12.9 EM Waves

Putting Faraday's law together with Ampere's law (magnetic field generates electric field and vice versa), we are naturally led to conjecture that there can be electromagnetic waves.

To see whether there can actually be EM waves, we check whether there are solutions to the Maxwell equations with vanishing charge and current densities ($\rho = \mathbf{J} = 0$) but non-vanishing propagating fields.

Consider the ansatz for the fields $\mathbf{E}$ and $\mathbf{B}$ of a plane wave propagating in the z direction:

$$\mathbf{E} = \hat{\mathbf{x}} E_x(t, z), \tag{12.129}$$

$$\mathbf{B} = \hat{\mathbf{y}} B_y(t, z). \tag{12.130}$$

The first two equations in Maxwell's equations are satisfied because E_x is independent of x and B_y is independent of y. Faraday's law has

$$\partial_z E_x = -\partial_t B_y, \tag{12.131}$$

and Ampere's law has

$$-\partial_z B_y = \mu_0 \epsilon_0 \partial_t E_x. \tag{12.132}$$

With the two equations above combined, we have

$$\partial_z^2 E_x = \mu_0 \epsilon_0 \partial_t^2 E_x \tag{12.133}$$

and a similar equation for B_y. This is a wave equation with the wave velocity

c is the speed of light.

$$\boxed{c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.} \tag{12.134}$$

As a special solution, a monochromatic sinusoidal plan wave is given by

$$\boxed{\begin{aligned} E_x(t, z) &= E_0 \cos(\omega t - kz + \phi), & (12.135)\\ B_y(t, z) &= B_0 \cos(\omega t - kz + \phi), & (12.136)\end{aligned}}$$

where

$$\boxed{\omega^2 = k^2 c^2.} \tag{12.137}$$

There are two independent solutions here for the two possibilities $k = \pm\frac{\omega}{c}$. For $k > 0$, it is a wave propagating in the $+z$ direction.

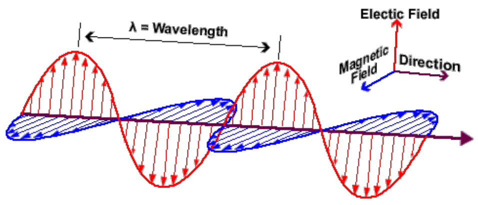
Plugging them back into Faraday's law, we find

$$k E_0 = \omega B_0. \tag{12.138}$$

That is,

$$B_0 = \pm\frac{1}{c}E_0. \tag{12.139}$$

Q 12.92: How do we know that light is a kind of EM waves?



We can now use the superposition principle to superpose electromagnetic waves with different frequencies ω, magnitudes E_0 and phases propagating in all directions.
Ex 12.84: Write down the fields for an EM wave with a generic frequency, magnitude, phase and direction.
Ex 12.85: (1) For a uniform surface current density $\mathbf{K} = K_0\hat{\mathbf{z}}$ on the $y-z$ plane, what is the magnetic field for (a) $x < 0$ and (b) $x > 0$? (2) If the current density is replaced by $\mathbf{K} = At\hat{\mathbf{z}}$ (A is a constant), what is electric field according to Faraday's law for (a) $x < 0$ and (b) $x > 0$?
Solution:
(1)

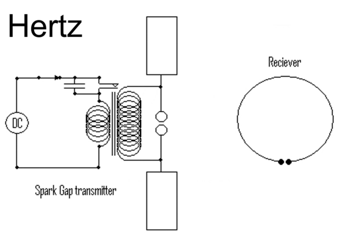
$$\mathbf{B}(x<0) = -\frac{\mu_0 K_0}{2}\hat{\mathbf{y}}, \qquad \mathbf{B}(x>0) = \frac{\mu_0 K_0}{2}\hat{\mathbf{y}}, \tag{12.140}$$

