Chapter 9

Special Relativity

9.1 Introduction

Until the end of 19th century, it is generally accepted the Newtonian idea of the complete separability of space and time, and the concept of the absoluteness of time is the fundamental basis for understanding the motion of matter. However, the formulation of the unified theory of electromagnetism by Maxwell, which is extraordinarily successful, challenges the foundation of Newtonian mechanics. The new Maxwell's equations of electromagnetism are not invariant under Galilean transformation. In a series of crucial experiments, culminating with the work of Michelson and Morley in 1881-1887, showed that the speed of light is independent of any relative uniform motion between source and observer.

In 1905, A. Einstein formulated the special theory of relativity based on two postulates:

1. The laws of physical phenomena are the same in all inertial reference frames (that is, only the relative motion of inertial frames can be measured; the concept of motion relative to "absolute rest" is meaningless).

2. The velocity of light in vacuum is a universal constant, independent of any relative motion of the source and the observer.

The first postulate is called the principle of relativity. The second postulate is the result of the first one and comes from the fact that Maxwell's equations are fundmental laws of physics.

9.2 Galilean Invariance

In Newtonian mechanics, the concepts of space and time are completely separable, and time is assumed to be an absolute quantity independent of the reference frame. Consider two inertial reference frames K and K', which move along their x_1- and x'_1-axes with a uniform relative velocity v. The transformation of the coordinates of a point from one system to the other is

$$\begin{aligned} x'_1 &= x_1 - vt \\ x'_2 &= x_2 \\ x'_3 &= x_3 \end{aligned} \tag{9.1}$$

and

$$t' = t \tag{9.2}$$

These equations define a **Galilean transformation**. The element of length in the two systems is the same:

$$\begin{aligned} ds^2 &= \sum_j dx_j^2 \\ &= \sum_j dx_j'^2 = ds'^2 \end{aligned} \tag{9.3}$$

Newton's laws are *invariant* under Galilean transformations. This is called **the principle of Newtonian relativity** or **Galilean invariance**. The terms in the law of motion are *covariant* under Galilean transformation as they transform according to the same scheme.

According to the Galilean transformation, the speed of light should be different in different frames,

$$\dot{x}'_1 = \dot{x}_1 - v \tag{9.4}$$

In system K' the velocity is measured as $\dot{x}'_1 = c$, and the speed of light in K will be $\dot{x}_1 = c + v$. This effect is not measured experimentally.

9.3 Lorentz Transformation

In 1904, H. A. Lorentz noticed the Maxwell equations are invariant under the transformation:

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}, \end{aligned} \tag{9.5}$$

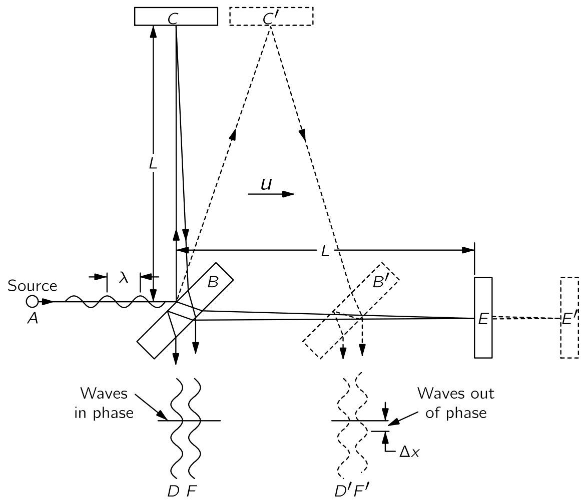

Figure 9.1: Michelso-Morley Interferometer.

namely, Maxwell's equations remain in the same form when this transformation is applied to them! Einstein, following a suggestion originally made by Poincaré, then proposed that all the physical laws should be of such a kind that they remain unchanged under a Lorentz transformation. In other words, we should change, not the laws of electrodynamics, but the laws of mechanics.

