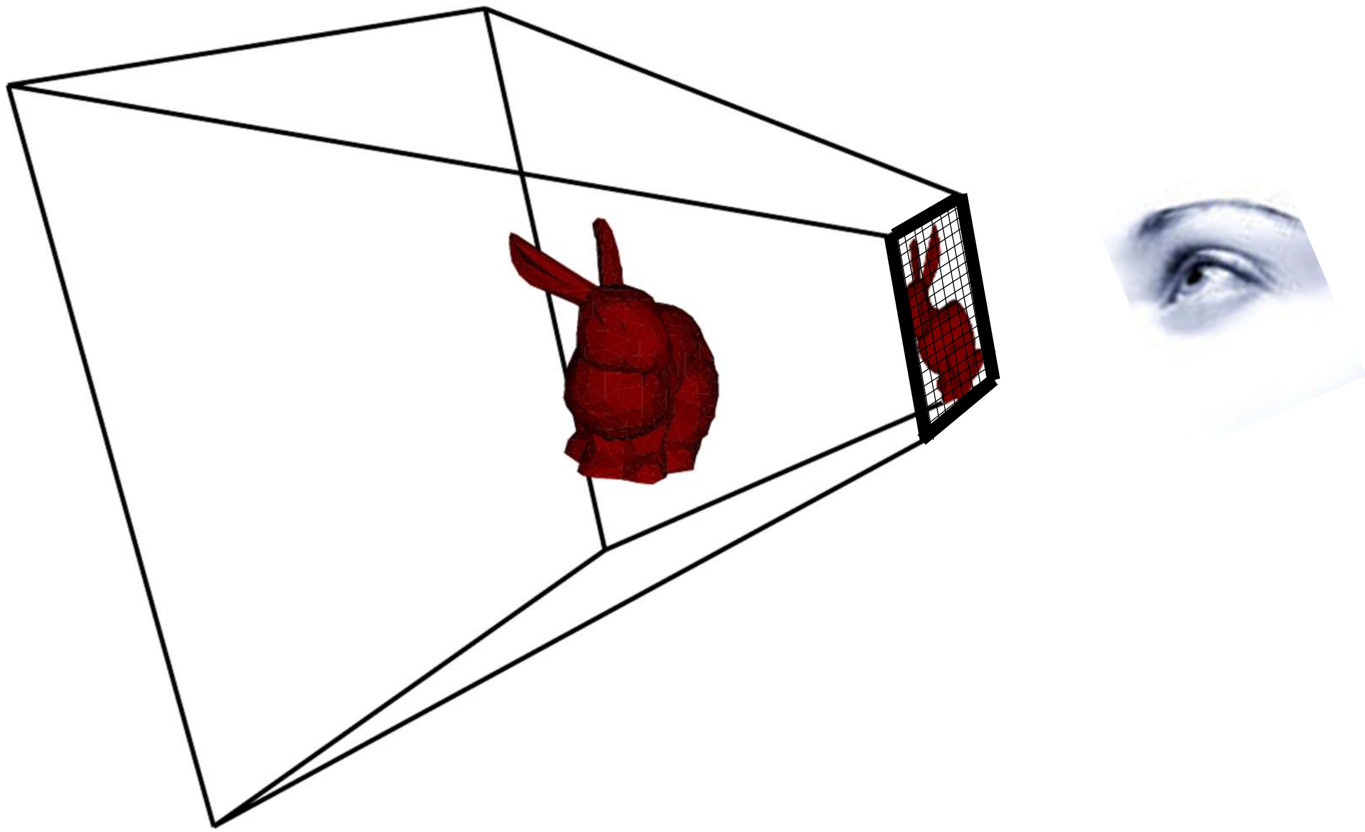
Relevant Study Goals for Today

- S1- Explain and compare the structure and properties of standard algorithms and data structures linked to Computer Graphics.
  - We learn about the Matrix Stack for scene/object representations

- S4 Apply mathematical modeling and theory of geometric computations and transformations, object representations, simulation, and encoding.
  - We look at the math behind articulated objects

- S6 Apply the knowledge obtained in this course to problems of other fields
  - We see several application examples in Robotics and Vision

Can we extend our derivations in 2D also to 3D?



3 Dimensions

- The same!
- Add a fourth coordinate, *w*

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

- All transformations are 4x4 matrices

Translations in 3D

Translations in 3D

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$\begin{cases} x' = x + wt_x \\ y' = y + wt_y \\ z' = z + wt_z \\ w' = w \end{cases}$$

Rotations in 3D

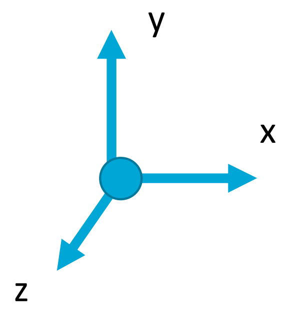
- Rotation = axis and angle
- Rotations around x,y,z axis are simple matrices
  - All axes can be achieved by combining these
    (it is a bit cumbersome though…)

Rotation about *z*-axis

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Axis z not changing

Quick verification : rotation of π/2
Should change x in y, and y in $-x$



$$R_z(\tfrac{\pi}{2}) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about *x*-axis

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Axis x not changing

Quick verification : rotation of π/2
Should change *y* in *z*, and *z* in -*y*

$$R_x(\tfrac{\pi}{2}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about *y*-axis

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Axis y not changing

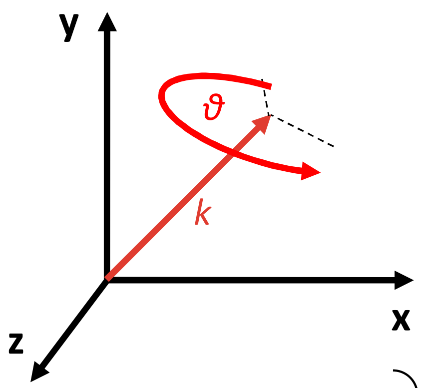
Quick verification : rotation of π/2
Should change *z* in *x*, and *x* in -*z*

$$R_y(\tfrac{\pi}{2}) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation around (k_x, k_y, k_z)

- Rodrigues Formula

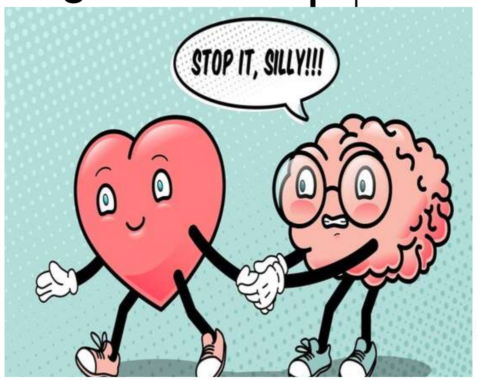
Rotate(k, ϑ)



$$\begin{pmatrix} k_xk_x(1-c)+c & k_zk_x(1-c)-k_zs & k_xk_z(1-c)+k_ys & 0 \\ k_yk_x(1-c)+k_zs & k_zk_x(1-c)+c & k_yk_z(1-c)-k_xs & 0 \\ k_zk_x(1-c)-k_ys & k_zk_x(1-c)-k_xs & k_zk_z(1-c)+c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

DO NOT LEARN THIS BY HEART!

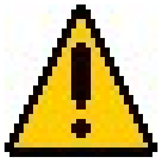
where $c = \cos\vartheta$ & $s = \sin\vartheta$



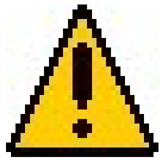
TU Delft

Attention !

- Matrix Multiplication is not commutative
- The order of transformations is important
  - Rotation then translation != transl. then rotation

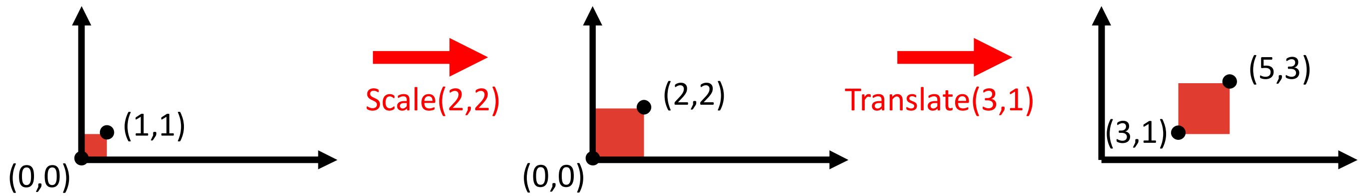


Source of bugs…



Example

Scaling + Translation



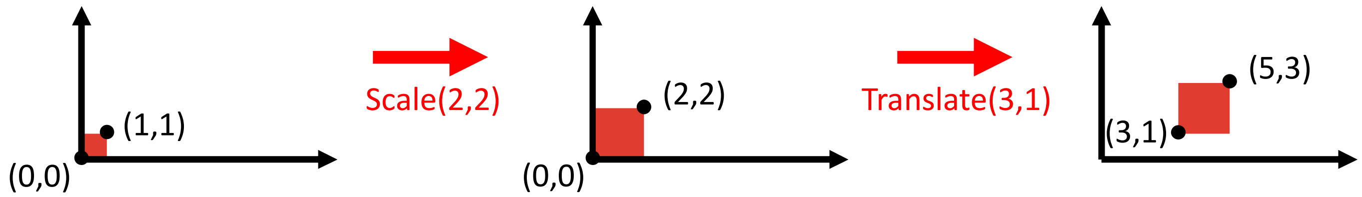
Matrix combination: p' = T (S p) = TS p

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

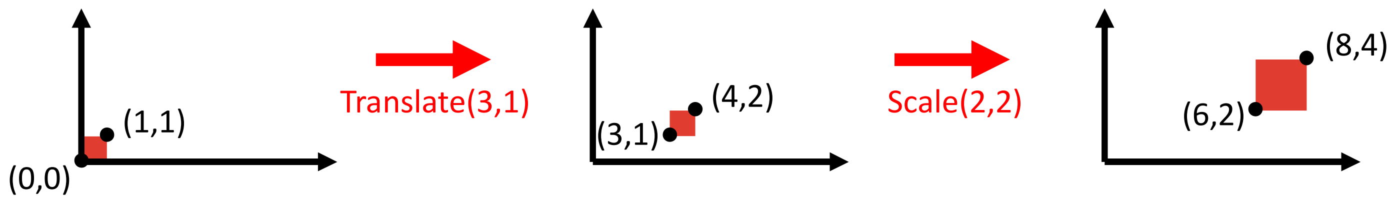
TUDelft

NOT commutative

Scaling + translation: $p' = T\,(\,S\,p\,) = TS\,p$



Translation + scaling: $p' = S\,(\,T\,p\,) = ST\,p$



NOT commutative

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$ST = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & \\ 0 & 2 & \\ 0 & 0 & 1 \end{pmatrix}$$

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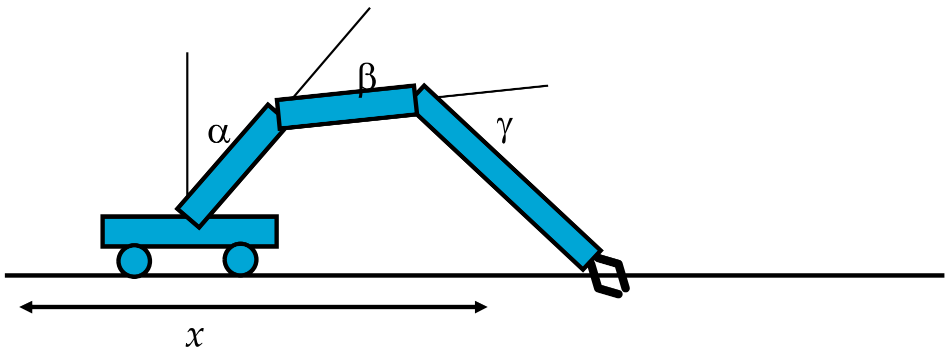
$$ST = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Complex Objects

- Objects are often defined via many components
  - E.g., wheels of cars, fingers on hands on arms …
- Concatenate matrices to place objects
  - Changing one matrix affects everyone hereafter



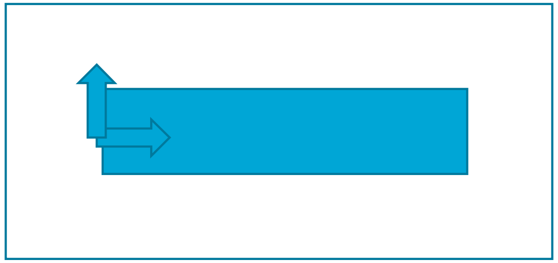
Complex Objects

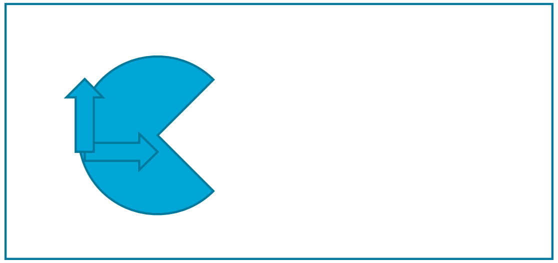
Example:

Robot arm consisting of two parts:

The arm itself and a hand

Both are designed independently
and are at the origin (shown below)





Why concatenate operations?

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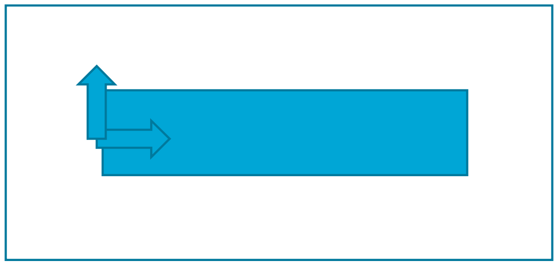
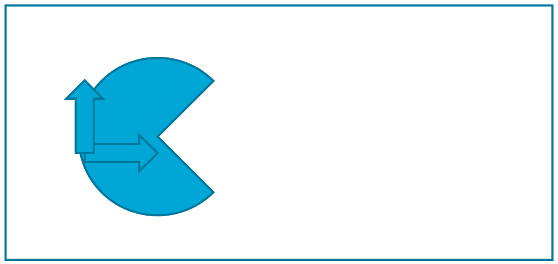
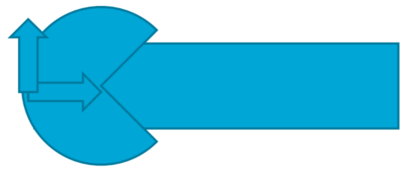




Why concatenate operations?

- Drawing first the arm, then the hand, you get:



- That does not look right!
- Instead: Produce matrix that when applied to the object shifts it to the wanted location

Why concatenate operations?

- Concatenate and apply matrices
  - S:=Translation matrix to position of arm (**S**houlder)

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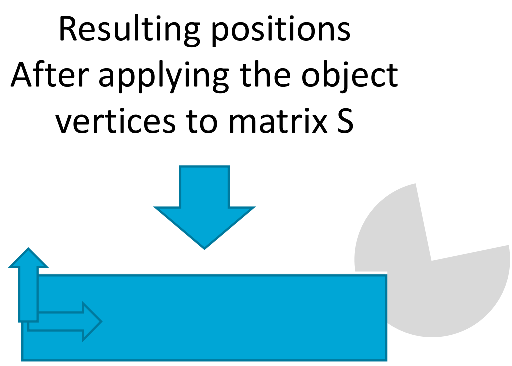

Why concatenate operations?

- Concatenate and apply matrices
  - S:=Translation matrix to position of arm (**S**houlder)
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  - T:=translation along arm (to the **J**oint of the hand)
  - J=S T

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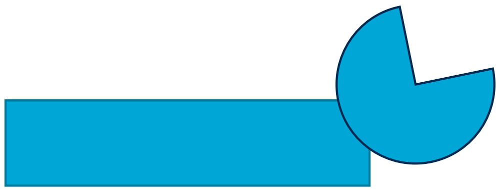

Why concatenate operations?

- Concatenate and apply matrices
  - S:=Translation matrix to position of arm (**S**houlder)
  - **S:=SN, with N a new rotation matrix**
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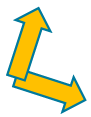

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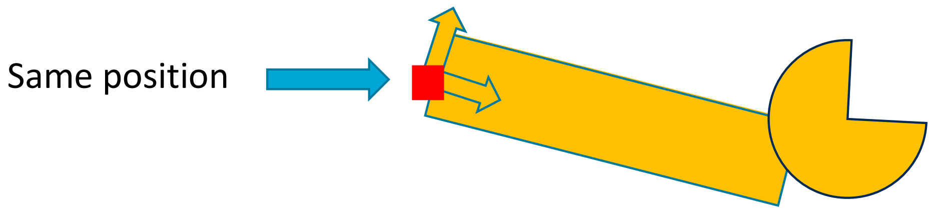
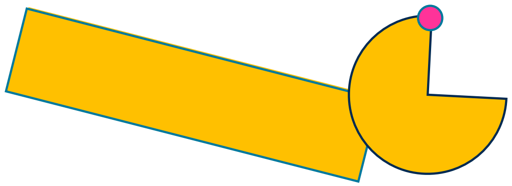
  - H:=J R
  - Apply H to all vertices of the hand



How to remember the order?



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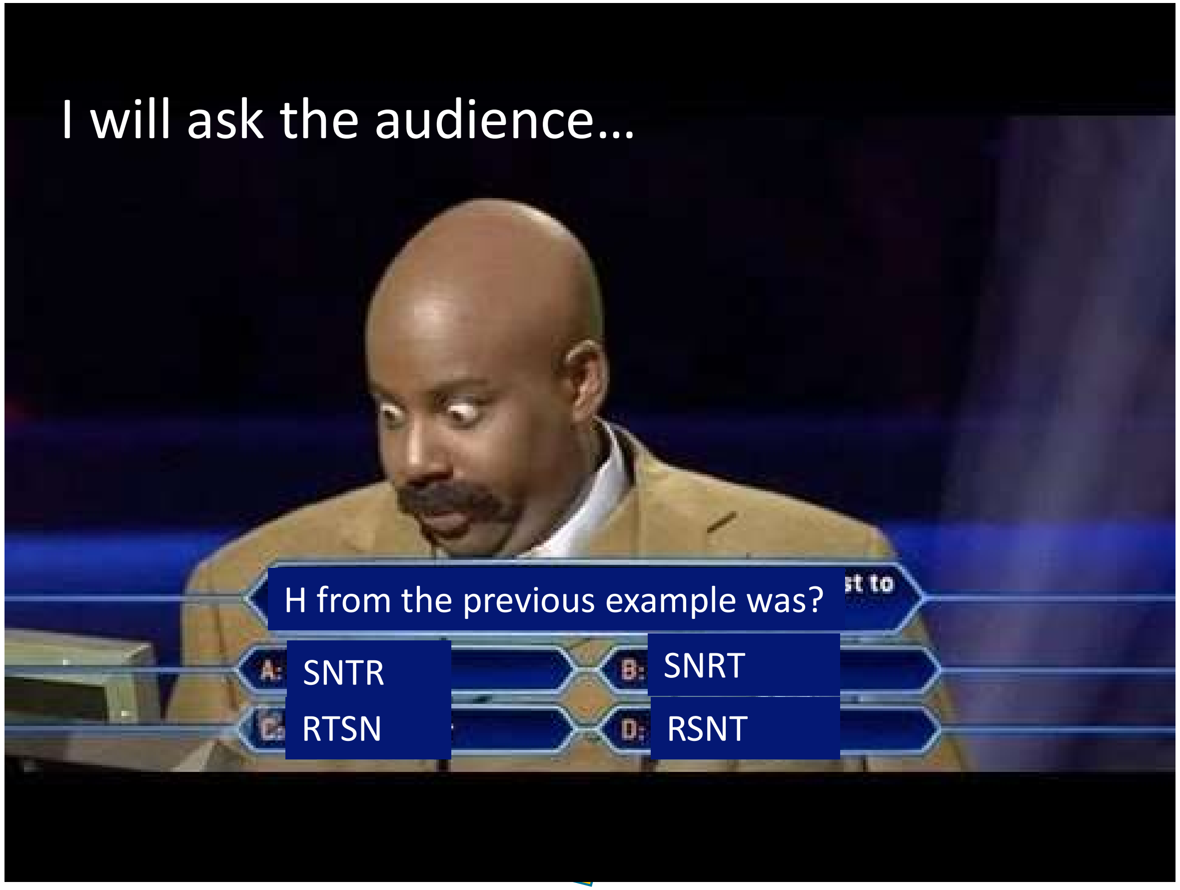

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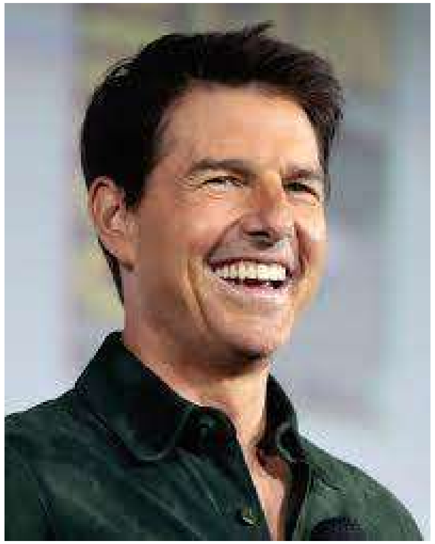
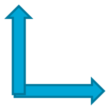
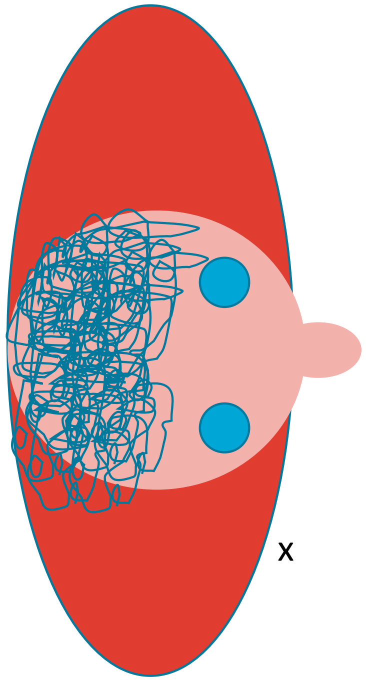




Tom/Penelope “Cross”

wikipedia

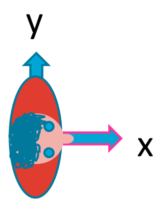
How to remember the order?



Tom/Penelope "Cross"

How to remember the order?

Matrix applications from **left to right** can be interpreted as moving the origin of our space.
If we depict the origin by “cross”, matrix operations will move “cross” (and in turn the origin) around.



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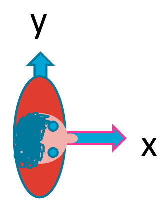

Tom/Penelope “Cross”

How to remember the order?

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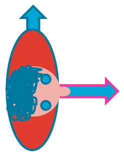
T=Translate(2,0)



Tom/Penelope “Cross”

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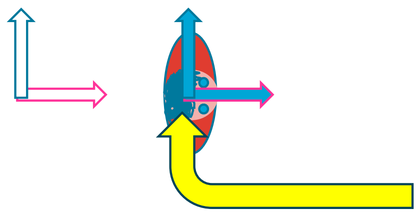
T=Translate(2,0)

T * “cross”



For example, if we apply the point $(0,0,1)^t$ to T, we end up here.

How to remember the order?

Matrix applications from **left to right** can be interpreted as moving the origin of our space.
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T * “cross”

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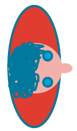

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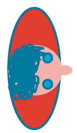
If we depict the origin by “cross”, matrix operations will move “cross” (and in turn the origin) around.

T * “cross”



T=Translate(2,0)
R=Rotate(45)

How to remember the order?

Matrix applications from **left to right** can be interpreted as moving the origin of our space.
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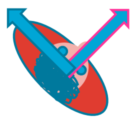
T * R * “cross”



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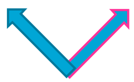

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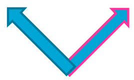

Let’s apply both operations to a point:
$T*R*(0,2,1)^t$

How to remember the order?

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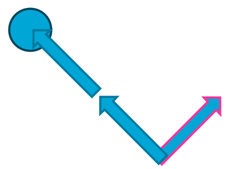
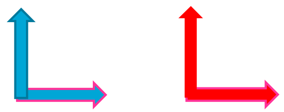


Let’s apply both operations to a point:
$T*R*(0,2,1)^t$

The point lands in the location that we would expect,
if the new frame described the origin of our space

How to remember the order?

T=translate (2,0)



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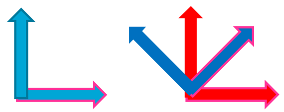
R=rotate(45)

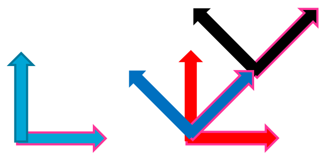


How to remember the order?

T=translate (2,0)

R=rotate(45)

T=translate (1,0)



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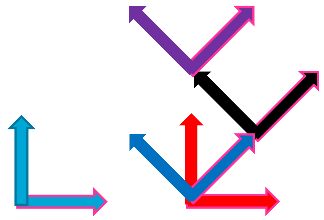
R=rotate(45)

T=translate (1,0)

T=translate (0,1)



How to remember the order?

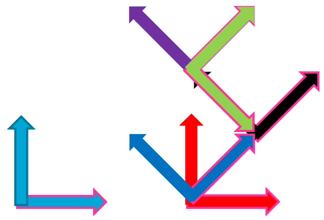
T=translate (2,0)

R=rotate(45)

T=translate (1,0)

T=translate (0,1)

R=rotate(-90)



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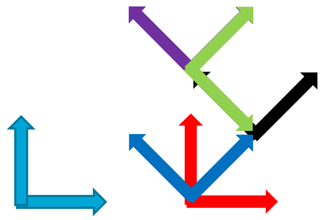

Matrix for the last position would be: TRTTR

Multiply left to right in the "playthrough" order

How to remember the order?

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T=translate (1,0)

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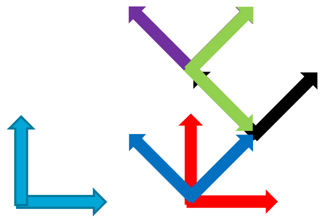

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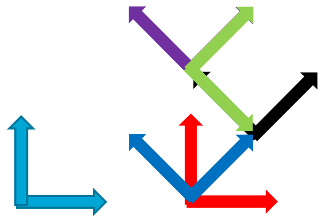

Matrix for the last position would be: TRTTR

Where is TRTTR $(0,0,1)^t$?

How to remember the order?

T=translate (2,0)

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T=translate (1,0)

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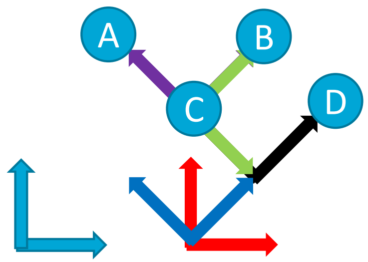

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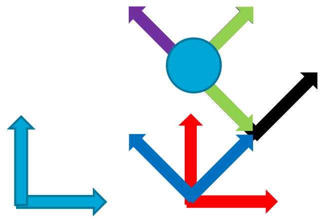

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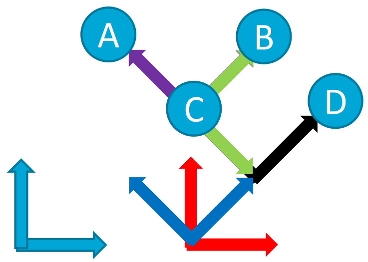

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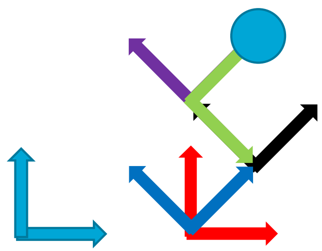

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T=translate (0,1)

R=rotate(-90)

(TRTTR) P

Multiply left to right in the “playthrough” order

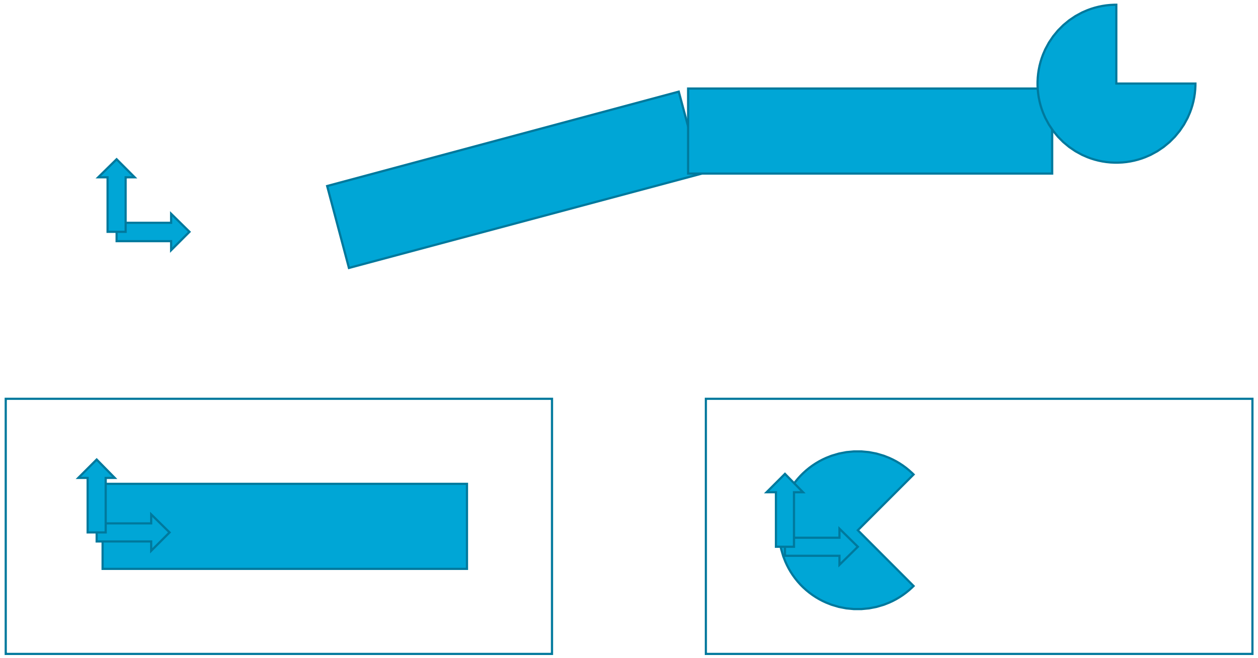
Why does this work? P multiplies mathematically from the right!

Local vs. Global interpretation

- Two interpretations of the same mathematics:
  - Points multiplied from the right:
    This transforms the points with respect to a GLOBAL coordinate system.
  - Multiplying matrices together on the left:
    This changes the origin/coordinate system, i.e., a LOCAL coordinate system is established.

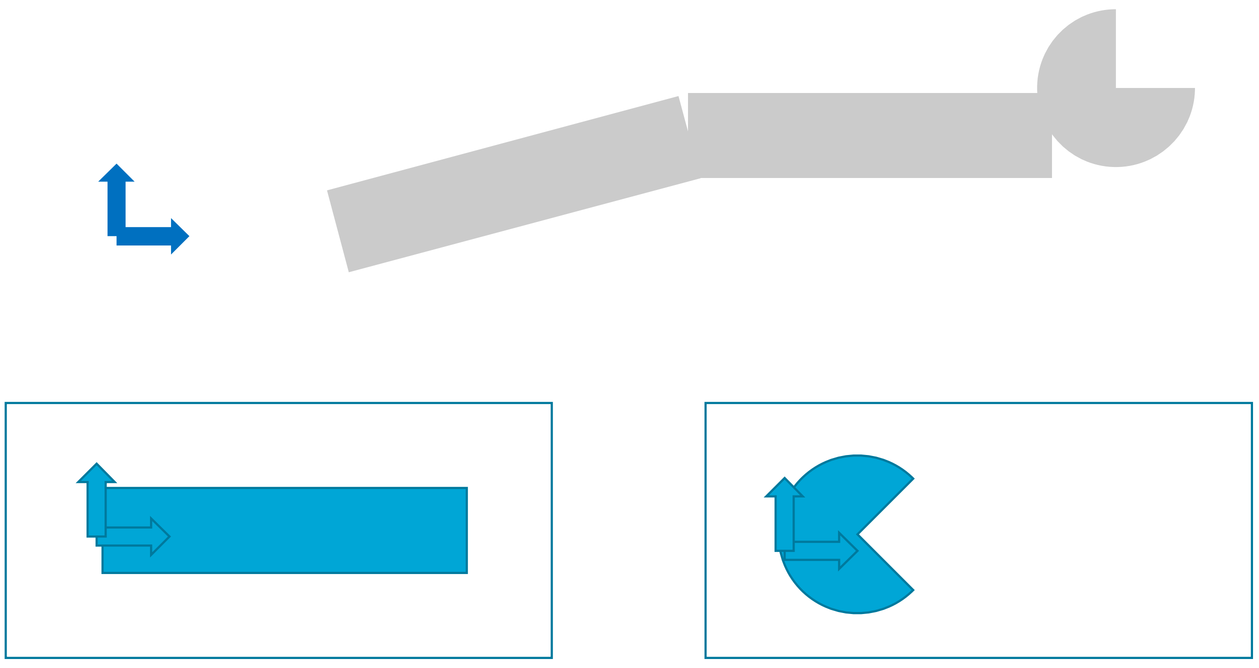
Local Interpretation

- Build complex object dependencies



Local Interpretation

- Build complex object dependencies



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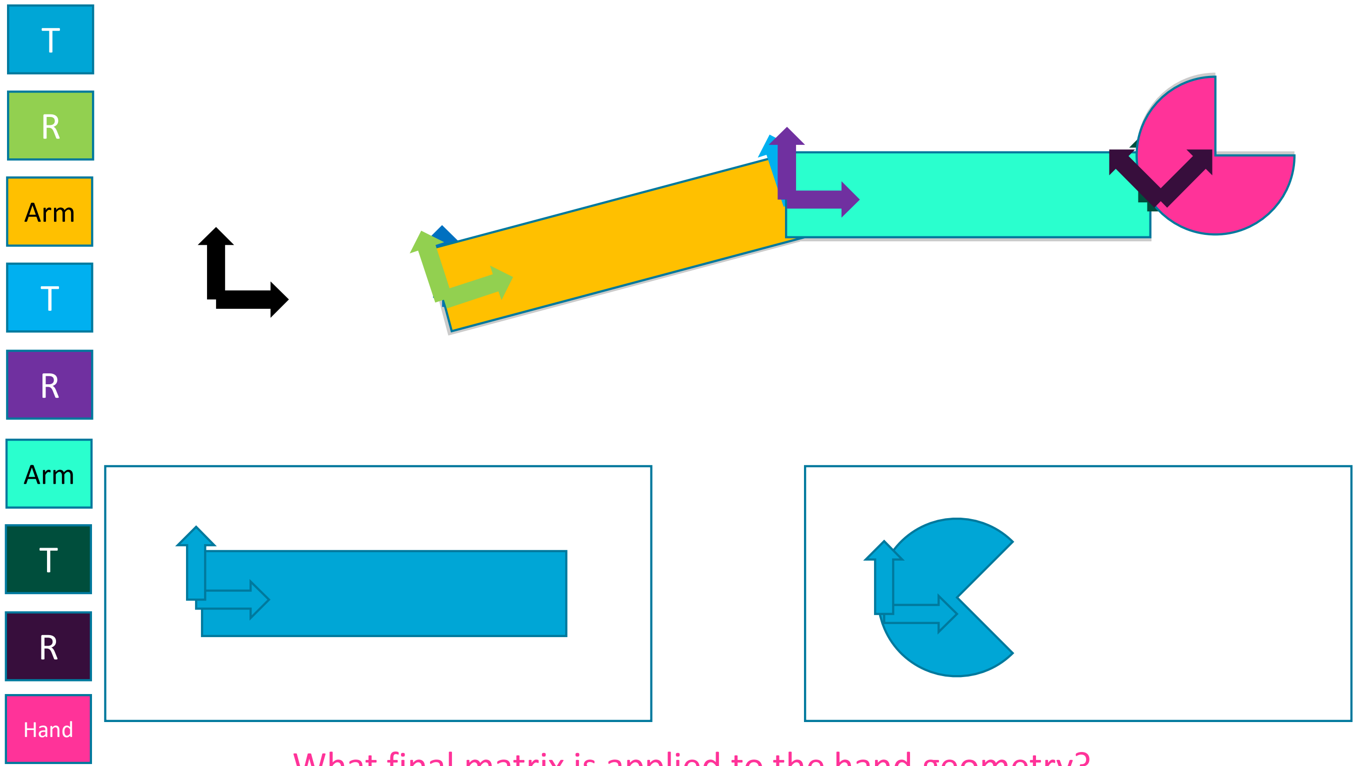
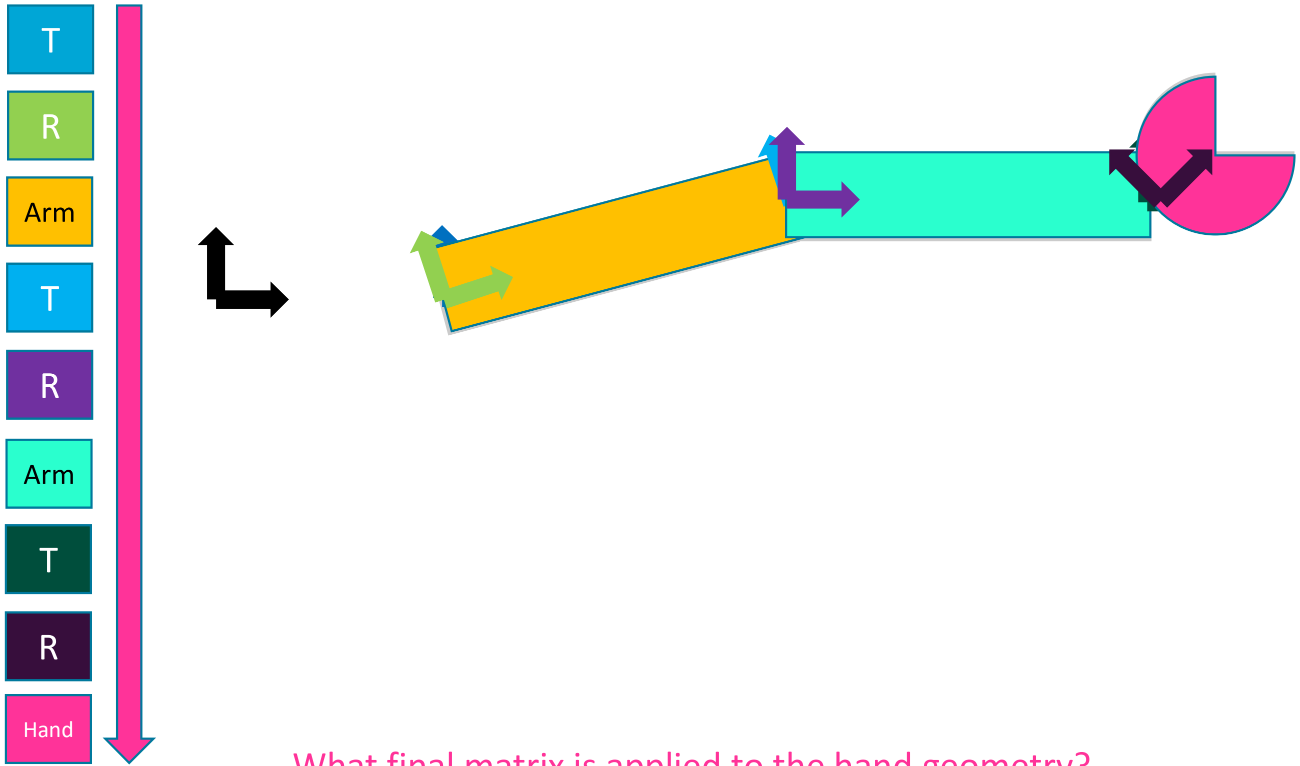


What final matrix is applied to the hand geometry?

Local Interpretation

- Build complex object dependencies



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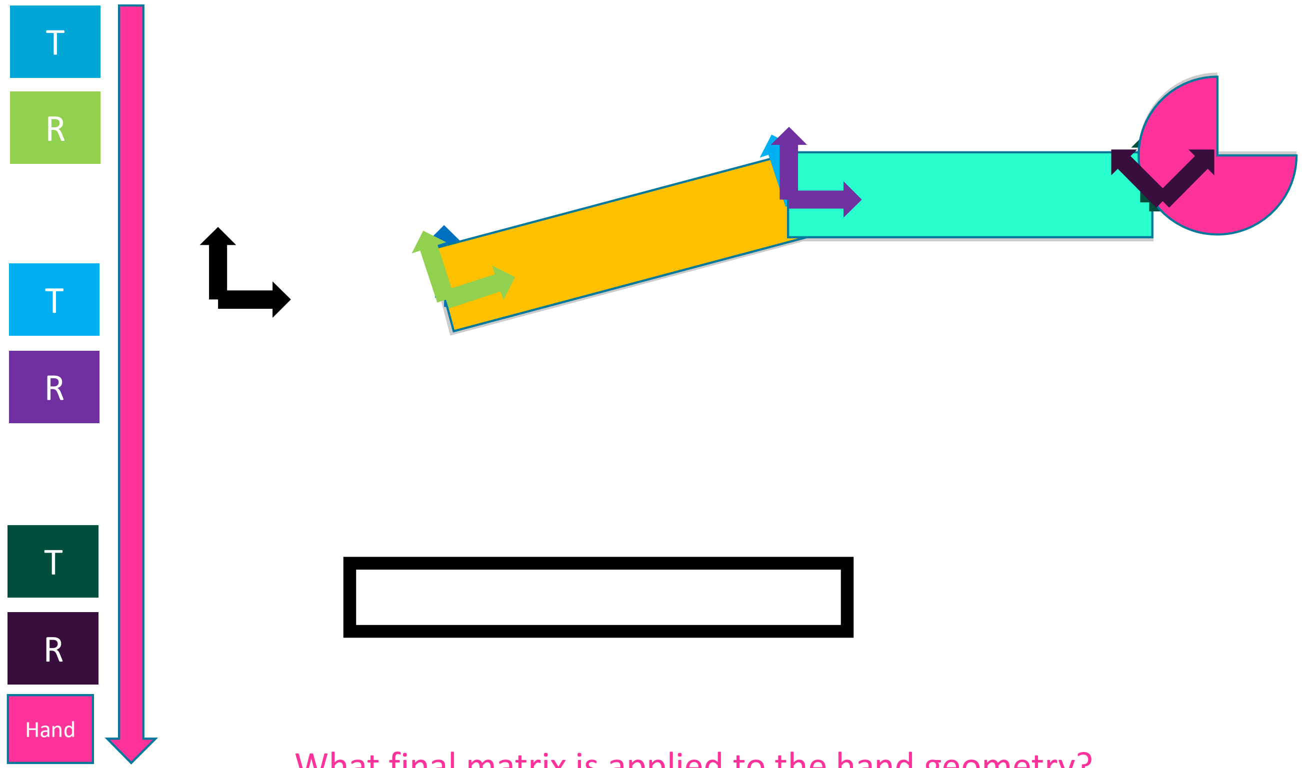

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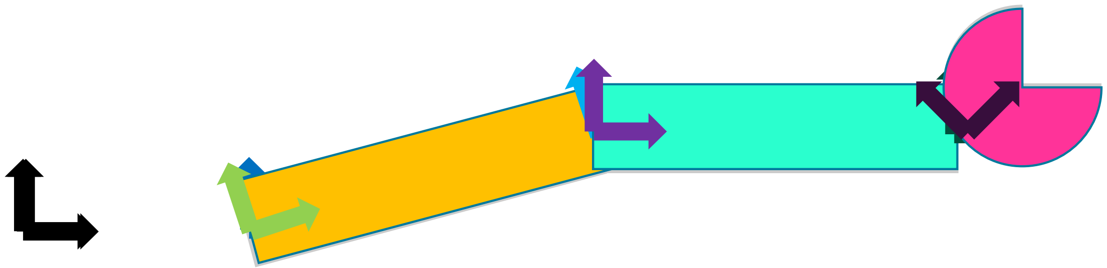
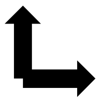
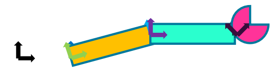


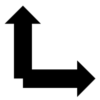
T R T R T R



T R T R T R

Global Interpretation





T R T R T R

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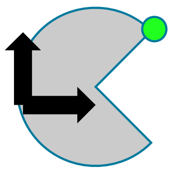

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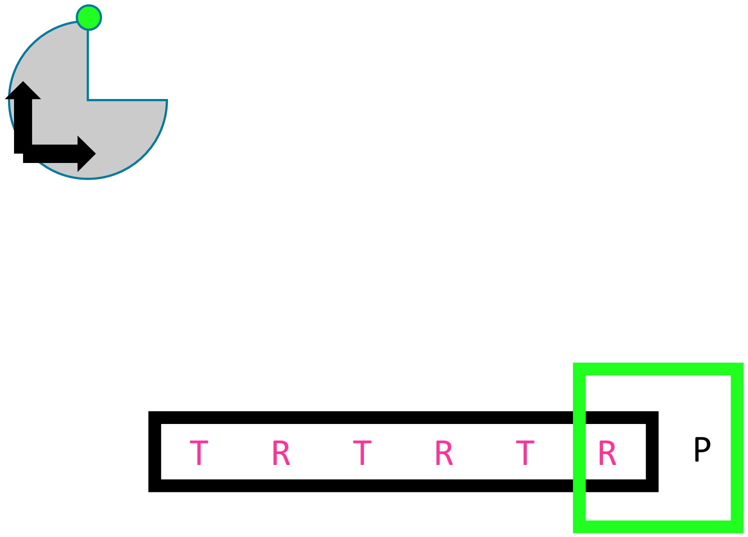

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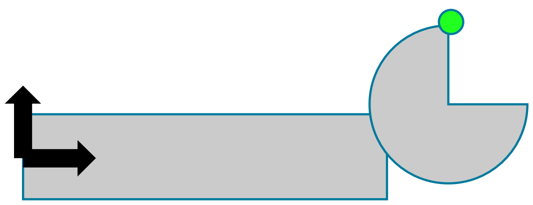

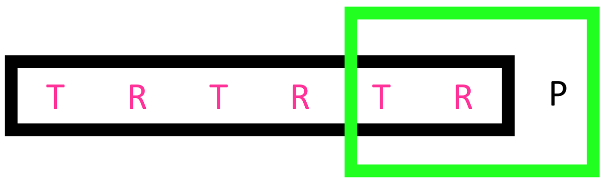

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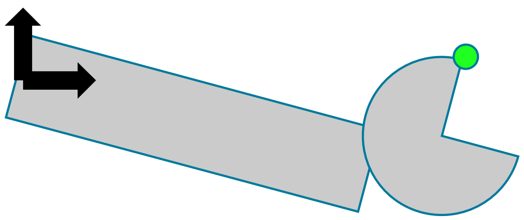

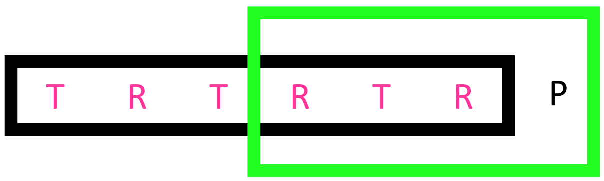

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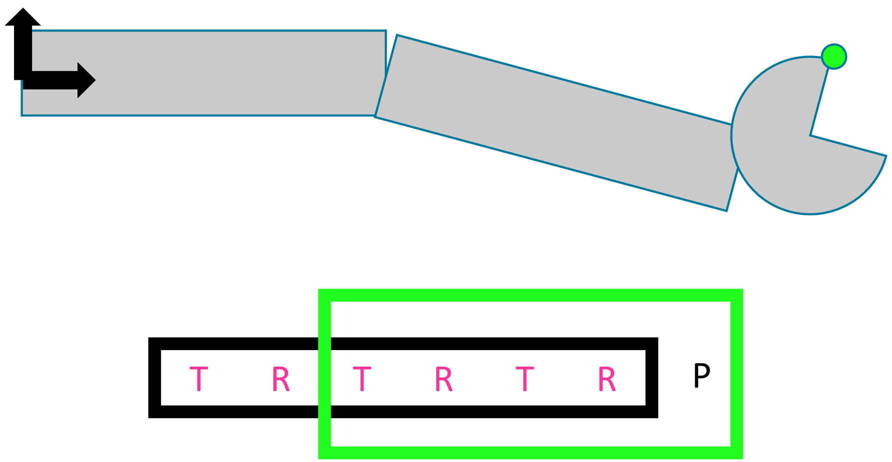

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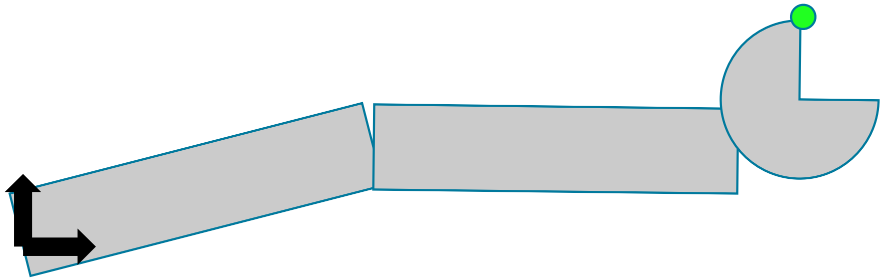

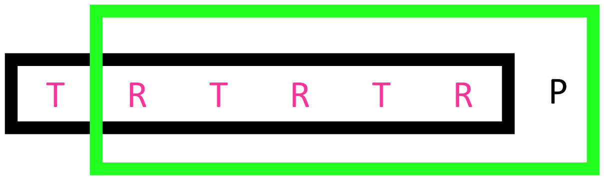

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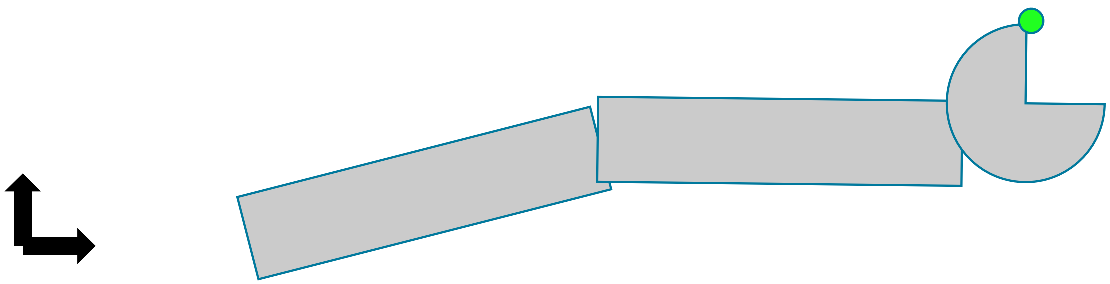

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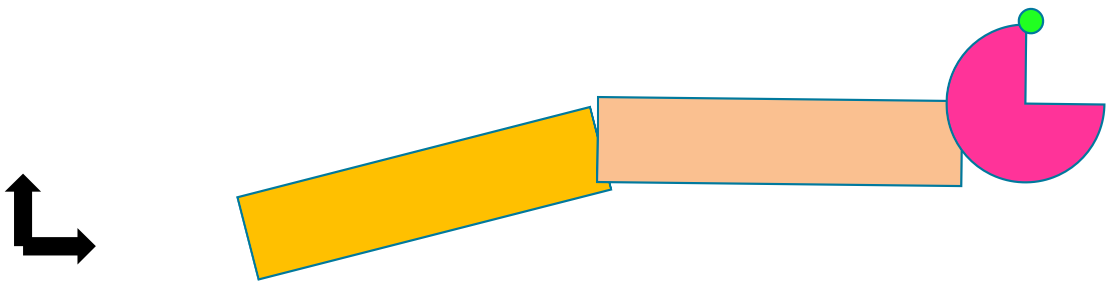


$\boxed{\boxed{T \quad R \quad T \quad R \quad T \quad R} \quad P}$

The point does end up in the expected location

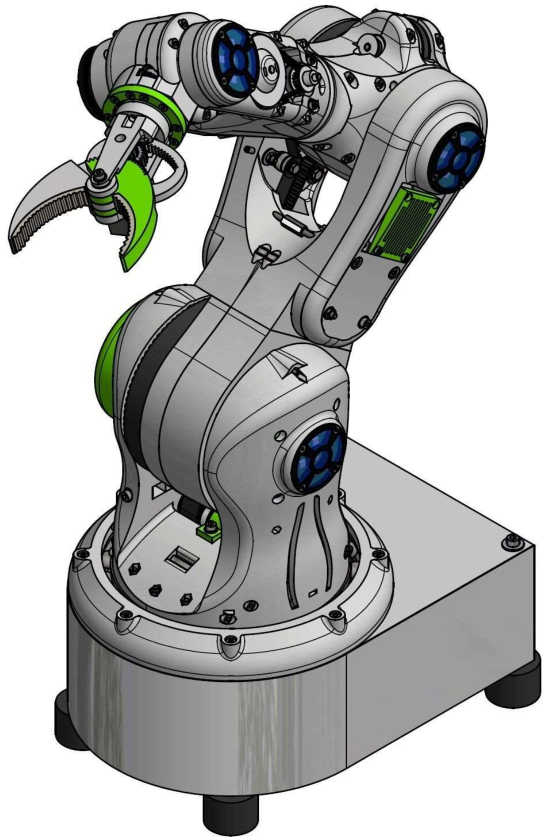
Please note: this is ONE matrix, which makes it very efficient to apply to all vertices of the hand.

Recap Video

Thanks to **Tim Huisman**

Let's try it out!

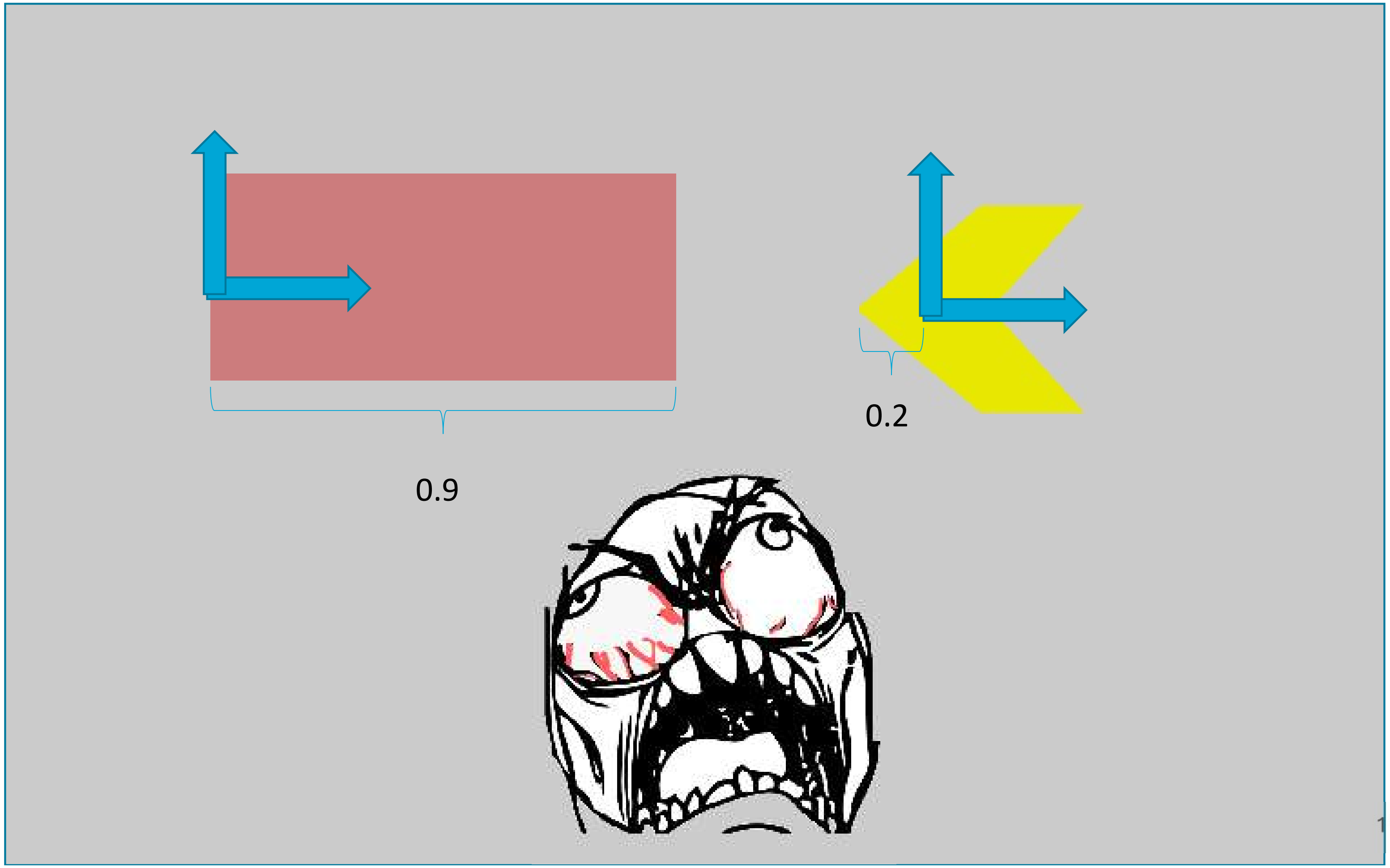
- Professional Robot Arm:



Let’s try it out!

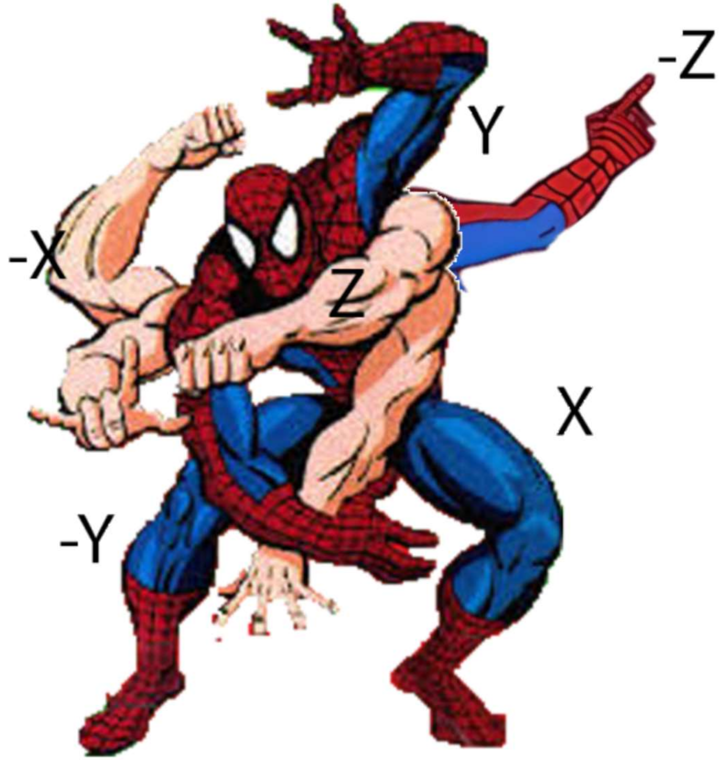
- Professional Robot Arm:

Let’s try it out!



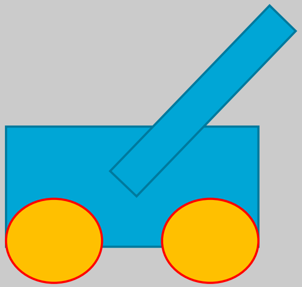
Questions?

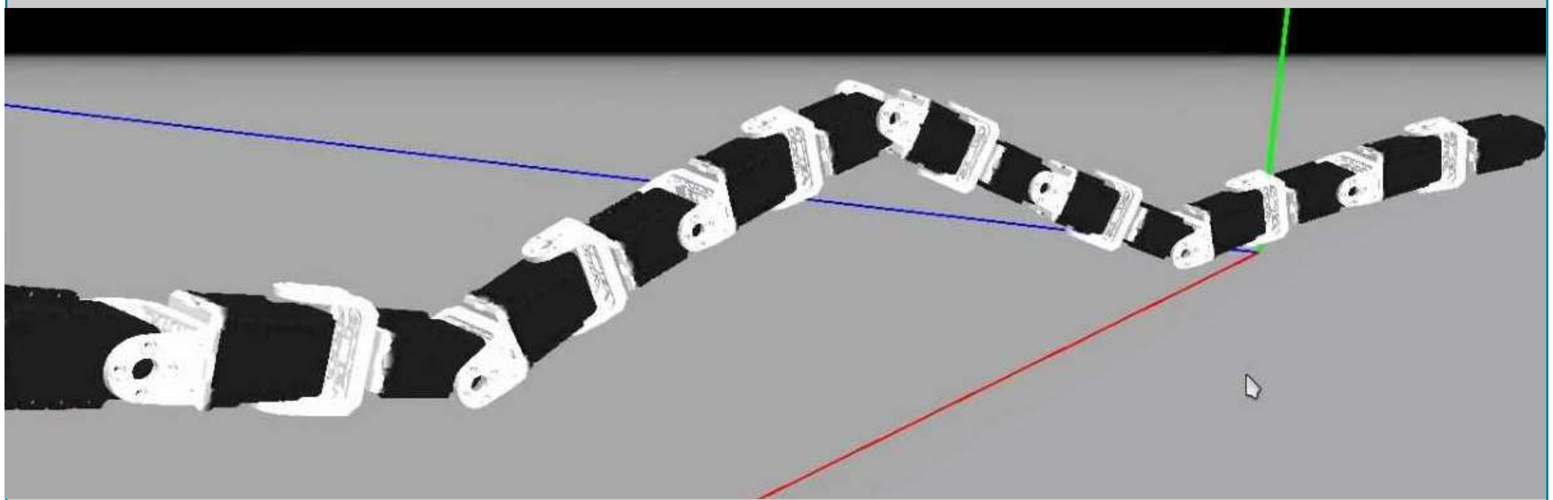
Before CSE2215



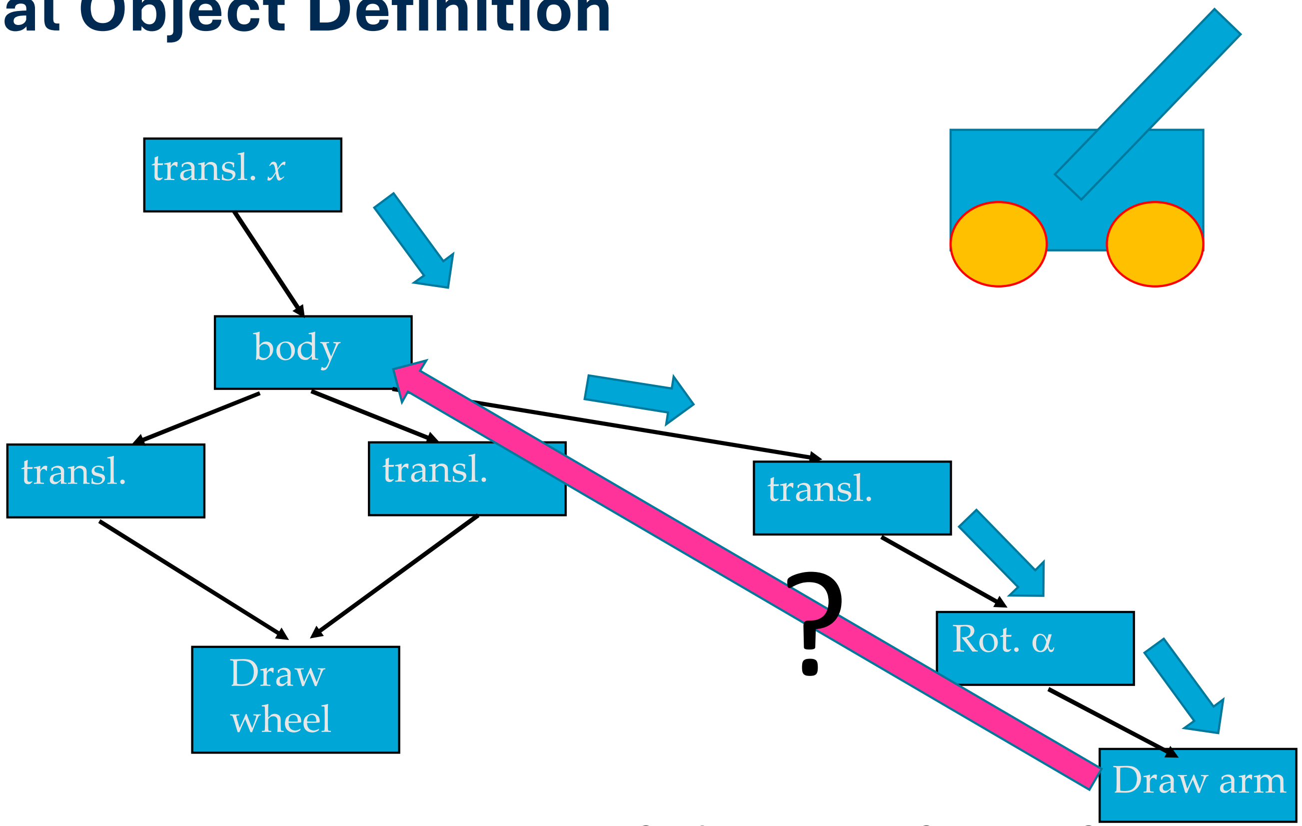
After CSE2215

Other examples?





Hierarchical Object Definition

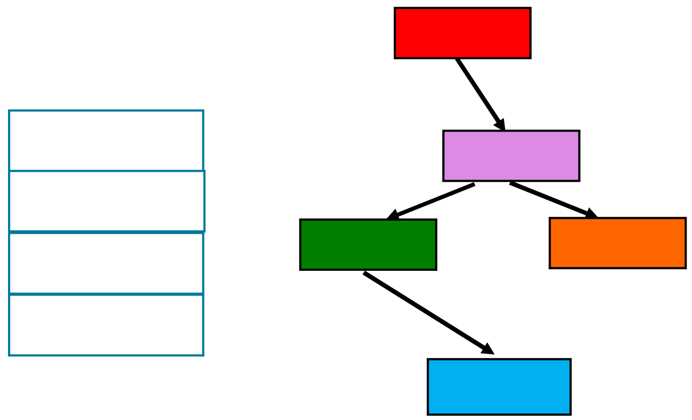


Common representation of objects in form of a tree
Geometry is reused (e.g., only one wheel is stored)

Excursion: Walking over a tree

Depth-first tree traversal

using a stack:



```
stack.push(root)

while (!stack.empty())
{
        node=stack.pop()

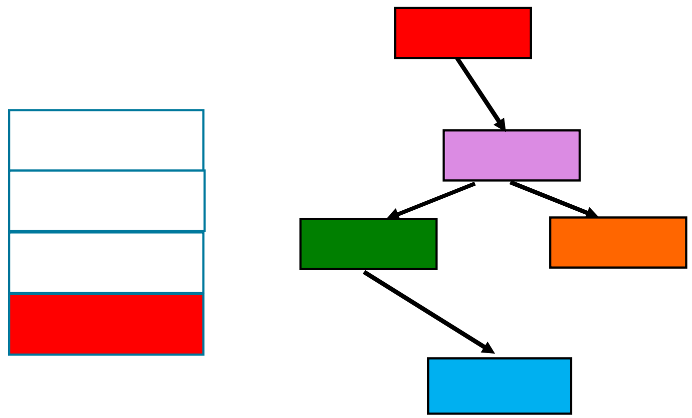
        process(node)//do what you need to do

        if (node.hasChildren()) stack.push(node.children())
}
```

Excursion: Walking over a tree

Depth-first tree traversal

using a stack:



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while (!stack.empty())
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    node=stack.pop()
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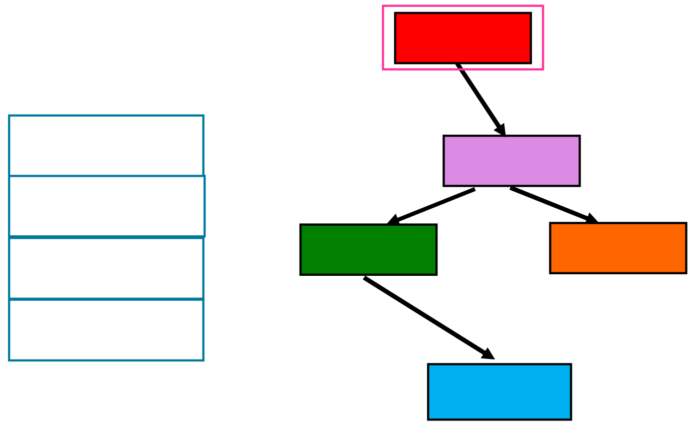
    if (node.hasChildren()) stack.push(node.children())
}
```

Excursion: Walking over a tree

Depth-first tree traversal

using a stack:



stack.push(root)

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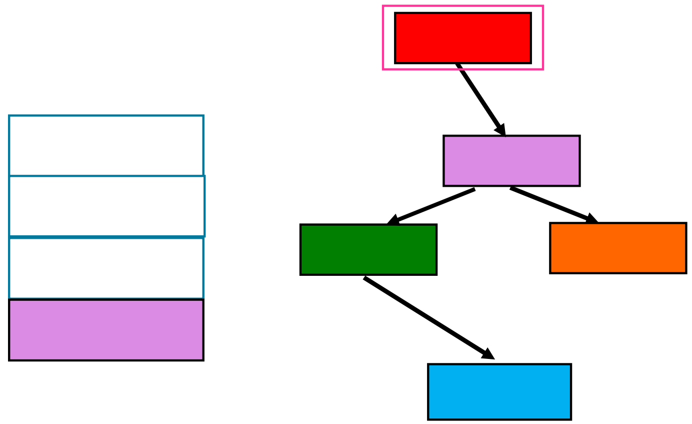

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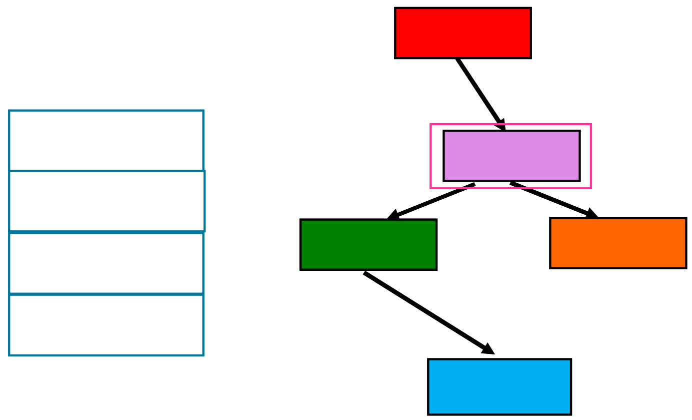

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stack.push(root)
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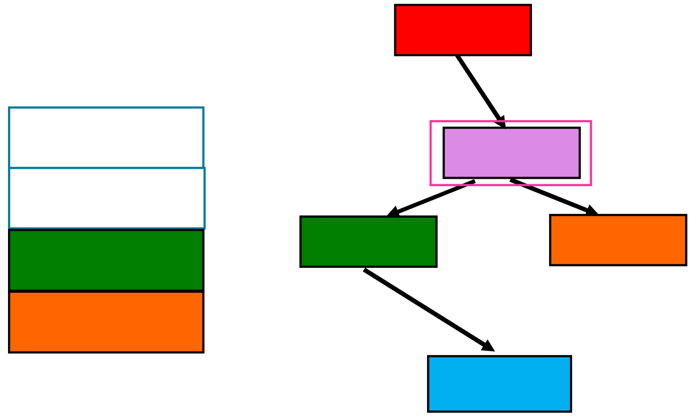

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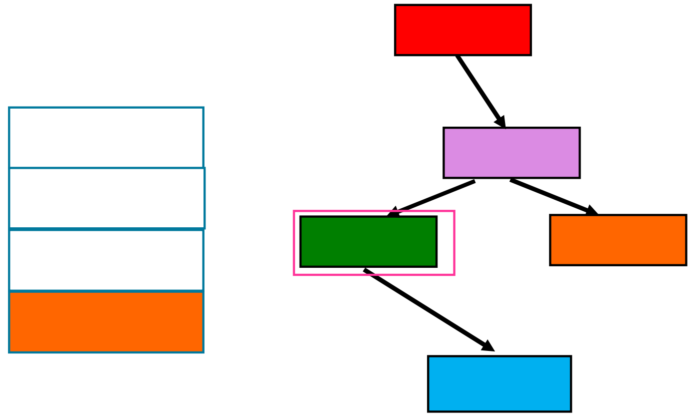

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Excursion: Walking over a tree

Depth-first tree traversal

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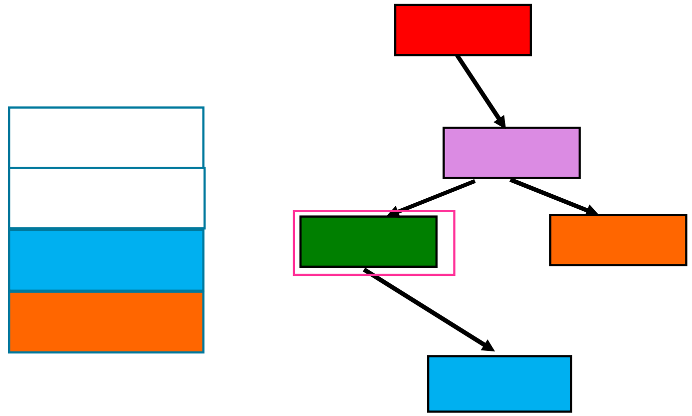

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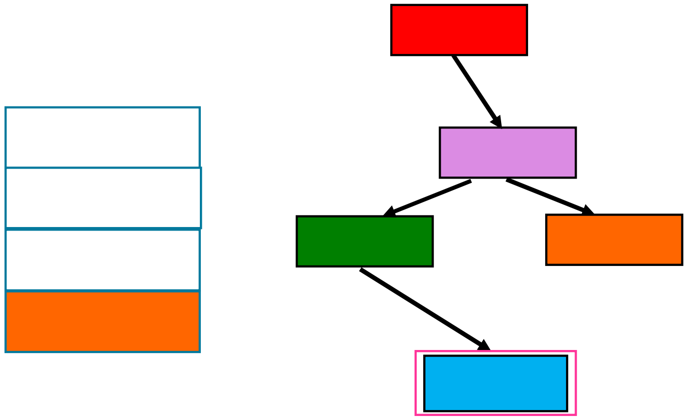

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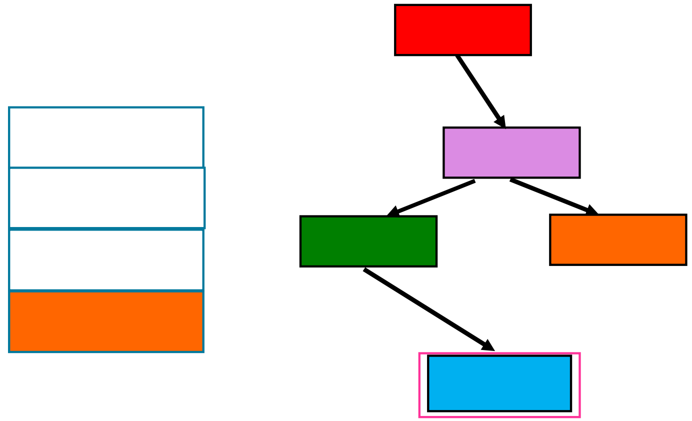

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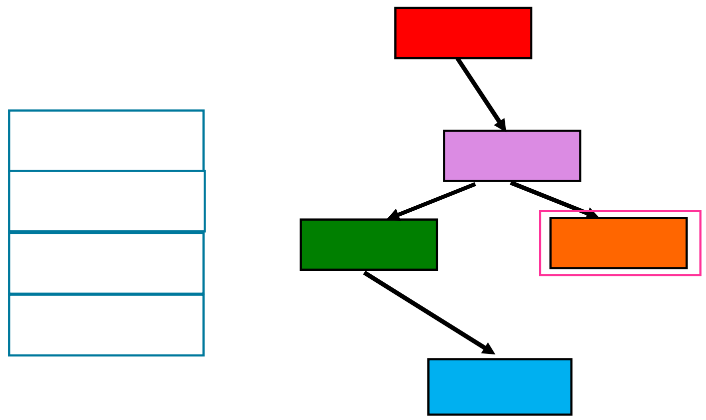
Excursion: Walking over a tree

Depth-first tree traversal

using a stack:



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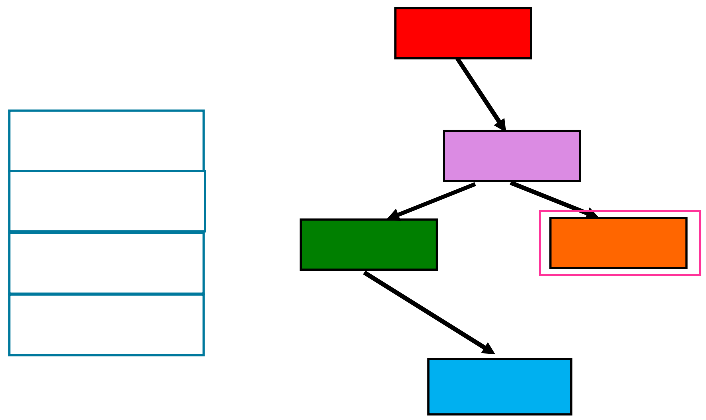

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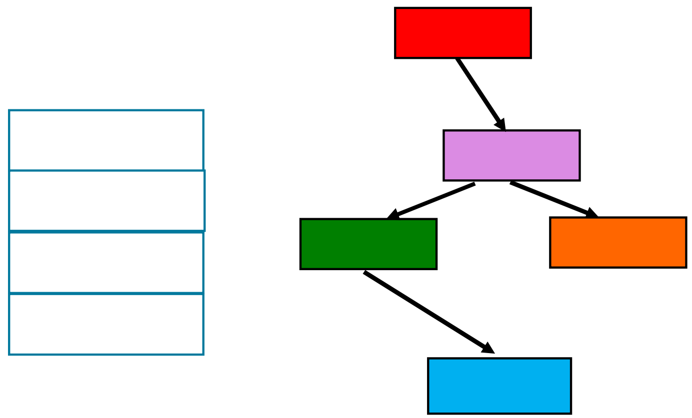
Excursion: Walking over a tree

Depth-first tree traversal

using a stack:



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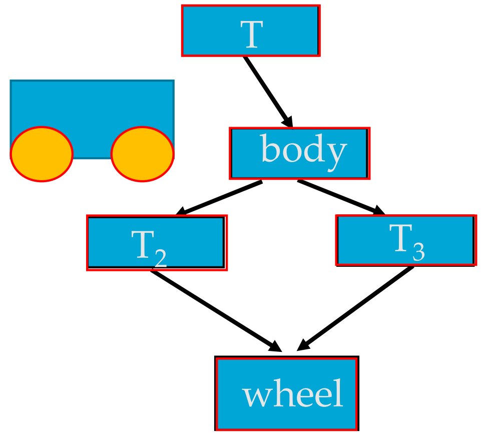
        if (node.hasChildren()) stack.push(node.children())
}
```

Hierarchical Objects

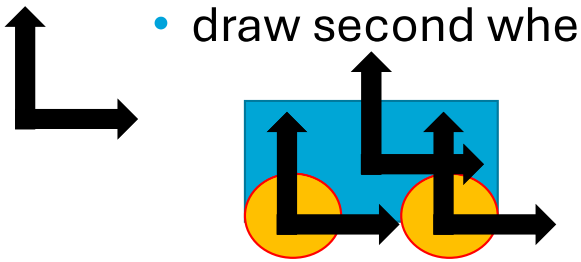
- Walk over tree using depth-first traversal

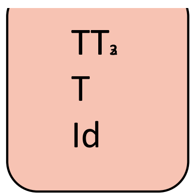
- Keep a “matrix” stack that maintains
  the concatenations (multiplications) of matrices along the way

Matrix Stack



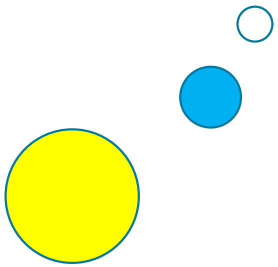
- Parallel to walking over the tree:
- Keep previous matrices on a stack
  - **T** (translation by x) pushMatrix idT=T
  - draw robot body
  - **T_2** (translation to center of 1st wheel) pushMatrix TT_2
  - draw first wheel as circle of center (0,0)
  - return to **T:** popMatrix
  - **T_3** (T_3 translation to center of 2nd wheel) pushMatrix TT_3
  - draw second wheel as circle of center (0,0)





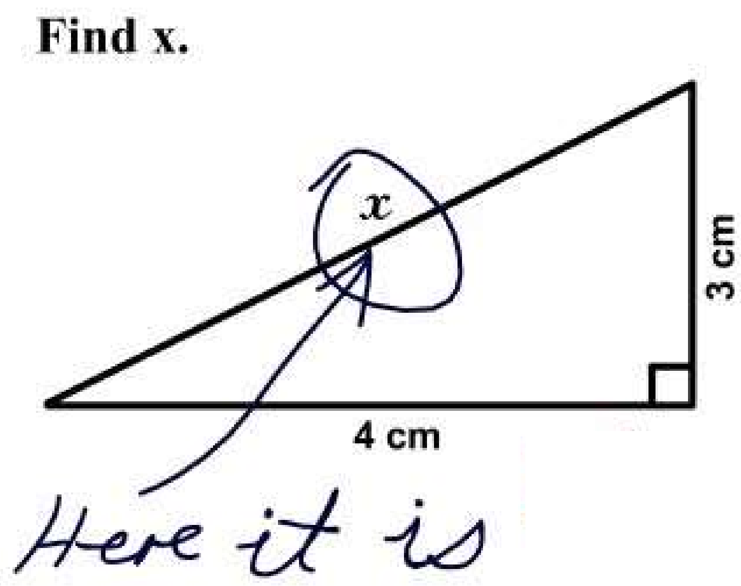
In the practical…

- Define the hierarchy for a solar system:
- Earth rotating around sun
- Moon rotating around earth



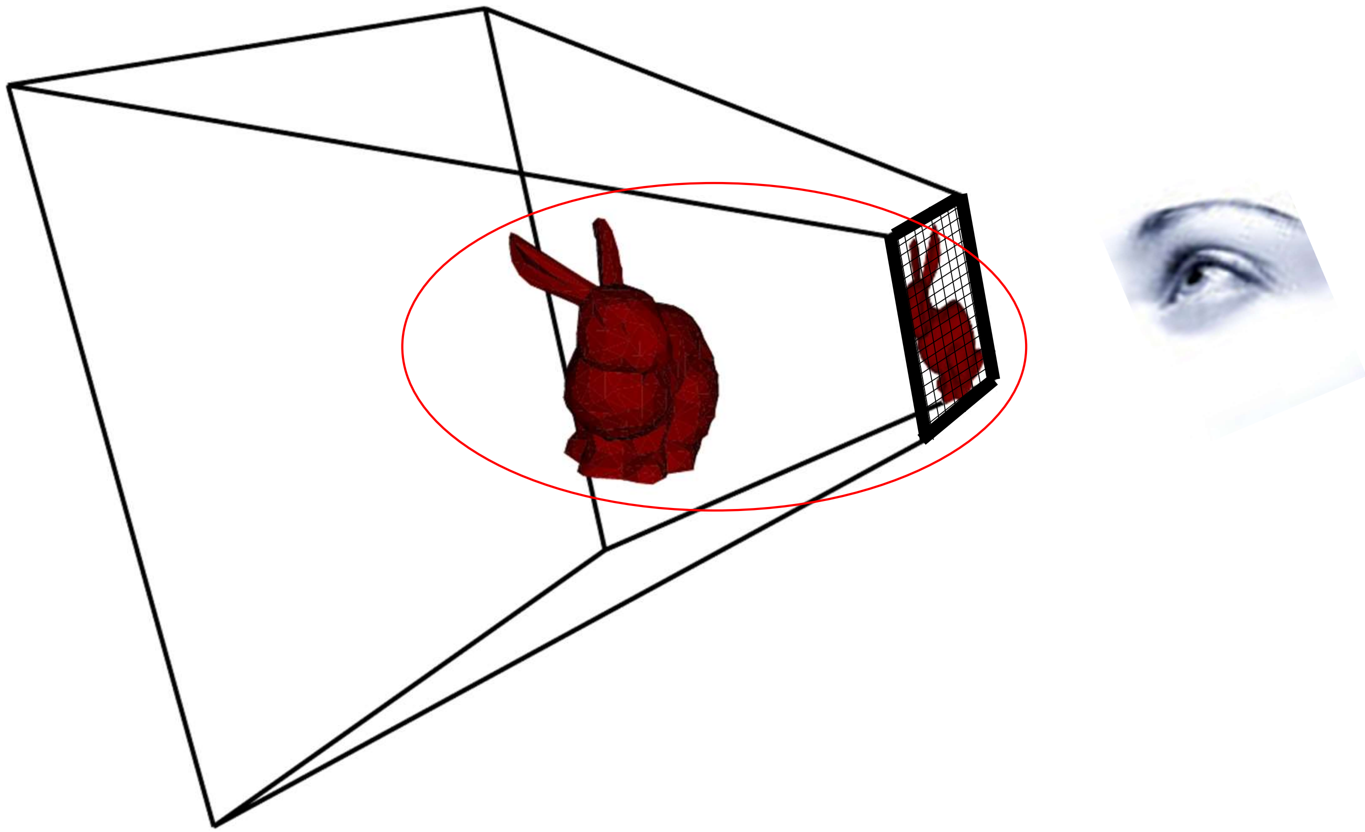
TU Delft

Questions?



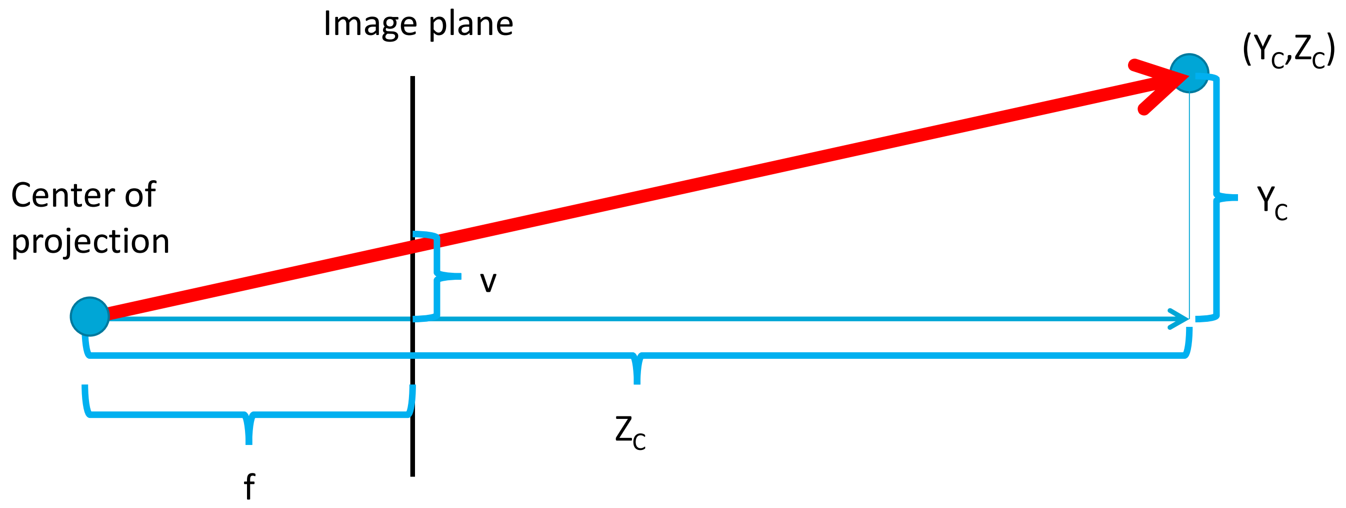
TUDelft

One pipeline step is still missing…



Perspective Projection

- sideview: Formula is simple if camera at origin



Similar triangles: $v / f = Y_C / Z_C$

Perspective Projection

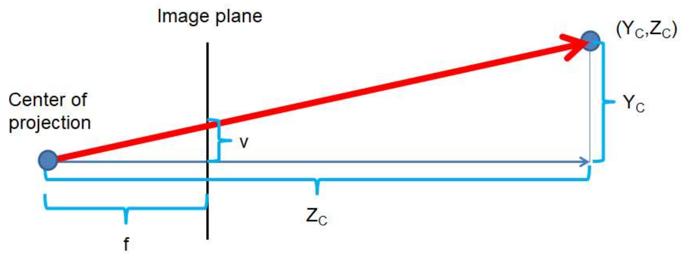
- Embed the point P in the projective space

$$P = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \Rightarrow \tilde{P} = \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

- Look for M such that:

$$M\tilde{P} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} fX/Z \\ fY/Z \\ 1 \end{pmatrix}$$

Impossible in standard R^n



$$u = f\,\frac{X_C}{Z_C} \qquad v = f\,\frac{Y_C}{Z_C}$$

What is the problem?

$$M\widetilde{P} = \begin{bmatrix} a & b & c & d \\ e & q & g & h \\ i & j & m & n \\ k & l & o & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

What is the problem?

$$M\widetilde{P} = \begin{bmatrix} a & b & c & d \\ e & q & g & h \\ i & j & m & n \\ k & l & o & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} ax + by + cz + dw \\ \cdots \\ \\ \end{bmatrix}$$

What is the problem?

$$M\widetilde{P} = \begin{bmatrix} a & b & c & d \\ e & q & g & h \\ i & j & m & n \\ k & l & o & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} ax + by + cz + dw \\ \dots \\ \\ \\ \end{bmatrix} \Rightarrow \frac{fx}{z}\ ?$$

$$u = f\,\frac{X_c}{Z_c}$$

Perspective Projection

- Hint: Think projective!

$$M\widetilde{P} = \begin{pmatrix} fX/Z \\ fY/Z \\ 1 \end{pmatrix}$$

Perspective Projection

- Hint: Think projective!

$$M\widetilde{P} = \begin{pmatrix} fX / Z \\ fY / Z \\ 1 \end{pmatrix} \qquad = \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix}$$

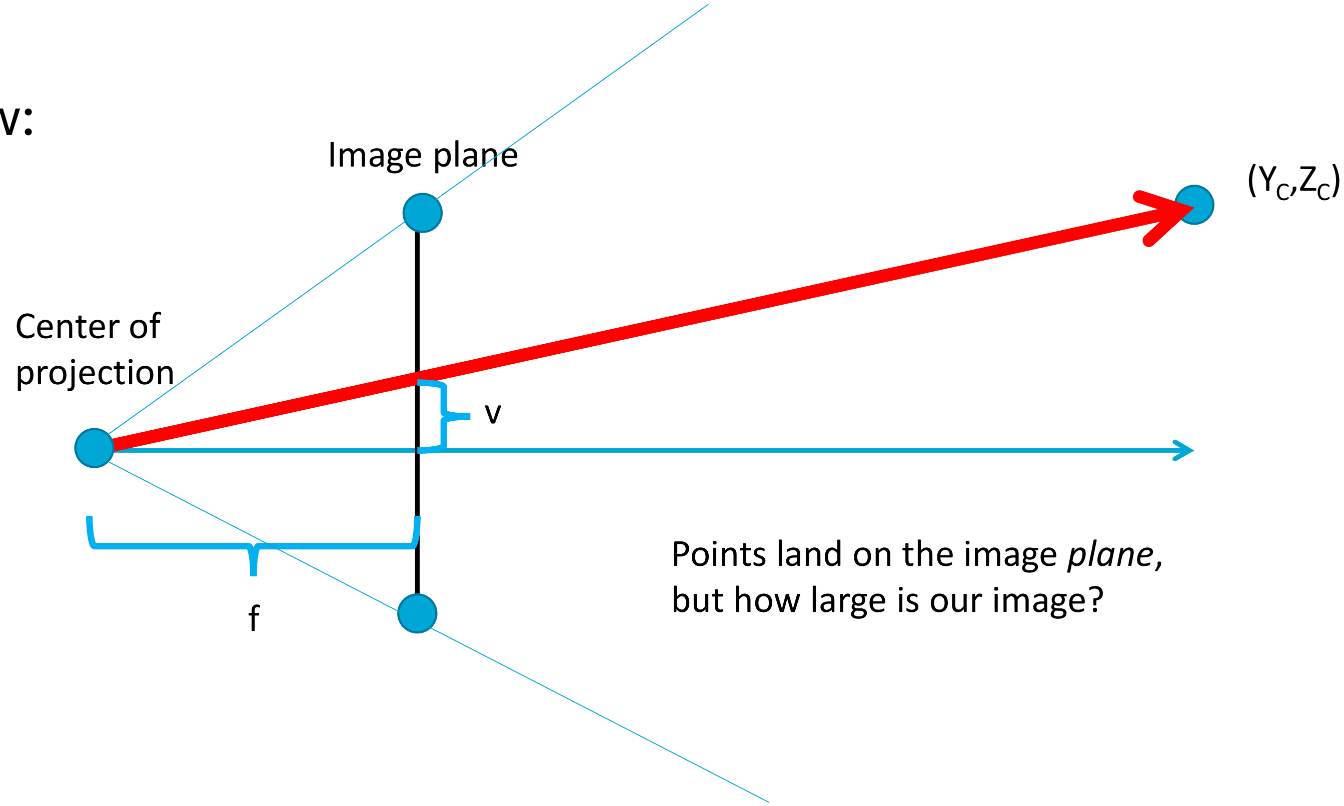
Perspective Projection

- Solution:

$$M = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Virtual Camera Model

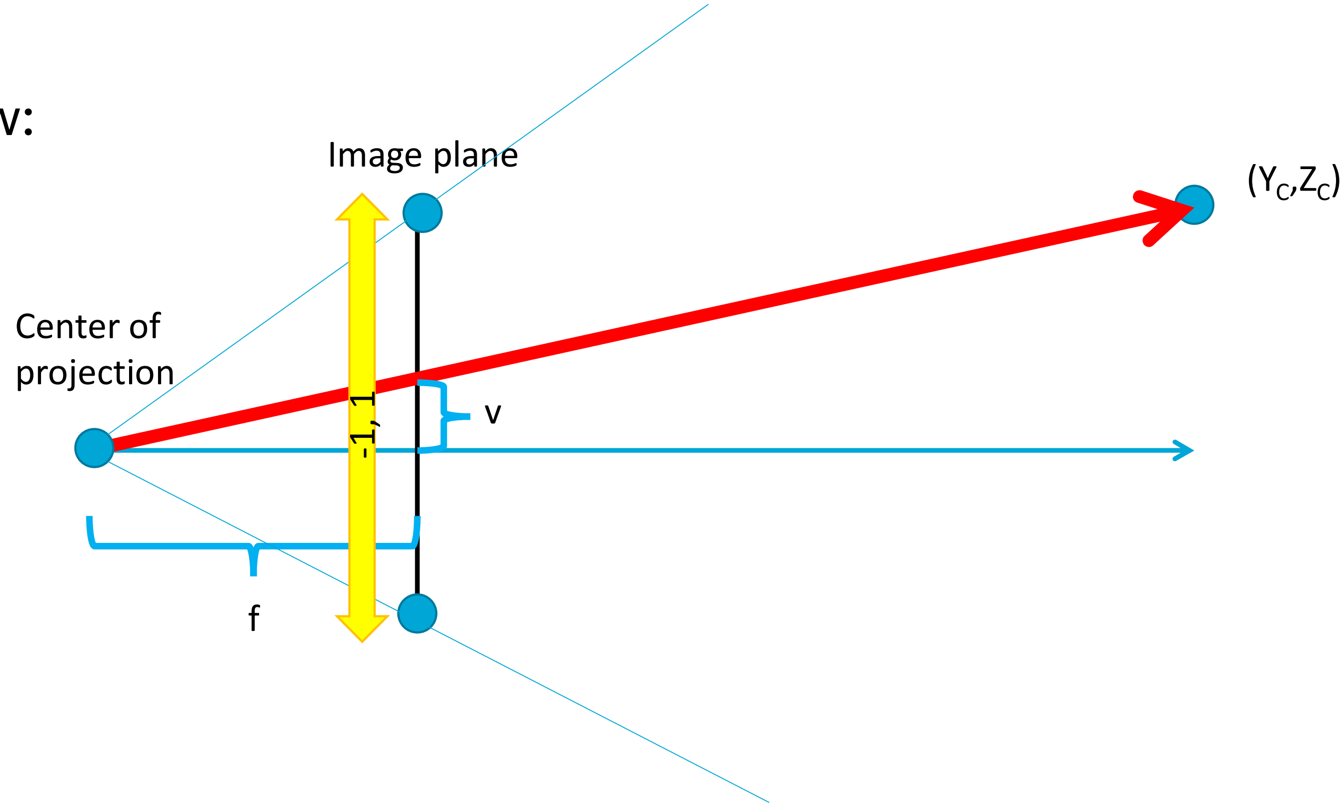
- sideview:



Points land on the image *plane*, but how large is our image?

