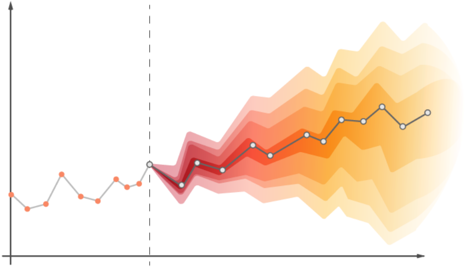
Time Series Regression Models

DS-5740 Advanced Statistics



Overview: Week 1

Goals for the Week

- Consider factors involved in forecasting
- Make your first forecast and see forecasts
- Cover linear regression models with time series
- Make forecasts and check their accuracy

Syllabus

- Meet-up Hours:
  - Alex: by appointment over Calendly
  - Danni: TBD
- Assignments:
  - Due on Sundays at 11:59pm (.Rmd or .html over Brightspace)
  - Late work policy: must be turned in 1 week after assignment due date
  - 12 assignments but only your *10* highest grades count

Forecasting Project Rubric

What can we forecast?

Difficulty of Forecasts

- daily electricity demand in three days
- time of sunrise this day next year
- Google stock price in 6 months (USD)
- maximum temperature tomorrow
- next week's gas prices (USD)

Difficulty of Forecasts

1. time of sunrise this day next year
2. maximum temperature tomorrow
3. daily electricity demand in three days
4. next week's gas prices (USD)
5. Google stock price in 6 months (USD)

Factors Affecting Difficulty

1. knowledge of variables involved
2. how much data are available
3. similarity of future to past
4. whether forecasts are useful

Consideration of Factors

1. time of sunrise this day next year

Consideration of Factors

1. time of sunrise this day next year
2. maximum temperature tomorrow

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Consideration of Factors

1. time of sunrise this day next year
2. maximum temperature tomorrow
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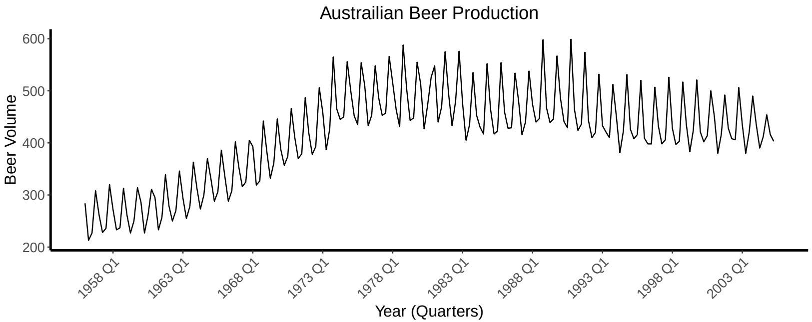
4. next week's gas prices (USD)
5. Google stock price in 6 months (USD)

Make a Forecast

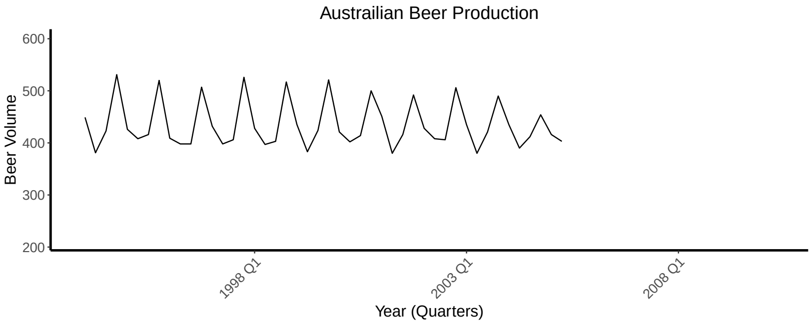
Your Chance to Forecast

forecast: an estimate of the probabilities of possible futures

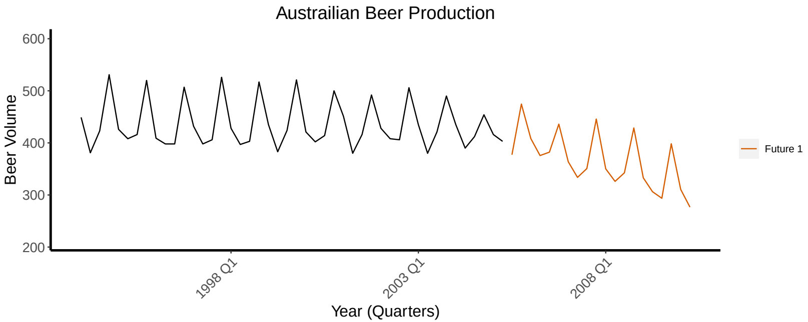


1. Problem definition
2. Gathering information
3. Preliminary (exploratory) analysis
4. Choosing and fitting models
5. Using and evaluating a forecasting model

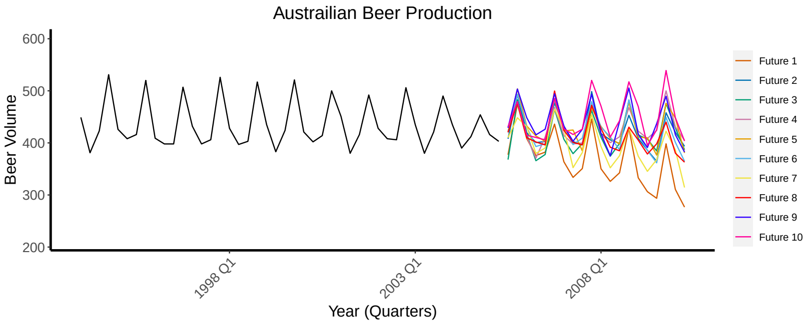
forecast: an estimate of the probabilities of possible futures



