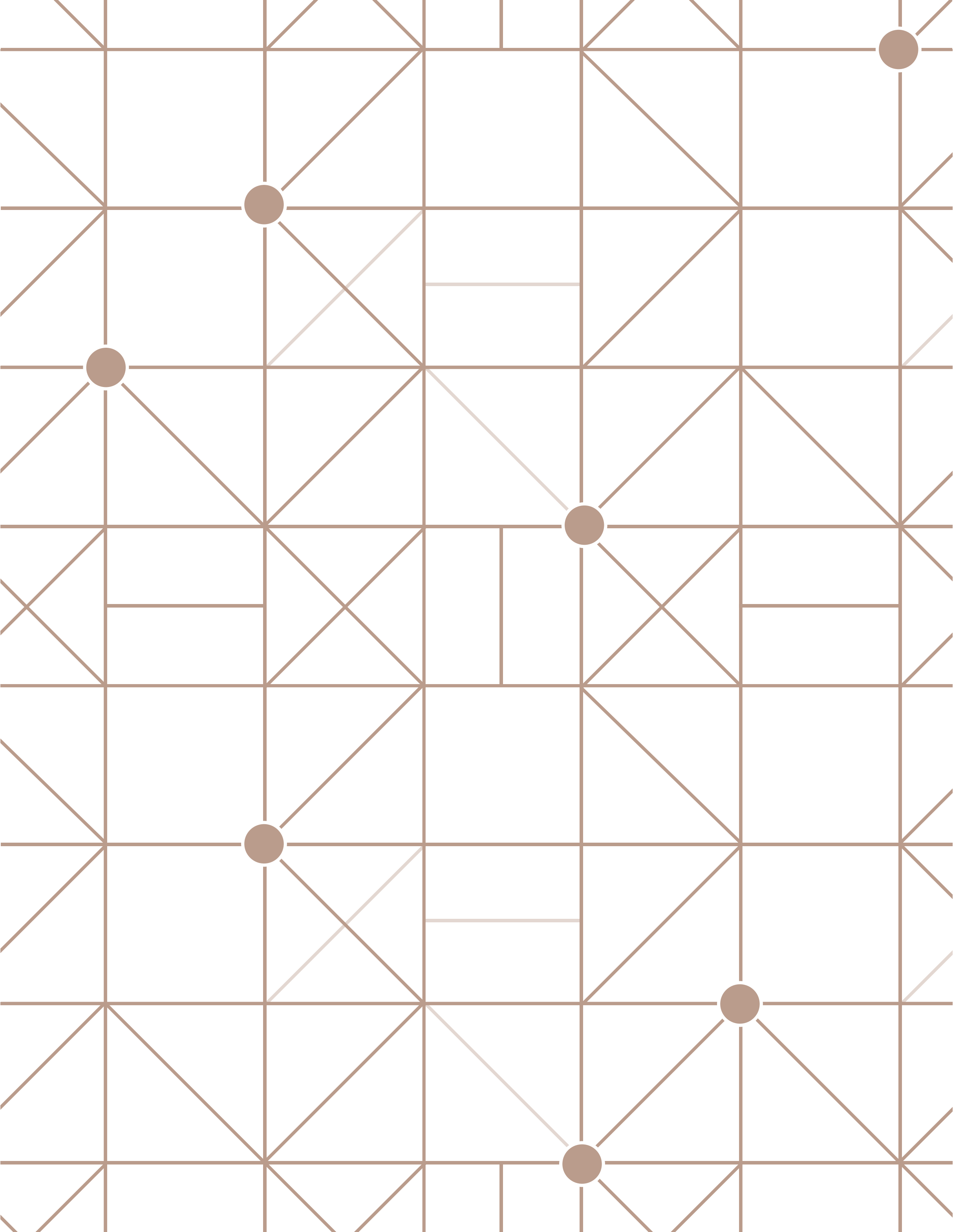
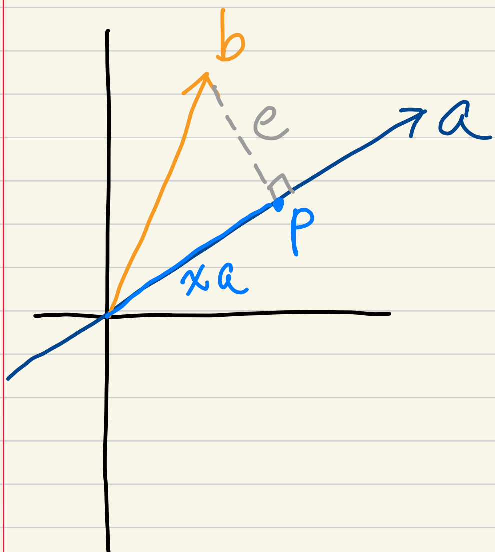


