Exercises for E&M

1. (30%) Which one (ones) of the following is (are) correct?

   (a) An infinite slab on the $x-y$ plane carries a surface current. Its magnetic field $\mathbf{B}$ is perpendicular to the current but in opposite directions for $z>0$ and $z<0$.

   (b) The force between two parallel currents in the same direction is repulsive.

   (c) The equivalent capacitance of two capacitors C_1 and C_2 in parallel is $C_1C_2/(C_1+C_2)$.

   (d) The potential energy of a capacitor of C with charge Q is $U = Q^2/(2C)$.

   (e) The energy density of the magnetic field is $u=\mu_0 B^2/2$.

   (f) The energy density of the electric and magnetic fields is negative when there are opposite charges in presence.

   (g) For a parallel plate conductor, the capacitance is larger when the permittivity ϵ of the substance betweeen the plates is larger.

   (h) Maxwell's equations imply charge conservation.

   (i) Maxwell's equations imply Lorentz force law.

   (j) Ampere's law is modified by Maxwell to be $\oint \mathbf{B}\cdot d\mathbf{a} = \mu_0 I_{encl} + \epsilon_0 \frac{d\Phi_E}{dt}$.

   Solution:

(a), (d), (g), (h).

2. A uniformly charged sphere of radius R (it has a constant volume charge density) with total charge Q is surrounded by a concentric spherical conducting shell of inner radius a and outer radius b. (See Fig.1.) The outer surface of radius b is connected to the ground. Find the electric field $\mathbf{E}$ and potential V for (1) $r>b$, (2) $b>r>a$, (3) $a>r>R$, (4) $R>r>0$.

Solution:

(1) If the conducting shell is not grounded, we would have charge $-Q$ on the inner surface at $r = a$ and charge Q on the outer surface at $r = b$. But since the conducting shell is grounded, the charge on the outer surface vanishes, while the charge on the inner surface remains $-Q$. As the total charge is now 0, we have

$$\boxed{\mathbf{E} = 0, \qquad V = 0.} \tag{12.135}$$

(2) In the conducting shell, the electric field vanishes, and the electric potential takes the same value as the outer surface which is grounded. So

$$\boxed{\mathbf{E} = 0, \qquad V = 0.} \tag{12.136}$$

(3) The charge enclosed in a Gaussian surface is Q if the Gaussian surface has its radius r in the range $r \in (R, a)$. So Gauss's law implies that

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r},} \tag{12.137}$$

and the potential can be obtained by integrating $\mathbf{E}$ from r to $r = a$

$$\boxed{V(r) = -\int_a^r dr'\, E_r(r') = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{a}\right).} \tag{12.138}$$

Another way to obtain the result is to realize that $V(r)$ should be

$$V(r) = \frac{Q}{4\pi\epsilon_0 r} + \text{const.}, \tag{12.139}$$

where the constant should be chosen to make $V(a) = 0$. Demanding that $V(a) = 0$, the constant is determined and the same result is obtained.

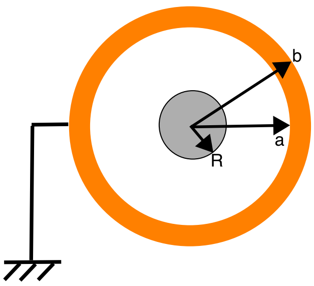


Fig.1

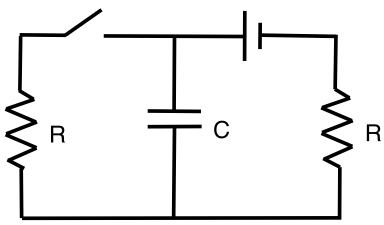


Fig.2

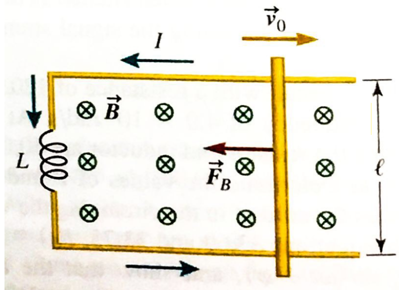


Fig.3

(4) For a Gaussian surface of radius r in the range $r \in (0, a)$, Gauss's law implies

