

## 1 Introduction

This report details the development of the two solution methodologies for the Constrained Shortest Path Problem (CSPP). The first methodology is based on a link/node-based Integer Programming (IP) formulation, employing a general IP solver such as Gurobi or HiGHS. The second methodology is the Branch-and-Price algorithm, which is emphasized more.

No AI tools were utilized for the development and experimental analysis. However, AI assistance was sought for refining the report's language and expression, specifically employing GPT-4.0 for sentence structure and expression enhancement and Grammarly for grammar checks.

Two Julia scripts are submitted:

- **CSPP.jl**: Key functions are defined in this script.
- **main.jl**: This script is the main file performing the experiment by importing functions defined in 'CSPP.jl.' It loads instances, solves them, and saves the results. The results are saved in the 'results' folder as .csv files.

The remainder of this report is organized into three sections. Section 2 introduces the Julia scripts and the methodologies they implement. Section 3 describes the results obtained by the developed codes.

## 2 Methodologies & codes summary

### 2.1 Link/node formulation with general IP solver

The Link/node formulation of CSPP is as follows:

$$\text{minimize } \sum_{(i,j) \in A} c_{ij} x_{ij} \quad (1)$$

$$\text{subject to } \sum_{j: (1,j) \in A} x_{1j} = 1, \quad (2)$$

$$\sum_{j: (i,j) \in A} x_{ij} - \sum_{j: (j,i) \in A} x_{ji} = 0, \quad i \in N \setminus \{1, d\} \quad (3)$$

$$\sum_{j: (j,d) \in A} x_{jd} = 1, \quad (4)$$

$$\sum_{(i,j) \in A} c_{ij} x_{ij} \leq T_{\max}, \quad (5)$$

$$x_{ij} \in \{0,1\}, \quad (i,j) \in A \quad (6)$$

where  $N$  is the node index set, and  $A$  is the arc index set. The decision variable,  $x_{ij}$ , indicates whether an arc is used or not. By writing a code throwing the whole formulation to a general IP solver, we can solve CSPP. It is implemented as the function named ‘IPSolver’ as follows:

---

```

34 function IPSolver(data, Tmax, origin, destination)
35
36 start_node = Int.(data[:, 1])
37 end_node = Int.(data[:, 2])
38 arc_cost = data[:, 3]
39 arc_time = data[:, 4]
40 n_nodes = maximum(vcat(start_node, end_node))
41 n_arcs = length(start_node)
42
43 model = Model(HiGHS.Optimizer)
44
45 @variable(model, x[i = 1:n_arcs], Bin)
46
47 @objective(model, Min, sum(arc_cost .* x))
48
49 @constraint(model, flow_origin, sum(x[start_node .== origin]) - sum(x[end_node .== origin]) == 1)
50 @constraint(model, flow_balance[i = 1:n_nodes; (i != origin) && (i != destination)], sum(x[start_node .== i]) - sum(x[end_node .== i]) == 0)
51 @constraint(model, flow_destination, sum(x[start_node .== destination]) - sum(x[end_node .== destination]) == -1)
52
53 @constraint(model, time, sum(arc_time .* x) <= Tmax)
54
55 optimize!(model)
56
57
58 if primal_status(model) == FEASIBLE_POINT
59     cost_opt, x_opt = objective_value(model), value.(x)
60 else
61     println("Infeasible instance!")
62     cost_opt, x_opt = nothing, nothing
63 end
64
65 path_opt = Set(findall(i->x_opt[i] > 1-EPS, 1:1:n_arcs))
66
67 return cost_opt, path_opt
68
69 end

```

---

The inputs are ‘data,’ ‘Tmax,’ ‘origin,’ and ‘destination.’ The data argument contains the information of the network as  $|N| \times 4$  matrix. Each row represents an arc, and each column represents the tail node index, the head node index, associated cost and time, respectively. The optimal cost, ‘cost\_opt,’ and the corresponding path, ‘path\_opt’ are provided as outputs.

## 2.2 Branch-and-Price

Implementation of Branch-and-Price, which is our main purpose, is written in the function named ‘MyCGSolver.’

