Prob. 1.1: (a)

| $j$ | $N(j)$ |
|---|---|
| 14 | 1 |
| 15 | 1 |
| 16 | 3 |
| 22 | 2 |
| 24 | 2 |
| 25 | 5 |

$$\Rightarrow \langle j \rangle = \frac{14+15+16\cdot 3+22\cdot 2+24\cdot 2+25\cdot 5}{1+1+3+2+2+5}$$

$$= 21 \Rightarrow \langle j \rangle^2 = 441$$

$$\langle j^2 \rangle = \frac{1}{14}\left(14^2+15^2+16^2\cdot 3+22^2\cdot 2+24^2\cdot 2+25^2\cdot 5\right)$$

$$= \frac{6434}{14} = \frac{3217}{7}$$

(b) $\Delta j$:

| $\Delta j$ | $N(j)$ | $\Delta j$ | $N(j)$ |
|---|---|---|---|
| -7 | 1 | 1 | 2 |
| -6 | 1 | 3 | 2 |
| -5 | 3 | 4 | 5 |

$$\sigma^2 = \langle (\Delta j)^2 \rangle = \frac{49+36+75+2+18+16\cdot 5}{14} = \frac{130}{7} \Rightarrow \sigma = \sqrt{\frac{130}{7}}$$

(c)

$$\frac{3217}{7} - 441 \stackrel{?}{=} \frac{130}{7} \Rightarrow \frac{3217-3087}{7} = \frac{130}{7}$$

Prob. 1.3 : (a)

$$\int_{-\infty}^{\infty} A e^{-\lambda(x-a)^2}\, dx, \quad \text{Let } \alpha = \sqrt{\lambda}\,(x-a)$$

$$d\alpha = \sqrt{\lambda}\, dx$$

$$= \frac{A}{\sqrt{\lambda}} \int_{-\infty}^{\infty} e^{-\lambda\alpha^2}\, d\alpha = A\sqrt{\frac{\pi}{\lambda}} = 1 \Rightarrow A = \sqrt{\frac{\lambda}{\pi}}$$

(b)

$$\langle x \rangle = \int_{-\infty}^{\infty} x\,\rho(x)\, dx = A\int_{-\infty}^{\infty} x\, e^{-\lambda(x-a)^2}\, dx, \quad \alpha = (x-a)$$

$$= A\int_{-\infty}^{\infty} (\alpha + a)\, e^{-\lambda\alpha^2}\, d\alpha$$

odd func.

$$= A\int_{-\infty}^{\infty} \alpha\, e^{-\lambda\alpha^2}\, d\alpha + a = a$$

$$\langle x^2 \rangle = A\int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2}\, dx, \quad \alpha = x - a$$

$$= A\int_{-\infty}^{\infty} (\alpha + a)^2 e^{-\lambda\alpha^2}\, d\alpha$$

$$= A\left[\int_{-\infty}^{\infty} \alpha^2 e^{-\lambda\alpha^2} d\alpha + \int_{-\infty}^{\infty} 2\alpha a\, e^{-\lambda\alpha^2} d\alpha + \int_{-\infty}^{\infty} a^2 e^{-\lambda\alpha^2} d\alpha\right]$$

$$I(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda x^2} dx = \sqrt{\frac{\pi}{\lambda}} \Rightarrow I'(\lambda) = \frac{-1}{2\lambda}\sqrt{\frac{\pi}{\lambda}}$$