9.3.1 Michelson-Morley Experiment

The Michelson-Morley experiment was performed with an apparatus like that shown schematically in Fig. 9.1. This apparatus is essentially comprised of a light source A, a partially silvered glass plate B, and two mirrors C and E, all mounted on a rigid base. The mirrors are placed at equal distances L from B. The plate B splits an oncoming beam of light, and the two resulting beams continue in mutually perpendicular directions to the mirrors, where they are reflected back to B. On arriving back at B, the two beams are recombined as two superposed beams, D and F. If the time taken for the light to go from B to E and back is the same as the time from B to C and back, the emerging beams D and F will be in phase and will reinforce each other, but if the two times differ slightly, the beams will be slightly out of phase and interference will result. If the apparatus is "at rest" in the ether, the times should be precisely equal, but if it is moving toward the right with a velocity u, there should be a difference in the times. If the time for light to go from plate B to mirror E is t_1, and the time for the return is t_2. Now, while the light is on its way from B to the mirror, the apparatus moves a distance ut_1, so the light must traverse a distance $L + ut_1$, at the speed c. We can also express this distance as $ct1$, so we have

$$ct_1 = L + ut_1, \quad \text{or} \quad t_1 = L/(c-u).$$

During t_2 the plate B advances a distance ut_2, so the return distance of the light is $L - ut_2$, and

$$ct_2 = L - ut_2, \quad \text{or} \quad t_2 = L/(c+u).$$

The total time is

$$t_1 + t_2 = \frac{2L/c}{1 - u^2/c^2}.$$

Denote the time for the light to go from B to the mirror C as t_3. As before, during time t_3 the mirror C moves to the right a distance ut_3 to the position C'; in the same time, the light travels a distance ct_3 along the hypotenuse of a triangle, which is BC', and we have

$$(ct_3)^2 = L^2 + (ut_3)^2,$$

and

$$t_3 = L/\sqrt{c^2 - u^2}.$$

The distance of the return trip is the same so the total time is

$$2t_3 = \frac{2L}{\sqrt{c^2 - u^2}} = \frac{2L/c}{\sqrt{1 - u^2/c^2}}.$$

Apparently, $2t_3 < t_1 + t_2$ for non-zero u. However, Michelson and Morley found no time difference– the velocity of the earth through the ether could not be detected. The result of the experiment was null.

Lorentz suggested that material bodies contract when they are moving, and that this foreshortening is only in the direction of the motion, and also, that if the length is L_0 when a body is at rest, then when it moves with speed u parallel to its length, the new length, which we call $L_{\parallel}$, is given by

$$L_{\parallel} = L_0\sqrt{1 - u^2/c^2}.$$

With this modification

$$t_1 + t_2 = \frac{(2L/c)\sqrt{1 - u^2/c^2}}{1 - u^2/c^2} = \frac{2L/c}{\sqrt{1 - u^2/c^2}} = 2t_3.$$

But why?

9.3.2 Derivation of Lorentz Transformation

Consider two inertial frames K and K', the reference frame K' moves relative to K with velocity v along the x axis. The coordinates y and z perpendicular to the velocity are the same in both frames: $y = y'$ and $z = z'$. We look for a linear transformation between (x, t) and (x', t'),

$$\begin{aligned} x' &= ax + bt \\ t' &= dx + et \end{aligned} \tag{9.6}$$

where a, b, d, e are coefficients depending on v. In matrix form:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{9.7}$$

The origin of the reference frame K' has the coordinate $x' = 0$ and moves with velocity v relative to the reference frame K, so that $x = vt$. Substituting these values into Eq. (9.6), we find $b = -va$. Thus, we have

$$x' = a(x - vt) \tag{9.8}$$

The origin of the reference frame K has the coordinate $x = 0$ and moves with velocity $-v$ relative to the reference frame K', so that $x' = -vt'$. Thus, we have $e = a$ and

$$t' = dx + at = a(fx + t) \tag{9.9}$$

where $f = d/a$. We will change into a more common notation $A = \gamma$. The equations now becomes

