12.10 Polarization

Roughly speaking, the polarization refers to the direction of the electric field in an EM wave. Given an EM plane wave propagating in a given direction (in vacuum), the electric field is constrained to the plane perpendicular to the direction of propagation, hence there are two linearly independent degrees of freedom. For example, for a wave propagating in the z-direction at frequency ω, the electric field is in general of the form ($k = \omega/c$)

In a wave guide, the electric field may not be perpendicular to the apparent direction of propagation.

$$
\begin{aligned}
\mathbf{E} &= \hat{\mathbf{x}} E_{0x} \cos(\omega t - kz + \phi_1) + \hat{\mathbf{y}} E_{0y} \cos(\omega t - kz + \phi_2) \\
&= \Re\left[\left(\hat{\mathbf{x}} E_{0x} e^{i\phi_1} + \hat{\mathbf{y}} E_{0y} e^{i\phi_2}\right) e^{i(\omega t - kz)}\right] \\
&= \Re\left[\left(\hat{\mathbf{x}} \tilde{E}_x + \hat{\mathbf{y}} \tilde{E}_y\right) e^{i(\omega t - kz)}\right] \\
&= \Re\left[\tilde{\mathbf{E}} e^{i(\omega t - kz)}\right].
\end{aligned}
\tag{12.152}
$$

The phase is included together with the polarization in this description.

We can use a 2-component column $\begin{pmatrix} \tilde{E}_x \\ \tilde{E}_y \end{pmatrix}$ to represent an EM wave if the frequency and direction of propagating are given.

Ex 12.88: Plot the electric field (12.152) along the z-axis at a fixed time t for the following possibilities: (a) $\tilde{\mathbf{E}} = \hat{\mathbf{x}} E_0$. (b) $\tilde{\mathbf{E}} = \hat{\mathbf{y}} E_0$. (c) $\tilde{\mathbf{E}} = \frac{1}{2}[\hat{\mathbf{x}} + \hat{\mathbf{y}}] E_0$. (d) $\tilde{\mathbf{E}} = \frac{1}{2}[\hat{\mathbf{x}} + i\hat{\mathbf{y}}] E_0$. (e) $\tilde{\mathbf{E}} = \frac{1}{2}[\hat{\mathbf{x}} - i\hat{\mathbf{y}}] E_0$. (f) $\tilde{\mathbf{E}} = \frac{1}{2}[i\hat{\mathbf{x}} + 2\hat{\mathbf{y}}] E_0$.

Q 12.99: What are the column representations of these waves?

The mathematics of polarizations is an example of the linear space (Hilbert space) in quantum mechanics. Superposing two plane waves of the same frequency with electric fields $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{E}}_2$ propagating in the z-direction, what we get is a plane wave of the same frequency with the electric field $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_1 + \tilde{\mathbf{E}}_2$.

In terms of $\tilde{E}_x$ and $\tilde{E}_y$, the energy density in a plane wave propagating in the z-direction is

$\langle u \rangle$ represents the time-averaged energy density for a plane wave.

$$
\begin{aligned}
& u = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} \frac{E^2}{c^2} = \epsilon_0 E^2 \\
\Rightarrow \quad & \langle u \rangle \propto |\tilde{\mathbf{E}}|^2 = \tilde{E}_x^* \tilde{E}_x + \tilde{E}_y^* \tilde{E}_y = \begin{pmatrix} \tilde{E}_x^* & \tilde{E}_y^* \end{pmatrix} \begin{pmatrix} \tilde{E}_x \\ \tilde{E}_y \end{pmatrix}.
\end{aligned}
\tag{12.153}
$$

12.10.1 Polarizers

An ideal (linear) polarizer is a filter through which EM waves of a certain polarization pass without change, and waves of the orthogonal polarization are absorbed.

Example: Sunglasses, LCD, ...

For convenience of discussion, let us consider EM waves propagating in the z-direction in all examples and exercises below.

This filter allows a wave $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to pass, but a wave $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to be blocked.

Example: If a polarizer is oriented such that an EM wave can freely pass if its electric field is in the x-direction, then the same polarizer would block an EM wave with its electric field in the y-direction.

