3. Magnetostatics

Charges give rise to electric fields. Current give rise to magnetic fields. In this section, we will study the magnetic fields induced by steady currents. This means that we are again looking for time independent solutions to the Maxwell equations. We will also restrict to situations in which the charge density vanishes, so $\rho = 0$. We can then set $\mathbf{E} = 0$ and focus our attention only on the magnetic field. We're left with two Maxwell equations to solve:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{3.1}$$

and

$$\nabla \cdot \mathbf{B} = 0 \tag{3.2}$$

If you fix the current density $\mathbf{J}$, these equations have a unique solution. Our goal in this section is to find it.

Steady Currents

Before we solve (3.1) and (3.2), let's pause to think about the kind of currents that we're considering in this section. Because $\rho = 0$, there can't be any net charge. But, of course, we still want charge to be moving! This means that we necessarily have both positive and negative charges which balance out at all points in space. Nonetheless, these charges can move so there is a current even though there is no net charge transport.

This may sound artificial, but in fact it's exactly what happens in a typical wire. In that case, there is background of positive charge due to the lattice of ions in the metal. Meanwhile, the electrons are free to move. But they all move together so that at each point we still have $\rho = 0$. The continuity equation, which captures the conservation of electric charge, is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Since the charge density is unchanging (and, indeed, vanishing), we have

$$\nabla \cdot \mathbf{J} = 0$$

Mathematically, this is just saying that if a current flows into some region of space, an equal current must flow out to avoid the build up of charge. Note that this is consistent with (3.1) since, for any vector field, $\nabla \cdot (\nabla \times \mathbf{B}) = 0$.

3.1 Ampère's Law

The first equation of magnetostatics,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{3.3}$$

is known as *Ampère's* law. As with many of these vector differential equations, there is an equivalent form in terms of integrals. In this case, we choose some open surface S with boundary $C = \partial S$. Integrating (3.3) over the surface, we can use Stokes' theorem to turn the integral of $\nabla \times \mathbf{B}$ into a line integral over the boundary C,

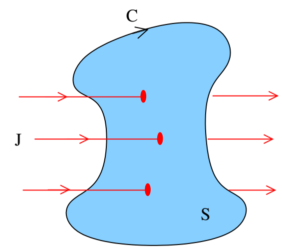


Figure 25:

$$\int_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}$$

Recall that there's an implicit orientation in these equations. The surface S comes with a normal vector $\hat{\mathbf{n}}$ which points away from S in one direction. The line integral around the boundary is then done in the right-handed sense, meaning that if you stick the thumb of your right hand in the direction $\hat{\mathbf{n}}$ then your fingers curl in the direction of the line integral.

The integral of the current density over the surface S is the same thing as the total current I that passes through S. Ampère's law in integral form then reads

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I \tag{3.4}$$

For most examples, this isn't sufficient to determine the form of the magnetic field; we'll usually need to invoke (3.2) as well. However, there is one simple example where symmetry considerations mean that (3.4) is all we need...

3.1.1 A Long Straight Wire

Consider an infinite, straight wire carrying current I. We'll take it to point in the $\hat{\mathbf{z}}$ direction. The symmetry of the problem is jumping up and down telling us that we need to use cylindrical polar coordinates, (r, φ, z), where $r = \sqrt{x^2 + y^2}$ is the radial distance away from the wire.

We take the open surface S to lie in the $x - y$ plane, centered on the wire. For the line integral in (3.4) to give something that doesn't vanish, it's clear that the magnetic field has to have some component that lies along the circumference of the disc.

But, by the symmetry of the problem, that's actually the only component that $\mathbf{B}$ can have: it must be of the form $\mathbf{B} = B(r)\hat{\boldsymbol{\varphi}}$. (If this was a bit too quick, we'll derive this more carefully below). Any magnetic field of this form automatically satisfies the second Maxwell equation $\nabla\cdot\mathbf{B} = 0$. We need only worry about Ampère's law which tells us

$$\oint_C \mathbf{B}\cdot d\mathbf{r} = B(r)\int_0^{2\pi} r\,d\varphi = 2\pi r B(r) = \mu_0 I$$

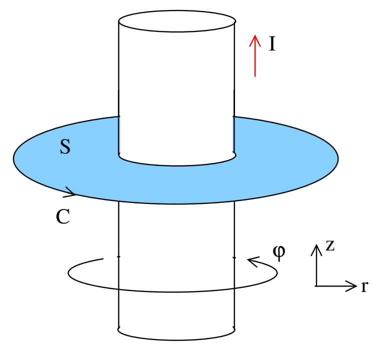


Figure 26:

We see that the strength of the magnetic field is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\varphi}} \tag{3.5}$$

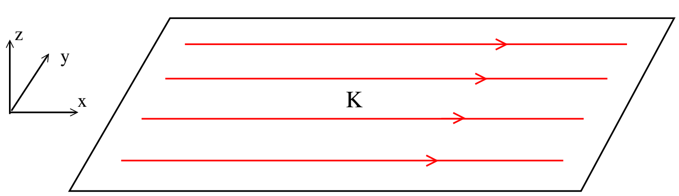
The magnetic field circles the wire using the "right-hand rule": stick the thumb of your right hand in the direction of the current and your fingers curl in the direction of the magnetic field.

Note that the simplest example of a magnetic field falls off as $1/r$. In contrast, the simplest example of an electric field – the point charge – falls of as $1/r^2$. You can trace this difference back to the geometry of the two situations. Because magnetic fields are sourced by currents, the simplest example is a straight line and the $1/r$ fall-off is because there are two transverse directions to the wire. Indeed, we saw in Section 2.1.3 that when we look at a line of charge, the electric field also drops off as $1/r$.

3.1.2 Surface Currents and Discontinuities

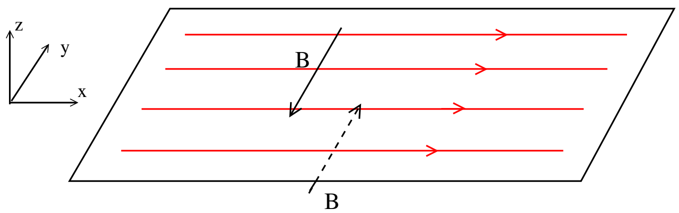
Consider the flat plane lying at $z = 0$ with a surface current density that we'll call $\mathbf{K}$. Note that $\mathbf{K}$ is the current per unit length, as opposed to $\mathbf{J}$ which is the current per unit area. You can think of the surface current as a bunch of wires, all lying parallel to each other.

We'll take the current to lie in the x-direction: $\mathbf{K} = K\hat{\mathbf{x}}$ as shown below.



From our previous result, we know that the $\mathbf{B}$ field should curl around the current in the right-handed sense. But, with an infinite number of wires, this can only mean that

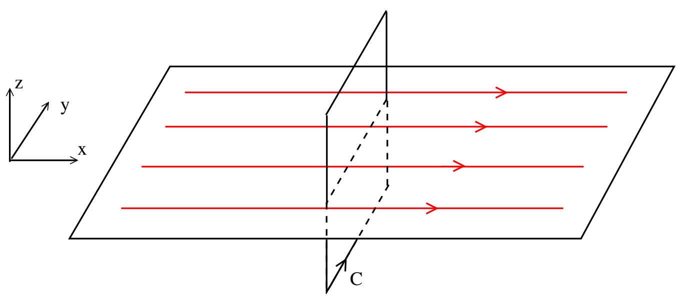
$\mathbf{B}$ is oriented along the $\mathbf{y}$ direction. In fact, from the symmetry of the problem, it must look like



with $\mathbf{B}$ pointing in the $-\hat{\mathbf{y}}$ direction when $z > 0$ and in the $+\hat{\mathbf{y}}$ direction when $z < 0$. We write

$$\mathbf{B} = -B(z)\hat{\mathbf{y}}$$

with $B(z) = -B(-z)$. We invoke Ampère's law using the following open surface:



with length L in the y direction and extending to $\pm z$. We have

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = LB(z) - LB(-z) = 2LB(z) = \mu_0 K L$$

so we find that the magnetic field is constant above an infinite plane of surface current

$$B(z) = \frac{\mu_0 K}{2} \qquad z > 0$$

This is rather similar to the case of the electric field in the presence of an infinite plane of surface charge.

