Chapter 16

Quantum Physics

16.1 Brief History

- 1900 (Energy in EM wave is quantized (when emitted).)
  Max Planck suggested that energy in EM waves could only be emitted in a quantized form $E = nhf$ $(n \in \mathbb{Z})$, in order to explain **black-body radiation** (1862).

- 1905 (EM waves are like particles (when absorbed).)
  Albert Einstein postulated that EM waves are composed of individual quantum particles (photons) based on Planck's hypothesis, to explain the **photoelectric effect** (discovered in 1887 by Heinrich Hertz). (Einstein explained Brownian motion and proposed Special Relativity also in 1905.)

  There is no threshold intensity, but there is a threshold frequency.

- 1909 (A single photon is still a wave.)
  Geoffrey Ingram Taylor demonstrated that the interference patterns of light appeared even when only one photon passed the slits at a time.

  Bohr did not like Einstein's particle interpretation of photons.

- 1913 (Electron orbits are quantized — Why?)
  Niels Bohr hypothesized that electrons revolve around a nucleus on spherical orbits with specific energies such that electron movements between orbits require "quantum" emissions or absorptions of energy.

  $mv^2/r = ke^2/r^2$, $2\pi r = n\lambda$.

- 1922 (EM waves are like particles (when scattered).)
  Arthur Compton found that X-ray wavelengths increased due to scattering by free electrons. The effect (**Compton scattering**) is in agreement with the calculation based on particle scattering.

  Energy-momentum conservation + quantized energy unit

- 1922 (Spin)
  Otto Stern and Walther Gerlach performed the **Stern-Gerlach experiment**, which detected discrete angular momentum of atoms through an inhomogeneous magnetic field, leading to the discovery of spins.

- 1923 (Matter waves — wave-particle duality)
  Louis de Broglie extended wave-particle duality to particles, postulating that electrons in motion are associated with waves of wavelength $\lambda = h/p$, as well as $E = hf$.

- 1924 (Bosons)
  Satyendra Nath Bose proposed a statistical model for **bosons**. Einstein generalized it to predict **Bose-Einstein condensate**.

Particles of the same kind are said to be *identical*.

> **Identical particles**
> The question is whether the system of particle #1 in state 1 and particle #2 in state 2 is exactly the same system as having particle #1 in state 2 and particle #2 in state 1. They are counted as two different states in classical statistics, but as a single state in quantum statistics.
> All particles are either bosons or fermions.

- 1924 (Fermions)
  Wolfgang Pauli proposed the **Pauli exclusion principle** for **fermions**. (Two identical fermions cannot occupy the same state.)

- 1925 (Matrix Mechanics)
  Werner Heisenberg, Max Born and Pascual Jordon developed the matrix mechanics formulation of quantum mechanics.

- 1926 (Schrödinger Equation)
  Erwin Schrödinger developed a wave equation based on de Broglie's wave postulates. His equation (16.27) gives the correct spectral lines of the hydrogen atom. He also introduced the Hamiltonian operator in Quantum Mechanics.

- 1926 (Spin-Statistics Theorem)
  Enrico Fermi proposed the **spin-statistics theorem**.

Spin-statistics theorem: Particles of integral (half-integral) spins are bosons (fermions).

- 1926 (Fermi-Dirac Statistics)
  Paul Dirac introduced the Fermi-Dirac Statistics.

- 1927 (Uncertainty Principle)
  Werner Heisenberg formulated the uncertainty principle.

- 1927 (Copenhagen interpretation — probability and wave collapse)
  Max Born developed the Copenhagen interpretation of the probabilistic nature of wave functions. The Copenhagen school proposed that a measurement leads to *wave collapse*.

The Copenhagen school thought then that the revolution is done. This annoyed Einstein.

Actually Born first proposed $|\psi(x)|dx$.

Born's rule: $|\psi(x)|^2 dx$ is the probability of finding (an observation that leads to wave collapse) the particle in the interval $(x, x + dx)$ if its wave function is $\psi(x)$.

- 1927 (Dirac equation)
  Paul Dirac stated his relativistic wave equation for electrons, which predicted the positrons (postulated in 1930).

- 1935 (EPR paradox)
  Albert Einstein, Boris Podolsky and Nathan Rosen described the EPR paradox which challenged the completeness of quantum mechanics. (They believed that there were hidden variables.)

- 1935 (Schrödinger's cat)
  Erwin Schrödinger's thought experiment illustrated what he saw as the problem of the Copenhagen interpretation.

- 1957 (Everettian interpretation)
  Hugh Everett formulated the many-worlds interpretation of quantum mechanics, which stated that every possible quantum outcome is realized in a superposition.

- 1964 (Bell's inequality)
  John Stewart Bell put forth **Bell's theorem**, which used testable inequality relations to show that no local hidden variables can ever reproduce all the results of QM. (Quantum entanglement)

The name "many-worlds" is misleading. Everett basically said that there is no need of wave collapse.

Local hidden variable theories cannot explain all experiments.

Comment:

Particle-wave duality is no longer a puzzle. Electrons and photons are just neither classical particles nor classical waves. They have to be addressed in terms of the mathematics of quantum physics. The spatial distribution can be described as waves, but with a different (probabilistic) interpretation.

There is no reason to expect that the microscopic physical world can be understood in terms of macroscopic intuitions.

16.2 Preliminary Ideas About Quantum Physics

Planck's constant

$$h \simeq 6.6 \times 10^{-34} Js. \tag{16.1}$$

This is the characteristic quantity which decides whether the quantum effect is important.

Q 16.1: What is the characteristic quantity that decides whether the special relativistic effect is important?

de Broglie's matter waves:

$$p = \frac{h}{\lambda}, \qquad E = hf. \tag{16.2}$$

It is perhaps more "covariant" to write $E = \frac{h}{T}$, where T is the period.

Example:

1. A ball of $m = 1g$ moving at $10^{-6}m/s$.
   ($\lambda \simeq 6.6 \times 10^{-25}m$.)

2. An atom of mass $m \sim 10^{-24}g$ moving at $10^{-6}m/s$.
   ($\lambda \sim 66cm$.)

3. An electron ($m \sim 10^{-30}kg$) moving at $10^{-6}m/s$.
   ($\lambda \simeq 660m$.)

4. What is the velocity of an electron for $\lambda = 10^{-10}m$?

5. What about photons?

Roughly speaking, quantum effect is more manifest for objects of tiny mass within a tiny space.