Q 12.100: How would you make a polarizer?

Ex 12.89: What would be the result of the wave $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ after it passes through the polarizer mentioned above? How much of its energy passes through the polarizer?

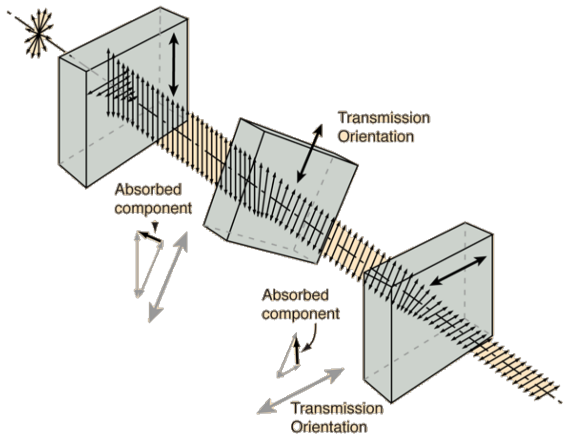
> For any linear device, for an arbitrary polarization represented by a 2-component column v, you can always find a 2×2 matrix M such that the column Mv represents the wave behind the linear device.
> Here we are making the assumption that the polarizer is a linear device, in the sense that if $v_1 \to v_1'$ and $v_2 \to v_2'$, then $v_1 + v_2 \to v_1' + v_2'$.

Ex 12.90: Consider two polarizers in series. The first one is oriented in the x-direction, followed by another oriented in the y-direction. What would happen to a wave after it passes both polarizers? Is the photon model compatible with this result?

Ex 12.91: Consider 3 polarizers in series. The first one is oriented in the x-direction, the 2nd one at 45 degrees between the x and y-directions, and the last one is in the y-direction. What is the fraction of the initial energy that passes through each polarizer? Is the photon model compatible with this result?

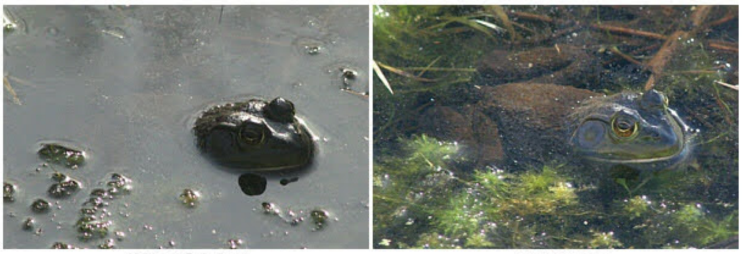
The difference between the exercises is the 3rd polarizer.

In contrast, if we replace the feature of polarization by color/frequency, with three color filters in series: blue, purple, red, we would have a quite different model to explain the results.



HW: (4-2) Consider a series of 3 polarizers. The first one is oriented in the x-direction, the 2nd one at 45 degrees between the x and y-directions, and the last one is in the x-direction again. Let an EM wave pass through all three polarizers at normal incidence. The electric field of the EM wave has $\tilde{\mathbf{E}} = \hat{\mathbf{x}}\tilde{E}_x + \hat{\mathbf{y}}\tilde{E}_y$ for given $\tilde{E}_x$ and $\tilde{E}_y$. (1) What is $\tilde{\mathbf{E}}'$ for the EM wave after passing the polarizers? (2) What is the fraction of the initial energy that passes through each polarizer?

Some animals (bees, octopuses, etc.) can *see* polarizations. This ability helps them survive. Some people claim that they can see polarizations.
Polarizers are also used in sunglasses, and glasses for 3D movies.
Photographers use filters (polarizers) for special effects.



Without Polarizer With Polarizer

12.10.2 A Little Math for Polarization

Given the frequency ω and the direction of propagation $\hat{\mathbf{z}}$, an monochromatic EM plane wave has the degrees of freedom encoded in the complex vector $\tilde{\mathbf{E}} = \tilde{E}_x\hat{\mathbf{x}} + \tilde{E}_y\hat{\mathbf{y}}$. This can be conveniently expressed as a two-component column $\begin{pmatrix} \tilde{E}_x \\ \tilde{E}_y \end{pmatrix}$

The space of $\tilde{\mathbf{E}}$ is a complex linear space of 2 dimensions. The inner product on this space can be defined via $|\tilde{\mathbf{E}}|^2 \equiv |\tilde{E}_x|^2 + |\tilde{E}_y|^2$.

