臺灣大學 109 學年第二學期 應用數學一 期中考試

系級(Department): ______ 學號(student No.): _______ 姓名(name): _______

1. Reduce this system to upper triangular form by two row operations:

$$\begin{aligned} 2x + 3y + z &= 8 \\ 4x + 7y + 5z &= 20 \\ -2y + 2z &= 0 \end{aligned}$$

Circle the pivots. Solve by back-substitution for z, y, x.

2. Solve by elimination and back-substitution:

$$\begin{aligned} u \quad + w &= 4 \\ u + v \quad &= 3 \\ u + v + w &= 6 \end{aligned} \qquad \text{and} \qquad \begin{aligned} v + w &= 0 \\ u \quad + w &= 0 \\ u + v \quad &= 6 \end{aligned}$$

3. Find the symmetric factorization $A = LDL^T$ of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 4 \\ 0 & 4 & 11 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

4. (a) $Ax = b$ has a solution under what conditions on b, for the following A and b?

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

(b) Find a basis for the nullspace of A.

(c) Find the general solution to $Ax = b$, when a solution exists.

(d) Find a basis for the column space of A.

(e) What is the rank of A^T?

5. Suppose **S** is spanned by the vectors (1,2,2,3) and (1,3,3,2). Find two vectors that span $\mathbf{S}^{\perp}$. This is the same as solving $Ax = 0$ for which A?

6. Draw the projection of b onto a and also compute it from $p = \hat{x}\, a$:

(a) $b = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (b) $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

7. (a) Write the four equations for fitting $y = C+Dt$ to the data

$$y = -4 \quad \text{at} \quad t = -2, \quad y = -3 \text{ at} \quad t = -1$$
$$y = -1 \quad \text{at} \quad t = 1, \quad y = 0 \text{ at} \quad t = 2$$

Show that the columns are orthogonal.

(b) Find the optimal straight line, draw its graph, and write E^2.

(c) Interpret the zero error in terms of the original system of four equations in two unknowns: The right-hand side $(-4, -3, -1, 0)$ is in the ____ space.

8. What is the angle between $a = (2, -2, 1)$ and $b = (1,2,2)$?

9. What is the projection p of $b = (1,2,2)$ onto $a = (2, -2, 1)$?

10. Let A = [3 1 1], and let **V** be the nullspace of A.

(a) Find a basis for **V** and a basis for $\mathbf{V}^{\perp}$.

(b) Write an orthonormal basis for $\mathbf{V}^{\perp}$, and find the projection matrix P_1 that projects vectors in $\mathbf{R}^3$ onto $\mathbf{V}^{\perp}$?.

(c) Find the projection matrix P_2 that projects vectors in $\mathbf{R}^3$ onto **V**.

11. The distance from a plane $a^T x = c$ (in m-dimensional space) to the origin is $|c|/||a||$.How far is the plane $x_1 + x_2 - x_3 - x_4 = 8$ from the origin, and what point on it is nearest?