The analogy with electrostatics continues. The magnetic field is not continuous across a plane of surface current. We have

$$B(z \to 0^+) - B(z \to 0^-) = \mu_0 K$$

In fact, this is a general result that holds for any surface current $\mathbf{K}$. We can prove this statement by using the same curve that we used in the Figure above and shrinking it until it barely touches the surface on both sides. If the normal to the surface is $\hat{\mathbf{n}}$ and $\mathbf{B}_\pm$ denotes the magnetic field on either side of the surface, then

$$\hat{\mathbf{n}} \times \mathbf{B}|_+ - \hat{\mathbf{n}} \times \mathbf{B}|_- = \mu_0 \mathbf{K} \tag{3.6}$$

Meanwhile, the magnetic field normal to the surface is continuous. (To see this, you can use a Gaussian pillbox, together with the other Maxwell equation $\nabla \cdot \mathbf{B} = 0$).

When we looked at electric fields, we saw that the normal component was discontinuous in the presence of surface charge (2.9) while the tangential component is continuous. For magnetic fields, it's the other way around: the tangential component is discontinuous in the presence of surface currents.

A Solenoid

A *solenoid* consists of a surface current that travels around a cylinder. It's simplest to think of a single current-carrying wire winding many times around the outside of the cylinder. (Strictly speaking, the cross-sectional shape of the solenoid doesn't have to be a circle – it can be anything. But we'll stick with a circle here for simplicity). To make life easy, we'll assume that the cylinder is infinitely long. This just means that we can neglect effects due to the ends.

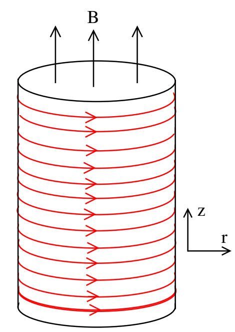


Figure 27:

We'll again use cylindrical polar coordinates, (r, φ, z), with the axis of the cylinder along $\hat{\mathbf{z}}$. By symmetry, we know that $\mathbf{B}$ will point along the z-axis. Its magnitude can depend only on the radial distance: $\mathbf{B} = B(r)\hat{\mathbf{z}}$. Once again, any magnetic field of this form immediately satisfies $\nabla \cdot \mathbf{B} = 0$.

We solve Ampère's law in differential form. Anywhere other than the surface of the solenoid, we have $\mathbf{J} = 0$ and

$$\nabla \times \mathbf{B} = 0 \quad \Rightarrow \quad \frac{dB}{dr} = 0 \quad \Rightarrow \quad B(r) = \text{constant}$$

Outside the solenoid, we must have $B(r) = 0$ since $B(r)$ is constant and we know $B(r) \to 0$ as $r \to \infty$. To figure out the magnetic field inside the solenoid, we turn to the integral form of Ampère's law and consider the surface S, bounded by the curve C shown in the figure. Only the line that runs inside the solenoid contributes to the line integral. We have

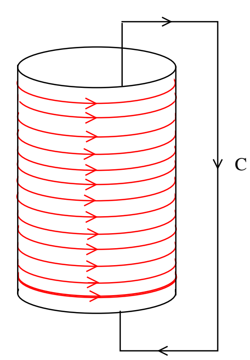


Figure 28:

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = BL = \mu_0 INL$$

where N is the number of windings of wire per unit length. We learn that inside the solenoid, the constant magnetic field is given by

$$\mathbf{B} = \mu_0 I N \,\hat{\mathbf{z}} \tag{3.7}$$

Note that, since $K = IN$, this is consistent with our general formula for the discontinuity of the magnetic field in the presence of surface currents (3.6).

3.2 The Vector Potential

For the simple current distributions of the last section, symmetry considerations were enough to lead us to a magnetic field which automatically satisfied

$$\nabla \cdot \mathbf{B} = 0 \tag{3.8}$$

But, for more general currents, this won't be the case. Instead we have to ensure that the second magnetostatic Maxwell equation is also satisfied.

In fact, this is simple to do. We are guaranteed a solution to $\nabla \cdot \mathbf{B} = 0$ if we write the magnetic field as the curl of some vector field,

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3.9}$$

Here $\mathbf{A}$ is called the *vector potential*. While magnetic fields that can be written in the form (3.9) certainly satisfy $\nabla \cdot \mathbf{B} = 0$, the converse is also true; any divergence-free magnetic field can be written as (3.9) for some $\mathbf{A}$.

(Actually, this previous sentence is only true if our space has a suitably simple topology. Since we nearly always think of space as $\mathbf{R}^3$ or some open ball on $\mathbf{R}^3$, we rarely run into subtleties. But if space becomes more interesting then the possible solutions to $\nabla \cdot \mathbf{B} = 0$ also become more interesting. This is analogous to the story of the electrostatic potential that we mentioned briefly in Section 2.2).

Using the expression (3.9), Ampère's law becomes

$$\nabla \times \mathbf{B} = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \mu_0 \mathbf{J} \tag{3.10}$$

where, in the first equality, we've used a standard identity from Vector Calculus. This is the equation that we have to solve to determine $\mathbf{A}$ and, through that, $\mathbf{B}$.

3.2.1 Magnetic Monopoles

Above, we dispatched with the Maxwell equation $\nabla \cdot \mathbf{B} = 0$ fairly quickly by writing $\mathbf{B} = \nabla \times \mathbf{A}$. But we never paused to think about what this equation is actually telling us. In fact, it has a very simple interpretation: it says that there are no magnetic charges. A point-like magnetic charge g would source the magnetic field, giving rise a $1/r^2$ fall-off

$$\mathbf{B} = \frac{g\hat{\mathbf{r}}}{4\pi r^2}$$

An object with this behaviour is usually called a *magnetic monopole*. Maxwell's equations says that they don't exist. And we have never found one in Nature.

However, we could ask: how robust is this conclusion? Are we sure that magnetic monopoles don't exist? After all, it's easy to adapt Maxwell's equations to allow for presence of magnetic charges: we simply need to change (3.8) to read $\nabla \cdot \mathbf{B} = \rho_m$ where ρ_m is the magnetic charge distribution. Of course, this means that we no longer get to use the vector potential $\mathbf{A}$. But is that such a big deal?

The twist comes when we turn to quantum mechanics. Because in quantum mechanics we're *obliged* to use the vector potential $\mathbf{A}$. Not only is the whole framework of electromagnetism in quantum mechanics based on writing things using $\mathbf{A}$, but it turns out that there are experiments that actually detect certain properties of $\mathbf{A}$ that are lost when we compute $\mathbf{B} = \nabla \times \mathbf{A}$. I won't explain the details here, but if you're interested then look up the "Aharonov-Bohm effect" in the lectures on Solid State Physics.

Monopoles After All?

To summarise, magnetic monopoles have never been observed. We have a law of physics (3.8) which says that they don't exist. And when we turn to quantum mechanics we need to use the vector potential $\mathbf{A}$ which automatically means that (3.8) is true. It sounds like we should pretty much forget about magnetic monopoles, right?

