Linear Algebra Notes

Book: Linear Algebra, by HOFFMAN,KUNZE

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Chapter 1.

Definition list

- If A and B are m X n matrices over the field F, we say that B is row-equivalent to A if B can be obtained from A by a finite sequence of elementary row operation.
- An m X n matrix R is called row-reduced if the first non-zero entry in each non-zero row of R is equal to 1 and each column of R which contains the leading non-zero entry of some row has all its other entries 0.
- An m X n matrix R is called a row-reduced echelon matrix if:
  1. R is row-reduced
  2. every row of R which has all its entries 0 occurs below every row which has a non-zero entry
  3. if row1,. . .,r are the non-zeros rows of R, and if the leading non-zero entry of row i occurs in column k_i, i =1,. . .,r, then $k_1 < k_2 < \cdots < k_r$
- Let A be an (m,n) matrix over the field F and let B be an (n,p) matrix over F. The product AB is the (m,p) matrix C whose i,j entry is

$$C_{ij} = \sum_{r=1}^{n} A_{ir}B_{rj} \implies C_{i*} = \sum^{r} A_{ir}B_{r*},\ C_{*j} = \sum^{r} A_{*r}B_{rj} \tag{1}$$

- An (m,n) matrix is said to be an elementary matrix if it can be obtained from the (m,m) identity matrix by means of a single elementary row operation.
- Let A be an (n,n) matrix over the field F. An (n,n) matrix B such that $BA = I$ is called a *left inverse* of A an (n,n) matrix B such that $AB = I$ is called a *right inverse* of A. $If AB = BA = I$, then B is called a *two-sided inverse* of A and A is said to be *invertible*.

  Lemma. If A has a left inverse B and a right inverse C, then $B = C$.

Theorem list

1. Equivalent systems of linear equations have exactly the same solutions.

2. To each elementary row operation e there corresponds an elementary row operation e_1 of the same type as e, s.t. $e_1(e(A)) = A$.

3. If A and B are row-equivalent m X n matrices, the homogeneous systems of linear equations $AX = 0$ and $BX = 0$ have exactly the same solutions.

4. Every m X n matrix over the field F is row-equivalent to a row-reduced matrix.

5. Every m X n matrix over the field F is row-equivalent to a row-reduced echelon matrix.

6. If A is an m X n matrix and $m < n$, then the homogeneous system of linear equations $AX = 0$ has a non-trivial solution.

7. If A is an n X N matrix, then A is row-equivalent to the identity matrix if and only if the system of equations $AX = 0$ has only the trivial solution.

8. If A,B,C are matrices over the field F such that the products BC and A (BC) are defined, then so are the products AB, (AB) C and

$$A(BC) = (AB)C \tag{2}$$

9. Let e be an elementary row operation and let E be the (m,m) elementary matrix $E = e(I)$. Then, for every (m,n) matrix A,

$$e(A) = EA \tag{3}$$

   Corollary. Let A and B be (m,n) matrices over the field F. Then B is row-equivalent to A if and only if $B = PA$, where P is a product of (m,m) elementary matrices.

10. Let A and B be (n,n) matrices over F.

   (a) If A is invertible, so is A^{-1} and $(A^{-1})^{-1} = A$.

   (b) If both A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.

   Corollary. A product of invertible matrices is invertible.

11. An elementary matrix is invertible.

12. If A is an (n,n) matrix, the following are equivalent.

   (a) A is invertible.

   (b) A is row-equivalent to the (n,n) identity matrix.

   (c) A is a product of elementary matrices.

   Corollary. If A is an invertible (n,n) matrix and if a sequence of elementary row operations reduces A to the identity, then that same sequence of operations when applied to I yields A^{-1}.

   Corollary. Let A and B be (m,n) matrices. Then B is row-equivalent to A *if and only if* $B = PA$ where P is an invertible (m,m) matrix.

13. For an (n,n) matrix A, the following are equivalent.

(a) A is invertible.
(b) The homogeneous system $AX = 0$ has only the trivial solution $X = 0$.
(c) The system of equations $AX = Y$ has a solution X for each (n,1) matrix Y.