Virtual Camera Model

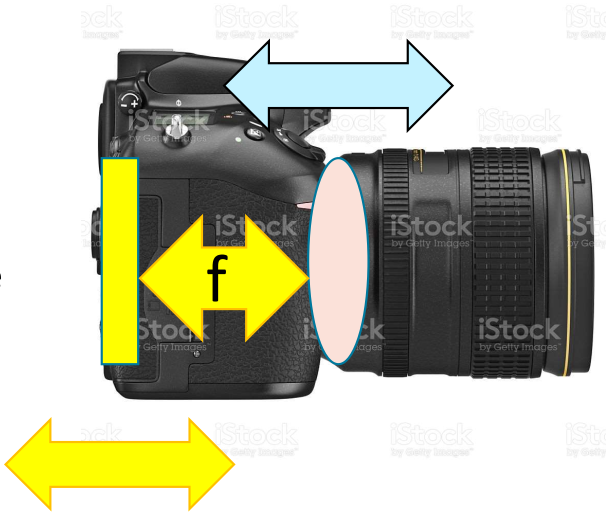
- sideview:



TUDelft

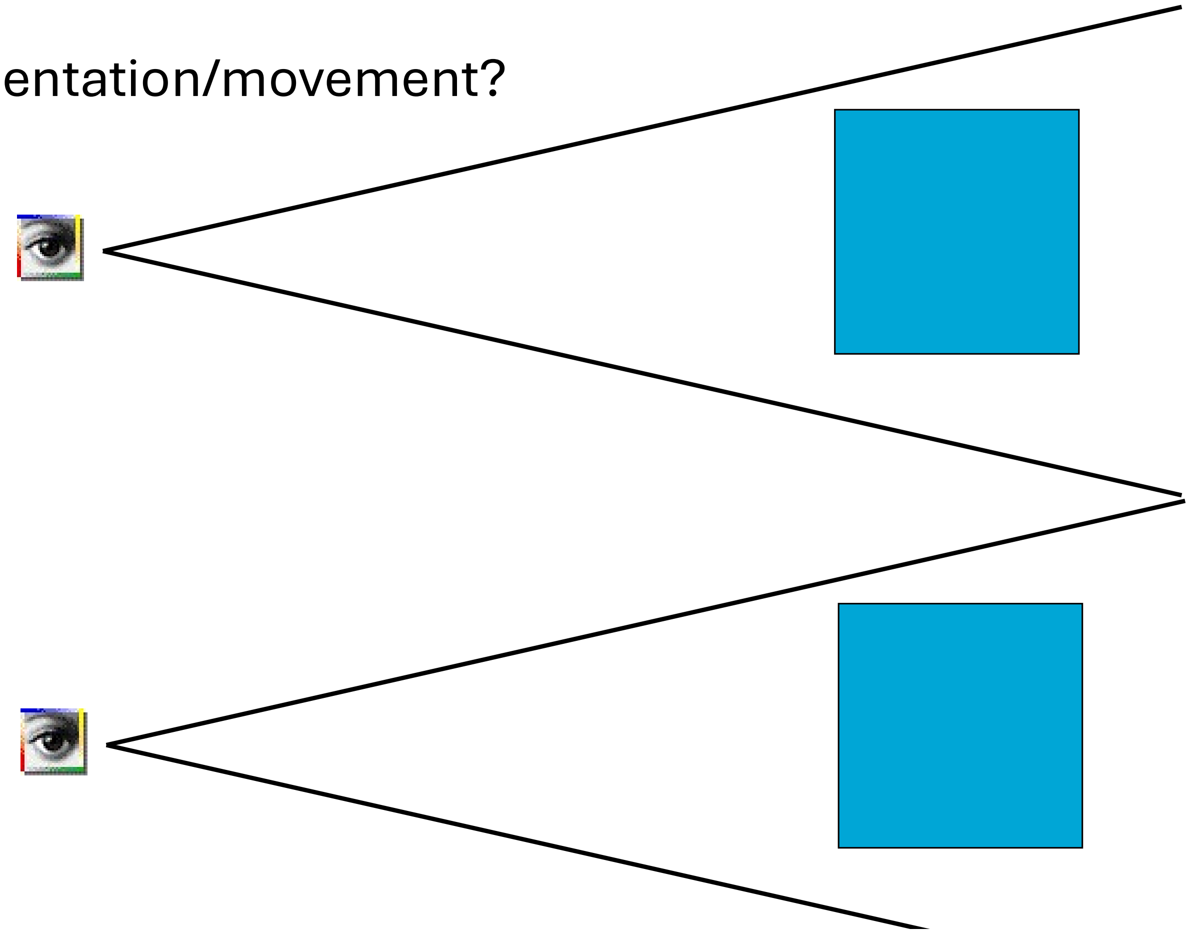
Real camera

Sensor has a fixed size
“[-1,1]”



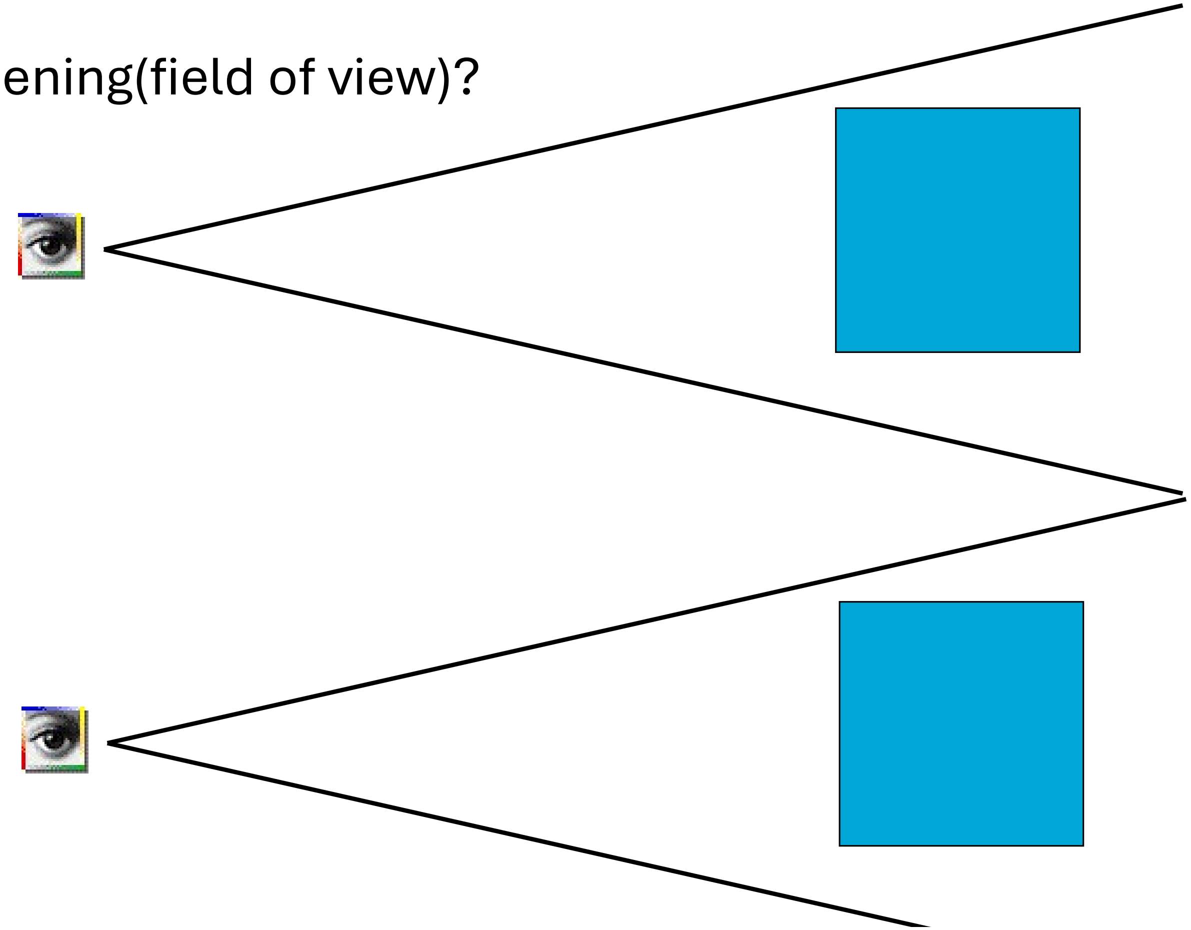
Virtual Camera Model

- Camera orientation/movement?



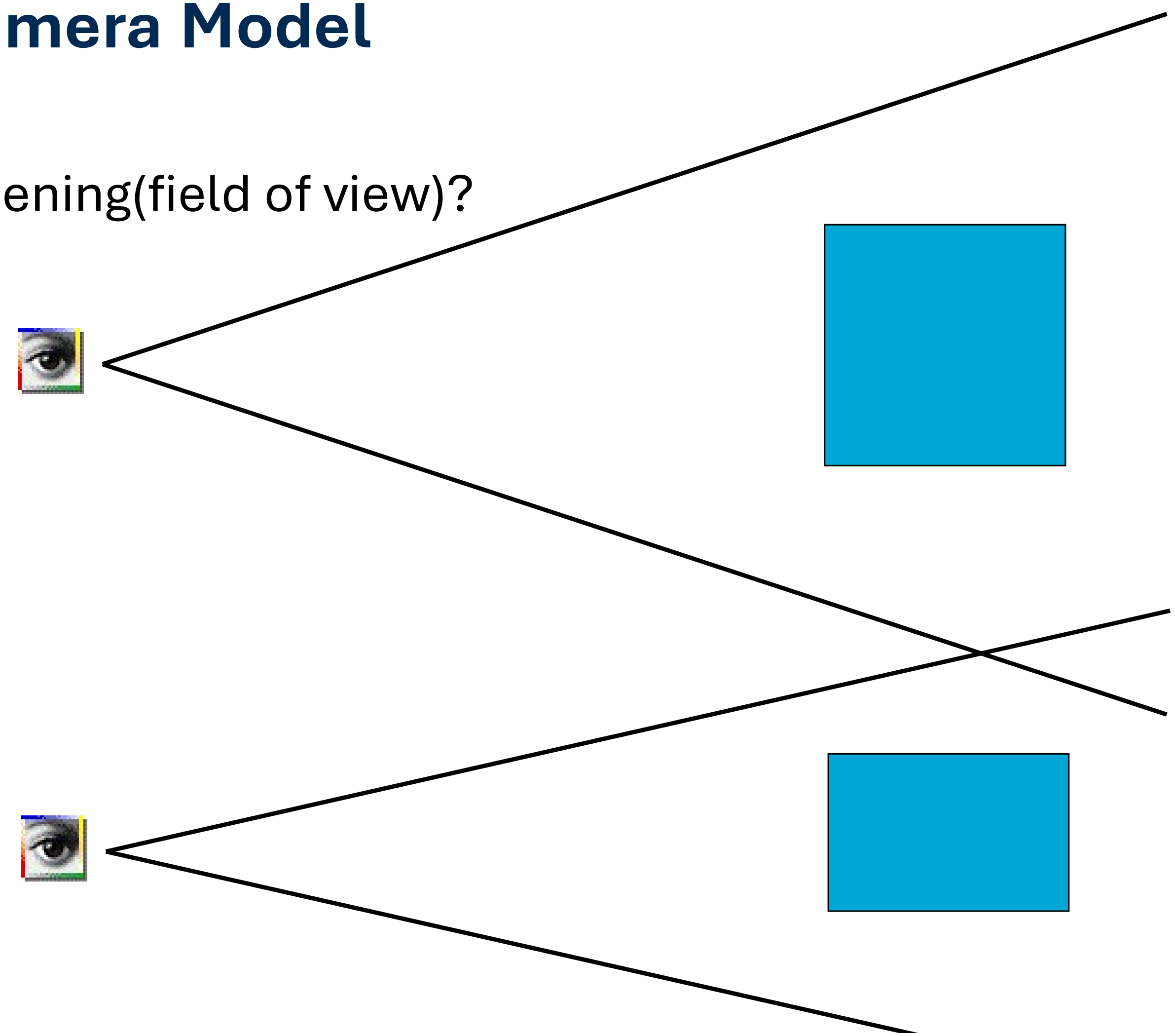
Virtual Camera Model

- Camera opening(field of view)?



Virtual Camera Model

- Camera opening(field of view)?



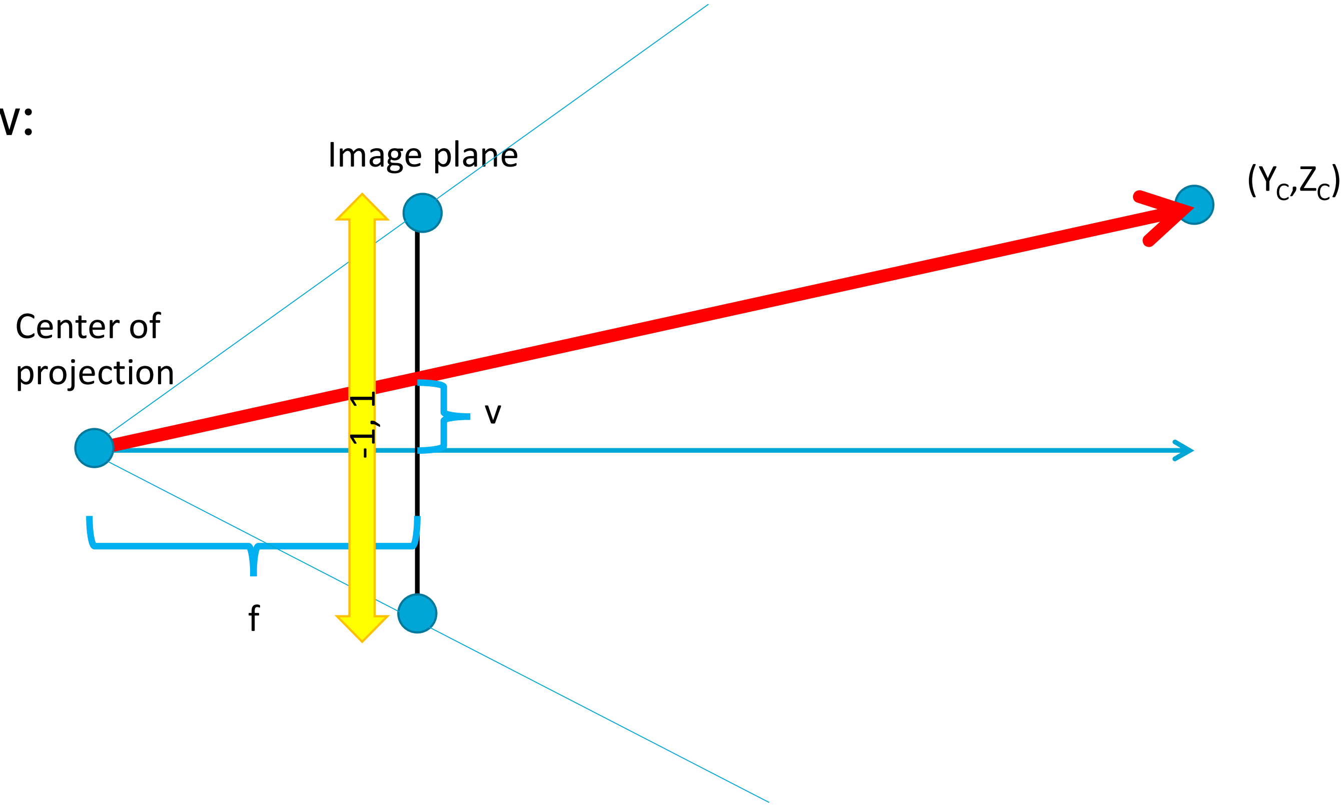
Virtual Camera Model

- The world literally revolves around the camera!

- Just deform the scene in the “right” way and we can always assume that
  - Camera centre (eye) is at the origin
  - Camera is oriented along the z-axis
  - Image aligned with x-y and has unit size $[-1,1]^2$
  - Only keep parameter f – focal length

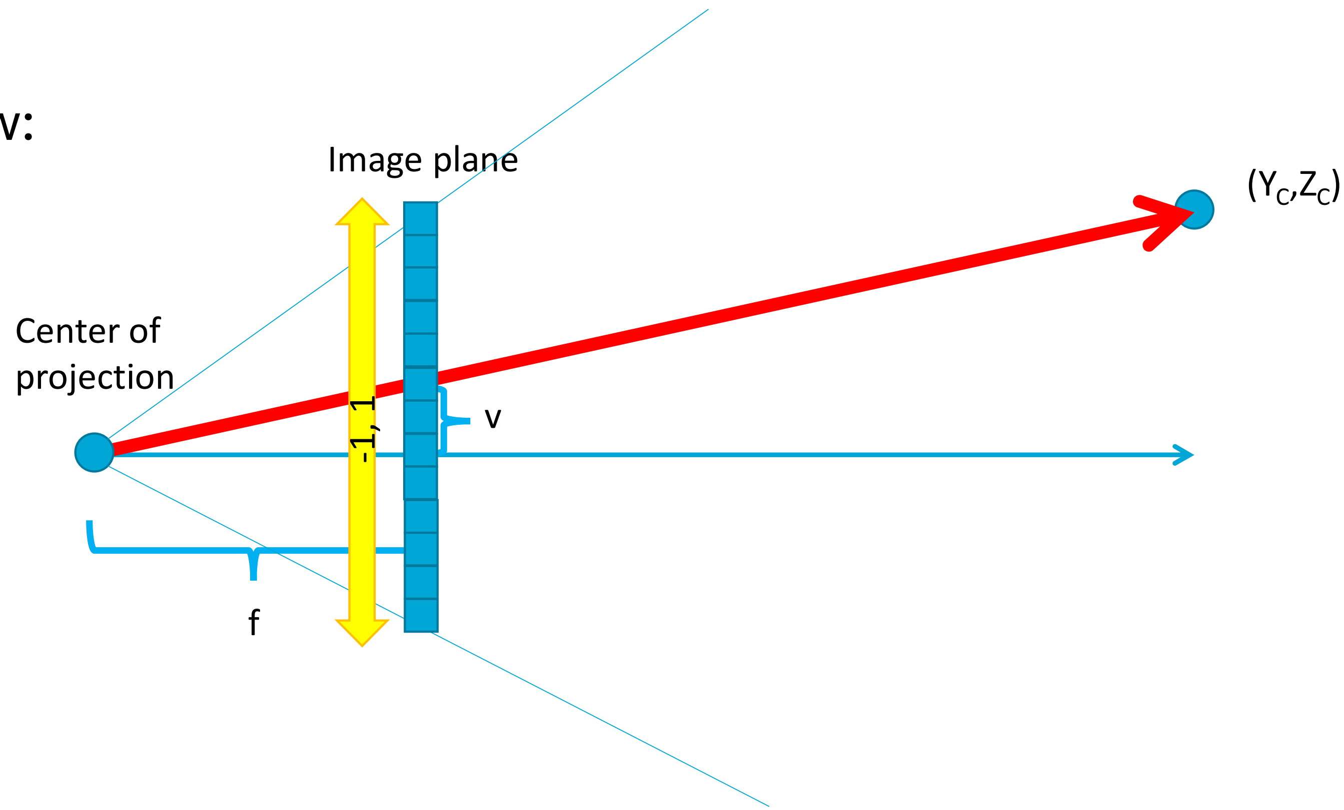
Virtual Camera Model

- sideview:



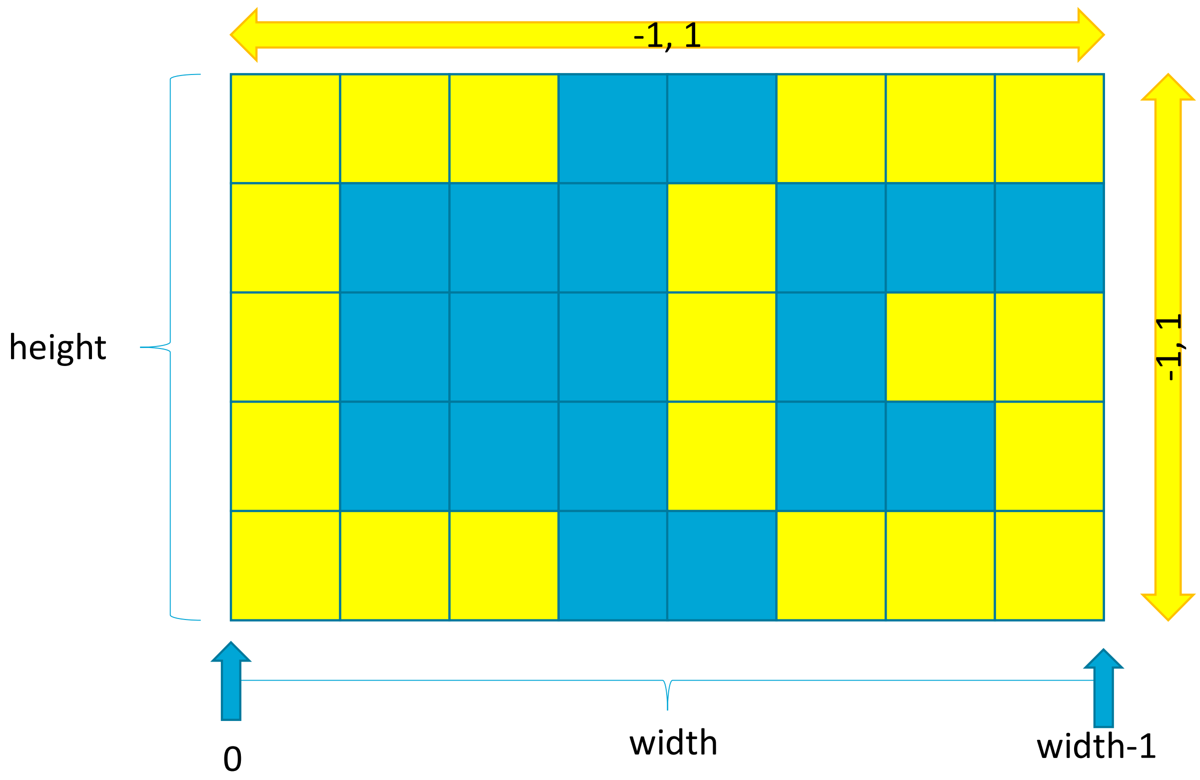
Virtual Camera Model

- sideview:



Viewport Transformation

- (-1,1) x (-1,1) ➡ [0, width-1]x[0,height-1]



Homework:
Find a matrix
to do this
mapping

Hint:
it has the form

$$\begin{pmatrix} k_x & 0 & x_0 \\ 0 & k_y & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

Complete Camera Model

- Finally:

$$\overbrace{\begin{pmatrix} k_x & 0 & x_0 \\ 0 & k_y & y_0 \\ 0 & 0 & 1 \end{pmatrix}}^{\text{Viewport}} \overbrace{\begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}^{\text{Projection}} \overbrace{\begin{pmatrix} r_{00} & r_{01} & r_{02} & t_0 \\ r_{10} & r_{11} & r_{12} & t_1 \\ r_{20} & r_{21} & r_{22} & t_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\text{Camera Matrix}}$$

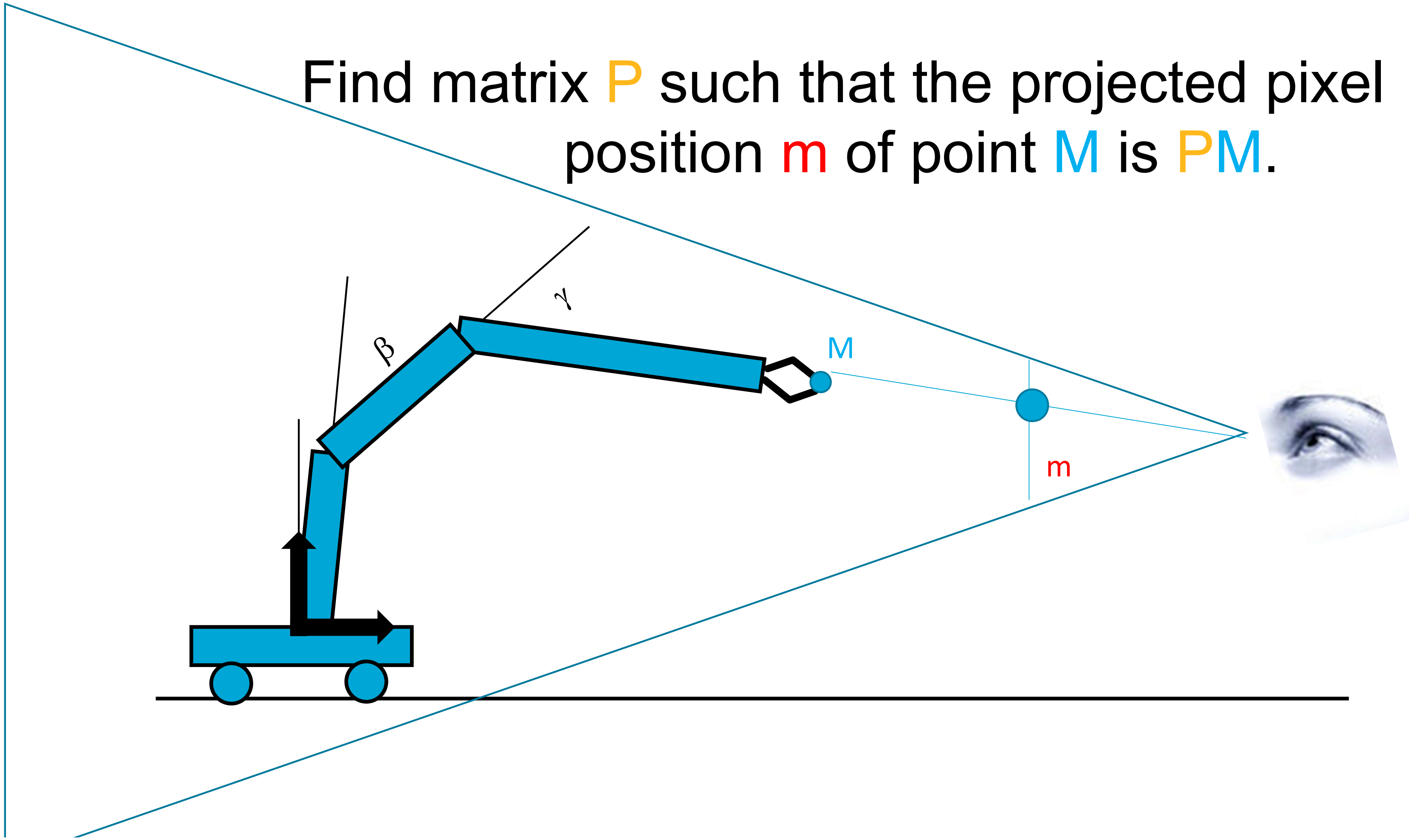
Pixel mapping

“standard camera” projection

Deforms scene so that a “standard camera” can be used

Today

Find matrix P such that the projected pixel position m of point M is PM.



Conclusion

- In this lecture, you have seen how to:
  - Transformations with homogeneous coordinates
  - Creation of complex objects via local frames
  - A virtual camera expressed as a matrix

...(almost) all ingredients for the geometric graphics pipeline

Reading

- Chapter 7 in the book
  - Derivation is in a slightly different order:
    - First scene deformation with respect to a camera frustum
    - Optional: As preparation for next time
      - viewport matrix
      - projection matrix

Exercises

- Define a 45 degree rotation around a point and verify the matrix is correct with examples
- Check the given rotation matrices on the slides

Thank you very much!

CSE2215 - Computer Graphics

Materials & Shading
All that glitters is not gold

Elmar Eisemann

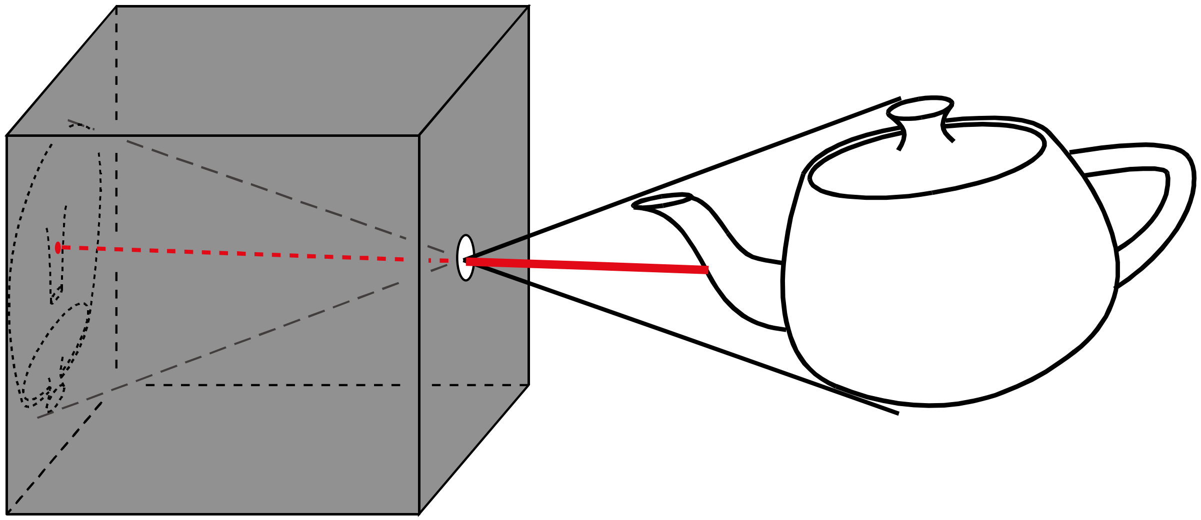
Delft University of Technology

The last week in a memory far,

far away....

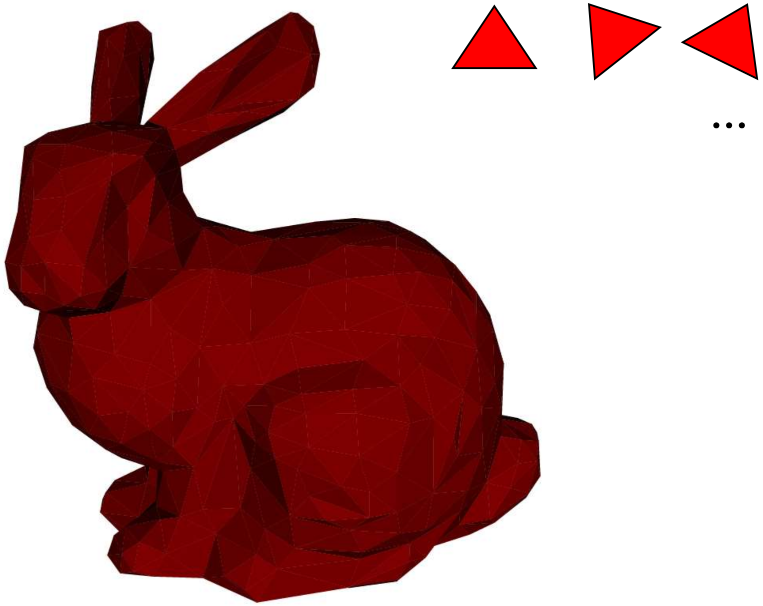
Pinhole camera

- Box with hole
- Perfect image for “point-sized” hole



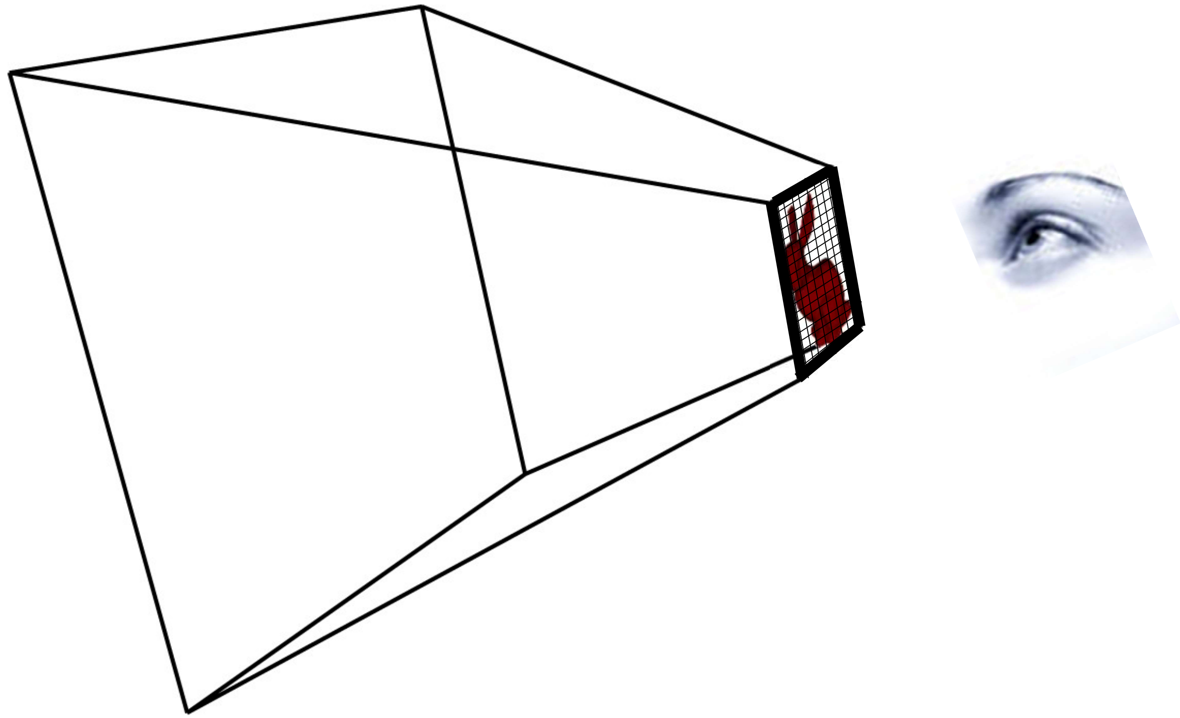
Triangle Mesh

- Models are typically lists of triangles



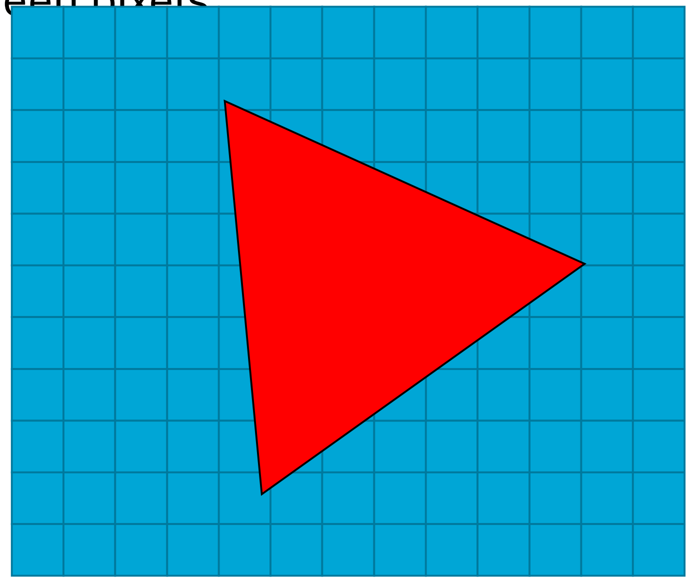
Simplified Graphics Pipeline

- Projection: Transform coordinates to screen



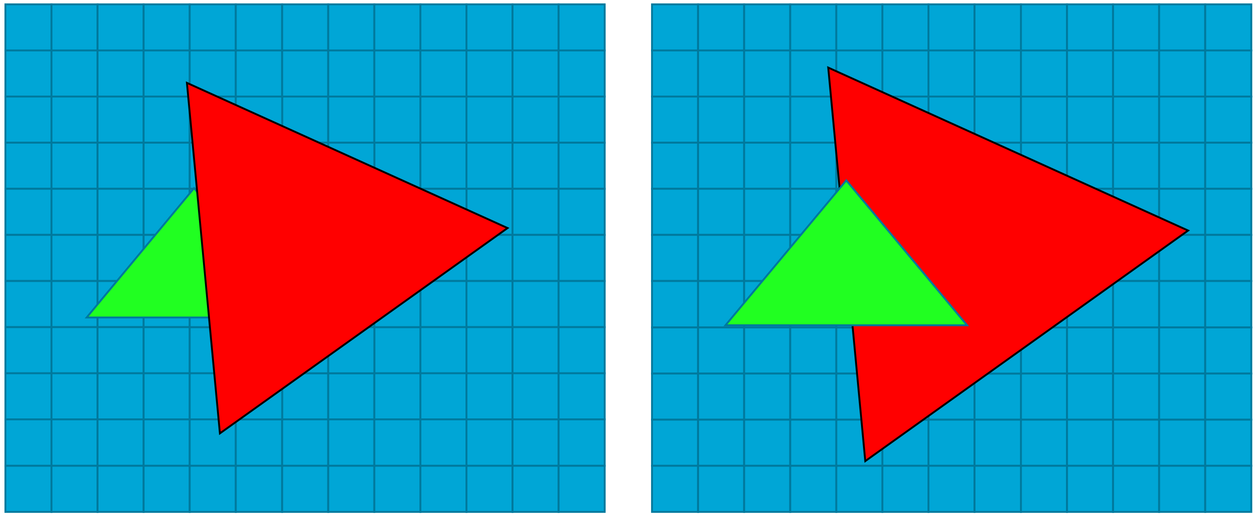
Simplified Graphics Pipeline

- Rasterization: Fill screen pixels



Simplified Graphics Pipeline

- Catch: Triangle order would change result

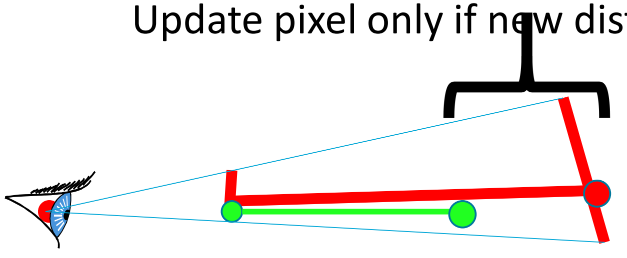


Need to keep the pixel of the closest triangle

Simplified Graphics Pipeline

- Solution to maintain drawing order:
  Compare Z values!

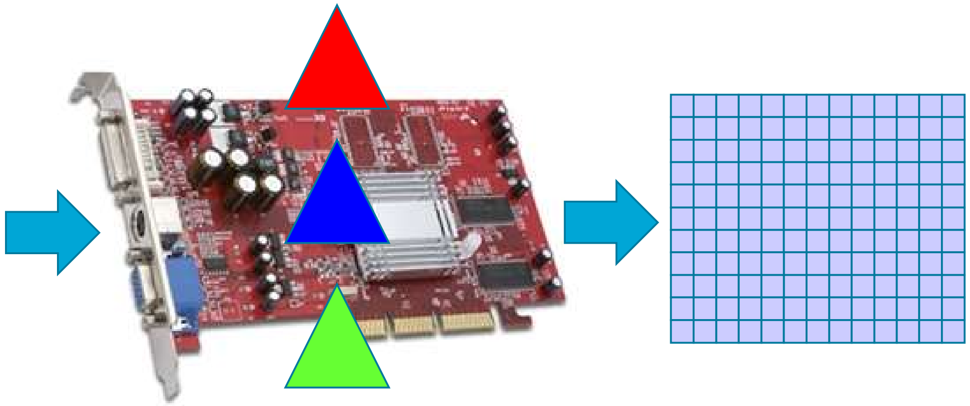
Compare new distance to stored distance
Update pixel only if new distance is nearer



- We need a “depth”!

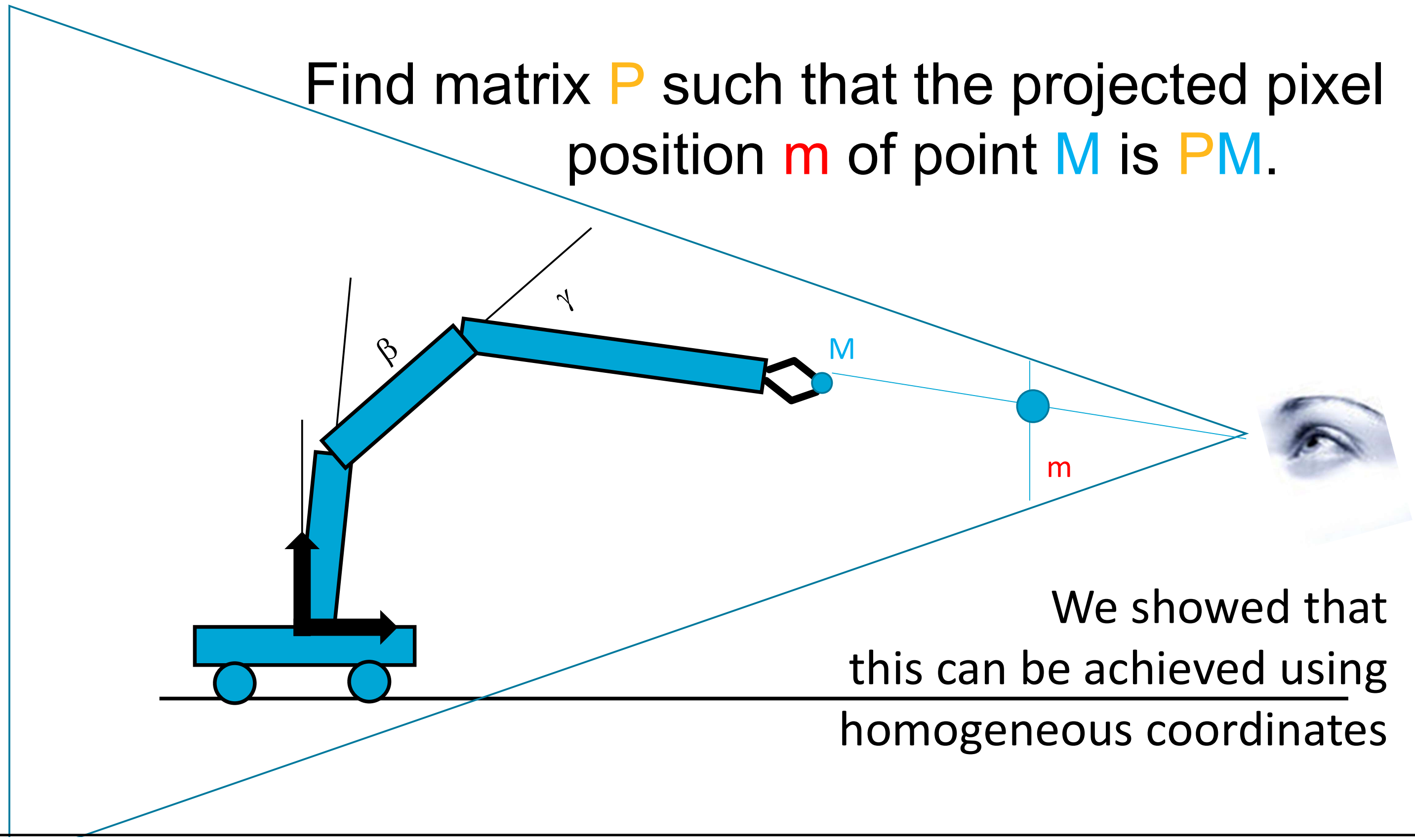
Graphics Pipeline

- Highly parallelizable GPUs



Last time:

Find matrix P such that the projected pixel position m of point M is PM.



We showed that
this can be achieved using
homogeneous coordinates

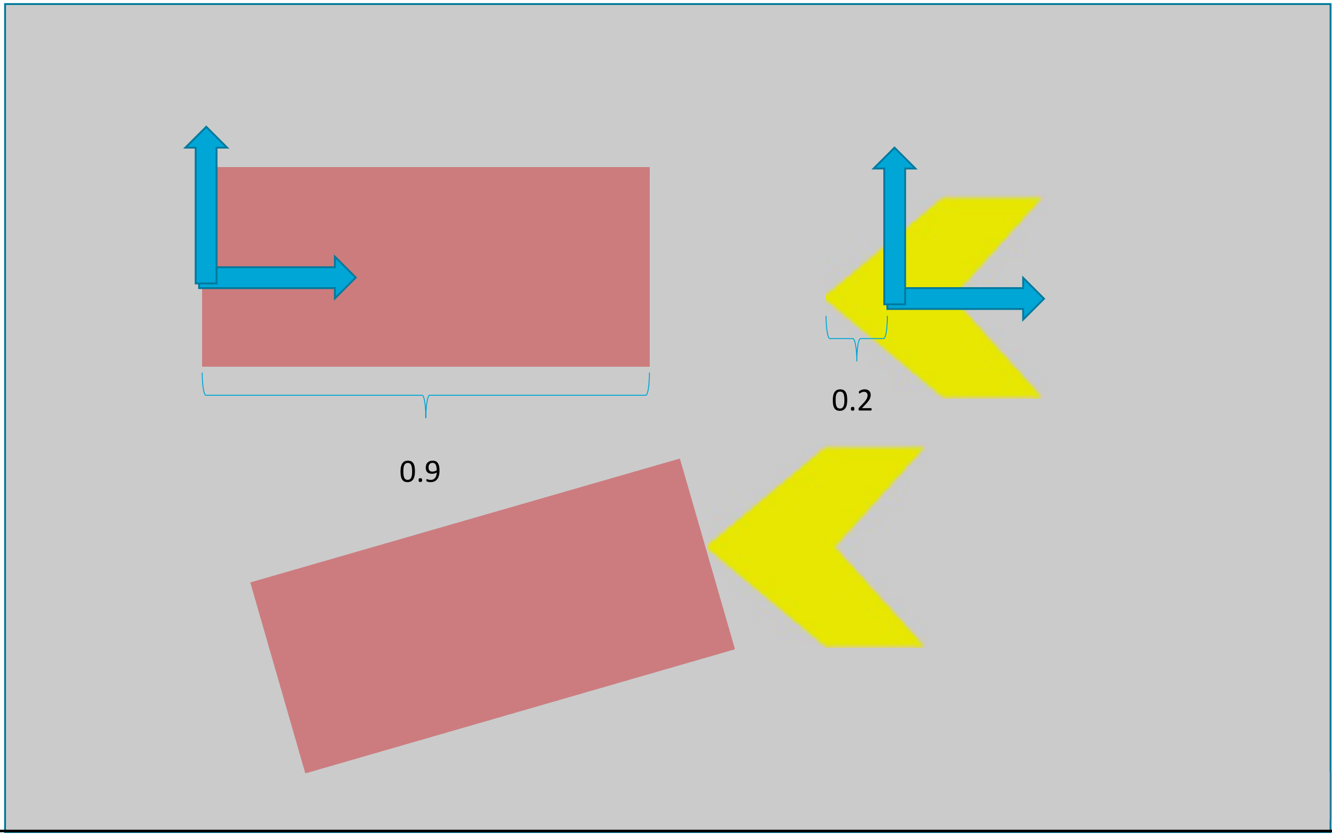
Last time:

- This matrix M is usually constructed as the multiplication of two matrices:
- A matrix to place the vertices of an object in the right location in 3D space
- A matrix that performs the projection from 3D via the virtual camera

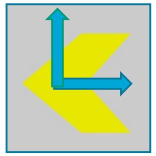
Last time:

- This matrix M is usually constructed as the multiplication of two matrices:
- **A matrix to place the vertices of an object in the right location in 3D space**
- A matrix that performs the projection from 3D via the virtual camera

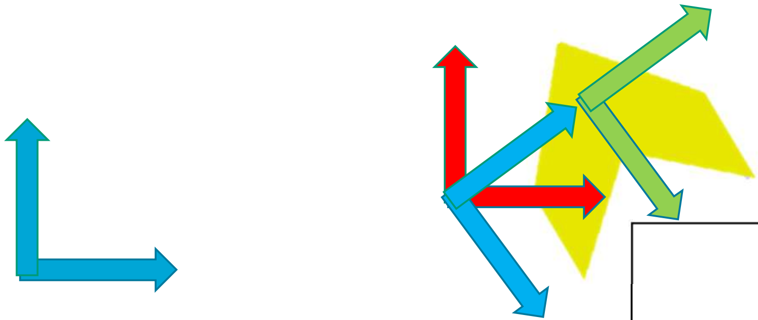
Object Placement

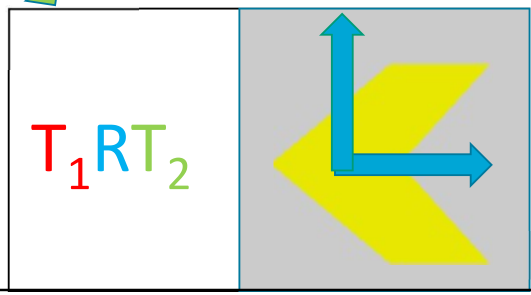


Object placement



- The matrix transformation to place the object can be derived by imagining a local frame that is moved by the matrices
- Imagine the following instructions:
- T_1
- R
- T_2





TU Delft

Last time:

- This matrix M is usually constructed as the multiplication of two matrices:
- A matrix to place the vertices of an object in the right location in 3D space
- **A matrix that performs the projection from 3D via the virtual camera**

Camera Model

$$\overbrace{\begin{pmatrix} k_x & 0 & x_0 \\ 0 & k_y & y_0 \\ 0 & 0 & 1 \end{pmatrix}}^{\text{Viewport}} \overbrace{\begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}^{\text{Projection}} \overbrace{\begin{pmatrix} r_{00} & r_{01} & r_{02} & t_0 \\ r_{10} & r_{11} & r_{12} & t_1 \\ r_{20} & r_{21} & r_{22} & t_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\text{Camera Matrix}}$$

Pixel mapping

“standard camera” projection

Deforms scene so that a “standard camera” can be used

Last time:

- Use of homogeneous coordinates

$$P = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \Rightarrow \widetilde{P} = \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

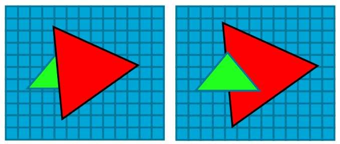
- Complex Object transformations
- Camera movement/orientation
- Projection
- Mapping to pixels

Single Matrix

Something is missing...

$$P = \overbrace{\begin{pmatrix} k_x & 0 & x_0 \\ 0 & k_y & y_0 \\ 0 & 0 & 1 \end{pmatrix}}^{\text{image}} \overbrace{\begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}^{\text{projection}} \overbrace{\begin{pmatrix} r_{00} & r_{01} & r_{02} & t_0 \\ r_{10} & r_{11} & r_{12} & t_1 \\ r_{20} & r_{21} & r_{22} & t_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\text{orientation/location}}$$

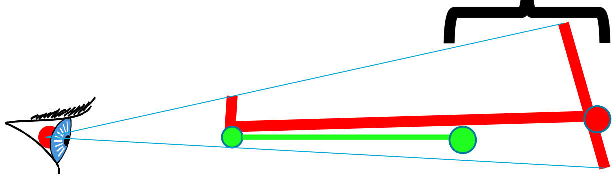
- What is the problem of this matrix for the Graphics Pipeline?



The Depth Test misses the depth...

- We need to keep a Z coordinate!

Compare new distance to stored distance
Update pixel only if new distance is nearer



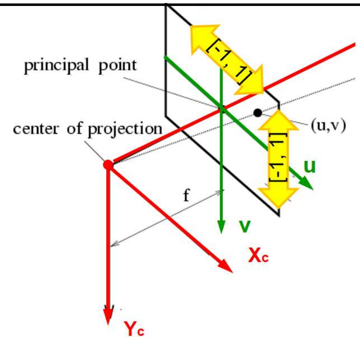
Something is missing... the depth

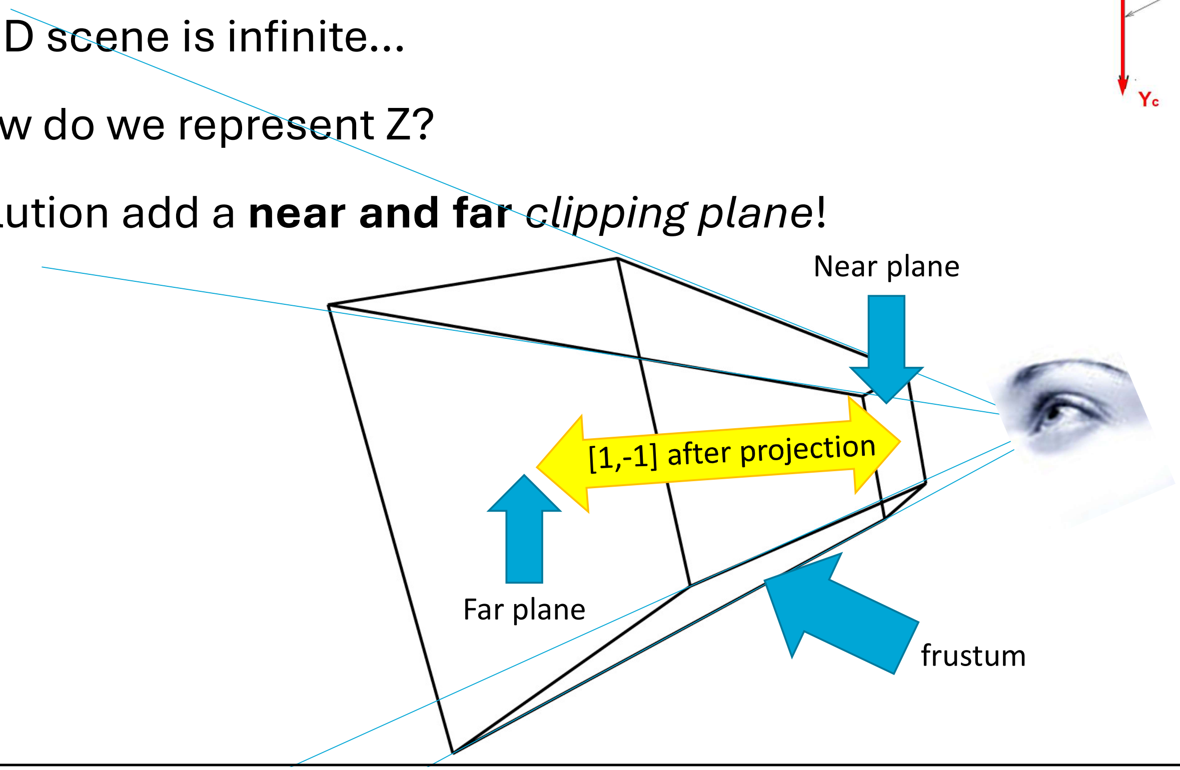
$$P = \overbrace{\begin{pmatrix} k_x & 0 & x_0 \\ 0 & k_y & y_0 \\ 0 & 0 & 1 \end{pmatrix}}^{\text{image}} \boxed{\overbrace{\begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}^{\text{projection}}} \overbrace{\begin{pmatrix} r_{00} & r_{01} & r_{02} & t_0 \\ r_{10} & r_{11} & r_{12} & t_1 \\ r_{20} & r_{21} & r_{22} & t_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\text{orientation/location}}$$

- It was eliminated in this projection matrix...
- To get it back, we need to extend this matrix

Defining a Depth

- A 3D scene is infinite...
- How do we represent Z?
- Solution add a **near and far** *clipping plane*!





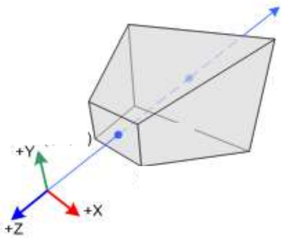
The OpenGL Projection Matrix

- Our projection matrix:

$$
\begin{array}{r}
\text{screen mapping} \\
\text{screen mapping} \\
\\
\\
\end{array}
\begin{bmatrix}
f & 0 & 0 & 0 \\
0 & f & 0 & 0 \\
\multicolumn{4}{c}{\text{New line – mapping the z value}} \\
0 & 0 & 1 & 0
\end{bmatrix}
$$

The OpenGL Projection Matrix

$$\begin{array}{l} \text{screen mapping} \\ \text{screen mapping} \\ \\ \\ \end{array} \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ \multicolumn{4}{c}{\text{New line – mapping the z value}} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

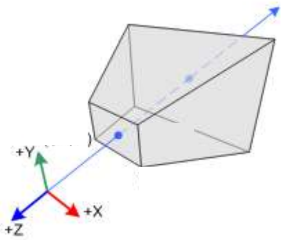


By OpenGL convention, camera looks along the negative z-Axis

…don't bother too much about it…

The OpenGL Projection Matrix

$$\begin{array}{r} \text{screen mapping} \\ \text{screen mapping} \\ \\ \\ \end{array} \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ \multicolumn{4}{c}{\text{New line – mapping the z value}} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$



The OpenGL Projection Matrix

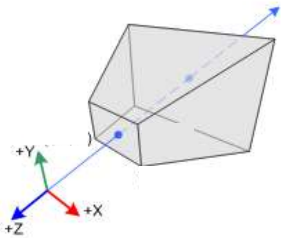
- Aspect ratio for non-square displays:

screen mapping

screen mapping

$$\begin{bmatrix} \boxed{\dfrac{f}{aspect}} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ \multicolumn{4}{c}{\text{New line – mapping the z value}} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Make a drawing:
How does a projection to $[-1,1]^2$ for a 16:9 image look like

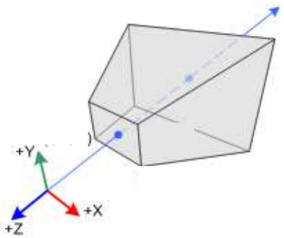


In other words:
The scene is scaled such that the square image appears correctly stretched

The OpenGL Projection Matrix

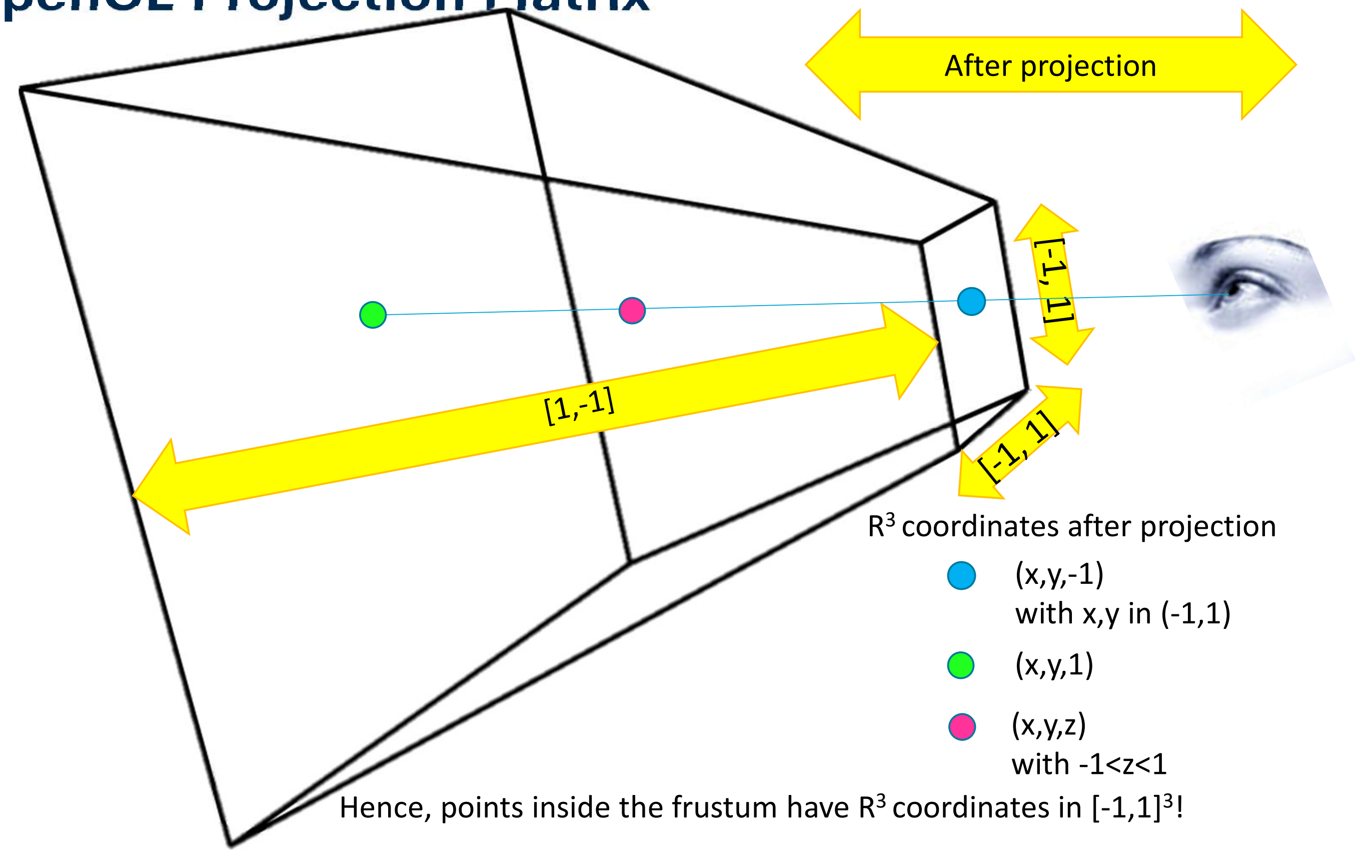
- Definition maps near and far to [-1,1]:

$$\begin{array}{r} \text{screen mapping} \\ \text{screen mapping} \\ \\ \\ \end{array} \begin{bmatrix} \frac{f}{aspect} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{near+far}{near-far} & \frac{2near\,far}{near-far} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \text{Z mapping} \\ \\ \end{array}$$



, where near and far are the distances of the planes to the origin.

The OpenGL Projection Matrix



R³ coordinates after projection

- (x,y,-1)
  with x,y in (-1,1)
- (x,y,1)
- (x,y,z)
  with -1<z<1

Hence, points inside the frustum have R^3 coordinates in $[-1,1]^3$!

Camera Space

- Content of Frustum is mapped inside a cube

$$\begin{bmatrix} \frac{f}{aspect} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{near+far}{near-far} & \frac{2near\,far}{near-far} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

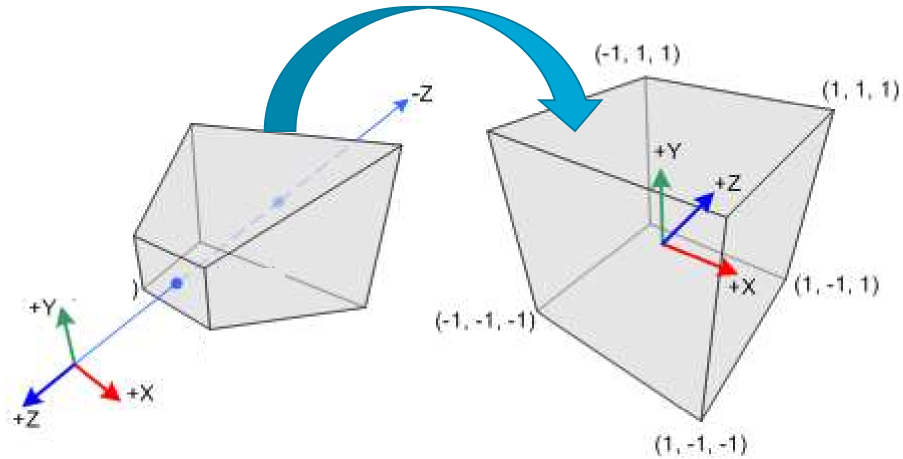


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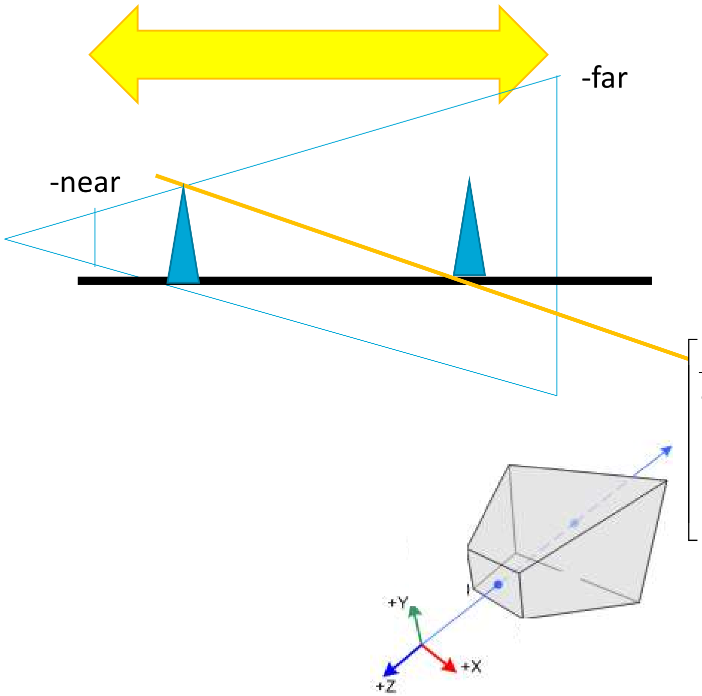
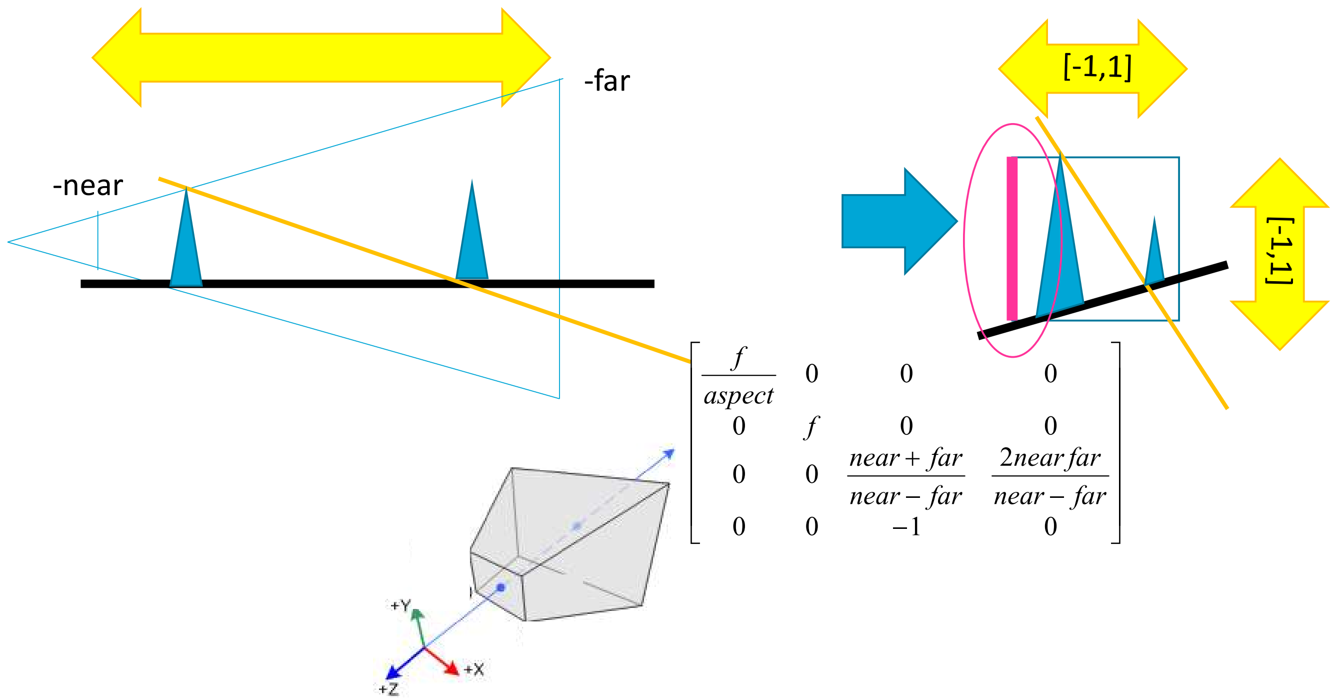
Illustration of the Camera Mapping



$$\begin{bmatrix} \frac{f}{aspect} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{near+far}{near-far} & \frac{2near\,far}{near-far} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

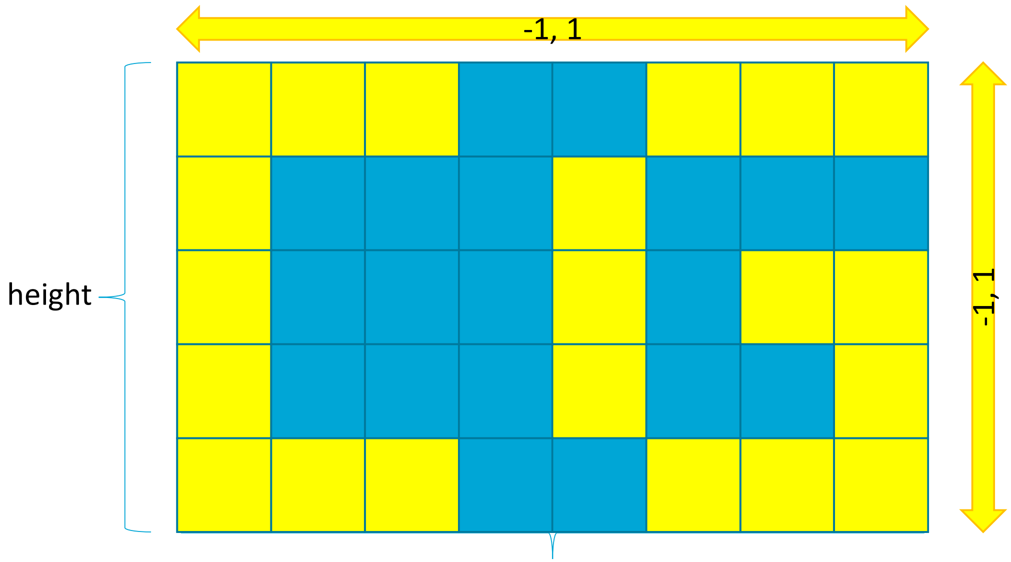
Illustration of the Camera Mapping



$$\begin{bmatrix} \frac{f}{aspect} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{near+far}{near-far} & \frac{2near\,far}{near-far} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Simplified Graphics Pipeline

- Map the projection onto pixels
- (-1,1) x (-1,1) ➡ [0, width-1]x[0,height-1]



$$\begin{pmatrix} k_x & 0 & x_0 \\ 0 & k_y & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

Viewport

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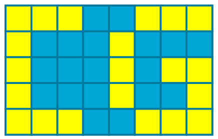
Simplified Graphics Pipeline

Traditional OpenGL Camera can be described as a single 4x4 camera matrix!

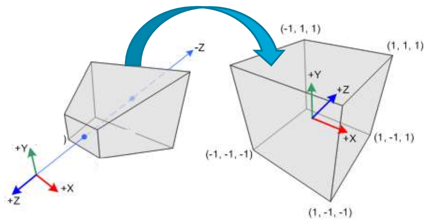
$$\begin{bmatrix} k_x & 0 & 0 & x_0 \\ 0 & k_y & 0 & y_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{f}{aspect} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{near+far}{near-far} & \frac{2near\,far}{near-far} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

4x4
ModelView Matrix

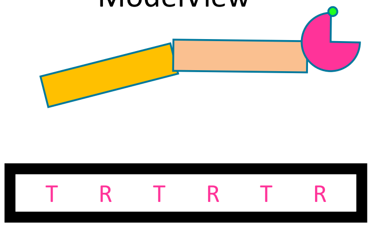
Viewport
(including depth)



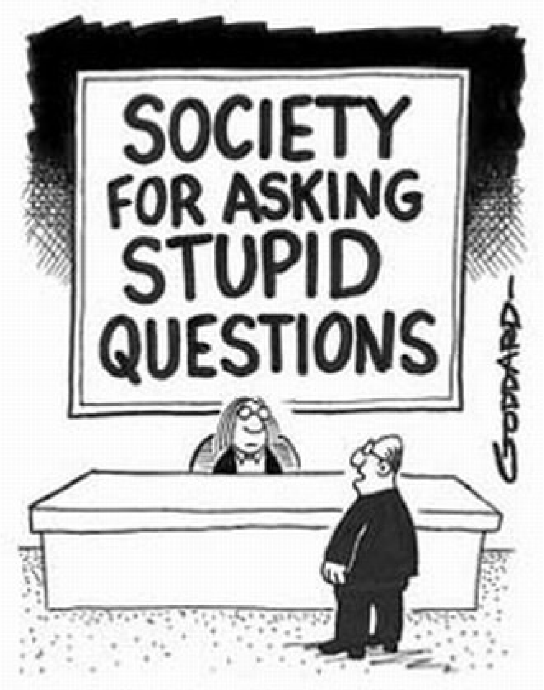
Projection



ModelView



Questions?



"Excuse me, is this the Society for Asking Stupid Questions?"

Episode IV

MATERIALS AND SHADING

Transformations have taken over. Homogeneous coordinates rule the 3D space. Only a few more questions remain...

I promise, this is the last point...

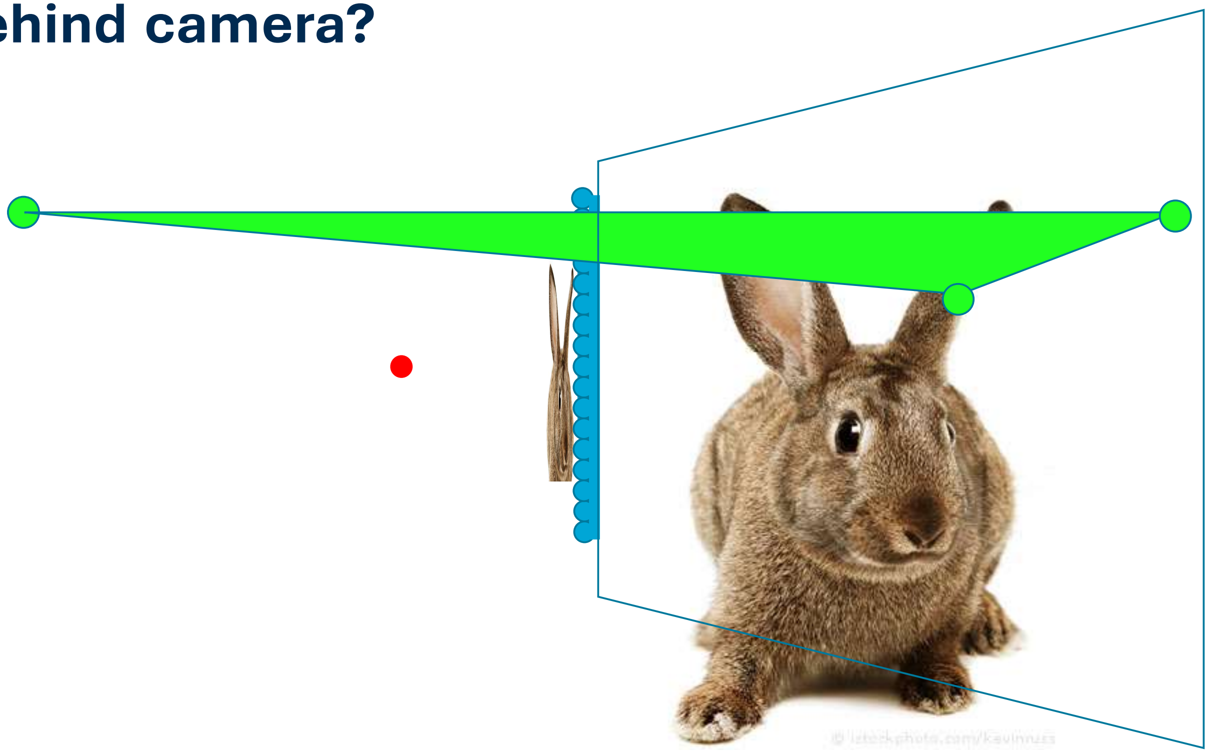
I promise, this is the last point…

- While all the math works out for individual points, it fails for triangles…

- Unfortunately, we need an additional step:

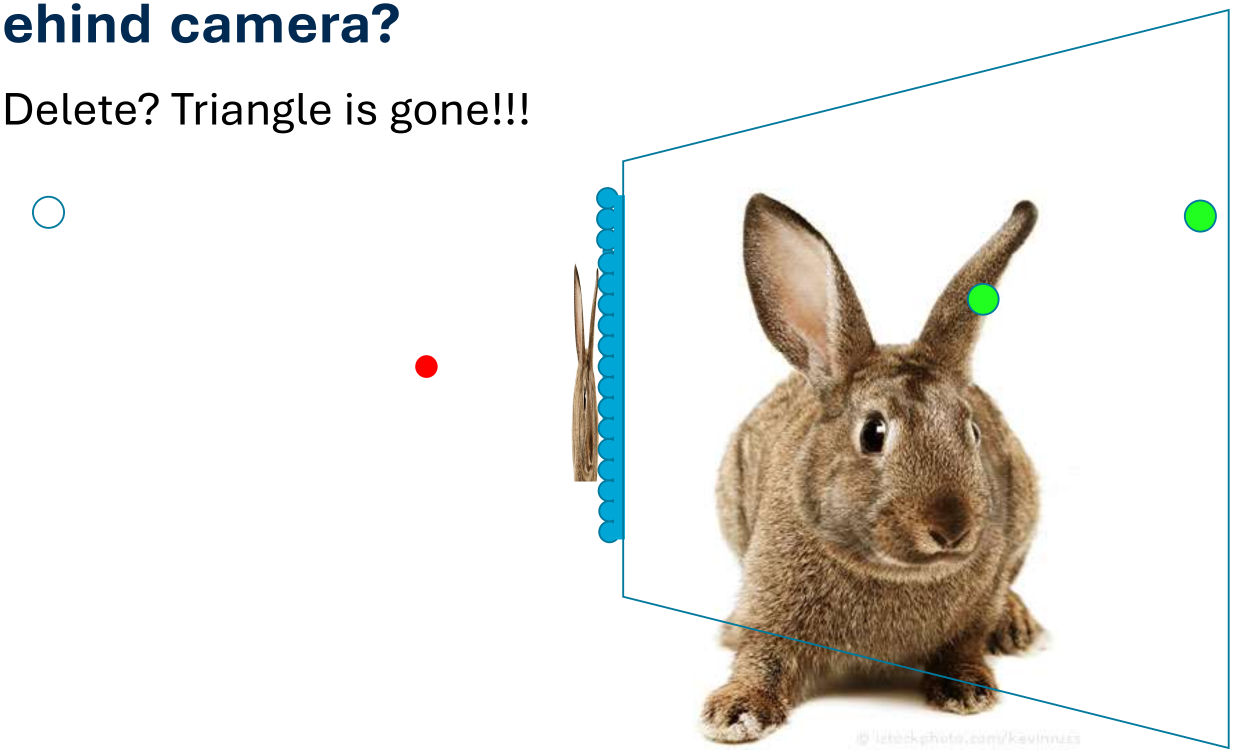
Clipping

Points behind camera?



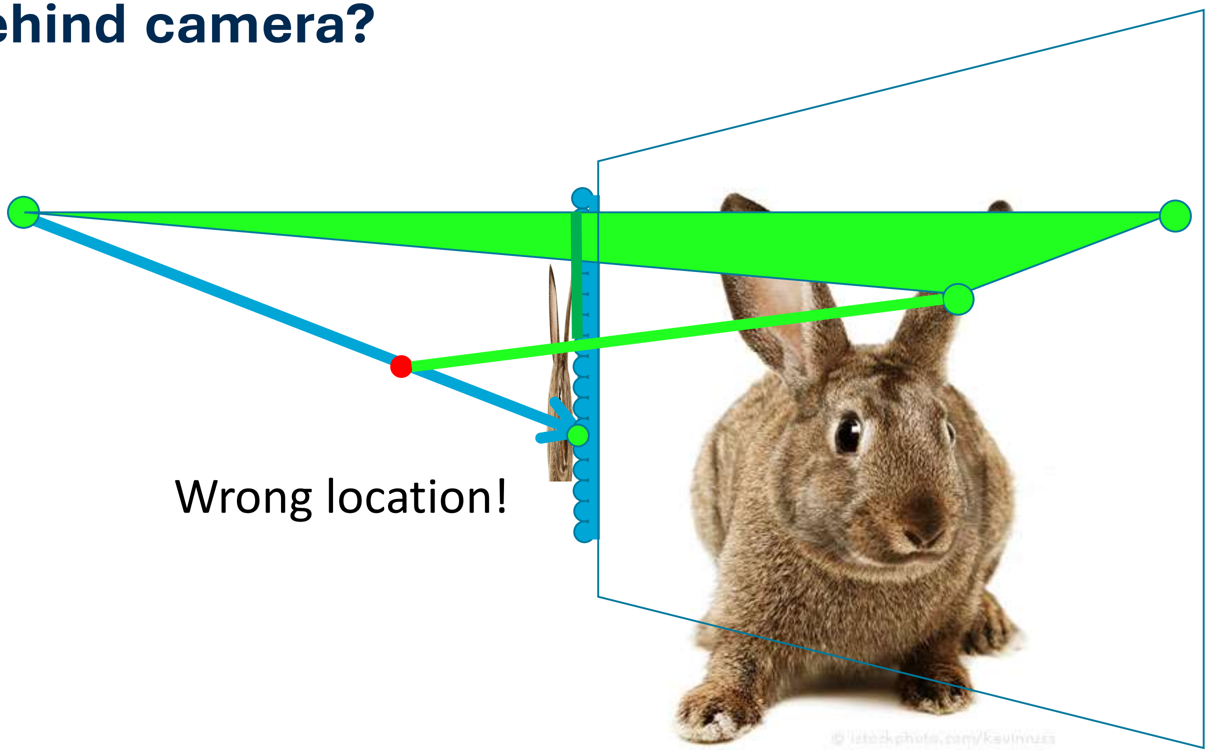
Points behind camera?

- Delete? Triangle is gone!!!



Points behind camera?

- Project?



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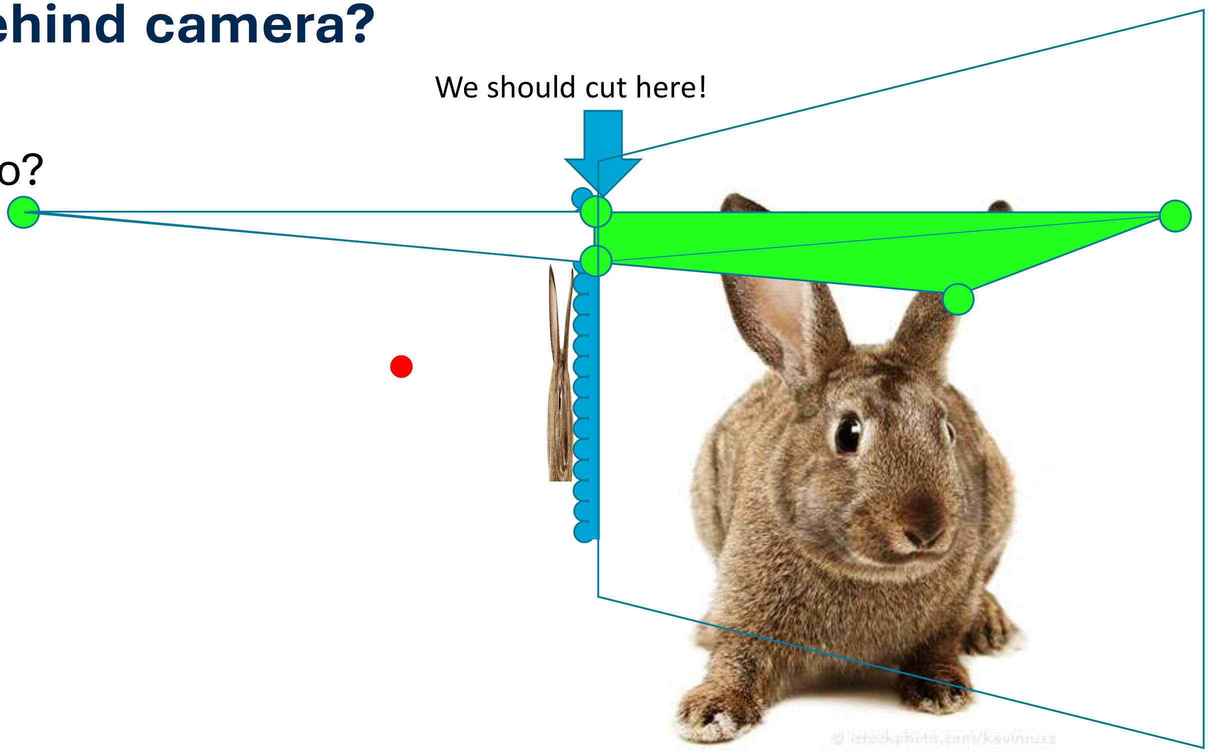
Points behind camera?

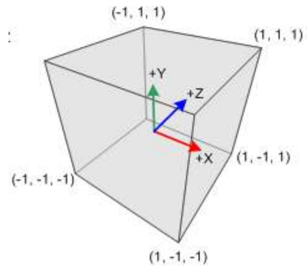
- What to do?



- Test triangle and **clip** if it crosses frustum!

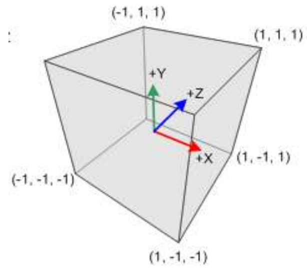
Efficient Clipping

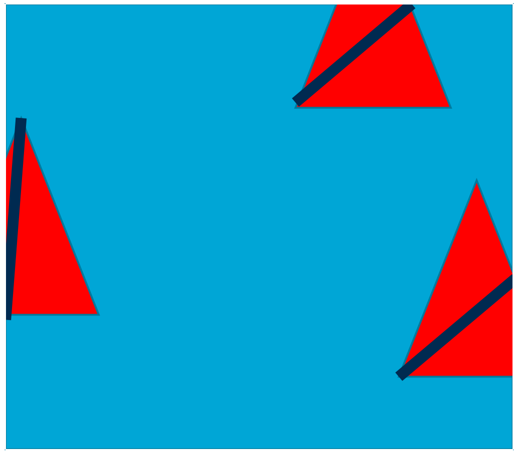
- Clip triangles against cube
  after applying projection matrix



Efficient Clipping

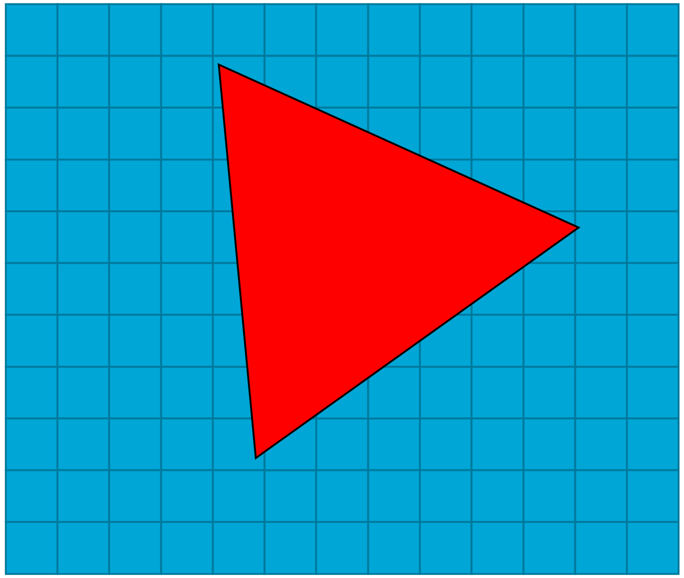
- Clip triangles against cube after applying projection matrix





Simplified Graphics Pipeline

- Rasterization: Fill screen pixels



+ Depth Test

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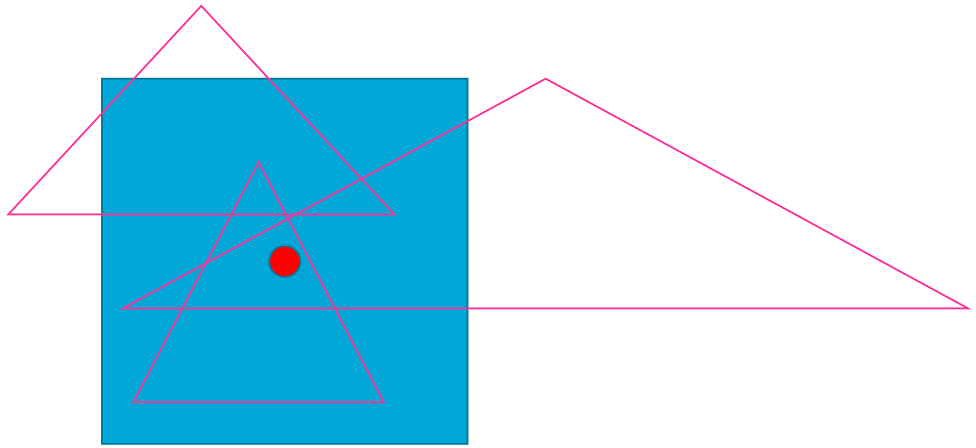
Simplified Graphics Pipeline

- Rasterization: Fill screen pixels
- Pixels are filled if the center is covered



+ Depth Test

Simplified Graphics Pipeline

Geometric Transformations: Completed

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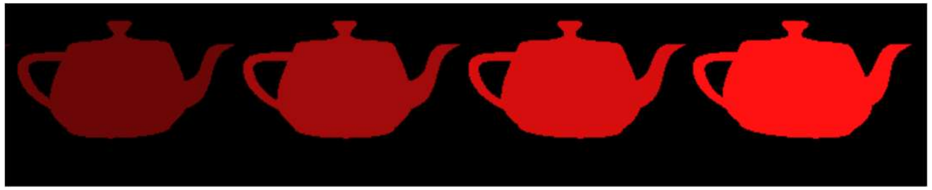
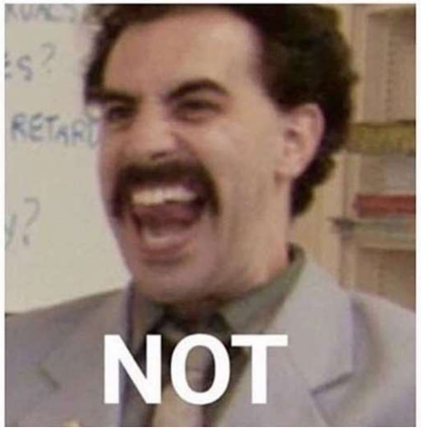
We can make beautiful pictures!





A first step towards beauty...

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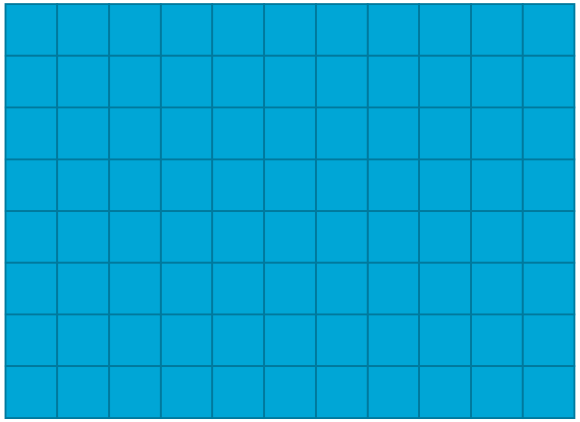
Rasterization

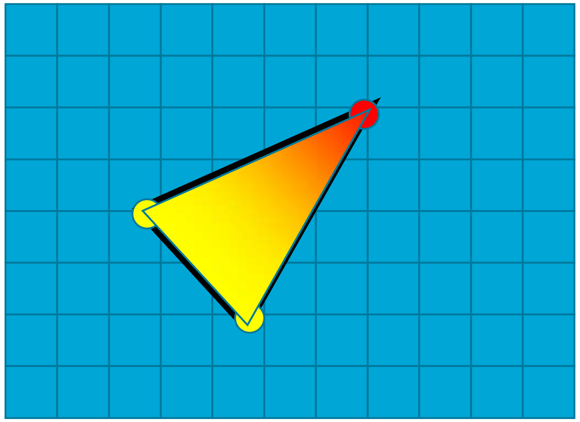
- Triangles can have different colors on vertices



- Values are extracted at pixel centers

Rasterization

- Triangles can have different colors on vertices



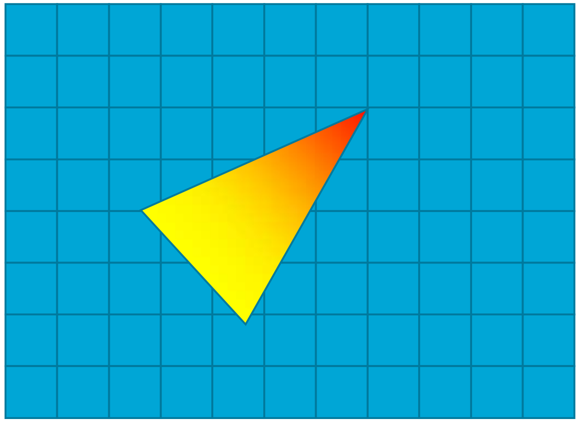
Two yellow,
one red vertex

Colors are
interpolated over
triangle

- Values are extracted at pixel centers

Rasterization

- Triangles can have different colors on vertices



Two yellow,
one red vertex

Colors are interpolated over triangle

- Values are extracted at pixel centers

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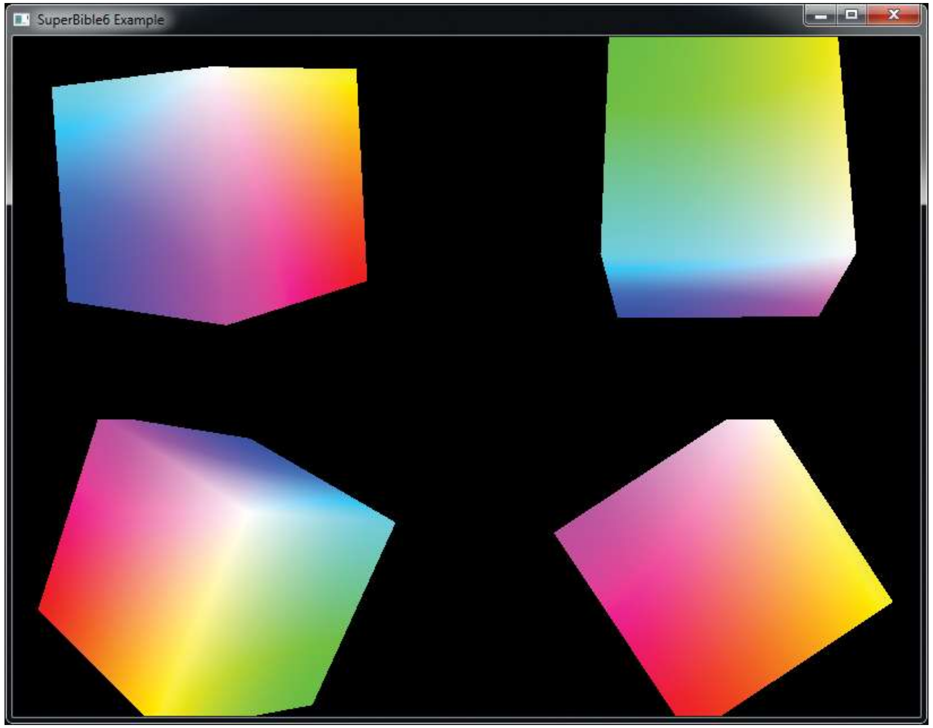
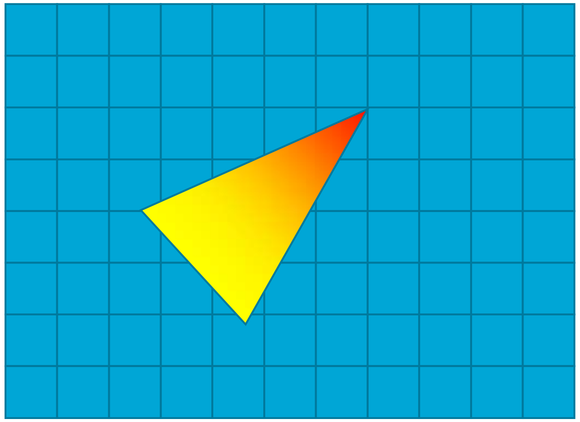
A first step towards beauty...



