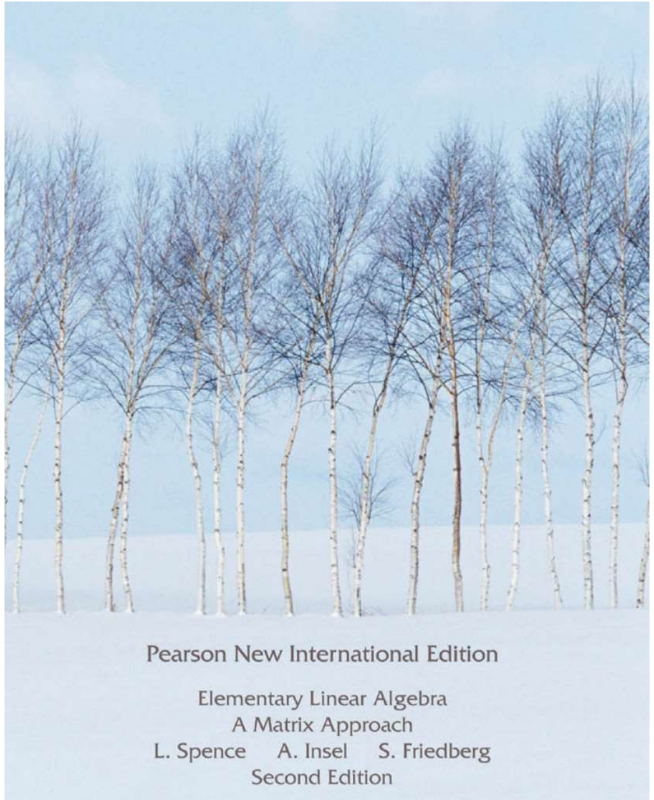
Elementary Linear Algebra A Matrix Approach

L. Spence A. Insel S. Friedberg Second Edition



Pearson New International Edition

Elementary Linear Algebra
A Matrix Approach
L. Spence A. Insel S. Friedberg
Second Edition

Chapter 1. Matrices, Vectors, and Systems of Linear Equations

1.1 MATRICES AND VECTORS

Example 1 Compute the matrices $A+B$, $3A$, $-A$, and $3A+4B$, where

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 2 & -3 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & 1 & 0 \\ 5 & -6 & 1 \end{bmatrix}.$$

1.1 MATRICES AND VECTORS

THEOREM 1.1

(Properties of Matrix Addition and Scalar Multiplication) Let A, B, and C be $m \times n$ matrices, and let s and t be any scalars. Then

(a) $A + B = B + A$. (commutative law of matrix addition)

(b) $(A + B) + C = A + (B + C)$. (associative law of matrix addition)

(c) $A + O = A$.

(d) $A + (-A) = O$.

(e) $(st)A = s(tA)$.

(f) $s(A + B) = sA + sB$.

(g) $(s + t)A = sA + tA$.

1.1 MATRICES AND VECTORS

THEOREM 1.2

(Properties of the Transpose) Let A and B be $m \times n$ matrices, and let s be any scalar. Then

(a) $(A+B)^T = A^T + B^T$.

(b) $(sA)^T = sA^T$.

(c) $(A^T)^T = A$.

1.1 MATRICES AND VECTORS

Example 2

Let $\mathbf{u} = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$. Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 7 \\ -1 \\ 7 \end{bmatrix}, \qquad \mathbf{u} - \mathbf{v} = \begin{bmatrix} -3 \\ -7 \\ 7 \end{bmatrix}, \qquad \text{and} \qquad 5\mathbf{v} = \begin{bmatrix} 25 \\ 15 \\ 0 \end{bmatrix}.$$

1.1 MATRICES AND VECTORS

EXERCISES

A square matrix A is called a **diagonal matrix** *if* $a_{ij} = 0$ *whenever* $i \neq j$. *Exercises 67–70 are concerned with diagonal matrices.*

67. Prove that a square zero matrix is a diagonal matrix.
68. Prove that if B is a diagonal matrix, then cB is a diagonal matrix for any scalar c.
69. Prove that if B is a diagonal matrix, then B^T is a diagonal matrix.
70. Prove that if B and C are diagonal matrices of the same size, then $B + C$ is a diagonal matrix.

