Chapter 2

Electrostatics

2.1 Charges and Fields

Charges:

1. There are two kinds of charges.
2. Charges are *locally* conserved.
3. Charges are quantized and additive.

Gravity has a single kind of charge (energy) parametrized by a non-negative real number. Electromagnetism has two kinds of charges parametrized by a real number. Weak interactions and strong interactions have charges that have to be described by more than one parameters.

> The word "charge" is now used to refer to all sorts of sources of force, e.g. charge for weak interaction or charge for strong interaction. It is also used to refer to all sorts of conserved quantities. Often they have both meanings simultaneously as the associated force arises from a gauge symmetry which implies the conservation of the charge.

The electric field at a point $\mathbf{r}_0$ is often defined as the electric force on a point charge q in the limit

$$\mathbf{E}(\mathbf{r}_0) = \lim_{q\to 0} \mathbf{F}/q. \tag{2.1}$$

The purpose of taking the limit is to ensure that the addition of the test charge does not affect the charge distribution ρ in space. However, since charges are quantized, strictly speaking the limit $q \to 0$ does not make sense, although practically it does for macroscopic phenomena.

The phrase "test charge" refers to a small charge that does not affect the original charge distribution.

Let us try the following definition of $\mathbf{E}$ to avoid taking the limit. If the initial charge distribution is $\rho_0(\mathbf{r})$, after introducing a finite charge q at $\mathbf{r}_0$, the charge distribution becomes $\rho(\mathbf{r}) = \rho_0(\mathbf{r}) + q\delta(\mathbf{r} - \mathbf{r}_0)$, assuming that the charge distribution ρ_0 is fixed by other forces. We define the electric field $\mathbf{E}$ *generated by* ρ_0 at $\mathbf{r}_0$ to be

$$\mathbf{E}(\mathbf{r}_0) = \mathbf{F}/q. \tag{2.2}$$

Note that $\mathbf{E}(\mathbf{r}_0)$ is defined by excluding the field generated by q. Including the field due to a point charge q at $\mathbf{r}_0$, the electric field at $\mathbf{r}_0$ always diverges.

We imagine that $\mathbf{E}$ is fully determined by the charge distribution ρ, and we only care about the final charge distribution ρ, i.e., we believe that $\mathbf{E}$ is independent of ρ_0 and also independent of the mechanism which determines how charges distribute. We can imagine that ρ is held fixed by mechanical forces so that it is independent of q and $\mathbf{r}_0$. However, **this definition of $\mathbf{E}$ is consistent only if $\mathbf{F}$ is proportional to q.**

> The physical meaning of $\mathbf{E}$ can be different for different physicists. There are at least 3 possibilities:
> (1) $\mathbf{E}$ is not a real physical entity. It is defined only as an intermediate step in the middle of a calculation. It can be completely avoided if we compute differently.
> (2) $\mathbf{E}$ corresponds to certain mechanical properties such as stress and strains in an invisible jellylike stuff called "ether". This viewpoint is already abandoned by physicists after the advent of special relativity.
> (3) $\mathbf{E}$ has its fundamental physical meaning and electromagnetism should be described as a field theory.
> Often physicists do not care as long as they know how to compute to get the correct prediction of a measurement.
> The nature of the definition (2.2) of $\mathbf{E}$ seems to suggest that it is fictitious, because the distinction between the test charge q and the rest of the charges ρ is artificial. Feynman and Wheeler wrote a paper claiming that the electric field is not a real physical entity in the context of classical eletrodynamics. They constructed a formulation in which there are only charges, and interactions between them can be described without referring to electric or magnetic fields. An alternative formulation is to use the fields to describe everything. The point charges are described as holes in space where the fields diverge, and the field equations determine how the holes move. When the electric field is viewed as physical entity by itself, the interaction between $\mathbf{E}$ and a charge q in principle can be more general than the linear relation $\mathbf{F} = q\mathbf{E}$. Perhaps this formula has to be modified when the electric field in very strong.
> Both viewpoints make sense. One can say that we never detect $\mathbf{E}$ directly without using a test charge. But we can say equally well that we never see a charge directly, but we only see the effect of the field created by the charge (e.g. we see the light, which is a wave of the field, emitted from the charge).
> The quantum field theory of electromagnetism (quantum electrodynamics – abbreviated as QED) is currently the most precise theory for electromagnetism (if we avoid talking about the unification of EM and weak interaction into the electroweak theory). In this theory both $\mathbf{E}$ and the charges appear as fields. (One can also choose to "integrate out" one of them and use only the other to describe physics.)

