We can now perform the elementary integral:

$$\int \frac{c\, du}{\sqrt{u^2 - c^2}} = x + \delta = c \cosh^{-1}\left(\frac{u}{c}\right)$$

$$\Rightarrow \quad u(x) = c \cosh\left(\frac{x+\delta}{c}\right), \sim \text{"Catenary"}$$

We can fix c & δ by $u(a) = \alpha$ and $u(b) = \beta$

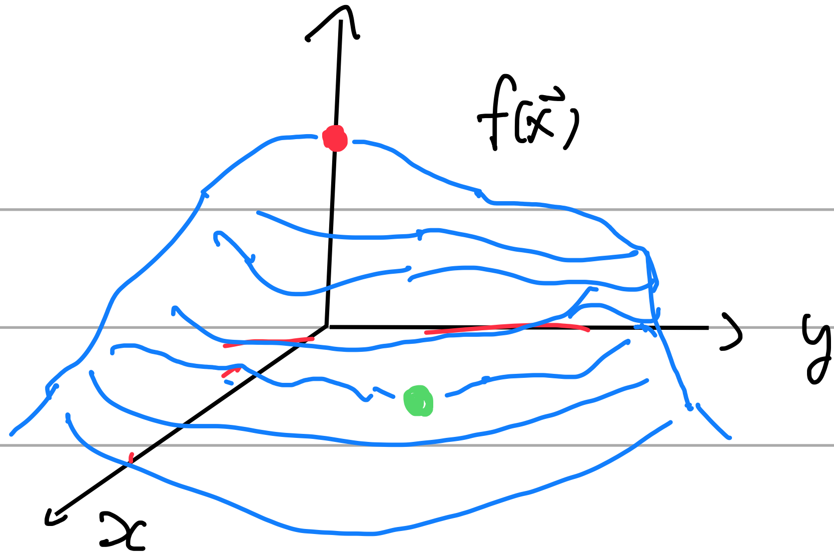
Variation with Constraints and Lagrangian Multiplier

We also mentioned earlier in isoperimetric problem that we sometimes need to consider variational problem subjected to certain constraints. To deal with these, we can introduce "Lagrangian Multiplier".

Before functional, think about extremizing a function

$f(\vec{x})$ subjected to some constraint: $g(\vec{x}) = c$

$\vec{x} \in \mathbb{R}^n$



e.g. Both red & green dots are extrema, but if we impose constraint $x^2 - y^2 = 0$ only red is allowed.

At extrema, we have $df = d\vec{x} \cdot \vec{\nabla} f(\vec{x}) = 0$

Now with constraint $g(\vec{x}) = C$ with defines a surface, only $d\vec{x}$ on such a surface is allowed.

$\Rightarrow$ $\vec{\nabla} f(\vec{x})$ is perpendicular $\perp$ to $g(\vec{x}) = C$

equivalently parallel $//$ to its normal vector $\vec{\nabla} g(\vec{x})$

For extrema we need $\vec{\nabla}(f(\vec{x}) - \lambda g(\vec{x})) = \vec{0}$ ~ n equations

λ: some undetermined real number

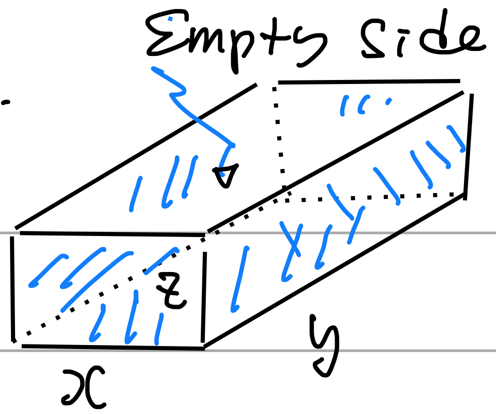
With one more constraint equation: $g(\vec{x}) = C$

$\Rightarrow$ We can view this as imposed by treating also as a variable, extremizing with respect to λ in

$\phi(\vec{x}, \lambda) = f(\vec{x}) - \lambda(g(\vec{x}) - C)$ ~ $n+1$ variables.

$\frac{\partial \phi}{\partial \lambda} = g(\vec{x}) - C = 0$

Ex.



A box with width & length and height (x, y, z), its volume is fixed at $L^3/2$, find (x, y, z) which minimise its surface area? (open)

$$V = xyz = L^3/2, \quad A(x,y,z) = xy + 2xz + 2yz$$

volume fixed surface area only one bottom no top

So we need consider

Constraint

$$\phi(x,y,z,\lambda) = A(x,y,z) - \lambda\left(xyz - L^3/2\right)$$

$$= 2z(x+y) + xy - \lambda\left(xyz - L^3/2\right)$$

Extremizing w.r.t z: $0 = 2(x+y) - \lambda xy \Rightarrow \lambda = \frac{2(x+y)}{xy}$

w.r.t. x: $0 = 2z + y - \lambda yz = \frac{y}{x}(x - 2z) \Rightarrow x = 2z$

w.r.t y: $0 = 2z + x - \lambda xz = \frac{x}{y}(y - 2z) \Rightarrow y = 2z$

Now we have $(x, y, z) = (2z, 2z, z)$

Finally, varying with respect to λ:

$$xyz = 4z^3 = \frac{L^3}{2} \Rightarrow \frac{L}{2} = z$$

$\Rightarrow$ This can be checked by directly solving the constraint $z = \frac{L^3}{2}\frac{1}{xy}$ and perform minimization.

Now if we extend our analysis to functionals, so we like to extremize some functional

Functional also

$J[u]$, subjected to constraint $P[u] = C$

So we construct:

$$\Phi[u, \lambda] = J[u] - \lambda(P[u] - C)$$

and extremize w.r.t λ & $u(x) \Rightarrow \lambda$ imposes $P[u] = C$

(w.r.t = with respect to)

Consider isoperimetric problem:

$$A[u] = \oint_C x(s)\frac{dy}{ds}\,ds\,, \quad P[u] = \oint_C ds\sqrt{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2}$$

$$\Rightarrow \quad \Phi[u,\lambda] = \oint_C ds\left(x(s)\frac{dy}{ds} - \lambda\left(\sqrt{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2} - 1\right)\right)$$

$$= \oint dy\,\underbrace{\left(x(y) - \lambda\sqrt{1+\left(\frac{dx}{dy}\right)^2}\right.}_{L\,\sim\,\text{No eplicit } y \text{ dependence}}\Bigg) + \lambda l$$

$$x' = \frac{dx}{dy}$$

From 1st integral earlier, we can deduce

$$\text{Constant} = -\left(x - \lambda\sqrt{1+(x')^2}\right) + x'\frac{\partial L}{\partial x'}$$

$$= \frac{\lambda}{\sqrt{1+(x')^2}} - x(y) \Rightarrow (x')^2 = \frac{\lambda^2}{(x-x_0)^2} - 1$$

Some Constant

$$\Rightarrow \frac{dx}{dy} = \frac{\lambda^2-(x-x_0)^2}{(x-x_0)^2} \Rightarrow x = x_0 \pm \sqrt{\lambda^2-(y-y_0)^2}$$

Some other Constant

We get $(x-x_0)^2+(y-y_0)^2 = \lambda^2 = L^2$ obtained from λ - E.O.M