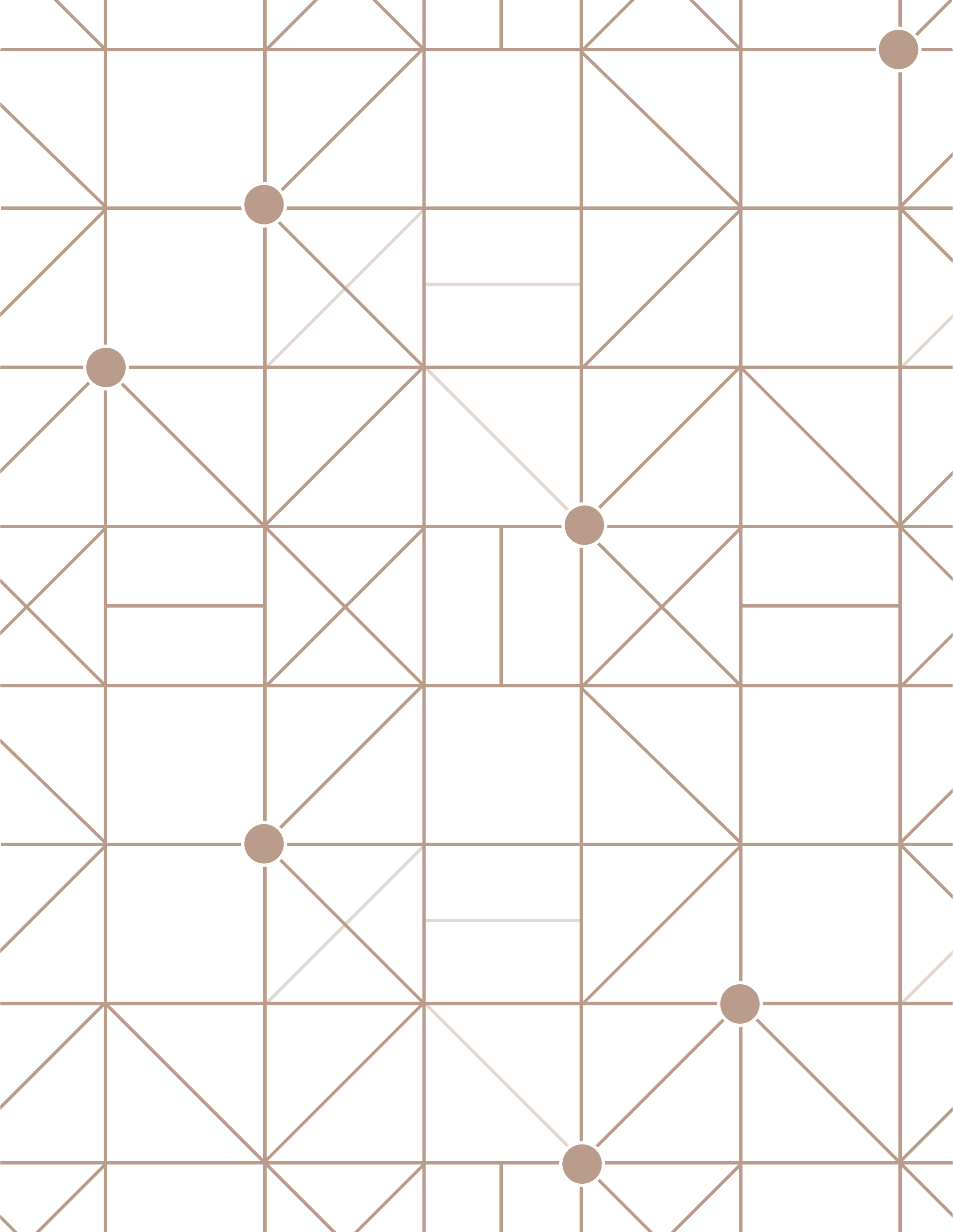
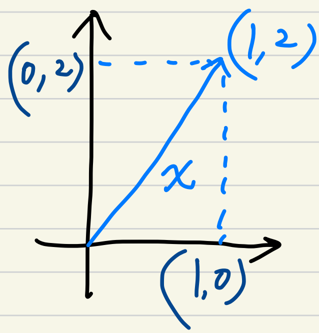