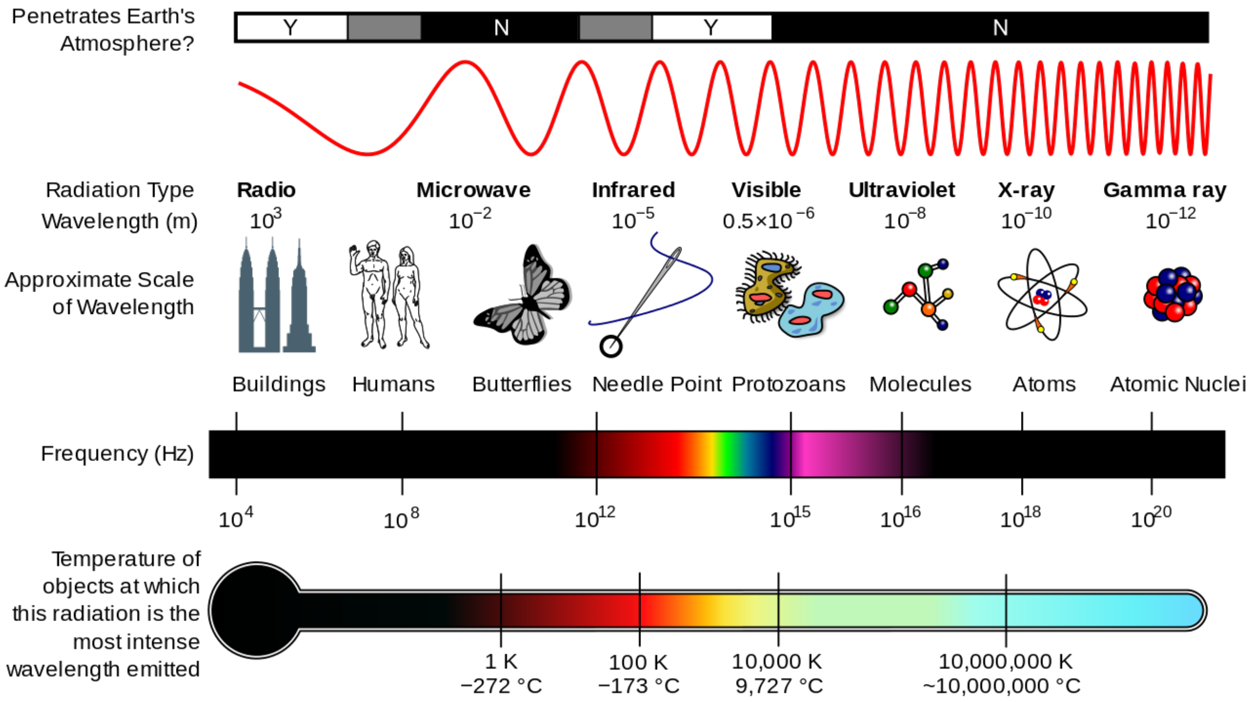
(2)

$$\mathbf{E}(x) = \frac{\mu_0 A}{2}|x|\hat{\mathbf{z}} + \text{constant vector}. \tag{12.141}$$

The additive constant part of $\mathbf{E}$ is not generated by the given current because $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ do not respect the symmetry of the current.
Q 12.93: In the exercise above, replace K_0 by $A_0\cos(\omega t)$, and describe the EM field in space.
Q 12.94: In the exercise above, replace the surface current by an oscillatory line current I on an infinite wire on the z-axis, and describe the generated EM field.

Hertz's Experiment





Q 12.95: What are exactly the EM waves? Can we see them?

Roughly speaking, the reason for the existence of the EM waves is that, according to Faraday's law and the modified Ampere's law, time-dependent **E** and **B** generate each other iteratively and indefinitely.

In the old days, all wave phenomena are oscillations of something. People imagine that there is something called "aether" (or "ether") responsible for the EM waves. Einstein urged us to think of spacetime-filling fields without assuming an underlying medium.

> With the complete Maxwell equations, we can estimate how good the quasi-static approximation is. In the quasi-static approximation, a time-dependent magnetic field is determined by the unmodified Ampere's law. This time-dependent magnetic field creates an electric field through Faraday's law, but the contribution of this time-dependent electric field in the modified Ampere's law is ignored.
> **Q 12.96:** Describe situations in which you expect the quasi-static approximation to be good.

12.9.1 Energy And Momentum in EM Waves

Conservation of Energy-Momentum

Charges interact with the EM fields through the Maxwell equations and the Lorentz force law. When the EM field does work, impulse and torque on the charges through Lorentz's force, the energy, momentum, and angular momentum of the charges change with time. Physicists ask this question: Is it possible to maintain the conservation of the total energy, total momentum, and total angular momentum of the system of EM fields and charges, by suitably defining the energy density, momentum density and angular momentum density of the EM fields?

We have defined the energy density in eq.(12.119) according to the requirement of energy conservation.

The task is to define the energy density, momentum density, and angular momentum density of the EM fields as suitable functions of **E** and **B** such that the Maxwell equations and the Lorentz force law guarantee that the work, impulse, and torque done to charges through Lorentz's force law are always compensated by the change in the energy, momentum and angular momentum of the EM fields.

