Electricity and Magnetism Problem Set 1

Problems from Chapter 1 Notes, and Introduction to Electrodynamics (Griffiths)

Due 2022/09/28

Problem 1 (Notes, Ex [1.27]). Suppose two functions $f(x, y)$ and $g(x, y)$ are both zero only when $(x, y) = (x_0, y_0)$, find A in the expression

$$\delta^{(2)}(f(x, y), g(x, y)) = A\delta^{(2)}(x - x_0, y - y_0).$$

Problem 2 (Notes, Sec 1.6, Prob 6). (Line Integral) Define a closed path $\mathcal{C}_1$ as the boundary of a rectangle on the x-y plane with the corners at $(0, 0), (L_1, 0), (L_1, L_2), (0, L_2)$ (in that order), and another closed path by $\mathcal{C}_2 = \{(x(t), y(t), 0) \mid t \in [0, 2\pi]\}$ with $x(t) = \cos t, y(t) = \sin t$. Let $\mathbf{A} = \hat{\mathbf{x}}y - \hat{\mathbf{y}}x$. Calculate directly the following quantities:

(a) $\displaystyle\oint_{\mathcal{P}} d\mathbf{l} \cdot \mathbf{A}(\mathbf{r})$, for $\mathcal{P} = \mathcal{C}_1$ and $\mathcal{P} = \mathcal{C}_2$.

(b) $\displaystyle\int_{\mathcal{S}} d\mathbf{a} \cdot (\nabla \times \mathbf{A})$, for $\mathcal{S}$ being the interior of $\mathcal{C}_1$ and $\mathcal{C}_2$.
(Use the right hand rule to determine the direction of $d\mathbf{a}$.)

Check that the results agree with the theorem (1.66):

$$\int_{\mathcal{S}} d\mathbf{a} \cdot (\nabla \times \mathbf{A}) = \oint_{\mathcal{C}} d\mathbf{l} \cdot \mathbf{A}.$$

Problem 3 (Notes, Sec 1.6, Prob 7). Let $\mathbf{A} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$. Calculate directly the following quantities:

(a) $\displaystyle\oint_{\mathcal{P}} d\mathbf{l} \cdot \mathbf{A}(\mathbf{r})$ for $\mathcal{P} = \mathcal{C}_1$ and $\mathcal{P} = \mathcal{C}_2$.

(b) Find V such that $\mathbf{A} = -\nabla V$.

Check that the results agree with the theorem (1.59):

$$\int_{\mathcal{P}} d\mathbf{l} \cdot (\nabla f) = f(\mathbf{r}_2) - f(\mathbf{r}_1).$$

Problem 4 (Griffiths, Prob 1.39). Check the divergence theorem for the function

(a) $\mathbf{v}_1 = r^2\hat{\mathbf{r}}$, using as your volume the sphere of radius R, centered at the origin.

(b) $\mathbf{v}_1 = \dfrac{1}{r^2}\hat{\mathbf{r}}$. (If the answer surprises you, look back at Prob 1.16: Sketch the function and compute its divergence.)

Problem 5 (Griffiths, Prob 1.64). In case you're not persuaded that $\nabla^2\frac{1}{r} = -4\pi\delta^{(3)}(\mathbf{r})$ (Equation 1.102: $\nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^{(3)}(\boldsymbol{\mathscr{r}})$, with $\mathbf{r}' = 0$ for simplicity), try replacing r by $\sqrt{r^2 + \epsilon^2}$, and watching what happens as $\epsilon \to 0$. Specifically, let

$$D(r, \epsilon) \equiv -\frac{1}{4\pi}\nabla^2\frac{1}{\sqrt{r^2 + \epsilon^2}}.$$

To demonstrate that this goes to $\delta^{(3)}(\mathbf{r})$ as $\epsilon \to 0$:

(a) Show that $D(r, \epsilon) = \dfrac{3\epsilon^2}{4\pi}(r^2 + \epsilon^2)^{-\frac{5}{2}}$.

(b) Check that $D(0, \epsilon) \to \infty$ as $\epsilon \to 0$.

(c) Check that $D(r, \epsilon) \to 0$ as $\epsilon \to 0$ for all $r \neq 0$.

(d) Check that the integral $D(r, \epsilon)$ over all space is 1.