CSE215
Foundations of Computer Science

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Exam info and related

- Midterm exam 10-06 (Thursday, here, the same time)
- Open books, open notes, open internet
- A physical copy of the exam will be provided. But bring your scanning device so you can upload PDF to Blackboard
- No chat; no e-communication
- Topics: all lectures to today, including today’s
- Difficulty: homework-like
- Review session 09-29
- No classes for 10-04
- No homework for this week
- Homework 03 will be explained tomorrow; homework 04 next Wednesday recitation

Today

- Examples of wrong proof
- Some proof techniques derived from what we have learned
- SBU exam problems

Some wrong “proof”

Wrong proof 1

Theorem: For all integers k, if $k > 0$ then $k^2 + 2k + 1$ is composite.

"**Proof:** For $k = 2$, $k^2 + 2k + 1 = 2^2 + 2 \cdot 2 + 1 = 9$. But $9 = 3 \cdot 3$, and so 9 is composite. Hence the theorem is true."

Do not prove a universal statement with a single case

Wrong proof 2

Theorem: The difference between any odd integer and any even integer is odd.

"**Proof:** Suppose n is any odd integer, and m is any even integer. By definition of odd, $n = 2k + 1$ where k is an integer, and by definition of even, $m = 2k$ where k is an integer. Then

$$n - m = (2k + 1) - 2k = 1.$$

But 1 is odd. Therefore, the difference between any odd integer and any even integer is odd."

Do not use the same "k" coming from two different "there exists k…"

Wrong proof 3

$\forall$ a and b $\in \mathbb{Z}$, If ab is even,
then a is even or b is even

$\exists$ a, b $\in \mathbb{Z}$, ab is even and a, b are odd

$ab = 2K$, $K \in \mathbb{Z}$.

$a = \frac{2K}{b} = 2\left(\frac{K}{b}\right)$ even

$b = 2\frac{K}{a} = 2\left(\frac{K}{a}\right)$ even.

$\neq$

$\therefore$ QED

b is even if b = 2*k for an integer k. Note “k” must be an integer

Wrong proof 4

- True or false: For any nonnegative integer n, n^2+3n+2 is a composite

- n^2+3n+2= (n+1)(n+2) therefore composite

c is a composite if c = a*b where a!=1 and b!=1. Note the “!=1” parts

Wrong proof 5

- True or false: Suppose r is rational, i is irrational. We have r*i is irrational

- “Proof: Since r is rational r = a/b for some integers a,b. Proof by Contradiction. If r*i is rational, then r*i = m/n for some integers m and n, then i = (mb)/(na), which contradicts with the fact that i is irrational”

When you do division p/q, always check q!=0

Some proof techniques derived from what we have learned

If-and-only-if

- Prove: The integer n is odd if and only if n^2 is odd.
- To prove A if and only if B, first prove A -> B, then prove B ->A

Solution

Proposition The integer n is odd if and only if n^2 is odd.

Proof. First we show that n being odd implies that n^2 is odd. Suppose n is odd. Then, by definition of an odd number, $n = 2a+1$ for some integer a. Thus $n^2 = (2a+1)^2 = 4a^2+4a+1 = 2(2a^2+2a)+1$. This expresses n^2 as twice an integer, plus 1, so n^2 is odd.

Conversely, we need to prove that n^2 being odd implies that n is odd. We use contrapositive proof. Suppose n is not odd. Then n is even, so $n = 2a$ for some integer a (by definition of an even number). Thus $n^2 = (2a)^2 = 2(2a^2)$, so n^2 is even because it's twice an integer. Thus n^2 is not odd. We've now proved that if n is not odd, then n^2 is not odd, and this is a contrapositive proof that if n^2 is odd then n is odd. ■

Equivalent statements

Theorem Suppose A is an $n \times n$ matrix. The following statements are equivalent:

- **(a)** The matrix A is invertible.
- **(b)** The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.
- **(c)** The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- **(d)** The reduced row echelon form of A is I_n.
- **(e)** $\det(A) \neq 0$.
- **(f)** The matrix A does not have 0 as an eigenvalue.

(How to actually prove this in details is not a part of our lecture)

$$\begin{array}{ccccc} (a) & \Longrightarrow & (b) & \Longrightarrow & (c) \\ \Uparrow & & & & \Downarrow \\ (f) & \Longleftarrow & (e) & \Longleftarrow & (d) \end{array}$$

$$\begin{array}{ccccc} (a) & \Longrightarrow & (b) & \Longleftrightarrow & (c) \\ \Uparrow & & \Downarrow & & \\ (f) & \Longleftarrow & (e) & \Longleftrightarrow & (d) \end{array}$$

$$\begin{array}{ccccc} (a) & \Longleftrightarrow & (b) & \Longleftrightarrow & (c) \\ & & \Updownarrow & & \\ (f) & \Longleftrightarrow & (e) & \Longleftrightarrow & (d) \end{array}$$

Uniqueness Proof

- Prove: there is a unique function f defined over R such that $f'(x)=2x$ and $f(0)=3$
- To prove there is a unique x such that P(x), first prove there exist an x, then prove “x and y both satisfy P, then x = y”

Solution

Proof. *Existence*: $f(x) = x^2 + 3$ works.