In theoretical physics, the conservation of energy, momentum and angular momentum is guaranteed by the *Noether theorem* if there are symmetries of time translation, space translation and rotation, whenever there is an action principle for the physical system. The action for Maxwell's EM fields is out of the scope of this course, but it is simply

$$S = \int d^4x \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{12.142}$$

As the EM field propagates, the energy, momentum, and angular momentum it carries also move around. Hence one can also define the "current/flux density" of these conserved quantities. The relationship between a conserved quantity (e.g. energy) and its corresponding "current" (Poynting vector), is the same as that between electric charge and the electric current.

While we have already defined the energy density u (12.119) of the EM field

$$u = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2, \tag{12.143}$$

as the energy density is given by eq.(12.119), and the EM wave propagates at the speed of light c, the density of the energy flux (that is, the amount of energy passing through an orthogonal plane per unit area per unit time) for a plane wave is uc.

Q 12.97: What is u for a plane wave? What is u in terms of the phasors?

More generally, one can use the **Poynting vector**

$$\mathbf{S} \equiv \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \tag{12.144}$$

for the density of the energy flux such that the energy conservation law holds, i.e.

$$\oint_{\mathcal{S}=\partial\mathcal{V}} \mathbf{S} \cdot d\mathbf{a} + \frac{d}{dt} \int_{\mathcal{V}} u \, d^3\mathbf{r} = 0. \tag{12.145}$$

Even though we motivated **S** from the plane waves, its definition is completely general.

Ex 12.86: Check that the Poynting vector gives the expected answer for a plane wave according to eq.(12.119), that is, the density of the energy flux should be uc in the direction of the propagation.

Ex 12.87: A current I passes uniformly through a cylindrical conductor of resistance R. Calculate the electric and magnetic fields on the surface of the cylindrical conductor and show that the power $\mathcal{P} = I^2R$ dissipated in it equals the energy entering its surface per unit time according to the Poynting vector.

Solution:

The magnitude of the electric field is $E = V/h = IR/h$ everywhere in the cylinder. The magnitude of the magnetic field at the surface is $B = \frac{\mu_0 I}{2\pi a}$. (Let a be the radius of the cylinder.) The Poynting vector gives the magnitude $S = \frac{EB}{\mu_0} = \frac{I^2R}{2\pi a h}$. The surface area of the cylinder is $A = 2\pi a h$, so $SA = I^2R = \mathcal{P}$.

12.9.2 Photons And Special Relativity

The energy, momentum and angular momentum of an EM wave can be visualized as the energy carried by particles called "photons". Photons are massless particles, so they cannot be properly described in Newtonian physics. We have to use Special Relativity.

Newton proposed that light is composed of particles, rather than waves.

We will discuss Quantum Mechanics, in which the wave-particle duality refers to the situation that sometimes light is like particles (photons) and sometimes light is like waves (EM waves). We will explain this in more detail later.

According to Special Relativity, the kinetic energy and momentum of a particle of mass m are

$$K = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{and} \quad P = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{12.146}$$

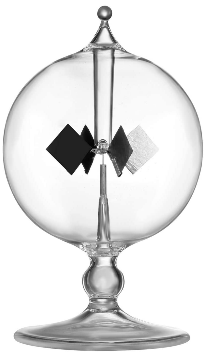
where v is the particle's velocity and c is the speed of light. For massless particles, $K = P = 0$ unless $v = c$, so we should assume that massless particles are always moving at the speed of light. Furthermore, notice that the ratio

$$\frac{P}{K} = \frac{v}{c^2} \tag{12.147}$$

is well defined in the limit $m \to 0, v \to c$, so we expect that the energy and momentum of a photon satisfy the relation

$$K = Pc. \tag{12.148}$$

Q 12.98: What causes the radiometer to turn?



A heuristic derivation by 林昆佑 goes like this:

$$dE = Fdx = \frac{dp}{dt}\frac{dx}{dt}dt = \frac{dp}{dt}cdt = cdp. \tag{12.149}$$

The pressure of EM waves can be derived from the momentum flux as

$$\text{Pressure} = \frac{F}{\text{Area}} = \frac{\Delta P}{\Delta t \text{Area}} = \frac{c\Delta P}{\Delta \text{volume}} = \frac{cu}{c} = u. \tag{12.150}$$

(c.f. textbook)

HW: (4-1) A plane wave has the electric field

$$\mathbf{E} = E_0 \cos(k(x - vt))\hat{\mathbf{y}}. \tag{12.151}$$

Find (a) the value of v, (b) the magnetic field, (c) the wave length, (d) the energy passing through the $y - z$ plane per unit area per unit time, (e) the pressure on a perfectly absorbing surface at normal incidence.