In macroscopic phenomena, typically the wavelength is too short to be observed. For a longer wavelength, we need h/p to be larger. If h is nearly 0 in the units convenient for describing the experiment, it is hard to see the wave nature of matter.

Notice that Quantum Mechanics is a general principle that should be applied to *everything*, large or small, including not only electrons and photons, but also footballs, tables, the earth and the sun.

Eq.(16.2) applies to the center-of-mass momentum and the total energy of anything.

Double-slit experiment for electrons

Interference pattern appears even if the electron is shot at the slits only one at a time. If a detector checks behind a slit, the interference pattern is changed.

The wave property of electrons does not come from the collective motion of electrons, unlike the wave property of water.

Particles or waves?

Q 16.2: What are the characteristic properties of particles?

Q 16.3: What are the characteristic properties of waves?

There are several types of discreteness. (1) The discreteness in the number of electrons. (2) The discreteness in the energies of the orbits for the electron in the hydrogen atom. (3) The discreteness in the energy of the photon in black-body radiation and photoelectric effect. This may be called the discreteness of the number of photons. ($E = hf$)

When we say that the photons behave like particles in the photoelectric effect, we are only referring to the discreteness of the energy. The photons do not really have to resemble particles in every way.

Electrons are stable, while photons can be easily absorbed or created. This distinction results in technical differences in our description of electrons and photons.

... is an ...ally?

Each electron is a wave governed by a wave equation. This probability wave exists as long as the electron still exists. The quantization of the electron's orbit around a nucleus is a result of the wave nature of the probability wave.

If a photon is represented by a wave, this wave is not always continuous in time as the photon appears and disappears. This is a new complication which is in principle shared by electrons, but only when there is enough energy for electrons to be created with positrons, or when an electron happens to meet a positron.

Strictly speaking, we need a **quantum field theory** to describe photons, or in general the creation and annihilation of particles.

Very roughly speaking, classical electromagnetism is already the quantum theory of photons, and the EM waves are the wave representation of photons, although the Maxwell equations are good approximations only when the number of photons is huge.

16.3 Examples of Quantum Systems

16.3.1 Polarization

The polarization of EM waves can be considered by itself as a quantum degree of freedom. We have discussed before how a polarizer projects the polarization of an EM wave onto a certain direction. In the mathematical description,

1. a 2-component column $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$ is used to represent the polarization in the direction of $\hat{\mathbf{n}} \equiv \hat{\mathbf{x}}a + \hat{\mathbf{y}}b$;
2. a 2×2 matrix $P = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$ is used to represent the effect of the polarizer;
3. the energy of the wave is proportional to $|\psi|^2 \equiv \psi^\dagger\psi$, so that the percentage of the energy passing through the polarizer is $|P\psi|^2/|\psi|^2$.

Check that the polarization $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ passes, and $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ is blocked.

Ex 16.1: Show that $P = \xi\xi^\dagger$, where $\xi = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$.

Ex 16.2: Show that for a generic polarization, the percentage of the energy passing through the polarizer is $\frac{|\xi^\dagger\psi|^2}{|\psi|^2}$. (Incidentally, for the sake of computing the percentage, it is convenient to apply the convention of normalizing ψ such that $|\psi|^2 = 1$.)

What is funny about the polarization is that (1) superficially there are two possible polarizations (x and y), and indeed a series of two polarizers in the x and y directions block all waves, however (2) inserting a polarizer neither in x nor y directions in between the two polarizers would allow waves to pass.

> A quantum mechanical interpretation of the polarizer:
> A polarizer can be viewed as an apparatus used to measure the polarization of a photon. For a polarizer in the x-direction, if a photon is polarized in the x-direction, the photon can pass the polarizer. If the photon is polarized in a perpendicular direction, it cannot pass the polarizer. If the photon is polarized in the $\hat{\mathbf{n}}$ direction (assuming $|\hat{\mathbf{n}}|^2 = 1$), it has the probability of $|\hat{\mathbf{n}} \cdot \hat{\mathbf{x}}|^2$ to pass the polarizer.

Conclusion: It is not appropriate to say that the polarization is either in the x or in the y direction; a polarizer at 45 degrees is not a filter that blocks both x and y polarizations half of the time.

Similar interpretation applies to all quantum degrees of freedom. This is a property of linear spaces.

16.3.2 Spin

An inhomogeneous magnetic field exerts a force on a magnetic dipole moment just like an inhomogeneous electric field exerts a force on an electric dipole moment. If $\mathbf{B} = \hat{\mathbf{z}}B(z)$ and $B'(z) > 0$, the force $\hat{\mathbf{m}} \cdot \nabla\mathbf{B}(z)$ is in the $+z$ $(-z)$ direction if the z-component of $\mathbf{m}$ is positive (negative).

Photons are spin-1 particles. Their spin is the origin of the polarization of the EM waves.

Stern-Gerlach experiment

It was found that an electron beam is divided into two deflected beams in an inhomogeneous magnetic field, implying that the electron has a magnetic dipole moment with two possible (vector) values (in any direction). This is because the electrons are spin-1/2 particles.

In general, **spin** refers to the intrinsic angular momentum of a particle. That is, even when a particle is at rest, it has an angular momentum equal to its spin. When the particle is charged, its non-zero spin also implies a non-zero magnetic dipole moment.

We can arrange the inhomogeneous magnetic field to measure the spin in any direction, and we can put electrons through a sequence of measurements in different directions.

An electron with its spin in the z direction has an equal probability to be deflected in the $+x$ and $-x$ directions. (The same statement holds for any two orthogonal directions.)

Now we try to mimic our calculation for the polarization problem to deal with the spin problem. Since the spin-up and spin-down states are mutually exclusive, we can choose to use the columns $|+z\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|-z\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to represent the spin-up state and the spin-down state. They satisfy the orthonomality condition:

Spin-up (spin-down) means $m_z > 0$ $(m_z < 0)$.