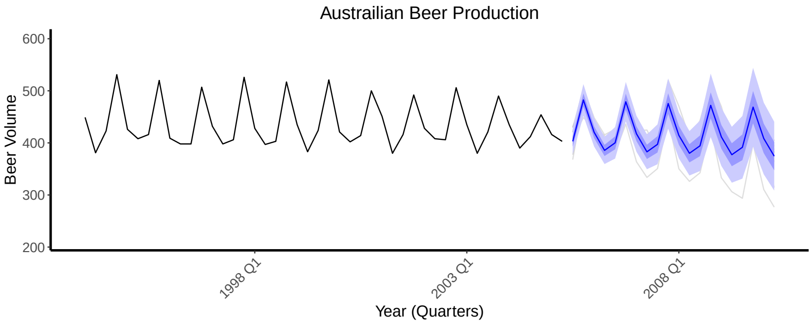
forecast: an estimate of the probabilities of possible futures



forecast: an estimate of the probabilities of possible futures

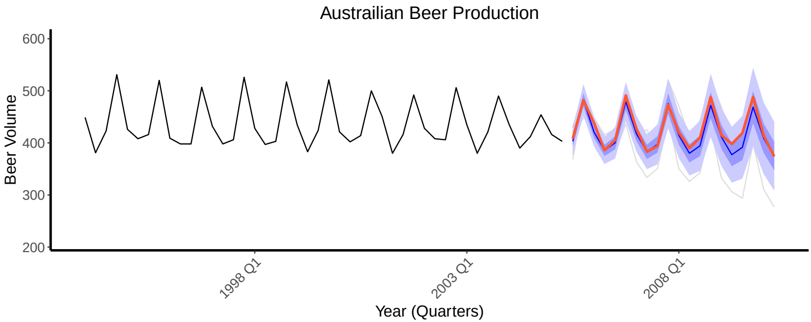


forecast: an estimate of the probabilities of possible futures



Make a Forecast | Ground Truth

forecast: an estimate of the probabilities of possible futures



Times Series Linear Model (TSLM)

$$y = \beta_0 + \sum_{k}^{n} \beta_k x_k + \epsilon$$

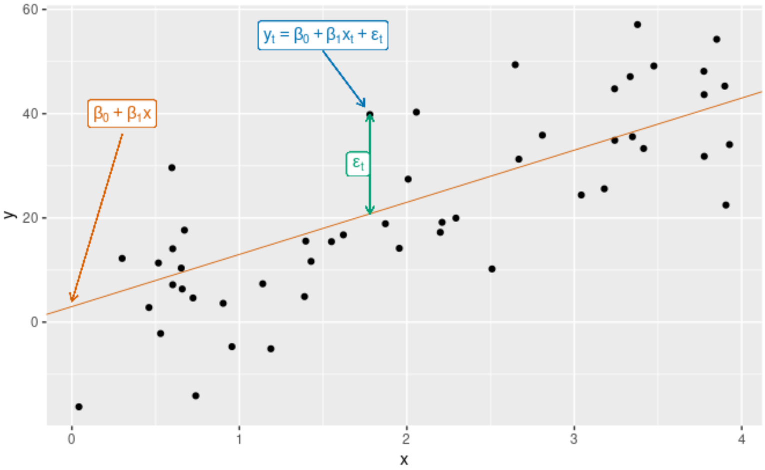
outcome

sum of weights by predictor

$$y = \beta_0 + \sum_k^n \beta_k x_k + \epsilon$$

intercept

error



$y_t = \beta_0 + \beta_1 x_t + \varepsilon_t$
$\beta_0 + \beta_1 x$
ε_t
60
40
20
0
y
0
1
2
3
4
x

$$y_t = \beta_0 + \sum_{k}^{n} \beta_k x_{k,t} + \epsilon_t$$

sum of weights by predictor (at time *t*)

outcome (at time *t*)

$$y_t = \beta_0 + \sum_k^n \beta_k x_{k,t} + \epsilon_t$$

intercept

error (at time *t*)

sum of weights by predictor (at time *t*)

outcome (at time *t*)

$$\boxed{y_t} = \boxed{\beta_0} + \boxed{\sum_k^n \beta_k x_{k,t}} + \boxed{\epsilon_t}$$

intercept

error (at time *t*)

- y_t = **outcome** or variable we want to predict

sum of weights by predictor (at time *t*)

outcome (at time *t*)

$$y_t = \beta_0 + \sum_k^n \beta_k x_{k,t} + \epsilon_t$$

intercept

error (at time *t*)

- y_t = **outcome** or variable we want to predict
- x_k, t = **predictor** or variable used to predict the outcome
  - Usually assumed to be known for all *past* and *future*

sum of weights by predictor (at time t)

outcome (at time t)

$$y_t = \beta_0 + \sum_k^n \beta_k x_{k,t} + \epsilon_t$$

intercept

error (at time t)

- y_t = **outcome** or variable we want to predict
- x_k, t = **predictor** or variable used to predict the outcome
  - Usually assumed to be known for all *past* and *future*
- β_k = **coefficients** that measure the effect of each predictor (after taking into account all other predictors)

sum of weights by predictor (at time *t*)

outcome (at time *t*)

$$y_t = \beta_0 + \sum_k^n \beta_k x_{k,t} + \epsilon_t$$

intercept

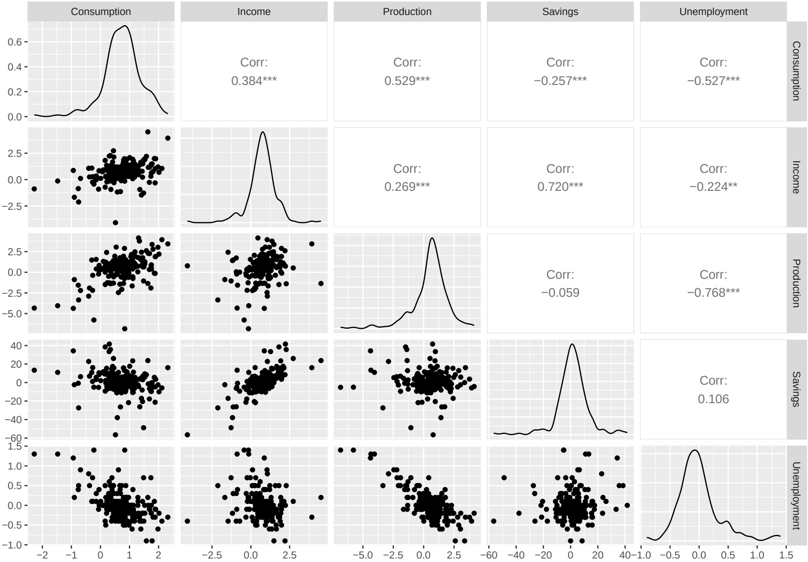
error (at time *t*)

