Chapter 10

Dynamics of a System of Particles

We will expand our discussion from the two-body systems to systems that consist out of many particles. In general, these particles are exposed to both external and internal forces. We will make the following assumptions about the internal forces:

- The forces exerted between any two particles are equal in magnitude and opposite in direction. $\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$.

- The forces exerted between any two particles are directed parallel or anti-parallel to the line joining the two particles.

These two requirements are fulfilled for many forces. However, there are important forces, such as the magnetic force, do not satisfy the second assumption.

10.1 The center-of-mass

It is useful to separate the motion of a system into the motion of its center of mass and the motion of its component relative to the center of mass. The definition of the position of the center of mass for a multi-particle system is given by (Fig. 10.1):

$$\mathbf{R}_{\mathrm{CM}} = \frac{\sum_\alpha m_\alpha \mathbf{r}_\alpha}{\sum_\alpha m_\alpha} = \frac{1}{M}\sum_\alpha m_\alpha \mathbf{r}_\alpha \tag{10.1}$$

If the mass distribution is a continuous distribution, the summation is replaced by an integration:

$$\mathbf{R}_{\mathrm{CM}} = \frac{1}{M}\int \mathbf{r}\, dm \tag{10.2}$$

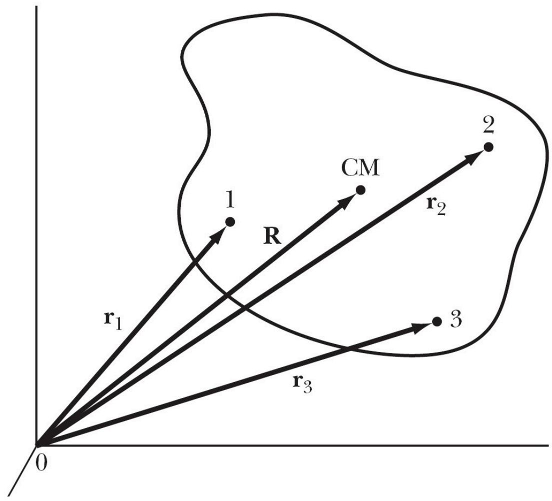

Figure 10.1: Position of the center of mass.

10.2 Linear Momentum

Consider a system of particles, of total mass M, exposed to *internal* and *external* forces. The linear momentum for this system is defined as

$$\mathbf{P} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha = \frac{d}{dt}\sum_\alpha m_\alpha \mathbf{r}_\alpha \tag{10.3}$$

The change in the linear momentum of the system can be expressed in terms of the forces acting on all the particles that make up the system:

$$\frac{d\mathbf{P}}{dt} = \sum_\alpha m\ddot{\mathbf{r}} = \sum_\alpha \mathbf{F}_\alpha = \sum_\alpha \mathbf{F}_{\alpha,\text{ext}} + \sum_\alpha \sum_{\beta\neq\alpha} \mathbf{f}_{\alpha\beta,} = \mathbf{F}_\text{ext} \tag{10.4}$$

The linear momentum is constant if the net external force acting on the system is zero. If there is an external force acting on the system, the component of the linear momentum in the direction of the net external force is not conserved, but the components in the directions perpendicular to the direction of the net external force are conserved. We conclude that the linear momentum of the system has the following properties:

1. The center of mass of a system moves as if it were a single particle with a mass equal to the total mass of the system, M, acted on by the total external force, and independent of the nature of the internal forces.

2. The linear momentum of a system of particles is the same as that of a single particle of mass M, located at the position of the center of mass, and moving in the manner the center of mass is moving.

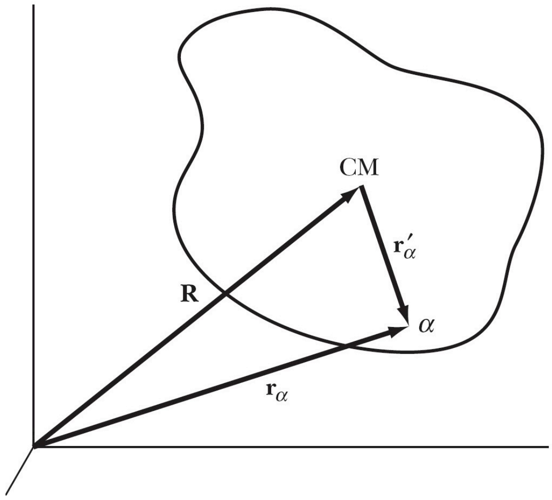

Figure 10.2: Coordinate system to describe a system of particles

3. The total linear momentum for a system free of external forces is constant and equal to the linear momentum of the center of mass.

