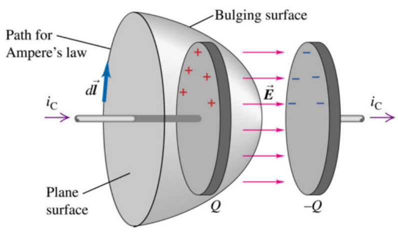
12.8 Maxwell's Equations

12.8.1 Ampere's Law Revisited

Maxwell considered the thought experiment depicted in the figure below.



The requirement of consistency implies that the current on the wire and the "displacement current" $\epsilon_0 \frac{d}{dt}\Phi_E$ should appear together. Hence Ampere's law should be modified as

$$\boxed{\oint_{\mathcal{C}} \mathbf{B} \cdot d\ell = \mu_0 I_{enc} + \mu_0 \epsilon_0 \frac{d}{dt}\Phi_E, \qquad (12.99)}$$

For a given closed loop $\mathcal{C}$, there are infinitely many choices of the surface for which $\mathcal{C}$ is its boundary.

where the electric flux is defined by

$$\Phi_E \equiv \int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a}. \qquad (12.100)$$

Ex 12.81: Consider the spherically symmetric current distribution

$$\mathbf{J}(\mathbf{r}) = \frac{A}{r^2}\hat{\mathbf{r}} \qquad (12.101)$$

This is one of the rare examples in which neither current nor time-dependent electric field creates any magnetic field.

for a given constant A. Charge conservation implies that there is a charge

$$Q(t) = Q(0) - 4\pi A t \qquad (12.102)$$

at the origin. Find the EM field.

Solution:

The spherical symmetry only allows electric and magnetic fields in the radial direction, but a current in the radial direction cannot create magnetic field in the radial direction, so we set $\mathbf{B} = 0$. The electric field is fixed by the (unmodified) Gauss law

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q(t)}{r^2} \qquad (12.103)$$

as in electrostatics, despite the time dependence of the charge. One can check that the modified Ampere's law as well as all other Maxwell's equations are satisfied.

In general, time-dependent charge/current generates time-dependent electric/magnetic fields, which creates electromagnetic waves to be discussed below in Sec.12.9.

12.8.2 Formal Properties of Maxwell's Equations

We have now the complete set of physical laws of classical electromagnetism:

1. **Maxwell's Equations**:

$$\oint_{\mathcal{S}=\partial\mathcal{V}} \mathbf{E}(t,\mathbf{r})\cdot d\mathbf{a} = \frac{Q_{\mathcal{V}}(t)}{\epsilon_0}, \tag{12.104}$$

$$\oint_{\mathcal{S}=\partial\mathcal{V}} \mathbf{B}(t,\mathbf{r})\cdot d\mathbf{a} = 0, \tag{12.105}$$

$$\oint_{\mathcal{C}=\partial\mathcal{S}} \mathbf{E}(t,\mathbf{r})\cdot d\ell = -\frac{d}{dt}\Phi^B_{\mathcal{S}}(t), \tag{12.106}$$

$$\oint_{\mathcal{C}=\partial\mathcal{S}} \mathbf{B}(t,\mathbf{r})\cdot d\ell = \mu_0 I_{\mathcal{S}}(t) + \mu_0\epsilon_0\frac{d}{dt}\Phi^E_{\mathcal{S}}(t), \tag{12.107}$$

The symbol ∂ here means "the boundary of".

Note that the surface $\mathcal{S}$ is typically closed in the first two equations, but open in the latter two equations.

2. **Lorentz force law**:

$$\mathbf{F}(t) = q\left[\mathbf{E}(t,\mathbf{r}(t)) + \mathbf{v}(t)\times\mathbf{B}(t,\mathbf{r}(t))\right], \tag{12.108}$$

where $\mathbf{r}(t)$ is the position of the charge q at time t, and

$$Q_{\mathcal{V}}(t) \equiv \int_{\mathcal{V}} \rho(t,\mathbf{r})d^3\mathbf{r}, \tag{12.109}$$

$$I_{\mathcal{S}}(t) \equiv \int_{\mathcal{S}} \mathbf{J}(t,\mathbf{r})\cdot d\mathbf{a}, \tag{12.110}$$

$$\Phi^B_{\mathcal{S}}(t) \equiv \int_{\mathcal{S}} \mathbf{B}(t,\mathbf{r})\cdot d\mathbf{a}, \tag{12.111}$$

$$\Phi^E_{\mathcal{S}}(t) \equiv \int_{\mathcal{S}} \mathbf{E}(t,\mathbf{r})\cdot d\mathbf{a}. \tag{12.112}$$

These 5 equations (Maxwell + Lorentz) are in principle all the physical laws we need for classical electromagnetism.