$$= \int_{-\infty}^{\infty} -x^2 e^{-\lambda x^2} dx$$

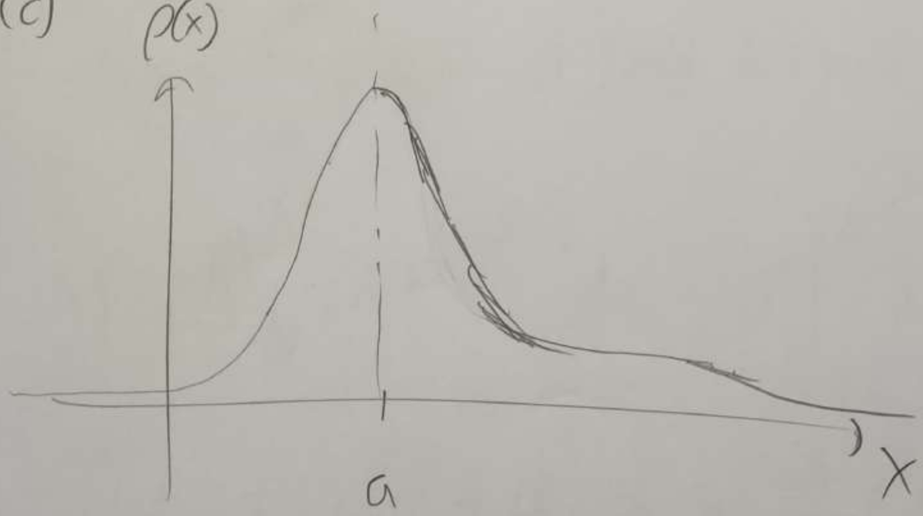
$$\Rightarrow \int x^2 e^{-\lambda x^2} dx = \frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}}$$

$$\langle x^2 \rangle = A \cdot \frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}} + a^2$$

$$= a^2 + \frac{1}{2\lambda} \#$$

$$\sigma^2 = a^2 + \frac{1}{2\lambda} - a^2 = \frac{1}{2\lambda} \Rightarrow \sigma = \sqrt{\frac{1}{2\lambda}} \#$$

(c)



Prob. 1.5

(a) $$1 = \int_{-\infty}^{\infty} \psi^* \psi \, dx = \int_{-\infty}^{\infty} A^2 e^{-2\lambda|x|} dx$$

$$= 2A^2 \int_0^{\infty} e^{-2\lambda x} dx$$

$$= 2A^2 \cdot \left(-\frac{1}{2\lambda}\right) e^{-2\lambda x} \Big|_0^{\infty}$$

$$= \frac{A^2}{\lambda} \Rightarrow A = \sqrt{\lambda} \Rightarrow \psi = \sqrt{\lambda} e^{-\lambda|x|} e^{-i\omega t}$$ #

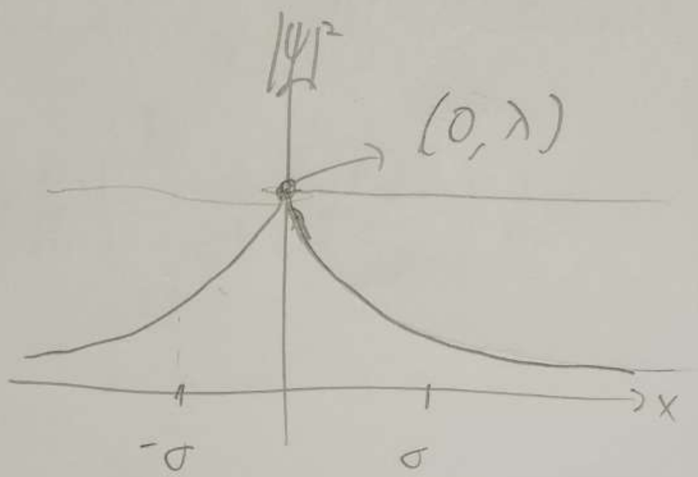
(b) $$\langle x \rangle = \int_{-\infty}^{\infty} x \psi^* \psi \, dx = 0$$ [odd integrand] #

$$\langle x^2 \rangle = A^2 \int_{-\infty}^{\infty} x^2 e^{-2\lambda|x|} dx = 2A^2 \int_0^{\infty} x^2 e^{-2\lambda x} dx$$

Let $t = +2\lambda x \Rightarrow \frac{dt}{2\lambda} = dx \Rightarrow \frac{2A^2}{2\lambda}\left(\frac{1}{2\lambda}\right)^2 \int_0^{\infty} t^2 e^{-t} dt$

$$= \frac{1}{4\lambda^2} \Gamma(3) = \frac{1}{2\lambda^2}$$ #

(c) $\sigma^2 = \frac{1}{2\lambda^2} \Rightarrow \sigma = \sqrt{\frac{1}{2}}\frac{1}{\lambda}$ #




$$2\int_{\sigma}^{\infty} \psi^* \psi \, dx = 2\lambda \int_{\sigma}^{\infty} e^{-2\lambda x} dx$$