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```

197 function MyCGSolver(data, Tmax, origin, destination)
198
199 start_node = Int.(data[:, 1])
200 end_node = Int.(data[:, 2])
201 arc_cost = data[:, 3]
202 arc_time = data[:, 4]
203 n_nodes = maximum(vcat(start_node, end_node))
204 n_arcs = length(start_node)
205
206 ## initialize
207 # containers for algorithms
208 pred_node = [0]

```

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```

209 zero_arcs_node::Vector{Vector{Int}} = [[]]
210 one_arcs_node::Vector{Vector{Int}} = [[]]
211 state_node = [0] # 0: unsolved and not pruned out, 1: integer, 2: fractional, 3: infeasible, 4: pruned out
212 obj_val_node = [0.0]
213 path_ind_node = [Int[]]
214 costs_node = [Number[]]
215 times_node = [Number[]]
216 paths_node = [Set{Int}[]]
217
218 curr_node_ind = 1
219 best_node_ind = 0
220 best_path = Int[]
221 best_obj_val = Inf
222
223 # get an initial feasible solution
224 g = SimpleWeightedDiGraph(n_nodes)
225 for i in 1:n_arcs
226     add_edge!(g, start_node[i], end_node[i], arc_cost[i])
227 end
228 bf_state = bellman_ford_shortest_paths(g, origin)
229 new_path = enumerate_paths(bf_state)[destination]
230 path_arcs = [findfirst(x->(start_node[x]==new_path[i]) && end_node[x]==new_path[i+1], 1:n_arcs) for i in 1:(length(new_path)-1)]
231 tpv = sum(data[path_arcs, 3:4], dims=1)
232 cost, time = tpv[1], tpv[2]
233
234 push!(path_ind_node[curr_node_ind], length(path_ind_node[curr_node_ind])+1)
235 push!(costs_node[curr_node_ind], cost)
236 push!(times_node[curr_node_ind], time)
237 push!(paths_node[curr_node_ind], Set{Int}(path_arcs))
238
239 # algorithm starts
240 while true
241     # solve node by column generation
242     lam_val, obj_val, y0_val, feasible = solve_node!(...)
243
244     # update the current node status & best node
245     if y0_val > EPS # 3: infeasible
246         state_node[curr_node_ind] = 3
247         obj_val_node[curr_node_ind] = Inf
248     elseif sum(lam_val .> 1-EPS) == 1 # 1: integer
249         state_node[curr_node_ind] = 1
250         obj_val_node[curr_node_ind] = obj_val
251         if obj_val < best_obj_val # best solution update
252             best_node_ind = curr_node_ind
253             tpv = findfirst(i->lam_val[i]>1-EPS, path_ind_node[best_node_ind])
254             best_path = paths_node[best_node_ind][tpv]
255             best_obj_val = obj_val
256         end
257     elseif sum(lam_val .> 1-EPS) != 1 # fractional
258         if best_obj_val - obj_val > EPS # 2: not pruned out
259             state_node[curr_node_ind] = 2
260             obj_val_node[curr_node_ind] = obj_val
261             ## branch, strategy: the first fractional arc
262             if feasible
263                 # find the first fractional arc
264                 ...
265             end
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```

290     for bInd in eachindex(state_node)
291         if state_node[bInd] == 0
292             curr_node_ind = bInd
293             break
294         end
295     end
296
297     if curr_node_ind == 0      # if no node to solve, terminate the algorithm
298         break
299     end
300 end
301
302 # solution history
303 solution_history = (pred_node, one_arcs_node, zero_arcs_node, state_node, obj_val_node)
304 best_solution = (best_node_ind, one_arcs_node[best_node_ind], zero_arcs_node[best_node_ind], best_obj_val, best_path)
305
306 return solution_history, best_solution
307
308 end

```

---

The inputs are the same as the IPSolver function. The overall process consists of 3 parts: 1) initialization (lines 206 to 237), 2) solve node (line 242), and 3) branch (lines 244 to 299).

In the initialization phase, the containers for the algorithm are defined, and an initial path to be included in the Restricted Master Problem (RMP) is obtained to start the algorithm. The containers are as follows:

- pred\_node: the predecessor node index
- zero\_arcs\_node: the arc index that is constrained to be 0, by branching
- one\_arcs\_node: the arc index that is constrained to be 1, by branching
- state\_node: state of nodes (0 - unsolved and not pruned out, 1 - integer solution, 2 - fractional solution, 3 - infeasible, 4 - pruned out)
- obj\_val\_node: the obtained optimal solution of each node
- path\_ind\_node: paths index inherited from the predecessor node and obtained from solving pricing sub-problems
- costs\_node: costs of paths in path\_ind\_node
- times\_node: times of paths in path\_ind\_node
- paths\_node: selected arcs set of paths in path\_ind\_node

As the Branch-and-Price algorithm progresses, the containers are updated to keep the information of each node. The initial path is obtained using the Bellman-Ford algorithm, which does not guarantee the feasibility of the path in the CSPP; however, starting from this solution is acceptable.