2.2 Coulomb's Law

The electrostatic force acting on a charge q by another charge Q is

$\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}$.

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}, \tag{2.3}$$

where ϵ_0 is the **permittivity of free space** and

$$\boldsymbol{\mathscr{r}} \equiv \mathbf{r}_q - \mathbf{r}_Q. \tag{2.4}$$

We interpret this result as the effect of the electric field

$$\mathbf{E}_Q = \frac{1}{4\pi\epsilon_0}\frac{Q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}, \tag{2.5}$$

created by Q at the location of the charge q. Eqs.(2.3) and (2.5) are in agreement with eq.(2.2).

superposition principle

Superposition principle applies to electrostatic forces, so

$$\mathbf{E}_Q(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{Q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i, \qquad \boldsymbol{\mathscr{r}}_i \equiv \mathbf{r} - \mathbf{r}_i. \tag{2.6}$$

Taking the continuum limit, we also have

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d\ell' \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}, \tag{2.7}$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int da' \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}, \tag{2.8}$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}, \tag{2.9}$$

where

$$\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'. \tag{2.10}$$

They can all be represented by the form

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}, \tag{2.11}$$

Q[2.1]: What is the electric field for a constant charge density $\rho =$ constant?

Via direct calculation, one can check that, for any charge distribution, the electric field always satisfy the following two differential equations

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \tag{2.12}$$

$$\nabla \times \mathbf{E} = 0. \tag{2.13}$$

Ex[2.1]: Prove that the superposition principle holds for the two differential equations above.

Q[2.2]: What are the integral equations corresponding to the two differential equations above?

Q[2.3]: Given a field $\mathbf{E}$ satisfying both differential equations above, is there always a charge density $\rho(\mathbf{r})$ so that $\mathbf{E}$ agrees with eq.(2.9)?

Q[2.4]: Let $\mathbf{E} = \mathbf{E}_0$ be a constant vector. What is the corresponding charge density?

Ex[2.2]: Griffiths Prob. 2.55

2.3 Gauss's Law

The integral form of the Gauss's law $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$,

$$\oint_S d\mathbf{a} \cdot \mathbf{E} = \frac{Q_{enc}}{\epsilon_0}, \tag{2.14}$$

allows us to derive the electric field $\mathbf{E}$ easily from the charge density $\rho(\mathbf{r})$ when there is spherical symmetry, cylindrical symmetry, or planar symmetry.

2.4 Potential

The other differential equation (2.13) allows us to define the potential V such that

$$\mathbf{E} = -\nabla V. \tag{2.15}$$

The definition of V is not unique.
Ex[2.3]: Check that the potential V for the electric field (2.9) produced by a charge distribution $\rho(\mathbf{r})$ can be chosen to be

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{\rho(\mathbf{r}')}{\mathscr{r}}. \tag{2.16}$$

2.5 $\mathbf{E}, V, \rho$ and Their Relations

Everything about electrostatics, including the formulas for boundary conditions, work and energy to be introduced below, can be derived from one of the many equivalent descriptions of electrostatics given in this section.

The laws of electrostatics as differential equations of $\mathbf{E}$:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho/\epsilon_0, & (2.17)\\ \nabla \times \mathbf{E} &= 0. & (2.18) \end{aligned}$$

The laws of electrostatics as integral equations of $\mathbf{E}$:

$$\begin{aligned} \oint_S d\mathbf{a} \cdot \mathbf{E} &= Q/\epsilon_0, & (2.19)\\ \oint_C d\mathbf{l} \cdot \mathbf{E} &= 0. & (2.20) \end{aligned}$$

Gauss's law.