$\langle\psi|$ represents the Hermitian conjugate of $|\psi\rangle$.

$$\langle +z|+z\rangle = \langle -z|-z\rangle = 1,$$
$$\langle -z|+z\rangle = \langle +z|-z\rangle = 0, \tag{16.3}$$

A generic spin state can be expressed as $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$. When it undergoes a Stern-Gerlach experiment in the z-direction, it has a probability of $|\langle +z|\psi\rangle|^2/\langle\psi|\psi\rangle = |a|^2/|\psi|^2$ to follow the spin-up trajectory, and a probability of $|\langle -z|\psi\rangle|^2/\langle\psi|\psi\rangle = |b|^2/|\psi|^2$ to follow the spin-down trajectory.

As a convention, we shall normalize ψ so that $|\psi|^2 \equiv \langle\psi|\psi\rangle = 1$ for all columns representing spins.

Construction of the Quantum Theory of Spin-$1/2$

These conditions imply $|a|^2 = |b^2| = |a'|^2 = |b'|^2 = 1/2$, and $a^*a' + b^*b' = 0$.

If we use $|+x\rangle \equiv \begin{pmatrix} a \\ b \end{pmatrix}$ and $|-x\rangle \equiv \begin{pmatrix} a' \\ b' \end{pmatrix}$ to represent the states with the spin in the $+x$ and $-x$ directions, we must have

$$|\langle +z|+x\rangle|^2 = |\langle -z|+x\rangle|^2 = |\langle +z|-x\rangle|^2 = |\langle -z|-x\rangle|^2 = \frac{1}{2},$$
$$\langle +x|+x\rangle = \langle -x|-x\rangle = 1,$$
$$\langle -x|+x\rangle = \langle +x|-x\rangle = 0, \tag{16.4}$$

The solution of a, b, a' and b' is not unique. A simple possibility is to have $|+x\rangle \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $|-x\rangle \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Ex 16.3: We wish to define $|+y\rangle$ and $|-y\rangle$ such that they are normalized and their inner product with a state of a spin in a perpendicular direction squares to $1/2$. Show that one can use $|+y\rangle \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$ and $|-y\rangle \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$

Superposition Principle

Sending an electron beam with spin in the $|+z\rangle$ state through a Stern-Gerlach experiment in the x-direction, we have 50% of the electrons deflected in the $+x$ direction and 50% in the $-x$ direction since

$$|+z\rangle = \frac{1}{\sqrt{2}}(|+x\rangle + |-x\rangle). \tag{16.5}$$

Q 16.4: Following the experiment just mentioned,

1. If we send the electron beam deflected in the $+x$ direction (their spins are in the $|+x\rangle$ state) through a Stern-Gerlach experiment in the z-direction, what would be the percentage of electrons deflected up?

2. Alternatively, if we combine the two deflected beams (with states $|+x\rangle$ and $|-x\rangle$) into a single beam and send it through a Stern-Gerlach experiment in the z-direction, what would be the percentage of electrons deflected up?

Q 16.5: What qualifies a measurement?

16.4 How to Define a Quantum System

16.4.1 Review of Classical Physics

In classical physics, a physical system is defined by two things. The 1st thing is the **configuration space**, i.e. the set of all possible configurations of the physical system. The 2nd thing is the **equations of motion**.

Example: For the system of N particles interacting with each other via springs or Coulomb force, the phase space is

$$\{(\mathbf{r}_1, \cdots, \mathbf{r}_N, \mathbf{v}_1, \cdots, \mathbf{v}_N) \mid \mathbf{r}_i \in \mathcal{V}, \mathbf{v}_i \in \mathbb{R}^3\}, \tag{16.6}$$

and the evolution equations are

$$\mathbf{F}_i(t, \{\mathbf{r}_1(t), \cdots, \mathbf{r}_N(t)\}, \{\dot{\mathbf{r}}_1(t), \cdots, \dot{\mathbf{r}}_N(t)\}) = m_i \ddot{\mathbf{r}}_i(t) \qquad (i = 1, 2, \cdots, N), \tag{16.7}$$

where the force $\mathbf{F}_i$ is assumed to be a given function of time t and the positions and velocities of the N particles. They can also be written in the form of the Hamilton-Jacobi equations

$$\frac{d}{dt} f = \{H, f\} \tag{16.8}$$

for any function f on the phase space if the Hamiltonian H and the Poisson bracket are given.

Example: For the electromagnetic field in the absence of the electric charges and currents, the phase space is

$$\{(\mathbf{E}(\mathbf{r}), \mathbf{B}(\mathbf{r})) \mid \nabla \cdot \mathbf{E} = 0,\ \nabla \cdot \mathbf{B} = 0\}, \tag{16.9}$$

and the evolution equations are

$$\frac{\partial \mathbf{E}}{\partial t} = -\nabla \times \mathbf{B}, \tag{16.10}$$
$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu_0 \epsilon_0} \nabla \times \mathbf{E}. \tag{16.11}$$

For a generic classical system, the phase space is the space of a set of variables q_α which can take different values at different times t. The specification of all these variables at a given time completely fixes the state of the system, such that the state of the system at any later time is uniquely determined by the equations of motion.

In the Hamiltonian formulation, The equations of motion are always first order differential equations. A physical system takes a "point" or "element" in the phase space at any given time. The time evolution of the system plots a curve in the phase space.

When we try to understand a new physical system, we try to find all "observables" (anything you can measure), and relations among them via experiments. A priori, it is not guaranteed that what we can measure is sufficient to describe the system such that the time evolution of these observables are uniquely determined by their initial values.

16.4.2 Formulation of Quantum Physics

In quantum physics, a physical system is defined by the following two things.

1. **Hilbert space $\mathcal{H}$**
   It has to be a **linear space** with an **inner product**. It is interpreted as the set of states of the system, while $|\psi\rangle$ and $c|\psi\rangle$ are identified with the same physical state for all $c \neq 0$.

In classical physics, the set of states of the system is the phase space.

2. **Hamiltonian operator H**
   It is a linear operator on the Hilbert space. It defines the evolution equation as

$\hbar \equiv \frac{h}{2\pi}$ and h is called the Planck constant.

$$\boxed{i\hbar \frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle, \qquad (16.12)}$$

   where $|\psi(t)\rangle \in \mathcal{H}$ represents the physical state at time t.