# Ch3-3 Projections and Least Squares




- difference

$$\boxed{e = b - p}$$

- $p$ = (multiple of a)

$$\boxed{p = xa}$$

- $a^T e = 0$

$$\therefore \quad \boxed{a^T(b - xa) = 0}$$

$$\therefore \quad x a^T a = a^T b$$

$$\therefore \quad \boxed{x = \frac{a^T b}{a^T a}}$$

- $p = ax$

$\therefore \; p = a\left(\frac{a^T b}{a^T a}\right)$

projection $p = \mathbb{P} b$

$$\therefore \; \boxed{\mathbb{P} = \frac{a a^T}{a^T a}}$$

- $\boxed{\mathbb{P}^T = \mathbb{P}}$ $\quad \mathbb{P}$ is symmetric

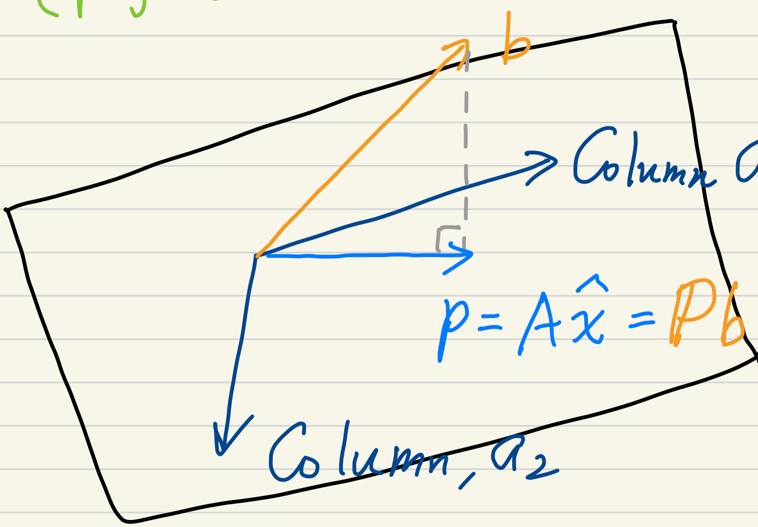
- $\boxed{\mathbb{P}^2 = \mathbb{P}}$ $\quad$ apply $\mathbb{P}$ second time is the same.

[A] **Q: why we need the P?**

Because $Ax=b$ may have no solution.

Slove $A\hat{x}=p$ instead.

(project $b$ onto the column space).




- plane of $a_1, a_2$ ⇒ column space of $A$.

$$A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$$

columns

- $\boxed{e=b-p}$ is $\perp$ to the plane.