The conservation of charges can be derived from Maxwell's equations.

Physical interpretation of the Maxwell equations:

- Gauss' law: Electric charges generate electric fields. (Electric field lines end on charges, although they can also be closed.)
- (A law without a name:) Magnetic field lines are always closed (There are no endpoints).
- Faraday's law: Changing magnetic fields generate electric fields.
- Ampere's law: Currents and changing electric fields generate magnetic fields.

Maxwell's equations can also be equivalently expressed as differential equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{12.113}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{12.114}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{12.115}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{12.116}$$

From these equations (Maxwell + Lorentz), one can deduce the superposition principle, that is, if $(\mathbf{E}_1, \mathbf{B}_1, \rho_1, \mathbf{J}_1)$ and $(\mathbf{E}_2, \mathbf{B}_2, \rho_2, \mathbf{J}_2)$ are two sets of solutions to the Maxwell equations and the Lorentz force law, then their superposition $(\mathbf{E}, \mathbf{B}, \rho, \mathbf{J})$ is also a solution, where $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ etc.

Q 12.86: For given charge density ρ and current density $\mathbf{J}$, is it possible to find two different sets of solutions $(\mathbf{E}_1, \mathbf{B}_1)$ and $(\mathbf{E}_2, \mathbf{B}_2)$? Is their superposition $(\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2, \mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2)$ another set of solution for the same ρ and $\mathbf{J}$?

From the Maxwell's equations, one can deduce the conservation of charge

$$\boxed{\oint_{\mathcal{S}=\partial\mathcal{V}} \mathbf{J} \cdot d\mathbf{a} + \frac{d}{dt} \int_{\mathcal{V}} \rho(\mathbf{r})\, d^3\mathbf{r} = 0.} \tag{12.117}$$

Consider Ampere's law with a closed surface $\mathcal{S}$ (so that $\mathcal{C} = \partial\mathcal{S} = 0$), then use Gauss' law.

or equivalently,

$$I_{\mathcal{S}=\partial\mathcal{V}}(t) + \frac{d}{dt} Q_{\mathcal{V}}(t) = 0. \tag{12.118}$$

The meaning of charge conservation is this: if the charge Q inside a volume $\mathcal{V}$ is changing over time, there must be electric currents flowing in or out of the volume through its surface.

Another implication of these equations is the flux rule (12.71) in the full generality (both the loop and the magnetic field can be time-dependent), but we will not prove it here.

Q 12.87: How would you modify these equations once magnetic monopoles are found?

One can also deduce from these equations (Maxwell + Lorentz) that the energy density in electromagnetic fields is in general

$$\boxed{u = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2.} \tag{12.119}$$

We will not prove it here, but eq.(12.119) applies to time-dependent electromagnetic fields.

This expression applies to electromagnetic fields in general, including electromagnetic waves.

Maxwell equations can be expressed more concisely in terms of the notation of special relativity as

$$\partial_\mu F^{\mu\nu} = -\mu_0 j^\nu, \tag{12.120}$$
$$\epsilon^{\mu\nu\lambda\rho}\partial_\mu F_{\nu\lambda} = 0, \tag{12.121}$$

where the anty-symmetric tensor $F_{\mu\nu} = -F_{\nu\mu}$ includes both $\mathbf{E}$ and $\mathbf{B}$, and j_μ includes both the current density $\mathbf{J}$ and the charge density ρ:

$$F^{0i} = \frac{1}{c}E_i, \qquad F_{ij} = \epsilon_{ijk}B_k, \qquad j^0 = c\rho, \qquad j^i = J_i. \tag{12.122}$$