$$4\pi r^2 E_r(r) = \frac{Q(r)}{\epsilon_0} = \frac{Q r^3}{\epsilon_0 R^3}, \tag{12.140}$$

so

$$\boxed{\mathbf{E} = \frac{Qr}{4\pi\epsilon_0 R^3}\hat{r}.} \tag{12.141}$$

Then

$$\boxed{V = V(R) - \int_R^r dr'\, E_r(r') = \frac{Q}{4\pi\epsilon_0}\left[-\frac{r^2}{2R^3} + \frac{3}{2R} - \frac{1}{a}\right],} \tag{12.142}$$

where $V(R)$ is obtained from the previous result eq.(12.138):

$$V(R) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{a}\right). \tag{12.143}$$

Again, the same result can be obtained in another way. First we realize that $V(r)$ must be of the form

$$V(r) = -\frac{Qr^2}{8\pi\epsilon_0 R^3} + \text{const.} \tag{12.144}$$

in order for $dV/dr = -E_r$ to agree with eq.(12.141). Then we demand that the constant here must ensure that eq.(12.144) agrees with eq.(12.143).

3. For the DC circuit in Fig.2, the battery has a constant voltage V_0. The capacitor has capacitance C and both resistors have the same resistance R. The circuit is closed at $t = 0$. (1) What is the voltage $V_c(0^+)$ of the capacitor immediately after the circuit is closed? (2) What is $V_c(t)$ for $t > 0$?

Solution:

(1) Before the circuit is closed, the capacitor is already charged by the battery to have the same potential:

$$\boxed{V_c(0^+) = V_0.} \tag{12.145}$$

(2) Kirchhoff's rules and the $V - I$ relations of the circuit elements imply

$$V_0 = V_c + V_{R'} = V_c + R(I_R + I_c), \tag{12.146}$$

$$V_c = V_R = RI_R, \tag{12.147}$$

$$I_c = C\frac{dV_c}{dt}. \tag{12.148}$$

They imply a differential relation on V_c:

$$V_0 = V_c + R\left(\frac{V_c}{R} + C\frac{dV_c}{dt}\right) = 2V_c + RC\frac{dV_c}{dt}. \tag{12.149}$$

This is solved by

$$V_c(t) = \frac{1}{2}V_0 + Ae^{-\frac{2t}{RC}}, \tag{12.150}$$

for an arbitrary constant A. The initial condition implies $A = V_0/2$, so

$$\boxed{V_c(t) = \frac{1}{2}V_0\left(1 + e^{-\frac{2t}{RC}}\right).} \tag{12.151}$$

The same result can be obtained in a second approach. First we find the decay rate of the circuit. When the battery is turned off, the capacitor can discharge through two resistors in parallel. The effective resistance is thus $R/2$. The current or voltage in an RC circuit has a time constant τ given by the product of the resistance and the capacitance, which is now $\tau = RC/2$. Thus we expect that, with the addition of the battery, the potential V_c is of the form

$$V_c(t) = A + Be^{-t/\tau} = A + Be^{-2t/RC}. \tag{12.152}$$

Now the two constants A and B can be determined by the initial condition (12.145) and the final state $V_c(t = \infty)$. But what is $V_c(t = \infty)$? When V_c reaches a fixed value as $t \to \infty$, there is no current passing through C. The current from the battery only passes through the two resistors, with each resistor having the voltage $V_0/2$. As the capacitor has the same voltage as the resistor on the left, the final voltage should be

$$V_c(t \to \infty) = V_0/2. \tag{12.153}$$

The two conditions (12.145) and (12.153) then determines the constant A and B.

4. For $V(r) = V_0 e^{-r/L}$ with given constants V_0 and L, find (1) the electric field $\mathbf{E}(r)$ and (2) the charge density $\rho(r)$.

