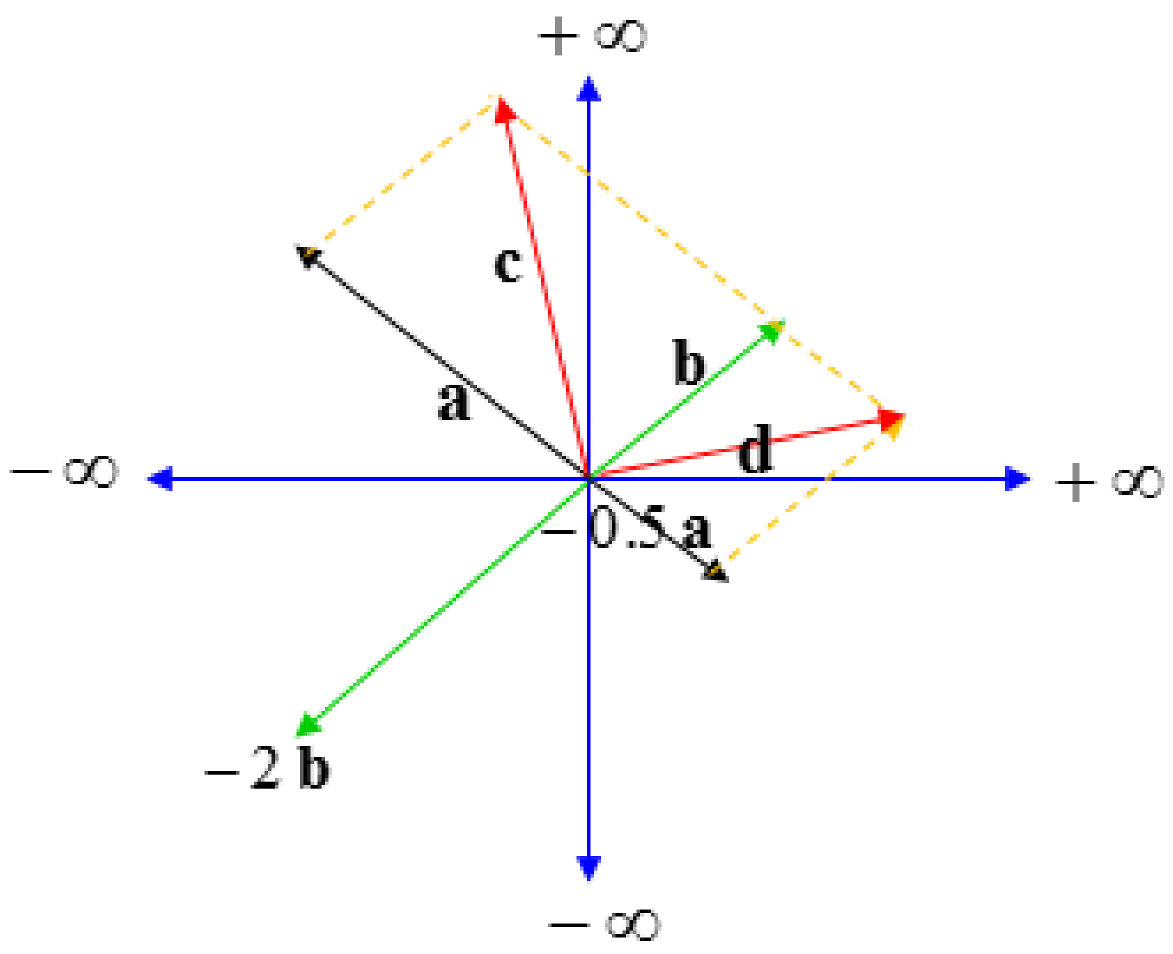
Chapter 2 Vector Space

2-1 Vector Space (向量空間) & Subspace (子空間)



2-1 Vector Space (向量空間) & Subspace (子空間)

Examples of Vector Spaces

$\mathbb{R}^3$

set of real vectors with three components

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \vec{a} + \vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$$

1) given $\vec{a} \in V$ and scalar c, then $c\vec{a} \in V$ ✓

➡ 2) given $\vec{a} \in V$ and $\vec{b} \in V$, then $\vec{a} + \vec{b} \in V$ ✓

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Definition of Vector Space

Let V be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and every scalar (real number) c and d, then V is called a **vector space.**

Addition:

