

1. (10 pts) Let the curve  $C$  be given by  $\mathbf{r}(t) = \frac{2}{3}(2+t)^{\frac{3}{2}}\mathbf{i} + \frac{2}{3}(2-t)^{\frac{3}{2}}\mathbf{j} + at\mathbf{k}$ ,  $a \neq 0$ ,  $t \in (-2, 2)$ . Find the vectors  $\mathbf{T}$ ,  $\mathbf{N}$ ,  $\mathbf{B}$ , the curvature  $\kappa$  and the torsion  $\tau$  of the curve  $C$  at  $t = 0$ .

**Solution:**

$$\mathbf{r}'(t) = \langle (2+t)^{1/2}, -(2-t)^{1/2}, a \rangle$$

$$\mathbf{r}''(t) = \frac{1}{2} \langle (2+t)^{-1/2}, (2-t)^{-1/2}, 0 \rangle$$

$$\mathbf{r}'''(t) = \frac{1}{4} \langle -(2+t)^{-3/2}, (2-t)^{-3/2}, 0 \rangle$$

$$|\mathbf{r}'(t)| = (4+a^2)^{1/2}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \frac{1}{2} \langle -a(2-t)^{-1/2}, a(2+t)^{-1/2}, 4(4-t^2)^{-1/2} \rangle$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{4+a^2}/\sqrt{4-t^2}, \quad (\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t) = \frac{a}{2}(4-t^2)^{-3/2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{4+a^2}} \langle (2+t)^{1/2}, -(2-t)^{1/2}, a \rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{|\mathbf{T}'|} = \frac{1}{2} \langle (2-t)^{1/2}, (2+t)^{1/2}, 0 \rangle$$

$$\mathbf{B}(t) = \mathbf{T} \times \mathbf{N}(t) = \frac{1}{2\sqrt{4+a^2}} \langle -a(2+t)^{1/2}, a(2-t)^{1/2}, 4 \rangle$$

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{1}{(4+a^2)\sqrt{4-t^2}}$$

$$\tau(t) = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{a}{2(4+a^2)\sqrt{4-t^2}}$$

At  $t = 0$  (每項 2 分)

$$\mathbf{T} = \frac{1}{\sqrt{4+a^2}} \langle \sqrt{2}, -\sqrt{2}, a \rangle$$

$$\mathbf{N} = \frac{1}{2} \langle 2, 2, 0 \rangle$$

$$\mathbf{B} = \frac{1}{2\sqrt{4+a^2}} \langle -a\sqrt{2}, a\sqrt{2}, 4 \rangle$$

$$\kappa = \frac{1}{2(4+a^2)}$$

$$\tau = \frac{a}{4(4+a^2)}$$

2. (12 pts) Consider the following function on  $\mathbf{R}^2$ :

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + (\sin y)^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

- (a) (6 pts) Is  $f(x, y)$  continuous at  $(x, y) = (0, 0)$ ? Justify your answer.
- (b) (4 pts) Do the partial derivatives  $f_x(0, 0)$  and  $f_y(0, 0)$  exist? (Compute them if you think they exist; otherwise, prove that they do not exist.)
- (c) (2 pts) Is  $f$  differentiable at  $(0, 0)$ ? Justify your answer.

**Solution:**

- (a) Let  $(x, y)$  tend to  $(0, 0)$  along special paths. One may gain 6 points if the students find such a path that along which the limit of  $f$  is different from  $0 = f(0, 0)$ . For example, one may consider  $(x, mx^2)$ : if  $x \neq 0$ ,

$$f(x, mx) = \frac{x^2 \cdot mx^2}{x^4 + \sin(mx)^2} = \frac{m}{1 + \left(\frac{m \sin(mx^2)}{mx^2}\right)^2} \rightarrow \frac{m}{1 + m^2} \neq 0 = f(0, 0) \text{ if } m \neq 0 \text{ as } x \rightarrow 0.$$

Therefore  $f$  is not continuous at  $(0, 0)$ . One may also consider for example the curve  $y = \sin^{-1}(mx^2)$ . 找尋路徑並說明上述極限不等於  $f(0, 0) = 0$  的任何一個環節錯誤可扣1至2分。

- (b) Compute by definition that  $f_x(0, 0) = 0 = f_y(0, 0)$ . 過程與計算結果均正確者得4分，否則得0分。
- (c) Since  $f$  is not continuous at  $(0, 0)$ , it is not differentiable at  $(0, 0)$ . One may also argue by definition. 理由充分者得2分，否則得0分。

3. (10 pts) Let  $g(x, y, z)$  be a function defined on  $\mathbb{R}^3$  with continuous partial derivatives. Suppose that

$$|\nabla g(2, 1, 3)|^2 = 24 \text{ and } g_z(2, 1, 3) > 0.$$

Moreover, the trajectories of the two curves

$$\mathbf{r}_1(s) = \langle 2s, s^2, 1 + 2s \rangle \text{ and } \mathbf{r}_2(t) = \langle 2e^t, \cos t, 3 + t + 5t^2 \rangle$$

lie on the level surface  $g(x, y, z) = 0$  completely.

