Chapter 1

Vector Analysis

1.1 Vector Algebra

1.1.1 Basic Definitions

Electric field is a vector, magnetic field is a pseudo-vector, electric potential is a scalar, and magnetic potential is a vector. But everything is changed in Special Relativity.

A vector space (or linear space) is a set $\mathcal{V}$ of elements that is closed under superpositions, that is,

$$\text{if} \quad \mathbf{A}, \mathbf{B} \in \mathcal{V}, \quad \text{then} \quad a\mathbf{A} + b\mathbf{B} \in \mathcal{V} \quad \forall a, b \in \mathbb{C}. \tag{1.1}$$

In the definition of a vector space, we need two operations: 1. $(a, \mathbf{A}) \to a\mathbf{A}$; 2. $(\mathbf{A}, \mathbf{B}) \to \mathbf{A}+\mathbf{B}$.

The two operations in a vector space must satisfy the following properties:

1. commutativity of $+$: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.
2. associativity of $+$: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.
3. associativity of multiplication: $a(b\mathbf{A}) = (ab)\mathbf{A}$.
4. distributive: $a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}$, $(a + b)\mathbf{A} = a\mathbf{A} + b\mathbf{A}$.

Ex[1.1]: Prove that $1\mathbf{A} = \mathbf{A}$ for any $\mathbf{A}$.
Ex[1.2]: Define $\mathbf{0}$ by $\mathbf{0} + \mathbf{A} = \mathbf{A}$ for any $\mathbf{A} \in \mathcal{V}$. Prove that $0\mathbf{A} = \mathbf{0}$ for any $\mathbf{A}$, and hence the vector $\mathbf{0}$ exists in any vector space.

By convention, $\mathbf{0}$ is simply denoted as 0.

A basis $\{\hat{\mathbf{e}}_i\}$ of a vector space $\mathcal{V}$ is a set of vectors in $\mathcal{V}$ chosen such that any vector in $\mathcal{V}$ can be expressed as a superposition of its elements.
In this course, unless otherwise specified, the term *vector* is reserved for vectors in the Euclidean 3 dimensional space of the physical world, and its tangent space. (See Sec. 1.1.3 below.) Furthermore, we will mostly consider **orthonormal** basis for which $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$.

A vector $\mathbf{A} \in \mathcal{V}$ can be specified in many ways:

We will also denote (A_1, A_2, A_3) as (A_x, A_y, A_z).

$$
\begin{aligned}
\mathbf{A} &= A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3 = \sum_{i=1}^{3} A_i\hat{\mathbf{e}}_i, && (1.2)\\
\mathbf{A} &= (A_1, A_2, A_3), \qquad (\mathbf{A})_i = A_i, && (1.3)
\end{aligned}
$$

where $\hat{\mathbf{e}}_i$ is a complete basis of the vector space $\mathcal{V}$. Roughly speaking, $\mathbf{A}$ can be considered as the information of a magnitude $|\mathbf{A}|$ and a direction $\hat{\mathbf{A}}$.

Einstein summation convention: repeated indices are summed over.
For example, $A_iB_{ij}C_{jk} = D_k$ is a short-hand for $\sum_i \sum_j A_iB_{ij}C_{jk} = D_k$.

Always denote vectors as A or $\vec{A}$ in your homework and exams.

1.1.2 Basic Operations

We will consider vector spaces equipped with more algebraic operations: **inner product** (or **dot product**) and **outer product** (or **cross product**).

The inner product maps a pair of vectors to a number. The cross product maps a pair of vectors to a vector.

By assumption, the inner product satisfy the following properties.

1. commutative: $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$.
2. distributive: $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$.
3. associative: $(a\mathbf{A})\cdot\mathbf{B} = a(\mathbf{A}\cdot\mathbf{B})$.
4. positive-definite: $\mathbf{A}\cdot\mathbf{A} \geq 0$, and we shall refer to $|\mathbf{A}| \equiv \sqrt{\mathbf{A}\cdot\mathbf{A}}$ as the **norm** of $\mathbf{A}$. We shall further assume that $|\mathbf{A}| = 0$ only when $\mathbf{A} = 0$.

The cross product satisfies

1. anti-commutative: $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$.
2. distributive: $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$.
3. associative: $(a\mathbf{A})\times\mathbf{B} = a(\mathbf{A}\times\mathbf{B})$.

Ex[1.3]: Prove that $\mathbf{A}\times\mathbf{A} = 0$ for any $\mathbf{A}$.

In an orthonormal basis, the basic operations are

$$
\begin{aligned}
(f\mathbf{A})_i &= fA_i, && (1.4)\\
(\mathbf{A}\cdot\mathbf{B}) &= A_iB_i, && (1.5)\\
\mathbf{A}\times\mathbf{B} &= \epsilon_{ijk}\hat{\mathbf{x}}_iA_jB_k, \qquad (\mathbf{A}\times\mathbf{B})_i = \epsilon_{ijk}A_jB_k, && (1.6)
\end{aligned}
$$

where ϵ_{ijk} is the totally antisymmetrized tensor with $\epsilon_{123} = 1$.

Make sure that you are familiar with all notations.

More explicitly, we have

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z, \tag{1.7}$$

$$\mathbf{A}\times\mathbf{B} = \hat{\mathbf{x}}(A_yB_z - A_zB_y) + \hat{\mathbf{y}}(A_zB_x - A_xB_z) + \hat{\mathbf{z}}(A_xB_y - A_yB_x). \tag{1.8}$$

Ex[1.4]: Try to use a determinant to express the cross product $\mathbf{A}\times\mathbf{B}$.
Ex[1.5]: Let $\mathbf{A} = \hat{\mathbf{x}} + 2\hat{\mathbf{y}}$, $\mathbf{B} = \hat{\mathbf{y}} - 3\hat{\mathbf{z}}$. What are $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{A}\times\mathbf{B}$?
Q[1.1]: What are the geometrical meanings of these operations?

The magnitude or norm of a vector $\mathbf{A}$ is

$$|\mathbf{A}| \equiv \sqrt{\mathbf{A}\cdot\mathbf{A}}. \tag{1.9}$$

Nice properties: (1) $|f\mathbf{A}| = |f||\mathbf{A}|$, (2) $|\mathbf{A}'| = |\mathbf{A}|$, where $\mathbf{A}'$ is any rotation of $\mathbf{A}$.

We will use the notation that a hat implies a unit vector

$$\hat{\mathbf{A}} \equiv |\mathbf{A}|^{-1}\mathbf{A}. \tag{1.10}$$

Combinations of basic operations:

The volume of the parallelepiped spanned by $\mathbf{A},\mathbf{B},\mathbf{C}$:

$$\boxed{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \epsilon_{ijk}A_iB_jC_k} \tag{1.11}$$

is totally antisymmetrized w.r.t. $\mathbf{A},\mathbf{B},\mathbf{C}$.
Ex[1.6]: Express the triple product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ as the determinant of a 3×3 matrix.
Ex[1.7]: Prove that the volume of a parallelepiped is given by the triple product above. What does it mean when the product is negative?
BAC-CAB rule:

$$\boxed{\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}).} \tag{1.12}$$

The RHS of this identity is fixed up to an overall constant from the requirements: (1) tri-linearity, (2) rotation symmetry, (3) skew-symmetry of B, C.

