

CSE215: Lecture 07

Foundations of Computer Science

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September 12, 2022

Many slides taken from Prof. Pramod Ganapathi (SBU). Thanks!

Things

- Attendance check on a weekly basis. Usually done on a day where content is not heavy
- Solutions to online classes will be provided by the end of the lecture day
- Homework grades will be available before the recitation after the due week
- Still seeking a wireless mic. Waiting for IT Department.

Previous lectures

Truth table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Equivalence laws

Laws	Formula	Formula
Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Propositional logic: A formal language to express facts and argue about them

Valid arguments

Premise ₁
Premise ₂
⋮
Premise _m
∴ Conclusion

Inference

Name	Rule	Name	Rule
Modus Ponens	$p \rightarrow q$ p $\therefore q$	Elimination	$p \vee q$ $\sim q$ $\therefore p$
Modus Tollens	$p \rightarrow q$ $\sim q$ $\therefore \sim p$	Transitivity	$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$

Today

- Study propositions with quantifiers: for all integers a , b , and c , $(a + b) + c = a + (b + c)$?
- Predicates & quantifiers
- Negation

Predicate & Quantifiers

Predicate

- A **propositional function** or **predicate** is a sentence that contains **one or more variables**
- A predicate is neither true nor false
- A predicate becomes a proposition when the variables are substituted with specific values
- The **domain** of a predicate variable is the set of all values that may be substituted for the variable

Examples

Symbol	Predicate	Domain	Propositions
$p(x)$	$x > 5$	$x \in \mathbb{R}$	$p(6), p(-3.6), p(0), \dots$
$p(x, y)$	$x + y$ is odd	$x \in \mathbb{Z}, y \in \mathbb{Z}$	$p(4, 5), p(-4, -4), \dots$

Universal quantifier (\forall)

Definition

- Let $p(x)$ be a predicate and D be the domain of x
- A **universal statement** is a statement of the form

$$\forall x \in D, p(x)$$

- Forms:
 - “ $p(x)$ is true for all values of x ”
 - “For all x , $p(x)$ ”
 - “For each x , $p(x)$ ”
 - “For every x , $p(x)$ ”
 - “Given any x , $p(x)$ ”

Existential quantifier (\exists)

Definition

- Let $p(x)$ be a predicate and D be the domain of x
- An **existential statement** is a statement of the form

$$\exists x \in D, p(x)$$

- Forms:
 - “There exists an x such that $p(x)$ ”
 - “For some x , $p(x)$ ”
 - “We can find an x , such that $p(x)$ ”
 - “There is some x such that $p(x)$ ”
 - “There is at least one x such that $p(x)$ ”

Examples

Universal st.s	Domain
$\exists x \in D, x^2 \geq x$	$D = \{1, 2, 3\}$
$\exists x \in \mathbb{R}, x^2 \geq x$	\mathbb{R}

Universal conditional statement (\forall, \rightarrow)

Definition

- A **universal conditional statement** is of the form

$$\forall x, \text{ if } p(x) \text{ then } q(x)$$

Examples

- $\forall x \in \mathbb{R}, \text{ if } x > 2 \text{ then } x^2 > 4$
- \forall real number x , if x is an integer then x is rational
 \forall integer x , x is rational
- $\forall x$, if x is a square then x is a rectangle
 \forall square x , x is a rectangle

Can be extended to **existential conditional statement** (\exists, \rightarrow)

Implicit quantifiers

Examples

- If **a number** is an integer, then it is a rational number
Implicit meaning: \forall number x , if x is an integer, x is rational
- **The number** 10 can be written as a sum of two prime numbers
Implicit meaning: \exists prime numbers p and q such that $10 = p + q$
- If $x > 2$, then $x^2 > 4$
Implicit meaning: \forall real x , if $x > 2$, then $x^2 > 4$

Definition

- Let $p(x)$ and $q(x)$ be predicates and D be the common domain of x . Then implicit quant. symbols $\Rightarrow, \Leftrightarrow$ are defined as:

$$p(x) \Rightarrow q(x) \equiv \forall x, p(x) \rightarrow q(x)$$

$$p(x) \Leftrightarrow q(x) \equiv \forall x, p(x) \leftrightarrow q(x)$$

Propositions with multiple quantifiers

$\forall x \in D, \exists y \in E$, such that $p(x, y)$

$\exists x \in D, \forall y \in E$, such that $p(x, y)$

$\forall x \in D, \forall y \in E$, such that $p(x, y)$

$\exists x \in D, \exists y \in E$, such that $p(x, y)$

Note:

the order of quantifiers matter

- Every lock has a key
- For any lock L , there exists a key K , such that K can unlock L .
- There is a key for every lock
- There exists a key K , such that for any lock L , K can unlock L .