- (a) (5 pts) Find the vector  $\nabla g(2, 1, 3)$ .  
(b) (5 pts) Suppose that  $f(x, y, z)$  is a function defined on  $\mathbb{R}^3$  with continuous partial derivatives such that

$$f(2, 1, 3) \geq f(x, y, z) \text{ for every point } (x, y, z) \text{ on the level surface } g(x, y, z) = 0.$$

If  $f(2, 1, 3) = 5$ ,  $|\nabla f(2, 1, 3)|^2 = 6$  and  $f_y(2, 1, 3) > 0$ , estimate the value of  $f(2.01, 0.9, 3.02)$  by the linear approximation of  $f$  at  $(2, 1, 3)$ .

**Solution:**

- (a) Note that  $\mathbf{r}_1(1) = (2, 1, 3) = \mathbf{r}_2(0)$ . Thus

$$(2, 2, 2) = \mathbf{r}'_1(1) \perp \nabla g(2, 1, 3) \perp \mathbf{r}'_2(0) = (2, 0, 1),$$

(到這裡兩個切向量都計算正確可得1分) and hence  $\nabla g(2, 1, 3)$  is parallel to  $(2, 2, 2) \times (2, 0, 1) = (2, 2, -4)$ . (指出 $\nabla g(2, 1, 3)$ 平行於兩切向量外積可得2分; 外積計算正確得1分) By (i), we see that  $\nabla g(2, 1, 3) = (-2, -2, 4)$ . (用到條件(i)來決定 $\nabla g(2, 1, 3)$ 的方向可得1分)

- (b) We need to find  $\nabla f(2, 1, 3)$  for the linear approximation. By (a) and by the Lagrange multiplier method we see that  $\nabla f(2, 1, 3) = \lambda \nabla g(2, 1, 3)$  for some  $\lambda \in \mathbf{R}$ . (說到要使用 Lagrange 得1分) By (b) we see that  $\lambda = \frac{-1}{2}$  and  $\nabla f(2, 1, 3) = (1, 1, -2)$ . (決定 $\nabla f(2, 1, 3) = (1, 1, -2)$ 的理由正確可得2分) Therefore

$$f(2.01, 0.9, 3.02) \approx f(2, 1, 3) + \nabla f(2, 1, 3) \cdot (0.01, -0.1, 0.02) = 4.87.$$

(線性逼近的形式正確得1分, 計算正確得1分)

4. (10 pts) Let  $f(x, y) = \frac{xy(x+y)}{e^{x+y}}$  be defined on the first quadrant  $D : x > 0$  and  $y > 0$  (without boundary). Find all critical points of  $f$  in  $D$  and classify them (as local maximum points, local minimum points, or saddle points). Please provide details of calculation.

**Solution:**

The first derivatives of  $f$  are

$$\frac{\partial f}{\partial x} = e^{-x-y} (2xy + y^2 - x^2y - xy^2),$$

$$\frac{\partial f}{\partial y} = e^{-x-y} (2xy + x^2 - x^2y - xy^2). \quad (2 \text{ points})$$

There is only one critical point of  $f$  in  $D$ , which is

$$(x, y) = \left(\frac{3}{2}, \frac{3}{2}\right). \quad (2 \text{ points})$$

The second partial derivatives of  $f$  are

$$\frac{\partial^2 f}{\partial x^2} = e^{-x-y} (2y - 4xy - 2y^2 + x^2y + xy^2),$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{-x-y} (2x + 2y - x^2 - 4xy - y^2 + x^2y + xy^2),$$

$$\frac{\partial^2 f}{\partial y^2} = e^{-x-y} (2x - 4xy - 2x^2 + x^2y + xy^2). \quad (3 \text{ points})$$

Since

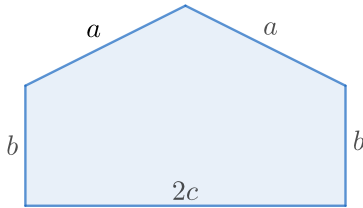
$$\det \nabla^2 f \left(\frac{3}{2}, \frac{3}{2}\right) = e^{-6} \begin{vmatrix} -\frac{15}{4} & -\frac{3}{4} \\ -\frac{3}{4} & -\frac{15}{4} \end{vmatrix} > 0$$

and

$$\frac{\partial^2 f}{\partial x^2} \left(\frac{3}{2}, \frac{3}{2}\right) = -\frac{15}{4}e^{-3} < 0 \quad (2 \text{ points}),$$

the critical point  $\left(\frac{3}{2}, \frac{3}{2}\right)$  is a local maximum point of  $f$  (1 point).

