12.5 Inductance And Magnetic Materials

12.5.1 Inductance

For two loops C_1, C_2, we define their **mutual Inductance** M_{12} by

$$M_{12} \equiv \frac{\Phi_{B_2}(S_1)}{I_2}, \tag{12.87}$$

where B_2 is the magnetic field created by the current I_2 on C_2, and S_1 is any surface whose boundary is C_1. Since B_2 is proportional to I_2, we expect that the value of M_{12} is independent of I_2, but it depends only on the geometric setting.

It can be proven that $M_{12} = M_{21}$.

Self Inductance L can be similarly defined when C_1 and C_2 are the same loop. That is, L characterizes how many of the field lines it generates pass through itself.

When the current I_2 changes over time, the magnetic flux $\Phi_{B_2}(S_1)$ changes over time, so that there is an induced emf on C_1 by I_2

$$\mathcal{E}_1 = -M_{12}\frac{dI_2}{dt}. \tag{12.88}$$

The minus sign is put there to remind us Lenz's law.

Lenz's Law

The induced emf is in the direction that, if C_1 is a conducting wire, it creates a current (to generate an additional magnetic field) to reduce the change in the total magnetic flux passing through C_1.

Nature abhors changes.

Example: Transformers.

Ex 12.69: What is the self-inductance L for a long straight solenoid of height h, cross-sectional area A and N turns of wire?

Solution:

$L = \mu N^2 A/h$.

The long straight solenoid is the analogue of the parallel plate capacitor.

Inductors

Inductors are circuit elements used for their inductances. The relation between V and I on an inductor is

$$\boxed{V = L\frac{dI}{dt}.} \tag{12.89}$$

Recall that $I = C\frac{dV}{dt}$ on a capacitor.

While a capacitor "dislikes" changes in V, inductors "dislike" changes in I.

V on a capacitor is always continuous in time and I is always continuous on inductors.

The potential difference V across an inductor (e.g. a solenoid) is not exactly the same as the electric potential.

Due to Lenz's law, we need to work against the induced emf to increase the current on a solenoid.

Make the analogy with capacitors.

Q 12.82: Where does the energy of the work used to increase the current on an inductor from 0 to I go?

Ex 12.70: For an inductor with inductance L, how much more energy does it have when it has a current I compared with its energy when there is no current?

Solution:

$$\boxed{U = \frac{L}{2}I^2 = \frac{\Phi^2}{2L} = \frac{1}{2}\Phi I.} \tag{12.90}$$

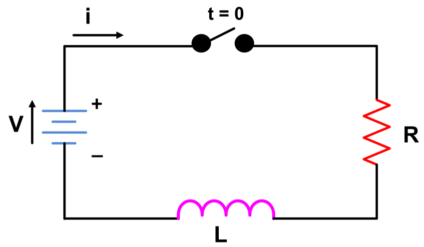
Ex 12.71: For a long straight solenoid, what is the energy density in the magnetic field, if we attribute the energy stored in it to the magnetic field?

Solution:

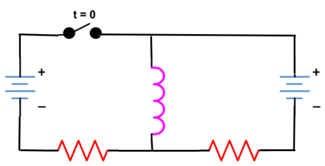
$$\boxed{u = \frac{B^2}{2\mu_0}.} \tag{12.91}$$

12.5.2 LR Circuits

Ex 12.72: The switch is turned on at $t = 0$. What is $V(t)$ and $I(t)$ on the inductor?



Ex 12.73: The switch is turned on at $t = 0$. (Before that the circuit is in a steady state.) What is $V(t)$ and $I(t)$ on either of the inductor?



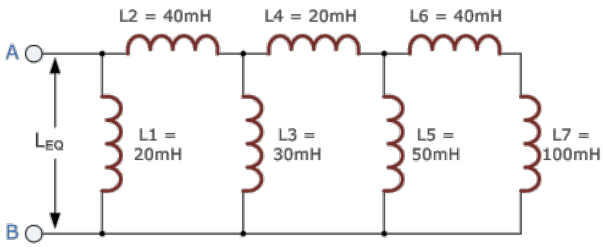
12.6 Exercises

HW: (3-3) For the given current density

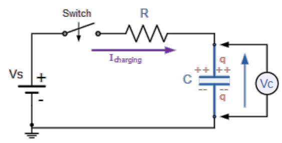
$$\mathbf{J} = \hat{\mathbf{z}} \frac{A}{r + c^2} \tag{12.92}$$

(for constants A and c), what is the magnetic field $\mathbf{B}$ in space?

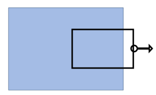
HW: (3-4) What is the effective inductance of the circuit below?



Ex 12.74: For the RC circuit (see the figure on the right), check the conservation of energy after the circuit is closed at $t = 0$. (What are the powers of each element in the circuit?)



HW: (3-5) A conducting loop of height h and width L is moving at a constant velocity v in a magnetic field background with $B(t) = At$ for a constant A. If the total resistance of the loop is R, what is the current $I(t)$? (Assume that the left side of the loop is at a distance s from the right edge of the area with magnetic field at $t = 0$. Let the magnetic field $B(t)$ be pointing out of the page, and the current $I(t)$ flowing clockwise around the loop.)

