14.8 Paradoxes

14.8.1 Twin Paradox

Sherlock Holmes: "Once you eliminate the impossible, whatever remains, no matter how improbable, must be the truth.

Q 14.29: The brother stayed on earth and the sister took a rocket flight. Due to time dilation, sister's clock ran slower, so that when the sister came back, the sister aged less than the brother. Why? Was not the brother also moving with respect to the sister? How can the arbitrarily short period of time for turning around the rocket has such a large effect?

Solution:

The paradox is this: the roles played by the brother and the sister are "symmetric", except when the sister's rocket accelerates to turn back. But the acceleration alone cannot explain why the sister is younger, because the age difference at the end is proportional to the distance of the tour, while the acceleration can be a fixed value within a fixed period of time.

There are three inertial frames involved. Let S be the inertial frame of the earth, S' the frame moving away from the earth, and S'' the frame moving back towards the earth. The sister was initially at rest with respect to S', but then she turned and became at rest with respect to S''. The sister aged slower than the brother according to S, but the brother aged slower than the sister according to S' and S''.

The crucial point is that, at the moment when the sister switched from S' to S'', the same moment on earth in S' is not the same moment on earth in S''. There is a gap of time on earth missing from the records of both S' and S'', which explains why the brother aged more.

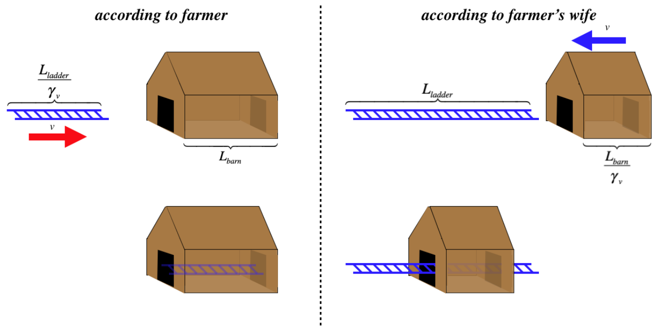
Notice that, when we say that the sister should find the brother aging slower (because her brother is moving with respect to herself), this finding relies on many clocks in her reference frame that are coincident with the brother at different times.

Even though the sister can turn her rocket around within a very short period of time, it makes a big difference. The same can be said about the spatial distance: The distance between two points A and B is shorter along the straight line but longer along another path that involves a change of direction.

Q 14.30: If the space of the universe is compactified so that the x-direction is periodic (with a huge periodicity such as 10^{10} light years), the sister would not have to turn around to meet her brother again. Would the sister appear younger when she meets her brother again?

14.8.2 Barn And Ladder Paradox

The farmer has a long ladder and a not-so-long barn. His daughter suggests that, according to Lorentz contraction, the farmer can run into the barn with the shortened ladder, and she will close the door quickly with the ladder inside. However, the farmer thinks that the barn will become shorter as the barn is moving with respect to himself. Who is right?



Solution:

Both could be right when the farmer is still running. The crucial points are the following.

1. Simultaneity is not universal. The farmer sees the head of the ladder hitting on the barn's wall before its tail fits inside the barn. The daughter sees the tail of the ladder inside the barn before its head hitting the wall.

2. It is inconsistent to have ideal rigid bodies whose lengths are constant.

There are various possibilities depending on what the ladder is made of and how the ladder is stopped from motion, but the key point is the relative simultaneity.

It is very interesting that Special Relativity, applied as a symmetry behind all physical laws, imposes a restriction on the materials that can exist in the universe through a consistency condition.

14.8.3 Ehrenfest Paradox

A disk rotates at a given angular velocity ω in an inertial frame S. With respect to both the inertial frame S and the (non-inertial) reference frame S' rotating with the disk, the radius is the same (denoted as R) because the radial direction is perpendicular to the direction of motion (the tangent direction). Consider the following questions.