5. (12 pts) A pentagon is formed by placing an isosceles triangle on a rectangle. The side lengths are denoted by  $a$ ,  $b$ , and  $c$  as shown in the figure.
- (a) (3 pts) Write down the area of pentagon in terms of  $a$ ,  $b$ , and  $c$ .
- (b) (9 pts) Find the maximum area of pentagon if the perimeter is fixed as 2.



**Solution:**

- (a) The height of the upper triangle equals  $\sqrt{a^2 - c^2}$  (2 points). Therefore the area is given by

$$A = c \cdot \sqrt{a^2 - c^2} + 2bc \quad (2 \text{ points}).$$

- (b) We use the method of Lagrange multiplier. We are looking for the maximum value of  $A$  under the conditions  $a, b, c > 0$  and  $g(a, b, c) = 1$  where  $g(a, b, c) = a + b + c$ . When  $A$  achieves the extremum, we have

$$\begin{aligned} \frac{ac}{\sqrt{a^2 - c^2}} &= \lambda \\ 2c &= \lambda \\ \frac{a^2 - 2c^2}{\sqrt{a^2 - c^2}} + 2b &= \lambda \\ a + b + c &= 1 \end{aligned} \quad (4 \text{ points}).$$

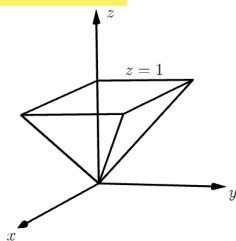
The first two equations imply  $3a^2 = 4c^2$  or equivalently  $c = \frac{\sqrt{3}}{2}a$ . Plugging into the third equation and replacing  $\lambda$  by  $2c = \sqrt{3}a$ , we have  $b = \frac{1+\sqrt{3}}{2}a$ . The last equation then reduces to  $\frac{3+2\sqrt{3}}{2}a = 1$  so we obtain

$$a = \frac{2}{3+2\sqrt{3}}, \quad b = \frac{1+\sqrt{3}}{3+2\sqrt{3}}, \quad c = \frac{\sqrt{3}}{3+2\sqrt{3}}.$$

In this circumstance,

$$A = \frac{6+3\sqrt{3}}{(3+2\sqrt{3})^2} \quad (4 \text{ points}).$$

6. (26 pts) (a) (6 pts) Find the average value of  $f(x) = \int_x^a e^{-t^2} dt$  on the interval  $[0, a]$ , where  $a > 0$  is a constant.
- (b) (10 pts) Compute  $\iiint_E e^{3y-y^3} dV$ , where  $E$  is the solid bounded by  $x = 0$ ,  $y = 0$ ,  $x = z$ ,  $y = z$ , and  $z = 1$ .



- (c) (10 pts) Compute  $\int_{-a}^a \int_0^{\sqrt{a^2-x^2}} \int_a^{a+\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{x^2+y^2+z^2}} dz dy dx$ .
- (Hint: Use Spherical coordinates.)

### Solution:

- (a) The average value of  $f(x)$  on  $[0, a]$  is  $\frac{1}{a} \int_0^a f(x) dx$ . (1 pt for the definition of average value.)

$$\begin{aligned} \frac{1}{a} \int_0^a f(x) dx &= \frac{1}{a} \int_0^a \int_x^a e^{-t^2} dt dx \\ &= \frac{1}{a} \int_0^a \int_0^t e^{-t^2} dx dt \quad (3 \text{ pts for changing the order of integration}) \\ &= \frac{1}{a} \int_0^a t e^{-t^2} dt = \frac{1}{a} \left( -\frac{1}{2} e^{-t^2} \right) \Big|_{t=0}^{t=a} = \frac{1}{2a} (1 - e^{-a^2}) \quad (2 \text{ pts for the final answer}) \end{aligned}$$

- (b) **Solution 1:**  $E = \{(x, y, z) | 0 \leq y \leq 1, y \leq z \leq 1, 0 \leq x \leq z\}$

$$\text{Hence } \iiint_E e^{3y-y^3} dV = \int_0^1 \int_y^1 \int_0^z e^{3y-y^3} dx dz dy$$