$$= e^{-2\lambda\sigma} = e^{-\sqrt{2}}$$ #

Prob. 1.7

$$\frac{d}{dt}\int_{-\infty}^{\infty}\psi^*(-i\hbar)\frac{\partial\psi}{\partial x}\,dx$$

$$\frac{\partial\psi}{\partial t}=\frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{i\hbar}V\psi$$

$$\frac{\partial\psi^*}{\partial t}=-\frac{i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2}-\frac{1}{i\hbar}V\psi^*$$

$$=-i\hbar\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial x}\right)dx$$

$$=-i\hbar\int_{-\infty}^{\infty}\left[\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial x}+\psi^*\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial t}\right)\right]dx$$

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$$\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial x}+\psi^*\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial t}\right)=-\frac{i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x}-\frac{1}{i\hbar}V\psi^*\frac{\partial\psi}{\partial x}$$

$$+\psi^*\frac{i\hbar}{2m}\frac{\partial^3\psi}{\partial x^3}+\frac{1}{i\hbar}\frac{\partial V}{\partial x}\psi^*\psi+\frac{1}{i\hbar}V\frac{\partial\psi}{\partial x}\psi^*$$

$$=\frac{i\hbar}{2m}\left(\psi^*\frac{\partial^3\psi}{\partial x^3}-\frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x}\right)+\frac{1}{i\hbar}\frac{\partial V}{\partial x}\psi^*\psi$$

$$\int_{-\infty}^{\infty}\psi^*\frac{\partial^3\psi}{\partial x^3}dx=-\int_{-\infty}^{\infty}\frac{\partial\psi^*}{\partial x}\frac{\partial^2\psi}{\partial x^2}dx+\cancel{\psi^*\frac{\partial^2\psi}{\partial x^2}\Big|_{-\infty}^{\infty}}\to 0$$

$$-\int_{-\infty}^{\infty}\frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x}=+\int_{-\infty}^{\infty}\frac{\partial\psi^*}{\partial x}\frac{\partial^2\psi}{\partial x^2}dx-\cancel{\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}\Big|_{-\infty}^{\infty}}\to 0$$

$$\frac{d\langle p\rangle}{dt} = -i\hbar\int_{-\infty}^{\infty}\frac{1}{i\hbar}\frac{\partial V}{\partial x}\psi^*\psi\,dx$$

$$= \left\langle -\frac{\partial V}{\partial x}\right\rangle \#$$

Prob. 1.9 $\psi = Ae^{-a[(\frac{m}{\hbar})x^2 + it]}$

(a)

$$1 = \int_{-\infty}^{\infty}\psi^*\psi\,dx = \int A^2 e^{-\frac{2am}{\hbar}x^2}dx = A^2\sqrt{\frac{\pi\hbar}{2am}} = 1$$

$$\Rightarrow A = \left(\frac{2am}{\pi\hbar}\right)^{\frac{1}{4}} \#$$

(b)

$$V\psi = i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2},\quad V = \frac{1}{\psi}\left(i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\right)$$

$$\frac{\partial\psi}{\partial t} = -ai\psi,\quad \frac{\partial^2\psi}{\partial x^2} = \left(\frac{2am}{\hbar}x\right)^2\psi - \frac{2am}{\hbar}\psi$$

$$V = a\hbar + \frac{\hbar^2}{2m}\cdot\left(\frac{2am}{\hbar}x\right)^2 - \frac{\hbar^2}{2m}\frac{2am}{\hbar}$$

$$= 2ma^2x^2 \#$$

(c)

$$\langle x\rangle = \int_{-\infty}^{\infty} x\psi^*\psi\, dx = \int_{-\infty}^{\infty} xA^2 e^{-2\lambda x^2} dx, \quad \lambda = \frac{am}{\hbar}$$

$$= 0 \quad (\text{odd func.})$$

$$\langle x^2\rangle = A^2\int_{-\infty}^{\infty} x^2 e^{-2\lambda x^2} dx, \quad \text{Let } +\alpha = +2\lambda x^2, \; x = \sqrt{\frac{\alpha}{2\lambda}}$$

