

Section 14.7 Maximum and Minimum Values

15. Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

$$f(x, y) = x^4 - 2x^2 + y^3 - 3y$$

Solution:

$$f(x, y) = x^4 - 2x^2 + y^3 - 3y \Rightarrow f_x = 4x^3 - 4x, f_y = 3y^2 - 3, f_{xx} = 12x^2 - 4, f_{xy} = 0, f_{yy} = 6y.$$

$$\text{Then } f_x = 0 \text{ implies } 4x(x^2 - 1) = 0 \Rightarrow x = 0 \text{ or } x = \pm 1, \text{ and } f_y = 0 \text{ implies } 3(y^2 - 1) = 0 \Rightarrow y = \pm 1.$$

Thus there are six critical points: $(0, \pm 1)$, $(\pm 1, 1)$, and $(\pm 1, -1)$.

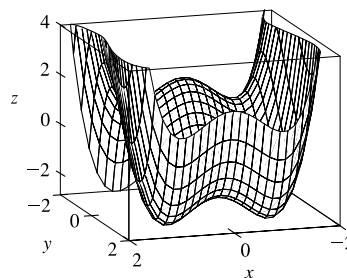
$$D(0, 1) = (-4)(6) - (0)^2 = -24 < 0 \text{ and}$$

$$D(\pm 1, -1) = (8)(-6) = -48 < 0, \text{ so } (0, 1) \text{ and } (\pm 1, -1) \text{ are saddle}$$

points. $D(0, -1) = (-4)(-6) = 24 > 0$ and $f_{xx}(0, -1) = -4 < 0$, so

$$f(0, -1) = 2 \text{ is a local maximum. } D(\pm 1, 1) = (8)(6) = 48 > 0 \text{ and}$$

$$f_{xx}(\pm 1, 1) = 8 > 0, \text{ so } f(\pm 1, 1) = -3 \text{ are local minima.}$$



21. Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

$$f(x, y) = y^2 - 2y \cos x, \quad -1 \leq x \leq 7$$

Solution:

$$f(x, y) = y^2 - 2y \cos x \Rightarrow f_x = 2y \sin x, f_y = 2y - 2 \cos x,$$

$$f_{xx} = 2y \cos x, f_{xy} = 2 \sin x, f_{yy} = 2. \text{ Then } f_x = 0 \text{ implies } y = 0 \text{ or}$$

$$\sin x = 0 \Rightarrow x = 0, \pi, \text{ or } 2\pi \text{ for } -1 \leq x \leq 7. \text{ Substituting } y = 0 \text{ into}$$

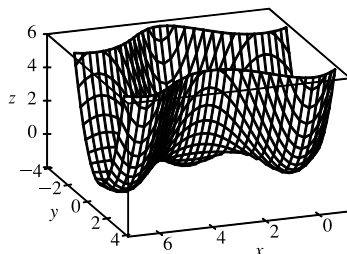
$$f_y = 0 \text{ gives } \cos x = 0 \Rightarrow x = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}, \text{ substituting } x = 0 \text{ or } x = 2\pi$$

$$\text{into } f_y = 0 \text{ gives } y = 1, \text{ and substituting } x = \pi \text{ into } f_y = 0 \text{ gives } y = -1.$$

Thus the critical points are $(0, 1)$, $(\frac{\pi}{2}, 0)$, $(\pi, -1)$, $(\frac{3\pi}{2}, 0)$, and $(2\pi, 1)$.

$$D(\frac{\pi}{2}, 0) = D(\frac{3\pi}{2}, 0) = -4 < 0 \text{ so } (\frac{\pi}{2}, 0) \text{ and } (\frac{3\pi}{2}, 0) \text{ are saddle points. } D(0, 1) = D(\pi, -1) = D(2\pi, 1) = 4 > 0 \text{ and}$$

$$f_{xx}(0, 1) = f_{xx}(\pi, -1) = f_{xx}(2\pi, 1) = 2 > 0, \text{ so } f(0, 1) = f(\pi, -1) = f(2\pi, 1) = -1 \text{ are local minima.}$$



39. Find the absolute maximum and minimum values of f on the set D .

$$f(x, y) = 2x^3 + y^4, \quad D = \{(x, y) | x^2 + y^2 \leq 1\}$$

Solution:

$f(x, y) = 2x^3 + y^4 \Rightarrow f_x(x, y) = 6x^2$ and $f_y(x, y) = 4y^3$. And so $f_x = 0$ and $f_y = 0$ only occur when $x = y = 0$.

Hence, the only critical point inside the disk is at $x = y = 0$ where $f(0, 0) = 0$. Now on the circle $x^2 + y^2 = 1$, $y^2 = 1 - x^2$

so let $g(x) = f(x, y) = 2x^3 + (1 - x^2)^2 = x^4 + 2x^3 - 2x^2 + 1$, $-1 \leq x \leq 1$. Then $g'(x) = 4x^3 + 6x^2 - 4x = 0 \Rightarrow$

$x = 0, -2$, or $\frac{1}{2}$. $f(0, \pm 1) = g(0) = 1$, $f\left(\frac{1}{2}, \pm \frac{\sqrt{3}}{2}\right) = g\left(\frac{1}{2}\right) = \frac{13}{16}$, and $(-2, -3)$ is not in D . Checking the endpoints, we

get $f(-1, 0) = g(-1) = -2$ and $f(1, 0) = g(1) = 2$. Thus the absolute maximum and minimum of f on D are $f(1, 0) = 2$

and $f(-1, 0) = -2$.

Another method: On the boundary $x^2 + y^2 = 1$ we can write $x = \cos \theta$, $y = \sin \theta$, so $f(\cos \theta, \sin \theta) = 2 \cos^3 \theta + \sin^4 \theta$,

$0 \leq \theta \leq 2\pi$.

55. A cardboard box without a lid is to have a volume of 32,000 cm³. Find the dimensions that minimize the amount of cardboard used.

Solution:

Let the dimensions be x , y and z , then minimize $xy + 2(xz + yz)$ if $xyz = 32,000$ cm³. Then

$$f(x, y) = xy + [64,000(x + y)/xy] = xy + 64,000(x^{-1} + y^{-1}), \quad f_x = y - 64,000x^{-2}, \quad f_y = x - 64,000y^{-2}.$$

And $f_x = 0$ implies $y = 64,000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64,000$ or $x = 40$ and then $y = 40$. Now

$D(x, y) = [(2)(64,000)]^2 x^{-3} y^{-3} - 1 > 0$ for $(40, 40)$ and $f_{xx}(40, 40) > 0$ so this is indeed a minimum. Thus the

dimensions of the box are $x = y = 40$ cm, $z = 20$ cm.