Unsolved n-dimensional problme

12.1.5 Gauss' Law

For an arbitrary 3D subspace $\mathcal{V}$ with the boundary S, which is a closed surface called the Gaussian surface, we have **Gauss' Law**

$$\boxed{\oint_S d\mathbf{a}\cdot\mathbf{E} = \frac{Q_{\mathcal{V}}}{\epsilon_0}. \qquad (12.21)}$$

The area element $d\mathbf{a} \equiv \hat{\mathbf{n}}da$ is defined to have its direction given by the normal direction $\hat{n}$ pointing outward from the volume $\mathcal{V}$.

The LHS is the electric flux through S:

$$\Phi_E(S) \equiv \oint_S d\mathbf{a}\cdot\mathbf{E}, \qquad (12.22)$$

and the RHS can be expressed in terms of the charge density $\rho(\mathbf{r})$ as

$$Q_{\mathcal{V}} \equiv \int_{\mathcal{V}} \rho(\mathbf{r})d^3\mathbf{r}. \qquad (12.23)$$

Example: If we choose $\mathcal{V}$ to be a spherical shell of inner radius R_1 and outer radius R_2, S is composed of two spherical surfaces of radii R_1 and R_2, with $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$ and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, respectively.

Q 12.42: For S being a spherical surface of radius R, can we choose $\mathcal{V}$ to be the space $r > R$?

Ex 12.13: (Application of Gauss' law #1)

Assuming that Gauss' law is correct, prove Coulomb's law.

Actually we also need to make an assumption about *symmetry*: For a source with spherical symmetry, the field respects spherical symmetry.

This assumption about symmetry is not always valid.

Electrostatics = Gauss' Law + Superposition Principle
+ "$\mathbf{F} = q\mathbf{E}$".

Ex 12.14: (Application of Gauss' law #2)

Prove the following theorem.

Theorem:

The electric field $\mathbf{E}(\mathbf{r})$ of a spherically symmetric charge distribution $\rho(r)$ is the same as if all charges enclosed within the sphere of radius $|\mathbf{r}|$ is replaced by a point charge at the origin.

Ex 12.15: (Application of Gauss' law #3)

Prove that it is impossible to stabilize an electron by a distribution of static charges.

Ex 12.16: (Application of Gauss' law #4)

Prove that, if the universe is *closed*, the total charge of the universe must be 0.

Ex 12.17: (Application of Gauss' law #5)

What would be the Coulomb's law if the space is n dimensional?

Gauss' law is expressed in its integral form in eq.(12.21), but we can take S to be infinitesimal, and derive its differential equivalence. We will talk more about this below in Sec.12.1.8.

12.1.6 From Charge to Field via Gauss' Law

To determine $\mathbf{E}(\mathbf{r})$ from given $\rho(\mathbf{r})$ when there are sufficient symmetries, it is much easier to use Gauss' law than carrying out the integral (12.14). There are 3 types of symmetries that are large enough so that Gauss' law is sufficient to determine electric field completely: spherical symmetry, cylindrical symmetry, and plane symmetry.

An intuition about how field lines must be distributed for a given symmetric $\rho(\mathbf{r})$ can be very helpful.

> When we apply Gauss's law to determine electric fields using symmetry, we assume that a symmetry of a charge distribution is also a symmetry of the electric field.
> **Q 12.43:** Why?
> More precisely, under a symmetry transformation of the physical law (Coulomb's law/Gauss's law), if the charge distribution is invariant, the electric field must also be invariant.
> There are actually exceptions to this assumption.

Spherical Symmetry

The spherical symmetry is composed of arbitrary rotations around any axis through the center, and their compositions.
Ex 12.18: A charge Q is uniformly distributed over a spherical surface of radius R. What is the electric field $\mathbf{E}(r)$ for $r > R$ and $r < R$, respectively?
Q 12.44: How to find the electric field for an arbitrary, spherically symmetric charge distribution $\rho(r)$?

The same exercise was given above. Is it easier to solve this problem using Gauss' law?

> Here we assume that if $\rho(\mathbf{r})$ remains the same after a transformation, $\mathbf{E}(\mathbf{r})$ must also remain the same. This is implied by the Coulomb's law (12.14), at least whenever the integral is well defined.

Cylindrical Symmetry

The cylindrical symmetry is composed of arbitrary rotations along the z-axis (which can be chosen arbitrarily) and arbitrary translations along the z-axis. The translation symmetry along the z-axis also implies the symmetry of inverting the z-axis.
Ex 12.19: What is the electric field $\mathbf{E}(\mathbf{r})$ of an infinite straight wire of uniform line charge density λ?
Ex 12.20: What is the electric field $\mathbf{E}(\mathbf{r})$ of a charge distribution $\rho(s)$ with cylindrical symmetry?

Planar Symmetry

The planar symmetry is composed of translations along the x and y axes.
Ex 12.21: What is the electric field $\mathbf{E}(\mathbf{r})$ of an infinite plane of uniform surface charge density σ?

If $\rho = \rho_1 + \rho_2$, and the fields $\mathbf{E}_1$ and $\mathbf{E}_2$ can be deduced from ρ_1 and ρ_2 from Gauss' law, respectively, the total field is $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ according to the superposition principle.

