Chapter 4 Orthogonality (正交)

4-1 內積 (inner product)

1) 向量 x 的長度

2) 二個向量 x 與 y 的距離

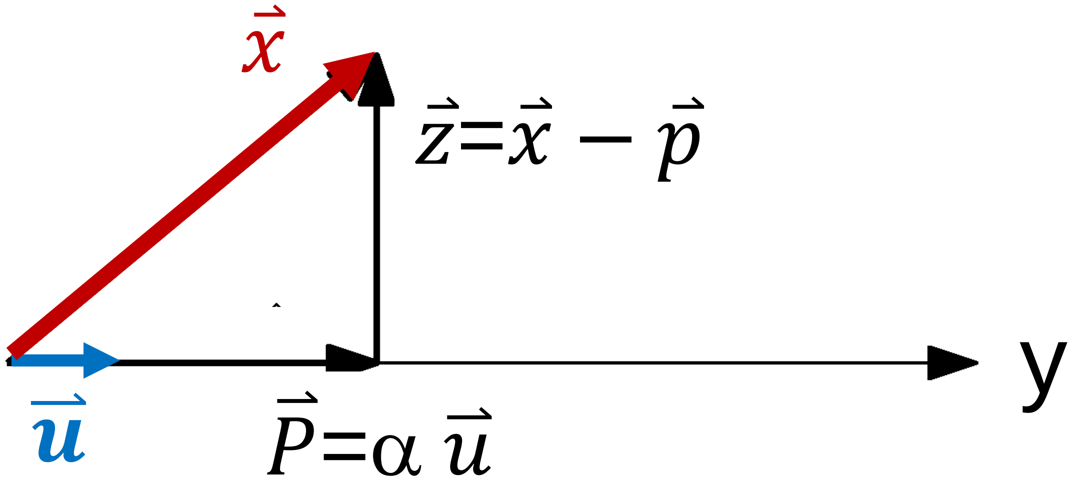
3) 內積 (inner product)

4) 正交 (orthogonality)

4-1 內積 (inner product)

4) 正交 (orthogonality)

--- $\boldsymbol{x^T y = 0}$



(a) 純量投影 (scalar projection)
(b) 向量投影 (vector projection)

4-1 內積 (inner product)