Ex[1.8]: Prove that

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = 0. \tag{1.13}$$

Q[1.2]: Can you make sense of $\mathbf{A}\times(\mathbf{B}\cdot\mathbf{C})$?

1.1.3 Scalar, Vector and Tensor Fields

The notion about tensor fields (including scalar and vector fields) is a combination of the notion about tensor and the notion about fields.

Rotations and Translations

The flat 3-dimensional space $\mathbb{R}^3$ is invariant under **rotation, translation, and inversion**. [1] They are transformations of the form

$$x_i \to x_i' = R_{ij}x_j + a_i, \tag{1.14}$$

[1] By this we mean that geometric notions such as distance, inner product of tangent vectors, etc. are invariant. But the cross product is not invariant under inversion.

where R_{ij} is a 3×3 matrix satisfying

$$R_{ik}R_{jk} = \delta_{ij}, \quad \text{or equivalently,} \quad R^T R = I = RR^T. \tag{1.15}$$

Eq.(1.15) is the condition for the transformation (1.14) to preserve the distance between any two points in $\mathbb{R}^3$.

From the relation above, one deduces that

$$\det(R) = \pm 1. \tag{1.16}$$

A matrix R satisfies (1.15) and $\det(R) = 1$ if and only if R corresponds to a 3 dimensional rotation. A matrix R satisfies (1.15) and $\det(R) = -1$ if and only if R corresponds to a 3 dimensional rotation plus an inversion $(x, y, z) \to (-x, y, z)$.

A rotation along the z-axis leads to the transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.17}$$

Ex[1.9]: Show that both transformations $(x, y, z) \to (-x, y, z)$ and $(x, y, z) \to (-x, -y, -z)$ have $\det(R) = -1$ and they differ by a 3D rotation.
Ex[1.10]: Show that eq.(1.15) is a necessary condition for the inner product of any two vectors to be invariant under the transformation (1.15).

Tensors

Tensors are objects that transform covariantly under rotations.

A column of 3 components is not necessarily a vector.

$$\begin{aligned} V &\to V' = V, && (1.18)\\ A_i &\to A'_i = R_{ij}A_j, && (1.19)\\ T_{ij} &\to T'_{ij} = R_{ik}R_{jl}T_{kl}, && (1.20)\\ S_{ijk} &\to S'_{ijk} = R_{il}R_{jm}R_{kn}S_{lmn}, && (1.21)\\ \vdots \;\; & \vdots \;\; \vdots && (1.22) \end{aligned}$$

Whether a physical quantity is a scalar, or a component of a vector or a tensor, is a physical question to be answered by experiments/observations.

Tensors transform *linearly* under rotations.

In general, for a rank-n tensor $T_{i_1\cdots i_n}$,

$$T_{i_1\cdots i_n} \to T'_{i_1\cdots i_n} = R_{i_1 j_1} \cdots R_{i_n j_n} T_{j_1\cdots j_n}. \tag{1.23}$$

Scalars and vectors are rank-0 and rank-1 tensors.

Under an *active* rotation, $\hat{\mathbf{x}}_i$'s are invariant, so that $\mathbf{A} = \hat{\mathbf{x}}_i A_i$ is changed upon a rotation to $\mathbf{A}' = \hat{\mathbf{x}}_i A'_i$. Under a *passive* rotation, $\hat{\mathbf{x}}_i$ is transformed like a vector, so that $\mathbf{A}$ is invariant. That is $\mathbf{A} = \hat{\mathbf{x}}_i A_i = \hat{\mathbf{x}}'_i A'_i$.
Q[1.3]: Is $\mathbf{A}$ invariant under a passive rotation? (What is the transformation law for $\mathbf{x}_i$?)

An active transformation changes the physical state, while a passive transformation is merely a change of coordinates.

Here are some examples.

Position vector in physical space:

$$\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z = \hat{\mathbf{x}}_i x_i. \tag{1.24}$$

Separation vector in physical space:

We will also use the notation $\mathscr{r} = |\boldsymbol{\mathscr{r}}|, \quad \hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}}$.

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}', \tag{1.25}$$

Tangent vector in physical space:

$$\mathbf{v} = \hat{\mathbf{x}} v_x + \hat{\mathbf{y}} v_y + \hat{\mathbf{z}} v_z = \hat{\mathbf{x}}_i v_i. \tag{1.26}$$

A pseudo-scalar and a pseudo-vector transform under (1.14) as

$$W \quad \to \quad W' = \det(R) W, \tag{1.27}$$
$$B_i \quad \to \quad B_i' = \det(R) R_{ij} B_j. \tag{1.28}$$

Q[1.4]: Check that $\mathbf{A} \cdot \mathbf{B}$ is a scalar and $\mathbf{A} \times \mathbf{B}$ is a pseudo-vector if $\mathbf{A}$ and $\mathbf{B}$ are both vectors.
Q[1.5]: Design an experiment to test whether the magnetic field is a pseudo-vector.

Invariant Tensors

Q[1.6]: Are the inner product and cross product all the operations we can have for 3D vectors?

A tensor $T_{i_1 \cdots i_n}$ is invariant under a transformation R if

$$T'_{i_1 \cdots i_n} \equiv R_{i_1 j_1} \cdots R_{i_n j_n} T_{j_1 \cdots j_n} = T_{i_1 \cdots i_n}. \tag{1.29}$$

There are two constant tensors invariant under rotations

δ_{ij} is a tensor but ϵ_{ijk} is a pseudo-tensor.

$$\delta_{ij}, \qquad \epsilon_{ijk},$$

which are defined by

$$\delta_{ij} = \begin{cases} 1 & (i = j), \\ 0 & (i \neq j), \end{cases} \tag{1.30}$$

$$\epsilon_{ijk} = \begin{cases} 1 & (i,j,k) = \text{cyclic perm. of } (1,2,3), \\ -1 & (i,j,k) = \text{cyclic perm. of } (2,1,3), \\ 0 & \quad i = j \text{ or } j = k \text{ or } k = i. \end{cases} \tag{1.31}$$

δ_{ij} is symmetric and ϵ_{ijk} is totally antisymmetrized

$$\delta_{ij} = \delta_{ji}, \tag{1.32}$$
$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{jik} = -\epsilon_{kji} = -\epsilon_{ikj}. \tag{1.33}$$

Ex[1.11]: Show that $\delta_{ij} A_j = A_i$, and $\epsilon_{ijk} A_j A_k = 0$ for arbitrary A_i.
Q[1.7]: What do you get by contracting the two indices of δ_{ij} with two vectors? What do you get by contracting two of the three indices of ϵ_{ijk} with two vectors, or by contracting all indices of ϵ_{ijk} with three vectors?

δ_{ij}, ϵ_{ijk} and their combinations are the only *constant* tensors.

There is an important relation that relates them [2]

$$\boxed{\epsilon_{ijm}\epsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}, \qquad (1.35)}$$

which leads to the BAC-CAB formula.