OpenGL SuperBible: Comprehensive Tutorial and Reference, 6th Edition

Rasterization

- Triangles can have different colors on vertices



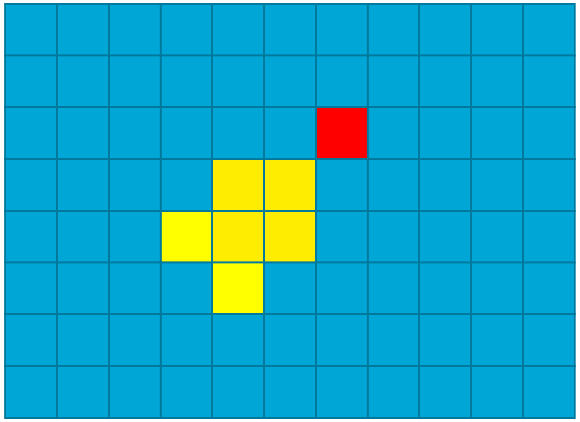
Two yellow,
one red vertex

Colors are
interpolated over
triangle

- Values are extracted at pixel centers

Rasterization

- Triangles can have different colors on vertices

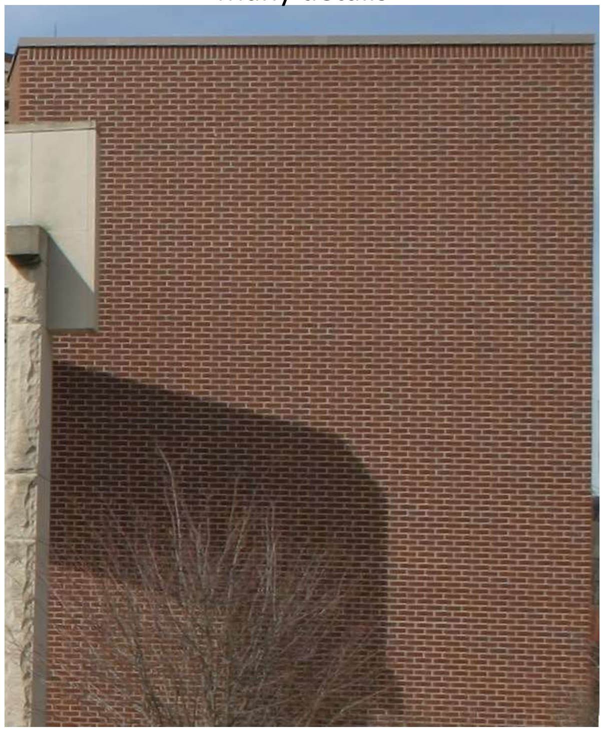


Colors sampled at pixel centers

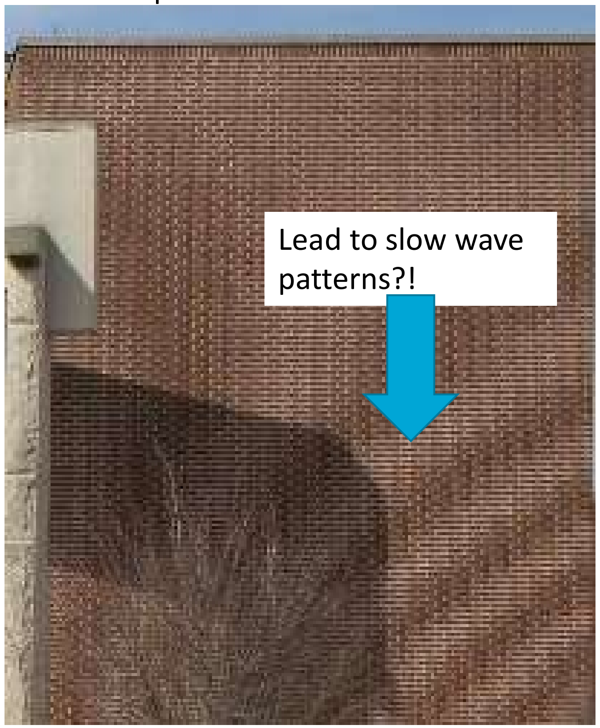
- Values are extracted at pixel centers
- Blocky appearance is referred to as “aliasing”

Excursion: Aliasing

Many details

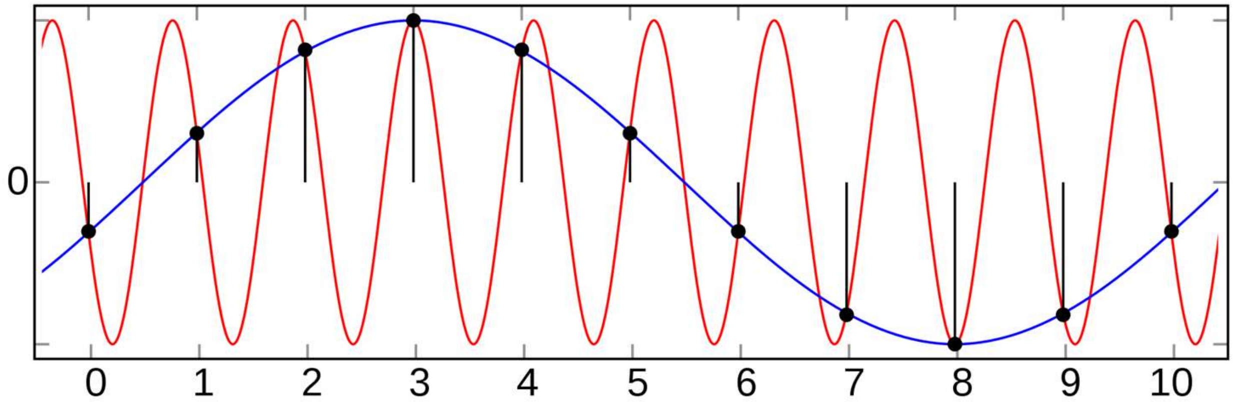


Captured at low resolution



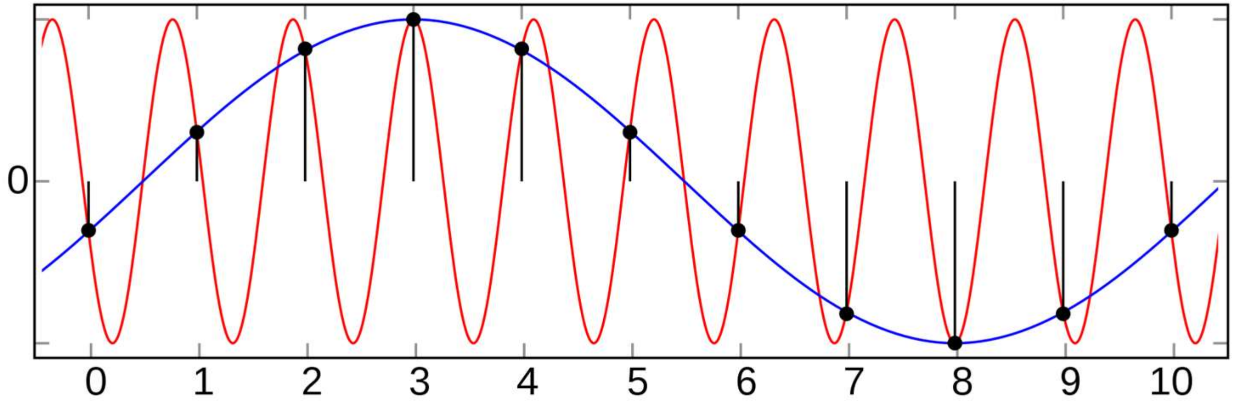
Excursion: Aliasing

- How would something with lot of details look like a low-frequency variation?



Excursion: Aliasing

- Two functions can have the same result after sampling



- Too high frequency creates artifacts...

Excursion: Aliasing

- If pixels are partially covered "aliasing" can occur.

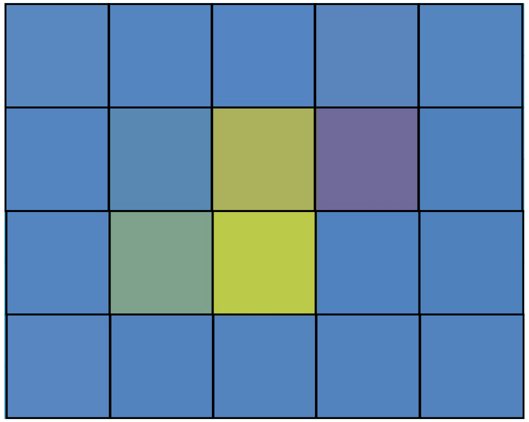
Broken wires because their triangles are too small

Dirty fix: render at higher resolution then reduce resolution by averaging pixels
It is called super sampling.

Images: http://www.humus.name/

Excursion: Aliasing

- Antialiasing via supersampling



- Reduce effective resolution by averaging neighboring pixels

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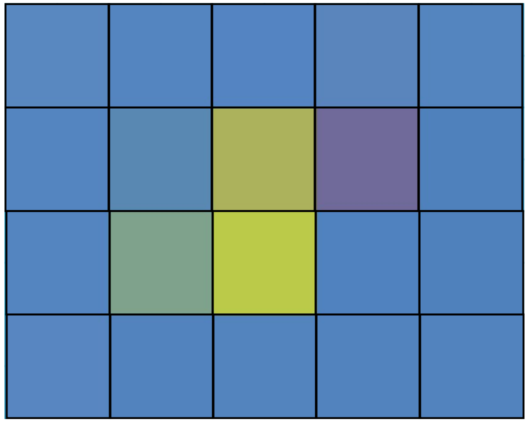
Excursion: Aliasing

- Antialiasing via supersampling

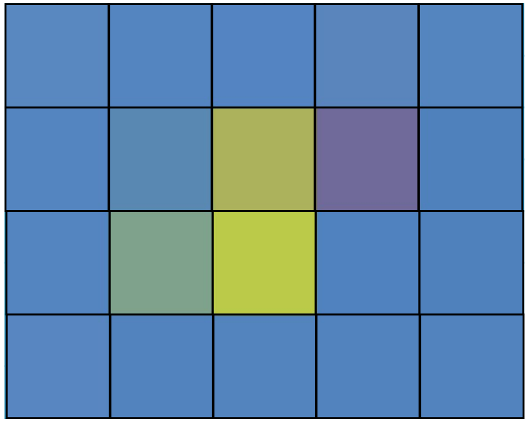


For this we **can** use:
a) Kung-fu Filter
b) Karate Filter
c) Box Filter

- Reduce effective resolution by averaging neighboring pixels

Excursion: Aliasing

- Antialiasing via supersampling

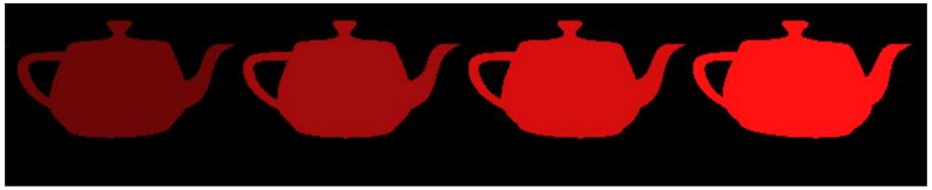


- Reduce effective resolution by averaging neighboring pixels

For this we **can** use:
a) Kung-fu Filter
b) Karate Filter
c) Box Filter

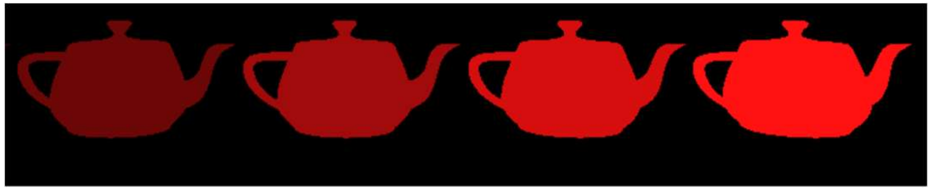
Although later in your studies, you will see that better filters exist
…

Images are not very exciting yet...

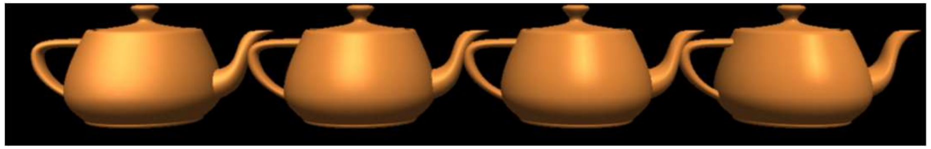


Shading

How to transform



in



?

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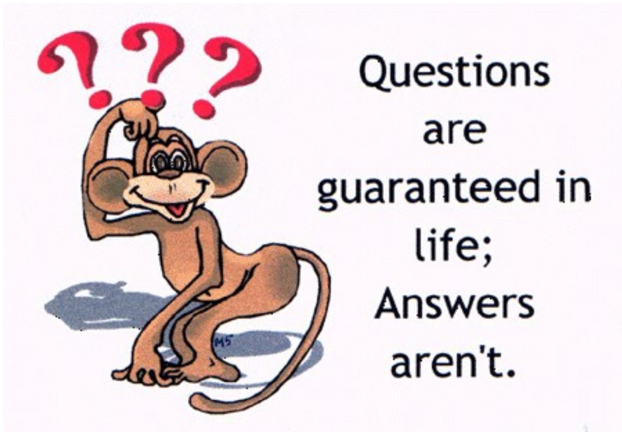
Questions?



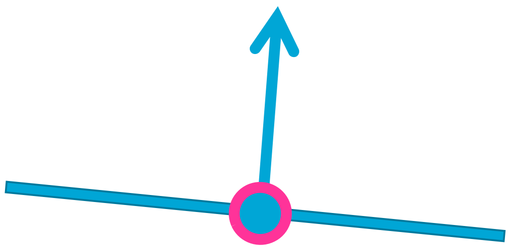
Today's Learning Goals

- S4- Apply mathematical modeling and theory of geometric computations and transformations, object representations, simulation, and encoding.
  - We have seen a transformation recap
  - You will learn how to derive and calculate a simple physically-oriented material model.
- S3- Use mathematical methods to analyze, create, apply algorithms and data structures, as well as understanding time and space complexity of image-generation algorithms
  - You will see various ways of calculating the effect of light on a mesh

Making beautiful pictures…

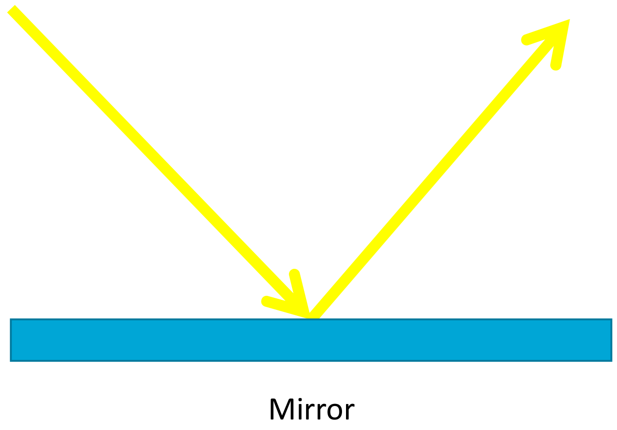
TODAY'S QUESTION:

- Given a surface point
  (position, its normal, and potential attributes)
  and a point emitting light…
- How to compute a realistic color???



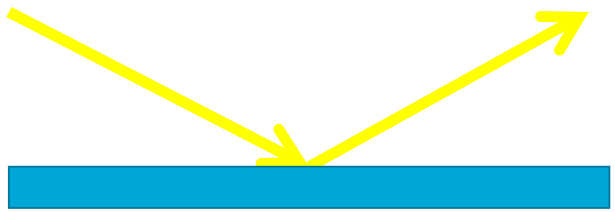
Reflection

- What happens when the light hits a surface?

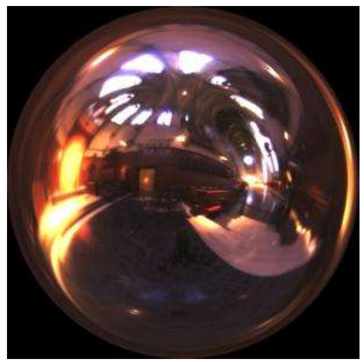


Reflection

- What happens when the light hits a surface?

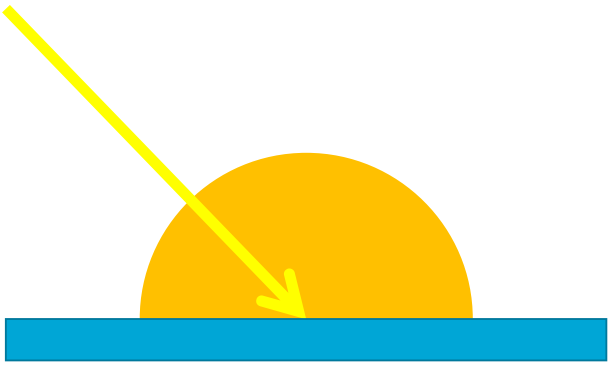


Mirror



Reflection

- What happens when the light hits a surface?



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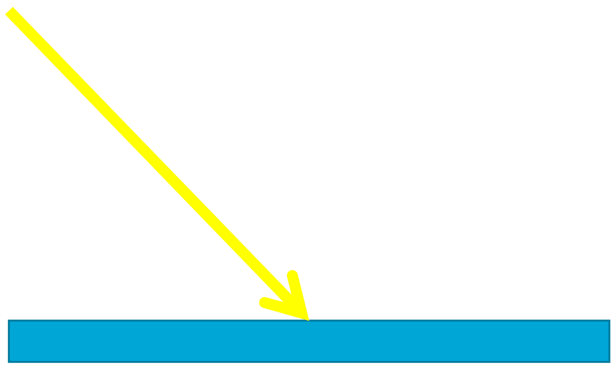
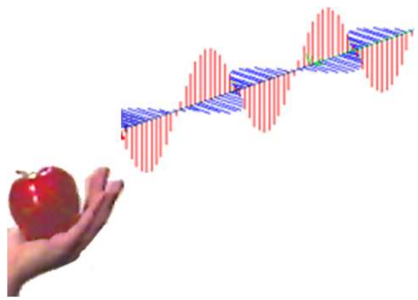
Reflection

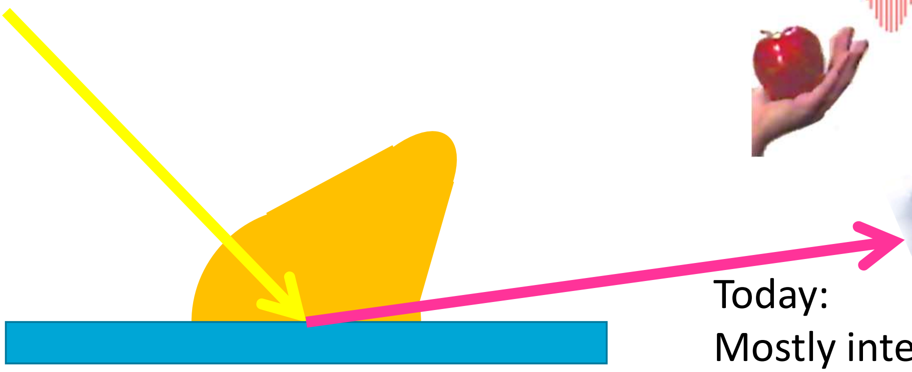
- What happens when the light hits a surface?



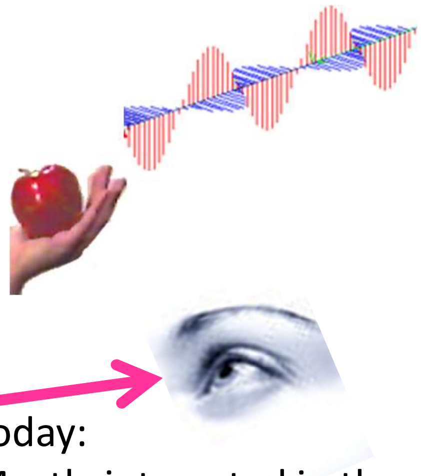


Reflection

- What happens when the light hits a surface?



Today:
Mostly interested in the direction towards the camera!

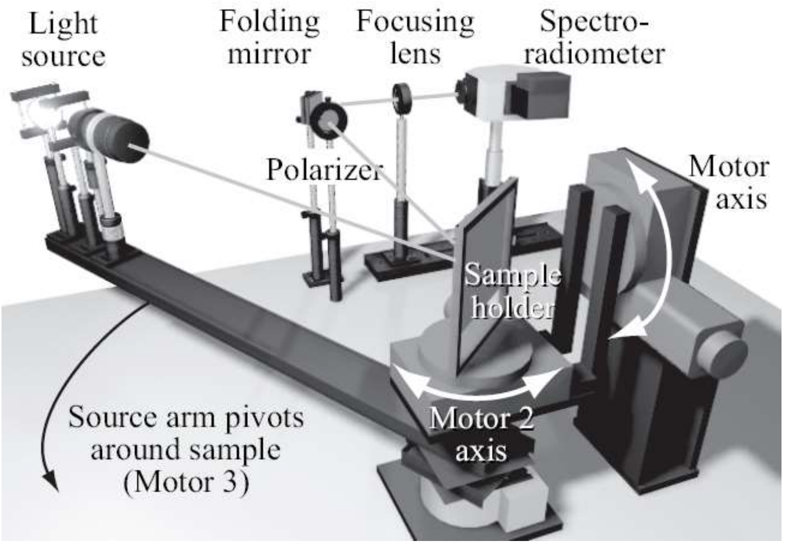


Acquired Materials

- Big databases

Material Acquisition

- Use a gonioreflectometer – yes, that is the name…



http://www.graphics.cornell.edu/~westin/

Acquired Materials

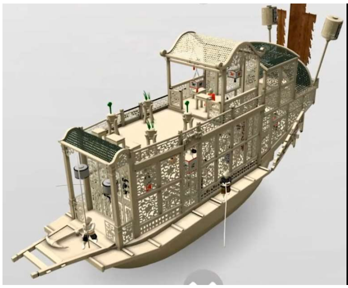
- Big databases

Often costly, much data

Hard to use for artists

e.g., “Can you make the blue darker?”

Very important when
acquiring real-world objects



Mathematical Models

- Describe light interaction as a function
- Usually more lightweight
- Has parameters to control appearance
- Acquired materials can be approximated

Published recently:
Effect of the additional glaze layer (left) that Vermeer placed over the black background.
graphics.tudelft.nl/WebPearl/

Girl With A Pearl Earring

Exhibition in the Mauritshuis

until 8th of January 24

World's largest 3D print

Time travel to the creation of the girl

vlnr: Elmar Eisemann (TUD),Ruben Wiersma (TUD), Lucie Oude Luttikhuis (TUD), Liselore Tissen (TUD/UL) Mane van Veldhuizen (UVA), Abbie Vandivere (MH), Mathijs Lefferts (TUD), Emilien Leonhardt (Hirox), Clemens Weijkamp (Canon), Camiel Rejhons (Canon)

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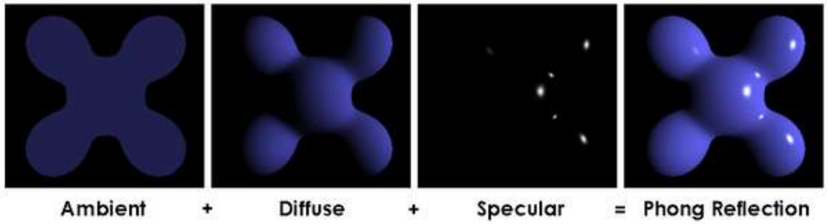
Simplified Models

- Mathematically describe Material Properties

- Phong Model: Sum of 3 terms
  - Ambient
  - Diffuse
  - Specular



Color - Recap

- Remember:

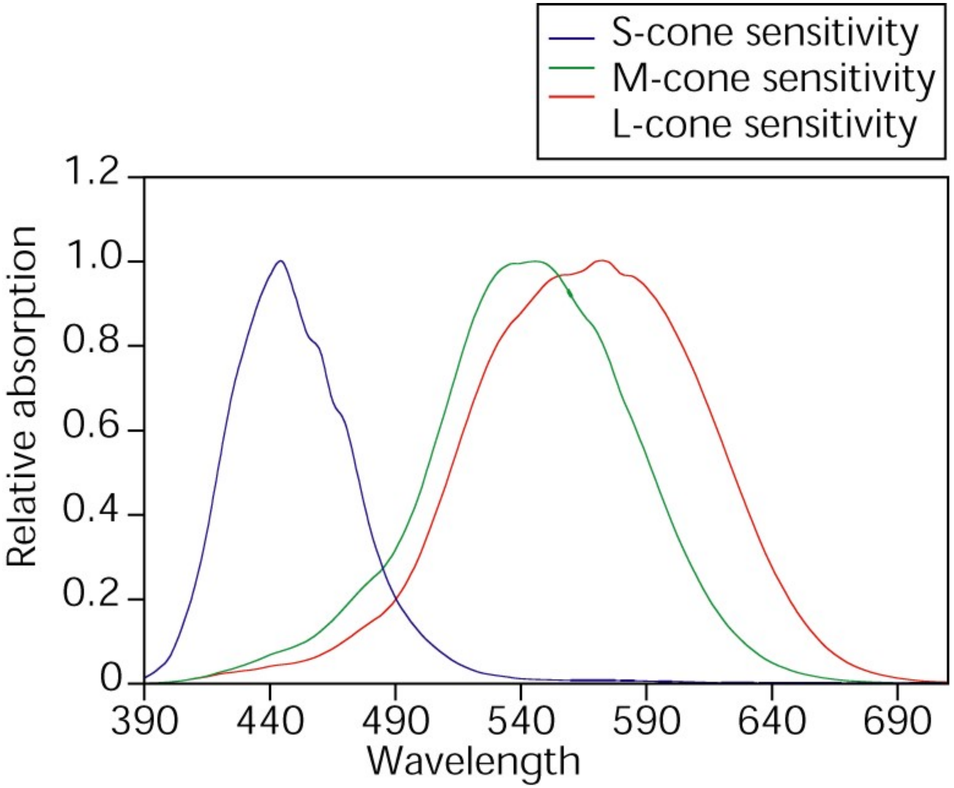
  Visual system uses 3 cone types for color

  In our model, we will treat wavelengths separately (in practice: **Red, Green, Blue**).

  In the following, we usually describe the model for a single wavelength/color channel

  (do it 3 times for red, green, blue...)

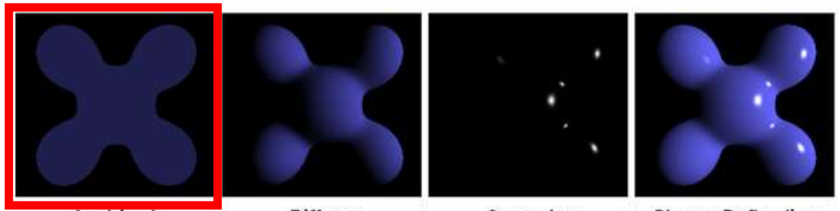
3 Cone types



S-cone sensitivity
M-cone sensitivity
L-cone sensitivity
1.2
1.0
0.8
0.6
0.4
0.2
0
Relative absorption
390
440
490
540
590
640
690
Wavelength

Phong Model

- Sum of 3 terms
  - Ambient
  - Diffuse
  - Specular



Ambient Term

- Is supposed to mimic “scene light”:
  - Skylight
  - Reflections from neighboring surfaces

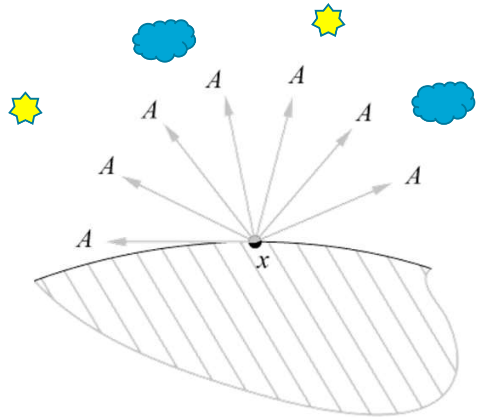
- Model:
  - Very simple:

Light property

Formula is used for Red, Green & Blue.

$$\boxed{A = I_a K_a}$$

Surface property



$$A = I_a K_a$$

Ambient Term

K_a increasing

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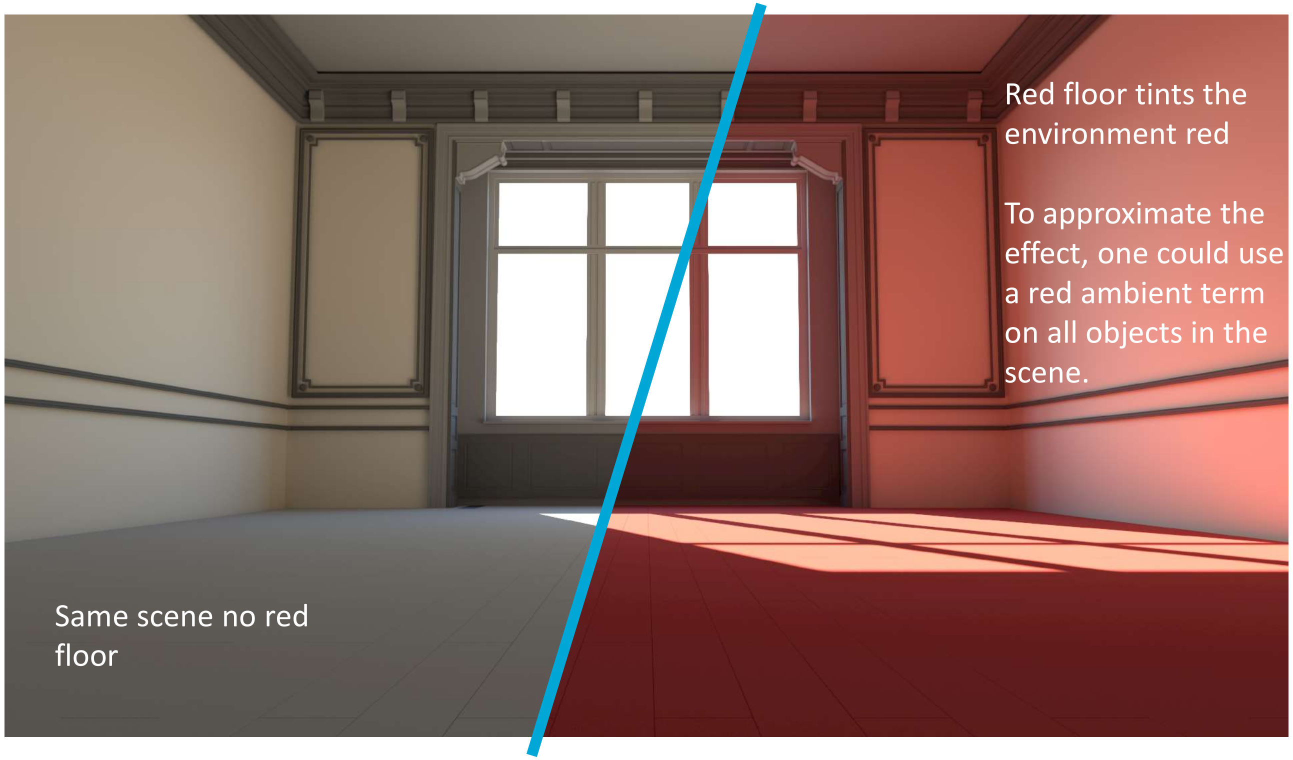
$$A = I_a K_a$$

Ambient Term

- Very simplistic
  - No indications on the shape of an object!

- Used often in practice as a strong approximation of indirect light

Example of Indirect Light



$$A = I_a K_a$$

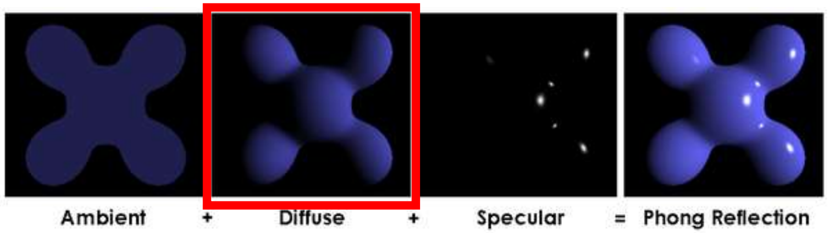
Example Ambient

- Typically, values between [0,1] are used
- $I_a = (0.9, 0.9, 0.9)$ “white light”
  (Red, green and blue are close to one)
- $K_a = (0.5, 0, 0)$ the surface is “dark red”
- The ambient term is?

$$A = I_a K_a = (0.9, 0.9, 0.9) * (0.5, 0, 0) = (0.45, 0, 0)$$ ...also dark red.

Phong Model

- Sum of 3 terms
  - Ambient
  - Diffuse
  - Specular

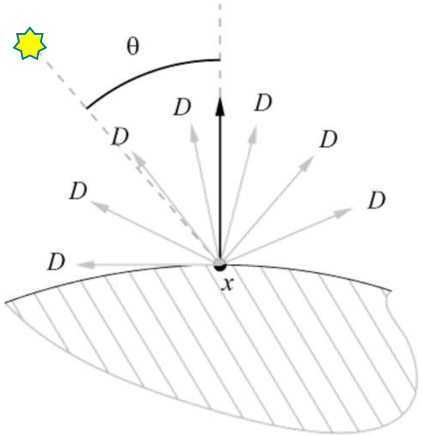


Diffuse Term

- Lambert Surfaces
  - Light is reflected uniformly in all directions
- Model:
  - Uses local surface orientation

$$D = I_d K_d \cos\theta$$

Light property (RGB) Surface property (RGB)



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Diffuse Term

$$D = I_d K_d \cos\theta$$

K_d increasing

Diffuse Term

$$D = I_d K_d \cos\theta$$

- Shading varies along surface
  - Gives information about shape

K_d increasing

  - Does not depend on observer position
    (looks the same from all positions)
  - **Careful:**
    the light should always come from above the surface, otherwise, it should stay black.
    What does this mean for the angle θ?

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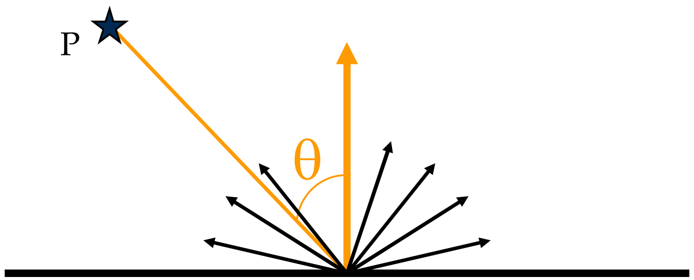
Diffuse Term

- Where does the cosine come from?

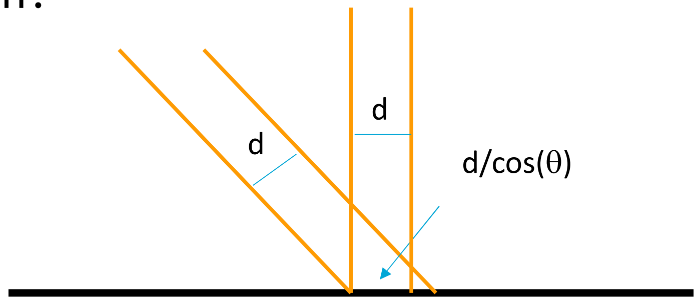
$$D = I_d K_d \cos\theta$$



Diffuse Term

- Where does the cosine come from?

$$D = I_d K_d \cos\theta$$

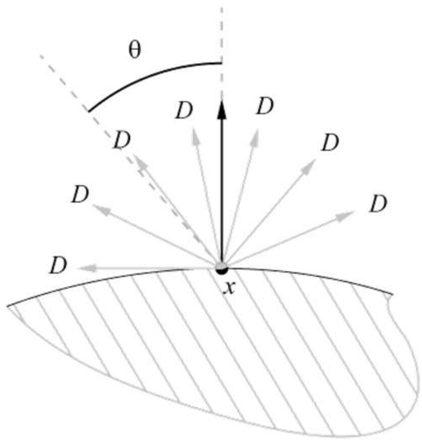


- What do you observe when tilting a flashlight?
- Imagine light arriving as parallel rays

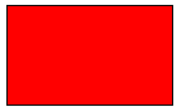
Example

$$A = I_a K_a \quad D = I_d K_d \cos\theta$$

- K_d= (0.9,0,0)
- I_d= (0.9,0.5,1.0)
- K_a= (0,0,0.1)
- I_a= (1,1,0.1)



- Normal at x: (0,0,1)
- Position x: (0,0,0)
- light position : (0,0,10)
- Resulting color: (0.81,0,0.01)



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Phong Model

- Sum of 3 terms
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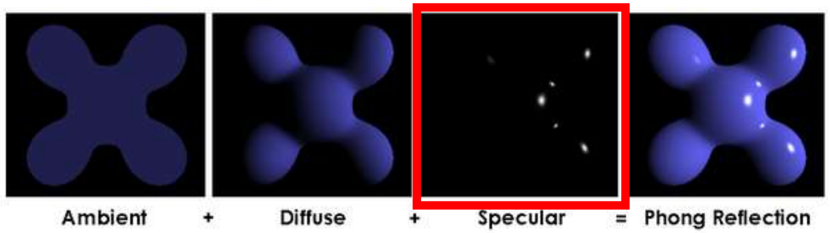
  - Specular

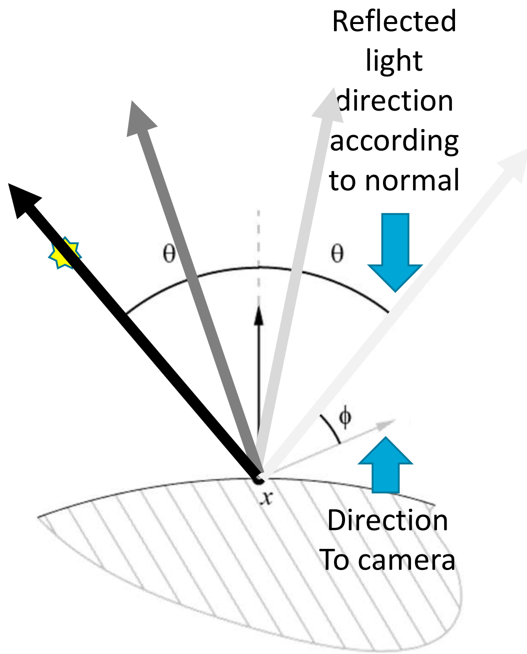


Specular Term

- Represent glossy surfaces
  - Ideal case: mirror
- Model:
  - reflection around mirrored ray
  - Falloff around perfect reflection

$$S(\phi) = I_s K_s \boxed{(\cos\phi)}^n$$

n : shininess



TUDelft

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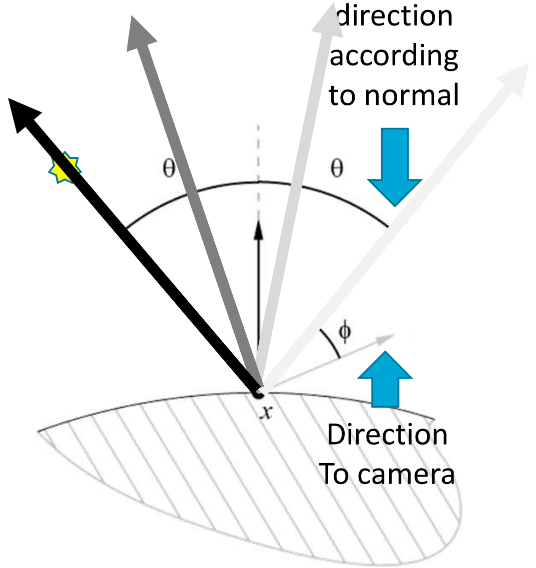

TUDelft

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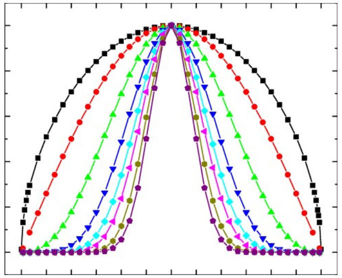
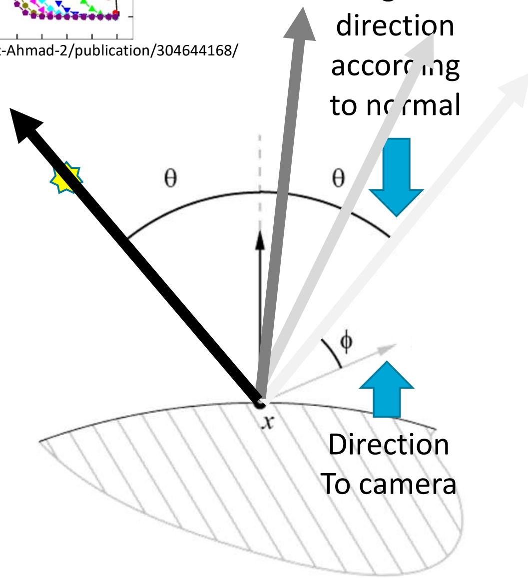
n : *shininess*



https://www.researchgate.net/profile/Fayyaz-Ahmad-2/publication/304644168/

Cosine over
[-pi/2,pi/2]
to the power of n



Specular Term

- Represent glossy surfaces
  - Ideal case: mirror
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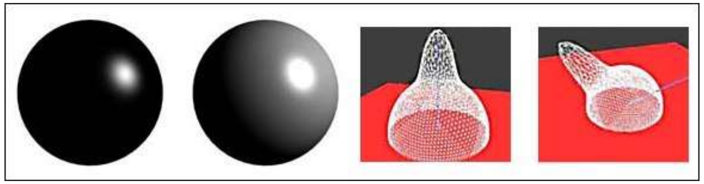
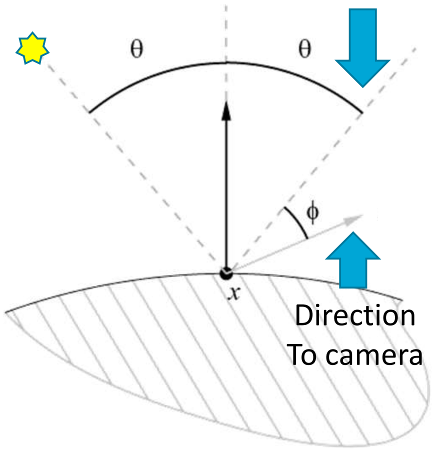
  - reflection around mirrored ray
  - Falloff around perfect reflection

$$S(\phi) = I_s K_s (\cos\phi)^n$$

n : *shininess*



specular   specular+diffuse   S(ϕ), called the *lobe*



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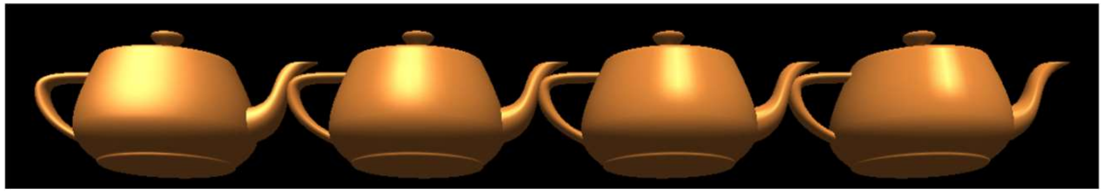
Specular Term

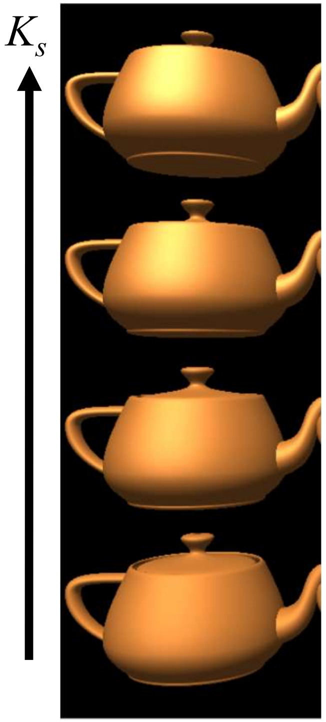
$$S(\phi) = I_s K_s (\cos\phi)^n$$



105

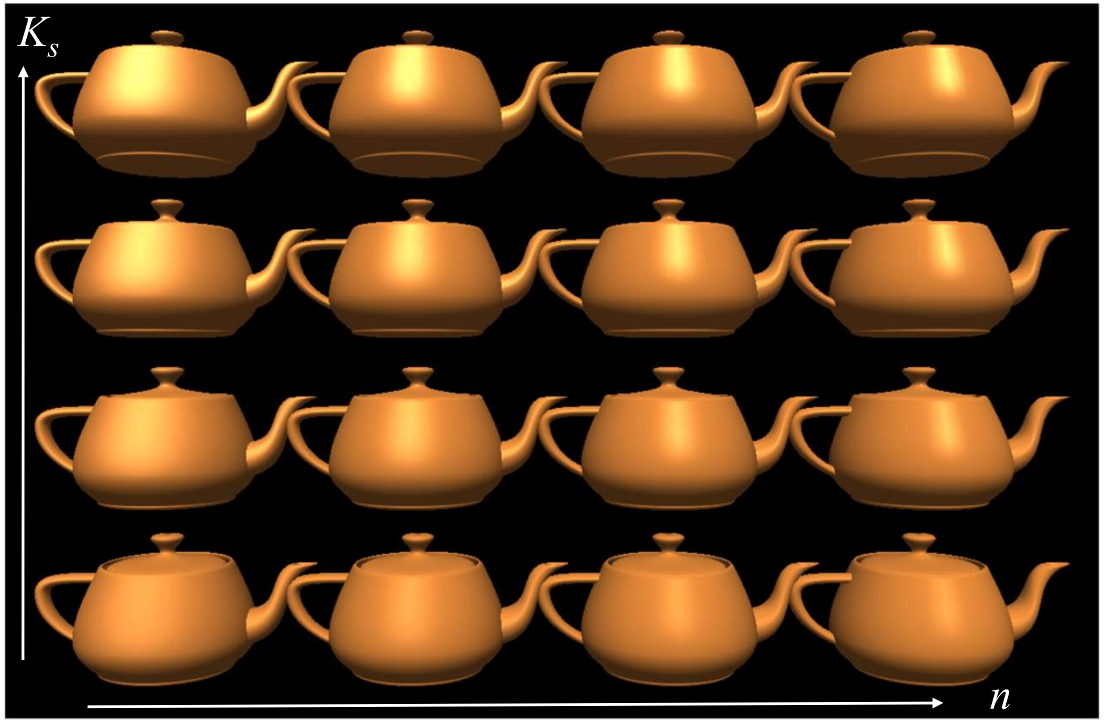
Specular Term

$$S(\phi) = I_s K_s \left(\cos\phi\right)^n$$



Specular Term

$$S(\phi) = I_s K_s (\cos\phi)^n$$



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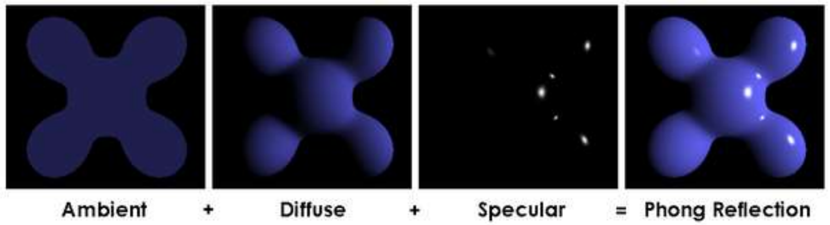
Phong Model

- Sum of 3 terms
  - Ambient
  - Diffuse
  - Specular



Phong Model

- Sum of 3 terms
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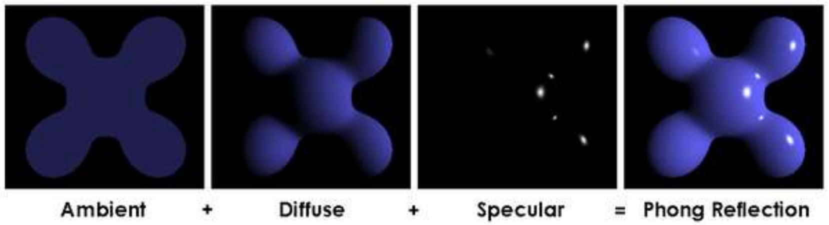

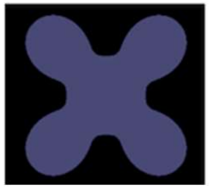

- Optional Extension:
  - Emission = Ambient with a Light set to 1
    - Idea: object is emitting light (e.g., hot glowing metal)

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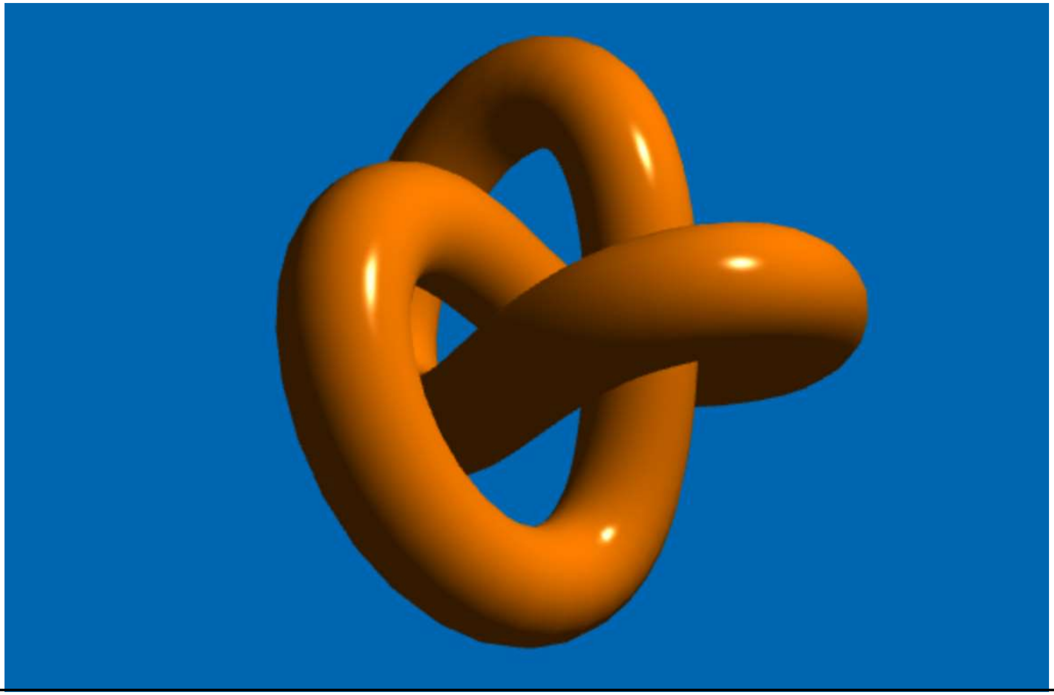
Demo Time

Demo Time

- http://multivis.net/lecture/phong.html

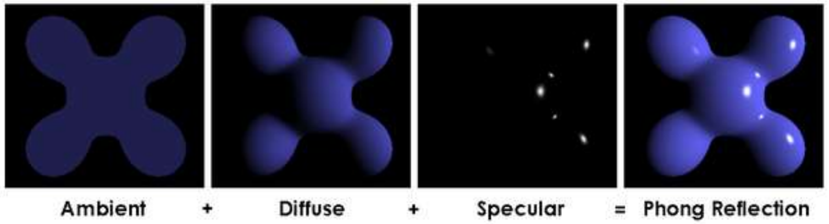


TUDelft

Mathematical Models

- Mathematically describe Material Properties

- Phong Model: Sum of 3 terms
  - Ambient
  - Diffuse
  - Specular



- In the literature: Many more material models

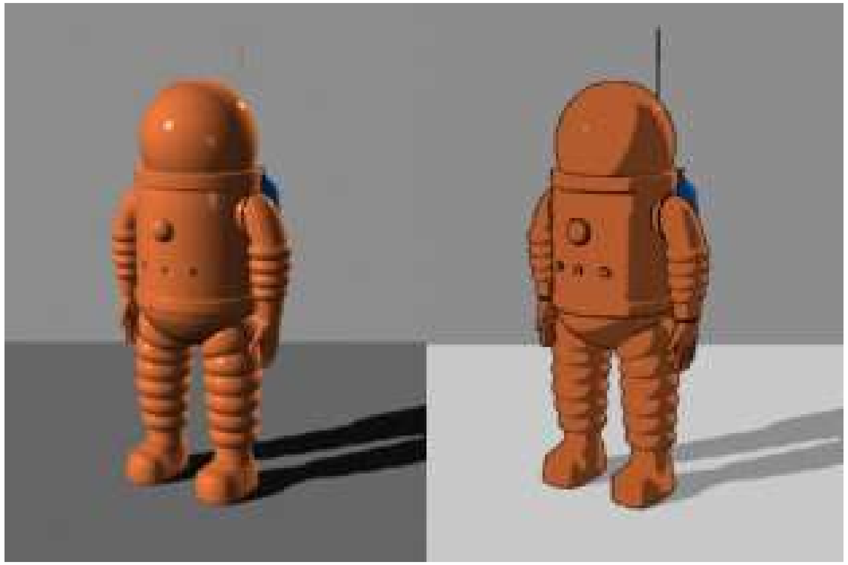
  Tradeoff between efficiency and accuracy

Advanced Material Models

https://blog.playcanvas.com/physically-based-rendering-comes-to-webgl/

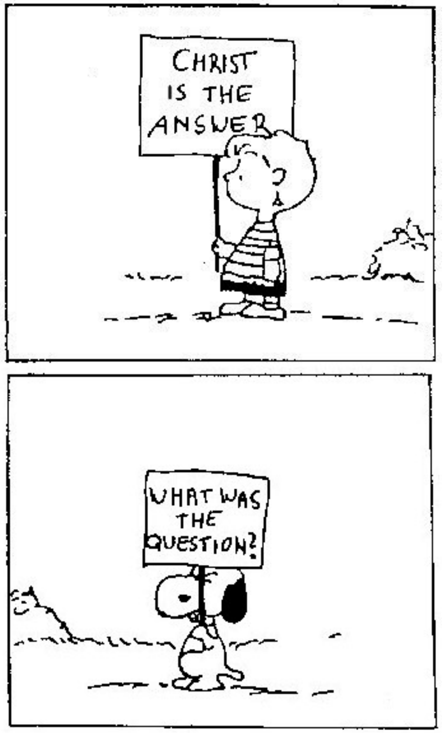
Extreme Example...

- Non-photorealistic materials



- Cel-shading: threshold on the diffuse shading

Questions?



How to apply Phong Model ?

- We know how to compute shading of a point, but how is it applied on a triangle mesh?

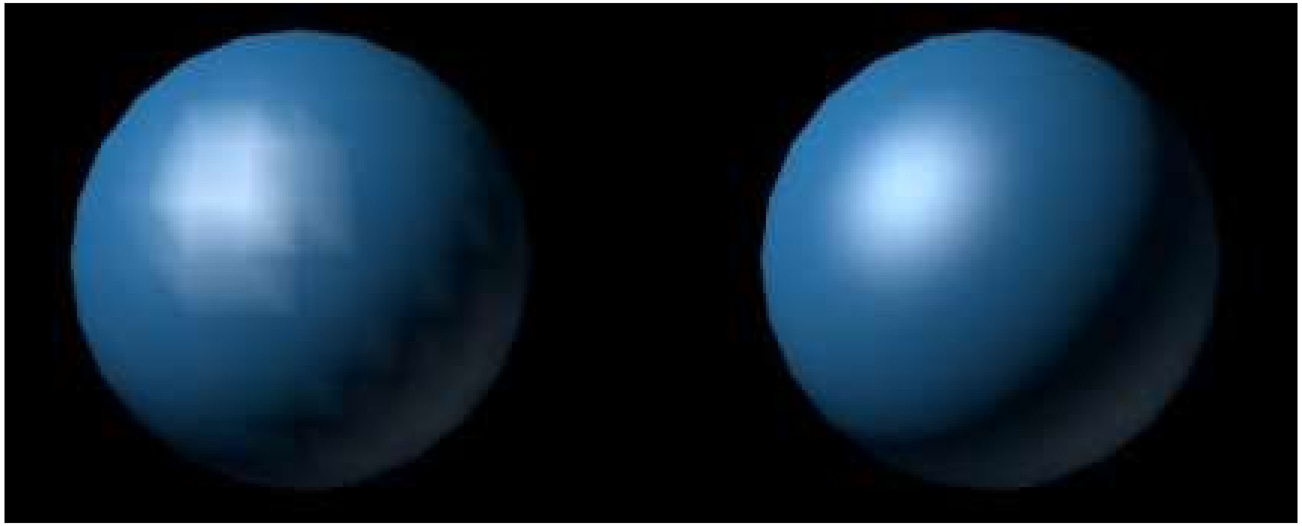
Shading

- *Early days - compute color per face:*
  Flat shading produces “facets”

- Later – compute color per vertex:
  produces *Gouraud Shading* produces a smooth look

Phong shading

- Today: compute result per pixel
- Phong Shading leads to smooth specularities

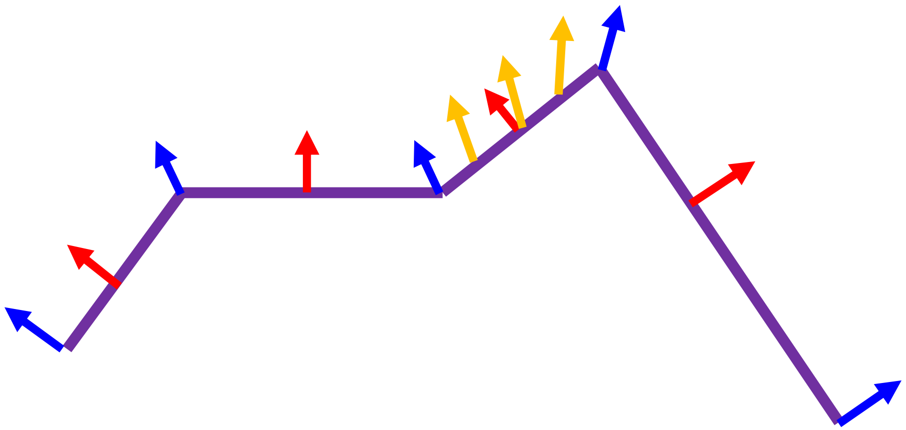


Diccan.com

- Phong interpolates normals from vertices over pixels

Normals on Meshes

- Face normals (normal of the plane containing triangle)
- Vertex normals (e.g., average neighboring face normal)
- Interpolated normal (interpolate vertex normals over triangle)

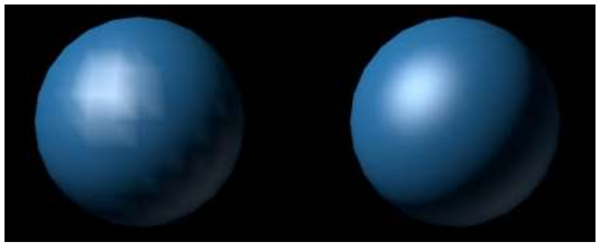


Per vertex

Per pixel

Gouraud *vs.* Phong shading

- Phong usually more expensive than Gouraud
  - Because there are often more pixels than vertices
- Phong is more beautiful and minimal standard
  - Captures specularities between faces



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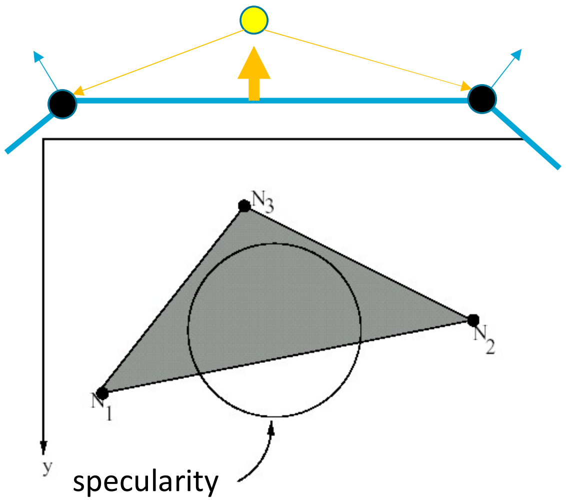
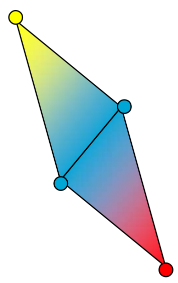


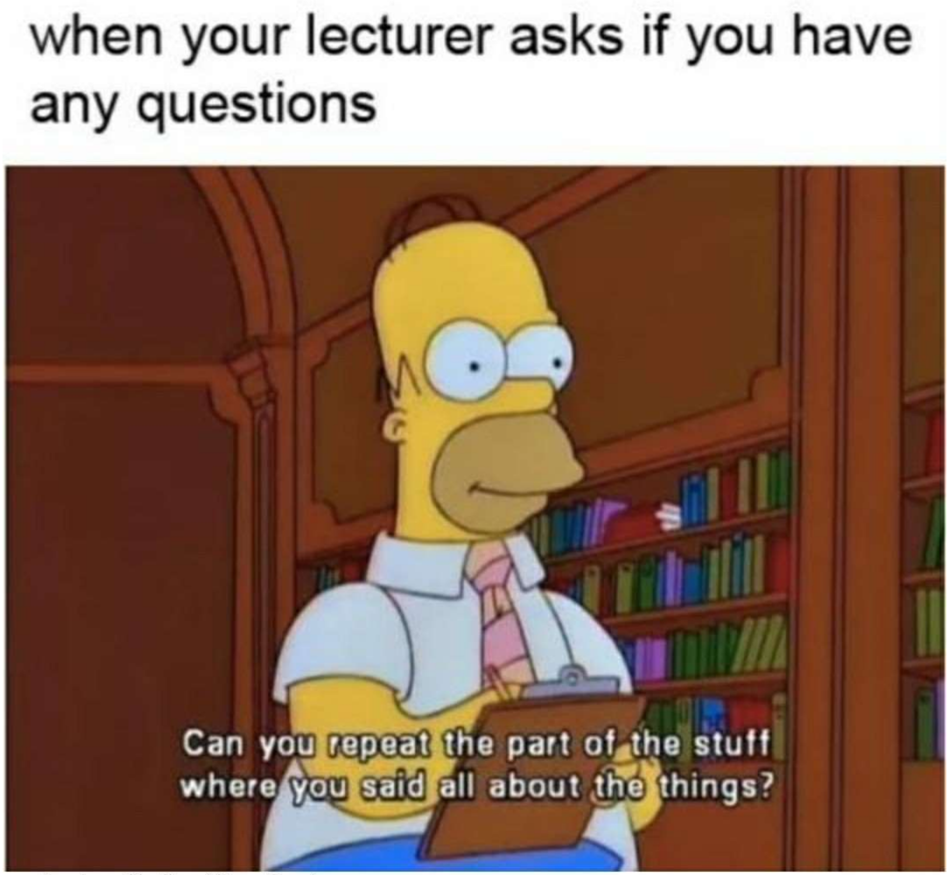
On the practical side: Shading types

- How are the three different types computed?
  - *Flat shading*
    - Applies Phong Model to produce a color per face
  - *Gouraud shading*
    - Applies Phong to produce a color per vertex
    - Interpolate color from vertices over triangle
  - *Phong shading*
    - Interpolate parameters of Phong model
    - Applies Phong to produce a color per pixel

2 MEANINGS!!!



Questions?



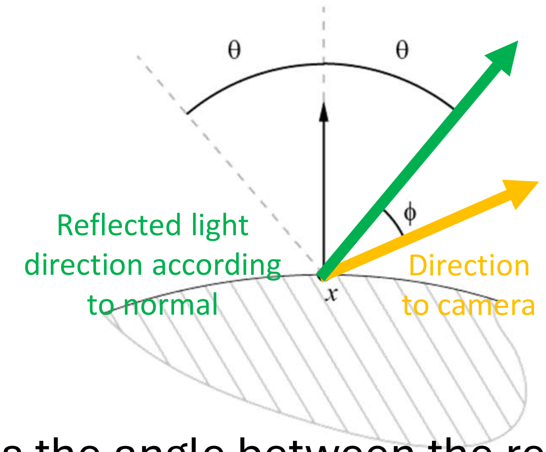
Jemima Skelley / BuzzFeed

Summary

- Graphics Pipeline Core:
  - Geometry
    - Transformation and Projection
  - Shading
    - Material model (Ambient, Diffuse, Specular)
    - Shading interpolation (Flat, Gouraud, Phong)

Additional Exercises

$$S(\phi) = I_s K_s (\cos\phi)^n$$

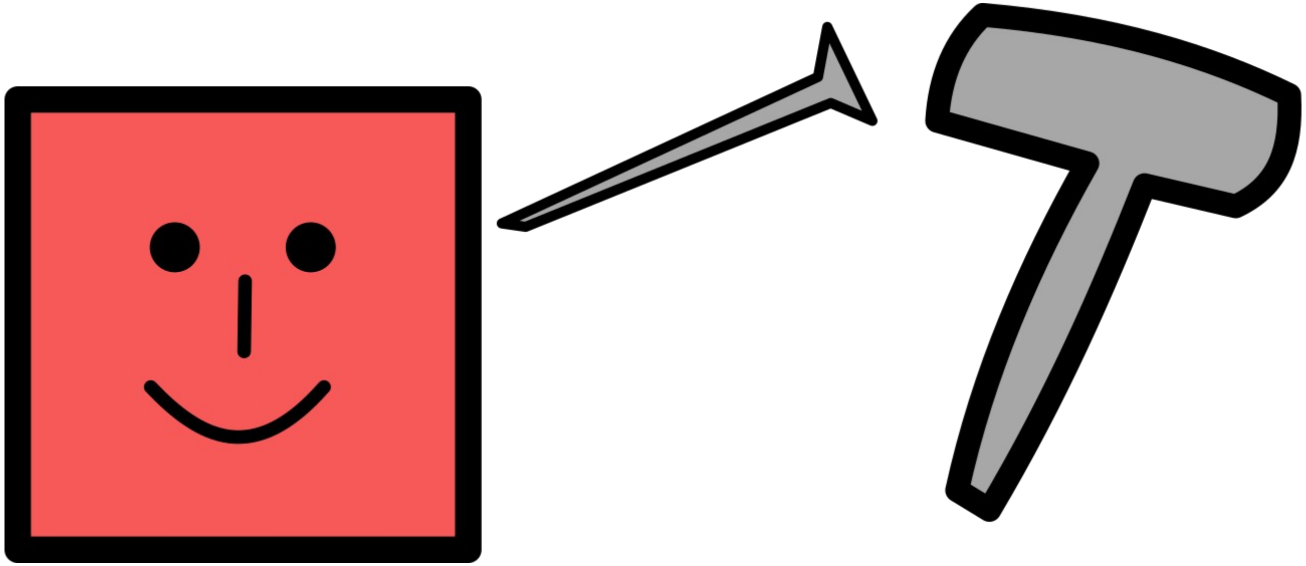


- The specular part of the Phong model uses the angle between the reflected light direction (according to the normal) and the direction to the camera.
- Calculate the reflected light direction.

Thank you very much for your attention!

OpenGL : how to make a pixel

CSE2215 Computer Graphics



Ricardo Marroquim

Delft University of Technology (TU Delft)

today

- **Graphics Pipeline** with **OpenGL**
  - model representation
  - transformation
  - rasterization
  - interpolation

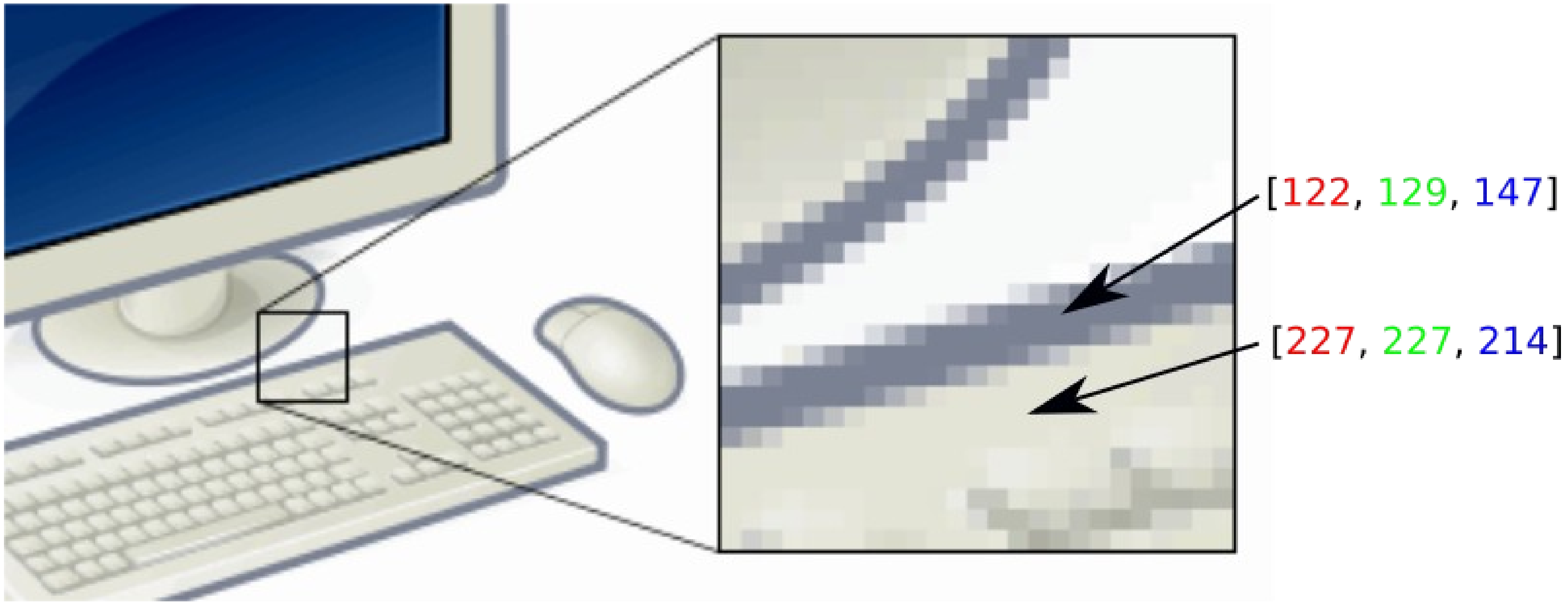
last lectures in a nutshell!

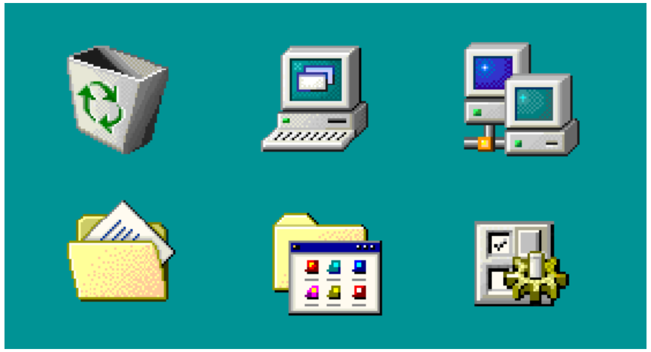
study goals

- S4- Apply mathematical modeling and theory of geometric computations and transformations, object representations, simulation, and encoding.
- S5- Implement algorithms and data structures using the C++ programming language and OpenGL.

in its simplest form

- our task is to color pixels RGB [0,255]





pixel art


PIXEL ART IS NOT REAL ART
PIXEL ART IS NOT REAL ART
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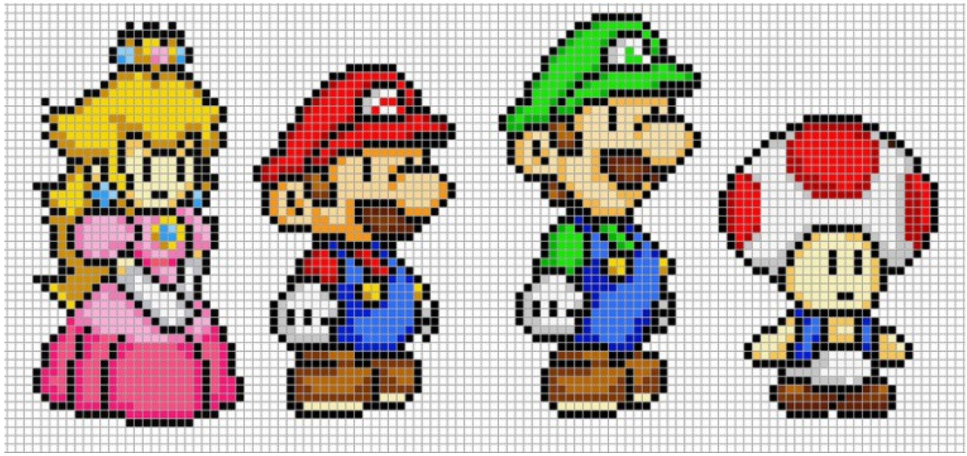
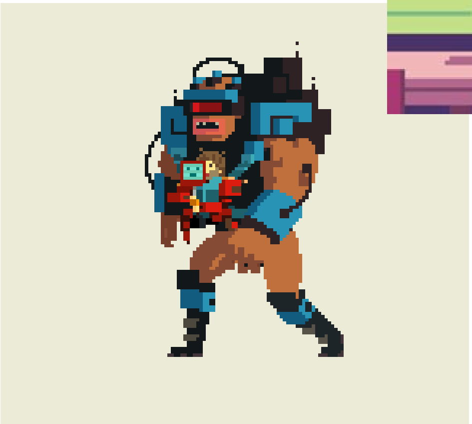
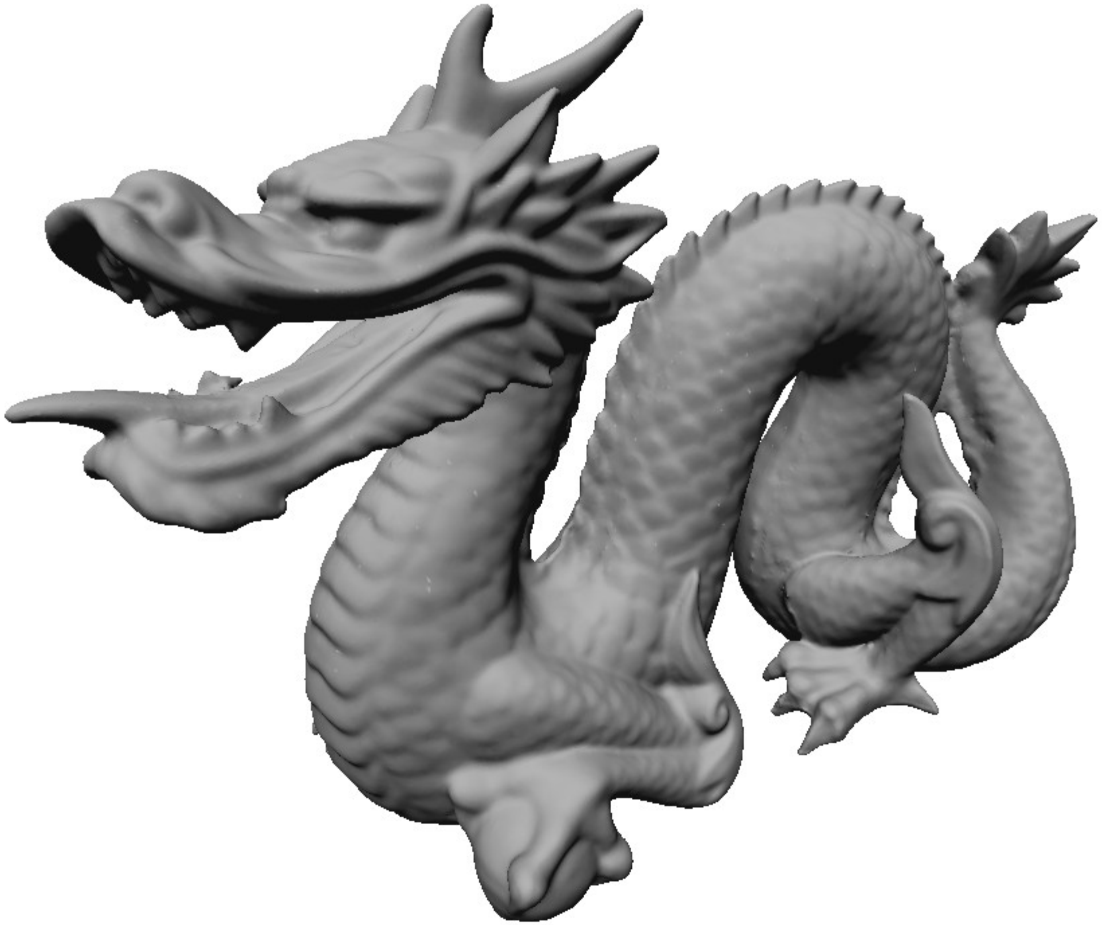


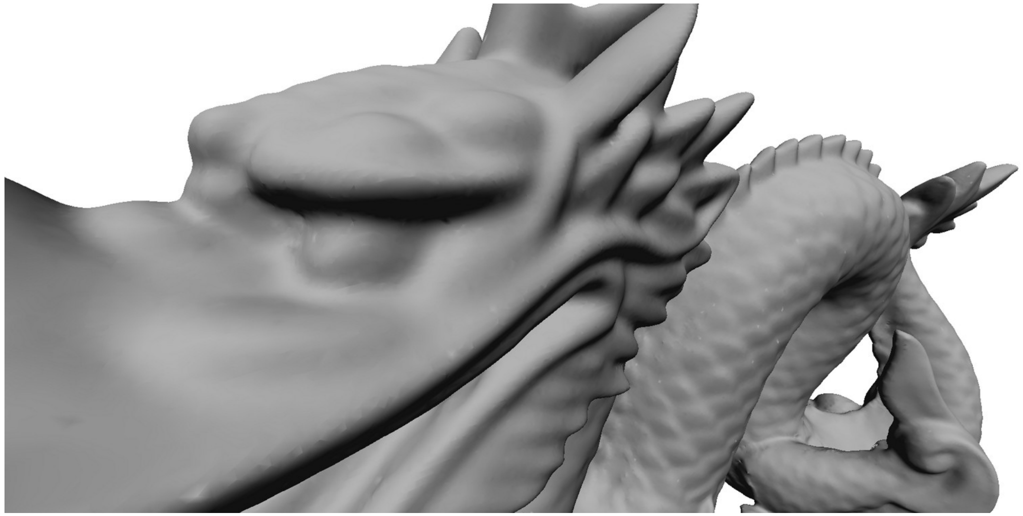


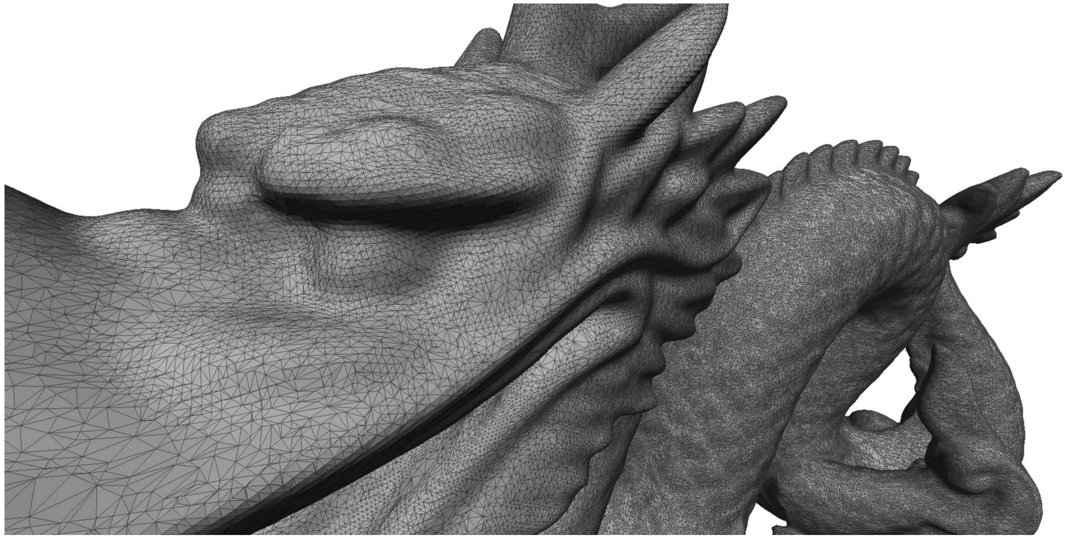
models



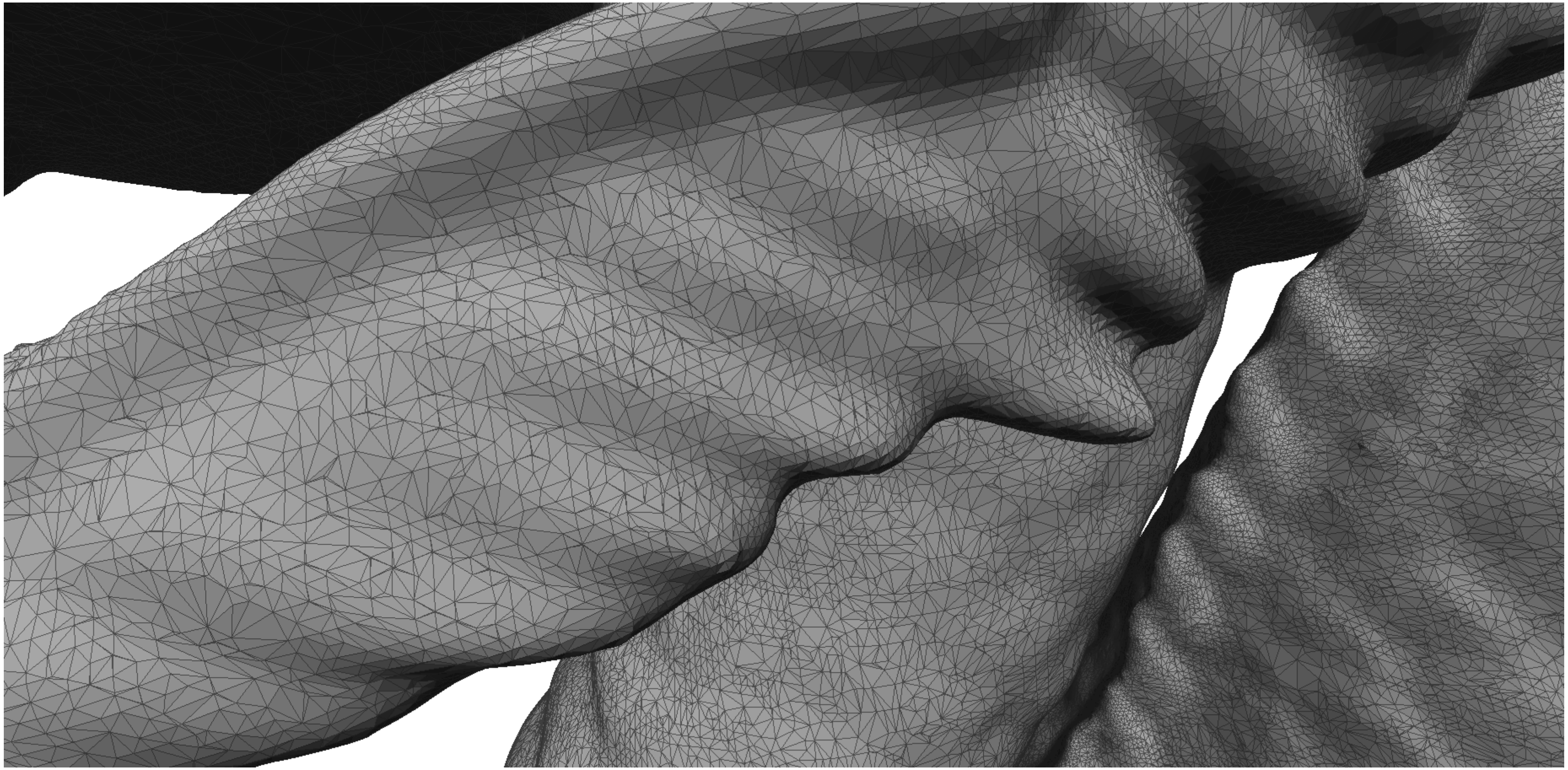
triangles

- **most** common representation

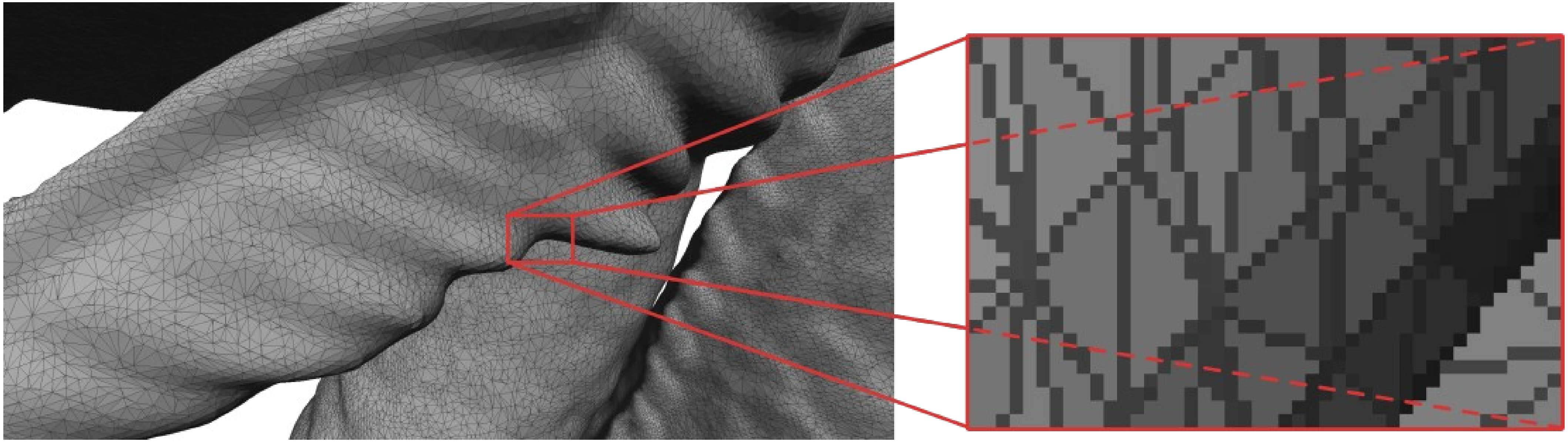




triangles

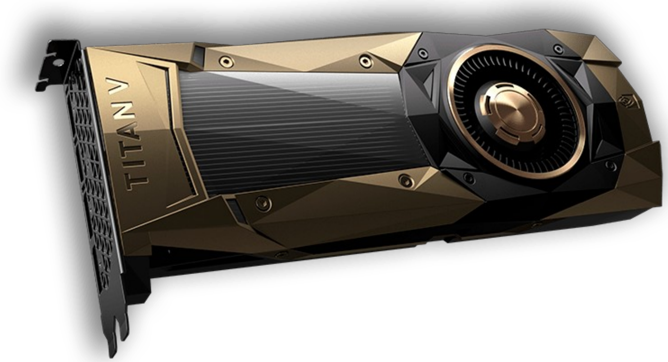


triangles



OpenGL®

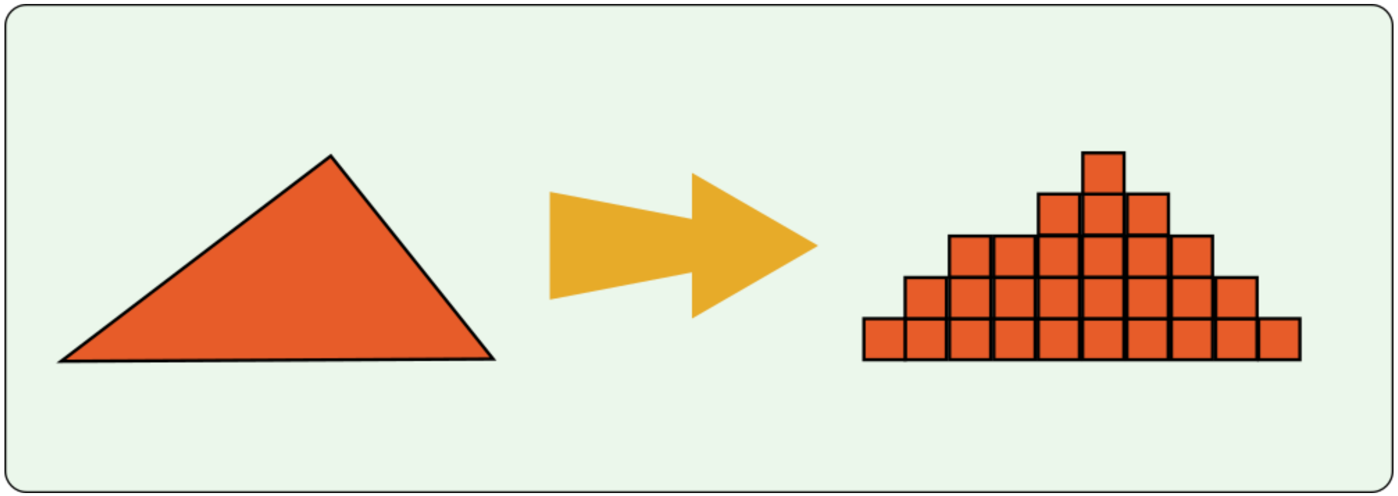
- **Open G**raphics **L**ibrary
  - **rendering** API : draw primitives on the screen (e.g. triangles)
  - actually it is an **abstract** API (specification)
    - usually implemented by hardware driver
    - OpenGL 1.0 (1992) → OpenGL 4.6 (2017)
  - does not handle your windows, needs libs (e.g. GLFW)



- **Open G**raphics **L**ibrary
  - around 1999 the first “programmable” GPU by NVIDIA
  - with new generations → more “programmability”
  - “modern” OpenGL: after 3.0 (2008)
    - more access to hardware features
  - but we stick to old school for now

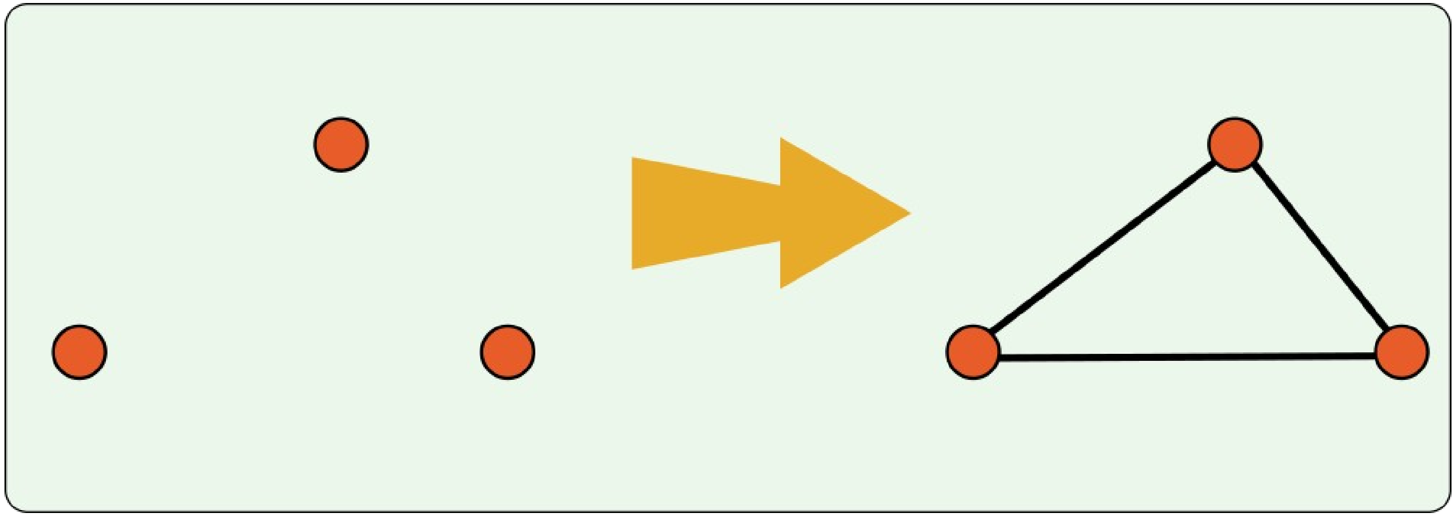
basic idea

- transform triangle to pixels
- **rasterization**



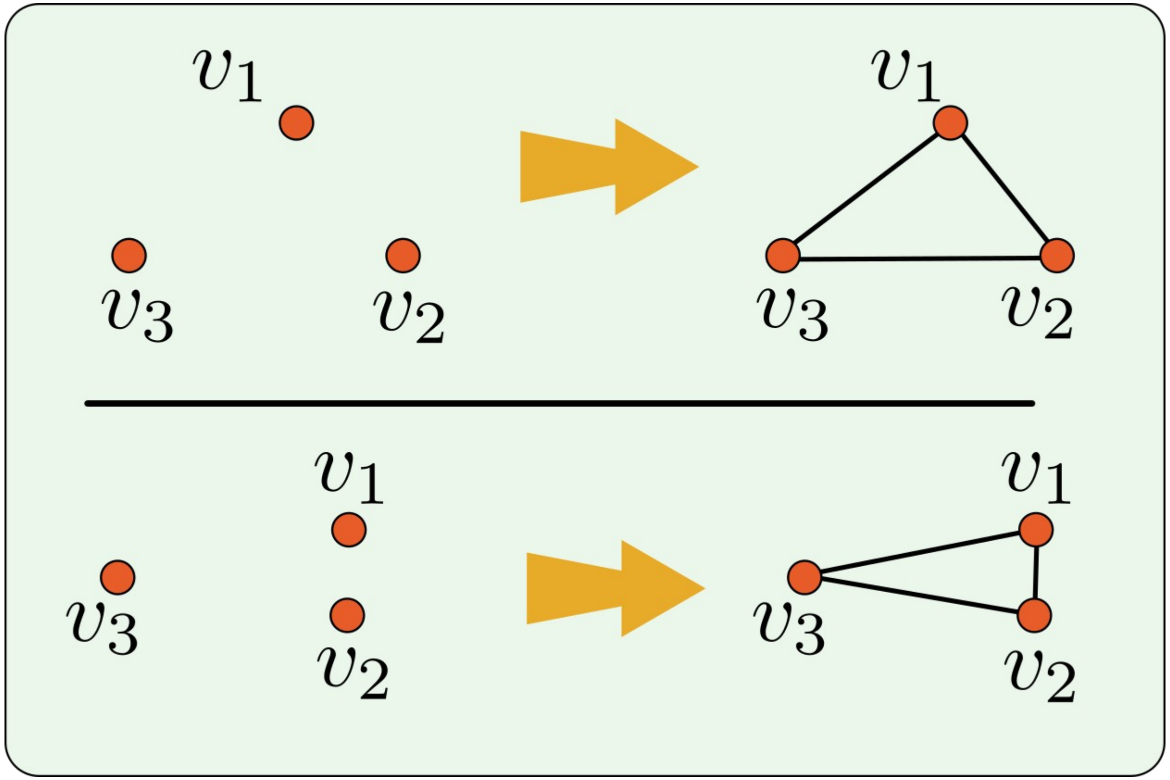
first point

- basic element is a **vertex**
  - triangles are just 3 connected vertices

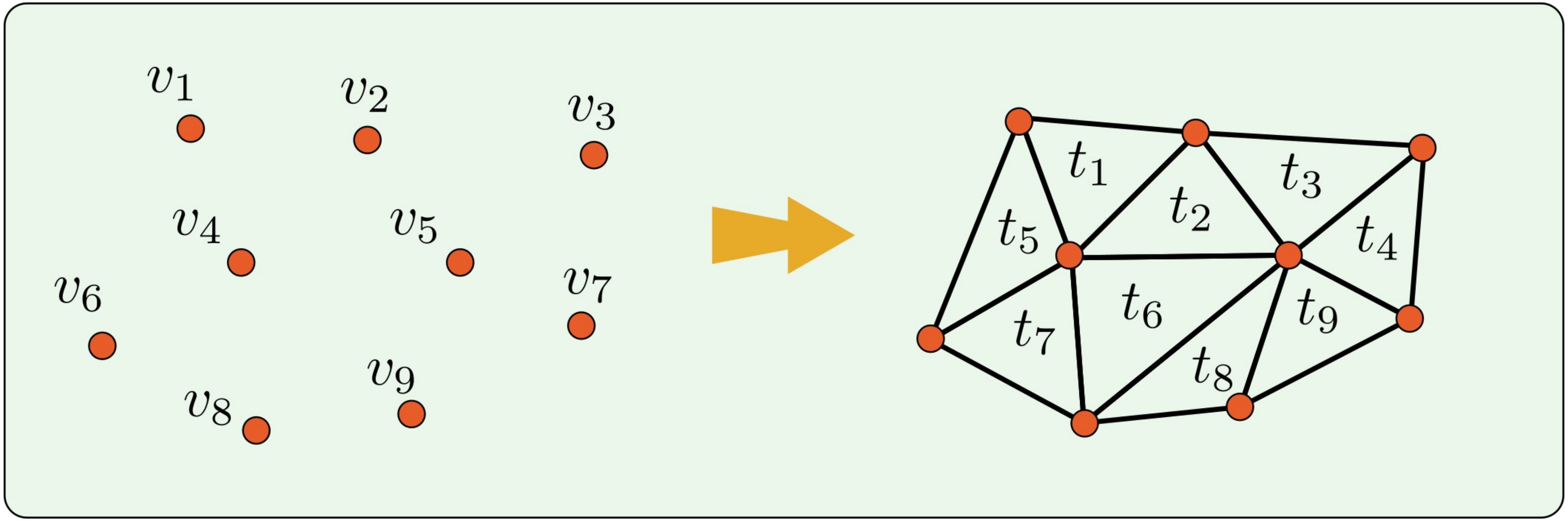


first point

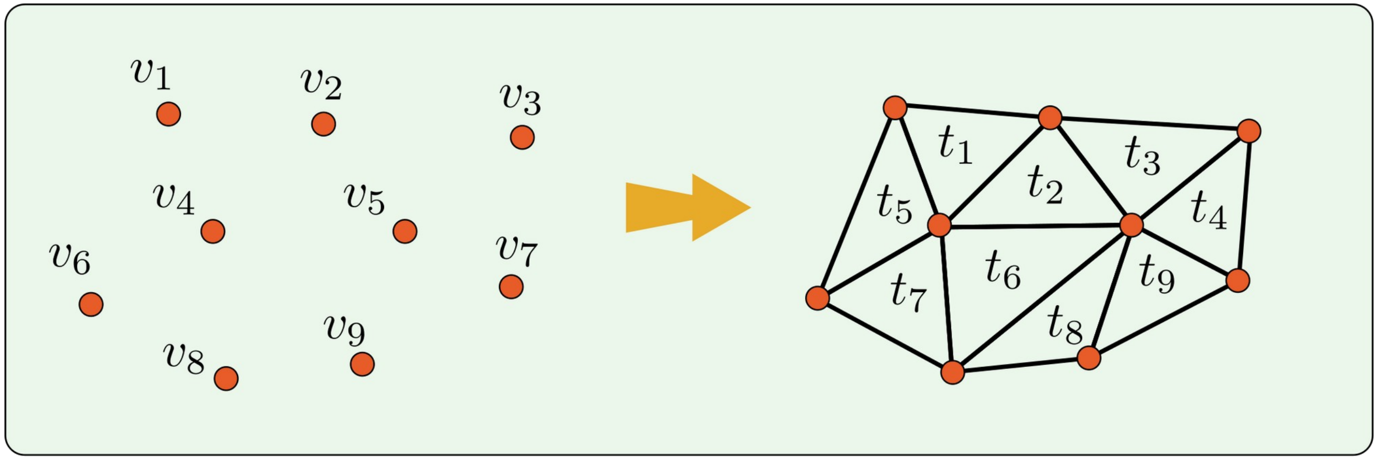
- modify vertices $\rightarrow$ new triangle is formed



basic structure



basic structure



vertices

$$v_1 = \{x_1, y_1, z_1\}$$
$$v_2 = \{x_2, y_2, z_2\}$$
$$\vdots$$
$$v_n = \{x_n, y_n, z_n\}$$

triangles

$$t_1 = \{v_1, v_2, v_4\}$$
$$t_2 = \{v_2, v_5, v_4\}$$
$$t_3 = \{v_2, v_3, v_5\}$$
$$\vdots$$