Eqs. (2.13) and (2.20) will be modified when we consider time-dependent magnetic field.

The laws of electrostatics as Coulomb's law plus superposition principle in terms of $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}, \quad + \quad \textit{superposition principle} \tag{2.21}$$

In the following, we will use the notation $\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'$, and $\mathscr{r} \equiv |\boldsymbol{\mathscr{r}}|$.

or equivalently,

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d\tau' \, \frac{\rho(\mathbf{r}')\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}.} \qquad (2.22)$$

For point charges q_i at $\mathbf{r}_i$,

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}.} \qquad (2.23)$$

Formulas for surface or line charge densities can be easily written down by noticing that the notion of a small quantity of charge dq can be expressed differently for different situations

$$dq \quad \sim \quad d\tau\rho \quad \sim \quad da\,\sigma \quad \sim \quad dl\,\lambda. \qquad (2.24)$$

Q[2.5]: What is the charge density $\rho(\mathbf{r})$ for a collection of point charges q_i at $\mathbf{r}_i$? Derive (2.23) from (2.22).

The notion of electric potential is not only convenient for many calculations, it will play a more fundamental role when we talk about gauge symmetry.

The definition of V requires the choice of a reference point.

The relations between $\mathbf{E}$ and V:

$$\mathbf{E} = -\nabla V, \qquad (2.25)$$

$$V = -\int d\mathbf{l} \cdot \mathbf{E}. \qquad (2.26)$$

The law of electrostatics as a differential equation of V:

$$\boxed{\nabla^2 V = -\rho/\epsilon_0.} \qquad (2.27)$$

The law of electrostatics as Coulomb's law plus superposition principle in terms of V:

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}}, \qquad + \qquad \textit{superposition principle}} \qquad (2.28)$$

or equivalently,

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d\tau' \, \frac{\rho(\mathbf{r}')}{\mathscr{r}}.} \qquad (2.29)$$

Finally, the law of electrostatics can also be derived from Gauss' law, 3D rotation and translation symmetry, together with superposition principle.

Ex[2.4]: Find $\mathbf{E}$ a distance z above the midpoint between two equal charges q, a distance d apart. Check that the result is consistent with what you would expect when $z \gg d$.

The equivalence among different descriptions of the laws of electrostatics is not rigorous. As an example, consider the configuration with a constant charge distribution $\rho(\mathbf{r}) = c$ constant. The electric field $\mathbf{E} = cx\hat{x}$ satisfies both differential equations (2.12) and (2.13), but the integral (2.22) is not well defined.
It is often assumed in a physical problem that the charge distribution decays to zero sufficiently fast as $r \to \infty$ so that the integral (2.22) is well defined. Throughout this course we will make a lot of physical assumptions unless otherwise states.

2.6 Field Lines

The invisible electric field can be visualized through the concept of **field lines**.

Interpretation of field lines:

1. The direction of the tangent vector of the field lines at a point $\mathbf{r}$ is $\hat{\mathbf{E}}(\mathbf{r})$.
2. The density of the field lines at a point $\mathbf{r}$ (the number of field lines crossing a unit area element with normal vector parallel to $\mathbf{E}(\mathbf{r})$) is $|\mathbf{E}(\mathbf{r})|$.

Following the 2nd point, the total number of field lines out of a surface $\mathcal{S}$ (more precisely, the number of field lines coming out minus the number of field lines going in) is given by

$$\#\text{ of field lines} = \int_{\mathcal{S}} d\mathbf{a} \cdot \mathbf{E} \equiv \text{electric flux}. \tag{2.30}$$

Rules about field lines:

1. Field lines do not intersect and they repel one another.
2. There is always a charge at an endpoint of a field line.
3. The number of field lines generated by a charge q is q/ϵ_0.

The notion of field lines (Gauss law) together with the requirement of certain symmetries are sufficient to determine the electric field in some cases.