$$d\alpha = 4\lambda x\, dx$$

$$= 2A^2\int_0^{\infty} \left(\frac{\alpha}{2\lambda}\right)\cdot\left(\frac{1}{4\lambda}\right)\sqrt{\frac{2\lambda}{\alpha}}\, e^{-\alpha} d\alpha$$

$$= 2^{-\frac{3}{2}}\lambda^{-\frac{3}{2}}A^2\int_0^{\infty} \alpha^{\frac{1}{2}} e^{-\alpha} d\alpha; \quad A = \left(\frac{2am}{\pi\hbar}\right)^{\frac{1}{4}} = \left(\frac{2\lambda}{\pi}\right)^{\frac{1}{4}}$$

$$= 2^{-\frac{3}{2}}\lambda^{-\frac{3}{2}}\cdot\left(\frac{2\lambda}{\pi}\right)^{\frac{1}{2}}\cdot\Gamma\left(\frac{3}{2}\right) = \frac{\hbar}{4am}$$

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$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = 0, \quad \langle p^2\rangle = \int \psi^*(-i\hbar)^2\frac{\partial^2\psi}{\partial x^2}\, dx$$

$$= -\hbar^2\int \psi^*\frac{\partial^2\psi}{\partial x^2}\, dx$$

(c

$$= -\hbar^2\int \psi^*\left(\frac{2m}{\hbar^2}\right)\left(2ma^2x^2\psi - i\hbar\frac{\partial\psi}{\partial t}\right) dx, \quad \frac{\partial\psi}{\partial t} = -ai\psi$$

$$\langle p \rangle = 2ma\hbar\left(1 - \frac{2ma}{\hbar}\int x^2 \psi^* \psi \, dx\right)$$

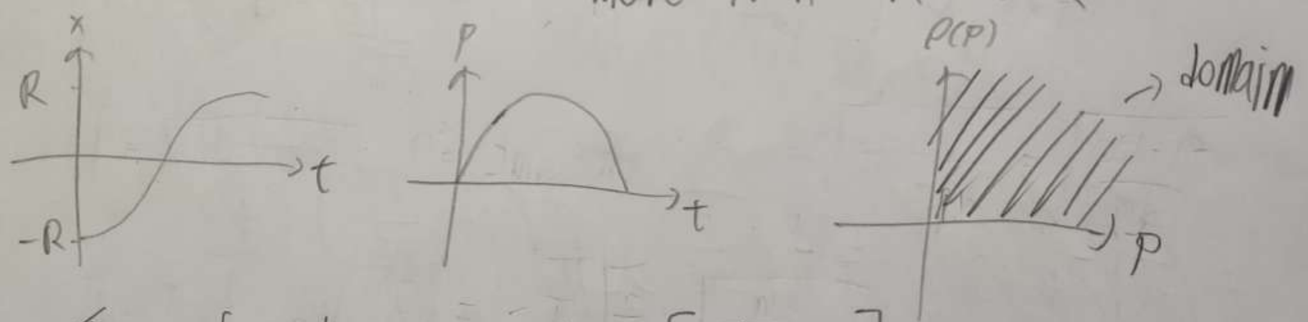
$$= 2ma\hbar\left(1 - \frac{1}{2}\right) = ma\hbar \; \#$$

(d) $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{\hbar}{4am}$, $\sigma_p^2 = am\hbar$

$$\sigma_x \sigma_p = \frac{\hbar}{2} \leq \frac{\hbar}{2}$$

Prob. 1.12 $\quad p = \sqrt{2m(E-V)}$, $\frac{1}{2}kR^2 = E \Rightarrow R = \sqrt{\frac{2E}{k}}$

(a) Consider the oscillator move from $-R$ to $R$




Specify the range $x \in [-R, 0]$, $p \in [0, \sqrt{2mE}]$

Since $\rho(p)$'s domain is $[-\sqrt{2mE}, \sqrt{2mE}]$

and the $\rho(p)$ is even function, $\rho(p) = \rho(-p)$

In the range I specified:

$$\rho(p)\,dp = 2 \cdot \frac{dt}{T} \Rightarrow \rho(p) = \frac{2}{-kxT}$$

$$p=\sqrt{2m\left(E-\frac{1}{2}kx^2\right)} \Rightarrow x=\pm\sqrt{\frac{2mE-p^2}{mk}}$$ ，取負

$$\rho(p) = +\frac{2}{kT}\sqrt{\frac{mk}{2mE-p^2}}, \quad T=2\pi\sqrt{\frac{m}{k}}$$

$$= \frac{1}{\pi}\sqrt{\frac{1}{2mE-p^2}}, \quad p\in\left[0,\sqrt{2mE}\right]$$

$$\rho(p)=\rho(-p) \Rightarrow \rho(p)=\frac{1}{\pi}\sqrt{\frac{1}{2mE-p^2}} \;\#, \quad p\in\left[-\sqrt{2mE},\sqrt{2mE}\right]$$

