Useful Results in Calculus

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1 Differentiation & Integration

1.1 Definition

Definition 1.1. *If a limit of a function, say $\lim_{x\to\theta} f(x)$,* ***exists****, then*

$$\lim_{x\to\theta^+} f(x) = \lim_{x\to\theta^-} f(x) = f(\theta) < +\infty \tag{1.1}$$

Therefore, it can be simply denoted as $\lim_{x\to\theta} f(x)$, which implies that it exists.

Theorem 1.2. ***(L'Hôpital's rule)***

Definition 1.3. *The* ***derivative*** *of a function $f(x)$ with respect to x at $x = u$ is defined as*

$$\boxed{\frac{d}{dx}f(x)\bigg|_{x=u} \equiv \lim_{x\to u}\frac{f(x)-f(u)}{x-u} \equiv \dot{f}(u) \equiv f'(u)} \tag{1.2}$$

The ***differential*** *of a function $f(x)$ is defined as $df(x) = \dot{f}(x)dx$*

Definition 1.4. *The function is called* ***differentiable*** *if its derivative exists, that is, the limit in equation (1.2) exists.*

Proposition 1.5. *It is easy to check that the derivative is a linear transformation, i.e. for two differentiable functions f and g, we have*

$$\begin{cases} [f(x)+g(x)]' = f'(x)+g'(x) \\ [af(x)]' = af'(x) \ \ , \ \ \text{where } a \text{ is a constant} \end{cases} \tag{1.3}$$

Proposition 1.6. ***(product rule)*** *For two differentiable functions, f and g, we have the following property :*

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) \tag{1.4}$$

Proposition 1.7. ***(chain rule)*** *For two differentiable functions, f and g, we have the following property :*

$$\frac{d}{dx}\left[f\left(g(x)\right)\right] = \frac{df(g)}{dg}\cdot\frac{dg(x)}{dx} \tag{1.5}$$

1.2 Differential and Integration of some basic Function

1.2.1 Polynomial function

Definition 1.8. *A **polynomial function** $P(x)$ is defined to have the form*

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \quad , \quad \textit{with constant } a_i \textit{ for } i = 1, ..., n$$

where $n < +\infty$.

Proposition 1.9. ***(power rule)***[1] *For a polynomial function $f(x) = x^n$, where $n \in \mathbb{C}$, we have*

$$\boxed{\frac{d}{dx} f(x) = n x^{n-1}} \tag{1.6}$$

If $n = 0$, x can't be 0.

Example 1.10.

$$\frac{d}{dx}\left(1 - \frac{x}{2}\right)^{-1/2} = \left(1 - \frac{x}{2}\right)^{-3/2} \cdot \left(-\frac{1}{2}\right) = -\sqrt{\frac{2}{(1-x)^3}}$$

1.2.2 Trigonometric function

Proposition 1.11. *The derivative of trigonometric functions are*

$$\boxed{\begin{cases} \frac{d}{dx} \sin x = \cos x \\ \frac{d}{dx} \cos x = -\sin x \\ \frac{d}{dx} \tan x = \sec^2 x \\ \frac{d}{dx} \sec x = \sec x \tan x \\ \frac{d}{dx} \csc x = -\csc x \cot x \\ \frac{d}{dx} \cot x = -\csc^2 x \end{cases}} \tag{1.7}$$

Proof: We just take $\sin x$ and $\tan x$ for example.

$$\begin{aligned} \frac{d}{dx} \sin x &= \lim_{\epsilon \to 0} \frac{\sin(x+\epsilon) - \sin x}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{\sin x \cos \epsilon + \cos x \sin \epsilon - \sin x}{\epsilon} \\ &= \lim_{\epsilon \to 0} \sin x \frac{\cos \epsilon - 1}{\epsilon} + \cos x \frac{\sin \epsilon}{\epsilon} \\ &= \cos x \\ \frac{d}{dx} \tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{\cos x}{\cos x} - \frac{\sin x}{\cos^2 x} \cdot (-\sin x) \\ &= \sec^2 x \end{aligned}$$

[1] For $n \in \mathbb{R}$, one can easily check this proposition by Binomial Theorem. However, for complex number n, you should need to care about the branches. Nevertheless, we will not discuss more details here.

Question : Prove that

$$\lim_{\epsilon \to 0} \frac{\cos\epsilon - 1}{\epsilon} = 0 \ , \ \lim_{\epsilon \to 0} \frac{\sin\epsilon}{\epsilon} = 1$$

[Hint : You can use the squeeze theorem or L'Hôpital's rule to prove them. Aside from these methods, you can also use Taylor expansion to obtain these results.]

1.2.3 Exponential function

Definition 1.12. *We define the* ***(natural) exponential function*** *with power x as the form*

$$\boxed{e^x \equiv \exp(x) \equiv \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}} \tag{1.8}$$

One can verify that by this definition,

$$e^x e^y = e^{x+y} \tag{1.9}$$

meaning it obeys the rules for exponent.

Proposition 1.13. *The derivative of the exponential function with respect to x is*

$$\boxed{\frac{d}{dx} e^x = e^x} \tag{1.10}$$

Proposition 1.14. ***(Euler Formula)*** *We can connect the exponential function with the trigonometric function by the Taylor expansion (see Section 1.4).*

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \tag{1.11}$$

1.2.4 Logarithm function

Definition 1.15. *The* ***natural logarithm function*** ln *is defined to be the inverse function of the natural exponential function, i.e.*

$$\ln(e) \equiv \log_e(e) = 1 \tag{1.12}$$

Proposition 1.16. *The derivative of the natural logarithm function with respect to x is*

$$\boxed{\frac{d}{dx} \ln(x) = \frac{1}{x}} \tag{1.13}$$

Proof: $x = \exp(\ln(x))$. We differentiate both sides, then we yield $1 = \exp(\ln(x)) \cdot (\ln x)' = x(\ln x)'$.

Example 1.17.

$$\frac{d}{dx} \log_n(x) = \frac{d}{dx}$$

1.3 Product, Chain Rules

1.4 Taylor Expansion

From great mathematicians' efforts, any well defined function can be approximated by linear function in first order,

$$f(x) = f(x_0) + A(x - x_0) + \mathcal{O}((x - x_0)^2) \tag{1.14}$$

where $\mathcal{O}(x^2)$ means higher terms of x which can be omitted if x is small enough. Differentiate both side, we have :

$$f'(x) = A + \mathcal{O}(x - x_0) \tag{1.15}$$

Hence, if we set $x = x_0$, we find $A = f'(x_0)$. With rigorous argument, one can show most of functions we encounter in physics can be expanded as:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{''}(x_0)}{2!}(x - x_0)^2 + \frac{f^{'''}(x_0)}{3!}(x - x_0)^3 \cdots \tag{1.16}$$

This is known as Taylor theorem.

1.5 Integral by Part in 1D

For two functions $u(x), v(x)$, by product rule, one has:

$$\begin{aligned} \frac{d(uv)}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx} \\ \longrightarrow uv' &= u'v - (uv)' \end{aligned}$$

Integrating both side w.r.t. x from $x = a$ to $x = b$, we obtain the useful formula:

$$\int_a^b uv' \, dx = \int_a^b u'v \, dx - (uv)|_a^b \tag{1.17}$$

2 Multiple Calculus

2.1 Partial derivative

2.2 Chain Rule

2.3 Multiple Integration

2.4 Change of coordinate and Jacobian

We are exhausted and tiresome so we save the rest part just for you.♡ If you find difficulty on these basic concepts, feel free to contact us.