Chapter 15

General Relativity

General Relativity is a theory of gravity. It is also a theory of space-time. It claims that it is possible to view gravity not as a force but as a property of the spacetime.

The mathematics of General Relativity is *Riemannian Geometry*. Riemannian Geometry studies the geometric properties based on the notion of distance, which is defined in terms of the *metric*.

General Relativity identifies physical notions related to gravity with (Riemannian) geometric notions of the spacetime. The reason why we feel the gravitational force is that the space is curved. Gravitational force is no longer viewed as a real force, but rather as a *fictitious force*.

15.1 Riemannian Geometry

Q 15.1: Why do we need the notion of distance? Can you imagine a universe without the notion of distance?

The primitive notion of space is just a set of points. Next we have the information about how points are connected with one another. The notion of distance can be considered as an extra construction (structure) on a space. The same space (as a set of points) can be equipped with all sorts of different definitions of distance (length). In General Relativity, for example, the notion of distance is determined dynamically by physical fields (the gravitational field – metric) over the 3+1 dimensional spacetime.

Imagine a piece of flexible cloth with marked points. The distance between two marked points depends on how the cloth is stretched.

> The distance L between two points on a piece of paper is often assumed to satisfy the relation $L^2 = (\Delta x)^2 + (\Delta y)^2$, where Δx and Δy are differences in the x and y Cartesian coordinates of the two points. However, we can also plot the world map on a piece of paper. The real distance between two points on the map no longer satisfies the Pythagorean theorem.

With the notion of distance, one can define *geodesics* as the curve/line connecting two given points on the space with the shortest length. They generalize the notion of straight lines in Euclidean geometry.

Q 15.2: What are the geodesics on a sphere? What happens to "parallel lines" and "triangles" on a sphere?

In Euclidean geometry, two parallel lines never intersect, and the sum of the three internal angles of a triangle is always π. These relations are modified for a generic curved space.

In Riemannian geometry, the notion of distance is given in the form of a line element ds defined by

The metric defines distance.

$$\boxed{ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu. \qquad (15.1)}$$

Einstein's summation convention is used here, and the functions $g_{\mu\nu}(x)$ are the components of a metric. The metric tensor $g_{\mu\nu}(x)$ is by definition symmetric

$$g_{\mu\nu}(x) = g_{\nu\mu}(x). \tag{15.2}$$

There are thus only 10 independent functions for the 16 components of the metric tensor for the $3+1$ dimensional spacetime.

Examples of Riemannian geometry:

1. 2D Euclidean space $\mathbb{R}^2$:

$$ds^2 = dx^2 + dy^2. \tag{15.3}$$

We do not need to use the Cartesian coordinates. In terms of a generic coordinate system (x', y'), if they are related to (x, y) via $x = x(x', y')$ and $y = y(x', y')$, the metric can be written in terms of the (x', y') coordinate system as

$$ds^2 = g_{x'x'}(x', y')dx'^2 + 2g_{x'y'}(x', y')dx'dy' + g_{y'y'}(x', y')dy'^2, \tag{15.4}$$

If we see a metric like eq.(15.4), it is not easy to tell immediately whether the notion of distance is the same as that on a flat Euclidean $\mathbb{R}^2$.

where

$$g_{x'x'} = \left(\frac{\partial x}{\partial x'}\right)^2 + \left(\frac{\partial y}{\partial x'}\right)^2, \tag{15.5}$$

$$g_{x'y'} = \frac{\partial x}{\partial x'}\frac{\partial x}{\partial y'} + \frac{\partial y}{\partial x'}\frac{\partial y}{\partial y'}, \tag{15.6}$$

$$g_{y'y'} = \left(\frac{\partial x}{\partial y'}\right)^2 + \left(\frac{\partial y}{\partial y'}\right)^2. \tag{15.7}$$

Ex 15.1: For each of the metric below,

$$ds^2 = u^2du^2 + \frac{dv^2}{v^2}, \tag{15.8}$$

$$ds^2 = 2du^2 + 2dudv + dv^2. \tag{15.9}$$

try to find a coordinate system (x, y) in which the metric is simply eq.(15.3).