Uniqueness: If $f_0(x)$ and $f_1(x)$ both satisfy these conditions, then $f_0'(x) = 2x = f_1'(x)$, so they differ by a constant, i.e., there is a C such that $f_0(x) = f_1(x) + C$. Hence, $3 = f_0(0) = f_1(0) + C = 3 + C$. This gives $C = 0$ and so $f_0(x) = f_1(x)$. ∎

Non constructive proof (not a part of exam)

Proposition There exist irrational numbers x, y for which x^y is rational.

Solution

Proposition There exist irrational numbers x, y for which x^y is rational.

Proof. Let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. We know y is irrational, but it is not clear whether x is rational or irrational. On one hand, if x is irrational, then we have an irrational number to an irrational power that is rational:

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^{2} = 2.$$

On the other hand, if x is rational, then $y^y = \sqrt{2}^{\sqrt{2}} = x$ is rational. Either way, we have a irrational number to an irrational power that is rational. ■

Break if time is okay

SBU exam problems

To finish before 12h50

- Prove: Given an integer a, then $a^3 + a^2 + a$ is even if and only if a is even.

Solution

- Suppose a is an integer.
- We first prove $a^3 + a^2 + a$ is even -> a is even.
  - We only need to show a is odd -> $a^3 + a^2 + a$ is odd
  - Suppose a is odd ….
- We then prove a is even -> $a^3 + a^2 + a$ is even
  - Suppose a is even ….

SBU 2021 Final

Problem 6. [5 points]

Prove that if $n^2 + 8n + 20$ is odd, then n is odd for natural numbers n.

Solution

- Proof.
  - We want to prove: for all natural numbers n, $n^2+8n+20$ is odd -> n is odd.
  - We use proof by contradiction to prove the statement above.
  - That is, we assume there exists a natural number n such that $n^2+8n+20$ is odd, and n is even.
  - From “n is even”, we know n^2 must be even, and $8n$ must be even
  - Therefore $n^2+8n+20$ must be even, which contradicts with the assumption above.
- QED.

SBU 2022 Midterm

Problem 6. [5 points]

Let $a_1, a_2, \ldots, a_n$ be real numbers for $n \geq 1$. Prove that at least one of these numbers is greater or equal to the average of the numbers.

Solution

- Proof.
  - We want to prove: for any real numbers a1,…a_n, there exists an a_i where 1<=i<=n, such that a_i>= (a1+a2+…a_n)/n.
  - We use proof by contradiction to prove the statement above.
  - That is, we assume: there exists some real numbers a1,…a_n, such that for all a_i, 1<=i<=n, a_i<(a1+a2+…a_n)/n.
  - From this assumption, we know (a1+a2+…a_n) < n * (a1+a2+…a_n)/n, which is a contradiction.
- QED.

SBU 2020 Midterm

Problem 8. [5 points]

Prove that for all integers a, if a^3 is even, then a is even.

- Proof.
  - We want to prove ________________
  - We use **proof by contradiction** to prove the statement above.
  - That is, we assume __________
  - From this assumption, we know…………, which is a contradiction.
- QED.

SBU 2022 Midterm

Problem 8. [5 points]

Prove that for any two integers a and b, if ab is odd, then a and b are both odd.

- Proof.
  - We want to prove ________________
  - We use **proof by contradiction** to prove the statement above.
  - That is, we assume __________
  - From this assumption, we know…………, which is a contradiction.
- QED.

Problem 6. [5 points]

Prove that the product of any four consecutive integers is a multiple of 4.

- Proof.
  - We want to prove ________________
  - We prove the statement by **division into cases.**
  - Case 1: ____
  - Case 2: _____
  - Thus, ___
- QED.

SBU 2021 Midterm

Problem 8. [5 points]

Prove by contradiction that there are no integers x and y such that $x^2 = 4y + 2$.

- Proof.
    - (formal statement of what we need to prove)
    - (our proof strategy)
    - core proof
- QED.

That is all for today

- Wrong proofs
- If-and-only-if, equivalent statements, uniqueness, non-constructive
- Exam problems. How to present your proof.

SBU 2022 Midterm

Problem 7. [5 points]

Prove or disprove the following statement. If x and y are rational, then x^y is rational.