Well, no. There are actually very good reasons to suspect that magnetic monopoles do exist. The most important part of the story is due to Dirac. He gave a beautiful argument which showed that it is in fact possible to introduce a vector potential $\mathbf{A}$ which allows for the presence of magnetic charge, but only if the magnetic charge g is related to the charge of the electron e by

$$ge = 2\pi\hbar n \qquad n \in \mathbf{Z} \tag{3.11}$$

This is known as the *Dirac quantization condition*.

Moreover, following work in the 1970s by 't Hooft and Polyakov, we now realise that magnetic monopoles are ubiquitous in theories of particle physics. Our best current theory – the Standard Model – does not predict magnetic monopoles. But every theory that tries to go beyond the Standard Model, whether Grand Unified Theories, or String Theory or whatever, always ends up predicting that magnetic monopoles should exist. They're one of the few predictions for new physics that nearly all theories agree upon.

These days most theoretical physicists think that magnetic monopoles probably exist and there have been a number of experiments around the world designed to detect them. However, while theoretically monopoles seem like a good bet, their future observational status is far from certain. We don't know how heavy magnetic monopoles will be, but all evidence suggests that producing monopoles is beyond the capabilities of our current (or, indeed, future) particle accelerators. Our only hope is to discover some that Nature made for us, presumably when the Universe was much younger. Unfortunately, here too things seem against us. Our best theories of cosmology, in particular inflation, suggest that any monopoles that were created back in the Big Bang have long ago been diluted. At a guess, there are probably only a few floating around our entire observable Universe. The chances of one falling into our laps seem slim. But I hope I'm wrong.

3.2.2 Gauge Transformations

The choice of $\mathbf{A}$ in (3.9) is far from unique: there are lots of different vector potentials $\mathbf{A}$ that all give rise to the same magnetic field $\mathbf{B}$. This is because the curl of a gradient is automatically zero. This means that we can always add any vector potential of the form $\nabla\chi$ for some function χ and the magnetic field remains the same,

$$\mathbf{A}' = \mathbf{A} + \nabla\chi \quad \Rightarrow \quad \nabla \times \mathbf{A}' = \nabla \times \mathbf{A}$$

Such a change of $\mathbf{A}$ is called a *gauge transformation*. As we will see in Section 5.3.1, it is closely tied to the possible shifts of the electrostatic potential ϕ. Ultimately, such gauge transformations play a key role in theoretical physics. But, for now, we're simply going to use this to our advantage. Because, by picking a cunning choice of χ, it's possible to simplify our quest for the magnetic field.

Claim: We can always find a gauge transformation χ such that $\mathbf{A}'$ satisfies $\nabla\cdot\mathbf{A}' = 0$. Making this choice is usually referred to as *Coulomb gauge*.

Proof: Suppose that we've found some $\mathbf{A}$ which gives us the magnetic field that we want, so $\nabla \times \mathbf{A} = \mathbf{B}$, but when we take the divergence we get some function $\nabla \cdot \mathbf{A} = \psi(\mathbf{x})$. We instead choose $\mathbf{A}' = \mathbf{A} + \nabla\chi$ which now has divergence

$$\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} + \nabla^2\chi = \psi + \nabla^2\chi$$

So if we want $\nabla \cdot \mathbf{A}' = 0$, we just have to pick our gauge transformation χ to obey

$$\nabla^2 \chi = -\psi$$

But this is just the Poisson equation again. And we know from our discussion in Section 2 that there is always a solution. (For example, we can write it down in integral form using the Green's function). □

Something a Little Misleading: The Magnetic Scalar Potential

There is another quantity that is sometimes used called the *magnetic scalar potential*, Ω. The idea behind this potential is that you might be interested in computing the magnetic field in a region where there are no currents and the electric field is not changing with time. In this case, you need to solve $\nabla \times \mathbf{B} = 0$, which you can do by writing

$$\mathbf{B} = -\nabla \Omega$$

Now calculations involving the magnetic field really do look identical to those involving the electric field.

However, you should be wary of writing the magnetic field in this way. As we'll see in more detail in Section 5.3.1, we can *always* solve two of Maxwell's equations by writing $\mathbf{E}$ and $\mathbf{B}$ in terms of the electric potential ϕ and vector potential $\mathbf{A}$ and this formulation becomes important as we move onto more advanced areas of physics. In contrast, writing $\mathbf{B} = -\nabla \Omega$ is only useful in a limited number of situations. The reason for this really gets to the heart of the difference between electric and magnetic fields: electric charges exist; magnetic charges don't!

3.2.3 Biot-Savart Law

We're now going to use the vector potential to solve for the magnetic field $\mathbf{B}$ in the presence of a general current distribution. From now, we'll always assume that we're working in Coulomb gauge and our vector potential obeys $\nabla \cdot \mathbf{A} = 0$. Then Ampère's law (3.10) becomes a whole lot easier: we just have to solve

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{3.12}$$

But this is just something that we've seen already. To see why, it's perhaps best to write it out in Cartesian coordinates. This then becomes three equations,

$$\nabla^2 A_i = -\mu_0 J_i \quad (i = 1, 2, 3) \tag{3.13}$$

and each of these is the Poisson equation.

It's worth giving a word of warning at this point: the expression $\nabla^2 \mathbf{A}$ is simple in Cartesian coordinates where, as we've seen above, it reduces to the Laplacian on each component. But, in other coordinate systems, this is no longer true. The Laplacian now also acts on the basis vectors such as $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$. So in these other coordinate systems, $\nabla^2 \mathbf{A}$ is a little more of a mess. (You should probably use the identity $\nabla^2 \mathbf{A} = -\nabla \times (\nabla \times \mathbf{A}) + \nabla(\nabla \cdot \mathbf{A})$ if you really want to compute in these other coordinate systems).

Anyway, if we stick to Cartesian coordinates then everything is simple. In fact, the resulting equations (3.13) are of exactly the same form that we had to solve in electrostatics. And, in analogy to (2.21), we know how to write down the most general solution using Green's functions. It is

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V d^3x' \, \frac{J_i(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

Or, if you're feeling bold, you can revert back to vector notation and write

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{3.14}$$

where you've just got to remember that the vector index on $\mathbf{A}$ links up with that on $\mathbf{J}$ (and not on $\mathbf{x}$ or $\mathbf{x}'$).

Checking Coulomb Gauge

We've derived a solution to (3.12), but this is only a solution to Ampère's equation (3.10) if the resulting $\mathbf{A}$ obeys the Coulomb gauge condition, $\nabla \cdot \mathbf{A} = 0$. Let's now check that it does. We have

$$\nabla \cdot \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V d^3x' \, \nabla \cdot \left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right)$$

where you need to remember that the index of ∇ is dotted with the index of $\mathbf{J}$, but the derivative in ∇ is acting on $\mathbf{x}$, not on $\mathbf{x}'$. We can write

$$\begin{aligned} \nabla \cdot \mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \int_V d^3x' \, \mathbf{J}(\mathbf{x}') \cdot \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= -\frac{\mu_0}{4\pi} \int_V d^3x' \, \mathbf{J}(\mathbf{x}') \cdot \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \end{aligned}$$

Here we've done something clever. Now our ∇' is differentiating with respect to $\mathbf{x}'$. To get this, we've used the fact that if you differentiate $1/|\mathbf{x} - \mathbf{x}'|$ with respect to $\mathbf{x}$ then you get the negative of the result from differentiating with respect to $\mathbf{x}'$. But since ∇' sits inside an $\int d^3x'$ integral, it's ripe for integrating by parts. This gives

$$\nabla \cdot \mathbf{A}(\mathbf{x}) = -\frac{\mu_0}{4\pi}\int_V d^3x' \, \left[\nabla' \cdot \left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) - \nabla' \cdot \mathbf{J}(\mathbf{x}') \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right]$$

The second term vanishes because we're dealing with steady currents obeying $\nabla \cdot \mathbf{J} = 0$. The first term also vanishes if we take the current to be localised in some region of space, $\hat{V} \subset V$ so that $\mathbf{J}(\mathbf{x}) = 0$ on the boundary ∂V. We'll assume that this is the case. We conclude that

$$\nabla \cdot \mathbf{A} = 0$$

and (3.14) is indeed the general solution to the Maxwell equations (3.1) and (3.2) as we'd hoped.

