CSE215
Foundations of Computer Science

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Agenda

- Proof by division into cases
- Disproof

$n^2 + 3n + 2$ is even

- Proof. We consider two cases.
  - Case 1: n is even. n^2 + 3n + 2 is even + even + even, therefore even.
  - Case 2: n is odd. n^2 + 3n + 2 is odd + odd + even, therefore even.
- QED.

Example: Prove the following statement

Proposition If $n \in \mathbb{N}$, then $1+(-1)^n(2n-1)$ is a multiple of 4.

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Proof. Suppose $n \in \mathbb{N}$.
Then n is either even or odd. Let's consider these two cases separately.

Case 1. Suppose n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$, and $(-1)^n = 1$.
Thus $1+(-1)^n(2n-1) = 1+(1)(2 \cdot 2k-1) = 4k$, which is a multiple of 4.

Case 2. Suppose n is odd. Then $n = 2k+1$ for some $k \in \mathbb{Z}$, and $(-1)^n = -1$.
Thus $1+(-1)^n(2n-1) = 1-(2(2k+1)-1) = -4k$, which is a multiple of 4.

These cases show that $1+(-1)^n(2n-1)$ is always a multiple of 4. ■

Disproof

Disproof

- We have been working on this:
  - Given a statement, prove it is true.
- How to do this:
  - Given a statement, prove it is false.
- The process of carrying out this procedure is called disproof.

Principle of disproof

- Suppose you want to disprove a statement P. In other words you want to prove that P is false. The way to do this is to prove that $\sim$ P is true, for if $\sim$ P is true, it follows immediately that P has to be false.

$$\boxed{\textbf{How to disprove } P\textbf{:} \quad \text{Prove} \sim P.}$$

Disproving Universal Statements: Counterexamples

- How to disprove a universally quantified statement such as
  - $\forall x \in S, P(x)$?
- To disprove this statement, we must prove its negation. Its negation is
  - $\sim(\forall x \in S, P(x)) = \exists x \in S, \sim P(x)$.

How to disprove $\forall x \in S, P(x)$.
Produce an example of an $x \in S$ that makes $P(x)$ false.

Disproving Conditional Statements

- How to disprove a conditional statement $P(x) \Rightarrow Q(x)$?
- This statement asserts that for every x that makes P(x) true, Q(x) will also be true.
- The statement can only be false if there is an x that makes P(x) true and Q(x) false.
- Formallya, to disprove this statement, we must prove its negation. Its negation is
  - $\sim(\forall x \in S,\ P(x) \Rightarrow Q(x)) = \exists x \in S, P(x) \land \sim Q(x)$.

How to disprove $P(x) \Rightarrow Q(x)$.

Produce an example of an x that makes $P(x)$ true and $Q(x)$ false.

There is a special name for an example that disproves a statement:
It is called a **counterexample.**

Example: Prove or disprove the following conjecture

Conjecture: For every $n \in \mathbb{Z}$, the integer $f(n) = n^2 - n + 11$ is prime.

- In resolving the truth or falsity of a conjecture, it's a good idea to gather as much information about the conjecture as possible. In this case let's start by making a table that tallies the values of f(n) for some integers n.

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
$f(n)$	23	17	13	11	11	13	17	23	31	41	53	67	83	101

- f (11) = 112 –11+11 = 112 is not prime. The conjecture is false because n = 11 is a counterexample. We summarize our disproof as follows:

Disproof. The statement "*For every $n \in \mathbb{Z}$, the integer $f(n) = n^2 - n + 11$ is prime,*" is **false**. For a counterexample, note that for $n = 11$, the integer $f(11) = 121 = 11 \cdot 11$ is not prime. ■

Disproving by contradiction

- suppose we wish to disprove a statement P.
- We know that to disprove P, we must prove ~ P.
- To prove ~ P with contradiction, we assume ~~ P is true and deduce a contradiction

How to disprove P with contradiction:

Assume P is true, and deduce a contradiction.

Disproving existential Statements

- How to disprove a existential statement $\exists x \in S, P(x)$?

- To disprove it, we have to prove its negation $\sim (\exists x \in S, P(x)) = \forall x \in S, \sim P(x)$.

- Note: this negation is universally quantified. Proving it involves showing that $\sim P(x)$ is true for all $x \in S$, and for this an example does not suffice.

Example: Prove or disprove the following conjecture

Conjecture: There is a real number x for which $x^4 < x < x^2$.