Conversely, if we are given a generic metric like eq.(15.4), we can always try to "simplify" the metric via change of coordinates. Since there are 3 independent functions $g_{x'x'}, g_{x'y'}, g_{y'y'}$, and there are two arbitrary functions $x = x(x', y')$, $y = y(x', y')$ to use, there is only one independent functional degree of freedom that really determines the notion of distance in 2D spaces.

2. 1D Euclidean space $\mathbb{R}$:

$$ds^2 = dx^2. \tag{15.10}$$

Ex 15.2: Show that a generic metric $ds^2 = g_{xx}(x)dx^2$ for a 1-dimensional space is always equivalent to the metric (15.10).

All 1-dimensional spaces are flat, including S^1!

3. Minkowski space:

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -(dx^0)^2 + (dx^i)^2. \tag{15.11}$$

4. Sphere S^2:

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 + r^2)^2}, \tag{15.12}$$

where $r^2 = x^2 + y^2$.

Ex 15.3: Check that the same ds^2 can be expressed as

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{15.13}$$

via

$$x = \tan(\theta/2)\cos\phi, \qquad y = \tan(\theta/2)\sin\phi. \tag{15.14}$$

This is the stereographic projection of a sphere, i.e., the projection of the sphere onto the tangent plane of the south pole from the north pole.

Ex 15.4: Show that eq.(15.13) is well approximated by that of a 2-dimensional Euclidean space for $0 \le \theta \ll 1$ around the north pole with (θ, ϕ) identified with the radial coordinate system.

Convince yourself that the minimal length between two points on S^2 is the length of a curve on a great circle. It is therefore natural to think of the great circles as the analogues of straight lines on S^2.

Ex 15.5: Find an example of a triangle on S^2 for which the sum of its three inner angles is larger than π.

5. Pseudosphere:

$$ds^2 = \frac{dx^2 + dy^2}{(1 - r^2)^2}, \tag{15.15}$$

where $r^2 = x^2 + y^2 \le 1$.

This is the Poincaré disk model (aka Lobacheski disk) of the pseudo-sphere.

6. Cylinder:

$$ds^2 = dz^2 + r^2 d\phi^2, \tag{15.16}$$

where r is a constant — the radius of the cylinder. Notice that this metric is equivalent to the metric of the 2D Euclidean space

$$ds^2 = dx^2 + dy^2 \tag{15.17}$$

via a coordinate transformation

$$x = z, \qquad y = r\phi. \tag{15.18}$$

> Apart from the periodic boundary condition on ϕ, the metric of the cylinder is the same as a flat 2D space. Therefore, locally, there is no way to distinguish the cylinder from the flat 2D space. Hence we must conclude that the cylinder (by itself, not as a subspace embedded in a flat 3D space) is flat in Riemannian geometry.

Notice that, for a generic curved space, there is no unique natural choice of the coordinate system.

> On the 2-dimensional Euclidean space $\mathbb{R}^2$, for any point A outside any straight line L, there is a unique straight line passing through A that has no intersection with L. This is a postulate of the Euclidean geometry. For S^2, there is no geodesic passing through A that has no intersection with a given geodesic L. This is an example of the *elliptic geometry*. As an example of the *hyperbolic geometry*, the pseudosphere has infinitely many geodesics passing through A that have no intersection with a given geodesic L.

If we assume that there is nothing beyond the universe, we should not think of the spacetime as a curved space embedded in a flat space. We must define the notion of "flatness" and "curvature", or other geometric properties of the spacetime in a way independent of how a spacetime can be embedded in a higher dimensional flat spacetime (which by definition has no physical meaning).

In other words, what we want is a notion of spacetime geometry which can be "observed" or "measured" by observers *within* this spacetime. Furthermore, we want to define geometric properties which are "objective", that is, independent of our choice of the coordinates.

Recall that, for a 1D space, the metric (15.1) is always locally $ds^2 = \pm dx^2$ or $ds^2 = 0$ with a proper choice of the coordinate x. Hence, all 1D spaces are flat in Riemannian geometry.

Riemannian geometry is asking the question: Assuming nothing but the notion of distance between any two neighboring events, and assuming that it is experimentally determined to be of the form of the metric (15.1) (equivalently, assuming that it is approximately flat in a small neighborhood), what are the "intrinsic properties" one can define?

