110-2 線性代數

物理系

2022_03_29

Matrices, Vectors, and Systems of Linear Equations

1) Elementary row operation:
REDUCED ROW ECHELON FORM (GAUSSIAN ELIMINATION)

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

會利用Gaussian 消去法，將A得到R

2) SOLVING SYSTEMS OF LINEAR EQUATIONS

Example 5 Solve the following system of linear equations:

$$\begin{aligned} x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2 \\ -x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\ 2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\ -3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9 \end{aligned}$$

Solution The augmented matrix of this system is

$$\left[\begin{array}{rrrrr|r} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{array}\right].$$

In Section 1.4, we show that the reduced row echelon form of this matrix is

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Because there is no row in this matrix in which the only nonzero entry lies in the last column, the original system is consistent. This matrix corresponds to the system of linear equations

$$\begin{aligned} x_1 + 2x_2 - x_5 &= -5 \\ x_3 &= -3 \\ x_4 + x_5 &= 2. \end{aligned}$$

In this system, the basic variables are x_1, x_3, and x_4, and the free variables are x_2 and x_5. When we solve for the basic variables in terms of the free variables, we obtain the following general solution:

$$\begin{aligned} x_1 &= -5 - 2x_2 + x_5 \\ x_2 &\quad \text{free} \\ x_3 &= -3 \\ x_4 &= 2 - x_5 \\ x_5 &\quad \text{free} \end{aligned}$$

This is the general solution of the original system of linear equations.

會利用R，並得到聯立方程式**一般解**。

3) An m × n matrix A:

RANK (n) and NULLITY of a MATRIX (n-rank(A))

Example 3 The reduced row echelon form of the matrix

$$B = \begin{bmatrix} 2 & 3 & 1 & 5 & 2 \\ 0 & 1 & 1 & 3 & 2 \\ 4 & 5 & 1 & 7 & 2 \\ 2 & 1 & -1 & -1 & -2 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & -1 & -2 & -2 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the latter matrix has two nonzero rows, the rank of B is 2. The nullity of B is $5 - 2 = 3$.

會計算出 **RANK 數** (PIVOT 數)。

If $A\mathbf{x} = \mathbf{b}$ is the matrix form of a consistent system of linear equations, then

(a) the number of basic variables in a general solution of the system equals the rank of A;

(b) the number of free variables in a general solution of the system equals the nullity of A.

Thus a consistent system of linear equations has a unique solution if and only if the nullity of its coefficient matrix equals 0. Equivalently, a consistent system of linear equations has infinitely many solutions if and only if the nullity of its coefficient matrix is positive.

Test of Consistency (有解)

(Test for Consistency) The following conditions are equivalent:

(a) The matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent.

(b) The vector $\mathbf{b}$ is a linear combination of the columns of A.

(c) The reduced row echelon form[11] of the augmented matrix $[A \;\; \mathbf{b}]$ has no row of the form $[0 \;\; 0 \;\; \cdots 0 \;\; d]$, where $d \neq 0$.

4) THE SPAN OF A SET OF VECTORS

Example 3 Is

$$\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 5 \\ -1 \end{bmatrix} \quad \text{or} \quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

a vector in the span of

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ -1 \\ 5 \end{bmatrix} \right\}?$$

If so, express it as a linear combination of the vectors in $\mathcal{S}$.

會判斷**任一向量**是否屬於 **SPAN 的空間**。

Solution Let A be the matrix whose columns are the vectors in $\mathcal{S}$. The vector $\mathbf{v}$ belongs to the span of $\mathcal{S}$ if and only if $A\mathbf{x} = \mathbf{v}$ is consistent. Since the reduced row echelon form of $[A \;\; \mathbf{v}]$ is

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$A\mathbf{x} = \mathbf{v}$ is consistent by Theorem 1.5. Hence $\mathbf{v}$ belongs to the span of $\mathcal{S}$.