Corollary. A square matrix with either a left or right inverse is invertible.

Corollary. Let $A = A_1A_2 \cdots A_k$, where $A_1 \ldots, A_k$ are (n,n) matrices. Then A is invertibl *if and only if* each A_j is invertible.

Chapter 2.

Old definition list

- A **vector space** consists of the following:
  1. a field F of scalars
  2. a set V of objects, called vectors
  3. a rule (or operation), called vector addition, which associates with each pair of vectors α, β in V a vector $\alpha + \beta$ in V, called the sum of α and β, in such a way thay
     (a) addition is commutative, $\alpha + \beta = \beta + \alpha$
     (b) addition is associativ, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
     (c) there is a unique vector 0 in V, called the zero vector, such that $\alpha + 0 = \alpha$ for all α in V
     (d) for each vector α in V there is a unique vector $-\alpha$ in V such that $\alpha + (-\alpha) = 0$
  4. a rule (or operation), called scalar multiplication, which associates with each scalar c in F and vector α in V, called the product of c and α, in such a way thay
     (a) $1\alpha = \alpha$
     (b) $(c_1c_2)\alpha = c_1(c_2\alpha)$
     (c) $c(\alpha + \beta) = c\alpha + c\beta$
     (d) $(c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$
- A vector β in V is said to be a **linear combination** of the vectors $\alpha_1, \ldots, \alpha_n$ in V provided there exist scalars $c_1, \ldots, c_n$ in F such that

$$\beta = \sum_{i=1}^{n} c_i\alpha_i \tag{4}$$

- Let V be a vector space over the field F. A **subspace** of V is a subset W of V which is itself a vector space over F with the operations of vector addition and scalar multiplication on V.
- Let S be a set of vectors in a vector space V. The **subspace spanned** by S is defined to be the intersection W of all subspaces of V which contain S. When S is a finite set of vectors, $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, we shall simply call W the subspace spanned by the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.

- *If* $S_1, S_2, \ldots, S_k$ are subsets of a vector space V, the set of all sums

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k \tag{5}$$

of vectors α_i in S_i is called the **sum** of the subsets S_1, $S_2, \ldots,$ S_k and is denoted by

$$\sum_{i=1}^{k} S_i \tag{6}$$

If W_1, $W_2, \ldots,$ W_k are subspaces of V, then the sum

$$W = W_1 + W_2 + \cdots + W_k \tag{7}$$

is easily seen to be a subspace of V which contains each of the subspaces W_i. From this it follows, as in the proof of Theorem 3, that W is the subspace spanned by the union of W_1, $W_2, \ldots,$ W_k

Definition list

Linearly dependnet

Let V be a vector space over F. A subset S of V is said to be **linearly dependent** if there exist distinct vectors α_1, $\alpha_2, \ldots,$ α_n in S and scalars c_1, $c_2, \ldots,$ c_n in F, not all of which are 0, such that

$$\sum_{i=1}^{n} c_i \alpha_i = 0 \tag{8}$$

A set which is not linearly dependent is called **linearly independent**. If the set S contains only finitely many vectors α_1, $\alpha_2, \ldots,$ α_n, we sometimes say that the vectors are dependent instead of saying S is dependent.

Basis

Let V be a vector space. A **basis** for V is a linearly independent set of vectors in V which spans the space V. The space V is finite-dimensional if it has a finite basis.

Ordered basis

If V is a finite-dimensional vector space, an ordered basis for V is a finite sequence of vectors which is linearly independent and spans V.

Theorem list

1. A non-empty subset W of V is a subspace of V *if and only if* for each pair of vectors α, β in W and each scalar c in F the vector $c\alpha + \beta$ is again in W.

2. Let V be a vector space over the field F. The intersection of any collection of subspace of V is a subspace of V.

3. The subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.

4. Let V be a vector space which is spanned by $\beta_1, \dots, \beta_m$. Then any independent set of vectors in V is finite and contains no more tahn m elements.