Ex[1.12]: Compute

$$\delta_{ij}\delta_{jk}, \quad \delta_{ij}\epsilon_{jkl}, \quad \delta_{ij}\epsilon_{ijk}, \quad \epsilon_{ijk}\epsilon_{jkl}. \tag{1.36}$$

Ex[1.13]: If A_{ij} is symmetric and B_{ij} is anti-symmetric, prove that $A_{ij}B_{ij} = 0$.

Ex[1.14]: Show that the determinant of a 3×3 matrix M can be expressed as

$$\det(M)\epsilon_{ijk} = M_{il}M_{jm}M_{kn}\epsilon_{lmn}. \tag{1.37}$$

(Hint: First show that $\det(M) = M_{1l}M_{2m}M_{3n}\epsilon_{lmn}$.)

Ex[1.15]: Use the relation above to check that

$$M_{il}M_{jm}\epsilon_{lmn} = \det(M)M^{-1}_{nk}\epsilon_{ijk}. \tag{1.38}$$

Fields

Google for "field" and you find pictures like this:

A *field* is a map that gives an element in $\mathcal{T}$ for every point in space (or spacetime). $\mathcal{T}$ is a set, e.g. $\mathbb{R}^n$, $\mathbb{C}^n$, S^2, $\mathbb{Z}_2$, etc. Often we assume that this map is smooth, in that case $\mathcal{T}$ is normally a continuous (rather than discrete) space.

The vector space for a vector field $\mathbf{A}(\mathbf{r})$ at a point $\mathbf{r}$ can not be identified with the 3D physical space. (In fact, unless our world is exactly flat, the physical space is not even a vector space.) To describe the configuration of a vector field $\mathbf{A}$, one should first associate a vector space $\mathcal{V}_\mathbf{r}$ to every point $\mathbf{r}$ in space-time, and then assign a vector $\mathbf{A}(\mathbf{r})$ on each $\mathcal{V}_\mathbf{r}$. In general the basis $\hat{\mathbf{e}}_i(\mathbf{r})$ of $\mathcal{V}_\mathbf{r}$ also depends on $\mathbf{r}$. For flat spaces, it is natural to use the same notation (as we did with $\hat{\mathbf{x}}_i$) for position, separation and tangent vectors.

One refers to a field as a scalar, vector, or tensor field, if it transforms under the transformation (1.14) as

$$f(\mathbf{r}) \rightarrow f'(\mathbf{r}') = f(\mathbf{r}), \tag{1.39}$$
$$A_i(\mathbf{r}) \rightarrow A'_i(\mathbf{r}') = R_{ij}A_j(\mathbf{r}), \tag{1.40}$$
$$T_{ij}(\mathbf{r}) \rightarrow T'_{ij}(\mathbf{r}') = R_{ik}R_{jl}T_{kl}(\mathbf{r}). \tag{1.41}$$

A pseudo-scalar field and a pseudo-vector field transform under (1.14) as

$$W(\mathbf{r}) \rightarrow W'(\mathbf{r}') = \det(R)W(\mathbf{r}), \tag{1.42}$$
$$B_i(\mathbf{r}) \rightarrow B'_i(\mathbf{r}') = \det(R)R_{ij}B_j(\mathbf{r}). \tag{1.43}$$

[2]The only other nontrivial relation that the epsilon tensor satisfies is the Plucker relation:

$$\epsilon_{i_1 i_2 j_1}\epsilon_{j_2 j_3 j_4} - \epsilon_{i_1 i_2 j_2}\epsilon_{j_1 j_3 j_4} + \epsilon_{i_1 i_2 j_3}\epsilon_{j_1 j_2 j_4} - \epsilon_{i_1 i_2 j_4}\epsilon_{j_1 j_2 j_3} = 0. \tag{1.34}$$

Q[1.8]: How do pseudo-scalar fields and pseudo-vector fields transform under the transformation $(x, y, z) \to (-x, y, z)$?

Covariance of Physical Laws

A physical theory respects a symmetry if all the equations of motion are *covariant*. That is, the equations of motion before and after the symmetry transformation must be satisfied by the same set of solutions.

Q[1.9]: Which of the following is (are) not covariant?

$$A_i = 1, \qquad T_{ij} = 0, \qquad A_i B_j = T_{ij}, \qquad A_{ij} B_j = C_i, \qquad A_{ij} B_j = C_j. \tag{1.44}$$

Maxwell Equations

Let

$$\nabla \equiv \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i}, \tag{1.45}$$

the Maxwell equations are

$$\nabla \cdot \mathbf{E} = \tfrac{\rho}{\epsilon_0}, \qquad \nabla \times \mathbf{E} = -\tfrac{\partial}{\partial t}\mathbf{B}, \tag{1.46}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \tfrac{\partial}{\partial t}\mathbf{E}. \tag{1.47}$$

Q[1.10]: Check that ∂_i is a vector, and that ∇ is invariant.

1.2 Differential Calculus

For a function $f(x_1, x_2, \cdots, x^n)$ of n variables, we define $\partial_i f(x)$ by

$$\boxed{df(x) = dx_1 \partial_1 f(x) + dx_2 \partial_2 f(x) + \cdots + dx_n \partial_n f(x) = dx_i \partial_i f(x).} \tag{1.48}$$

The definition of $\partial_i f(x)$ depends on the choice of other variables $x_{j\neq i}$.

That is, $\partial_i f$ is the ratio of the change in f and the change in x_i when all other variables $x_{j\neq i}$ are held fixed.

1.2.1 General Properties of Derivations

Important algebraic properties of derivations include the following.

1. distribution: $\partial_i(f(x) + g(x)) = \partial_i f(x) + \partial_i g(x)$.
2. Leibniz rule: $\partial_i f(x) = f(x)\partial_i + (\partial_i f(x))$.
3. chain rule: $\partial_i f(g(x)) = (\partial_i g(x)) f'(g(x))$.
4. commutativity: $\partial_i \partial_j = \partial_j \partial_i$.

$f(x)$ in $\partial_i f(x)$ on the LHS is viewed as an operator acting on functions on the right (by multiplication).

When we change coordinates, the derivatives change according to the chain rule. Upon the change of coordinates by

$$x^i \to x'^i = x'^i(x), \tag{1.49}$$

their derivatives change by

$$\frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j}, \tag{1.50}$$

where $\frac{\partial x^j}{\partial x'^i}$ can be computed using the inverse function $x^j(x')$ of the transformation (1.49), or it can be computed as the inverse matrix of $\frac{\partial x'^j}{\partial x^i}$.

Notice that even if some of the variables are not changed, their partial derivatives may change due to the change of other variables.

Ex[1.16]: Prove that

$$\frac{\partial x'^j}{\partial x^i}\frac{\partial x^k}{\partial x'^j} = \delta_i^k. \tag{1.51}$$

Ex[1.17]: For

$$x' = f(x, y), \qquad y' = y, \tag{1.52}$$

show that the partial derivatives with respect to y and y' are different:

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} - \left(\frac{\partial f}{\partial x}\right)^{-1}\left(\frac{\partial f}{\partial y}\right)\frac{\partial}{\partial x}. \tag{1.53}$$

In this section we will discuss how partial derivatives ∂_i can act on scalar and vector fields. We will focus on the following operations:
Gradient, divergence and curl:

$$\begin{aligned} \nabla f &= \hat{\mathbf{x}}_i(\partial_i f), & (1.54)\\ \nabla\cdot\mathbf{A} &= (\partial_i A_i), & (1.55)\\ \nabla\times\mathbf{A} &= \hat{\mathbf{x}}_i\epsilon_{ijk}\partial_j A_k. & (1.56)\end{aligned}$$

These operations are special because they are the first derivatives whose action on scalars/vectors always produce tensors.