1. $\mathbf{u} + \mathbf{v}$ is in V. — Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutative property
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ — Associative property
4. V has a **zero vector** $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u} + \mathbf{0} = \mathbf{u}$. — Additive identity
5. For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. — Additive inverse

Scalar Multiplication:

6. $c\mathbf{u}$ is in V. — Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ — Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$ — Associative property
10. $1(\mathbf{u}) = \mathbf{u}$ — Scalar identity

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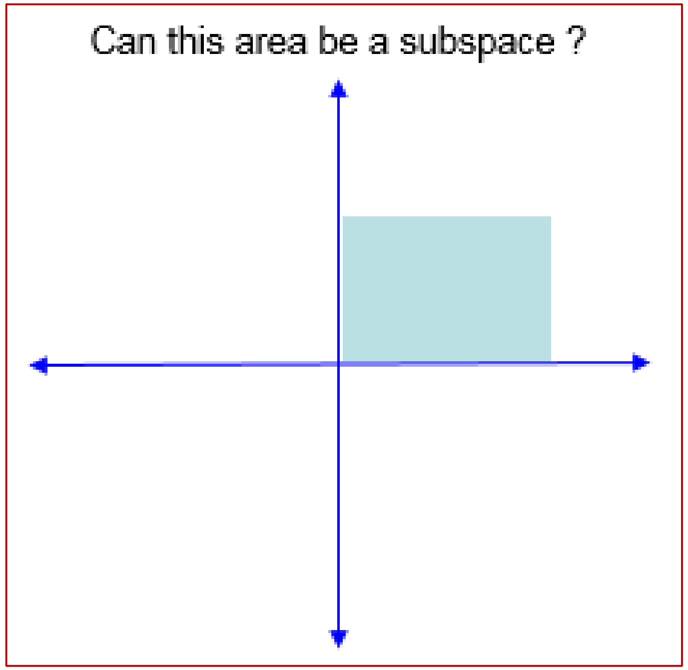
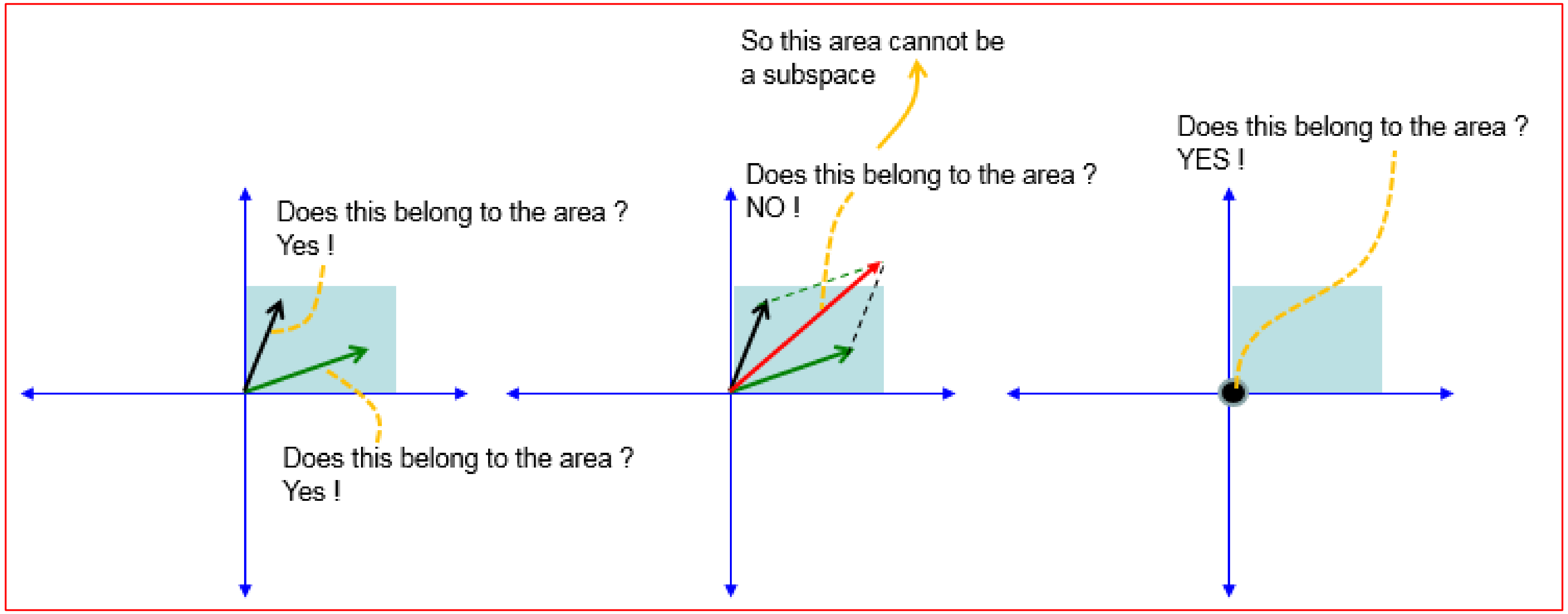
$\mathbf{a}+\mathbf{b} \in W$, $where\ \mathbf{a} \in W, \mathbf{b} \in W$ ---------------- (1)

