Finish

12.1.9 Conductors

Remember that in electrostatics the electric field only depends on the "final" charge distribution.

Electric field and potential in a conductor satisfy the following rules.

1. $\mathbf{E} = 0$ in the bulk of a conductor.

2. $\mathbf{E}_{||} = 0$ on the surface of a conductor.

3. $V =$ constant on a conductor, including the bulk and the surface.

Q 12.54: Can you plot the field lines by intuition for the following configurations? (i) a conducting ball of a given total charge Q. (ii) a conducting sphere inside an otherwise uniform electric field.

Ex 12.33: A spherical conducting shell of inner and outer radii R_1 and R_2 has a total charge Q. What is the electric field and potential everywhere in space?

Ex 12.34: (Screening Effect)

Argue that the charge distribution on the surface of a closed conducting shell (of any shape) is independent of the charge distribution inside.

HW: (1-5) A conducting shell of inner radius a and outer radius b is concentric with a conducting ball of radius R inside the shell. ($b > a > R > 0$.) The total charge on the shell is q and the total charge on the ball is Q. Find the electric field and potential for (1) $r > b$, (2) $b > r > a$, (3) $a > r > R$, (4) $R > r > 0$, respectively.

12.1.10 Capacitor and Electric Potential Energy

Capacitors are objects used in electric circuits that store electrostatic potential energy.

A capacitor is typically composed of two pieces of conductors connected to the rest of a circuit. Each piece of the conductor has a position-independent potential, and their difference is denoted by V. These two pieces of conductors are typically oppositely charged of equal magnitude. We define the "charge of a capacitor" Q to be the absolute value of the charge on either piece of the conductor.

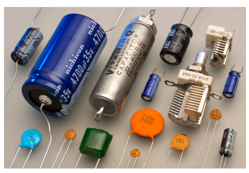
If a capacitor is made of a single piece of conductor, one can imagine that the other piece of conductor of this capacitor is located at spatial infinity.

Capacitance

The **Capacitance** of a capacitor is defined as the charge stored per unit voltage:

$$C \equiv \frac{Q}{V}. \tag{12.36}$$

The SI unit of capacitance is Faraday = Coulomb/Volt.



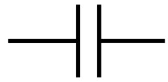
Q 12.55: Why do we define the capacitance this way? Do we expect C to be a constant independent of Q and V?

Ex 12.35: Find the capacitance of a parallel plate capacitor. What are the parameters that the capacitance depends on?

Solution:
For a parallel plate capacitor of area A and separation d, assuming that d is much smaller than any other dimension (width, length), we find

$$C = \frac{\epsilon_0 A}{d}. \tag{12.37}$$

The symbol for a capacitor:



Ex 12.36: Find the capacitance of two concentric spherical shells of radii R_1 and R_2.
Ex 12.37: Find the equivalent capacitance if we put two capacitors of capacitances C_1 and C_2 in parallel or in series.
Solution:
In parallel, V is the same and Q adds up, so $C = C_1 + C_2$.

In series, V adds up and Q is the same, so $1/C = 1/C_1 + 1/C_2$.

Potential Energy of Capacitors

Q 12.56: What is the potential energy U of a capacitor for a given V?
Q 12.57: Can you argue that U is proportional to QV before calculation?
Ex 12.38: How much energy is needed to charge a capacitor of capacitance C from 0 to Q?

See eq.(12.31).

Q 12.58: How do we know that energy is conserved through the charging process?

By energy conservation, the energy needed to charge a capacitor is the energy stored in it. The potential energy U stored in a capacitor is thus

$$\boxed{U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV. \tag{12.38}}$$

Example: Calculate the potential energy of a parallel plate capacitor by computing the energy it takes to separate the plates assuming that the plates are initially on top of each other.

When the plates are on top of each other, the electrostatic energy is zero.

