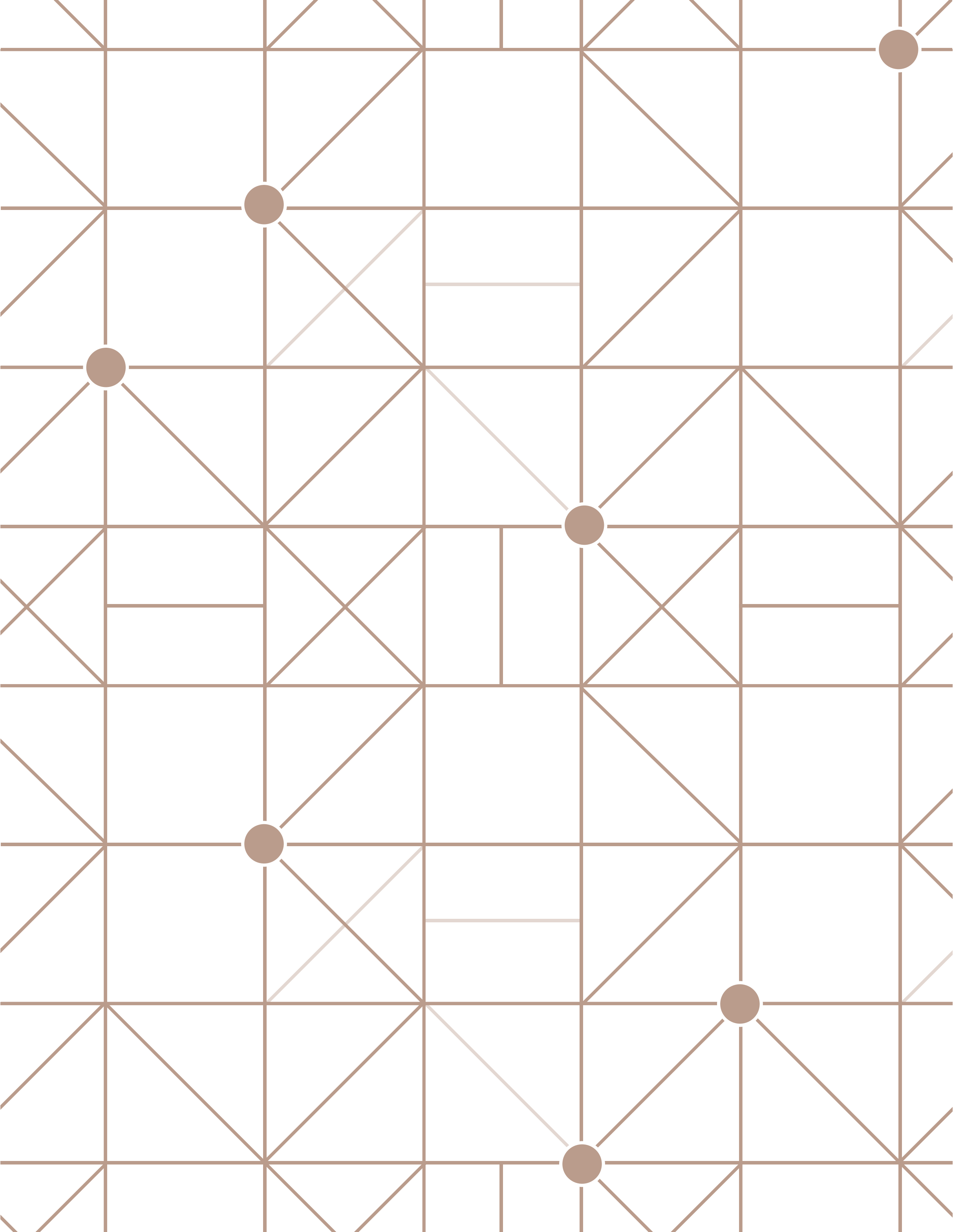
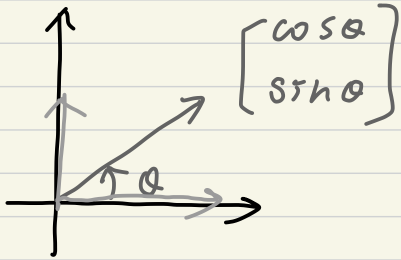
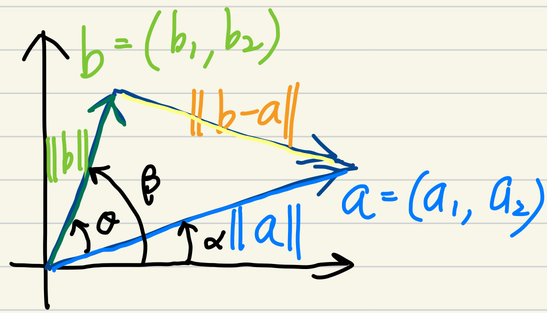


Ch3-2 Cosines and Projections onto Lines

[A] Inner Products and Cosines





$$\sin\alpha = \frac{a_2}{\|a\|}, \quad \cos\alpha = \frac{a_1}{\|a\|}$$

$$\sin\beta = \frac{b_2}{\|b\|}, \quad \cos\beta = \frac{b_1}{\|b\|}$$

$$(\theta = \beta - \alpha)$$

$$\therefore \cos\theta = \cos(\beta - \alpha)$$

$$= \cos\beta\cos\alpha + \sin\beta\sin\alpha$$

$$= \frac{a_1 b_1 + a_2 b_2}{\|a\|\ \|b\|}$$

$$\Rightarrow \boxed{\cos\theta = \frac{a^T b}{\|a\| \|b\|}}$$

[B] Law of Cosines:

