Calculus

王弘禹

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16.8

8.

$$\mathbf{r}(u,v) = \langle 2uv, u^2 - v^2, u^2 + v^2 \rangle \implies \mathbf{r}_u \times \mathbf{r}_v = \langle 8uv, 4u^2 - 4v^2, -4v^2 - 4u^2 \rangle \tag{1}$$
$$u^2 + v^2 \le 1 \implies 0 \le v^2 \le 1 - u^2 \tag{2}$$
$$\tag{3}$$

then

$$\iint_S x^2 + y^2 \, dS = \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_{-1}^{1} (u^2 + v^2)^2 \mid \mathbf{r}_u \times \mathbf{r}_v \mid \, dudv \tag{4}$$
$$= 4\sqrt{2} \int_0^{2\pi} \int_0^1 r^7 \, drd\theta = \sqrt{2}\pi \tag{5}$$

20.

Side:

Consider a map γ

$$\gamma : (z, \theta) \longmapsto (3\cos\theta, 3\sin\theta, z) \implies \gamma_z \times \gamma_\theta = \langle -3\cos\theta, -3\sin\theta, 0 \rangle \tag{6}$$

The magnitude of $\gamma_z \times \gamma_\theta = 3$

$$\iint_{S_1} (x^2 + y^2 + z^2) \, dS = \int_0^{2\pi} \int_0^2 3(9 + z^2) \, dzd\theta = 124\pi \tag{7}$$

Top:

Consider a map γ

$$\gamma : (r,\theta) \longmapsto (r\cos\theta, r\sin\theta, 2) \implies |\ \gamma_r \times \gamma_\theta\ | = r \tag{8}$$

then

$$\iint_{S_2} (x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^3 r(r^2+4)\,drd\theta = 2\pi(\frac{3^4}{4}+2\cdot 3^2) = \frac{153}{2}\pi \tag{9}$$

Bottom:

Consider a map γ

$$\gamma : (r,\theta) \longmapsto (r\cos\theta, r\sin\theta, 0) \implies |\ \gamma_r \times \gamma_\theta\ | = r \tag{10}$$

then

$$\iint_{S_3} (x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^3 r^3\,drd\theta = \frac{81}{2}\pi \tag{11}$$

the area S

$$\iint_{S} (x^2+y^2+z^2) = 124\pi + \frac{153}{2}\pi + \frac{81}{2}\pi = 241\pi \tag{12}$$

44.

Consider a map γ

$$\gamma : (\theta,\phi) \longmapsto (3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi),\ \theta\in[0,2\pi]\ \phi\in[0,\pi/2] \tag{13}$$

$$\implies \gamma_\theta \times \gamma_\phi = \langle -9\sin^2\phi\cos\theta, -9\sin^2\phi\sin\theta, -9\sin\phi\cos\phi\rangle \tag{14}$$

the rate of flow outward through the hemisphere S is

$$\iint_S \mathbf{v}\cdot(-\gamma_\theta\times\gamma_\phi)\,dS = \int_0^{2\pi}\int_0^{\pi/2} 54\sin^3\phi\sin\theta\cos\theta\,d\phi d\theta = 0 \tag{15}$$

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10.

Consider a map γ

$$\gamma : (r,\theta) \longmapsto (r\cos\theta, r\sin\theta, r\sin\theta+2) \implies \gamma_r\times\gamma_\theta = \langle 0, -r, r\rangle \tag{16}$$

then $\nabla \times \mathbf{F} = \langle 1-x, -1, z-2 \rangle$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 [r + r(r\sin\theta)]\, dr d\theta = \pi \tag{17}$$

14.

Consider a map γ

$$\gamma : (x, y) \longmapsto (x, y, x) \implies \gamma_x \times \gamma_y = \langle -1, 0, 1 \rangle \tag{18}$$

Find the domain of x and y

$$x = x^2 + y^2 \implies y^2 = x - x^2 \implies y \in [-\sqrt{x-x^2}, \sqrt{x-x^2}],\ x \in [0,1] \tag{19}$$

then $\nabla \times \mathbf{F} = \langle 1, -1, y \rangle$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} \nabla \times \mathbf{F} \cdot (\gamma_x \times \gamma_y)\, dy dx \tag{20}$$

$$= \int_0^1 -2\sqrt{x-x^2}\, dx = -2\int_0^1 \sqrt{\frac{1}{4} - (x - \frac{1}{2})^2}\, dx \tag{21}$$

Let $x - \frac{1}{2} = \frac{1}{2}\cos\theta$

$$\int_{-\pi}^0 \sqrt{\frac{1}{4} - \frac{1}{4}\cos^2\theta} \sin\theta d\theta = \int_{-\pi}^0 -\frac{1}{2}\sin^2\theta d\theta = -\frac{1}{4}\pi \tag{22}$$

22.

Consider a map γ

$$\gamma : (r, t) \longmapsto (r\cos t, r\sin t, 2r^2 \sin t\cos t) \implies \gamma_r \times \gamma_t = \langle -2r^2 \sin t, -2r^2\cos t, r \rangle \tag{23}$$

then $\nabla \times \mathbf{F} = \langle -2z, -3x^2, -1 \rangle$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_0^{2\pi} \int_0^1 (8r^4 \sin^2 t \cos t + 6r^4 \cos t \cos^2 t - r)\, dr dt \tag{24}$$

$$= \int_0^{2\pi} \int_0^1 r\, dr d\theta = \pi \tag{25}$$

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14.

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,dV = \int_0^1\int_{-\sqrt{x}}^{\sqrt{x}}\int_0^{1-x}(y+2z)\,dzdydx \tag{26}$$

$$= \int_0^1\int_{-\sqrt{x}}^{\sqrt{x}}[y(1-x)+(1-x)^2]\,dydx \tag{27}$$

$$= \int_0^1 2\sqrt{x}(1-x)^2\,dx \tag{28}$$

Let $u=\sqrt{x} \implies 2udu = dx$

$$\int_0^1 2\sqrt{x}(1-x)^2\,dx = \int_0^1(4u^6-8u^4+4u^2)\,du = \frac{32}{105} \tag{29}$$

20.

$\boldsymbol{\nabla}\cdot F = 0+0+1 = 1,\ x^{+}y^2+z=2 \implies z = 2-x^2-y^2$,and use the cylinderical coordinate:

$$\iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,dV = \int_0^{2\pi}\int_0^1\int_1^{2-r^2} r\,dzdrd\theta = \frac{\pi}{2} \tag{30}$$

For bottom

Consider a map γ

$$\gamma : (r,\theta)\longmapsto(r\cos\theta, r\sin\theta, 1) \implies \gamma_r\times\gamma_\theta = \langle 0,0,r\rangle \tag{31}$$

then calculate the flux

$$\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = -\int_0^{2\pi}\int_0^1 r\,drd\theta = -\pi \tag{32}$$

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \frac{\pi}{2}+\pi = \frac{3}{2}\pi \tag{33}$$