10.3 Angular Momentum

Consider a system of particles that are distributed as shown in Fig. 10.2. We can specify the location of the center of mass of this system by specifying the vector $\mathbf{R}$. This position vector may be time dependent. The location of each component of this system can be specified by either specifying the position vector, $\mathbf{r}_\alpha$, with respect to the origin of the coordinate system, or by specifying the position of the component with respect to the center of mass, $\mathbf{r}'_\alpha$,

$$\mathbf{r}_\alpha = \mathbf{R} + \mathbf{r}'_\alpha \tag{10.5}$$

The angular momentum of this system with respect to the origin of the coordinate system is equal to

$$\mathbf{L} = \sum_\alpha \mathbf{L}_\alpha = \sum_\alpha \mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha = \sum_\alpha (\mathbf{R} + \mathbf{r}'_\alpha) \times m_\alpha (\mathbf{R} + \mathbf{r}'_\alpha) \tag{10.6}$$

Since

$$\sum_\alpha m_\alpha \mathbf{r}'_\alpha = \sum_\alpha m_\alpha (\mathbf{r}_\alpha - \mathbf{R}) = \sum_\alpha m_\alpha \mathbf{r}_\alpha - \mathbf{R} \sum_\alpha m_\alpha = M\mathbf{R} - M\mathbf{R} = 0 \tag{10.7}$$

We have

$$\mathbf{L} = \mathbf{R} \times \dot{\mathbf{R}} \sum_\alpha m_\alpha + \sum_\alpha \mathbf{r}'_\alpha \times m_\alpha \mathbf{r}'_\alpha = \mathbf{R} \times \mathbf{P} + \sum_\alpha \mathbf{r}'_\alpha \times \mathbf{p}'_\alpha = \mathbf{L}_{\text{cm}} + \mathbf{L}_{\text{wrt,cm}} \tag{10.8}$$

The rate of change in the angular momentum of the system is

$$
\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_{\alpha,\text{ext}} + \sum_\alpha \sum_{\beta\neq\alpha} \mathbf{r}_\alpha \mathbf{f}_{\alpha\beta} \\
&= \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_{\alpha,\text{ext}} + \sum_\alpha \sum_{j<i} (\mathbf{r}_\alpha - \mathbf{r}_j) \times \mathbf{f}_{\alpha\beta} \\
&= \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_{\alpha,\text{ext}} \qquad (10.9)
\end{aligned}
$$

The last step in this derivation is only correct of the internal force between i and j is parallel or anti-parallel to the relative position vector. Since the vector product between the position vector and the force vector is the torque associated with this force, we can rewrite the rate of change of the angular momentum of the system as

$$\frac{d\mathbf{L}}{dt} = \sum_\alpha \mathbf{N}_{\alpha,\text{ext}} \qquad (10.10)$$

We conclude that the angular momentum of the system has the following properties:

1. The total angular momentum about an origin is the sum of the angular momentum of the center of mass about that origin and the angular momentum of the system about the position of the center of mass.

2. If the net resultant torques about a given axis vanish, then the total angular momentum of the system about that axis remained constant in time.

3. The total internal torque must vanish if the internal forces are central, and the angular momentum of an isolated system can not be altered without the application of external forces.