The node solution procedure is written as the supporting function named 'solve\_node!'

---

```

72 function solve_node!(...)
73
74     ## select paths satisfying the node conditions
75     path_ind_rmp = Int[]
76
77     if isempty(one_arcs)
78         path_ind_rmp = copy(path_ind)
79     else

```

```

80     for pInd in path_ind, arc in paths[pInd]
81         if (arc in one_arcs) && !(pInd in path_ind_rmp)
82             push!(path_ind_rmp, pInd)
83         end
84     end
85 end
86
87 # delete zero arcs
88 if !isempty(zero_arcs)
89     for pInd in path_ind_rmp, arc in paths[pInd]
90         if (arc in zero_arcs) && (pInd in path_ind_rmp)
91             deleteat!(path_ind_rmp, findfirst(x->x==pInd, path_ind_rmp))
92         end
93     end
94 end
95
96 M = 100.0          # Big-M
97 while true
98     # restricted master problem
99     rmp = Model(HiGHS.Optimizer)
100
101     @variable(rmp, y0 >= 0)                # artificial variable facilitating the solution procedure
102     @variable(rmp, lam[path_ind_rmp] >= 0) # path variables
103     @objective(rmp, Min, M*y0 + sum(costs[path_ind_rmp] .* lam)) # cost minimization
104     @constraint(rmp, resource, sum(times[path_ind_rmp] .* lam) <= Tmax) # resource constraint, 1
105     @constraint(rmp, convexity, y0 + sum(lam) == 1) # convexity constraint, 0
106     optimize!(rmp) # solve the relaxed RMP
107
108     # results
109     y0_val = value(y0)
110     lam_val = value(lam)
111     obj_val = objective_value(rmp)
112     pi0 = dual(convexity)
113     pi1 = dual(resource)
114
115     # if infeasible
116     if y0_val > EPS
117         pi0 = M
118     end
119
120     # pricing sub-problem
121     psp = Model(HiGHS.Optimizer)
122     @variable(psp, x[i = 1:n_arcs], Bin)
123     @objective(psp, Min, sum((arc_cost.-arc_time.*pi1) .* x))
124     @constraint(psp, flow_origin, sum(x[start_node .== origin] - sum(x[end_node .== origin]) == 1)
125     @constraint(psp, flow_balance[i = 1:n_nodes; (i != origin) && (i != destination)], sum(x[start_node .== i]) - sum(x[end_node .== i]) == 0)
126     @constraint(psp, flow_destination, sum(x[start_node .== destination]) - sum(x[end_node .== destination]) == -1)
127     if !isempty(one_arcs)
128         @constraint(psp, node_ones[i = one_arcs], x[i] == 1)
129     end
130     if !isempty(zero_arcs)
131         @constraint(psp, node_zeros[i = zero_arcs], x[i] == 0)
132     end
133     @constraint(psp, time, sum(arc_time .* x) <= Tmax)
134     optimize!(psp)
135
136     # no feasible path within node
137     if primal_status(psp) != FEASIBLE_POINT
138         lam_val_full = zeros(Float64, length(path_ind))
139         lam_val_full[path_ind_rmp] = lam_val.data
140         return lam_val_full, obj_val, y0_val, false
141     end
142
143     reduced_cost = objective_value(psp) - pi0
144
145     if y0_val > EPS # RMP is infeasible
146         x_opt = value(x)
147         new_path = Set(findall(i->x_opt[i]>(1-EPS), collect(1:1:n_arcs)))
148         if new_path in paths[path_ind_rmp] # no column to generate
149             lam_val_full = zeros(Float64, length(path_ind))
150             lam_val_full[path_ind_rmp] = lam_val.data
151             return lam_val_full, obj_val, y0_val, true
152         elseif reduced_cost < -EPS
153             x_opt = value(x)
154             new_path = Set(findall(i->x_opt[i]>(1-EPS), collect(1:1:n_arcs)))
155             new_cost = sum(arc_cost[collect(new_path)])
156             new_time = sum(arc_time[collect(new_path)])
157             # add column to the RMP
158             push!(path_ind, length(path_ind)+1)
159             push!(costs, new_cost)
160             push!(times, new_time)