Solution:

(1) Due to the spherical symmetry, we expect that $\mathbf{E} = \hat{r}E_r$, so

$$\begin{aligned} dV &= -\frac{V_0}{L}e^{-r/L}dr \\ &= -\mathbf{E}\cdot d\ell = -E_r dr, \end{aligned} \tag{12.154}$$

from which we deduce

$$\boxed{\mathbf{E} = \hat{r}\frac{V_0}{L}e^{-r/L}.} \tag{12.155}$$

(2) For the charge density $\rho(r)$, Gauss law implies

$$\oint \mathbf{E}\cdot d\mathbf{a} = \frac{Q_{encl}}{\epsilon_0}. \tag{12.156}$$

$$\Rightarrow \quad 4\pi r^2 E_r(r) = \frac{1}{\epsilon_0}\int_0^r dr'\, 4\pi r'^2 \rho(r'). \tag{12.157}$$

$$\Rightarrow \quad \frac{d}{dr}\left(r^2 E_r(r)\right) = \frac{1}{\epsilon_0} r^2 \rho(r). \tag{12.158}$$

$$\Rightarrow \quad \rho(r) = \frac{\epsilon_0}{r^2}\frac{d}{dr}\left(r^2 E_r(r)\right). \tag{12.159}$$

So

$$\rho(r) = \frac{V_0}{L}\frac{\epsilon_0}{r^2}\frac{d}{dr}\left(r^2 e^{-r/L}\right) = \frac{V_0}{L}\frac{\epsilon_0}{r^2}\left(2r - \frac{r^2}{L}\right)e^{-r/L} \qquad \Rightarrow \tag{12.160}$$

$$\boxed{\rho(r) = \frac{\epsilon_0 V_0}{L}\left(\frac{2}{r} - \frac{1}{L}\right)e^{-r/L}.} \tag{12.161}$$

5. A conducting bar of mass M and length ℓ is given an initial speed v_0 on a smooth horizontal conducting rail as shown in Fig.3. (Initially $I = 0$.) Assume the system has a constant inductance L and there is a magnetic field B into the page. Find (1) the differential equation that determines the motion of the conducting bar, (2) the maximal distance the bar travels before it stops.

Solution:

(1) The flux rule gives the emf:

$$B\ell v(t) = L\frac{dI(t)}{dt}. \tag{12.162}$$

Newton's law gives

$$-B\ell I(t) = M\frac{dv(t)}{dt}. \tag{12.163}$$

They imply

$$\boxed{(B\ell)^2 v(t) = -LM\frac{d^2 v(t)}{dt^2}.} \tag{12.164}$$

(2) The solution (up to a shift of t) is

$$v(t) = A\cos(\omega t), \tag{12.165}$$

for constant A and

$$\omega = \frac{B\ell}{\sqrt{LM}}. \tag{12.166}$$

The initial condition fixes A:

$$v(t) = v_0 \cos(\omega t). \tag{12.167}$$

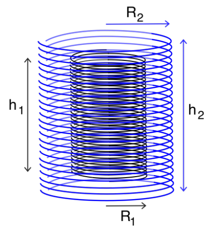
Thus

$$x(t) = x_0 + \frac{v_0}{\omega} \sin(\omega t), \tag{12.168}$$

and the maximal displacement is

$$\boxed{\Delta x = \frac{v_0}{\omega} = \frac{v_0\sqrt{LM}}{B\ell}.} \tag{12.169}$$

1. A long solenoid of radius R_1, length h_1, with N_1 turns of wire is placed inside a larger long solenoid of radius R_2 and length h_2, with N_2 turns of wire along the same axis. The current on the smaller solenoid is $I_1(t) = At$ for a constant $A > 0$. What is the magnitude of the induced *emf* on the larger solenoid?



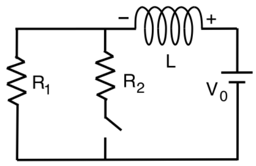
Solution:
The magnitude of the emf on the second solenoid is $|\mathcal{E}_2| = \left|\frac{d\Phi_{21}}{dt}\right| = M_{21}\left|\frac{dI_1}{dt}\right| = M_{21}A$. Since $M_{21} = M_{12}$, we calculate it as

$$M_{21} = M_{12} = \frac{\Phi_1}{I_2} = \frac{N_1 \pi R_1^2 B_2}{I_2} = \frac{N_1 \pi R_1^2 \mu_0 N_2 I_2}{h_2 I_2} = \frac{\mu_0 \pi N_1 N_2 R_1^2}{h_2}. \tag{12.170}$$

Hence,

$$\boxed{|\mathcal{E}_2| = \frac{\mu_0 \pi N_1 N_2 R_1^2 A}{h_2}} \tag{12.171}$$

2. The circuit on the right is in a time-independent state for $t < 0$. The switch is closed at $t = 0$. Define the current $I_L(t)$ according to the notation in the diagram. Find the current on the inductor L for (1) $t < 0$, (2) $t > 0$.