To express $\mathbf{v}$ as a linear combination of the vectors in $\mathcal{S}$, we need to find the actual solution of $A\mathbf{x} = \mathbf{v}$. Using the reduced row echelon form of $[A \;\; \mathbf{v}]$, we see that the general solution of this equation is

$$\begin{aligned} x_1 &= \;\; 1 - 3x_3 \\ x_2 &= -2 - 2x_3 \\ x_3 & \;\; \text{free.} \end{aligned}$$

For example, by taking $x_3 = 0$, we find that

$$1\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 8 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 5 \\ -1 \end{bmatrix} = \mathbf{v}.$$

In the same manner, $\mathbf{w}$ belongs to the span of $\mathcal{S}$ if and only if $A\mathbf{x} = \mathbf{w}$ is consistent. Because the reduced row echelon form of $[A \;\; \mathbf{w}]$ is

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

Theorem 1.5 shows that $A\mathbf{x} = \mathbf{w}$ is not consistent. Thus $\mathbf{w}$ does not belong to the span of $\mathcal{S}$.

5) LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

A set of k vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is called **linearly independent** if the only scalars $c_1, c_2, \ldots, c_k$ such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}$$

are $c_1 = c_2 = \cdots = c_k = 0$. In this case, we also say that **the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly independent**.

Example 1 Show that the sets

$$\mathcal{S}_1 = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{S}_2 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

are linearly dependent.

Solution The equation

$$c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is true with $c_1 = 2$, $c_2 = -1$, and $c_3 = 1$. Since not all the coefficients in the preceding linear combination are 0, $\mathcal{S}_1$ is linearly dependent.

Because

$$1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and at least one of the coefficients in this linear combination is nonzero, $\mathcal{S}_2$ is also linearly dependent.

6) LINEAR DEPENDENCE

Example 2 Determine whether the set

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

is linearly dependent or linearly independent.

會判斷向量是否為 **L.D.**。

Solution We must determine whether $Ax = \mathbf{0}$ has a nonzero solution, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 4 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

is the matrix whose columns are the vectors in $\mathcal{S}$. The augmented matrix of $Ax = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 2 & 0 \\ 1 & 1 & 1 & 3 & 0 \end{bmatrix},$$

and its reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Hence the general solution of this system is

$$\begin{aligned} x_1 &= -2x_3 \\ x_2 &= x_3 \\ x_3 &\quad \text{free} \\ x_4 &= 0. \end{aligned}$$

Because the solution of $Ax = \mathbf{0}$ contains a free variable, this system of linear equations has infinitely many solutions, and we can obtain a nonzero solution by choosing any nonzero value of the free variable. Taking $x_3 = 1$, for instance, we see that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

is a nonzero solution of $Ax = \mathbf{0}$. Thus $\mathcal{S}$ is a linearly dependent subset of $\mathcal{R}^3$ since

$$-2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is a representation of $\mathbf{0}$ as a linear combination of the vectors in $\mathcal{S}$.

6) LINEAR INDEPENDENCE

Example 3 Determine whether the set

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly dependent or linearly independent.

會判斷向量是否為 **L.I.** 。

Solution As in Example 2, we must check whether $A\mathbf{x} = \mathbf{0}$ has a nonzero solution, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}.$$

There is a way to do this without actually solving $A\mathbf{x} = \mathbf{0}$ (as we did in Example 2). Note that the system $A\mathbf{x} = \mathbf{0}$ has nonzero solutions if and only if its general solution contains a free variable. Since the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the rank of A is 3, and the nullity of A is $3 - 3 = 0$. Thus the general solution of $A\mathbf{x} = \mathbf{0}$ has no free variables. So $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions, and hence S is linearly independent.

LINEAR INDEPENDENCE

THEOREM 2.4

The following statements are true for any matrix A:

(a) The pivot columns of A are linearly independent.

(b) Each nonpivot column of A is a linear combination of the previous pivot columns of A, where the coefficients of the linear combination are the entries of the corresponding column of the reduced row echelon form of A.

Matrices and Linear Transformations

1) INVERTIBILITY MATRICES: A^{-1}

Definitions An $n \times n$ matrix A is called **invertible** if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. In this case, B is called an **inverse** of A.

2) AN ALGORITHM FOR MATRIX INVERSION

1) INVERTIBILITY MATRICES: A^{-1}

Example 2 Use a matrix inverse to solve the system of linear equations

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 + 5x_2 &= 7. \end{aligned}$$

Solution This system is the same as the matrix equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

We saw in Example 1 that A is invertible. Hence we can solve this equation for $\mathbf{x}$ by multiplying both sides of the equation on the left by

$$A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

as follows:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$$

會利用A^{-1}得到解答$\mathbf{X}$。

Therefore $x_1 = -6$ and $x_2 = 5$ is the unique solution of the system.

