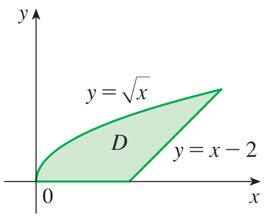
Section 15.2 Double Integrals over General Regions

9. (a) Express the double integral $\iint_D f(x,y)dA$ as an iterated integral for the given function f and region D.

   (b) Evaluate the iterated integral.



Solution:

(a) We express the iterated integral as a Type II. A Type I would require the sum of two integrals. The curves intersect when

$\sqrt{x}=x-2 \quad\Rightarrow\quad x=x^2-4x+4 \quad\Leftrightarrow\quad 0=x^2-5x+4 \quad\Leftrightarrow\quad (x-4)(x-1)=0 \quad\Leftrightarrow\quad x=1$ or $x=4$. The point for $x=1$ is not in D. Thus, the point of intersection of the curves is $(4,2)$ and the integral is $\int_0^2\int_{y^2}^{y+2} xy\,dx\,dy$.

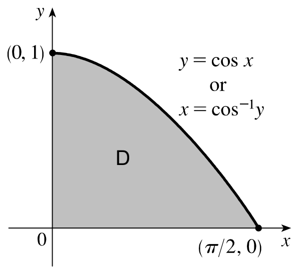
(b) $$\int_0^2\int_{y^2}^{y+2} xy\,dx\,dy=\int_0^2 y\left[\frac{x^2}{2}\right]_{x=y^2}^{x=y+2}dy=\frac{1}{2}\int_0^2 y\left[(y+2)^2-(y^2)^2\right]dy=\frac{1}{2}\int_0^2\left[y^3+4y^2+4y-y^5\right]dy$$

$$=\tfrac{1}{2}\left[\tfrac{1}{4}y^4+\tfrac{4}{3}y^3+2y^2-\tfrac{1}{6}y^6\right]_0^2=\tfrac{1}{2}\left(4+\tfrac{32}{3}+8-\tfrac{32}{3}\right)=6$$

21. Set up iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it’s easier.

$$\iint_D \sin^2 x dA,\ \ D \text{ is bounded by } y=\cos x,\ 0\le x\le\frac{\pi}{2},\ y=0,\ x=0$$

Solution:



If we describe D as a type I region, $D=\{(x,y)\mid 0\le x\le\pi/2, 0\le y\le\cos x\}$ and $\iint_D\sin^2 x\,dA=\int_0^{\pi/2}\int_0^{\cos x}\sin^2 x\,dy\,dx$. As a type II region, $D=\{(x,y)\mid 0\le x\le\cos^{-1}y, 0\le y\le 1\}$ and $\iint_D\sin^2 x\,dA=\int_0^1\int_0^{\cos^{-1}y}\sin^2 x\,dx\,dy$. Evaluating $\int_0^{\cos^{-1}y}\sin^2 x\,dx$ will result in a very difficult integral. Therefore, we evaluate the iterated integral that describes D as a type I region because integrating $\sin^2 x$ with respect to y is easy.

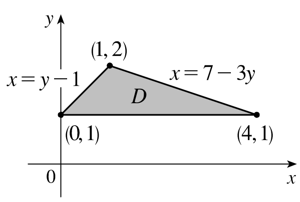
$$\int_0^{\pi/2}\int_0^{\cos x}\sin^2 x\,dy\,dx=\int_0^{\pi/2}\sin^2 x\left[y\right]_{y=0}^{y=\cos x}dx=\int_0^{\pi/2}\cos x\sin^2 x\,dx$$

$$=\int_0^1 u^2\,du\quad\begin{bmatrix}u=\sin x,\\ du=\cos x\,dx\end{bmatrix}\quad=\left[\frac{u^3}{3}\right]_0^1=\frac{1}{3}$$

25. Evaluate the double integral.

$$\iint_D y^2dA,\ \ D \text{ is the triangular region with vertices } (0,1),(1,2),(4,1)$$

Solution:



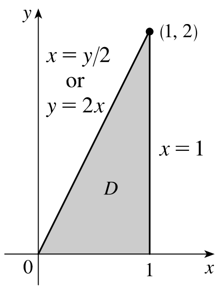
$$\iint_D y^2\,dA=\int_1^2\int_{y-1}^{7-3y}y^2\,dx\,dy=\int_1^2\left[xy^2\right]_{x=y-1}^{x=7-3y}dy$$

$$=\int_1^2\left[(7-3y)-(y-1)\right]y^2\,dy=\int_1^2(8y^2-4y^3)\,dy$$

$$=\left[\tfrac{8}{3}y^3-y^4\right]_1^2=\tfrac{64}{3}-16-\tfrac{8}{3}+1=\tfrac{11}{3}$$

64. Evaluate the integral by reversing the order of integration. $\int_0^2 \int_{y/2}^1 y\cos(x^3-1)dxdy$

Solution:



$$\begin{aligned}\int_0^2 \int_{y/2}^1 y\cos(x^3-1)\,dx\,dy &= \int_0^1 \int_0^{2x} y\cos(x^3-1)\,dy\,dx \\ &= \int_0^1 \cos(x^3-1)\,\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=2x}\,dx \\ &= \int_0^1 2x^2\cos(x^3-1)\,dx = \tfrac{2}{3}\sin(x^3-1)\Big]_0^1 \\ &= \tfrac{2}{3}\,[0-\sin(-1)] = -\tfrac{2}{3}\sin(-1) = \tfrac{2}{3}\sin 1\end{aligned}$$