- Projection $p = \hat{x}_1 a_1 + \hat{x}_2 a_2$

$$\boxed{p = A\hat{x}}$$

- Finding $\boxed{\hat{x}}$ and $\boxed{p = A\hat{x}}$

$\boxed{\text{Key}}$: $\underbrace{(b - A\hat{x})}_{e}$ is $\perp$ to the plane

$$\therefore \begin{cases} a_1^T (b - A\hat{x}) = 0 \\ a_2^T (b - A\hat{x}) = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{A^T(b - A\hat{x}) = 0}$$

$$(b - A\hat{x}) = e$$

$\boxed{e \text{ is in } N(A^T)}$

$\boxed{e \perp C(A)}$

$$\Rightarrow \boxed{A^T A \hat{x} = A^T b}$$

what's the $\hat{x}$ ?

$$\boxed{\hat{x} = (A^TA)^{-1}A^Tb}$$

$$p = A\hat{x}$$

$$\therefore p = A\left[(A^TA)^{-1}A^Tb\right]$$

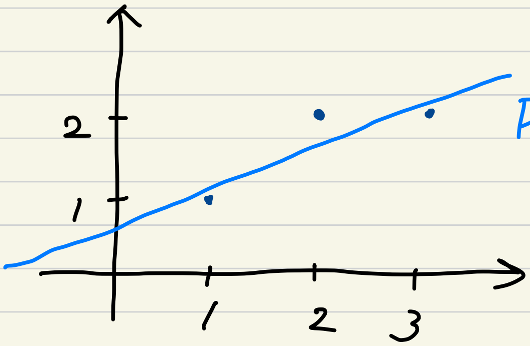
$\therefore$ Matrix $\boxed{P = A(A^TA)^{-1}A^T}$

Note: 1D of $P$

$$\boxed{P = \frac{aa^T}{a^Ta}} \quad \therefore 1D$$

Note: $\boxed{P^T = P}$ $\boxed{P^2 = P}$

## [B] Least Squares




(1,1) (2,2) (3,2)

Fitting by a line

$b = C + Dt$

$$\begin{cases} C + D = 1 \\ C + 2D = 2 \\ C + 3D = 2 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$\underline{Ax = b}$

can't solve!

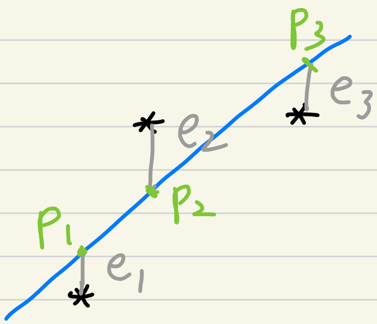
– The best solution is to solve

$$\boxed{A^T A \hat{x} = A^T b}$$

$\therefore$ Best solution for $Ax = b$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

(Column space doesn't include the b vector.)



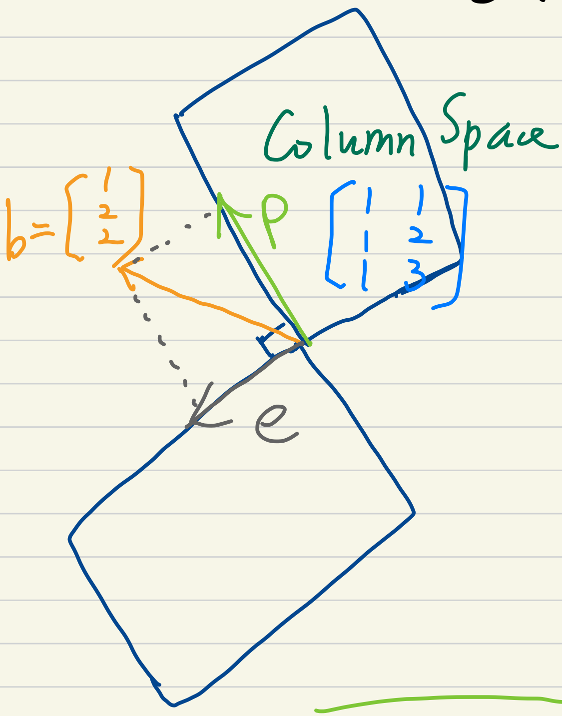

- Minimize

$$\|Ax - b\|^2 = \|e\|^2$$

(Small length of error vector)

$$\|e\|^2 = e_1^2 + e_2^2 + e_3^2$$

- $p_1$, $p_2$, $p_3$ are combinations of column space.




