Calculus HW2

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16.4

22.

$$\mathbf{F}(x, y) = \langle \sin x, \sin y + xy^2 + \frac{1}{3}x^3 \rangle \tag{1}$$

By green theorem and polar coordinate:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^5 \int_0^{\pi/2} r^3 \, d\theta dr = \frac{\pi}{2} \frac{625}{4} \tag{2}$$

31.

$$\frac{\partial}{\partial y} F_x = \frac{\partial}{\partial x} F_y = \frac{2x^2 - 6xy^2}{x^2 + y^2} \tag{3}$$

Therefore, by the method of example 5, we know that all the path which is closed to origin have the same integral. So, we chose a unit circle C_1 and use polar coordinate:

$$\mathbf{F} = \langle 2 \sin t \cos t, \sin^2 t - \cos^2 t \rangle \tag{4}$$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-2 \sin^2 t \cos t + \sin^2 t \cos t - \cos^3 t) \, dt = \int_0^{2\pi} - \cos t \, dt = 0 = \int_C \mathbf{F} \cdot d\mathbf{r} \tag{5}$$

16.5

20.

The domain $D \in \mathbf{R}$ is star-shaped, therefore, we just need to check these conditions

$$\underset{i \neq j}{\forall}\, i, j \implies \frac{\partial F_i}{\partial j} = \frac{\partial F_j}{\partial i} \tag{6}$$

in this case

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0 - 0 = 0 \tag{7}$$
$$\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = e^z \cos x - e^z \cos x = 0 \tag{8}$$
$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = -e^y \sin z - (-e^y \sin z) = 0 \tag{9}$$

29.

Use the Levi-Civita symbol, use Kroneker delta,,and x_1, x_2, x_3 represent x, y, z

$$\mathrm{div}(\mathbf{F} \times \mathbf{G}) = \mathrm{div}(\sum_{ijk=1}^{3} \epsilon_{i,j,k} \hat{x}_i F_j \cdot G_k) = \sum_{l,i,j,k=1}^{3} \delta_{li} \epsilon_{ijk} \frac{\partial}{x_l}(F_j \cdot G_k) \tag{10}$$
$$= \sum_{l,i,j,k=1}^{3} \delta_{li} \epsilon_{ijk} (\frac{\partial F_j}{\partial x_l} G_k + F_j \frac{\partial G_k}{\partial x_l}) \tag{11}$$
$$= \mathbf{G} \cdot \sum_{i,j,k=1}^{3} \epsilon_{ijk} \hat{x}_k \frac{\partial}{\partial x_i} F_j + \mathbf{F} \cdot \sum_{i,j,k=1}^{3} \hat{x}_j \frac{\partial}{\partial x_i} G_k \tag{12}$$
$$= \mathbf{G} \cdot \mathrm{curl}\ \mathbf{F} - \mathbf{F} \cdot \mathrm{curl}\ \mathbf{G} \tag{13}$$

34.

$$r = \sqrt{x^2 + y^2 + z^2},\ \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r} \tag{14}$$
$$\tag{15}$$

then, $\frac{\partial}{\partial x} \frac{x}{r^p} = \frac{r^p - x \cdot p \cdot r^{p-1} \cdot \frac{x}{r}}{r^{2p}}$

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} \frac{x}{r^p} + \frac{\partial}{\partial y} \frac{y}{r^p} + \frac{\partial}{\partial z} \frac{z}{r^p} \tag{16}$$
$$= \frac{3r^p - p \cdot r^{p-1} \cdot \frac{r^2}{r}}{r^{2p}} = \frac{3-p}{r^p} \to \nabla \cdot F = 0 \implies p = 3 \tag{17}$$

35.

By EQ13

$$\oint_C \mathbf{f}(\nabla g)\cdot\mathbf{n}\,ds = \iint_D \boldsymbol{\nabla}\cdot\mathbf{f}(\nabla g)\,dA \tag{18}$$

then

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{f}(\nabla g) =& \frac{\partial}{\partial x}(f(\nabla g)_x) + \frac{\partial}{\partial y}(f(\nabla g)_y) & (19)\\ =&(\nabla g)_x\frac{\partial}{\partial x}f + f\frac{\partial}{\partial x}(\nabla g)_x + (\nabla g)_y\frac{\partial}{\partial y}f + f\frac{\partial}{\partial}(\nabla g)_y & (20)\\ =&(\nabla g)\cdot\nabla f + f\nabla^2 g & (21)\end{aligned}$$

then

$$\oint_C \mathbf{f}(\nabla g)\cdot\mathbf{n} - \iint_D \nabla f\cdot\nabla g\,dA = \iint_D f\nabla^2 g\,dA \tag{22}$$

16.6

62.

Consider $x \geq 0,\ z \geq 0$, and let $z(x,y) = \sqrt{1-x^2} \implies \dfrac{\partial z}{\partial x} = \dfrac{-x}{\sqrt{1-x^2}},\ \dfrac{\partial z}{\partial y} = 0$

$$\sqrt{1 + (\frac{x^2}{\sqrt{1-x^2}})} = \sqrt{\frac{1}{1-x^2}} \tag{23}$$

then

$$\frac{1}{8}\cdot\text{Area} = \int_0^1\int_{-x}^{x}\frac{1}{\sqrt{1-x^2}}\,dydx = 2 \implies \text{A} = 16 \tag{24}$$

64.

(a)

$$r(\theta,\alpha) = (x,y,z),\ x = (b + a\cos\alpha)\cos\theta,\ y = (b + a\cos\alpha)\sin\theta,\ z = a\sin\alpha \tag{25}$$

(c)

$$\begin{aligned} r_\theta &= \langle -(b + a\cos\alpha)\sin\theta, (b + a\cos\alpha)\cos\theta\rangle & (26)\\ r_\alpha &= \langle -a\sin\alpha\cos\theta, -a\sin\alpha\sin\theta, a\cos\alpha\rangle & (27)\end{aligned}$$

and $r\theta \times r_\alpha = a(b + a\cos\alpha)$

$$\text{Area} = \int_0^{2\pi} \int_0^{2\pi} (ab + a^2 \cos\alpha)\, d\alpha d\theta = 4\pi^2 ab \tag{28}$$