The Magnetic Field

From the solution (3.14), it is simple to compute the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. Again, we need to remember that the ∇ acts on the $\mathbf{x}$ in (3.14) rather than the $\mathbf{x}'$. We find

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V d^3x' \, \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \tag{3.15}$$

This is known as the *Biot-Savart law*. It describes the magnetic field due to a general current density.

There is a slight variation on (3.15) which more often goes by the name of the Biot-Savart law. This arises if the current is restricted to a thin wire which traces out a curve C. Then, for a current density $\mathbf{J}$ passing through a small volume δV, we write $\mathbf{J}\delta V = (JA)\delta\mathbf{x}$ where A is the cross-sectional area of the wire and δx lies tangent to C. Assuming that the cross-sectional area is constant throughout the wire, the current $I = JA$ is also constant. The Biot-Savart law becomes

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\int_C \frac{d\mathbf{x}' \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \tag{3.16}$$

This describes the magnetic field due to the current I in the wire.

An Example: The Straight Wire Revisited

Of course, we already derived the answer for a straight wire in (3.5) without using this fancy vector potential technology. Before proceeding, we should quickly check that the Biot-Savart law reproduces our earlier result. As before, we'll work in cylindrical polar coordinates. We take the wire to point along the $\hat{\mathbf{z}}$ axis and use $r^2 = x^2 + y^2$ as our radial coordinate. This means that the line element along the wire is parametrised by $d\mathbf{x}' = \hat{\mathbf{z}}dz$ and, for a point $\mathbf{x}$ away from the wire, the vector $d\mathbf{x}' \times (\mathbf{x}-\mathbf{x}')$ points along the tangent to the circle of radius r,

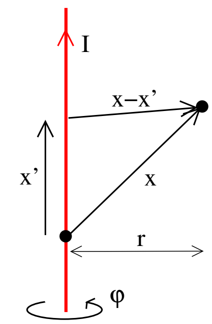


Figure 29:

$$d\mathbf{x}' \times (\mathbf{x} - \mathbf{x}') = r\hat{\boldsymbol{\varphi}}\, dz$$

So we have

$$\mathbf{B} = \frac{\mu_0 I \hat{\boldsymbol{\varphi}}}{4\pi} \int_{-\infty}^{+\infty} dz\ \frac{r}{(r^2 + z^2)^{3/2}} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\varphi}}$$

which is the same result we found earlier (3.5).

3.2.4 A Mathematical Diversion: The Linking Number

There's a rather cute application of these ideas to pure mathematics. Consider two closed, non-intersecting curves, C and C', in $\mathbf{R}^3$. For each pair of curves, there is an integer $n \in Z$ called the *linking number* which tells you how many times one of the curves winds around the other. For example, here are pairs of curves with linking number $|n| = 0, 1$ and 2.

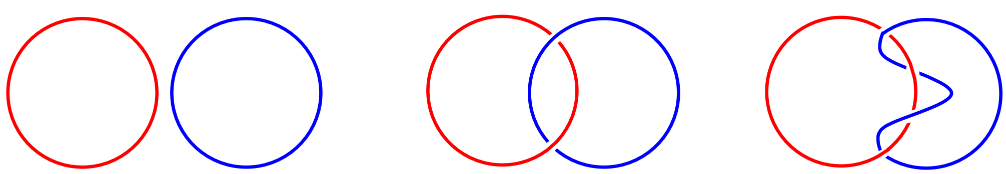


Figure 30: Curves with linking number $n = 0$, $n = 1$ and $n = 2$.

To determine the sign of the linking number, we need to specify the orientation of each curve. In the last two figures above, the linking numbers are negative, if we traverse both red and blue curves in the same direction. The linking numbers are positive if we traverse one curve in a clockwise direction, and the other in an anti-clockwise direction.

Importantly, the linking number doesn't change as you deform either curve, provided that the two curves never cross. In fancy language, the linking number is an example of a topological invariant.

There is an integral expression for the linking number, first written down by Gauss during his exploration of electromagnetism. The Biot-Savart formula (3.16) offers a simple physics derivation of Gauss' expression. Suppose that the curve C carries a current I. This sets us a magnetic field everywhere in space. We will then compute $\oint_{C'} \mathbf{B} \cdot d\mathbf{x}'$ around another curve C. (If you want a justification for computing $\oint_{C'} \mathbf{B} \cdot d\mathbf{x}'$ then you can think of it as the work done when transporting a magnetic monopole of unit charge around C, but this interpretation isn't necessary for what follows.) The Biot-Savart formula gives

$$\oint_{C'} \mathbf{B}(\mathbf{x}') \cdot d\mathbf{x}' = \frac{\mu_0 I}{4\pi} \oint_{C'} d\mathbf{x}' \cdot \oint_C \frac{d\mathbf{x} \times (\mathbf{x}' - \mathbf{x})}{|\mathbf{x} - \mathbf{x}'|^3}$$

where we've changed our conventions somewhat from (3.16): now $\mathbf{x}$ labels coordinates on C while $\mathbf{x}'$ labels coordinates on C'.

Meanwhile, we can also use Stokes' theorem, followed by Ampère's law, to write

$$\oint_{C'} \mathbf{B}(\mathbf{x}') \cdot d\mathbf{x}' = \int_{S'} (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \mu_0 \int_{S'} \mathbf{J} \cdot d\mathbf{S}$$

where S' is a surface bounded by C'. The current is carried by the other curve, C, which pierces S' precisely n times, so that

$$\oint_{C'} \mathbf{B}(\mathbf{x}') \cdot d\mathbf{x}' = \mu_0 \int_{S'} \mathbf{J} \cdot d\mathbf{S} = n\mu_0 I$$

Comparing the two equations above, we arrive at Gauss' double-line integral expression for the linking number n,

$$n = \frac{1}{4\pi} \oint_{C'} d\mathbf{x}' \cdot \oint_C \frac{d\mathbf{x} \times (\mathbf{x}' - \mathbf{x})}{|\mathbf{x} - \mathbf{x}'|^3} \tag{3.17}$$

Note that our final expression is symmetric in C and C', even though these two curves played a rather different physical role in the original definition, with C carrying a current, and C' the path traced by some hypothetical monopole. To see that the expression is indeed symmetric, note that the triple product can be thought of as the determinant $\det(\mathbf{x}', \mathbf{x}, \mathbf{x}' - \mathbf{x})$. Swapping $\mathbf{x}$ and $\mathbf{x}'$ changes the order of the first two vectors and changes the sign of the third, leaving the determinant unaffected.

The formula (3.17) is rather pretty. It's not at all obvious that the right-hand-side doesn't change under (non-crossing) deformations of C and C'; nor is it obvious that the right-hand-side must give an integer. Yet both are true, as the derivation above shows. This is the first time that ideas of topology sneak into physics. It's not the last.

