Chapter 12

Electromagnetism

Einstein's favorite subject in college.

The notion of *field* is perhaps the most important concept behind electromagnetism. Furthermore, quantum fields are the basic building blocks of the universe in the most up-to-date theories of fundamental physics. We can learn about the notion of fields through studying E&M.

Q 12.1: Is there anything not related to electromagnetism?

Plan: We start with static electricity, followed by static magnetism, and then introduce Maxwell's theory of classical electromagnetism.

Q 12.2: How do we describe a field?

12.1 Electrostatics

Electrostatics is the study of the interactions among static electric charges. (Imagine that the charges are held fixed by other much stronger forces.)

By "static", we mean a sufficiently slow change of state, e.g. all charges are moving much slower than light. Extra-fast oscillations are averaged out.

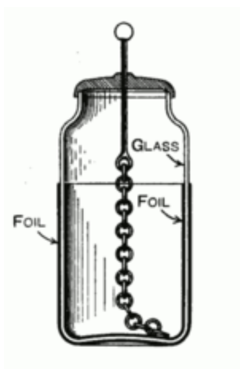
Concepts:

1. Charges Q
2. Electric field $\mathbf{E}(\mathbf{r})$
3. Electric Potential $V(\mathbf{r})$
4. Coulomb's law: $\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$.
5. Gauss' law: $\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\epsilon_0}$.

12.1.1 Charges

Static electricity is created by rubbing different materials against each other.
Q 12.3: Why can we generate charges by rubbing?

Q 12.4: Amber rubbed by fur carries positive charges. Can amber carry negative charges?
Q 12.5: What decides the results of rubbing?



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Q 12.6: How many types of charges are there?
Q 12.7: How do we know that there are two types of charges?
Q 12.8: Can electric charges respond to other force fields, e.g. gravity or nuclear interactions?
Q 12.9: Is it possible for us to find a 3rd type of charges in the future?
Ex 12.1: How many types of charges are there if we find the following chart?

	A	B	C
A	a	a	r
B	a	r	r
C	r	r	a

Q 12.10: Why is the chart symmetrical along the diagonal?

Solution:
There are *at least* 3 types of charges.

Q 12.12: How in general will you determine how many types of charges for a given chart like the one above?

Q 12.11: Is it possible that there are different types of charges indistinguishable from this kind of chart?

HW: (1-1) At least how many types of charges are there if we find the following chart?

	A	B	C	D	E	F
A	a	a	r	r	a	a
B	a	r	a	a	a	r
C	r	a	a	r	r	a
D	r	a	r	r	r	a
E	a	a	r	r	a	a
F	a	r	a	a	a	r

Q 12.13: What happens if the interaction depends on 3 bodies at the same time? Can you imagine an interaction like that?

We are lucky that the universe obeys simple physics. (Or that harder physics is hidden at smaller scales.)

Properties of electric charges:

1. A charge can be represented by a real number (actually an integer is enough). (No need of complex numbers!)

2. They are **additive** and **locally** conserved: the total charge

$$Q_{tot} \equiv \sum_i Q_i \tag{12.1}$$

(by summation, i.e. superposition) in a finite region changes only if a charge crosses its boundary.

Q 12.14: Is it possible to confirm a conservation law if it is only globally conserved?

3. Electric charge is a property of particles that cannot be disassociated, i.e. there is no such thing as an electron with its charge stripped off.

Q 12.15: Is it possible for some sort of "charges" to be conserved but not additive?

4. Charges are only associated with massive particles.

5. So far charges are found to be *quantized*, i.e. always appearing in integral multiples of (1/3) of the charge of an electron. ($q_e \simeq 1.6 \times 10^{-19}$ Coulomb.)

Q 12.16: Why?

The law of **charge conservation** says that it is impossible to have a charge to be created or annihilated by itself. It always has to be created or annihilated with other charges such that the total charge is conserved.

Q 12.17: Is it possible that a particle of charge 0 decays into two different particles of charge 0 and +1?

Q 12.18: Is it possible that two particles of charges 0 and +1 interchanges their charges without touching each other?

This would be possible if the charge is only required to be "globally" conserved. But it would not make sense in Special Relativity because simultaneity is ill-defined.

Conductors and insulators

Depending on whether charges are allowed or not allowed to move in the bulk, materials are classified as **insulators** and **conductors**.

