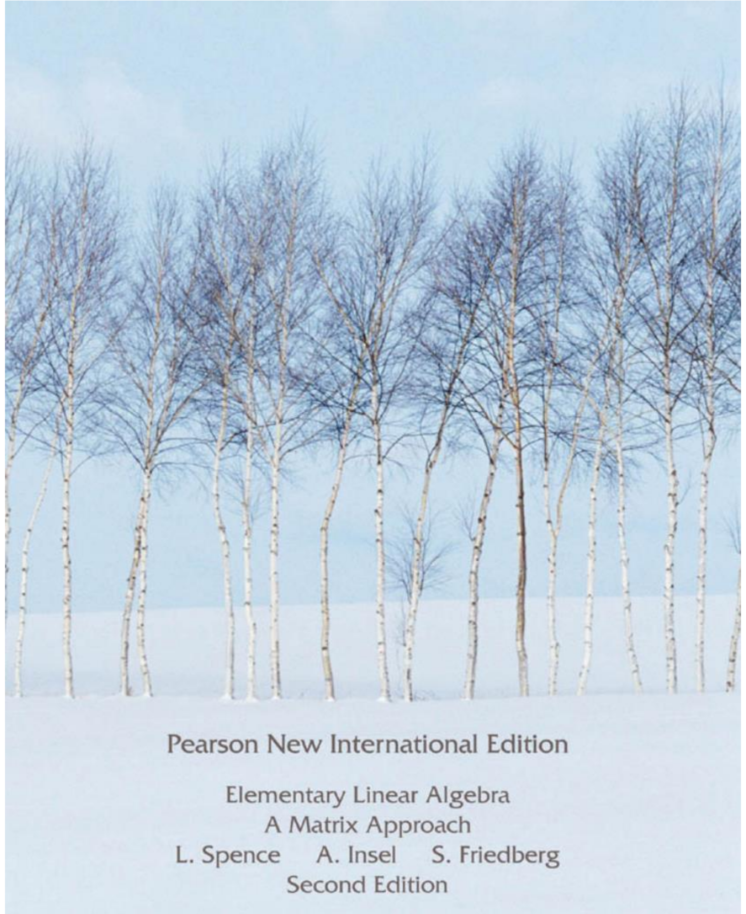
Elementary Linear Algebra A Matrix Approach

L. Spence A. Insel S. Friedberg Second Edition



Pearson New International Edition

Elementary Linear Algebra
A Matrix Approach
L. Spence A. Insel S. Friedberg
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Chapter 1. Matrices, Vectors, and Systems of Linear Equations

1.1 MATRICES AND VECTORS

Example 1 Compute the matrices $A + B$, $3A$, $-A$, and $3A + 4B$, where

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 2 & -3 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & 1 & 0 \\ 5 & -6 & 1 \end{bmatrix}.$$

1.1 MATRICES AND VECTORS

THEOREM 1.1

(Properties of Matrix Addition and Scalar Multiplication) Let A, B, and C be $m \times n$ matrices, and let s and t be any scalars. Then

(a) $A + B = B + A$. (commutative law of matrix addition)

(b) $(A + B) + C = A + (B + C)$. (associative law of matrix addition)

(c) $A + O = A$.

(d) $A + (-A) = O$.

(e) $(st)A = s(tA)$.

(f) $s(A + B) = sA + sB$.

(g) $(s + t)A = sA + tA$.

1.1 MATRICES AND VECTORS

THEOREM 1.2

(Properties of the Transpose) Let A and B be $m \times n$ matrices, and let s be any scalar. Then

(a) $(A+B)^T = A^T + B^T$.

(b) $(sA)^T = sA^T$.

(c) $(A^T)^T = A$.

1.1 MATRICES AND VECTORS

Example 2

Let $\mathbf{u} = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$. Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 7 \\ -1 \\ 7 \end{bmatrix}, \qquad \mathbf{u} - \mathbf{v} = \begin{bmatrix} -3 \\ -7 \\ 7 \end{bmatrix}, \qquad \text{and} \qquad 5\mathbf{v} = \begin{bmatrix} 25 \\ 15 \\ 0 \end{bmatrix}.$$

1.1 MATRICES AND VECTORS

EXERCISES

A square matrix A is called a **diagonal matrix** *if $a_{ij} = 0$ whenever $i \neq j$. Exercises 67–70 are concerned with diagonal matrices.*

67. Prove that a square zero matrix is a diagonal matrix.
68. Prove that if B is a diagonal matrix, then cB is a diagonal matrix for any scalar c.
69. Prove that if B is a diagonal matrix, then B^T is a diagonal matrix.
70. Prove that if B and C are diagonal matrices of the same size, then $B + C$ is a diagonal matrix.