   Corollary If V is a finite-dimensional vector space, then any two bases of V have same number of elements.

   Corollary Let V be a finite-dimensional vector space and let $n = \dim V$. Then

   - any subset of V which contains more than n vectors is linearly dependent
   - no subset of V which contains fewer than n vectors can span V.

   Lemma. Let S be a linearly independent subset of a vector space V. Suppose β is a vector in V which is not in the subspace spanned by S. Then the set obtained by adjoining β to S is linearly independent.

5. If W is a subspace of a finite-dimensional vector space V, every linearly independent subset of W is finite and is part of a finite basis for W.

   Corollary If W is a proper subspace of a finite-dimensional vector space V, then W is finite-dimensional and dim W < dim V

   Corollary In a finite-dimensional vector space V every non-empty linearly independent set of vectors is part of a basis.

   Corollary Let A be an (n,n) matrix over a field F, and suppose the row vectors of A form a linearly independent set of vectors in F^n. Then A is invertible.

6. If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then $W_1 + W_2$ is finite dimensional and

$$\dim W_1 + \dim W_2 = \dim (W_1 \cap W_2) + \dim(W_1 + W_2) \tag{9}$$

7. Let V be an n-dimensional vector space over the field F, and let $\mathscr{B}$ and $\mathscr{B}'$ be two ordered bases of V. Then there is a unique, necessarily invertible, (n,n) matrix P with entries in F such that

$$[\alpha]_{\mathscr{B}} = P[\alpha]_{\mathscr{B}'} \tag{10}$$
$$[\alpha]_{\mathscr{B}'} = P^{-1}[\alpha]_{\mathscr{B}} \tag{11}$$

   where $\alpha'_j = \sum\limits_{i=1}^{n} P_{ij}\alpha_i$.The columns of P are given by

$$P_j = [\alpha'_j]_{\mathscr{B}},\ j = 1, \dots, n. \tag{12}$$

8. Suppose P is an (n,n) invertible matrix over F and $\mathscr{B}$ be an ordered basis of V. Then there is a unique ordered $\mathscr{B}$ of V such that

$$[\alpha]_{\mathscr{B}} = P[\alpha]_{\mathscr{B}'} \tag{13}$$
$$[\alpha]_{\mathscr{B}'} = P^{-1}[\alpha]_{\mathscr{B}} \tag{14}$$

9. Row-equivalent matrices have the same row space.
10. Let R be a non-zero row-reduced echelon matrix. Then the non-zero row vectors of R form a basis for the row space of R.
11. Let m and n be postive integers and lef F be a field. Suppose W is a subspace of F^n and $\dim W \leq m$. Then there is precisely one (m,n) row-reduced echelon

Chapter 3.

Theorem list

1. Let V be a finite-dimensional vector space over the field F and let $\{\alpha_1, \dots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F and let $\beta_1, \dots, \beta_n$ be any vectors in W. Then there is precisely one linear transforamtion T from V into W such that
$$T\alpha_j = \beta_j, \quad j = 1, \dots, n. \tag{15}$$
2. Let V and W be vector space over the field F and let T be a linear transforamtion from V into W. Suppose that V is finite-dimensional.
$$\text{rank}(T) + \text{nullity}(T) = \dim V. \tag{16}$$
3. If A is an (m,n) matrix with entries in the field F, then
$$\text{row rank}(A) = \text{column rank}(A) \tag{17}$$
4. T and U are linear transforamtion from V into W. Define
$$(T + U)(\alpha) = T\alpha + U\alpha \tag{18}$$
$$(cT)(\alpha) = c(T\alpha) \tag{19}$$
then, the set of all linear transforamtions from V into W, is a vector space over the field F.
5. Let V be an n-dimensional vector space over the field F, and let W be an m-dimensional vector space over F. Then the space L(V,W) is finite-dimensional and has dimension mn.
6. Let V, W, and Z be vector spaces over the field F. Let T be a linear transforamtion from V into W and U a linear transforamtion from W into Z. Then the composed function UT defined by (UT) (α) = U(T(α)) is a linear transforamtion from V into Z.