$k\,\mathbf{a} \in W$, $where\ \mathbf{a} \in W$ ---------------- (2)

$zero\ vector \in W$ ---------------- (3)



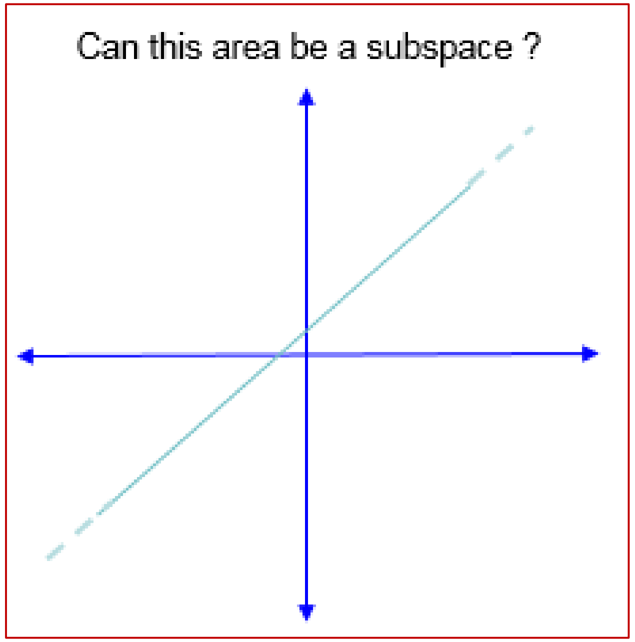


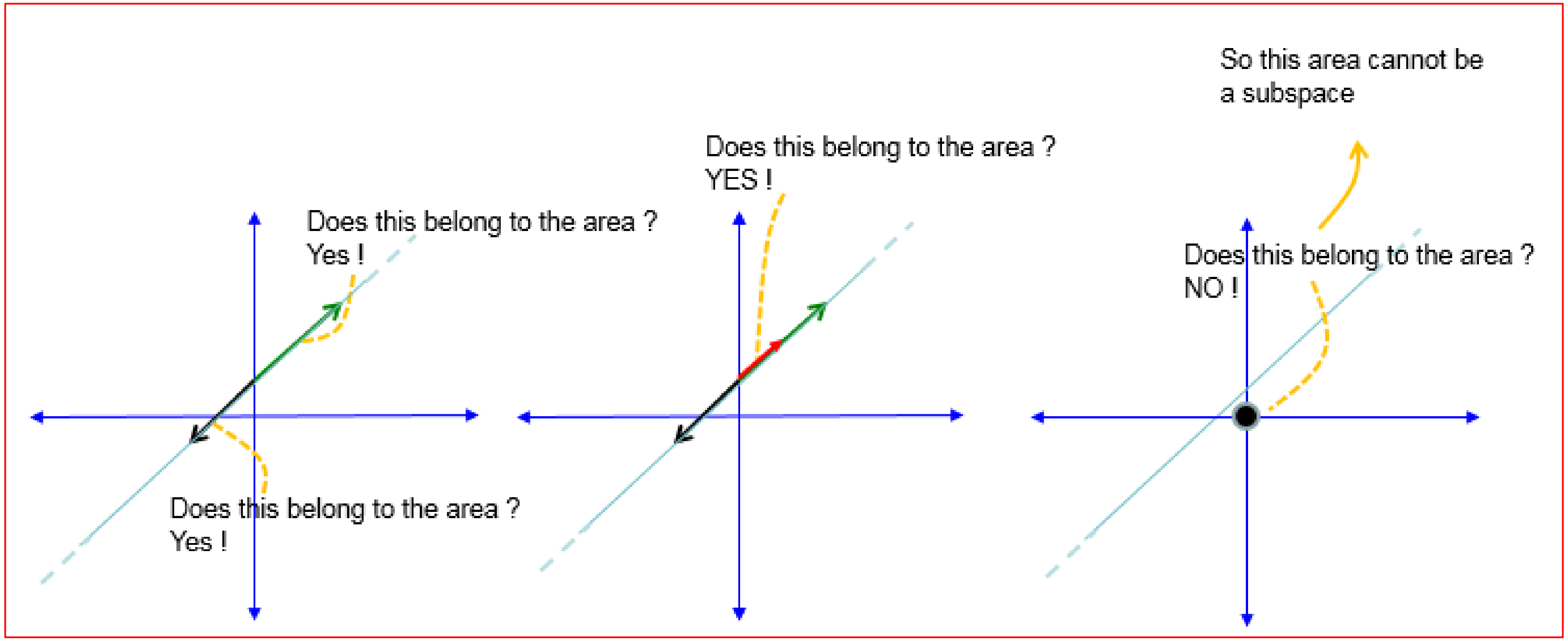
2-1 Vector Space (向量空間) & Subspace (子空間)