1.1 MATRICES AND VECTORS

EXERCISES

A (square) matrix A is said to be **symmetric** *if $A = A^T$. Exercises 71–78 are concerned with symmetric matrices.*

71. Give examples of 2×2 and 3×3 symmetric matrices.
72. Prove that the (i, j)-entry of a symmetric matrix equals the (j, i)-entry.
73. Prove that a square zero matrix is symmetric.
74. Prove that if B is a symmetric matrix, then so is cB for any scalar c.
75. Prove that if B is a square matrix, then $B + B^T$ is symmetric.
76. Prove that if B and C are $n \times n$ symmetric matrices, then so is $B + C$.
77. Is a square submatrix of a symmetric matrix necessarily a symmetric matrix? Justify your answer.
78. Prove that a diagonal matrix is symmetric.

1.1 MATRICES AND VECTORS

EXERCISES

A (square) matrix A is called **skew-symmetric** *if $A^T = -A$. Exercises 79–81 are concerned with skew-symmetric matrices.*

79. What must be true about the (i, i)-entries of a skew-symmetric matrix? Justify your answer.
80. Give an example of a nonzero 2×2 skew-symmetric matrix B. Now show that every 2×2 skew-symmetric matrix is a scalar multiple of B.
81. Show that every 3×3 matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

1.2 LINEAR COMBINATIONS, MATRIX–VECTOR PRODUCTS, AND SPECIAL MATRICES

Example 1

(a) Determine whether $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(b) Determine whether $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(c) Determine whether $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

1.2 LINEAR COMBINATIONS, MATRIX–VECTOR PRODUCTS, AND SPECIAL MATRICES

IDENTITY MATRICES

Suppose we let $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{v}$ be any vector in $\mathcal{R}^2$. Then

$$I_2\mathbf{v} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{v}.$$

So multiplication by I_2 leaves every vector $\mathbf{v}$ in $\mathcal{R}^2$ unchanged. The same property holds in a more general context.

Definition For each positive integer n, the $n \times n$ **identity matrix** I_n is the $n \times n$ matrix whose respective columns are the standard vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in $\mathcal{R}^n$.

ROTATION MATRICES

Example 4

To rotate the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ by 30°, we compute $A_{30^\circ}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$; that is,

$$\begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \dfrac{3\sqrt{3}}{2} - \dfrac{4}{2} \\ \dfrac{3}{2} + \dfrac{4\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3\sqrt{3} - 4 \\ 3 + 4\sqrt{3} \end{bmatrix}.$$

Thus when $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is rotated by 30°, the resulting vector is $\dfrac{1}{2}\begin{bmatrix} 3\sqrt{3} - 4 \\ 3 + 4\sqrt{3} \end{bmatrix}$.

1.2 LINEAR COMBINATIONS, MATRIX–VECTOR PRODUCTS, AND SPECIAL MATRICES

THEOREM 1.3

(Properties of Matrix–Vector Products) Let A and B be $m \times n$ matrices, and let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathcal{R}^n$. Then

(a) $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$.

(b) $A(c\mathbf{u}) = c(A\mathbf{u}) = (cA)\mathbf{u}$ for every scalar c.

(c) $(A+B)\mathbf{u} = A\mathbf{u} + B\mathbf{u}$.

(d) $A\mathbf{e}_j = \mathbf{a}_j$ for $j = 1, 2, \ldots, n$, where $\mathbf{e}_j$ is the jth standard vector in $\mathcal{R}^n$.

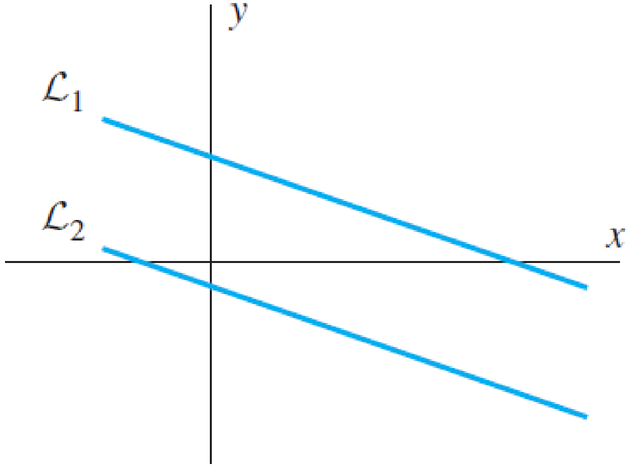
(e) If B is an $m \times n$ matrix such that $B\mathbf{w} = A\mathbf{w}$ for all $\mathbf{w}$ in $\mathcal{R}^n$, then $B = A$.