- y_t = **outcome** or variable we want to predict
- x_k, t = **predictor** or variable used to predict the outcome
  - Usually assumed to be known for all *past* and *future*
- β_k = **coefficients** that measure the effect of each predictor (after taking into account all other predictors)
- ϵ_t = white noise error term (we'll talk more on this later)

Regression Example

Forecasting | US Consumption Expediture

Forecasting | US Consumption Expediture



```
Series: Consumption
Model: TSLM

Residuals:
     Min       1Q   Median       3Q      Max
-0.90555 -0.15821 -0.03608  0.13618  1.15471

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.253105   0.034470   7.343 5.71e-12 ***
Income        0.740583   0.040115  18.461  < 2e-16 ***
Production    0.047173   0.023142   2.038   0.0429 *
Unemployment -0.174685   0.095511  -1.829   0.0689 .
Savings      -0.052890   0.002924 -18.088  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.3102 on 193 degrees of freedom
Multiple R-squared: 0.7683, Adjusted R-squared: 0.7635
F-statistic:   160 on 4 and 193 DF, p-value: < 2.22e-16
```

```r
# Load {fpp3}
library(fpp3)

# Load US Consumption data
data("us_change")

# Length of time series
ts_length <- nrow(us_change)

# Remove last five years (we'll make a prediction later)
us_prediction <- us_change[
  -c((ts_length - 19):ts_length), # remove last 5 years
]

# Save last five years (we'll compare with prediction)
us_actual <- us_change[
  c((ts_length - 19):ts_length), # keeps last 5 years
]
```

```r
# Fit linear model
fit_us_lm <- us_prediction %>% # our data
  model( # model for time series
    tslm = TSLM( # time series linear model
      Consumption ~ Income + Production + Savings + Unemployment
    )
  )
```

```r
# Report fit
report(fit_us_lm)
```

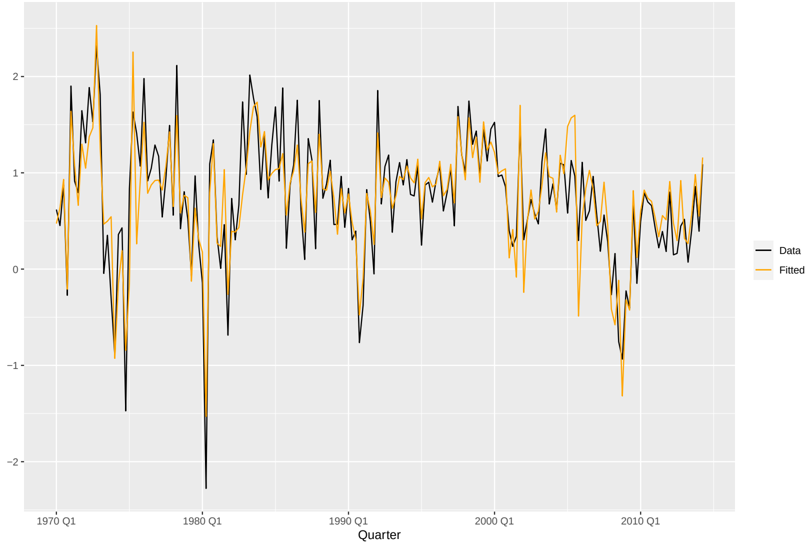
```
Series: Consumption
Model: TSLM

Residuals:
     Min       1Q   Median       3Q      Max
-0.89952 -0.16879 -0.03979  0.13944  1.14909

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.261795   0.037847   6.917 8.56e-11 ***
Income        0.737779   0.042300  17.442  < 2e-16 ***
Production    0.044788   0.026403   1.696   0.0916 .
Savings      -0.052416   0.003091 -16.960  < 2e-16 ***
Unemployment -0.191468   0.107811  -1.776   0.0775 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.3251 on 173 degrees of freedom
Multiple R-squared: 0.768,   Adjusted R-squared: 0.7627
F-statistic: 143.2 on 4 and 173 DF, p-value: < 2.22e-16
```

Forecasting with Regression

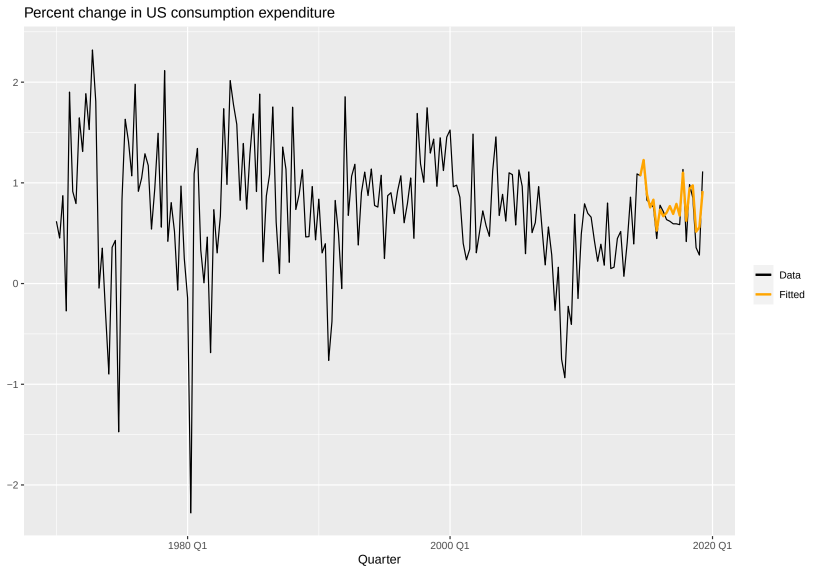
```r
# Plot model
augment(fit_us_lm) %>%
  # Plot quarter on x-axis
  ggplot(aes(x = Quarter)) +
  # Plot actual values
  geom_line(aes(y = Consumption, colour = "Data")) +
  # Plot fit values
  geom_line(aes(y = .fitted, colour = "Fitted")) +
  labs(
    # No y-axis label
    y = NULL,
    # Change title
    title = "Percent change in US consumption expenditure"
  ) +
  # Change colors
  scale_colour_manual(
    values = c(
      Data = "black", # Make data line black
      Fitted = "orange" # Make fitted line orange
    )
  ) +
  # No title for legend
  guides(colour = guide_legend(title = NULL))
```