$$\mathbf{a}+\mathbf{b} \in W \text{ , where } \mathbf{a} \in W, \mathbf{b} \in W \quad \text{-----------} \quad (1)$$

$$k\,\mathbf{a} \in W \text{ , where } \mathbf{a} \in W \quad \text{-----------} \quad (2)$$

$$\text{zero vector} \in W \quad \text{-----------} \quad (3)$$



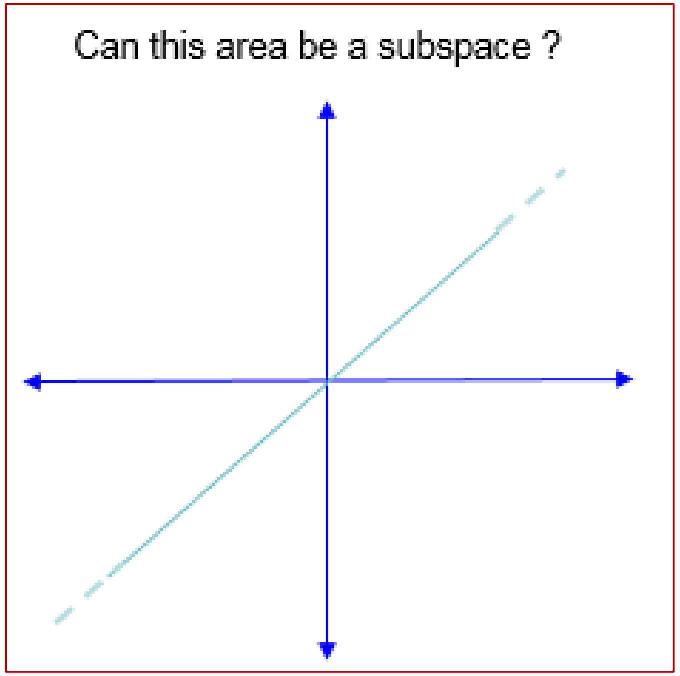


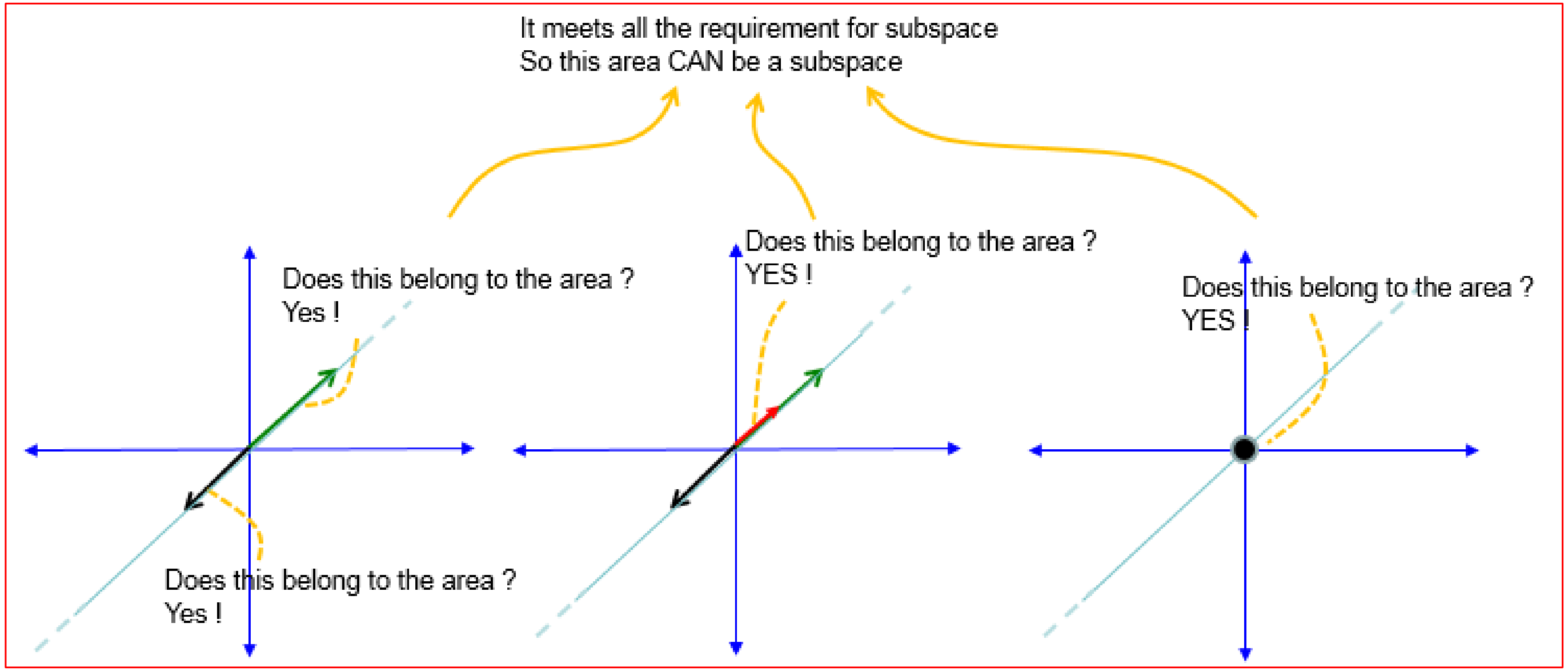
2-1 Vector Space (向量空間) & Subspace (子空間)

$$\mathbf{a}+\mathbf{b} \in W \text{ , } where\, \mathbf{a} \in W, \mathbf{b} \in W \quad \text{- - - - - - - - - - - - - - -} \quad (1)$$