10.4 Energy

The total energy of a system of particles is equal to the sum of the kinetic and the potential energy. The kinetic energy of the system is equal to the sum of the kinetic energy of each of the components. The kinetic energy of particle i can either be expressed in terms of its velocity with respect to the origin of the coordinate system, or in terms of its velocity with respect to the center of mass:

$$\dot{r}_\alpha^2 = \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}} = (\dot{\mathbf{R}} + \dot{\mathbf{r}}'_\alpha) \cdot (\dot{\mathbf{R}} + \dot{\mathbf{r}}'_\alpha) = \dot{R}^2 + 2\dot{\mathbf{r}}'_\alpha \cdot \dot{\mathbf{R}} + \dot{r}'^2_\alpha \qquad (10.11)$$

The kinetic energy of the system is thus equal to

$$T = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2 = \sum_\alpha \frac{1}{2} m_\alpha (\dot{R}^2 + 2\dot{\mathbf{r}}'_\alpha \cdot \mathbf{R} + \dot{r}'^2_\alpha) = \frac{1}{2} M V^2 + \frac{1}{2} \sum_\alpha m_\alpha v'^2_\alpha \qquad (10.12)$$

where the second term vanishes since

$$\sum_\alpha m_\alpha \mathbf{r}'_\alpha = 0 \tag{10.13}$$

The change in the potential energy of the system when it moves from a configuration 1 to a configuration 2 is related to the work done by the forces acting on the system:

$$W_{12} = \sum_\alpha \int_1^2 \mathbf{F}_{\alpha,\mathrm{ext}} \cdot d\mathbf{r}_\alpha + \sum_\alpha \int_1^2 \sum_{\beta \neq \alpha} \mathbf{f}_{\alpha\beta} \cdot d\mathbf{r}_\alpha \tag{10.14}$$

If we make the assumption that the forces, both internal and external, are derivable from potential functions, we can rewrite this expression as

$$W_{12} = -\sum_\alpha \int_1^2 \nabla U_{\alpha,\mathrm{ext}} \cdot d\mathbf{r}_\alpha - \sum_\alpha \int_1^2 \sum_{\beta \neq \alpha} \nabla U_{\alpha\beta} \cdot d\mathbf{r}_\alpha \tag{10.15}$$

The first term on the right-hand side can be evaluated easily:

$$\sum_\alpha \int_1^2 \nabla U_{\alpha,\mathrm{ext}} \cdot d\mathbf{r}_\alpha = \sum_\alpha (U_{\alpha,\mathrm{ext}}(2) - U_{\alpha,\mathrm{ext}}(1)) \tag{10.16}$$

The second term can be rewritten as

$$\begin{aligned} &\sum_\alpha \int_1^2 \sum_{\beta \neq \alpha} \nabla_\alpha U_{\alpha\beta} \cdot d\mathbf{r}_\alpha \\ &= \int_1^2 \sum_{\alpha,\beta<\alpha} \nabla_\alpha U_{\alpha\beta} \cdot d\mathbf{r}_\alpha + \int_1^2 \sum_{\beta,\alpha<\beta} \nabla_\beta U_{\beta\alpha} \cdot d\mathbf{r}_\beta \\ &= \int_1^2 \sum_{\alpha,\beta<\alpha} \nabla_\alpha U_{\alpha\beta} \cdot d\mathbf{r}_{\alpha\beta} = \sum_{\alpha,\beta<\alpha} (U_{\alpha\beta}(2) - U_{\alpha\beta}(1)) \end{aligned} \tag{10.17}$$

where $\mathbf{r}_{\alpha\beta} = \mathbf{r}_\alpha - \mathbf{r}_\beta$ Here we have used the fact that the internal force between α and β satisfy the following relation

$$\mathbf{f}_{\alpha\beta} = -\nabla_\alpha U_{\alpha\beta} = -\mathbf{f}_{\beta\alpha} = \nabla_\beta U_{\beta\alpha} \tag{10.18}$$

The total potential energy of the system U is defined as the sum of the internal and the external potential energy and is equal to

$$U = \sum_\alpha U_{\alpha,\mathrm{ext}} + \sum_{\alpha,\beta<\alpha} U_{\alpha\beta} \tag{10.19}$$

The work done by al the force to make the transition from configuration 1 to configuration 2 is

$$W_{12} = U_1 - U_2 \tag{10.20}$$

Therefore we conclude

$$W_{12} = U_1 - U_2 = T_2 - T_1 \tag{10.21}$$

or

$$E_1 = T_1 + U_1 = T_2 + U_2 = E_2 \tag{10.22}$$

The total energy is conserved. If the system of particles is a rigid object, the components of the system will retain their relative positions, and the internal potential energy of the system will remain constant.