In Riemannian geometry, there is a notion of *curvature*, as an intrinsic property, that is completely determined by the metric.

This is how we can talk about *curved* spacetime in General Relativity without having to think of a higher dimensional flat space in which the universe is embedded. (This is also the revolutionary contribution of Riemann.) You should learn to imagine a curved space without embedding it in a flat space. The notion of distance itself is sufficient to tell whether and how the space is curved.

Ex 15.6: Argue that the metric of any 2D Riemannian space can be put into the form

$$ds^2 = e^{\phi(x,y)}(dx^2 + dy^2) \tag{15.19}$$

for some function $\phi(x, y)$.

According to Riemannian geometry, the scalar curvature of a 2D space with the metric (15.19) is, up to a numerical constant factor, $-e^{-\phi}(\partial_x^2 + \partial_y^2)\phi$.

Ex 15.7: Consider a 2-sphere of radius R and express its metric in the form of eq.(15.19). Show that $-e^{-\phi}(\partial_x^2 + \partial_y^2)\phi$ is constant on the 2-sphere and it is larger for smaller R.

At any given point $x = x_0$ in the 4D spacetime, the spacetime metric $g_{\mu\nu}(x_0)$ is a symmetric 4×4 matrix. One can try to diagonalize the matrix so that $g(x_0) = M^T DM$, where D is diagonal. We can then define a new coordinate system x' by $x' = Mx$ so that ds^2 is, at this point $x = x_0$, given by $dx^T g(x_0)dx = dx'^T Ddx'$. Clearly, whenever this is possible, one can scale the coordinates x' so that the elements of D are either $+1, -1$ or 0's. If $D = diag(-1, 1, 1, 1)$, a small neighborhood of x_0 resembles the Minkowski space.

A sufficiently small neighborhood of any point on a (smooth) curved space can be approximated by a small region of the tangent space at that point.

In the examples considered above, $\mathbb{R}^2$, S^2 and the pseudosphere, have Euclidean signature ($D = diag(1, 1, 1, 1)$). The Minkowski space has the Lorentzian signature ($D = diag(-1, 1, 1, 1)$).

Physicists refer to pseudo-Riemannian space just as Riemannian space.

Some more terminology: A space (or a manifold) with a metric $g_{\mu\nu}$ that is positive definite is called a Riemannian space (Riemannian manifold), or a space with Euclidean signature. A space with a metric that has 1 positive and 3 negative (or 3 positive and 1 negative, depending on your convention) eigenvalues is called a pseudo-Riemannian space, or a space with a Lorentzian signature.

The notion of *curvature* in geometry is defined for a given notion of *parallel transport*. If you carry a vector along a closed curve by this "parallel transport", the vector may be changed when you return to the starting point. Roughly speaking, there is a larger curvature when there is a larger change in the parallel-transported vector over a given closed loop.

In Riemannian geometry, there is a natural notion of parallel transport when the notion of distance is given. This parallel transport is defined such that if you parallel-transport the tangent vector of a geodesic along itself, it remains being the tangent vector of the geodesic.

15.2 The notion of inertial frame – – from special to general relativity

The definition of inertial frames is ambiguous.

Usually, the term *inertial frame* refers to those reference frames in which Newton's first law holds: objects at rest remain at rest, objects in motion move at constant velocities, assuming that the objects are not subject to external forces.

But how do we know whether there is external force or not? If you ask a friend this question (without bringing up the definition of inertial frame first), very likely his/her answer is this: You know there is an external force if the object is not moving at constant velocity. Is this tautology?

Usually, we assume that interactions in nature are *local*. Longer distance implies weaker force. If a particle is far from everything else, one can assume that its external force can be ignored. When the particle is moving with constant velocity, you are justified to believe that your reference frame is inertial.

Is it possible that there are unknown nonlocal forces that can not be ignored at large distances? How do we know that there are no such forces on earth?

However, how far is far enough? There can always be an extremely massive star far away from you so you can't see it, but it makes everything around you accelerating towards it.