When the Hamiltonian H is time-independent, eq.(16.12) is solved by

$$\boxed{|\psi(t)\rangle = e^{-\frac{i}{\hbar}H(t-t_0)}|\psi(t_0)\rangle. \qquad (16.13)}$$

Typically the Hamiltonian is assumed to be **Hermitian**.
Let us now define the phrases used above.

- **Linear Space**
  A **linear space** is a set of elements $\mathcal{V}$ endowed with the operation of superposition. That is, if $|\psi\rangle, |\phi\rangle \in \mathcal{V}$, then $a|\psi\rangle + b|\phi\rangle \in \mathcal{V}$ as well.

The assumption that the space of states is a linear space is equivalent to the **Superposition Principle** in QM.

- **Inner Product**
  An **inner product** is a map from two elements in a linear space $\mathcal{V}$ to a number.

We denote the inner product of $|\psi\rangle$ with $|\phi\rangle$ as $\langle\psi|\phi\rangle$. It is assumed to satisfy the relations

$$\langle\psi|\phi\rangle^* = \langle\phi|\psi\rangle, \tag{16.14}$$

$$\langle\psi|(a|\phi\rangle + b|\xi\rangle) = a\langle\psi|\phi\rangle + b\langle\psi|\xi\rangle \quad (\forall a, b \in \mathbb{C}), \tag{16.15}$$

where $*$ denotes the complex conjugation of a number.

- **Linear Operator**
  For a given linear space $\mathcal{V}$, a **linear operator** M is a linear map from $\mathcal{V}$ to $\mathcal{V}$. That is, for any element $|\psi\rangle \in \mathcal{V}$, $M|\psi\rangle$ is also an element in $\mathcal{V}$, and

$$M(a|\psi\rangle + b|\phi\rangle) = a(M|\psi\rangle) + b(M|\phi\rangle). \tag{16.16}$$

- **Hermiticity**
  A linear operator M is **Hermitian** if

$$\langle\psi|M|\phi\rangle^* = \langle\phi|M|\psi\rangle. \tag{16.17}$$

Having learned how to define mathematically a quantum system, we still do not know how to make physical sense out of it. We need to learn how to use this mathematical formulation to describe measurements.

Measurement

1. Every physical quantity q corresponds to a linear operator $\hat{q}$ (which is called an **observable**) on the Hilbert space $\mathcal{H}$.

2. The eigenvalues $\{\lambda_A\}$ of the observable $\hat{q}$ are the possible outcomes of the measurement of this physical quantity q, and an eigenvector $|\lambda_A\rangle$ of $\hat{q}$ is a state which have the well-defined value λ_A for q, i.e. $\hat{q}|\lambda_A\rangle = \lambda_A|\lambda_A\rangle$.

3. When a measurement of q is done and the value of q is found to be λ_A, the state of the quantum system is changed to the eigenstate $|\lambda_A\rangle$.

4. The expectation value of q (after repeating the measurement on a given state $|\psi\rangle$ many times) is

$$\frac{\langle\psi|\hat{q}|\psi\rangle}{\langle\psi|\psi\rangle}. \tag{16.18}$$

The physical quantity corresponding to the Hamiltonian H is the energy E.

This is called "wave collapse". (More about this later.)

The factor $\langle\psi|\psi\rangle$ is obviously necessary as $|\psi\rangle$ and $c|\psi\rangle$ are identified with the same state.

The last statement about the expectation of a physical quantity q is equivalent to the following statement: The probability of finding $q = \lambda_A$ in a measurement of a system in the state $|\psi\rangle$ is

$$\boxed{\mathcal{P}(|\psi\rangle \to |\lambda_A\rangle) = \frac{|\langle\lambda_A|\psi\rangle|^2}{\langle\lambda_A|\lambda_A\rangle\langle\psi|\psi\rangle}.} \tag{16.19}$$

This is called **Born's rule**.

This can be easily understood as follows. The probability of seeing an outcome $q = \lambda_A$ for a given state $|\psi\rangle$ equals the expectation value of the observable $|\lambda_A\rangle\langle\lambda_A|$, assuming that $|\lambda_A\rangle$ is normalized, i.e. $\langle\lambda_A|\lambda_A\rangle = 1$.

An example is the matrix P for a polarizer.

Check that the following information can be used to define the quantum physics for a physical system. Define the Hilbert space as the space of columns with N complex numbers

Ex 16.4: How to use this formulation to describe a specific physical system is another issue.

$$\mathcal{H} = \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} \mid c_i \in \mathbb{C} \right\}. \tag{16.20}$$

Define the Hamiltonian to be any given $N \times N$ Hermitian matrix

$$H^*_{ij} = H_{ji}. \tag{16.21}$$

Ex 16.5: The Hamiltonian of a system is $H \equiv \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}$. Assume that $a \neq b$ and answer the following questions.

1. What is the Hilbert space?
2. What is the state that has the energy equal to a?
3. What is the expectation value of the operator $A \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for the state in the answer to the previous question?
4. What are the eigenvalues of the operator A?
5. If the state $|\lambda\rangle$ is one of the eigenstates of A at $t = 0$, what will be the state at $t = T$?
6. What are the states that has the expectation value of the energy equal to $(a + b)/2$.
7. Let $B \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $C \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $E \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Can $\Delta B \Delta C = 0$ for any state? Can $\Delta A \Delta E = 0$ for any state?

Ex 16.6: Check that the following information can be used to define the quantum physics for a certain physical system. Define the Hilbert space as the space of smooth functions on $\mathbb{R}$:

"Normalizable" means that $\int |f(x)|^2 dx$ is finite.

$$\mathcal{H} = \{ f(x) \mid f(x) = \text{smooth, normalizable complex function on } \mathbb{R} \}. \tag{16.22}$$

Define the inner product of $f(x)$ and $g(x)$ as

$$\langle f(x)|g(x)\rangle = \int_{-\infty}^{\infty} dx\ f^*(x)g(x). \tag{16.23}$$

Define the Hamiltonian to be

This comes from $H = \frac{p^2}{2m} + V(x)$ and $p = -i\hbar\frac{\partial}{\partial x}$.

$$H \equiv -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x). \tag{16.24}$$

where $V(x)$ is a real smooth finite function of x.