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Conjecture: There is a real number x for which $x^4 < x < x^2$.

Disproof. Suppose for the sake of contradiction that this conjecture is true. Let x be a real number for which $x^4 < x < x^2$. Then x is positive, since it is greater than the non-negative number x^4. Dividing all parts of $x^4 < x < x^2$ by the positive number x produces $x^3 < 1 < x$. Now subtract 1 from all parts of $x^3 < 1 < x$ to obtain $x^3 - 1 < 0 < x - 1$ and reason as follows:

$$\begin{array}{rcccl} x^3 - 1 & < & 0 & < & x - 1 \\ (x-1)(x^2+x+1) & < & 0 & < & (x-1) \\ x^2+x+1 & < & 0 & < & 1 \end{array}$$

Now we have $x^2 + x + 1 < 0$, which is a contradiction because x is positive. Thus the conjecture must be false. ■

Break if time allows

Exercises

To finish by

Prove or disprove

- If $x,y \in R$, then $|x+y| = |x| + |y|$.
- For every natural number n, the integer $2n^2 - 4n + 31$ is prime.
- If $a,b \in N$, then $a + b < ab$
- Every odd integer is the sum of three odd integers.
- Rational + Irrational = Irrational
- Rational * Irrational = Irrational

Solution

- If x, y∈R, then |x+y| = |x| + |y|.
  - False. Counterexample: x = 2, y = -2
- For every natural number n, the integer 2n^2 – 4n + 31 is prime.
  - False. Counterexample: n = 31
- If a,b ∈ N, then a + b < ab
  - False counterexample: a=1, b=1.
- Every odd integer is the sum of three odd integers.
  - True. Any odd number n can be written as (n-2=+1+1
- Rational + Irrational = Irrational
  - True Proof by contradiction. …
- Rational * Irrational = Irrational
  - False. Counterexample: 0 * sqrt(2) = 0

Prove the following statement

- Suppose $x, y \in \mathbb{R}$. If $x^2 + 5y = y^2 + 5x$, then $x = y$ or $x + y = 5$.

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Proof. Suppose $x^2 + 5y = y^2 + 5x$.
Then $x^2 - y^2 = 5x - 5y$, and factoring gives $(x-y)(x+y) = 5(x-y)$.
Now consider two cases.
Case 1. If $x - y \neq 0$ we can divide both sides of $(x-y)(x+y) = 5(x-y)$ by the non-zero quantity $x - y$ to get $x + y = 5$.
Case 2. If $x - y = 0$, then $x = y$. (By adding y to both sides.)
Thus $x = y$ or $x + y = 5$. ■

Prove the following statement

If $n \in \mathbb{Z}$, then $n^2 + 3n + 4$ is even.

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Proof. Suppose $n \in \mathbb{Z}$. We consider two cases.
Case 1. Suppose n is even. Then $n = 2a$ for some $a \in \mathbb{Z}$.
Therefore $n^2 + 3n + 4 = (2a)^2 + 3(2a) + 4 = 4a^2 + 6a + 4 = 2(2a^2 + 3a + 2)$.
So $n^2 + 3n + 4 = 2b$ where $b = 2a^2 + 3a + 2 \in \mathbb{Z}$, so $n^2 + 3n + 4$ is even.
Case 2. Suppose n is odd. Then $n = 2a + 1$ for some $a \in \mathbb{Z}$.
Therefore $n^2 + 3n + 4 = (2a+1)^2 + 3(2a+1) + 4 = 4a^2 + 4a + 1 + 6a + 3 + 4 = 4a^2 + 10a + 8$
$= 2(2a^2 + 5a + 4)$. So $n^2 + 3n + 4 = 2b$ where $b = 2a^2 + 5a + 4 \in \mathbb{Z}$, so $n^2 + 3n + 4$ is even.

In either case $n^2 + 3n + 4$ is even. ■

Problem 4. [5 points]

Prove that $n^2+9n+27$ is odd for all natural numbers n. You can use any proof technique.

Solution

- Proof.
    - We want to prove
        - for any natural number n, $n^2+9n+27$ is odd
    - $n^2+9n+27= n(n+9) + 27$
    - We consider two cases
        - If n is even, $n(n+9) + 27$ is even+odd, thus odd
        - if n is odd, $n(n+9) + 27$ is also even+odd, thus odd
    - Thus $n^2+9n+27$ is odd for whatever n.
- QED.

That is all for today

- Proof by division into cases
- Disproof

Thank you!