3.3 Magnetic Dipoles

We've seen that the Maxwell equations forbid magnetic monopoles with a long-range $B \sim 1/r^2$ fall-off (3.11). So what is the generic fall-off for some distribution of currents which are localised in a region of space? In this section we will see that, if you're standing suitably far from the currents, you'll typically observe a dipole-like magnetic field.

3.3.1 A Current Loop

We start with a specific, simple example. Consider a circular loop of wire C of radius R carrying a current I. We can guess what the magnetic field looks like simply by patching together our result for straight wires: it must roughly take the shape shown in the figure However, we can be more accurate. Here we restrict ourselves only to the magnetic field far from the loop.

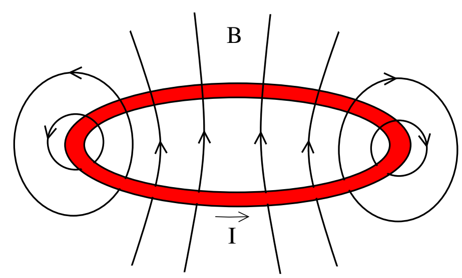


Figure 31:

To compute the magnetic field far away, we won't start with the Biot-Savart law but instead return to the original expression for $\mathbf{A}$ given in (3.14). We're going to return to the notation in which a point in space is labelled as $\mathbf{r}$ rather than $\mathbf{x}$. (This is more appropriate for long-distance distance fields which are essentially an expansion in $r = |\mathbf{r}|$). The vector potential is then given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V d^3r' \; \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Writing this in terms of the current I (rather than the current density $\mathbf{J}$), we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

We want to ask what this looks like far from the loop. Just as we did for the electrostatic potential, we can Taylor expand the integrand using (2.22),

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \ldots$$

So that

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint_C d\mathbf{r}' \left(\frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \ldots \right) \tag{3.18}$$

The first term in this expansion vanishes because we're integrating around a circle. This is just a reflection of the fact that there are no magnetic monopoles. For the second term, there's a way to write it in slightly more manageable form. To see this, let's introduce an arbitrary constant vector $\mathbf{g}$ and use this to look at

$$\oint_C d\mathbf{r}' \cdot \mathbf{g}\,(\mathbf{r}\cdot\mathbf{r}')$$

Recall that, from the point of view of this integral, both $\mathbf{g}$ and $\mathbf{r}$ are constant vectors; it's the vector $\mathbf{r}'$ that we're integrating over. This is now the kind of line integral of a vector that allows us to use Stokes' theorem. We have

$$\oint_C d\mathbf{r}' \cdot \mathbf{g}\,(\mathbf{r}\cdot\mathbf{r}') = \int_S d\mathbf{S}\cdot\nabla\times(\mathbf{g}\,(\mathbf{r}\cdot\mathbf{r}')) = \int_S dS_i\,\epsilon_{ijk}\partial'_j(g_k r_l r'_l)$$

where, in the final equality, we've resorted to index notation to help us remember what's connected to what. Now the derivative ∂' acts only on the r' and we get

$$\oint_C d\mathbf{r}' \cdot \mathbf{g}\,(\mathbf{r}\cdot\mathbf{r}') = \int_S dS_i\,\epsilon_{ijk} g_k r_j = \mathbf{g}\cdot\int_S d\mathbf{S}\times\mathbf{r}$$

But this is true for all constant vectors $\mathbf{g}$ which means that it must also hold as a vector identity once we strip away $\mathbf{g}$. We have

$$\oint_C d\mathbf{r}'\,(\mathbf{r}\cdot\mathbf{r}') = \boldsymbol{\mathcal{S}}\times\mathbf{r}$$

where we've introduced the vector area $\boldsymbol{\mathcal{S}}$ of the surface S bounded by C, defined as

$$\boldsymbol{\mathcal{S}} = \int_S d\mathbf{S}$$

If the boundary C lies in a plane – as it does for us – then the vector $\boldsymbol{\mathcal{S}}$ points out of the plane.

Now let's apply this result to our vector potential (3.18). With the integral over $\mathbf{r}'$, we can treat $\mathbf{r}$ as the constant vector $\mathbf{g}$ that we introduced in the lemma. With the first term vanishing, we're left with

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3} \tag{3.19}$$

where we've introduced the *magnetic dipole moment*

$$\mathbf{m} = I\boldsymbol{\mathcal{S}}$$

This is our final, simple, answer for the long-range behaviour of the vector potential due to a current loop. It remains only to compute the magnetic field. A little algebra gives

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}\right) \tag{3.20}$$

Now we see why $\mathbf{m}$ is called the magnetic dipole; this form of the magnetic field is exactly the same as the dipole electric field (2.19).

I stress that the $\mathbf{B}$ field due to a current loop and $\mathbf{E}$ field due to two charges don't look the same close up. But they have identical "dipole" long-range fall-offs.

3.3.2 General Current Distributions

We can now perform the same kind of expansion for a general current distribution $\mathbf{J}$ localised within some region of space. We use the Taylor expansion (2.22) in the general form of the vector potential (3.14),

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\ \frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0}{4\pi}\int d^3r'\ \left(\frac{J_i(\mathbf{r}')}{r} + \frac{J_i(\mathbf{r}')\,(\mathbf{r}\cdot\mathbf{r}')}{r^3} + \ldots\right) \tag{3.21}$$

where we're using a combination of vector and index notation to help remember how the indices on the left and right-hand sides match up.

The first term above vanishes. Heuristically, this is because currents can't stop and end, they have to go around in loops. This means that the contribution from one part must be cancelled by the current somewhere else. To see this mathematically, we use the slightly odd identity

$$\partial_j(J_j r_i) = (\partial_j J_j)\, r_i + J_i = J_i \tag{3.22}$$

where the last equality follows from the continuity condition $\nabla\cdot\mathbf{J} = 0$. Using this, we see that the first term in (3.21) is a total derivative (of $\partial/\partial r'_i$ rather than $\partial/\partial r_i$) which vanishes if we take the integral over $\mathbf{R}^3$ and keep the current localised within some interior region.

For the second term in (3.21) we use a similar trick, now with the identity

$$\partial_j(J_j r_i r_k) = (\partial_j J_j) r_i r_k + J_i r_k + J_k r_i = J_i r_k + J_k r_i$$

Because $\mathbf{J}$ in (3.21) is a function of $\mathbf{r}'$, we actually need to apply this trick to the $J_i r'_j$ terms in the expression. We once again abandon the boundary term to infinity.

Dropping the argument of $\mathbf{J}$, we can use the identity above to write the relevant piece of the second term as

$$\int d^3r'\; J_i\, r_j r'_j = \int d^3r'\; \frac{r_j}{2}(J_i r'_j - J_j r'_i) = \int d^3r'\; \frac{1}{2}\left(J_i\,(\mathbf{r}\cdot\mathbf{r}') - r'_i(\mathbf{J}\cdot\mathbf{r})\right)$$

But now this is in a form that is ripe for the vector product identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$. This means that we can rewrite this term as

$$\int d^3r'\; \mathbf{J}\,(\mathbf{r}\cdot\mathbf{r}') = \frac{1}{2}\mathbf{r}\times\int d^3r'\; \mathbf{J}\times\mathbf{r}' \tag{3.23}$$

With this in hand, we see that the long distance fall-off of any current distribution again takes the dipole form (3.19)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}$$

now with the magnetic dipole moment given by the integral,

$$\mathbf{m} = \frac{1}{2}\int d^3r'\; \mathbf{r}'\times\mathbf{J}(\mathbf{r}') \tag{3.24}$$

Just as in the electric case, the multipole expansion continues to higher terms. This time you need to use vector spherical harmonics. Just as in the electric case, if you want further details then look in Jackson.