OBJ file

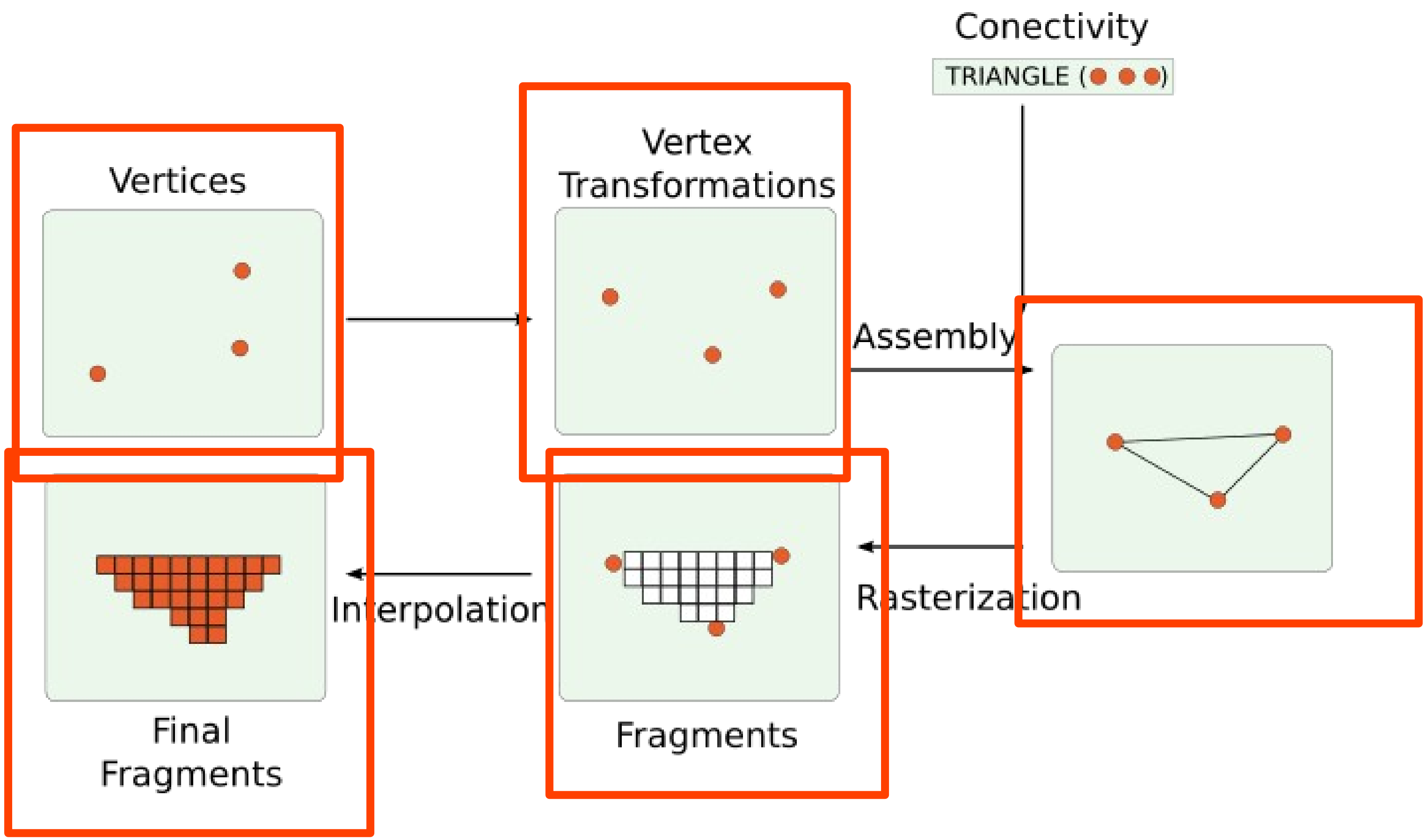
```
# triangle.obj
#

v  0.0  0.0  0.0
v  0.0  0.0  1.0
v  0.0  1.0  0.0

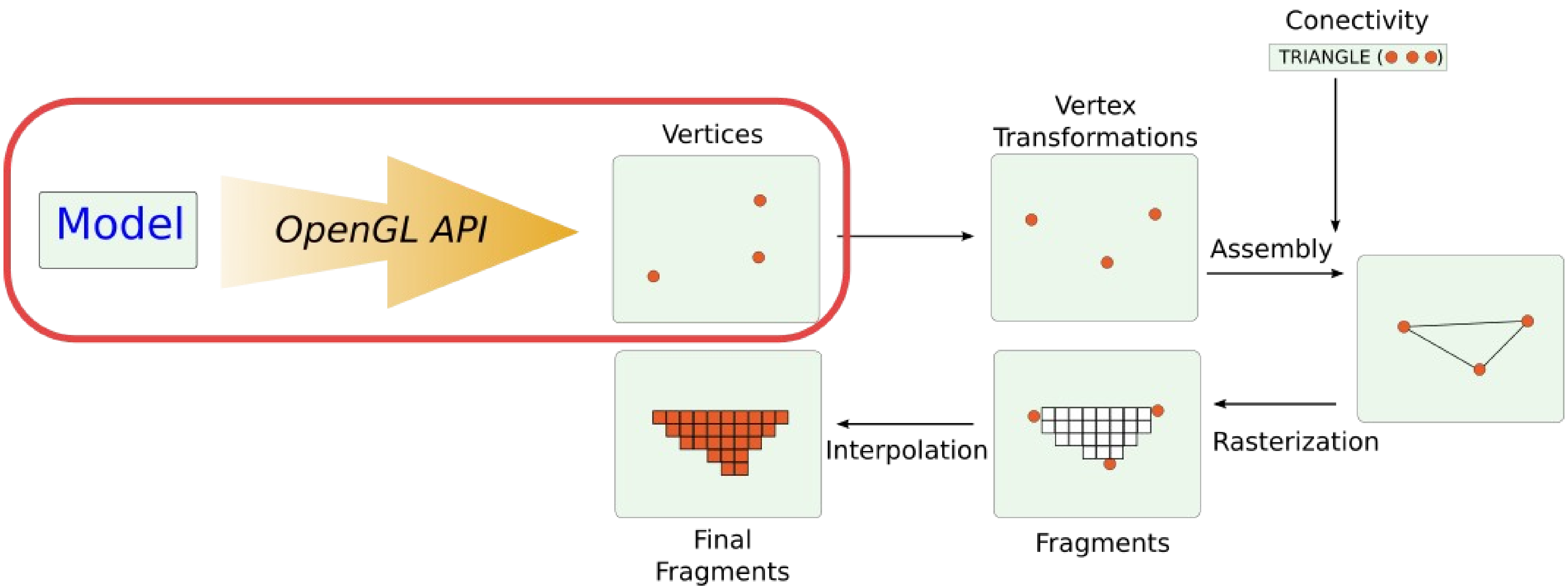
f  1  2  3
```

```
# Blender v2.78 (sub 0) OBJ File: ''
# www.blender.org
o root
v 0.207195 -0.263764 0.214438
v 0.206990 -0.264675 0.211045
v 0.207187 -0.260728 0.213590
v 0.206078 -0.264410 0.217782
v 0.207410 -0.260020 0.216926
v 0.206488 -0.259670 0.220876
v 0.206412 -0.267725 0.208776
v 0.205247 -0.269258 0.207019
v 0.205058 -0.267527 0.205096
v 0.204569 -0.270174 0.210299
v 0.205506 -0.267026 0.214885
v 0.206529 -0.265218 0.208719
v 0.205088 -0.265985 0.205913
v 0.203961 -0.261230 0.207378
v 0.203722 -0.262400 0.222970
v 0.203432 -0.259140 0.226576
v 0.204938 -0.257517 0.211903
v 0.205295 -0.255813 0.226379
v 0.205062 -0.254126 0.216151
v 0.206667 -0.256820 0.218732
v 0.206375 -0.256206 0.222311
v 0.205382 -0.251990 0.221828
```

graphics pipeline (simplified)

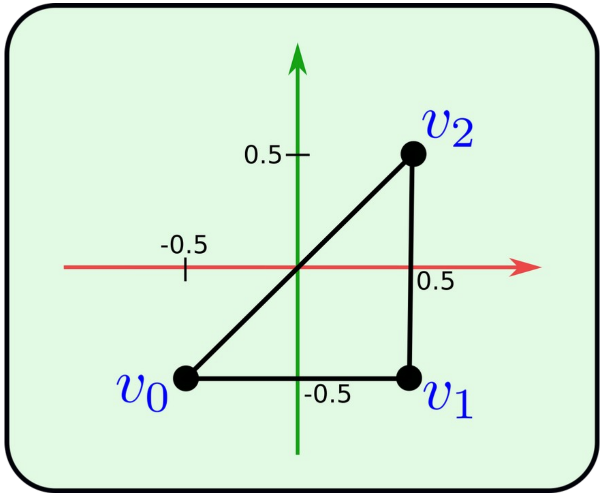


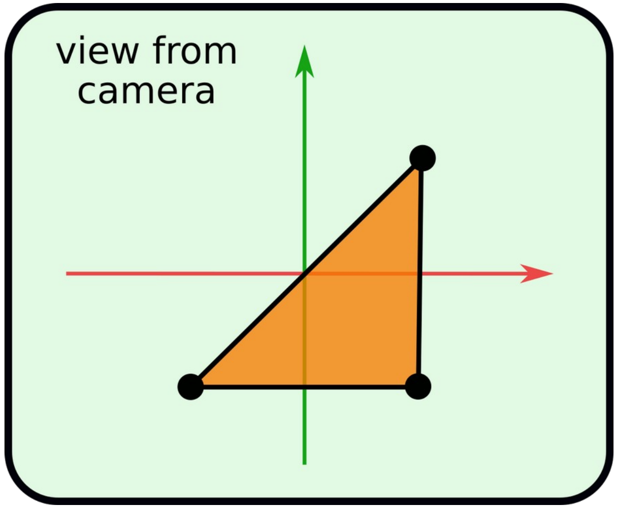
graphics pipeline (simplified)



simple triangle

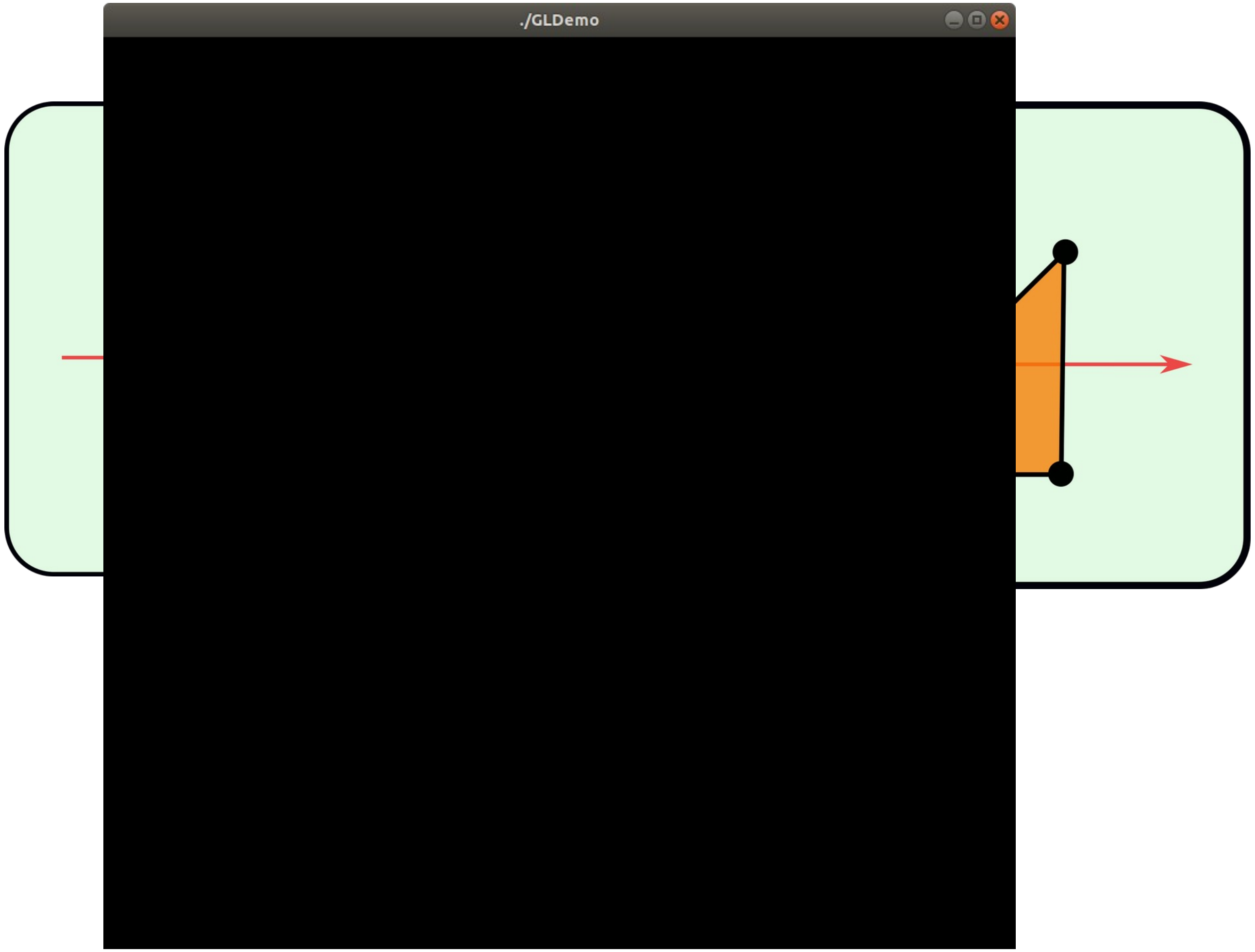
```
void drawTriangle()
{
    glBegin(GL_TRIANGLES);
    glColor3f(1.0, 0.5, 0.0);
    glVertex3f (-0.5, -0.5, 0.0);
    glVertex3f ( 0.5, -0.5, 0.0);
    glVertex3f ( 0.5,  0.5, 0.0);
    glEnd();
}
```



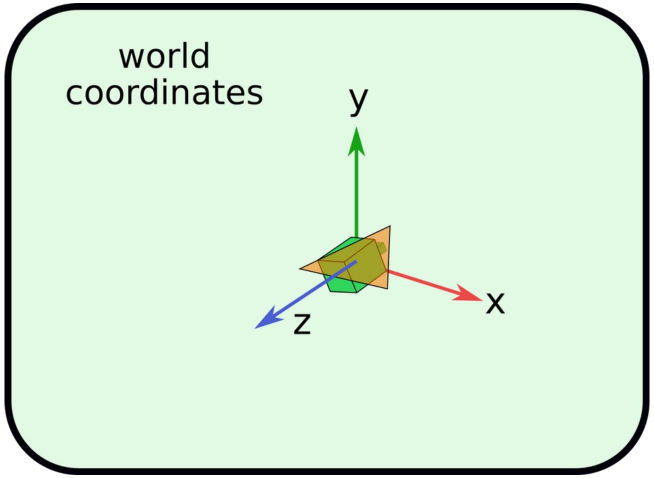


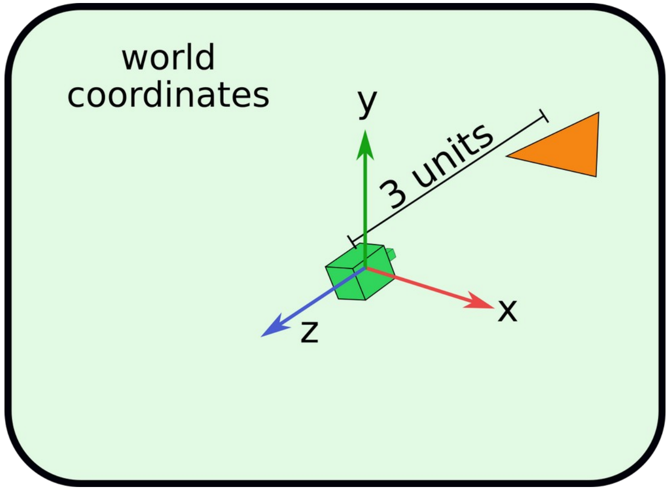
simple triangle

```
void drawTriangle()
{
    glBegin(GL_TRIANGLES);
    glColor3f(1.0, 0.5, 0.0);
    glVertex3f (-0.5, -0.5, 0.0);
    glVertex3f ( 0.5, -0.5, 0.0);
    glVertex3f ( 0.5,  0.5, 0.0);
    glEnd();
}
```



simple triangle

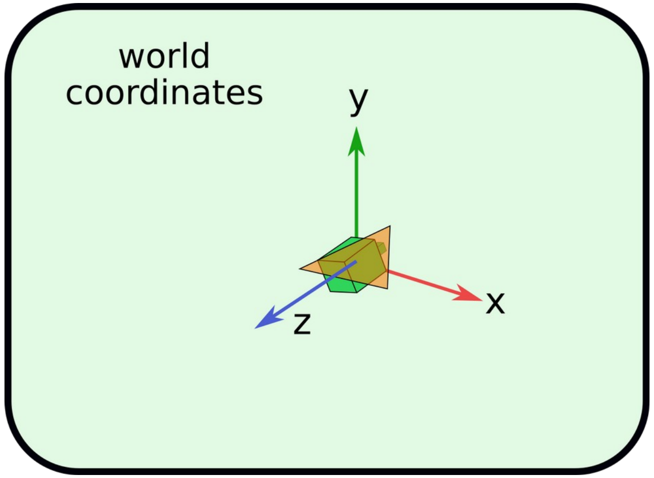


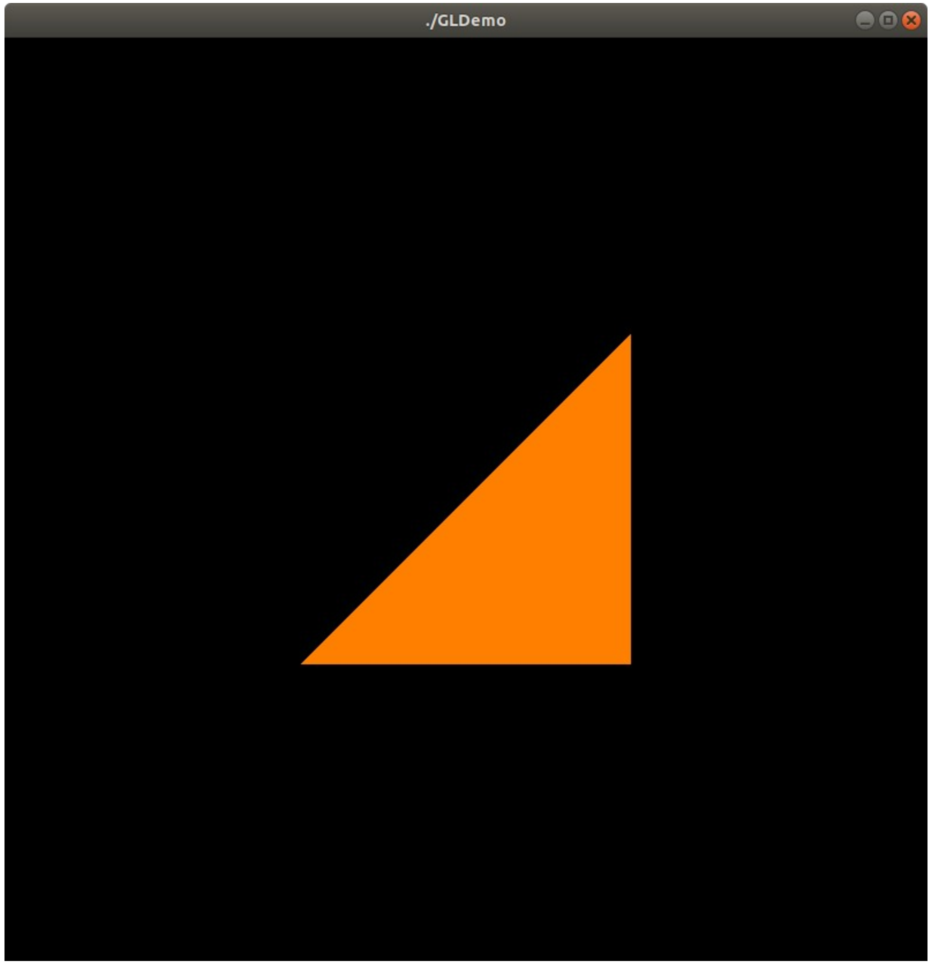



```

void drawTriangle()
{
    glBegin(GL_TRIANGLES);
    glColor3f(1.0, 0.5, 0.0);
    glVertex3f (-0.5, -0.5, -3.0);
    glVertex3f ( 0.5, -0.5, -3.0);
    glVertex3f ( 0.5,  0.5, -3.0);
    glEnd();
}
```

simple triangle





color per vertex

```
void drawTriangle()
{
    glBegin(GL_TRIANGLES);
    glColor3f(1.0, 0.5, 0.0);
    glVertex3f (-0.5, -0.5, -3.0);
    glVertex3f ( 0.5, -0.5, -3.0);
    glVertex3f ( 0.5,  0.5, -3.0);
    glEnd();
}
```

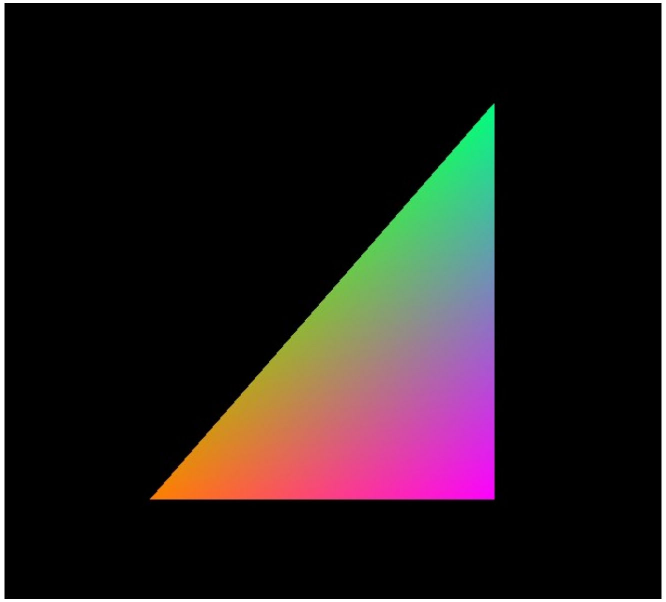
```
void drawTriangle()
{
    glBegin(GL_TRIANGLES);

    glColor3f(1.0f, 0.5f, 0.0f);
    glVertex3f (-0.5f, -0.5f, -3.0f);

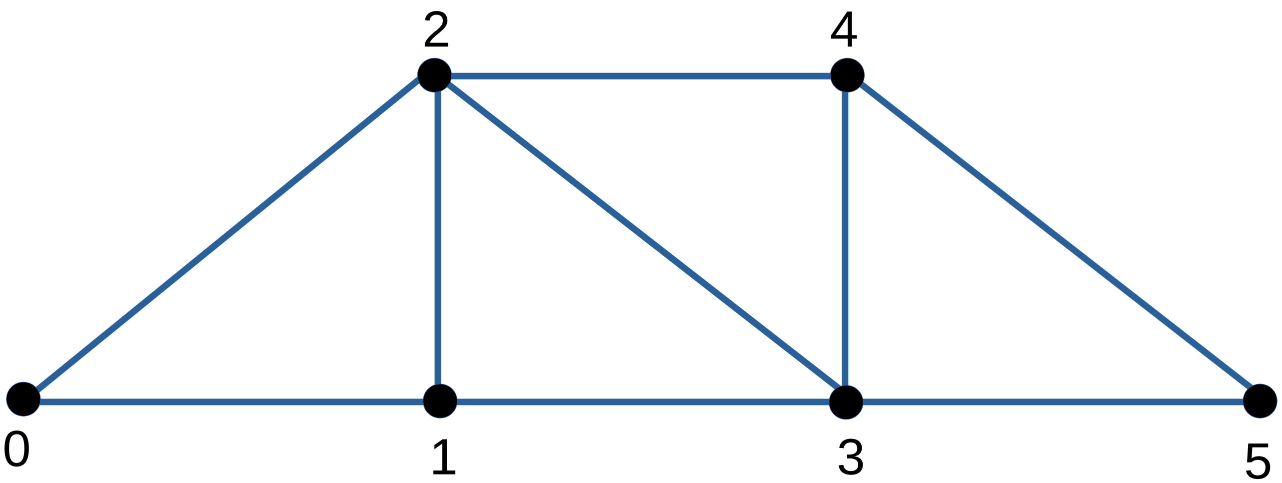
    glColor3f(1.0f, 0.0f, 1.0f);
    glVertex3f ( 0.5f, -0.5f, -3.0f);

    glColor3f(0.0f, 1.0f, 0.5f);
    glVertex3f ( 0.5f,  0.5f, -3.0f);

    glEnd();
}
```

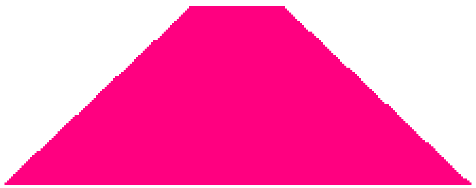


triangle strips




```
glBegin(GL_TRIANGLE_STRIP);
glColor3f(1.0f, 0.0f, 0.5f);
glVertex3f (-0.5f, -0.5f, 0.0f);
glVertex3f ( 0.5f, -0.5f, 0.0f);
glVertex3f ( 0.5f,  0.5f, 0.0f);
glVertex3f ( 1.0f, -0.5f, 0.0f);
glVertex3f ( 1.0f,  0.5f, 0.0f);
glVertex3f ( 2.0f, -0.5f, 0.0f);

glEnd();
```

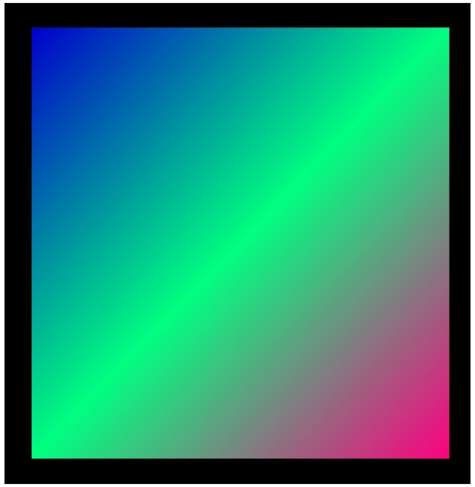


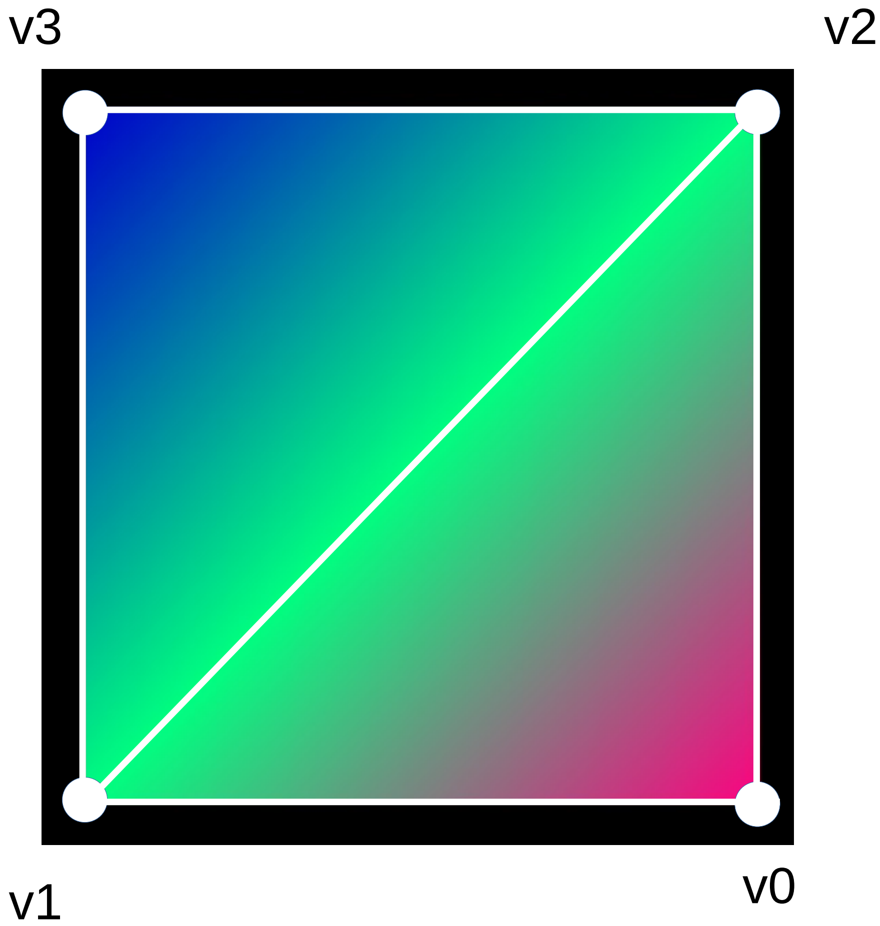
triangle strip

```
glBegin(GL_TRIANGLE_STRIP);
glColor3f(1.0f, 0.0f, 0.5f);
glVertex3f ( 0.5f, -0.5f, 0.0f);

glColor3f(0.0f, 1.0f, 0.5f);
glVertex3f (-0.5f, -0.5f, 0.0f);
glVertex3f ( 0.5f,  0.5f, 0.0f);

glColor3f(0.0f, 0.0f, 0.8f);
glVertex3f (-0.5f,  0.5f, 0.0f);
glEnd();
```

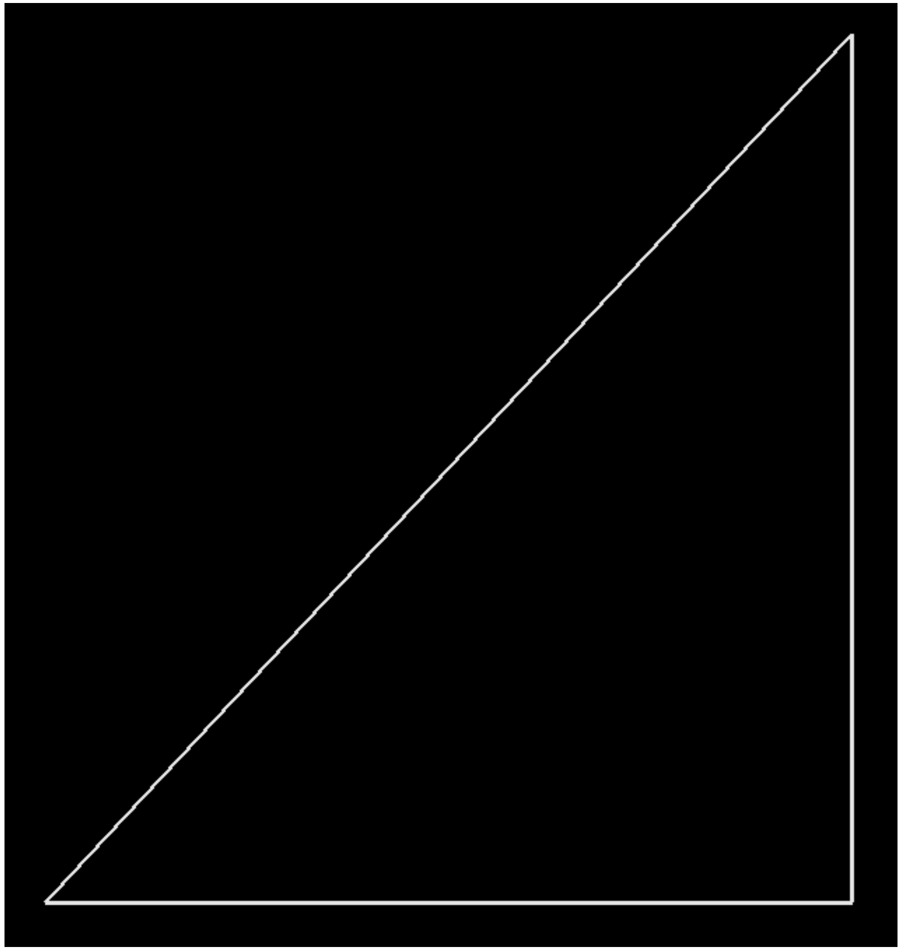




lines

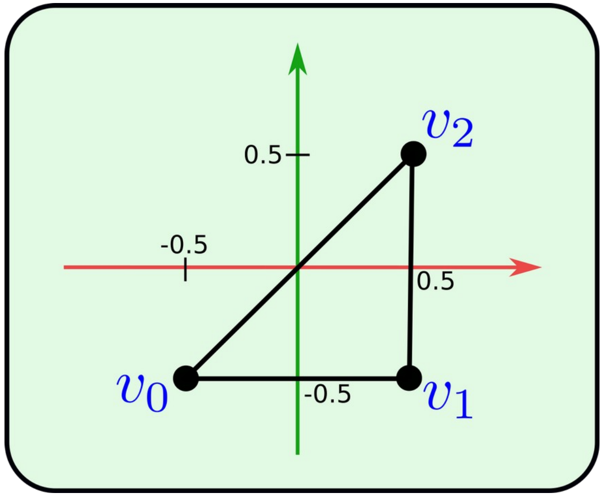
```
void drawTriangleWireframe()
{
    glLineWidth(3.0f);
    glBegin(GL_LINES);
    glColor3f(0.9f, 0.9f, 0.9f);
    glVertex3f (-0.5f, -0.5f, -3.0f);
    glVertex3f ( 0.5f, -0.5f, -3.0f);
    glVertex3f ( 0.5f, -0.5f, -3.0f);
    glVertex3f ( 0.5f,  0.5f, -3.0f);
    glVertex3f ( 0.5f,  0.5f, -3.0f);
    glVertex3f (-0.5f, -0.5f, -3.0f);
    glEnd();
}
```

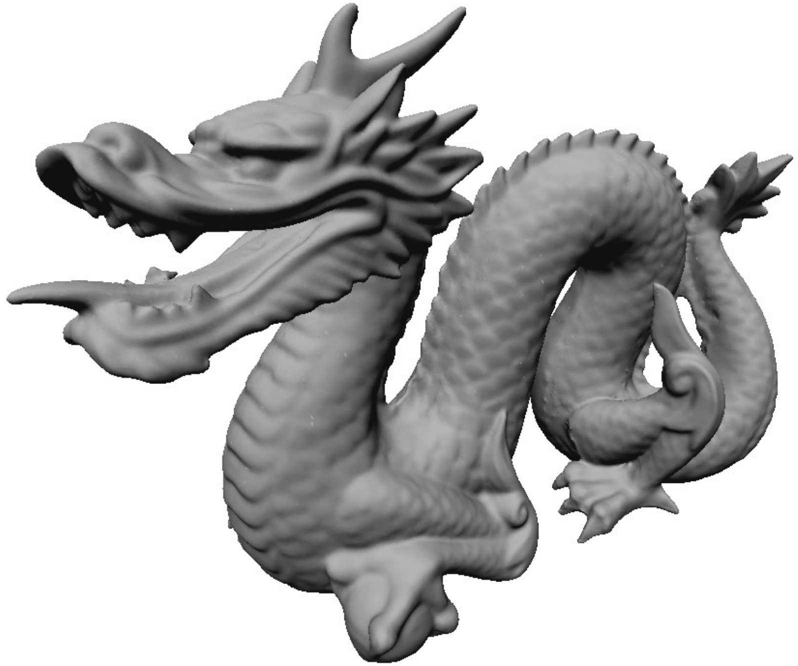
```
void drawTriangleWireframe()
{
    glLineWidth(3.0f);
    glBegin(GL_LINE_STRIP);
    glColor3f(0.9f, 0.9f, 0.9f);
    glVertex3f (-0.5f, -0.5f, -3.0f);
    glVertex3f ( 0.5f, -0.5f, -3.0f);
    glVertex3f ( 0.5f,  0.5f, -3.0f);
    glVertex3f (-0.5f, -0.5f, -3.0f);
    glEnd();
}
```



transformations

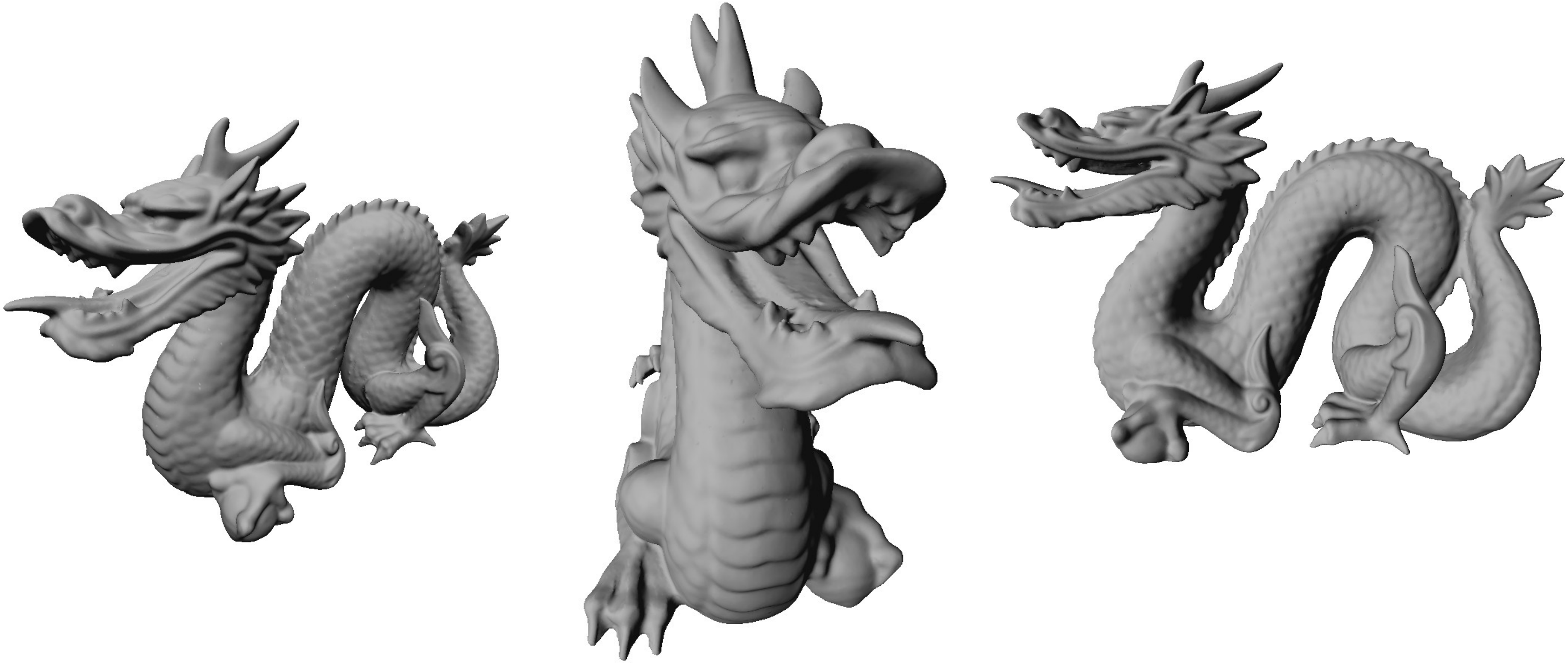
```
void drawTriangle()
{
    glBegin(GL_TRIANGLES);
    glColor3f(1.0, 0.5, 0.0);
    glVertex3f (-0.5, -0.5, -3.0);
    glVertex3f ( 0.5, -0.5, -3.0);
    glVertex3f ( 0.5,  0.5, -3.0);
    glEnd();
}
```





changing the values of
thousands of vertices is costly!!

transformations



transformations

remember the camera model?

$$\underbrace{\begin{bmatrix} k_x & 0 & 0 & x_0 \\ 0 & k_y & 0 & y_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{image} \underbrace{\begin{bmatrix} \frac{f}{\text{aspect}} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{\text{near}+\text{far}}{\text{near}-\text{far}} & \frac{2*\text{near}*\text{far}}{\text{near}-\text{far}} \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{projection} \underbrace{\begin{bmatrix} m_{00} & m_{01} & m_{02} & t_0 \\ m_{10} & m_{11} & m_{12} & t_1 \\ m_{20} & m_{21} & m_{22} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{modelview}$$

OpenGL uses the same model!

ModelView

rotation, scale, shear

translation

$$\begin{bmatrix} X_{eye} \\ Y_{eye} \\ Z_{eye} \\ 1 \end{bmatrix} = \begin{bmatrix} m_{00} & m_{01} & m_{02} & t_0 \\ m_{10} & m_{11} & m_{12} & t_1 \\ m_{20} & m_{21} & m_{22} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{obj} \\ Y_{obj} \\ Z_{obj} \\ 1 \end{bmatrix}$$

eye coordinates

object coordinates

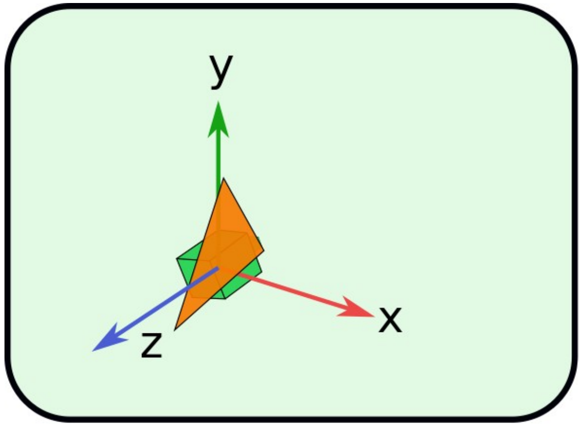
ModelView

translate vertex -3 units along the z axis?

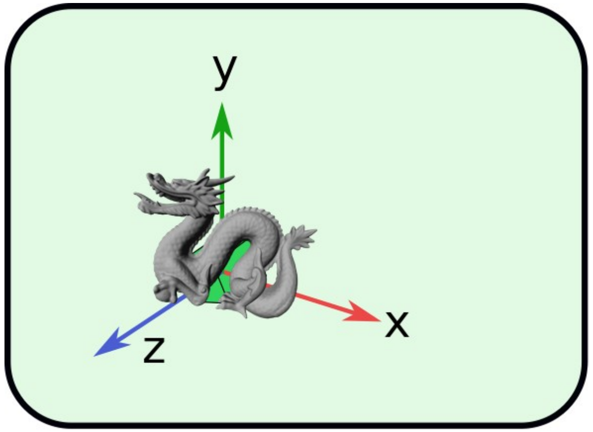
$$\begin{bmatrix} m_{00} & m_{01} & m_{02} & t_0 \\ m_{10} & m_{11} & m_{12} & t_1 \\ m_{20} & m_{21} & m_{22} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{obj} \\ Y_{obj} \\ Z_{obj} \\ 1 \end{bmatrix}$$

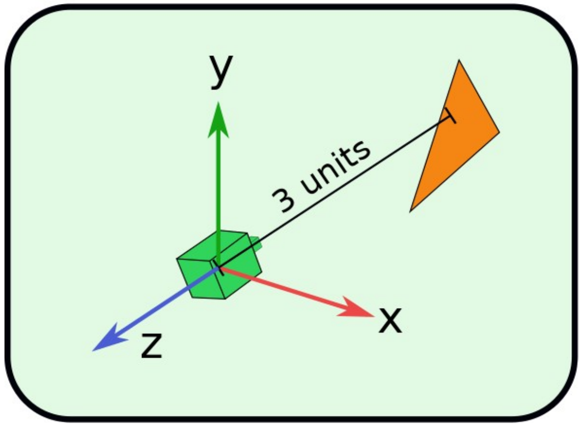
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{obj} \\ Y_{obj} \\ Z_{obj} \\ 1 \end{bmatrix}$$

ModelView

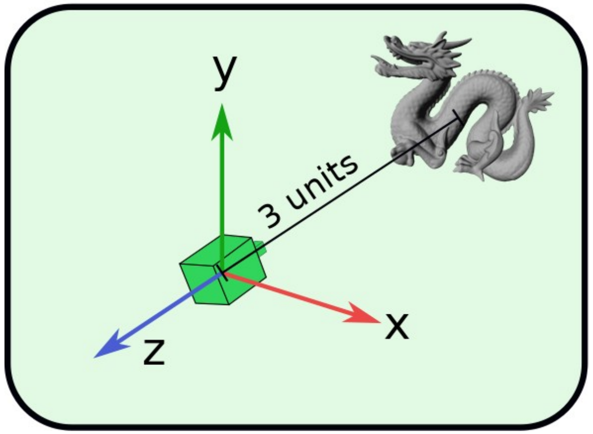


$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



ModelViewMatrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
void drawTriangle()
{
    glBegin(GL_TRIANGLES);
    glColor3f(1.0, 0.5, 0.0);
    glVertex3f (-0.5, -0.5, -3.0);
    glVertex3f ( 0.5, -0.5, -3.0);
    glVertex3f ( 0.5,  0.5, -3.0);
    glEnd();
}
```

```
void drawTriangle()
{
    glMatrixMode(GL_MODELVIEW);
    float mm [16] = {1, 0, 0, 0,
                     0, 1, 0, 0,
                     0, 0, 1, 0,
                     0, 0, -3, 1 };
    glLoadMatrixf(mm);

    glBegin(GL_TRIANGLES);
    glColor3f(1.0, 0.5, 0.0);
    glVertex3f (-0.5, -0.5, 0.0);
    glVertex3f ( 0.5, -0.5, 0.0);
    glVertex3f ( 0.5,  0.5, 0.0);
    glEnd();
}
```

why is the matrix transposed?

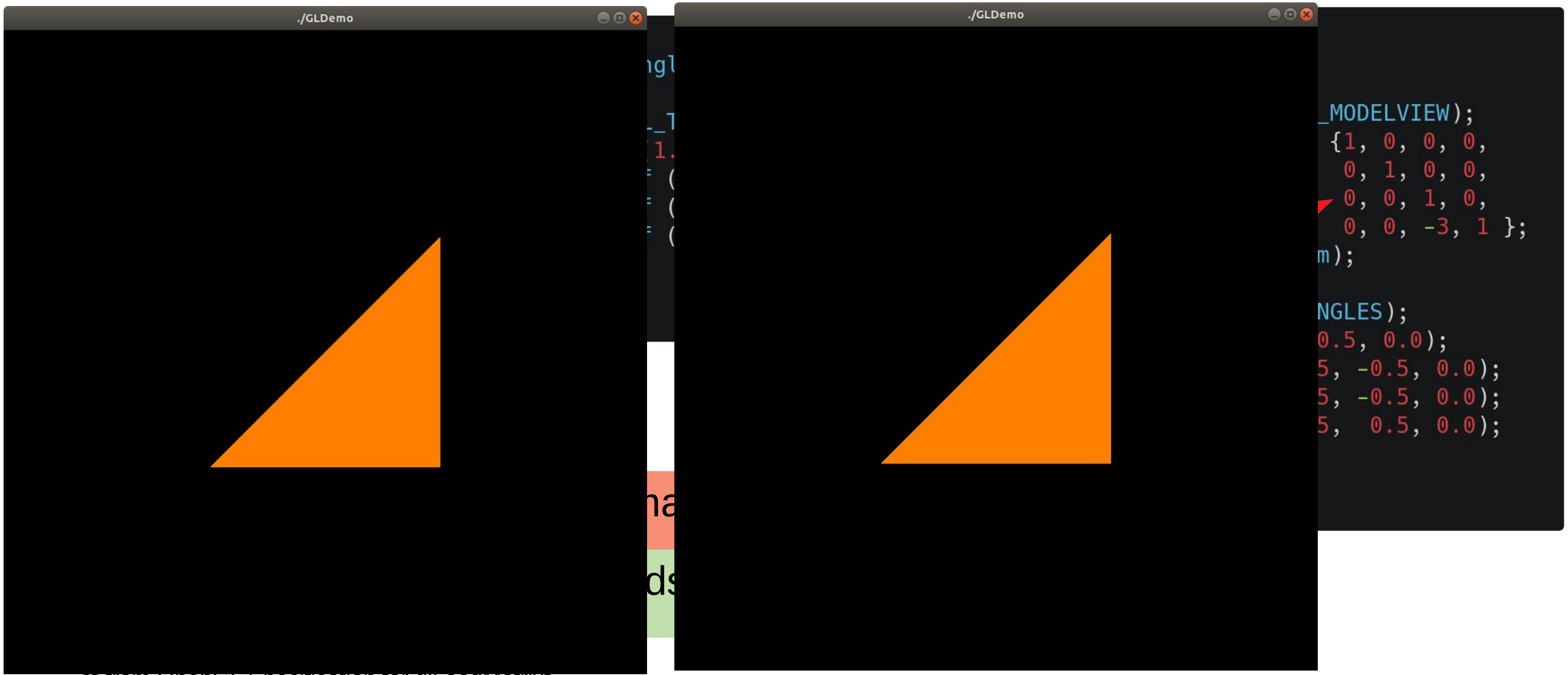
OpenGL reads in column order!

** from OpenGL Specification for glLoadMatrixf

Specifies a pointer to 16 consecutive values, which are used as the elements of a 4 × 4 column-major matrix.

ModelViewMatrix

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



** from OpenGL Specification for glLoadMatrixf

Specifies a pointer to 16 consecutive values, which are used as the elements of a 4 × 4 column-major matrix.

OpenGL API

glMatrixMode (mode)
mode = GL_MODELVIEW, GL_PROJECTION ...

State Machine
any operation after glMatrixMode affects only the current *mode*

glMultMatrix (*m)

glLoadMatrix (*m)

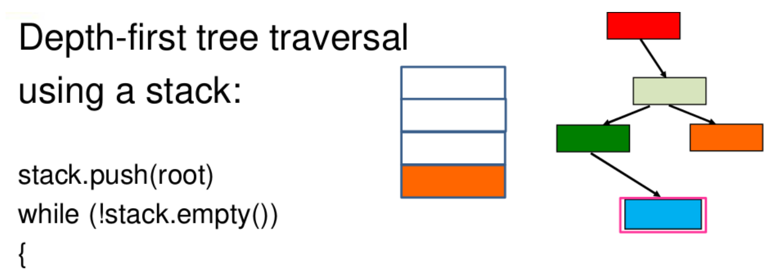
glLoadIdentity ()

matrix stack (for each mode)
save and restore matrix

glPushMatrix()

glPopMatrix()

you can also write your own stack!



ModelView

```
void drawTriangle()
{
    glMatrixMode(GL_MODELVIEW);
    float mm [16] = {1, 0, 0, 0,
                     0, 1, 0, 0,
                     0, 0, 1, 0,
                     0, 0, -3, 1 };
    glLoadMatrixf(mm);

    glBegin(GL_TRIANGLES);
    glColor3f(1.0, 0.5, 0.0);
    glVertex3f (-0.5, -0.5, 0.0);
    glVertex3f ( 0.5, -0.5, 0.0);
    glVertex3f ( 0.5,  0.5, 0.0);
    glEnd();
}
```

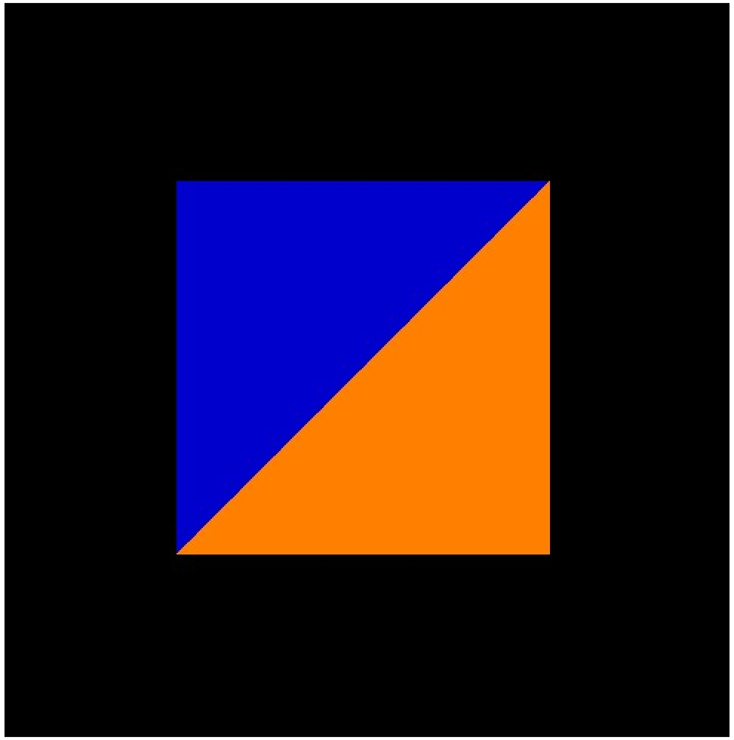
```
void drawTriangle()
{
    glMatrixMode(GL_MODELVIEW);
    glm::mat4 mm (1, 0, 0, 0,
                  0, 1, 0, 0,
                  0, 0, 1, 0,
                  0, 0, -3, 1 );
    glLoadMatrixf(glm::value_ptr(mm));

    glBegin(GL_TRIANGLES);
    glColor3f(1.0, 0.5, 0.0);
    glVertex3f (-0.5, -0.5, 0.0);
    glVertex3f ( 0.5, -0.5, 0.0);
    glVertex3f ( 0.5,  0.5, 0.0);
    glEnd();
}
```

using glm

```
void drawTriangle()
{
    glMatrixMode(GL_MODELVIEW);
    glm::mat4 mm (1, 0, 0, 0,
                     0, 1, 0, 0,
                     0, 0, 1, 0,
                     0, 0, -3, 1 );
    glMultMatrixf(glm::value_ptr(mm));

    glBegin(GL_TRIANGLES);
    glColor3f(1.0, 0.5, 0.0);
    glVertex3f (-0.5, -0.5, 0.0);
    glVertex3f ( 0.5, -0.5, 0.0);
    glVertex3f ( 0.5,  0.5, 0.0);
    glColor3f(0.0, 0.0, 0.8);
    glVertex3f (-0.5, -0.5, 0.0);
    glVertex3f ( 0.5,  0.5, 0.0);
    glVertex3f (-0.5,  0.5, 0.0);
    glEnd();
}
```



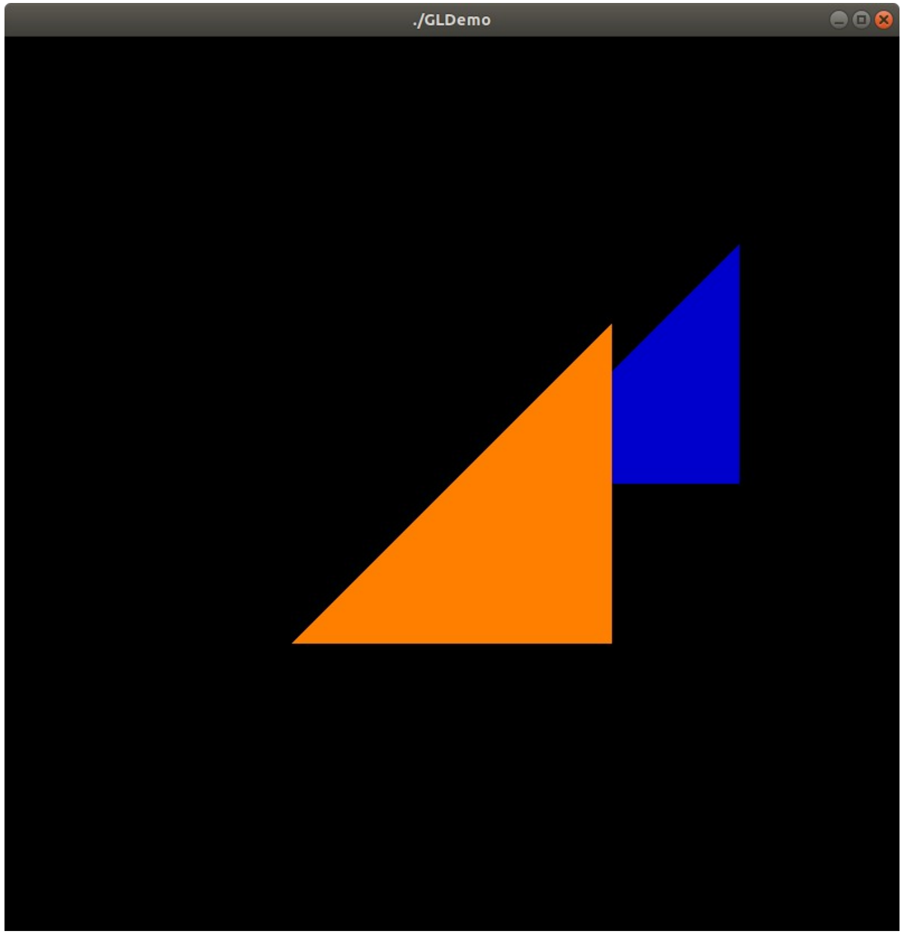
ModelView

```
void drawTriangle()
{
    glMatrixMode(GL_MODELVIEW);
    glm::mat4 mm (1, 0, 0, 0,
                  0, 1, 0, 0,
                  0, 0, 1, 0,
                  0, 0, -3, 1 );
    glLoadMatrixf(glm::value_ptr(mm));

    glBegin(GL_TRIANGLES);
    glColor3f(1.0, 0.5, 0.0);
    glVertex3f (-0.5, -0.5, 0.0);
    glVertex3f ( 0.5, -0.5, 0.0);
    glVertex3f ( 0.5,  0.5, 0.0);
    glEnd();


    mm = glm::mat4(1, 0, 0, 0,
                   0, 1, 0, 0,
                   0, 0, 1, 0,
                   0.7, 0.5, -4, 1 );
    glLoadMatrixf(glm::value_ptr(mm));

    glBegin(GL_TRIANGLES);
    glColor3f(0.0, 0.0, 0.8);
    glVertex3f (-0.5, -0.5, 0.0);
    glVertex3f ( 0.5, -0.5, 0.0);
    glVertex3f ( 0.5,  0.5, 0.0);
    glEnd();
}
```



OpenGL API

you can load your
matrices directly

glLoadMatrix (*m)

glLoadIdentity ()

glMultMatrix (*m)

or use helper functions that
act on top of stack

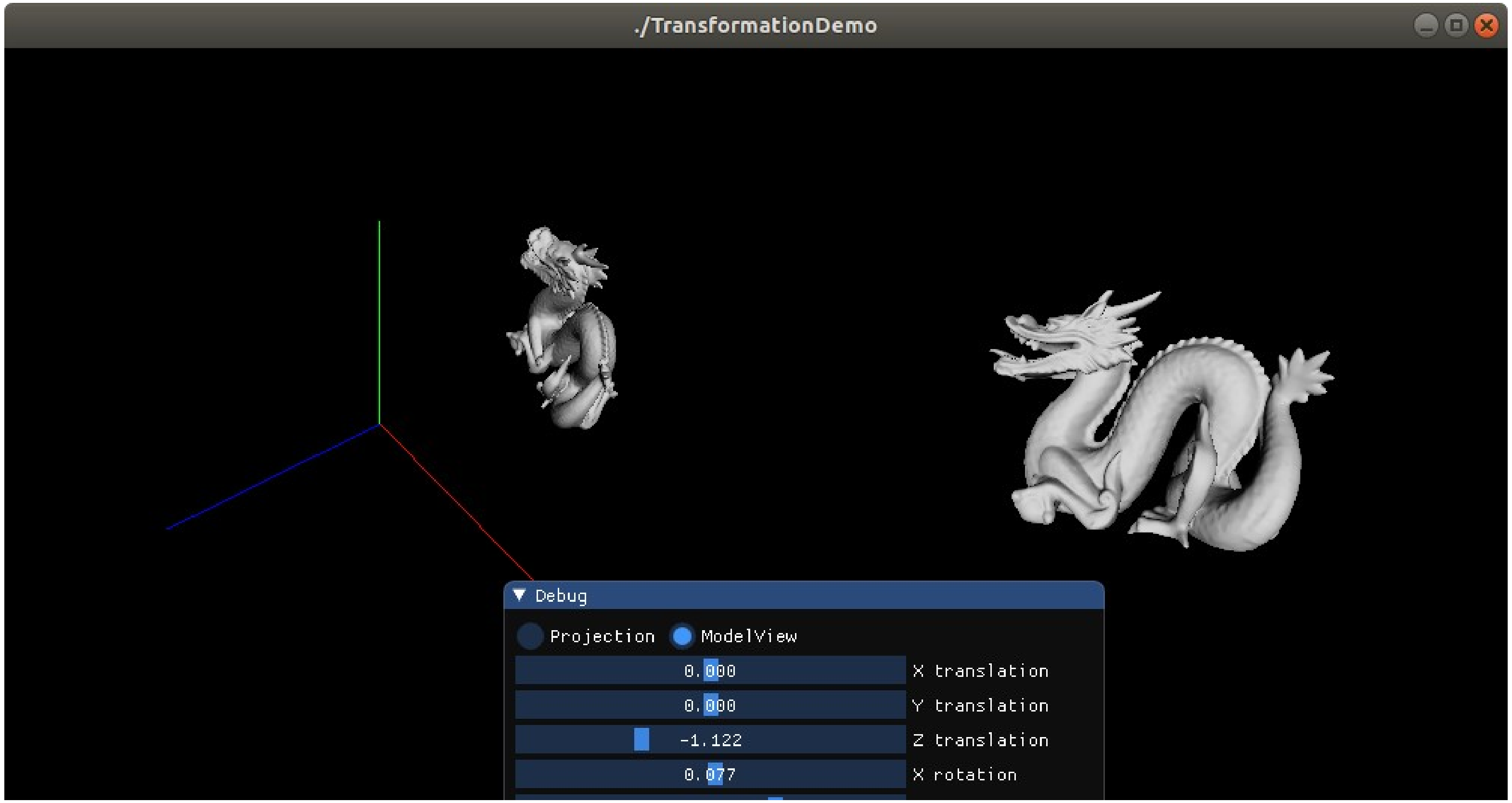
~~**glTranslate (x, y, z)**~~

~~**glRotate (angle, x, y, z)**~~

~~**glScale (x, y, z)**~~

you don’t need this, you KNOW how to create the transformation you want!

demo


./TransformationDemo
Debug
Projection
ModelView
0.000
X translation
0.000
Y translation
-1.122
Z translation
0.077
X rotation

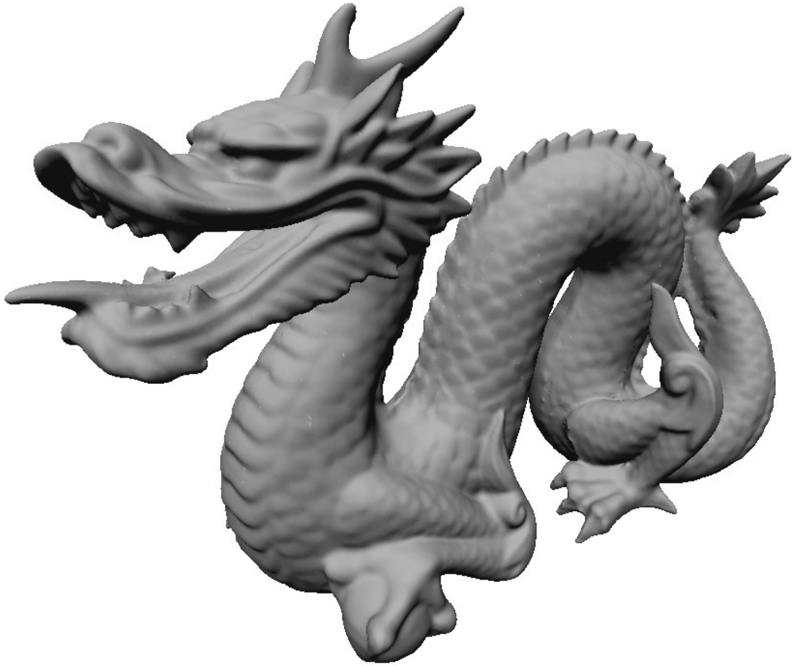
note: normals

```
void drawTriangle()
{
    glBegin(GL_TRIANGLES);
    glColor3f (1.0f, 0.5f, 0.0f);

    glNormal3f ( 0.0f, 0.0, 1.0);
    glVertex3f (-0.5f, -0.5f, -3.0f);

    glNormal3f ( 1.0f, 0.0, 0.0);
    glVertex3f ( 0.5f, -0.5f, -3.0f);