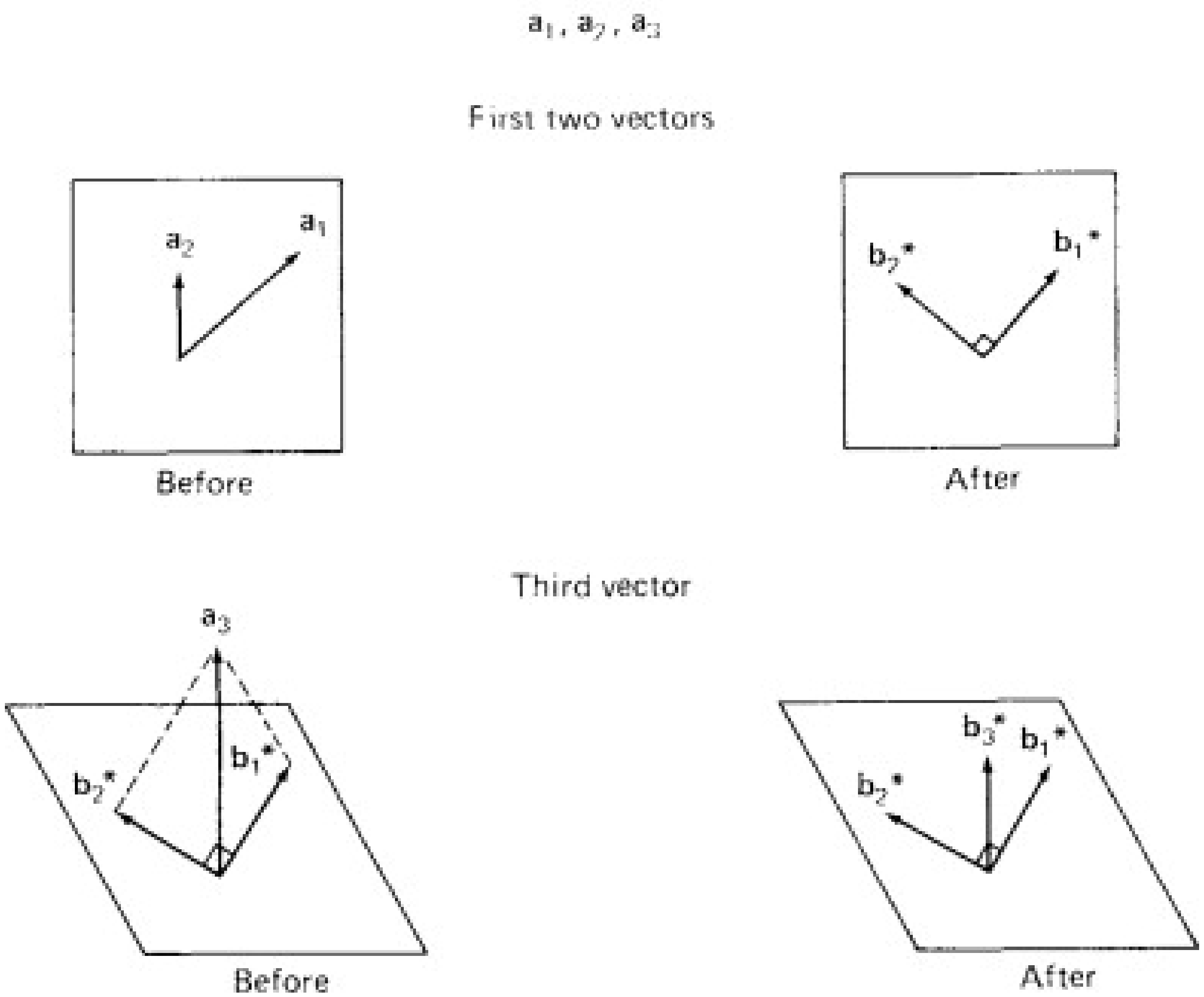
定理 一：$\mathbf{R}^n$ 的畢氏定理 (Pythagorean theorem in R^n)

定理 二：**Cauchy-Schwarz** 不等式

定理 三：三角不等式 (Triangle inequality)

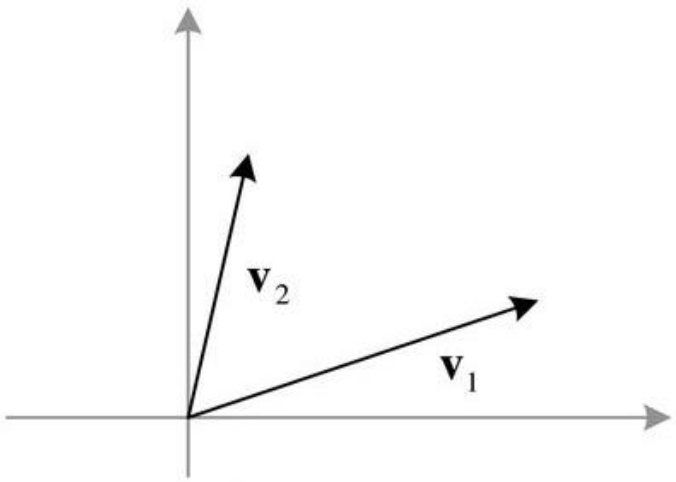
定理 四：平行四邊形定律 (Parallelogram law)