Q 14.31: From the perspective of observers in S, the circumference $2\pi R$ is a length-contracted result. When the disk stops rotating, the circumference should be longer than $2\pi R$. How can a disk at rest have a circumference $> 2\pi R$?

Q 14.32: From the perspective of observers in S, a disk of radius R should have a circumference $2\pi R$. But the circumference should appear to be longer than $2\pi R$ when people use length contracted rulers in S' to measure it. How is it possible that the circumference-to-radius ratio is different from (larger than) 2π for S'?

Can you think of other questions?

Q 14.33: From the perspective of observers in S', the disk is at rest, so its circumference should be $2\pi R$. This prediction is contradictory to the prediction in **Q**14.32. Who is right?

To answer the questions above, note the following crucial points:

Einstein drew intuitions about General Relativity from this paradox.

1. The reference frame S' has a space slice defined by all events at the same instant

of time t'. This space slice can be curved, although the full space-time is flat by assumption.

2. The disk may expand in the tangential direction when it starts to rotate (like a pizza does when a chef is tossing it). How much it expands depends on the material it is made from. A restriction on the rigidity of materials is imposed as in the barn-and-ladder paradox.

Q 14.34: Can you come up with your own paradox for Special Relativity?

14.9 Covariance and Invariance

This section is a straightforward generalization of the notion of vectors and tensor in a 3 dimensional space which we are familiar with.

Often people use the convention in which they distinguish upper (contravariant) indices from lower (covariant) indices. Upper indices transform like the infinitesimal displacement 4-vector

$$dx^\mu \to dx'^\mu = \Lambda^\mu{}_\nu dx^\nu. \tag{14.28}$$

By definition, A^μ is a *contravariant* 4-vector if it transforms like

$$A^\mu \to A'^\mu = \Lambda^\mu{}_\nu A^\nu. \tag{14.29}$$

An object A_μ that transforms like

$$A_\mu \to A'_\mu = A_\nu \Lambda^{-1\nu}{}_\mu. \tag{14.30}$$

is called a *covariant* 4-vector.

Q 14.35: How do we know whether 4 quantities (a, b, c, d) transform like a contravariant/covariant vector or not?

Q 14.36: Is $\frac{dx^\mu}{dt}$ a contravariant/covariant vector?

Conventionally we use $\eta^{\mu\nu}$ to denote the inverse of $\eta_{\mu\nu}$, instead of writing $\eta^{-1\mu\nu}$.

Ex 14.9: Check that $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ can be used to raise and lower Lorentz indices to turn a contravariant index into a covariant index, and vise versa.

That is, if A_μ is a covariant vector and B^μ a contravariant vector, then we can define a contravariant vector A^μ and a covariant vector B_μ by

$$A^\mu \equiv \eta^{\mu\nu} A_\nu, \tag{14.31}$$

$$B_\mu \equiv \eta_{\mu\nu} B^\nu. \tag{14.32}$$

Since the information contained in A^μ is the same as that in A_μ, the information of a 4-vector can be equally represented either as a contravariant 4-vector or a covariant 4-vector.

Even if you decide to label 4 quantities as A^0, A^1, A^2, A^3, whether they form a contravariant vector is a physical question that in principle has to be checked by observations.

We use the same symbol for both its covariant and contravariant representations of a 4-vector.

Ex 14.10: Show that the latter 3 components of a contra-variant/covariant vector is a spatial vector. That is, given any rotation R, for which the corresponding Lorentz transformation is

$$\Lambda = \begin{pmatrix} 1 & 0_{1\times 3} \\ 0_{3\times 1} & R_{3\times 3} \end{pmatrix}, \tag{14.33}$$

Recall that a rotation matrix R satisfies $R^T R = I$.

show that the transformation $A^\mu \to \Lambda^\mu{}_\nu A^\nu$ and $A_\mu \to A_\nu \Lambda^{-1\nu}{}_\mu$ both imply that $A_i \to R_{ij} A_j$.