(f) $A\mathbf{0}$ is the $m \times 1$ zero vector.

(g) If O is the $m \times n$ zero matrix, then $O\mathbf{v}$ is the $m \times 1$ zero vector.

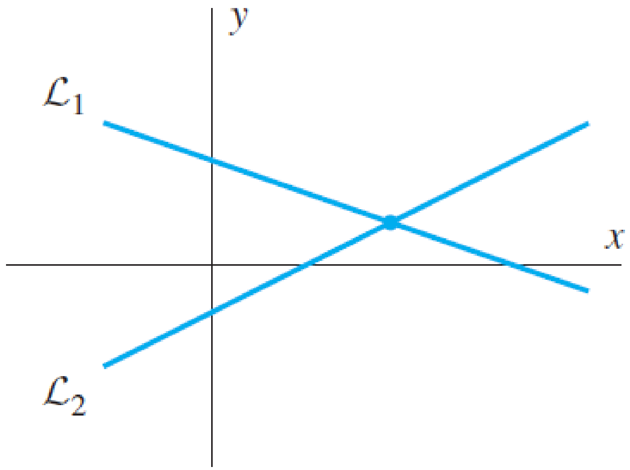
(h) $I_n\mathbf{v} = \mathbf{v}$.

1.3 SYSTEMS OF LINEAR EQUATIONS



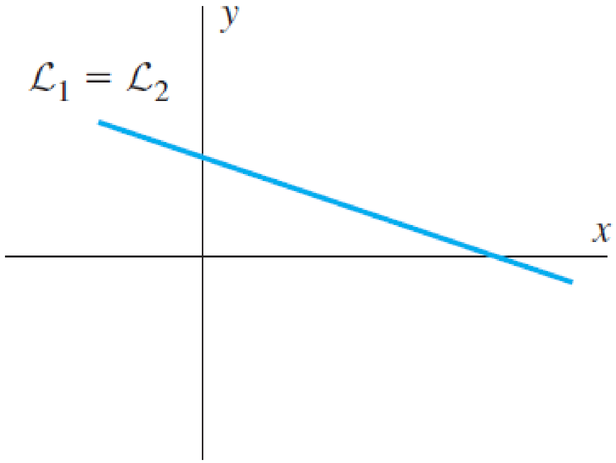
$\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel.
No solution

Figure 1.16



$\mathcal{L}_1$ and $\mathcal{L}_2$ are different but not parallel.
Exactly one solution

Figure 1.17



$\mathcal{L}_1$ and $\mathcal{L}_2$ are the same.
Infinitely many solutions

Figure 1.18

ELEMENTARY ROW OPERATIONS

Definition Any one of the following three operations performed on a matrix is called an **elementary row operation**:

1. Interchange any two rows of the matrix. (**interchange operation**)
2. Multiply every entry of some row of the matrix by the same nonzero scalar. (**scaling operation**)
3. Add a multiple of one row of the matrix to another row. (**row addition operation**)

ELEMENTARY ROW OPERATIONS

$$\boxed{\begin{aligned} x_1 - 2x_2 - x_3 &= 3 \\ 3x_1 - 6x_2 - 5x_3 &= 3 \\ 2x_1 - x_2 + x_3 &= 0 \end{aligned}} \qquad (2)$$

$$\begin{array}{rl} -3x_1 + 6x_2 + 3x_3 = -9 & (-3 \text{ times equation } 1) \\ 3x_1 - 6x_2 - 5x_3 = 3 & (\text{equation } 2) \\ \hline -2x_3 = -6 & \end{array}$$

$$\begin{array}{rl} -2x_1 + 4x_2 + 2x_3 = -6 & (-2 \text{ times equation } 1) \\ 2x_1 - x_2 + x_3 = 0 & (\text{equation } 3) \\ \hline 3x_2 + 3x_3 = -6 & \end{array}$$

