1. (15 pts) Consider the function

$$f(x,y)=\begin{cases}\dfrac{x^2+y^2}{4}\cdot\ln(x^2+y^2) & ,\text{if } (x,y)\neq(0,0),\\ 0 & ,\text{if } (x,y)=(0,0).\end{cases}$$

   (a) (3 pts) Is $f(x,y)$ continuous at $(0,0)$?
   (b) (4 pts) Find $f_x(0,0), f_y(0,0)$, and $f_x(x,y), f_y(x,y)$ for $(x,y)\neq(0,0)$.
   (c) (4 pts) Find $f_{xy}(0,0)$ and $f_{xy}(x,y)$ for $(x,y)\neq(0,0)$.
   (d) (4 pts) Is $f_{xy}(x,y)$ continuous at $(0,0)$?

Solution:

(a) Set $x=r\cos\theta$ and $r\sin\theta$. Then

$$\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{4}\ln(x^2+y^2)=\lim_{r\to0^+}\frac{r^2}{4}\ln(r^2)=\frac{1}{2}\lim_{r\to0^+}r^2\ln r=\frac{1}{2}\lim_{r\to0^+}\frac{\ln r}{r^{-2}}$$
$$=\frac{1}{2}\lim_{r\to0^+}\frac{r^-1}{-2r^{-3}}=\frac{1}{2}\lim_{r\to0^+}\frac{r^2}{-2}=0=f(0,0).$$

Therefore, $f(x,y)$ is continuous at $(0,0)$.

$f(0,0)=0$ (1%), Computation of the limit (2%).

(b) For $(x,y)\neq(0,0)$,

$$f_x(x,y)=\frac{x}{2}\cdot\ln(x^2+y^2)+\frac{x^2+y^2}{4}\cdot\frac{1}{x^2+y^2}\cdot2x=\frac{x}{2}(\ln(x^2+y^2)+1).\ (1\%)$$
$$f_y(x,y)=\frac{y}{2}\cdot\ln(x^2+y^2)+\frac{x^2+y^2}{4}\cdot\frac{1}{x^2+y^2}\cdot2y=\frac{y}{2}(\ln(x^2+y^2)+1).\ (1\%)$$

$$f_x(0,0)=\lim_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{\frac{h^2}{4}\ln h^2}{h}=\frac{1}{4}\lim_{h\to0}h\cdot\ln h^2$$
$$=\frac{1}{2}\lim_{h\to0}\frac{\ln|h|}{h^{-1}}=\frac{1}{2}\lim_{h\to0}\frac{h^{-1}}{-h^{-2}}=\frac{1}{2}\lim_{h\to0}-h=0\ (1\%)$$
$$f_y(0,0)=\lim_{h\to0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to0}\frac{\frac{h^2}{4}\ln h^2}{h}=0\ (1\%).$$

(c) For $(x,y)\neq(0,0)$,

$$f_{xy}(x,y)=\frac{x}{2}\cdot\frac{2y}{x^2+y^2}=\frac{xy}{x^2+y^2}.\ (2\%)$$

$$f_{xy}(0,0)=\lim_{h\to0}\frac{f_x(0,h)-f_x(0,0)}{h}=\lim_{h\to0}\frac{\frac{0}{2}(\ln(h^2)+1)-0}{h}=0\ (2\%)$$

(d) **Solution 1.**

Set $x = r\cos\theta$ and $r\sin\theta$. Then

$$\lim_{(x,y)\to(0,0)} f_{xy}(x,y) = \lim_{r\to 0^+} \frac{r^2\cos\theta\sin\theta}{r^2} = \lim_{r\to 0^+} \cos\theta\sin\theta. \ (2\%)$$

For different θ, the limit value is different. So the limit does not exist. (1%) Therefore, $f_{xy}(x,y)$ is not continuous at $(0,0)$. (1%)

Solution 2.

