12.3 Magnetostatics

12.3.1 Magnetic Materials

Magnets are found in nature. There are two types of magnetic charges (North and South).

Faramagnetic, **Paramagnetic**, **Diamagnetic**:

These magnetic properties can be explained by the two sources of magnetic field in matter: the spin and orbital motion of electrons, together with Pauli's exclusion principle.

There are many other kinds of materials with interesting magnetic properties, e.g. antiferromagnetism, superconductor, etc.

Q 12.65: When a ferromagnetic material is magnetized by a permanent magnet, how is the energy conserved?

12.3.2 Magnetic Field

It is tempting, as Newton did, to conjecture that an analogue of Coulomb's law also applies to the magnetic force between magnetic charges. However, **magnetic monopoles** have never been (repeatedly) found. A magnet always comes with both charges of equal magnitude. If you break it in halves, each piece will still have its own North Pole and South Pole. An analogue of Coulomb's law seems to be missing something fundamental. The theory of magnetostatics resembles electrostatics in a less obvious way.

Valentine's Day monopole: 1982, Cabrera, Stanford U.

Instead of thinking of the magnetic field as something generated by magnetic charges, it is more proper to think of the magnetic field as relativistic partner of the electric field. It interacts with and is generated by the motion of electric charges.

Q 12.66: What is the connection between A interacts with B and A is generated by B?

The sources of magnetic field: electric current and spin.

Like the electric field $\mathbf{E}(\mathbf{r})$, the magnetic field $\mathbf{B}(\mathbf{r})$ is a vector field,

What are the counterparts of Coulomb's law, Gauss's law, electric potential, etc. for the magnetic field?

> When we learn Special Relativity, we will no longer call them vector fields because the definition of vector fields will refer to Lorentz transformations. They are vector fields only when we restrict our considerations to spatial rotations.

The magnetic field $\mathbf{B}$ can also be represented by **magnetic field lines**.

Corresponding to the fact that magnetic monopoles have never been found, we assume that the magnetic field lines are always **closed**. Following the ideas behind Gauss' law, we should have

$$\oint_S \mathbf{B} \cdot d\mathbf{a} = 0. \tag{12.51}$$

12.3.3 Lorentz Force

A charge q in motion with velocity $\mathbf{v}(t)$ at the position $\mathbf{r}(t)$ $(\mathbf{v}(t) = \dot{\mathbf{r}}(t))$ experiences the force

$$\mathbf{F}(t) = q\mathbf{v}(t) \times \mathbf{B}(t, \mathbf{r}(t)) \tag{12.52}$$

in a magnetic field $\mathbf{B}(t, \mathbf{r})$. If both electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are present, the electromagnetic force is given by the *Lorentz force law*

$$\mathbf{F} = q\left[\mathbf{E}(t, \mathbf{r}) + \mathbf{v}(t) \times \mathbf{B}(t, \mathbf{r})\right]. \tag{12.53}$$

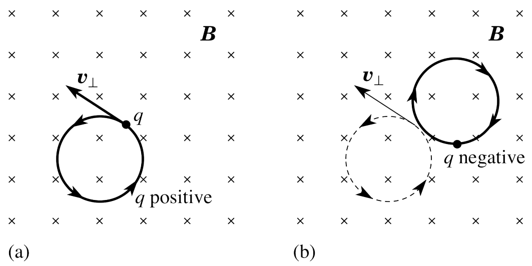
The Lorentz force law remains valid for time-dependent fields.

Q 12.67: Is eq.(12.52) invariant under spatial or temporal inversion?

Under spatial inversion, $\mathbf{F}$ and $\mathbf{v}$ changes sign, and q remains the same, so $\mathbf{B}$ should remain the same if eq.(12.52) continues to hold.

In the context of Special Relativity, neither $\mathbf{B}$ nor $\mathbf{E}$ is a 4-vector, because they do not transform like a vector under Lorentz transformations.