There is of course a spectrum of all possible resistances. But in electrostatics the question is whether charges move around beyond a certain *length scale* within a certain *time scale* of observation for a given electric field.

Q 12.19: What is the difference between an insulator and a conductor with a huge resistance?

Example: The electrons and nuclei can still move within a molecule for an insulator. It is possible for electrons to move away from or to move onto an insulator.

(Electric) Induction

Q 12.20: Does a charged insulator attract or repel neutral conductors?

Q 12.21: Does a charged insulator attract or repel neutral insulators? (Does a charged plastic ruler attract tiny paper shreds?)

Q 12.22: Does a charged insulator attract or repel charged conductors?

Q 12.23: Two nearby conductors have opposite charges attracting each other. What happens when you connect the two conductors by a conducting wire? What happens when the conducting wire is extremely long (e.g. wound around the globe)?

semi-conductors, superconductors, etc, are beyond the scope of our discussion.

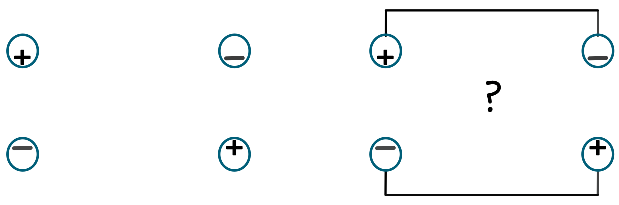

Figure 12.1: (a) (b)

Q 12.24: Paradox: If you have two pairs of nearby conductors with opposite charges attracting each other, and the two pairs of conductors are separated by a great distance. (See fig.12.1(a).) Connect oppositely charged conductors across the two pairs with extremely long conducting wires. (See fig.12.1(b).) What will happen? Will the charges on the connected conductors annihilate?
Q 12.25: You have an insulator with charge Q and a conductor without charge. How do you charge the conductor without changing the charge on the insulator?

Ground is effectively an infinitely large conductor whose charge density remains 0 regardless of its total charge.

Electrostatic force:

1. The electrostatic force between two bodies is uniquely fixed by their charges and relative displacement (according to Coulomb's law). It is an amazing fact that it is independent of the presence of everything else, such as the composition of the material (whether the charge is carried by electrons or nuclei, whether the charges are held fixed or free to move, whether there are other objects around ...).
2. Like all forces in Newton's mechanics, electrostatic forces obey the superposition law of forces, i.e. they add up as vectors.

Q 12.26: Does the force between q_1 and q_2 depend on whether there is a 3rd charge q_3 near by?

12.1.2 Coulomb's Law

The electrostatic force on a charge by another charge is given by Coulomb's law.

The electrostatic force on a charge by many other charges is then determined by superposition.

Coulomb's Law:

$$\boxed{\mathbf{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{\mathscr{r}_{12}^2}\hat{\boldsymbol{\mathscr{r}}}_{12}, \qquad (12.2)}$$

Q 12.27: Is $\mathbf{F}_{12}$ the force on q_1 due to q_2 or vise versa?

where

$$\boldsymbol{\mathscr{r}}_{12} \equiv \mathbf{r}_1 - \mathbf{r}_2. \qquad (12.3)$$

Coulomb's constant

$$k_e \equiv \frac{1}{4\pi\epsilon_0} \simeq 9 \times 10^9\, Nm^2/C^2, \qquad (12.4)$$

where

$$\epsilon_0 \simeq 9 \times 10^{-12} \qquad (12.5)$$

in SI units.

Coulomb's law is unlikely to apply to the case when charges are moving fast if physical effect (causality) has a finite speed. (Why?)

Q 12.28: Is Newton's 3rd law satisfied?

Properties of Electrostatic force:

1. The force is inversely proportional to r^2.
2. The force is attractive or repulsive for opposite or like charges.
3. The magnitude can only be determined by experiments. The unit of charge is arbitrary. The Coulomb's constant changes when a different unit of charge is used.

Q 12.29: What is the Coulomb's constant if the charge of an electron is defined as one unit of charge?

Superposition Principle:
The force on a charge q at $\mathbf{r}$ is given by

$$\mathbf{F} = q\frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i, \tag{12.6}$$

where $\mathscr{r}_i \equiv \mathbf{r} - \mathbf{r}_i$. This equation holds regardless of the states of the charges.