HW: (1-2) What is the electric field and potential of two infinite planes (both parallel to the $x-y$ plane) of uniform surface charge densities σ_1 and σ_2 located at $z = 0$ and $z = h$, respectively? (Assume that the charged planes are of zero thickness.) What is the pressure on either plane?

Q 12.45: What is the electric field for $\rho = c$ (a constant)?

After we discuss the electric potential, you should return to the exercises above and find the electric potential for each problem.

12.1.7 Electric Potential and Potential Energy

Analogous to the relation between conservative force and potential energy, we define the electric potential $V(\mathbf{r})$ from the electric field $\mathbf{E}(\mathbf{r})$ by

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}), \tag{12.24}$$

The SI unit of the electric potential V is Volt = N/(C×m), so that the SI unit of electric field is Volt/m.

or conversely,

$$V(\mathbf{r}) = V(\mathbf{r}') - \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot \mathbf{E}(\mathbf{r}''). \tag{12.25}$$

For any function $f(\mathbf{r})$, $\nabla f(\mathbf{r}) \equiv (\hat{\mathbf{x}}\partial_x + \hat{\mathbf{y}}\partial_y + \hat{\mathbf{z}}\partial_z)f(\mathbf{r})$. It will be convenient to identify the symbol ∇ with $\hat{\mathbf{x}}\partial_x + \hat{\mathbf{y}}\partial_y + \hat{\mathbf{z}}\partial_z$.

$V(\mathbf{r})$ can always be shifted by a constant without changing its relation with a given $\mathbf{E}(\mathbf{r})$, eqs.(12.24) and (12.25). But, whenever possible, as a convention, we shall always assume that $V \to 0$ at spatial infinity.

Ex 12.22: What is the electric potential for a point charge?

Solution:

Due to the spherical symmetry, V depends only on r, so

$$V(\mathbf{r}) = V(\infty) - \int_{\infty}^{r} dr' E_r(r') = 0 - \int_{\infty}^{r} dr' \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2} = \frac{q}{4\pi\epsilon_0 r}. \tag{12.26}$$

The electric potential $V(\mathbf{r})$ for a point charge at $\mathbf{r}'$ is

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 \boldsymbol{\mathscr{r}}}, \tag{12.27}$$

where $\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'$.

Both the electric field $\mathbf{E}$ and the electric potential V diverges at the location of a point charge.

Using the superposition principle and taking the continuum limit, we get

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3\mathbf{r}' \, \frac{\rho(\mathbf{r}')}{\mathscr{r}}, \tag{12.28}$$

up to a constant, assuming that the integral is well defined.

When this integral is not well defined (e.g. it diverges), V may still be solved from eq.(12.24) with suitable boundary conditions.

As $V(\mathbf{r})$ is a scalar field and $\mathbf{E}(\mathbf{r})$ a vector field, it is typically easier to deal with $V(\mathbf{r})$ than $\mathbf{E}(\mathbf{r})$. In particular, for a given $\rho(\mathbf{r})$, it is easier to first find $V(\mathbf{r})$ from eq.(12.28) and then derive $\mathbf{E}(\mathbf{r})$ from eq.(12.24) rather than deriving $\mathbf{E}(\mathbf{r})$ directly from $\rho(\mathbf{r})$ using eq.(12.14).

However, when $\mathbf{E}(\mathbf{r})$ can be easily determined by Gauss' law, it would be easier to deduce $V(\mathbf{r})$ from $\mathbf{E}(\mathbf{r})$ from eq.(12.25) rather than using eq.(12.28).

Q 12.46: To compute the electric field $\mathbf{E}(\mathbf{r})$ for a given charge distribution $\rho(\mathbf{r})$, one can either use eq.(12.14) or use eq.(12.28) followed by eq.(12.24). Which approach is easier?

Electric Potential Energy

The electric potential energy U of a charge q at a position $\mathbf{r}$ is $qV(\mathbf{r})$, where $V(\mathbf{r})$ is the electric potential due to charges other than q.

Ex 12.23: What is the electric potential energy of two charges q_1 and q_2 separated by a distance R?

Solution:

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{R} = q_1 V_2(\mathbf{r}_1) = q_2 V_1(\mathbf{r}_2) = \frac{1}{2}\left[q_1 V_2(\mathbf{r}_1) + q_2 V_1(\mathbf{r}_2)\right]. \tag{12.29}$$

Notice that this potential energy belongs to the system including both q_1 and q_2.

Note the difference between the "electric potential" and "electric potential energy".

Q 12.47: What is the electric potential energy of a single point charge q?

Ex 12.24: What is the potential energy of a system of N charges q_i $(i = 1, 2, \cdots, N)$ fixed at positions $\mathbf{r}_i$?

Solution:

$$U_{tot} = \sum_{i=1}^{N}\sum_{j=i+1}^{N} \frac{1}{4\pi\epsilon_0}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} = \frac{1}{2}\sum_{i\neq j}\frac{1}{4\pi\epsilon_0}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}. \tag{12.30}$$

The energy of a system is defined up to a total additive constant until we talk about gravity. In General Relativity, the energy is a source of gravity, so a constant added to the energy changes the spacetime geometry.