```

```

161         push!(paths, new_path)
162         push!(path_ind_rmp, length(path_ind))
163     else
164         M *= 10.0
165     end
166 else # RMP is feasible
167     if reduced_cost < -EPS # find a new column
168         x_opt = value.(x)
169         new_path = Set(findall(i->x_opt[i]>(1-EPS), collect(1:n_arcs)))
170         new_cost = sum(arc_cost[collect(new_path)])
171         new_time = sum(arc_time[collect(new_path)])
172         # add column to the RMP
173         push!(path_ind, length(path_ind)+1)
174         push!(costs, new_cost)
175         push!(times, new_time)
176         push!(paths, new_path)
177         push!(path_ind_rmp, length(path_ind))
178     else # no column to generate
179         lam_val_full = zeros{Float64, length(path_ind)}
180         lam_val_full[path_ind_rmp] = lam_val.data
181         return lam_val_full, obj_val, y0_val, true
182     end
183 end
184 end
185 end
186 end

```

---

The inputs of this function are omitted in this report due to its extensive length of codes. In this function, the RMP and pricing sub-problem are solved iteratively. To keep generating columns even if the RMP is infeasible with the current columns, an artificial decision variable ‘y0’ is introduced in the RMP.

This process has two terminal conditions: First, it terminates if no feasible path satisfies the branching constraints. Specifically, if branching mandates that certain  $x_{ij}$  should be kept 0 or 1, and no path satisfies all those conditions, then the node solution process terminates. Second, the process also terminates if there is no column with a negative reduced cost.

Finally, after solving a node, the branch process is initiated. Each node is labeled following the results obtained by the solution process. If the solution is infeasible, integer, or dominated by the current best solution, the node does not branch. However, if the solution is fractional and not dominated by the best solution, the node does branch. To branch out, the first fractional arc in the data that has not been selected to branch is chosen. Two branches are generated, whose selected arc values are 0 and 1, respectively. The details of the codes (lines 263 to 280) can be found in the Julia script.

After the branching process is over, the algorithm finds the next node to solve. If no node remains unsolved, the whole algorithm terminates.

### 3 Results: test problems

Two given problems in the assignment guide were solved using the IPSolver and the MyCGSolver functions. Both solvers provided the same optimal solution for each problem. Since the Branch-and-Price algorithm is our main target, this section will show its results.

The experiments were performed on my personal computer located in my laboratory, N7-2 3332. The specifications of the computer, language, and used packages are in Table 1.

Table 1: Experimental Environment

Resource	Specification
CPU	13th Gen Intel(R) Core(TM) i9-13900K
RAM	128GB
OS	Windows 11
Language	Julia 1.10.0
Used packages	Graphs (v1.9.0) HiGHS (v1.9.0) JuMP (v1.20.0) JLD (v0.13.5) SimpleWeightedGraphs (v1.4.0)

The results of the first problem are provided as Figure 1. The root node (Node 1) yielded a fractional solution. Following the branching strategy outlined in the previous section, the first fractional arc, (1,2), was chosen for branching. Both Nodes 2 and 3 resulted in integer solutions, eliminating the need for further branching. Node 3 presented the optimal solution with a cost of 13 and the path 1->3->2->4->6

