12.4 Electromagnetic Induction

Now we consider time-dependent properties of electromagnetism, in which electricity and magnetism are unified. Some of the physical laws we have introduced above,

$$\oint_S \mathbf{E}\cdot d\mathbf{a} = \frac{Q_{enc}}{\epsilon_0}, \qquad \oint_{\mathcal{C}} \mathbf{E}\cdot d\ell = 0, \qquad \oint_S \mathbf{B}\cdot d\mathbf{a} = 0, \qquad \oint_{\mathcal{C}} \mathbf{B}\cdot d\ell = \mu_0 I_{enc}$$

The 2nd and 4th will be modified.

will have to be modified.

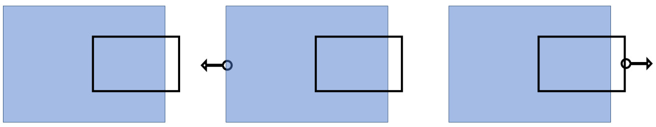
Define the magnetic flux through a surface S by

$$\Phi_B(S) \equiv \int_S \mathbf{B}\cdot d\mathbf{a}. \tag{12.69}$$

If S is closed, Φ_B(S) must be 0.

For the integral on the right to be well defined, a normal direction has to be chosen for S. If the normal direction is chosen to be in the opposite direction, the flux changes by sign.

Faraday carried out a series of experiments in 1825 – 1831. Schematically, the experiments are of the three types in Fig.12.2. (Note that experiment (b) and (c) are related to each other by relative motion.)



(a) (b) (c)

Figure 12.2: Faraday's experiments: The blue square represents a spatial region of magnetic field perpendicular to the paper. In (a), the magnetic field is time-dependent (e.g. by turning on and off the current on a solenoid that generates the magnetic field). In (b), the spatial region with magnetic field is changing with time. In (c), the conducting loop is moving.

As an explanation of these experiments, we have **Faraday's law**:

$$\boxed{\oint_{\mathcal{C}} \mathbf{E}\cdot d\ell = -\frac{d\Phi_B(S)}{dt},} \tag{12.70}$$

For a given closed curve $\mathcal{C}$, there are infinitely many ways to choose S so that $\mathcal{C}$ is the boundary of S.

where $\mathcal{C}$ is the boundary of S. When S is open, it has a boundary $\mathcal{C} \equiv \partial S$. By definition the direction of the boundary $\mathcal{C}$ is related to the normal direction of S by the **right-hand rule**.

Faraday's law (12.70) holds for arbitrary choices of $\mathcal{C}$ and S at any instant of time.

Q 12.77: What happens to Faraday's law if S is closed?

Q 12.78: Which ones of Faraday's experiments are about Faraday's law, Lorentz force law, or other laws?

All of Faraday's experiments can be summarized by the **flux rule**

$$\boxed{\mathcal{E} = -N\frac{d\Phi_B}{dt}, \qquad (12.71)}$$

N is the winding number of the loop around $\mathcal{C}$.

where $\mathcal{E}$ is (unfortunately) called the **electromotive force** (*emf*)

$$\mathcal{E} = \oint_{\mathcal{C}} \mathbf{f} \cdot d\boldsymbol{\ell}. \qquad (12.72)$$

The loop $\mathcal{C}$ is arbitrary.

Here $\mathbf{f}$ is the force on a unit charge. (If the only force is electromagnetic, $\mathbf{f} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$.) It can be electric force, magnetic force, or other forces. When the Lorentz force law is used to compute $\mathbf{f}$, the instant velocity of a charge on each point on the loop is by definition given by the instant velocity of the loop.

The flux rule incorporates both the effect of the Lorentz force and that of Faraday's law.

Unfortunately, we will not justify the flux rule for its maximal generality here. (Please take the course of Electromagnetism.)

Q 12.79: Can we still define the electric potential by $V(\mathbf{r}_2) - V(\mathbf{r}_1) = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E} \cdot d\boldsymbol{\ell}$?

Notice that the relation between the electric field and the electric potential $V(\mathbf{r}_2) - V(\mathbf{r}_1) = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E} \cdot d\boldsymbol{\ell}$, which was correct in electrostatics, must be modified now, because it is inconsistent with Faraday's law. On the other hand, the *emf* can be viewed as a kind of potential difference in an electric circuit, from the perspective of the rest of the circuit.

We will talk about how to define V in time-dependent cases later.

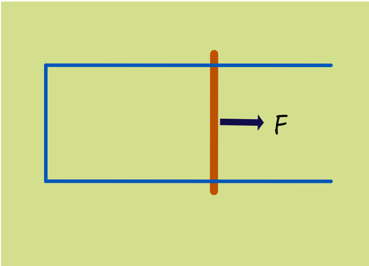
Q 12.80: Will the emf $\mathcal{E}$ on a conducting loop with the resistance R satisfy Ohm's law as $\mathcal{E} = RI$?

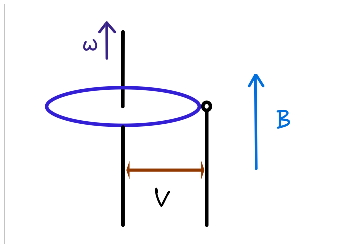
Notice also that Faraday's law holds regardless of whether a conducting wire coincides with the loop $\mathcal{C}$. But when there is a conducting wire, the charges on the wire can be redistributed so that the charge density is no longer zero everywhere, in such a way that the resulting electric field is consistent with the motion of charges (e.g. Ohm's law is observed).