Ex 12.25: What is the electric potential $V(\mathbf{r})$ of a uniformly charged spherical surface of radius R with total charge Q? How much energy does it take to bring a charge q from infinity to r assuming $r > R$?

Ex 12.26: What is the potential energy of a uniformly charged spherical surface of radius R with total charge Q?

Q 12.48: What is the potential energy in the limit $R \to 0$? If it does not vanish, why can we assume that the potential energy of a point charge vanishes?

Ex 12.27: What is the potential energy of a given charge distribution $\rho(\mathbf{r})$?

Solution:

$$\boxed{U = \frac{1}{2}\left[\frac{1}{4\pi\epsilon_0}\int d^3\mathbf{r}\int d^3\mathbf{r}'\,\frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{\mathscr{r}}\right],} \tag{12.31}$$

where $\mathscr{r} \equiv \mathbf{r} - \mathbf{r}'$, and we assume that $\rho(\mathbf{r})$ is finite everywhere so that the self-energy contribution due to the integration over the points $\mathbf{r} = \mathbf{r}'$ can be ignored.

In electrostatics, the electric potential energy is included as part of the conserved energy in Newton's mechanics. Similarly, the momentum is conserved when electrostatic forces are involved, because Newton's 3rd law is observed.

Here we assume that the motion of charges are so slow that the Coulomb's law is still a good description of the electromagnetic interaction.

Q 12.49: What about the energy and momentum of electromagnetic waves? (We will discuss it later. But a complete description will only be given in the course *Electromagnetism.*

Q 12.50: Charges and electrostatic forces described by Coulomb's law and superposition principle provide all the information we need for every problem in electrostatics. Why do we need to understand $\mathbf{E}(\mathbf{r})$ or $V(\mathbf{r})$?

HW: (1-3) For a volume density distribution $\rho(\mathbf{r}) = cr^2$ for $0 < r < R$ with a given constant c, find (1) the electric field $\mathbf{E}(\mathbf{r})$, (2) the electric potential $V(\mathbf{r})$, and (3) the total potential energy U. (For (1) and (2), give answers for both (a) $r > R$ and (b) $R > r > 0$.)

12.1.8 From Field to Charge via Gauss' Law

Ex 12.28: Using the electric field $\mathbf{E}$ found for a spherically symmetric charge distribution $\rho(r)$, express $\rho(r)$ in terms of the derivatives of the electric field.

Ex 12.29: Using the electric field $\mathbf{E}$ found for a planar symmetric charge distribution $\rho(z)$, express $\rho(z)$ in terms of the derivatives of the electric field.

The "inverse" of the integral form of Gauss' law (12.21) is the differential form of Gauss' law:

We can derive eq.(12.32) from eq.(12.21) by considering an infinitesimal volume $\mathcal{V}$.

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\epsilon_0}.} \tag{12.32}$$

In Cartesian coordinates,

$$\nabla \cdot \mathbf{E} \equiv \partial_x E_x + \partial_y E_y + \partial_z E_z. \tag{12.33}$$

∇ is simultaneously a vector and an operator.

> As a statment equivalent to eq.(12.21), we expect the first derivative of $\mathbf{E}(\mathbf{r})$ to be related to $\rho(\mathbf{r})$. Demanding that the equation is invariant under rotations and translations, and the superposition principle ($\mathbf{E}(\mathbf{r})$ be linearly related to $\rho(\mathbf{r})$), is there any option different from eq.(12.32)?

Ex 12.30: What is the expression of $\nabla \cdot \mathbf{E}$ in terms of the cylindrical coordinates (s, ϕ, z)? Express it in terms of E_s, E_ϕ, E_z.
Ex 12.31: For the potential $V(r) = cr^n$, where c, n are constants and r is the radial coordinate in the spherical coordinate system, find $\mathbf{E}(\mathbf{r})$ and $\rho(\mathbf{r})$.
Q 12.51: Is it possible to have a non-zero electric field with no charge in space?
Ex 12.32: For the electric field $\mathbf{E}(\mathbf{r}) = c\hat{\mathbf{x}}$ ($c =$ a constant), find the potential $V(\mathbf{r})$ and charge density $\rho(\mathbf{r})$.
Q 12.52: Given one of the three $(\rho(\mathbf{r}), \mathbf{E}(\mathbf{r}), V(\mathbf{r}))$, can you always find the other two? What is the easiest way to do that?

HW: (1-4) What is the electric field $\mathbf{E}(\mathbf{r})$ for the potential $V(\mathbf{r}) = e^{-ar}$ with a constant a? What is the charge density $\rho(\mathbf{r})$?
Q 12.53: How to derive $\rho(\mathbf{r})$ from a given $V(\mathbf{r})$? (What is the differential equation relating $V(\mathbf{r})$ to $\rho(\mathbf{r})$?)

Using eqs.(12.24) and (12.32), we find

$$\boxed{\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0},} \tag{12.34}$$

where

$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{12.35}$$