1.2.2 Gradient

Recall Taylor expansion of a function with multi-variables

$$f(\mathbf{r}+d\mathbf{l}) = f(\mathbf{r}) + dx_i\,\partial_i f(\mathbf{r}) + \cdots, \tag{1.57}$$

where $d\mathbf{l} = \hat{\mathbf{x}}_i dx_i$.

Thus

$$f(\mathbf{r}+d\mathbf{l}) - f(\mathbf{r}) = \nabla f(\mathbf{r})\cdot d\mathbf{l}. \tag{1.58}$$

This is the change in the value of f over an infinitesimal displacement $d\mathbf{l}$ at $\mathbf{r}$.

Patching infinitesimal displacements $d\mathbf{l}$ together to form a path $\mathcal{P}$ from $\mathbf{r}_1$ to $\mathbf{r}_2$, we obtain

$$\boxed{\int_{\mathcal{P}} d\mathbf{l}\cdot(\nabla f) = f(\mathbf{r}_2) - f(\mathbf{r}_1). \qquad (1.59)}$$

Ex[1.18]: Argue that $\nabla r = \hat{\mathbf{r}}$ using geometric notions. Then check it by explicit computation.

As a result of (1.59), $\oint d\mathbf{l}\cdot(\nabla f) = 0.$

Ex[1.19]: Show that if f is a scalar, ∇f is a vector.

Ex[1.20]: Find the most general solution of $f(x, y, z)$ for

$$\nabla f(x, y, z) = a\hat{\mathbf{r}} \tag{1.60}$$

for a given constant a.

1.2.3 Curl

$$\sum \mathbf{A}(\mathbf{r}) \cdot d\mathbf{l} = (\nabla \times \mathbf{A}) \cdot d\mathbf{a}. \tag{1.61}$$

This gives how much the vector field $\mathbf{A}$ curls around an infinitesimal area element $d\mathbf{a}$ at $\mathbf{r}$.

For the area element spanned by two vectors $d\mathbf{l}_1$, $d\mathbf{l}_2$, we have

$$d\mathbf{a} = d\mathbf{l}_1 \times d\mathbf{l}_2, \tag{1.62}$$

and it is bounded by 2 pairs of linear elements $\pm d\mathbf{l}_1$, $\pm d\mathbf{l}_2$.

> If $\mathbf{A}$ represents the flow of a fluid, i.e., $\hat{\mathbf{A}}$ is the direction of the flow and $|\mathbf{A}|$ the velocity, then $\nabla \times \mathbf{A}$ is an attempt to account for the flow by superposing infinitely many infinitesimal vortices together with a certain distribution. The direction of $\nabla \times \mathbf{A}$ is determined by the right hand rule, and $|\nabla \times \mathbf{A}|$ is proportional to the density of the vortices (imagining that each vortex are of the same magnitude).

The sum $\mathbf{A} \cdot d\mathbf{l}$ over the first pair $\pm d\mathbf{l}_1$ is

$$\mathbf{A}(\mathbf{r} + d\mathbf{l}_2) \cdot (-d\mathbf{l}_1) + \mathbf{A}(\mathbf{r}) \cdot d\mathbf{l}_1 = -\partial_j A_i(\mathbf{r})(d\mathbf{l}_1)_i(d\mathbf{l}_2)_j. \tag{1.63}$$

Similarly, the sum $\mathbf{A} \cdot d\mathbf{l}$ over the 2nd pair $\pm d\mathbf{l}_2$ is

$$\mathbf{A}(\mathbf{r} + d\mathbf{l}_1) \cdot d\mathbf{l}_2 + \mathbf{A}(\mathbf{r}) \cdot (-d\mathbf{l}_2) = \partial_i A_j(\mathbf{r})(d\mathbf{l}_1)_i(d\mathbf{l}_2)_j. \tag{1.64}$$

The total sum is therefore

$$\sum \mathbf{A}(\mathbf{r}) \cdot d\mathbf{l} = (\partial_i A_j - \partial_j A_i)(d\mathbf{l}_1)_i(d\mathbf{l}_2)_j = \epsilon_{ijk}\partial_i A_j \, \epsilon_{kmn}(d\mathbf{l}_1)_m(d\mathbf{l}_2)_n = (\nabla \times \mathbf{A}) \cdot d\mathbf{a}. \tag{1.65}$$

Patching infinitesimal area elements together to form a surface $\mathcal{S}$ bounded by a closed curve $\mathcal{C}$, we have

$$\boxed{\int_{\mathcal{S}} d\mathbf{a} \cdot (\nabla \times \mathbf{A}) = \oint_{\mathcal{C}} d\mathbf{l} \cdot \mathbf{A}. \tag{1.66}}$$

1.2.4 Divergence

$$\sum \mathbf{A}(\mathbf{r}) \cdot d\mathbf{a} = \nabla \cdot \mathbf{A}(\mathbf{r}) \, d\tau. \tag{1.67}$$

This is the amount of "flux" generated by sources within an infinitesimal volume element $d\tau$ nearby the point $\mathbf{r}$.

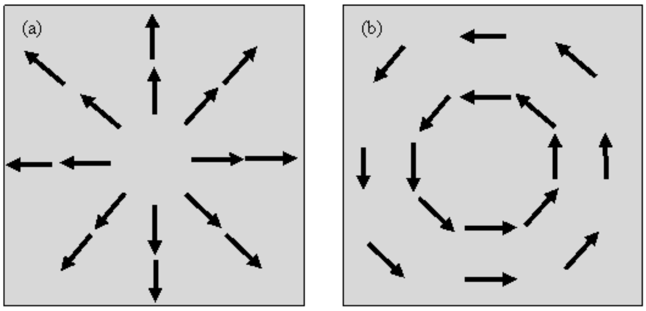


Figure 1.1: (a) a vector field with positive divergence at the center. (b) a vector field with positive curl at the center.

> Again, if $\mathbf{A}$ represents a flow, then $\nabla \cdot \mathbf{A}$ is an attempt to account for the flow by infinitely many infinitesimal openings (water sources) where fluid goes out (+) or in (-). $\nabla \cdot \mathbf{A}$ gives the density of such infinitesimal sources, assuming that each source is of the same magnitude.