# Chapter 3-1

- Basis is a set of indep. vectors that span a space.
- Orthogonal Basis
  1. Length $\|x\|$ of a vector
  2. $\boxed{x^T y = 0}$ for perpendicular vectors.
- Subspaces can also be perpendicular.

## [A] The Length of a vector. $\|x\|$

(2D)




$$\|x\|^2 = x_1^2 + x_2^2$$

$$= 1^2 + 2^2$$

$$= 5$$

$$\therefore \boxed{\|x\| = \sqrt{5}}$$

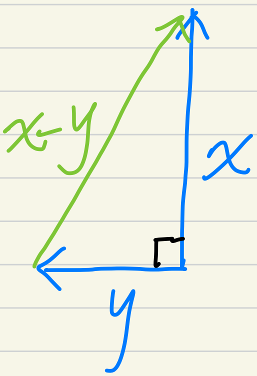
The length $\|x\|$ in $R^n$ is the positive square root of $\boxed{x^T x}$.

$$\|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x^T x$$

## [B] Orthogonal Vectors




$$\|x\|^2 + \|y\|^2 = \|x-y\|^2$$

$$\Rightarrow (x_1^2 + x_2^2 + \cdots + x_n^2) + (y_1^2 + y_2^2 + \cdots + y_n^2)$$

$$= (x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2$$

$$= (x_1^2 + x_2^2 + \cdots + x_n^2) + (y_1^2 + y_2^2 + \cdots + y_n^2)$$
$$- 2(x_1 y_1 + \cdots + x_n y_n)$$

$\therefore$ $\boxed{\text{Orthogonal Vectors}}$

$$(x_1 y_1 + \cdots + x_n y_n) = 0 \qquad \boxed{x^T y = 0}$$

## [C] Orthogonal Subspace

- Every vector in one subspace must be orthogonal to every vector in the other subspace.

- A line can be orthogonal to another line or orthogonal to a plane

- But a plane cannot be orthogonal to a plane.

- Orthogonal Subspaces are the Fundamental Subspaces.

$\boxed{\text{Fundamental theorem of orthogonality}}$ ($A_{mn}$)

$1^{st}$ - The $\boxed{\text{row space}}$ is orthogonal to the $\boxed{\text{nullspace}}$ (in $\boxed{\mathbb{R}^n}$).

$2^{nd}$ - The $\boxed{\text{column space}}$ is orthogonal to the $\boxed{\text{left nullspace}}$ (in $\boxed{\mathbb{R}^m}$).

## $\boxed{\text{The } 1^{st} \text{ pair}}$ $(A_{mn})$

$$\boxed{Ax=0}$$

$$\begin{bmatrix} \cdots \text{row } 1 \cdots \\ \cdots \text{row } 2 \cdots \\ \vdots \\ \cdots \text{row } m \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- row 1 is orthogonal to $x$.
- $x$ is orthogonal to every combination of the rows.
- Every <u>$x$ in the nullspace</u> is orthogonal to every vector in the row space.

$$\boxed{N(A) \perp C(A^T)}$$

**The $2^{nd}$ pair** $\quad A_{mn}$

$\boxed{A^T y = 0}$ or $\boxed{y^T A = 0}$

$$y^T A = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \begin{bmatrix} C & & C \\ O & & O \\ L & & L \\ U & \cdots & U \\ M & & M \\ N & & N \\ 1 & & n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$$

- The vector $y$ is orthorgonal to every column.