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma(fx + t) \end{aligned} \tag{9.10}$$

From Postulate II, we know that the interval is invariant

$$\sum_j x_j^2 - c^2 t^2 = \sum_j x_j'^2 - c^2 t'^2 \tag{9.11}$$

Substituting Eq. (9.10) into the above equation and solve for E and γ,

$$x^2 - c^2 t^2 = \gamma^2 (1 - c^2 f^2) x^2 - \gamma^2 (2v + 2fc^2) xt - \gamma^2 (c^2 - v^2) t^2 \tag{9.12}$$

Finally we have

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1 - v^2/c^2}} \\ f &= -\frac{v}{c^2}. \end{aligned} \tag{9.13}$$

Substituting back to Eq. (9.10), we have

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned} \tag{9.14}$$

These equations give the coordinates as seen by K' in terms of those of K. If we want the coordinates as seen by K in terms of those of K', then we let $\beta \to -\beta$ and we have

$$t = \gamma\,(t' + \beta x') \tag{9.15}$$
$$x = \gamma\,(x' + \beta t') \tag{9.16}$$
$$y = y' \tag{9.17}$$
$$z = z'. \tag{9.18}$$

Note that $0 \leq \beta \leq 1$ so that $1 \leq \gamma < \infty$. We also see that

$$\gamma^2 = \frac{1}{1 - \beta^2},$$

so

$$\gamma^2 - \gamma^2\beta^2 = 1.$$

Define a parameter θ called the *rapidity* such that

$$\begin{aligned} \beta &= \tanh\theta, \\ \gamma &= \cosh\theta, \\ \gamma\beta &= \sinh\theta. \end{aligned}$$

And the Lorentz transform can be written as

$$\begin{aligned} t' &= (\cosh\theta)t - (\sinh\theta)x \\ x' &= -(\sinh\theta)t + (\cosh\theta)x \\ y' &= y \\ z' &= z \end{aligned} \tag{9.19}$$

9.3.3 Spacetime Diagram

We can represent any physical phenomenon taking place at a space position and at a certain time, an event, as a point in a four dimensional spacetime, with coordinates

(x, y, z, ct). All the clocks at rest in a given inertial frame are synchronized. Minkowski spacetime diagram (Fig. 9.2) is a graphical representation of events and sequences of events in spacetime as "seen" by the observer at rest(in K). Such sequences are named *wordlines*. A light ray emitted at the origin along the x-axis towards positive values of x in space is represented by a straight line $ct = x$. The light worldline bisects the quadrant formed by ct-axis and x-axis; $ct = -x$represents a light ray emitted at the origin traveling along the negative part of the x-axis. A particle at rest is represented by a vertical line. An object traveling with moderate constant speed as compared to c is represented by a straight line with slope $c/v = 1/\beta$. The ct-axis itself represents anything standing still at point O: $(x, ct) = (0, ct)$; the x-axis represents simultaneous events at $(x, ct) = (x, 0)$. The light worldline is called *lightline*. The locus of all lightlines starting at the origin (the present) constitute a *lightcone*. The upper light cone is all the *future* events can be reached from the present, and the lower light cone is all the *past* events that can reach the present.

The spacetime interval between two events is invariant under Lorentz transformation,

$$\begin{aligned}(\Delta s)^2 &= -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\ &= -c^2(\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 \\ &= (\Delta s')^2 \qquad (9.20)\end{aligned}$$

The interval can be positive, negative or zero

- If $(\Delta s)^2 < 0$, we called the interval **timelike**, as we can find a inertial frame such that the two events occur at the same place.
- If $(\Delta s)^2 > 0$, we called the interval **spacelike**, as we can find a inertial frame such that the two events occur at the same time.
- If $(\Delta s)^2 = 0$, we called the interval **lightlike**, as the two events are connected by a signal traveling at the speed of light.