Solution:

The Hilbert space is obviously a linear space since the superposition of two smooth normalizable functions is a smooth normalizable function.

If $f(x)$ is normalizable, we must have $f(x) \to 0$ as $x \to \pm\infty$.

Now we have to check the Hermiticity of the Hamiltonian. We first check the Hermiticity of the first term in the Hamiltonian:

$$\begin{aligned}
\langle f|\frac{\partial^2}{\partial x^2}|g\rangle^* &= \int dx\ f(x)(g''(x))^* \\
&= \int dx\ \{\frac{d}{dx}[(g'(x))^* f(x)] - (g'(x))^* f'(x)\} \\
&= \int dx\ \{-\frac{d}{dx}[g^*(x)f'(x)] + g^*(x)f''(x)\} \\
&= \langle g|\frac{\partial^2}{\partial x^2}|f\rangle.
\end{aligned} \tag{16.25}$$

It is straightforward to prove that the 2nd term in the Hamiltonian is Hermitian, and that the sum of two Hermitian operators is also Hermitian.

The evolution equation for the exercise above is the Schrödinger equation in 1D:

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(t,x). \tag{16.26}$$

The Schrödinger equation in 3D is

How did Schrödinger came up with this equation?

$$\boxed{i\hbar\frac{\partial}{\partial t}\psi(t,\mathbf{r}) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right]\psi(t,\mathbf{r}),} \tag{16.27}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

The expectation values of the coordinate x and the momentum p_x are

$$\langle\psi(t)|\hat{x}|\psi(t)\rangle = \int d^3\mathbf{r}\ \psi^*(t,\mathbf{r})x\psi(t,\mathbf{r}), \tag{16.28}$$

$$\langle\psi(t)|\hat{p}_x|\psi(t)\rangle = \int dx\ \psi^*(t,\mathbf{r})\left[-i\hbar\frac{\partial}{\partial x}\right]\psi(t,\mathbf{r}). \tag{16.29}$$

For the wave function $\psi(t,x)$ of a particle in 1D, the probability of finding the particle in the infinitesimal interval $(x, x+dx)$ is $|\psi(t,x)|^2 dx$ at t.

What is impossible in classical physics is sometimes possible in quantum mechanics.

Tunneling Effect

Classically, a particle of energy E can never come out of a potential well with its highest potential energy at $V_0 > E$. In quantum mechanics, typically, the solution of the Schrödinger equation has non-zero values $\psi(x)$ outside the potential barrier. This means that, quantum mechanically, it is possible to find the particle outside the potential well.

16.5 Uncertainty Relation

Heisenberg proposed the uncertainty relation

$$\boxed{\Delta x \Delta p_x \geq \frac{\hbar}{2}. \qquad (16.30)}$$

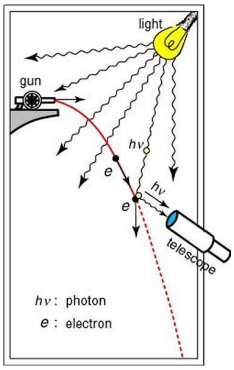
The mathematical origin of this equation is the commutation relation

$$[\hat{p}_x, \hat{x}] = -i\hbar. \qquad (16.31)$$

Equivalently, the mathematical origin is the property of Fourier transform that if $f(x)$ has a narrow profile, its Fourier transform $\tilde{f}(k)$ has a broad profile, and vice versa.

This widely-spread thought experiment is misleading.

The thought experiment of observing the electron via a photon (see figure) that is often used as an argument for the uncertainty principle has actually nothing to do with the uncertainty principle of the electron. A common misconception about this thought experiment is to imagine that the uncertainty about the electron is originated from the wave nature of the photon. In fact, the electron has its own intrinsic uncertainty relation which is independent of how it is measured or observed. The intrinsic uncertainty about the electron and the uncertainty originated from the uncertainty of the photon through the electron-photon interaction should "add up" to the final uncertainty about the electron.



It is therefore incorrect to say that the meaning of the uncertainty relation is that an observation on a quantum state changes that state. The proper way to understand the uncertainty relation is that the values of x and p_x cannot be simultaneously well defined.

> Katz: "Uncertainty is inherent. Our inability precisely to know both the position and the momentum of a (free) particle is an example of Heisenberg's uncertainty principle. This uncertainty has nothing to do with the human inability to design an experiment; it is fundamental to nature."
> Benson: "This inability has nothing to do with experimental skill or equipment; it is a fundamental restriction imposed on us by nature." (However, Benson quoted the misleading thought experiment as a derivation of the uncertainty principle.)

Via Special Relativity, we also expect that $\Delta t \Delta E \geq \hbar/2$, which is also supported by the identification of the Hamiltonian and $i\hbar\frac{\partial}{\partial t}$ in the Schrödinger equation.

An implication of the uncertainty relation is that we need higher energies to probe physics at shorter length scales.

This is why the study of fundamental physics is call high-energy physics.

The commutation relation $[x, p_x] = i\hbar$ is merely one of many other commutation relations in quantum mechanics. There are thus many other uncertainty relations for other physical quantities.

16.6 Composite Systems And Entanglement

Given two physical systems with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and Hamiltonians H_1 and H_2, the composite system has a Hilbert space that is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and a Hamiltonian that is $H = H_1 \otimes I + I \otimes H_2 + H_{int}$, where H_{int} is an operator on $\mathcal{H}$ that represents the interaction, assuming that the interaction is not so strong as to change the nature of the two systems.

The **tensor product** $\otimes$ of two linear spaces is defined as follows. Let $\{e_i\}$ and $\{e'_a\}$ be the bases of two linear spaces $\mathcal{V}$ and $\mathcal{V}'$. The tensor product $\mathcal{V} \otimes \mathcal{V}'$ is then defined as the linear space with the basis $\{\hat{e}_{ia} \equiv e_i \otimes e'_a\}$.

For example, the spin of particle #1 and the spin of particle #2 can be collectively viewed as a composite system. There are 4 states in this system

$$|+z\rangle \otimes |+z\rangle, \quad |+z\rangle \otimes |-z\rangle, \quad |-z\rangle \otimes |+z\rangle, \quad |-z\rangle \otimes |-z\rangle. \tag{16.32}$$

An arbitrary state in this system is

$$|\psi\rangle = a|+z\rangle \otimes |+z\rangle + b|+z\rangle \otimes |-z\rangle + c|-z\rangle \otimes |+z\rangle + d|-z\rangle \otimes |-z\rangle \tag{16.33}$$

for $a, b, c, d \in \mathbb{C}$.