Forecasting | Regression Example

Percent change in US consumption expenditure



```r
# Forecast
fc <- forecast(fit_us_lm, new_data = us_actual)

# Plot forecast
us_change %>%
  # Plot quarter on x-axis
  ggplot(aes(x = Quarter)) +
  # Plot actual values
  geom_line(aes(y = Consumption, colour = "Data")) +
  # Plot predicted values
  geom_line(
    data = fc,
    aes(y = .mean, colour = "Fitted"),
    size = 1
  ) +
  labs(
    # No y-axis label
    y = NULL,
    # Change title
    title = "Percent change in US consumption expenditure"
  ) +
  # Change colors
  scale_colour_manual(
    values = c(
      Data = "black", # Make data line black
      Fitted = "orange" # Make fitted line orange
    )
  ) +
  # No title for legend
  guides(colour = guide_legend(title = NULL))
```


Percent change in US consumption expenditure
2
1
0
−1
−2
1980 Q1
2000 Q1
2020 Q1
Quarter
Data
Fitted

Measures of Accuracy

- R-squared: proportion of variance explained

$$R^2 = \frac{\sum(\hat{y}_t - \bar{y})^2}{\sum(y_t - \bar{y})^2}$$

- Mean absolute error: average error

$$MAE = \frac{\sum|\hat{y}_t - y_t|}{T}$$

- Root mean square error: standard deviation of error

$$RMSE = \sqrt{\frac{\sum(\hat{y}_t - y_t)^2}{T}}$$

- Mean bias error: tendency to over- (+) or underestimate (-)

$$MBE = \frac{\sum \hat{y}_t - y_t}{T}$$

```r
# R-squared
cor(fc$.mean, us_actual$Consumption)^2
```

```
[1] 0.8647245
```

```r
# MAE
mean(abs(fc$.mean - us_actual$Consumption))
```

```
[1] 0.1000182
```

```r
# RMSE
sqrt(mean((fc$.mean - us_actual$Consumption)^2))
```

```
[1] 0.1235474
```

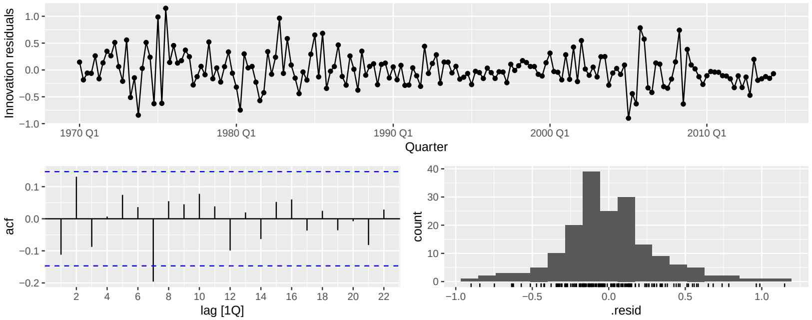
```r
# MBE
mean(fc$.mean - us_actual$Consumption)
```

```
[1] 0.06020543
```

```r
# General function for many measures
accuracy(fc, us_change)
```

```
# A tibble: 1 x 10
  .model .type       ME  RMSE   MAE   MPE  MAPE  MASE RMSSE   ACF1
  <chr>  <chr>    <dbl> <dbl> <dbl> <dbl> <dbl> <dbl> <dbl>  <dbl>
1 tslm   Test   -0.0602 0.124 0.100 -15.1  19.2 0.152 0.141 -0.180
```