$$\begin{aligned} x_1 - 2x_2 - x_3 &= 3 \\ -2x_3 &= -6 \\ 3x_2 + 3x_3 &= -6 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 - x_3 &= 3 \\ 3x_2 + 3x_3 &= -6 \\ -2x_3 &= -6 \end{aligned} \qquad (3)$$

$$\begin{aligned} x_1 - 2x_2 - x_3 &= 3 \\ 3x_2 + 3x_3 &= -6 \\ x_3 &= 3. \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 &= 6 \\ 3x_2 &= -15 \\ x_3 &= 3. \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 &= 6 \\ x_2 &= -5 \\ x_3 &= 3. \end{aligned}$$

$$\begin{aligned} x_1 &= -4 \\ x_2 &= -5 \\ x_3 &= 3, \end{aligned}$$

$$\boxed{\begin{bmatrix} -4 \\ -5 \\ 3 \end{bmatrix} \text{ is a solution}}$$

the system of equations performed on matrices

$$\begin{aligned} x_1 - 2x_2 - x_3 &= 3 \\ 3x_1 - 6x_2 - 5x_3 &= 3 \\ 2x_1 - x_2 + x_3 &= 0 \end{aligned}$$

as the matrix equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 3 & -6 & -5 \\ 2 & -1 & 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}.$$

$$A\mathbf{u} = \begin{bmatrix} 1 & -2 & -1 \\ 3 & -6 & -5 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = \mathbf{b}.$$

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{bmatrix},$$

augmented matrix of the system.

ELEMENTARY ROW OPERATIONS

Example 2 Let

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 1 & 3 \\ 3 & 1 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & -5 & -3 & -7 \end{bmatrix}.$$

The following sequence of elementary row operations transforms A into B:

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 1 & 3 \\ 3 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{\mathbf{r}_1 \leftrightarrow \mathbf{r}_2} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & -1 & 3 \\ 3 & 1 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{-2\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & -3 & -3 \\ 3 & 1 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{-3\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}_3} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & -3 & -3 \\ 0 & -5 & -3 & -7 \end{bmatrix} \xrightarrow{-\frac{1}{3}\mathbf{r}_2 \to \mathbf{r}_2} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & -5 & -3 & -7 \end{bmatrix} = B.$$

REDUCED ROW ECHELON FORM

Definitions A matrix is said to be in **row echelon form** if it satisfies the following three conditions:

1. Each nonzero row lies above every zero row.
2. The leading entry of a nonzero row lies in a column to the right of the column containing the leading entry of any preceding row.
3. If a column contains the leading entry of some row, then all entries of that column below the leading entry are 0.[5]

If a matrix also satisfies the following two additional conditions, we say that it is in **reduced row echelon form**.[6]

4. If a column contains the leading entry of some row, then all the other entries of that column are 0.
5. The leading entry of each nonzero row is 1.

REDUCED ROW ECHELON FORM

Example 3 The following matrices are *not* in reduced row echelon form:

$$A = \begin{bmatrix} 1 & 0 & 0 & 6 & 3 & 0 \\ 0 & 0 & 1 & 5 & 7 & 0 \\ 0 & 1 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 7 & 2 & -3 & 9 & 4 \\ 0 & 0 & 1 & 4 & 6 & 8 \\ 0 & 0 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REDUCED ROW ECHELON FORM

Example 4 Find a general solution of the system of linear equations

$$\begin{aligned} x_1 \qquad\quad + 2x_4 &= 7 \\ x_2 \quad - 3x_4 &= 8 \\ x_3 + 6x_4 &= 9. \end{aligned}$$

$$\begin{aligned} x_1 &= 7 - 2x_4 \\ x_2 &= 8 + 3x_4 \\ x_3 &= 9 - 6x_4 \\ x_4 & \quad \text{free.} \end{aligned}$$

We can write the general solution in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 3 \\ -6 \\ 1 \end{bmatrix}.$$

REDUCED ROW ECHELON FORM

Example 5 Solve the following system of linear equations:

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2 \\
-x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\
2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\
-3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9
\end{aligned}
$$

1.4 GAUSSIAN ELIMINATION

- Write the augmented matrix [A b] of the system.
- Find the reduced row echelon form [R c] of [A b].

