Section 14.7 Maximum and Minimum Values

15. Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

$$f(x, y) = x^4 - 2x^2 + y^3 - 3y$$

Solution:

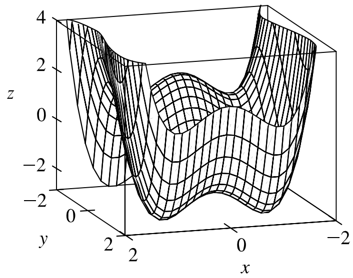
$f(x,y) = x^4 - 2x^2 + y^3 - 3y \quad \Rightarrow \quad f_x = 4x^3 - 4x,\ f_y = 3y^2 - 3,\ f_{xx} = 12x^2 - 4,\ f_{xy} = 0,\ f_{yy} = 6y.$

Then $f_x = 0$ implies $4x(x^2-1) = 0 \quad \Rightarrow \quad x = 0$ or $x = \pm 1$, and $f_y = 0$ implies $3(y^2-1) = 0 \quad \Rightarrow \quad y = \pm 1$.

Thus there are six critical points: $(0, \pm 1)$, $(\pm 1, 1)$, and $(\pm 1, -1)$.

$D(0,1) = (-4)(6) - (0)^2 = -24 < 0$ and

$D(\pm 1, -1) = (8)(-6) = -48 < 0$, so $(0,1)$ and $(\pm 1, -1)$ are saddle points. $D(0,-1) = (-4)(-6) = 24 > 0$ and $f_{xx}(0,-1) = -4 < 0$, so $f(0,-1) = 2$ is a local maximum. $D(\pm 1, 1) = (8)(6) = 48 > 0$ and $f_{xx}(\pm 1, 1) = 8 > 0$, so $f(\pm 1, 1) = -3$ are local minima.

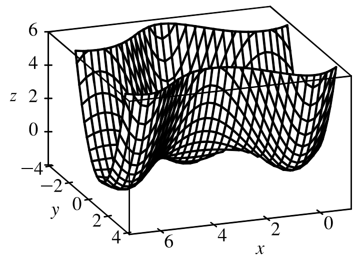


21. Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

$$f(x, y) = y^2 - 2y \cos x, \quad -1 \le x \le 7$$

Solution:

$f(x,y) = y^2 - 2y\cos x \quad \Rightarrow \quad f_x = 2y \sin x$, $f_y = 2y - 2\cos x$, $f_{xx} = 2y\cos x$, $f_{xy} = 2\sin x$, $f_{yy} = 2$. Then $f_x = 0$ implies $y = 0$ or $\sin x = 0 \quad \Rightarrow \quad x = 0, \pi$, or 2π for $-1 \le x \le 7$. Substituting $y = 0$ into $f_y = 0$ gives $\cos x = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, substituting $x = 0$ or $x = 2\pi$ into $f_y = 0$ gives $y = 1$, and substituting $x = \pi$ into $f_y = 0$ gives $y = -1$. Thus the critical points are $(0,1)$, $\left(\frac{\pi}{2}, 0\right)$, $(\pi, -1)$, $\left(\frac{3\pi}{2}, 0\right)$, and $(2\pi, 1)$.



$D\left(\frac{\pi}{2}, 0\right) = D\left(\frac{3\pi}{2}, 0\right) = -4 < 0$ so $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$ are saddle points. $D(0,1) = D(\pi,-1) = D(2\pi, 1) = 4 > 0$ and $f_{xx}(0,1) = f_{xx}(\pi,-1) = f_{xx}(2\pi,1) = 2 > 0$, so $f(0,1) = f(\pi,-1) = f(2\pi,1) = -1$ are local minima.

39. Find the absolute maximum and minimum values of f on the set D.

$$f(x,y) = 2x^3 + y^4, \quad D = \{(x,y) | x^2 + y^2 \leq 1\}$$

Solution:

$f(x,y) = 2x^3 + y^4 \quad \Rightarrow \quad f_x(x,y) = 6x^2$ and $f_y(x,y) = 4y^3$. And so $f_x = 0$ and $f_y = 0$ only occur when $x = y = 0$. Hence, the only critical point inside the disk is at $x = y = 0$ where $f(0,0) = 0$. Now on the circle $x^2 + y^2 = 1$, $y^2 = 1 - x^2$ so let $g(x) = f(x,y) = 2x^3 + (1-x^2)^2 = x^4 + 2x^3 - 2x^2 + 1$, $-1 \leq x \leq 1$. Then $g'(x) = 4x^3 + 6x^2 - 4x = 0 \quad \Rightarrow$ $x = 0, -2$, or $\frac{1}{2}$. $f(0, \pm 1) = g(0) = 1$, $f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = g\left(\frac{1}{2}\right) = \frac{13}{16}$, and $(-2,-3)$ is not in D. Checking the endpoints, we get $f(-1,0) = g(-1) = -2$ and $f(1,0) = g(1) = 2$. Thus the absolute maximum and minimum of f on D are $f(1,0) = 2$ and $f(-1,0) = -2$.

Another method: On the boundary $x^2 + y^2 = 1$ we can write $x = \cos\theta$, $y = \sin\theta$, so $f(\cos\theta, \sin\theta) = 2\cos^3\theta + \sin^4\theta$, $0 \leq \theta \leq 2\pi$.

55. A cardboard box without a lid is to have a volume of 32,000 cm^3. Find the dimensions that minimize the amount of cardboard used.

Solution:

Let the dimensions be x, y and z, then minimize $xy + 2(xz + yz)$ if $xyz = 32{,}000$ cm^3. Then $f(x,y) = xy + [64{,}000(x+y)/xy] = xy + 64{,}000(x^{-1} + y^{-1})$, $f_x = y - 64{,}000x^{-2}$, $f_y = x - 64{,}000y^{-2}$. And $f_x = 0$ implies $y = 64{,}000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64{,}000$ or $x = 40$ and then $y = 40$. Now $D(x,y) = [(2)(64{,}000)]^2 x^{-3} y^{-3} - 1 > 0$ for $(40, 40)$ and $f_{xx}(40,40) > 0$ so this is indeed a minimum. Thus the dimensions of the box are $x = y = 40$ cm, $z = 20$ cm.