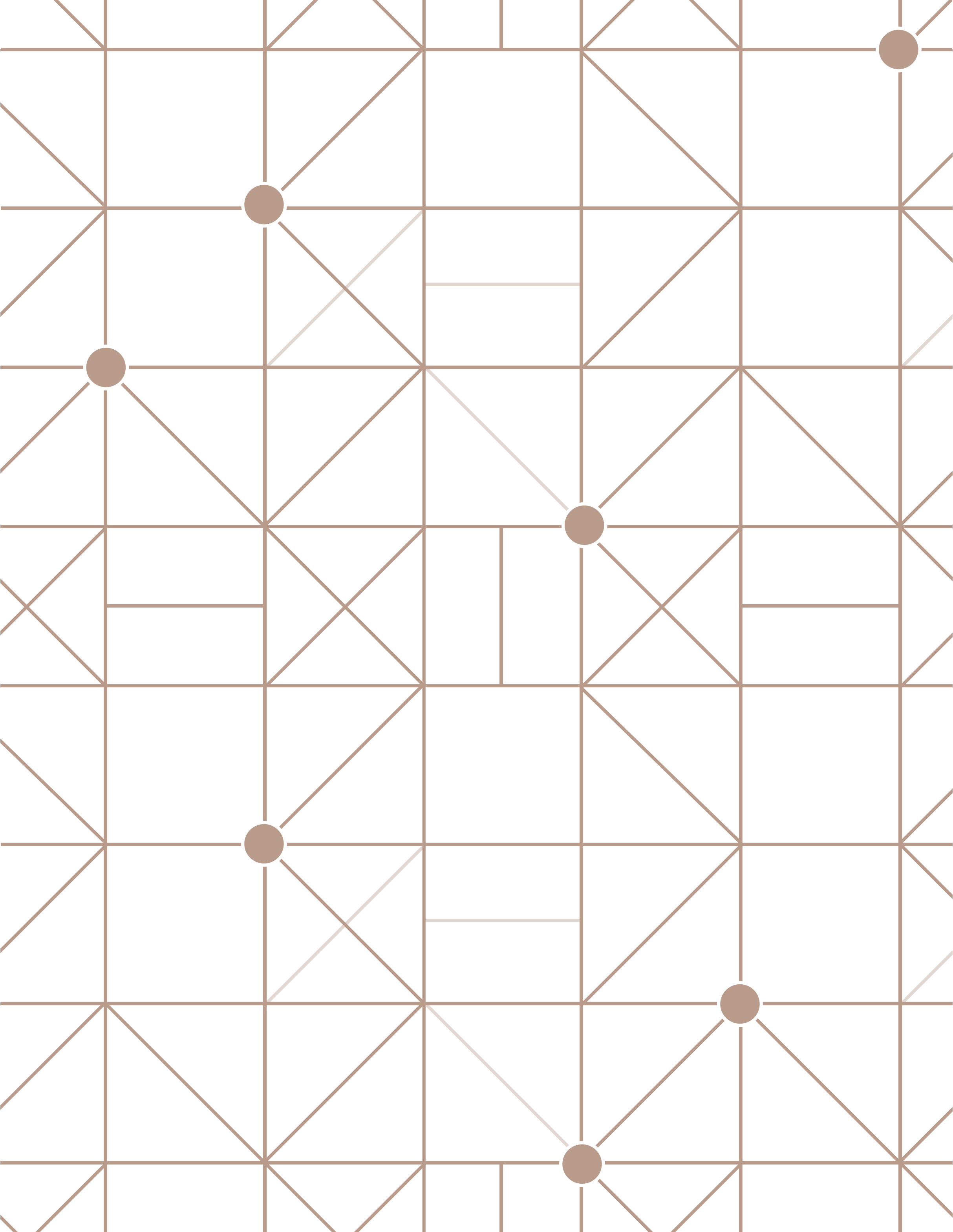


Ch 2.4 The Four Fundamental Subspaces.

[A] Four Subspaces

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{bmatrix}$$

① Column Space $C(A)$ in R^m

② Nullspace $N(A)$ in R^n
(All vectors x such that $\boxed{Ax=0}$)

③ Row Space
= all combinations of rows
= all combinations of columns of A^T in R^n
$= C(A^T)$