Ex 16.7: Show that any state in this composite system can also be written as

$$|\psi\rangle = A|+x\rangle \otimes |+x\rangle + B|+x\rangle \otimes |-x\rangle + C|-x\rangle \otimes |+x\rangle + D|-x\rangle \otimes |-x\rangle \tag{16.34}$$

for $A, B, C, D \in \mathbb{C}$.

For a composite system with the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, a state is **entangled** if it cannot be written in the form $|\psi_1\rangle \otimes |\psi_2\rangle$ for any $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$.

Ex 16.8: Which of the following is (are) entangled? (a) $|+z\rangle \otimes |-z\rangle - 2|+z\rangle \otimes |+z\rangle$. (b) $|+z\rangle \otimes |-z\rangle - 2|+z\rangle \otimes |+z\rangle - 3|-z\rangle \otimes |-z\rangle + 6|-z\rangle \otimes |+z\rangle$. (c) $|+z\rangle \otimes |-z\rangle + 2|+z\rangle \otimes |+z\rangle - 3|-z\rangle \otimes |-z\rangle + 6|-z\rangle \otimes |+z\rangle$. (d) $|+z\rangle \otimes |+z\rangle + |-z\rangle \otimes |-z\rangle$.

Simple examples of entangled states include

$$|\pm\rangle \equiv |+z\rangle \otimes |-z\rangle \pm |-z\rangle \otimes |+z\rangle. \tag{16.35}$$

These states are interesting because of the correlation between the two spins. Imagine two particles separated by a great distance. If we measure the spin of the particle #1 and find the spin-up state, we know instantly that the particle #2 is in the spin-down state, no matter how far it is from particle #1.

16.7 Interpretation of QM

There is obviously a conflict between QM and the classical world around us.

Consider the following problem. A particle can be in states $|1\rangle$ and $|2\rangle$. A machine that we can use to measure the state of the particle should work like this: When the particle in the state $|1\rangle$ into the machine in its initial state $|M0\rangle$, the composite system is in the state $|1\rangle \otimes |M0\rangle$. The assumption that the machine can detect the state of the particle means that this state evolves to another state $|1'\rangle \otimes |M1\rangle$, where $|M1\rangle$ involves, say, a screen on the machine showing "1". Similarly, the state $|2\rangle \otimes |M0\rangle$ evolves to the state $|2'\rangle \otimes |M2\rangle$, with "2" showing on the screen.

If we put a particle in the state $|1\rangle + |2\rangle$ in the machine, superposition principle implies that we get the final state $|1'\rangle \otimes |M1\rangle + |2'\rangle \otimes |M2\rangle$ for the composite system. The problem is that we have never seen the superposition of "1" and "2" on the screen of a machine except when the machine is broken.

If QM is more fundamental than classical mechanics, the predictions of QM should apply to the macroscopic world as well. Why have we not observed superposition in our everyday lives?

16.7.1 EPR paradox

Prepare a system of two particles in the state $|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle$. Separate the particles by a large distance, and then measure the state of one of the two particles. The state of the other particle light-years away is suddenly instantaneously fixed by this measurement.

This kind of entanglement can be used for *quantum teleportation*, which allows us to transfer message secretly.

Q 16.6: Does it mean that the influence of the measurement of the particle #1 propagates at a superluminal speed to the particle #2? Is causality violated?

Q 16.7: In Special Relativity, if you can travel faster than light, you can go back to the past. Can you design a way utilizing this property of entanglement between two particles to win a lottery by sending the winning numbers from the future to the past?

If we think of the change of state from $|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle$ to $|+\rangle \otimes |-\rangle$ or $|-\rangle \otimes |+\rangle$, it looks like the state of the 2nd particle is changed by the measurement on the 1st particle. But this simply tells us that the notion of "wave collapse" is strictly speaking not valid in general. We will explain more below.

16.7.2 Prof. X's Equality

Q 16.8: We use probabilistic language in classical physics, too. Is it possible that the uncertainty in QM is merely a reflection of uncontrolled microscopic classical degrees of freedom (hidden variables)?

The story below tells us that QM cannot be explained by hidden variables.

This is an enhanced version of Bell's Inequality.

Prof. X has a machine that produces 3 objects at a time. He sends one object to each of his 3 students. Each object has two types of features A and B to be measured, and each feature can only be either one of two possibilities (e.g. A = black/white, and B = hard/soft). Each student can choose at will which feature to measure independently for each object. However, because the objects are extremely fragile, when A is measured, the object is broken so that B can no longer be measured, and vice versa.

Prof. X told them that the features of the 3 objects in a sample are statistically correlated in a certain way, and their task is to find the correlation pattern. After recording their measurement results of a huge number of samples (each sample contains 3 objects with correlated features), the three students meet and compare their results.

They find that, by coding the results in numbers ± 1 for both A and B, whenever the measurements are $A_1B_2B_3$, $B_1A_2B_3$ or $B_1B_2A_3$, the product of the three numbers is always $+1$.

Since $A_i^2 = B_i^2 = 1$, this implies that

$$1 = (A_1B_2B_3)(B_1A_2B_3)(B_1B_2A_3) = A_1A_2A_3. \tag{16.36}$$

The hidden assumption in this mathematical derivation is that the feature not measured has a well defined value.

However, to their great surprise, they found that whenever they all measured feature A, the product $A_1A_2A_3$ is in fact always -1!

This situation is impossible in classical physics, but possible in quantum physics.