For the parallelogram spanned by $\hat{\mathbf{x}}dx$, $\hat{\mathbf{y}}dy$, $\hat{\mathbf{z}}dz$, we have the infinitesimal volume element $d\tau = dxdydz$ bounded by 3 pairs of rectangular infinitesimal area elements $(\pm\hat{\mathbf{x}}dydz)$, $(\pm\hat{\mathbf{y}}dzdx)$, $(\pm\hat{\mathbf{z}}dxdy)$. The sum of $\mathbf{A}\cdot d\mathbf{a}$ over the pair of area elements $\hat{\mathbf{x}}dydz$ is

$$\mathbf{A}(\mathbf{r}+\hat{\mathbf{x}}dx)\cdot(\hat{\mathbf{x}}dydz)+\mathbf{A}(\mathbf{r})\cdot(-\hat{\mathbf{x}}dydz) = \partial_x A_x(\mathbf{r})\, dxdydz. \tag{1.68}$$

Apparently, the sum over all 3 pairs of area elements is

$$(\partial_x A_x + \partial_y A_y + \partial_z A_z)\, dxdydz = (\nabla\cdot\mathbf{A})\, d\tau. \tag{1.69}$$

Q[1.11]: Generalize the derivation above to a parallelepiped spanned by 3 vectors $d\mathbf{l}_1$, $d\mathbf{l}_2$, $d\mathbf{l}_3$.
Ex[1.21]: Compute $\nabla\cdot\mathbf{A}$ and $\nabla\times\mathbf{A}$ for (1) $\mathbf{A}=\hat{\mathbf{r}}f(r)$, and (2) $\mathbf{A}=\hat{\phi}f(r)$.
Ex[1.22]: Compute $\nabla V(\mathbf{r})$ and $\nabla^2 V(\mathbf{r})$ for (1) $V(\mathbf{r}) = f(r)$, and (2) $V(\mathbf{r}) = f(\cos\phi_0 x + \sin\phi_0 y)$.

Patching infinitesimal volume elements together to form a volume $\mathcal{V}$ with a boundary $\mathcal{S}$, we get

$$\boxed{\int_{\mathcal{V}} d\tau\, (\nabla\cdot\mathbf{A}) = \oint_{\mathcal{S}} d\mathbf{a}\cdot\mathbf{A}. \tag{1.70}}$$

Q[1.12]: Check the following identities:

$$\nabla\times(\nabla V) = 0, \tag{1.71}$$
$$\nabla\cdot(\nabla\times\mathbf{A}) = 0, \tag{1.72}$$
$$\nabla(fg) = (\nabla f)g + f(\nabla g), \tag{1.73}$$
$$\nabla\cdot(f\mathbf{A}) = f\nabla\cdot\mathbf{A} + (\nabla f)\cdot\mathbf{A}, \tag{1.74}$$
$$\nabla\times(f\mathbf{A}) = f\nabla\times\mathbf{A} + (\nabla f)\times\mathbf{A}. \tag{1.75}$$
$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}). \tag{1.76}$$

These identities can be quickly derived whenever you need them by noting that ∇ is at the same time a vector and an operator. Inner product or cross product with ∇ refers to its vectorial nature, and it acts on the right as an operator via Lebniz rule.

Q[1.13]: Expand $\nabla\times(\mathbf{A}\times\mathbf{B})$ so that cross product is avoided.

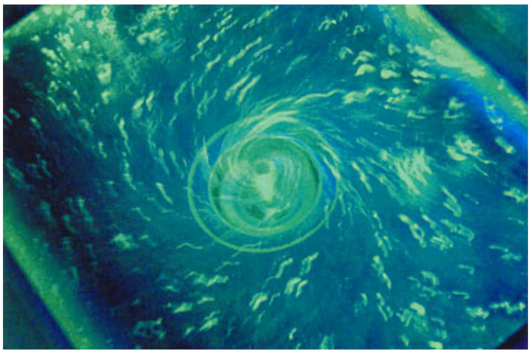


Figure 1.2: The velocity field corresponding to a vortex in a sink has negative divergence at the center (sink) and positive or negative curl (depending on the orientation of the flow) at the center. The curl off the center can be positive or negative (depending on whether the vortex is forced) The divergence off the center is zero, assuming that the fluid is incompressible.

1.2.5 Algebraic Properties of ∇

Ex[1.23]: Check the following algebraic properties of ∇.

1. $\nabla(fg) = (\nabla f)g + f(\nabla)g$.
2. $\nabla \cdot (f\mathbf{A}) = (\nabla f) \cdot \mathbf{A} + f(\nabla \cdot \mathbf{A})$.
3. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B})$.
4. $\nabla \times (f\mathbf{A}) = (\nabla f) \times \mathbf{A} + f(\nabla \times \mathbf{A})$.
5. $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$.
6. $\nabla \times (\nabla f) = 0$ for any function f.
7. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ for any vector field $\mathbf{A}$.
8. $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$ for any vector field $\mathbf{A}$.

Define the Laplacian operator by

$$\boxed{\nabla^2 \equiv \partial_x^2 + \partial_y^2 + \partial_z^2. \qquad (1.77)}$$

Ex[1.24]: Prove that

$$\nabla^2 f = \nabla \cdot (\nabla f) \qquad (1.78)$$

for any function f.

1.3 Curvilinear Coordinates

The only curvilinear coordinates we will use in this course are spherical coordinates and cylindrical coordinates.

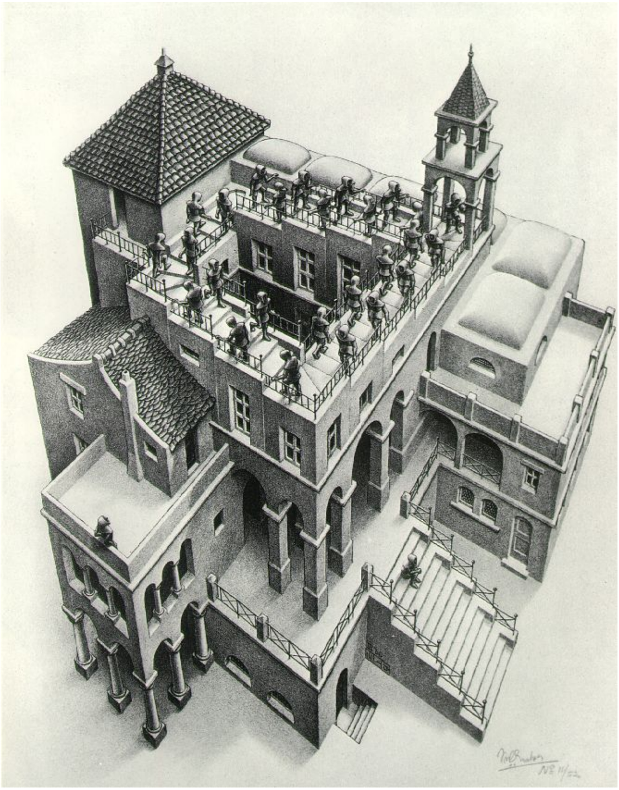


Figure 1.3: The relation $\nabla \times \nabla V = 0$ is violated in the imaginary world of Escher.