First, let's approach $(0,0)$ along the $y = x$. Then $y = x$ gives $f_{xy}(x,x) = 1/2$ for all $x \neq 0$, so $f_{xy}(x,y) \to 1/2$ as $(x,y) \to (0,0)$ along the line $y = x$. (1%)

Next, we approach $(0,0)$ along the $y = -x$. Then $y = -x$ gives $f_{xy}(x,-x) = -1/2$ for all $x \neq 0$, so $f_{xy}(x,y) \to -1/2$ as $(x,y) \to (0,0)$ along the line $y = -x$. (1%)

So $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ does not exist. (1%)

Therefore, $f_{xy}(x,y)$ is not continuous at $(0,0)$. (1%)

2. (8 pts) $f(x,y)$ is a differentiable function on R^2. Consider two points $P_0 = (x_0, y_0) \neq P_1 = (x_1, y_1)$ and define a function $g(t) = f(x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0))$.

   (a) (2 pts) Compute $g'(t)$ by the chain rule.

   (b) (6 pts) Suppose that $f(x_0, y_0) = f(x_1, y_1)$ and $\nabla f \neq \vec{0}$. Prove that the line segment $\overline{P_0P_1}$ is tangent to at least one level curve $f(x,y) = c$ for some c.

Solution:

(a) Define $g(t) = f(x_0 + t(x_1 - x_0),\, y_0 + t(y_1 - y_0))$. Then

$$\begin{aligned}
(1\%)\quad g'(t) &= f_x(x_0 + t(x_1 - x_0),\, y_0 + t(y_1 - y_0))\, x_t \\
&\quad + f_y(x_0 + t(x_1 - x_0),\, y_0 + t(y_1 - y_0))\, y_t \\
(1\%)\quad\quad\quad &= f_x(x_0 + t(x_1 - x_0),\, y_0 + t(y_1 - y_0))\,(x_1 - x_0) \\
&\quad + f_y(x_0 + t(x_1 - x_0),\, y_0 + t(y_1 - y_0))\,(y_1 - y_0)
\end{aligned}$$

(b) $P_0 \neq P_1$ and $f(x_0,\, y_0) = f(x_1,\, y_1) \;\Rightarrow$

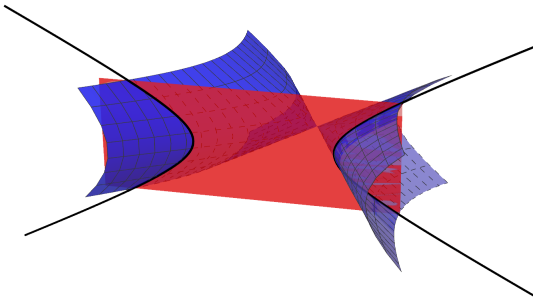
(2%) $g(0) = f(x_0,\, y_0) = f(x_1,\, y_1) = g(1) \;\Rightarrow$
There exists a $0 \le t^* \le 1$ such that $g'(t^*) = \dfrac{g(1) - g(0)}{1 - 0} = 0$

(2%) There exists a point $P^*(x_0 + t^*(x_1 - x_0),\, y_0 + t^*(y_1 - y_0))$
lying on $\overline{P_0P_1}$ with $f(x,\, y) = c = g(t^*)$

(1%) At this point P^*, $\nabla f \cdot (x_1 - x_0,\, y_1 - y_0) = g'(t^*) = 0$

(1%) $\Rightarrow \overline{P_0P_1}$ tangent to the level curve $f(x,\, y) = c = g(t^*)$

3. (10 pts) Let C be the hyperbola formed by the intersection of the cone $x^2+3z^2=4y^2$ and the plane $2x+y=5$. Find the maximum and the minimum distance between the origin and the point on C (if exist) by the method of Lagrange multipliers.



Solution:

Let $f(x,y,z)=x^2+y^2+z^2$, $g(x,y,z)=x^2-4y^2+3z^2$, $h(x,y,z)=2x+y$. We want to find extreme values of f under constraints $g=0$ and $h=5$. By the method of Lagrange multiplizers, we solve the system of equations:

$$\begin{cases} f_x=\lambda g_x+\mu h_x \\ f_y=\lambda g_y+\mu h_y \\ f_z=\lambda g_z+\mu h_z \\ g=0 \\ h=5 \end{cases} \Rightarrow \begin{cases} 2x=\lambda 2x+2\mu\cdots\text{①} \\ 2y=-\lambda 8y+\mu\cdots\cdots\text{②} \\ 2z=\lambda 6z\cdots\cdots\text{③} \\ x^2-4y^2+3z^2=0\cdots\cdots\text{④} \\ 2x+y=5\cdots\cdots\text{⑤} \end{cases}$$

(3pts for correct setting and equations.)

