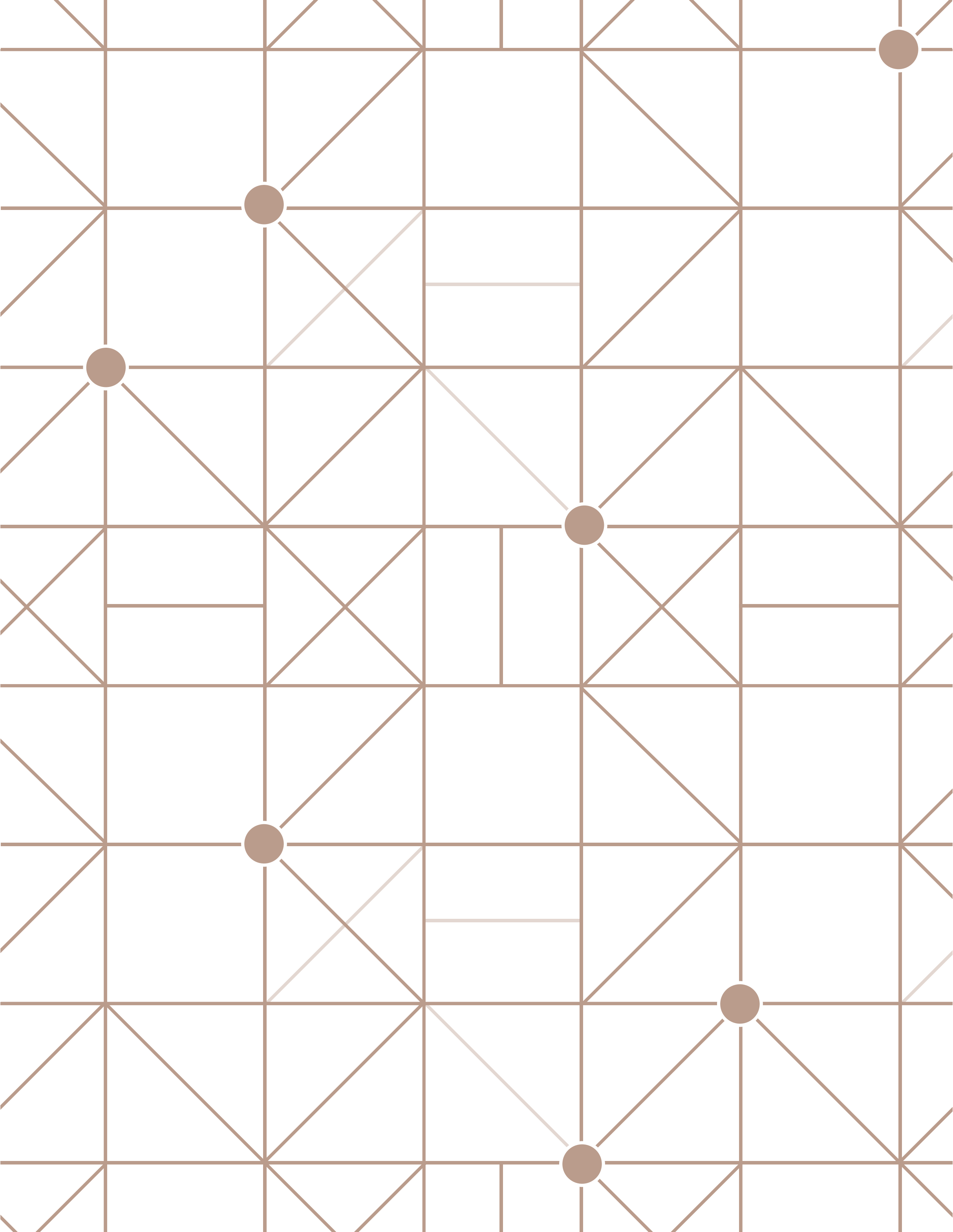


Ch 2.2 Solving $Ax=0$ and $Ax=b$

[A] Key concepts:

1) For an invertible matrix (A^{-1} exists), the nullspace ($N(A)=0$) contains only $\boxed{x=0}$.

($\because$ $Ax=0$ $\quad A^{-1}Ax=0$ $\quad \therefore x=0$)

2) For an invertible matrix, the column space is $\boxed{\text{the whole space}}$.