Exercise:

translate to formal logic

Every nonzero real number has a reciprocal

The reciprocal of 4 is $1/4$ (namely 0.25)

- for any nonzero real number r , r has a reciprocal
- for any nonzero real number r , there exists a real number s , such that $r * s = 1$

Exercise:

translate to formal logic

- “There is a person supervising every detail of the production process.”
- \exists person p such that \forall detail d , p supervises d

Negation

Negation of quantified statements (\sim)

Definition

- Formally,

$$\sim (\forall x \in D, p(x)) \equiv \exists x \in D, \sim p(x)$$

$$\sim (\exists x \in D, p(x)) \equiv \forall x \in D, \sim p(x)$$

- Negation of a **universal** statement (“all are”) is logically equivalent to an **existential** statement (“there is at least one that is not”)

Negation of an **existential** statement (“some are”) is logically equivalent to a **universal** statement (“all are not”)

Negation of quantified statements (\sim)

Examples

- All mathematicians wear glasses

Negation (**incorrect**): No mathematician wears glasses

Negation (**correct**): There is at least one mathematician who does not wear glasses

- Some snowflakes are the same

Negation (**incorrect**): Some snowflakes are different

Negation (**correct**): All snowflakes are different

Negation of quantified statements (\sim)

Examples

- \forall primes p , p is odd

Negation: \exists primes p , p is even

- \exists triangle T , sum of angles of T equals 200°

\forall triangles T , sum of angles of T does not equal 200°

- No politicians are honest

Formal statement: \forall politicians x , x is not honest

Formal negation: \exists politician x , x is honest

Informal negation: Some politicians are honest

- 1357 is not divisible by any integer between 1 and 37

Formal statement: $\forall n \in [1, 37]$, 1357 is not divisible by n

Formal negation: $\exists n \in [1, 37]$, 1357 is divisible by n

Informal negation: 1357 is divisible by some integer between 1 and 37

Negation of universal conditional statements

Definition

- Formally,

$$\begin{aligned}\sim (\forall x, p(x) \rightarrow q(x)) &\equiv \exists x, \sim (p(x) \rightarrow q(x)) \\ &\equiv \exists x, (p(x) \wedge \sim q(x))\end{aligned}$$

Examples

- \forall real x , if $x > 10$, then $x^2 > 100$.
Negation: \exists real x such that $x > 10$ and $x^2 \leq 100$.
- If a computer program has more than 100,000 lines, then it contains a bug.
Negation: There is at least one computer program that has more than 100,000 lines and does not contain a bug.

Negation of propositions with multiple quantifiers

$$\begin{aligned} & \sim (\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)) \\ & \equiv \exists x \text{ in } D \text{ such that } \sim (\exists y \text{ in } E \text{ such that } P(x, y)) \\ & \equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y) \end{aligned}$$

$$\begin{aligned} & \sim (\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)) \\ & \equiv \forall x \text{ in } D, \sim (\forall y \text{ in } E, P(x, y)) \\ & \equiv \forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y) \end{aligned}$$

Break if time is good;

Adapted exam exercises

To finish by 1h50pm

Exercise: 2021 midterm-1

Problem 3. [10 points]

Give negations of the following statements. Reasoning is not required.

- (a) [1 point] $p \wedge q$
- (b) [1 point] $p \vee q$
- (c) [1 point] $p \oplus q$
- (d) [1 point] $p \rightarrow q$
- (e) [1 point] $p \leftrightarrow q$
- (f) [1 point] $\forall x, \forall y$ such that $p(x, y)$
- (g) [1 point] $\forall x, \exists y$ such that $p(x, y)$
- (h) [1 point] $\exists x, \forall y$ such that $p(x, y)$
- (i) [1 point] $\exists x, \exists y$ such that $p(x, y)$
- (j) [1 point] $\exists x, \forall y, \exists z$ such that $p(x, y, z)$

Final 2021

Problem 4. [5 points]

Prove that $n^2 + 9n + 27$ is odd for all natural numbers n . You can use any proof technique.

- Students: Express the propositions we need to prove here
- Instructor: Prove

Final 2021

Problem 6. [5 points]

Prove that if $n^2 + 8n + 20$ is odd, then n is odd for natural numbers n .

- Students: Express the propositions we need to prove here
- Instructor: Prove

Final 2021

Problem 5. [5 points]

Prove using contradiction that the cube root of an irrational number is irrational.

- Students: Express the propositions we need to prove here
- Instructor: Prove

Final 2021

Problem 7. [10 points]

Use mathematical induction to prove the following identities.

(a) [5 points] For all natural numbers n ,

$$1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2 \times (n + 1) = \frac{n(n + 1)(n + 2)(3n + 1)}{12}$$

- Students: Express the propositions we need to prove here
- Instructor: Prove

Summary

- Predicates
- Universal and existential quantifiers
- Negations

Thank you!