③$\Rightarrow \lambda=\frac{1}{3}$ or $z=0$.

Case 1: $z=0$, ④$\Rightarrow x=\pm 2y$. If $x=2y$, ⑤$\Rightarrow y=1,\ x=2\Rightarrow \lambda=0,\ \mu=2$.
There is one solution $(x,y,z)=(2,1,0)$, $(\lambda,\mu)=(0,2)$.
If $x=-2y$ ⑤$\Rightarrow y=-\frac{5}{3},\ x=\frac{10}{3}\Rightarrow \lambda=-\frac{2}{3},\ \mu=\frac{50}{9}$.
There is another solution $(x,y,z)=(\frac{10}{3},\frac{-5}{3},0)$, $(\lambda,\mu)=(\frac{-2}{3},\frac{50}{9})$. 2pts.

Case 2: $\lambda=\frac{1}{3}$ but $z\neq 0$, ①$\Rightarrow x=\frac{3}{2}\mu$, ②$\Rightarrow y=\frac{3}{14}\mu$
However, ④$\Rightarrow \left(\frac{3}{2}\mu\right)^2-\left(\frac{3}{14}\mu\right)^2+3z^2=0$
$\Rightarrow 3z^2=-\left(\frac{9}{4}-\left(\frac{3}{14}\right)^2\right)\mu^2<0\ldots\ldots(\rightarrow\leftarrow)$ 2pts.

Hence the only solutions are $(x,y,z)=(2,1,0)$, $(x,y,z)=(\frac{10}{3},-\frac{5}{3},0)$
$\because f(2,1,0)=5<f(\frac{10}{3},-\frac{5}{3},0)=\frac{125}{9}$ $\therefore f$ obtains minimum value at $(2,1,0)$. 1pt
i.e the minimum distance between $(0,0,0)$ and C is $\sqrt{5}$. 1pt
The C is unbounded and the maximum distance doesn't exist. 1pt **<u>Sol 2:</u>**

$\begin{cases} x^2+3z^2=4y^2 \\ 2x+y=5 \end{cases}$ $\Rightarrow$ Let $x=t$, $y=5-2x=5-2t$, $3z^2=4y^2-x^2=4(5-2t)^2-t^2$

$x^2+y^2+z^2=t^2+(5-2t)^2+\frac{4}{3}(5-2t)^2-\frac{1}{3}t^2=f(t)$.
Find t such that $f(t)$ is minimized.
At most 3pts for this solution.

4. (12 pts) Find the center of mass of a lamina

$$D = \{(x,y) \in \mathbf{R}^2 | x \geq 0, y \geq 0, x^2 + 9y^2 \leq 1\}$$

whose density function at any point is proportional to the square of its distance from the y-axis.

Solution:

The density function is $\rho(x,y) = x^2$(1 point). Let $x = r\cos\theta$ and $y = \frac{r}{3}\sin\theta$ where $0 \leq r \leq 1$ and $0 \leq \theta \leq \frac{\pi}{2}$(1 point). Thus

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \frac{1}{3}\sin\theta & \frac{1}{3}r\cos\theta \end{vmatrix} = \frac{r}{3}.\text{(1 points)}$$

$$\begin{aligned} m &= \iint_D \rho(x,y)dA = \iint_D x^2 dA \\ &= \int_0^{\pi/2}\int_0^1 r^2\cos^2\theta \frac{r}{3} dr d\theta \\ &= \frac{1}{3}[\int_0^{\pi/2}\cos^2\theta d\theta][\int_0^1 r^3 dr] \\ &= \frac{1}{3}\cdot\frac{\pi}{4}\cdot\frac{1}{4} = \frac{\pi}{48}\text{(2 points)}. \\ m\bar{X} &= \iint_D x\rho(x,y)dA \\ &= \iint_D x^3 dA\text{(1 point)} \\ &= \int_0^{\pi/2}\int_0^1 \frac{1}{3}r^4\cos^3\theta dr d\theta \\ &= \frac{1}{3}\cdot[\int_0^{\pi/2}\cos^3\theta d\theta][\int_0^1 r^4 dr] \\ &= \frac{1}{3}[\int_0^{\pi/2}(1-\sin^2\theta)d(\sin\theta)]\cdot[\frac{1}{5}] \\ &= \frac{2}{45}\text{(2 points)}. \end{aligned}$$