AN ALGORITHM FOR MATRIX INVERSION

Example 2 We use the algorithm for matrix inversion to compute A^{-1} for the invertible matrix A of Example 1. This algorithm requires us to transform $[A \;\; I_3]$ into a matrix of the form $[I_3 \;\; B]$ by means of elementary row operations. For this purpose, we use the Gaussian elimination algorithm in Section 1.4 to transform A into its reduced row echelon form I_3, while applying each row operation to the entire row of the 3×6 matrix.

$$[A \;\; I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 4 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3\mathbf{r}_1+\mathbf{r}_3 \to \mathbf{r}_3]{-2\mathbf{r}_1+\mathbf{r}_2 \to \mathbf{r}_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array}\right]$$

A

$$\xrightarrow{2\mathbf{r}_2+\mathbf{r}_3 \to \mathbf{r}_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & -7 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{-\mathbf{r}_3 \to \mathbf{r}_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right]$$

$$\xrightarrow{-3\mathbf{r}_3+\mathbf{r}_1 \to \mathbf{r}_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -20 & 6 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right]$$

$$\xrightarrow{-2\mathbf{r}_2+\mathbf{r}_1 \to \mathbf{r}_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16 & 4 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right] = [I_3 \;\; B]$$

A^{-1}

會利用Gaussian 消去法，得到A^{-1}。

Determinants

COFACTOR EXPANSION
CRAMER'S RULE

COFACTOR EXPANSION

Practice Problem 3 ▶ Evaluate the determinant of

$$\begin{bmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{bmatrix}$$

by using the cofactor expansion along the second row.

會利用**Cofactor**，得到 **determinate**。

CRAMER'S RULE

Example 4 Use Cramer's rule to solve the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 2 \\ x_1 + x_3 &= 3 \\ x_1 + x_2 - x_3 &= 1. \end{aligned}$$

Solution The coefficient matrix of this system is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Since $\det A = 6$, A is invertible by Theorem 3.4(a), and hence Cramer's rule can be used. In the notation of Theorem 3.5, we have

$$M_1 = \begin{bmatrix} 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \; M_2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \; \text{and} \; M_3 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}.$$

Therefore the unique solution of the given system is the vector with components

$$u_1 = \frac{\det M_1}{\det A} = \frac{15}{6} = \frac{5}{2}, \; u_2 = \frac{\det M_2}{\det A} = \frac{-6}{6} = -1, \; \text{and} \; u_3 = \frac{\det M_3}{\det A} = \frac{3}{6} = \frac{1}{2}.$$

It is readily checked that these values satisfy each of the equations in the given system.

會利用**Cramer's rule**，得到 **解答 x**。

Subspaces and Their Properties

Null Space / Column Space

Definition The **null space** of a matrix A is the solution set of $A\mathbf{x} = \mathbf{0}$. It is denoted by Null A.

If A is an $m \times n$ matrix, then Null A is a subspace of $\mathcal{R}^n$.

Definition The **column space** of a matrix A is the span of its columns. It is denoted by Col A.

Column Space

Example 7 Find a generating set for the column space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix}.$$

Is $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ in Col A? Is $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ in Col A?

會判斷任一項量是否在 **C(A)**。

Solution The column space of A is the span of the columns of A. Hence one generating set for Col A is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -8 \\ 6 \end{bmatrix} \right\}.$$

To see whether the vector $\mathbf{u}$ lies in the column space of A, we must determine whether $A\mathbf{x} = \mathbf{u}$ is consistent. Since the reduced row echelon form of $[A \;\; \mathbf{u}]$ is

$$\begin{bmatrix} 1 & 2 & 0 & -4 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

we see that the system is inconsistent, and hence $\mathbf{u}$ is not in Col A. (See Figure 4.4.) On the other hand, the reduced row echelon form of $[A \;\; \mathbf{v}]$ is

$$\begin{bmatrix} 1 & 2 & 0 & -4 & 0.5 \\ 0 & 0 & 1 & 3 & 1.5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the system $A\mathbf{x} = \mathbf{v}$ is consistent, so $\mathbf{v}$ is in Col A. (See Figure 4.4.)