Ex 12.39: In the example above, if the potential energy of a parallel plate conductor is attributed to the electric field, what is the electrostatic energy in electric field per unit volume?
Solution:

$$\boxed{u = \frac{\epsilon_0}{2}E^2. \tag{12.39}}$$

Note that this equation is independent of the geometry of the parallel plate capacitor.

It turns out that this equation actually holds in general, even when the electric field is time and space dependent.

Change of concept:

Clearly, we cannot attribute energy to both charges and field at the same time, otherwise the energy would be over-counted. If something physical must carry energy,

To whom — the charge or the field — do we attribute the energy?

either we think of the charges as physical objects carrying energy, with the electric field as merely a calculation tool, or we think of the electric field as a physical entity carrying energy, and the charges are merely outlets of electric fluxes.

A problem with attributing potential energy to charges is that we cannot specify the location of the potential energy. The electric potential energy between two charges does not belong to either charge.

> In particular, in General relativity, the energy needs to be localized as it is the source to the gravitational field.

Q 12.59: What about the gravitational potential energy of a point mass?

Imagine a charge q_1 being static except moving over a short period of time. Right after its motion, before its electromagnetic wave reaches another charge q_2, some energy input to q_1 seems to have disappeared. If energy is locally conserved, we need to imagine a wave carrying energy.

Ex 12.40: What is the electric potential energy of a uniform charge distribution over a spherical surface of radius R of total charge Q? Compute the energy in two different ways: (1) by integrating the energy in electric field per unit volume over space, (2) by calculating the work it takes to bring all charges from spatial infinity to their final location.

HW: (2-1) Repeat the exercise above for two concentric conducting shells of radii R_1, R_2 ($R_2 > R_1$) with charges q_1, q_2, respectively.

Q 12.60: What is the electric potential energy of a point charge? If we use Einstein's equation $m = E/c^2$, what is the mass of a point charge?

12.1.11 Dielectrics

Insert a slab of a certain insulating material inside a parallel-plate capacitor. The atoms on the slab are expected to be polarized, and the induced charges on the surface of the slab would change the electric field.

Ex 12.41: How would the electric field change? How would the capacitance change?

Solution:

$$C = \epsilon \frac{A}{d}, \tag{12.40}$$

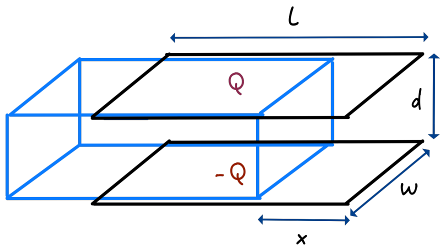
where ϵ is the permittivity of the material (called a dielectric).

> ϵ_0 is the vacuum permittivity, and the ratio $\kappa \equiv \epsilon/\epsilon_0$ is called the dielectric constant

In the definition of the capacitance, Q only counts the free charges on the conductors that can be discharged through an external circuit.

HW: (2-2) A dielectric slab of dielectric constant κ is inserted in a parallel plate capacitor. The dielectric slab is exactly the same size as the gap between the plates, of length L, width w and thickness d, but is displaced so that there is an empty gap of length x. Find the force on the slab for both cases: (1) if the plates are not connected to a closed circuit, with potential difference V; (2) if the plates are connected to a battery of potential V. (The x-axis points to the right.) *Hint:* Find the potential energy of the capacitor.

It is easier to use the potential energy to compute the force than using Coulomb's law. But it is helpful to keep Coulomb's law in mind to understand how the two cases are related.



12.1.12 Pressure on Surface Charges

The surface charge density σ on an arbitrary surface S generates its electric field so that there is a discontinuity in the electric field across S.