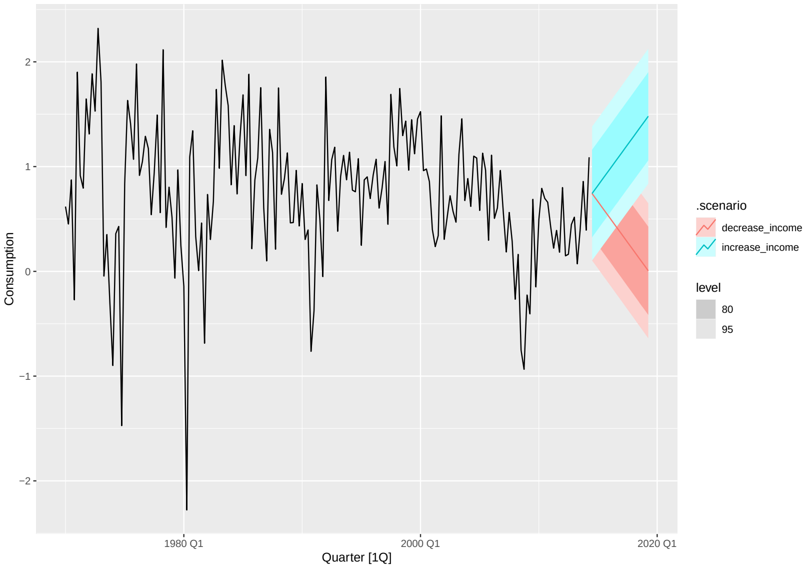
```r
# Check residuals
gg_tsresiduals(fit_us_lm)
```



```r
# Future scenarios
future_scenarios <- scenarios( # Create future scenarios
  increase_income = new_data( # Create new data
    us_prediction,  # Original data
    nrow(us_actual) # Number of new data
  ) %>%
    mutate(
      Income = mean(us_prediction$Income) + # Add to mean Income
        seq(0, 1, length = nrow(us_actual)), # Increase from 0 to 1
      # with a length equal to the number of actual data
      Production = mean(us_prediction$Production) +
        rep(0, nrow(us_actual)), # No increase/decrease
      # Repeat 0 with a length equal to the number of actual data
      Savings = mean(us_prediction$Savings) +
        rep(0, nrow(us_actual)),
      Unemployment = mean(us_prediction$Unemployment) +
        rep(0, nrow(us_actual))
    ),
  decrease_income = new_data(
    us_prediction, nrow(us_actual)
  ) %>%
    mutate(
      Income = mean(us_prediction$Income) +
        seq(0, -1, length = nrow(us_actual)),
      Production = mean(us_prediction$Production) +
        rep(0, nrow(us_actual)),
      Savings = mean(us_prediction$Savings) +
        rep(0, nrow(us_actual)),
      Unemployment = mean(us_prediction$Unemployment) +
        rep(0, nrow(us_actual))
    )
)
```

```r
# Forecast
fc_us <- fit_us_lm %>%
  forecast(new_data = future_scenarios)

# Plot
autoplot(us_prediction, Consumption) +
  autolayer(fc_us)
```

Forecasting | Regression Forecast (no actual data)



Prediction Intervals (Confidence Bands/Intervals)

$$\hat{y} \pm 1.96\,\hat{\sigma}_e \sqrt{1 + \frac{1}{T} + \frac{(x - \bar{x})^2}{(T - 1)s_x^2}}$$

Regression with Trend and Seasonal Components outcome (at time *t*)

trend

$$y_t = \beta_0 + \beta_1 t + \epsilon_t$$

intercept

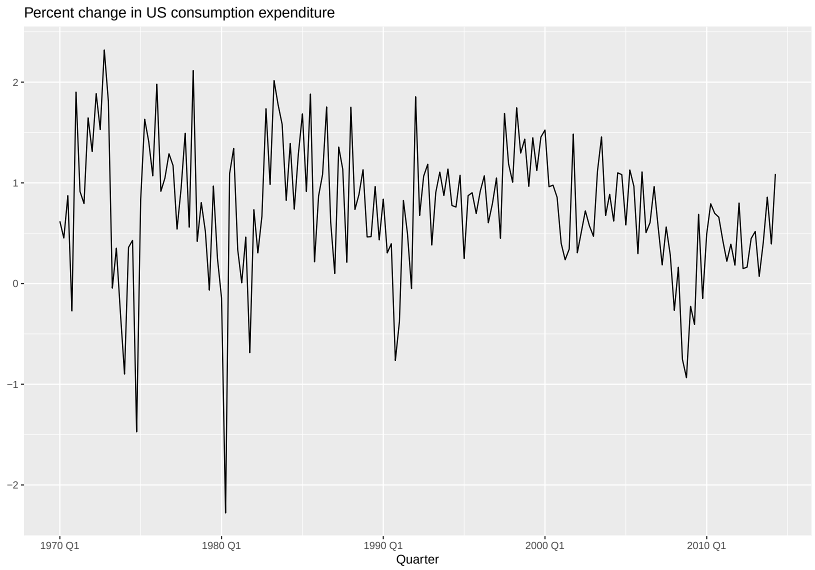
error (at time *t*)

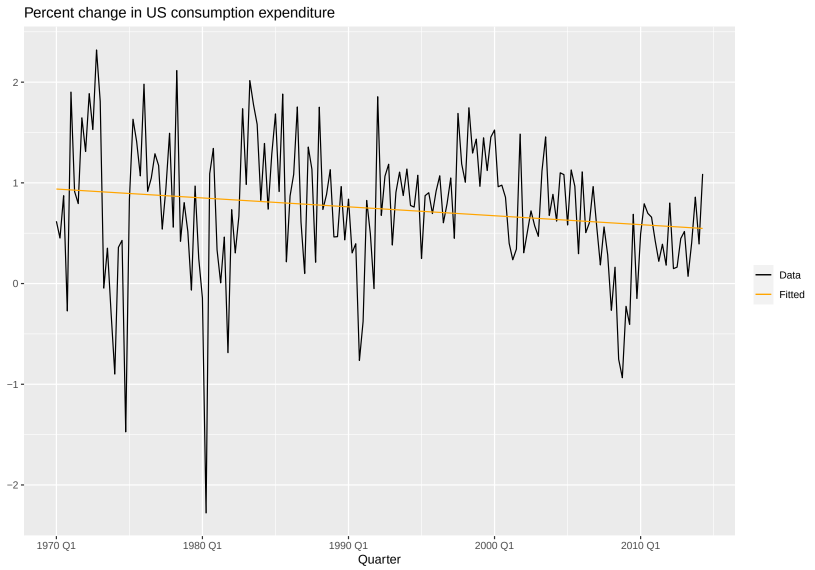
```
# Fit linear model with trend
  fit_us_trend <- us_prediction %>%
  model( # model for time series
    tslm = TSLM( # time series linear model
      Consumption ~ trend() # trend component
    )
  )
```

```r
# Report fit
report(fit_us_trend)
```

```
Series: Consumption
Model: TSLM

Residuals:
    Min      1Q  Median      3Q     Max
-3.1258 -0.3403  0.0366  0.3867  1.4053

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.9408053  0.0992577   9.478   <2e-16 ***
trend()     -0.0022103  0.0009618  -2.298   0.0227 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.6593 on 176 degrees of freedom
Multiple R-squared: 0.02913,    Adjusted R-squared: 0.02362
F-statistic: 5.281 on 1 and 176 DF, p-value: 0.022733
```