$$\begin{aligned} m\bar{Y} &= \iint_D y\rho(x,y)dA \\ &= \iint_D x^2 y dA\text{(1 point)} \\ &= \frac{1}{9}\int_0^{\pi/2}\int_0^1 r^4\cos^2\theta\sin\theta dr d\theta \\ &= \frac{1}{9}[\int_0^{\pi/2}\cos^2\theta d(-\cos\theta)][\int_0^1 r^4 dr] \\ &= \frac{1}{9}\cdot[\frac{-1}{3}\cos^3\theta|_0^{\pi/2}]\cdot\frac{1}{5} \\ &= \frac{1}{135}\text{(2 points)}. \end{aligned}$$

Then the center of mass is $(\bar{X},\bar{Y}) = (\frac{32}{15\pi}, \frac{16}{45\pi})$(1 point).

5. (10 pts) Evaluate the integral

$$\iiint_U \frac{1}{x^2+y^2+z^2}\,\mathrm{d}V$$

where $U = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 36 \text{ and } z \ge 3\}$.

Solution:

Use spherical coordinate, $z \ge 3$ can be expressed as $\rho\cos\phi \ge 3$. We can obtain $\rho \ge 3\sec\phi$. The bound for ρ is given by $3\sec\phi \le \rho \le 6$. Consider the ϕ value when the upper bound and the lower bound meet, we have $\sec\phi = 2$, $\phi = \frac{\pi}{3}$. The region is rotationally symmetric with respect to z axis, we have $0 \le \theta \le 2\pi$.

The integral can be expressed in spherical coordinates.

$$\begin{aligned}
\iiint_U \frac{1}{x^2+y^2+z^2}dV &= \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{3\sec\phi}^{6} \frac{1}{\rho^2}\cdot\rho^2\sin\phi d\rho d\phi d\theta = 2\pi\cdot\int_0^{\frac{\pi}{3}} \rho\sin\phi|_{3\sec\phi}^{6} d\phi \\
&= 2\pi\int_0^{\frac{\pi}{3}} \sin\phi(6-3\sec\phi)d\phi = 6\pi\int_0^{\frac{\pi}{3}} 2\sin\phi - \tan\phi d\phi \\
&= 6\pi(-2\cos\phi + \ln\cos\phi)_0^{\frac{\pi}{3}} = 6\pi[(-1-\ln 2)+2] = 6\pi\ln\frac{e}{2}.
\end{aligned}$$

Grading policies:

- Knowing the formula $dV = \rho^2\sin\phi d\rho d\phi d\theta$ for spherical coordinates. No partial credit is allowed for this part. **3pts**.
- Writing the domain in spherical coordinates. **4pts**
  - Correct upper bound for ρ. **1pt**
  - Correct lower bound for ρ. **1pt**
  - Correct upper and lower bound for ϕ. **1pt**
  - Correct upper and lower bound for θ. **1pt**
- Evaluating the integral. Partial credits is allowed in this part.**3pts**

It is possible to use cylindrical coordinate. The equation would be like this

$$\begin{aligned}
\iiint_U \frac{1}{x^2+y^2+z^2}dV &= \int_0^{2\pi}\int_3^{6}\int_0^{\sqrt{36-z^2}} \frac{1}{r^2+z^2}\cdot rdrdzd\theta \\
&= 2\pi\int_3^{6}\int_0^{\sqrt{36-z^2}} \frac{1}{2}\frac{1}{r^2+z^2}\cdot dr^2dz = \pi\int_3^6 \ln(r^2+z^2)_0^{\sqrt{36-z^2}}dz \\
&= \pi\int_3^6 (\ln(36)-2\ln z)dz = \pi[\ln(36)z - 2z\ln z + 2z]_3^6 = 6\pi\ln\frac{e}{2}
\end{aligned}$$

The grading policies are the same:

- Knowing the formula $dV = rdrdzd\theta$. No partial credit is allowed for this part. **3pts**.
- Writing the domain in cylindrical coordinates. **4pts**
- Evaluating the integral. Partial credits is allowed in this part.**3pts**