Using the notation of differential calculus, the Maxwell equations are expressed as

$$*d*F = \mu_0 j, \tag{12.123}$$
$$dF = 0. \tag{12.124}$$

Note that two of the Maxwell equations involve the source ρ and $\mathbf{J}$ of the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$, while the other two are free from the source terms. The latter can be solved in general in terms of the potentials V and $\mathbf{A}$

$$\mathbf{E} = -\nabla V - \tfrac{\partial \mathbf{A}}{\partial t}, \tag{12.125}$$
$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{12.126}$$

V is the electric potential. $\mathbf{A}$ is called the magnetic potential or vector potential.

In terms of the potentials V and $\mathbf{A}$, only two of the four Maxwell equations are nontrivial. Any $\mathbf{E}$ and $\mathbf{B}$ satisfying eqs.(12.114) and (12.115) can be expressed as eqs.(12.125) and (12.126) for some V and $\mathbf{A}$.

The correspondence between the potentials and the fields is not 1-1. There are infinitely many sets of potentials $(V, \mathbf{A})$ corresponding to the same set of fields $(\mathbf{E}, \mathbf{B})$. If $(V, \mathbf{A})$ is a set of potentials for given $(\mathbf{E}, \mathbf{B})$, then $(V', \mathbf{A}')$ defined by

$$V' = V + \tfrac{\partial \lambda}{\partial t}, \tag{12.127}$$
$$\mathbf{A}' = \mathbf{A} - \nabla\lambda \tag{12.128}$$

We can measure $\mathbf{E}$ and $\mathbf{B}$ directly by observing the behavior of a point charge (Lorentz force). The values of V and $\mathbf{A}$ at a point in space at a given time are however not directly measurable.

is always another set of potentials for the same $(\mathbf{E}, \mathbf{B})$ for an arbitrary function λ.

Ex 12.82: Check that both $(V, \mathbf{A})$ and $(V', \mathbf{A}')$ give the same fields $(\mathbf{E}, \mathbf{B})$.

Q 12.88: In electrostatics, we said that a shift of the electric potential by a constant does not change any physics. Is it a special case of the gauge transformations?

Q 12.89: As V is a scalar, there is some advantage of using V instead of $\mathbf{E}$ to describe the electric field. What is the point of using the vector potential $\mathbf{A}$ in place of B?

The change from $(V, \mathbf{A})$ to $(V', \mathbf{A}')$ for an arbitrary function λ is called a *gauge* *transformation*. The invariance of something under gauge transformations is called a *gauge symmetry*.

Ex 12.83: Check that the effect of two consecutive gauge transformations λ_1, λ_2 is equivalent to another gauge transformation λ_3. What is λ_3 in terms of λ_1 and λ_2?

The gauge symmetry of electromagnetism is generalized to other interactions. The transformation laws (12.128) for electromagentism is replaced by different transformation laws acting on potentials defined for different interactions. Most of the fundamental interactions in nature, including gravity, strong, weak and electromagnetic interactions, are all dictated by gauge symmetries.

C. N. Yang: "*Symmetry dictates interaction.*"

While gauge symmetries and global symmetries (e.g. translation, rotation symmetry) are both symmetries, they differ by whether a physical state is changed to another physical state after a symmetry transformation. Gauge transformations do not change any physical state. Global transformations change a generic state, although a particular state may happen to be invariant under a particular global transformation.

Q 12.90: What is the difference between treating $x \to x + L$ (for a given constant L) as a gauge symmetry and a global symmetry?

The circle S^1 can be defined as the result of identifying points separated by nL ($n \in \mathbb{Z}$) on the real line.

In general relativity, general coordinate transformations are gauge transformations. The transformation $x \to x + L$ is thus a global symmetry in Newtonian physics but a gauge symmetry in GR.

Nowadays, some people regard global symmetries as unnatural (compared with gauge symmetries) for a fundamental theory.

Q 12.91: If experiments are conducted with much higher precision, how do you think Maxwell's equations and the Lorentz force law could be modified? Would the superposition principle be violated? Would the law of charge conservation be violated or modified?