In **spherical coordinates**,

$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \tan^{-1}(\sqrt{x^2+y^2}/z), \quad \phi = \tan^{-1}(y/x),$$
$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$

$$\begin{aligned}
\nabla V &= \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}, \\
\nabla \cdot \mathbf{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}A_\phi, \\
\nabla \times \mathbf{A} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial}{\partial \phi}A_\theta\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}A_r - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta} \\
&\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial \theta}A_r\right]\hat{\phi}, \\
\nabla^2 V &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}.
\end{aligned}$$

In **cylindrical coordinates**,

$$s = \sqrt{x^2+y^2}, \quad \phi = \tan^{-1}(y/x), \quad z = z,$$
$$x = s\cos\theta, \quad y = s\sin\theta, \quad z = z. \tag{1.79}$$

One should have tried to avoid using the same symbol z in both Cartesian and cylindrical coordinates.

$$
\begin{aligned}
\nabla V &= \frac{\partial V}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{\mathbf{z}}, \\
\nabla \cdot \mathbf{A} &= \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial}{\partial \phi}A_\phi + \frac{\partial}{\partial z}A_z, \\
\nabla \times \mathbf{A} &= \left[\frac{1}{s}\frac{\partial}{\partial \phi}A_z - \frac{\partial}{\partial z}A_\phi\right]\hat{\mathbf{s}} + \left[\frac{\partial}{\partial z}A_s - \frac{\partial}{\partial s}A_z\right]\hat{\phi} \\
&\quad + \frac{1}{s}\left[\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial}{\partial \phi}A_s\right]\hat{\mathbf{z}}, \\
\nabla^2 V &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}.
\end{aligned}
$$

1.4 Dirac Delta Function

A *distribution* $D(x)$ is to be defined by

$$
\int_{-\infty}^{\infty} dx\ f(x)D(x) \in \mathbb{C}, \tag{1.80}
$$

that is, it defines a linear map from a well defined smooth function $f(x)$ to a number in $\mathbb{C}$.

Change of variables and integration by parts are always assumed to be legitimate manipulations.

Distributions are not necessarily well defined functions.

Products of distributions may not be well defined.

Two distributions D_1, D_2 are considered the same distribution if

$$
\int dx\ f(x)D_1 = \int dx\ f(x)D_2 \tag{1.81}
$$

for all well defined functions $f(x)$.

The function $F(x)$ that equals 1 at $x = 0$ and vanishes everywhere else is considered the same distribution as the constant 0.

The Dirac Delta function is a distribution defined by

$$
\boxed{\int_{-\infty}^{\infty} f(x)\delta(x) = f(0).} \tag{1.82}
$$

Ex[1.25]: The Gaussian function is defined as

$$
G_a(x) = \frac{1}{\sqrt{2\pi a}}e^{-x^2/2a}. \tag{1.83}
$$

For $a > 0$, show that $\int_{-\infty}^{\infty} dx\, G_a(x) = 1$. Argue that

$$
\delta(x) = \lim_{a\to 0} G_a(x) \tag{1.84}
$$

by evaluating $\int dx\ f(x)G_a(x)$ for an analytic function

$$
f(x) = f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + \cdots . \tag{1.85}
$$

Q[1.14]: Write an expression for the electric charge density $\rho(\mathbf{r})$ of a point charge q with the trajectory $\mathbf{R}(t)$. What is the current density?

Ex[1.26]: For a positive integer n,

$$\int_{-\infty}^{\infty} dx\ f(x)\frac{d^n}{dx^n}\delta(x-x_0) = ? \tag{1.86}$$

Q[1.15]: For a function $f(x)$ with n zeros at $x = x_1, \cdots, x_n$, find A_n's in the expression

$$\delta(f(x)) = \sum_{i=1}^{n} A_n \delta(x - x_n). \tag{1.87}$$

Ex[1.27]: Suppose two functions $f(x, y)$ and $g(x, y)$ are both zero only when $(x, y) = (x_0, y_0)$, find A in the expression

$$\delta^{(2)}(f(x,y), g(x,y)) = A\delta^{(2)}(x - x_0, y - y_0). \tag{1.88}$$

Dirac delta functions can be defined in higher dimensions

$$\delta^{(3)}(\mathbf{r}) \equiv \delta(x)\delta(y)\delta(z). \tag{1.89}$$

The following identities will become important in the next chapter

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^{(3)}(\mathbf{r}), \tag{1.90}$$

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta^{(3)}(\mathbf{r}). \tag{1.91}$$

To prove (1.90), first show that $\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 0$ everywhere $r > 0$. Next show that for the inside $\mathcal{V}$ of a sphere of radius R,

$$\int_{\mathcal{V}} d\tau\ f(r)\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi f(0) \tag{1.92}$$

independent of R for any well defined function $f(r)$. Since $\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right)$ is invariant under rotations, one can argue that (1.90) is correct.

Eq.(1.91) is straightforward to prove.

Due to (1.91), there is (almost) always a solution V to the equation

$$\nabla^2 V = -\rho \tag{1.93}$$

for a given function ρ. First note that after a translation by a vector $\mathbf{r}'$, eq.(1.91) becomes

$$\nabla^2 \left(\frac{1}{\mathscr{r}}\right) = -4\pi\delta^{(3)}(\boldsymbol{\mathscr{r}}), \tag{1.94}$$

where

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}', \qquad \mathscr{r} = |\mathbf{r} - \mathbf{r}'|. \tag{1.95}$$

This allows us to write the solution to the equation above as

$$V(\mathbf{r}) = \frac{1}{4\pi}\int d^3\tau'\ \frac{\rho(\mathbf{r}')}{\mathscr{r}}. \tag{1.96}$$

This is one of the useful applications of delta functions.

You can check that this is indeed a solution by plugging it back into the differential equation for V. Thus we can always solve that differential equation at least whenever the integral (1.96) is well defined (does not diverge).

1.5 The Theory of Vector Fields

Read Griffiths Sec. 1.6 and Appendix B.

Helmholtz theorem:

For any (sufficiently smooth) vector field $\mathbf{F}$ in $\mathbb{R}^3$,

$$\mathbf{F} = -\nabla U + \nabla \times \mathbf{W}, \tag{1.97}$$

for some U and $\mathbf{W}$ (not unique).

Special case:

Let

$$\mathbf{C} \equiv \nabla \times \mathbf{F}, \qquad D \equiv \nabla \cdot \mathbf{F}. \tag{1.98}$$

If $\mathbf{F}$ goes to zero, and $\mathbf{C}$, D go to zero faster than $1/r^2$ as $r \to \infty$, F is uniquely determined by $\mathbf{C}$ and D via

$$U(\mathbf{r}) \equiv \frac{1}{4\pi} \int \frac{D(\mathbf{r}')}{\boldsymbol{\mathscr{r}}} d\tau', \tag{1.99}$$

$$\mathbf{W}(\mathbf{r}) \equiv \frac{1}{4\pi} \int \frac{\mathbf{C}(\mathbf{r}')}{\boldsymbol{\mathscr{r}}} d\tau'. \tag{1.100}$$

In the analogy with the fluid dynamics, in general the flow $\mathbf{A}$ can not be completely described by infinitesimal vortices alone, or by infinitesimal sources alone. Rather it is a superposition, but no more than that.

Theorem 1:

$\nabla \times \mathbf{A} = 0$ iff there exists a function V such that $\mathbf{A} = -\nabla V$.

The $\Leftarrow$ part of the statement is trivial to prove.