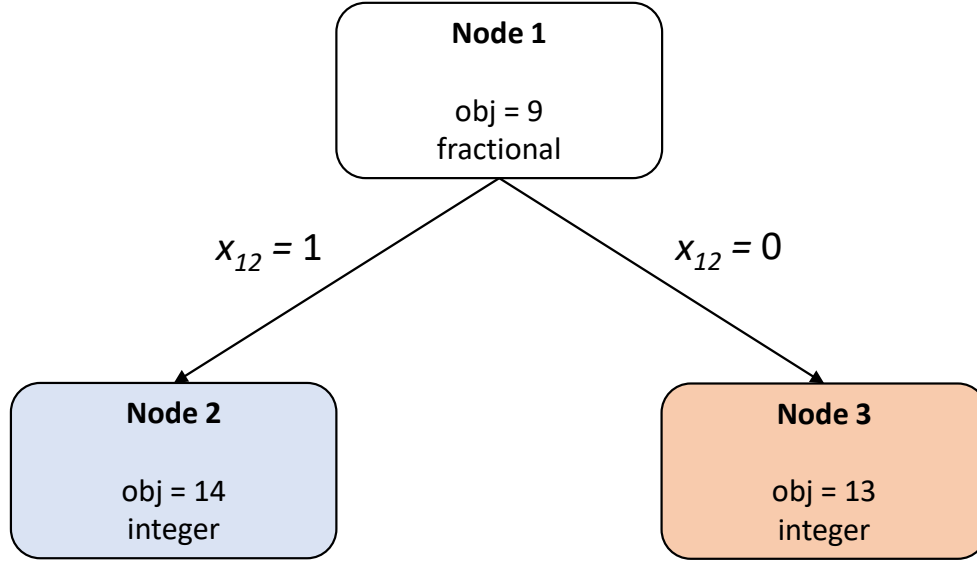


Figure 1: Branch-and-Price Results of Problem 1

The results of the second problem are provided as Figure 2. It gives a more interesting scenario than the first problem. Given that the root node yielded a fractional solution, the first fractional arc (1,2) was chosen to branch. Node 2 resulted in another fractional solution, leading to the creation of Nodes 4 and 5 by respectively setting the decision variable for arc (2,5) to 1 and 0. Subsequently, Node 3 achieved an integer solution. Although Node 4 also produced a fractional solution, it did not undergo further branching as it was dominated by Node 3. Similarly, Node 5 yielded an integer solution and hence did not branch further. Finally, Node 3 emerged as providing the optimal

solution, with a cost of 14 and a path of 1->3->6->10.

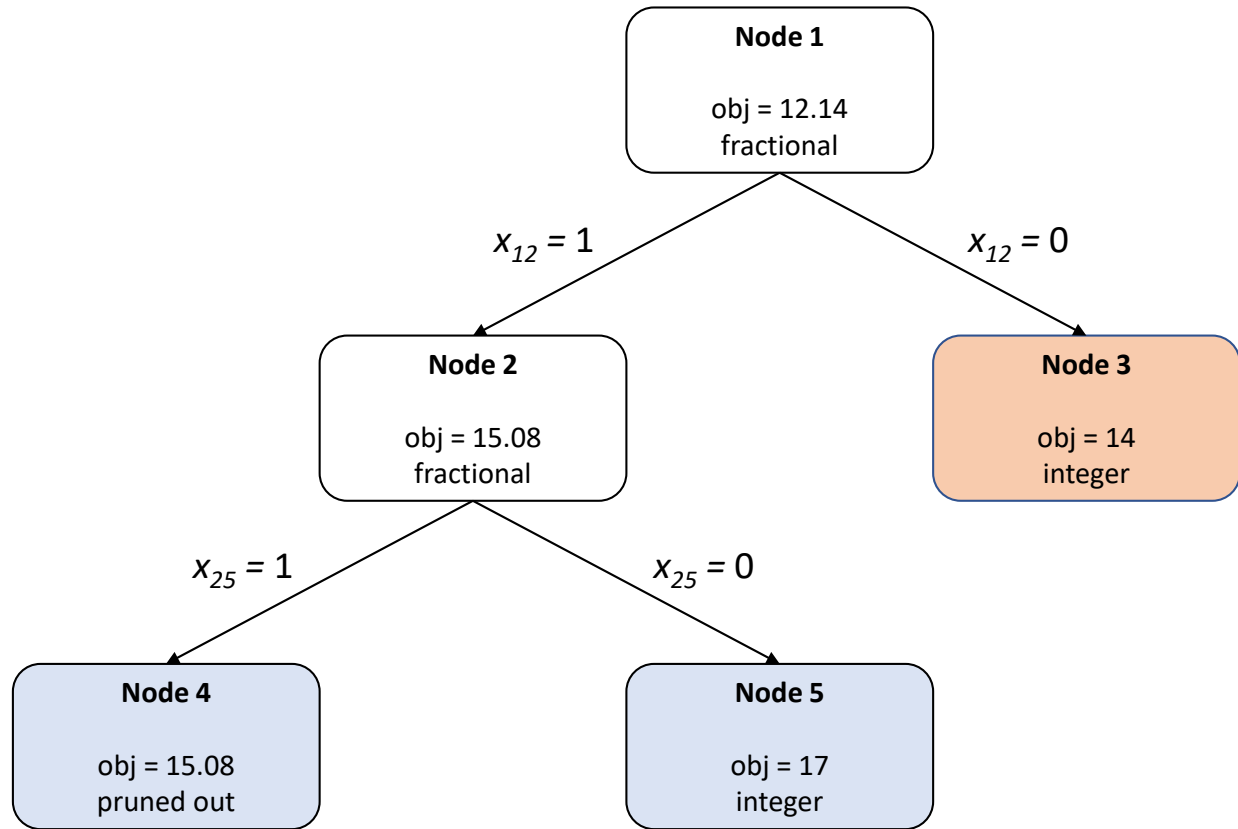


Figure 2: Branch-and-Price Results of Problem 2

## 4 Conclusions

To summarize,

- Julia functions for solving CSPP were developed in two ways: 1) link/node-based formulation with a general IP solver, 2) Branch-and-Price algorithm
- The solution procedure of Branch-and-Price algorithm was visualized

## References

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