For an inertial frame K' moving relative to K at velocity v, the worldline of the origin O', which is at rest in K', is represented by the ct'-axis. The x'-axis represents simultaneous events at $(x', ct') = (x', 0)$ (Fig. 9.3). As we can see, the lines of simultaneity in K and K' are different. So simultaneity is not an absolute concept, on the contrary, simultaneity is relative; the judgments of simultaneity of events will vary according to the state of motion of the observer.

9.3.4 Doppler effect and time dilation

The spacetime diagram (Fig. 9.4) K', moving with constant speed v to the right, is represented as "seen" in K. The initial conditions are $(x, ct) = (0, 0) = (x', ct')$.

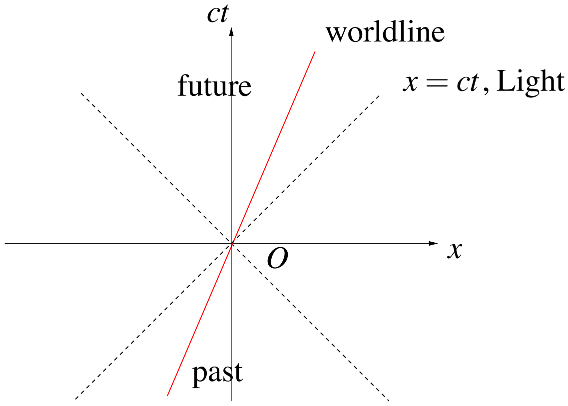

Figure 9.2: Spacetime diagram

Suppose you (in K) have a light source at the origin which emits a pulse towards K' (your friend) where exists a mirror that reflects it back. Your clock says that the pulse is emitted at event $(0, cT_1)$ and is back at event $(0, cT_2)$. From Fig. reffig:timedilation you can get that the event $(0, cT)$ where

$$cT = \frac{cT_1 + cT_2}{2} \tag{9.21}$$

since the worldlines of light corresponding to stright lines of slope ± 1, and

$$vT = c(T - T_1) \tag{9.22}$$

This event is simultaneous to the event $(0, cT')$, in K', at which the pulse is reflected. In K, this event has the spacetime coordinates (vT, cT). We define two factors: one is the ratio between T' and T

$$\frac{T}{T'} = \gamma$$

which relates the time components of the same event in two different inertial frames. The second one is the ratio between the emission time cT_1 measured in K and the reflection time cT' measured in K'

$$\frac{cT'}{cT_1} = k \tag{9.23}$$

In K', the reflection from the mirror corresponds to emission of light at cT' in K' and reception of the light pulse at cT_2 in K. From the principle of relativity that all physical laws should be the same in all frames, we should have

$$\frac{cT_2}{cT'} = k \tag{9.24}$$

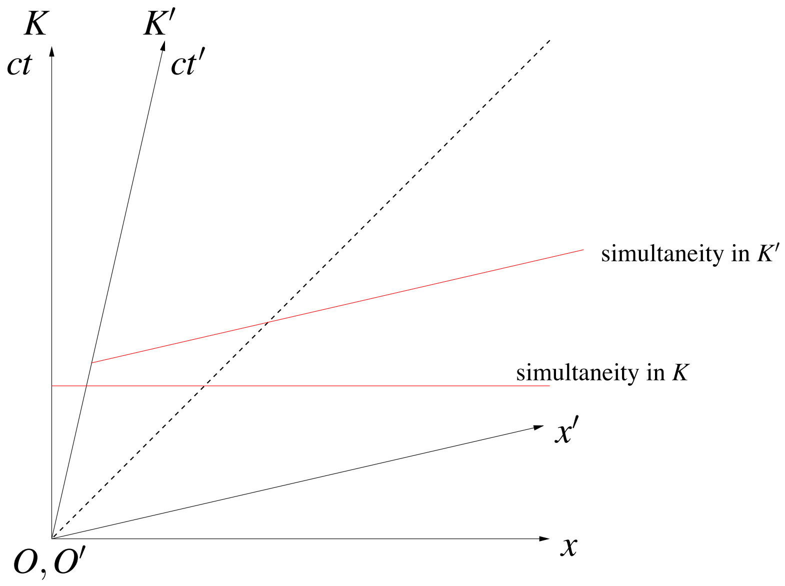

Figure 9.3: Lines of simultaneity

Finally, we have $cT_2 = k^2 cT_1$. Substituting it into Eq. (9.24) and (9.25), we obtain