1. spherical symmetry (invariant under 3D rotations)
2. cylindrical symmetry (invariant under 2D rotation of the $x-y$ plane and translation along the z-axis)
3. 2D translational symmetry (translation along the $x - y$ plane)

Note that there is a symmetry of 2 degrees of freedom in each case, so that the electric field depends on a single variable in 3D.

Ex[2.5]: Compute the electric field due to a spherical shell of uniform charge distribution in two approaches: (1) Notion of field lines and symmetry. (2) Superposition of Coulomb's law.

Ex[2.6]: Find the electric field in space due to two parallel infinite plane of uniform charge distribution σ and $\pm\sigma$.

2.7 Boundary Conditions

Across a surface charge density σ, there is a discontinuity in the electric field

$$\boxed{\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}},} \tag{2.31}$$

where $\hat{\mathbf{n}}$ is the normal vector of the surface.

The discontinuity is due to the field generated by the surface charge σ.

On the other hand, the electric potential is continuous unless the electric field diverges on the interface

$$\boxed{V_{\text{above}} - V_{\text{below}} = 0.} \tag{2.32}$$

Eq. (2.31) gives the discontinuity of the first derivative of V.

On the boundary, consider a point p and a small patch surrounding the point. The electric field at the point (slightly above or below the boundary) is generated by charges on the patch and charges outside the patch. The field due to charges outside the patch is always continuous at p. Let us denote this field by $\mathbf{E}_o$. The field due to charges on the patch is discontinuous at p. At a scale much smaller than the size of the small patch, the charges near p can be approximated by uniformly distributed charges over a large plate. Hence the field due to the small patch at p immediately above/below the boundary is $\pm\frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$. Thus we have

The patch is "small" in the sense that (1) the surface charge density does not change significantly over the patch, and that (2) the total charge on the small patch is negligible compared with the rest of the charges, so that $\mathbf{E}_o$ does not change significantly when the size of the patch is further reduced.

$$\mathbf{E}_{\text{above}} = \mathbf{E}_o + \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}, \qquad \mathbf{E}_{\text{below}} = \mathbf{E}_o - \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}. \tag{2.33}$$

This implies (2.31). Furthermore, we get

$$\mathbf{E}_o = \frac{1}{2}(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}}), \tag{2.34}$$

which is the electric field used to compute the force on the boundary charges. The pressure (force per unit area) on the boundary due to electrostatic force is thus

$$\textbf{Press.} = \sigma\mathbf{E}_o. \tag{2.35}$$

2.8 Work and Energy in Electrostatics

The work needed to move a charge q across a potential difference ΔV is $W = q\Delta V$. This is path-independent, and so the electrostatic force is conservative.

A simple process for which the work is easy to compute is to slowly change the charge distribution so that it is $\rho(\alpha, \mathbf{r}) = \alpha\rho(\mathbf{r})$, with α slowly tuned from 0 to 1. The linearity of electrostatics implies that $V(\alpha, \mathbf{r}) = \alpha V(\mathbf{r})$.

To compute the potential energy stored in the configuration with a given charge distribution $\rho(\mathbf{r})$, we can consider an arbitrary process that brings charges from infinity to specified locations until the charge distribution is $\rho(\mathbf{r})$. The work done will be independent of the process.

The result of the computation is that, for a continuous distribution of charges, the electrostatic energy W is

$$\boxed{W = \frac{1}{2}\int d\tau\ \rho V = \frac{\epsilon_0}{2}\int d\tau\ E^2.} \tag{2.36}$$

This is the energy it takes to collect charges from spatial infinity and relocate them to construct the given charge distribution ρ.

For point charges, the expressions above diverge, because the energy it takes to concentrate a finite amount of charge on a single point is infinity. We should not count the energy used to create an electron as part of the work we need to do, so one has to take out the divergent contribution due to self interactions. It is

$$\boxed{W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i\neq j}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}. \qquad (2.37)}$$

Ex[2.7]: Compute the potential energy for a charge q uniformly distributed over a spherical surface of radius R.
Q[2.6]: What formula will you use for W if the system consists of both point charges and continuous charge distributions?
Q[2.7]: Is the energy to be associated with the charge or the field or both?
Q[2.8]: Check that (2.36) is different from (2.37) when $\rho(\mathbf{r}) = \sum_i q_i \delta(\mathbf{r} - \mathbf{r}_i)$. Why?