Solution:
(1) For $t < 0$, the current I_L is constant by assumption, so $V_L = 0$. As a result, the current on R_1, which is the same as I_L, is

$$\boxed{I_L(t < 0) = \frac{V_0}{R_1}.} \tag{12.172}$$

(2) For $t > 0$, the voltage on L is $V_L = V_0 - \frac{R_1 R_2}{R_1+R_2} I_L$, which should satisfy $V_L = L dI_L/dt$, so

$$V_0 - \frac{R_1 R_2}{R_1 + R_2} I_L = L \frac{dI_L}{dt}. \tag{12.173}$$

Use the ansatz $I_L(t) = A + Be^{-ct}$, we find

$$A = \frac{R_1 + R_2}{R_1 R_2} V_0, \qquad c = \frac{R_1 R_2}{(R_1 + R_2)L}. \tag{12.174}$$

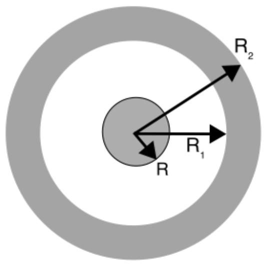
Since I_L must be continuous at $t = 0$, we have $I_L(0_+) = V_0/R_1$, so B is fixed as

$$B = -\frac{V_0}{R_2}. \tag{12.175}$$

The current is thus

$$\boxed{I_L(t > 0) = \frac{R_1 + R_2}{R_1 R_2} V_0 - \frac{V_0}{R_2} e^{-\frac{R_1 R_2 t}{(R_1+R_2)L}}.} \tag{12.176}$$

3. Given the spherically symmetric charge density $\rho(r) = \frac{A}{r}$ for a constant A for $r < R$. (1) Assuming that $\rho(r) = 0$ for $r > R$, what is the electric potential energy U of this charged ball of radius R? Next, we surround the charged ball by a thick concentric conducting shell of inner radius R_1 and outer radius R_2, and fill the gap between R and R_1 by a linear dielectric with the electric permittivity ϵ. The conducting shell has a total charge Q. (2) What are charges on the inner and outer surfaces, and the bulk of the thick conducting shell? (3) What is the electric field $\mathbf{E}(r)$ in the regions (a) $R > r > 0$, (b) $R_1 > r > R$, (c) $R_2 > r > R_1$, (d) $r > R_2$. (4) What is the electric potential at the origin? (The potential is 0 at spatial infinities.)



Solution:

(1) The total charge in the ball of radius r (for $r < R$) is

$$q(r) = \int_0^r dr' 4\pi r'^2 \rho(r') = 4\pi A \int_0^r dr' r' = 2\pi A r^2. \tag{12.177}$$

Using Gauss's law, the electric field is

$$\mathbf{E}(r) = \hat{\mathbf{r}} \frac{q(r)}{4\pi\epsilon_0 r^2} = \hat{\mathbf{r}} \frac{A}{2\epsilon_0} \tag{12.178}$$

for $r < R$ and it is $\mathbf{E}(r) = \hat{\mathbf{r}} \frac{q(R)}{4\pi\epsilon_0 r^2}$ for $r > R$. The electric potential for $r < R$ is

$$V(r) = \int_r^\infty dr' \, E_r(r') = \frac{A}{2\epsilon_0}(R - r) + \frac{q(R)}{4\pi\epsilon_0 R} = \frac{A}{2\epsilon_0}(2R - r). \tag{12.179}$$

The potential energy is

$$U = \frac{1}{2}\int_0^R dr 4\pi r^2 \rho(r) V(r) = \pi \frac{A^2}{\epsilon_0}\int_0^R drr(2R - r) = \frac{2\pi A^2 R^3}{3\epsilon_0}. \tag{12.180}$$

That is

$$\boxed{U = \frac{2\pi A^2 R^3}{3\epsilon_0}.} \tag{12.181}$$

(2)

There is no charge in the bulk of the conducting shell.