Recall that a vector $\mathbf{v} = \hat{\mathbf{x}}^i v_i$ is invariant under rotations in 3 dimensional space. One can check that the components v_i and vector basis $\hat{\mathbf{x}}^i$ transform simultaneously under a rotation such that $\mathbf{v}$ is invariant. The reason behind is that the vector $\mathbf{v}$ is a single entity that has an invariant meaning. Similarly, we should think of a 4-vector A as a single entity with some invariant meaning, and its components A_μ (or A^μ) as a particular way to represent it when a reference frame is chosen.

The inner product of a covariant vector A_μ and a contravariant vector B^μ is defined as

$$A \cdot B \equiv A_\mu B^\mu, \tag{14.34}$$

and it is invariant:

$$A_\mu B^\mu \to A'_\mu B'^\mu = A^T \Lambda^{-1} \Lambda B = A_\mu B^\mu. \tag{14.35}$$

Ex 14.11: Check that $A_\mu B^\mu = A^\mu B_\mu$.

The norm of a 4-vector A is then naturally defined via $A^2 \equiv A_\mu A^\mu$.

This definition depends on the convention of $\eta_{\mu\nu}$.

A 4-vector A is said to be

- *time-like* if $A^2 < 0$.
- *light-like* if $A^2 = 0$.
- *space-like* if $A^2 > 0$.

On the trajectory of a particle moving at a speed slower than light, the coordinate difference dx^μ between two neighboring points is time-like. On the trajectory of a particle moving at the speed of light (say, a massless particle), the displacement 4-vector dx^μ is light-like. For any two events that are separated by a space-like vector, one can find an inertial frame in which these two events are simultaneous.

We can extend the definition of 4-vectors to tensors. For example, a rank-2 tensor $A_{\mu\nu}$ can be expressed as $A^{\mu\nu} \equiv \eta^{\mu\lambda}\eta^{\nu\rho}A_{\lambda\rho}$, or $A^\mu{}_\nu \equiv \eta^{\mu\lambda}A_{\lambda\nu}$, or $A_\mu{}^\nu \equiv A_{\mu\rho}\eta^{\rho\nu}$. They transform as

Recall that the ordering in a product is not important when we spell out all the indices to be summed over.

$$A_{\mu\nu} \to A'_{\mu\nu} = A_{\lambda\rho}\Lambda^{-1\lambda}{}_\mu \Lambda^{-1\rho}{}_\nu, \tag{14.36}$$

$$A^{\mu\nu} \to A'^{\mu\nu} = \Lambda^\mu{}_\lambda \Lambda^\nu{}_\rho A^{\lambda\rho}, \tag{14.37}$$

$$A^\mu{}_\nu \to A'^\mu{}_\nu = \Lambda^\mu{}_\lambda A^\lambda{}_\rho \Lambda^{-1\rho}{}_\nu, \tag{14.38}$$

$$A_\mu{}^\nu \to A'_\mu{}^\nu = \Lambda^\nu{}_\rho A_\lambda{}^\rho \Lambda^{-1\lambda}{}_\mu. \tag{14.39}$$

In general, a tensor with m contravariant indices and n covariant indices is denoted as

$$T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n}. \tag{14.40}$$

We can use the metric $\eta_{\mu\nu}$ and its inverse to raise or lower the indices to express the tensor with an arbitrary combination of k contravariant and $m+n-k$ covariant indices.

The ordering of the indices on a tensor should not be changed when the indices are raised or lowered.

In special relativity, all physical laws are invariant/covariant under Lorentz transformations.

A set of equations representing a physical law can be "covariant" under Lorentz transformations, that is, they transform into linear combinations of themselves like individual components of a tensor.

14.9.1 Lorentz transformation as matrix multiplication

Some of the calculations involving Lorentz transformations can be simplified if we use the notation of matrix multiplication. For objects with two indices, such as $\Lambda^\mu{}_\nu$, we can use the notation in which the index on the left labels the row number and the index on the right labels the column number, so that a matrix multiplication AB is implemented by contracting the neighboring indices between A and B. For example,

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}. \tag{14.41}$$

This rule is applied regardless of whether the indices are contravariant (upper) or covariant (lower).