$$k\,\mathbf{a} \in W \text{ , } where\, \mathbf{a} \in W \quad \text{- - - - - - - - - - - - - - -} \quad (2)$$

$$zero\, vector \in W \quad \text{- - - - - - - - - - - - - - -} \quad (3)$$



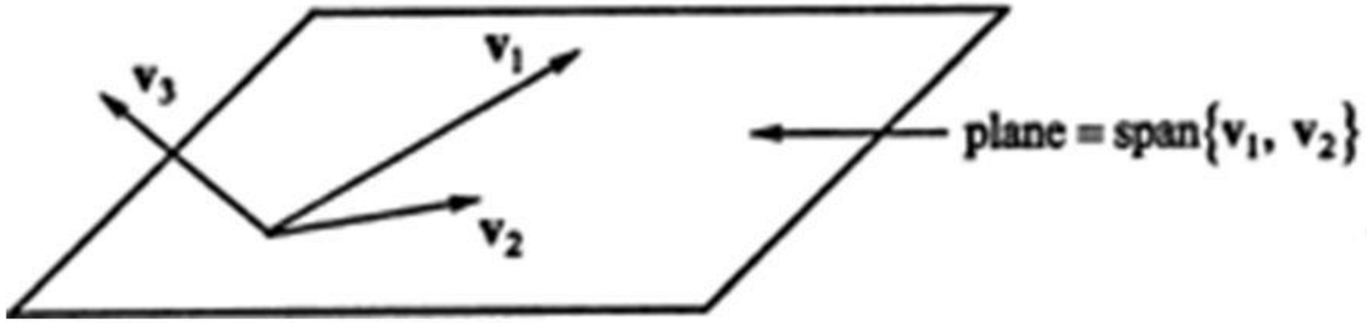


2-1 Vector Space (向量空間) & Subspace (子空間)

Recall: A Subspace

A subspace of a linear space V is called a subspace if:

a) W contains the neutral element 0 of V
b) W is closed under addition
c) W is closed under scalar multiplication



2-2 線性獨立 與 伸展

- 基本名詞和內容：

1) 線性獨立 (L.I.) 與 線性相依 (L.D.)