The $\Rightarrow$ part can be proved by computation as follows. Let $V(\mathbf{r}) = V_0 - \int_{\mathcal{P}(\mathbf{r})} d\mathbf{l}' \cdot \mathbf{A}(\mathbf{r}')$, where V_0 is an arbitrary constant, and $\mathcal{P}(\mathbf{r})$ is an arbitrary path from the origin to $\mathbf{r}$. Since $\nabla \times \mathbf{A} = 0$, the integral of the 2nd term is independent of the choice of the path, as long as it starts at the origin and ends on $\mathbf{r}$. Thus $V(\mathbf{r})$ is well defined, and one can check that $\nabla V = -\mathbf{A}$.

If $\nabla \cdot \mathbf{A}$ goes to 0 at infinities sufficiently fast, the special case of the Helmholtz theorem is already a proof.

Ex[1.28]: Prove that the following statements are equivalent in $\mathbb{R}^3$:

1. $\nabla \times \mathbf{A} = 0$.
2. $\mathbf{A} = -\nabla V$ for some V.
3. $\oint \mathbf{A} \cdot d\ell = 0$.
4. $\int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{A} \cdot d\ell =$ independent of the path from $\mathbf{r}_a$ to $\mathbf{r}_b$.

Theorem 2:

$\nabla \cdot \mathbf{A} = 0$ iff there exists $\mathbf{B}$ such that $\mathbf{A} = \nabla \times \mathbf{B}$.

The $\Leftarrow$ part of the statement is trivial to prove.

If $\nabla \times \mathbf{A}$ goes to zero at infinities sufficiently fast, the $\Rightarrow$ part of the statement is included in the special case of the Helmholtz theorem. We will not give the proof for the general case here.

Given a flow $\mathbf{B}$, $\mathbf{A} = \nabla \times \mathbf{B}$ corresponds to the direction and density of infinitesimal vortices. The fact that $\nabla \cdot \mathbf{A} = 0$ implies that the center of the vortices form continuous lines and there is no "source for the vortices".

Ex[1.29]: Prove that the following statements are equivalent in $\mathbb{R}^3$.

1. $\nabla \cdot \mathbf{A} = 0$.
2. $\mathbf{A} = \nabla \times \mathbf{B}$ for some $\mathbf{B}$.
3. $\oint \mathbf{A} \cdot d\mathbf{a} = 0$.
4. $\int_S \mathbf{A} \cdot d\mathbf{a} =$ independent of the surface S for a given boundary ∂S.

1.6 Exercises

1. (tensor) Let A_i, B_j, C_k be vectors, which of the following is a tensor? (1) $A_1 B_i + A_2 B_j + A_3 B_k$; (2) $(A_1 + A_2 + A_3) B_i C_j$; (3) $\partial_1 B_2 + \partial_2 B_3 + \partial_3 B_1$; (4) $(A_1 B_1 + A_2 B_2 + A_3 B_3) A_i B_j C_k$.

2. (rotation) Upon a 3D rotation along the axis $(\hat{\mathbf{y}} + \hat{\mathbf{z}})$ by an angle θ, all vectors $\mathbf{A}$ are transformed as
$$A_i \to A'_i = R_{ij} A_j. \tag{1.101}$$
Find R_{ij}.

3. (δ-fx) Find $A(s_0, \phi_0)$ defined by
$$\delta(x - s_0 \cos\phi_0)\delta(y - s_0 \sin\phi_0) = A(s_0, \phi_0)\delta(s - s_0)\delta(\phi - \phi_0), \tag{1.102}$$
where x, y are 2D Cartesian coordinates, and s, ϕ are 2D polar coordinates. $(s_0 > 0, 0 \leq \phi_0 < 2\pi)$.

4. (δ-fx) For given $a, b, c \in \mathbb{R}$, find A, B, C in the expression
$$\delta\left((x-a)(x-b)(x-c)\right) = A\delta(x-a) + B\delta(x-b) + C\delta(x-c). \tag{1.103}$$

5. (δ-fx) Suppose D functions $f_i(x_1, \cdots, x_D)$ vanish only when $x_i = y_i$ $(i = 1, 2, \cdots, D)$ for given real numbers y_i. Find A in the expression
$$\delta^{(D)}(\mathbf{f}) = A\delta^{(D)}(\mathbf{x} - \mathbf{y}). \tag{1.104}$$

6. (line integral) Define a closed path $\mathcal{C}_1$ as the boundary of a rectangle on the $x-y$ plane with the corners at $(0,0), (L_1, 0), (L_1, L_2), (0, L_2)$ (in that order), and another closed path by $\mathcal{C}_1 = \{(x(t), y(t), 0) | t \in [0, 2\pi]\}$ with $x(t) = \cos t, y(t) = \sin t$. Let $\mathbf{A} = \hat{\mathbf{x}} y - \hat{\mathbf{y}} x$. Calculate directly the following quantities:

   (a) $\oint_{\mathcal{P}} d\mathbf{l} \cdot \mathbf{A}(\mathbf{r}) =?$
   for $\mathcal{P} = \mathcal{C}_1$ and $\mathcal{P} = \mathcal{C}_2$.

   (b) $\int_{\mathcal{S}} d\mathbf{a} \cdot (\nabla \times \mathbf{A}) =?$
   for $\mathcal{S}$ being the interior of $\mathcal{C}_1$ and $\mathcal{C}_2$. (Use the right hand rule to determine the direction of $d\mathbf{a}$.)

   Check that the results agree with the theorem (1.66).

7. Let $\mathbf{A} = \hat{\mathbf{x}} x + \hat{\mathbf{y}} y$. Calculate directly the following quantities:

   (a) $\oint_{\mathcal{P}} d\mathbf{l} \cdot \mathbf{A}(\mathbf{r}) =?$
   for $\mathcal{P} = \mathcal{C}_1$ and $\mathcal{P} = \mathcal{C}_2$.

   (b) Find V such that $\mathbf{A} = -\nabla V$.

   Check that the results agree with the theorem (1.59).

8. (surface integral) Let a surface be defined by

$$\mathcal{S} = \{(x(\alpha,\beta), y(\alpha,\beta), z(\alpha,\beta))|\alpha \in [0,1], \beta \in [0,2\pi)\} \tag{1.105}$$

with

$$x(\alpha,\beta) = \alpha\cos\beta, \qquad y(\alpha,\beta) = \alpha\sin\beta, \qquad z(\alpha,\beta) = \frac{1}{2}c\alpha^2, \tag{1.106}$$

for a constant $c \in \mathbb{R}$. Evaluate the surface integral

$$\int_{\mathcal{S}} d\mathbf{a}\cdot\hat{\mathbf{z}}\, ze^{-(x^2+y^2)/2}. \tag{1.107}$$

(Choose the area element $d\mathbf{a}$ to have a positive projection on $\hat{\mathbf{z}}$.)

9. (differential calculus) Define $\mathcal{V}$ as the solid sphere of radius R centered at the origin. Evaluate

$$\int_{\mathcal{V}} d^3\tau \frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla f(r). \tag{1.108}$$

(Express the result in terms of $f(0)$ and $f(R)$.