Forecasting | Regression Trend Example




Percent change in US consumption expenditure
2
1
0
−1
−2
1970 Q1
1980 Q1
1990 Q1
2000 Q1
2010 Q1
Quarter
Data
Fitted

$$\overbrace{y_t}^{\text{outcome (at time } t\text{)}} = \underbrace{\beta_0}_{\text{intercept}} + \underbrace{\beta_1 t}_{\text{trend}} + \underbrace{\beta_2 d_{2,t} + \beta_3 d_{3,t} + \beta_4 d_{4,t}}_{\text{season}} + \underbrace{\epsilon_t}_{\text{error (at time } t\text{)}}$$

outcome (at time t)

$$y_t = \beta_0 + \beta_1 t + \beta_2 d_{2,t} + \beta_3 d_{3,t} + \beta_4 d_{4,t} + \epsilon_t$$

intercept

trend

season

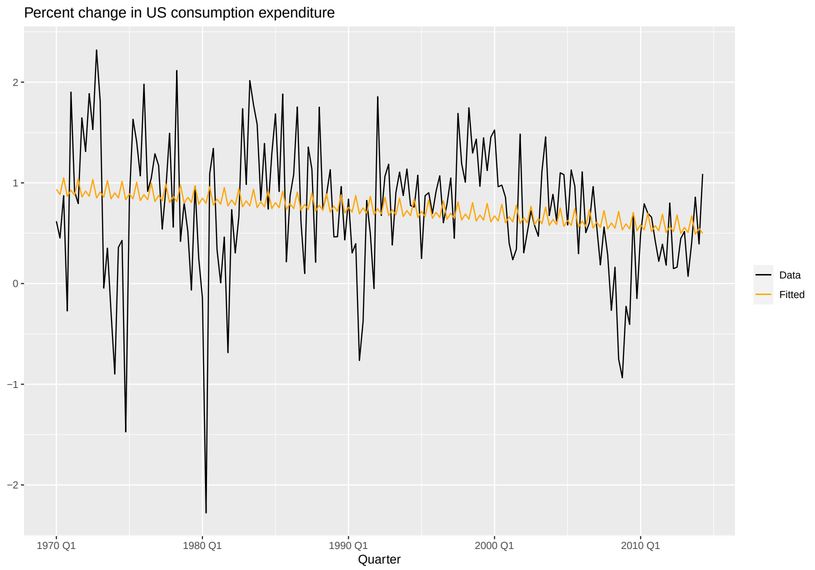
error (at time t)

	$d_{2,t}$	$d_{3,t}$	$d_{4,t}$
Quarter 1	0	0	0
Quarter 2	1	0	0
Quarter 3	0	1	0
Quarter 4	0	0	1
Quarter 1	0	0	0
...	...	...	...

```r
# Fit linear model with trend and season
fit_us_season <- us_prediction %>%
  model( # model for time series
    tslm = TSLM( # time series linear model
      Consumption ~ trend() + # trend component
        season() # season component
    )
  )
```

```r
# Report fit
report(fit_us_season)
```

```
Series: Consumption
Model: TSLM

Residuals:
     Min       1Q   Median       3Q      Max
-3.07488 -0.33612  0.00766  0.41042  1.46950

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)    0.9380208  0.1306974   7.177 2.01e-11 ***
trend()       -0.0021995  0.0009645  -2.281   0.0238 *
season()year2 -0.0485962  0.1393858  -0.349   0.7278
season()year3  0.1186395  0.1401721   0.846   0.3985
season()year4 -0.0615712  0.1401754  -0.439   0.6610
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.6611 on 173 degrees of freedom
Multiple R-squared: 0.04045,    Adjusted R-squared: 0.01826
F-statistic: 1.823 on 4 and 173 DF, p-value: 0.12648
```


Percent change in US consumption expenditure
2
1
0
−1
−2
1970 Q1
1980 Q1
1990 Q1
2000 Q1
2010 Q1
Quarter
Data
Fitted

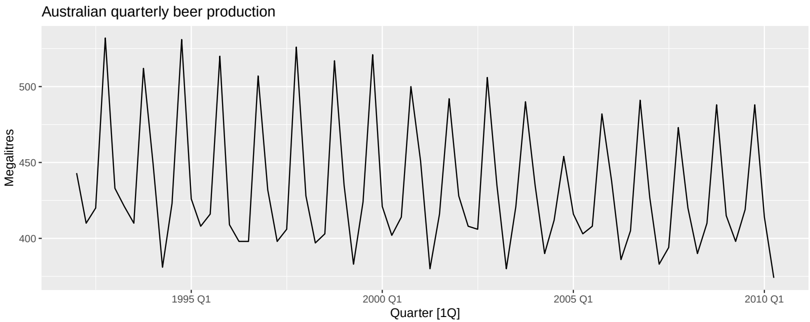
What happened..?

What happened..?

Let's look at a beer-ter example…

Forecasting | A Beer-ter Example

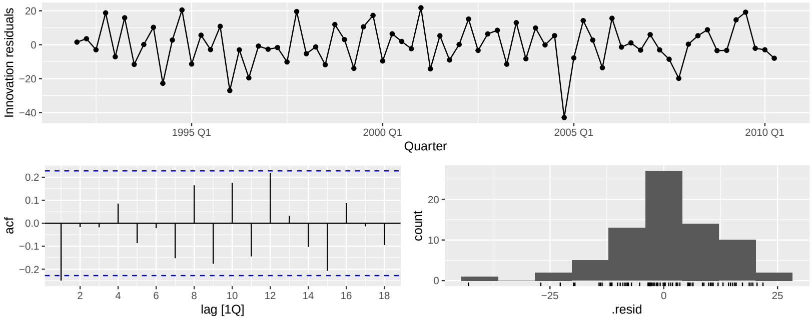
```r
# Australian beer production
recent_production <- aus_production %>% filter(year(Quarter) >= 1992)
recent_production %>% autoplot(Beer) +
  labs(y="Megalitres",title ="Australian quarterly beer production")
```



```r
# Fit model
fit_beer <- recent_production %>% model(TSLM(Beer ~ trend() + season()))
fit_beer %>% report()
```