An object with one index can be either a column or a row depending on how its index is to be contracted. For example, applying two Lorentz transformations (first Λ_1 and then Λ_2) to a contravariant vector gives

$$A^\mu \to (\Lambda_1 A)^\mu \to (\Lambda_2 \Lambda_1 A)^\mu = (\Lambda_2 \Lambda_1)^\mu{}_\nu A^\nu. \tag{14.42}$$

Similarly, for a covariant vector, it is

$$A_\mu \to (A\Lambda_1^{-1})_\mu \to (A\Lambda_1^{-1}\Lambda_2^{-1})_\mu = (A(\Lambda_2\Lambda_1)^{-1})_\mu. \tag{14.43}$$

This notation allows us to manipulate Lorentz transformations as matrices and can sometimes simplify calculations.

However, we cannot apply the matrix idea to higher rank tensors.

HW: (5-3) Let $B(\mathbf{v})$ denote the Lorentz transformation of a boost by the velocity $\mathbf{v}$ (the new reference frame moves at the velocity $\mathbf{v}$ with respect to the old reference frame). $B(\mathbf{v}_2)B(\mathbf{v}_1)$ means the Lorentz transformation of first boosting by $\mathbf{v}_1$ and then boosting by $\mathbf{v}_2$. (1) What are the 4×4 Lorentz transformation matrices for the boosts $B(\hat{\mathbf{x}}v)$, $B(\hat{\mathbf{y}}v)$ when $|v| \ll c$? (Expand $B(\hat{\mathbf{x}}v)$ and $B(\hat{\mathbf{y}}v)$ in powers of v to the 2nd order – ignoring 3rd and higher order terms.) (2) What is the 4×4 Lorentz transformation matrix for a rotation along the z-axis by an angle θ when θ is small? (Expand the matrix to the 1st order in θ.) (3) Check that

Carry out the calculation to the first nontrivial order in v_1 and v_2.

$$B(-\hat{\mathbf{y}}v_2)B(-\hat{\mathbf{x}}v_1)B(\hat{\mathbf{y}}v_2)B(\hat{\mathbf{x}}v_1) \tag{14.44}$$

is approximately equivalent to an infinitesimal rotation when $|v_1|, |v_2| \ll c$. (Expand the product of matrix in powers of v_1 and v_2, to the 2nd order terms: v_1^2, v_1v_2, v_2^2.) Express the rotation as a 4×4 Lorentz transformation matrix.

14.9.2 Lorentz transformation of fields

We will use x to represent (x^0, x^1, x^2, x^3). The Lorentz transformations of a scalar field $\phi(x)$ is given by

$$\phi(x) \quad \to \quad \phi'(x) \quad \ni \quad \phi'(x'(p)) = \phi(x(p)), \tag{14.45}$$

where $x'(p)$ and $x(p)$ are the coordinates of the same spacetime point p. This relation is usually abbreviated as

$$\phi'(x') = \phi(x). \tag{14.46}$$

Occasionally we just write $\phi' = \phi$.

It says that the value of the scalar field at a point p is independent of the coordinate system. This definition can be extended to arbitrary coordinate transformations.

Similarly, for a contravariant vector filed A^μ and covariant vector field A_μ, we have

$$A^\mu \quad \to \quad A'^\mu(x'(p)) = \Lambda^\mu{}_\nu A^\nu(x(p)), \tag{14.47}$$
$$A_\mu \quad \to \quad A'_\mu(x'(p)) = A_\nu(x(p))(\Lambda^{-1})^\nu{}_\mu \tag{14.48}$$

Q 14.37: How does a rank-2 tensor $T_{\mu\nu}(x)$ transform?

under a Lorentz transformation.