While eq.(2.36) is positive-definite, eq.(2.37) is not, they are not exactly the same. Why?

2.9 Conductor

Charges are free to move around in a conductor. In electrostatics, we only consider static configurations. When all charges are settled to fixed locations, all free charges inside the conductor feel no electrostatic force.

Properties of a conductor in electrostatics (after everything reaches a static state):

1. $\mathbf{E} = 0$ in the bulk.
2. $\rho = 0$ in the bulk.
3. $V =$ constant on the boundary.
4. $\mathbf{E} = \sigma\hat{\mathbf{n}}/\epsilon_0$ on the boundary.

Ex[2.8]: A charge q is placed inside a conducting cavity. The outer surface of the conductor is a sphere of radius R. Find the electric field outside the sphere.

Griffiths Prob. 2.38, screening effect

Ex[2.9]: What is the pressure on a conducting surface when the surface charge density is σ and the electric field on (slightly outside) the surface is $\mathbf{E}$?

2.10 Capacitor

The capacitance of a capacitor is defined as

Farad = Coulomb/Volt.

$$C \equiv \frac{Q}{V}. \qquad (2.38)$$

Its electrostatic energy is

$$W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}. \qquad (2.39)$$

Since W can be computed in other ways, this expression allows us to find C.

This is the work to be done to charge the capacitor.

Often the capacitor is composed of two disconnected pieces of conductor, so that the total charge on the capacitor is 0. But sometimes we also regard a single piece of conductor as a capacitor by imagining that the other piece of conductor is at spatial infinity.

Ex[2.10]: For a parallel plate capacitor with area A and separation d, compute (1) the capacitance C, (2) the change in the potential energy when the separation is changed to d', and the work needed, and check the conservation of energy. (Consider separately the cases of (i) fixed charges and (ii) fixed electric potential difference on the plates.)

$C = \epsilon_0 \frac{A}{d}$.

Exercises

1. Assume that the electron is a spherical shell of radius R with uniform charge density. According to the theory of special relativity, the electrostatic energy $E(R)$ contributes to the rest mass of an electron. That is, $m = m_0 + E(R)/c^2$, where m is the observed mass of an electron, and m_0 is the "real mass" of the electron. It is known from experiments that the mass and charge of an electron is approximately

$$m \simeq 9 \times 10^{-31} \text{ kg}, \qquad q_e \simeq -1.6 \times 10^{-19} \text{ C}. \tag{2.40}$$

If $m_0 > 0$, what is the lower bound of R? What is the energy of a photon if its wave length equals the lower bound of R?

For a photon, the energy is given by $E = hf$, where f is the frequency and $h \simeq 6.6 \times 10^{-34}$ kg m^2/s.

2. (a) Find the electric field at $\mathbf{r} = \hat{\mathbf{z}}z$ due to a charged circular disk of radius R and surface charge density $\rho(r) = \alpha r^2 + \beta r$ for given constants α, β. The disk is centered at the origin on the $x - y$ plane.

   (b) Consider the limit $R \to 0$ with the total charge on the disk fixed (so we need $\alpha \to \infty$). Check whether the result agrees with that of a point charge.

3. Find the charge density corresponding to the electric field $\mathbf{E} = ax\hat{\mathbf{x}}$ (in Cartesian coordinates), and that for $\mathbf{E} = ar\hat{\mathbf{r}}$ (in spherical coordinates). Compare and comment.

4. Two charged spherical conductors of radii a and b separated at a large distance are connected through a very thin conducting wire. Ignoring the charges on the wire, what is the ratio of the electric field strengths near the surface of the two spheres?

