5 Vector Calculus

5.1 Gradient, Divergent, Curl

In this section, we assume everyone is familiar with all calculus technics mentioned before. Consider a 2-dim function $f(x, y)$ with a parametrization: $x = x(t)$, $y = y(t)$, where $t \in [a, b]$ for some constants $a, b \in \mathbb{R}$. To put into a simpler way, we are now considering a curved line, C, which starts at $t = a$ and ends at $t = b$, with a given function $f(x, y)$. (see fig 1.)

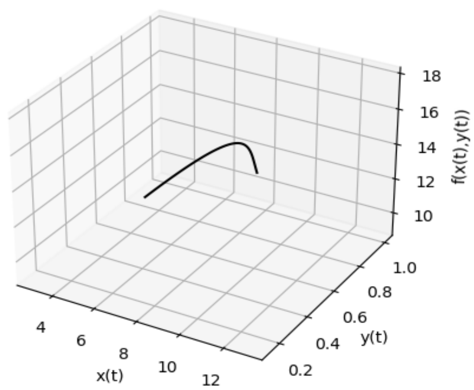


Figure 1:
In this example, we consider $x = t^2 + t + 1$, $y = \sin(t)$ and the function $f(x, y) = x + y + 5$ with the region $t \in [1, 3]$

So, you might have a straight-forward thought: this is an 1-dimensional problem. But, a problem immediately might come to your sight: what can we connect $f(x, y)$ into $f(t)$ while we are dealing with the calculus? Okay, we have no idea, so we start from the easiest theorem, the fundamental theorem of calculus.

$$f(x(b), y(b)) - f(x(a), y(a)) = \int_a^b \frac{df}{dt}\, dt \tag{5.1}$$

And the easiest rule, the chain rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{5.2}$$

It's very natural to let $\mathbf{r(t)} = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$ and make eq(5.2) be a more professional one (maybe):

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}}\right) \cdot \frac{d\mathbf{r(t)}}{dt} \tag{5.3}$$

and eq(5.1) becomes:

$$f(x(b),y(b)) - f(x(a),y(a)) = \int_a^b \left(\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}}\right) \cdot \frac{d\mathbf{r(t)}}{dt}\, dt = \int \left(\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}}\right) \cdot d\mathbf{r} \tag{5.4}$$

where $d\mathbf{r}$ is defined as $d\mathbf{r}/dt \cdot dt$.

Therefore, it's more natural to define a quantity called gradient of f:

$$\nabla f := \left(\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}}\right)$$

Ah ha! We finally get a very tidy equation:

$$f(x(b),y(b)) - f(x(a),y(a)) = \int_C \nabla f \cdot d\mathbf{r} \tag{5.5}$$

This is the famous **gradient theorem**, also known as **fundamental theorem of calculus for line integrals**.

Question 1 : You may immediately ask if this theorem only exists in 1-dim (of course, this is just the fundamental theory of calculus) and 2-dim. The answer is it can be applied into the function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ with any continuous curve C in U. Consider a function $f(x_1, x_2, \ldots, x_n)$, if we still define $\nabla f = \frac{\partial f}{\partial x_1}\hat{x}_1 + \frac{\partial f}{\partial x_2}\hat{x}_2 + \cdots + \frac{\partial f}{\partial x_n}\hat{x}_n$, $\mathbf{r} = x_1\hat{x}_1 + x_2\hat{x}_2 + \ldots x_n\hat{x}_n$ and every variable is parametrized by a single parameter t in a given closed interval, then **please** prove the gradient theorem still holds. (Hint: You might think this is a trivial question as long as you totally get the point mentioned before.)

Question 2 : Please use the example at figure 1 to check the gradient theorem.