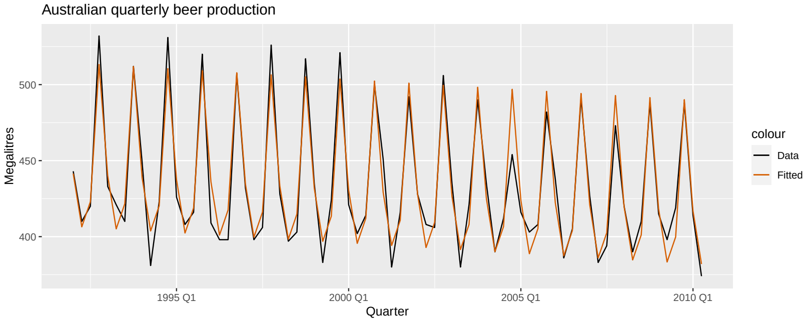
```
Series: Beer
Model: TSLM

Residuals:
     Min       1Q   Median       3Q      Max
-42.9029  -7.5995  -0.4594   7.9908  21.7895

Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)   441.80044    3.73353 118.333  < 2e-16 ***
trend()        -0.34027    0.06657  -5.111 2.73e-06 ***
season()year2 -34.65973    3.96832  -8.734 9.10e-13 ***
season()year3 -17.82164    4.02249  -4.430 3.45e-05 ***
season()year4  72.79641    4.02305  18.095  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 12.23 on 69 degrees of freedom
Multiple R-squared: 0.9243, Adjusted R-squared: 0.9199
F-statistic: 210.7 on 4 and 69 DF, p-value: < 2.22e-16
```

```r
# Residuals
fit_beer %>%
  gg_tsresiduals()
```

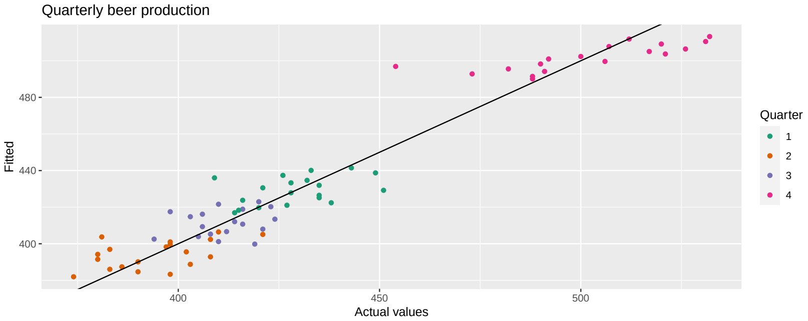


Forecasting | A Beer-ter Example

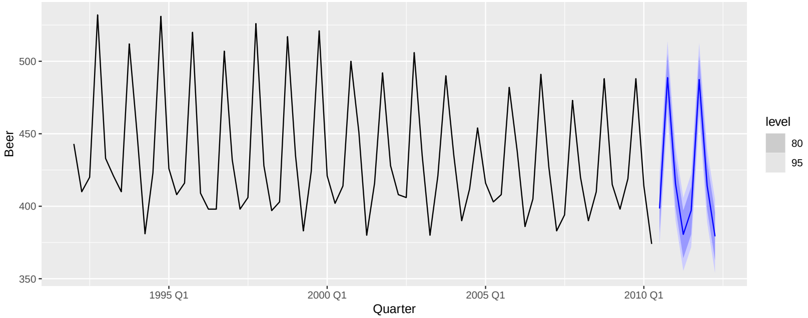
```r
# Plot fitted model
augment(fit_beer) %>%
  ggplot(aes(x = Quarter)) +
  geom_line(aes(y = Beer, colour = "Data")) +
  geom_line(aes(y = .fitted, colour = "Fitted")) +
  labs(y="Megalitres",title ="Australian quarterly beer production") +
  scale_colour_manual(values = c(Data = "black", Fitted = "#D55E00"))
```



```r
# Examining seasonality
augment(fit_beer) %>%
  ggplot(aes(x=Beer, y=.fitted, colour=factor(quarter(Quarter)))) +
    geom_point() +
    labs(y="Fitted", x="Actual values", title = "Quarterly beer production") +
    scale_colour_brewer(palette="Dark2", name="Quarter") +
    geom_abline(intercept=0, slope=1)
```



```r
# Forecasting prediction
fc <- fit_beer %>% forecast
# Plot forecast
fc %>% autoplot(recent_production)
```



Measures of Fit

- Adjusted R-squared: proportion of variance explained

$$\bar{R}^2 = 1 - (1 - R^2)\frac{T-1}{T-k-1}$$

- Cross-validation:

1. Remove time point t, fit model, and compute error $e_t^* = y_t - \hat{y}_t$
2. Repeat for each time point T
3. Compute MSE

$$MSE = \frac{\sum(\hat{y}_t - y_t)^2}{T}$$

Measures of Fit

- Akaike's Information Criterion

$$AIC = T \log\left(\frac{SSE}{T}\right) + 2(k+2)$$

$$SSE = \sum_{t=1}^{T} e_t^2$$

- Corrected Akaike's Information Criterion

$$AIC_c = AIC + \frac{2(k+2)(k+3)}{T-k-3}$$

- Schwarz's Bayesian Information Criterion

$$BIC = T \log\left(\frac{SSE}{T}\right) + (k+2)\log(T)$$

```r
# Report fit measures
glance(fit_beer) %>%
  select(
    adj_r_squared, CV, AIC, AICc, BIC
  )
```

```
# A tibble: 1 x 5
  adj_r_squared    CV   AIC  AICc   BIC
          <dbl> <dbl> <dbl> <dbl> <dbl>
1         0.920  160.  377.  379.  391.
```