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

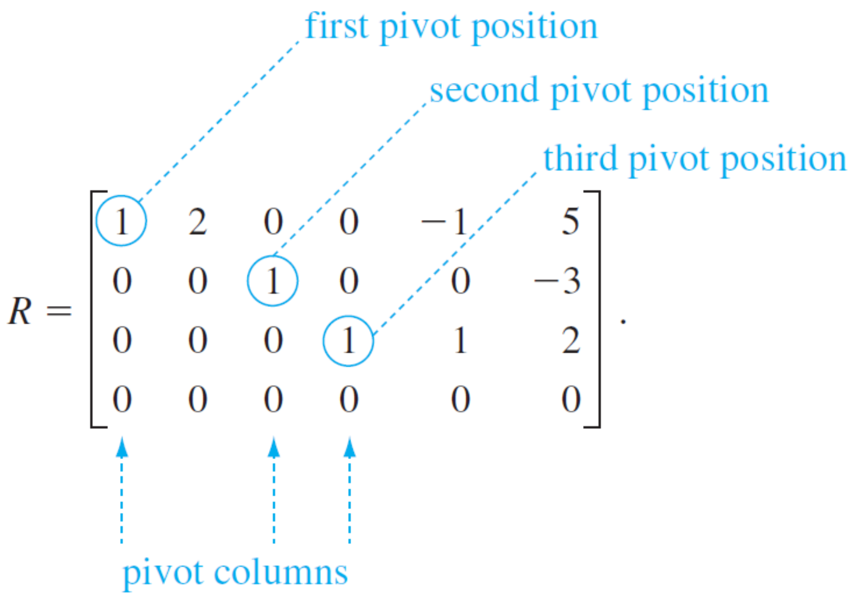


Figure 1.19 The pivot positions of the matrix *R*

1.4 GAUSSIAN ELIMINATION

Example 1 Solve the following system of linear equations:

$$\begin{aligned}
x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2 \\
-x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\
2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\
-3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9
\end{aligned}$$

Solution The augmented matrix of this system is

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
x_1 + 2x_2 \qquad\qquad - x_5 &= -5 \\
x_3 \qquad\qquad &= -3 \\
x_4 + x_5 &= 2.
\end{aligned}$$

$$\begin{aligned}
x_1 &= -5 - 2x_2 + x_5 \\
x_2 &\quad \text{free} \\
x_3 &= -3 \\
x_4 &= 2 \qquad - x_5 \\
x_5 &\quad \text{free.}
\end{aligned}$$

THE RANK AND NULLITY OF A MATRIX

The rank of a matrix equals the number of pivot columns in the matrix.
The nullity of a matrix equals the number of nonpivot columns in the matrix.

Example 3 The reduced row echelon form of the matrix

$$B = \begin{bmatrix} 2 & 3 & 1 & 5 & 2 \\ 0 & 1 & 1 & 3 & 2 \\ 4 & 5 & 1 & 7 & 2 \\ 2 & 1 & -1 & -1 & -2 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & -1 & -2 & -2 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the latter matrix has two nonzero rows, the rank of B is 2. The nullity of B is $5 - 2 = 3$.

If $A\mathbf{x} = \mathbf{b}$ is the matrix form of a consistent system of linear equations, then

(a) the number of basic variables in a general solution of the system equals the rank of A;

(b) the number of free variables in a general solution of the system equals the nullity of A.

Thus a consistent system of linear equations has a unique solution if and only if the nullity of its coefficient matrix equals 0. Equivalently, a consistent system of linear equations has infinitely many solutions if and only if the nullity of its coefficient matrix is positive.

EXERCISES

- *In Exercises 1–16, determine whether the given system is consistent,*
- *and if so, find its general solution.*

- *In Exercises 17–26, determine the values of r, if any, for which*
- *the given system of linear equations is inconsistent.*

- *In Exercises 27–34, determine the values of r and s for which the*
- *given system of linear equations has (a) no solutions, (b) exactly*
- *one solution, and (c) infinitely many solutions.*

- *In Exercises 35–42, find the rank and nullity of the given matrix.*

1.5 APPLICATIONS OF SYSTEMS OF LINEAR EQUATIONS

- CURRENT FLOW IN ELECTRICAL CIRCUITS

Kirchhoff's Voltage Law

In a closed path within an electrical circuit, the sum of the voltage drops in any one direction equals the sum of the voltage sources in the same direction.