14.9.3 Invariance/Covariance of Physical Laws

Relativistic equations of motion for a physical system must be covariant under Lorentz transformations such that the complete set of equations have the same set of solutions before and after the transformation. Or, equivalently, we can say that given any solution of the equations of motion, you get another solution via Lorentz transformation.
Q 14.38: Which of the following can not be a relativistic equation of motion?
(1) $A_\mu = B_\mu{}^\nu C_\nu$, (2) $A_\mu = C_\nu B_\mu{}^\nu$, (3) $A_\mu = B_\nu{}^\lambda C_\lambda$, (4) $A_\mu = B_\nu C^\nu$, (5) $A_\mu B_\nu = 0$, (6) $A_\mu = 1$.

This notion of Lorentz invariance/covariance is a generalization of the invariance/covariance of physical laws under spatial rotation in Newtonian mechanics.

In principle, it is possible to write a set of equations such that its mathematical meaning is invariant under Lorentz transformation, but the equations do not transform linearly. However, it seems to be almost always possible to define physical quantities such that physical laws transform linearly under Lorentz transformations. For the sake of convenience in dealing with relativistic analysis, we always try to define physical quantities as covariant tensors in the context of Relativity.

An exception is the density.

14.10 Relativistic Mechanics

When dx^μ is time-like on a trajectory, define the *proper time* for dx^μ by

$$d\tau = \frac{\sqrt{-ds^2}}{c} = \sqrt{dt^2 - \frac{d\mathbf{r}^2(t)}{c^2}} = \gamma^{-1}(\mathbf{v})dt, \tag{14.49}$$

You can imagine that the proper time for the trajectory of a particle is the time defined by a clock moving with the particle.

where $\mathbf{v}(t) \equiv d\mathbf{r}(t)/dt$, for a time-like trajectory in spacetime.

Q 14.39: Is τ invariant under Lorentz transformations?

> In non-relativistic physics, we are used to use the time t as the variable so that the trajectory is given by $(x^1(t), x^2(t), x^3(t))$. In special relativity, we should include $x^0 = ct$. While one can continue to use t to parametrize the points on the trajectory, one can use any real parameter λ to parametrize the trajectory of a particle as $(x^0(\lambda), x^1(\lambda), x^2(\lambda), x^3(\lambda))$. A reparametrization $\lambda \to \lambda'(\lambda)$ does not change the trajectory. The proper time τ is in a sense the most natural way to parametrize a trajectory.

For a particle with rest mass m, define 4-velocity, 4-momentum, Minkowski force and 4-acceleration as

$$\boxed{\begin{aligned} \eta^\mu &= \frac{dx^\mu(\tau)}{d\tau}, && (14.50)\\ p^\mu &= m\frac{dx^\mu(\tau)}{d\tau}, && (14.51)\\ K^\mu &= \frac{dp^\mu(\tau)}{d\tau}, && (14.52)\\ \alpha^\mu &= \frac{d^2x^\mu(\tau)}{d\tau^2}. && (14.53) \end{aligned}}$$

$$\begin{aligned} p^0 &= m\gamma c = E/c,\\ p^i &= m\gamma v^i,\\ K^0 &= \frac{1}{c}\frac{dE}{d\tau},\\ K^i &= \frac{dp^i}{d\tau}. \end{aligned}$$

> In the old days, people like to define "effective mass" $m_{eff} \equiv m\gamma$. The advantage of this redundant definition is slim so it is no longer a popular terminology.

These are natural modifications of the Newtonian definitions of velocity, momentum and force defined such that they are 4-vectors (i.e. so that they transform linearly under Lorentz transformations).

Q 14.40: Check the following identities

$$\eta^2 \equiv \eta^\mu \eta_\mu = -c^2, \tag{14.54}$$

$$p^2 \equiv p^\mu p_\mu = -m^2c^2, \tag{14.55}$$

$$v^\mu \alpha_\mu = 0. \tag{14.56}$$

Ex 14.12: Check that the time-component p^0 of the 4-momentum of a particle is approximately a constant plus the kinetic energy of the particle when $v \ll c$. Notice that this additive constant is mc^2, which is naturally viewed as the energy of the particle when it is at rest.