Faraday "saw" magnetic flux through iron shreds in 1832, and imagined force mediated by a field. Einstein viewed Faraday's contribution in the idea of *field* much greater than his discovery of electromagnetic induction.

Ex 12.65: A bar is moving under external force F on a conducting rail of width L in the background of a constant magnetic field B (pointing out of the paper in the figure below). The resistance of the bar is R, and that of the rail can be neglected. What is the direction and magnitude of the current? What is the terminal force of the bar?

$v = \frac{RF}{L^2B^2}$.





Ex 12.66: A conducting disk of radius R rotates in the background of a constant magnetic field B. perpendicular to the disk. What is the emf V between the center of the disk and a fixed point in space touching the edge of the disk?

Solution:

There are two ways to find the answer.

(1) Use Lorentz force law. The work done on a charge q by Lorentz force when it moves from the center to the rim is

$$W = \int_0^R dr\; q(\omega r)B = \frac{q\omega BR^2}{2}, \tag{12.73}$$

so

$$V = W/q = \frac{\omega BR^2}{2}. \tag{12.74}$$

(2) Use the flux rule.

$$V = \frac{d\Phi_B}{dt} = B\frac{d(Area)}{dt} = B\frac{d}{dt}\left(\frac{R^2}{2}\theta\right) = \frac{\omega BR^2}{2}. \tag{12.75}$$

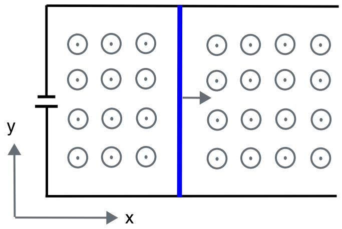
Example: Electric generator (vs. electric motor).

Example: eddy currents: induction cooker/stove, a magnet falling through a metal tube, etc.

Q 12.81: How to turn a motor into a generator?

HW: (3-2) A long solenoid of n turns per unit length with a given current $I(t)$ has radius R with the z-axis at its center. What is the emf on a concentric circle of radius r? Consider both cases $r > R$ and $r < R$. (Assume that the time-dependence of $I(t)$ has a characteristic time scale $\Delta t \gg R/c$, where c is the speed of light.)

Ex 12.67: A conducting bar of mass M and length L in the y direction is placed in the background of a constant magnetic field B in the z direction. It moves on a conducting rail that connects it to a source of electricity such that the bar carries a constant current I in the y direction at all times. The bar is released at $t = 0$ from rest to move in the x-direction without friction due to the Lorentz force. Find the emf induced between the two ends of the bar due to the magnetic field as a function of time for $t > 0$.



Ex 12.68: A conducting bar of mass M and length L in the y-direction is placed in the background of a constant magnetic field B in the z-direction. It moves on a conducting rail that connects it to a battery of a constant voltage V_0 through a resistance R. (Ignore other resistance in the circuit.) The bar is initially at rest at $t = 0$ with $x = 0$. (a) (5%) What is the terminal velocity $v(t \to \infty)$ of the bar? (b) (5%) What is the trajectory $x(t)$ of the bar? (c) (5%) How much energy is wasted on the resistor? (d) (5%) What is the total energy taken from the battery from $t = 0$ to $t = \infty$?

Solution:

The magnetic force on the bar s $F = ILB$ (in the x-direction), and so the acceleration of the bar is

$$a(t) = \frac{F(t)}{M} = \frac{I(t)LB}{M} \tag{12.76}$$

(in the x-direction). The velocity $v(t)$ of the bar leads to a magnetic force $f = v(t)B$ on a unit charge on the bar, producing the emf of $\mathcal{E} = BLv(t)$. (The flux rule gives the same result.) The current $I(t)$ thus satisfies Kirchhoff's loop rule:

$$V_0 = \mathcal{E} + RI(t) = BLv(t) + RI(t). \tag{12.77}$$

The time derivative of this equation is

$$0 = BLa(t) + R\frac{dI}{dt}. \tag{12.78}$$

Combined with the expression of $a(t)$ above, we find

$$0 = \frac{L^2B^2}{M}I(t) + R\frac{dI(t)}{dt}, \tag{12.79}$$

which is solved by

$$I(t) = I_0 e^{-\frac{L^2B^2}{RM}t}. \tag{12.80}$$

Since the initial velocity $v(0) = 0$, the initial current is $I_0 = \frac{V_0}{R}$. Hence eq.(12.77) above implies that at $t > 0$,

$$v(t) = \frac{V_0}{BL}\left[1 - e^{-\frac{L^2B^2}{RM}t}\right], \tag{12.81}$$

which implies that the terminal velocity is

$$v(\infty) = \frac{V_0}{BL}, \tag{12.82}$$

and that the trajectory is given by

$$x(t) = x(0) + \int_0^t v(t')dt' = \frac{V_0}{BL}\left[t + \frac{RM}{L^2B^2}\left(e^{-\frac{L^2B^2}{RM}t} - 1\right)\right]. \tag{12.83}$$

The energy wasted on the resistor is

$$W_R = \int_0^\infty RI^2(t)dt = \frac{V_0^2 M}{2L^2B^2}. \tag{12.84}$$

The energy provided by the battery is

$$W_{batt} = \int_0^\infty V_0 I(t)dt = \frac{V_0^2 M}{L^2B^2}. \tag{12.85}$$

Check that the rest of the battery's energy is indeed given to the bar:

$$K = \frac{1}{2}Mv^2(\infty) = \frac{V_0^2 M}{2L^2B^2}. \tag{12.86}$$