(5 pts. If students correctly project  $E$  onto the  $yz$ -plane and write down the correct range of  $x$ , they get 2 pts. 3 pts for correct iterated integrals.)

$$\begin{aligned} \int_0^1 \int_y^1 \int_0^z e^{3y-y^3} dx dz dy &= \int_0^1 \int_y^1 z e^{3y-y^3} dz \quad (1 \text{ pt}) \\ &= \int_0^1 \frac{1}{2} (1-y^2) e^{3y-y^3} dy \quad (1 \text{ pt}) \\ &\stackrel{\substack{\text{let } u=3y-y^3 \\ du=(3-3y^2)dy}}{\quad} \int_0^2 \frac{1}{6} e^u du \\ &\quad (1 \text{ pt for substitution. 1 pt for correct upper and lower limit.}) \\ &= \frac{1}{6} (e^2 - 1) \quad (1 \text{ pt for final answer.}) \end{aligned}$$

**Solution 2:**  $E = \{(x, y, z) | 0 \leq y \leq 1, y \leq x \leq 1, x \leq z \leq 1\} \cup \{(x, y, z) | 0 \leq y \leq 1, 0 \leq x \leq y, y \leq z \leq 1\}$

$$\iiint_E e^{3y-y^3} dV = \int_0^1 \int_t^1 \int_x^1 e^{3y-y^3} dz dx dy + \int_0^1 \int_0^y \int_y^1 e^{3y-y^3} dz dx dy$$

(5 pts. If students correctly project  $E$  onto the  $xy$ -plane and write down the correct range of  $x$ , they get 2 pts. 3pts for correct iterated integrals.)

The integral is

$$\begin{aligned} & \int_0^1 \int_y^1 (1-x)e^{3y-y^3} dx dy + \int_0^1 \int_0^y (1-y)e^{3y-y^3} dx dy \quad (1 \text{ pt}) \\ &= \int_0^1 (1-y) - \frac{1}{2}(1-y^2)e^{3y-y^3} dy + \int_0^1 y(1-y)e^{3y-y^3} dy \quad (1 \text{ pt}) \\ &= \int_0^1 \frac{1}{2}(1-y^2)e^{3y-y^3} dy = \frac{1}{6}(e^2 - 1) \quad (3 \text{ pts}) \end{aligned}$$

(c) **Solution 1:** The integral is  $\iiint_E \frac{1}{\sqrt{x^2 + y^2 + z^2}} dV$ , where  $E = \{(\rho, \theta, \varphi) | 0 \leq \theta \leq \pi, 0 \leq \varphi \leq \frac{\pi}{4}, \frac{a}{\cos \varphi} \leq \rho \leq 2a \cos \varphi\}$ .

$$\begin{pmatrix} 1 \text{ pt for the range of } \theta \\ 2 \text{ pts for the range of } \varphi \\ 2 \text{ pts for the range of } \rho \end{pmatrix}$$

$$\begin{aligned} \iiint_E \frac{1}{\sqrt{x^2 + y^2 + z^2}} &= \int_0^\pi \int_0^{\frac{\pi}{4}} \int_{\frac{a}{\cos \varphi}}^{2a \cos \varphi} \frac{1}{\rho} \rho^2 \sin \varphi d\rho d\varphi d\theta \quad (1 \text{ pt for Jacobian}) \\ &= \pi \int_0^{\frac{\pi}{4}} \frac{1}{2} \left( 4a^2 \cos^2 \varphi - \frac{a^2}{\cos^2 \varphi} \right) \sin \varphi d\varphi \quad (1 \text{ pt}) \\ &= \frac{\pi}{2} a^2 \int_0^{\frac{\pi}{4}} \left( 4 \cos^2 \varphi - \frac{1}{\cos^2 \varphi} \right) \sin \varphi d\varphi \\ &\stackrel{u=\cos \varphi}{du=-\sin \varphi d\varphi} \frac{\pi}{2} a^2 \int_1^{\frac{1}{\sqrt{2}}} \left( 4u^2 - \frac{1}{u^2} \right) (-du) \quad (2 \text{ pts for substitution}) \\ &= \frac{\pi}{2} a^2 \left[ \frac{4}{3} u^3 + \frac{1}{u} \right] \Big|_{u=\frac{1}{\sqrt{2}}}^{u=1} \\ &= \frac{\pi}{2} a^2 \left[ \frac{7}{3} - \frac{4}{3} \sqrt{2} \right] \quad (1 \text{ pt for final answer.}) \end{aligned}$$

**Solution 2:** Use cylindrical coordinates. The integral is  $\iiint_E \frac{1}{\sqrt{x^2 + y^2 + z^2}} dV$ , where  $E = \{(r, \theta, z) | 0 \leq \theta \leq \pi, 0 \leq r \leq a, a \leq z \leq a + \sqrt{a^2 - r^2}\}$ .