    glNormal3f ( 0.0f, 1.0, 0.0);
    glVertex3f ( 0.5f,  0.5f, -3.0f);
    glEnd();
}
```

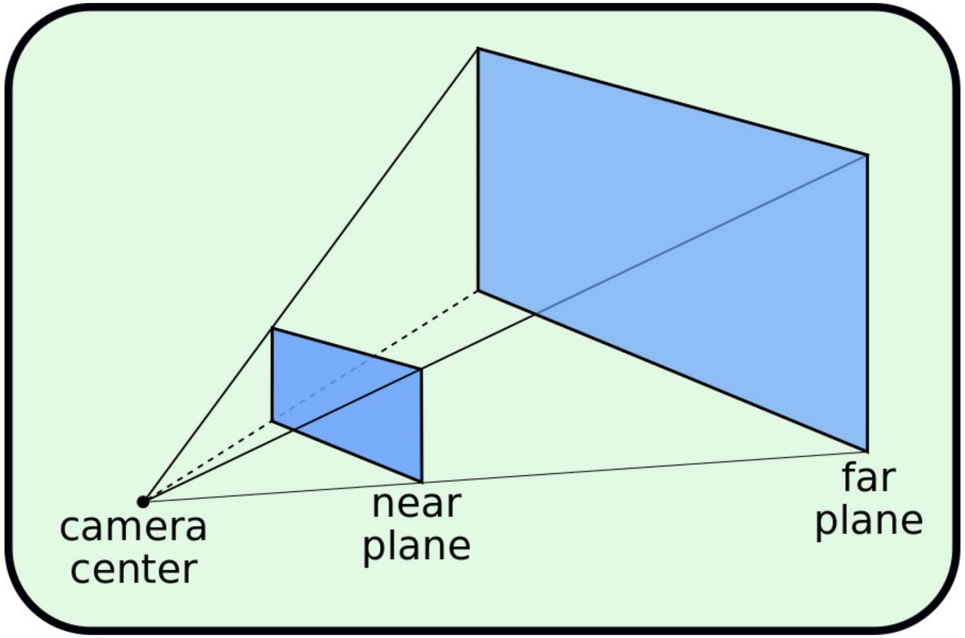


OpenGL transformations

$$\underbrace{\begin{bmatrix} k_x & 0 & 0 & x_0 \\ 0 & k_y & 0 & y_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{image}\underbrace{\begin{bmatrix} \frac{f}{\text{aspect}} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{\text{near}+\text{far}}{\text{near}-\text{far}} & \frac{2*\text{near}*\text{far}}{\text{near}-\text{far}} \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{projection}\underbrace{\begin{bmatrix} m_{00} & m_{01} & m_{02} & t_0 \\ m_{10} & m_{11} & m_{12} & t_1 \\ m_{20} & m_{21} & m_{22} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{modelview}$$

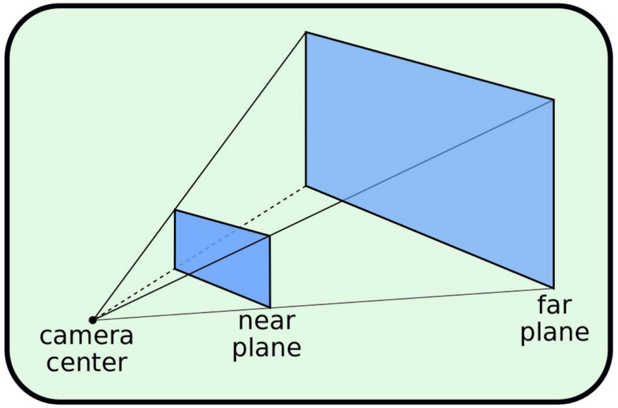
$$\begin{bmatrix} X_{eye} \\ Y_{eye} \\ Z_{eye} \\ 1 \end{bmatrix} = \begin{bmatrix} m_{00} & m_{01} & m_{02} & t_0 \\ m_{10} & m_{11} & m_{12} & t_1 \\ m_{20} & m_{21} & m_{22} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{obj} \\ Y_{obj} \\ Z_{obj} \\ 1 \end{bmatrix}$$

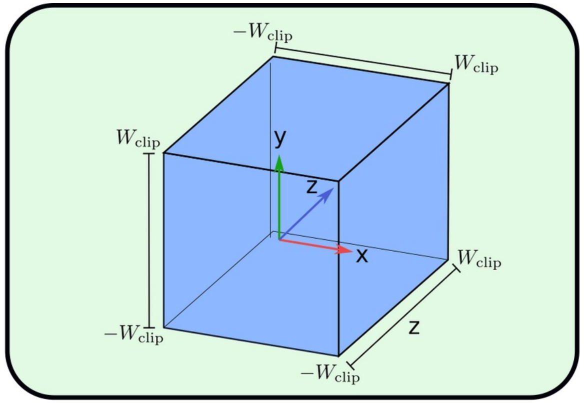
Projection



$$\begin{bmatrix} \frac{f}{\text{aspect}} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{\text{near}+\text{far}}{\text{near}-\text{far}} & \frac{2*\text{near}*\text{far}}{\text{near}-\text{far}} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Projection





we do not actually project to a plane in order to keep the z value

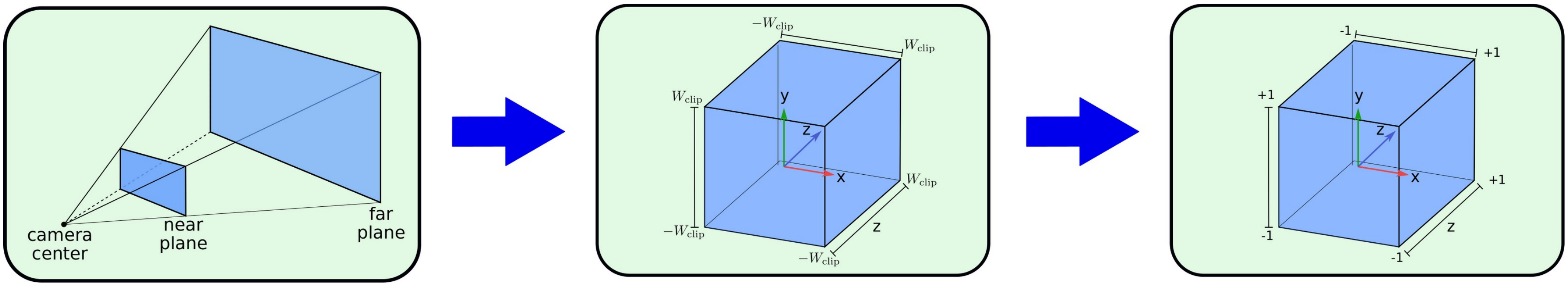
$$\begin{bmatrix} X_{clip} \\ Y_{clip} \\ Z_{clip} \\ W_{clip} \end{bmatrix} = \begin{bmatrix} \frac{f}{\text{aspect}} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{\text{near}+\text{far}}{\text{near}-\text{far}} & \frac{2*\text{near}*\text{far}}{\text{near}-\text{far}} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} X_{eye} \\ Y_{eye} \\ Z_{eye} \\ 1 \end{bmatrix}$$

clip coordinates

OpenGL clip space is in range
$[-W_{\text{clip}}, +W_{\text{clip}}]$

eye coordinates

Perspective division



$$\begin{bmatrix} X_{\text{clip}} \\ Y_{\text{clip}} \\ Z_{\text{clip}} \\ W_{\text{clip}} \end{bmatrix} = \begin{bmatrix} \frac{f}{\text{aspect}} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{\text{near}+\text{far}}{\text{near}-\text{far}} & \frac{2*\text{near}*\text{far}}{\text{near}-\text{far}} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} X_{\text{eye}} \\ Y_{\text{eye}} \\ Z_{\text{eye}} \\ 1 \end{bmatrix}$$

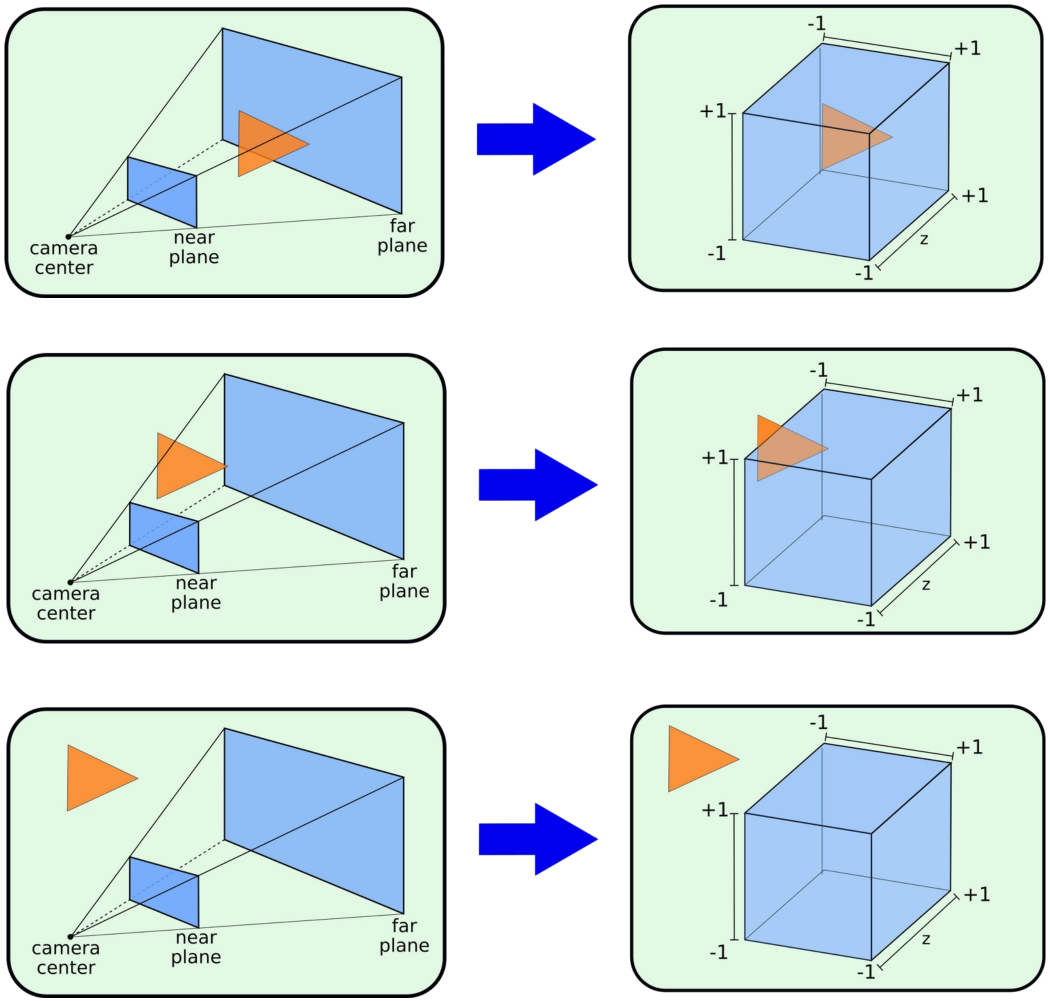
$$\begin{bmatrix} X_{\text{ndc}} \\ Y_{\text{ndc}} \\ Z_{\text{ndc}} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{X_{\text{clip}}}{W_{\text{clip}}} \\ \frac{Y_{\text{clip}}}{W_{\text{clip}}} \\ \frac{Z_{\text{clip}}}{W_{\text{clip}}} \\ \frac{W_{\text{clip}}}{W_{\text{clip}}} \end{bmatrix}$$

clip coordinates

eye coordinates

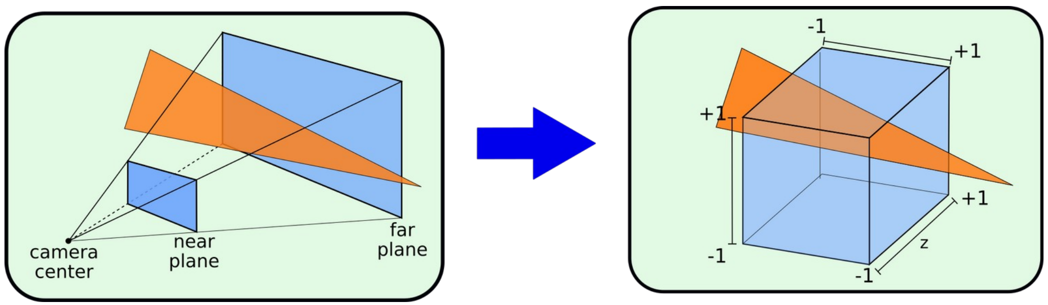
normalized device coordinates

Clipping

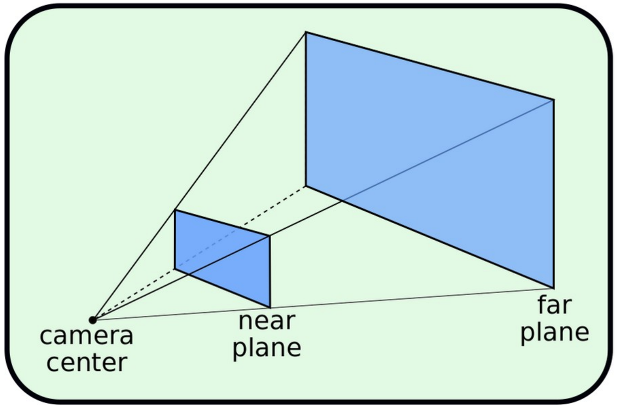


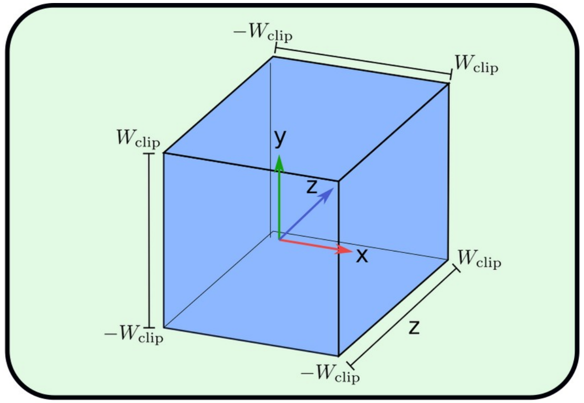
can we discard triangles with all vertices outside the cube?

no!



Projection





Question:
OpenGL performs clipping in Clip Space, not DNC.
Why??

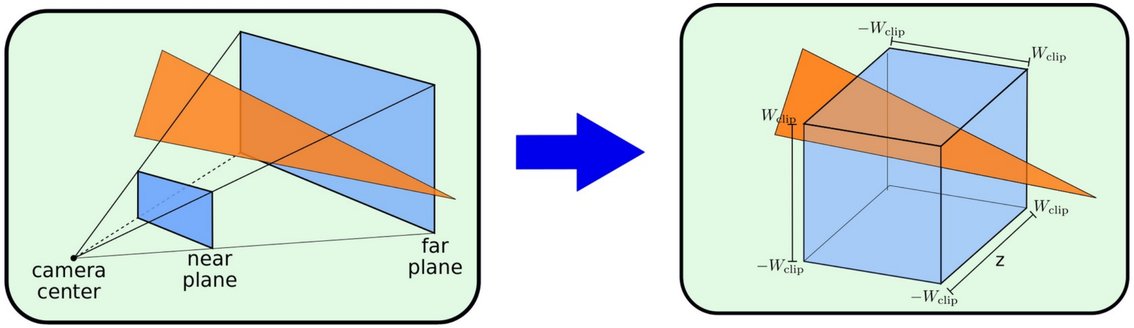
$$\begin{bmatrix} X_{clip} \\ Y_{clip} \\ Z_{clip} \\ W_{clip} \end{bmatrix} = \begin{bmatrix} \frac{f}{\text{aspect}} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{\text{near}+\text{far}}{\text{near}-\text{far}} & \frac{2*\text{near}*\text{far}}{\text{near}-\text{far}} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} X_{eye} \\ Y_{eye} \\ Z_{eye} \\ 1 \end{bmatrix}$$

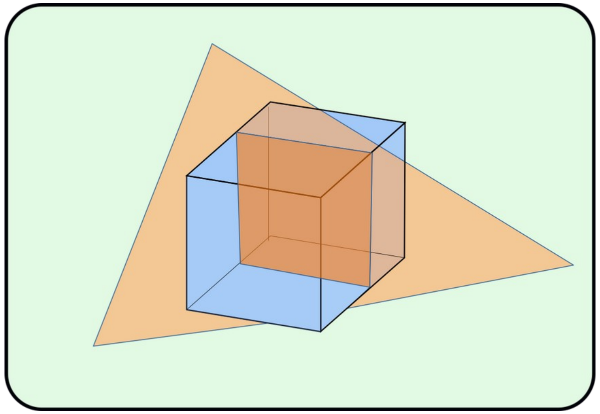
OpenGL clips primitives using the range $[-W_{\text{clip}}, +W_{\text{clip}}]$

clip coordinates

eye coordinates

question





1) with the knowledge so far, how would you implement a function to test if a triangle needs to be clipped?
2) where would you clip the triangle?

obs1: work in normalized device coordinates.
obs2: It does not have to be efficient
obs3: think about the first lecture

bonus case: can your method detect a triangle that intersects the cube, but no triangle edge intersects the cube?
Would you need another test for this case?

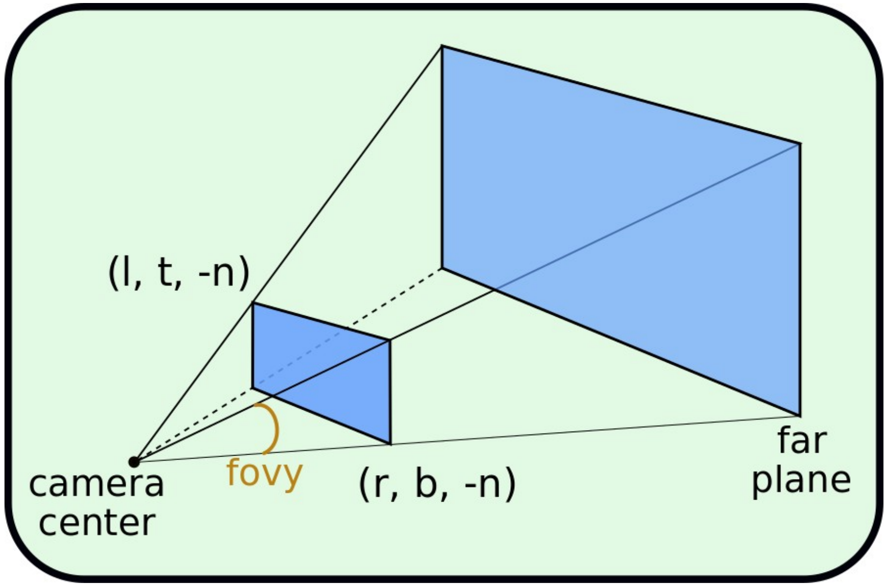
OpenGL API

glFrustum (left, right, bottom, top, near , far)

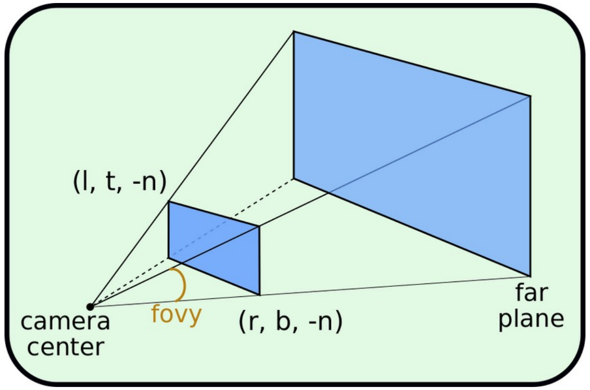
gluPerspective (fovy, aspect, near , far)

glm::perspective (fovy, aspect, near , far)

$$\begin{bmatrix} \frac{f}{\text{aspect}} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{\text{near}+\text{far}}{\text{near}-\text{far}} & \frac{2*\text{near}*\text{far}}{\text{near}-\text{far}} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

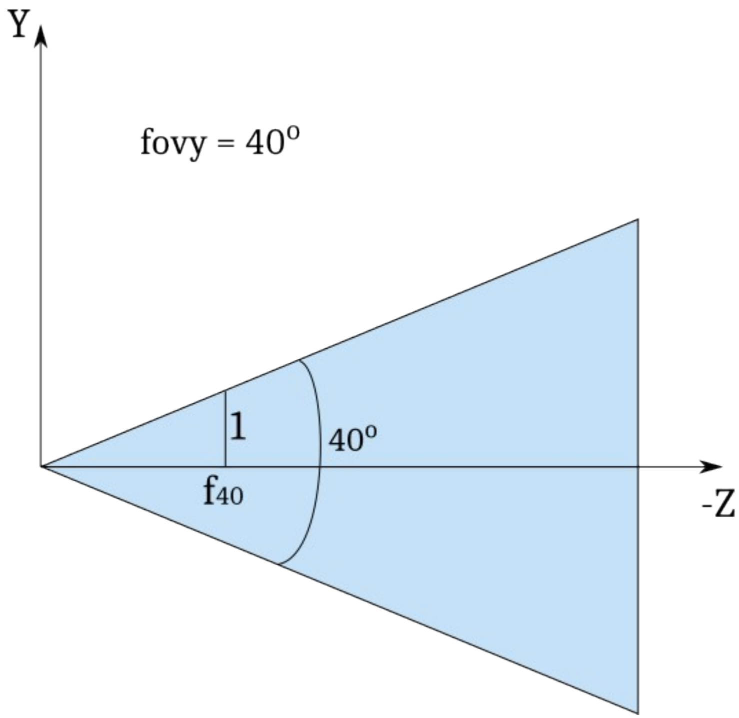


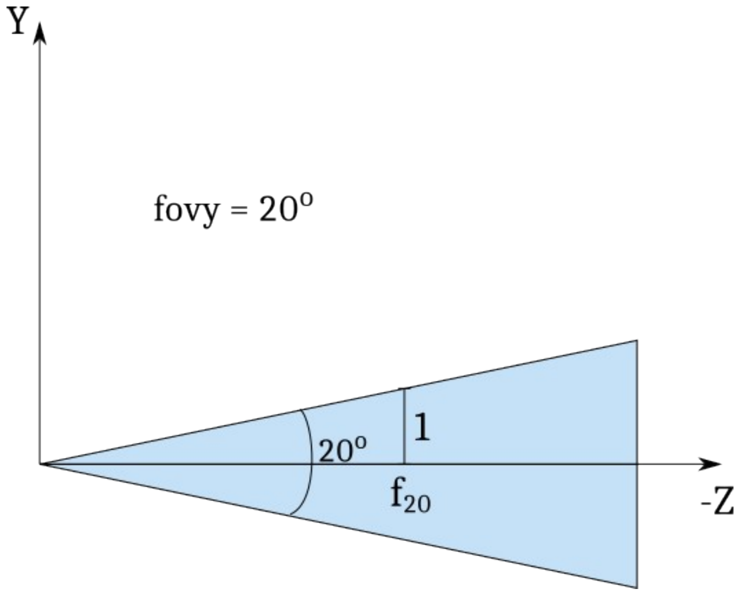
attention: specify ||near|| and ||far||

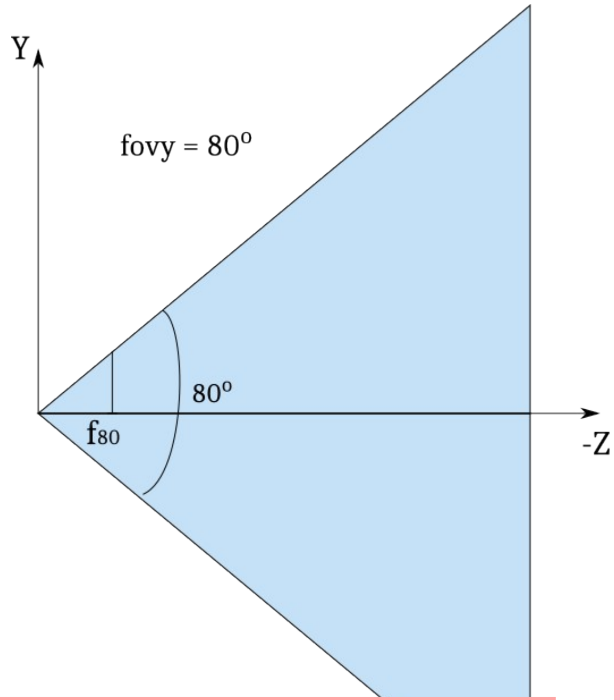


FOV

Field of View
(section 7.5)



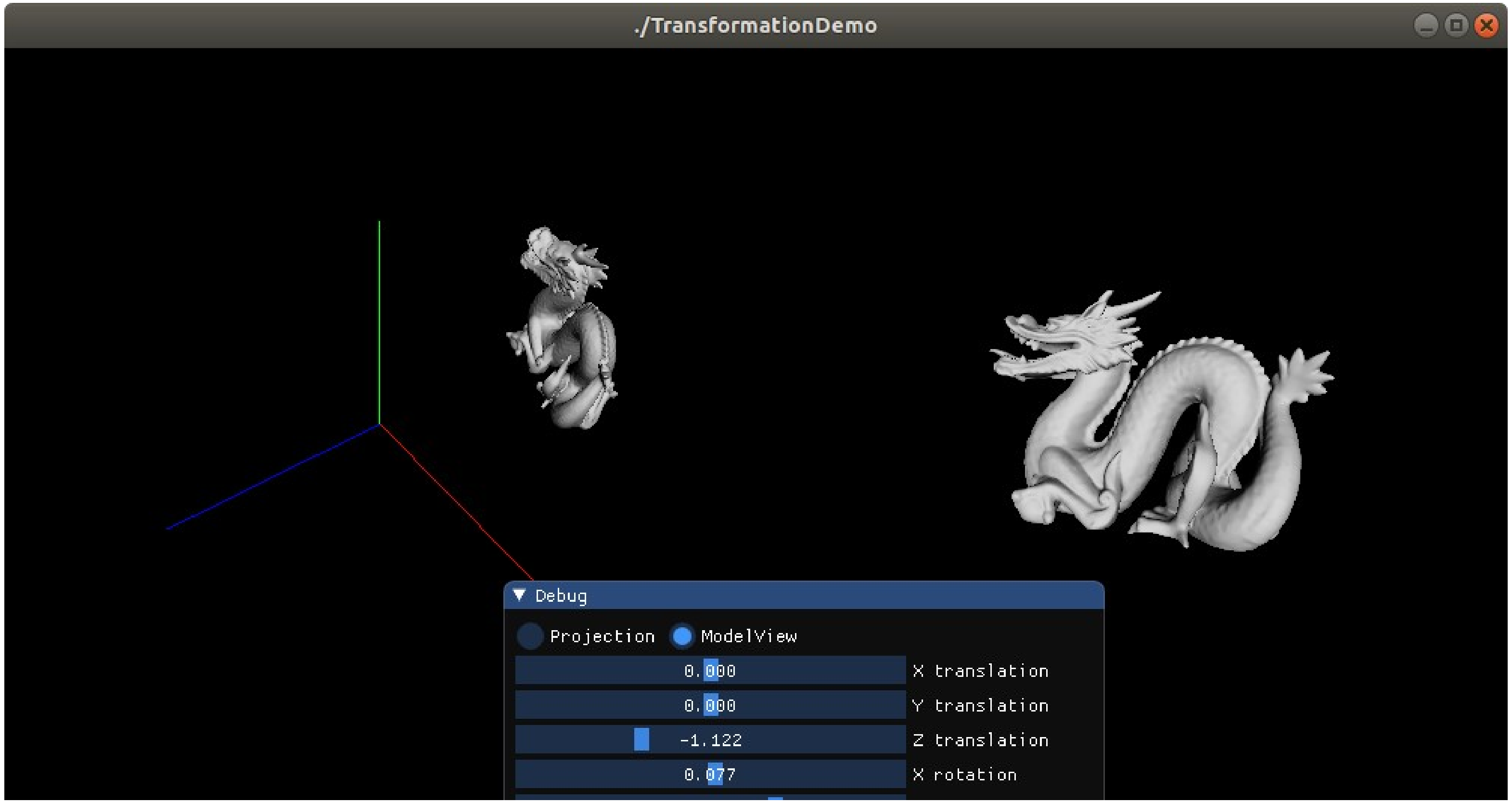




$$f_{80} < f_{40} < f_{20}$$

focal length (f) has inverse relation with field of view (fov)

demo


./TransformationDemo
Debug
Projection
ModelView
0.000
X translation
0.000
Y translation
-1.122
Z translation
0.077
X rotation

question

$$\begin{bmatrix} \frac{f}{\text{aspect}} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{\text{near}+\text{far}}{\text{near}-\text{far}} & \frac{2*\text{near}*\text{far}}{\text{near}-\text{far}} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

can we simulate a fisheye lens with this matrix? Why or why not?

OpenGL transformations

$$\underbrace{\begin{bmatrix} k_x & 0 & 0 & x_0 \\ 0 & k_y & 0 & y_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{image} \underbrace{\begin{bmatrix} \frac{f}{\text{aspect}} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{\text{near}+\text{far}}{\text{near}-\text{far}} & \frac{2*\text{near}*\text{far}}{\text{near}-\text{far}} \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{projection} \underbrace{\begin{bmatrix} m_{00} & m_{01} & m_{02} & t_0 \\ m_{10} & m_{11} & m_{12} & t_1 \\ m_{20} & m_{21} & m_{22} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{modelview}$$

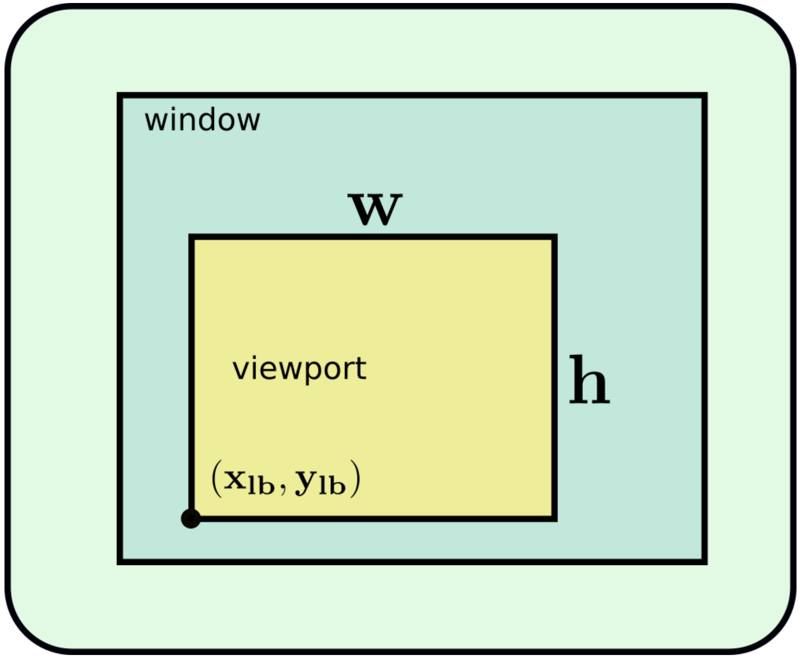
$$\begin{bmatrix} X_{eye} \\ Y_{eye} \\ Z_{eye} \\ 1 \end{bmatrix} = \begin{bmatrix} m_{00} & m_{01} & m_{02} & t_0 \\ m_{10} & m_{11} & m_{12} & t_1 \\ m_{20} & m_{21} & m_{22} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{obj} \\ Y_{obj} \\ Z_{obj} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_{\text{clip}} \\ Y_{\text{clip}} \\ Z_{\text{clip}} \\ W_{\text{clip}} \end{bmatrix} = \begin{bmatrix} \frac{f}{\text{aspect}} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{\text{near}+\text{far}}{\text{near}-\text{far}} & \frac{2*\text{near}*\text{far}}{\text{near}-\text{far}} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} X_{\text{eye}} \\ Y_{\text{eye}} \\ Z_{\text{eye}} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_{\text{ndc}} \\ Y_{\text{ndc}} \\ Z_{\text{ndc}} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{X_{\text{clip}}}{W_{\text{clip}}} \\ \frac{Y_{\text{clip}}}{W_{\text{clip}}} \\ \frac{Z_{\text{clip}}}{W_{\text{clip}}} \\ \frac{W_{\text{clip}}}{W_{\text{clip}}} \end{bmatrix}$$

object coordinates → ModelView → eye coordinates → Projection → clip coordinates → Perspective division → NDC

Viewport



defines part of the window to use
(for most applications, use entire window)

windows position (pixels)

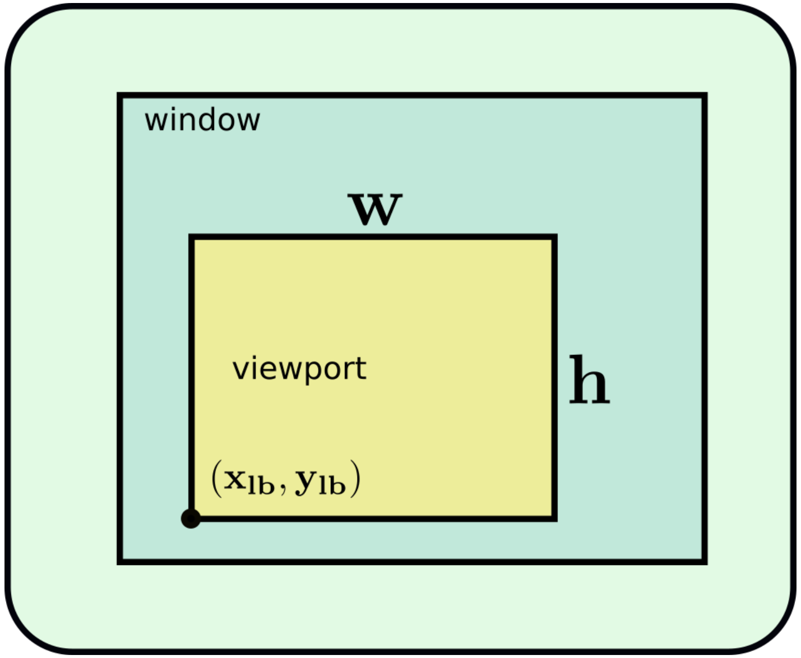
$$X_{\text{window}} = \left(\frac{X_{\text{ndc}}+1}{2}\right)\mathbf{w} + \mathbf{x_{lb}}$$

$$Y_{\text{window}} = \left(\frac{Y_{\text{ndc}}+1}{2}\right)\mathbf{h} + \mathbf{y_{lb}}$$

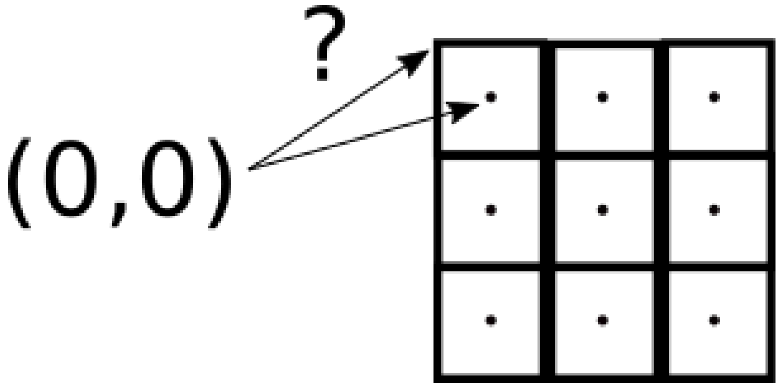
$$\mathbf{x_{lb}} = 0, \mathbf{y_{lb}} = 0$$

$$X_{\text{ndc}} = +1 \rightarrow X_{\text{window}} = ?$$

Viewport



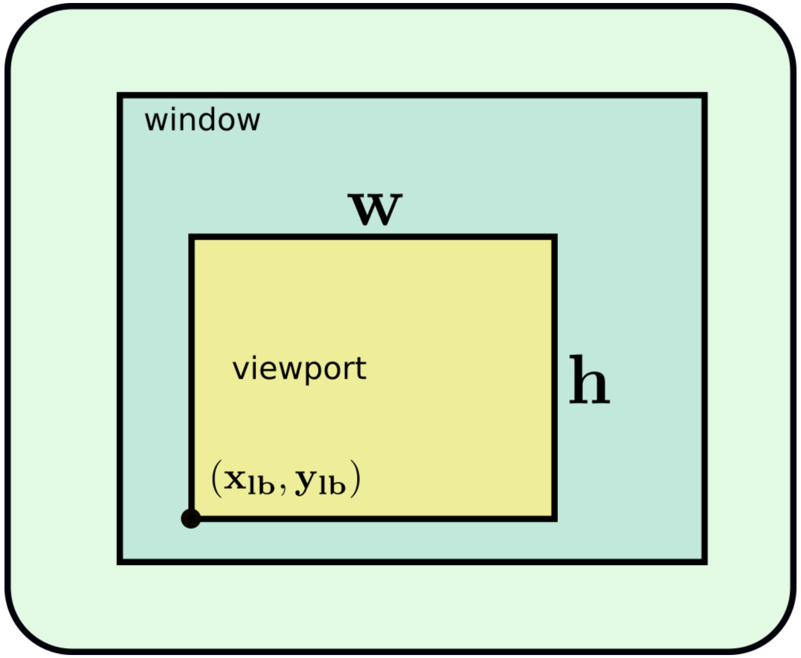
defines part of the window to use
(for most applications, use entire window)



- if top-left corner is (0,0) the range is [0,w]
- if pixel center is (0,0) the range is [-0.5,w-0.5]

note: different implementation choices (here vs book)

Viewport

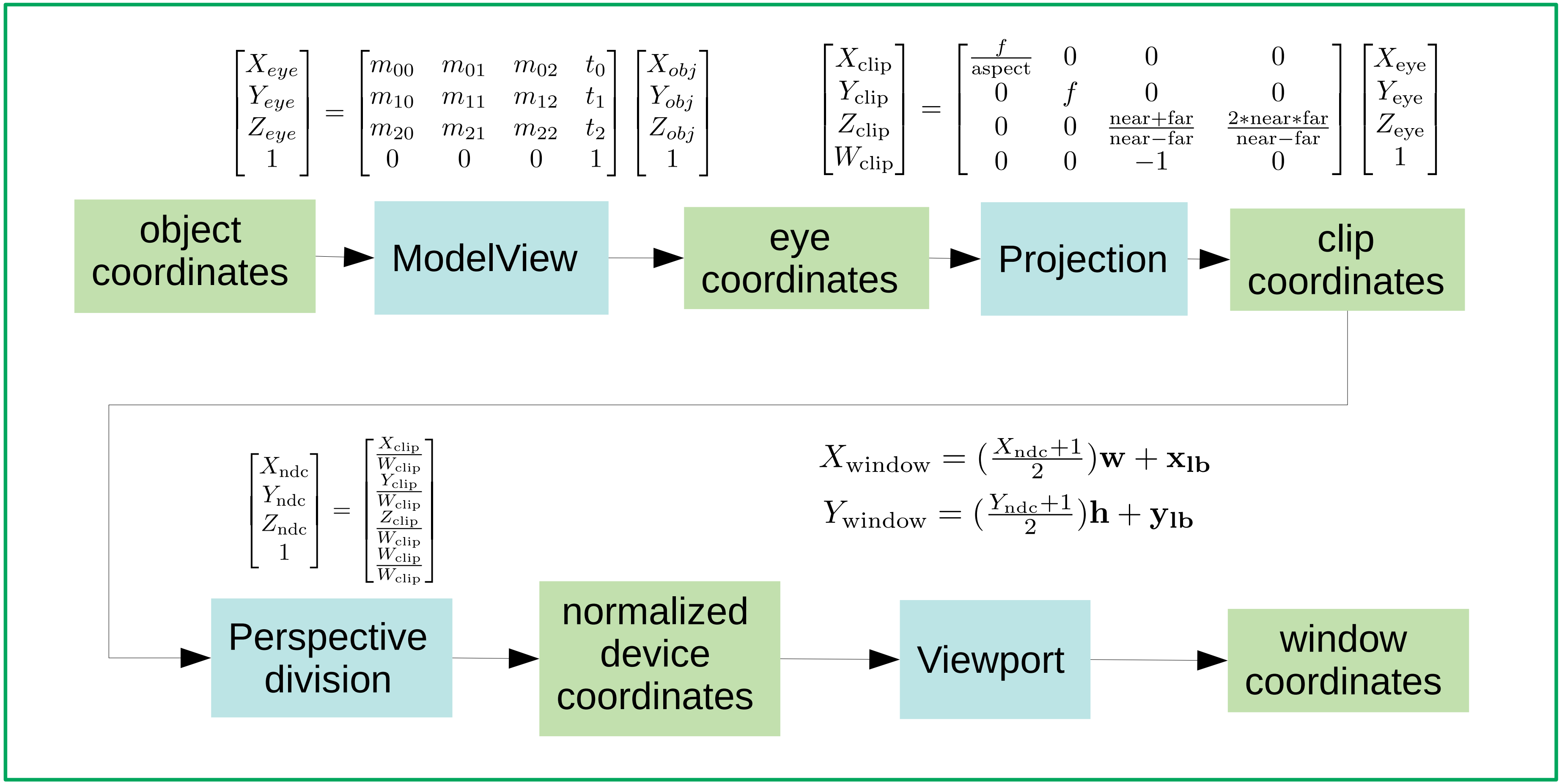


defines part of the window to use
(for most applications, use entire window)

void glViewport(x, y, width, height);

Defines lower-left corner and size in pixels

OpenGL transformations



$$\begin{bmatrix} X_{eye} \\ Y_{eye} \\ Z_{eye} \\ 1 \end{bmatrix} = \begin{bmatrix} m_{00} & m_{01} & m_{02} & t_0 \\ m_{10} & m_{11} & m_{12} & t_1 \\ m_{20} & m_{21} & m_{22} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{obj} \\ Y_{obj} \\ Z_{obj} \\ 1 \end{bmatrix}$$

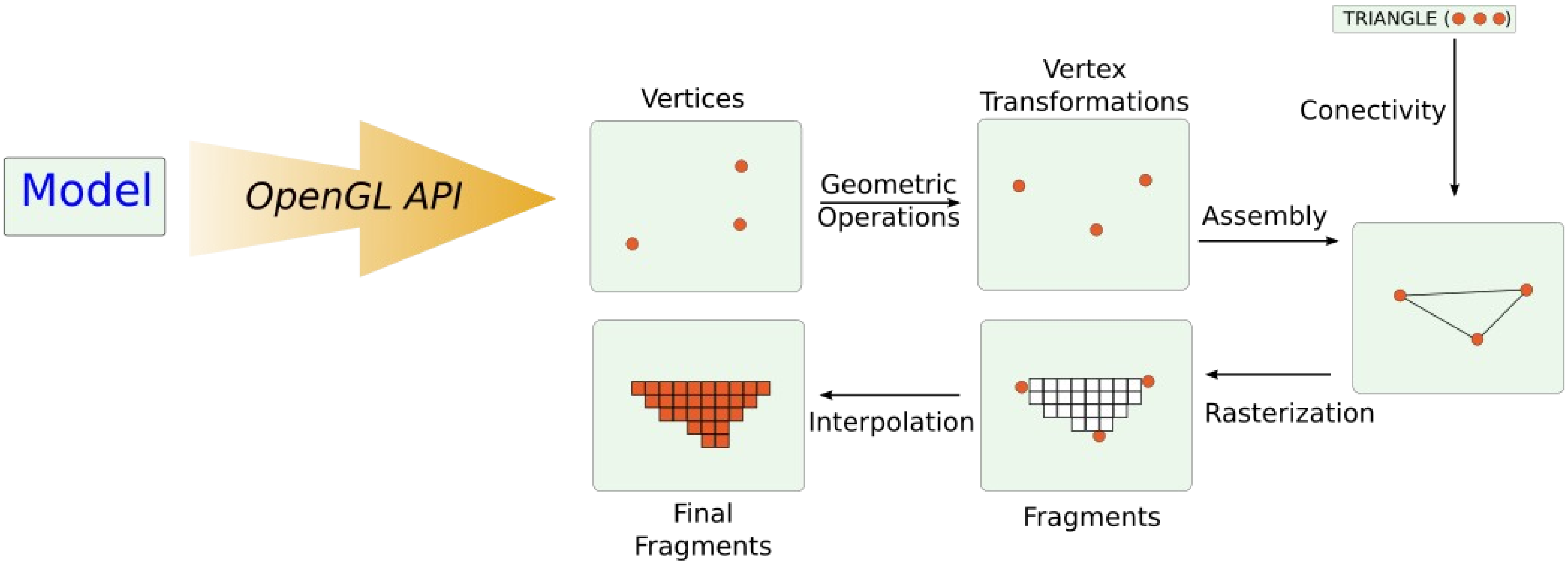
$$\begin{bmatrix} X_{\text{clip}} \\ Y_{\text{clip}} \\ Z_{\text{clip}} \\ W_{\text{clip}} \end{bmatrix} = \begin{bmatrix} \frac{f}{\text{aspect}} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{\text{near}+\text{far}}{\text{near}-\text{far}} & \frac{2*\text{near}*\text{far}}{\text{near}-\text{far}} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} X_{\text{eye}} \\ Y_{\text{eye}} \\ Z_{\text{eye}} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_{\text{ndc}} \\ Y_{\text{ndc}} \\ Z_{\text{ndc}} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{X_{\text{clip}}}{W_{\text{clip}}} \\ \frac{Y_{\text{clip}}}{W_{\text{clip}}} \\ \frac{Z_{\text{clip}}}{W_{\text{clip}}} \\ \frac{W_{\text{clip}}}{W_{\text{clip}}} \end{bmatrix}$$

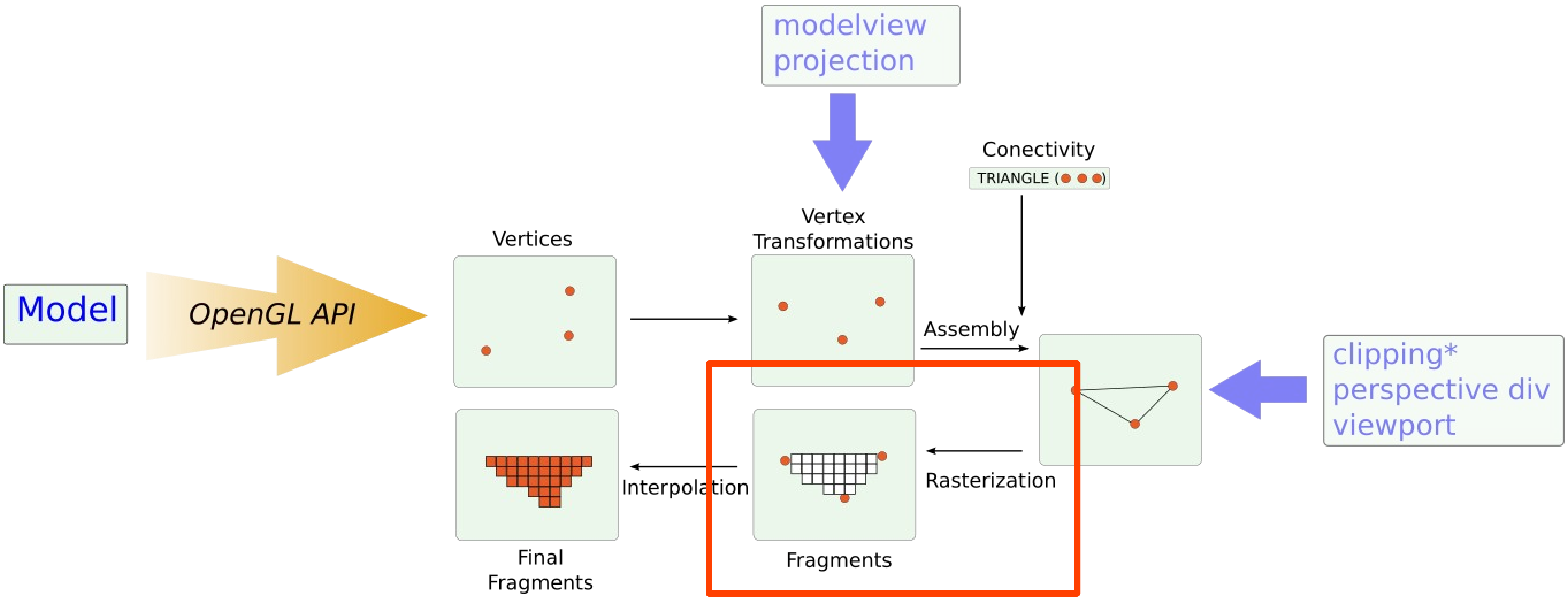
$$X_{\text{window}} = \left(\frac{X_{\text{ndc}}+1}{2}\right)\mathbf{w} + \mathbf{x_{lb}}$$

$$Y_{\text{window}} = \left(\frac{Y_{\text{ndc}}+1}{2}\right)\mathbf{h} + \mathbf{y_{lb}}$$

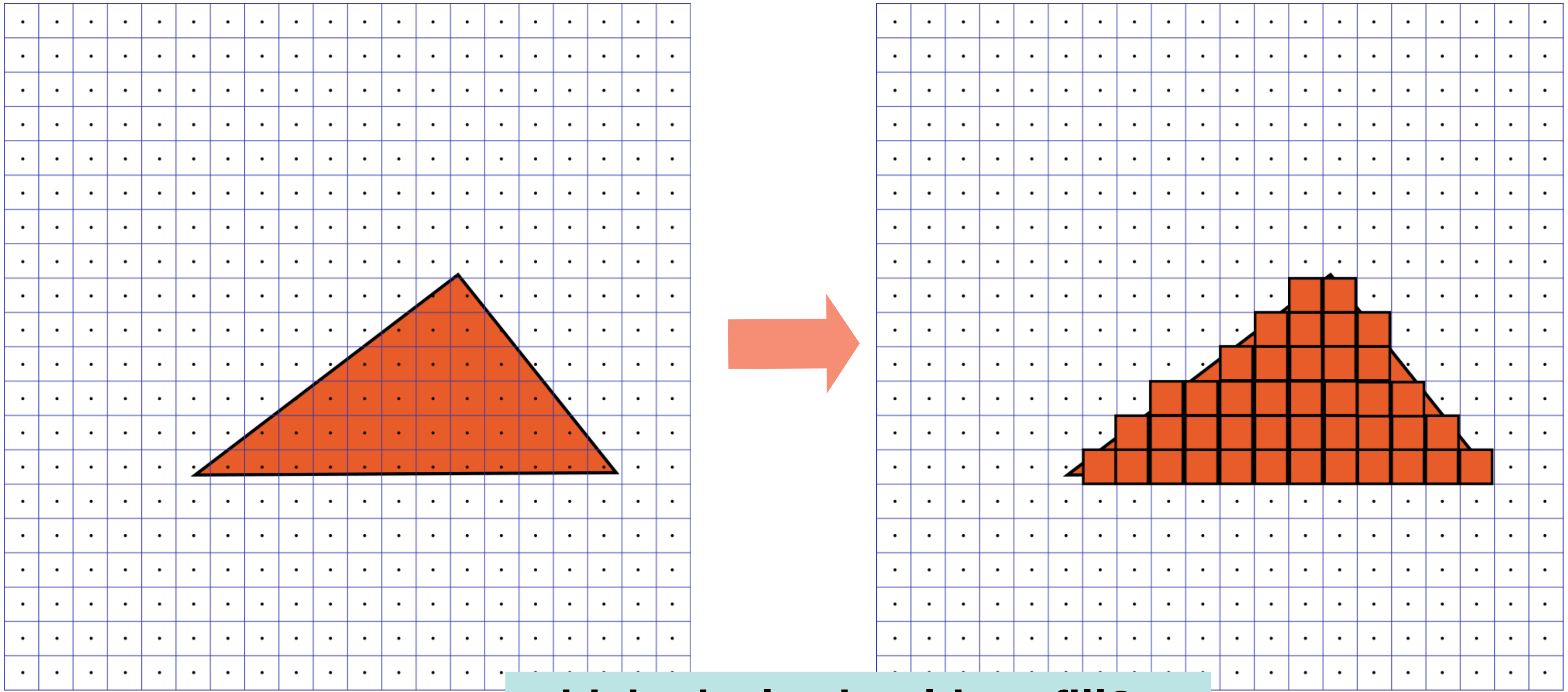
graphics pipeline



graphics pipeline: transformations



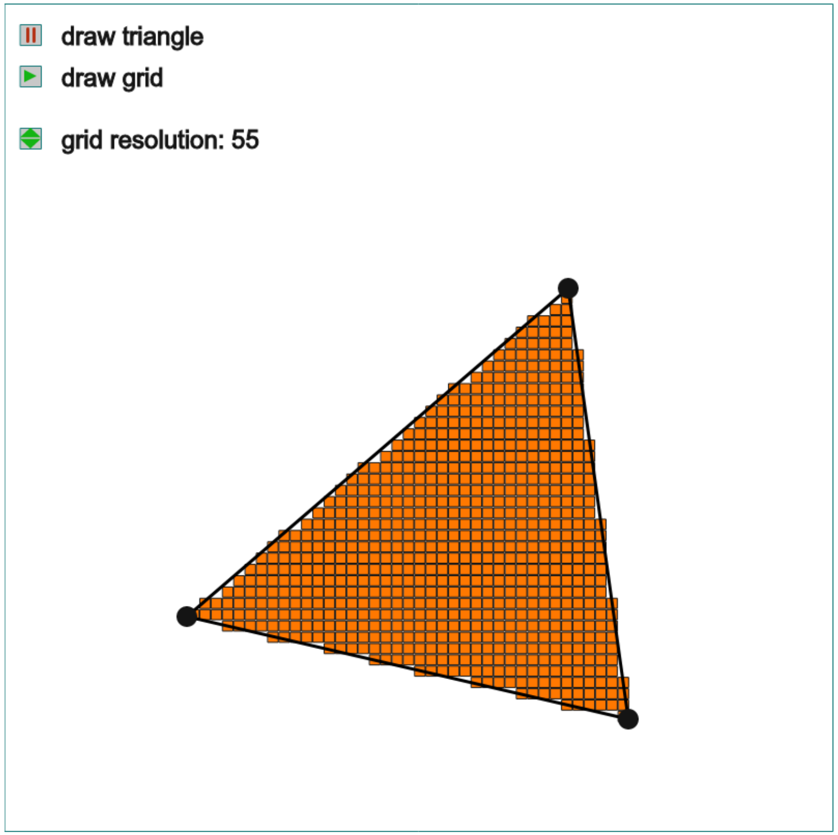
rasterization



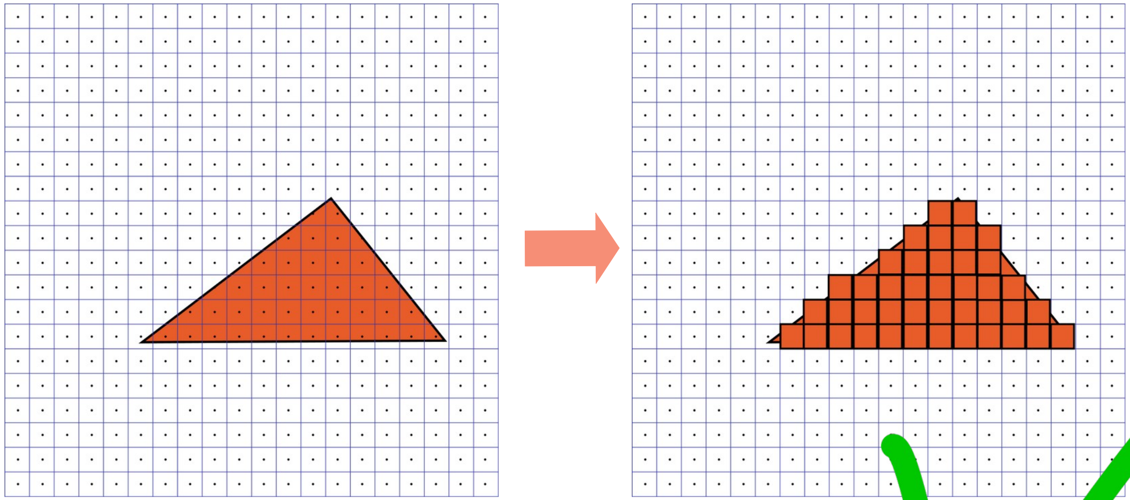
which pixels should we fill?

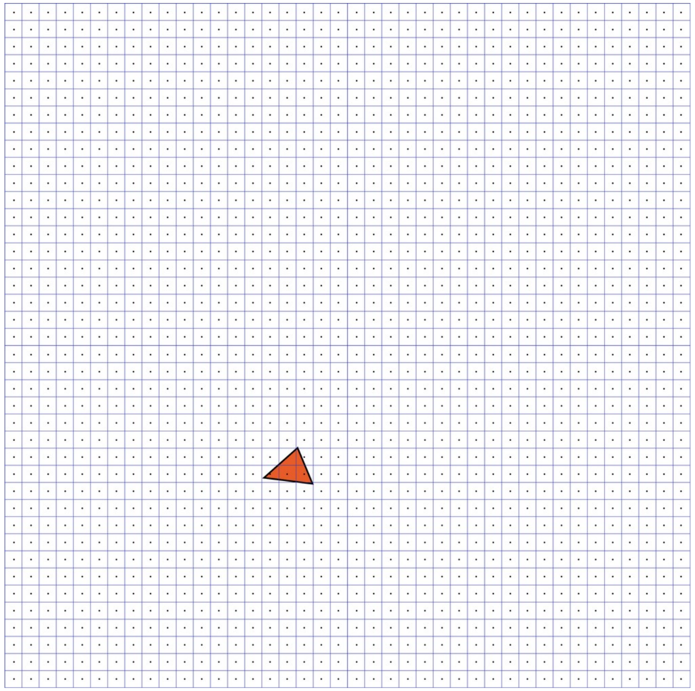
check if centre of pixel is inside triangle

demo



rasterization





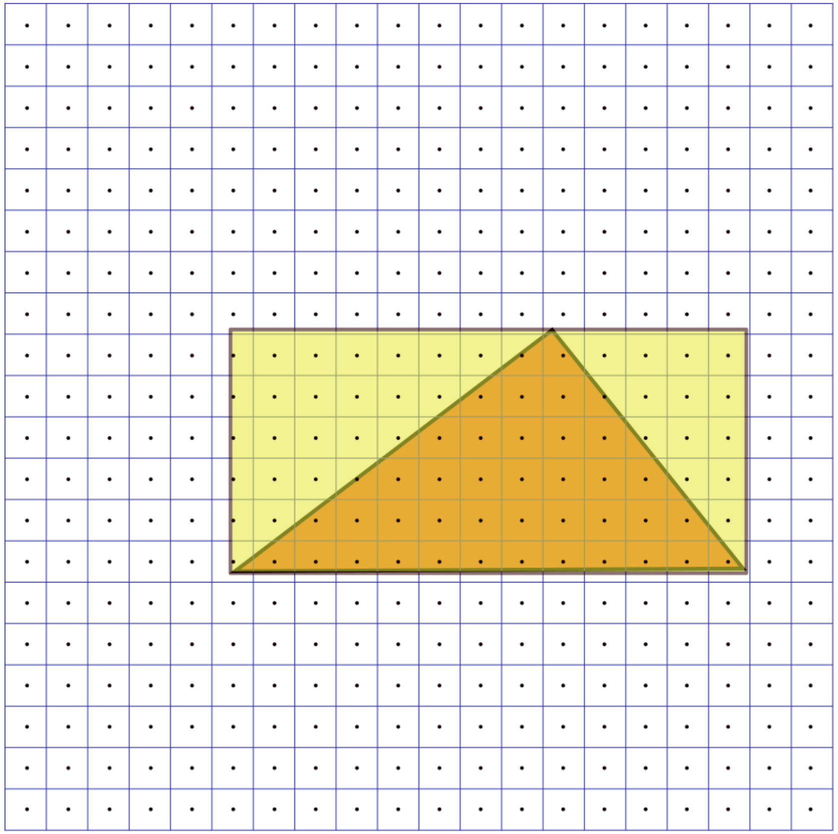
which pixels should we fill?

test if each pixel centre is inside/outside triangle

issues?

check all pixels for each triangle is not a good idea!

question

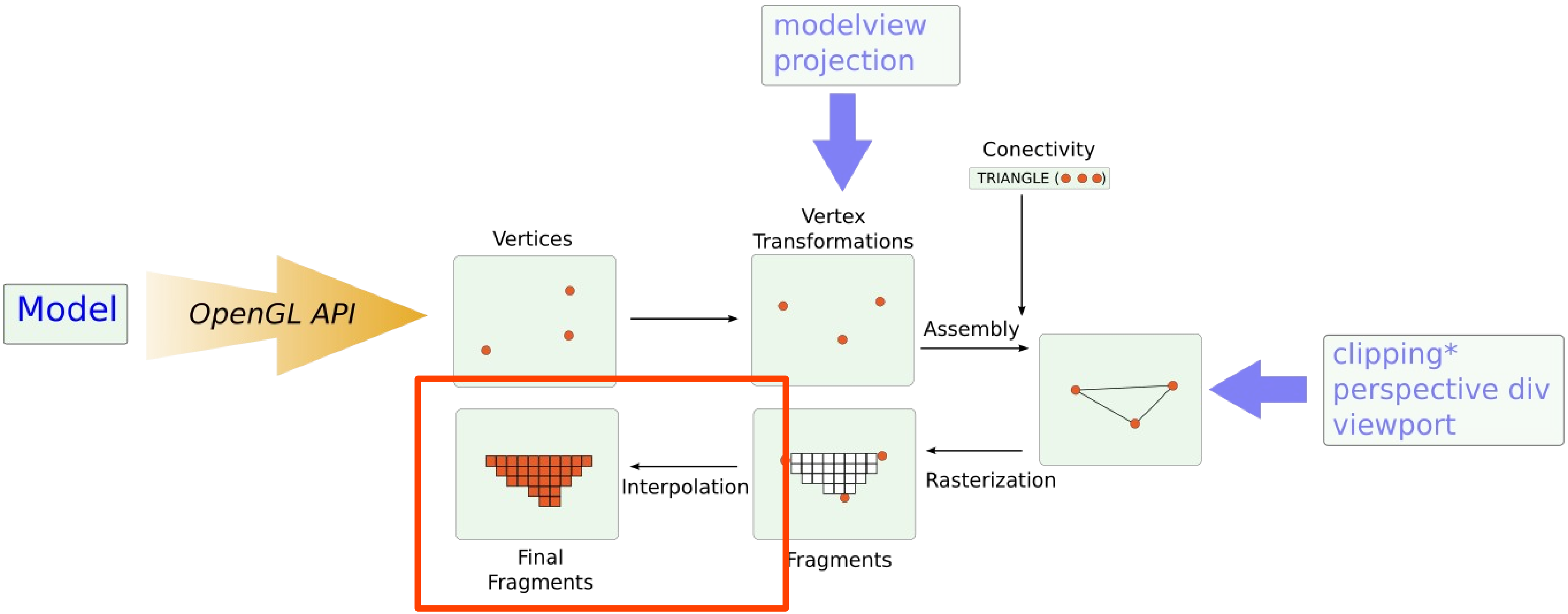


Axis-Aligned Bounding Box (AABB)

if pixel inside triangle → inside box

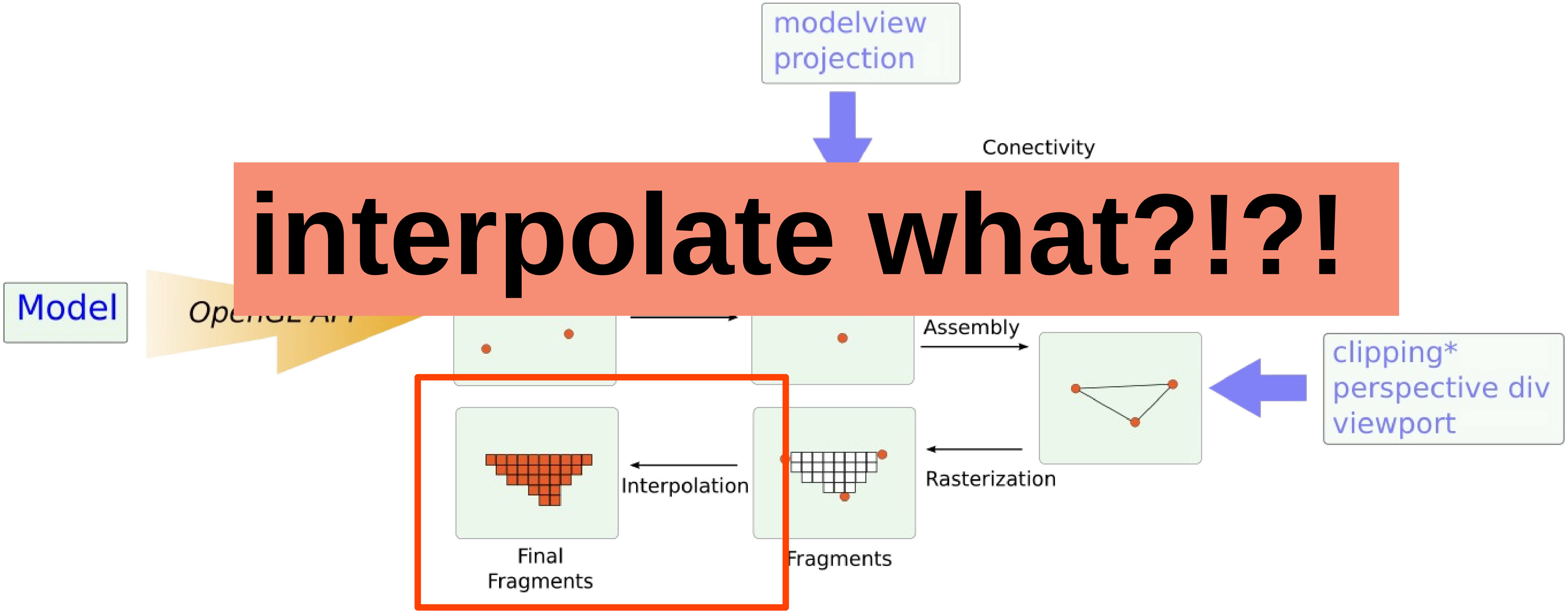
given the window coordinates (x,y) of the three vertices, how do we find the AABB?
With the AABB which pixels do we need to check (design a loop to iterate over the pixels)

graphics pipeline

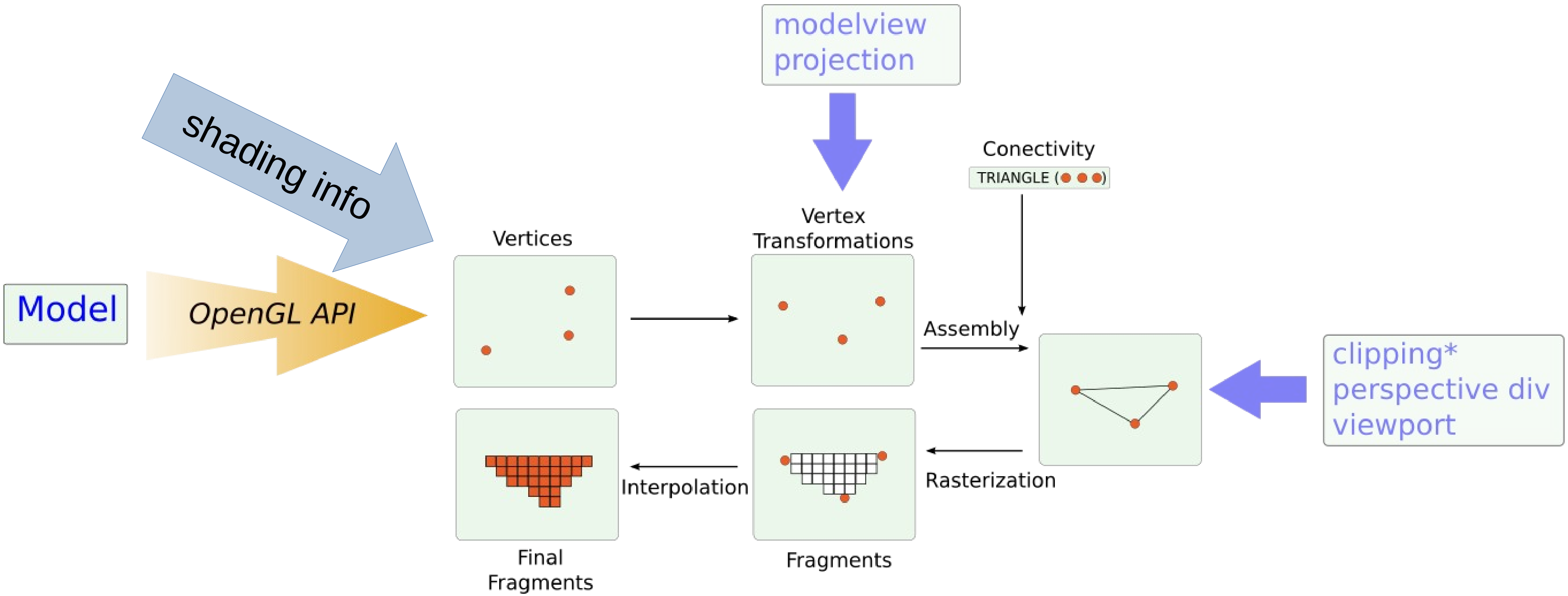


graphics pipeline

interpolate what?!?!



e.g. colours!!!

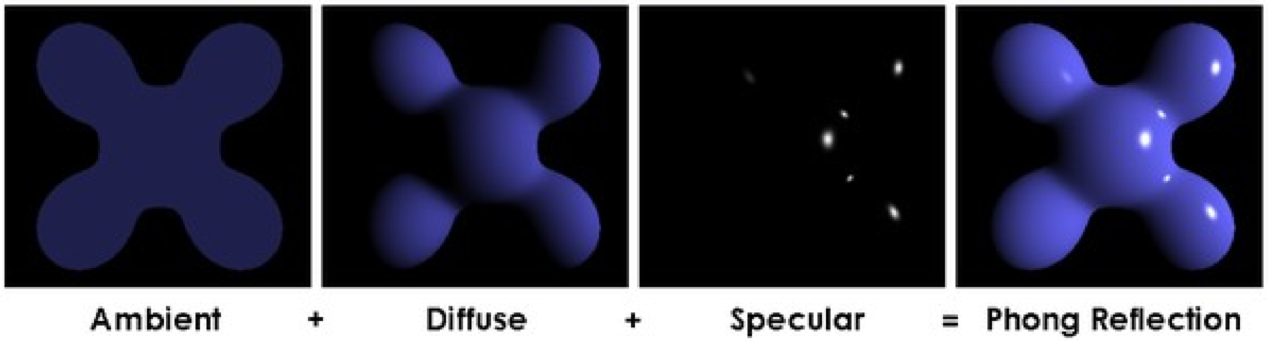


modelview
projection
shading info
Model
OpenGL API
Vertices
Vertex
Transformations
Conectivity
TRIANGLE (● ● ●)
Assembly
clipping*
perspective div
viewport
Rasterization
Interpolation
Final
Fragments
Fragments

shading (Phong model)

Phong Model: Sum of 3 terms

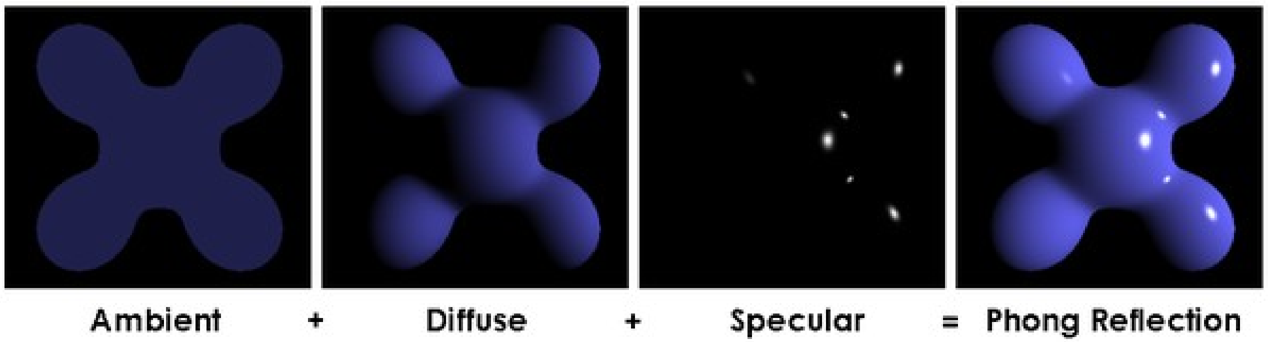
- Ambient
- Diffuse
- Specular



shading (Phong model)

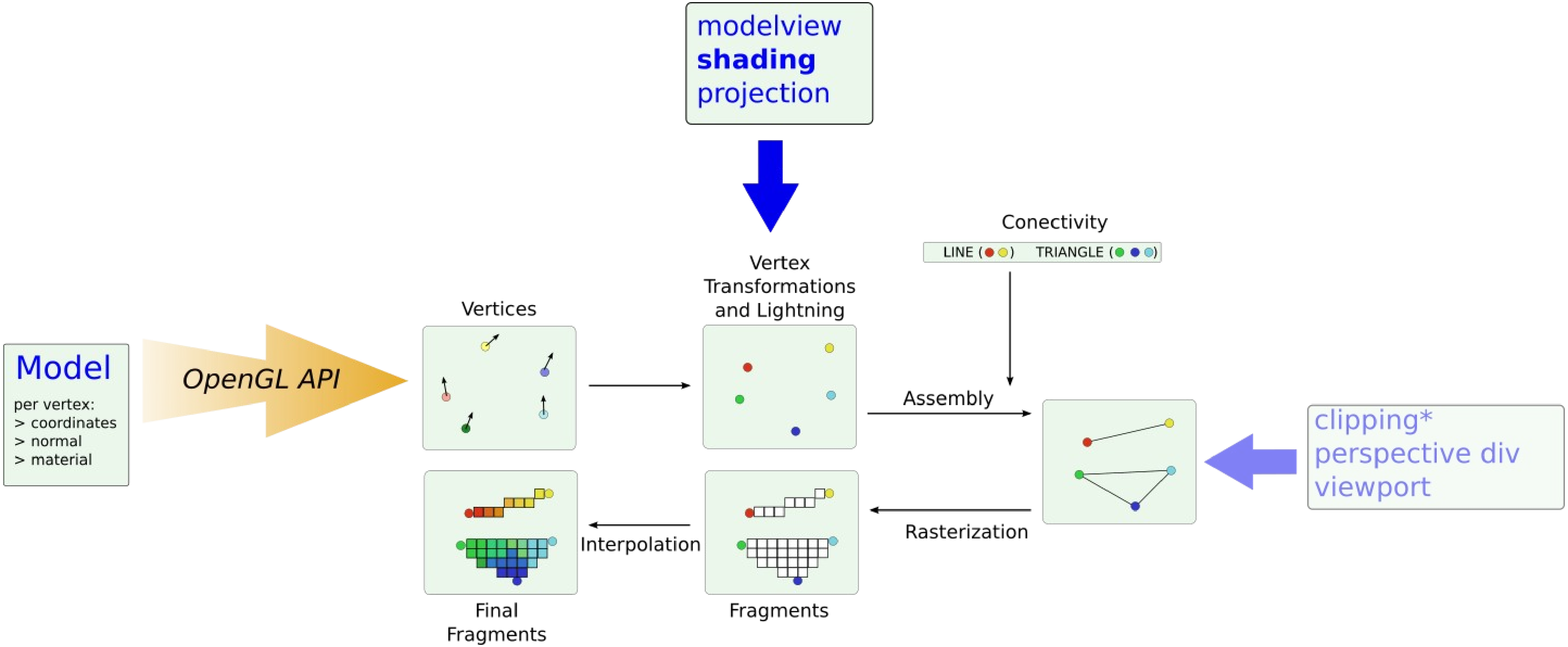
Phong Model: Sum of 3 terms

- Ambient
- Diffuse
- Specular



- light and view info
- surface material (ka, kd, ks)
- normal

e.g., shading information



lighting (**Phong Model**)

define each light source

```
glLightfv(GL_LIGHT0, GL_AMBIENT, light0_ambient)
glLightfv(GL_LIGHT0, GL_DIFFUSE, light0_diffuse)
glLightfv(GL_LIGHT0, GL_SPECULAR, light0_specular)
glLightfv(GL_LIGHT0, GL_POSITION, light0_position)

glEnable(GL_LIGHT0)
```

```
glEnable ( GL_LIGHTING )
```

```
glDisable ( GL_LIGHTING )
```

instead of glColor define the complete material
obs: Normal is needed for shading

```
glMaterialfv(GL_FRONT, GL_SPECULAR, mat_specular)
glMaterialfv(GL_FRONT, GL_SHININESS, mat_shininess)
glMaterialfv(GL_FRONT, GL_DIFFUSE, mat_diffuse)
```

use only RGB to fake material

```
glEnable ( GL_COLOR_MATERIAL )
```

this is for completeness!

For Visual Debug you can mostly use Color Material

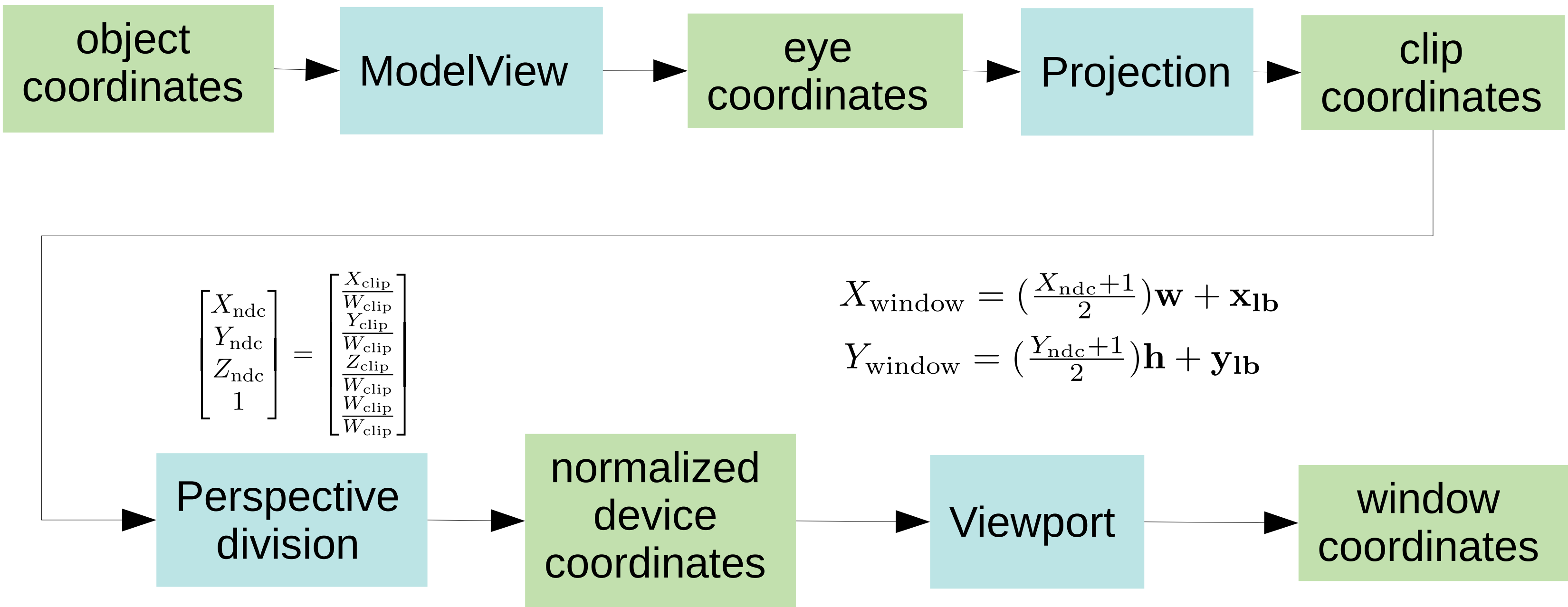
today:
OpenGL transformation pipeline

$$\begin{bmatrix} X_{eye} \\ Y_{eye} \\ Z_{eye} \\ 1 \end{bmatrix} = \begin{bmatrix} m_{00} & m_{01} & m_{02} & t_0 \\ m_{10} & m_{11} & m_{12} & t_1 \\ m_{20} & m_{21} & m_{22} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{obj} \\ Y_{obj} \\ Z_{obj} \\ 1 \end{bmatrix}$$

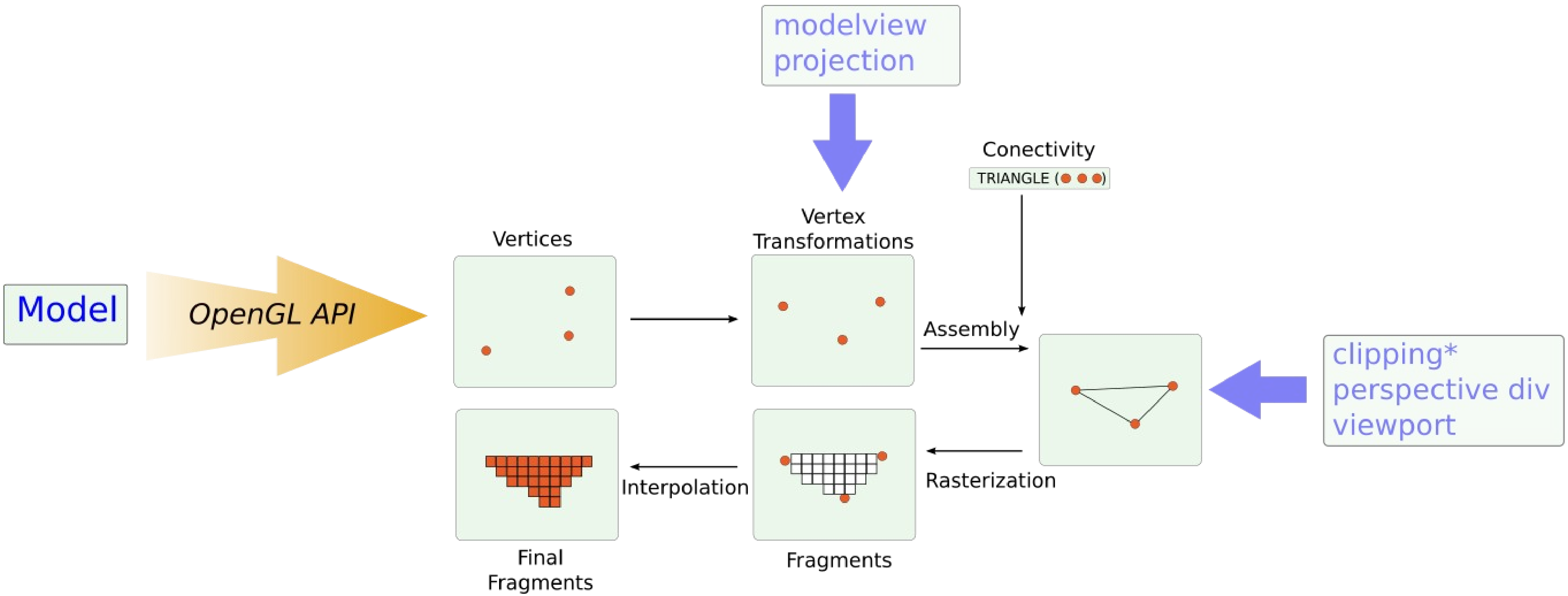
$$\begin{bmatrix} X_{\text{clip}} \\ Y_{\text{clip}} \\ Z_{\text{clip}} \\ W_{\text{clip}} \end{bmatrix} = \begin{bmatrix} \frac{f}{\text{aspect}} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{\text{near}+\text{far}}{\text{near}-\text{far}} & \frac{2*\text{near}*\text{far}}{\text{near}-\text{far}} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} X_{\text{eye}} \\ Y_{\text{eye}} \\ Z_{\text{eye}} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_{\text{ndc}} \\ Y_{\text{ndc}} \\ Z_{\text{ndc}} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{X_{\text{clip}}}{W_{\text{clip}}} \\ \frac{Y_{\text{clip}}}{W_{\text{clip}}} \\ \frac{Z_{\text{clip}}}{W_{\text{clip}}} \\ \frac{W_{\text{clip}}}{W_{\text{clip}}} \end{bmatrix}$$

$$X_{\text{window}} = \left(\frac{X_{\text{ndc}}+1}{2}\right)\mathbf{w} + \mathbf{x_{lb}}$$
$$Y_{\text{window}} = \left(\frac{Y_{\text{ndc}}+1}{2}\right)\mathbf{h} + \mathbf{y_{lb}}$$

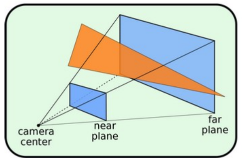


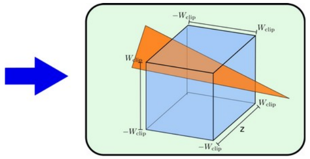
graphics pipeline



questions

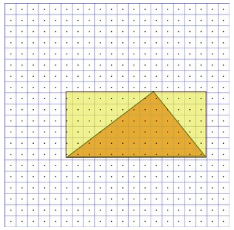
- design a simple clipping algorithm



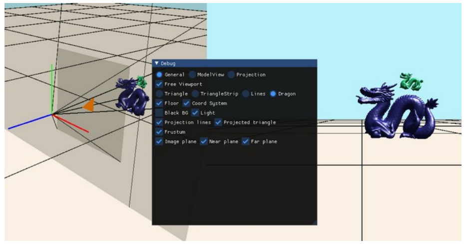


- explain if fisheye lens is possible with our camera model

- find triangle’s AABB

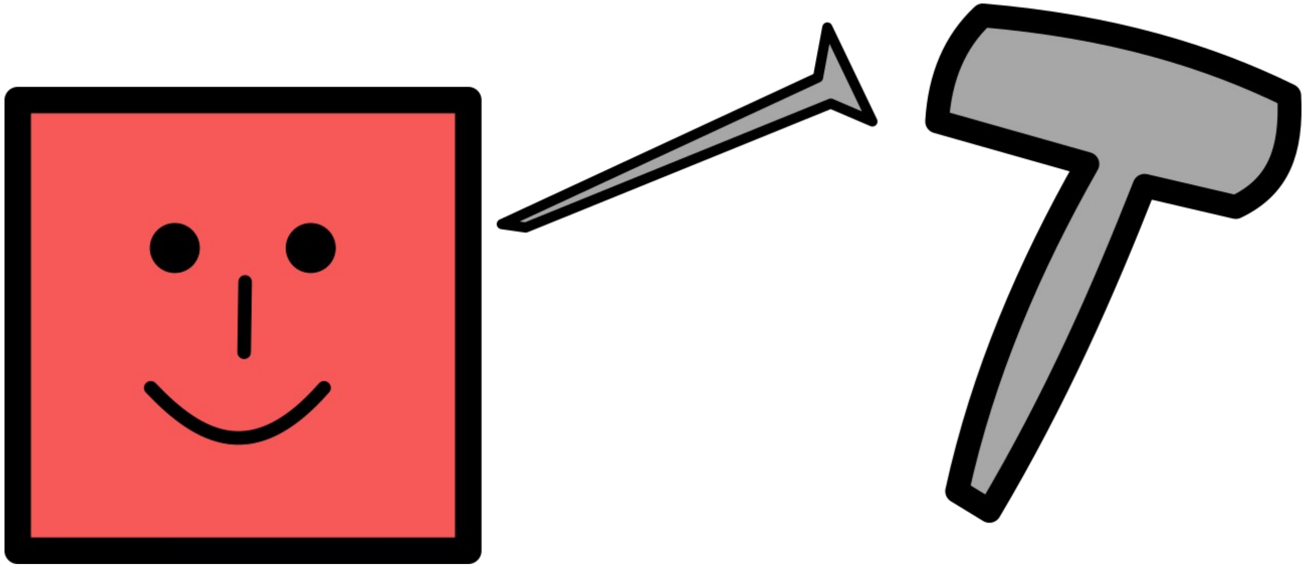


- Rebuild the demo and introduce new features



how to make a pixel

CSE2215 Computer Graphics



Ricardo Marroquim

Delft University of Technology (TU Delft)

CSE2215 - Computer Graphics

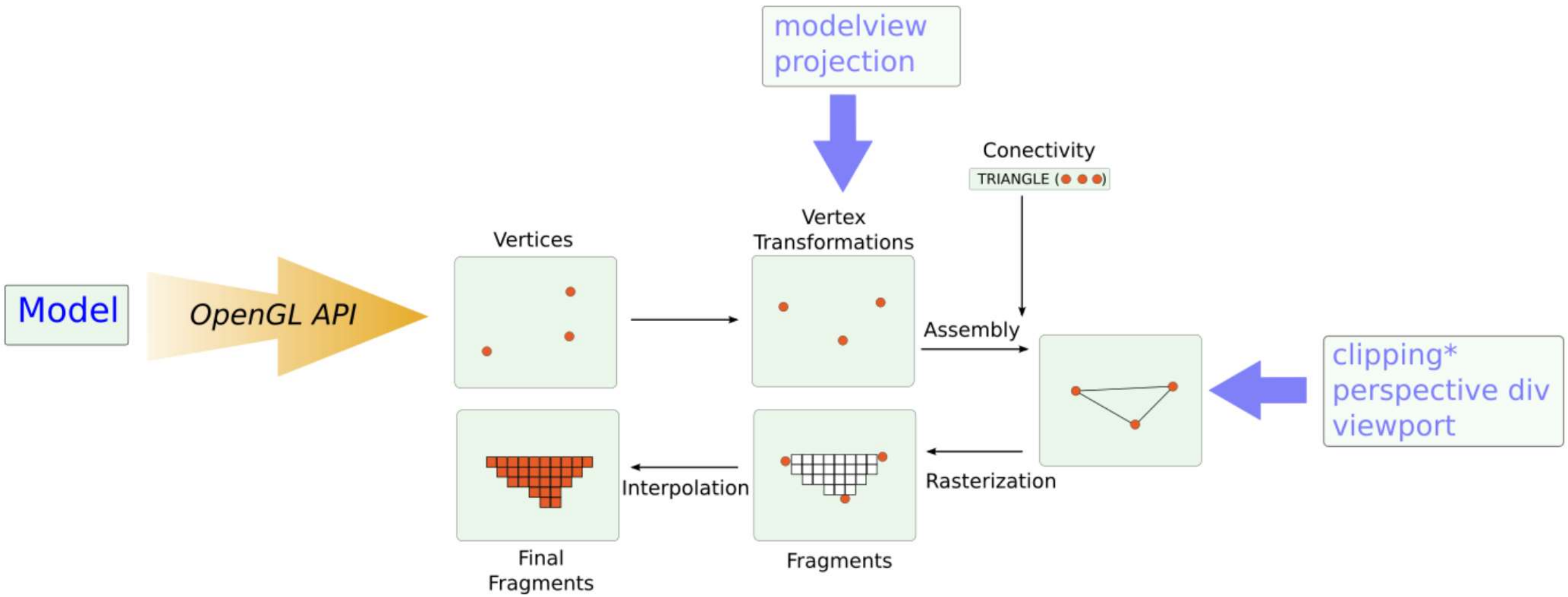
Textures

An image says more than...

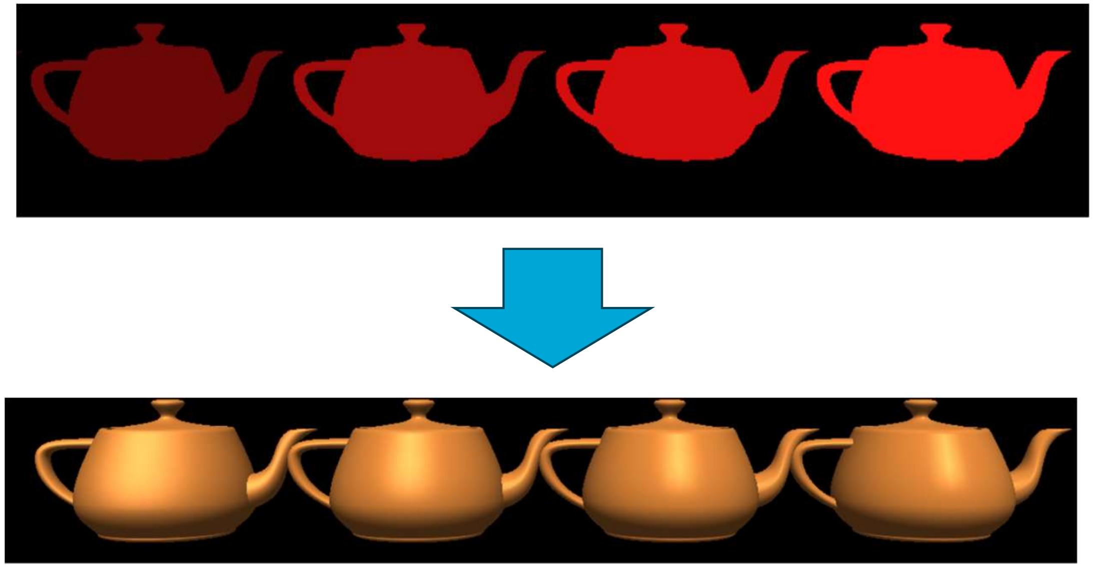
Elmar Eisemann

Delft University of Technology

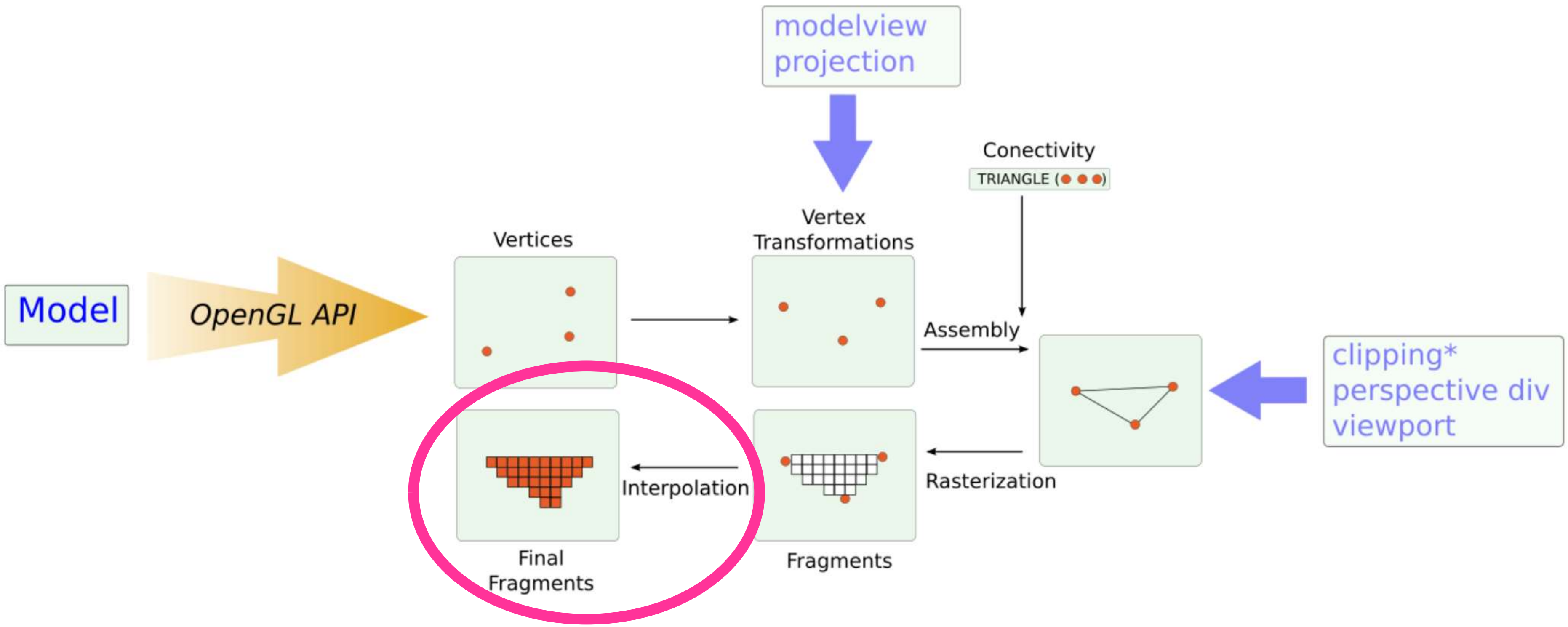
Graphics Pipeline



Shading



Graphics Pipeline



Reminder: How to apply Phong Model ?

- We know how to compute shading of a point, but how is it applied on a triangle mesh?

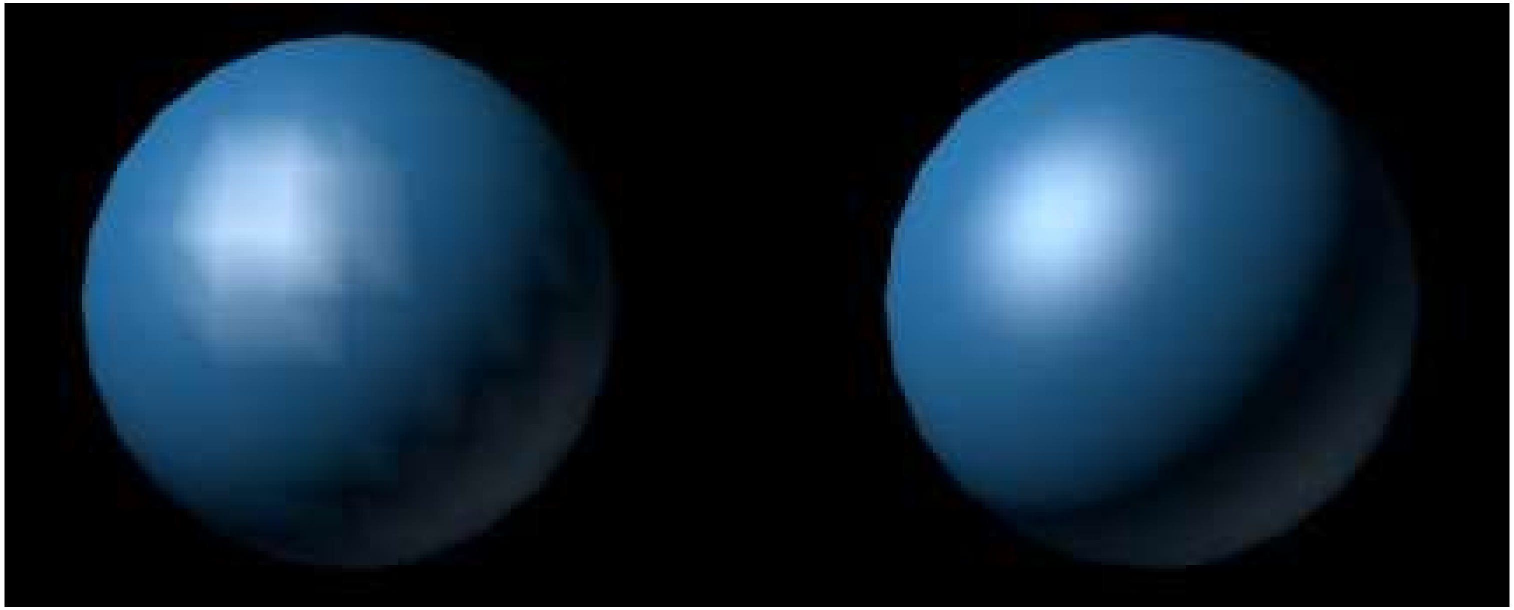
Shading

- *Early days - compute color per face:*
  Flat shading produces “facets”


- Later – compute color per vertex:
  produces *Gouraud Shading* produces a smooth look


Phong shading

- Today: compute result per pixel
- Phong Shading leads to smooth specularities

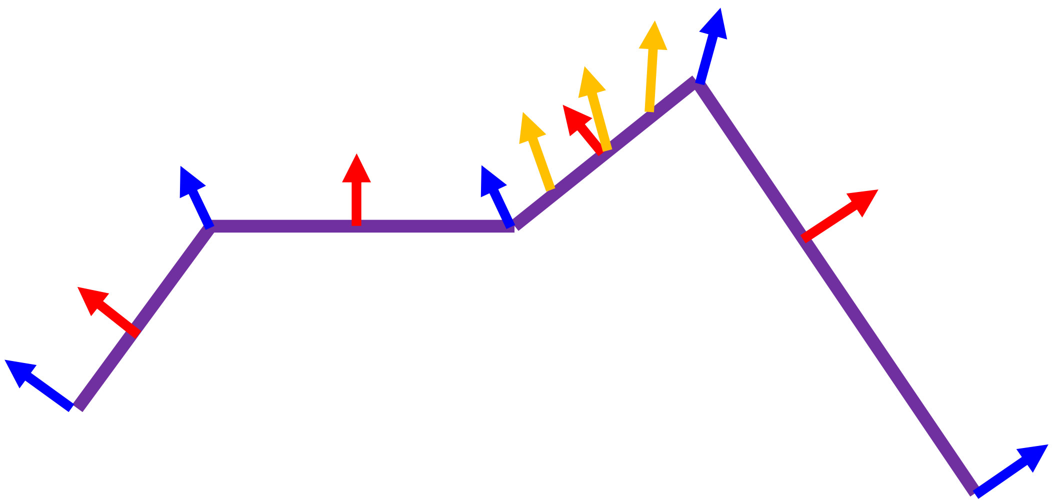


Diccan.com

- Phong interpolates normals from vertices over pixels

Normals on Meshes

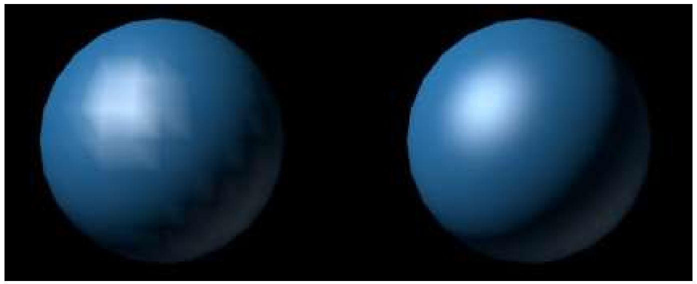
- Face normals (normal of the plane containing triangle)
- Vertex normals (e.g., average neighboring face normal)
- Interpolated normal (interpolate vertex normals over triangle)



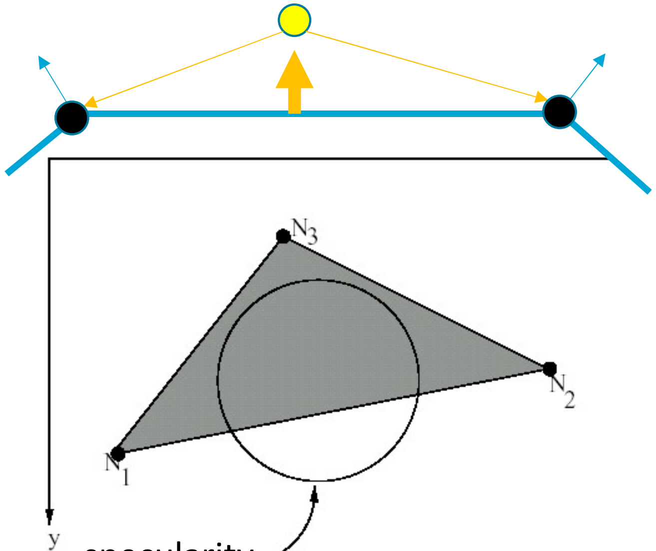
Gouraud *vs.* Phong shading

Per vertex *Per pixel*

- Phong usually more expensive than Gouraud
  - Because there are often more pixels than vertices
- Phong is more beautiful and minimal standard
  - Captures specularities between faces



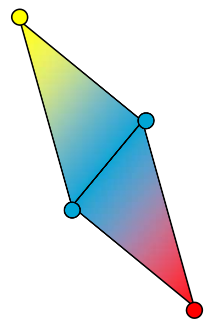
Diccan.com



On the practical side: Shading types

- How are the three different types computed?
  - *Flat shading*
    - Applies Phong Model to produce a color per face
  - *Gouraud shading*
    - Applies Phong to produce a color per vertex
    - Interpolate color from vertices over triangle
  - *Phong shading*

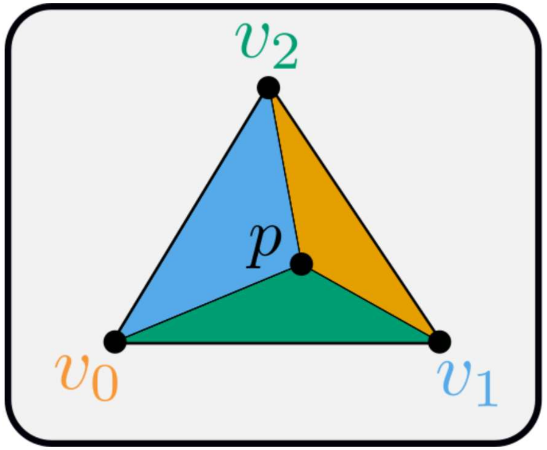
    2 MEANINGS!!!
    - Interpolate parameters of Phong model
    - Applies Phong to produce a color per pixel



TUDelft

Interpolation on a Triangle

- Last time you went through barycentric coordinates



weights are related to areas

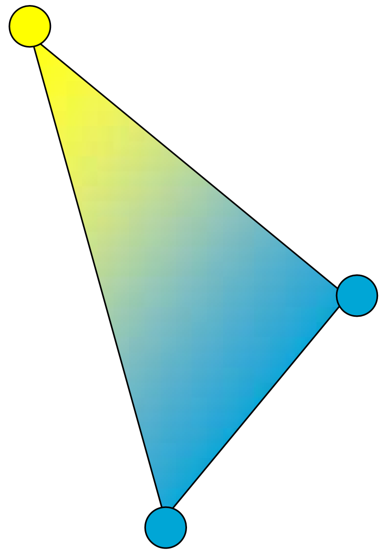
$$p = \alpha v_0 + \beta v_1 + \gamma v_2$$

$$\alpha = \frac{A(pv_1v_2)}{A(v_0v_1v_2)}$$

$$\beta = \frac{A(pv_0v_2)}{A(v_0v_1v_2)}$$

$$\gamma = \frac{A(pv_0v_1)}{A(v_0v_1v_2)}$$

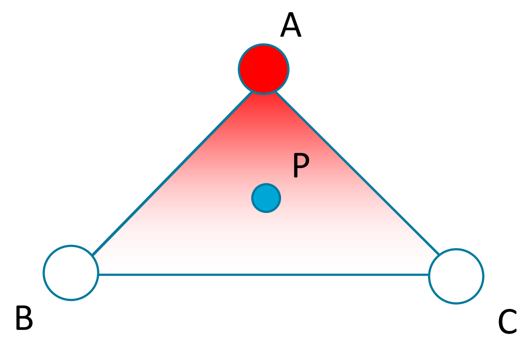
$$\alpha + \beta + \gamma = 1$$
$$\alpha, \beta, \gamma > 0$$



Interpolation on a Triangle

- Imagine P=1/3 A + 1/3 B + 1/3 C
- If the colors at A, B, C are red (1,0,0), white (1,1,1), white (1,1,1) respectively
- The interpolated color at P would be:

  (1, 2/3, 2/3) pink



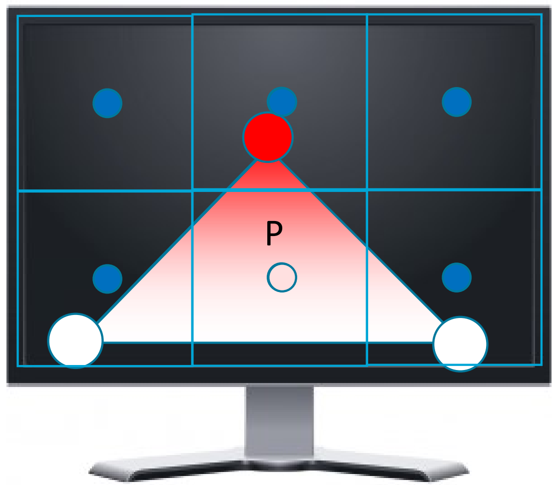
Do you know how this color is called?

Rasterization

- Interpolated values are used for rasterization. Here is an amazing 6x2 pixel display…

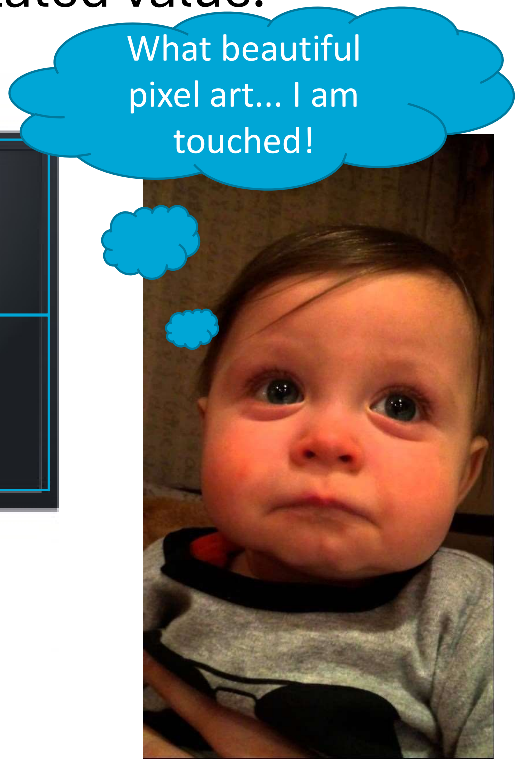
Rasterization

- For the pixel center, find the underlying interpolated value.



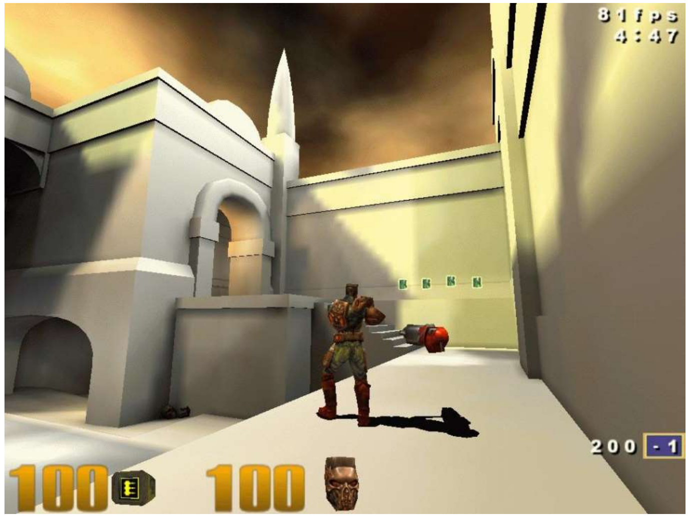
Rasterization

- For the pixel center, find the underlying interpolated value.



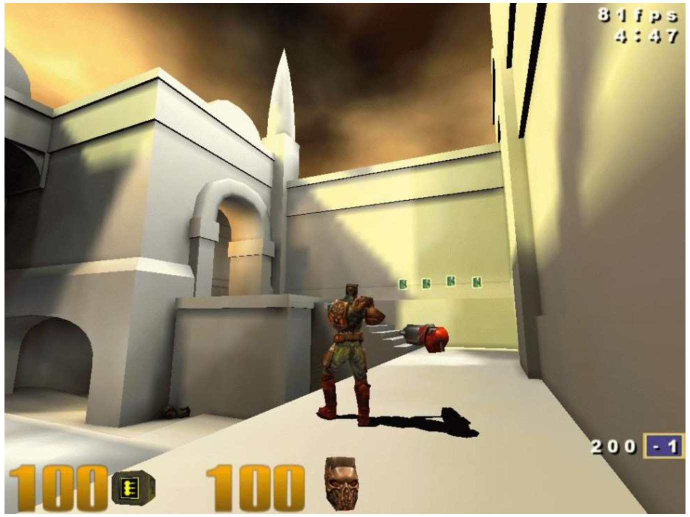
Joke aside…

- There is some truth to it…
- For now, everything looks “kind of clean”…



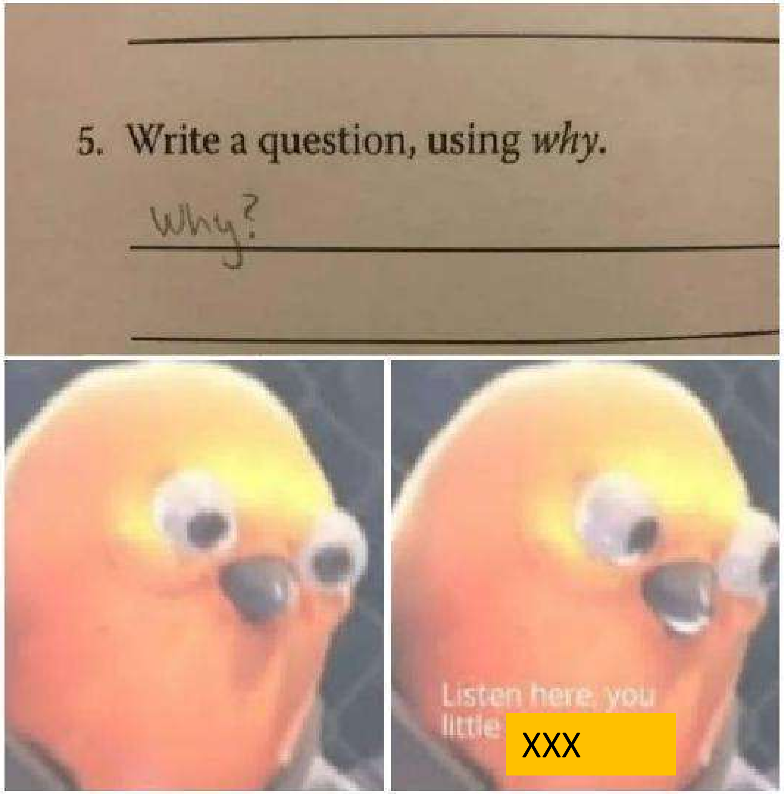
Textures: Adding small scale details

- No textures



- Textures

Questions?



Study Goals

- S1- Explain and compare the structure and properties of standard algorithms and data structures linked to Computer Graphics.

- S3- Use mathematical methods to analyze, create, apply algorithms and data structures, as well as understanding time and space complexity of image-generation algorithms

You could represent all color changes with triangles… but it is too expensive because it generates a lot of geometry…
Textures avoid this problem!

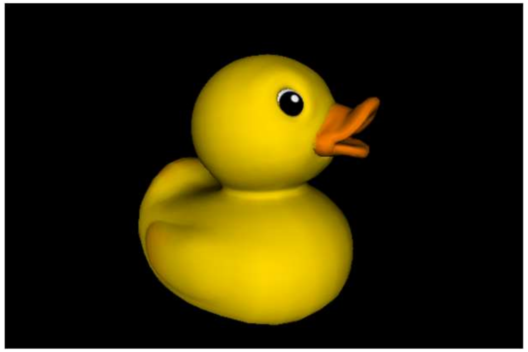
Textures

- Mapping an image onto a surface

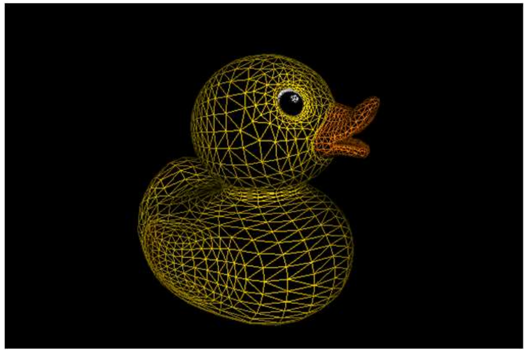
Texture

consisting of *texels* (**tex**ture pix**els**)*

Texture mapped triangles



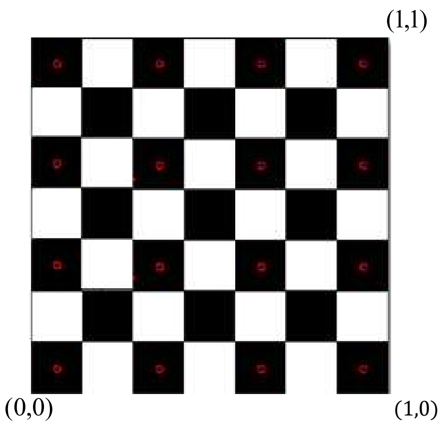
Wireframe of triangles



* Some say **tex**ture **el**ements

Textures

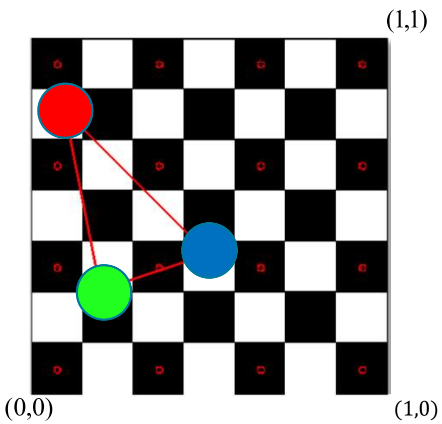
- Map image via texture coordinates



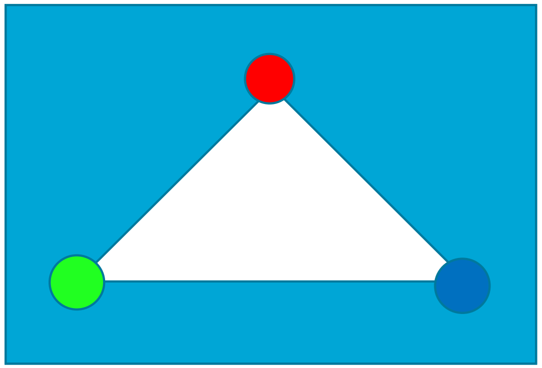
texture

Textures

- Map image via texture coordinates
  - Specify a texture coordinate at each vertex



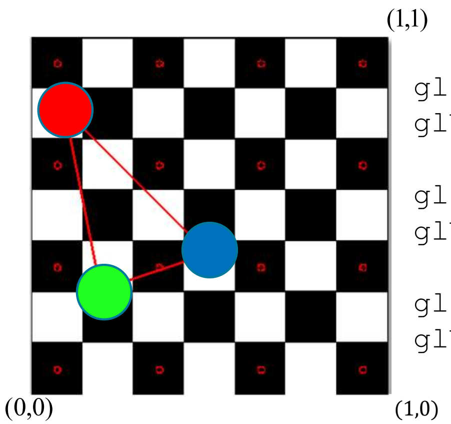
texture



Camera View

Textures

- Map image via texture coordinates
  - Specify a texture coordinate at each vertex
  - Vertex texture coordinates are interpolated over triangle

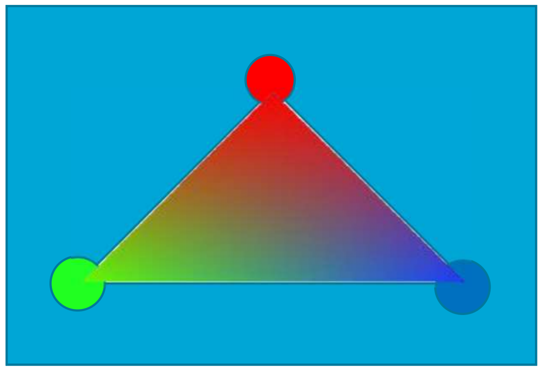


texture

```
glTextureCoord2f(0.2f,0.3f);
glVertex2f(-1.0f,0.0f);

glTextureCoord2f(0.5f,0.4f);
glVertex2f(+1.0f,0.0f);

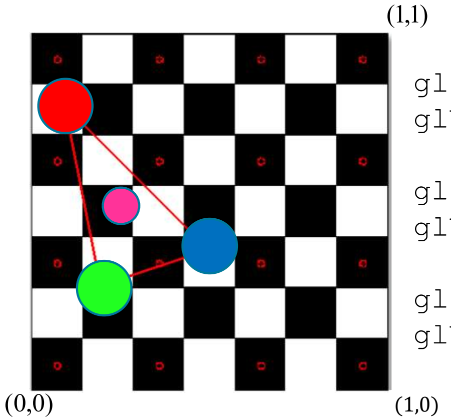
glTextureCoord2f(0.1f,0.8f);
glVertex2f( 0.0f,1.0f);
```



Camera View

Textures

- Map image via texture coordinates
  - Specify a texture coordinate at each vertex
  - Vertex texture coordinates are interpolated over triangle
  - Drawn pixels use interpolated coordinates to retrieve texel values

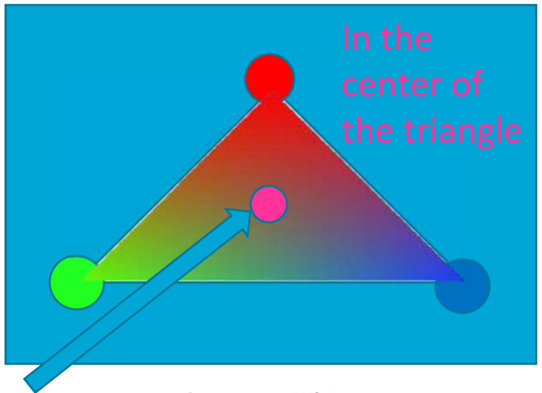


texture