$$cT = \frac{k^2+1}{2} cT_1$$

and

$$vT = v\frac{k^2+1}{2} T_1 = c\frac{k^2-1}{2} T_1$$

Finally, we have

$$\beta = \frac{v}{c} = \frac{k^2-1}{k^2+1}$$

or

$$k = \sqrt{\frac{1+\beta}{1-\beta}} \tag{9.25}$$

This is the Doppler factor. The event R has the time component T in K and T' in K', which correspond to the elapsed time from O, O' measured by clocks in K and K' respectively. And

$$\gamma = \frac{T}{T'} = \frac{\frac{k^2+1}{2} T_1}{kT_1} = \frac{1}{\sqrt{1-\beta^2}} \tag{9.26}$$

or

$$\Delta t = \gamma \Delta t' \tag{9.27}$$

To an observer in motion relative to the clock at rest in K', the time intervals appear to be lengthed. This is the origin of the phrase "moving clocks run more slowly", as the measured time interval on the moving clock is lengthened, the clock actually ticks slower. This is the time dilation effect.

If an observer in K sends out light pulses seperated by time T_0, (Fig. 9.5) and an observer in K' moving away at velocity v ($\overline{O'R}$) will receive the pulses separated by time T_r, and we have

$$T_r = kT_0 = \sqrt{\frac{1+\beta}{1-\beta}} T_0 \tag{9.28}$$

In terms of frequency,

$$\nu_r = \frac{1}{k}\nu_0 = \sqrt{\frac{1-\beta}{1+\beta}} \nu_0 \tag{9.29}$$

for the source and receiver moving away from each other. For the source and receiver approaching each other($\bar{O'A}$), similar derivation gives

$$\nu_a = \sqrt{\frac{1+\beta}{1-\beta}} \nu_0 \tag{9.30}$$

If we choose the convetion that a $+$ sign for β when the source and receiver are approaching each other, and a $-$ sign for β when they are receding, we can summarize the relativistic Doppler effect as

$$\nu = \sqrt{\frac{1+\beta}{1-\beta}} \nu_0 \tag{9.31}$$

9.3.5 Proper time

The time measured on a clock at rest is called the **proper time**,

$$d\tau = \sqrt{1-\beta^2} dt = dt/\gamma \tag{9.32}$$

where dt is the time measured in a frame moving at speed v. The proper time is always the minimum measurable time difference between two events. The proper time is also invariant under Lorentz transformation.

9.3.6 Length Contraction

Consider the length of a rod is L_0 at rest. By length we mean that we perform measurements of the end points of the rod *simultaneously*. Another way to perform the measurement is to send a light pulse from one end (called it A), and measure the time τ

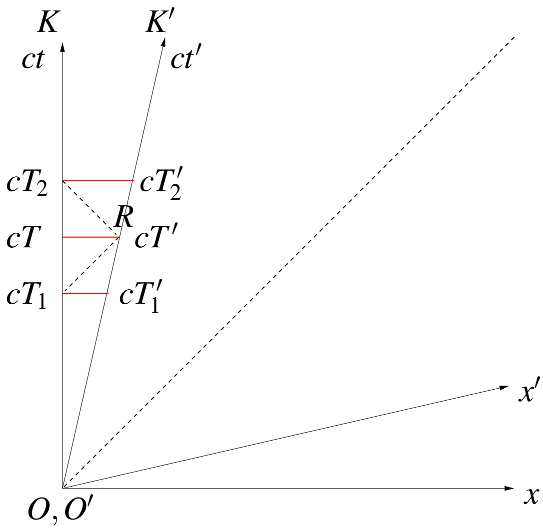

Figure 9.4: Time Dilation

it takes for the light pulse to be reflected back by a mirror at the other end (B) and back to A.