Q 15.3: If you wake up in a small room with no window, can you tell whether you are in a rocket floating in space or inside an elevator in free fall?

Q 15.4: If you wake up in a space rocket with windows, can you tell whether the rocket is floating in space or in free fall towards a giant dark star very very far away?

Let us assume that the notion of inertial frame can be defined for an infinitesimal chunk of spacetime. Physical laws should look the same in all infinitesimal inertial frames, e.g. in a free-falling elevator as well as a floating space rocket. The technical issue involved in this proposal is that you have to know how to carry out the transition from one infinitesimal local patch of spacetime to the next patch.

When we talk about the universe at the length scale of 10^{10} light years, a reference frame defined in a region of diameter 10 light years can be practically considered "infinitesimal".

Equivalence Principle

Einstein proposed that an infinitesimal region around an observer in free fall can be viewed (approximately) as an inertial frame within a sufficiently short time, and physical laws in this frame should look the same as they do in other inertial frames.

> Physical laws are used to make predictions, which are conditional statements such as *if certain conditions $A_1, A_2, \cdots, A_n$ are satisfied, we will observe certain results $B_1, B_2, \cdots, B_m$*. Therefore, according to Einstein's equivalence principle, all physical laws that apply to a small spacetime region in an inertial frame should also apply to a small space region in the freely falling frame.
> One can check that Newton's law, Maxwell's equations, etc. are in agreement with the equivalence principle.

Thus we can "undo" gravity by choosing the reference frames of free falls.

Conversely, the reference frame of a lab staying on the surface of the earth is an accelerating frame (even if the earth is floating in space without rotating), since objects under no external force (apart from gravity) move with acceleration. Gravity is identified with the fictitious force in this accelerating frame.

free-falling frame = inertial frame

Gravity is no longer considered a "real" force. **Gravity is a fictitious force** experienced in an accelerated frame. If you are not in a free-falling frame, physical laws in your reference frame looks different from those in an inertial frame (e.g. a lab in a floating space rocket), due to the fictitious force called gravity.

Q 15.5: Is a lab in a satellite circulating around the earth an inertial frame?

Lorentz force law $K^\mu = qF^{\mu\nu}\eta_\nu$ is expected to hold in both rockets floating in space and elevators in free fall.

In Special Relativity, the notion of an absolute inertial frame is abandoned. Inertial frames exist by themselves. The only clue about whether a reference frame is inertial or not is Newton's 1st law of motion.

Physical laws are all purely "if A then B" relations. Einstein's claim: If Newton's first law of motion is satisfied in a reference frame, physical laws in this reference frame must look exactly the same as physical laws in any other inertial frame.

For instance, if physical quantities A, B^μ, C^μ are found to satisfy the relation $A = B_\mu C^\mu$ in the labs of rockets floating in space, we would expect the same relation in the labs of free-falling elevators.

In general, if the physical laws (at a point in spacetime) are expressed in terms of vectors and tensors of the tangent space (at that point), it would be easy (with the help of Riemannian geometry) to express the physical laws in an arbitrary coordinate system. We can therefore write down the physical laws in general coordinate systems. The feature of physical laws that they look the same in general coordinate systems (in general reference frames) is called **general covariance**.

Q 15.6: How do you express Lorentz force law in General Relativity?

Geodesics = Trajectories of Free Particles

In Newtonian mechanics, free particles move in straight lines with constant velocities (straight lines in spacetime diagrams). In General Relativity, free particles move along geodesics of spacetime, which is the natural generalization of straight lines.

The great circles on the surface of a sphere, for example, can be viewed as curves composing of infinitely many infinitesimal straight lines in each infinitesimal (approximately flat) region on the sphere.

Gravitational Mass = Inertial Mass

For Newtonian gravity, it was unknown why gravitational charges (which is defined by the strength of the gravitational force between objects) happen to agree with the inertial masses (which is defined by $\mathbf{F} = m\mathbf{a}$ for any force $\mathbf{F}$) for everything. This puzzle is solved in Einstein's theory of gravity, where the gravitational force is a fictitious force. Like all fictitious forces, it is proportional to the inertial mass.