```
glTextureCoord2f(0.2f,0.3f);
glVertex2f(-1.0f,0.0f);

glTextureCoord2f(0.5f,0.4f);
glVertex2f(+1.0f,0.0f);

glTextureCoord2f(0.1f,0.8f);
glVertex2f( 0.0f,1.0f);
```

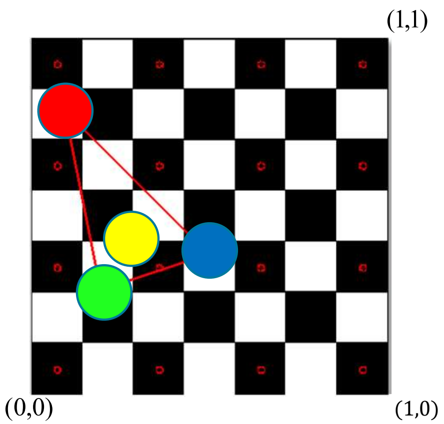


Camera View

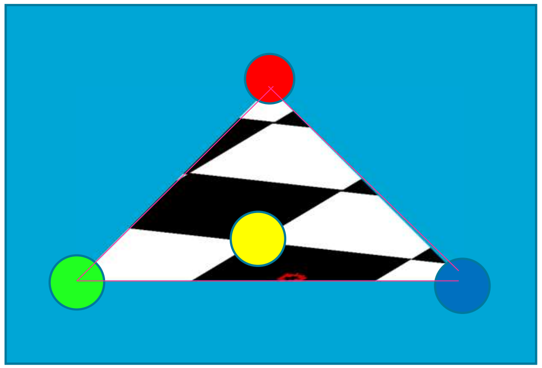
1/3 (0.2,0.3)+1/3 (0.5,0.4)+1/3 (0.1,0.8)
= 1/3*(0.8,1.5) = (0.8/3,0.5)

Textures

- Map image via texture coordinates
  - Specify a texture coordinate at each vertex
  - Vertex texture coordinates are interpolated over triangle
  - Drawn pixels use interpolated coordinates to retrieve texel values



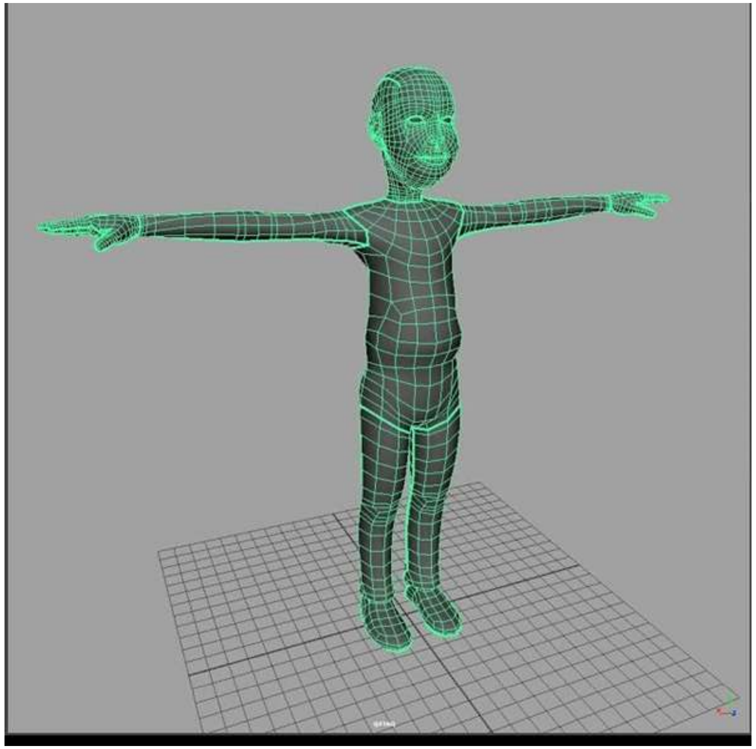
texture



Camera View

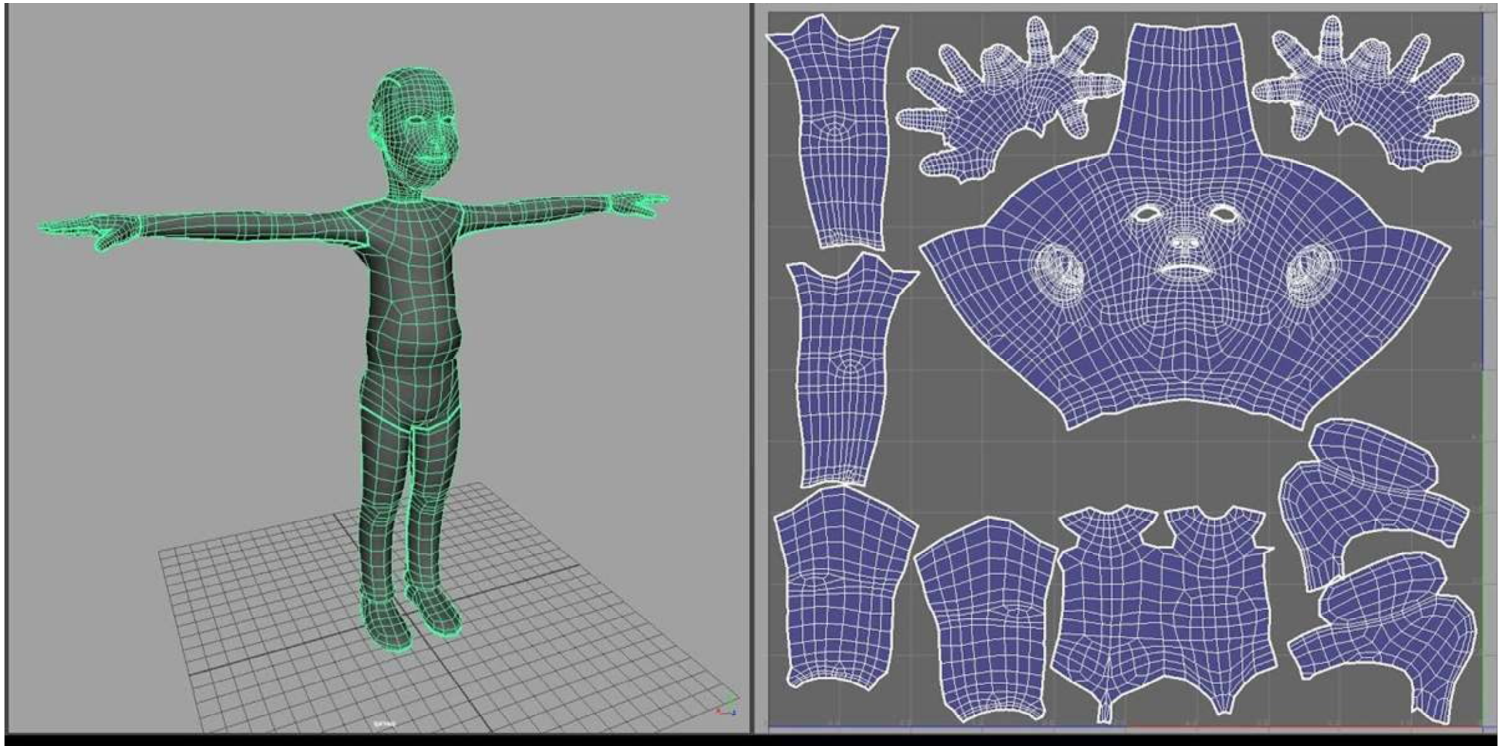
How to define Texture Coordinates?

- Common start: Mesh Unwrapping



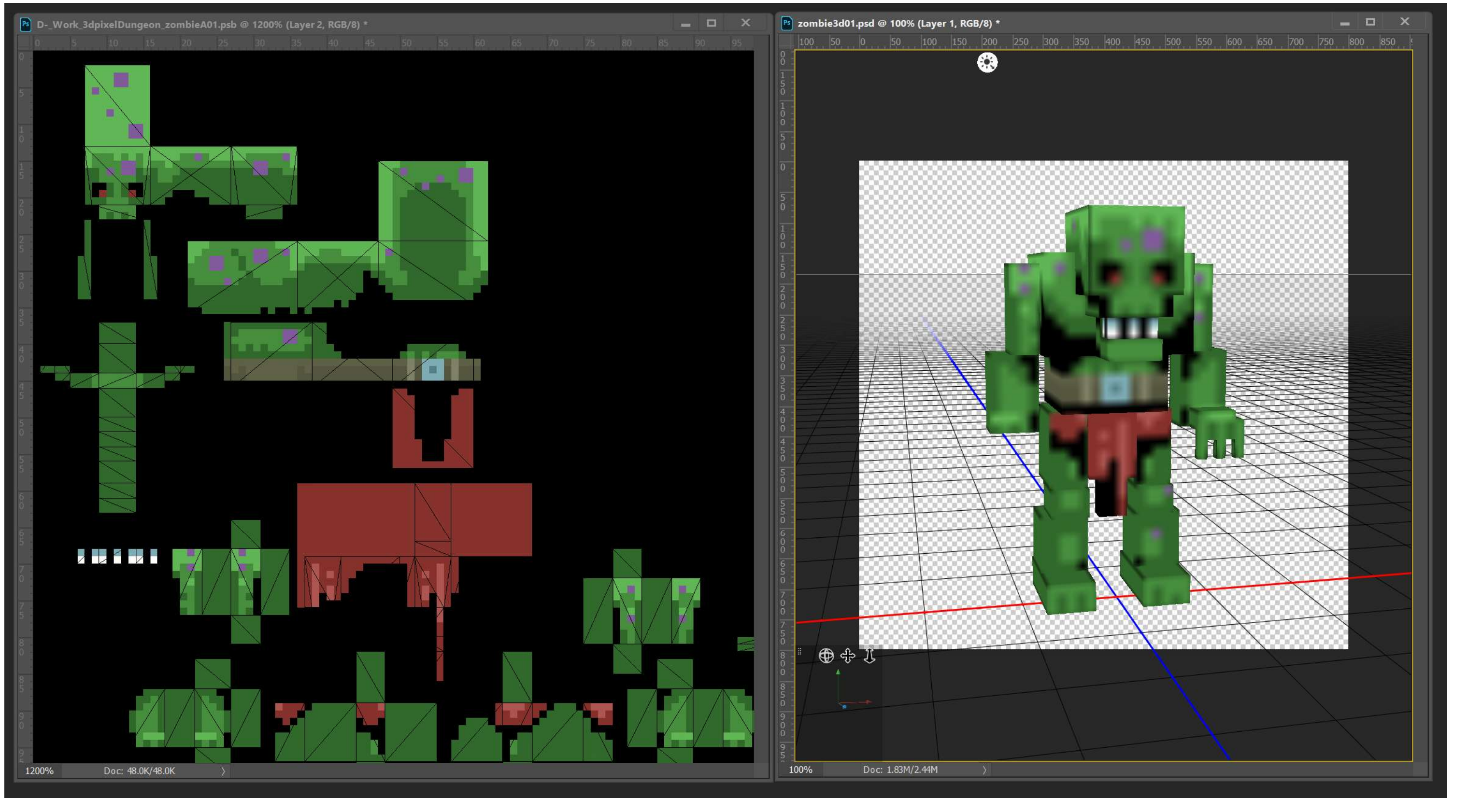
How to define Texture Coordinates?

- Common start: Mesh Unwrapping



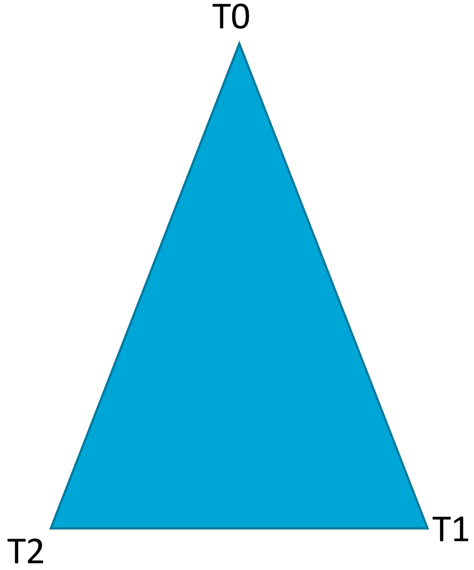
Specialized Software

Specialized Software

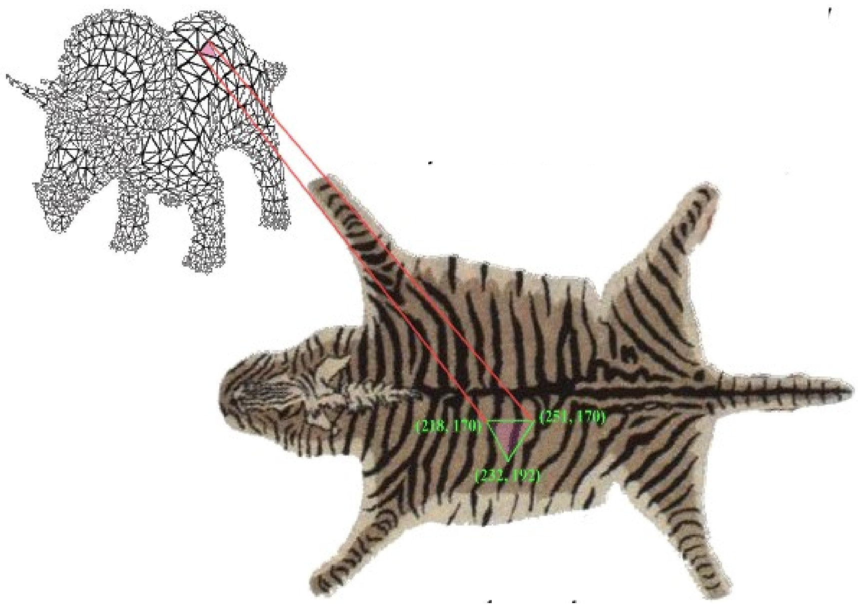


How to define Texture Coordinates?

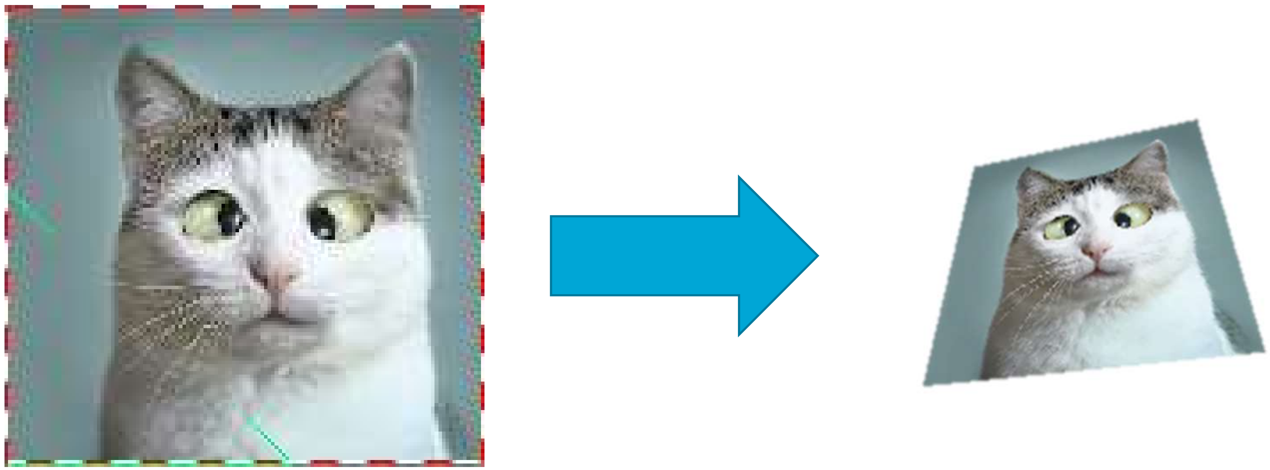
- Often involves a lot of hand tweaking!



Texture coordinates need to be defined for each vertex but are then interpolated over the triangle



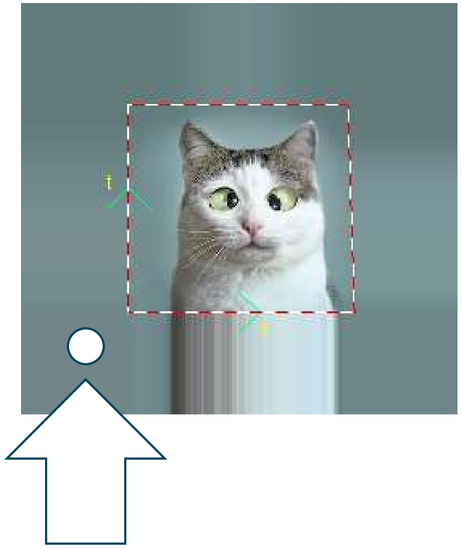
Example of a mapped texture



What happens outside the texture?

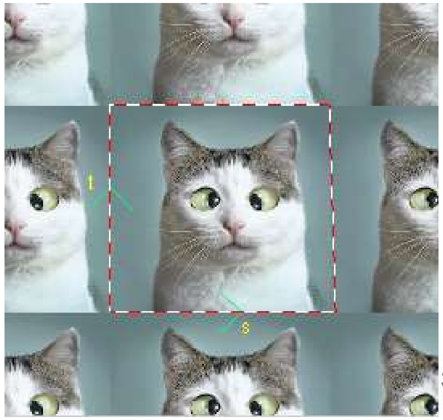
- Different modes can be defined **per axis:**

Border = constant color



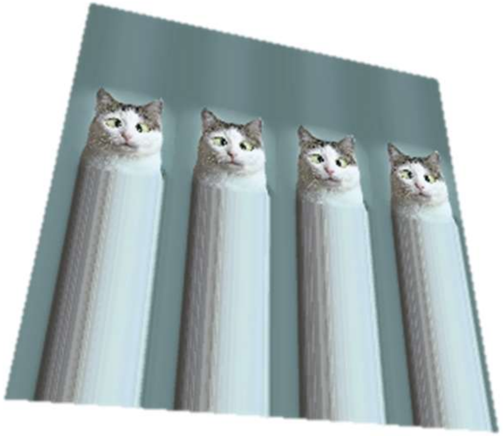
Clamp = keep texel value on the border

Repeat = repeat at borders



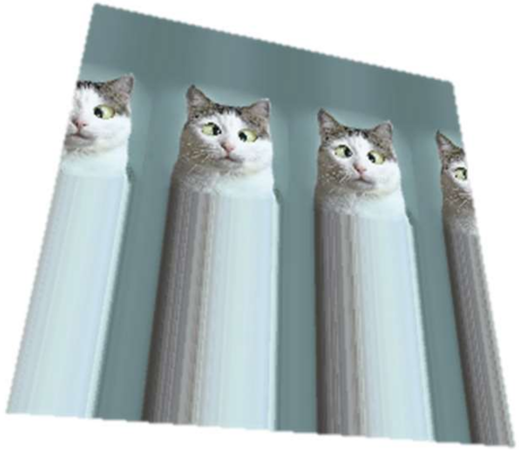
Demo Time!!!

What happened here?



- Vertex texture coordinates are (-2,-2), (-2,2), (2,-2), (2,2)
- X axis: repeat
- Y axis: clamp

- Vertex texture coordinates: same with -1.5 and 1.5
- X axis: repeat
- Y axis: clamp

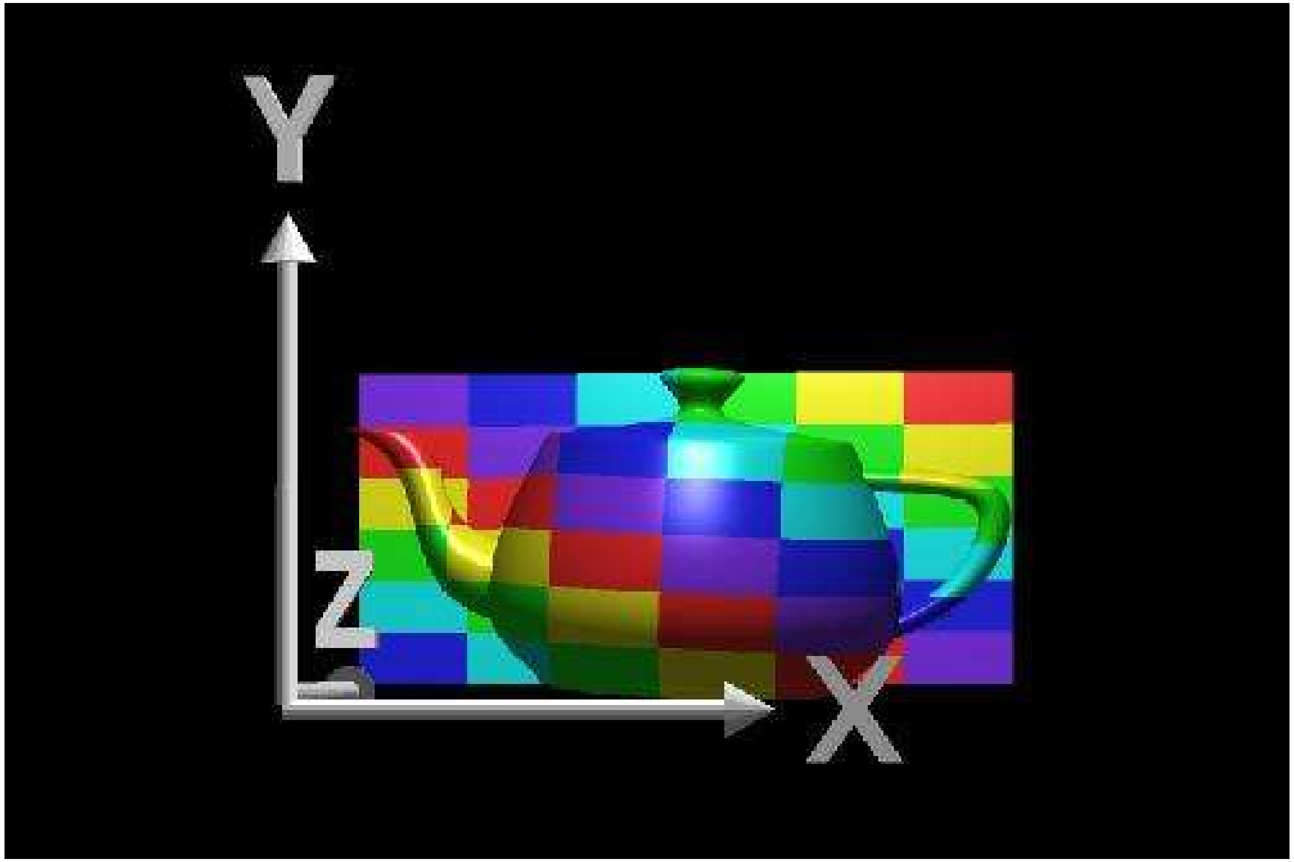


How else to define Tex Coords?

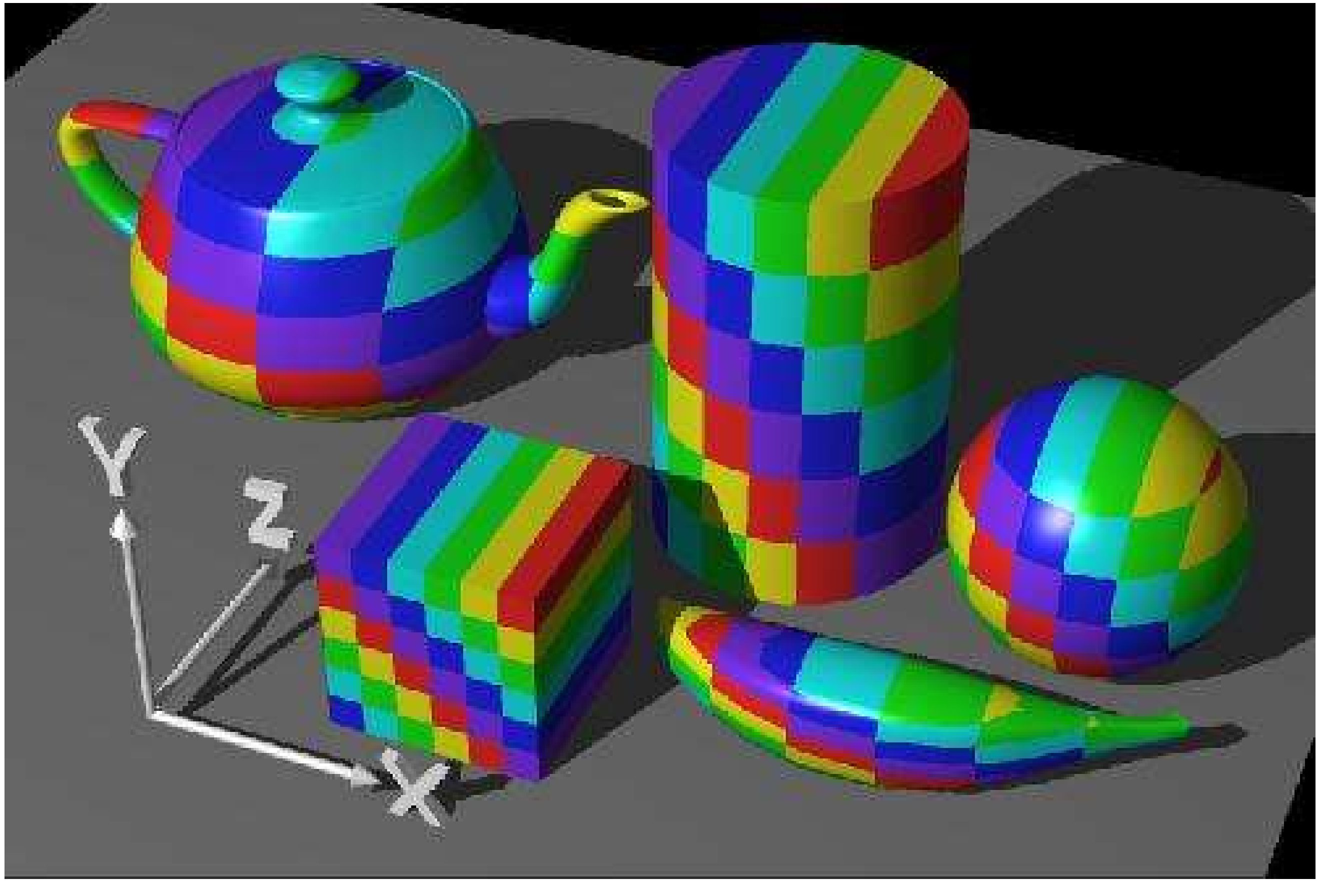
- Geometric Texture Mapping:
- Define a function T that takes
  a vertex position (x,y,z) in R^3
  and maps it to texture coordinates $(u,v)=T(x,y,z)$
- For example:

$$T(x,y,z)=(x,y)$$

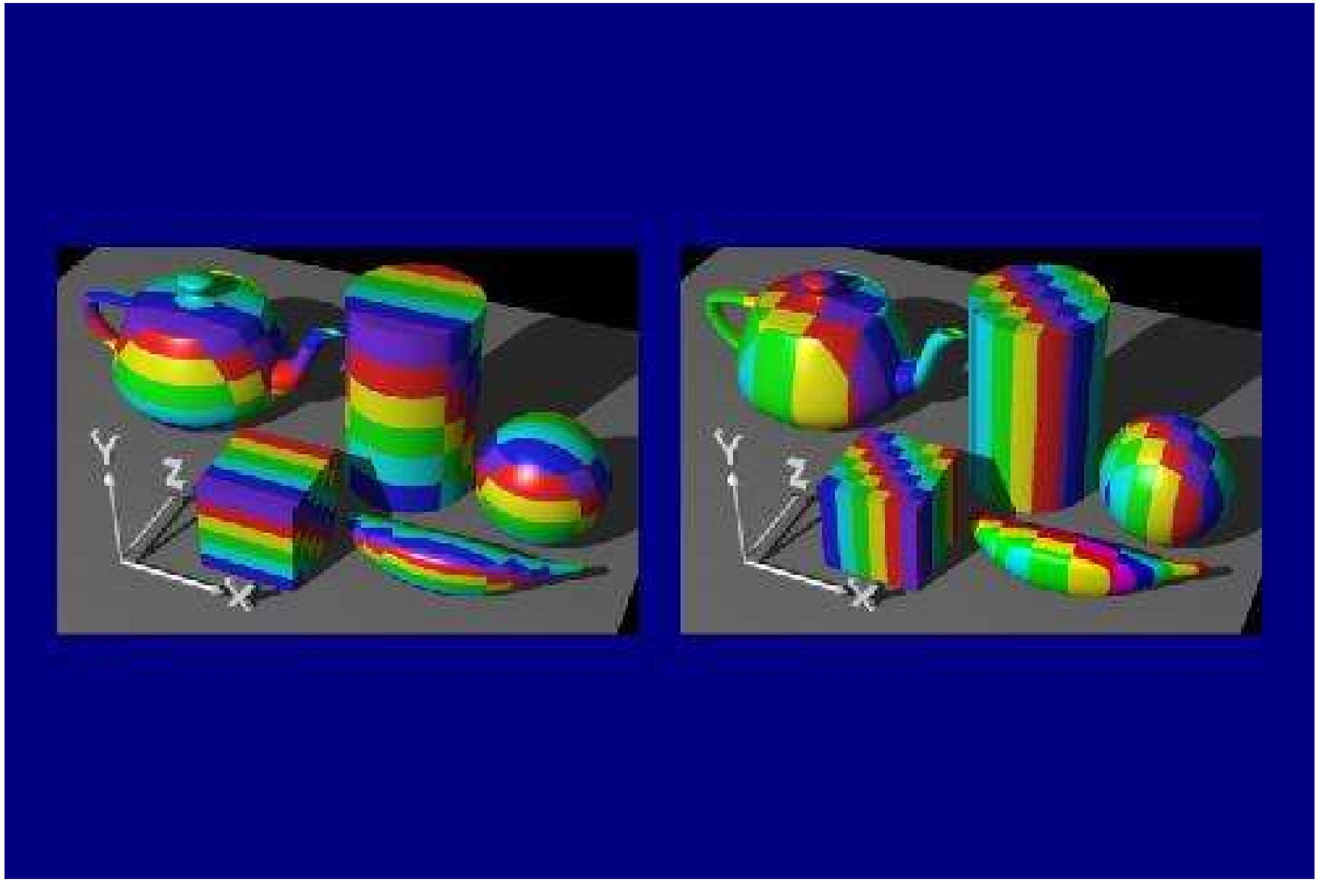
XY



XY

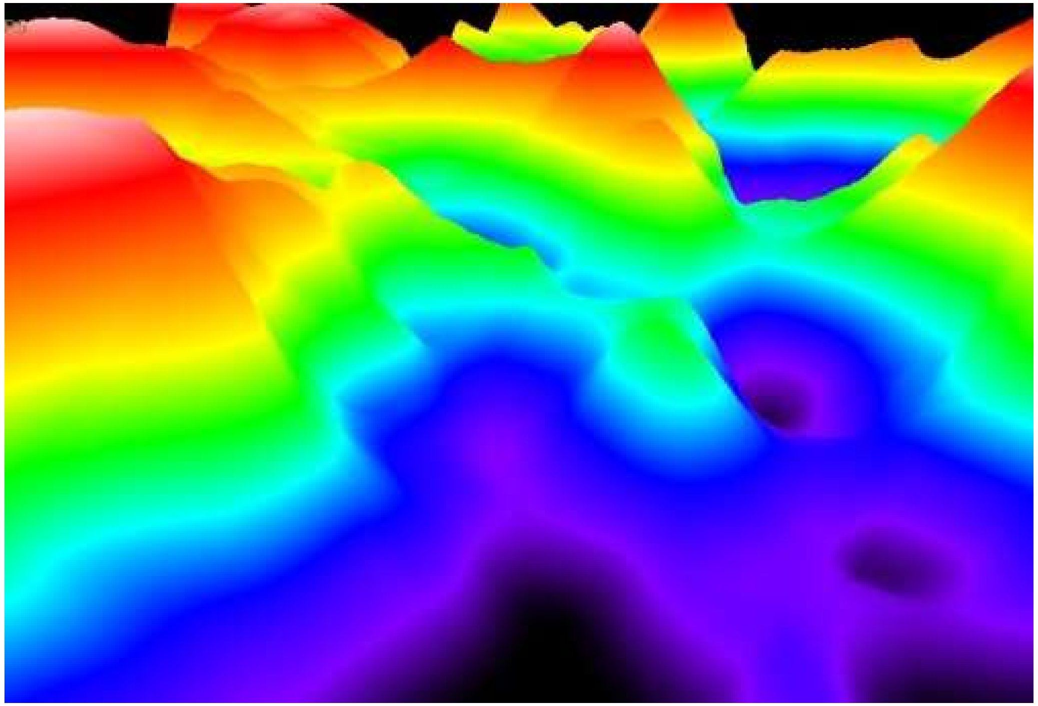


YZ/ZX



How could you do this?

- E.g., map to (0,z)

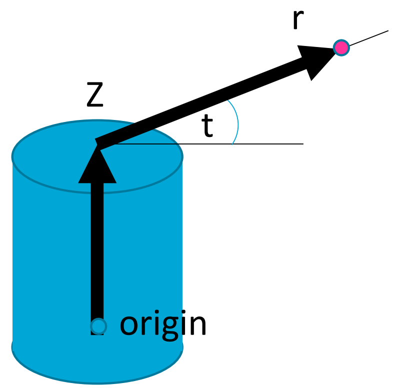


Cylinder

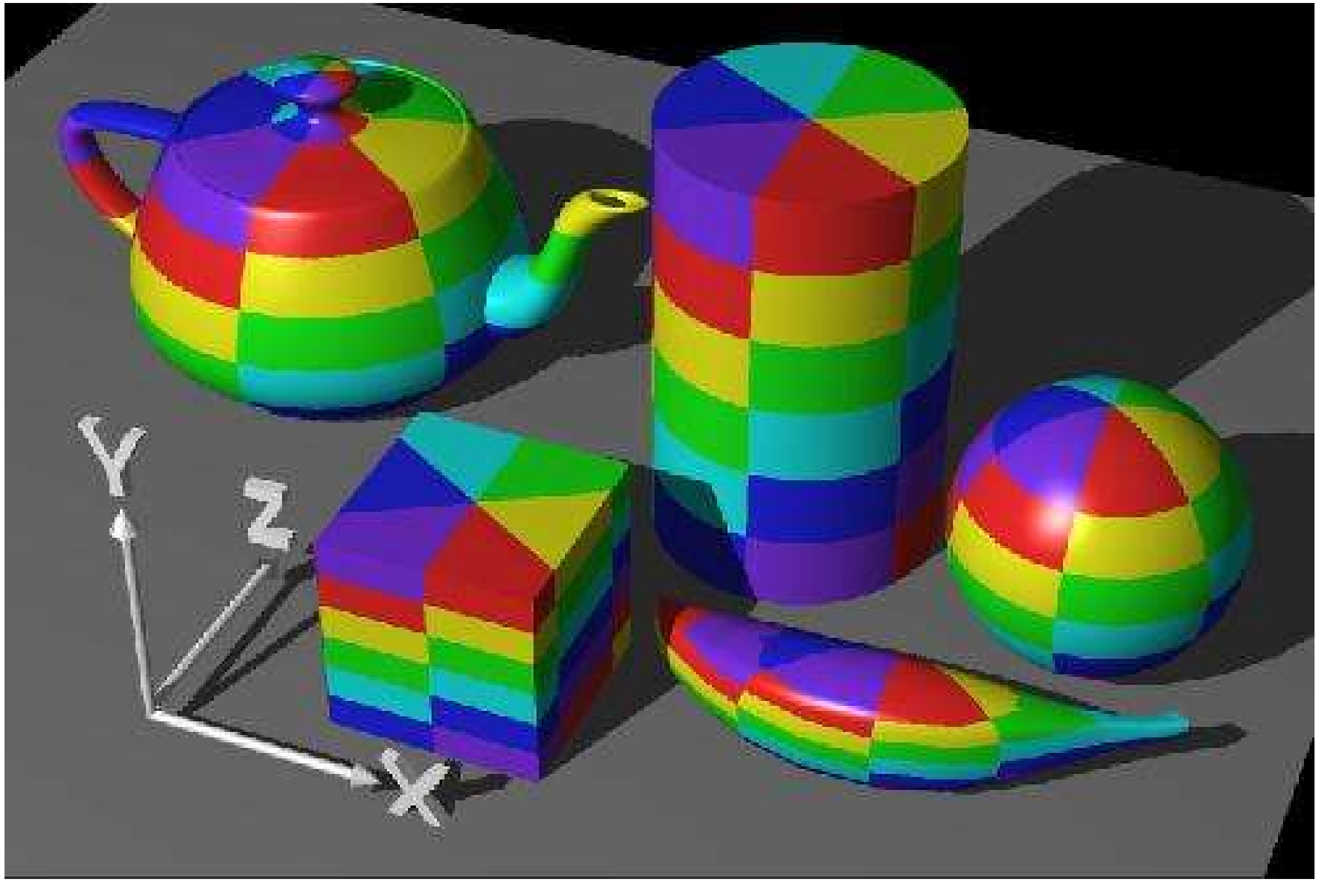
Cylinder

- A bit harder to find T:
- Let position
  (x,y,z)=(r*cos(t),r*sin(t),z)
  for suitable r,t,z
  (you can write any 3D position like this)

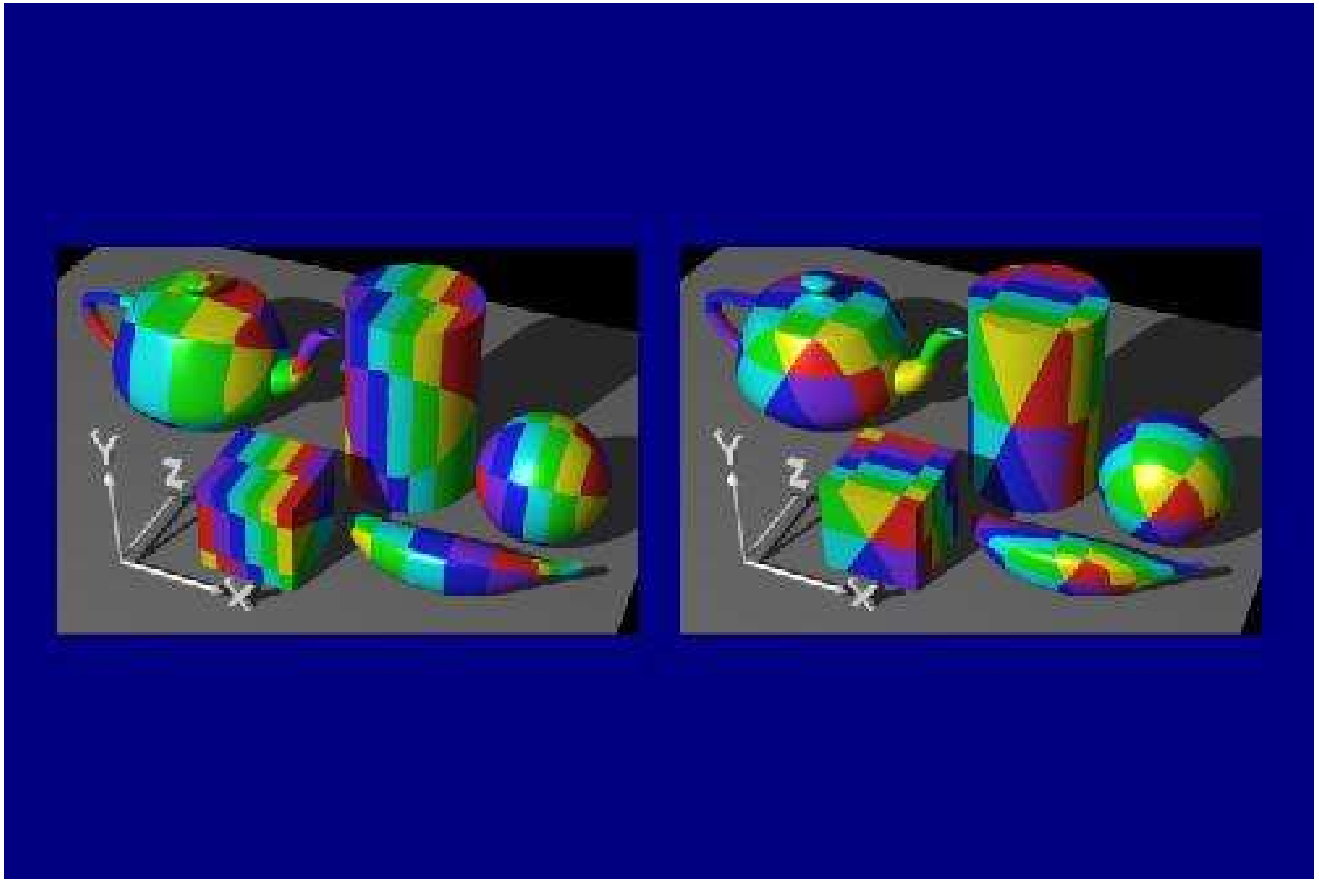
- Then T(r*cos(t), r*sin(t),z)= (t/2π, z)



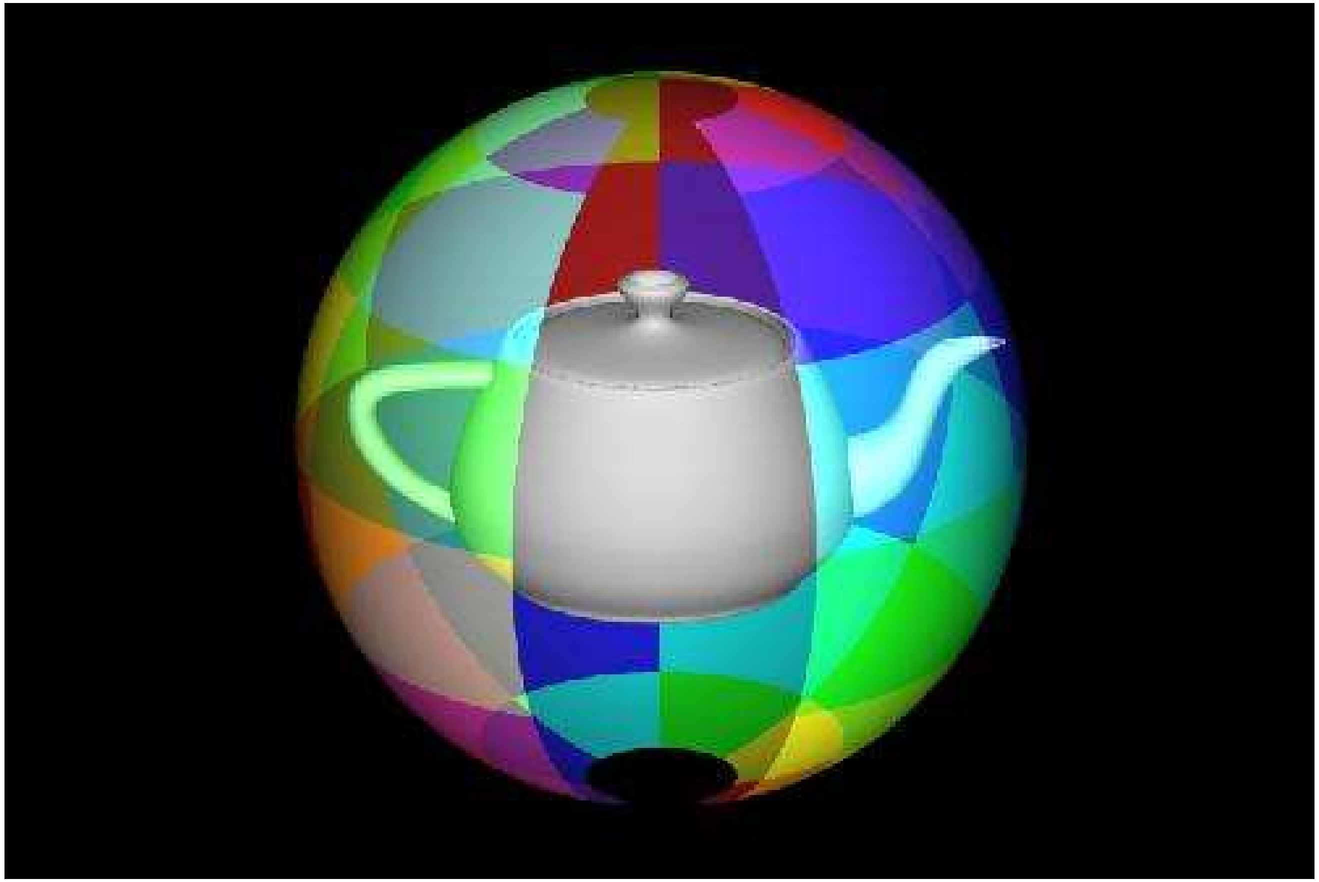
Cylinder



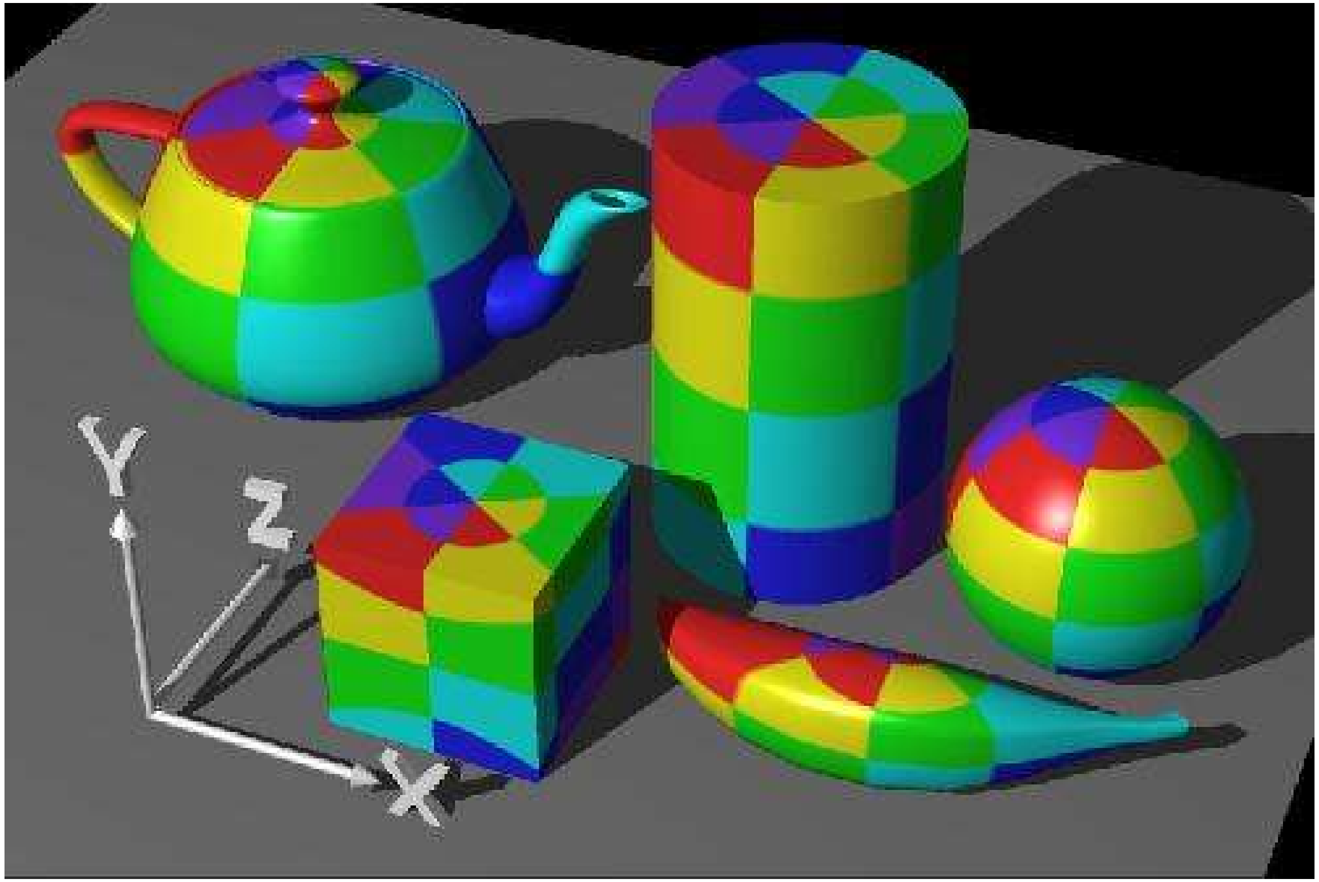
Cylinder



Sphere

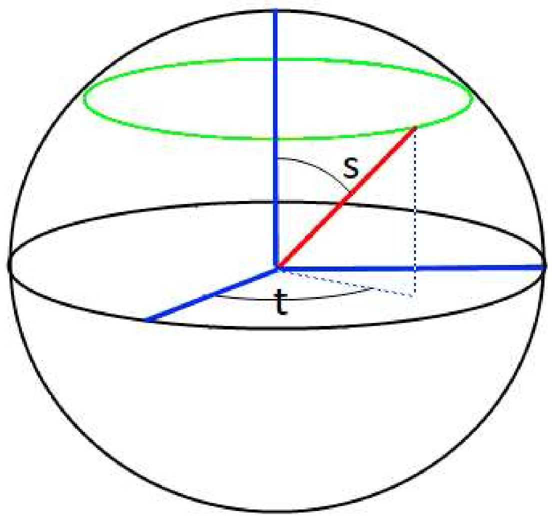


Sphere



Sphere

- A bit harder to find T:
- Let position
  (x,y,z)=(r*cos(t)sin(s),r*sin(t)sin(s),r*cos(s))
  for suitable r in [0,infinity),t in [0,2π),s in [0,π]
  (you can write any 3D position like this,
  if r is constant, points are on a sphere)

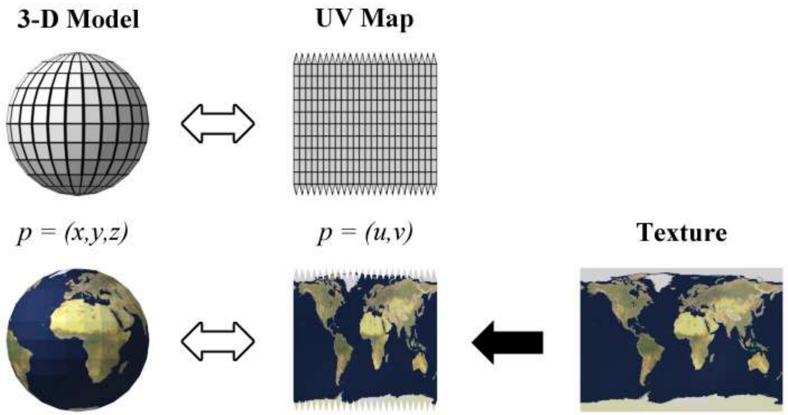


- Then T((r*cos(t)sin(s),r*sin(t)sin(s),r*cos(s)))
  = (t/2π, s/π)

Sphere is useful for Environment Mapping

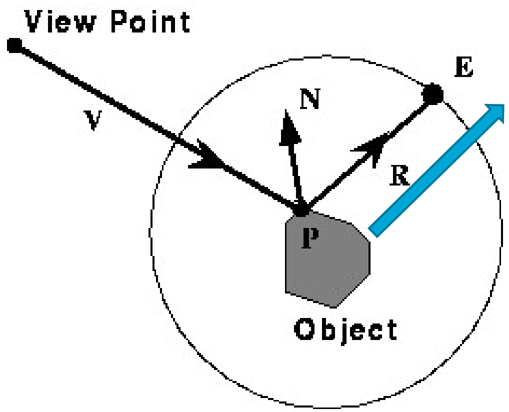
- Textures can encode an environment
  An environment map (approximation of scene)

Spherical mapping



Environment Mapping

- Textures can encode an environment
  An environment map (approximation of scene)
  is useful for, e.g., reflections (texcoords = ray)

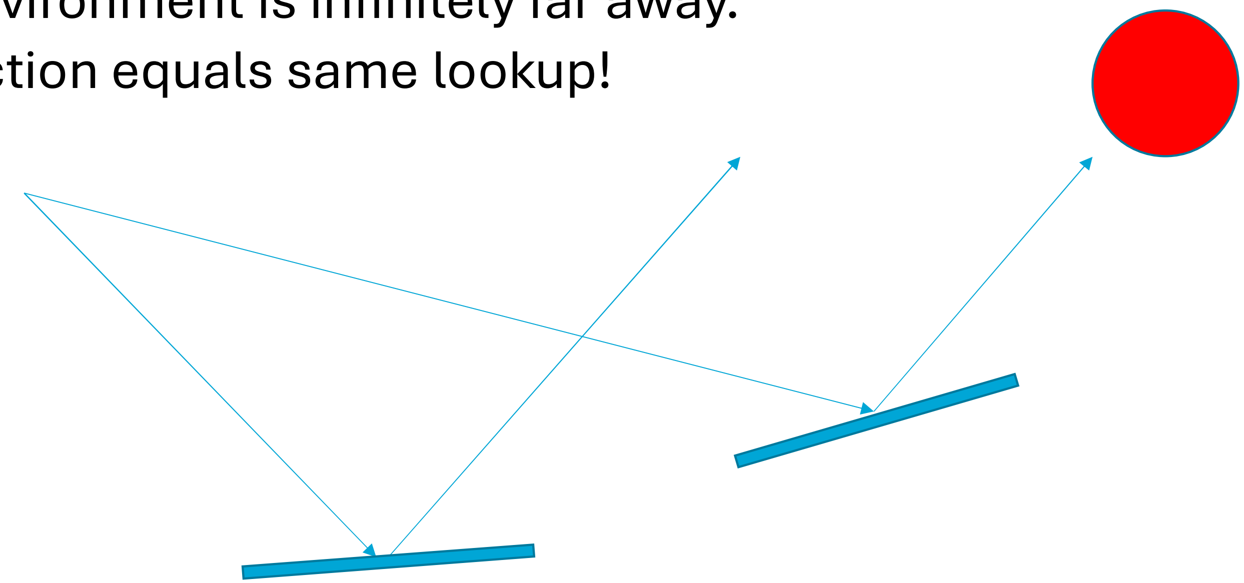


Ray direction determines lookup position

- This *environment mapping* is an approximation!

Environment Mapping

- Typical approximation:
  Assume environment is infinitely far away.
  Same direction equals same lookup!



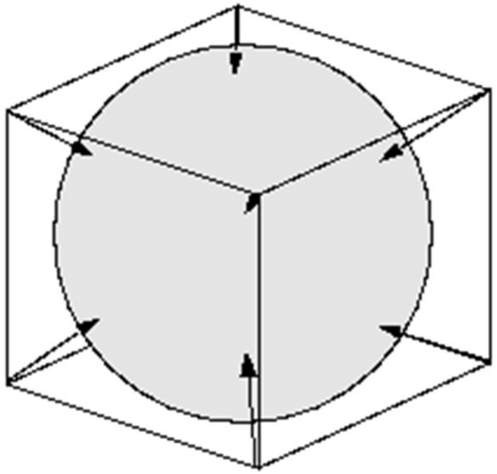
- Red ball reflection seen by both or none.

Environment Mapping

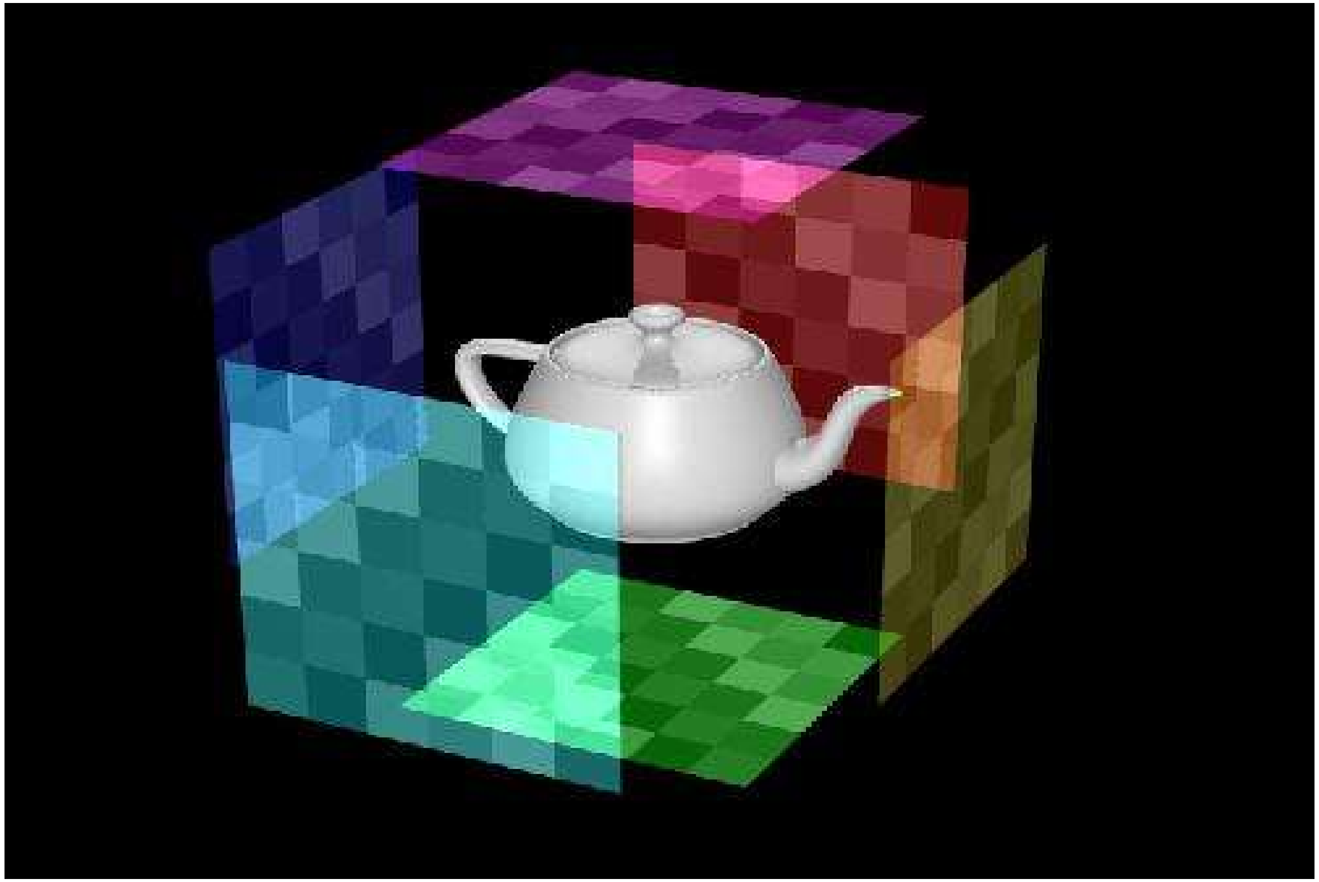
Environment Mapping

- Alternatively, a sphere can be mapped to a cube
- Less distortions if images are used for the cube faces

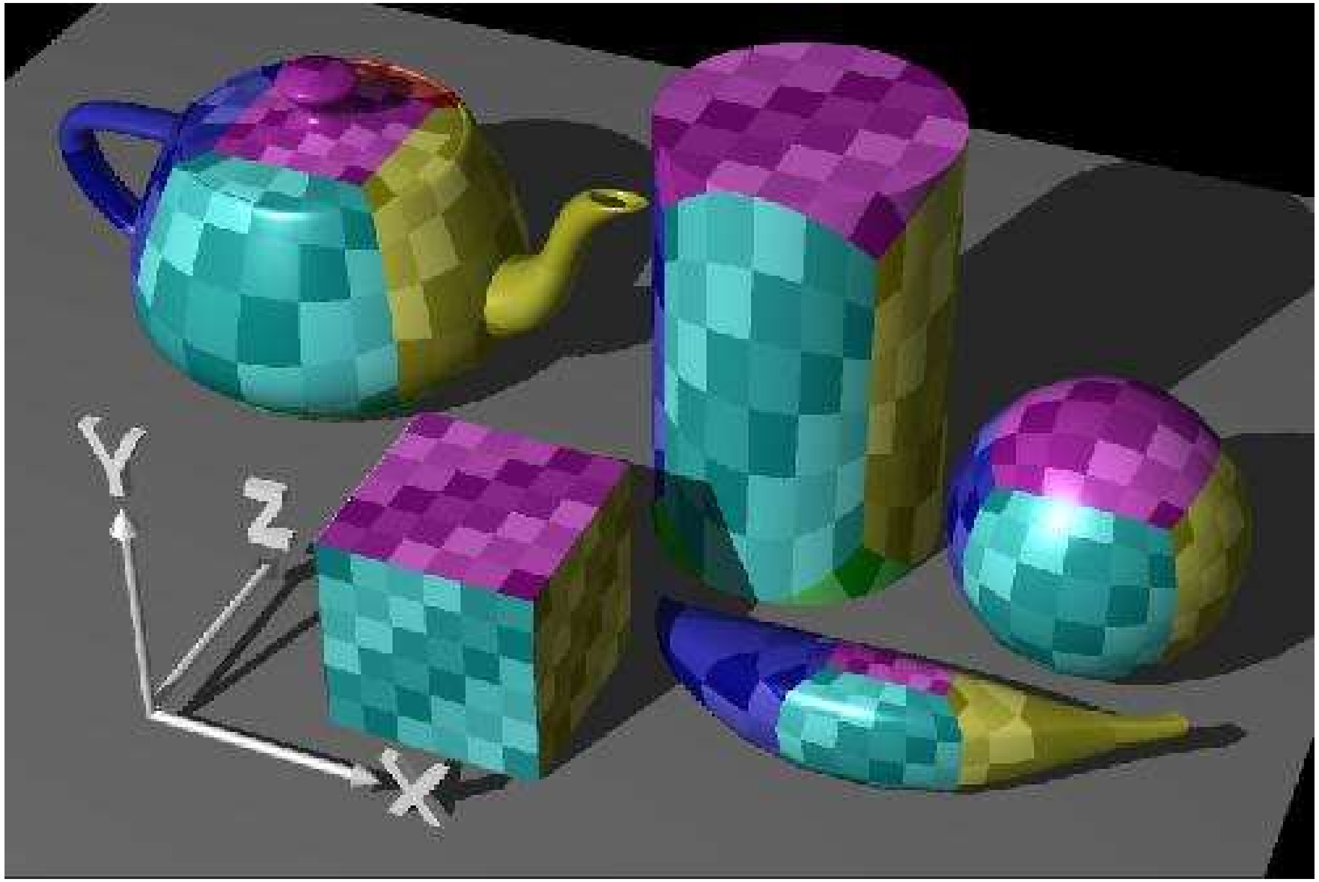
Cube mapping



Cube Map



Cube Map



Questions?


It is our duty as scientists to question everything.
Says who?
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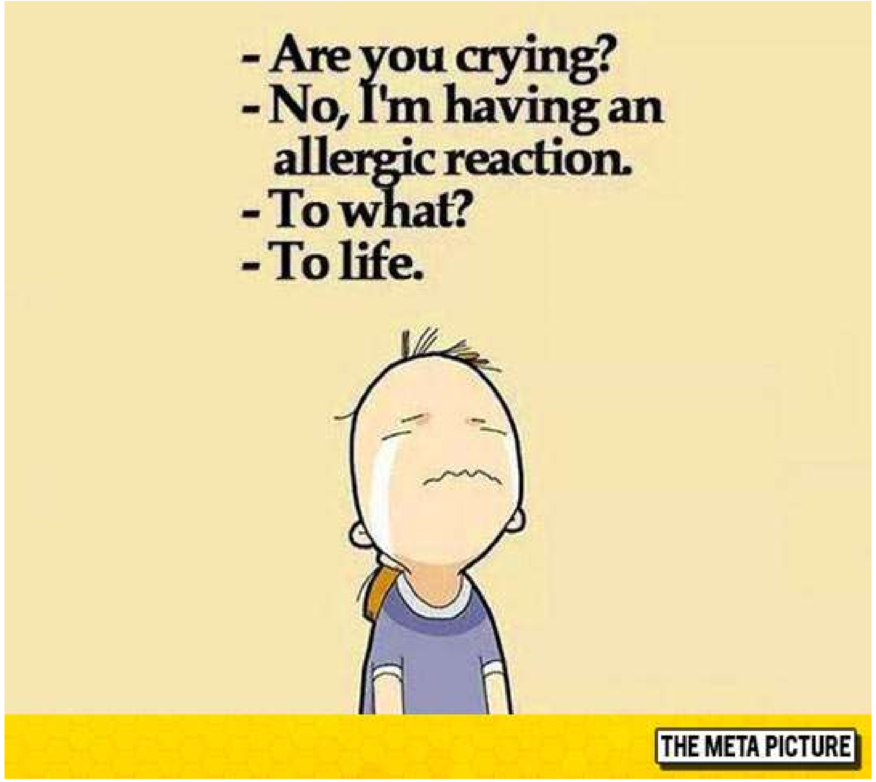
Textures

- Image content mapped on surfaces
- Increase detail level without geometric cost

- Many applications for color textures

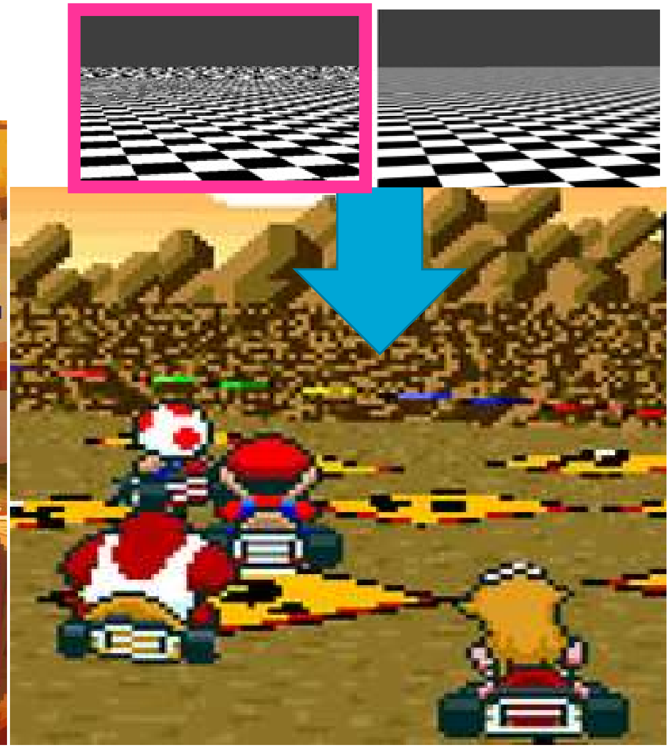
Texture Issues?

- What are limitations of textures?



Texture issues: Aliasing

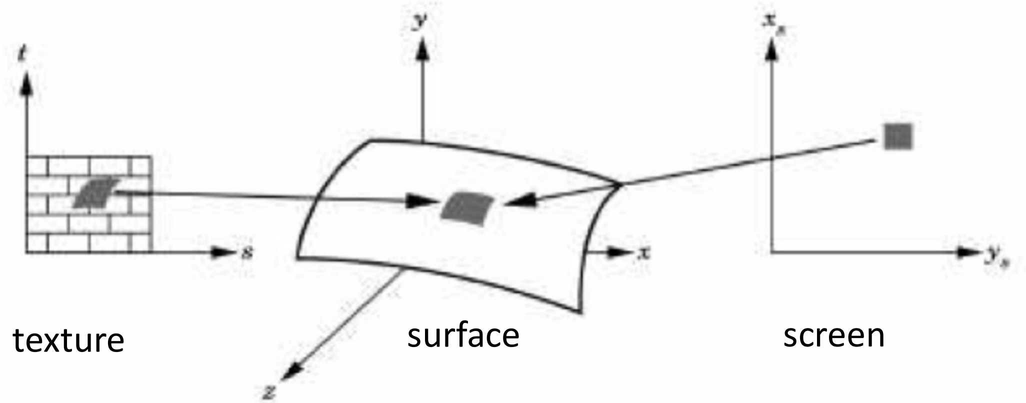
- Not always completely beautiful…



Aliasing

- From screen to texture:

  Not a one-to-one pixel mapping!



Texture problems: Aliasing

- Two types of aliasing!

- Oversampling

- Undersampling

1. Oversampling

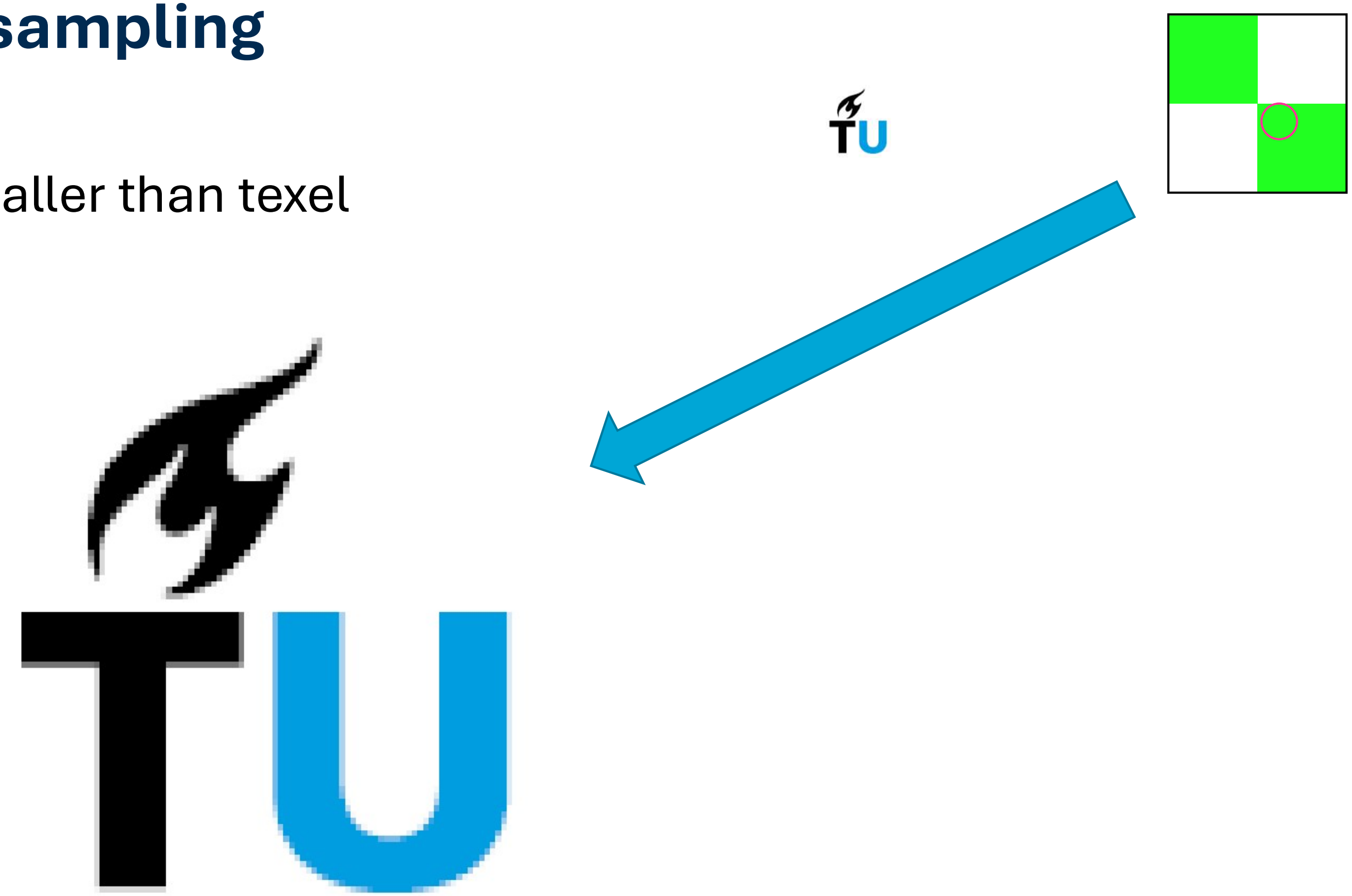
- Due to limited texture resolution

Minecraft

Resident Evil 4

1. Oversampling

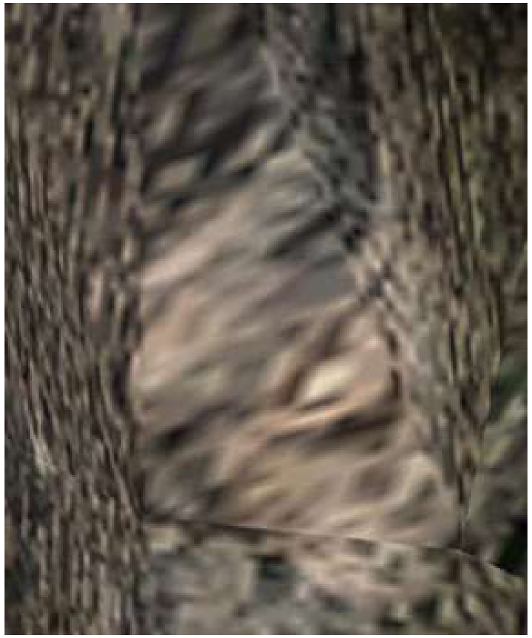
- Pixel smaller than texel



Nearest Neighbor

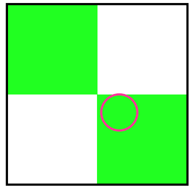
1. Oversampling

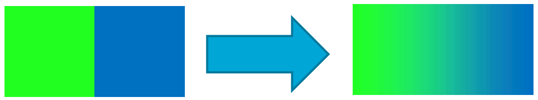
- In practice:
  Often less blocky but washed out (most applications use texel interpolation)



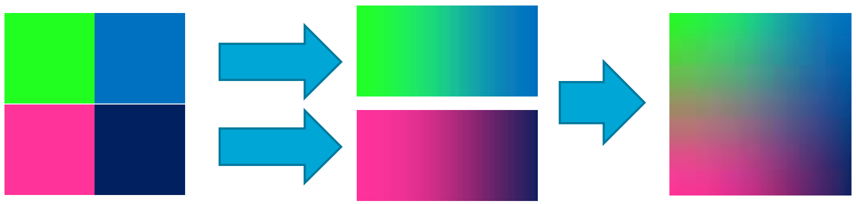
1. Oversampling

- Bilinear interpolation
  - Linear interpolation: (1-alpha) * col1 + alpha * col2



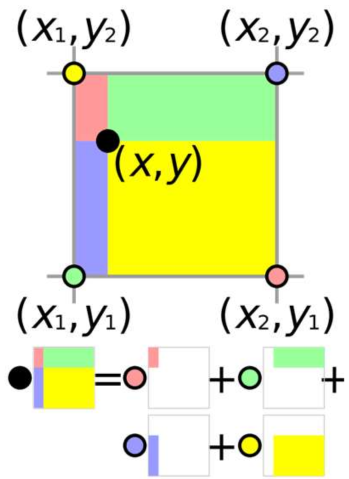


  - Bilinear interpolation:



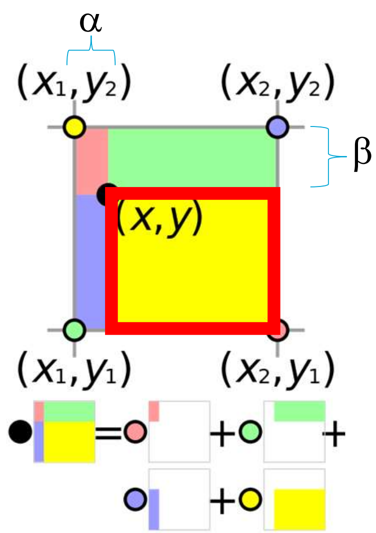
Linear in
x direction

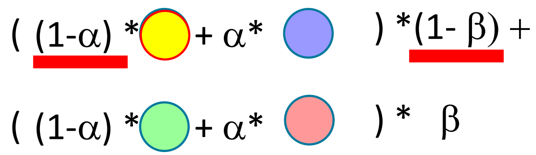
Linear in
y direction



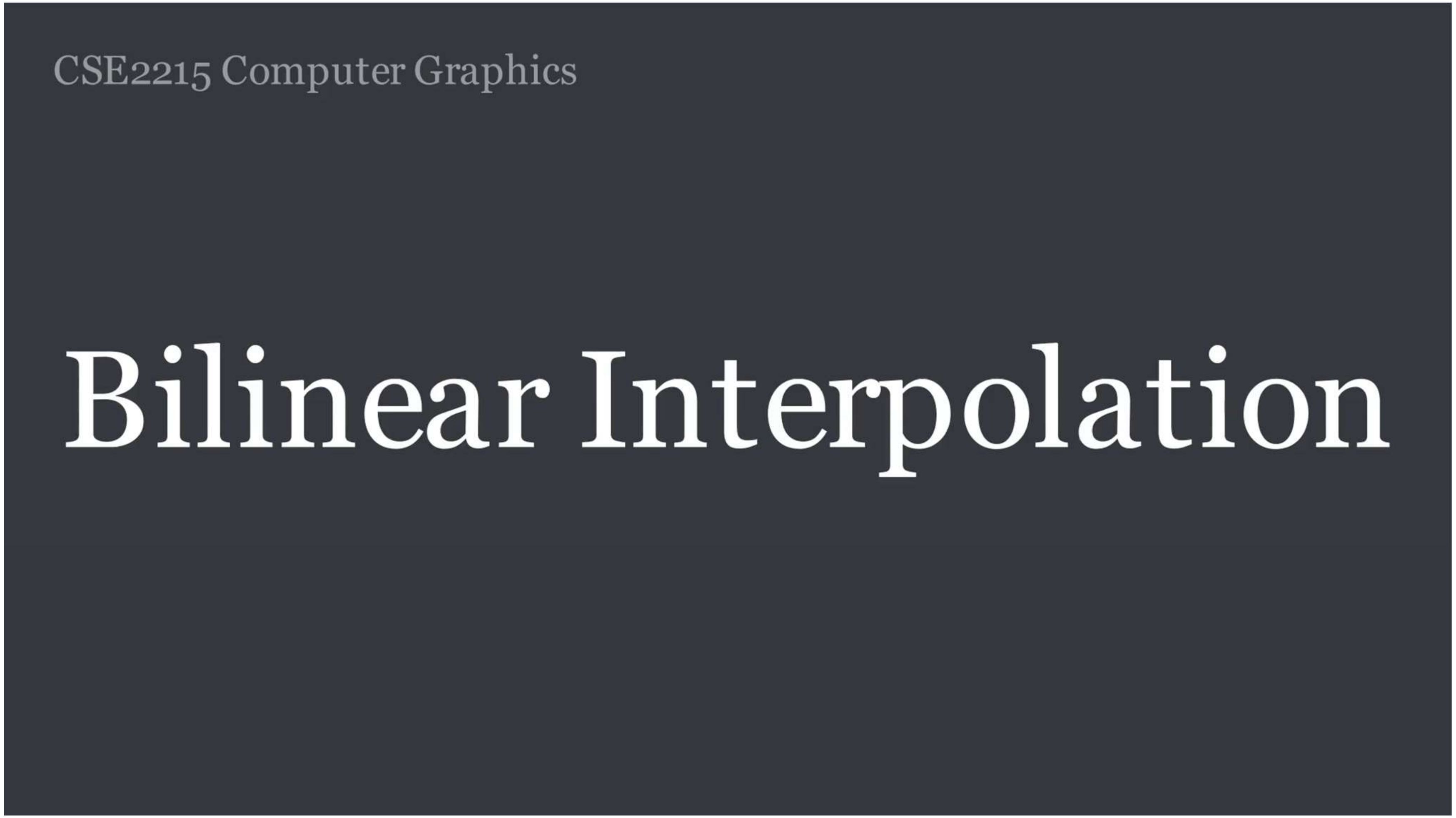
1. Oversampling

- Why does this work?





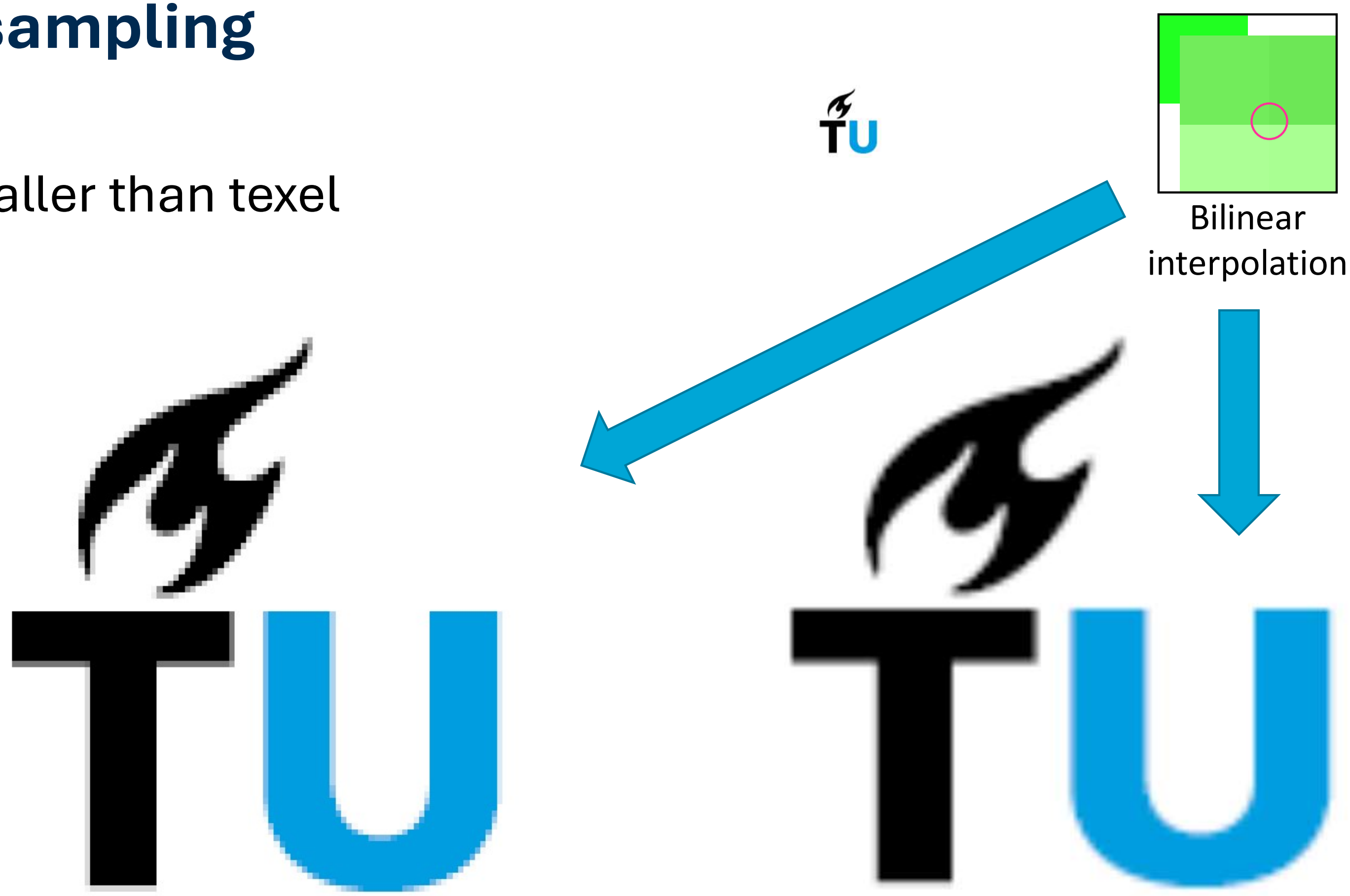
Recap Video – Bilinear Interpolation



Tim Huisman

1. Oversampling

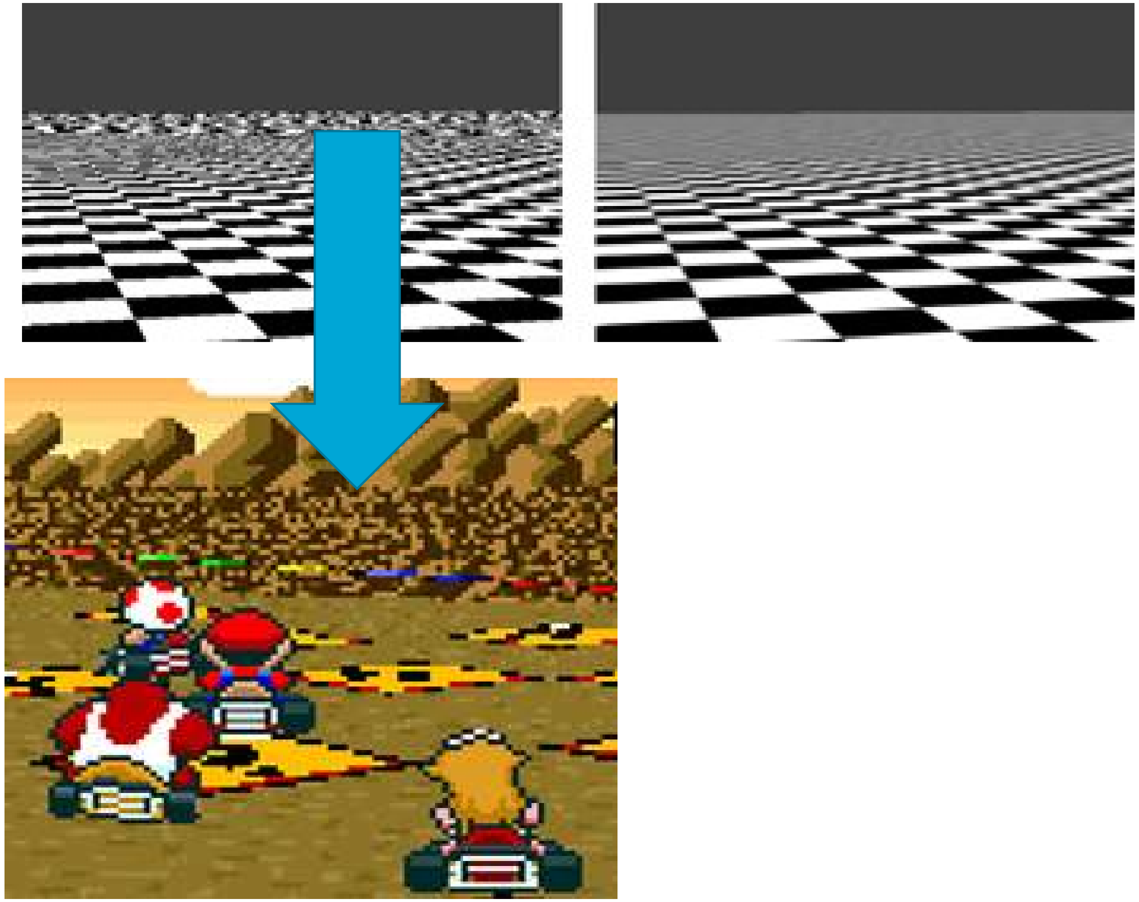
- Pixel smaller than texel



Nearest Neighbor

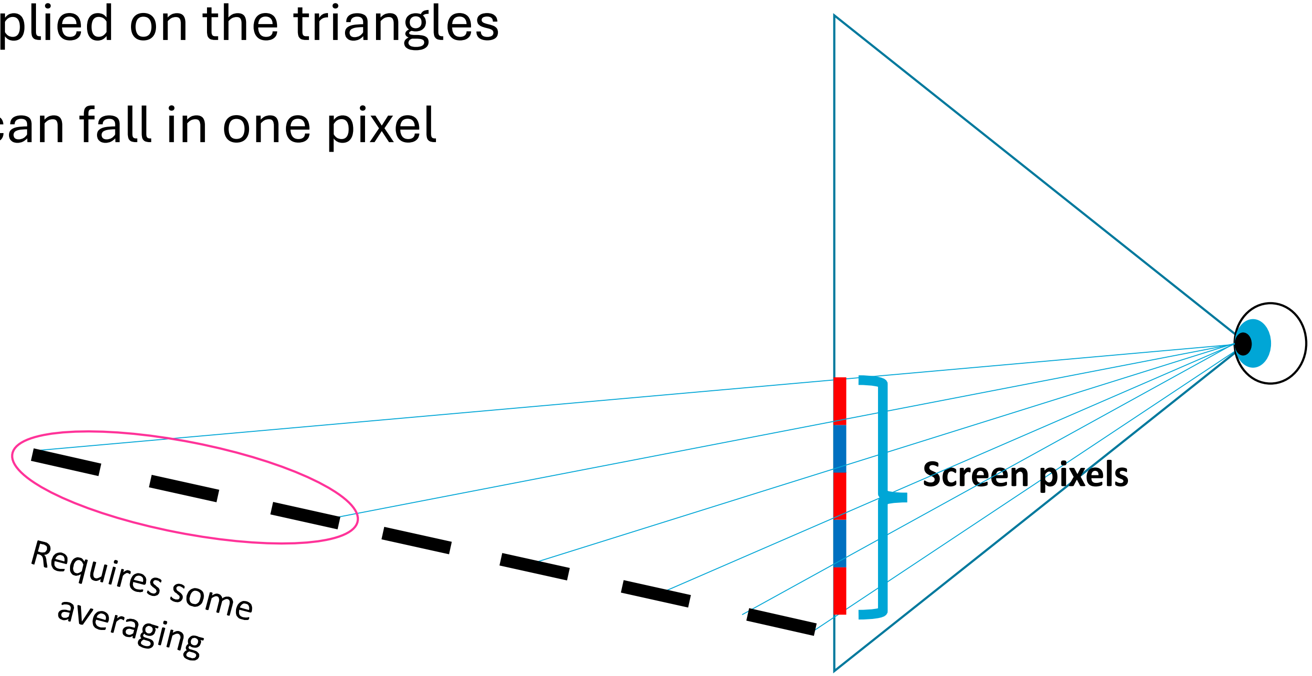
Bilinear Interpolation

2. Undersampling



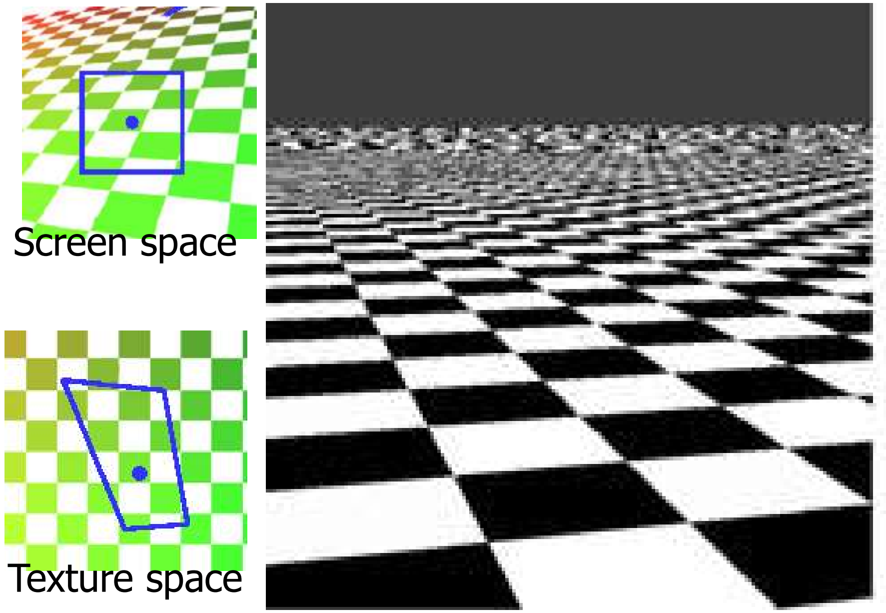
2. Undersampling

- Textures are applied on the triangles
- Several texels can fall in one pixel



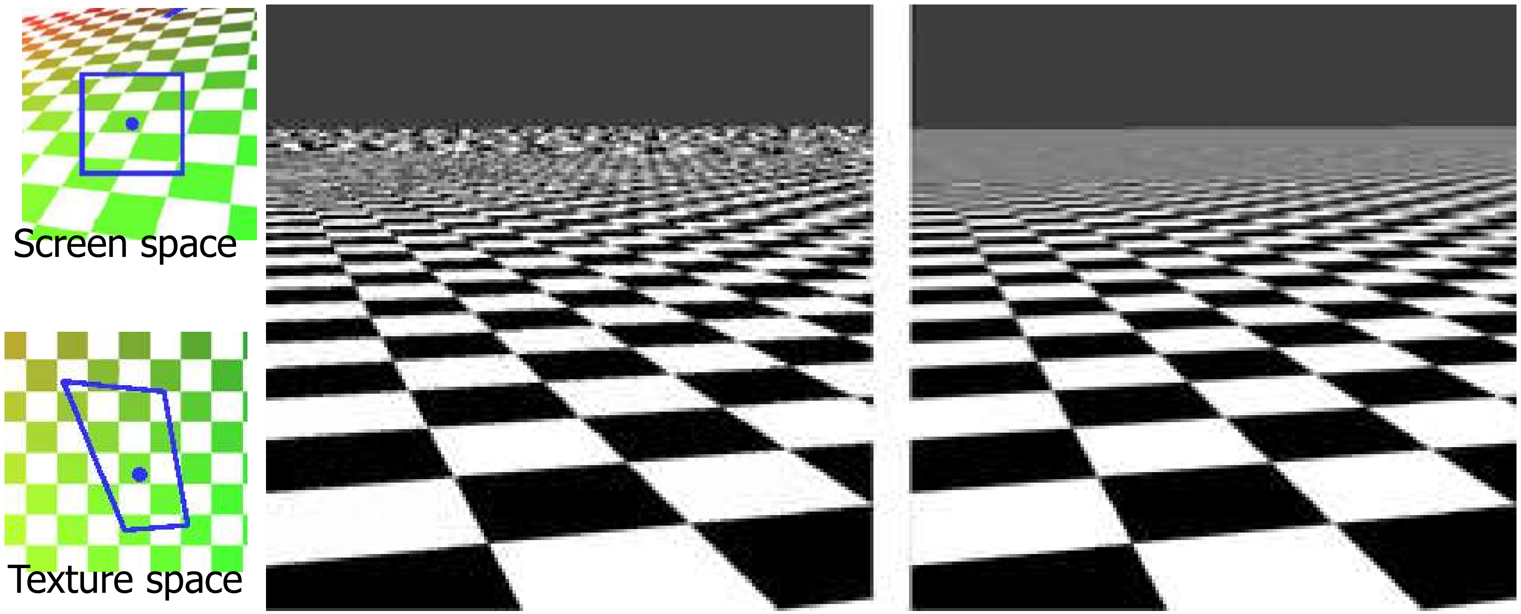
2. Undersampling

- One screen pixel does not necessarily correspond to one texel