$$L_0 = \frac{1}{2} c\tau \tag{9.33}$$

If now we consider the rod is moving at velocity v. For an observer in K', the rod is not moving. If it takes $2T'_0$ in K' for the light to travel from end A to end B and reflected back, we have $\overline{AP} = \overline{PQ} = T'_0$

$$L_0 = cT'_0 \tag{9.34}$$

On the other hand, the event R has time component cT_R in K. From the geometry and previous discussions, we know that

$$T_R = kT'_0 = \sqrt{\frac{1+\beta}{1-\beta}} T'_0 \tag{9.35}$$

The event P has time component cT in K and cT'_0 in K', and they are related by

$$T = \gamma T'_0 \tag{9.36}$$

The measured length in K is related to T_R as

$$vT_R + L = cT_R$$

or

$$L = (c - v)T_R = c(1 - \beta)\sqrt{\frac{1+\beta}{1-\beta}} T'_0 = \frac{1}{\gamma} cT'_0 \tag{9.37}$$

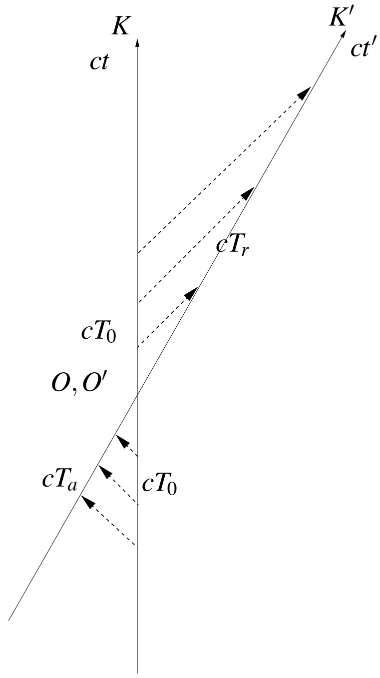

Figure 9.5: Relativistic Doppler effects

Finally we have

$$L = \frac{1}{\gamma} L_0 \tag{9.38}$$

The length of a moving rod appears to be contracted to an observer in K.

9.4 Four-vectors

If we express the time component as the

$$x^0 \equiv ct \tag{9.39}$$

The Lorentz transformation can be expressed as

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) \\ x'^1 &= \gamma(x^1 - \beta x^0) \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned} \tag{9.40}$$

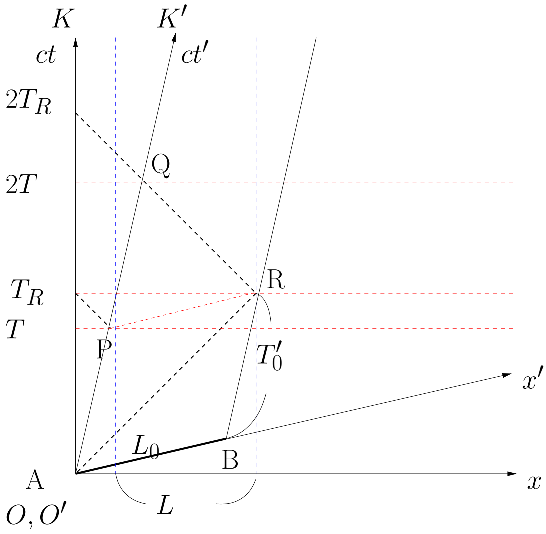

Figure 9.6: Relativistic Doppler effects

or in the matrix form

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{9.41}$$

We can write this in a simplified form

$$x'^\mu = \sum_\nu \Lambda^\mu_\nu x^\nu \tag{9.42}$$

where Λ^μ_ν is a 4×4 matrix, the Lorentz transformation matrix. This form is similar to rotation of a vector in 3-dimensional space. The matrix form allows us to handle a more general transformation in the same manner, where the relative motion are not in the x, x' direction. In this case, the matrix becomes more complicated.