3.4 Magnetic Forces

We've seen that a current produces a magnetic field. But a current is simply moving charge. And we know from the Lorentz force law that a charge q moving with velocity $\mathbf{v}$ will experience a force

$$\mathbf{F} = q\mathbf{v}\times\mathbf{B}$$

This means that if a second current is placed somewhere in the neighbourhood of the first, then they will exert a force on one another. Our goal in this section is to figure out this force.

3.4.1 Force Between Currents

Let's start simple. Take two parallel wires carrying currents I_1 and I_2 respectively. We'll place them a distance d apart in the x direction.

The current in the first wire sets up a magnetic field (3.5). So if the charges in the second wire are moving with velocity $\mathbf{v}$, they will each experience a force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = q\mathbf{v} \times \left(\frac{\mu_0 I_1}{2\pi d}\right)\hat{\mathbf{y}}$$

where $\hat{\mathbf{y}}$ is the direction of the magnetic field experienced by the second wire as shown in the Figure. The next step is to write the velocity $\mathbf{v}$ in terms of the current $\mathbf{I}_2$ in the second wire. We did this in Section 1.1 when we first introduced the idea of currents: if there's a density n of these particles and each carries charge q, then the current density is

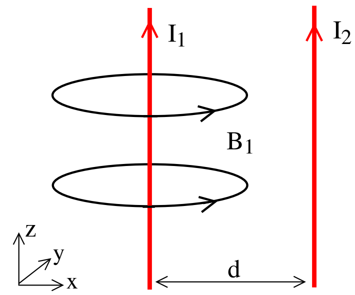


Figure 32:

$$\mathbf{J}_2 = nq\mathbf{v}$$

For a wire with cross-sectional area A, the total current is just $I_2 = J_2 A$. For our set-up, $\mathbf{J}_2 = J_2\hat{\mathbf{z}}$.

Finally, we want to compute the force on the wire per unit length, $\mathbf{f}$. Since the number of charges per unit length is nA and $\mathbf{F}$ is the force on each charge, we have

$$\mathbf{f} = nA\mathbf{F} = \left(\frac{\mu_0 I_1 I_2}{2\pi d}\right)\hat{\mathbf{z}} \times \hat{\mathbf{y}} = -\left(\frac{\mu_0 I_1 I_2}{2\pi d}\right)\hat{\mathbf{x}} \tag{3.25}$$

This is our answer for the force between two parallel wires. If the two currents are in the same direction, so that $I_1 I_2 > 0$, the overall minus sign means that the force between two wires is attractive. For currents in opposite directions, with $I_1 I_2 < 0$, the force is repulsive.

The General Force Between Currents

We can extend our discussion to the force experienced between two current distributions $\mathbf{J}_1$ and $\mathbf{J}_2$. We start by considering the magnetic field $\mathbf{B}(\mathbf{r})$ due to the first current $\mathbf{J}_1$. As we've seen, the Biot-Savart law (3.15) tells us that this can be written as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\ \frac{\mathbf{J}_1(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

If the current $\mathbf{J}_1$ is localised on a curve C_1, then we can replace this volume integral with the line integral (3.16)

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I_1}{4\pi}\oint_{C_1}\frac{d\mathbf{r}_1 \times (\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3}$$

Now we place a second current distribution $\mathbf{J}_2$ in this magnetic field. It experiences a force per unit area given by (1.3), so the total force is

$$\mathbf{F} = \int d^3r \ \mathbf{J}_2(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) \tag{3.26}$$

Again, if the current $\mathbf{J}_2$ is restricted to lie on a curve C_2, then this volume integral can be replaced by the line integral

$$\mathbf{F} = I_2 \oint_{C_2} d\mathbf{r} \times \mathbf{B}(\mathbf{r})$$

and the force can now be expressed as a double line integral,

$$\mathbf{F} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} d\mathbf{r}_2 \times \left(d\mathbf{r}_1 \times \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \right)$$

In general, this integral will be quite tricky to perform. However, if the currents are localised, and well-separated, there is a somewhat better approach where the force can be expressed purely in terms of the dipole moment of the current.

3.4.2 Force and Energy for a Dipole

We start by asking a slightly different question. We'll forget about the second current and just focus on the first: call it $\mathbf{J}(\mathbf{r})$. We'll place this current distribution in a magnetic field $\mathbf{B}(\mathbf{r})$ and ask: what force does it feel?

In general, there will be two kinds of forces. There will be a force on the centre of mass of the current distribution, which will make it move. There will also be a torque on the current distribution, which will want to make it re-orient itself with respect to the magnetic field. Here we're going to focus on the former. Rather remarkably, we'll see that we get the answer to the latter for free!

The Lorentz force experienced by the current distribution is

$$\mathbf{F} = \int_V d^3r \ \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})$$

We're going to assume that the current is localised in some small region $\mathbf{r} = \mathbf{R}$ and that the magnetic field $\mathbf{B}$ varies only slowly in this region. This allows us to Taylor expand

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{R}) + (\mathbf{r} \cdot \nabla)\mathbf{B}(\mathbf{R}) + \ldots$$

We then get the expression for the force

$$\mathbf{F} = -\mathbf{B}(\mathbf{R}) \times \int_V d^3r \; \mathbf{J}(\mathbf{r}) + \int_V d^3r \; \mathbf{J}(\mathbf{r}) \times [(\mathbf{r} \cdot \nabla)\mathbf{B}(\mathbf{R})] + \ldots$$

The first term vanishes because the currents have to go around in loops; we've already seen a proof of this following equation (3.21). We're going to do some fiddly manipulations with the second term. To help us remember that the derivative ∇ is acting on $\mathbf{B}$, which is then evaluated at $\mathbf{R}$, we'll introduce a dummy variable $\mathbf{r}'$ and write the force as

$$\mathbf{F} = \int_V d^3r \; \mathbf{J}(\mathbf{r}) \times [(\mathbf{r} \cdot \nabla')\mathbf{B}(\mathbf{r}')]\Big|_{\mathbf{r}'=\mathbf{R}} \tag{3.27}$$

Now we want to play around with this. First, using the fact that $\nabla \times \mathbf{B} = 0$ in the vicinity of the second current, we're going to show, that we can rewrite the integrand as

$$\mathbf{J}(\mathbf{r}) \times [(\mathbf{r} \cdot \nabla')\mathbf{B}(\mathbf{r}')] = -\nabla' \times [(\mathbf{r} \cdot \mathbf{B}(\mathbf{r}'))\mathbf{J}(\mathbf{r})]$$

To see why this is true, it's simplest to rewrite it in index notation. After shuffling a couple of indices, what we want to show is:

$$\epsilon_{ijk} J_j(r) \, r_l \, \partial'_l B_k(r') = \epsilon_{ijk} J_j(r) \, r_l \, \partial'_k B_l(r')$$

Or, subtracting one from the other,

$$\epsilon_{ijk} J_j(r) \, r_l \, \left(\partial'_l B_k(r') - \partial'_k B_l(r')\right) = 0$$

But the terms in the brackets are the components of $\nabla \times \mathbf{B}$ and so vanish. So our result is true and we can rewrite the force (3.27) as

$$\mathbf{F} = -\nabla' \times \int_V d^3r \; (\mathbf{r} \cdot \mathbf{B}(\mathbf{r}')) \, \mathbf{J}(\mathbf{r})\Big|_{\mathbf{r}'=\mathbf{R}}$$