Question 3 : From eq(5.3), we know that $f'(t) = \nabla f \cdot \mathbf{r}'(t)$. By Cauchy inequality, we have $|f'(t)| \leq |\nabla f||\mathbf{r}'(t)|$, the equal sign holds only when ∇f is parallel to $\mathbf{r}'(t)$. As you all know $\mathbf{r}'(t)$ represents the tangential vector when we give the function a fixed point. (One can compare it to the velocity which is always the tangential vector of the path) **The problem is as followed**, for one person standing at a given point (x_0, y_0) with height $f(x_0, y_0)$, please prove the direction of $\nabla f(x_0, y_0)$ is the **steepest slope** for the person to go to the **higher** place. (Or, for those who are experts to the calculus, the direction of $\nabla f(x_0, y_0)$ has the biggest directional derivative.)

It's very nice. I guess everyone has understood well what the gradient is. In fact, we can take a deeper look at the strange operator ∇, which is called the "del operator" or the "nabla operator". It's very straight-forward that one can write $\nabla = \hat{\mathbf{x}}\partial/\partial x + \hat{\mathbf{y}}\partial/\partial y + \hat{\mathbf{z}}\partial/\partial z$ in 3-dim case. (where I assume everyone has overcome the question 1.) Notice that it is similar to a vector, so why not define its inner product and cross product?

Definition 5.1. *The divergence of a vector function (or vector field)* $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ *is defined by* $\nabla \cdot \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$, *where* F_x *is the x component of* $\boldsymbol{F}$ *and* F_y *is the y component of* $\boldsymbol{F}$*, etc.*

Definition 5.2. *The curl of a vector function (or vector field) $\boldsymbol{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $\nabla \times \boldsymbol{F} = \hat{x}(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}) + \hat{y}(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}) + \hat{z}(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})$, where F_x is the x component of $\boldsymbol{F}$ and F_y is the y component of $\boldsymbol{F}$, etc.*

If one indeed views ∇ operator as a vector and does the inner and outer product, then one can find the same results as definition 5.1 and 5.2. (However, strictly speaking, ∇ is still an opeator not a vector.)

5.2 Line Integral and Surface Integral

Welcome to section 5.2. During this section, I am going to introduce you about the line integral and surface integral. For simplicity and practicality, I will just deal with the 3-dim case. But I encourage all of you to discover higher-dim cases. All right, I think someone has noticed that there was a interesting integral existing at equation (5.5), which is called the line integral:

$$\int_C \nabla f \cdot d\mathbf{r}$$

In general, we can replace ∇f from any vector field (vector function). Here is the definition:

Definition 5.3. *Consider a vector field $\boldsymbol{F}: U \subset \mathbb{R}^3 \to \mathbb{R}^3$ and a continuous curve $C \subset U$. We set a parametrization $t: (x(t), y(t), z(t))$, $t \in [a, b]$ to the point on C. (Of course, consider they are all smooth) Define the line integral of $\boldsymbol{F}$ on C to be:*

$$\int_C \boldsymbol{F} \cdot\, d\boldsymbol{r} = \int_a^b \boldsymbol{F}(x(t), y(t), z(t)) \cdot \boldsymbol{r}'(t)\, dt$$

Notice that, in this definition, we admit curve C to be a closed one. In this case, we have $x(a) = x(b)$, $y(a) = y(b)$, $z(a) = z(b)$ and denote this kind of closed integral to be:

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

Question 1 : Consider $\mathbf{F} = 3y\hat{x} + 5\hat{y} + x^2 e^x \hat{z}$ and the curve C to be the close unit circle on $z = 1$ plane, which starts at $(1, 0, 1)$ and makes one rotation counterclockwise. ($x = cos(t)$, $y = sin(t)$, $z = 1$, for $t \in [0, 2\pi]$). Please evaluate the line integral of $\mathbf{F}$ on C.
Question 2 : If we change $\mathbf{F}$ in question 1 to Columbs Law: $\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\hat{r}$ ($r := \sqrt{x^2 + y^2 + z^2}$ and $\hat{r} := \frac{x}{r}\hat{x} + \frac{y}{r}\hat{y} + \frac{z}{r}\hat{z}$) with the same curve C in question 1, please evaluate the line integral of $\mathbf{F}$ on C.
Question 3 : Back to the gradient theorem (equation (5.5)), we can easily know that:

$$\oint_C \nabla f \cdot d\mathbf{r} = 0 \quad \forall \text{C is a close curve}$$

However, for general vector field $\mathbf{F}$, the close integral may not be zero as you showed in question 1. For those $\mathbf{F}$, which can be written as ∇f, we call them conservative vector fields. You can easily check that the line integral of the conservative vector field is path-independent. (as you learned in the high school) Please prove the Columbs Force (vector field in question 2) is a conservative force by finding a scalar function f s.t $\mathbf{F} = \nabla f$. (Hint: You would find a familiar scalar function as you finish.)