Einstein Equations

In Newton's theory of gravity, the source of gravity is mass, or rather mass density. The mass density can be viewed as part of the energy density (up to a factor of c^2), which is included as a component of the *energy-momentum tensor* (or *stress tensor*) in Special Relativity. Hence, in General Relativity, the source of gravity is the energy-momentum tensor $T_{\mu\nu}$. Einstein's theory of gravity first identifies the effect of gravitational force with the curvature of spacetime, and then it determines the curvature of spacetime via the Einstein equation $G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$, which relates the curvature of spacetime to the energy-momentum tensor.

G_μ is called the Einstein tensor. It is a function of the metric and its first and second derivatives.

The energy-momentum tensor (also called stress energy tensor, or just stress tensor) has components representing energy density, momentum density and pressure, etc.

The energy-momentum tensor is symmetric: $T_{\mu\nu} = T_{\nu\mu}$. T_{00} corresponds to the energy density, and $T_{0\mu}$ the momentum density.

"Biggest blunder of my life"
Einstein added a term called the *cosmological constant* to the Einstein equation so that it has static solutions. He learned later that the universe is actually not static. There was no need of the cosmological constant. Einstein regretted that he did not predict the expansion of the universe from his equations. On the other hand, in modern cosmology, the cosmological constant is considered as a candidate of the dark energy.

If the universe is static and infinite, there should be starlight in our eye sight where ever we look, and the night sky would have been bright.

15.3 Applications

Einstein's theory of gravity has many successful applications.

1. Perihelion precession of Mercury.

   Perihelion precession of Mercury $\sim$ 5600 arc seconds per century. Deviation from Newton's theory by $\sim$ 43 arc seconds per century. (1 arc minute = $1°/60$, 1 arc second = 1 arc minute/60)

2. Red-shift of light (Pound-Rebka experiment)

   γ rays at Jefferson Lab, Harvard University.

   The experiment used the technique of Nuclear Magnetic Resonance.
   The result is also consistent with Quantum Mechanics in Newton's theory of gravity.

3. Gravitational lensing
   e.g. deflection of light by the sun

   Eddington experiment:
   total solar eclipse of May 29, 1919
   two expeditions: West African island of Príncipe, and Sobral, Brazil.

   Cavendish and Soldner pointed out that star light bends around the sun in Newtonian physics. Einstein got the same answer in the beginning, but later found that the correct answer differs by a factor of 2. Eddington led two teams in 1919 to verify Einstein's theory.

4. Cosmology
   Q 15.7: Why is the sky dark at night?
   Q 15.8: What is the age of the universe?
   Q 15.9: What happens before the big bang? Is there no time before the big bang?

The metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{15.20}$$

can be used to describe a spatially flat, isotropic spacetime. The scale factor $a(t)$ describes how the spatial part expands or contracts. Through observations about the geometry of the universe, Einstein equations tell us the energy-momentum tensor $T_{\mu\nu}$. It is deduced that $T_{\mu\nu}$ is composed of dark energy $\sim 70\%$, dark matter $\sim 25\%$, and the baryonic matter we understand is only $\sim 5\%$.
The *big bang* ($a(t) \sim 0$) happened $\sim 1.4 \times 10^{10}$ yr ago, followed by *baryogenesis*, *nucleosynthesis*, *recombination*, ant then the formation of large scale structures such as galaxies, etc.
Assuming an era of *inflation* before the big bang, quantum field theory predicts the *Cosmic Microwave Background Radiation* $\sim 3°K$ that has been observed.

5. Gravitational waves

Using the arrival times of pulses from the pulsar in a binary system to give the first evidence of gravitational waves. (Nobel Prize 1993)
Gravitational waves from binary black hole merger have been directly observed by LIGO. (Nobel Prize 2017)

6. GPS

SR: The atomic clocks tick more slowly than stationary ground clocks by a factor of $\frac{v^2}{c^2} \approx 10^{-10}$, where the orbital velocity is $v = 4km/s$.
GR: a clock closer to a massive object runs slower than a clock farther away. The GPS receivers are much closer to Earth than the satellites, causing the GPS clocks to be faster by a factor of 5×10^{-10}.

