

1082 Calculus A 01-10, Mode 01-02 Makeup Midterm

1. (15 pts) Consider the function

$$f(x, y) = \begin{cases} \frac{\sin(xy)}{\sqrt{x^2 + y^2}}, & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) (3 pts) Is $f(x, y)$ continuous at $(0, 0)$?

(b) (4 pts) Find $f_x(0, 0)$, $f_y(0, 0)$, and $f_x(x, y)$, $f_y(x, y)$ for $(x, y) \neq (0, 0)$.

(c) (2 pts) Write down the linear approximation of $f(x, y)$ at $(0, 0)$, $L(x, y)$.

(d) (4 pts) Compute $\lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, y) - L(x, y)}{\sqrt{x^2 + y^2}}$ if it exists.

(e) (2 pts) Is $f(x, y)$ differentiable at $(0, 0)$?

Solution:

(a) We consider polar coordinate, let $x = r \cos \theta$ and $y = r \sin \theta$, then

$$f(x, y) = f(r, \theta) = \begin{cases} \frac{\sin(r^2 \cos \theta \sin \theta)}{\sqrt{r^2}} & , \text{if } r \neq 0, \\ 0 & , \text{if } r = 0. \end{cases}$$

By definition of continuous, we can consider the continuity of $f(x, y)$ at $(x, y) = (0, 0)$ by taking $r \rightarrow 0^+$. $\lim_{r \rightarrow 0^+} f(r, \theta) = \lim_{r \rightarrow 0^+} \frac{\sin(r^2 \cos \theta \sin \theta)}{r} = \lim_{r \rightarrow 0^+} \frac{\sin(r^2 \cos \theta \sin \theta)}{r^2 \cos \theta \sin \theta} \cdot r \cos \theta \sin \theta$.

Since both $\lim_{r \rightarrow 0^+} \frac{\sin(r^2 \cos \theta \sin \theta)}{r^2 \cos \theta \sin \theta}$ and $\lim_{r \rightarrow 0^+} r \cos \theta \sin \theta$ are exist with $\lim_{r \rightarrow 0^+} \frac{\sin(r^2 \cos \theta \sin \theta)}{r^2 \cos \theta \sin \theta} = 1$ and $\lim_{r \rightarrow 0^+} r \cos \theta \sin \theta = 0$, then

$$\lim_{r \rightarrow 0^+} \frac{\sin(r^2 \cos \theta \sin \theta)}{r^2 \cos \theta \sin \theta} \cdot r \cos \theta \sin \theta = \lim_{r \rightarrow 0^+} \frac{\sin(r^2 \cos \theta \sin \theta)}{r^2 \cos \theta \sin \theta} \cdot \lim_{r \rightarrow 0^+} r \cos \theta \sin \theta = 1 \cdot 0 = 0.$$

That is $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = f(0, 0)$, so we have $f(x, y)$ is continuous at $(0, 0)$.

(b) By definition of partial derivative at $(0, 0)$.

$$\begin{cases} f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{\sin(h \cdot 0)}{\sqrt{h^2}} - 0}{h} = 0 \\ f_y(0, 0) = \lim_{k \rightarrow 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \rightarrow 0} \frac{\frac{\sin(0 \cdot k)}{\sqrt{k^2}} - 0}{k} = 0 \end{cases}$$

and we can derivative for $(x, y) \neq (0, 0)$ since $f(0, 0)$ is differentiable except $(x, y) = (0, 0)$.

$$\begin{cases} f_x(x, y) = \frac{\cos(xy) \cdot y \cdot \sqrt{x^2 + y^2} - \sin(xy) \cdot \frac{1}{2} \frac{2x}{\sqrt{x^2 + y^2}}}{(\sqrt{x^2 + y^2})^2} = \frac{y(x^2 + y^2) \cos(xy) - x \sin(xy)}{(x^2 + y^2)^{\frac{3}{2}}} \\ f_y(x, y) = \frac{\cos(xy) \cdot x \cdot \sqrt{x^2 + y^2} - \sin(xy) \cdot \frac{1}{2} \frac{2y}{\sqrt{x^2 + y^2}}}{(\sqrt{x^2 + y^2})^2} = \frac{x(x^2 + y^2) \cos(xy) - y \sin(xy)}{(x^2 + y^2)^{\frac{3}{2}}} \end{cases}$$

(c) Linear approximation of $f(x, y)$ at $(0, 0)$ is

$$\begin{aligned} L(x, y) &= f(0, 0) + f_x(0, 0)x + f_y(0, 0)y \\ &= 0 + 0 + 0 = 0 \end{aligned}$$

(d)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x^2 + y^2}.$$

Consider two straight line $x = y$ and $x = -y$ then

$$\begin{cases} \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{\sin(x^2)}{2x^2} = \frac{1}{2}, & \text{if } x = y \\ \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{\sin(-x^2)}{2x^2} = -\frac{1}{2}, & \text{if } x = -y \end{cases}$$

(d)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x^2 + y^2}.$$

That is the limit $\lim_{(x,y) \rightarrow (0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2 + y^2}}$ does not exist.