There is a uniform charge distribution on the inner and outer surfaces of the conducting shell of total charges

$$Q_{in} = -2\pi A R^2, \tag{12.182}$$

$$Q_{out} = Q + 2\pi A R^2. \tag{12.183}$$

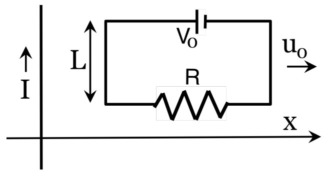
(3)

$$\begin{aligned}
(a) \qquad & \mathbf{E}(r) = \tfrac{1}{4\pi\epsilon_0}\tfrac{q(r)}{r^2}\hat{\mathbf{r}} = \tfrac{A}{2\epsilon_0}\hat{\mathbf{r}}, \\
(b) \qquad & \mathbf{E}(r) = \tfrac{1}{4\pi\epsilon}\tfrac{q(R)}{r^2}\hat{\mathbf{r}} = \tfrac{AR^2}{2\epsilon r^2}\hat{\mathbf{r}}, \\
(c) \qquad & \mathbf{E}(r) = 0, \\
(d) \qquad & \mathbf{E}(r) = \tfrac{1}{4\pi\epsilon_0}\tfrac{Q_{out}}{r^2}\hat{\mathbf{r}} = \tfrac{1}{4\pi\epsilon_0}\tfrac{Q+2\pi AR^2}{r^2}\hat{\mathbf{r}},
\end{aligned} \tag{12.184}$$

(4)

$$V(0) = \int_0^\infty E_r(r)dr = \frac{AR}{2\epsilon_0} + \frac{AR(R_1 - R)}{2\epsilon R_1} + \frac{Q + 2\pi AR^2}{4\pi\epsilon_0 R_2}. \tag{12.185}$$

4. An infinite straight wire along the z-axis (at $x = 0, y = 0$) carries a constant current I. A rectangular loop on the $x - z$ plane moves at a constant velocity $\mathbf{u} = \hat{\mathbf{x}}u_0$. The left side of the loop has $x(t) = x_1 + u_0 t$, and the right side has $x(t) = x_2 + u_0 t$ for constants $x_1 < x_2$. (The lengths of the sides of the loop are L and $(x_2 - x_1)$.)



(1) What is the *induced emf* in the loop? Give both its direction and magnitude.

(2) Let the loop be a conducting wire including a battery of voltage V_0 and a resistance R. What is the external force needed on the loop to keep its velocity at u_0?

Solution:

(1) Due to the current I, we have $\mathbf{B} = \hat{\phi}\frac{\mu_0 I}{2\pi r}$, and so

$$\Phi_B = L\int_{x_1+u_0t}^{x_2+u_0t} dx \frac{\mu_0 I}{2\pi x} = \frac{\mu_0 I L}{2\pi}\log\left(\frac{x_2+u_0t}{x_1+u_0t}\right). \tag{12.186}$$

The emf is

$$\boxed{\mathcal{E} = -\frac{d}{dt}\Phi_B = -\frac{\mu_0 I L u_0}{2\pi}\left(\frac{1}{x_2+u_0t} - \frac{1}{x_1+u_0t}\right) = \frac{\mu_0 I L u_0}{2\pi}\frac{(x_2-x_1)}{(x_2+u_0t)(x_1+u_0t)} \quad \text{It points into the paper.}} \tag{12.187}$$

(2) The total emf of the loop of wire is $\mathcal{E} - V_0$. The current in the loop is $(\mathcal{E} - V_0)/R$. The magnetic field created by the long straight wire exerts a force on this current:

$$\hat{\mathbf{x}}(B(x_2+u_0t) - B(x_1+u_0t))L\frac{(\mathcal{E}-V_0)}{R} = -\frac{\mu_0^2 I^2 L^2 u_0}{4\pi^2 R}\frac{(x_2-x_1)^2}{(x_2+u_0t)^2(x_1+u_0t)^2}\hat{\mathbf{x}}. \tag{12.188}$$

We need to balance it with an external force

$$\boxed{\mathbf{F}_{ext} = \frac{\mu_0 I L}{2\pi R}\left(-V_0 + \frac{\mu_0 I L u_0}{2\pi}\frac{(x_2-x_1)}{(x_2+u_0t)(x_1+u_0t)}\right)\frac{(x_2-x_1)}{(x_2+u_0t)(x_1+u_0t)}\hat{\mathbf{x}}} \tag{12.189}$$