7. Let V and W be vector spaces over the field F and let T be a linear transforamtion from V into W. If T is invertible, then the inverse function T^{-1} is a linear transforamtion from W onto V.

8. Let T be a linear transforamtion from V into W. Then T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.

9. Let V and W be finite-dimensional vector spaces over the field F such that $\dim V = \dim W$. If T is a linear transforamtion from V into W, the following are equivalent:

   (a) T is invertible.

   (b) T is non-singular.

   (c) T is onto, that is, the range of T is W.

10. Every n-dimensional vector space over tyhe field F is isomorphic to the space F^n.

11. Let V be an n-dimensional vector space over the field F and W an m-dimensional vector space over F. Let $\mathscr{B}$ be an ordered basis for V and $\mathscr{B}'$ an ordered basis for W. For each linear transformation T from V into W, there is an (m,n) matrix A with entries in F such that

$$[T\alpha]_{\mathscr{B}'} = A[\alpha]_{\mathscr{B}} \tag{20}$$

for every vector α in V. Furthermore, $T \to A$ is a one-one correspondence between the set of all linear transforamtions from V into W and the set of all (m,n) matrices over the field F.

12. Let V be an n-dimensional vector space over the field F and let W be an m-dimensional vector space over F. For each pair of ordered bases $\mathscr{B}$, $\mathscr{B}'$ for V and W respectively, the function which assigns to a linear transforamtion T its matrix relative to $\mathscr{B}$, $\mathscr{B}'$ is an isomorphism between the space L(V, W) and the space of all (m,n) matrices over the field F.

13. Let V, W, and Z be finite-dimensional vector spaces over the field F; let T be a linear transforamtion from V into W and U a linear transforamtion from W into Z. If β, β', β'' are ordered bases for the space V, W, and Z, respectively, if A is the matrix of T relative to the pair β, β', and B is the matrix of U relative to the pair β', β'', then the matrix of the composition UT relative to the pair β, β'' is the product matrix C=BA.

14. Let V be a finite-dimensional vector space over the field F, and let

$$\beta = \{\alpha_1, \ldots, \alpha_n\} \quad \beta' = \{\alpha'_1, \ldots, \alpha'_n\} \tag{21}$$

be ordered bases for V. Suppose T is a linear operator on V. If $P = [P_1, \ldots, P_n]$ is the (n,n) matrix with columns $P_j = [\alpha'_j]_\beta$, then

$$[T]_{\beta'} = P^{-1}[T]_\beta P \tag{22}$$

Alternatively, if $U\alpha_j = \alpha'_j$

$$[T]_{\beta'} = [U]_\beta{}^{-1}[T]_\beta[U]_\beta \tag{23}$$

15. Let V be a finite-dimensional vector space over the field F, and let $\beta = \{\alpha_1, \dots, \alpha_n\}$ be a basis for V. Then there is a unique dual basis $\beta^* = \{f_1, \dots, f_n\}$ for V^* such that $f_i(\alpha_j) = \delta_{ij}$. For each linear functional f on V we have

$$f = \sum_{i=1}^{n} f(\alpha_i) f_i \tag{24}$$

and for each vector α in V we have

$$\alpha = \sum_{i=1}^{n} f_i(\alpha) \alpha_i \tag{25}$$

16. Let V be a finite-dimensional vector space over the field F, and let W be a subspace of V. Then

$$\dim W + \dim W^0 = \dim V. \tag{26}$$

17. Let V be a finite-dimensional vector space over the field F.
For each vector α in V define

$$L_\alpha(f) = f(\alpha), \qquad f \in V^* \tag{27}$$

The mapping $\alpha \to L_\alpha$ is then an isomorphism of V onto V^{**}.
Corollary.
Let V be a finite-dimensional vector space over the field F. Each basis for V^* is the dual of some basis for V.
Corollary.
Let V be a finite-dimensional vector space over the fiel F. If L is a linear functional on the dual space V^* of V, then there is a unique vector α in V such that

$$L(f) = f(\alpha) \tag{28}$$

for every f in V^*