(b)

$$\langle p\rangle = \int_{-\sqrt{2mE}}^{\sqrt{2mE}} p\rho(p)\,dp = 0 \;\#$$ $p\cdot\rho(p)$ is odd func.

$$\langle p^2\rangle = \int_{-\sqrt{2mE}}^{\sqrt{2mE}} \frac{p^2}{\pi}\sqrt{\frac{1}{2mE-p^2}}\,dp$$

Let $p=\sqrt{2mE}\sin\theta$, $\langle p^2\rangle = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2mE\sin^2\theta\,d\theta$

$$= \frac{1}{2}(2mE) = mE \;\#$$

(c) $\sigma_p=\sqrt{mE}\;\#$, $\quad \sigma_x^2=\langle x^2\rangle = \int_{-R}^{R} x^2\rho(x)\,dx$

$$\rho(x) = \frac{1}{v(x)\,T}, \quad \frac{1}{2}mv^2 + V(x) = E$$

$$\Rightarrow v = \sqrt{\frac{2E - kx^2}{m}}$$

取 $T = \frac{1}{2}(2\pi)\sqrt{\frac{m}{k}} = \pi\sqrt{\frac{m}{k}}, \quad \rho(x) = \frac{1}{\pi}\sqrt{\frac{k}{m}} \cdot \sqrt{\frac{m}{2E - kx^2}}$

$$= \frac{1}{\pi}\sqrt{\frac{k}{2E - kx^2}}$$

$$\sigma_x^2 = \int_{-R}^{R} \frac{1}{\pi} x^2 \sqrt{\frac{k}{2E - kx^2}}\, dx = \frac{1}{\pi}\sqrt{\frac{k}{2E}} \int_{-R}^{R} x^2 \left(1 - \frac{k}{2E}x^2\right)^{-\frac{1}{2}} dx$$

Let $\sin\theta = \sqrt{\frac{k}{2E}}\,x, \quad \sigma_x^2 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2E}{k}\sin^2\theta\, d\theta$

$$\sigma_x = \sqrt{\frac{E}{k}}$$

$$\sigma_x \sigma_p = \sqrt{mE} \cdot \sqrt{\frac{E}{k}} = E \cdot \sqrt{\frac{m}{k}} \quad \#$$

Let $E = \frac{\hbar}{2}\sqrt{\frac{k}{m}}, \quad \sigma_x \sigma_p = \frac{\hbar}{2} \geq \frac{\hbar}{2}$ (uncertainty principle)

Prob. 1.17

(a) $$\frac{\partial \Psi}{\partial t}\Psi^* = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2}\Psi^* - \frac{i}{\hbar}V\Psi\Psi^*$$

$$\frac{\partial \Psi^*}{\partial t}\Psi = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2}\Psi + \frac{i}{\hbar}V^*\Psi^*\Psi$$

$$\frac{dP}{dt} = \int_{-\infty}^{\infty}\left(\frac{\partial \Psi^*}{\partial t}\Psi + \Psi^*\frac{\partial \Psi}{\partial t}\right)dx$$

$$= \int_{-\infty}^{\infty} -\frac{2\Gamma}{\hbar}\Psi^*\Psi\, dx = -\frac{2\Gamma}{\hbar}P \quad \#$$

(b) $$\frac{dP}{dt} = -\frac{2\Gamma}{\hbar}P \Rightarrow \ln P = -\frac{2\Gamma}{\hbar}t$$

$$P = e^{-\frac{2\Gamma}{\hbar}t} = e^{-\frac{t}{\tau}} \quad \#$$

$$\tau = \frac{\hbar}{2\Gamma} \quad \#$$