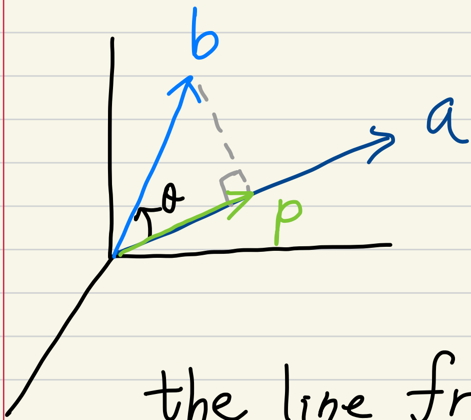
$$\|b-a\|^2 = \|b\|^2 + \|a\|^2 - 2\|b\|\|a\|\cos\theta$$

$$\because \|b-a\|^2 = (b-a)^T(b-a)$$

$$\therefore b^Tb - 2a^Tb + a^Ta = b^Tb + a^Ta - 2\|b\|\|a\|\cos\theta$$

$$a^Tb = \|a\| \|b\| \cos\theta$$

[C] Projection onto a Line



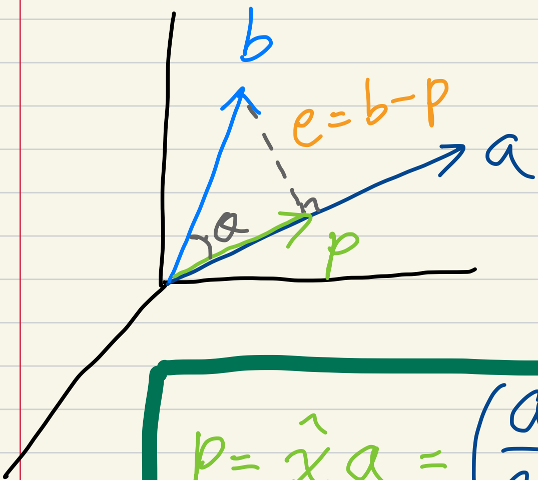
the line from b to the closet point $\boxed{p = \hat{x} a}$ is $\perp$ to the vector a.

$(b - \hat{x} a) \perp a$ or $a^T(b - \hat{x} a) = 0$

or $\boxed{\hat{x} = \dfrac{a^T b}{a^T a}}$

Projection onto a Line

$$\boxed{p = \hat{x} a = \left(\frac{a^T b}{a^T a}\right) a}$$



$$\boxed{p = \hat{x} a = \left(\frac{a^T b}{a^T a}\right) a}$$

The projection p of b onto a

with $\boxed{\cos\theta = \frac{a^T b}{\|a\| \, \|b\|}}$

- Schwarz inequality

$$\boxed{\|e\|^2 = \|b-p\|^2 \text{ cannot be negative.}}$$

$$\therefore \left\| b - \left(\frac{a^Tb}{a^Ta}\right) a \right\|^2$$

$$= b^Tb - 2\frac{(a^Tb)^2}{a^Ta} + \left(\frac{a^Tb}{a^Ta}\right)^2 a^Ta$$

$$= \frac{(b^Tb)(a^Ta) - (a^Tb)^2}{(a^Ta)} \geq 0$$

$$\Rightarrow (b^Tb)(a^Ta) \geq (a^Tb)^2$$

$$\Rightarrow \boxed{|a^Tb| \leq \|a\| \, \|b\|}$$

$\therefore$ Schwarz inequality

$$\left(\cos\theta = \frac{a^T b}{\|a\| \|b\|} , \quad |\cos\theta| \le 1 \right)$$

(Ex1) Project $b = (1, 2, 3)$ onto the line through $a = (1, 1, 1)$ to get $\hat{x}$ and p :

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{6}{3} = 2$$

$$p = \hat{x} a = (2, 2, 2)$$

$$\cos\theta = \frac{\|p\|}{\|b\|} = \frac{\sqrt{12}}{\sqrt{14}}$$

$$\cos\theta = \frac{a^T b}{\|a\|\,\|b\|} = \frac{6}{\sqrt{3}\,\sqrt{14}}$$

Schwarz inequality:

$$|a^T b| \leq \|a\|\,\|b\|$$

$$\therefore \quad 6 \leq \sqrt{3}\,\sqrt{14}$$

[D] <u>Projection Matrix</u> of Rank 1.

P is the matrix that multiplies b and produces p.

$$\boxed{p = a\frac{a^T b}{a^T a}}$$

So the projection matrix is

$$\boxed{P = \frac{a a^T}{a^T a}}$$

(Ex 2) Project onto the "θ-direction" in the x-y plane.

The line goes through $a = (\cos\theta, \sin\theta)$ and matrix is symmetric:

$$\boxed{P^2 = P}$$

ex: The matrix that projects onto the line through $a = (1,1,1)$ is

$$\mathbb{P} = \frac{aa^T}{a^Ta} = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix}$$

- 1) $\mathbb{P}$ is a symmetric matrix.
- 2) $\mathbb{P}^2 = \mathbb{P}$
- 3) $\because \mathbb{P}$ is symmetric, its column and row space are the same!