4-2 Gram-Schmidt 正交法

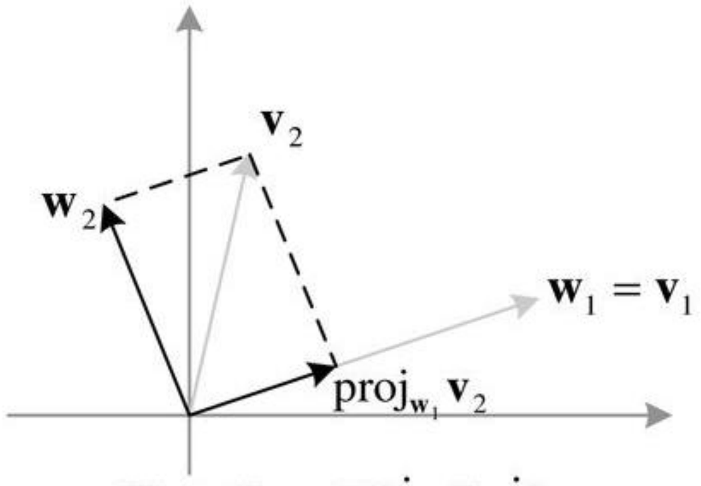


4-2 Gram-Schmidt 正交法

- The geometric intuition of the Gram-Schmidt process to find an orthonormal basis in R^2



$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for R^2



$\mathbf{w}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{w}_1}\mathbf{v}_2$ is orthogonal to $\mathbf{w}_1 = \mathbf{v}_1$

$$\Rightarrow \left\{\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}\right\} \text{ is an orthonormal basis for } R^2$$

4-2 Gram-Schmidt 正交法

Let $W = \text{span}(\vec{x}_1, \vec{x}_2, \vec{x}_3)$, where $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}.$

Construct an orthogonal basis for W.

Gram-Schmidt Process:

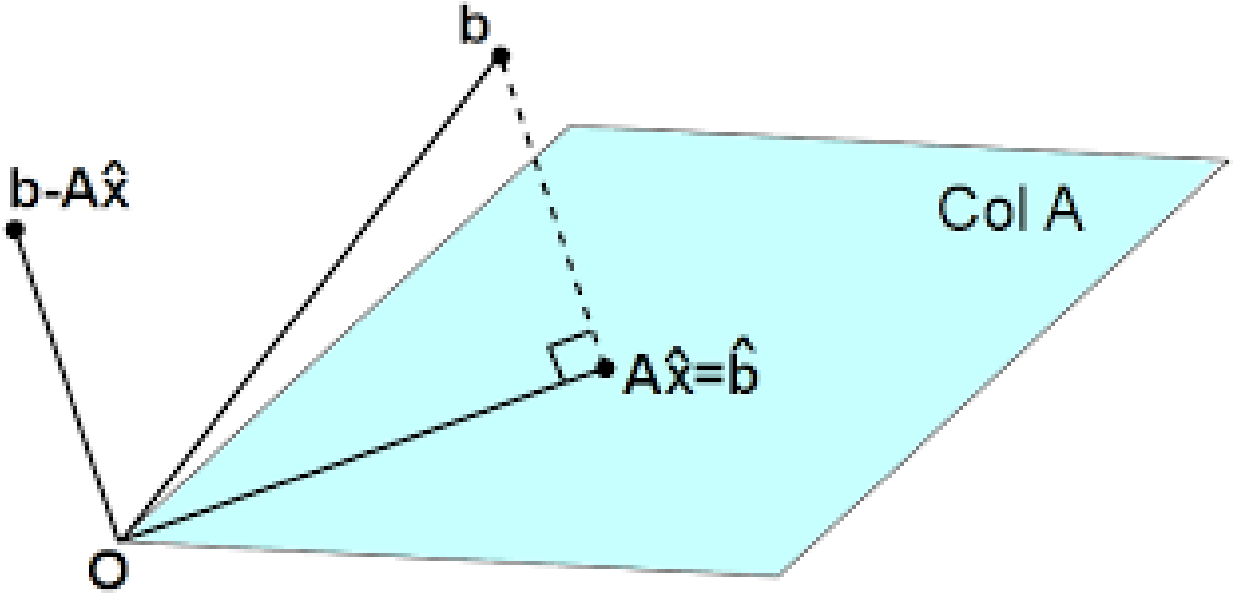
$\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = \vec{x}_2 - \frac{(\vec{v}_1\cdot\vec{x}_2)}{(\vec{v}_1\cdot\vec{v}_1)}\vec{v}_1$