Electrostatics = Coulomb's Law + Superposition Principle.

Q 12.30: How is electrostatics different from gravity?
Ex 12.2: What is the total electrostatic force on a system of charges q_i at positions $\mathbf{r}_i$ $(i = 1, 2, \cdots, N)$ by another system of charges $q'_{j'}$ at positions $\mathbf{r}'_{j'}$ $(j' = 1, 2, \cdots, N')$?

When there are too many charges too close together, in practice we often describe them approximately by a charge density $\rho(\mathbf{r})$:

$$\mathbf{r}_i \to \mathbf{r}', \tag{12.7}$$
$$\mathscr{r}_i \to \mathscr{r} \equiv \mathbf{r} - \mathbf{r}', \tag{12.8}$$
$$q_i \to \rho(\mathbf{r}')d^3\mathbf{r}', \tag{12.9}$$
$$\sum_i \to \int. \tag{12.10}$$

In this continuum limit, eq.(12.6) is turned into

$$\mathbf{F} = q\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} d^3\mathbf{r}'\,\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}, \tag{12.11}$$

where $\mathscr{r} \equiv \mathbf{r} - \mathbf{r}'$ and $\mathcal{V}$ is the region where $\rho \neq 0$.

When the region $\mathcal{V}$ of the charge distribution ρ is well separated from the charge q, the force (12.11) is well approximated by the Coulomb's law (12.2), with q_1 and q_2 being q and $\int_{\mathcal{V}} \rho d^3\mathbf{r}$.
Ex 12.3: What is the electrostatic force on a point charge q due to a uniform charge distribution of total charge Q over a spherical surface of radius R, whose center is at a distance r from q?

Ex 12.4: What is the total electrostatic force on a uniform charge distribution of total charge Q over a spherical surface of radius R due to a point charge q, which is at a distance r from the center of the sphere?

Ex 12.5: What is the total electrostatic force on a charge distribution $\rho(\mathbf{r})$ by another charge distribution $\rho'(\mathbf{r})$?

Ex 12.6: Prove that the total force by a system of charges with an arbitrary charge distribution on itself is always 0.

Q 12.31: Does the answer depend on whether the charges are distributed on conductors or insulators?

12.1.3 Electric Field

Q 12.32: Electrostatics is already "completely determined" by Coulomb's law and the superposition principle, why do we care about the electric field?

Notice that $\mathbf{F}$ above is linearly proportional to q, so we are allowed to define the analogue of the gravitational acceleration $\mathbf{g}$ in gravity: the electric field

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i, \tag{12.12}$$

where $\boldsymbol{\mathscr{r}}_i \equiv \mathbf{r} - \mathbf{r}_i$, so that the force on any charge q at the same location is given by $\mathbf{F} = q\mathbf{E}(\mathbf{r})$.

Notice that here $\mathbf{E}(\mathbf{r})$ is the electric field due to all the charges except the charge q.

The SI unit for electric field is V/m = Newton/Coulomb.
The dielectric breakdown of air happens around $3\times10^6 V/m$.
The Schwinger limit (related to the Schwinger effect): $10^{18}V/m$, beyond which nonlinear effect becomes important.

Clearly, you can think of the electric field of a single charge q' at $\mathbf{r}'$ as

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q'}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}},} \tag{12.13}$$

where $\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'$, and eq.(12.12) is the superposition of the electric fields generated by all the charges.

Using eqs.(12.7)–(12.10), in the continuum limit, eq.(12.12) becomes

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\mathbf{r}'\,\frac{\rho(\mathbf{r})}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}},} \tag{12.14}$$

where

$$\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'. \tag{12.15}$$

Q 12.33: Is the electrostatic force on a charge q at $\mathbf{r}$ equal to q times the electric field $\mathbf{E}$ at $\mathbf{r}$?

Point charges are imaginations.

Not really, because the field due to charge q needs to be excluded. Its electric field is ill-defined at its position.

Ex 12.7: What is the electrostatic force on a charge distribution $\rho(\mathbf{r})$ by an electric field $\mathbf{E}(\mathbf{r})$?

Q 12.34: In the exercise above, should we include or exclude the charge distribution $\rho(\mathbf{r})$ when we compute the electric field $\mathbf{E}(\mathbf{r})$?