OK, let's move on to the surface integral. First, for a given curved surface, one needs to know how to parametrize it just as curves we did before. Well, immediately, you come out an idea: if one parameter can only generate a line, why not use 2 parameters. That's right! Two parameters can generate a surface and in this lecture note, we would denote it as $\mathbf{r}(u, v)$.

Example: If we want to parametrize the surface of a unit ball centered at the origin, we can let $\mathbf{r}(u, v) = sin(u)cos(v)\hat{x} + sin(u)sin(v)\hat{y} + cos(u)\hat{z}$, with $u \in [0, \pi]$, $v \in [0, 2\pi]$

Question 4 : Please find a parametrization for a flat plane on x-y plane with the vertices $(3, 6, 0)$, $(3, -6, 0)$, $(-3, 6, 0)$, $(-3, -6, 0)$.

Then, we have enough knowledge to introduce the surface integral.

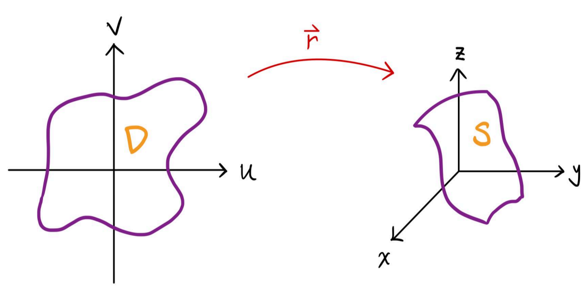
Definition 5.4. *Consider a vector field $\boldsymbol{F} : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ and a curved surface $S \subset U$. We set a parametrization $(u, v) : (x(u, v), y(u, v), z(u, v))$. Consider there are lots of partitions on the curved surface, each is denoted by S_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. (i:along the u-direction, j:along the v-direction) The area of each divided patches is ΔS_{ij} and S^*_{ij} means a random point inside S_{ij}. Denote the normal vector as $\hat{\boldsymbol{n}}$. We finally define the surface integral of $\boldsymbol{F}$ on S to be:*

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (\boldsymbol{F}(S^*_{ij}) \cdot \hat{\boldsymbol{n}}(S_{ij}))\, \Delta S_{ij}$$

if the limit exists.

In fact, we still have no idea about how to calculate the surface integral from definition 5.4. Thus, here comes the problem.
Question 5 : In order to make the definition 5.4 meaningful, we first consider the parametrized map:

$$\begin{aligned} \mathbf{r} : \quad & \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \\ & (u, v) \mapsto (x(u, v), y(u, v), z(u, v)). \end{aligned}$$



if we can in some sense change our integral target from S into D, we are able to do it. (learning from multiple integration) Starting from the small region in D near point (u_0, v_0), say ΔA (or you can do it in $\Delta u \Delta v$), first, please show that it would be approximately mapped into S with area $\Delta A \left|\mathbf{r}_u \times \mathbf{r}_v\right|_{(u_0,v_0)}$. (Hint: because it is small, you can approximate the mapped region to be a flat plane in S.) Second, please show that for a given point, the normal vector of the point is paralleled to the direction of $\mathbf{r}_u \times \mathbf{r}_v$. Therefore, you can conclude that $d\mathbf{S} = \pm(\mathbf{r}_u \times \mathbf{r}_v)dA$, where dA represents a small region in D. And get:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \pm \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.$$

(If you have confused with the notation $\mathbf{r}_u$, it just represents $\frac{\partial \mathbf{r}}{\partial u}$. And in fact, the $\pm$ sign just depends on your chosen orientation of the curved surface. You would see most poeple just write the positive sign, for they pick a good parametrization.)