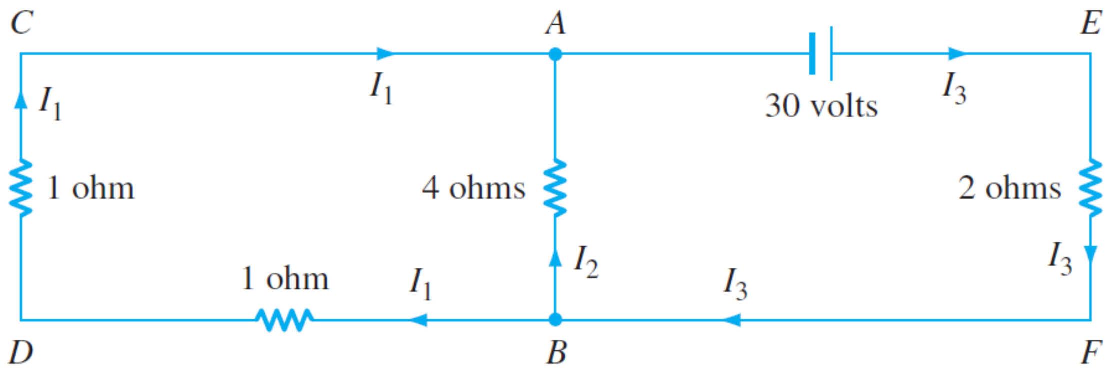


Figure 1.22 An electrical circuit

$$
\begin{aligned}
2I_1 - 4I_2 \qquad &= 0 &\quad (7)\\
4I_2 + 2I_3 &= 30 &\quad (8)\\
I_1 + I_2 - I_3 &= 0. &\quad (10)
\end{aligned}
$$

1.6 THE SPAN OF A SET OF VECTORS

Example 1 Describe the spans of the following subsets of $\mathcal{R}^2$:

$$\mathcal{S}_1 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \qquad \mathcal{S}_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}, \qquad \mathcal{S}_3 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\},$$