We conclude that the total energy of the system has the following properties:

1. The total kinetic energy of the system is equal to the sum of the kinetic energy of a particle of mass M moving with the velocity of the center of mass and the kinetic energy of the motion of the individual particles relative to the center of mass.

2. The total energy for a conservative system is constant.

10.5 Elastic and Inelastic Collisions

When two particles interact, the outcome of the interaction will be governed by the force law that describes the interaction. Consider an interaction force Fint that acts on a particle. The result of the interaction will be a change in the momentum of the particle since

$$\mathbf{F}_{\text{int}} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} \tag{10.23}$$

If the interaction occurs over a short period of time, we expect to a change in the linear momentum of the particle:

$$\int_{t_1}^{t_2} \mathbf{F}dt = \Delta\mathbf{p} = \mathbf{P} \tag{10.24}$$

This relation shows us that if we know the force we can predict the change in the linear momentum, or if we measure the change in the linear momentum we can extract information about the force. We note that the change in the linear momentum provides us with information about the time integral of the force, not the force. We called it the impulse $\mathbf{P}$. If we consider the effect of the interaction force on both particles we conclude that the change in the linear momentum is 0:

$$\Delta\mathbf{p} = \Delta\mathbf{p}_1 + \Delta\mathbf{p}_2 = \int_{t_1}^{t_2} (\mathbf{F}_{12} + \mathbf{F}_{21})dt = 0 \tag{10.25}$$

This of course should be no surprise since when we consider both particles, the interaction force becomes an internal force and in the absence of external forces, linear momentum will be conserved. The conservation of linear momentum is an important conservation law that restricts the possible outcomes of a collision. No matter what the nature of the collision is, if the initial linear momentum is non-zero, the final linear momentum will also be non-zero, and the system can not be brought to rest as a result of the collision. If the system is at rest after the collision, its linear momentum is zero, and the initial linear momentum must therefore also be equal to zero. Note that a zero linear momentum does not imply that all components of the system will be at rest; it only requires that the two object have linear momenta that are equal in magnitude but directed in opposite directions. The most convenient way to look at the collisions is in the center-of-mass frame. In the center-of-mass frame, the total linear momentum is equal to zero, and the objects will always travel in a co-linear fashion. (Fig. 10.3)

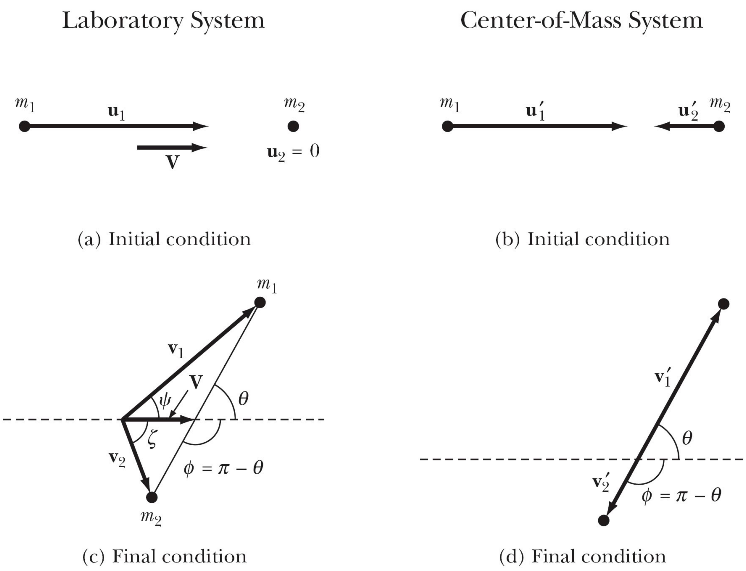

Figure 10.3: Two-dimensional collisions in the laboratory frame (left) and the center-of-mass frame (right).

We frequently divide collisions into two distinct groups:

- Elastic collisions: collisions in which the total kinetic energy of the system is conserved. The kinetic energy of the objects will change as a result of the interaction, but the total kinetic energy will remain constant. The kinetic energy of one of the objects in general is a function of the masses of the two objects and the scattering angle.