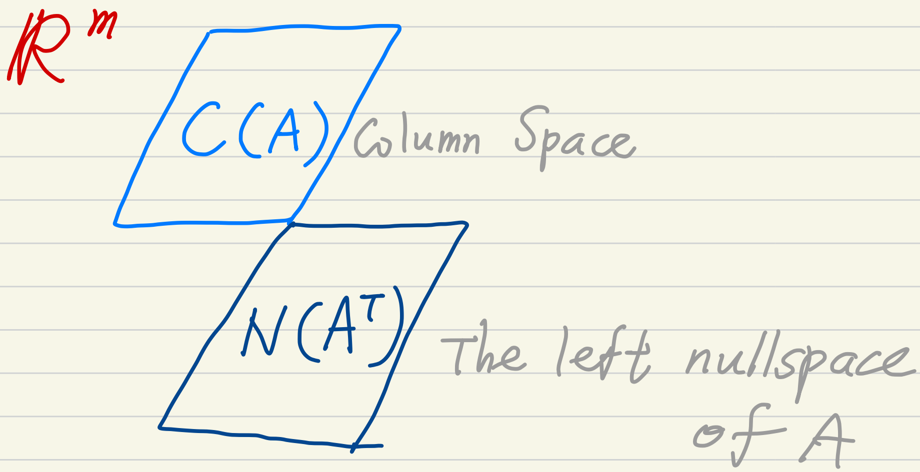
④ Null space of A^T

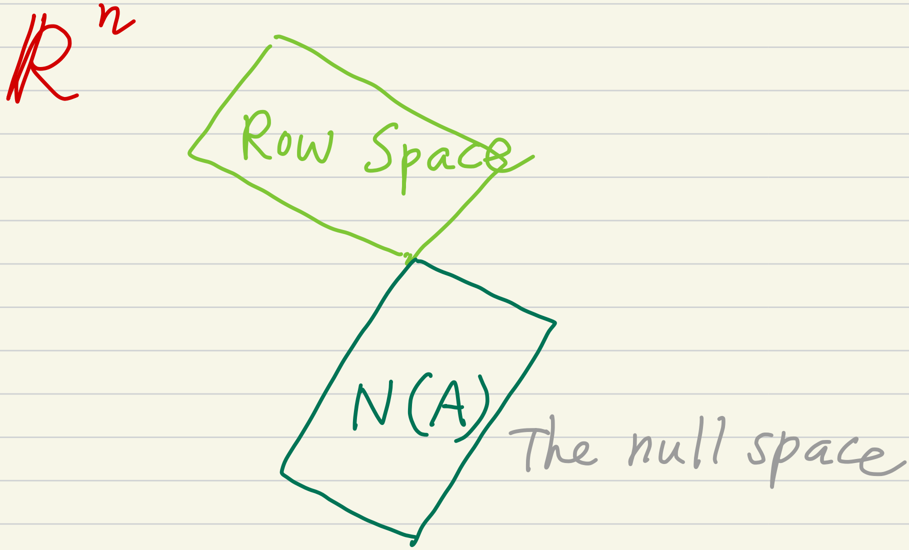
$= N(A^T)$ in $\mathbb{R}^m$

$=$ The left nullspace of A

(All vectors y such that $\boxed{A^T y = 0}$)

[B] Four Subspaces





[C] Dimension of 4 subspaces

$A_{m \times n}$

①

– $\boxed{\text{Column Space}}$, dim. $C(A)$

$\mathbb{R}^m$ $= \text{rank } \boxed{r}$

- $\boxed{\text{Null space}}$, dim. $N(A)$

$\mathbb{R}^n$ = special solutions

= free variables

= $\boxed{(n-r)}$

- $\boxed{\text{Row Space}}$, dim. $C(A^T)$

$\mathbb{R}^n$ = rank $\boxed{r}$

- $\boxed{\text{The left nullspace}}$,

$\mathbb{R}^m$ dim. $N(A^T) = \boxed{(m-r)}$

Ex:

$$A = U = R = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot, $\boxed{r=1}$

pivot row

pivot column

① Column Space :

A line through $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in R^2

$\because r=1$, dim. of $C(A) = 1$

② Row Space :

A line through $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in R^3

$\because r=1$, dim. of Row Space $= 1$

③ Null space :

A plane in R^3

$\because r=1$, dim. of Nullspace $= 3-1=2$

④ Left nullspace :

A line in R^2.

$\because r=1$, dim. of left nullspace $= 2-1=1$

Ex:

$$A = U = R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{N(A)}$ contains $\boxed{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$ and $\boxed{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$.

$\boxed{N(A^T)}$ contains $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Notes:

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \longrightarrow U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The nonzero rows are formed a basis, and the row space has dim. r.

- The row space $C(A^T)$ has the same dimension r as the row space of $C(U^T)$, and it has the same bases because the row space of A and U (and R) are the same.

$\Rightarrow$ A and U have different rows, but the combinations of the rows are identical:

$$\boxed{\text{Row Space: } C(A^T) = C(U^T)}$$

$$Ax=0 \quad \boxed{\text{Null space: } N(A) = N(U)}$$

$$\boxed{\text{Column Space: } C(A) \neq C(U)}$$

Notes

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}_{2\times 2} \quad \text{rank } r = 1$$

① Column Space:

$\therefore$ $\boxed{\text{All multiples of } \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$

② Null Space :

$\boxed{AX = 0}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 + 2x_2 = 0$$

$\therefore$ $\boxed{\text{All multiples of } \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$

③ Row Space:

$$\boxed{\text{All multiples of } \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

Row Space: $C(A^T)$

$=$ Column Space of A^T.