Ex 16.9: Consider the state $|\psi\rangle \equiv |+\rangle|+\rangle|+\rangle - |-\rangle|-\rangle|-\rangle$, measurements corresponding to the operators $\sigma_x \equiv |+\rangle\langle -|+|-\rangle\langle +|$ and $\sigma_y \equiv i|+\rangle\langle -|-i|-\rangle\langle +|$ on each particle. Check that

Thus we can identify $A = \sigma_x$ and $B = \sigma_y$.

$$\sigma_x\sigma_y\sigma_y|\psi\rangle = |\psi\rangle, \tag{16.37}$$
$$\sigma_y\sigma_x\sigma_y|\psi\rangle = |\psi\rangle, \tag{16.38}$$
$$\sigma_y\sigma_y\sigma_x|\psi\rangle = |\psi\rangle, \tag{16.39}$$
$$\sigma_x\sigma_x\sigma_x|\psi\rangle = -|\psi\rangle. \tag{16.40}$$

This example tells us that, before a measurement, a physical property is not only unknown, but it is not even defined. Strictly speaking, you cannot say, e.g. the spin of an electron is either up or down. Before a measurement, it could be neither up nor down.

Recall the proof that the smallest real number is 1.

16.7.3 Copenhagen Interpretation

The Copenhagen interpretation of QM states that

1. a measurement leads to a "wave collapse", that is why macroscopic world looks classical;
2. a measurement is done by a "classical" apparatus, that is why the interactions among quantum particles do not lead to wave collapse.

These ideas are conceptually problematic.

1. After the advent of Special Relativity, we cannot imagine that the wave collapse happen instantaneously, so it has to be a dynamical process that takes time. How does a wave collapse happen? Can we carry out experiments to observe the dynamical process of wave collapse? Is it something that cannot be understood?
2. How can a classical apparatus be distinguished from a large number of interacting quantum particles?

Wave collapse has to introduce deviation from the superposition principle. There should be equations to decide how and when the superposition principle is violated.

Experimentally, there has been no evidence of any deviation from the superposition principle, even when the number of particles is $\sim 10^5$, which is as many as there are in a nano-apparatus.

In addition, with the decoherence process explained below, there is no need of the "wave collapse" in order to explain why everything looks classical.

16.7.4 Decoherence

The superposition of states is manifest only if there is *coherence*. Recall the double-slit experiment. Let $|1\rangle$ be the particle state to pass through the 1st slit, and $|2\rangle$ the state for the 2nd slit. The interference pattern of a state like $|\psi\rangle \equiv |1\rangle + |2\rangle$ tells us about the wave nature of the particle.

Recall that Newton thought light was particles because it was hard to generate a coherent light source.

After passing through the slits, the particles propagate in space until they hit the screen. Denote the new state just before hitting the screen as $|\psi'\rangle \equiv |1'\rangle + |2'\rangle$, and the state with the particle caught at a certain point q on the screen as $|q\rangle$. The probability of finding the particle at q is thus proportional to

$$\langle\psi'|q\rangle\langle q|\psi'\rangle = |\langle q|1'\rangle|^2 + |\langle q|2'\rangle|^2 + 2\Re(\langle 1'|q\rangle\langle q|2'\rangle). \tag{16.41}$$

This differs from the classical result without interference by

$$2\Re(\langle 1'|q\rangle\langle q|2'\rangle). \tag{16.42}$$

As long as this term is non-zero for some $|q\rangle$, the wave nature (superposition) can be observed.

This happens, for example, when there is a detector behind slit 1 to check if a particle passes through it.

Now consider the case when the state of the particle is entangled with the environment, so that we need to consider the states of the environment at the same time. Hence we define $|\psi\rangle \equiv |1\rangle \otimes |E_1\rangle + |2\rangle \otimes |E_2\rangle$, etc, where $|E_1\rangle, |E_2\rangle$ specify the states of all the degrees of freedom in the environment interacting with the particles in the double-slit experiment. Then the analogue of eq.(16.41) is

$$\langle\psi'|(|q\rangle\langle q| \otimes \mathbb{I})|\psi'\rangle = |\langle q|1'\rangle|^2 + |\langle q|2'\rangle|^2 + 2\Re(\langle 1'|q\rangle\langle q|2'\rangle\langle E_1'|E_2'\rangle). \tag{16.43}$$

The analogue of eq.(16.42) is thus

$$2\Re(\langle 1'|q\rangle\langle q|2'\rangle\langle E_1'|E_2'\rangle). \tag{16.44}$$

Notice that $\langle E_1'|E_2'\rangle$ is almost always 0 in practice because the environment typically consists of a huge number of degrees of freedom.

> If each degree of freedom in $|E_1'\rangle$ and $|E_2'\rangle$ has an inner product of absolute value 0.99999, then 10^{23} such degrees of freedom in $|E_i'\rangle$ leads to $|\langle E_1'|E_2'\rangle| \sim 10^{-10^{17}}$. That is, the difference between the quantum and classical calculation is practically 0.

Decoherence occurs naturally as any change at a place typically leads to many changes everywhere else. In fact, avoiding decoherence is one of the major challenges in making quantum computers work. Due to decoherence, quantum effect is washed away and only classical effects remains. This is the major reason why our macroscopic world looks classical even though the microscopic world is quantum.

16.7.5 Everettian Interpretation

The Everettian interpretation — also (inappropriately) called many-worlds interpretation — of quantum mechanics simply states that we do not need anything more than the usual QM calculation to explain why the macroscopic world looks classical. (All we need to do is to imagine that the abstract mathematical formulation of QM is the fundamental reality, while our perception only reflects certain distorted partial aspects of the reality.) The seemingly "classical" nature of phenomena is a result of decoherence. As long as we include everything in the calculation, from the quantum system under investigation to every particle in the apparatus and the environment, we would come to the conclusion that the macroscopic world looks classical.

Everett originally called his approach the "Correlation Interpretation", where "correlation" refers to quantum entanglement. The phrase "many-worlds" is due to De Witt.

The phrase "many worlds" often give people the wrong idea that the universe branches out into several (non-communicating) parallel universes every time a measurement is done. That would be an unfalsifiable theory worse than the Copenhagen interpretation. It is not what we refer to as the Everettian interpretation (although others may do so).

In the following, we apply the Everettian interpretation to the paradox of "Schrödinger's cat" as an example, and explain how a single universe obeying the superposition principle may appear to be classical.

16.7.6 Schrödinger's Cat

A cat in a box is dead or alive depending on whether an atom decays to trigger the release of a poisonous gas.

Q 16.9: Is the cat in a superposition state $|O\rangle + |X\rangle$ before the box is opened for us to "measure" whether it is still alive?

Let $|O\rangle$ and $|X\rangle$ represent the states of the cat being alive and dead.

