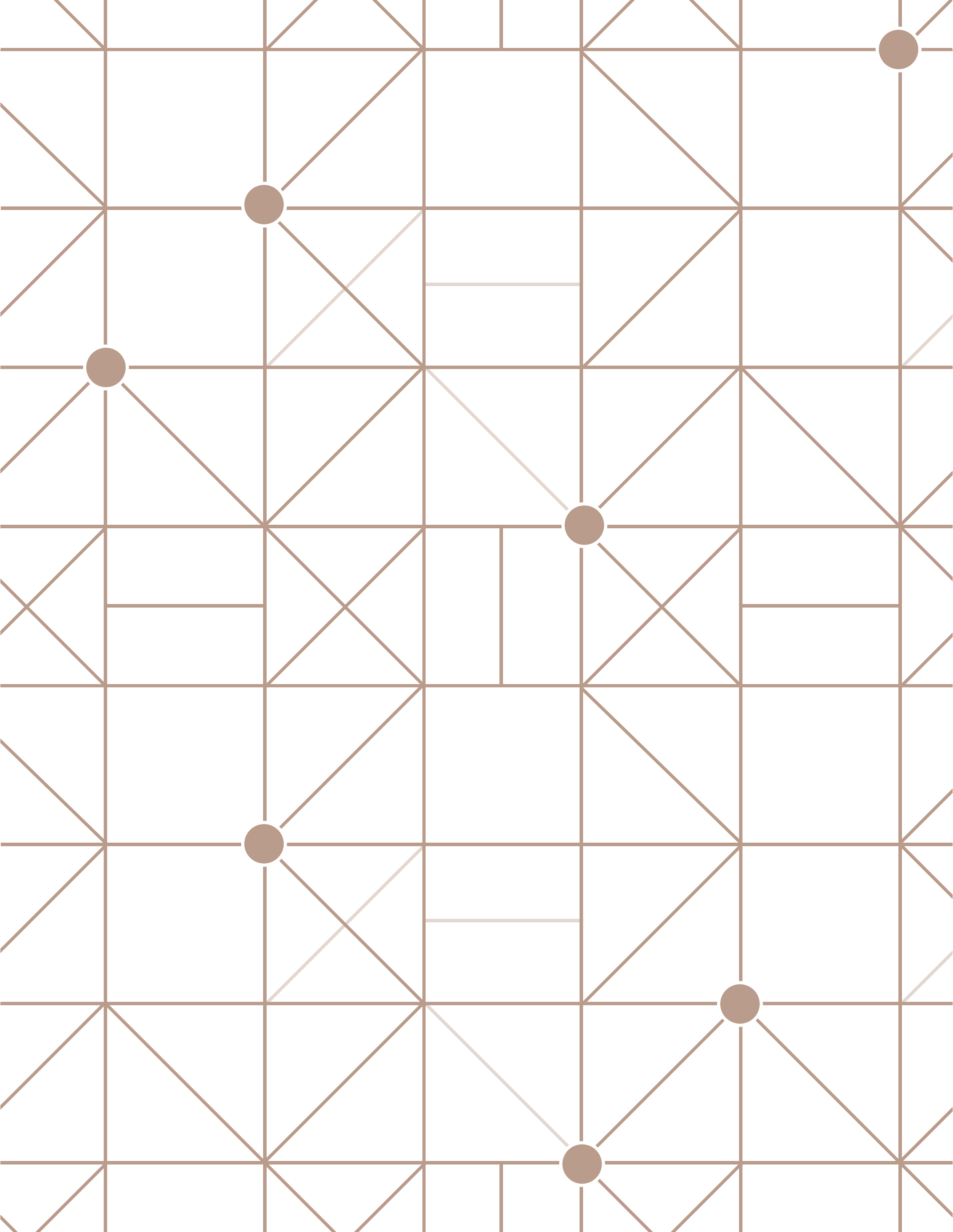


Ch.3.4 Orthogonal Bases and Gram-Schmidt

$$\begin{cases} \text{Orthogonal basis}: & q_1 \cdots q_n \\ \text{Orthogonal matrix}: & Q = [q_1 \cdots q_n] \\ \text{Gram-Schmidt}: & A \to Q \end{cases}$$

[A] Orthonormal Vectors

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{bmatrix} = I$$

- If Q is square, then

$\boxed{Q^TQ = I}$ tell us $\boxed{Q^T = Q^{-1}}$

[B] What's good for a Q?

- Q has orthonormal columns
- Project onto its column space

$$P = Q\underbrace{(Q^TQ)^{-1}}_{I}Q^T$$

$$\therefore \boxed{P = QQ^T}$$

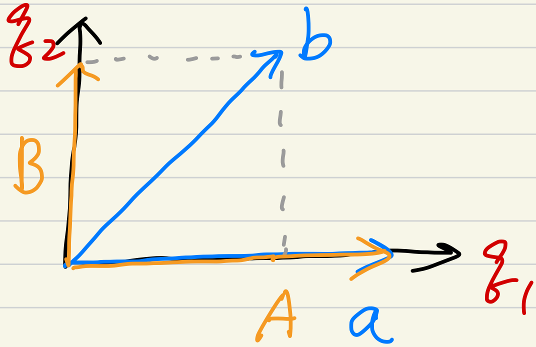
- $\boxed{A^T A \hat{x} = A^T b}$ Now A is Q.

$$\underbrace{Q^T Q}_{I} \hat{x} = Q^T b$$

$$\therefore \boxed{\hat{x} = Q^T b}$$

$$\therefore \boxed{\hat{x}_i = q_i^T b}$$

[C] Gram-Schmidt



indep. vectors a, b

$\Rightarrow$ orthogonal vectors A, B

$\Rightarrow$ orthonormal vector q_1, q_2

$$\left(q_1 = \frac{A}{\|A\|} \; ; \; q_2 = \frac{B}{\|B\|} \right)$$

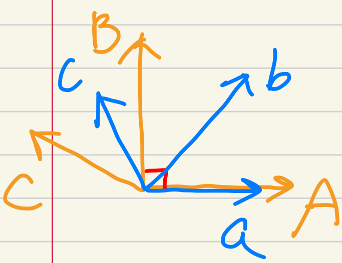
- 2 vectors A, B

$$\begin{cases} A \\ B = b - \left(\dfrac{A^T b}{A^T A}\right) A \end{cases}$$

(Proof $A \perp B$:

$$A^T B = A^T \left[b - \frac{A^T b}{A^T A} A \right] = 0$$

- 3 vectors A, B, C



$$\begin{cases} A \\ B = b - \dfrac{A^T b}{A^T A} A \\ C = c - \dfrac{A^T c}{A^T A} A - \dfrac{B^T c}{B^T B} B \end{cases}$$

$(\because C \perp A, \ C \perp B)$

(Ex) $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$$

Factorization $\boxed{A = QR}$

QR factors :

$$A = \begin{bmatrix} a & b & c \end{bmatrix}$$

$$= \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ & q_2^T b & q_2^T c \\ & & q_3^T c \end{bmatrix}$$

$$= Q\,R$$

(Ex)

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & \sqrt{2} \\ & 1/\sqrt{2} & \sqrt{2} \\ & & 1 \end{bmatrix}$$

$$= Q\,R$$