2) 伸展 (span)

3) 基底 (basis)

4) 維度 (dimension)

2-2 線性獨立

定理

- 已知$\boldsymbol{a}_i$為 R^n之向量，若矩陣 A=[$\boldsymbol{a}_1$ $\boldsymbol{a}_2$ $\boldsymbol{a}_3$ …… $\boldsymbol{a}_n$]$_{nxn}$ ，若且唯若 $\boldsymbol{a}_1$ $\boldsymbol{a}_2$ $\boldsymbol{a}_3$ …… $\boldsymbol{a}_n$ 為 L.D.， 則$|A| = 0$，即 A 為奇異矩陣。

2-2 伸展

定理：伸展 (span)

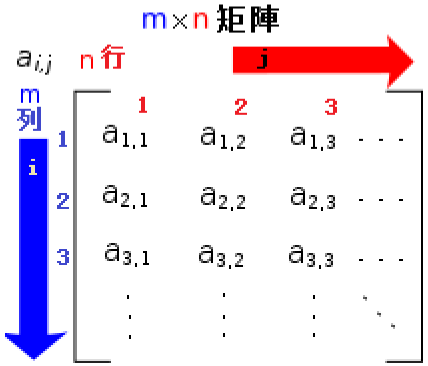
2-3 基底 與維度

定理：基底 (Basis)

定理 : 維度 (Dimension)

- 以物理意義而言，維度就是自由度 (degrees of freedom) 的數目。

2-4 矩陣的四個基本子空間



A_{mxn}

$\mathbf{R^n}$

$\mathbf{R^m}$

定理：列空間 (Row Space)

R(A)

定理：行空間 (Column Space)

C(A)

定理：零空間 (Null Space)