Physical laws in Newtonian Mechanics are naturally extended to Relativistic Mechanics.

Newton's 2nd law is now just eq.(14.52).

Q 14.41: What is the physical interpretation of the case $\mu = 0$ in the equation (14.52)?

The laws of energy conservation and momentum conservation are now unified into one equation

$$\sum_i p^\mu_{(i)} = \text{time-independent}, \tag{14.57}$$

where i is the label of particles in a closed system.

Ex 14.13: Define

$$L^{\mu\nu} \equiv x^\mu p^\nu - x^\nu p^\mu \tag{14.58}$$

for a particle with coordinate x and momentum p. Check that L^{ij}'s $(i, j = 1, 2, 3)$ correspond to the angular momentum of the particle. While the angular momentum of a free particle is conserved, is L^{0i} $(i = 1, 2, 3)$ also conserved? What is the physical interpretation of L^{0i}?

14.11 Relativistic Formulation of Electrodynamics

Maxwell's equations are already covariant under Lorentz transformations. (Lorentz transformations were found from Maxwell's equations.) There is no need to modify them, but we can rewrite Maxwell equations so that their covariance is manifest. That is, we can redefine all physical quantities in classical electromagnetism in terms of 4-vectors and tensors so that they transform linearly under Lorentz transformations.

The gauge potentials V and $\mathbf{A}$ are unified in a 4-vector

$$A^\mu = (V/c, A_x, A_y, A_z). \tag{14.59}$$

The gauge transformation of A_μ is given by

$$\boxed{A_\mu \to A'_\mu = A_\mu + \partial_\mu \lambda.} \tag{14.60}$$

The field strength is defined as

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,} \tag{14.61}$$

which is antisymmetric

$$F_{\mu\nu} = -F_{\nu\mu}, \tag{14.62}$$

and thus there are 6 independent components in $F_{\mu\nu}$, and it is invariant under gauge transformations (14.60). The electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are unified in $F_{\mu\nu}$ as

$$F^{0i} = E^i/c, \qquad F^{ij} = \epsilon^{ijk} B_k. \tag{14.63}$$

The charge density ρ and current density $\mathbf{J}$ are combined into a current density 4-vector J_μ as

$$J^0 = c\rho, \qquad J^i = (J_x, J_y, J_z), \tag{14.64}$$

and the continuity equation can be expressed as

$$\partial_\mu J^\mu = 0. \tag{14.65}$$

Maxwell's equations can be rewritten as

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu, \tag{14.66}$$

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0. \tag{14.67}$$

The 2nd equation is called *Bianchi identity*. It is trivially valid if $F_{\mu\nu}$ is given by (14.61). Conversely, for any $F_{\mu\nu}$ that satisfies the Bianchi identity, one can always find A_μ locally such that $F_{\mu\nu}$ is given by (14.61).

It should be clear from the presence of the source term how these two equations are matched with the usual expressions of Maxwell's equations.

The Lorentz force law is

$$K^\mu = qF^{\mu\nu}\eta_\nu. \tag{14.68}$$

Ex 14.14: Find the trajectory of a point charge q in the background of a constant electric field $\mathbf{E} = \hat{\mathbf{x}}E$. (At $t = 0$, the charge is at rest at the origin.)

HW: (5-4) For the electric and magnetic fields in the inertial frame $\mathcal{S}$ given by

$$\mathbf{E}(x^\mu) = \hat{\mathbf{x}}E_0 \cos(kz - \omega t), \tag{14.69}$$

$$\mathbf{B}(x^\mu) = \hat{\mathbf{y}}\frac{E_0}{c}\cos(kz - \omega t), \tag{14.70}$$

where $k = \omega/c$, find the electric and magnetic fields in the reference frame $\mathcal{S}'$ which is moving at the velocity $\hat{\mathbf{x}}v$ with respect to $\mathcal{S}$.