and

$$\mathcal{S}_4 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$$

1.6 THE SPAN OF A SET OF VECTORS

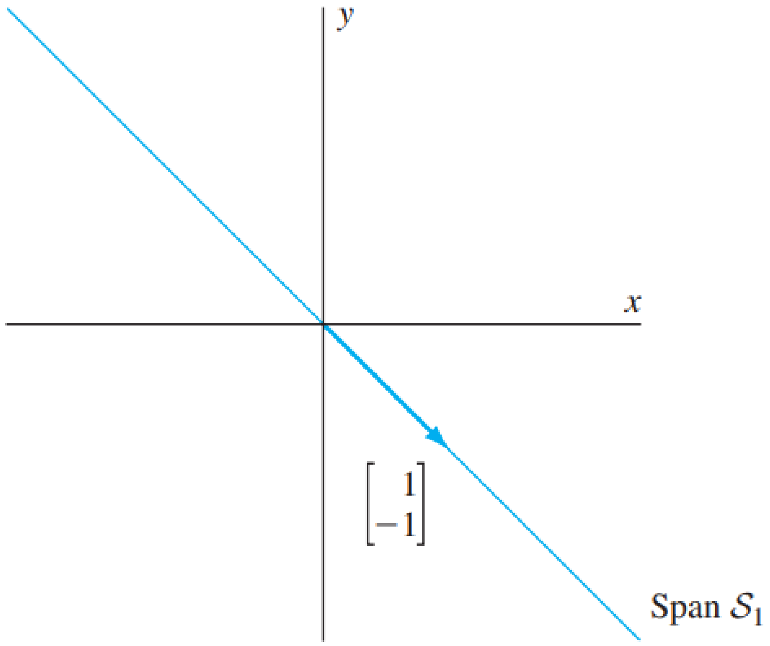


Figure 1.23 The span of $\mathcal{S}_1$

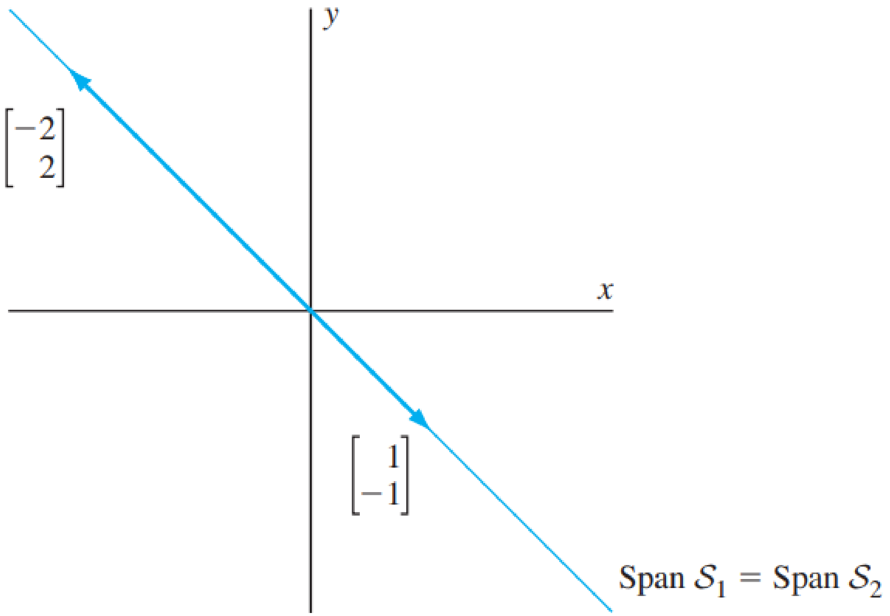


Figure 1.24 The span of $\mathcal{S}_2$

1.6 THE SPAN OF A SET OF VECTORS

Suppose that $\mathbf{v} = a\begin{bmatrix} 1 \\ -1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for some scalars a and b. Then

$$\mathbf{v} = a\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0\begin{bmatrix} -2 \\ 2 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

so every vector in $\mathcal{R}^2$ is a linear combination of the vectors in $\mathcal{S}_3$. It follows that the span of $\mathcal{S}_3$ is $\mathcal{R}^2$.

Finally, since every vector in $\mathcal{R}^2$ is a linear combination of the nonparallel vectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, every vector in $\mathcal{R}^2$ is also a linear combination of the vectors in $\mathcal{S}_4$. Therefore the span of $\mathcal{S}_4$ is again $\mathcal{R}^2$.

1.6 THE SPAN OF A SET OF VECTORS

Example 3 Is

$$\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 5 \\ -1 \end{bmatrix} \quad \text{or} \quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

a vector in the span of

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ -1 \\ 5 \end{bmatrix} \right\}?$$

If so, express it as a linear combination of the vectors in $\mathcal{S}$.

1.6 THE SPAN OF A SET OF VECTORS

Solution Let A be the matrix whose columns are the vectors in $\mathcal{S}$. The vector $\mathbf{v}$ belongs to the span of $\mathcal{S}$ if and only if $A\mathbf{x} = \mathbf{v}$ is consistent. Since the reduced row echelon form of $[A\ \ \mathbf{v}]$ is

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$A\mathbf{x} = \mathbf{v}$ is consistent by Theorem 1.5. Hence $\mathbf{v}$ belongs to the span of $\mathcal{S}$.

To express $\mathbf{v}$ as a linear combination of the vectors in $\mathcal{S}$, we need to find the actual solution of $A\mathbf{x} = \mathbf{v}$. Using the reduced row echelon form of $[A\ \ \mathbf{v}]$, we see that the general solution of this equation is

$$\begin{aligned} x_1 &= \ \ 1 - 3x_3 \\ x_2 &= -2 - 2x_3 \\ x_3 &\ \ \text{free.} \end{aligned}$$

For example, by taking $x_3 = 0$, we find that

$$1\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 8 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 5 \\ -1 \end{bmatrix} = \mathbf{v}.$$

In the same manner, $\mathbf{w}$ belongs to the span of $\mathcal{S}$ if and only if $A\mathbf{x} = \mathbf{w}$ is consistent. Because the reduced row echelon form of $[A\ \ \mathbf{w}]$ is

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

Theorem 1.5 shows that $A\mathbf{x} = \mathbf{w}$ is not consistent. Thus $\mathbf{w}$ does not belong to the span of $\mathcal{S}$.

1.6 THE SPAN OF A SET OF VECTORS

THEOREM 1.6

The following statements about an $m \times n$ matrix A are equivalent:

(a) The span of the columns of A is $\mathcal{R}^m$.

(b) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution (that is, $A\mathbf{x} = \mathbf{b}$ is consistent) for each $\mathbf{b}$ in $\mathcal{R}^m$.

(c) The rank of A is m, the number of rows of A.

(d) The reduced row echelon form of A has no zero rows.

(e) There is a pivot position in each row of A.

1.6 THE SPAN OF A SET OF VECTORS

In Exercises 29–36, an $m \times n$ matrix A is given. Determine whether the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathcal{R}^m$.