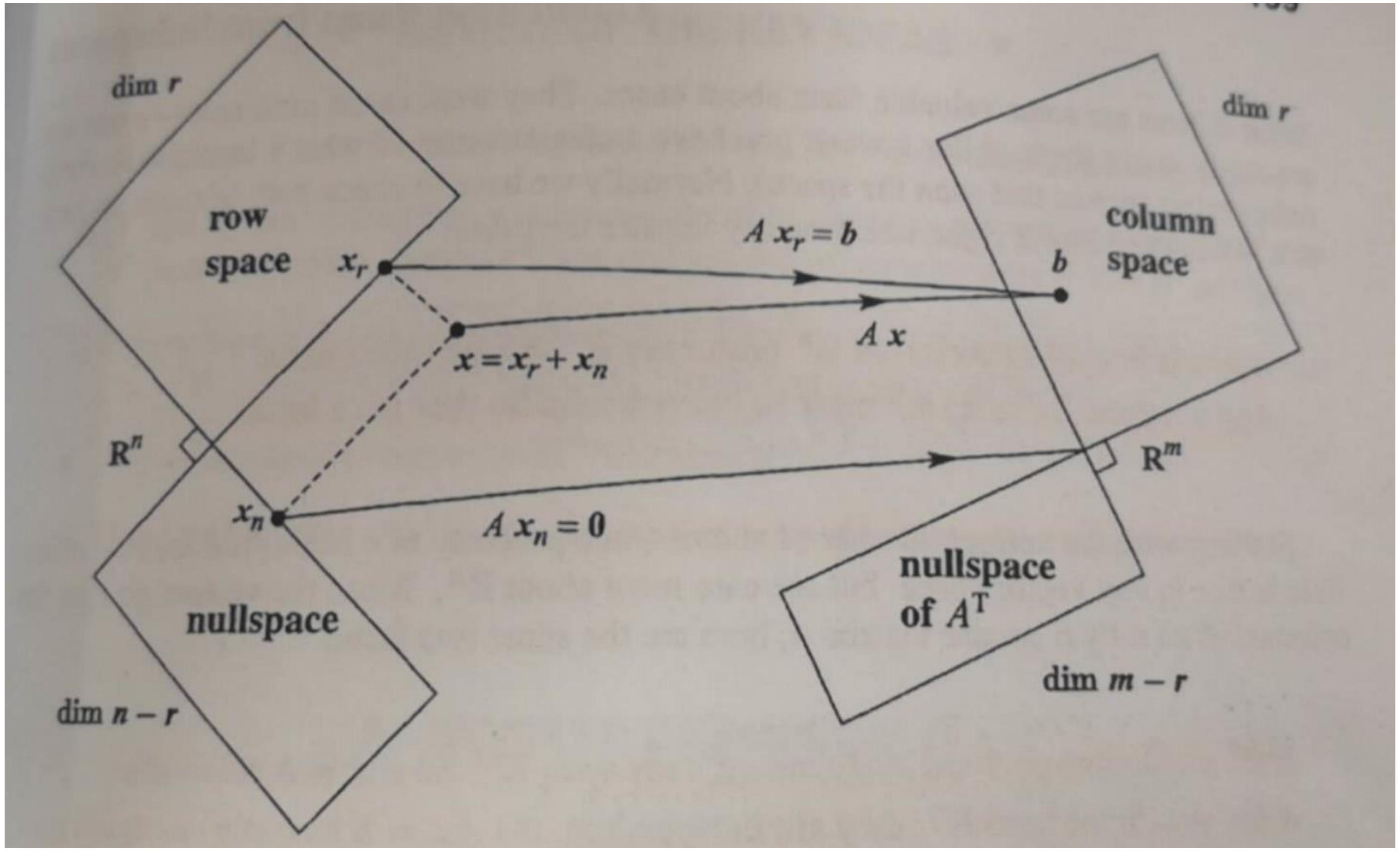
方程組 $Ax=0$ 之解 x 所形成的集合，**N(A)**

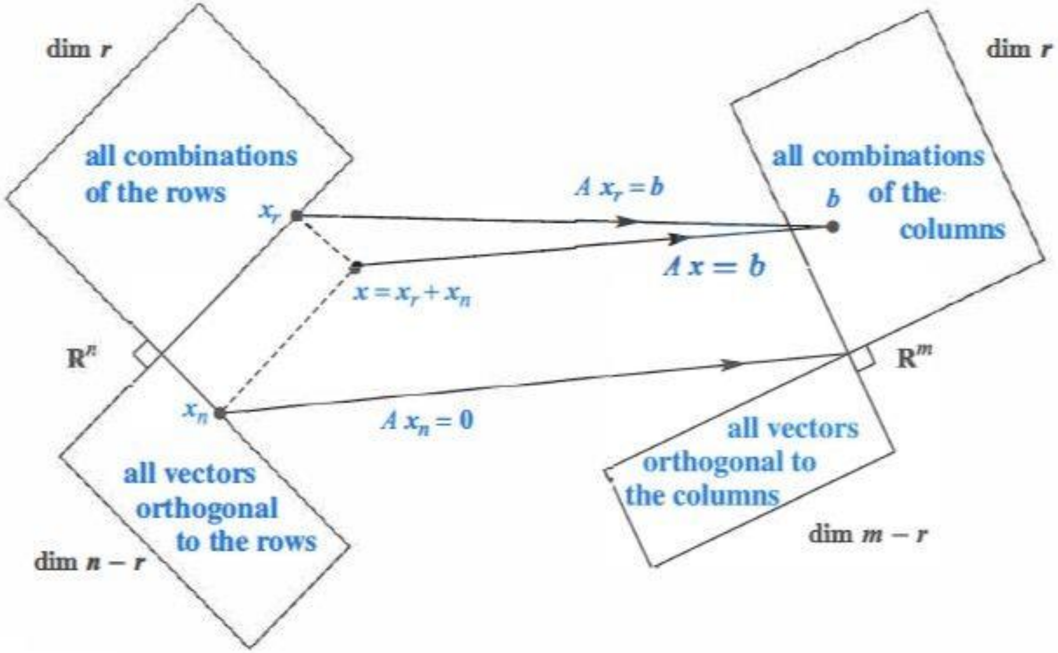
定理：左零空間 (Left Null Space)

方程組 $A^Tx=0$ 之解 x 所形成的集合，**$N(A^T)$**

$(x^TA=0)$

2-4 矩陣的四個基本子空間





...: This update of Figure 4.2 shows the true action of A on $x = x_r + x_n$. Row space vector x_r to column space, nullspace vector x_n to zero.