(e) No, $f(x,y)$ is not differentiable at $(0,0)$ since $\lim_{(x,y) \rightarrow (0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2 + y^2}}$ does not exist.

2. (15 pts) Consider the level surface S defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 : z^5 - xz^4 + yz^3 = 1\}.$$

This level surface defines $z = z(x, y)$ implicitly as a differentiable function of x and y .

- (a) (4 pts) Find an equation of the tangent plane at the point $P(0, 0, z(0, 0))$.
- (b) (4 pts) Use linear approximation at P to estimate the value $z(0.02, -0.03)$.
- (c) (5 pts) Find $\frac{\partial^2 z}{\partial x \partial y} \Big|_{(x,y)=(0,0)}$.
- (d) (2 pts) Find the directional derivative $D_{\vec{u}}z(0, 0)$ along the direction \vec{u} parallel to the vector $(-3, 4)$.

Solution:

(a) Let $F(x, y, z) = z^5 - xz^4 + yz^3 - 1$, then $S = \{(x, y, z) \in \mathbb{R}^3 : F(x, y, z) = 0\}$, then the normal vector of tangent plane at each point (x, y, z) is $\nabla F(x, y, z) = (-z^4, z^3, 5z^4 - 4xz^3 + 3yz^2)$, that is the normal vector at $(0, 0, z(0, 0)) = (0, 0, 1)$ is $(-1, 1, 5)$. Then the tangent plane at the point $P(0, 0, 1)$ is $-(x - 0) + (y - 0) + 5(z - 1) = 0$.

(b) Linear approximate at P is $L(x, y) = z(0, 0) + z_x(0, 0)(x - 0) + z_y(0, 0)(y - 0) = z(0, 0) - \frac{F_x(0, 0, 0)}{F_z(0, 0, 0)}(x - 0) - \frac{F_y(0, 0, 0)}{F_z(0, 0, 0)}(y - 0) \Rightarrow z(0.02, -0.03) = 1 + \frac{1}{5} \cdot 0.02 - \frac{1}{5} \cdot (-0.03) = 1.01$
 Actually, by tangent plane at $P(0, 0, 1)$, $-(0.02 - 0) + (-0.03 - 0) + 5(z(0.02, -0.03) - 1) = 0 \Rightarrow 5(z(0.02, -0.03) - 1) = 0.05 \Rightarrow z(0.02, -0.03) = 1 + \frac{0.05}{5} = 1.01$.

(c) Since $\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = -\frac{z^3}{5z^4 - 4xz^3 + 3yz^2} = -\frac{z}{5z^2 - 4xz + 3y}$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} \Big|_{(x,y)=(0,0)} = \frac{\partial}{\partial x} \left(\frac{-z}{5z^2 - 4xz + 3y} \right) \Big|_{(x,y)=(0,0)}$$

$$= \frac{-z_x(5z^2 - 4xz + 3y) + z(10zz_x - 4(z + xz_x))}{(5z^2 - 4xz + 3y)^2} \Big|_{(x,y)=(0,0)}$$

$$= \frac{-\frac{1}{5} \cdot 5 + 1 \cdot (10 \cdot 1 \cdot \frac{1}{5} - 4)}{5^2} = \frac{-1 + (-2)}{25} = -\frac{3}{25}$$

(d) $D_{\vec{u}}z(0, 0) = \nabla z(0, 0) \cdot \vec{u} = \left(\frac{1}{5}, -\frac{1}{5} \right) \cdot \left(\frac{-3}{5}, \frac{4}{5} \right) = \pm \frac{7}{25}$

3. (14 pts) Let $f(x, y) = y(x + 1)^2$.

(a) (6 pts) Find and classify critical point(s) of $f(x, y)$.

(b) (8 pts) Find the extreme values of $f(x, y)$ on the region $R = \{(x, y) | x^2 - 1 \leq y \leq 3\}$.