Dummy Variables

- Interventions (one time): An effect that lasts only one period. Add a dummy variable with 1 at time point (t)
- Interventions (permanent): An effect that continues. Add a dummy variable with 1 at time point (t) and each time point there after (t, t_{+1}, …, t_n)
- Number of days: Use number of days in each month as a regressor
- Lags: Inclusion of previous time points to predict current time point
- Holidays: Adjust placement of 1 with each year
- Fourier series (alternative to season): sine and cosine based on m periods (e.g., m = 52 for weeks in a year)

Fourier Example

Periodic seasonality can be handled using pairs of Fourier terms

$$s_k(t) = \sin\left(\frac{2\pi kt}{m}\right) \qquad c_k(t) = \cos\left(\frac{2\pi kt}{m}\right)$$

$$y_t = a + bt + \sum_{k=1}^{K}\left[\alpha_k s_k(t) + \beta_k c_k(t)\right] + \varepsilon_t$$

- Every periodic function can be approximated by sums of sin and cos terms for large enough K.
- Choose K by minimizing AICc
- Called "harmonic regression"

```
TSLM(y ~ trend() + fourier(K))
```

```r
# Harmonic regression
fourier_beer <- recent_production %>%
  model( # model for time series
    tslm = TSLM( # time series linear model
      Beer ~ trend() + # trend component
        fourier(K = 2) # harmonic regression
    )
  )

# Report fit
report(fourier_beer)
```

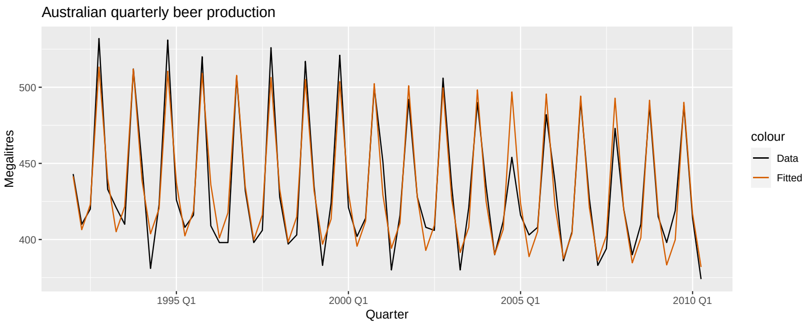
```
Series: Beer
Model: TSLM

Residuals:
     Min       1Q   Median       3Q      Max
-42.9029  -7.5995  -0.4594   7.9908  21.7895

Coefficients:
                    Estimate Std. Error t value Pr(>|t|)
(Intercept)        446.87920    2.87321 155.533  < 2e-16 ***
trend()             -0.34027    0.06657  -5.111 2.73e-06 ***
fourier(K = 2)C1_4   8.91082    2.01125   4.430 3.45e-05 ***
fourier(K = 2)S1_4 -53.72807    2.01125 -26.714  < 2e-16 ***
fourier(K = 2)C2_4 -13.98958    1.42256  -9.834 9.26e-15 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 12.23 on 69 degrees of freedom
Multiple R-squared: 0.9243, Adjusted R-squared: 0.9199
F-statistic: 210.7 on 4 and 69 DF, p-value: < 2.22e-16
```

Selecting a model:

```
# Fit multiple models
fit <- recent_production %>%
  model(
    K1 = TSLM(Beer ~ trend() + fourier(K = 1)),
    K2 = TSLM(Beer ~ trend() + fourier(K = 2)),
    K3 = TSLM(Beer ~ trend() + fourier(K = 3)),
    K4 = TSLM(Beer ~ trend() + fourier(K = 4)),
    K5 = TSLM(Beer ~ trend() + fourier(K = 5)),
    K6 = TSLM(Beer ~ trend() + fourier(K = 6))
  )

# Check fit
glance(fit) %>% select(.model, r_squared, adj_r_squared, AICc)
```

```
# A tibble: 2 x 4
  .model r_squared adj_r_squared  AICc
  <chr>      <dbl>         <dbl> <dbl>
1 K1         0.818         0.810  441.
2 K2         0.924         0.920  379.
```

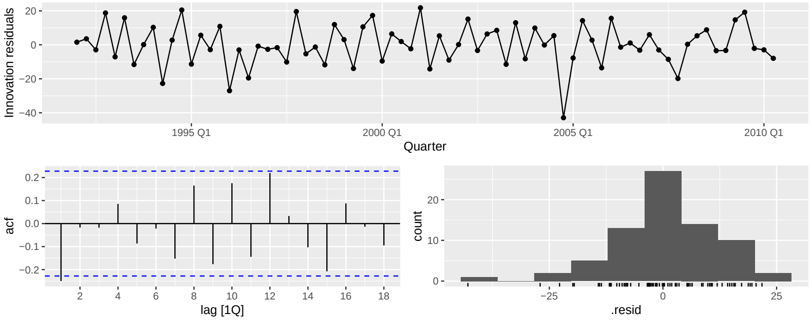
```r
# Plot fitted model
augment(fourier_beer) %>%
  ggplot(aes(x = Quarter)) +
  geom_line(aes(y = Beer, colour = "Data")) +
  geom_line(aes(y = .fitted, colour = "Fitted")) +
  labs(y="Megalitres",title ="Australian quarterly beer production") +
  scale_colour_manual(values = c(Data = "black", Fitted = "#D55E00"))
```



Residuals

Residual Diagnostics

```
# Plot fitted model
fourier_beer %>%
  gg_tsresiduals()
```



- ϵ_t have zero mean, uncorrelated, and uncorrelated with each $x_{k,t}$
- Normal distribution ($\epsilon_t \sim N(0, \sigma^2)$) **useful** for prediction intervals and statistical tests
- If there is a pattern:
    - predictor used: possible *nonlinear* relationship between residual and predictor
    - predictor *not* used: predictor should be added to model