1.1 MATRICES AND VECTORS

EXERCISES

A (square) matrix A is said to be **symmetric** *if $A = A^T$. Exercises 71–78 are concerned with symmetric matrices.*

71. Give examples of 2×2 and 3×3 symmetric matrices.
72. Prove that the (i, j)-entry of a symmetric matrix equals the (j, i)-entry.
73. Prove that a square zero matrix is symmetric.
74. Prove that if B is a symmetric matrix, then so is cB for any scalar c.
75. Prove that if B is a square matrix, then $B + B^T$ is symmetric.
76. Prove that if B and C are $n \times n$ symmetric matrices, then so is $B + C$.
77. Is a square submatrix of a symmetric matrix necessarily a symmetric matrix? Justify your answer.
78. Prove that a diagonal matrix is symmetric.

1.1 MATRICES AND VECTORS

EXERCISES

A (square) matrix A is called **skew-symmetric** *if* $A^T = -A$. *Exercises 79–81 are concerned with skew-symmetric matrices.*

79. What must be true about the (i, i)-entries of a skew-symmetric matrix? Justify your answer.
80. Give an example of a nonzero 2×2 skew-symmetric matrix B. Now show that every 2×2 skew-symmetric matrix is a scalar multiple of B.
81. Show that every 3×3 matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

1.2 LINEAR COMBINATIONS, MATRIX–VECTOR PRODUCTS, AND SPECIAL MATRICES

Example 1

(a) Determine whether $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(b) Determine whether $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(c) Determine whether $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

1.2 LINEAR COMBINATIONS, MATRIX–VECTOR PRODUCTS, AND SPECIAL MATRICES

IDENTITY MATRICES

Suppose we let $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{v}$ be any vector in $\mathcal{R}^2$. Then

$$I_2\mathbf{v} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{v}.$$

So multiplication by I_2 leaves every vector $\mathbf{v}$ in $\mathcal{R}^2$ unchanged. The same property holds in a more general context.

Definition For each positive integer n, the $n \times n$ **identity matrix** I_n is the $n \times n$ matrix whose respective columns are the standard vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in $\mathcal{R}^n$.

ROTATION MATRICES

Example 4

To rotate the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ by 30°, we compute $A_{30^\circ}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$; that is,

$$\begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \dfrac{3\sqrt{3}}{2} - \dfrac{4}{2} \\ \dfrac{3}{2} + \dfrac{4\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3\sqrt{3} - 4 \\ 3 + 4\sqrt{3} \end{bmatrix}.$$

Thus when $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is rotated by 30°, the resulting vector is $\dfrac{1}{2}\begin{bmatrix} 3\sqrt{3} - 4 \\ 3 + 4\sqrt{3} \end{bmatrix}$.

1.2 LINEAR COMBINATIONS, MATRIX–VECTOR PRODUCTS, AND SPECIAL MATRICES

THEOREM 1.3

(Properties of Matrix–Vector Products) Let A and B be $m \times n$ matrices, and let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathcal{R}^n$. Then

(a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$.

(b) $A(c\mathbf{u}) = c(A\mathbf{u}) = (cA)\mathbf{u}$ for every scalar c.

(c) $(A + B)\mathbf{u} = A\mathbf{u} + B\mathbf{u}$.

(d) $A\mathbf{e}_j = \mathbf{a}_j$ for $j = 1, 2, \ldots, n$, where $\mathbf{e}_j$ is the jth standard vector in $\mathcal{R}^n$.

(e) If B is an $m \times n$ matrix such that $B\mathbf{w} = A\mathbf{w}$ for all $\mathbf{w}$ in $\mathcal{R}^n$, then $B = A$.

(f) $A\mathbf{0}$ is the $m \times 1$ zero vector.

(g) If O is the $m \times n$ zero matrix, then $O\mathbf{v}$ is the $m \times 1$ zero vector.

(h) $I_n\mathbf{v} = \mathbf{v}$.