$$\begin{pmatrix} 1 \text{ pt for the range of } \theta \\ 1 \text{ pt for the range of } r \\ 1 \text{ pt for the range of } z \end{pmatrix}$$

$$\iiint_E \frac{1}{\sqrt{x^2 + y^2 + z^2}} = \int_0^\pi \int_0^a \int_a^{a+\sqrt{a^2-r^2}} \frac{1}{\sqrt{r^2 + z^2}} r dz dr d\theta \quad (1 \text{ pt for Jacobian})$$

Note that  $\int \frac{1}{\sqrt{a^2 + t^2}} dt = \ln(t + \sqrt{a^2 + t^2}) + c. \quad (2 \text{ pts})$

7. (10 pts) Let  $D$  be an  $xy$ -plane region bounded by one loop of  $r^2 = \cos 2\theta$ . Find the area of the part of the upper half sphere  $z = \sqrt{1 - x^2 - y^2}$  that is above  $D$ .

**Solution:**

The first derivatives of  $f(x, y) = \sqrt{1 - x^2 - y^2}$  are

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{1 - x^2 - y^2}},$$

$$\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{1 - x^2 - y^2}}. \quad (2 \text{ points})$$

The area of the graph of  $f$  above  $D$  is given by

$$\iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \, dx \, dy = \iint_D \frac{1}{\sqrt{1 - x^2 - y^2}} \, dx \, dy \quad (2 \text{ points})$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{\cos(2\theta)}} \frac{r}{\sqrt{1 - r^2}} \, dr \, d\theta \quad (3 \text{ points})$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(1 - \sqrt{1 - \cos(2\theta)}\right) \, d\theta$$

$$= 2 \int_0^{\frac{\pi}{4}} \left(1 - \sqrt{2} \sin \theta\right) \, d\theta = 2 \left(\frac{\pi}{4} + 1 - \sqrt{2}\right) \quad (3 \text{ points}).$$

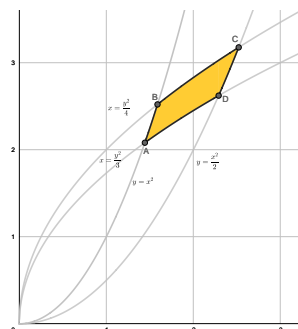


8. (10 pts) Let  $D = \{(x, y) | x > 0, y > 0, y \leq x^2 \leq 2y, 3x \leq y^2 \leq 4x\}$ . Evaluate  $\iint_D xy \, dA$ .

**Solution:**

Method 1:

$$A(3^{1/3}, 3^{2/3}) \quad B(4^{1/3}, 4^{2/3}) \quad C(2^{4/3}, 2^{5/3}) \quad D(2^{2/3}3^{1/3}, 2^{1/3}3^{2/3})$$



$$I = \int_{3^{1/3}}^{4^{1/3}} \int_{\sqrt{3x}}^{x^2} xy \, dy \, dx + \int_{4^{1/3}}^{2^{4/3}3^{1/3}} \int_{\sqrt{3x}}^{\sqrt{4x}} xy \, dy \, dx + \int_{2^{2/3}3^{1/3}}^{2^{4/3}} \int_{x^2/2}^{\sqrt{4x}} xy \, dy \, dx = \frac{7}{4}$$

6 個積分上下限，每個上、下全對給 1 分，獨立給分，共 6 分。不看積分的計算，核驗最後答案 ( $\frac{7}{4}$ )，4 分

Method 2:

Make a change of variables  $u = \frac{x^2}{y}$ ,  $v = \frac{y^2}{x}$ , which transforms  $D$  to  $E = \{(u, v) | 1 \leq u \leq 2, 3 \leq v \leq 4\}$ .

Observe that  $uv = xy$ ,  $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = 4 - 1 = 3$ ,  $\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{3}$  (4 分，若沒取倒數扣 1 分)

$$I = \iint_E \underbrace{uv}_{\text{變換後的函數 2 分}} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = \frac{1}{3} \underbrace{\int_1^2 u \, du \int_3^4 v \, dv}_{\text{2 組上、下限各 1 分，共 2 分}} = \frac{1}{3} \left( \frac{u^2}{2} \right)_1^2 \left( \frac{v^2}{2} \right)_3^4 = \frac{7}{4}$$

任何有效或無效的變換，皆如下獨立個別給分：2 組上、下限各 1 分，共 2 分；Jacobian 4 分；變換後的被積分函數 2 分；答案 2 分。(計算過程不必看)