Solution:

(a) $\nabla f(x, y) = (2y(x + 1), (x + 1)^2) = (0, 0)$ iff $(x, y) = (-1, y)$

$$\Rightarrow \begin{cases} f_{xx} = 2y \\ f_{xy} = 2(x + 1) = f_{yx} \\ f_{yy} = 0 \end{cases} \Rightarrow D|_{(x,y)=(-1,y)} = f_{xx}f_{yy} - f_{xy}^2|_{(x,y)=(-1,y)} = 0 \quad \forall y \in \mathbb{R}.$$

If $y > 0$, $f(x, y) \geq 0$ for all $x \neq -1$, then $(-1, y)$ is local minimum. With same argument for $y < 0$, $(-1, y)$ is local maximum. The last one is $(-1, 0)$ is a saddle point.

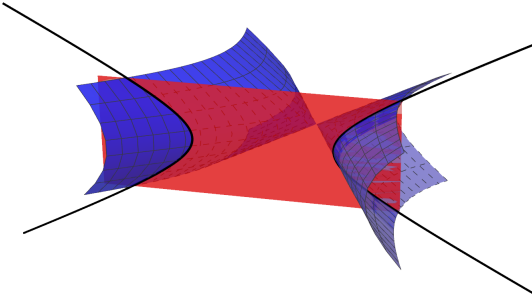
(b) $R = \{(x, y) | x^2 - 1 \leq y \leq 3\}$.

First, we have the local minimum in the region equal to 0 when $(x, y) = (-1, y)$ and $y > 0$.

Second, we consider the boundary $x^2 - 1 \leq y \leq 3 \Rightarrow x^2 - 1 = 3$ with $-2 \leq x \leq 2$ and $y = 3$ then we have the local maximum when $(x, y) = (2, 3)$ is equal to 27.

The last, $y = x^2 - 1 \Rightarrow f(x, y) = f(x, x^2 - 1) = (x^2 - 1)(x + 1)^2 = g(x)$ with $-2 \leq x \leq 2$, then $g'(x) = 2x(x + 1)^2 + 2(x^2 - 1)(x + 1) = 2(x + 1)(x^2 + x + x^2 - 1) = 2(x + 1)(2x^2 + x - 1) = 2(x + 1)^2(2x - 1) = 0$ for $x = -1$, $\frac{1}{2} \Rightarrow (x, y) = \left(\frac{1}{2}, -\frac{3}{4}\right)$ has local minimum is equal to $-\frac{27}{16}$.

4. (10 pts) Let C be the hyperbola formed by the intersection of the cone $4x^2 = y^2 + 3z^2$ and the plane $x + y = 6$. Find the maximum and the minimum distance between the origin and the point on C (if exist) by the method of Lagrange multipliers.



Solution:

That is to find the extreme of $d(x, y, z) = x^2 + y^2 + z^2$ with the restriction $x + y = 6$ and $4x^2 = y^2 + 3z^2$
By Lagrange Multipliers

$$\begin{cases} 2x = \lambda + \mu 8x \\ 2y = \lambda + \mu(-2y) \\ 2z = 0 + \mu(-6z) \end{cases} \Rightarrow \begin{cases} \lambda = (2 - 8\mu)x = (2 + 2\mu)y \\ \mu = -\frac{1}{3} \text{ or } z = 0 \end{cases} \Rightarrow \begin{cases} \lambda = \frac{14}{3}x = \frac{4}{3}y \\ \text{or} \\ \lambda = (2 - 8\mu)x = (2 + 2\mu)y, z = 0 \end{cases}$$

- (i) If $\lambda = \frac{14}{3}x = \frac{4}{3}y$, $x + y = 6$ and $4x^2 = y^2 + 3z^2$

$$\Rightarrow \frac{3}{14}\lambda + \frac{3}{4}\lambda = 6 \Rightarrow \frac{27}{28}\lambda = 6 \Rightarrow \lambda = \frac{56}{9} \Rightarrow (x, y) = \left(\frac{4}{3}, \frac{14}{3}\right) \Rightarrow (x, y, z) \text{ not exists in } \mathbb{R}^3.$$

- (ii) If $\lambda = (2 - 8\mu)x = (2 + 2\mu)y$, $z = 0$, $x + y = 6$ and $4x^2 = y^2 + 3z^2$

$\Rightarrow 4x^2 = y^2 \Rightarrow 2x = \pm y$ and $x + y = 6 \Rightarrow (x, y) = (2, 4)$ or $(-6, 12)$ Since there is no maximum distance between origin and the point on hyperbola C , we have the minimum at $(2, 4, 0)$ is equal to $\sqrt{20}$.