5. A hollow spherical shell carries charge density $\rho = kr$ (for a given constant k) in the region $a \leq r \leq b$. Find the electric field and the electric potential in the three regions (i) $r < a$, (ii) $a < r < b$, (iii) $b < r$.

6. Two spheres, each of radius R and carrying uniform charge density $+\rho$ and $-\rho$, respectively, are placed so that they partially overlap. Call the vector from the

positive center to the negative center $\mathbf{d}$. Show that the field in the region of overlap is constant and find its value.

7. For a flat d dimensional space ($d > 1$) it is natural to generalize Gauss' law to d dimensional space as

$$\partial_i E_i = \rho/\epsilon_0, \tag{2.41}$$

and replace $\nabla \times \mathbf{E} = 0$ by

$$\partial_i E_j - \partial_j E_i = 0. \tag{2.42}$$

The divergence theorem can be generalized to higher dimensions as

$$\int_V d\tau\, \partial_i E_i = \oint_S E_i\, da_i, \tag{2.43}$$

where V is a d-dimensional volume and S is its $(d-1)$ dimensional boundary. Similarly, (2.42) implies that locally there exists a function V such that

$$E_i = -\partial_i V. \tag{2.44}$$

(a) Find the electric field $\mathbf{E}$ generated by a point charge q at the origin.

(b) Define the electric potential V by (2.44) and find the electric potential of a point charge q at the origin.

(c) Check that the $d = 2$ result agrees with that of an infinite straight line of linear charge density q.

In your answer, you can use $A(S^n)$ to denote the area of a unit n dimensional sphere.

In d dimensions with Cartesian coordinates $x_1, \cdots, x_d$, let $r = \sqrt{x_1^2 + \cdots + x_d^2}$.

See the appendix if you are interested in computing the area of a unit n-sphere.

8. Assume that the law of electrostatics is changed to

$$\nabla \cdot \mathbf{E} + \lambda \mathbf{E} \cdot \mathbf{E} = \rho/\epsilon_0, \tag{2.45}$$

$$\nabla \times \mathbf{E} = 0, \tag{2.46}$$

where λ is a constant.

(a) Which of the following statement(s) is (are) correct?

i. One can still define the electric potential V as $\mathbf{E} = -\nabla V$.

ii. The superposition principle is still valid.

iii. These two equations are inconsistent.

(b) In order to measure the value of λ, you should do an experiment with ... (1) a very strong electric field, (2) a very weak electric field, (3) a very high potential, (4) a fast changing electric field.

9. A spherical insulator of radius R is charged with a charge density $\rho(r) = qr/(\pi R^4)$. A spherical conducting shell with inner radius a and outer radius b surrounds the insulator with coincident centers at the origin. ($R < a < b$.) The total charge on the conducting shell is Q.

(a) Find the total charges on the spherical surface at (1) $r = a$ and (2) $r = b$.

(b) Find the electric field $\mathbf{E}$ and the electric potential V for (1) $b < r$, (2) $a < r < b$, (3) $R < r < a$ and (4) $0 < r < R$.

(c) Find the total electrostatic energy of the system.

(d) After we connect the outer surface of the conducting shell to the ground (so that its electric potential becomes 0), (i) find the the total charges at (1) $r = a$ and (2) $r = b$, and (ii) find the electric potential $V(r)$ for $0 < r < a$.

10. Two infinite plates extending in the $y - z$ directions are located at $x = 0$ and $x = a$. The surface charge density at $x = 0$ is σ_1 and that at $x = a$ is σ_2.

    (a) Find the electric field $\mathbf{E}(x)$ and electric potential $V(\mathrm{x})$ for (1) $x < 0$, (2) $0 < x < a$, and (3) $a < x$.

    (b) Find the surface charge densities on both sides of the surface for each plate. That is, find the surface charge densities at (1) $x = 0^-$, (2) $x = 0^+$, (3) $x = a^-$, (4) $x = a^+$.

    (c) For $\sigma_1 = -\sigma_2$, find the capacitance per unit area.