$\vec{v}_3 = \vec{x}_3 - \frac{(\vec{v}_1\cdot\vec{x}_3)}{(\vec{v}_1\cdot\vec{v}_1)}\vec{v}_1 - \frac{(\vec{v}_1\cdot\vec{x}_3)}{(\vec{v}_2\cdot\vec{v}_2)}\vec{v}_2$

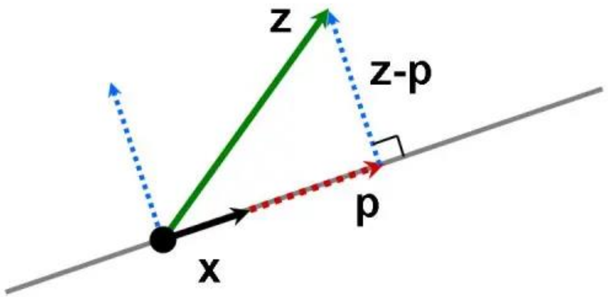
4-3 應用

- 1) 最小方差解 (Least square solution)



4-3 應用

- 2) 正交投影矩陣 (Orthogonal projection matrix)



When $Ax = b$ is inconsistent, its least-squares solution minimizes $\|Ax -$

:

Normal equations $A^TA\hat{x} = A^Tb$. (1)

A is invertible exactly when the columns of A are linearly independent!

en,

Best estimate $\hat{x}$ $\hat{x} = (A^TA)^{-1}A^Tb$. (2)

projection of b onto the column space is the nearest point $A\hat{x}$:

Projection $p = A\hat{x} = A(A^TA)^{-1}A^Tb$. (3)

Normal equations

$$A^T\left(\vec{b} - A\hat{x}\right) = \vec{0} \qquad \boxed{A^T A\hat{x} = A^T\vec{b}}$$

normal equations

- OK, we've got the equation, let's solve it.
- A^TA is n by n matrix.
- As in the line case, we must get answers to three questions:

1. What is $\hat{x}$?
2. What is projection p?
3. What is projection matrix P?

Projection matrix

$$A^T(b - A\hat{x}) = 0$$

$$A^T A\hat{x} = A^T b$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$p = A\hat{x} = A(A^T A)^{-1} A^T b$$

$$\boxed{x = \frac{a^T b}{a^T a} \Rightarrow p = ax}$$

$$P = A(A^T A)^{-1} A^T \longrightarrow \boxed{\text{Projection matrix}}$$

$$\boxed{AA^{-1}(A^T)^{-1}A^T = I}$$

Case 1. A is not a square matrix so this equation is not true
Case 2. A is a square matrix and invertible, means b is in C(A) than the projection is identity

Projection matrix

6.4 question 26

(a) $W = \text{span}\{(2,-1,4)\}$ so that the vector $(2,-1,4)$ forms a basis for W (its linear independence follows from Theorem 4.3.2(b))

(b) Letting $A = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, Formula (11) yields

$$P = A(A^T A)^{-1} A^T = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \left(\begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} [21]^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 4 & -2 & 8 \\ -2 & 1 & -4 \\ 8 & -4 & 16 \end{bmatrix}.$$

4-4 矩陣的 QR 分解

$$\overset{\mathbf{A}}{\begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{bmatrix}} = \overset{\mathbf{Q}}{\underbrace{\begin{bmatrix} | & | & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ | & | & | \end{bmatrix}}_{\substack{\text{Orthogonal} \\ \text{Unit vectors}}}} \overset{\mathbf{R}}{\underbrace{\begin{bmatrix} \mathbf{e}_1^T \cdot \mathbf{a}_1 & \mathbf{e}_1^T \cdot \mathbf{a}_2 & \mathbf{e}_1^T \cdot \mathbf{a}_3 \\ 0 & \mathbf{e}_2^T \cdot \mathbf{a}_2 & \mathbf{e}_2^T \cdot \mathbf{a}_3 \\ 0 & 0 & \mathbf{e}_3^T \cdot \mathbf{a}_3 \end{bmatrix}}_{\substack{\text{Upper Diagonal} \\ \text{Matrix}}}}$$