Q 12.36: While the electric field $\mathbf{E}$ defined to compute the electrostatic force on a point charge q excludes the contribution of q, is the electric field $\mathbf{E}$ a real thing, or is it merely a computational tool?

Q 12.35: Do we measure $\mathbf{E}$ by observing charges, or do we observe charges through their field $\mathbf{E}$?

Ex 12.8: What is the electric field $\mathbf{E}(\mathbf{r})$ due to a uniform surface charge distribution of total charge Q over a spherical surface of radius R?

Ex 12.9: What is the pressure on the uniform surface charge distribution of total charge Q over a spherical surface of radius R?

12.1.4 Visualization of Electric Field (Field Lines)

To visualize the electric field, one may imagine electric *field lines*.

The electric field will be viewed as a real physical entity rather than merely a computational tool.

For a single point charge q at the origin, the electric field is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \tag{12.16}$$

The $1/r^2$ law is reminiscent of the inverse square law of luminosity. This allows us to imagine field lines starting from the charge in the radial direction. The density of the field lines decreases according to the inverse square law.

We identify the direction of the *field lines* with $\hat{\mathbf{E}}$, and the density of them with $|\mathbf{E}|$.

The "field lines" are also called "lines of force".

> In case you wonder how to make sure that the number of field lines is always an integer according to this identification, note that the value of $|\mathbf{E}|$ depends on the choice of the unit system. This question has no physical significance.

Q 12.37: Is this a consistent definition applicable to generic charge distributions?

Q 12.38: In general, how to choose the cross-sectional area to define the density of field lines?

Field lines in electrostatics:

We can also imagine magnetic *field lines*. But gravitational field lines are different. (Why?)

1. Field lines are directional.
2. Field lines are originated from positive charges and ending on negative charges.
3. In a suitable unit system (in which $\epsilon_0 = 1$), the number of field lines emerging/ending on a charge equals that charge.
4. Field lines cannot intersect with each other.
5. Field lines repel each other.

Ex 12.10: Draw the field lines for (i) a positive point charge, (ii) a negative point charge, (iii) a pair of equal positive point charges, (iv) a positive point charge q and a negative point charge $-q$.

Since E_r is the density of field lines in the radial direction, the number of field lines coming out of a sphere of radius R is

$$\Phi_E(S^2(R)) = \oint_{S^2(R)} da\ E_r(R) = 4\pi R^2 \frac{q}{4\pi\epsilon_0 R^2} = \frac{q}{\epsilon_0}. \tag{12.17}$$

Here is the reason for the factor of 4π in the Coulomb constant $k_e = \frac{1}{4\pi\epsilon_0}$.

Q 12.39: What is Φ_E for a cube of side L with the charge q at the center?
Q 12.40: What is Φ_E for an arbitrary closed surface enclosing a point charge q?
Q 12.41: What is Φ_E for a closed surface if the point charge resides outside the surface?
Ex 12.11: Argue that, for the electric field $\mathbf{E}$ due to a single point charge q and an arbitrary closed surface S,

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{q}{\epsilon_0} \tag{12.18}$$

of 0 depending on whether the charge is enclosed within S.

Given the fact that $\oint \mathbf{E} \cdot d\mathbf{a} = q/\epsilon_0$ for a point charge q, the case of an arbitrary charge distribution can be deduced via the superposition principle.
Ex 12.12: Prove that for any charge distribution in space and an arbitrary closed surface S, the electric field $\mathbf{E}$ always satisfies

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\epsilon_0}, \tag{12.19}$$

where Q_{enc} is the charge enclosed within S.
Solution:
The superposition principle and the linear property of integrals imply that, for an arbitrary closed surface S, the electric flux is

$$\oint_S d\mathbf{a} \cdot \mathbf{E} = \oint_S d\mathbf{a} \cdot \left(\sum_i \mathbf{E}_i\right) = \sum_i \oint_S d\mathbf{a} \cdot \mathbf{E}_i = \sum_i \frac{q_i}{\epsilon_0} = \frac{Q_{enc}}{\epsilon_0}, \tag{12.20}$$

where Q_{enc} is the total charge enclosed within S.

This general result is called Gauss' law. Gauss' law is the coolest thing in electrostatics. It represents an important idea in physics that goes well beyond electromagnetism. We will focus on Gauss's law in the next section.