Example: Cyclotron Motion



Q 12.68: What is the frequency of the cyclotron motion?
Q 12.69: How would a charge q move in a background of constant $\mathbf{B}$ if its initial velocity is not perpendicular to $\mathbf{B}$?
Q 12.70: How would a charge q move in a background of constant $\mathbf{E}$ and constant $\mathbf{B}$ perpendicular to each other?

Exercise: Derive the formula for **magnetic force on a current**

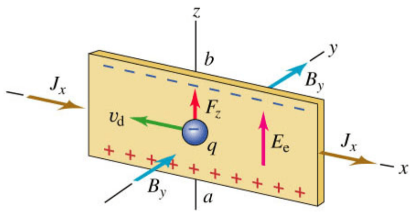
$$d\mathbf{F} = I d\boldsymbol{\ell} \times \mathbf{B} \tag{12.54}$$

from the magnetic force on a charge (12.52).

Note that $d\mathbf{F}$ in eq.(12.54) is independent of the sign of the free charges carrying the current.

Example: Hall Effect

It could be either positive charges, negative charges, or both, carrying electric currents in a conductor. How do we tell them apart experimentally?



Exercise: Show that the potential difference between the top and bottom of the slab is

$$V = \frac{IB}{nqt}, \tag{12.55}$$

where q is the charge of the conducting particles in the material, n is the number density of them, and t is the thickness of the slab (in the y direction).

The magnetic force changes the direction of a charge, but does zero work because

"Magnetic force does not work."

$$dW = \mathbf{F} \cdot d\mathbf{r} = \mathbf{F} \cdot \mathbf{v} dt, \tag{12.56}$$

and the magnetic force is always perpendicular to $\mathbf{v}$.

Q 12.71: When an electromagnet is used to lift up a heavy metal box, who is responsible for the work done to the box?

Solution:

To do work, the conducting loop has to move up, and the upward velocity of the charges lead to a Lorentz force opposing the current. ($F = qwB$, where w is the upward velocity.) Hence there must be a battery at work. Since we have $FqL = qV$ for energy conservation of each charge around the wire loop, the work done by the battery (Power $= IV$) matches the work done in the box lifting ($ILBw$).

12.3.4 Ampère's Law

Ampère's law:

The right-hand rule applies.

$$\boxed{\oint_{\mathcal{C}} \mathbf{B} \cdot d\ell = \mu_0 I_{enc},} \tag{12.57}$$

Ampere's law (1822) would be modified by Maxwell.

where the electric current enclosed in the closed loop $\mathcal{C}$ can be determined by integrating the current density $\mathbf{J}$ over a surface $\mathcal{S}$ enclosed by $\mathcal{C}$ (i.e. $\mathcal{C}$ is the boundary of $\mathcal{S}$) as

$$I_{enc} \equiv \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}. \tag{12.58}$$

Before 2019, the vacuum permeability (magnetic constant) is defined by

$$\mu_0 \equiv 4\pi \times 10^{-7} H/m. \tag{12.59}$$

After 2019, it is defined by

$$\mu_0 \equiv \frac{2\alpha h}{e^2 c}, \tag{12.60}$$

where α is the fine-structure constant.

Notice that

1. The direction of the area element $d\mathbf{a}$ is determined by the right-hand rule according to the direction of $\mathcal{C}$.

2. There are infinitely many possible surfaces $\mathcal{S}$ that share the same boundary $\mathcal{C}$.

Q 12.72: How is Ampere's law different from but similar to Gauss's law?

Exercise: If both $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same boundary $\mathcal{C}$, show that Ampére's law is consistent only if

$$\int_{\mathcal{S}_1} \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{S}_2} \mathbf{J} \cdot d\mathbf{a}. \tag{12.61}$$

Exercise: Following the above, show that Ampére's law is consistent only if

$$\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} = 0 \tag{12.62}$$

for any closed surface $\mathcal{S}$.

Q 12.73: Why is eq.(12.62) valid in magnetostatics?

Exercise: What is $\mathbf{B}$ for a long straight wire along the z-axis?

Solution:

First, we have to use the symmetry of rotating the z-axis by 180 degrees (along the x or y-axis) to argue that $B_r = 0$ (in cylindrical coordinates).