($\because$ For an invertible matrix, $\det|A| \neq 0$.

$\therefore$  $m \times m$ $\quad$ # of pivots $= m$)

∴ The linear combination of all columns spans the whole space!)

(∴ $Ax=b$ has a solution for every b.)

3) Complete solution of $\boxed{Ax=b}$:

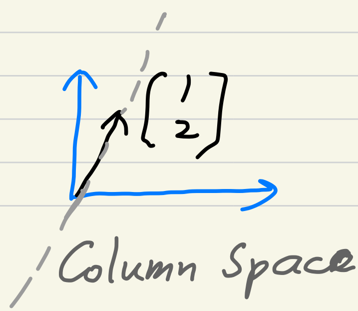
$Ax_p = b$ and $Ax_n = 0$

$$\Rightarrow \boxed{A(x_p + x_n) = b}$$

4) $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not invertible.

— Column Picture:

R^2



— For example, $Ax = b$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$\therefore$ the equation is solvable, unless b lie in the Column space $\Rightarrow$ $\boxed{b_2 = 2b_1}$

— Once $\boxed{b_2 = 2b_1}$, the equation has infinitely many solutions,

$$\begin{cases} x + y = 2 \\ 2y + 2z = 4 \end{cases}$$

① $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $\therefore \boxed{x_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

② $N(A)$ contains solutions satisfy $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \boxed{x_n = c\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

$\therefore$ Complete solution:

$$\boxed{x_p + x_n = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-c \\ 1+c \end{bmatrix}}$$

#

[B] Echelon Form $\boxed{U}$ & Row Reduced Form $\boxed{R}$

1)

$$U = \begin{bmatrix} \bullet & x & x & x & x & x & x \\ 0 & \bullet & x & x & x & x & x \\ 0 & 0 & 0 & \bullet & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{pivots}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 \\ & 0 & & & & & 1 \\ & & & & & & 0 \end{bmatrix} \quad \text{pivots}$$

2)

$$\underbrace{\begin{bmatrix}1&3&3&2\\2&6&9&7\\-1&-3&3&4\end{bmatrix}}_{A} \to \begin{bmatrix}1&3&3&2\\0&0&3&3\\0&0&0&0\end{bmatrix} \to$$

$$\begin{bmatrix}1&3&3&2\\0&0&1&1\\0&0&0&0\end{bmatrix} \to \underbrace{\begin{bmatrix}1&3&0&-1\\0&0&1&1\\0&0&0&0\end{bmatrix}}_{R}$$

[C] Pivot Variables & Free Variables

1) $\boxed{RX = 0}$

pivot

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Columns with Pivots : $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

pivot variables : $u \cdot w$

free variables : $v \cdot y$

2) $\boxed{Rx=0}$

$$\begin{cases} u+3v-y=0 \\ w+y=0 \end{cases}$$

$$\therefore \boxed{u = -3v+y}$$

$$\boxed{w = \qquad -y}$$

3) $\boxed{Rx=0}$

Nullspace contains all combinations of special solutions

$$\begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -3v+y \\ v \\ -y \\ y \end{bmatrix}$$

$$x = \begin{bmatrix} -3v+y \\ v \\ -y \\ y \end{bmatrix} = v\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ Special solutions:

$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ to $\underline{Rx=0}$ and $\underline{Ax=0}$

4) In $\boxed{R^4}$ space,

$N(A)=0$ is a 2-dim. subspace,

generated by $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

∴ Nullspace matrix

$$N = \begin{bmatrix} -3 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$$

5) $\boxed{N(A)}$ has the same "dim." as the # of $\boxed{\text{free variables}}$ and $\boxed{\text{special solutions}}$.

6) The "dim." of $\boxed{C(A)}$ is counted by the # of $\boxed{\text{pivot variables}}$.

[D] Summary of Ch. 2.2

If there are r pivots,
there are r pivot variables
and $(n-r)$ free variables.

r: the rank of the matrix

$\boxed{\text{rank}: r}$ $(A_{m\times n})$

(i) # of r pivot rows in the row space.

(ii) # of r pivot columns in the $C(A)$.

(iii) $(n-r)$ special solution in the $N(A)$

(iv) $(m-r)$ solvability conditions on b.