We define a **four-vector** which transform as (x^0, x^1, x^2, x^3) under Lorentz transformation

$$a^\mu = \mathbb{A} = (a^0, a^1, a^2, a^3) \tag{9.43}$$

We define the four-dimensional scalar product (dot product) of four-vectors as

$$\mathbb{A} \cdot \mathbb{B} = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \tag{9.44}$$

(compared with $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$). This quantity is invariant under Lorentz transformation.

To keep track of the minus sign, it is convenient to introduce the **covariant vector** $a_\mu = (a_0, a_1, a_2, a_3) = (-a^0, a^1, a^2, a^3)$, which differs from the **contravariant vector** by a sign in the zeroth component. The upper indices designate *contravariant* vectors and the lower indices designate *covariant* vectors. Raising or lowering the temporal index costs a minus sign ($a_0 = -a^0$), while raising or lowering the spatial index changes nothing ($a_1 = a^1, a_2 = a^2, a_3 = a^3$). And we will introduce the Einstein summation convention, where repeated Greek indices are summed over.

$$a^\mu b_\mu = a_\mu b^\mu = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \tag{9.45}$$

The space-time interval, which is a four-scalar, and is invariant under Lorentz transformation

$$ds = \sqrt{dx_\mu dx^\nu} = \sqrt{-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2} \tag{9.46}$$

and the proper time is just the interval divided by c

$$d\tau = \frac{ds}{c} \tag{9.47}$$

9.4.1 Velocity addition rule

Consider there intertial frames K, K', K'' which are in collinear motion along x_1-axis. Let the velocity of K' relative to K be v_1 and K'' relative to K' be v_2. Using the Lorentz transformation matrix, we have

$$\Lambda_{K\to K'} = \begin{pmatrix} \gamma_1 & -\gamma_1\beta_1 & 0 & 0 \\ -\gamma_1\beta_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9.48}$$

and

$$\Lambda_{K'\to K''} = \begin{pmatrix} \gamma_2 & -\gamma_2\beta_2 & 0 & 0 \\ -\gamma_2\beta_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9.49}$$

The transformation from K to K” is the product of these two matrices

$$\begin{aligned} \Lambda_{K\to K"} &= \Lambda_{K\to K'}\Lambda_{K'\to K"} \\ &= \begin{pmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & -\gamma_1\gamma_2(\beta_1+\beta_2) & 0 & 0 \\ -\gamma_1\gamma_2(\beta_1+\beta_2) & \gamma_1\gamma_2(1+\beta_1\beta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{9.50}$$

if K'' is moving at velocity v relative to K. Finally we have

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} \tag{9.51}$$

This is the velocity addition rule.

9.4.2 Proper velocity

We can define a proper velocity as

$$\boldsymbol{\eta} = \frac{d\mathbf{l}}{d\tau} \tag{9.52}$$

and is related to the ordinary velocity by

$$\boldsymbol{\eta} = \frac{1}{\sqrt{1-\beta^2}}\mathbf{u} = \gamma\mathbf{u} \tag{9.53}$$

The proper velocity is the spatial part of a four-vector

$$\eta^\mu \equiv \frac{dx^\mu}{d\tau}$$

whose zeroth component is

$$\eta^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \gamma c \tag{9.54}$$

The numerator transforms as a four-vector under Lorentz transformation, and the denominator $d\tau$ is invariant. We can easily write down the Lorentz transformation of the proper velocity

$$\begin{aligned}
\eta'^0 &= \gamma(\eta^0 - \beta\eta^1) \\
\eta'^1 &= \gamma(\eta^1 - \beta\eta^0) \\
\eta'^2 &= \eta^2 \\
\eta'^3 &= \eta^3
\end{aligned} \tag{9.55}$$

More generally

$$\eta'^\mu = \Lambda^\mu_\nu \eta^\nu \tag{9.56}$$

η^μ is called the **4-velocity**.