Ex 14.15: A stationary magnetic dipole $\mathbf{m} = m\hat{\mathbf{z}}$ is situated above an infinite uniform surface current $\mathbf{K} = \hat{\mathbf{x}}K$.

(1) Find the torque of the dipole, using $\mathbf{N} = \mathbf{m} \times \mathbf{B}$.

(2) Suppose that the surface current consists of a uniform surface charge σ, moving at velocity $\mathbf{v} = \hat{\mathbf{x}}v$, so that $\mathbf{K} = \sigma\mathbf{v}$, and the magnetic dipole consists of a uniform line charge λ, circulating at speed v (same v) around a square loop of side l, so that $m = \lambda v l^2$. Examine the same configuration from the point of view of system $\bar{\mathcal{S}}$,

moving in the x-direction at speed v. In $\bar{S}$ the surface charge is at rest, so it generates no magnetic field. Show that in this frame the current loop carries an electric dipole moment, and calculate the resulting torque.

Ex 14.16: A point charge q moves with the velocity $\mathbf{v} = v\hat{\mathbf{x}}$ in a constant electric field $\mathbf{E} = E\hat{\mathbf{y}}$ and magnetic field background $\mathbf{B} = B\hat{\mathbf{z}}$.

1. Find the condition on E and B such that the magnetic field is absent in an inertial frame $\bar{S}$ moving in the x-direction with respect to the original frame S.

2. Assuming that the condition in the previous question is satisfied, find the relative velocity of $\bar{S}$ with respect to S.

3. Find the Lorentz force on the charge in $\bar{S}$.

4. Show that the force in $\bar{S}$ is related to that in S through suitable Lorentz transformations.

Solution:

1. In another reference frame $\bar{S}$ which moves at velocity v' with respect to S, the field configuration described in the problem corresponds to the field configuration

$$\bar{\mathbf{E}} = \gamma(v')(E - v'B)\hat{\mathbf{y}}, \qquad \bar{\mathbf{B}} = \gamma(v')(B - \frac{v'}{c^2}E)\hat{\mathbf{z}}. \tag{14.71}$$

Hence $\bar{\mathbf{B}} = 0$ if $v'/c = cB/E$. This is possible only if

$$cB < E. \tag{14.72}$$

2. The velocity of $\bar{S}$ relative to S is

$$v' = c^2 B/E. \tag{14.73}$$

3. The Lorentz force on the charge in $\bar{S}$ is

$$\bar{\mathbf{F}} = q\bar{\mathbf{E}} = \gamma(v')(E - v'B)\hat{\mathbf{y}} = \sqrt{E^2 - c^2B^2}\,\hat{\mathbf{y}}. \tag{14.74}$$

4. The Lorentz force F in S is $\mathbf{F} = q(E - vB)\hat{\mathbf{y}}$. The Minkowski force K in S is

$$K^0 = \frac{dp^0}{d\tau} = 0, \qquad \mathbf{K} = \gamma(v)\mathbf{F}. \tag{14.75}$$

(K^0 vanishes because the power is zero as the force is perpendicular to the velocity.) As the only non-zero component of K^μ is K^2, the only non-zero component of $\bar{F}$ is

$$\bar{F}^2 = \gamma^{-1}(\bar{v})\bar{K}^2 = \gamma^{-1}(\bar{v})K^2 = \gamma^{-1}(\bar{v})\gamma(v)F^2 = \gamma^{-1}(\bar{v})\gamma(v)q(E - vB), \tag{14.76}$$

where $\bar{v}$ is the velocity of the charge in $\bar{S}$. Plugging in v' from (14.73), we find exactly