- Inelastic collisions: collisions in which the total kinetic energy of the system is not conserved. A totally inelastic collision is a collision in which the two objects after the collision stick together. The loss in kinetic energy is usually expressed in terms of the Q value, where $Q = K_f - K_\alpha$:

  1. $Q < 0$: endoergic collision (kinetic energy is lost)
  2. $Q > 0$: exoergic collision (kinetic energy is gained)
  3. $Q = 0$: elastic collision (kinetic energy is conserved).

In most inelastic collisions, a fraction of the initial kinetic energy is transformed into internal energy (for example in the form of heat, deformation, etc.). Another parameter that is frequently used to quantify the inelasticity of an inelastic collision is the coefficient of restitution

$$\epsilon = \frac{v_2 - v_1}{u_2 - u_1} \tag{10.26}$$

where u are the velocities after the collision and v are the velocities before the collision. For a perfect elastic collision $\epsilon = 1$ and for a totally inelastic collision $\epsilon = 0$.

One important issue we need to address when we focus on collisions is the issue of predictability. We will assume that we looking at a collision in the center-of-mass frame. Define the x axis to be the axis parallel to the direction of motion of the incident objects, and assume that the masses of the objects do not change. The unknown parameters are the velocities of the object; for the n-dimensional case, there will be $2n$ unknown. And the following are the known relations:

- Conservation of linear momentum: this conservation law provides us with n equations with $2n$ unknowns.

- Conservation of energy: if the collision is elastic, this conservation law will provide us with 1 equation with $2n$ unknowns.

For elastic collisions we thus have $n + 1$ equations with $2n$ unknown. We immediately see that only for $n = 1$ the final state is uniquely defined. For inelastic collisions we have n equations with $2n$ unknown and we conclude that even for $n = 1$ the final state is undefined. When the final state is undefined we need to know something about some of the final-state parameters to fix the others.

10.6 Relativistic Kinematics

In the event that the velocities in a collision process are not negligible with respect to the velocity of light, it becomes necessary to use relativistic kinematics. Because mass and energy are interrelated in relativity theory, it no longer is meaningful to speak

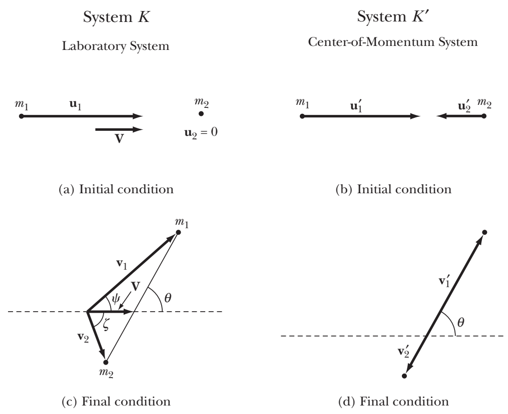

Figure 10.4: Scattering of relativistic particles.

of a *center-of-mass* system; in relativistic kinematics, one uses a *center-of-momentum* coordinate system instead. Such a system possesses the same essential property as the previously used center-of-mass system-the total linear momentum in the system is zero. Therefore, if a particle of mass m_1 collides elastically with a particle of mass m_2, then in the center-of-momentum system we have

$$p_1' = p_2'$$

The space components of the momentum four-vector can be written as

$$m_1 u_1' \gamma_1' = m_2 u_2' \gamma_2'$$

where, as before, $\gamma \equiv 1/\sqrt{1-\beta^2}$ and $\beta \equiv u/c$.

In a collision problem, it is convenient to associate the laboratory coordinate system with the inertial system K and the center-of-momentum system with K'. A simple Lorentz transformation then connects the two systems. To derive the relativistic kinematic expressions, the procedure is to obtain the center-of-momentum relations and then perform a Lorentz transformation back to the laboratory system. We choose the coordinate axes so that m_1 moves along the x-axis in K with speed u_1. Because m_2 is initially at rest in $K, u_2 = 0$. In K', m_2 moves with speed u_2' and so K' moves with respect to K also with speed u_2' and in the same direction as the initial motion of m_1. Using the fact that $\beta\gamma = \sqrt{\gamma^2 - 1}$, we have

$$\begin{aligned}
p_1' &= m_1 u_1' \gamma_1' = m_1 c \beta_1' \gamma_1' \\
&= m_1 c \sqrt{\gamma_1'^2 - 1} = m_2 c \sqrt{\gamma_2'^2 - 1} \\
&= p_2'
\end{aligned}$$

which expresses the equality of the momenta in the center-of-momentum system.

