The derivation of black body radiation

王弘禹

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主題

討論實驗十九、光譜分析的【思考問題】為何黑體輻射所產生的是連續光譜？以前高中就學過是普朗克引入能量量子化才解決黑體輻射、紫外災變，但其實我撰寫此報告前也不知道普朗克具體是怎麼做的。因此，為回答該思考問題，本報告將推導一次黑體輻射的公式。

Black body and wave in a box

Consider a box with perfectly conducting walls, and a electromagnetic wave inside it. Since the electric field *must be zero at the walls of the box*, we can expect that only the standing wave can fit this boundary condition.

Assume it's a cubic box with length L, thus, the modes of oscillation can be written

$$R(x, y, z) = R_0 \sin(k_x x) \sin(k_y y) \sin(k_z z) \tag{1}$$

and $l, m, n \in \mathbb{N}$

$$k_x = \frac{\pi l}{L},\ k_y = \frac{\pi m}{L},\ k_z = \frac{\pi n}{L} \tag{2}$$

Applied wave equation

$$\nabla^2 A = \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} \implies {k_x}^2 + {k_y}^2 + {k_z}^2 = \frac{\pi^2}{L^2}(l^2 + m^2 + n^2) = \frac{\omega^2}{c^2} \tag{3}$$

The number of modes of oscillation is equal to the number of latice points in l-m-n coordinate. Let the number of modes denote as N_T and a postive integer M

$$l, m, n \leq M \implies N_T = l \cdot m \cdot n \text{ (combination num.)} = V_{lmn} \text{ (Volume)} \tag{4}$$

Let

$$p^2 = l^2 + m^2 + n^2 \to p^2 = \frac{L^2}{\pi^2}\frac{\omega^2}{c^2} \tag{5}$$

$$k^2 = {k_x}^2 + {k_y}^2 + {k_z}^2 \to k = \frac{\pi p}{L} \tag{6}$$

$$d(N_T) = N(p)\, dp = N(k)\, dk \tag{7}$$

And derive $N(k)$

$$d(N_T) = dV = \frac{1}{8}(4\pi p^2)\, dp = N(p)\, dp \implies N(k)\, dk = \frac{L^3}{2\pi^2}k^2\, dk \tag{8}$$

Denote $V = L^3$ and $\mathbf{k} = \langle k_x, k_y, k_z \rangle$

$$\mid \mathbf{k} \mid = k = \frac{2\pi}{\lambda} = \frac{2\pi}{c}\nu \tag{9}$$

$$N(k)\, dk = N(\nu)\, d\nu = \frac{4\pi V}{c^3}\nu^2\, d\nu \tag{10}$$

Since the polarisation of electromagnetic wave, the number of states *per unit volume* is

$$dN_T = 2 \cdot \frac{4\pi\nu^2}{c^3}\, d\nu = \frac{8\pi\nu^2}{c^3}\, d\nu \tag{11}$$

Ultraviolet Catastrophe

Denote the energy density be $u(\nu)$ and average energy to each mode of oscillation $\overline{E}$

$$u(\nu) = \frac{8\pi\nu^2}{c^3}\overline{E} \tag{12}$$

Because the average energy of a harmonic oscillator $\overline{E} = kT$

$$u(\nu) = \frac{8\pi\nu^2 kT}{c^3} \tag{13}$$

It results *ultraviolet catastrophe*, the total energy diverges

$$\int_0^\infty u(\nu) d\nu = \int_0^\infty \frac{8\pi\nu^2 kT}{c^3} d\nu \to \infty \tag{14}$$

Solution of the catastrophe

In above section, the waves are constrained to fit into the box. Now we have a further constraint. The energy of the mode is $E(\nu) = nh\nu$, where h is planck constant and n is postive number.