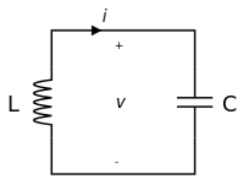


12.7 AC Circuits

Here we shall discuss mostly sinusoidal AC currents at a single frequency at a time.

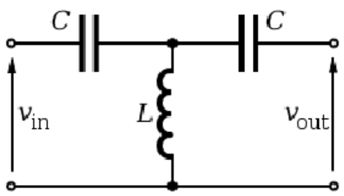
12.7.1 LC Oscillators

Ex 12.75: What is the general solution to $I(t)$ on the LC circuit?



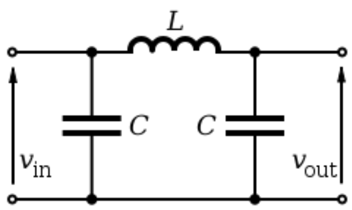
Q 12.83: What is the oscillation frequency of the LC circuit for given capacitance C and inductance L?

high pass T filter



Q 12.84: Why does the high pass T filter work as a high pass filter?

low pass T filter

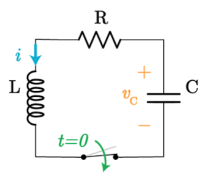


Q 12.85: Why does the low pass T filter work as a low pass filter?

Damped LC oscillation (RLC circuit)

Ex 12.76: What is the general solution to the current $I(t)$ for the RLC circuit?

Recall damped simple harmonic oscillators.



12.7.2 Phasor

For AC circuits composed of linear circuit elements (i.e. the voltage and current on an element satisfy a linear relation, as resistors, capacitors and inductors do), often the voltages and currents on a circuit is better described via its Fourier modes, i.e. as a superposition of waves of different frequencies.

It is thus a good idea to understand a circuit through its behavior at a given frequency. At a given angular frequency ω, the voltage on an element is of the form

$$V(t) = V_0 \cos(\omega t + \phi) = \Re\left(V_0 e^{i(\omega t+\phi)}\right) = \Re\left(\tilde{V} e^{i\omega t}\right), \tag{12.93}$$

where $\tilde{V} \equiv V_0 e^{i\phi}$. For a given frequency ω, we only need a complex number $\tilde{V}$ to keep track of the time-dependence of a voltage.

Similarly, all currents can be represented by a complex number.

> The choice of taking the real part (as opposed to the imaginary part) of the complex number is clearly merely a convention. The choice of the factor i (as opposed to $(-i)$) in the exponent $e^{i\omega t}$ is also a convention. In engineering, often people use the symbol j instead of i, and their convention often allows us to identify j with $-i$.

This description – using complex numbers to represent waves – will also be useful in our discussion on electromagnetic waves, or any kind of waves.

The modulus V_0 of the complex number $\tilde{V}$ gives the amplitude of the oscillation, and the phase ϕ of the complex number $\tilde{V}$ gives the phase of the oscillation. |

Using this convention of bookkeeping, the relations between voltages and currents for the resistors, capacitors and inductors are, respectively,

$$\text{R:} \qquad \tilde{V} = R\tilde{I}, \tag{12.94}$$

$$\text{C:} \qquad \tilde{I} = i\omega C\tilde{V}, \tag{12.95}$$

$$\text{L:} \qquad \tilde{V} = i\omega L\tilde{I}. \tag{12.96}$$

It is now much easier to understand the high-pass or low-pass filters.

In terms of the phasor notation, the description of the properties of R, C, and L are formally the same: the ratio V/I is a constant, like a resistor, although the ratio could be imaginary. We call this generalized notion of resistance **impedance**:

$$Z_R = R, \qquad Z_C = \frac{1}{i\omega C}, \qquad Z_L = i\omega L. \tag{12.97}$$

Analysis of a circuit involving R, C, and L becomes as simple as analyzing a circuit involving resistors only.

The differences among R, C, and L rely on how the impedance Z is related to the frequency ω, and whether it implies a relative phase between the current I and the voltage V.

Ex 12.77: What is the effective impedance of a capacitor C and an inductor L in series or in parallel? How does the effective impedance change with the angular frequency ω?

Ex 12.78: For a given $\tilde{V}_{in}$ at a certain angular frequency ω, what is $\tilde{V}_{out}$ for the high-pass filter mentioned above?

Ex 12.79: The power P on a circuit element is $P(t) = V(t)I(t)$. In terms of the complex numbers $\tilde{V}$ and $\tilde{I}$ representing $V(t)$ and $I(t)$, what is the time-averaged power $\bar{P}$ at a given angular frequency ω?

Solution:

The complex conjugate is defined by $(a+bi)^* = a - bi$ for real numbers a, b.

$$\boxed{\bar{P} = \frac{1}{2}\Re\left(\tilde{V}^*\tilde{I}\right). \qquad (12.98)}$$

Ex 12.80: Find the time-averaged power consumed by a circuit element with voltage $V(t) = V_0 \cos(\omega t + \phi)$ if the circuit element is (a) a resistor R, (b) a capacitor C or (c) an inductor L. Express your result in terms of the complex number $\tilde{V}$ representing the oscillatory voltage $V(t)$. How do you interpret your result?