In physics, the surface integral is usually regarded as the flux of a physical quantitites $\mathbf{F}$ flowing through the curved surface S. For example, the magnetic flux, electric flux, mass flux, etc.

5.3 Divergence Theorem and Stoke's Theorem

Yes! It's time to connect section 5.1 and section 5.2, which can be done by magic Stoke's Theorem and Divergence Theorem. Are you ready? Let's go!

Theorem 5.5. *Let V be a simple solid region and denote ∂V as the boundary of V with positive orientation. Let F be a vector field, then we have:*

$$\oiint_{\partial V} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_V (\nabla \cdot \boldsymbol{F})\, dV.$$

First, let me explain some technical terms. "Boundary" is straight-forward, which means the most outside surface. For example, the boundary of a ball is the spherical shell, the boundary of a box is the combination of six planes outside. "Positive Orientation" means the $\hat{n}$ (defined in definition 5.4) points outward. In fact, if we write the notation ∂V, we directly mean it's positive orientation.

The proof of this theorem is a little bit tricky. We need to first deal with a lemma.

Lemma 5.6. *With the same assumption in theorem 5.5, we have the following identity (WLOG: I would take z-axis as an example. The result should be the same as you change 'z' into 'x' and 'y'.):*

$$\oiint_{\partial V} F_z \hat{\boldsymbol{z}} \cdot d\boldsymbol{S} = \iiint_V \frac{\partial F_z}{\partial z}\, dV. \tag{5.6}$$

Proof: Let us assume in solid V the range of z is $f_1(x,y) \leq z \leq f_2(x,y)$. Therefore, we can integrate the right hand side.

$$\iint_D \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial F_z}{\partial z}\, dz\, dA = \iint_D (F_z(x,y,f_2) - F_z(x,y,f_1)\, dA = \oiint_{\partial V} F_z \hat{\mathbf{z}} \cdot d\mathbf{S}.$$

The last equal sign is a little bit non-trivial. You can imagine it by the help of the toy model at right hand side in the following figure.

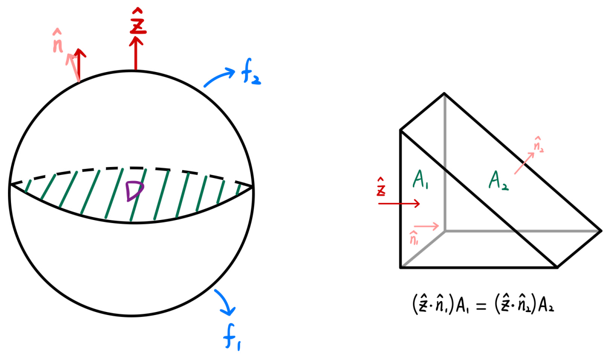


Figure 2:
We are dealing with the problem at L.H.S. You can first regard it as lots of flat planes (pink one) and use the method at R.H.S to confirm the result.

With this lemma, you can prove the divergence theorem. I would save it as a question for you.

Question 1: Please prove the divergence theorem with lemma 5.6.

Next, for the Stoke's Theorem:

Theorem 5.7. *Let S be an oriented (with a chosen direction of $\hat{n}$) smooth surface that is bounded by a simple, close, smooth boundary positive orientedcurve ∂S. Let F be a vector field, then we have:*

$$\oint_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}.$$

In this case, the "positive oriented" means the direction of the curve ∂S would abide right-hand rule. (where the thumb points to the normal vector of S.)

This proof is a little bit tricky, for you have to use the Green's Theorem. Therefore, it's time for you to figure out by yourself. (Better read out the Green's Theorem before you start to prove. You can also go through the proof from Griffith E&M if you just want to get a simple, special case.)

In fact, you would find the simple proof of the Gradient Theorem, Divergence Theorem and Stoke's Theorem in some physical books (such as Griffith E&M). Although they are not strict enough, I encourage you to go through them. You would find more concrete images of why we can connect the differentiation and the integration in vector calculus instead of an abstract proof like us doing right now.