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Modelling and analysis of rumour propagation based on stochastic optimal control

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Abstract Considering the decay of information with time, this paper proposes a rumour propagation model based on information intervention. Applying optimal control and stochastic noise propagation, we explain the influence of the information intervention mechanism on the dynamics of rumour propagation. First, we calculate the equilibrium points and basic reproduction number of our model. Second, this paper selects the intensity of information intervention and the control rate of spreaders as control variables and explores the dynamics of rumour propagation in deterministic and stochastic systems under optimal control. Finally, to support the theoretical results, we verify the stationarity condition through numerical simulation and analyse the changes in different populations. Additionally, we compare the performance of the deterministic and stochastic models to each other, as well as with existing models.

1. Introduction

A rumour is a product of the spread of topics or time of interest to people in a certain range through various channels that has not been proven true or false [1,2]. At present, the development of social media provides a new medium for rumour dissemination. A large number of online rumours can be quickly spread on large social media platforms such as Facebook, Twitter and Sina Weibo. The spread of internet rumours has gradually developed into a serious social problem. It has the characteristics of fast propagation speed, wide coverage and deep harm. The spread of rumours on the internet has not only caused a certain degree of damage to social media, but it has also promoted the generation of online public opinion, increased the possibility of cyber crime [3], and affected the stability of society. Meanwhile, rumours often accompany the arrival of emergencies, causing panic to the masses [4,5].

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Research on rumour propagation has been conducted by many researchers. The early DK model [6] and MT model [7] cover the theory of rumour propagation and give representative results. After that, some researchers began to consider the topology of social networks and apply it to the study of rumour propagation. Meng et al. [8] proposed the SISR model, analysed the interactions in the network and completed a simulation. Zhu et al. [9] completed the dynamic analysis of rumour propagation based on the delayed SIS epidemic model, adding psychological factors and network topology. With the deepening of the research on rumour propagation, some scholars have found that the behavioural characteristics of memory and forgetting will have an impact on rumour propagation. Kawachi studied the influence of memory mechanisms on the spread of rumours [10]. Zhao et al. considered the influence of the forgetting mechanism in the process of rumour propagation [11]. In addition, some scholars extended the model and combined the uncertainty method to create an uncertain SIR rumour propagation model driven by the Liu process to study the influence of disturbance on the rumour propagation mechanism [12]. In [13], the author established a SEIR rumour propagation model under the condition of a variable total number of users and verified the internal characteristics of the rumour propagation process through actual data. For the time delay in the process of rumour propagation, some scholars also proposed a modified rumour propagation model [14], discussed its local and global stability, and verified the correctness of the theory through numerical simulation.

Although the abovementioned articles have made important contributions to rumour propagation [1–14], these models still suffer from the following problems. First, current models rarely take into account the intervening effect of external information on rumour spreading. The process of spreading rumours is often accompanied by many informational factors. Policies, reports and disclosures will have a significant impact on the process of spreading rumours. These external information interventions will affect the psychology of different groups in the process of rumours, and different psychological effects will affect the process of rumour propagation [15]. Therefore, the number of relationships between different groups changes, altering the evolution of rumour propagation to some extent. Second, information dissemination in the current social network is complex, and the information transmission in the communication process will also be attenuated by the interference of various factors [16].Information in the process of rumour spreading will also have a similar attenuation effect. Existing models barely account for the impact of this effect. With the spread of rumours, the attenuation effect of information is strengthened, which will affect the change in the number of relationships between different groups of people in the process of rumour spread. Third, the transmission of internet user information data is affected by noise [17], and the information dissemination process is full of noise. Therefore, the rumour propagation model we study needs to consider the effect of noise. There are many noise effects in the existing mathematical biological dynamics system [18–21]. The stochastic dynamic system under the action of noise is more in line with the actual development law. At present, a small number of scholars consider the impact of noise on the spread of rumours. Some studies have proposed a new homogeneous social network rumour propagation model, which considers the randomness of user behaviour, and finally comes to the conclusion that improving the noise intensity and cutting off the dissemination path can effectively control rumour propagation [22]. Jia et al. studied a stochastic rumour propagation model, explained the sufficient conditions of rumour mean dissipation and persistence, and obtained the threshold of rumour dissipation [23]. The above shows the practicability and feasibility of establishing a stochastic rumour propagation model.