$$\bar{F}^2 = \sqrt{E^2 - c^2B^2} \tag{14.77}$$

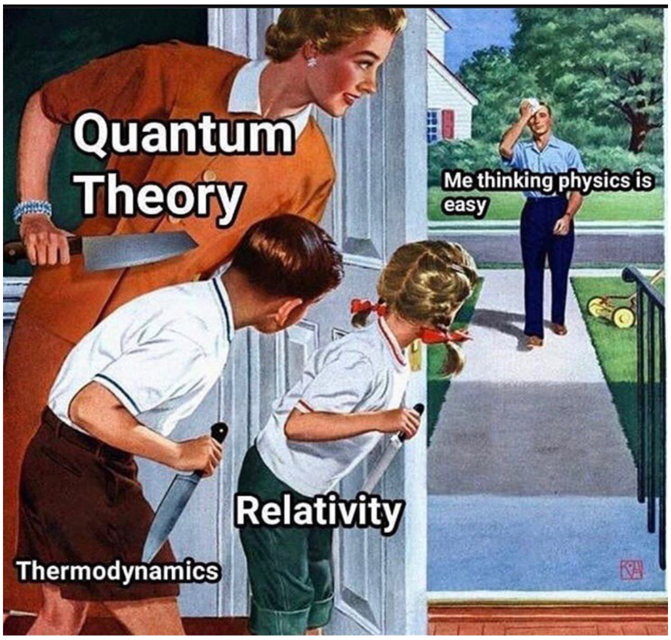
as expected.

14.12 Exercises

HW: (5-5) The trajectory of a particle of mass m in free fall is described by $x(t) = y(t) = 0$, $z(t) = h - \frac{1}{2}gt^2$, where g is the gravitational acceleration constant. (a) Find the expression $x'(t'), y'(t'), z'(t')$ for the particle in another reference frame with coordinates x', y', z', t' which moves in the x-direction at speed v relative to the first reference frame. (b) Find the final velocities of the particle in the z direction when it hits the ground at $z = 0$ for both frames. (c) If the particle carries a watch, how much time does it take for it to go from $z = h$ to $z = 0$ according to its watch?
Ex 14.17: Bob on earth sees Alice on a rocket at speed v moving away from the earth at $t = 0$, and then Alice turns around at $t = T$ with respect to the inertial frame of the earth. She returns to the earth at the same speed at $t = 2T$. (Ignore the time it takes Alice to turn the rocket around.) (a) What is the time t' for Alice when she turns around the rocket? (Assume that $t' = 0$ for Alice when she leaves Bob.) (b) At the moment when Alice is turning her rocket around, if she can see Bob's watch, what time does Bob's watch read? (c) After Alice turns around, she is at rest with respect to a new inertial frame. Let the time coordinate of that frame be denoted t''. Assume that $t'' = T'$ when she turns around the rocket. What is the time t on earth when $t'' = T'$? (d) What Alice is moving towards the earth, how must faster does Bob's watch go than the time t'' for Alice's new inertial frame? (That is, what is dt/dt''?) (e) When Alice is moving towards the earth, how much faster does Bob's watch appear to go than her own watch, when she looks directly at Bob's watch?
Ex 14.18: A particle of mass m moves under a constant force F in the x-direction. We have $F = dp/dt$ and $p = m\gamma v$. (a) Find $v(t) \equiv dx/dt$ as a function of t. Is it always less than c? (b) Find $dx/d\tau$ as a function of t, where τ is the proper time of the particle. Is it always less than c?
Ex 14.19: A wave propagating in the x-direction is described as $f(t, x) = A\sin(\omega t - kx)$. Find the angular frequency ω' and wavelength λ' of this wave observed in another inertial frame at velocity v in the x-direction with respect to the original frame. How is the new wave velocity related to the original wave velocity?


I WENT BACK IN TIME AND KILLED ADOLF HITLER!
WHO?


$E = \cancel{ma}^2$
$E = \cancel{mb}^2$
S.Harris


Quantum Theory
Me thinking physics is easy
Relativity
Thermodynamics