The transformation of the momentum p_1 (from K to K') is

$$p_1' = \left(p_1 - \frac{u_2'}{c^2}E_1\right)\gamma_2'$$

We also have

$$\left.\begin{array}{l} p_1 = m_1 u_1 \gamma_1 \\ E_1 = m_1 c^2 \gamma_1 \end{array}\right\}$$

$$\begin{aligned} m_1 c\sqrt{\gamma_1'^2 - 1} &= \left(m_1 c\beta_1\gamma_1 - \beta_2' m_1 c\gamma_1\right)\gamma_2' \\ &= m_1 c\left(\gamma_2'\sqrt{\gamma_1^2 - 1} - \gamma_1\sqrt{\gamma_2'^2 - 1}\right) \\ &= m_2 c\sqrt{\gamma_2'^2 - 1} \end{aligned}$$

These equations can be solved for γ_1' and γ_2' in terms of γ_1 :

$$\gamma_1' = \frac{\gamma_1 + \frac{m_1}{m_2}}{\sqrt{1 + 2\gamma_1\left(\frac{m_1}{m_2}\right) + \left(\frac{m_1}{m_2}\right)^2}}$$

$$\gamma_2' = \frac{\gamma_1 + \frac{m_2}{m_1}}{\sqrt{1 + 2\gamma_1\left(\frac{m_2}{m_1}\right) + \left(\frac{m_2}{m_1}\right)^2}}$$

Next, we write the equations of the transformation of the momentum components from K' back to K after the scattering. We now have both x - and y-components:

$$\begin{aligned} p_{1,x} &= \left(p_{1,x}' + \frac{u_2'}{c^2}E_1'\right)\gamma_2' \\ &= \left(m_1 c\beta_1'\gamma_1'\cos\theta + m_1 c\beta_2'\gamma_1'\right)\gamma_2' \\ &= m_1 c\gamma_1'\gamma_2'\left(\beta_1'\cos\theta + \beta_2'\right) \end{aligned}$$

Also,

$$p_{1,y} = m_1 c\beta_1'\gamma_1'\sin\theta$$

The tangent of the laboratory scattering angle ψ is given by $p_{1,y}/p_{1,x}$, and

$$\tan\psi = \frac{1}{\gamma_2'}\frac{\sin\theta}{\cos\theta + (\beta_2'/\beta_1')}$$

Since $\boldsymbol{\beta}_2'/\beta_1'$, we have

$$\tan\psi = \frac{1}{\gamma_2'}\frac{\sin\theta}{\cos\theta + (m_1\gamma_1'/m_2\gamma_2')}$$

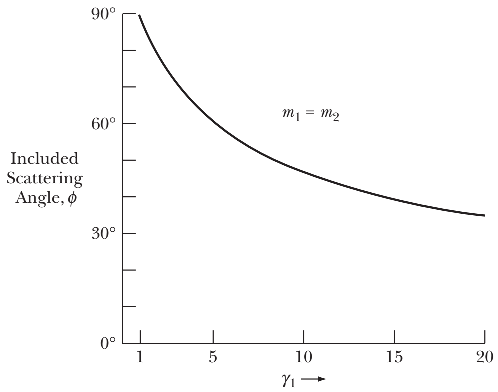

Figure 10.5: The included scattering angle, $\phi = \psi + \zeta$, is shown as a function of the relativistic parameter γ_1 for $m_1 = m_2$. For nonrelativistic scattering $(\gamma_1 = 1)$, this angle is always $90°$.