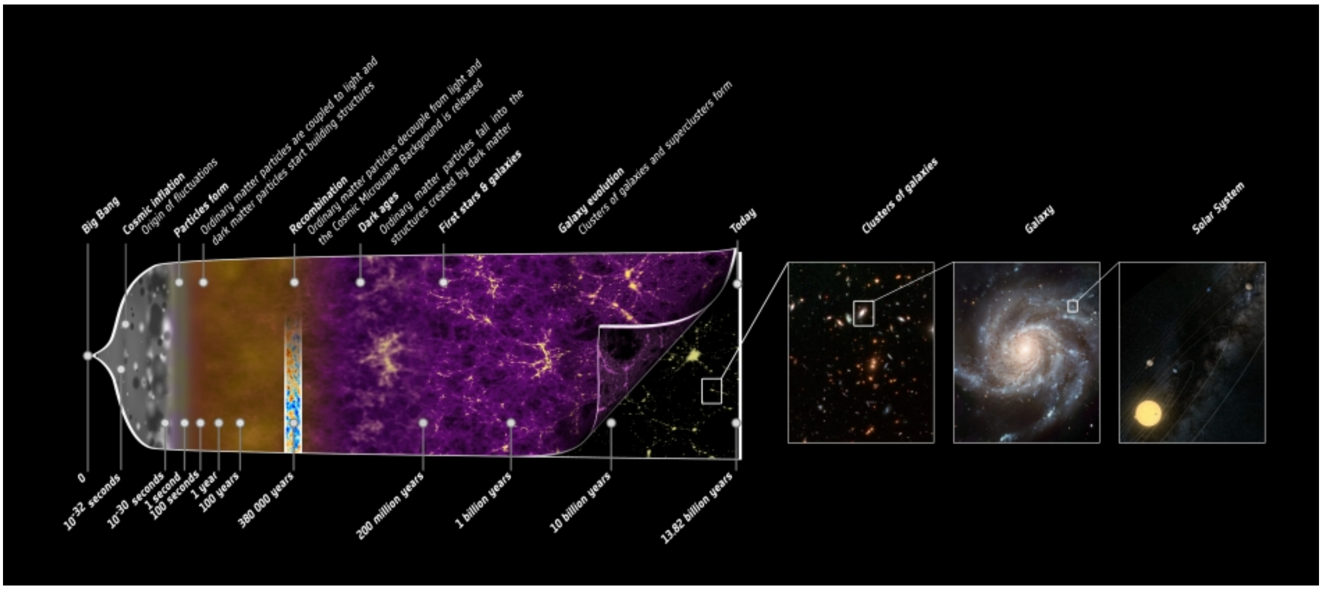
7. Black Holes
Q 15.10: What is a black hole?
Q 15.11: Does a black hole have an event horizon?

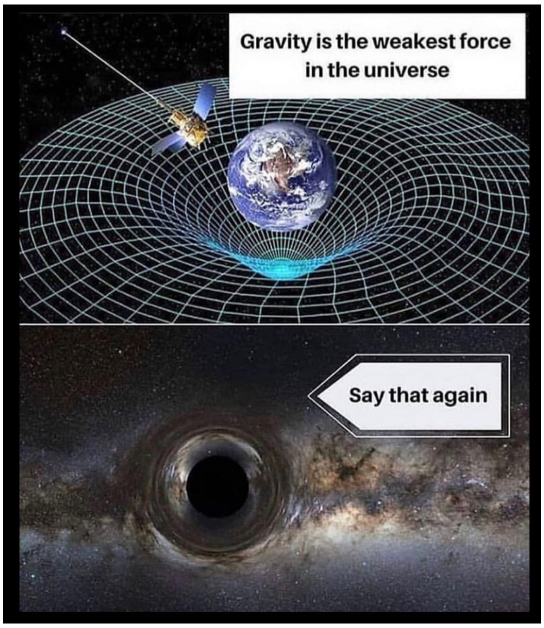
Making the analogy between space (light) and fluid (sound), a black hole can be understood as a dumb Hole.

Some sort of BHs are also predicted in Newton's gravity.

Is the title "Event Horizon Telescope" appropriate?

8. wormholes, warp drive, ...?





"Oh, no—Karen baked a cake so dense that not even light can escape."