The cat interacts with everything inside the box, which is collectively represented by $|E\rangle$. The composite system evolves over time as

$$\begin{aligned}(|O\rangle + |X\rangle) \otimes |E\rangle &= |O\rangle \otimes |E\rangle + |X\rangle \otimes |E\rangle \\ &\to |O'\rangle \otimes |E_1\rangle + |X'\rangle \otimes |E_2\rangle. \end{aligned} \tag{16.45}$$

$|E_1\rangle$ and $|E_2\rangle$ include the states of a huge number of particles that are correlated with the difference between a live cat and a dead cat. For instance, the molecules of the poison, the box, the air, and the cat are all in different states correlated with whether the cat is alive or dead. The decoherence between $|O\rangle$ and $|X\rangle$ due to their entanglement with the environment removes any signal about their superposition.

16.7.7 Sense of Certainty

Ex 16.10: There is a state $|\psi\rangle \equiv |q = +\rangle + |q = -\rangle$ going through a measurement of $q = \pm$. After the measurement, the universe is in a state of superposition

$$|\Psi\rangle \equiv |q = +\rangle \otimes |E_+\rangle + |q = -\rangle \otimes |E_-\rangle, \tag{16.46}$$

where $|E_\pm\rangle$ is the state of the environment, including, e.g. the display of the apparatus, the digital record of the measurement left on a hard drive, the image of the display reflected on a nearby mirror, the brain of the observer, etc.. Define a physical quantity s of your brain so that it equals $+1$ when you feel that the outcome of the measurement is definite (either $q = +$ or $q = -$, as long as you feel very sure about the result), and it equals -1 when you are uncertain of the result. What is the expectation value of $\hat{s}$ for the state $|\Psi\rangle$?

The expectation value of $\hat{s}$ for a state indicates how certain one is about something.

Solution:

The definition of s implies that, whenever you are in your right mind,

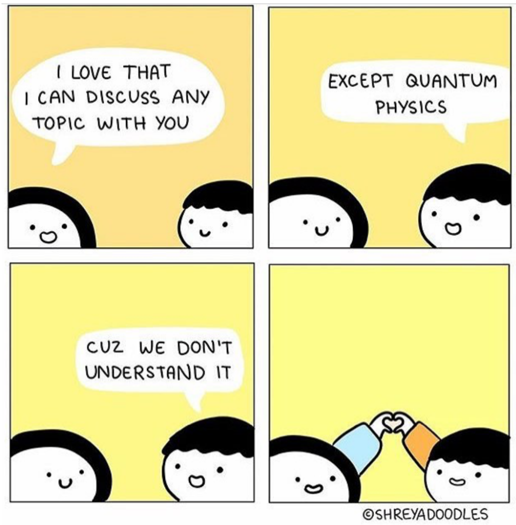
$$\hat{s}|a = +\rangle \otimes |E_+\rangle = (+1)|a = +\rangle \otimes |E_+\rangle, \tag{16.47}$$

$$\hat{s}|a = -\rangle \otimes |E_-\rangle = (+1)|a = -\rangle \otimes |E_-\rangle. \tag{16.48}$$

As a result, $\frac{\langle\Psi|\hat{s}|\Psi\rangle}{\langle\Psi|\Psi\rangle} = +1$ for any state $|\Psi\rangle$. That is, you always feel that you are sure about q taking a definite value even when it is a superposition of $q = +$ and $q = -$ states.

Q 16.10: How much can we really be sure about ever having been sure about anything?


If I never open it, it might be what I want...
Schrödinger's Lunchbox.


I LOVE THAT I CAN DISCUSS ANY TOPIC WITH YOU
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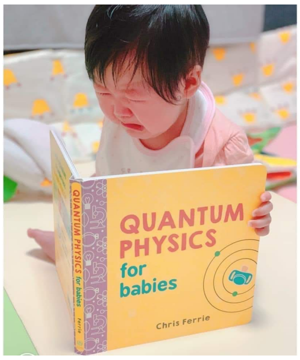
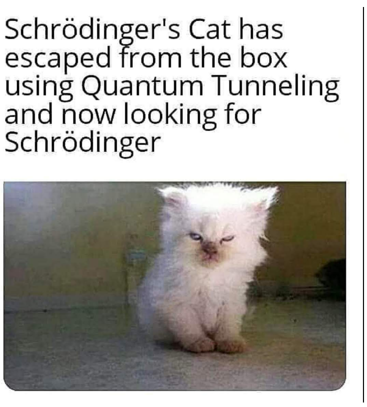
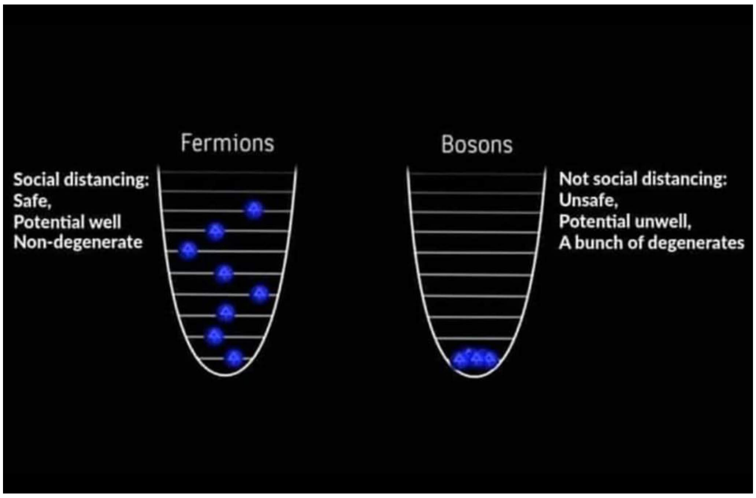
SCHRÖDINGER PLATES
THEY'RE BOTH BROKEN AND NOT BROKEN UNTIL YOU OPEN THE DOOR


QUANTUM PHYSICS
for babies
Chris Ferrie


Schrödinger's Cat has escaped from the box using Quantum Tunneling and now looking for Schrödinger


Fermions
Bosons
Social distancing:
Safe,
Potential well
Non-degenerate
Not social distancing:
Unsafe,
Potential unwell,
A bunch of degenerates