2. Undersampling

- One screen pixel does not necessarily correspond to one texel



2. Undersampling

- Naïve solution:
  Render at high resolution
  then average the result

- Very costly....

MipMapping: Approximate Filtering

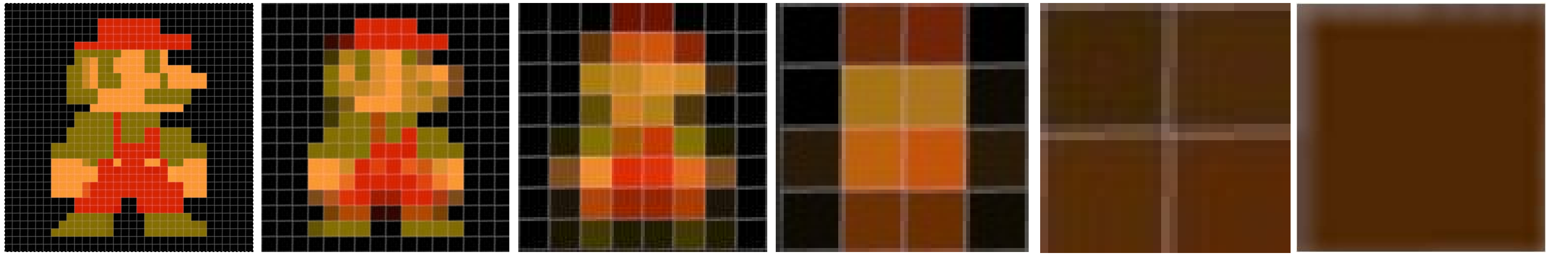
- ***Mip** = **m**ultum **i**n **p**arvo*

- meaning "much in little"

MipMapping: Approximate Filtering

- Idea: Compute a filtered texture beforehand

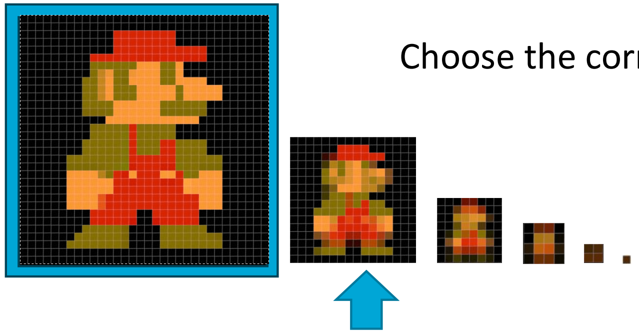
16x16 level 1 4x4 level 3 1x1 level 5



32x32 level 0 8x8 level 2 2x2 level 4

Choose the correct level depending on pixel-to-texel matching

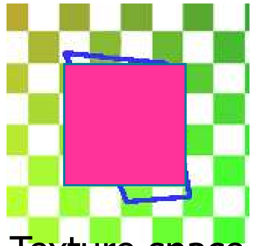
Imagine
A 32x32
screen



e.g., 4 texels per pixel

MipMapping: Approximate Filtering

- Precompute *Mipmap pyramid:*
  - *Hierarchical texture*
  - *Reduce resolution by 2x2 on each level*



Per pixel, approximate region with a square of size 2^k
k is then the mipmap level

For example:
4x4 pixels covered

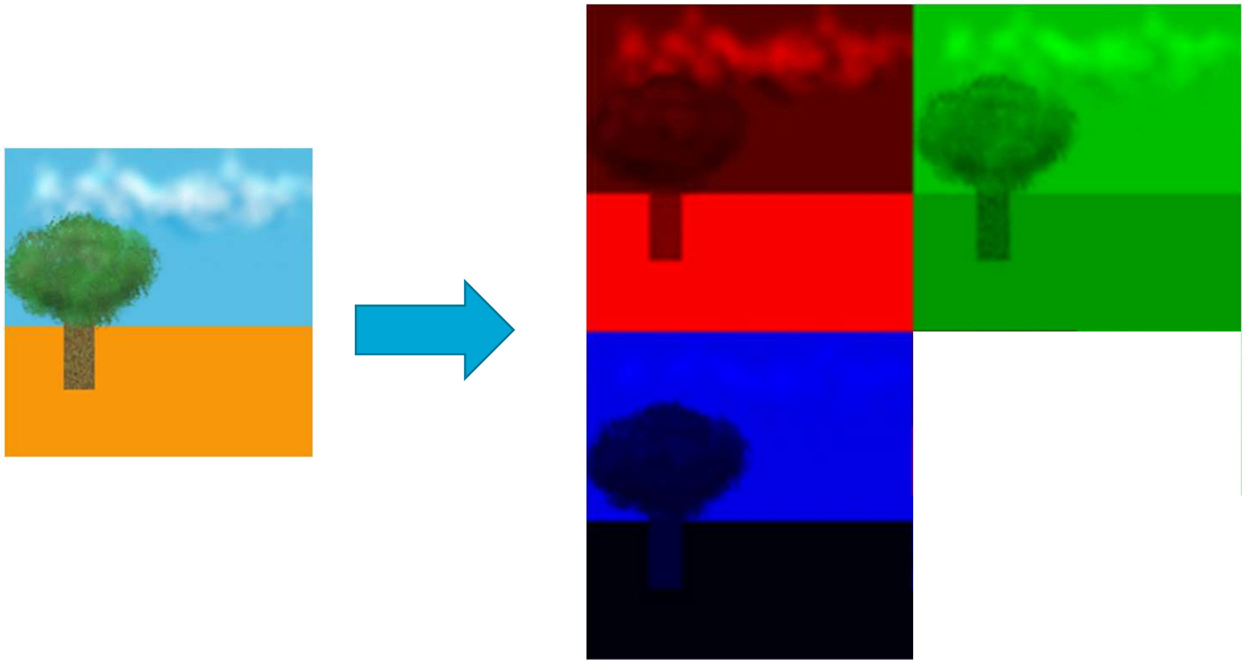
Then 16 lookups needed to average

Here: One pixel on LOD2 is a 4x4 pixel area in the original texture

A single texture lookup is enough in LOD2.

MipMapping: Memory Requirements

- Visual Proof that it is <1/3 extra memory



MipMapping: Memory Requirements

- We assume memory of the original texture is 1
- Texture resolution 2^n x 2^n
- Then the memory cost for all mipmap levels is:

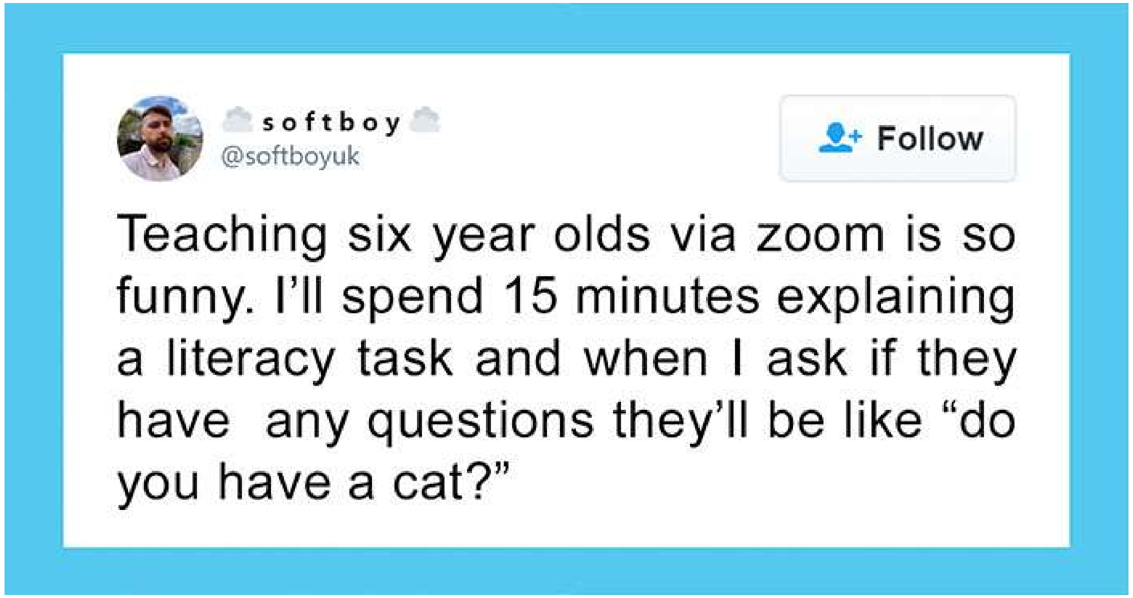
$$\sum_{i=0}^{n} \left(\frac{1}{4}\right)^{i}$$

$$= \frac{1 - 1/4^{n+1}}{1 - 1/4}$$

$$= \frac{4}{3}(1 - 1/4^{n+1})$$

In the limit the cost is 1/3 higher.

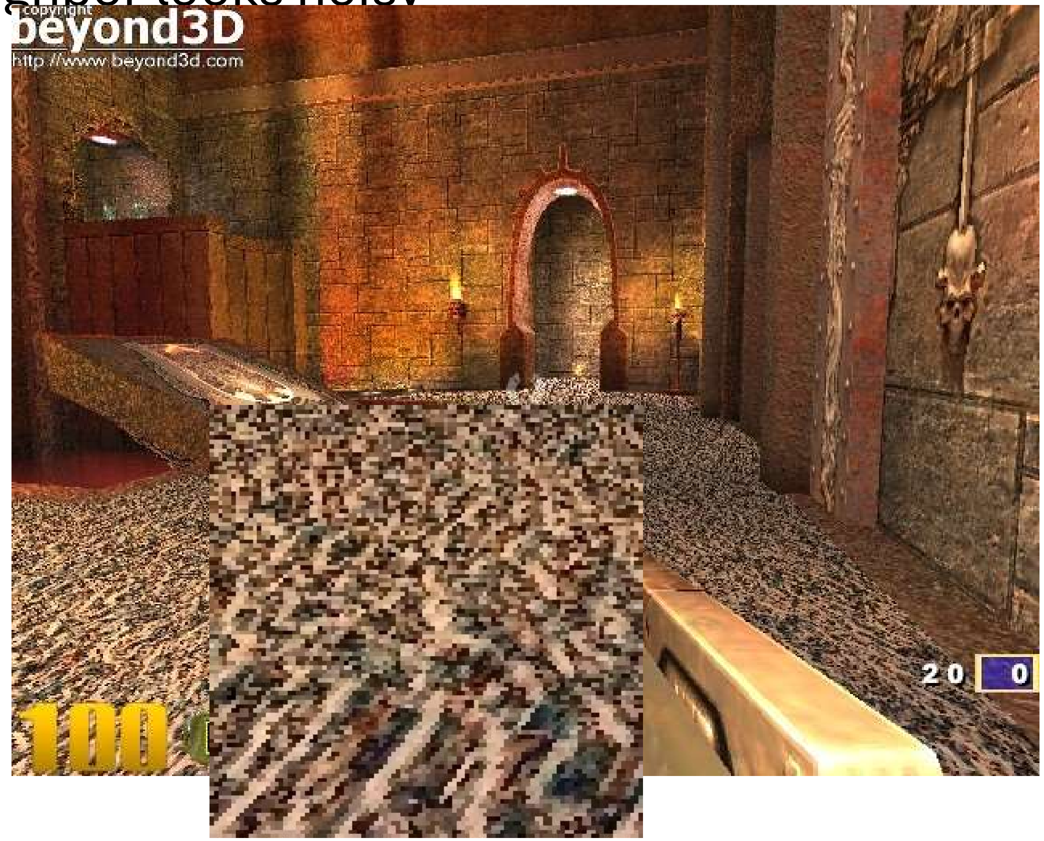
Questions?


softboy
@softboyuk
Follow
Teaching six year olds via zoom is so funny. I'll spend 15 minutes explaining a literacy task and when I ask if they have any questions they'll be like "do you have a cat?"

Let's look at an example…

MipMapping: OFF

- Just Nearest Neighbor looks noisy



MipMapping: Off + Linear Filtering

Similarly bad!

Better!

MipMapping On (use nearest level)

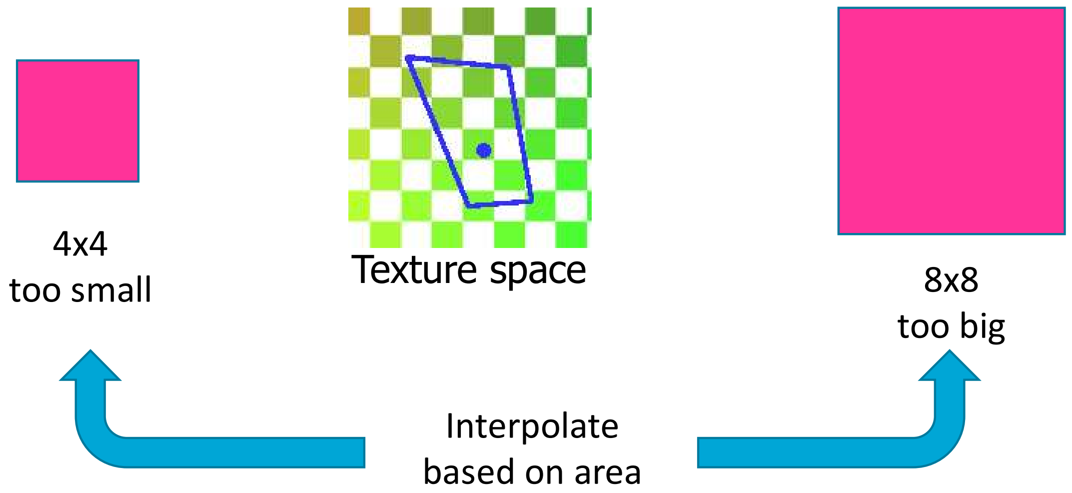
- Discontinuities at jumps from one level to next

Much nicer!

Bad: Different levels visible!

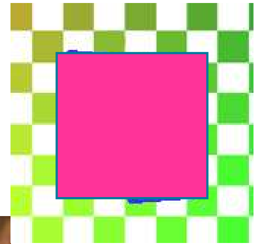
MipMapping

- Discontinuities come from changing region size



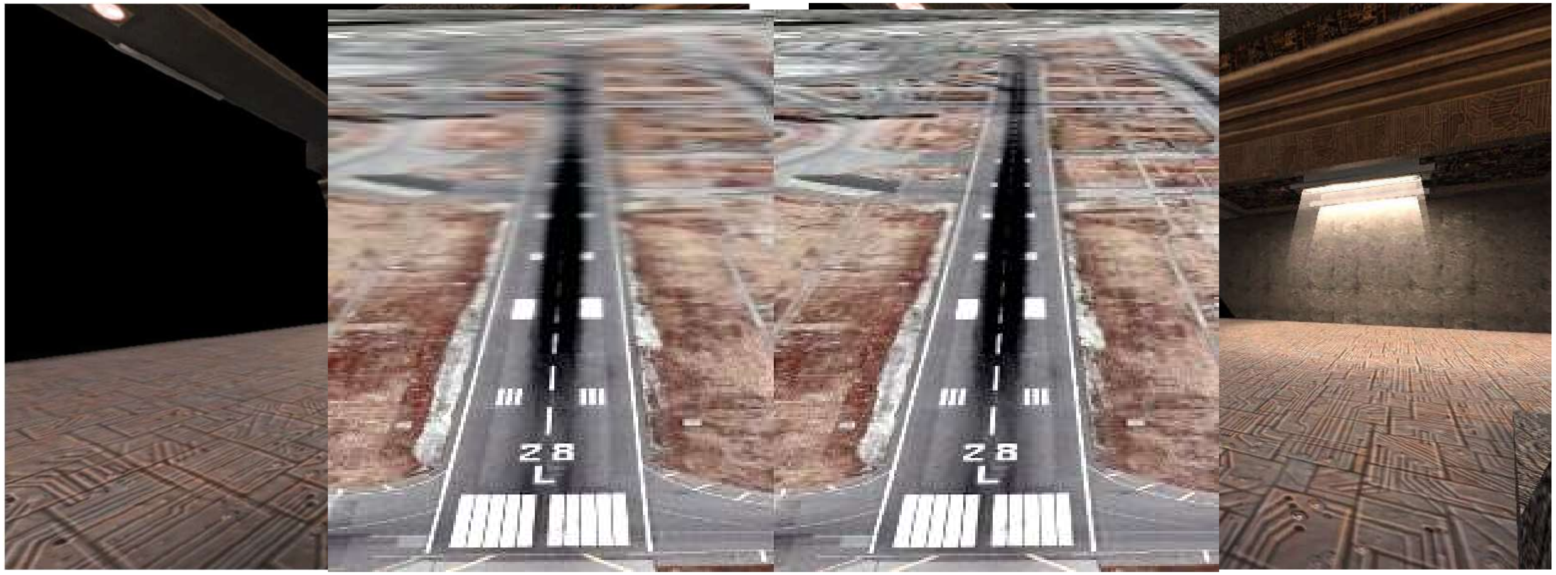
MipMapping: TriLinear Filtering

- Blend (mix) between different levels



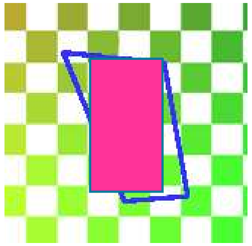
Anisotropic: Comparison

MipMapping Anisotropic



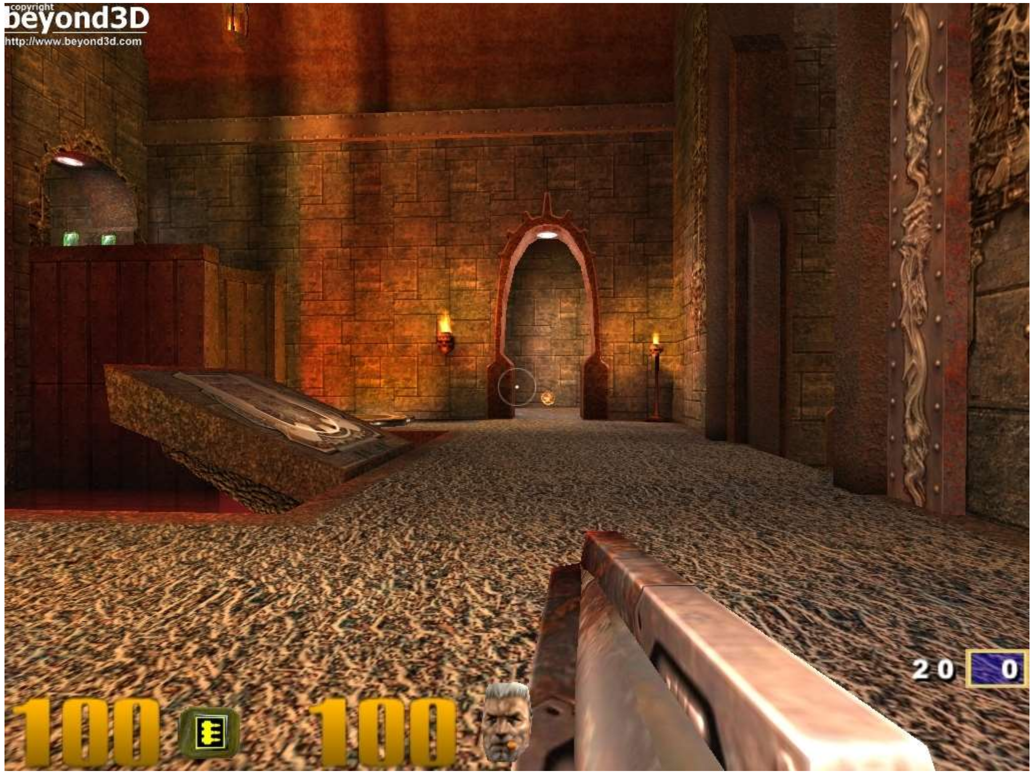
- Better approximates pixel projection in texture

Anisotropic Filtering



- Better approximate real region

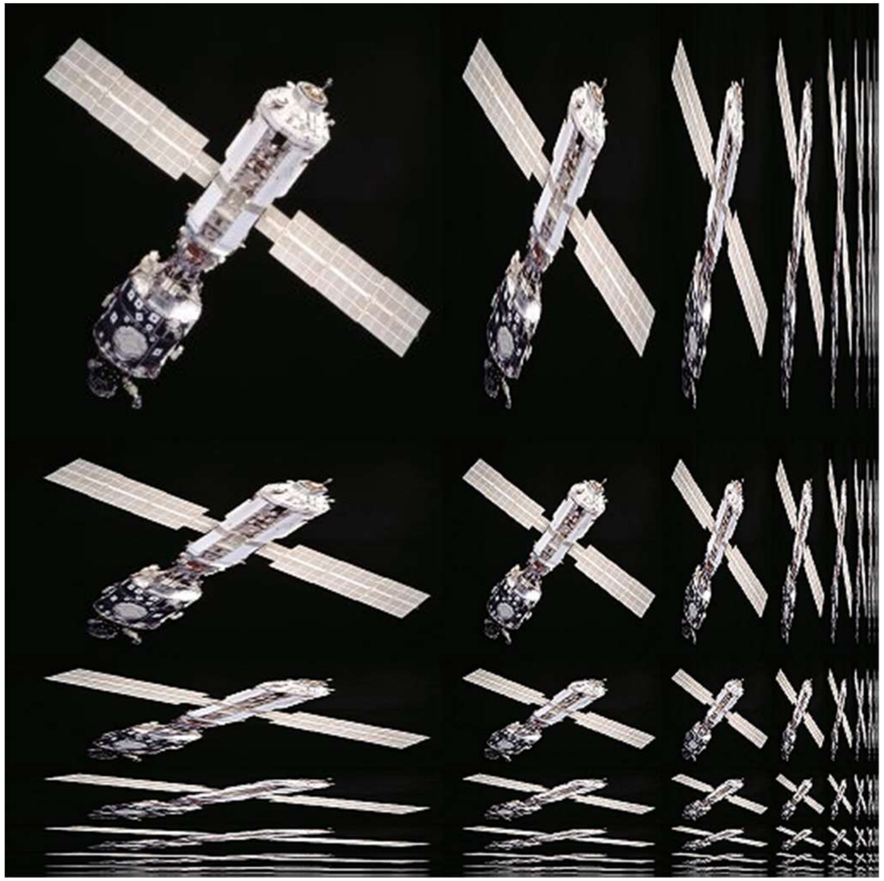
Beautiful!



TUDelft

Anisotropic MipMaps

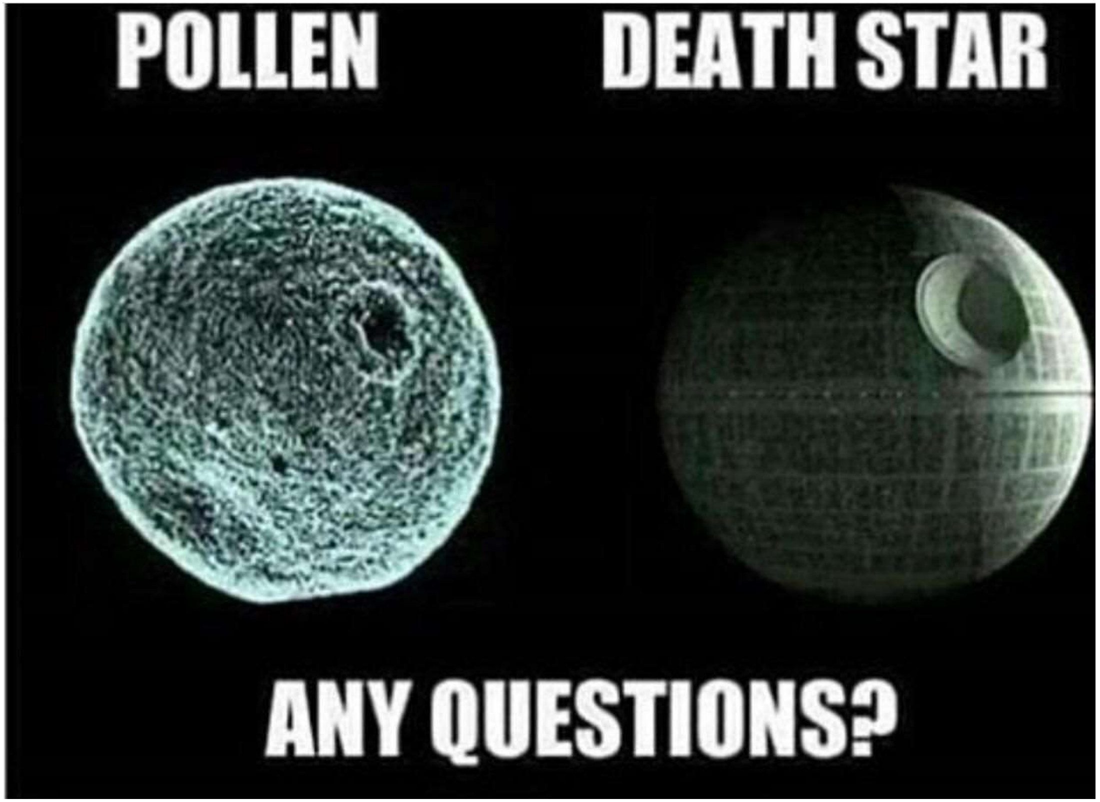
- Storage:
  - Power of two scaling on each axis
  - Fits into 4 times the original memory



MipMaps

- Absolute standard!
- Memory cost relatively low
- Computational cost very low

Questions



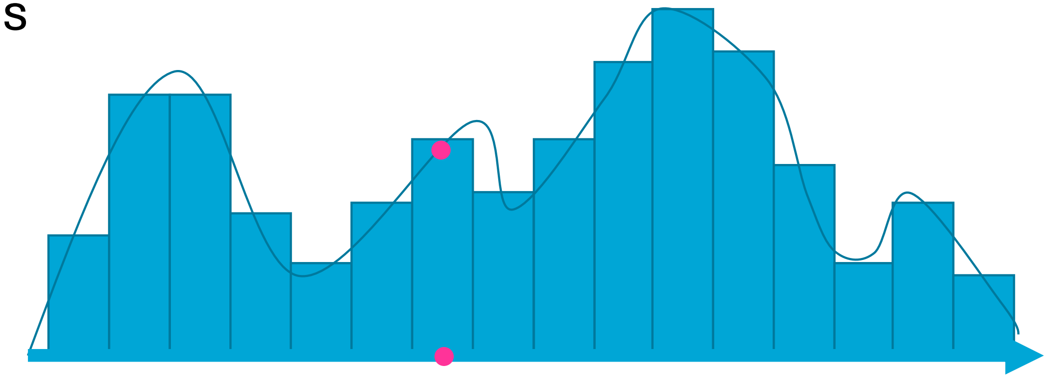
Dissection of a professional example:

You can store more than color in a texture!

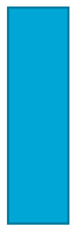
-NOT EXAM RELEVANT-

Textures as Lookup Table

- Approximate complex computations via texture lookups

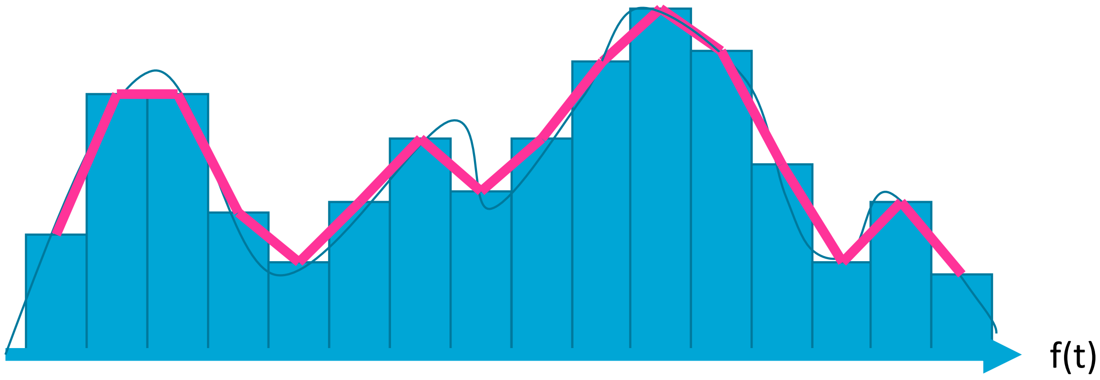


Replace computation by a lookup in a texture



Textures as Lookup Table

- Activating linear texture interpolation leads to a piecewise linear approximation

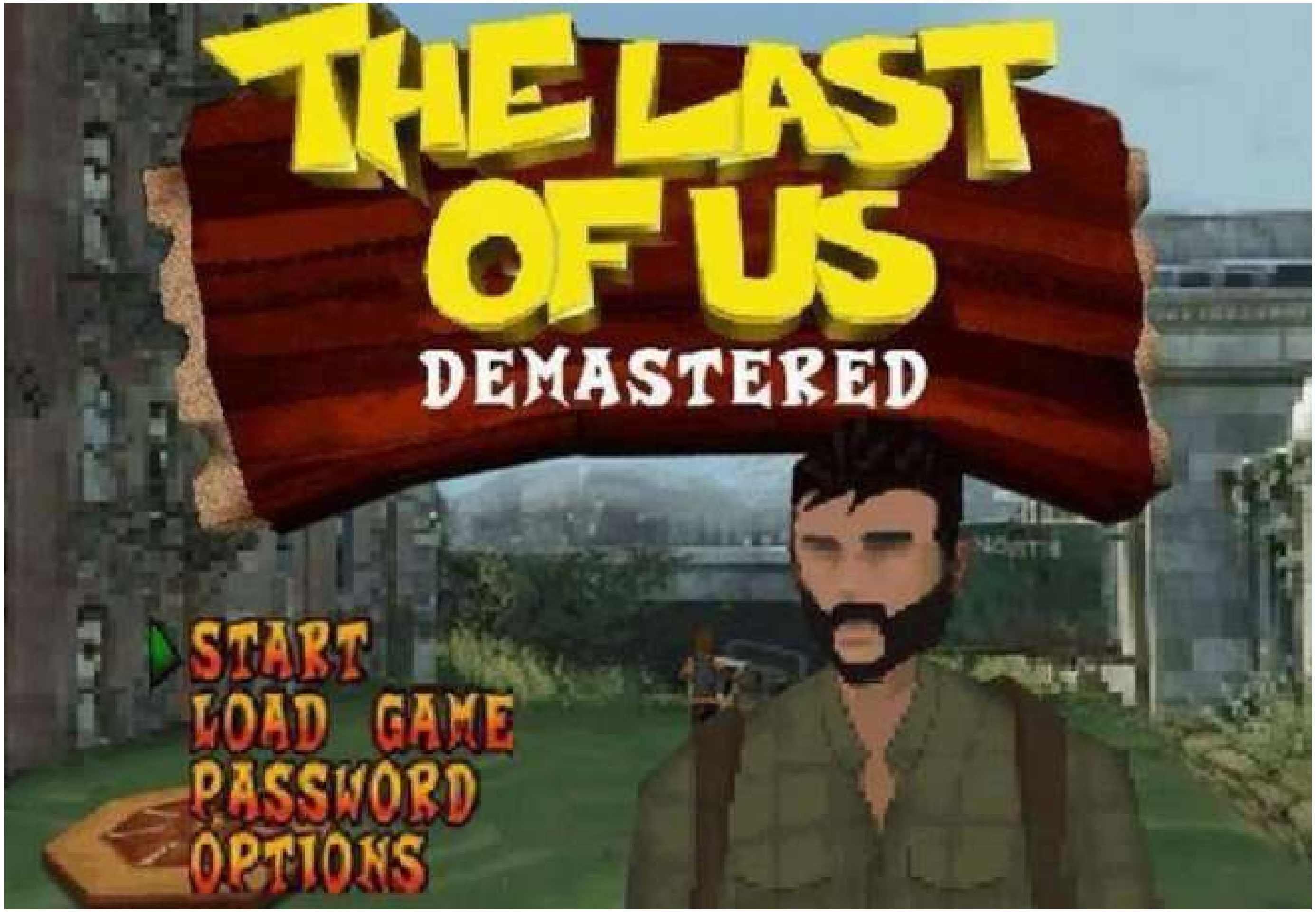


- This type of representation is very common in high-performance computing and simulations

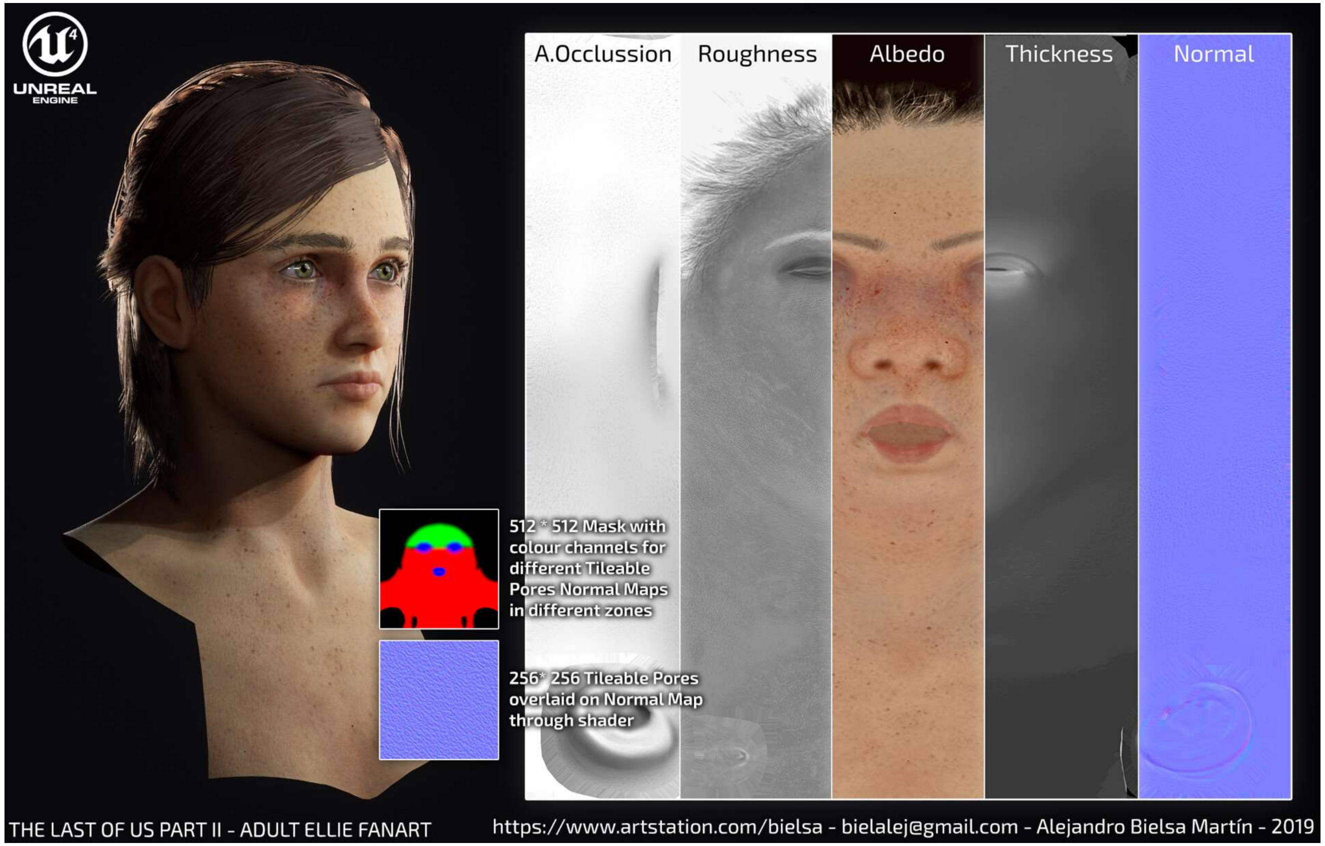
Textures to the top!

Last of us

Last of Us on PS1

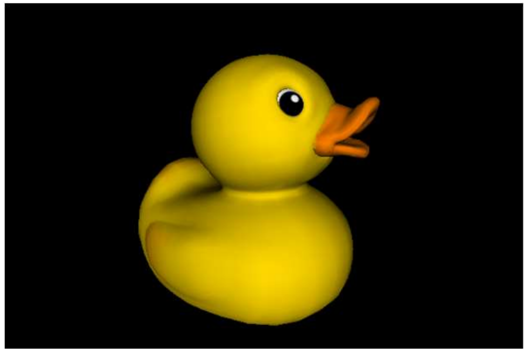


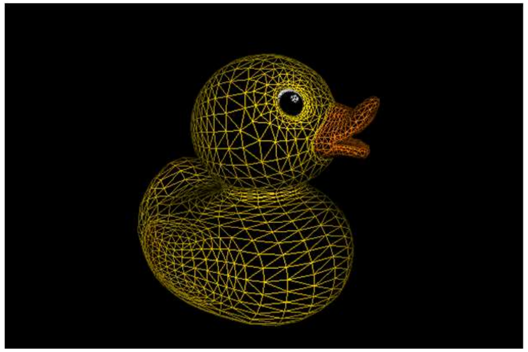
Last of Us – Professional Example



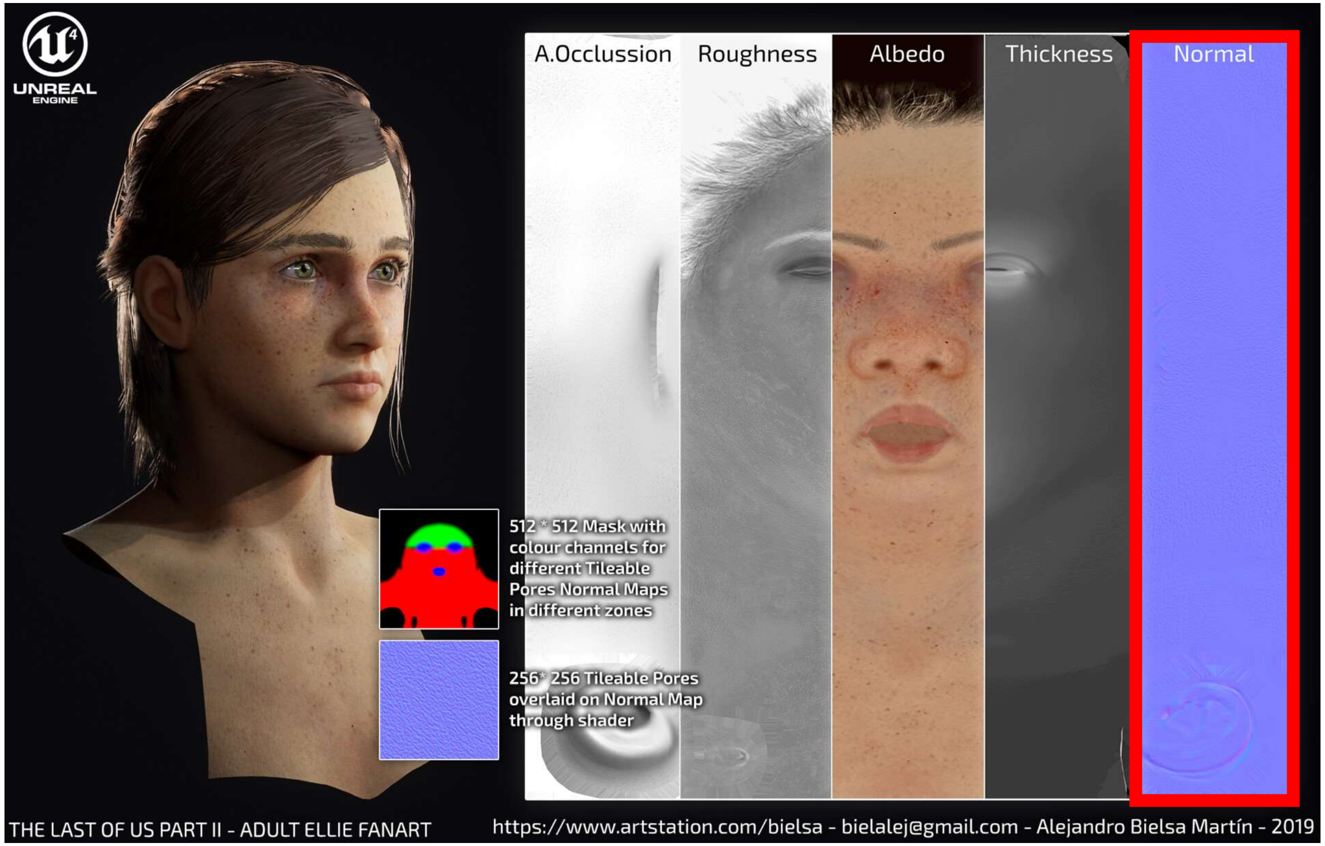
Textures: Materials

- Can be used to provide, e.g.,
  - parameters for material models (ambient, diffuse...)



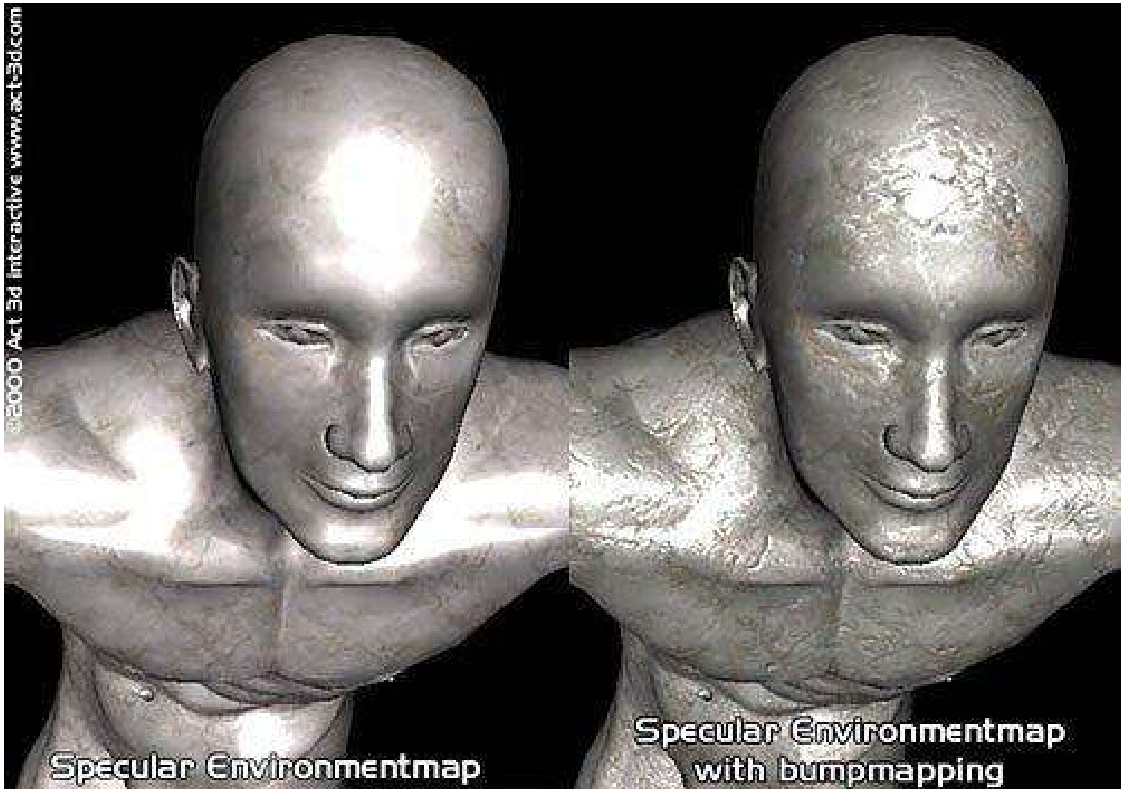


Last of Us – Professional Example



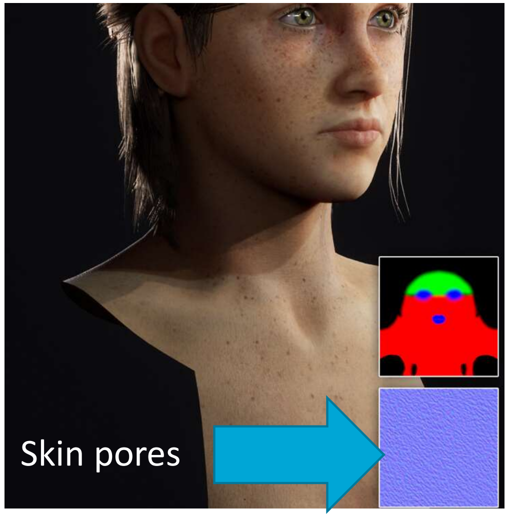
Textures: Normal Mapping

- Encode Normals (*bump/normal mapping*)



Textures

- Normal Mapping



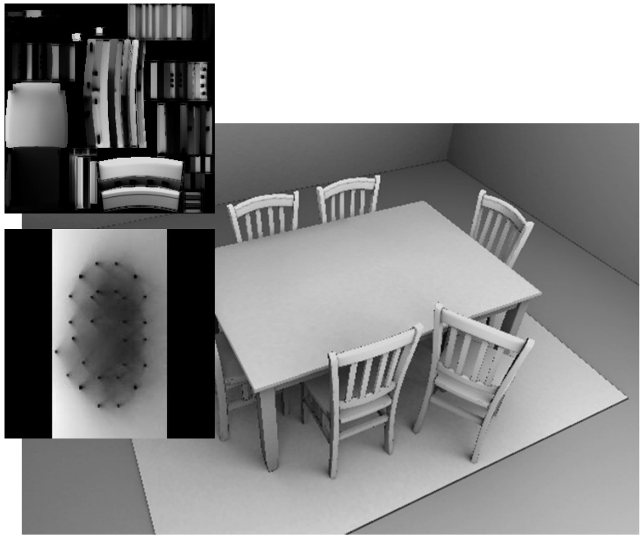
100,000+ polygons

=

Textures: Light Maps

- Precomputed illumination (*light map*)


Diffuse mapped on scene


Lightmap mapped on scene


Combined

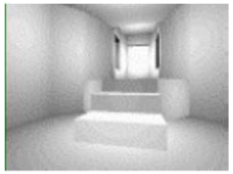
Textures: Light Maps

- Precomputed illumination (*light map*)

Quake – id Software

Materials

*



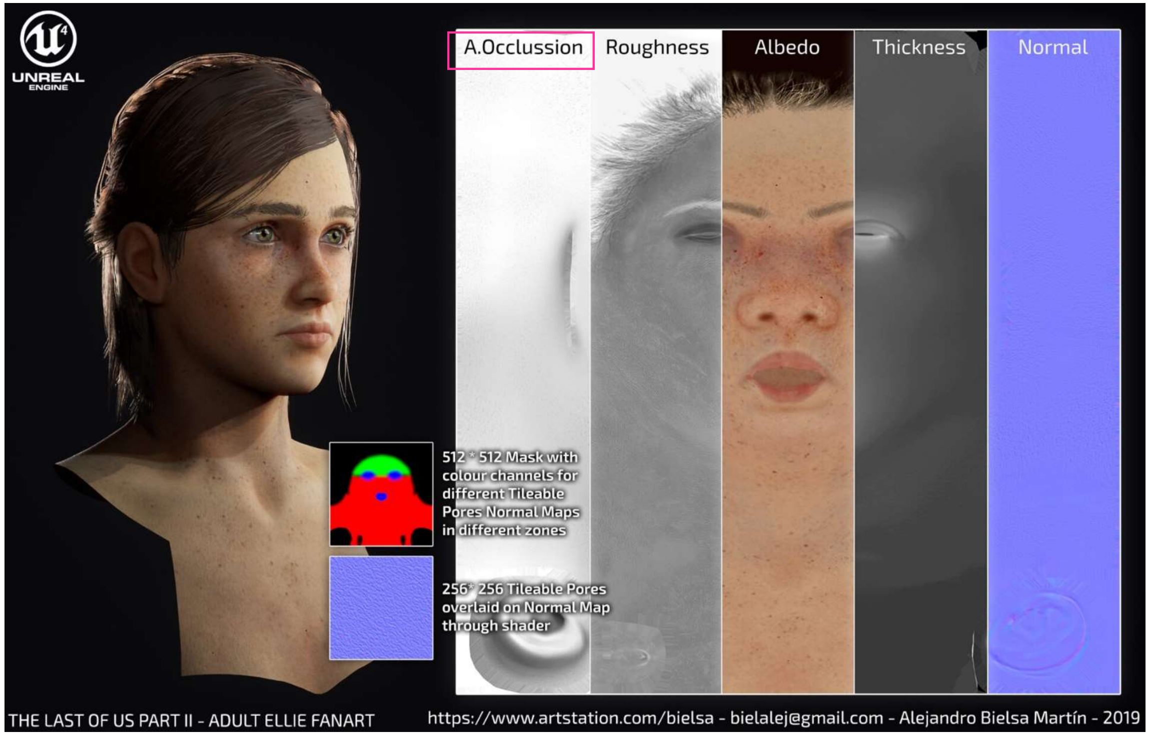
Light Map

=

Final result

Textures: Light Maps

- Precomputed illumination (*light map*)



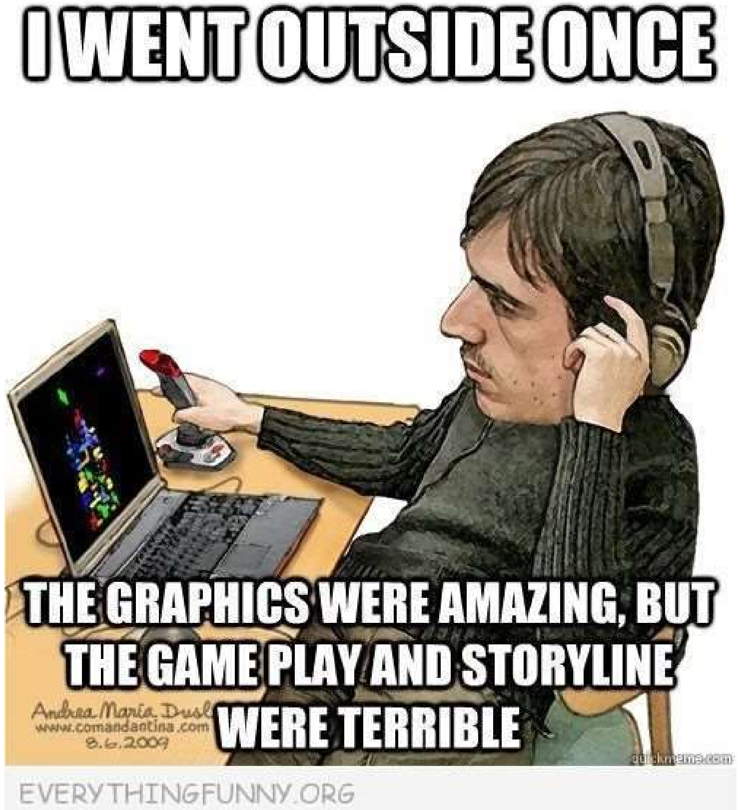
Texture Summary

- Textures are used to attach data to scene points
- Mapping performed via Interpolation
- Filtering is crucial for high quality

- Broad range of applications
  - Actually many more things

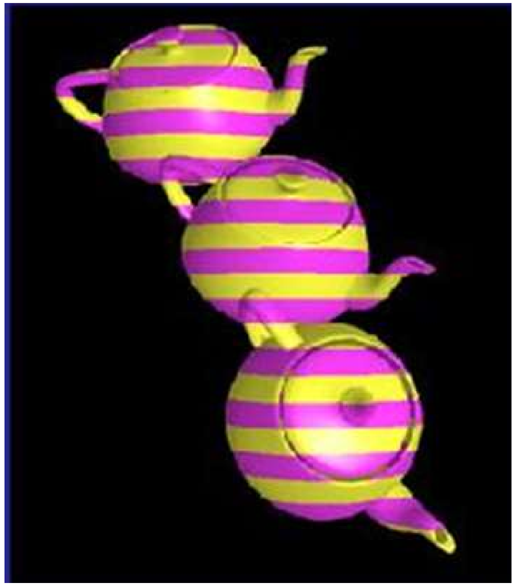
  ... a little “foreshadowing” ;)

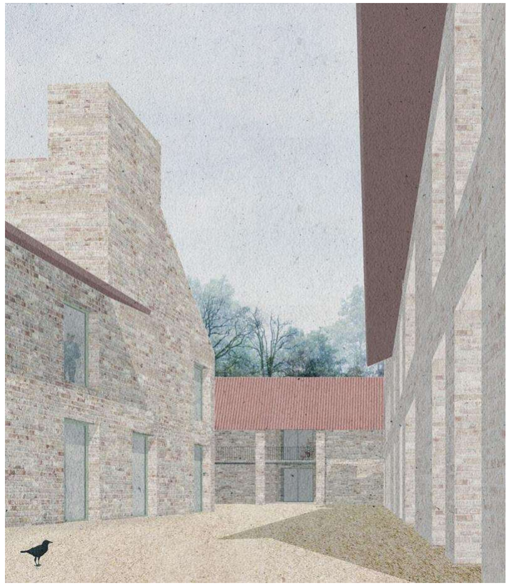
Thank you very much!



Exercise: Texture Coordinates in Screen Space

- Example application:
  Add paper grain over your rendered image





CGArchitect Forums

How would you define the texture mapping T to obtain screen-space texture coordinates?
Hint: Use the projected vertex position!

CSE2215 - Computer Graphics

Shadows
Separate Light & Darkness

Elmar Eisemann

Delft University of Technology

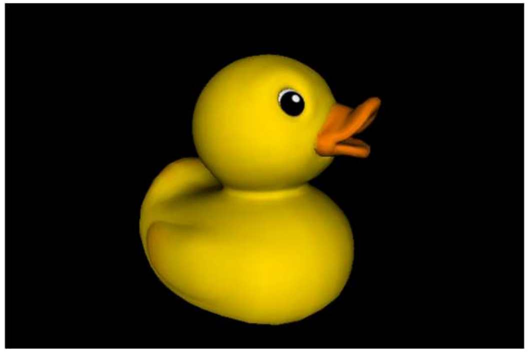
Textures

- Mapping an image onto a surface

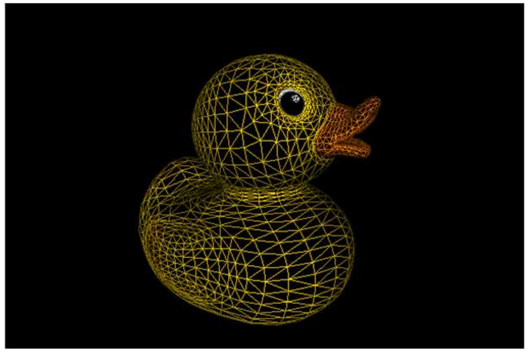
Texture

consisting of *texels*
(**tex**ture **el**ements)

Texture mapped triangles

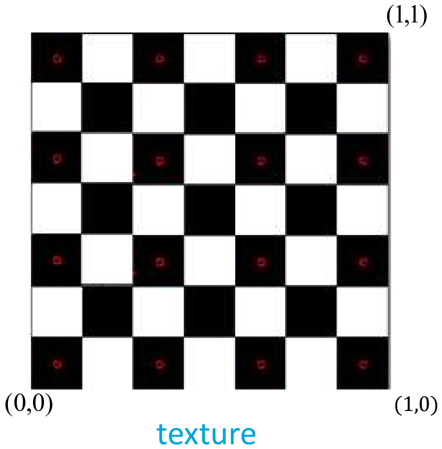


Wireframe of triangles



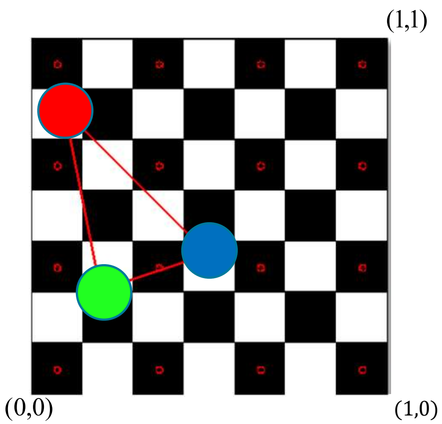
Textures

- Map image via texture coordinates

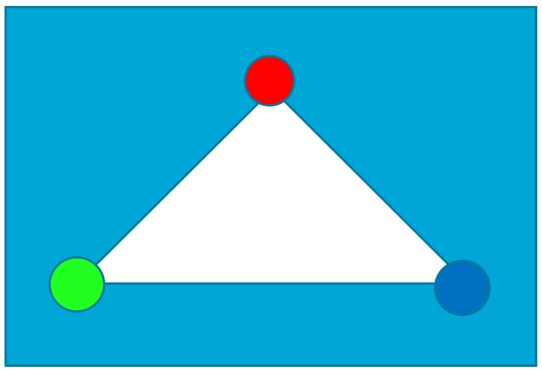


Textures

- Map image via texture coordinates
  - Specify a texture coordinate at each vertex



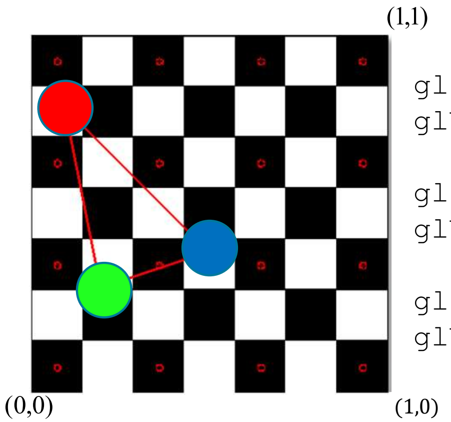
texture



Camera View

Textures

- Map image via texture coordinates
  - Specify a texture coordinate at each vertex
  - Vertex texture coordinates are interpolated over triangle

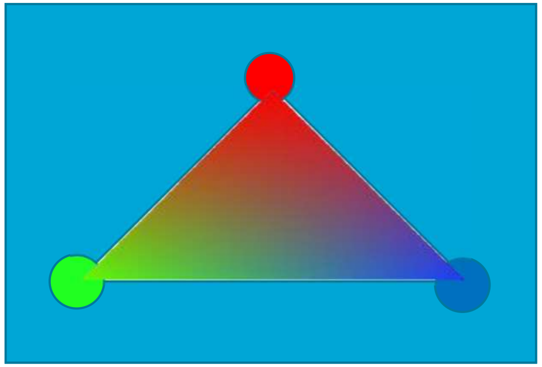


texture