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

④ Left Nullspace:

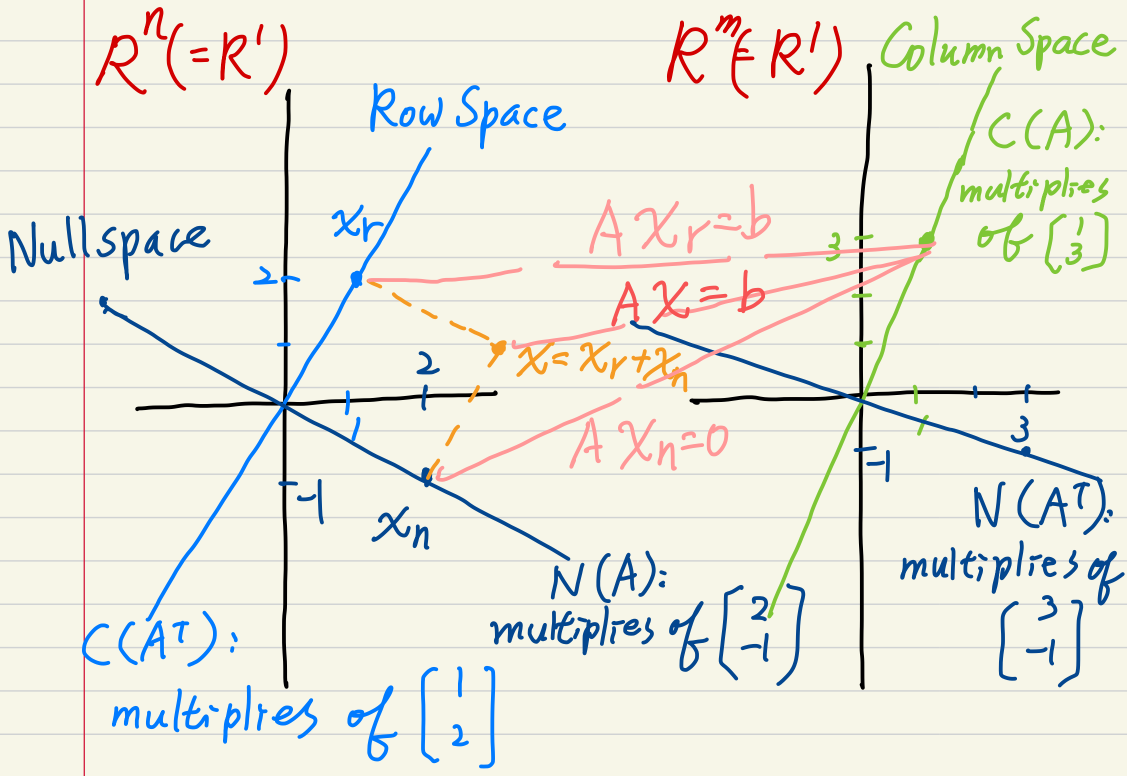
$$\boxed{A^T y = 0} \quad \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

$$\therefore y_1 + 3y_2 = 0$$

All multiples of $y = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

⑤ All Four Subspaces are Lines.

$$\left[\begin{array}{l} \because\ r=1 \\ \therefore\ n-1=1 \\ \quad m-1=1 \end{array}\right.$$



$\boxed{A_{m\times n}}$

Note:

Dimension of $C(A)$ (r)

$+$ Dimension of $N(A)$ $(n-r)$

$=$ Number of Columns (n)

$(r)+(n-r)=n$

(rank + nullity $= n$)

Dimension of $C(A^T)$ (r)

$+$ Dimension of $N(A^T)$ $(m-r)$

$=$ Number of Rows m

$(r) + \dim.(N(A^T)) = m$

$\therefore \dim.(N(A^T)) = (m-r).$

Note: Existence and Uniqueness

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}_{m \times n} \quad (r = m = 2)$$

$\boxed{\text{Righ- inverse } C}$: $AC = I_{m \times m}$

$$\therefore AC = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ C_{31} & C_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

→ There are many C_{31}, C_{32}

→ A case of existence but not uniqueness.

① Existence :

Full row rank, $r = m$.

$\boxed{Ax = b}$ has at least one solution x of every b if and only if the columns span $\boxed{R^m}$. Then A has a right-inverse C such that

$$\boxed{AC = I_{m\times m}}$$

(This is possible only if $m \le n$)

$\boxed{\text{Left-inverse } B}$: $BA^T = I_{n\times n}$

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \quad (r = n = 2)$$

$$BA^T = \begin{bmatrix} \frac{1}{4} & 0 & b_{13} \\ 0 & \frac{1}{5} & b_{23} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$b_{13} = b_{23} = 0$ Uniqueness Case.

② Uniqueness:

Full column rank, $r=n$.

$\boxed{Ax=b}$ has at most one solution x for every b if and only if the columns are L.I. The A has a left-inverse B such that

$$\boxed{BA = I_{n\times n}}$$

(This is possible only if $m \geq n$).