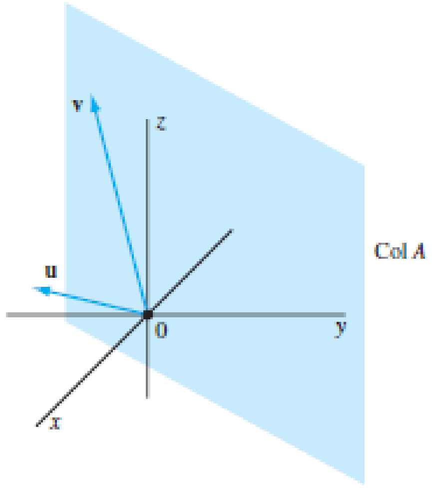

Figure 4.4 The vector **v** is in the column space of A, but **u** is not.

Null Space

Example 8

Find a generating set for the null space of the matrix A in Example 7. Is $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 3 \\ -1 \end{bmatrix}$ in Null A? Is $\mathbf{v} = \begin{bmatrix} 5 \\ -3 \\ 2 \\ 1 \end{bmatrix}$ in Null A?

會判斷任一項量是否在 **N(A)**。

Solution Unlike the calculation of a generating set for the column space of A in Example 7, there is no easy way to obtain a generating set for the null space of A directly from the entries of A. Instead, we must solve $A\mathbf{x} = \mathbf{0}$. Because the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the vector form of the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + 4x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

It follows that

$$\text{Null } A = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

So the span of the set of vectors in the vector form of the general solution of $A\mathbf{x} = \mathbf{0}$ equals Null A.

To see if the vector $\mathbf{u}$ lies in the null space of A, we must determine whether $A\mathbf{u} = \mathbf{0}$. An easy calculation confirms this; so $\mathbf{u}$ belongs to Null A. On the other hand,

$$A\mathbf{v} = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}.$$

Since $A\mathbf{v} \neq \mathbf{0}$, we see that $\mathbf{v}$ is not in Null A.

BASIS AND DIMENSION

Definition Let V be a nonzero subspace of $\mathcal{R}^n$. A $\boxed{\textbf{basis}}$ (*plural, bases*) for V is a linearly independent generating set for V.

The $\boxed{\text{pivot columns}}$ of a matrix form a $\boxed{\text{basis}}$ for its column space.

BASIS

Example 1 Find a basis for Col A if

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}.$$

會判斷 **C(A)** 的基底 **basis**。

Solution In Example 1 of Section 1.4, we showed that the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the leading ones of this matrix are in columns one, three, and four, the pivot columns of A are

$$\begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

As previously mentioned, these vectors form a basis for Col A. Note that it is the pivot columns of A, and not those of the reduced row echelon form of A, that form a basis for Col A.

DIMENSION

Example 1 For the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

in Example 1 of Section 4.2, we saw that the set of pivot columns of A,

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\},$$

is a basis for Col A. Hence the dimension of Col A is 3.

The dimension of the column space of a matrix equals the rank of the matrix.

The dimension of the null space of a matrix equals the nullity of the matrix.

DIMENSION

Example 3 Recall that for the matrix

$$A = \begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ -5 & -2 & 5 & -5 & -3 \\ -2 & -1 & 3 & 2 & -10 \end{bmatrix}$$

in Example 2, the reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -5 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence a basis for Row A is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -5 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

Thus the dimension of Row A is 3, which is the rank of A.

The dimension of the row space of a matrix equals its rank.

The rank of any matrix equals the rank of its transpose.

DIMENSION

The Dimensions of the Subspaces Associated with an $m \times n$ Matrix A

Subspace	Containing Space	Dimension
Col A	$\mathcal{R}^m$	rank A
Null A	$\mathcal{R}^n$	nullity $A = n -$ rank A
Row A	$\mathcal{R}^n$	rank A

Bases for the Subspaces Associated with a Matrix A

Col A: The pivot columns of A form a basis for Col A.

Null A: The vectors in the vector form of the solution of $A\mathbf{x} = \mathbf{0}$ constitute a basis for Null A. (See page 80.)

Row A: The nonzero rows of the reduced row echelon form of A constitute a basis for Row A. (See page 257.)