We may choose a different basis of polarization for a mathematically equivalent expression $\tilde{\mathbf{E}} = \tilde{E}_{x'}\hat{\mathbf{x}}' + \tilde{E}_{y'}\hat{\mathbf{y}}'$ for a different choice of the $x - y$ axes on the same $x - y$ plane. The same EM wave would be expressed as a different two-component column $\begin{pmatrix} \tilde{E}_{x'} \\ \tilde{E}_{y'} \end{pmatrix}$.

Ex 12.92: Show that

$$\begin{pmatrix} \tilde{E}_{x'} \\ \tilde{E}_{y'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \tilde{E}_x \\ \tilde{E}_y \end{pmatrix}, \tag{12.154}$$

where θ is the angle between the x- and x'-axes.

Q 12.101: What is the relation between $(\hat{\mathbf{x}}', \hat{\mathbf{y}}')$ and $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$?

Instead of the linear polarizations $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$, we may also use the circular polarizations $(\mathbf{e}_+, \mathbf{e}_-) = \left(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}), \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - i\hat{\mathbf{y}})\right)$ as the basis.

In general,

$$\tilde{\mathbf{E}} = (\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)\begin{pmatrix} \tilde{E}_1 \\ \tilde{E}_2 \end{pmatrix}$$

Ex 12.93: What is the matrix U in the relation

$$\begin{pmatrix} \tilde{E}_+ \\ \tilde{E}_- \end{pmatrix} = U \begin{pmatrix} \tilde{E}_x \\ \tilde{E}_y \end{pmatrix}. \tag{12.155}$$

A polarizer detects whether the polarization of a wave is in a certain polarization. We shall focus on linear polarizers, whose effect on the EM wave is a linear operation. If the output is $\tilde{\mathbf{E}}^{(i)}_{out}$ when the input be $\tilde{\mathbf{E}}^{(i)}_{in}$ $(i = 1, 2)$, the output would be $a\tilde{\mathbf{E}}^{(1)}_{out} + b\tilde{\mathbf{E}}^{(2)}_{out}$ when the input is $a\tilde{\mathbf{E}}^{(1)}_{in} + b\tilde{\mathbf{E}}^{(2)}_{in}$ for any $a, b \in \mathbb{C}$. This means that we can use a matrix to represent its action when the fields are represented as columns in any given basis.

Ex 12.94: Show that a polarizer that allows the $\hat{\mathbf{x}}$-polarized light to pass is represented by the matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{12.156}$$

when we use the basis $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ for polarization.

The matrix M would look different for different choices of the basis. But if the bases are all properly normalized, the eigenvalues of M is always 1 and 0.

Ex 12.95: For a polarizer in the direction $\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}$ (i.e. a polarization in this direction passes through), in terms of the basis $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$, what is the output for the following inputs? (1) $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, (2) $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$, (3) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, (4) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Ex 12.96: How would the matrix representing a polarizer differ for a new basis if the new basis is related to the old basis via a matrix U as in eqs.(12.154) and (12.155)?

If we wish to exclude the information of the amplitude from the polarization, we would identify $\tilde{\mathbf{E}}$ with $\tilde{\mathbf{E}}' = A\tilde{\mathbf{E}}$ for any $A > 0$. It is then natural to normalize $\tilde{\mathbf{E}}$ by dividing it by $\sqrt{|\tilde{E}_x|^2 + |\tilde{E}_y|^2}$.

The crucial idea behind is the superposition principle.

In Dirac's famous book "*Principles of Quantum Mechanics*", the polarization of EM wave is used to motivate the general definition of quantum mechanical systems. A polarization is matched with the concept of a quantum state, and the polarizer matrix M with an observable (or a measurement).