- Find $\boxed{\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}}$ , $\boxed{p}$

$\because \quad A^TA\hat{x} = A^Tb$

$$\therefore \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

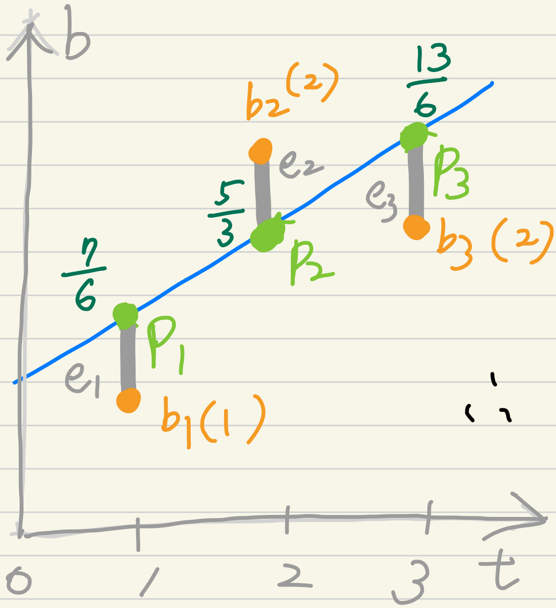
$$\therefore \begin{cases} 3C + 6D = 5 \\ 6C + 14D = 11 \end{cases}$$

$$\boxed{D = \frac{1}{2} \quad C = \frac{2}{3}}$$

$\therefore$ The best line is $\left(\frac{2}{3} + \frac{1}{2}t\right)$

$(C + Dt)$

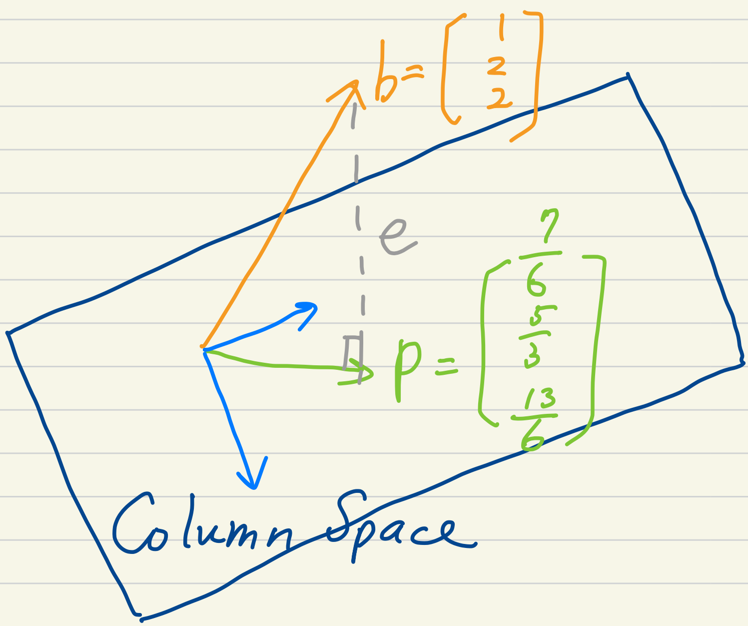
$$\therefore y = C + Dt = \frac{2}{3} + \frac{1}{2}t$$




$$\therefore\ e_1 = -1/6$$

$$e_2 = +2/6$$

$$e_3 = -1/6$$



$$- \quad p + e = b$$

$$e = (I - P)b$$

$$\therefore \; b = p + e$$

[check] $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix} + \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}$

$$- \quad p \perp e$$

[check]

$$\begin{bmatrix} 7/6 & 5/3 & 13/6 \end{bmatrix} \cdot \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix} = 0$$

- $e$ is in $N(A^T)$

$\therefore$ $e$ also perpendicular to column space.

$\Rightarrow$ $e$ also $\perp$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.