Orthogonality

The length of a vector

Example 1 Find $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and the distance between $\mathbf{u}$ and $\mathbf{v}$ if

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}.$$

Solution By definition,

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, \quad \|\mathbf{v}\| = \sqrt{2^2 + (-3)^2 + 0^2} = \sqrt{13},$$

and the distance between $\mathbf{u}$ and $\mathbf{v}$ is

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{(1-2)^2 + (2-(-3))^2 + (3-0)^2} = \sqrt{35}.$$

ORTHOGONAL PROJECTION OF A VECTOR ON A LINE

Example 3 Find the distance from the point $(4, 1)$ to the line whose equation is $y = \frac{1}{2}x$.

Solution Following our preceding derivation, we let

$$\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{9}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Then the desired distance is $\left\| \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \frac{9}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\| = \frac{1}{5}\left\| \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\| = \frac{2}{5}\sqrt{5}.$

求出點到線的最近距離。

Gram–Schmidt Process

Example 2 Let W be the span of $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

are linearly independent vectors in $\mathcal{R}^4$. Apply the Gram–Schmidt process to $\mathcal{S}$ to obtain an orthogonal basis $\mathcal{S}'$ for W.

會利用 G-S 方法，得到**Orthogonal Basis**。

Solution Let

$$\mathbf{v}_1 = \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix},$$

and

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} - \frac{5}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{(-1)}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Then $\mathcal{S}' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for W.

ORTHOGONAL PROJECTIONS

For any matrix A, the orthogonal complement of the row space of A is the null space of A; that is,

$$(\text{Row}\,A)^{\perp} = \text{Null}\,A.$$

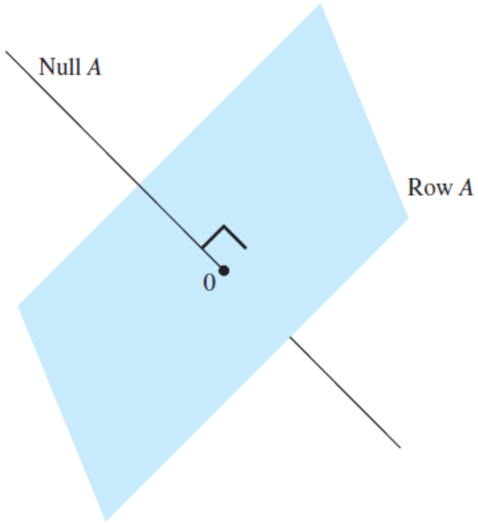

Figure 6.12 The null space of A is the orthogonal complement of the row space of A.

For any subspace W of $\mathcal{R}^n$,

$$\dim W + \dim W^{\perp} = n.$$

Projection matrix P

Example 4 Find P_W, where W is the 2-dimensional subspace of $\mathcal{R}^3$ with equation

$$x_1 - x_2 + 2x_3 = 0.$$

Solution Observe that a vector $\mathbf{w}$ is in W if and only

$$\mathbf{w} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix},$$

and hence

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for W. Let

$$C = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

the matrix whose columns are the basis vectors just computed. Then

$$P_W = C(C^T C)^{-1} C^T = \frac{1}{6} \begin{bmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}.$$