- $y$ is orthogonal to every combination of the columns.

$$\boxed{N(A^T) \perp C(A)}$$

Ex:

Rank-1 matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix}$$

- The rows are multiples of $\begin{bmatrix} 1 & 3 \end{bmatrix}$.

- The nullspace contains $x = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, which is orthogonal to all rows.

- The row space and nullspace are perpendicular lines in $R^2$.

- The nullspace is perpendicular to the row space.

- $N(A)$ contains every vector orthogonal to the row space.

$$N(A) = (C(A^T))^{\perp}$$

$$C(A^T) = (N(A))^{\perp}$$

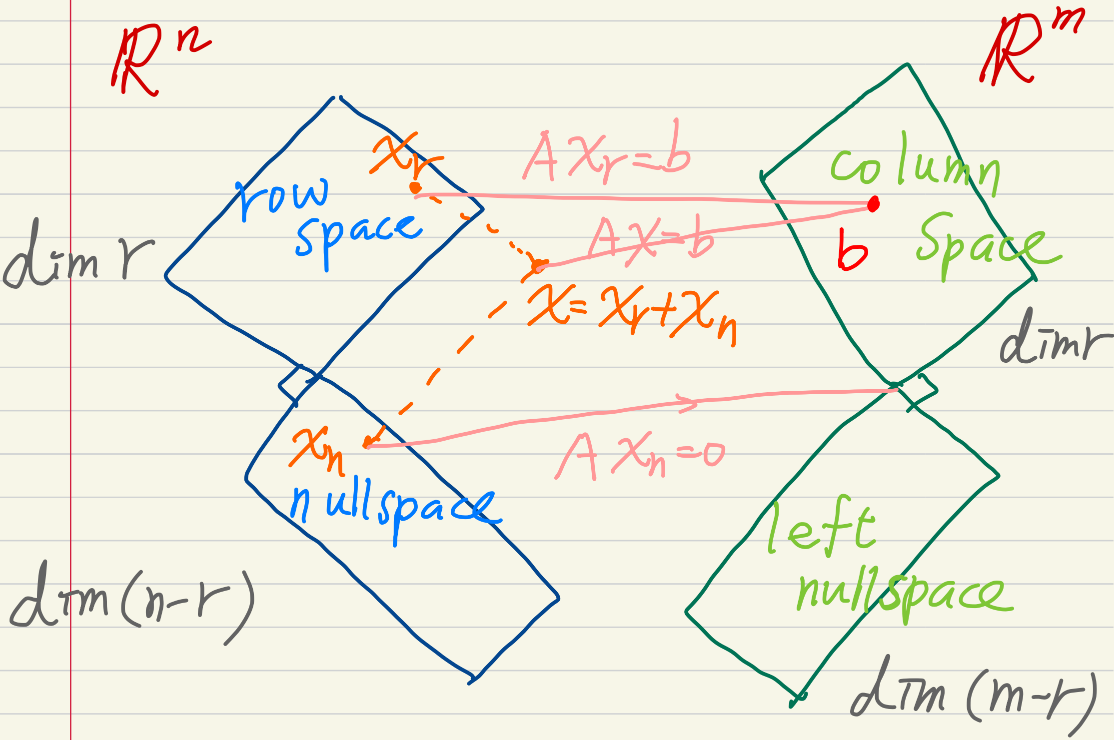
- The column space is the line through $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

- The left nullspace must be perpendicular plane

$$\boxed{y_1 + 2y_1 + 3y_3 = 0} \qquad y^T A = 0$$

- The nullspace is "orthogonal complement" of the row space in $\mathbb{R}^n$.

- The left nullspace is "orthogonal complement" of the column space in $\mathbb{R}^m$.



R^n
R^m
dim r
row space
x_r
Ax_r=b
column space
b
Ax=b
x=x_r+x_n
dim r
x_n
Ax_n=0
nullspace
dim (n-r)
left nullspace
dim (m-r)