Now we need to manipulate this a little more. We make use of the identity (3.23) where we replace the constant vector by $\mathbf{B}$. Thus, up to some relabelling, (3.23) is the same as

$$\int_V d^3r \; (\mathbf{B} \cdot \mathbf{r})\mathbf{J} = \frac{1}{2}\mathbf{B} \times \int_V d^3r \; \mathbf{J} \times \mathbf{r} = -\mathbf{B} \times \mathbf{m}$$

where $\mathbf{m}$ is the magnetic dipole moment of the current distribution. Suddenly, our expression for the force is looking much nicer: it reads

$$\mathbf{F} = \nabla \times (\mathbf{B} \times \mathbf{m})$$

where we've dropped the $\mathbf{r}' = \mathbf{R}$ notation because, having lost the integral, there's no cause for confusion: the magnetic dipole $\mathbf{m}$ is a constant, while $\mathbf{B}$ varies in space. Now we invoke a standard vector product identity. Using $\nabla \cdot \mathbf{B} = 0$, this simplifies and we're left with a simple expression for the force on a dipole

$$\mathbf{F} = \nabla(\mathbf{B} \cdot \mathbf{m}) \tag{3.28}$$

After all that work, we're left with something remarkably simple. Moreover, like many forces in Newtonian mechanics, it can be written as the gradient of a function. This function, of course, is the energy U of the dipole in the magnetic field,

$$U = -\mathbf{B} \cdot \mathbf{m} \tag{3.29}$$

This is an important expression that will play a role in later courses in Quantum Mechanics and Statistical Physics. For now, we'll just highlight something clever: we derived (3.29) by considering the force on the centre of mass of the current. This is related to how U depends on $\mathbf{r}$. But our final expression also tells us how the energy depends on the orientation of the dipole $\mathbf{m}$ at fixed position. This is related to the torque. Computing the force gives us the torque for free. This is because, ultimately, both quantities are derived from the underlying energy.

The Force Between Dipoles

As a particular example of the force (3.28), consider the case where the magnetic field is set up by a dipole $\mathbf{m}_1$. We know that the resulting long-distance magnetic field is (3.24),

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_1}{r^3}\right) \tag{3.30}$$

Now we'll consider how this affects the second dipole $\mathbf{m} = \mathbf{m}_2$. From (3.28), we have

$$\mathbf{F} = \frac{\mu_0}{4\pi}\nabla\left(\frac{3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}}) - \mathbf{m}_1 \cdot \mathbf{m}_2}{r^3}\right)$$

where $\mathbf{r}$ is the vector from $\mathbf{m}_1$ to $\mathbf{m}_2$. Note that the structure of the force is identical to that between two electric dipoles in (2.30). This is particularly pleasing because we used two rather different methods to calculate these forces. If we act with the derivative, we have

$$\mathbf{F} = \frac{3\mu_0}{4\pi r^4}\left[(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\mathbf{m}_2 + (\mathbf{m}_2 \cdot \hat{\mathbf{r}})\mathbf{m}_1 + (\mathbf{m}_1 \cdot \mathbf{m}_2)\hat{\mathbf{r}} - 5(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}\right] \tag{3.31}$$

First note that if we swap $\mathbf{m}_1$ and $\mathbf{m}_2$, so that we also send $\mathbf{r} \to -\mathbf{r}$, then the force swaps sign. This is a manifestation of Newton's third law: every action has an equal and opposite reaction. Recall from Dynamics and Relativity lectures that we needed Newton's third law to prove the conservation of momentum of a collection of particles. We see that this holds for a bunch of dipoles in a magnetic field.

But there was also a second part to Newton's third law: to prove the conservation of angular momentum of a collection of particles, we needed the force to lie parallel to the separation of the two particles. And this is *not* true for the force (3.31). If you set up a collection of dipoles, they will start spinning, seemingly in contradiction of the conservation of angular momentum. What's going on?! Well, angular momentum is conserved, but you have to look elsewhere to see it. The angular momentum carried by the dipoles is compensated by the angular momentum carried by the magnetic field itself.

Finally, a few basic comments: the dipole force drops off as $1/r^4$, quicker than the Coulomb force. Correspondingly, it grows quicker than the Coulomb force at short distances. If $\mathbf{m}_1$ and $\mathbf{m}_2$ point in the same direction and lie parallel to the separation $\mathbf{R}$, then the force is attractive. If $\mathbf{m}_1$ and $\mathbf{m}_2$ point in opposite directions and lie parallel to the separation between them, then the force is repulsive. The expression (3.31) tells us the general result.

3.4.3 So What is a Magnet?

Until now, we've been talking about the magnetic field associated to electric currents. But when asked to envisage a magnet, most people would think if a piece of metal, possibly stuck to their fridge, possibly in the form of a bar magnet like the one shown in the picture. How are these related to our discussion above?

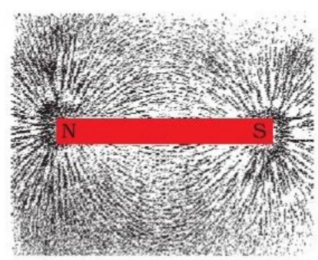


Figure 33:

These metals are permanent magnets. They often involve iron. They can be thought of as containing many microscopic magnetic dipoles, which align to form a large magnetic dipole $\mathbf{M}$. In a bar magnet, the dipole $\mathbf{M}$ points between the two poles. The iron filings in the picture trace out the magnetic field which takes the same form that we saw for the current loop in Section 3.3.

This means that the leading force between two magnets is described by our result (3.31). Suppose that $\mathbf{M}_1$, $\mathbf{M}_2$ and the separation $\mathbf{R}$ all lie along a line. If $\mathbf{M}_1$ and $\mathbf{M}_2$

point in the same direction, then the North pole of one magnet faces the South pole of another and (3.31) tells us that the force is attractive. Alternatively, if $\mathbf{M}_1$ and $\mathbf{M}_2$ point in opposite directions then two poles of the same type face each other and the force is repulsive. This, of course, is what we all learned as kids.

The only remaining question is: where do the microscopic dipole moments $\mathbf{m}$ come from? You might think that these are due to tiny electric atomic currents but this isn't quite right. Instead, they have a more fundamental origin. The electric charges — which are electrons — possess an inherent angular momentum called *spin*. Roughly you can think of the electron as spinning around its own axis in much the same way as the Earth spins. But, ultimately, spin is a quantum mechanical phenomenon and this classical analogy breaks down when pushed too far. The magnitude of the spin is:

$$s = \frac{1}{2}\hbar$$

where, recall, $\hbar$ has the same dimensions as angular momentum.

We can push the classical analogy of spin just a little further. Classically, an electrically charged spinning ball would give rise to a magnetic dipole moment. So one may wonder if the spinning electron also gives rise to a magnetic dipole. The answer is yes. It is given by

$$\mathbf{m} = g\,\frac{e}{2m}\mathbf{s}$$

where e is the charge of the electron and m is its mass. The number g is dimensionless and called, rather uninspiringly, the *g-factor*. It has been one of the most important numbers in the history of theoretical physics, with several Nobel prizes awarded to people for correctly calculating it! The classical picture of a spinning electron suggests $g = 1$. But this is wrong. The first correct prediction (and, correspondingly, first Nobel prize) was by Dirac. His famous relativistic equation for the electron gives

$$g = 2$$

Subsequently it was observed that Dirac's prediction is not quite right. The value of g receives corrections. The best current experimental value is

$$g = 2.00231930419922 \pm (1.5 \times 10^{-12})$$

Rather astonishingly, this same value can be computed theoretically using the framework of quantum field theory (specifically, quantum electrodynamics). In terms of precision, this is one of the great triumphs of theoretical physics.