The ordinary velocities transform in a more complicated way as

$$\begin{aligned}
u'_x &= \frac{dx'}{dt'} = \frac{u_x - v}{1 - vu_x/c^2} \\
u'_y &= \frac{dy'}{dt'} = \frac{u_y}{\gamma(1 - vu_x/c^2)} \\
u'_z &= \frac{dz'}{dt'} = \frac{u_z}{\gamma(1 - vu_x/c^2)}
\end{aligned} \tag{9.57}$$

The reason for this complexity is due to the fact that we have to transform both the denominator and numerator, while in the proper velocity, $d\tau$ is invariant under transformation.

9.5 Relativistic Momentum and Energy

We define the relativistic momentum as

$$\mathbf{p} = m\boldsymbol{\eta} = \gamma m\mathbf{u} \tag{9.58}$$

which is the spatial part of a 4-vector

$$p^\mu = m\eta^\mu \tag{9.59}$$

The temporal component is

$$p^0 = m\eta^0 = \frac{mc}{\sqrt{1-\beta^2}} = E/c \tag{9.60}$$

where E is the **relativistic energy**

$$E = \frac{mc^2}{\sqrt{1-\beta^2}} \tag{9.61}$$

p^μ is called **energy-momenum 4-vector**.

The rest energy of an object ($u = 0$) is

$$E_{\text{rest}} \equiv mc^2 \tag{9.62}$$

The kinetic energy is given by

$$E_{\text{kin}} = E - E_{\text{rest}} = mc^2(\gamma - 1) \tag{9.63}$$

In the non-relativistic regime ($u \ll c$), we can expand γ in powers of $\beta^2 = u^2/c^2$ and we have

$$E_{\text{kin}} = \frac{1}{2}mu^2 + \frac{3}{8}\frac{mu^4}{c^2} + \cdots \tag{9.64}$$

The energy and momentum conservation laws are now stated as a single law:

> **In a closed system, the relativistic energy and momentum are conserved.**

The scalar product of p^μ with itself is

$$p^\mu p_\mu = -(p^0)^2 + \mathbf{p}\cdot\mathbf{p} = -m^2c^2 \tag{9.65}$$

This is invariant under Lorentz transformation. In terms of relativistic energy and momentum,

$$E^2 - p^2c^2 = m^2c^4 \tag{9.66}$$

9.6 Lagrangian in Special Relativity

For a particle moving in a velocity-independent potential, the momentum can be written as

$$p_i = \frac{\partial L}{\partial u_i} \tag{9.67}$$

The relativistic expression for the momentum is

$$p_i = \frac{m u_i}{\sqrt{1 = \beta^2}} = \frac{\partial L}{\partial u_i} \tag{9.68}$$

We can then write the relativistic Lagrangian as

$$L = -mc^2\sqrt{1-\beta^2} - U(r) \tag{9.69}$$

Notice the velocity dependent part $T^* = -mc^2\sqrt{1-\beta^2}$ is no longer the kinetic energy $T = \frac{mc^2}{\sqrt{1-\beta^2}} - mc^2$.

The Hamiltonian can be calculated from

$$\begin{aligned} H &= \sum_i u_i p_i - L \\ &= \sum_i \frac{p_i^2 c^2}{\gamma m c^2} + \frac{mc^2}{\gamma} + U \\ &= \frac{p^2 c^2}{\gamma m c^2} + \frac{mc^2}{\gamma} + U \\ &= \frac{E^2}{\gamma m c^2} + U \\ &= E + U = T + U + E_0 \end{aligned} \tag{9.70}$$

The relativistic Hamiltonian is equal to the total energy defined previously plus the potential energy.