```
glTextureCoord2f(0.2f,0.3f);
glVertex2f(-1.0f,0.0f);

glTextureCoord2f(0.5f,0.4f);
glVertex2f(+1.0f,0.0f);

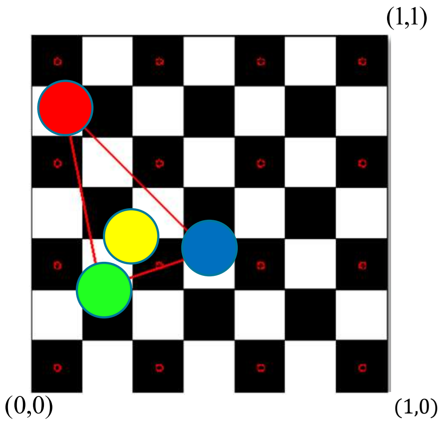
glTextureCoord2f(0.1f,0.8f);
glVertex2f( 0.0f,1.0f);
```



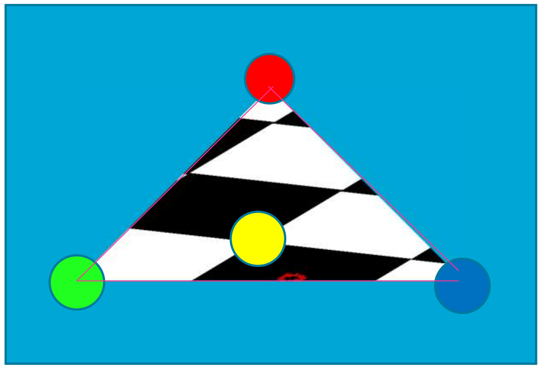
Camera View

Textures

- Map image via texture coordinates
  - Specify a texture coordinate at each vertex
  - Vertex texture coordinates are interpolated over triangle
  - Drawn pixels use interpolated coordinates to retrieve texel values



texture

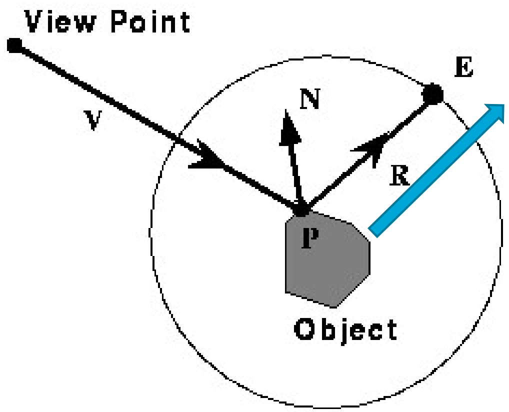


Camera View

Texture Mapping: Special Software

Environment Mapping

- Textures can encode an environment
  An environment map (approximation of scene)
  is useful for, e.g., reflections (texcoords = ray)



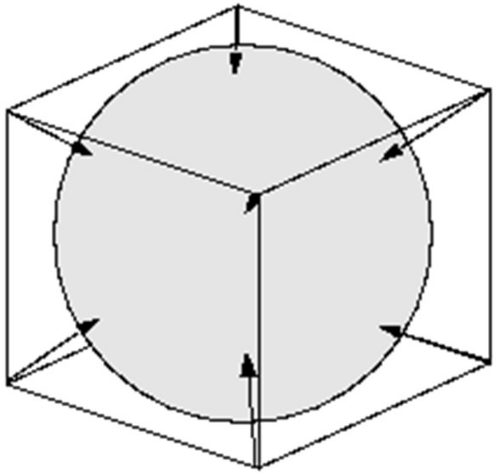
Ray direction determines lookup position

- This *environment mapping* is an approximation!

Environment Mapping

- Alternatively, a sphere can be mapped to a cube
- Less distortions if images are used for the cube faces

Cube mapping



It is costly, but you can even update such textures in every frame!
Render an image (or here, it would be 6) and use it as a texture.

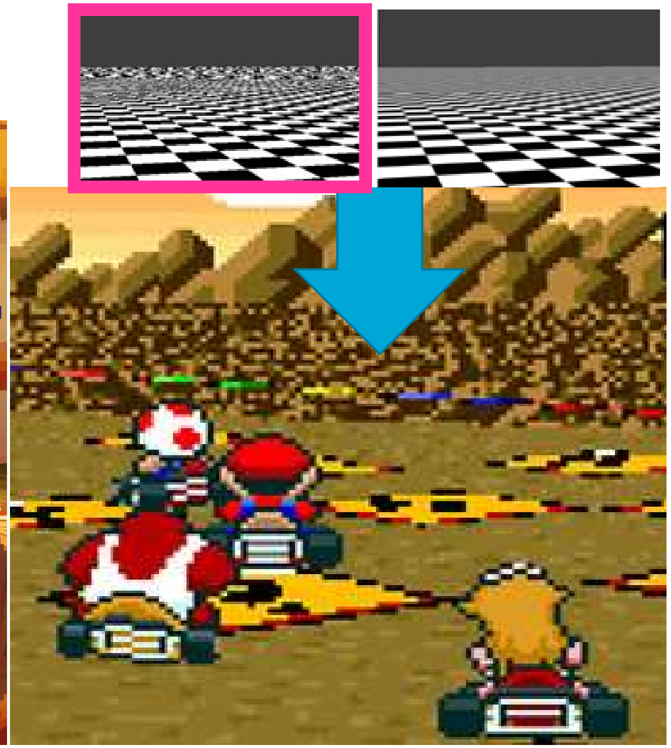
Textures

- Image content mapped on surfaces
- Increase detail level without geometric cost

- Many applications for color textures

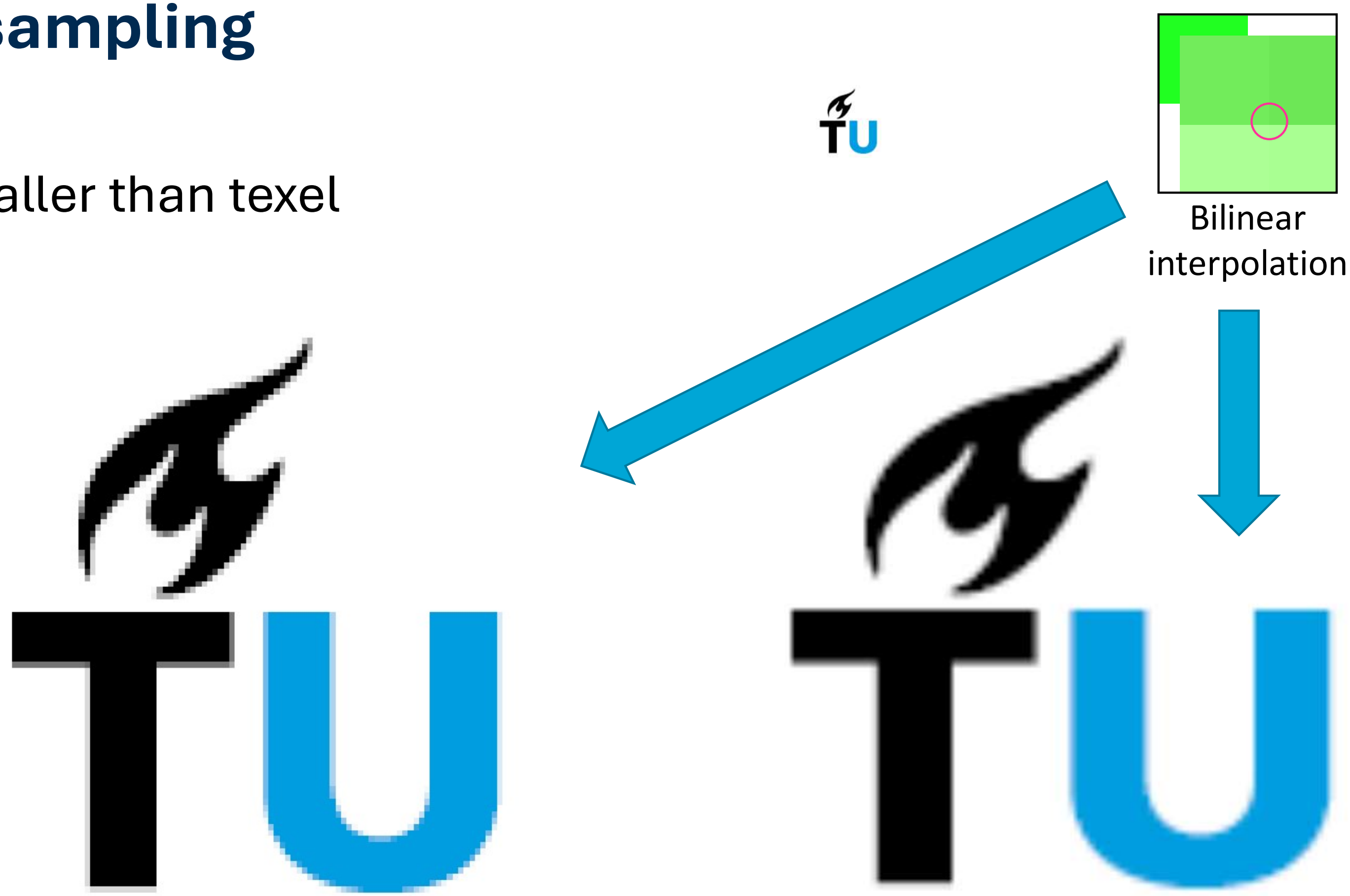
Texture issues: Aliasing

- Not always completely beautiful…



1. Oversampling

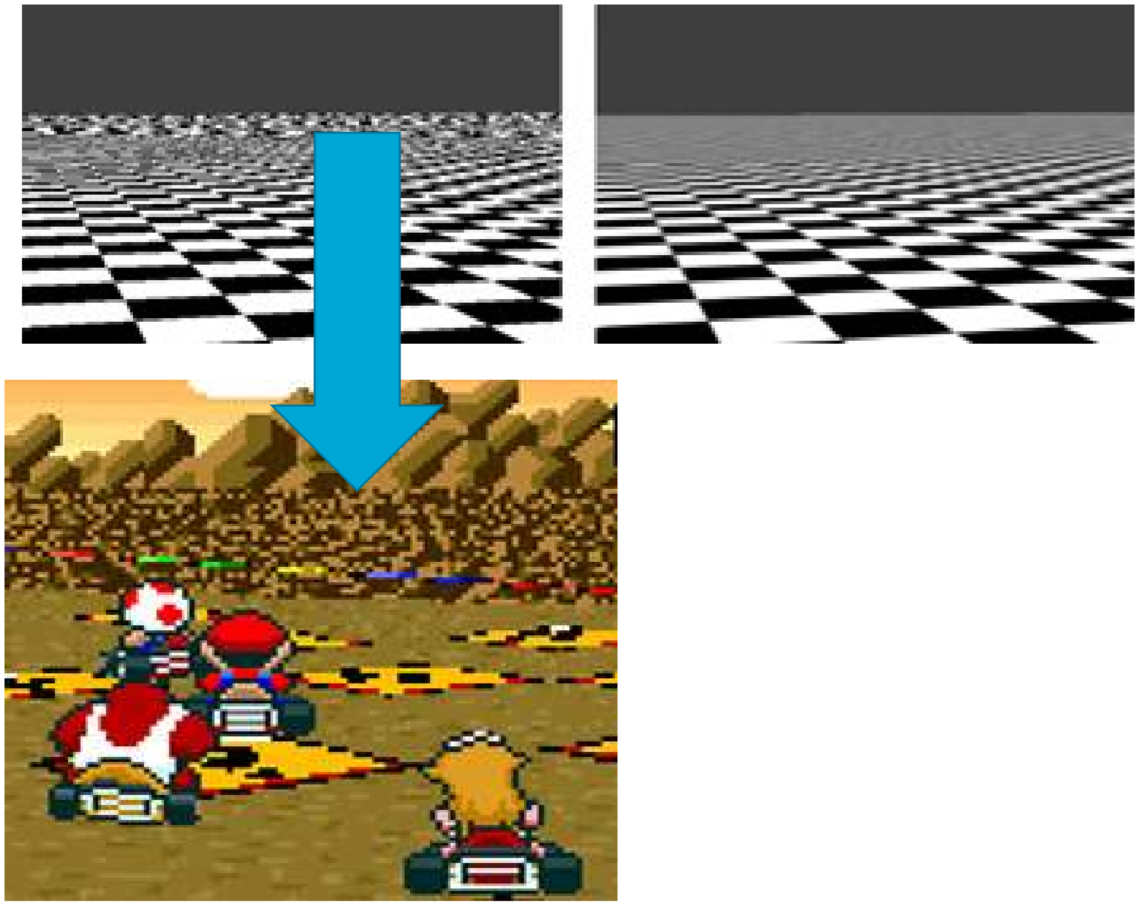
- Pixel smaller than texel



Nearest Neighbor

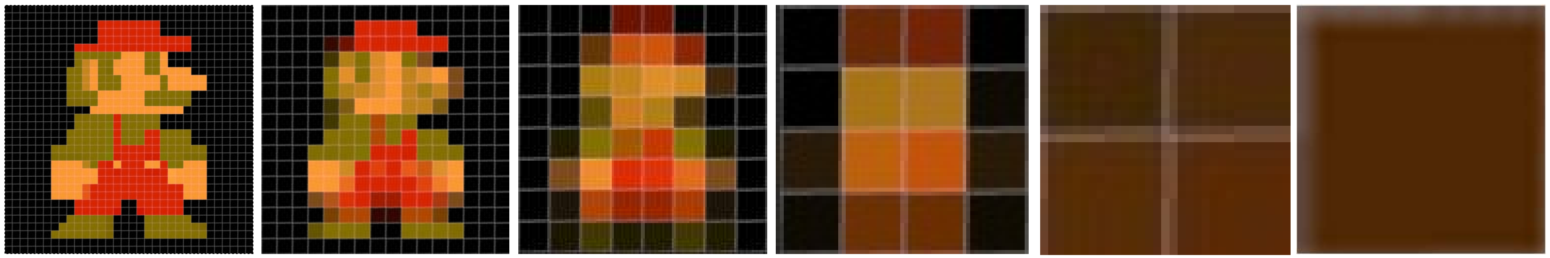
Bilinear Interpolation

2. Undersampling



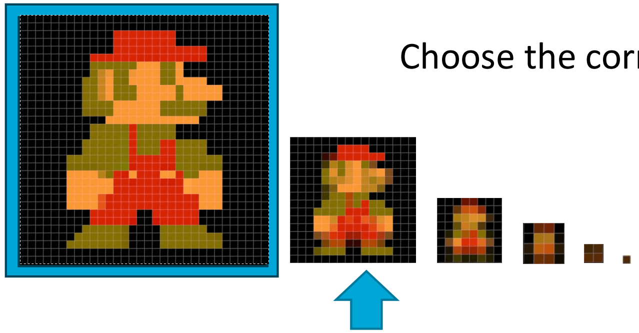
MipMapping: Approximate Filtering

- Idea: Compute a filtered texture beforehand



Choose the correct level depending on pixel-to-texel matching

Imagine
A 32x32
screen



e.g., 4 texels per pixel

Texture filtering: Off

Texture filtering: On

Advanced Texture Representations

- Texture Data can represent more than just colors of an image
- Modern definition:
  A texture stores values associated to a domain

Advanced Texture Representations

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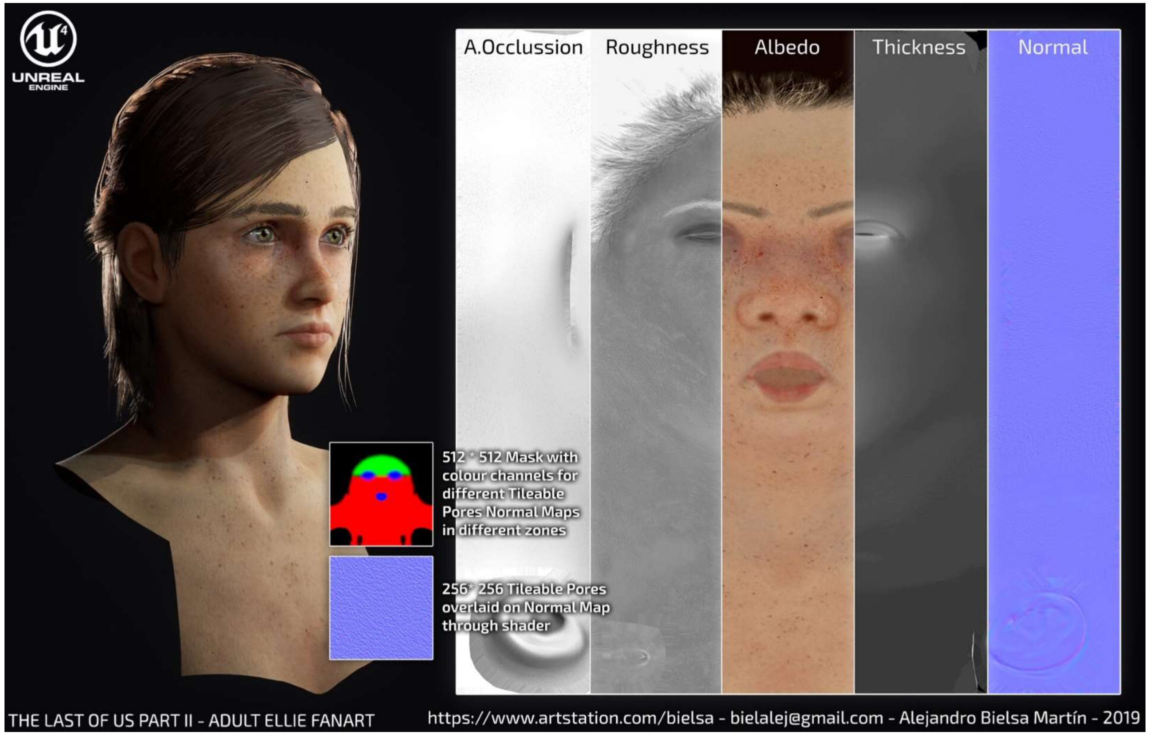
Multiple Texture Maps store Surface Appearance



Demo Time…

Questions?

CAN I ASK 1 QUESTION?
YOU JUST DID
HAHA.. OK CAN I ASK 2?
YOU ALREADY DID
OK CAN I ASK 4 QUESTIONS?
YOU JUST DID
WHEN ?
NOW

Today's Study Goals

- S1- Explain and compare the structure and properties of standard algorithms and data structures linked to Computer Graphics.
  - 1D and 3D textures, and Shadow Mapping.

- S4- Apply mathematical modeling and theory of geometric computations and transformations, object representations, simulation, and encoding.
  - We discuss appearance, derive the formulas, and compute shadows.

Advanced Texture Representations

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Advanced Texture Representations

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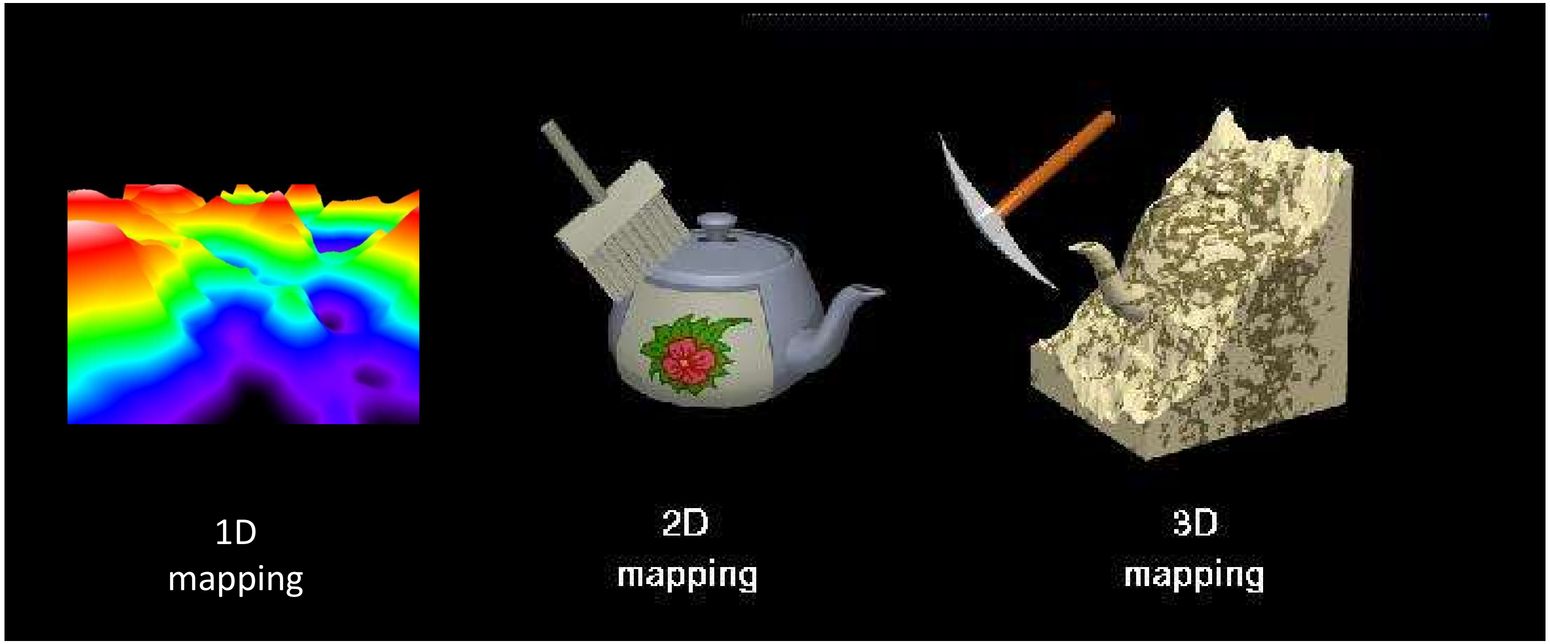
- Modern definition:
  A texture stores values associated to a **domain**

domain is accessed via the texture coordinates,
so far, we had 2 coordinates to access a 2D image.

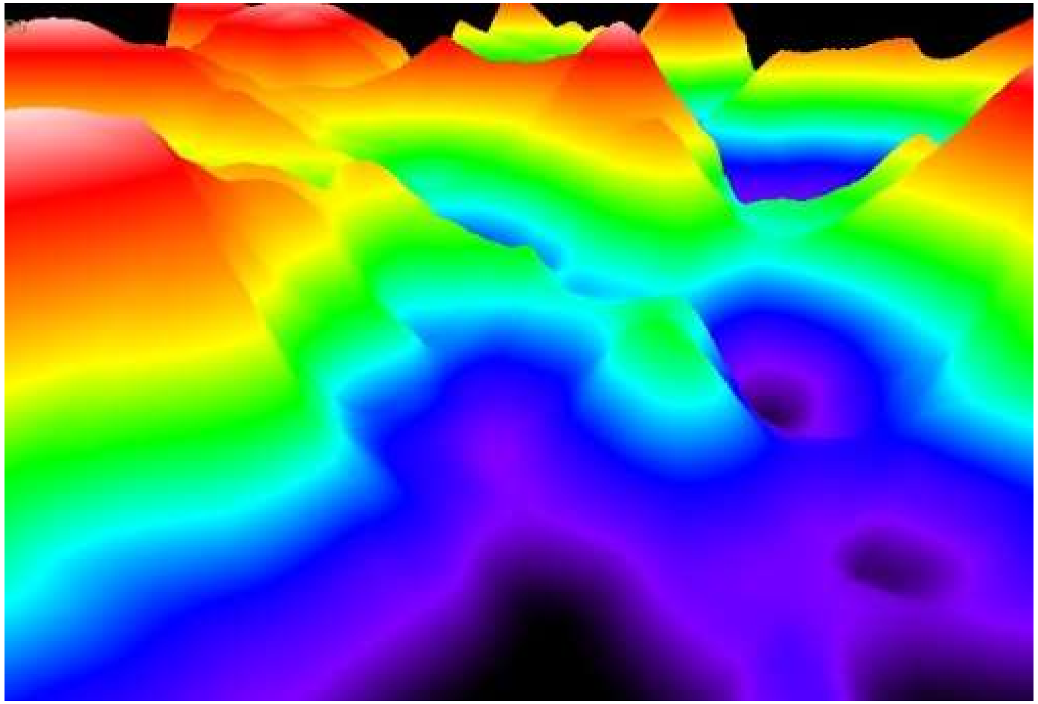
Advanced Texture Representations

- Textures can be 1D, 2D, 3D... using glTexCoord1f, glTexCoord2f, glTexCoord3f. GPUs provide native support up to 3D



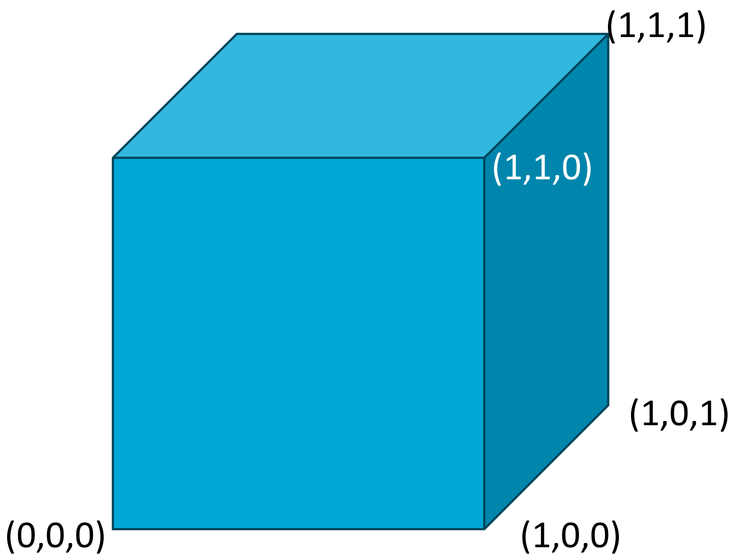
We already saw one example of a 1D Texture

- We can use a 1-pixel wide 2D Texture accessed with glTexCoord2f(0,z).
- A 1D Texture always has 1-pixel width. Access with glTexCoord1f(z).



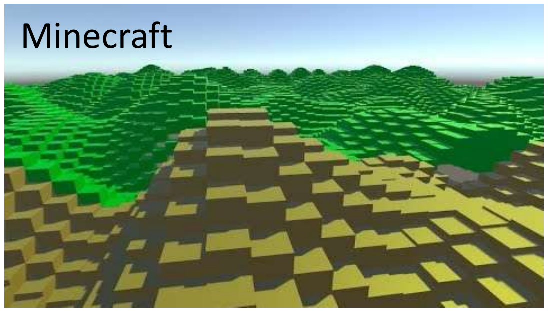
What are 3D Textures?

- They are accessed with 3 texture coordinates (u,v,w)
- Bilinear interpolation in 2D textures becomes trilinear with 3D textures



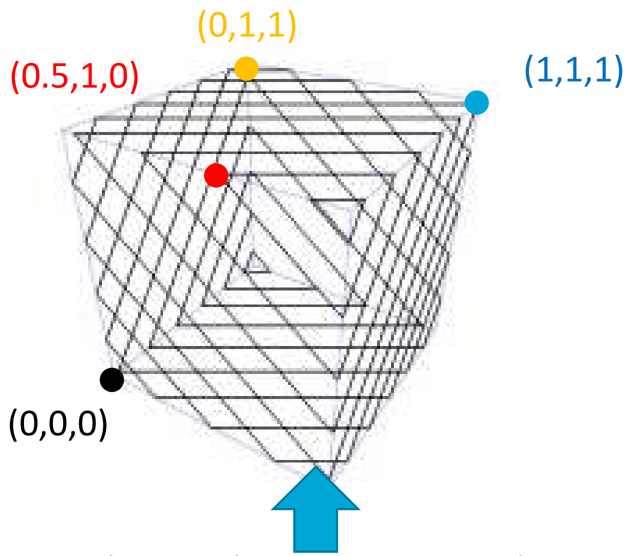
3D Textures are Volumes

- The equivalent of a texels is a small cubic volume (**vo**lume **el**ement=**voxel**)



Volume Rendering

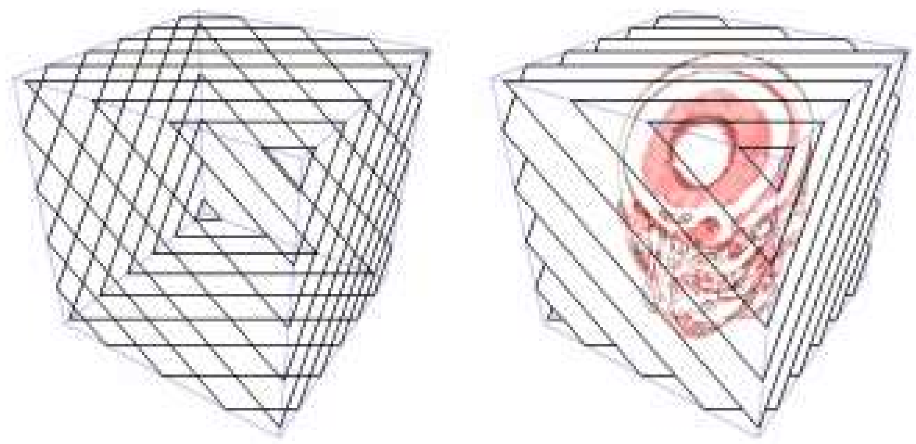
- How to render volumes?
  - One often-employed method is slicing



Draw slices with 3D texture coordinates corresponding to the position of the vertices in the volume

Volume Rendering

- How to render volumes?
  - One often-employed method is slicing



Volume Rendering

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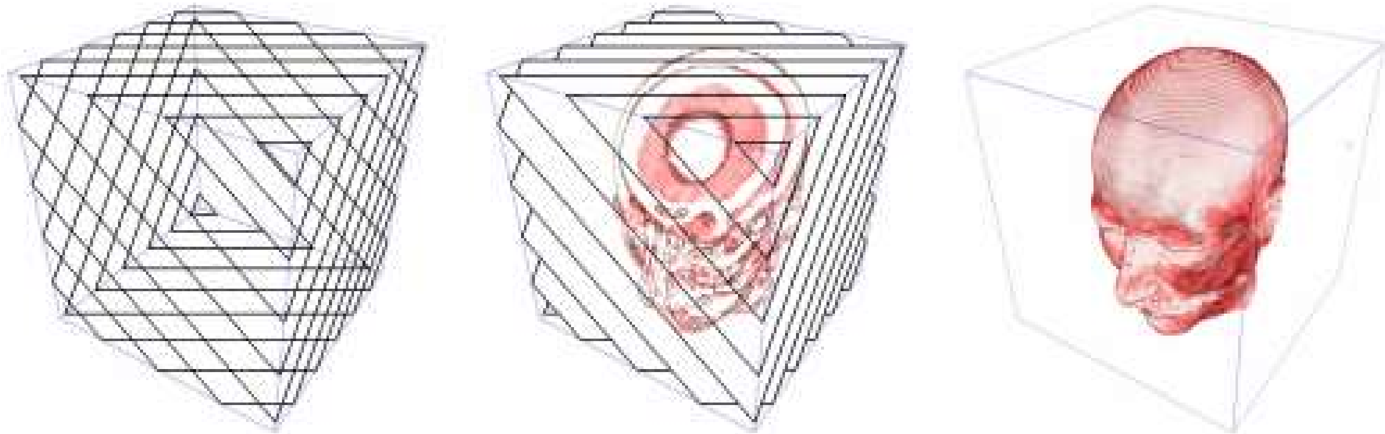
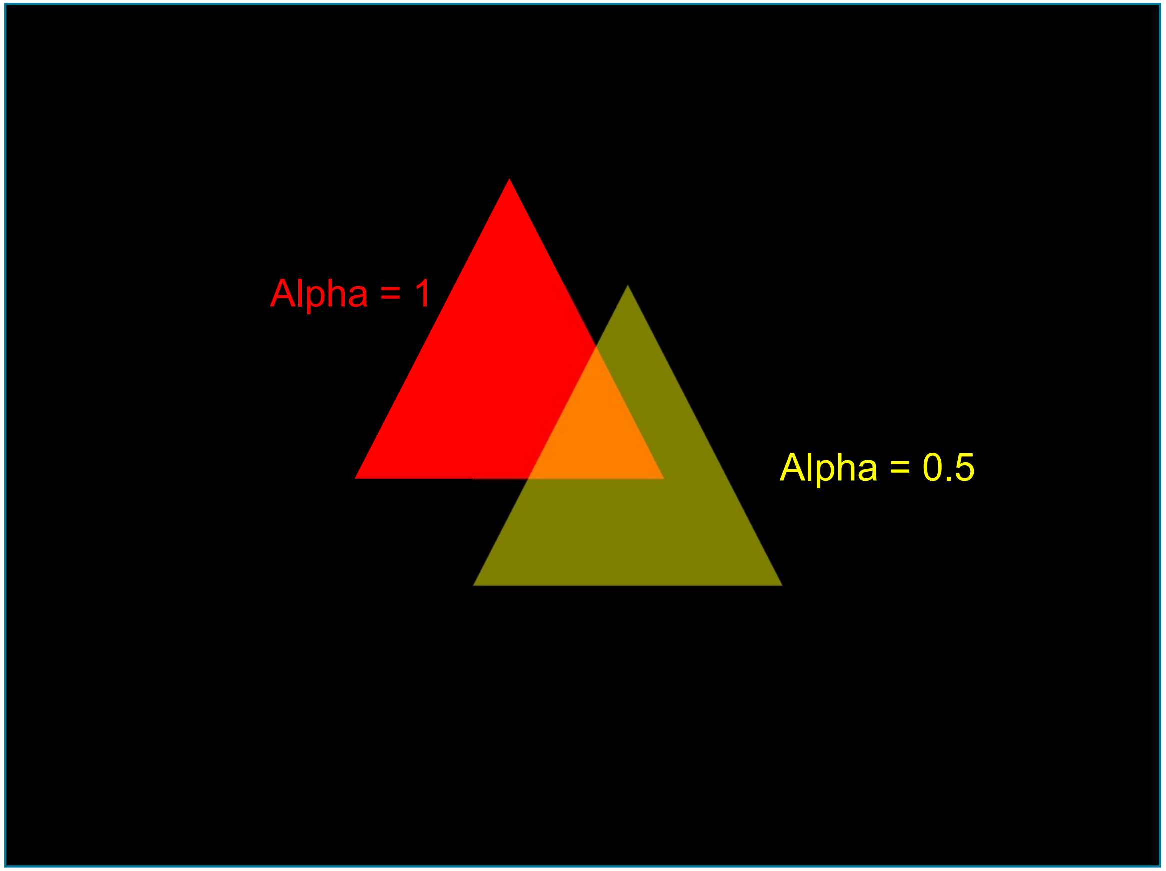


But wait? Why are there parts that are transparent?

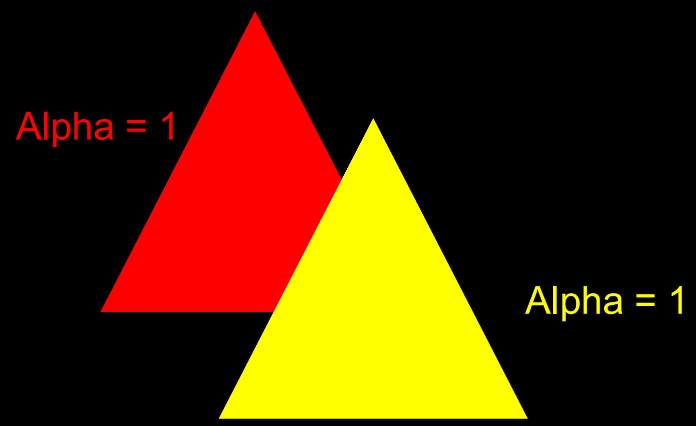
Alpha Blending

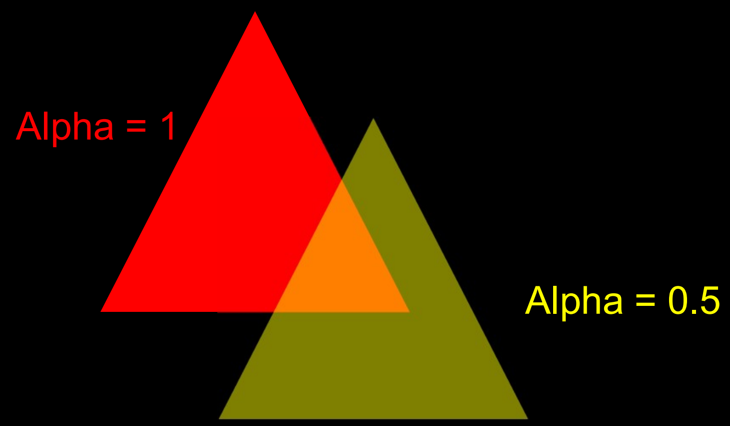


Standard Alpha Blending

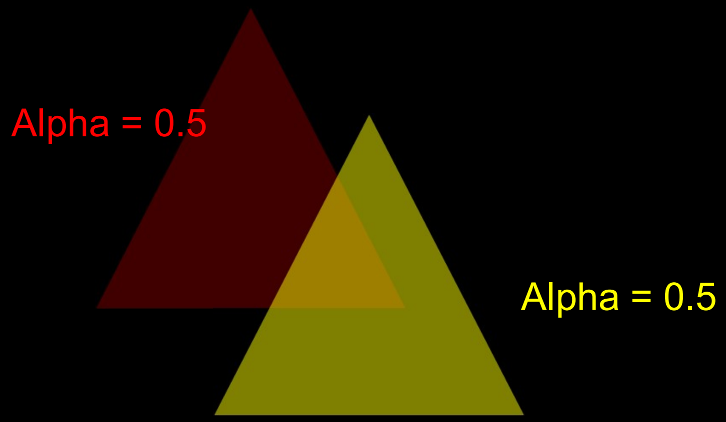
- Color with four components: RGB and an alpha value
- Alpha describes the "opacity"
  - 0 : object is completely transparent
  - 1 : object is completely opaque
  - 0.5: 50% of the stored color and 50% background

- In general, drawn (R,G,B,A) and in a pixel with color ($R_B G_B B_B$) results in:
  A * RGB+ (1-A) * $R_B G_B B_B$

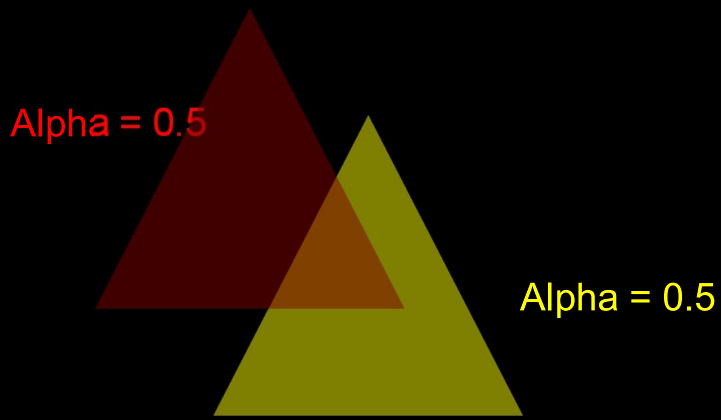

Alpha = 1
Alpha = 1



Alpha = 1
Alpha = 0.5

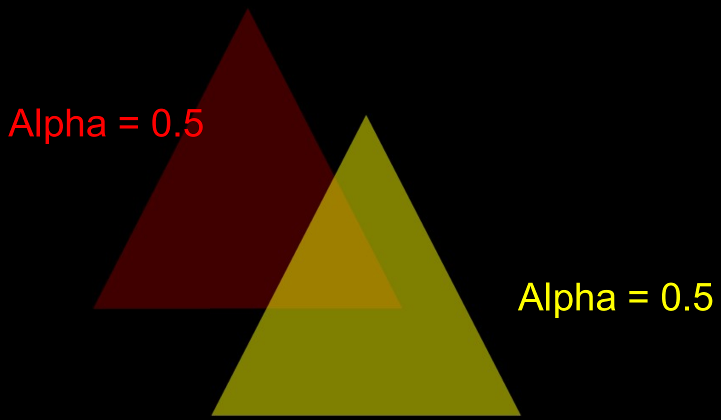

Alpha = 0.5
Alpha = 0.5

Attention: ORDER is important



RED drawn last

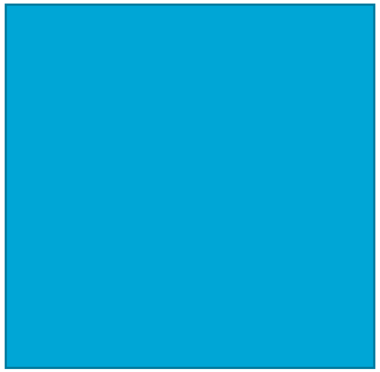
Attention: ORDER is important



YELLOW drawn last

Why this formula for Alpha Blending?

- Alpha is the opacity of the object
- Imagine the object to have tiny holes with probability Alpha
- 0.5 could be imagined as a checkerboard



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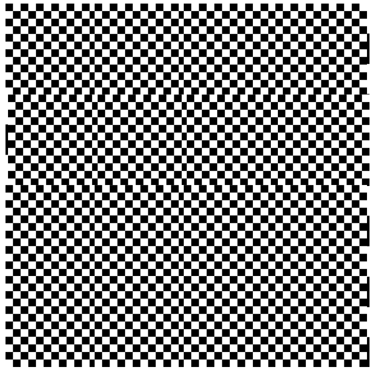

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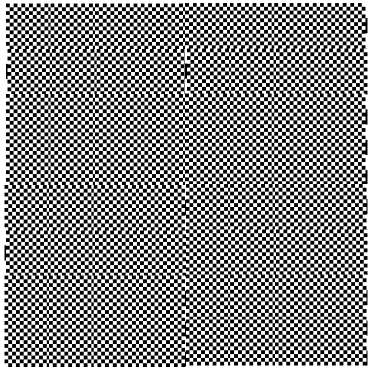

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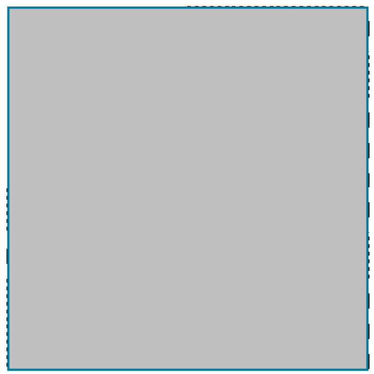
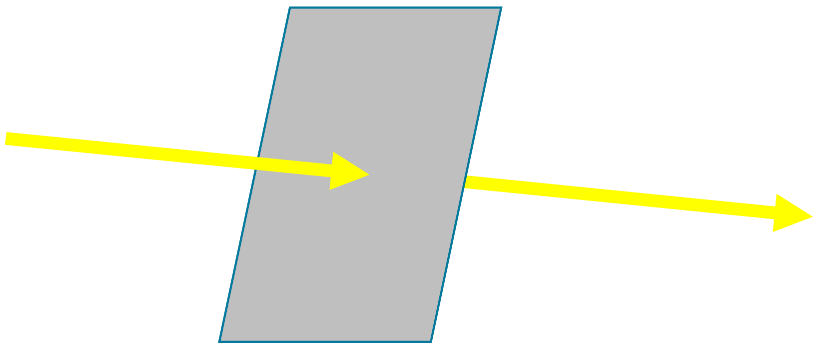


Why this formula for Alpha Blending?

- A light ray hits with probability A
  and passes with probability (1-A),
  hence, A * color + (1-A) * Background



Volume Rendering

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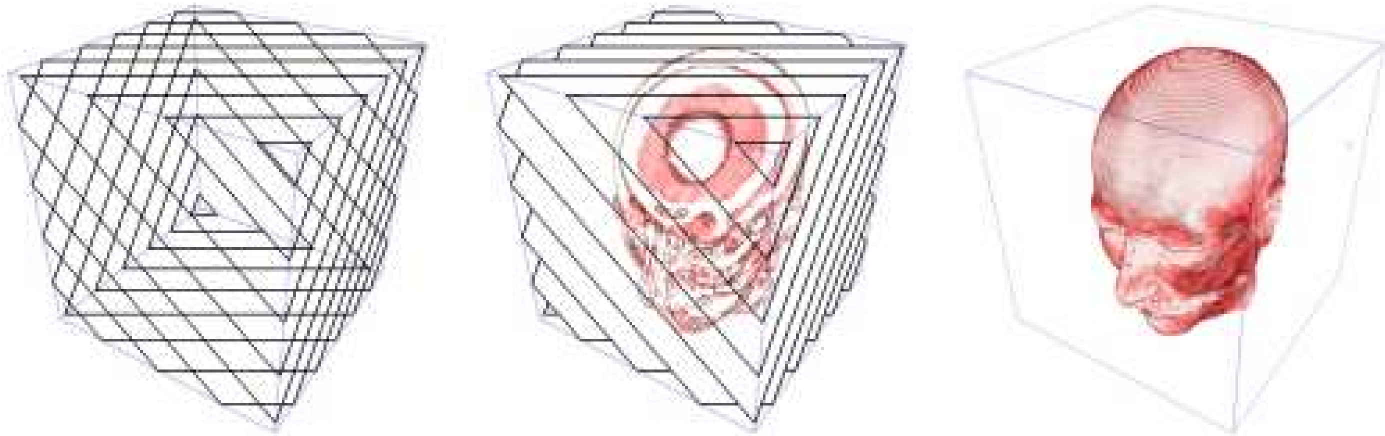
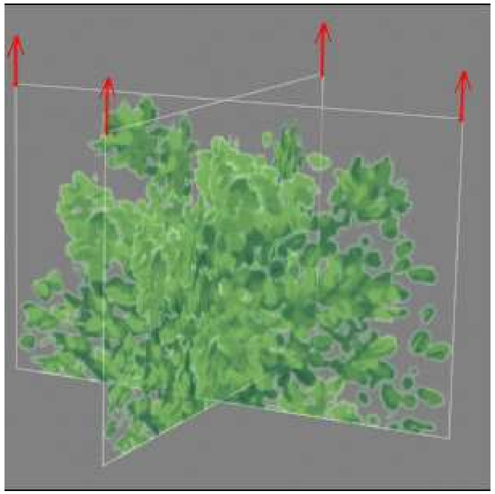
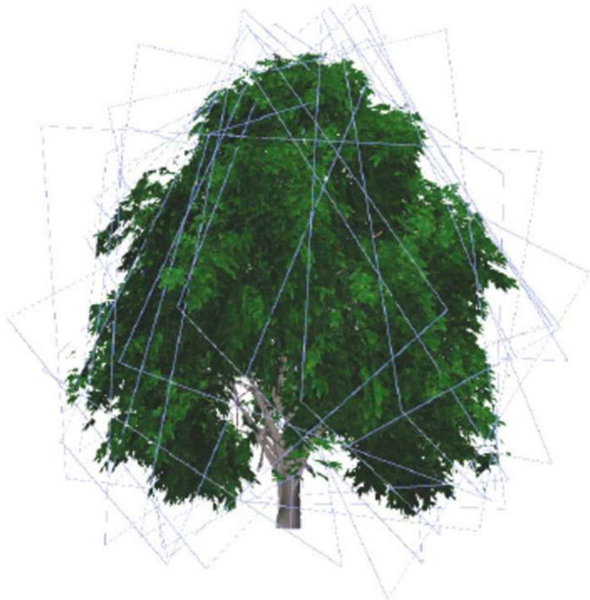


Alpha Blending

- Used to create transparency effects
- Textures can have (R,G,B,A) values:
  - If A=0 : texel is considered transparent – not drawn
  - If A=1 : texel is drawn

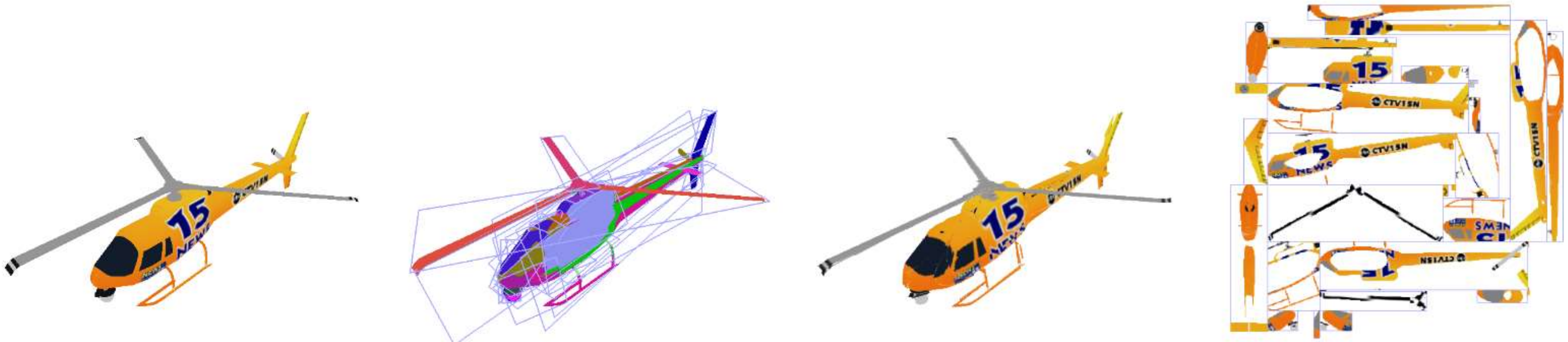


Alpha Blending - Stacked Polygons



- Source: irrlicht engine, artlplants.blogspot.com

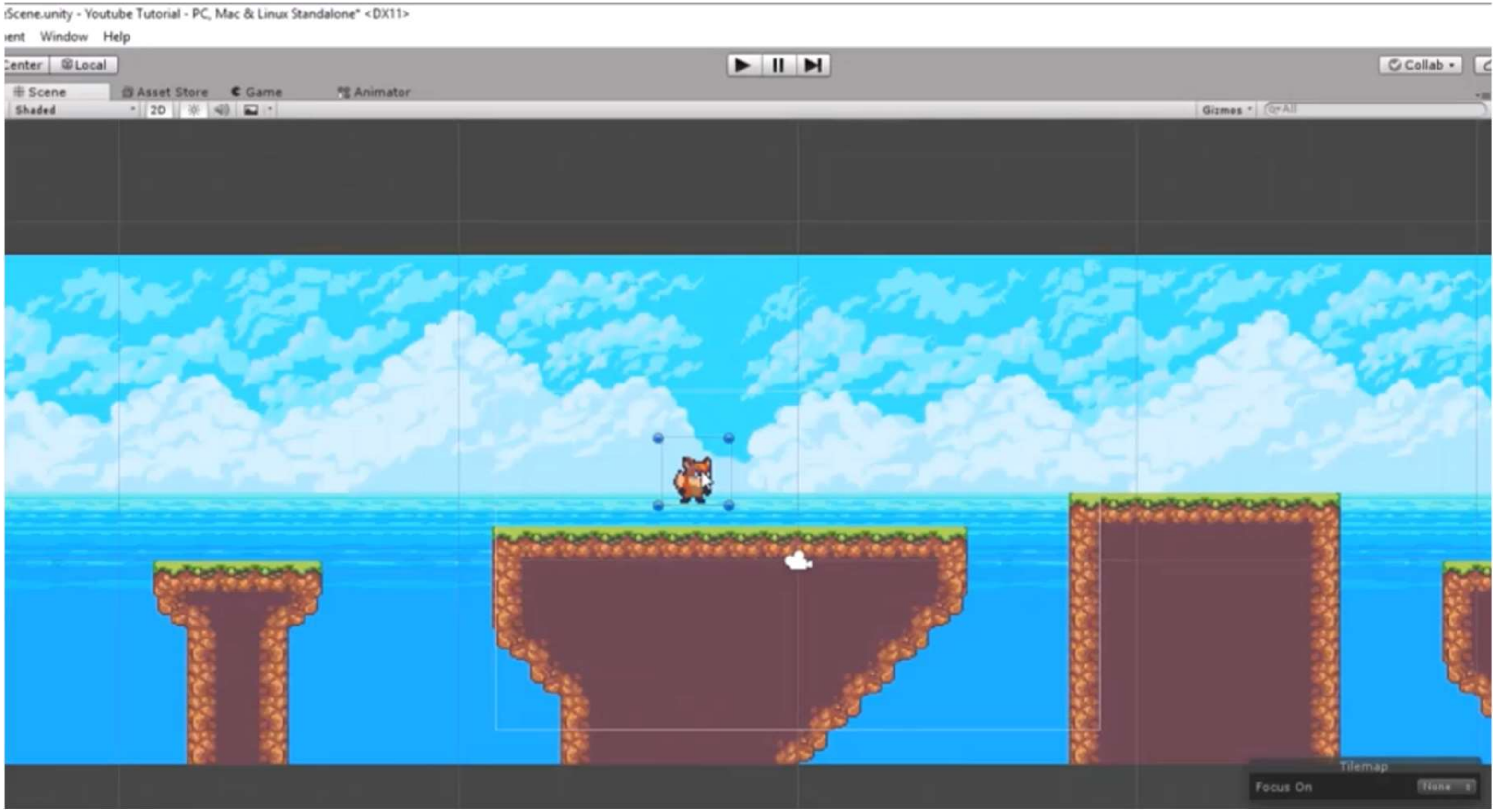
Billboard Clouds



(a) (b) (c) (d)

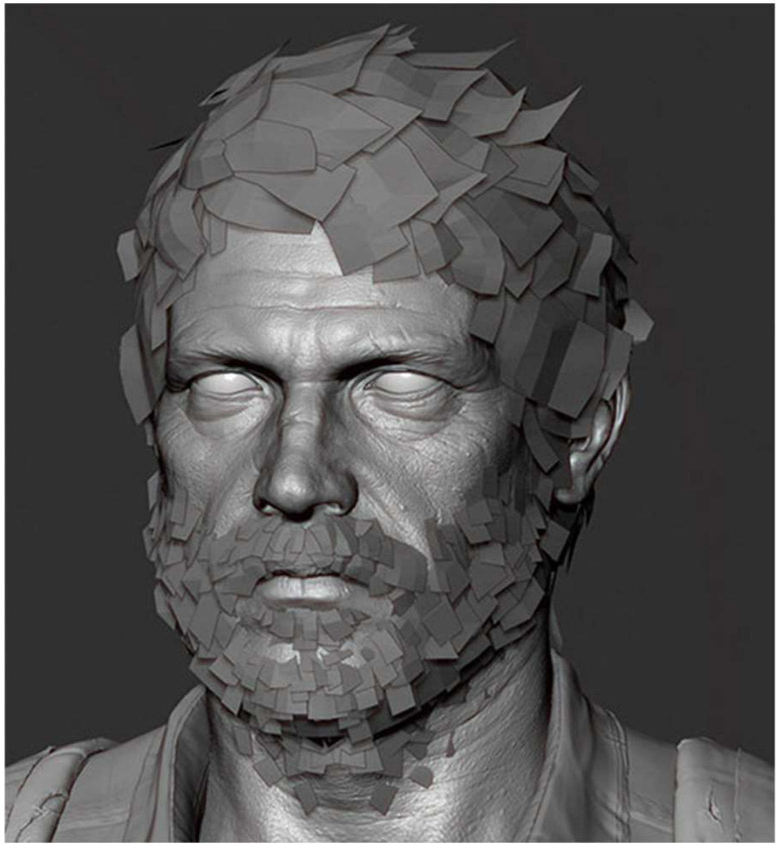
Figure 1: Example of a billboard cloud: (a) Original model (5,138 polygons) (b) false-color rendering using one color per billboard to show the faces that were grouped on each (c) View of the (automatically generated) 32 textured billboards (d) the billboards side by side.

Alpha Blending - Sprites (here Unity)



Use pixel discard based on alpha to make the object background transparent

Alpha Blending - Hair



Alpha Blending - Hair

Last of us

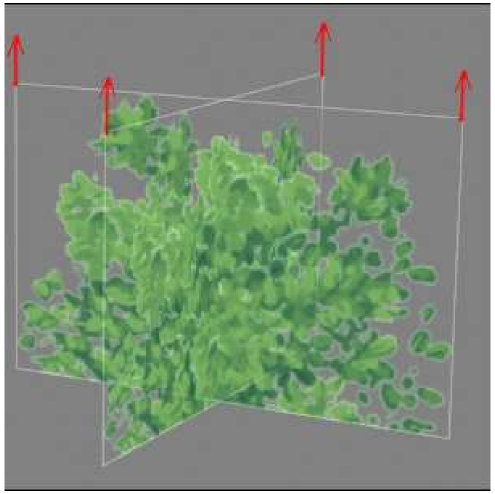
Alpha Blending - Water Particles

- Textured quads used as water particles

Alpha Blending

Pros:

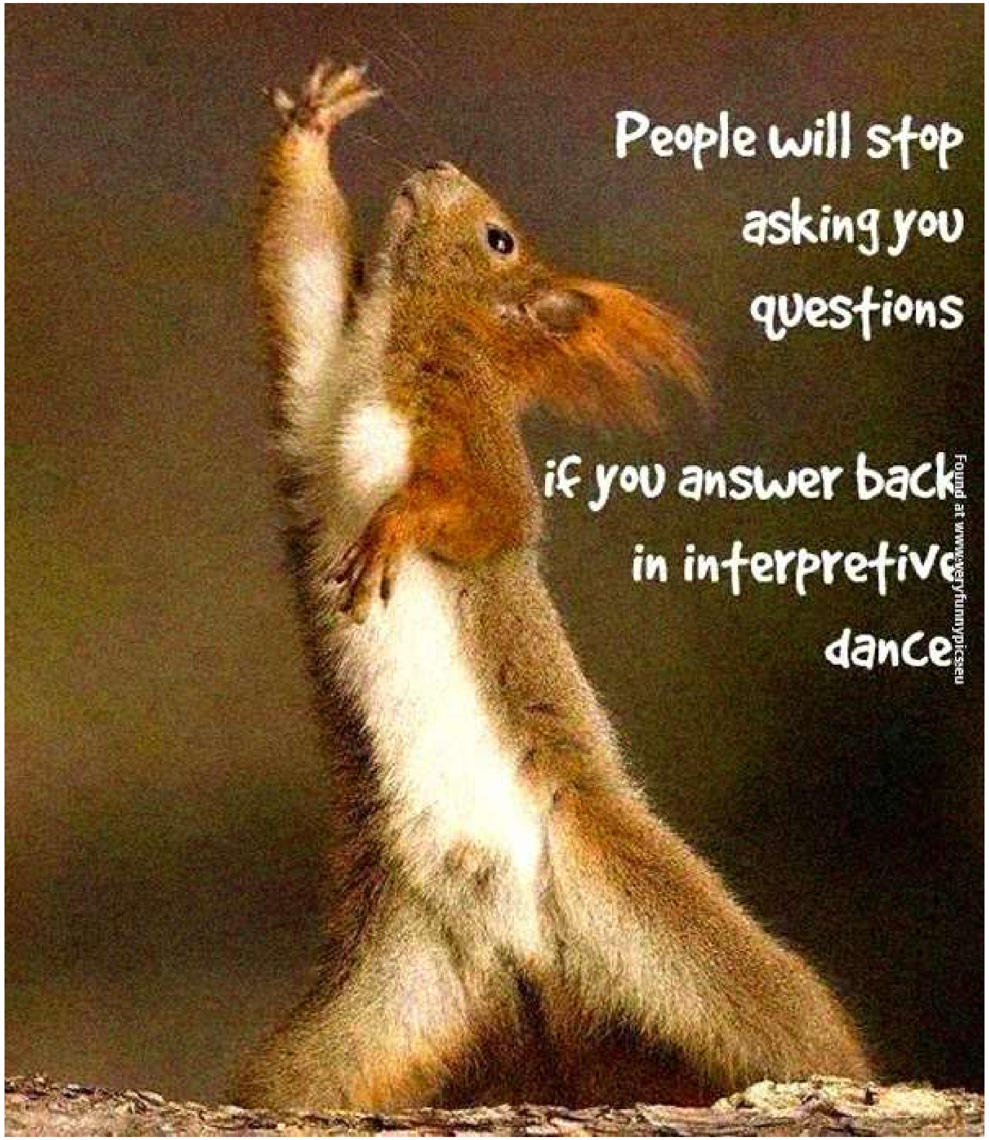
- Give the illusion of very complex geometry
- Can mimic transparent materials (glas, smoke, ...)



Cons:

- Additional cost:
  - Generally need to sort triangles
  - If only binary alpha values (0 and 1), sorting can be omitted

Questions?



TU Delft

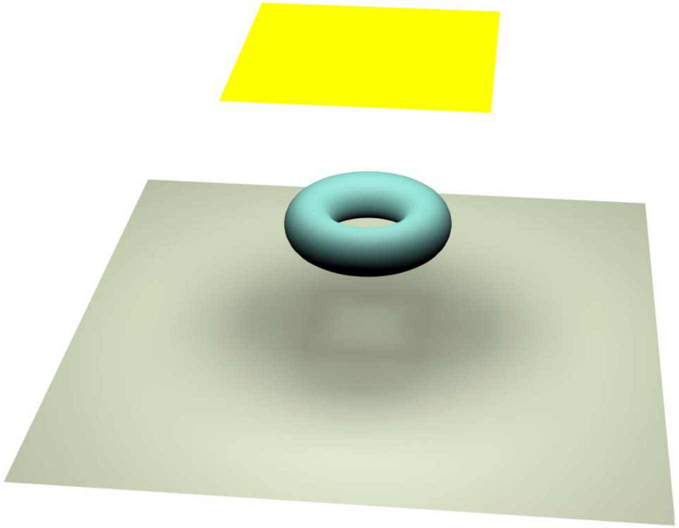
What is a Shadow?

- WordNet:

  Shade within clear boundaries

  or

  An unilluminated area.

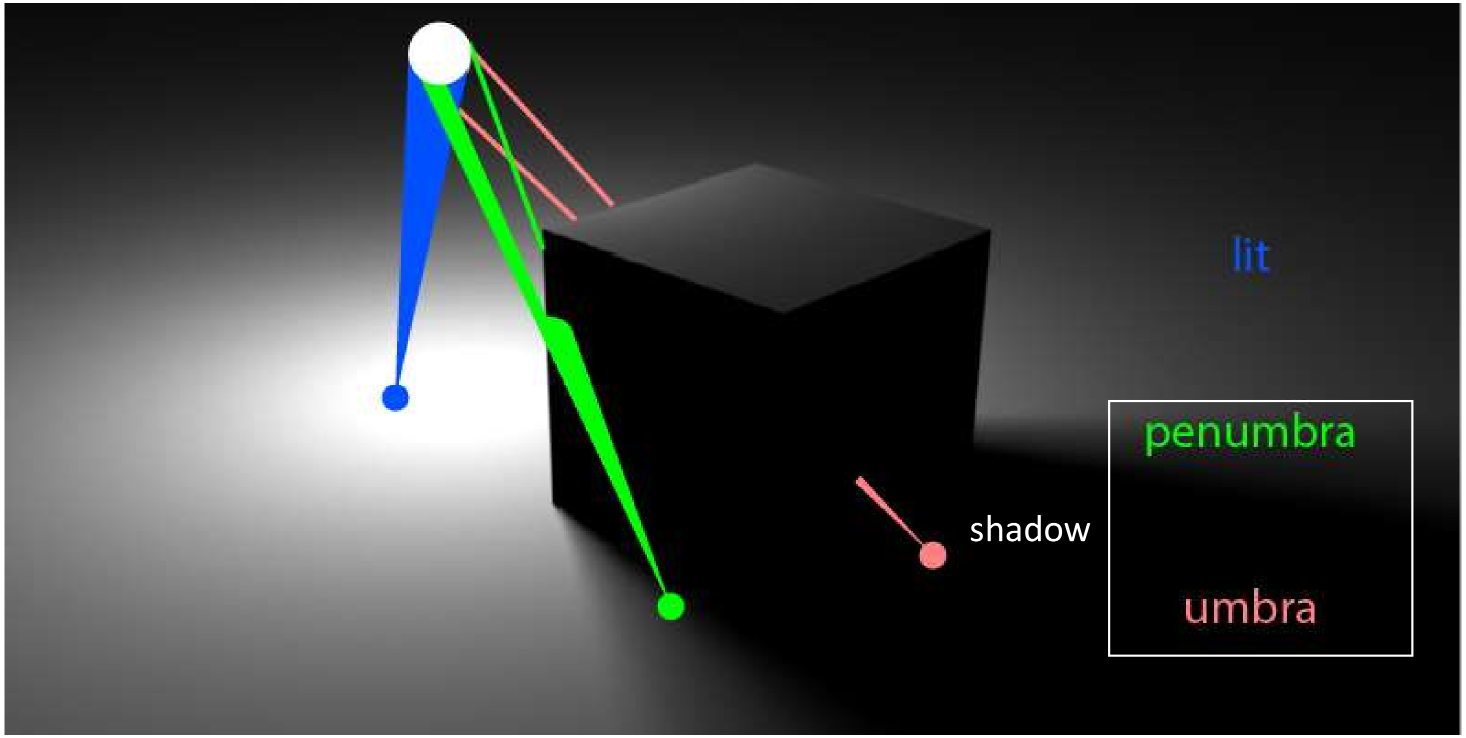


What is a Shadow?

- Hasenfratz et al. [2003]:

Shadow [is] the region of space
for which at least one point of the light source is occluded.

What is a Shadow?



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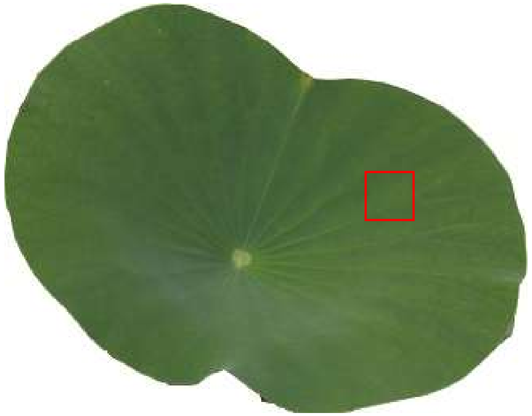
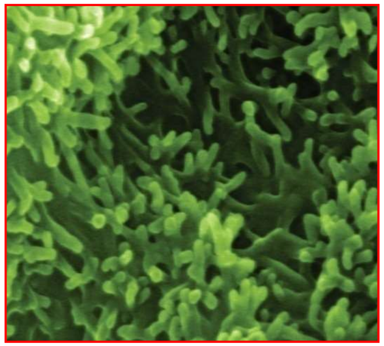
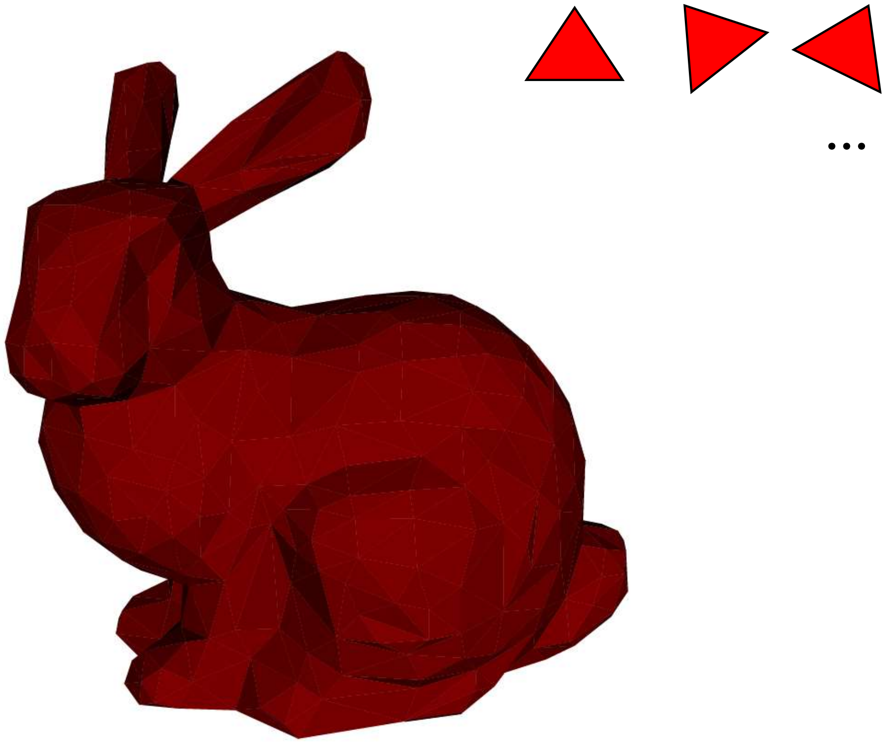




Our models are simple...

- Typically a list of triangles



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How to draw shadows?

- Artists know how to draw shadows!

  Or not?

Fra Carnevale
(1467)

How to draw shadows?

- Artists know how to draw shadows!

  Or not?

Signorelli
(1488)

How to draw shadows?

- Artists know how to draw shadows!

  Or not?

Moore (1893)

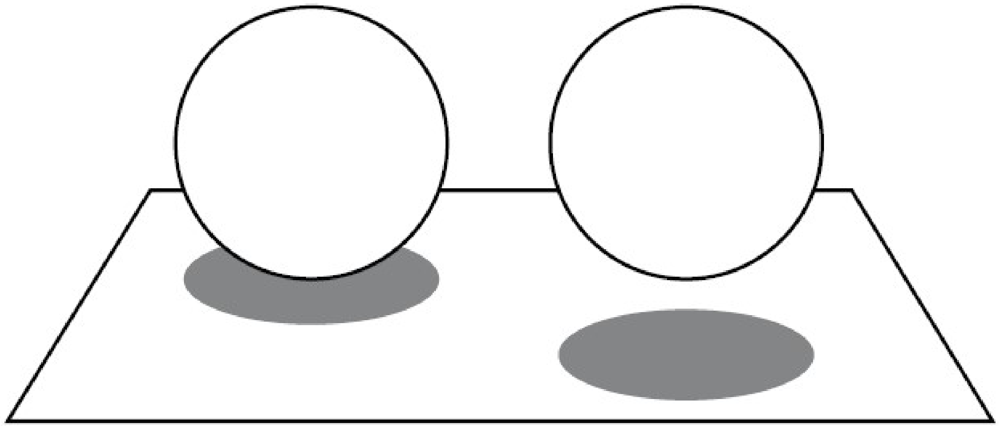
How to draw shadows?

- Drawing shadows is apparently difficult…

So why not just ignore shadows?

- Shadow of the Colossus, Sony

So why not just ignore shadows?



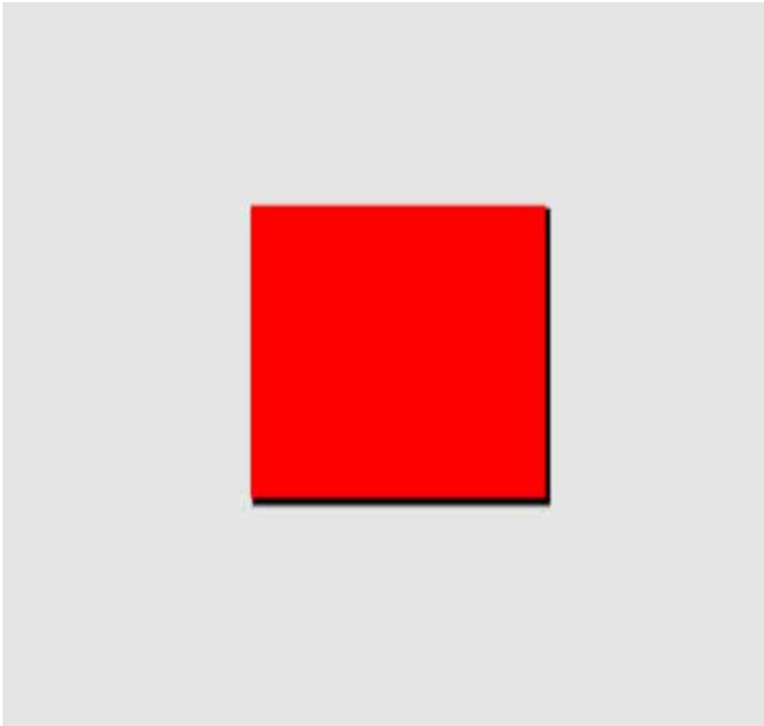
So why not just ignore shadows?

- Shadow of the Colossus, Sony

TU Delft

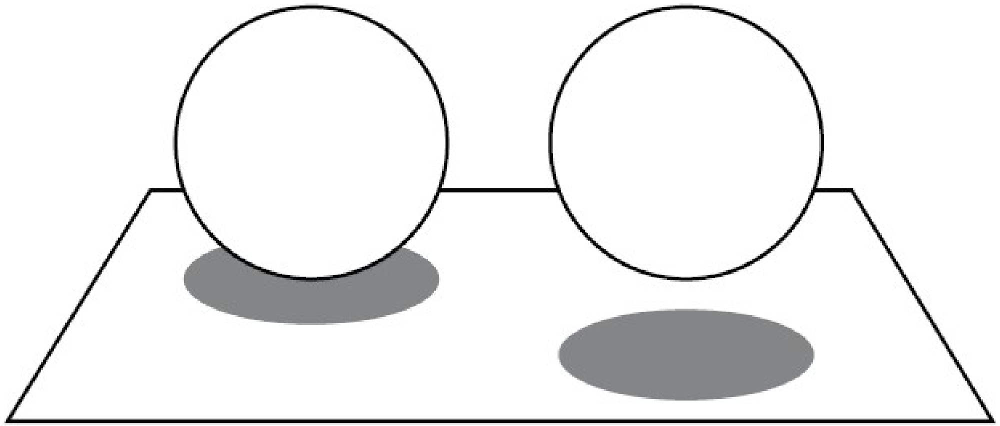
Psychophysical-Experiments

[Kersten et al. 96]



So why not just ignore shadows?

- But this is not a good argument for realistic shadows…



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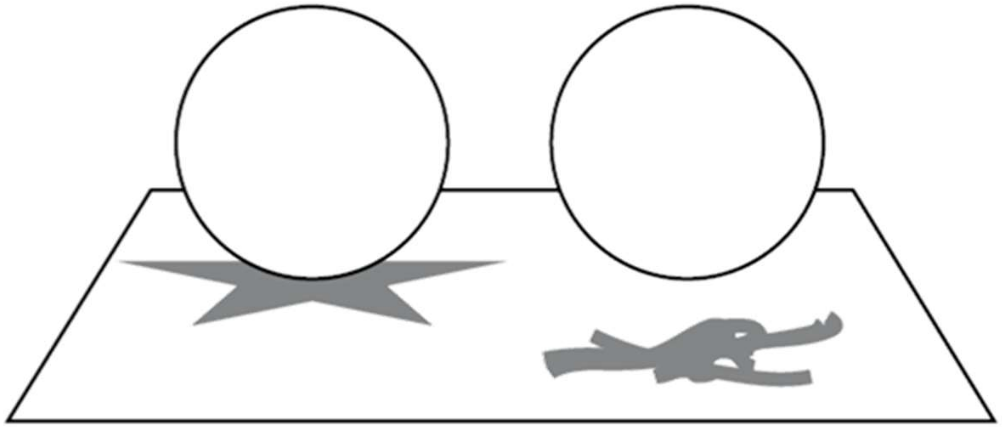
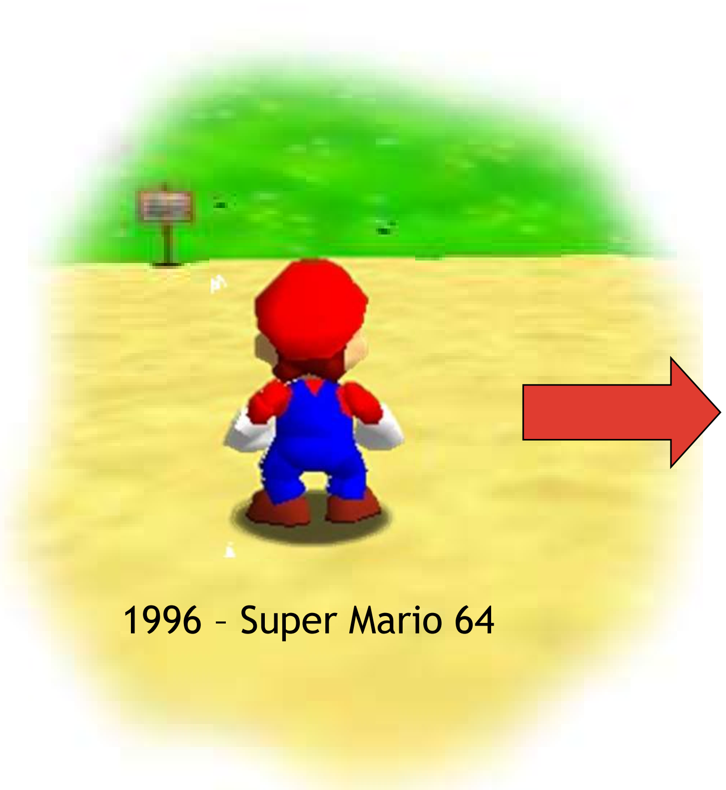
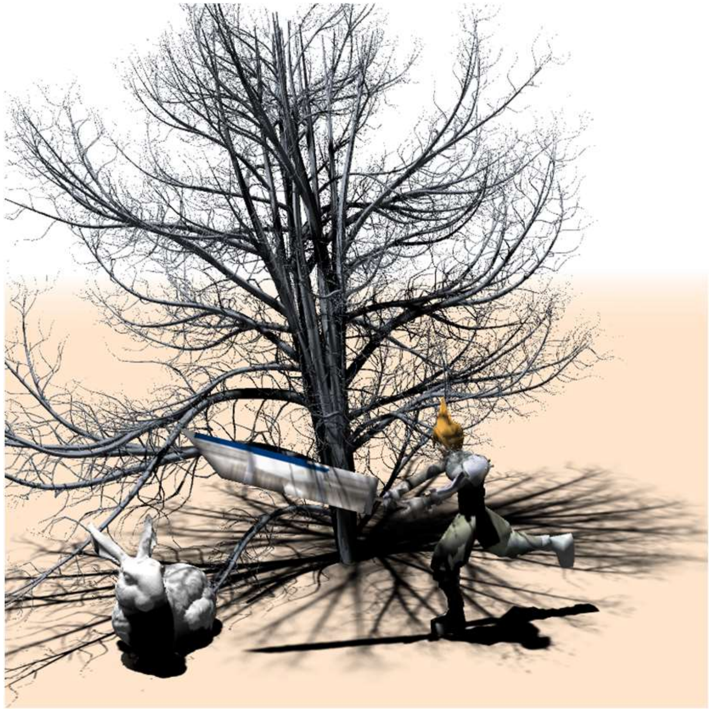


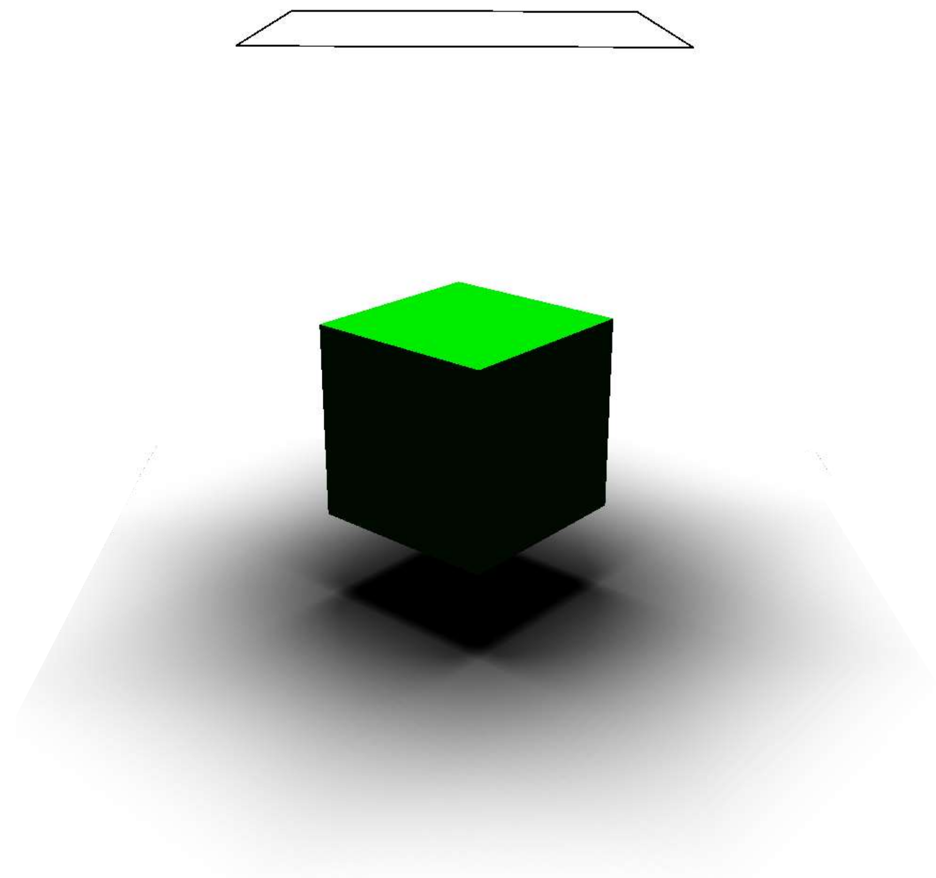
Simple shadows can be sufficient



1996 – Super Mario 64

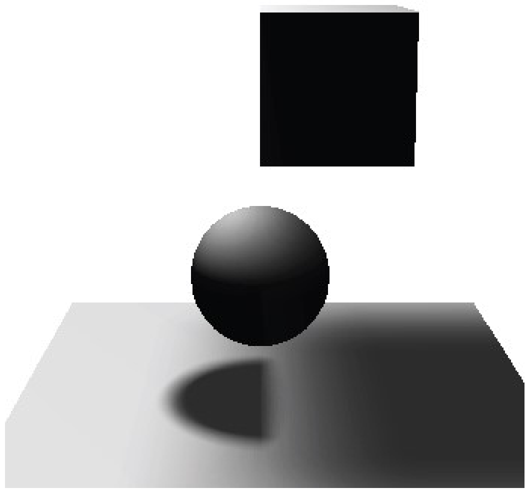


Plausible shadows

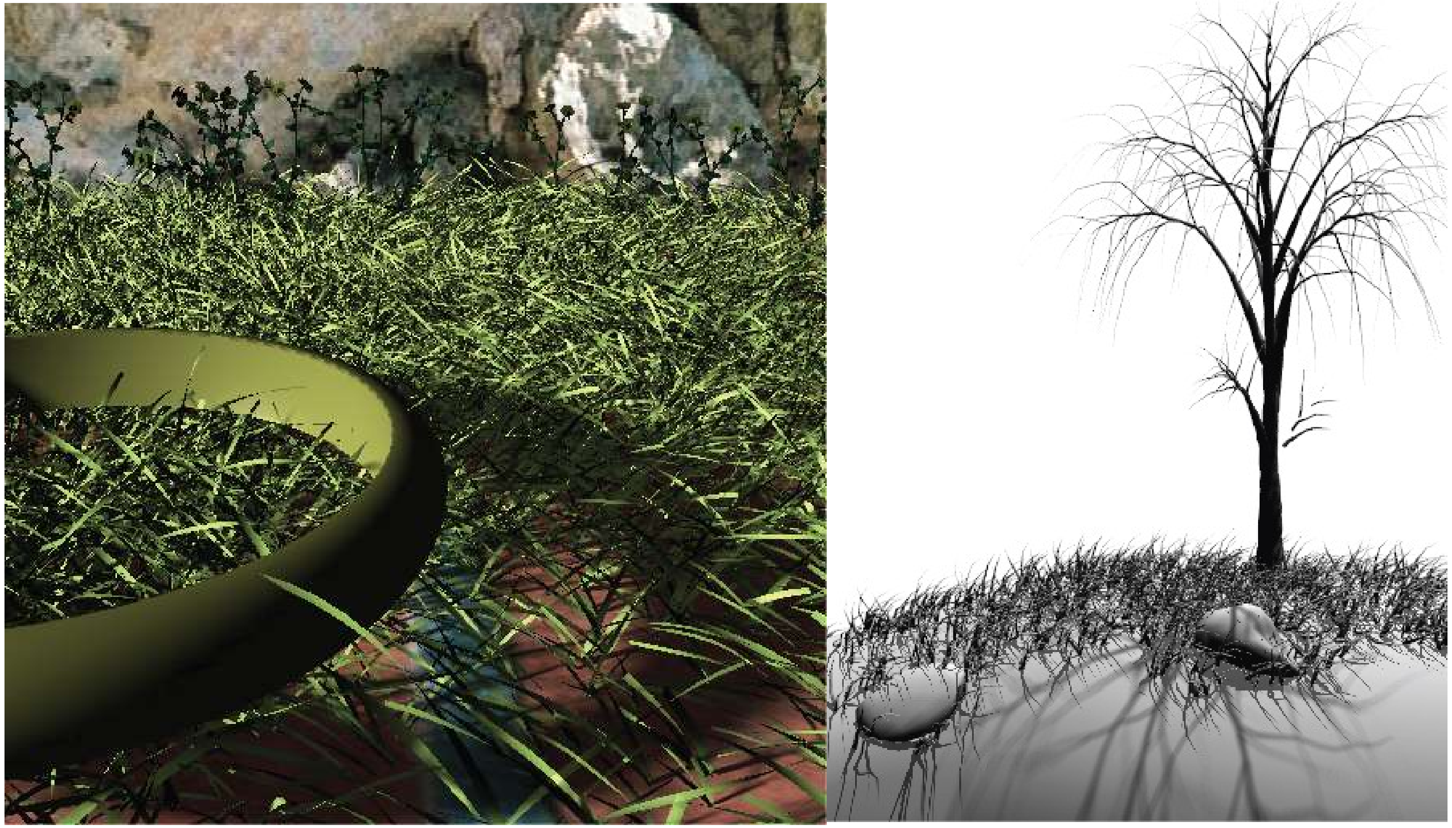


Plausible shadows

- Attention: “plausible” can often fail



Realistic shadows are important



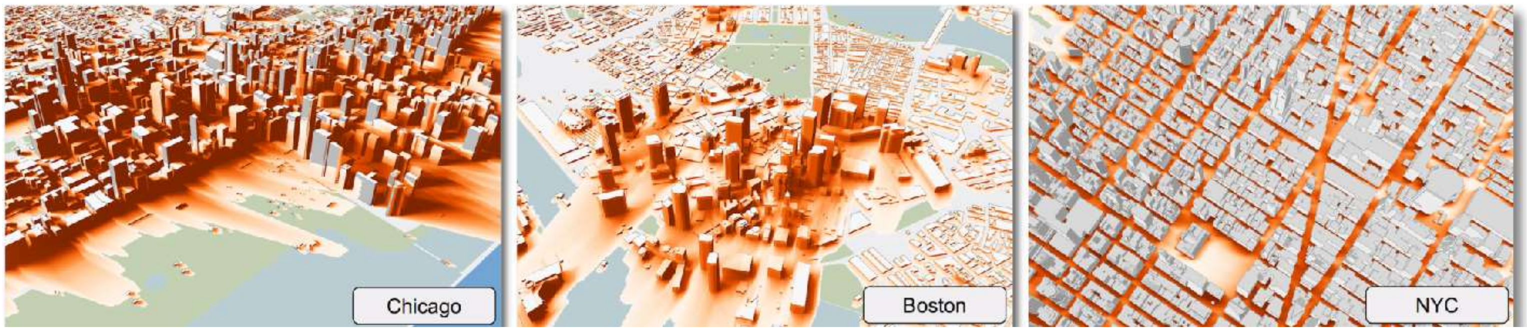
Shadows can give information!

- Courtesy of Hasenfratz et al.

Realistic shadows are important

- Many contexts in which accuracy is needed:
  - Architecture
  - Simulation
  - Movies

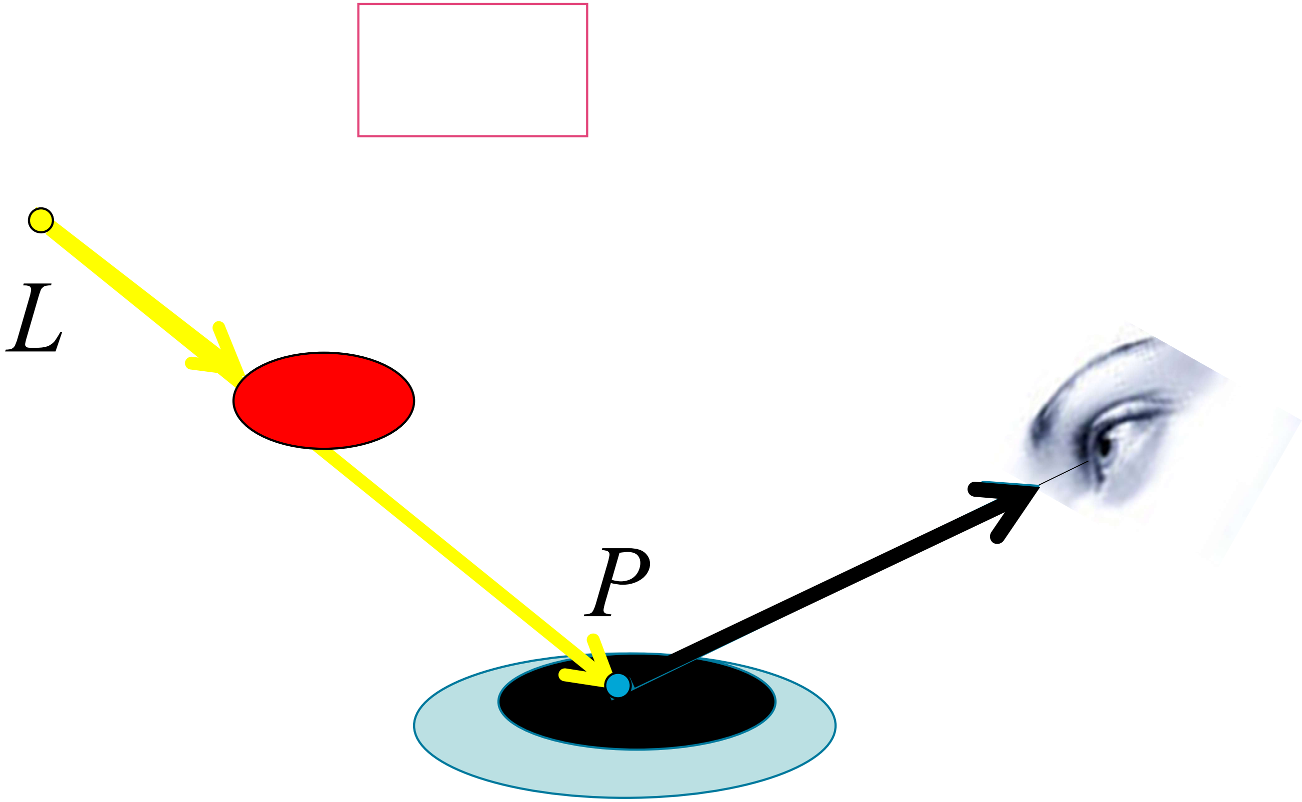
  …



Miranda et al. TVCG2019

Desert villa by Studio Aiko

How to compute shadows?



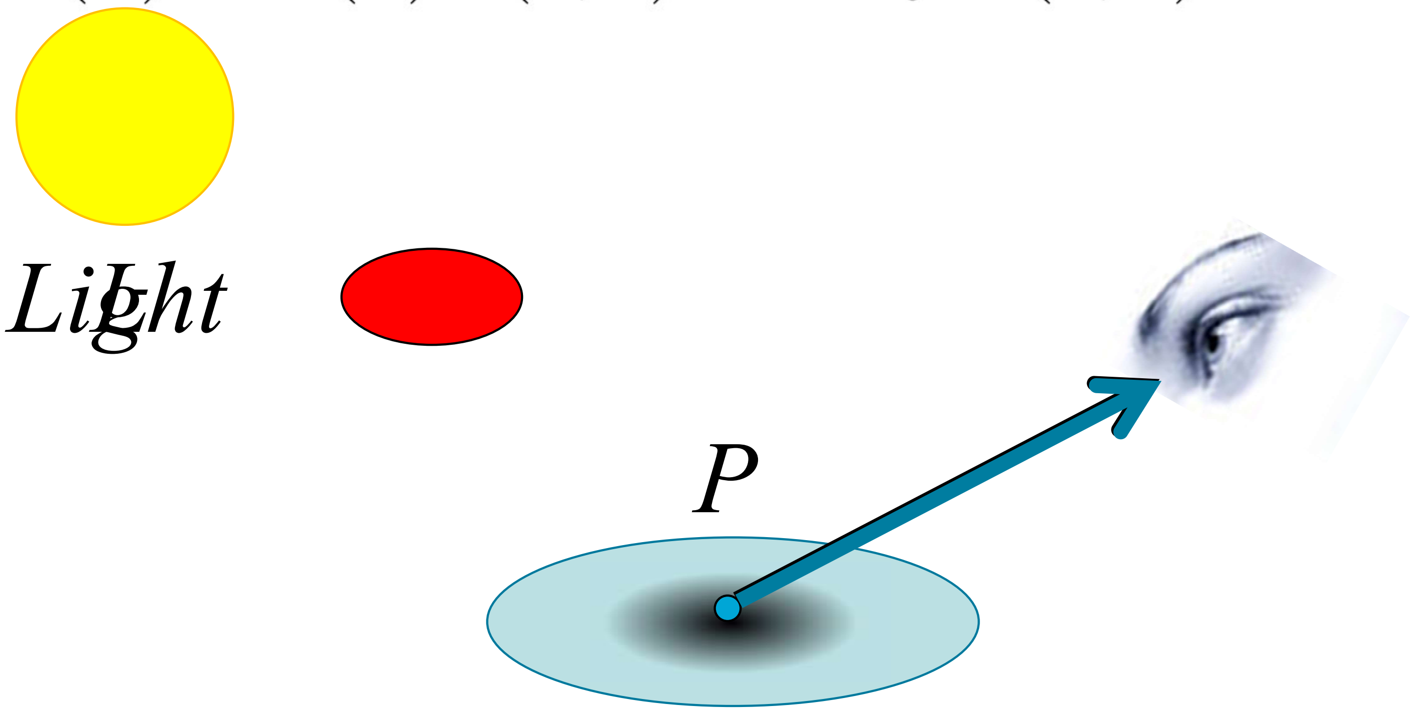
Hard Shadows

Point lights do not create penumbrae

Soft Shadows

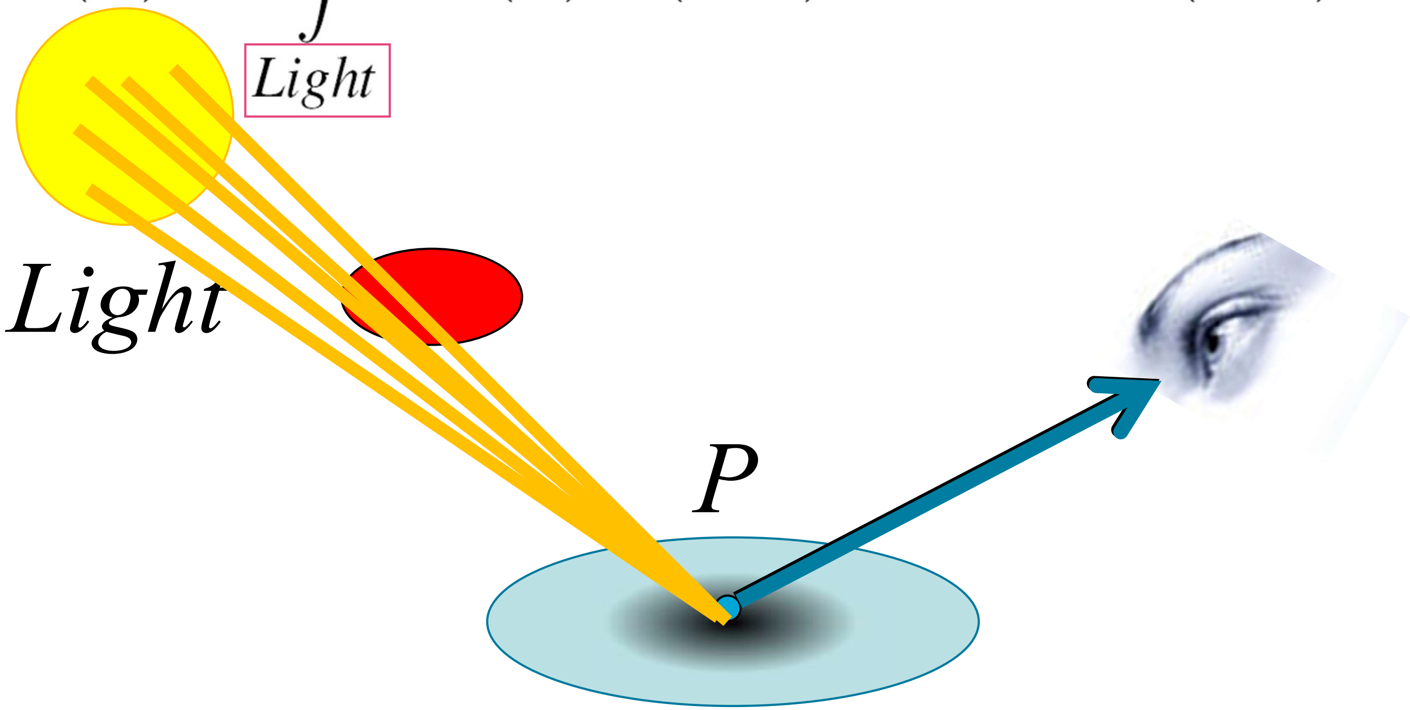
What is a Shadow ? – Part II

$$B(P) = E(L)\ v(P,L)\ Transfer(P,L)$$

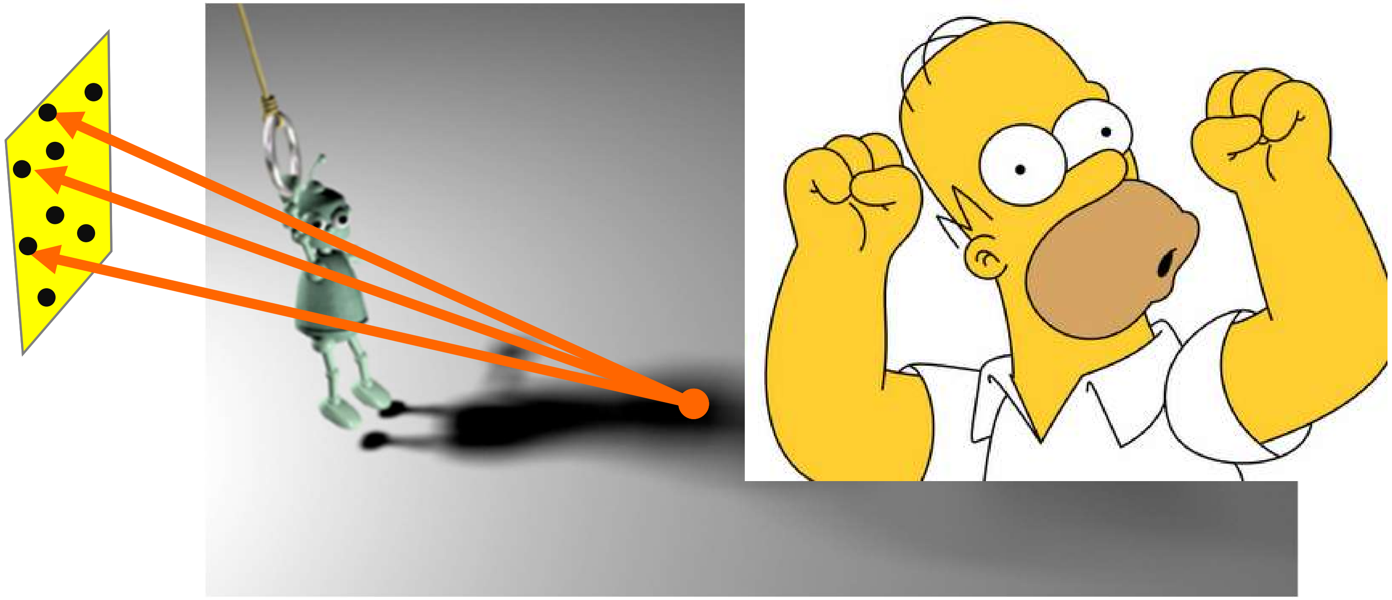


What is a Shadow ? – Part II

$$B(P) = \int_{\boxed{Light}} E(L)\; v(P,L)\; Transfer(P,L)\; \boxed{dL}$$



Lecture 1? Ray Tracing?



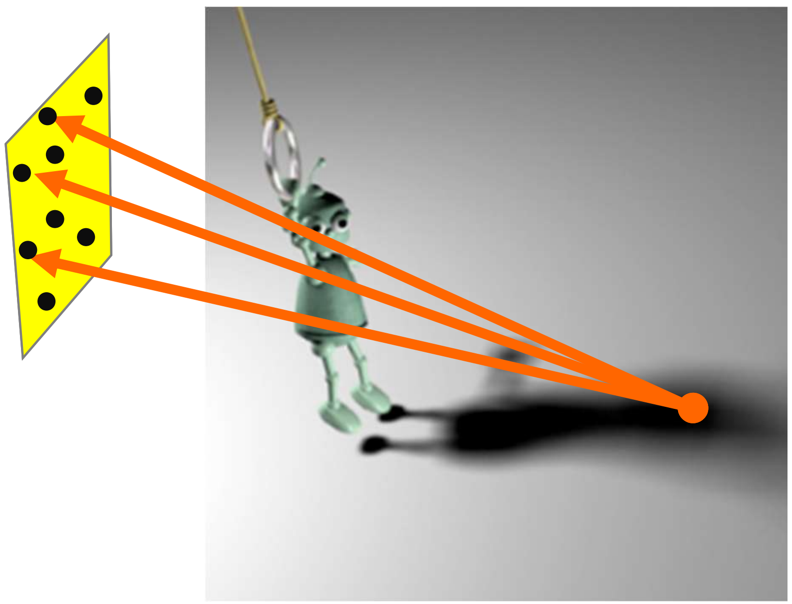
- Physically correct (...enough)
- Robust
- Easy to implement

Ray Tracing?

1 sample 32 sample 512 samples

TU Delft

Ray Tracing



Even on modern GPUs (RTX) relatively costly...

SLOW!

Today: Hard Shadows within the Rendering Pipeline

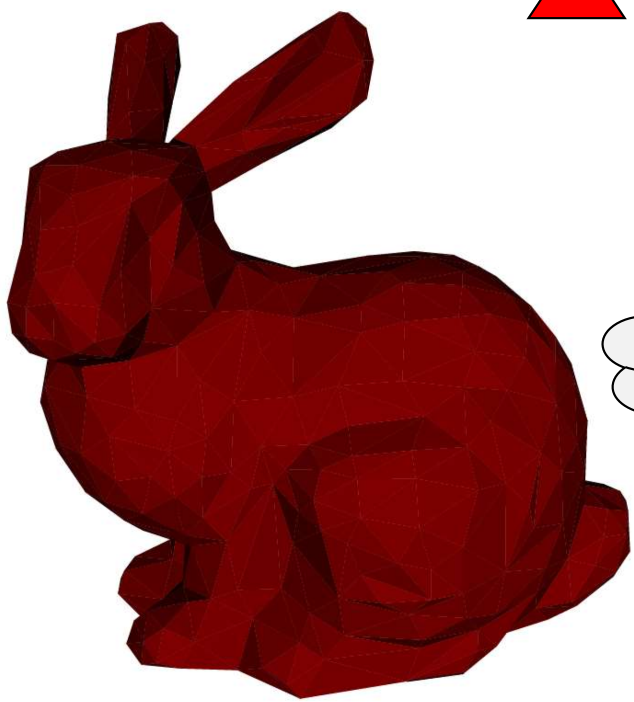
- Meaning: light source L is a point

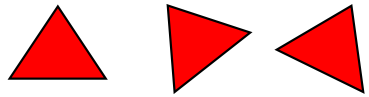
L

How to accelerate the process?

Simplified Graphics Pipeline

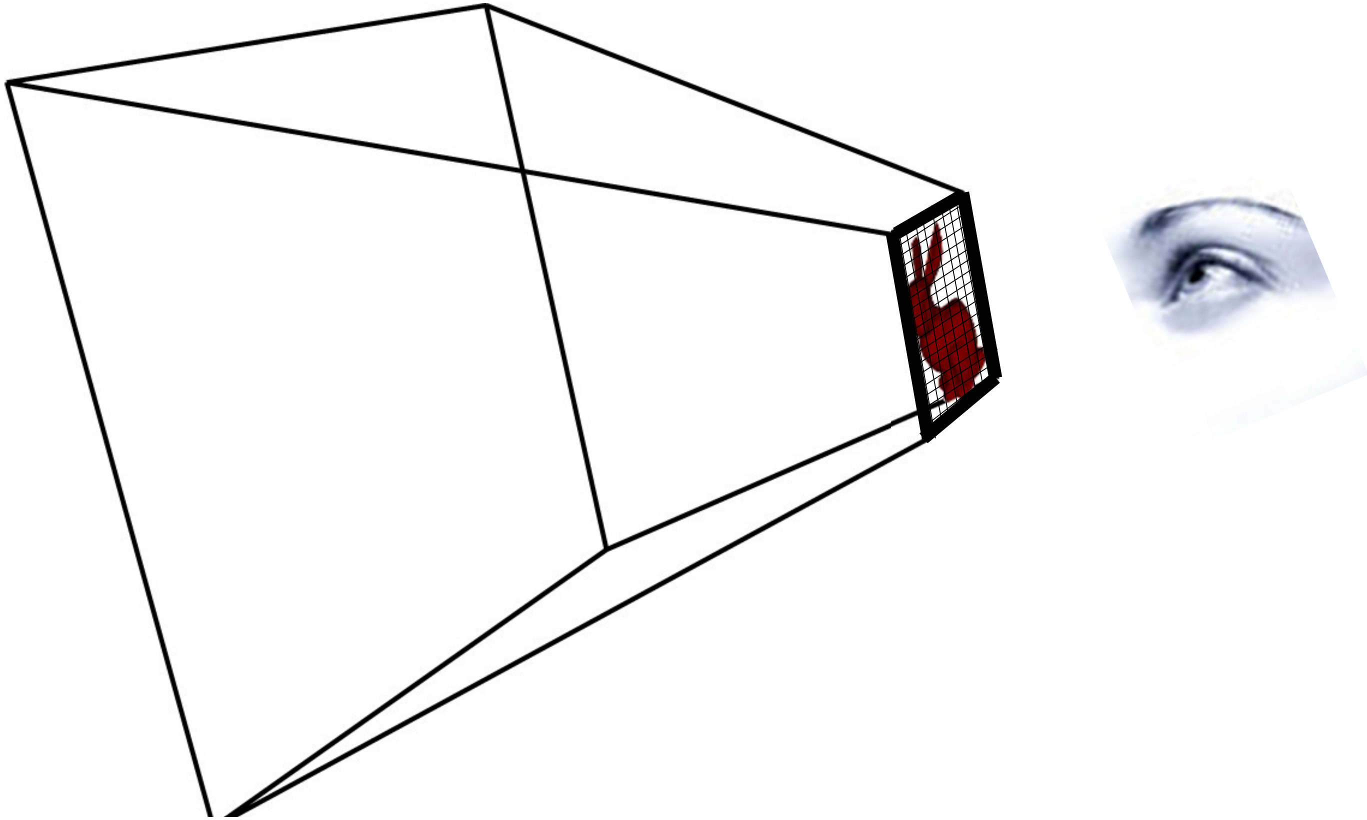
- Models are typically lists of triangles





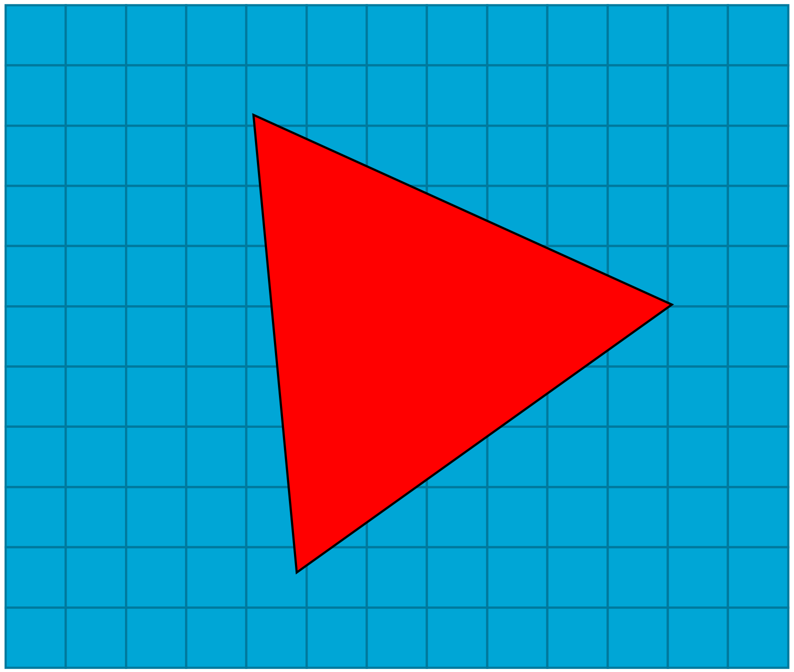
Simplified Graphics Pipeline

- Projection: Transform coordinates to screen



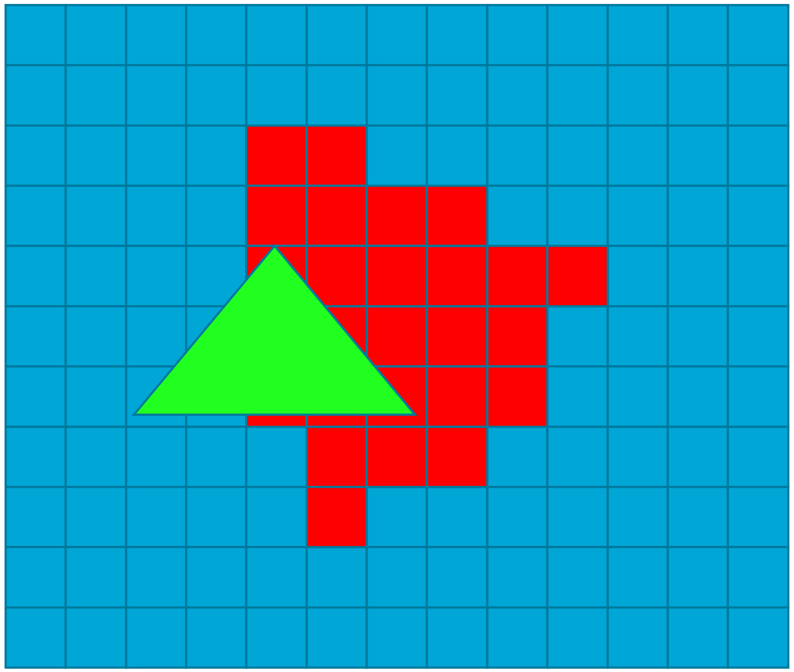
Simplified Graphics Pipeline

- Rasterization: Fill screen pixels



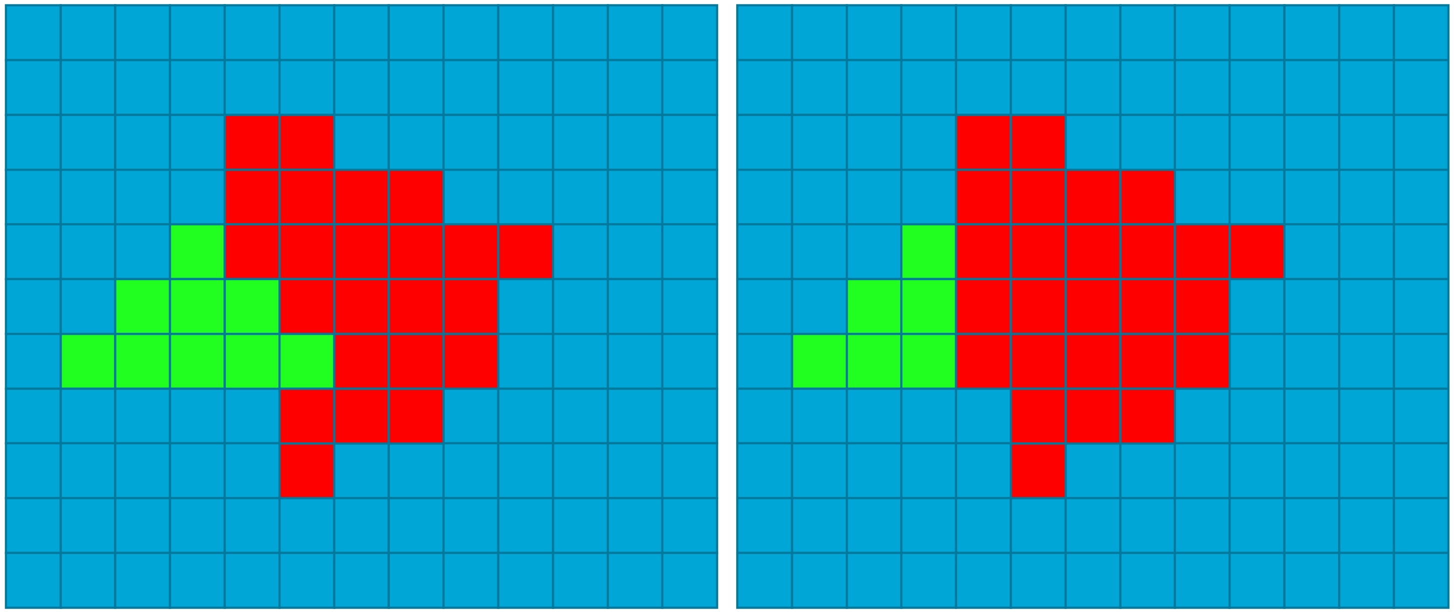
Simplified Graphics Pipeline

- Catch: Let's look at a second triangle...



Simplified Graphics Pipeline

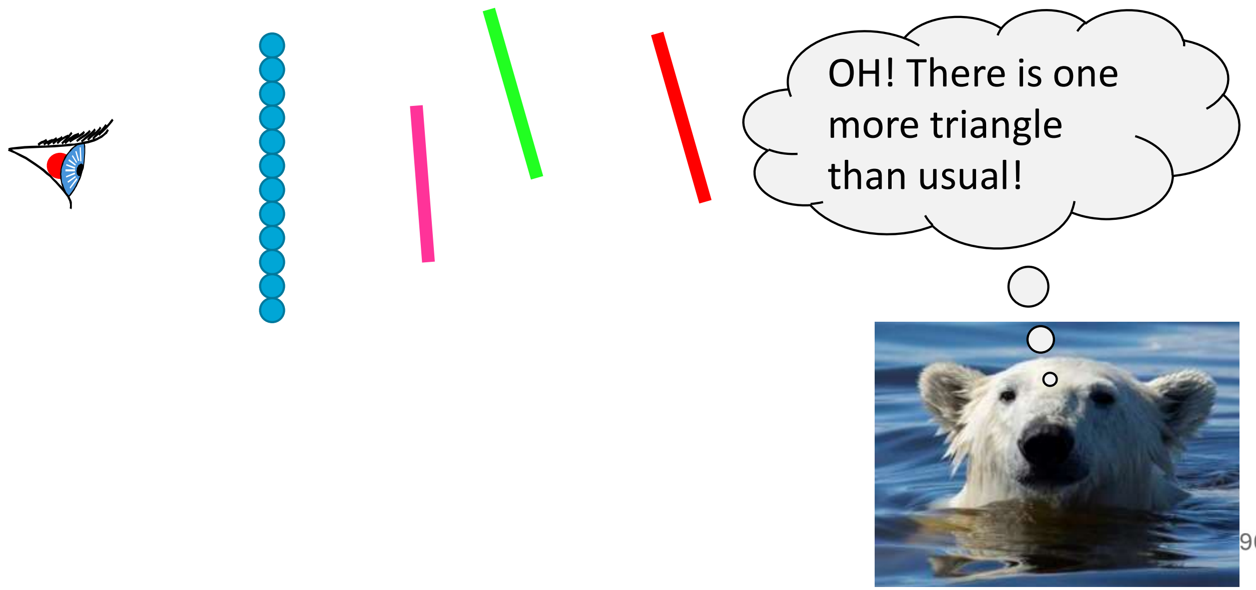
- Catch: Drawing order changes result



Need to keep nearest pixels

Simplified Graphics Pipeline [Catmull74] , [Strasser74]

- Depth Buffer: Avoid sorting triangles!
- Store a color and depth in each pixel



Simplified Graphics Pipeline [Catmull74] , [Strasser74]

- Depth Buffer: Avoid sorting triangles!
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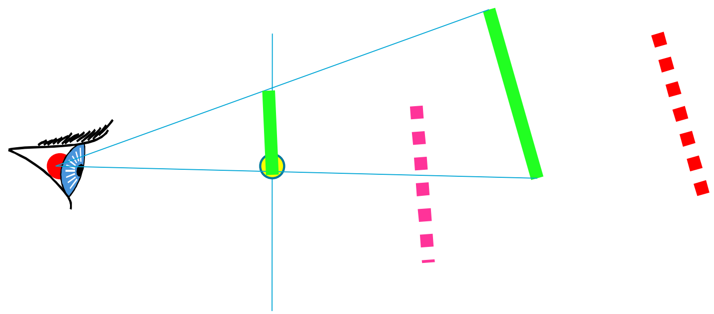

Simplified Graphics Pipeline [Catmull74] , [Strasser74]

- Depth Buffer: Avoid sorting triangles!
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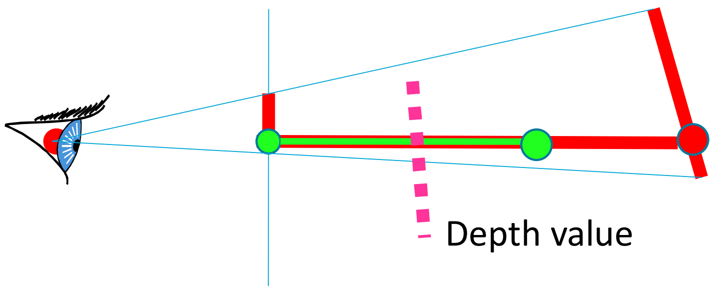

Simplified Graphics Pipeline [Catmull74] , [Strasser74]

- Depth Buffer: Avoid sorting triangles!
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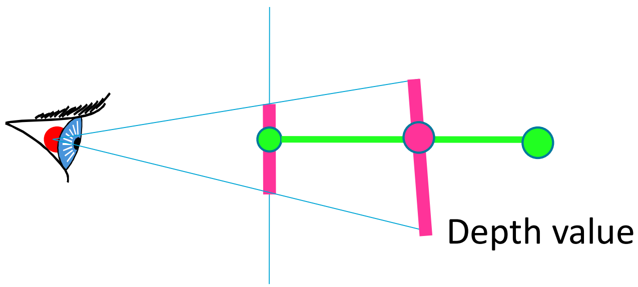
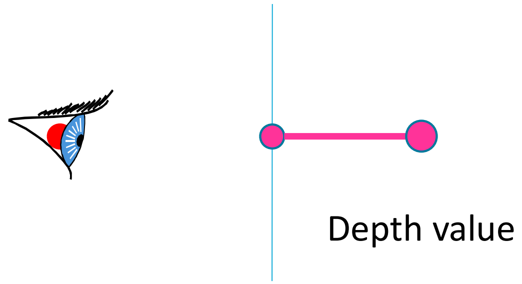


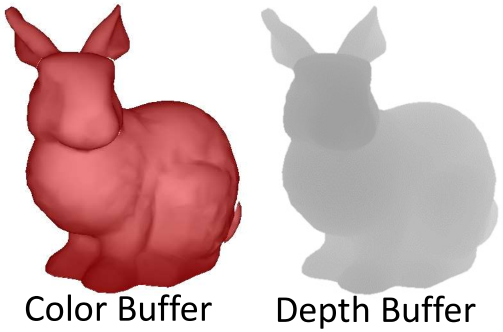
Simplified Graphics Pipeline [Catmull74] , [Strasser74]

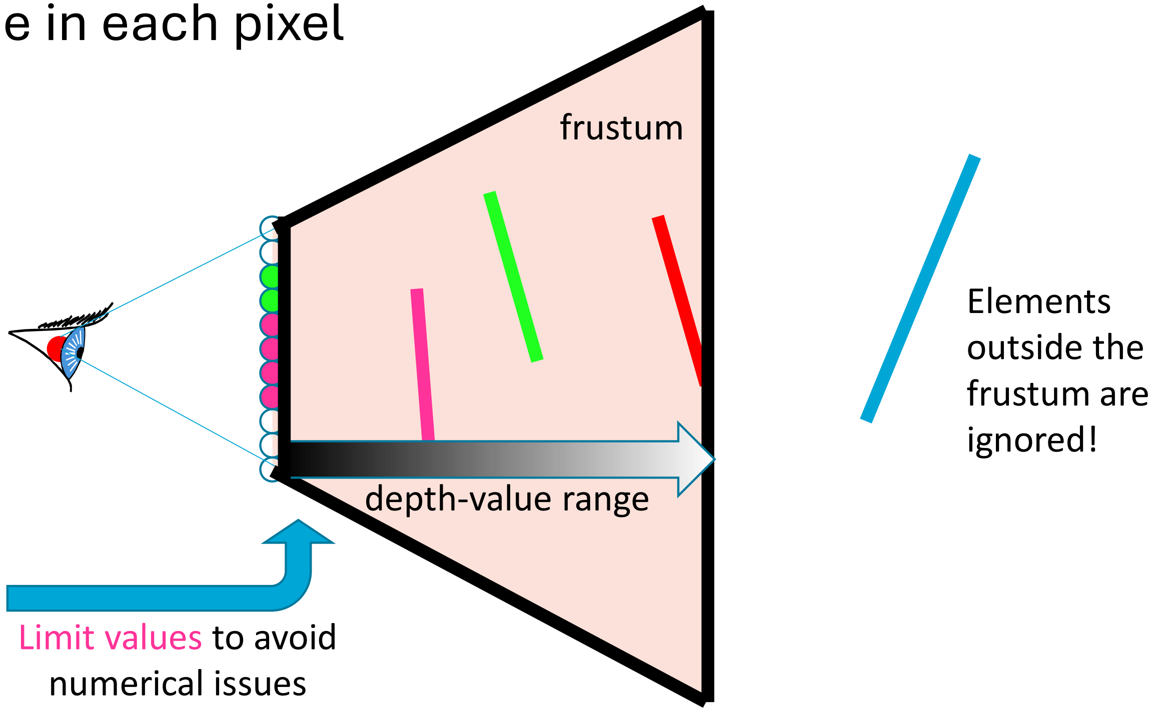
- Depth Buffer: Avoid sorting triangles!
- Store a color and depth in each pixel



Simplified Graphics Pipeline [Catmull74] , [Strasser74]

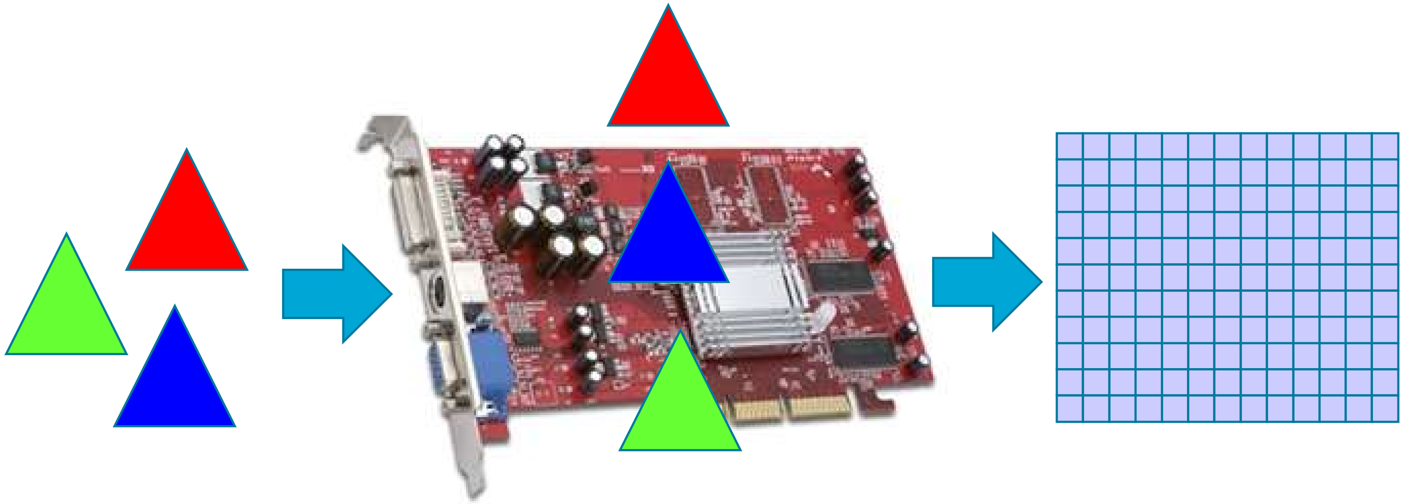
- Depth Buffer: Avoid sorting triangles!
- Store a depth value in each pixel





Simplified Graphics Pipeline

- Highly parallelizable
  ➡ Graphics Processing Units (GPUs)

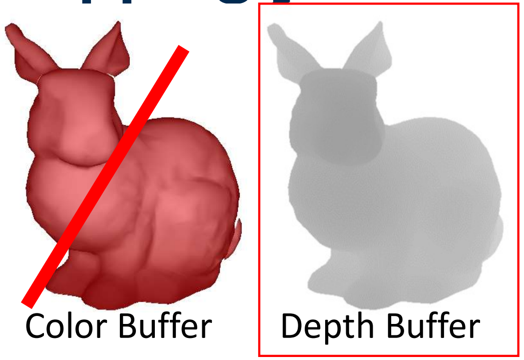


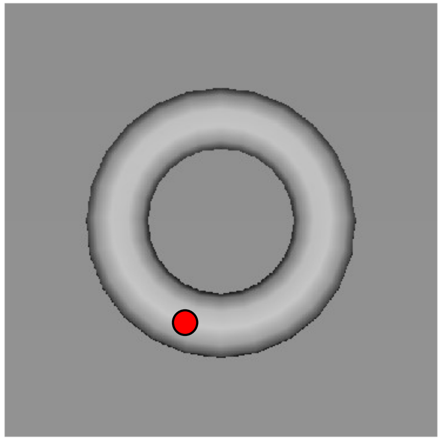
Simplified Graphics Pipeline

- Local computations:
  - Processor only knows its current triangle

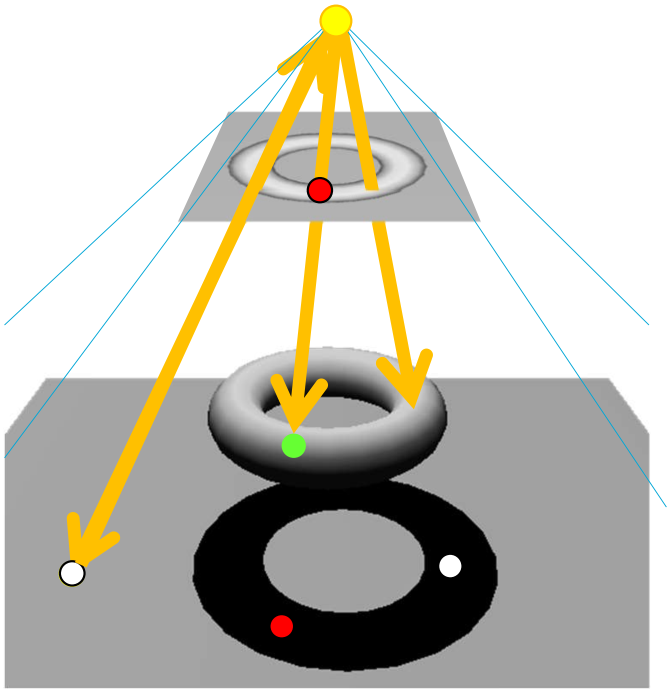
this is NOT enough for shadows

Shadow Mapping [Williams78]





1. render view from light
use depth buffer

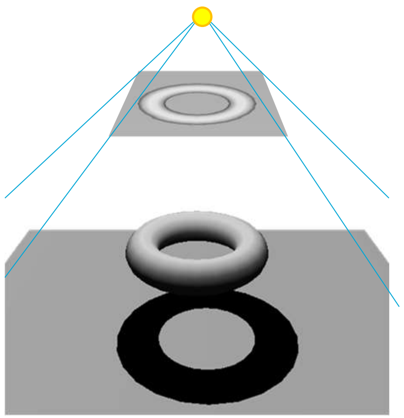


2. render from viewpoint

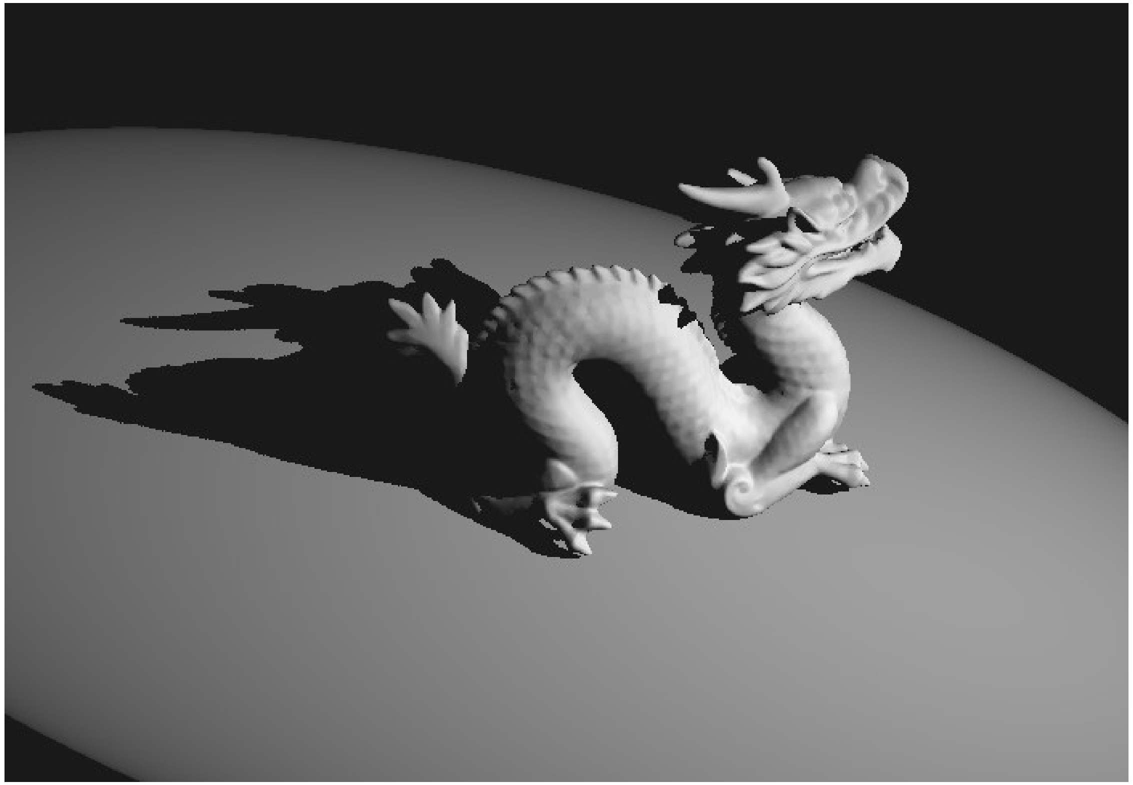
Shadow Mapping [Williams78]

1. Render depth buffer from light (**Shadow Map**)
   - Store result in a texture

2. Render from viewpoint
   - For each drawn pixel:
     Compare its depth in the light's view
     to the stored depth at this texel location
     in the Shadow Map
     - Equal: pixel is lit
     - Farther: pixel is in shadow

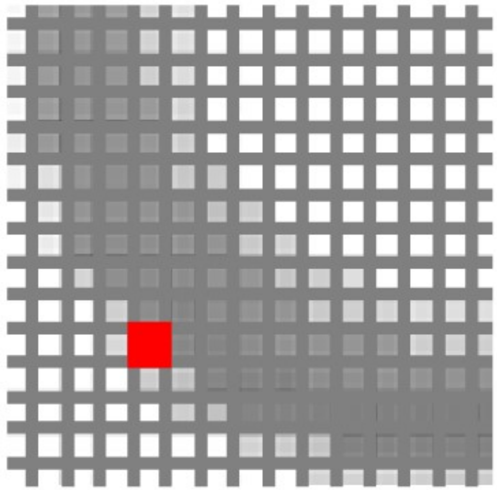


Demo



Problem 1: Discretization

from viewpoint



from light

Shadow Mapping [Williams78]

1. Render depth buffer from light (**Shadow Map**)
   - Store result in a texture

2. Render from viewpoint
   - For each drawn pixel:
     Compare its depth in the light view
     to the stored depth at the texel location
     in the Shadow Map
     - Equal: pixel is lit
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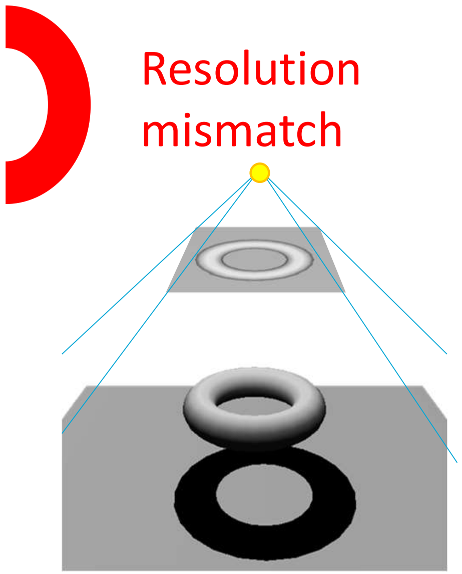

Lord of the Rings
Electronic Arts

Shadow Mapping [Williams78]

1. Render depth buffer from light (**Shadow Map**)
   - Store result in a texture

2. Render from viewpoint
   - For each drawn pixel:
     Compare its depth in the light view
     to the stored depth at the texel location
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     - Equal: pixel is lit
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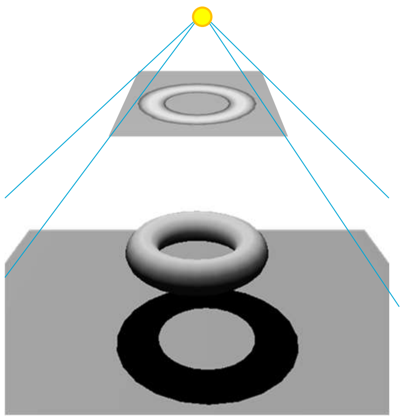
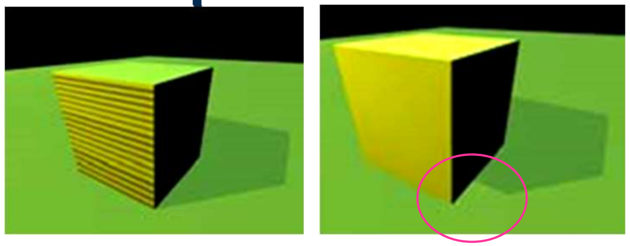
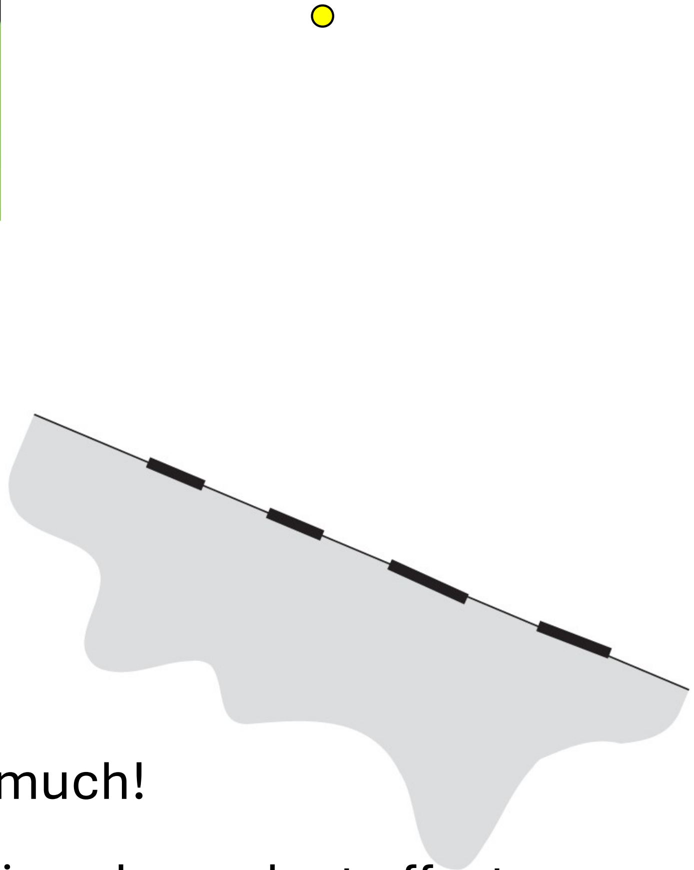


Problem 2: Depth Bias



Too small Too big

- Self-shadowing
  - Discretisation
  - Limited precision



- Solution:
  add an offset, but not too much!
- OpenGL supports orientation-dependent offsets

Problem 2: Depth Bias

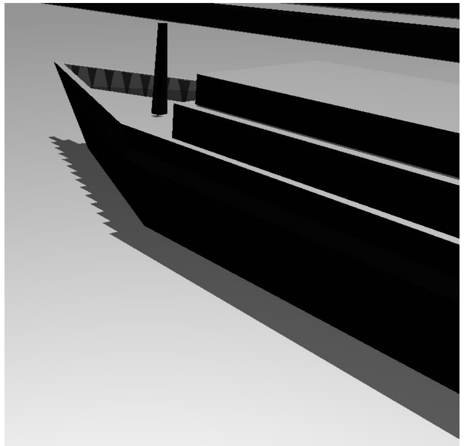
- “Real-world” example in Crysis by Crytek

Problem 3:

- Does this ring a bell?
  Unfortunately, no easy solution...

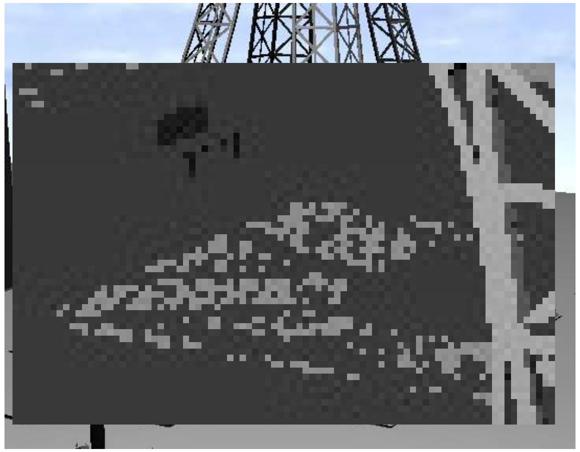
Reconstruction

- Staircase artifacts



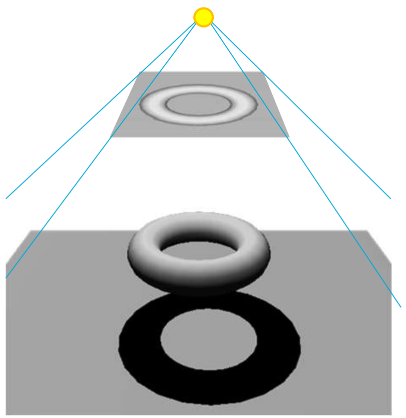
Undersampling

of the shadow map



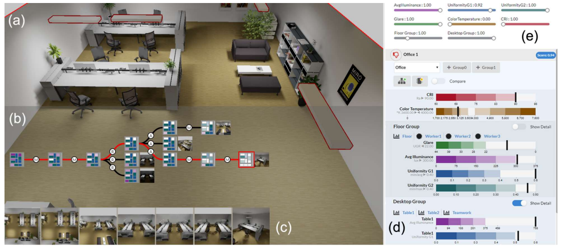
Conclusion: Shadow Mapping [Williams78]

- Simple, efficient, and easy to implement
- Compatible to most object representations
- Additional hardware support
- Variants are common
  - (games, movies...)



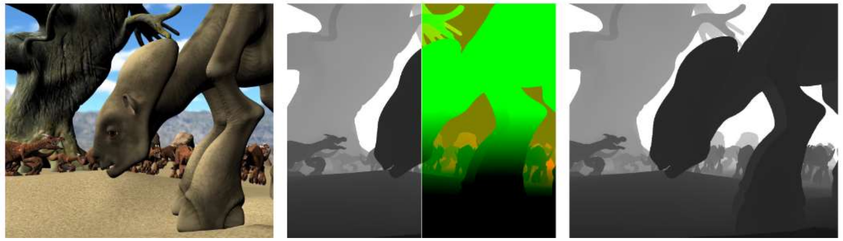
Shadow Maps have a widespread use!

Light Design



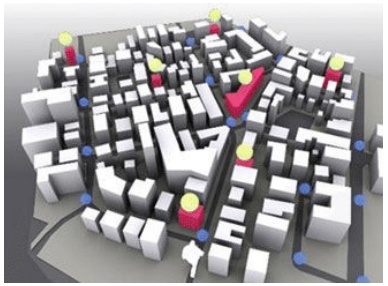
Schwaerzler et al. VIS2020

Real-time Visibility Testing of Objects



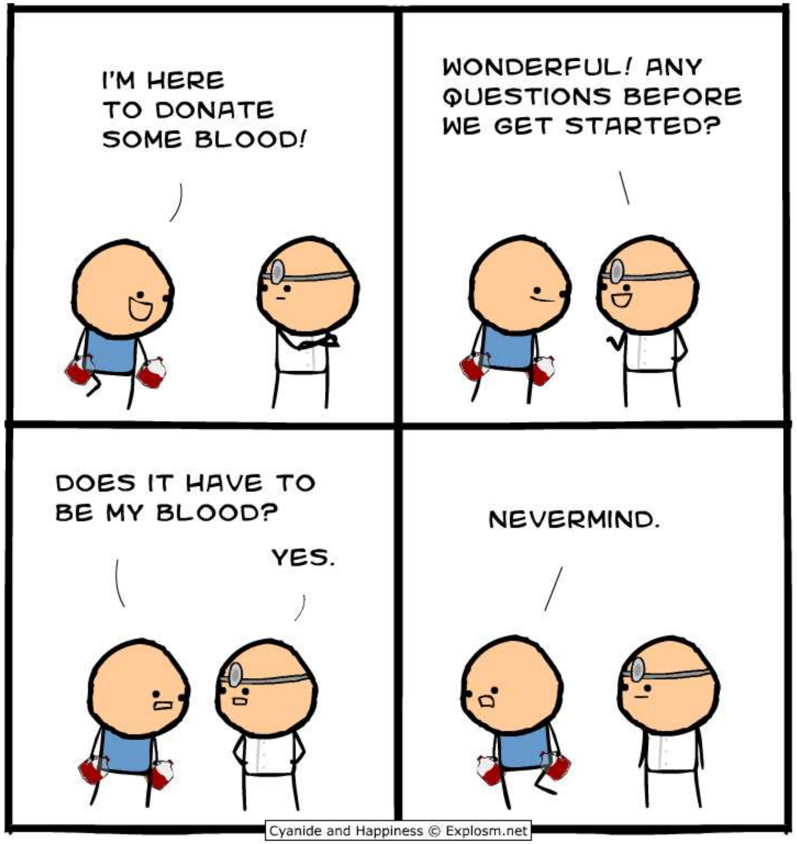
Lee et al. ToG2019

Urban Design



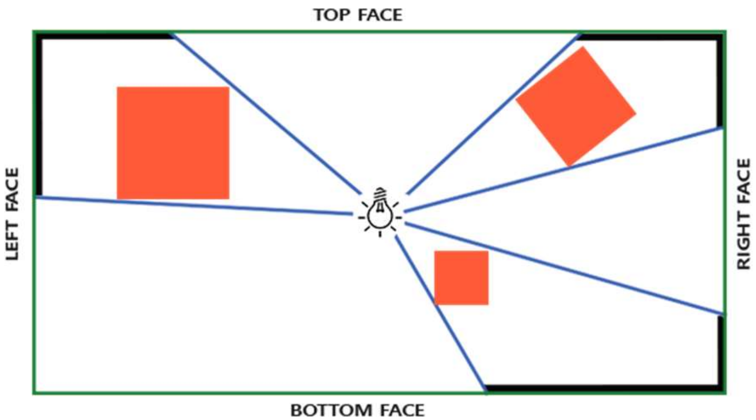
many many more...

Questions?



Talking about questions:
Consider submitting a potential exam question! See Brightspace for details!

Exercise: How to make an omnidirectional Shadow Map?



Answer: Environment Mapping

- Render a shadow map for each view.
- Could use Geometry Shader to avoid 6 loops

Cube mapping