In this paper, we propose an improved SIR rumour propagation model based on information intervention. The proposed model considers the influence of the forgetting mechanism and information intervention mechanism on the spread of rumours. The rest of this paper is organized as follows. In Section 2, we describe a rumour propagation model with an information intervention mechanism and forgetting mechanism. In Section 3, we discuss and present the positivity and bounds of stochastic differential equations. In Section4, we analyse the system equilibrium point and basic reproduction number of the model. In Section 5, we analyse the stability of the equilibrium point of the system. In Section 6, we consider the optimal control strategy of the model under deterministic and stochastic systems. In Section 7, we complete the theoretical verification of Sections 4, 5 and 6 through numerical simulation. Finally, conclusions are shown in Section 8.

2. Formulation of the model

In this section, we will develop the SIR rumour propagation model according to some characteristics of rumour propagation [14].The whole population in social networks is divided into three categories: ignorants$S(t)$, spreaders $I(t)$ and stiflers$R(t)$.The total number of individuals at time t is$N(t) = S(t) + I(t) + R(t)$. The density of information intervention at time t is$Z(t)$. With the passage of time, the dynamics of the model may change with the change in the environment and satisfy the following assumptions:

a_1 : All states and parameters of the system in question are nonnegative.

a_2 : These are four standard Brownian motions with noise $W_i(t)$i $= \{1, 2, 3, 4\}$ that are independent.

a_3 : Individuals who become stiflers will no longer spread rumours.

Based on the above assumptions($a_1 - a_3$), we include white noise type disturbance into the model [24], which is proportional to the S, I, R, Z classes. Therefore, the rumour dissemination system is as follows:

$$\begin{aligned} dS(t) &= \left[\beta - \frac{\lambda S(t)I(t)}{1+Z(t)} - (\alpha+\kappa)S(t) - wmZ(t)S(t)\right]dt \\ &\quad + \Phi_1 S(t)dW_1(t)dI(t) \\ &= \left[\frac{\lambda S(t)I(t)}{1+Z(t)} - (\gamma+\varepsilon+\kappa)I(t)\right]dt \\ &\quad + \Phi_2 I(t)dW_2(t)dR(t) \\ &= [\gamma I(t) + \alpha S(t) + \varepsilon I(t) - \kappa R(t) + wmZ(t)S(t)]dt \\ &\quad + \Phi_3 R(t)dW_3(t)dZ(t) \\ &= \left[\frac{aI(t)}{1+bI(t)} - a_0 Z(t)\right]dt + \Phi_4 Z(t)dW_4(t) \end{aligned} \tag{1}$$

where $\Phi_i, (i = 1, 2, 3, 4)$ represents the intensity of the environmental white noise.

The specific rumour propagation rules are as follows. A schematic diagram of model propagation is shown in Fig. 1.

b_1 : When ignorants (S) come into contact with spreaders(I), they will become spreaders at rateλ. $\frac{1}{1+Z}$ indicates the impact of external information intervention on the contact between ignorant and spreaders.

b_2 : When spreaders (I) come into contact with stiflers(R), they will become stiflers at rateγ.

b_3 : When ignorants (S) come into contact with stiflers(R), they will become stiflers at rateα.It satisfies$\alpha = 1 - \lambda$.

b_4 : According to the existence of the forgetting mechanism, spreaders (I) will no longer believe rumours at rate ε and will become stiflers(R).