Next, we use Ampere's law to prove that $B_z = 0$.

Then we can determine B_ϕ by choosing a suitable Amperian loop. The result is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}. \tag{12.63}$$

Exercise: What is $\mathbf{B}$ for a long straight solenoid of n turns of current I per unit length? (Assume that the solenoid stands along the z-axis and the current flows from bottom to top in the ϕ-direction.

Solution:

Implicitly, in such a problem, we assume that the solenoid is so tightly packed such that it is equivalent to parallel rings carrying currents of their own. Apart from argument of

What is $\oint \mathbf{B} \cdot d\ell$ for the Amperian loop on the $x-y$ plane encircling the solenoid?

symmetries, we also need to use the boundary condition that $\mathbf{B} = 0$ at $r = \infty$. The result is

$$\mathbf{B} = \mu_0 n I \hat{\mathbf{z}}. \tag{12.64}$$

Example: Motors, magnetic train

HW: (3-1) A solenoid in the shape of a torus has a total of N turns of wire carrying a constant current I. Find the magnitude of the magnetic field inside the solenoid assuming that the magnetic field is in the circular direction.

> In magnetostatics, the current is assumed to be time-independent. Even though microscopically there must be moving charges when there is a current, hence some time-dependence, macroscopically the current can be time-independent and the charge density can vanish.

Exercise: What is the differential form of Ampere's law?
Solution:
Consider the Amperian loop of a rectangle on the $x-y$ plane with sides dx and dy, the LHS of eq.(12.57) is

$$\begin{aligned}\oint_{\mathcal{C}} \mathbf{B} \cdot d\ell &\simeq B_x(x,y,z)dx + B_y(x+dx,y,z)dy - B_x(x,y+dy,z) - B_y(x,y,z) \\ &\simeq -[\partial_y B_x(\mathbf{r}) - \partial_x B_y(\mathbf{r})]dxdy,\end{aligned} \tag{12.65}$$

and the RHS is $\mu_0 J_z(\mathbf{r})dxdy$. Hence we have $(\nabla \times \mathbf{B})_z = \mu_0 J_z$. The same consideration applies to other choices of $\mathcal{C}$.

The differential form of Ampere's law (12.57) is

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}), \tag{12.66}$$

where $\mathbf{J}(\mathbf{r})$ is the current density.

Exercise: Given $\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi r}$ in cylindrical coordinates, what is the corresponding current density $\mathbf{J}(\mathbf{r})$?

Exercise: Given $\mathbf{B}(\mathbf{r}) = B_0 \hat{\mathbf{z}}$, what is the current density?

12.3.5 Biot-Savart Law

Although we will skip the derivation here, Ampere's law (12.57) together with eq. (12.51) imply the Biot-Savart Law,

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{I d\ell \times d\boldsymbol{\mathscr{r}}}{\boldsymbol{\mathscr{r}}^2} \tag{12.67}$$

which is the counterpart of Coulomb's law.

We will not use the Biot-Savart law extensively. But it is good to know that in principle you can compute $\mathbf{B}$ from the current directly.

After Special Relativity is introduced, we can see that the Biot-Savart law can be derived from Coulomb's law via Lorentz transformations.

Exercise: Derive eq.(12.63) using the Biot-Savart law.

The physical laws for electrostatics and magnetostatics are analogous.

Q 12.74: What is $\oint \mathbf{E} \cdot d\ell$?

Q 12.75: What is the interpretation of $\oint \mathbf{E} \cdot d\ell$ in terms of the electric field lines?

Strictly speaking, we also needed two equations, Gauss' law and

$$\oint \mathbf{E} \cdot d\ell = 0 \tag{12.68}$$

in order to uniquely determine the electric field from a given charge distribution (to derive Coulomb's law). But we circumvented the trouble of introducing (12.68) by resorting to superposition principle and spherical symmetry.

Eq.(12.68) is the condition for the potential V to be consistently defined.

Q 12.76: Can we define a magnetic potential?