10. (theory of vector fields) If a vector field $\mathbf{A}$ satisfies

$$\nabla\cdot\mathbf{A} = 0, \qquad \text{and} \qquad \nabla\times\mathbf{A} = 0, \tag{1.109}$$

prove that $\mathbf{A}$ can be solved in terms of a scalar V which satisfies the Laplace equation

$$\nabla^2 V = 0. \tag{1.110}$$

11. (Helmholtz theorem) By finding infinitely many explicit solutions to the equations

$$\nabla\cdot\mathbf{A} = 0, \qquad \nabla\times\mathbf{A} = 0, \tag{1.111}$$

prove that there are infinitely many solutions to (1.98) for given $\mathbf{C}$ and D if $\mathbf{F}$ is not assumed to vanish at infinity.

Homework Set 1

1. Ex.[1.27].

2. Prob. 6 in this section.

3. Prob. 7 in this section.

4. Griffiths Prob. 1.39

5. Griffiths Prob. 1.64

Appendix: More about the Helmholtz Theorem

The Helmholtz theorem states that (almost) any vector field $\mathbf{F}$ can be expressed as

$$\mathbf{F} = -\nabla U + \nabla \times \mathbf{W} \tag{1.112}$$

for some U and $\mathbf{W}$.

Instead of aiming at the highest generality, we prove this theorem via direct computation, using the solution (1.96) to the Poisson equation (1.93).

A crucial ingredient in proving the theorem is to notice that a vector field $\mathbf{V}$ can be shifted

$$\mathbf{V} \to \mathbf{V}' = \mathbf{V} + \nabla f \tag{1.113}$$

for an arbitrary function f without affecting $\nabla \times \mathbf{V}$. This degeneracy allows us to choose $\mathbf{V}$ to be divergenceless if we only care about $\nabla \times \mathbf{V}$ because if $\mathbf{V}$ is not divergenceless, we can choose f to satisfy

$$\nabla^2 f = -\nabla \cdot \mathbf{V} \tag{1.114}$$

f can (almost) always be solved because the equation for f is of the same form as (1.93).

so that the new vector field $\mathbf{V}'$ is divergenceless.

To prove that we can always find U and $\mathbf{W}$ for given $\mathbf{F}$, we note that if U is already known, we should choose $\mathbf{W}$ such that

$$\nabla \times \mathbf{W} = \mathbf{F} + \nabla U. \tag{1.115}$$

Let

$$\mathbf{W} = \nabla \times \mathbf{V}, \tag{1.116}$$

the equation above becomes

$$\nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V} = \mathbf{F} + \nabla U. \tag{1.117}$$

As mentioned above, we can always choose $\mathbf{V}$ to be divergenceless, and so we want $\mathbf{V}$ to satisfy

$$\nabla^2 \mathbf{V} = -(\mathbf{F} + \nabla U). \tag{1.118}$$

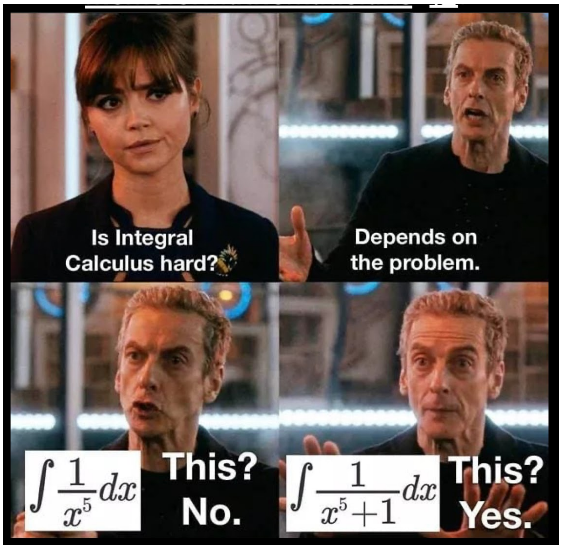
This equation can be solved because for each component of $\mathbf{V}$ it is of the same form as (1.93). But this is consistent only if the right hand side is also divergenceless

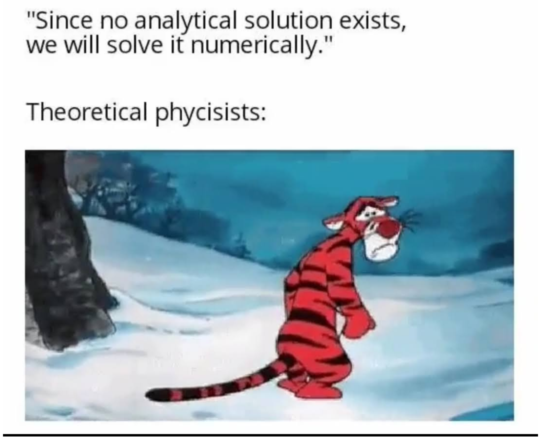
$$\nabla \cdot \mathbf{F} + \nabla^2 U = 0. \tag{1.119}$$

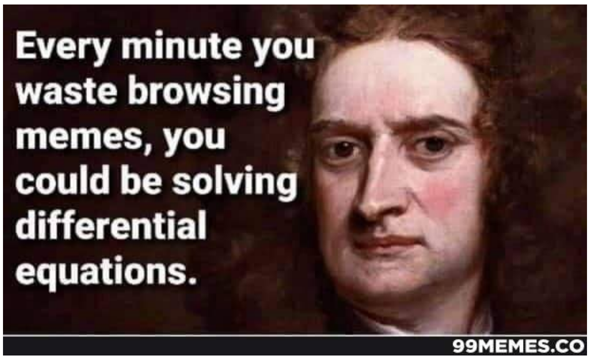
This condition can be achieved by choosing U to solve this equation, which is also of the form (1.93). This completes the proof.

Q[1.16]: Prove (1.99) and (1.100).


Maryam
@MaryamSheikh__
Forget about beards, six packs and good looks. Your kids deserve a dad, who can solve complex differential calculus and integrations.


Is Integral Calculus hard?
Depends on the problem.
$\int \frac{1}{x^5}dx$
This? No.
$\int \frac{1}{x^5+1}dx$
This? Yes.


"Since no analytical solution exists, we will solve it numerically."
Theoretical phycisists: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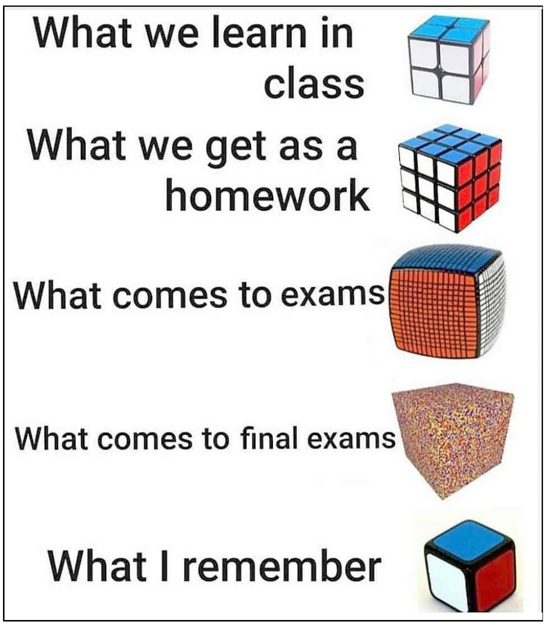
What we learn in class
What we get as a homework
What comes to exams
What comes to final exams
What I remember