Without loss of generality, let us consider an infinitesimal area element da centered at the origin with its normal vector $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. There is then a discontinuity in the electric field at the origin:

$$\mathbf{E}(z = 0^+) - \mathbf{E}(z = 0^-) = \frac{\sigma(0)}{\epsilon_0}. \tag{12.41}$$

We can derive this equation as follows. We decompose the electric field $\mathbf{E}$ into the part generated by the charges on this area element da and the part generated by everything else as

$$\mathbf{E} = \mathbf{E}^{da} + \mathbf{E}^{ext}. \tag{12.42}$$

For sufficiently small da, it is almost flat, so we have

$$\mathbf{E}^{da}(z = 0^\pm) = \pm\frac{\sigma}{2\epsilon_0(0)}\hat{\mathbf{z}}. \tag{12.43}$$

As $\mathbf{E}^{ext}$ should be continuous, we find eq.(12.41). Furthermore, we find that

$$\mathbf{E}^{ext} = \frac{1}{2}\left[\mathbf{E}(z = 0^+) + \mathbf{E}(z = 0^-)\right]. \tag{12.44}$$

This is also the "democratic" result.

As a charge cannot exert force on itself, the pressure on the area element is

$$\mathbf{P} = \sigma(0)\mathbf{E}^{ext} = \frac{1}{2}\sigma(0)\left[\mathbf{E}(z = 0^+) + \mathbf{E}(z = 0^-)\right]. \tag{12.45}$$

Ex 12.42: What is the pressure on a uniformly charged shell of charge density σ.

12.1.13 Superposition Principle

The laws of electrostatics is linear, so it is clear that the following superposition principle is true: If a given charge distribution $\rho_1(\mathbf{r})$ is associated with a field $\mathbf{E}_1(\mathbf{r})$, and $\rho_2(\mathbf{r})$ with $\mathbf{E}_2(\mathbf{r})$, the charge distribution $\rho(\mathbf{r}) = \rho_1(\mathbf{r}) + \rho_2(\mathbf{r})$ is associated with the field $\mathbf{E}(\mathbf{r}) = \mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r})$.

Ex 12.43: Consider 3 fixed points A, B, and C in space. When a charge of 1 unit is placed at A (no charge at B), the electric field at C is $\mathbf{E} = \hat{\mathbf{x}} + \hat{\mathbf{y}}$. When a charge of -1 unit is placed at A, and a charge of $1/2$ unit at B, the electric field at C is $\mathbf{E} = \hat{\mathbf{x}} - \hat{\mathbf{y}}$. What is the electric field at C when the charge at A is q_A and the charge at B is q_B?

Solution:

A unit charge at A gives $\mathbf{E} = \hat{\mathbf{x}}+\hat{\mathbf{y}}$ at C. A unit charge at B gives $\mathbf{E} = 2((\hat{\mathbf{x}}-\hat{\mathbf{y}})+(\hat{\mathbf{x}}+\hat{\mathbf{y}})) = 4\hat{\mathbf{x}}$. For q_A at A and q_B at B,

$$\mathbf{E} = q_A(\hat{\mathbf{x}} + \hat{\mathbf{y}}) + q_B(4\hat{\mathbf{x}}) = (q_A + 4q_B)\hat{\mathbf{x}} + q_A\hat{\mathbf{y}}. \tag{12.46}$$

Superposition principle also applies when an electrostatic system is composed of only linear media (including conductors) with linear boundary conditions, whenever the problem is specified unambiguously.

For instance, let there be N pieces of conductors in space, and the total charge on each conductor is Q_i $(i = 1, 2, \cdots, N)$. Notice that we do not have to specify how charges are distributed on each conductor in an experiment to make sure that the result of the experiment is well defined. Then the relation between $\{Q_1, Q_2, \cdots, Q_N\}$ and the field $\mathbf{E}(\mathbf{r})$ obeys the superposition principle.

Ex 12.44: Consider three pieces of conductors: A, B, C. When the total charge on A is 1 and no charge on B, C, the potential difference between A and B is V_1. When the total charge on B is 1 and no charge on A, C, the potential difference between A and B is V_2. What is the potential difference between A and B when there is charge Q_A on A and Q_B on B and no charge on C?