會求出矩陣的 **Projection Matrix** 。

LEAST-SQUARES APPROXIMATION

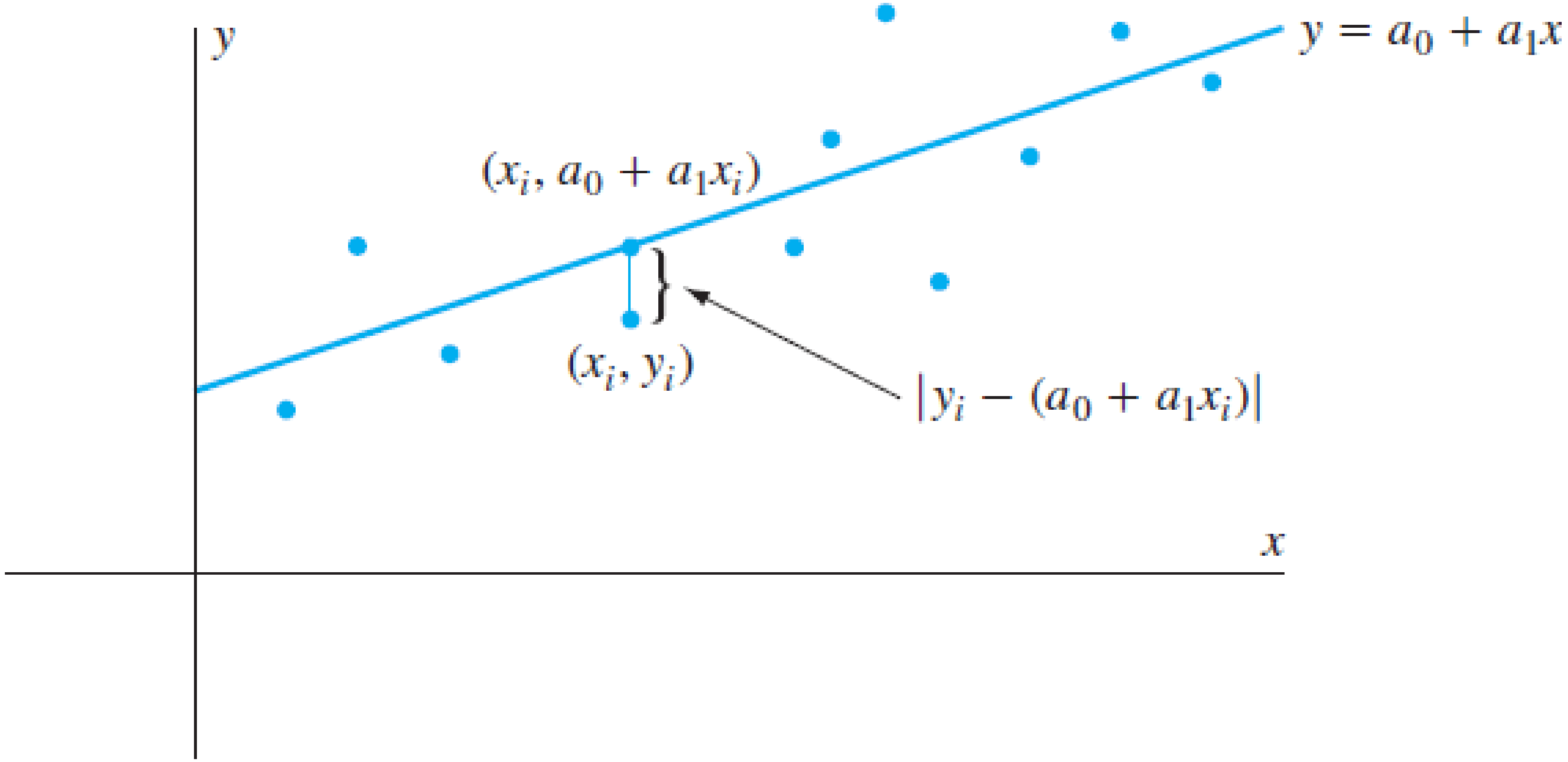

Figure 6.16 A plot of the data

LEAST-SQUARES APPROXIMATION

Example 3 Given the inconsistent system of linear equations $A\mathbf{x} = \mathbf{b}$, with

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -3 \\ 3 & 2 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 7 \\ -4 \\ 8 \end{bmatrix},$$

use the method of least squares to describe the vectors $\mathbf{z}$ for which $\|A\mathbf{z} - \mathbf{b}\|$ is a minimum.

Solution By computing the reduced row echelon form of A, we see that the rank of A is 2 and that the first two columns of A are linearly independent. Thus the first two columns of A form a basis for $W = \operatorname{Col} A$. Let C be the 4×2 matrix with these two vectors as its columns. Then

$$P_W\mathbf{b} = C(C^TC)^{-1}C^T\mathbf{b} = \frac{1}{3}\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ -4 \\ 8 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5 \\ 19 \\ -14 \\ 24 \end{bmatrix}.$$

As noted, the vectors that minimize $\|A\mathbf{z} - \mathbf{b}\|$ are the solutions to $A\mathbf{x} = P_W\mathbf{b}$. The general solution of this system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 14 \\ -9 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

So these are the vectors that minimize $\|A\mathbf{z} - \mathbf{b}\|$. Note that, for each of these vectors, we have

$$\|A\mathbf{z} - \mathbf{b}\| = \|P_W\mathbf{b} - \mathbf{b}\| = \left\| \frac{1}{3}\begin{bmatrix} 5 \\ 19 \\ -14 \\ 24 \end{bmatrix} - \begin{bmatrix} 1 \\ 7 \\ -4 \\ 8 \end{bmatrix} \right\| = \frac{2}{\sqrt{3}}.$$

會求出 minimum 的 least square。