5. (12 pts)

(a) Compute $\int_0^1 \int_0^{\cos^{-1} y} \sin x (1 + \sin^2 x)^{\frac{1}{3}} dx dy$.

(b) Let $f(x) = \int_x^1 e^{-t^2} dt$. Find the average value of f on the interval $[0, 1]$.

Solution:

(a)

$$\begin{aligned} \int_0^1 \int_0^{\cos^{-1} y} \sin x (1 + \sin^2 x)^{\frac{1}{3}} dx dy &= \int_0^{\frac{\pi}{2}} \int_0^{\cos x} \sin x (1 + \sin^2 x)^{\frac{1}{3}} dy dx = \int_0^{\frac{\pi}{2}} \sin x (1 + \sin^2 x)^{\frac{1}{3}} \cos x dx \\ &= \frac{1}{2} (1 + \sin^2 x)^{\frac{4}{3}} \cdot \frac{3}{4} \Big|_0^{\frac{\pi}{2}} = \frac{3 \cdot 2^{\frac{4}{3}}}{8} - \frac{3}{8} \end{aligned}$$

(b) Average value of $f = \frac{\int_0^1 f dx}{1 - 0} = \int_0^1 \int_x^1 e^{-t^2} dt dx$

$$\Rightarrow \int_0^1 \int_x^1 e^{-t^2} dt dx = \int_0^1 \int_0^t e^{-t^2} dx dt = \int_0^1 t e^{-t^2} dt = \frac{-1}{2} e^{-t^2} \Big|_0^1 = \frac{1}{2} (1 - e^{-1})$$

6. (12 pts) Find the center of mass of a lamina

$$D = \left\{ (x, y) \in \mathbf{R}^2 \mid \frac{(x-2)^2}{4} + y^2 \leq 1 \text{ and } x \leq 2 \right\}$$

whose density function at any point is proportional to the square of its distance from the line $x = 2$.

Solution:

Assume the coordinate of center of mass is (\bar{x}, \bar{y}) , then obvious $\bar{y} = 0$.

$$\Rightarrow \bar{x} \cdot \iint_D C(x^2 + y^2) dA = \iint_D Cx(x^2 + y^2) dA \Rightarrow \bar{x} = \frac{\iint_D x(x^2 + y^2) dA}{\iint_D (x^2 + y^2) dA}$$

$$\begin{cases} \iint_D (2-x)^2 dA = \int_0^2 \int_{-\sqrt{1-\frac{(x-2)^2}{4}}}^{\sqrt{1-\frac{(x-2)^2}{4}}} (2-x)^2 dy dx = \int_0^2 2\sqrt{1-\frac{(x-2)^2}{4}} (2-x)^2 dx \\ \iint_D x(2-x)^2 dA = \int_0^2 \int_{-\sqrt{1-\frac{(x-2)^2}{4}}}^{\sqrt{1-\frac{(x-2)^2}{4}}} x(2-x)^2 dy dx = \int_0^2 2\sqrt{1-\frac{(x-2)^2}{4}} x(2-x)^2 dx \end{cases}$$

$$\text{Let } t = \frac{x-2}{2} \Rightarrow dx = 2dt$$

$$\Rightarrow \begin{cases} \int_0^2 2\sqrt{1-\frac{(x-2)^2}{4}} (2-x)^2 dx = \int_{-1}^0 2\sqrt{1-t^2} 4t^2 \cdot 2dt \\ \int_0^2 2\sqrt{1-\frac{(x-2)^2}{4}} x(2-x)^2 dx = \int_{-1}^0 2\sqrt{1-t^2} \cdot (2t+2) \cdot 4t^2 \cdot 2dt \end{cases}$$

$$\text{Let } t = \cos \theta \Rightarrow dt = -\sin \theta d\theta$$

$$\Rightarrow \begin{cases} -\int_{-1}^0 2\sqrt{1-t^2} 4t^2 \cdot 2dt = \int_{\frac{\pi}{2}}^{\pi} 16 \sin \theta \cos^2 \theta \sin \theta d\theta = 16 \int_{\frac{\pi}{2}}^{\pi} \sin^2 \theta \cos^2 \theta d\theta \\ -\int_{-1}^0 2\sqrt{1-t^2} \cdot (2t+2) \cdot 4t^2 \cdot 2dt = \int_{\frac{\pi}{2}}^{\pi} 32 \sin \theta \cos^2 \theta \sin \theta (\cos \theta + 1) d\theta = 32 \int_{\frac{\pi}{2}}^{\pi} \sin^2 \theta \cos^3 \theta + \sin^2 \theta \cos^2 \theta d\theta \end{cases}$$