There is much much more to the story of magnetism, not least what causes the magnetic dipoles $\mathbf{m}$ to align themselves in a material. The details involve quantum mechanics and are beyond the scope of this course.

3.5 Units of Electromagnetism

More than any other subject, electromagnetism is awash with different units. In large part this is because electromagnetism has such diverse applications and everyone from astronomers, to electrical engineers, to particle physicists needs to use it. But it's still annoying. Here we explain the basics of SI units.

The SI unit of charge is the *Coulomb*. As of 2019[2], the Coulomb is defined in terms of the charge $-e$ carried by the electron. This is taken to be exactly

$$e = 1.602176634 \times 10^{-19}\ C$$

If you rub a balloon on your sweater, it picks up a charge of around $10^{-6}\ C$ or so. A bolt of lightening deposits a charge of about 15 C. The total charge that passes through an AA battery in its lifetime is about 5000 C.

The SI unit of current is the *Ampere*, denoted A. It is defined as one Coulomb of charge passing every second. The current that runs through single ion channels in cell membranes is about $10^{-12}\ A$. The current that powers your toaster is around 1 A to 10 A. There is a current in the Earth's atmosphere, known as the Birkeland current, which creates the aurora and varies between $10^5\ A$ and $10^6\ A$. Galactic size currents in so-called Seyfert galaxies (particularly active galaxies) have been measured at a whopping $10^{18}\ A$.

The electric field is measured in units of NC^{-1}. The electrostatic potential ϕ has units of *Volts*, denoted V, where the 1 Volt is the potential difference between two infinite, parallel plates, separated by 1 m, which create an electric field of 1 NC^{-1}.

[2]Prior to 2019, a reluctance to rely on fundamental physics meant that the definitions were a little more tortuous. The Ampere was taken to be the base unit, and the Coulomb was defined as the amount of charge transported by a current of 1 A in a second. The Ampere, in turn, was defined to be the current carried by two straight, parallel wires when separated by a distance of 1 m, in order to experience an attractive force-per-unit-length of $2 \times 10^{-7}\ Nm^{-1}$. (Recall that a Newton is the unit of force needed to accelerate 1 Kg at 1 ms^{-1}.) From our result (3.25), we see that if we plug in $I_1 = I_2 = 1\ A$ and $d = 1\ m$ then this force is $f = \mu_0/2\pi\ A^2m^{-1}$. This definition is the reason that μ_0 has the strange-looking value $\mu_0 = 4\pi \times 10^{-7}\ m\,Kg\,C^{-2}$. The new definitions of SI units means that we can no longer say with certainty that $\mu_0 = 4\pi \times 10^{-7}\ m\,Kg\,C^{-2}$, but this only holds up to the experimental accuracy of a dozen significant figures or so. For our purposes, the main lesson to draw from this is that, from the perspective of fundamental physics, SI units are arbitrary and a little daft.

A nerve cell sits at around $10^{-2}\ V$. An AA battery sits at $1.5\ V$. The largest man-made voltage is $10^7\ V$ produced in a van der Graaf generator. This doesn't compete well with what Nature is capable of. The potential difference between the ends of a lightening bolt can be $10^8\ V$. The voltage around a pulsar (a spinning neutron star) can be $10^{15}\ V$.

The unit of a magnetic field is the *Tesla*, denoted T. A particle of charge $1\ C$, passing through a magnetic field of $1\ T$ at $1\ ms^{-1}$ will experience a force of $1\ N$. From the examples that we've seen above it's clear that $1\ C$ is a lot of charge. Correspondingly, $1\ T$ is a big magnetic field. Our best instruments (SQUIDs) can detect changes in magnetic fields of $10^{-18}\ T$. The magnetic field in your brain is $10^{-12}\ T$. The strength of the Earth's magnetic field is around $10^{-5}\ T$ while a magnet stuck to your fridge has about $10^{-3}\ T$. The strongest magnetic field we can create on Earth is around $100\ T$. Again, Nature beats us quite considerably. The magnetic field around neutron stars can be between $10^6\ T$ and $10^9\ T$. (There is an exception here: in "heavy ion collisions", in which gold or lead nuclei are smashed together in particle colliders, it is thought that magnetic fields comparable to those of neutron stars are created. However, these magnetic fields are fleeting and small. They are stretch over the size of a nucleus and last for a millionth of a second or so).

As the above discussion amply demonstrates, SI units are based entirely on historical convention rather than any deep underlying physics. A much better choice is to pick units of charge such that we can discard ϵ_0 and μ_0. There are two commonly used frameworks that do this, called *Lorentz-Heaviside* units and *Gaussian* units. I should warn you that the Maxwell equations take a slightly different form in each.

To fully embrace natural units, we should also set the speed of light $c = 1$. (See the rant in the Dynamics and Relativity lectures). However we can't set everything to one. There is one combination of the fundamental constants of Nature which is dimensionless. It is known as the *fine structure constant*,

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$$

and takes value $\alpha \approx 1/137$. Ultimately, this is the correct measure of the strength of the electromagnetic force. It tells us that, in units with $\epsilon_0 = \hbar = c = 1$, the natural, dimensionless value of the charge of the electron is $e \approx 0.3$.

3.5.1 A History of Magnetostatics

The history of magnetostatics, like electrostatics, starts with the Greeks. The fact that magnetic iron ore, sometimes known as "lodestone", can attract pieces of iron was apparently known to Thales. He thought that he had found the soul in the stone. The word "magnetism" comes from the Greek town Magnesia, which is situated in an area rich in lodestone.

It took over 1500 years to turn Thales' observation into something useful. In the 11^{th} century, the Chinese scientist Shen Kuo realised that magnetic needles could be used to build a compass, greatly improving navigation.

The modern story of magnetism begins, as with electrostatics, with William Gilbert. From the time of Thales, it had been thought that electric and magnetic phenomenon are related. One of Gilbert's important discoveries was, ironically, to show that this is not the case: the electrostatic forces and magnetostatic forces are different.

Yet over the next two centuries, suspicions remained. Several people suggested that electric and magnetic phenomena are intertwined, although no credible arguments were given. The two just smelled alike. The following unisightful quote from Henry Elles, written in 1757 to the Royal Society, pretty much sums up the situation: "There are some things in the power of magnetism very similar to those of electricity. But I do not by any means think them the same". A number of specific relationships between electricity and magnetism were suggested and all subsequently refuted by experiment.

When the breakthrough finally came, it took everyone by surprise. In 1820, the Danish scientist Hans Christian Ørsted noticed that the needle on a magnet was deflected when a current was turned on or off. After that, progress was rapid. Within months, Ørsted was able to show that a steady current produces the circular magnetic field around a wire that we have seen in these lectures. In September that year, Ørsted's experiments were reproduced in front of the French Academy by Francois Arago, a talk which seemed to mobilise the country's entire scientific community. First out of the blocks were Jean-Baptiste Biot and Félix Savart who quickly determined the strength of the magnetic field around a long wire and the mathematical law which bears their name.

Of those inspired by Arago's talk, the most important was André-Marie Ampère. Skilled in both experimental and theoretical physics, Ampère determined the forces that arise between current carrying wires and derived the mathematical law which now bears his name: $\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$. He was also the first to postulate that there exists an atom of electricity, what we would now call the electron. Ampère's work was published in 1827 a book with the catchy title "*Memoir on the Mathematical Theory of Electrodynamic Phenomena, Uniquely Deduced from Experience*". It is now viewed as the beginning of the subject of electrodynamics.