For the recoil particle, we have

$$\begin{aligned} p_{2,x} &= \left(p'_{2,x} + \frac{u'_2}{c^2}E'_2\right)\gamma'_2 \\ &= \left(-m_2 c\beta'_2\gamma'_2 \cos\theta + m_2 c\beta'_2\gamma'_2\right)\gamma'_2 \\ &= m_2 c\beta'_2\gamma'^2_2(1-\cos\theta) \end{aligned}$$

By using the transformation properties of the fourth component of the momentum four-vector (i.e., the total energy), it is possible to obtain the relativistic analogs of all the energy equations we have previously derived in the nonrelativistic limit.

10.7 Scattering Cross Section

Properties of atoms and nuclei can be learned using elastic scattering of projectiles to probe the properties of the target elements. A schematic of the scattering process is shown in Fig. 10.6, with a repulsive force between m_1 and m_2. The parameter b is called the impact parameter. The impact parameter is related to the angular momentum of the projectile m_1 with respect to the target m_2,

$$l = m_1 u_1 b \tag{10.27}$$

When we study the scattering process we in general measure the intensity of the scattered particles as function of the scattering angle. The intensity distribution is ex-

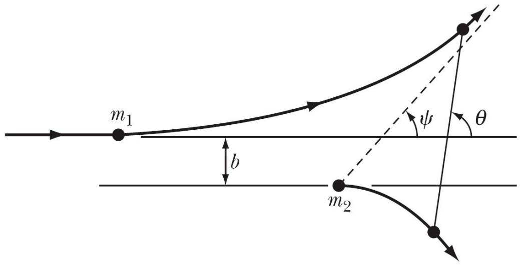

Figure 10.6: Scattering of projectile particle m_1 from target m_2.

pressed in terms of the *differential cross section*, which is defined as

$$\sigma(\theta)d\Omega' = \frac{\text{\# of interactions per target particle that lead to scattering into } d\Omega' \text{at the angle } \theta}{\text{\# of incident particles per unit area}} \tag{10.28}$$

There is a one-to-one correlation between the impact parameter b and the scattering angle θ (Fig. 10.7). The one-to-one correlation is a direct effect of the conservation of angular momentum. Assuming that the number of incident particles is conserved, the flux of incident particles with an impact parameter between b and $b + db$ is equal to

$$dI_{\text{in}} = I2\pi b db \tag{10.29}$$

and this must be the same as the number of particles scattered in the cone that is specified by the angle θ and width $d\theta$. The solid angle of this cone is equal to

$$d\Omega' = 2\pi \sin\theta d\theta \tag{10.30}$$

The number of particles scattered into this cone will be

$$dI_{\text{out}} = -I\sigma(\theta)2\pi \sin\theta d\theta \tag{10.31}$$

The minus sign is due to the fact that if $db > 0, d\theta < 0$, that is, the force law is such that the amount of angular deflection decreases monotonically with increasing impact parameter.

Finally, we have

$$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{10.32}$$

The scattering angle θ is related to the impact parameter b. This relation can be obtained using the orbital motion in a central-force field:

$$\Delta\Theta = \int_{r_{\min}}^{r_{\max}} \frac{(l/r^2)dr}{\sqrt{2\mu[E - U - (l^2/2\mu r^2)]}} \tag{10.33}$$

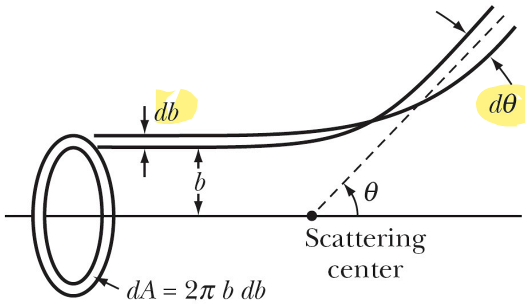

Figure 10.7: Correlation between impact parameter and scattering angle.