Boltzmann distribution

The probability that a single mode has energy $E_n = nh\nu$

$$p(n) = \frac{\exp(-E_n/kT)}{\sum\limits_{n=0}^{\infty} \exp(-E_n/kT)} \tag{15}$$

The mean energy become a function of ν

$$\overline{E_\nu} = \sum_{n=0}^{\infty} E_n p(n) = \frac{\sum\limits_{n=0}^{\infty} nh\nu \exp(-nh\nu/kT)}{\sum\limits_{n=0}^{\infty} \exp(-nh\nu/kT)} \tag{16}$$

It looks like power series, let us substitute $x = \exp(-hv/kT)$.

$$\overline{E_\nu} = h\nu \frac{\sum\limits_{n=0}^{\infty} nx^n}{\sum\limits_{n=0}^{\infty} x^n} = h\nu x \frac{(1 + 2x + 3x^2 + \dots)}{(1 + x + x^2 + \dots)} \tag{17}$$

Since $\frac{1}{1-x} = 1 + x + x^2 + \dots$ and $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots$

$$\overline{E_\nu} = h\nu \frac{x}{1-x} = \frac{h\nu}{\exp(h\nu/kT) - 1} \tag{18}$$

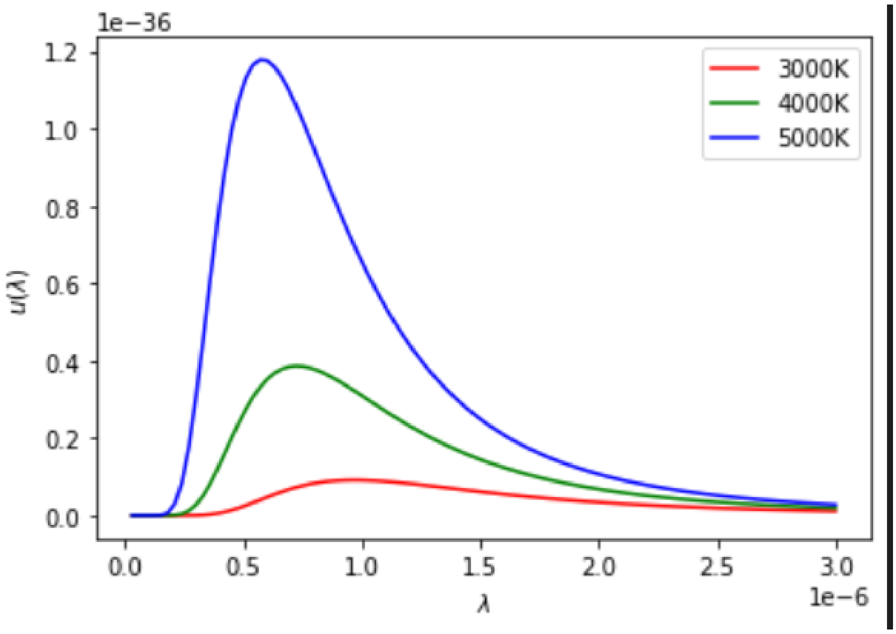
Remenber (13) tell us $u(\nu) = \frac{8\pi\nu^2}{c^3}\overline{E}$, then we get *Planck distribution function*

$$u(\nu)d\nu = \frac{8\pi\nu^2}{c^3}\overline{E_\nu} = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp(h\nu/kT) - 1} d\nu \tag{19}$$

where $u(v)$ is energy density. Change the variable to λ

$$u(\nu)d\nu = u(\lambda)d\lambda = -\frac{8\pi h}{\lambda^3} \frac{1}{\exp(hc/kT\lambda) - 1} (\frac{h}{\lambda^2})\, d\lambda \tag{20}$$

Draw the graph $\mid u(\lambda) \mid - \lambda$



後記

會用英文寫是因為我發現我不太會用中文寫，畢竟參考資料本來就是英文，要把一些敘述精確的轉成中文反而好難。

主要參考的資料是文章"The Derivation of the Planck Formula"以及另一個系上的强者同學。

由於推導過程都是現有的知識，我僅僅做的是將其學會後，用自己的理解重新編排推導過程罷了，感謝閱讀。