「Electromagnetism」Homework Sheet No. 2

Issued 13 Oct. 2021

1. Evaluate the integral $J = \int_V e^{-r} (\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2}) d\tau$ (where V is a sphere of radius R, centered at the origin) by two different methods, as in Ex. 1.16.
(Textbook, p. 52, Problem 1.49).

2. (a) Let $\mathbf{F}_1 = x^2 \hat{\mathbf{z}}$ and $\mathbf{F}_2 = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Calculate the divergence and curl of $\mathbf{F}_1$ and $\mathbf{F}_2$.

Which one can be written as the gradient of a scalar? Find a scalar potential that does the job. Which one can be written as the curl of a vector? Find a suitable vector potential.
(b) Show that $\mathbf{F}_3 = yz\hat{\mathbf{x}} + zx\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$ can be written both as the gradient of a scalar and as the curl of a vector. Find scalar and vector potentials for this function.
(Textbook, p. 52, Problem 1.50).

3. Check the divergence theorem for the function

$\mathbf{v} = r^2 \cos\theta \hat{\mathbf{r}} + r^2 \cos\phi \hat{\boldsymbol{\theta}} - r^2 \cos\theta \sin\phi \hat{\boldsymbol{\phi}}$, using as your volume one octant of the sphere of radius R (Fig. 1). Make sure you include the entire surface. [Answer: $\pi R^4/4$]
(Textbook, p. 55, Problem 1.54).

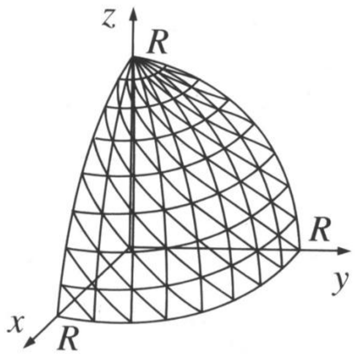
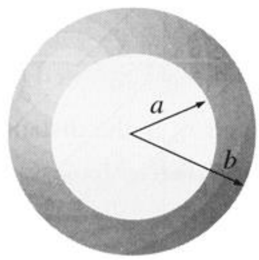


Fig. 1 Figure for problem 3.

4. A thick spherical shell carriers charge density $\rho = k / r^2$ $(a \le r \le b)$ (Fig. 2).
(a) Find the electric field in the three regions:
(i) $r < a$, (ii) $a < r <$ b, (iii) $r > b$.
(b) Plot $|\mathbf{E}|$ as a function of r, for the case $b = 2a$.
(c) Find the potential at the center, using infinity as your reference point.
(Textbook, p. 76, Problem 2.15; p. 83, Problem 2.23).

Fig. 2 Figure for problem 4.



5. A long coaxial cable (Fig. 3) carries a uniform volume charge density ρ on the inner cylinder (radius a), and a uniform surface charge density on the outer cylindrical shell (radius b). This surface charge is negative and is of just the right magnitude that the cable as a whole is electrically neutral.

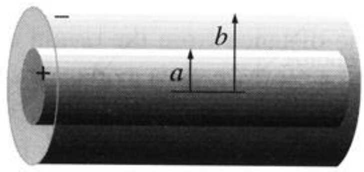


Fig. 3 Figure for problem 5.

(a) Find the electric field in each of the three regions: (i) inside the inner cylinder ($s < a$),

(ii) between the cylinders ($a < s < b$), (iii) outside the cable ($s > b$).
(b) Plot $|\mathbf{E}|$ as a function of s.
(c) Find the potential difference between a point on the axis and a point on the outer cylinder.
(Textbook, p. 76, Problem 2.16; p. 83, Problem 2.24).

6. Consider an infinite chain of point charges, $\pm q$ (with alternating signs), strung out along the x axis, each a distance a from its nearest neighbors. Find the work per particle required to assemble this system. [Partial answer: $-\alpha q^2/(4\pi\varepsilon_0 a)$, for some dimensionless number α; your problem is to determine α. It is known as the Madelung constant. Calculating the Madelung constant for 2- and 3-dimensional arrays is much more subtle and difficult. See "Principles of the Theory of Solids" by J. M. Ziman, Sec. 2.3, (Cambridge U. Press, 1972)].
(Textbook, p.94, Problem 2.33)

7. A metal sphere of radius R carries a total charge Q. What is the force of repulsion between the "northern" hemisphere and the "southern" hemisphere?
(Textbook, p. 104, Problem 2.42).

8. The electrical potential of some configuration is given by the expression $V(\mathbf{r}) = A(e^{-\lambda r}/r)$, where A and λ are constants. Find the electric field $\mathbf{E}(\mathbf{r})$, the charge density $\rho(r)$, and the total charge Q. [Answer: $\rho = \varepsilon_0 A(4\pi\delta^3(\mathbf{r}) - \lambda^2 e^{-\lambda r}/r)$]

(Textbook, p. 108, Problem 2.50).