Using the fact that $r_{\max} = \infty$, and $l = b\sqrt{2\mu T_0'}$ where $T_0' = \frac{1}{2}\mu u_1^2$, and $E = T_0'$ as $U = 0$ at $r = \infty$, we have

$$\Theta = \int_{r_{\min}}^{\infty} \frac{(b/r^2)dr}{\sqrt{1 - (b^2/r^2) - (U/T_0')}} \tag{10.34}$$

$r_{\min}$ is the turning point of the motion and corresponds to the distance of closest approach of the particle to the force center. The relation between the scattering angle θ and the impact parameter b depends on the potential U. The above derivation is in the CM frame, as we treat the problem as a particle of effective mass μ scattering off the scattering center. To transform back to the LAB frame, we notice that the total number of particles scattered into a unit solid angle must be the same in the LAB system and CM system, that is

$$\begin{aligned} \sigma(\theta)d\Omega' &= \sigma(\psi)d\Omega \\ \sigma(\psi) &= \sigma(\theta)\frac{\sin\theta}{\sin\psi}\frac{d\theta}{d\psi} \end{aligned} \tag{10.35}$$

Using the geometrical relation $x = \frac{m_1}{m_2} = \frac{\sin(\theta-\psi)}{\sin\psi}$, we have

$$\sigma(\psi) = \sigma(\theta)\frac{[x\cos\psi + \sqrt{1 - x^2\sin^2\psi}]^2}{\sqrt{1 - x^2\sin^2\psi}} \tag{10.36}$$

10.8 Rutherford Scattering Formula

For the important case of nuclear scattering, the potential varies as k/r. For this potential we can carry out the integration and determine the following correlation between the scattering angle and the impact parameter:

$$\cos\Theta = \frac{\kappa/b}{\sqrt{1 + (\kappa/b)^2}} \tag{10.37}$$

where $\kappa = \frac{k}{2T_0'}$. We can use this relation to calculate $db/d\theta$ and get the following differential cross section

$$\sigma(\theta) = \frac{k^2}{(4T_0')^2} \frac{1}{\sin^4(\theta/2)} \tag{10.38}$$

We conclude that the intensity of scattered projectile nuclei will decrease when the scattering angle increases. If the energy of projectile nuclei is low enough, the measured angular distribution will agree with the so-called Rutherford distribution over the entire angular range, as was first shown by Geiger and Marsden in 1913 (Fig. 10.8).

Each trajectory of the projectile can be characterized by distance of closest approach and there is a one-to-one correspondence between the scattering angle and this distance of closest approach r_{min}. The smallest distance of closest approach occurs when the projectile is scattered backwards ($\theta = 180°$). The distance of closest approach decreases with increasing incident energy and the Rutherford formula indicates that the intensity should decrease as $1/T^2$. This was indeed observed, up to a maximum incident energy, beyond which the intensity dropped of much more rapidly than predicted by the Rutherford formula . At this point, the nuclei approach each other so close that the strong attractive nuclear force starts to play a role, and the scattering is no longer elastic (the projectile nuclei may for example merge with the target nuclei). This deviation can be used to estimate the radius of the nucleus.

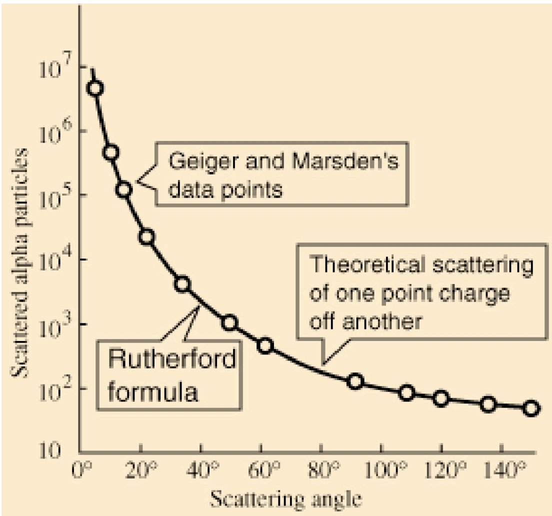

Figure 10.8: Comparison of Rutherford scattering experiment (symbols) and theory (line).

10.9 Total cross section

By integrating $\sigma(\theta)$, we will have the total scattering cross section σ_t, which corresponding to the effective area of the target particle for producing a scattering event:

$$\sigma_t = 2\pi \int_0^{\pi} \sigma(\theta) \sin\theta d\theta \tag{10.39}$$

It should be note that the total cross section is the same in the LAB as in the CM system.