29. $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

30. $\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 0 & -3 \\ -1 & 0 & 3 \end{bmatrix}$

32. $\begin{bmatrix} 1 & 1 & 2 \\ -1 & -3 & 4 \end{bmatrix}$

33. $\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -2 & 2 \end{bmatrix}$

34. $\begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 0 & 3 & -2 \\ 1 & 1 & -3 \end{bmatrix}$

35. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 4 & 5 \end{bmatrix}$

1.7 LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

Definitions A set of k vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ in $\mathcal{R}^n$ is called **linearly dependent** if there exist scalars $c_1, c_2, \ldots, c_k$, not all 0, such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}.$$

In this case, we also say that **the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly dependent**.

A set of k vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is called **linearly independent** if the only scalars $c_1, c_2, \ldots, c_k$ such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}$$

are $c_1 = c_2 = \cdots = c_k = 0$. In this case, we also say that **the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly independent**.

1.7 LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

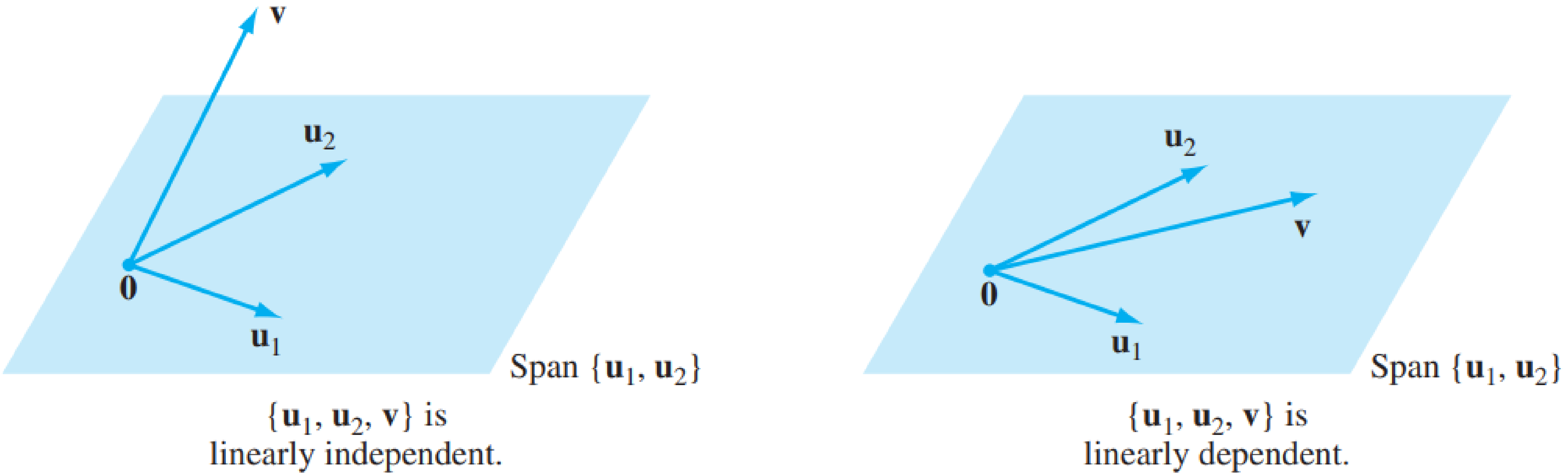


Figure 1.27 Linearly independent and linearly dependent sets of 3 vectors

1.7 LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

Example 3 Determine whether the set

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly dependent or linearly independent.

1.7 LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

THEOREM 1.8

The following statements about an $m \times n$ matrix A are equivalent:

(a) The columns of A are linearly independent.

(b) The equation $A\mathbf{x} = \mathbf{b}$ has at most one solution for each $\mathbf{b}$ in $\mathcal{R}^m$.

(c) The nullity of A is zero.

(d) The rank of A is n, the number of columns of A.

(e) The columns of the reduced row echelon form of A are distinct standard vectors in $\mathcal{R}^m$.

(f) The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$.

(g) There is a pivot position in each column of A.

Exercises

In Exercises 23–30, determine whether the given set is linearly independent.

23. $\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

24. $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

25. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \end{bmatrix} \right\}$

26. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$

27. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

28. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ -3 \end{bmatrix} \right\}$

29. $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

30. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \end{bmatrix} \right\}$