$$\Rightarrow \begin{cases} 16 \int_{\frac{\pi}{2}}^{\pi} \sin^2 \theta \cos^2 \theta d\theta = 2 \int_{\frac{\pi}{2}}^{\pi} \sin^2 2\theta d2\theta = 2 \int_{\frac{\pi}{2}}^{\pi} \cos^2 2\theta d2\theta = 2 \int_{\frac{\pi}{2}}^{\pi} \frac{\sin^2 2\theta + \cos^2 2\theta}{2} d2\theta = 2\pi - \pi = \pi \\ 32 \int_{\frac{\pi}{2}}^{\pi} \sin^2 \theta \cos^3 \theta d\theta = 32 \int_{\frac{\pi}{2}}^{\pi} \sin^2 \theta (1 - \sin^2 \theta) \cos \theta d\theta = 32 \left(\frac{1}{3} \sin^3 \theta - \frac{1}{5} \sin^5 \theta \right)_{\frac{\pi}{2}}^{\pi} = -\frac{64}{15} \end{cases}$$

$$\Rightarrow \iint_D (2-x)^2 dA = \pi \quad \text{and} \quad \iint_D x(2-x)^2 dA = -\frac{64}{15} + 2\pi$$

$$\therefore \bar{x} = \frac{2\pi - \frac{64}{15}}{\pi} \Rightarrow \text{center of mass : } (\bar{x}, \bar{y}) = \left(2 - \frac{64}{15\pi}, 0 \right)$$

7. (12 pts) Use spherical coordinates to evaluate $\int_0^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} \int_1^{\sqrt{4-x^2-y^2}} \frac{1}{\sqrt{x^2+y^2+z^2}} dz dx dy$.

Solution:

By spherical coordinates, let $\begin{cases} x = \rho \cos \theta \sin \phi \\ y = \rho \sin \theta \sin \phi \\ z = \rho \cos \phi \end{cases}$, where $0 \leq \rho \leq 2$, $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$.

$$\begin{aligned} \Rightarrow \int_0^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} \int_1^{\sqrt{4-x^2-y^2}} \frac{1}{\sqrt{x^2+y^2+z^2}} dz dx dy &= \int_0^{\pi} \int_0^{\frac{\pi}{3}} \int_{\sec \phi}^2 \frac{\rho^2 \sin \phi}{\rho} d\rho d\phi d\theta = \frac{\pi}{2} \int_0^{\frac{\pi}{3}} (4 - \sec^2 \phi) \sin \phi d\phi \\ &= \frac{\pi}{2} \int_0^{\frac{\pi}{3}} 4 \sin \phi - \sec \phi \tan \phi d\phi = \frac{\pi}{2} (-4 \cos \phi - \sec \phi) \Big|_0^{\frac{\pi}{3}} = \frac{\pi}{2} \end{aligned}$$

8. (10 pts) Compute $\iint_R \ln(x^2y + x) dA$, where R is the region bounded by curves $xy = 1$, $xy = 3$, $x = 1$ and $x = e$.

Solution:

By change variables, let $\begin{cases} u = xy, 1 \leq u \leq 3 \\ v = x, 1 \leq v \leq e \end{cases} \Rightarrow |J| = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} \right| = \frac{1}{v}$

$$\begin{aligned} \Rightarrow \iint_R \ln(x^2y + x) dA &= \int_1^e \int_1^3 \ln(uv + v) |J| du dv = \int_1^e \int_1^3 \ln[(u + 1)v] \frac{1}{v} du dv \\ &= \int_1^e \int_1^3 \frac{\ln(u + 1) + \ln v}{v} du dv \\ &= \int_1^e [(u + 1) \ln(u + 1) - (u + 1)]_1^3 \frac{1}{v} + (3 - 1) \frac{\ln v}{v} dv \\ &= \ln e [(3 + 1) \ln(3 + 1) - (3 + 1) - 2 \ln 2 + 2] + \frac{(3 - 1)}{2} (\ln e)^2 \\ &= 8 \ln 2 - 4 - 2 \ln 2 + 2 + 1 = 6 \ln 2 - 1 \end{aligned}$$