11. Consider two coaxial cylindrical conducting shells of radii a and b ($a < b$), respectively. (Their thickness is negligible.) The potential difference between the shells is V_0.

    (a) Find the charges per unit length on each shell.

    (b) Find the capacitance per unit length.

    (c) Find the electrostatic energy per unit length.

    (d) Find the pressure at $r = a$ and that at $r = b$ by (1) computing the Coulomb force (charge density times electric field), (2) computing the changes in electrostatic energy due to changes in a or b.

    (e) If the shells are connected to a battery of constant potential V_0, will the pressure at $r = a$ or $r = b$ be different from the answer to the previous question?

12. Given a material of constant charge density ρ of volume V. What shape should it take in order to create the maximal possible electric field at a given point?

13. Consider a system of conductors as in the figure below. The concentric conducting shells have inner radii a_1, a_2 and outer radii b_1, b_2. The sphere of radius a has charge Q. There is no net charge on the inner shell, and the outer shell has total charge $-Q$. Find (a) the potential $V(r)$ in the region $a < r < a_1$. (b) the surface charge density σ at $r = b_1$. (c) the pressure on the surface at $r = a_2$. (d) the capacitance C of the system. (e) the total electrostatic energy.

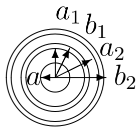

Figure for Prob. 13

14. For the charge distribution $\rho(\mathbf{r}) = q\delta^{(3)}(\mathbf{r}) + k\delta(r-R)$ for given constants q, k, find the total electrostatic energy.

15. Griffiths Prob's. 2.32, 2.37, 2.38, 2.40, 2.46, 2.47, 2.49.

Homework Set 2

1. Griffiths Prob. 2.18.

2. Exercise 1 in this section.

3. Exercise 11 in this section.

4. Exercise 13 in this section.

5. Griffiths Prob. 2.54

Appendix: Area of S^n

A unit n-dimensional sphere (a unit n-sphere, or S^n) can be defined as an n-dimensional subspace in an $(n+1)$-dimensional Euclidean space with Cartesian coordinates $x_1, \cdots, x_n, x_{n+1}$ by

$$S^n = \{(x_1, \cdots, x_{n+1}) | \sum_{i=1}^{n+1} x_i^2 = 1\}. \tag{2.47}$$

To derive the area of S^n, we start with the computation of

$$A_0^2 \equiv \int dxdy\ e^{-(x^2+y^2)/2} = \int_0^{2\pi} d\theta \int_0^{\infty} dr\ re^{-r^2/2} = 2\pi. \tag{2.48}$$

Now consider

$$A_0^{n+1} = A(S^n) \int_0^{\infty} dr\ r^n e^{-r^2/2}, \tag{2.49}$$

where $r = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$ is the radial coordinate in $(n+1)$-dimensional space, and $A(S^n)$ is the area of S^n. The area of S^n is thus

$$A(S^n) = \frac{A_0^{n+1}}{B_n}, \tag{2.50}$$

where

$$B_n \equiv \int_0^{\infty} dr\ r^n e^{-r^2/2}. \tag{2.51}$$

Using integration by part, we find the recursion relation for B_n

$$B_n = (n-1)B_{n-2} \tag{2.52}$$

for $n > 1$. For $n = 1$, it is easy to compute directly

$$B_1 = 1. \tag{2.53}$$

For $n = 0$, we get

$$B_0 = \frac{1}{2} A_0 = \sqrt{\frac{\pi}{2}}. \tag{2.54}$$

Thus for $n > 1$,

$$B_n = \begin{cases} (n-1)(n-3)\cdots 3\cdot 1\cdot \sqrt{\frac{\pi}{2}}, & n = \text{even}, \\ (n-1)(n-3)\cdots 2\cdot 1, & n = \text{odd}. \end{cases} \tag{2.55}$$

We list a few examples of the final result:

$$A(S^1) = 2\pi, \quad A(S^2) = 4\pi, \quad A(S^3) = 2\pi^2, \quad A(S^4) = \frac{8}{3}\pi^2. \tag{2.56}$$