By setting$\Phi_i = 0$, where$i = 1, 2, 3, 4$, we can then convert the corresponding stochastic system (1) into a deterministic form as follows:

$$\begin{aligned}
\frac{dS(t)}{dt} &= \beta - \lambda \frac{S(t)I(t)}{1+Z(t)} - (\alpha+\kappa)S(t) - wmZ(t)S(t)\\
\frac{dI(t)}{dt} &= \lambda \frac{S(t)I(t)}{1+Z(t)} - (\gamma+\varepsilon+\kappa)I(t)\\
\frac{dR(t)}{dt} &= \gamma I(t) + \alpha S(t) + \varepsilon I(t) - \kappa R(t) + wmZ(t)S(t)\\
\frac{dZ(t)}{dt} &= \frac{aI(t)}{1+bI(t)} - a_0 Z(t)
\end{aligned} \tag{2}$$

Table 1 gives the detailed information and complete description of the parameters used in Model (2).

3. Positivity and bounds of stochastic differential equations

In this section, we demonstrate the positivity and bounds of Model (1) by proving the following theorem.

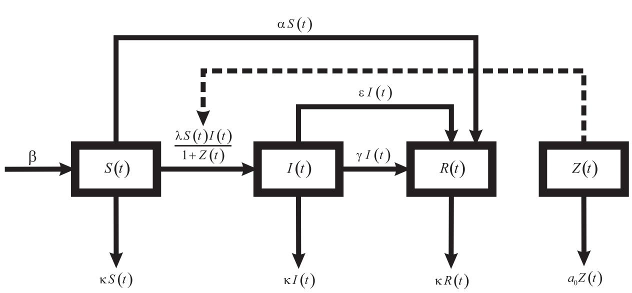


Fig. 1 Basic diagram of the improved SIR rumour propagation model under information intervention.

Table 1 Description of the parameters used in Model (2).

Notation	Parameter description
β	Unit time registration rate of users in social networks
λ	The rate of ignorants becoming spreaders
κ	Unit time offline rate of users in social networks
γ	The rate of spreaders becoming stiflers
ε	Forgetting rate
α	The rate of ignorants becoming stiflers
m	Intervention rate of information
w	Response intensity of information
a	Growth rate of information
b	Saturation constant
a_0	Natural decline rate of information

Theorem 3.1. *The solution* $(S(t), I(t), R(t), Z(t))$ *of problem* (1) *is unique for the initial condition*$\xi(0) = (S(0), I(0), R(0), Z(0)) \in R^4_+$, *where*$t \geqslant 0$. *At the same time, it can be proven that* $\xi(t) = (S(t), I(t), R(t), Z(t)) \in R^4_+$ *for all* $t \geqslant 0$ *i.e., all solutions lie in* R^4_+ *with probability one a.s (almost surely).*

Let$\xi(0) \in R^4_+$; then, the locally Lipschitz property obviously satisfies the problem requirements. There is a locally unique solution $\xi(t)$ on$t \in [0, \tau_h)$, where the explosion time is denoted byτ_h; see [25] for details. Furthermore, we verify that the solution exists globally. For this analysis, we prove that $\tau_h = \infty$ a.s.Therefore, for sufficiently large values$p_0 \geqslant 0$, $S(0), I(0), R(0)$ and $Z(0)$ remain in$\left[\frac{1}{p_0}, p_0\right]$. For all integers$p \geqslant p_0$, Suppose the stop time is:

$$\tau_p = \left\{ t \in [0, \tau_h) : \max\{S(t), I(t), R(t), Z(t)\} \geqslant p \text{ or } \min\{\mathrm{S(t), I(t), R(t), Z(t)}\} \leqslant \frac{1}{\mathrm{p}} \right\} \tag{3}$$

Below, we refer to the work proposed in [25]. The notion $\inf \varphi = \infty$ is used in this study, where φ is an empty set. As long as p tends to∞, τ_p will increase. Using $\tau_\infty = \lim_{p\to\infty}$ with $\tau_\infty \leqslant \tau_h$ a.s shows that τ_∞ become ∞ a.s., so the solution of (1) lies in the set of R^4_+ a.s., for all$t \geqslant 0$.For the completeness of the conclusion, we need to prove $\tau_h = \infty$ a.s. If this is not true, then the constant pair exists for $T > 0$ and$\theta \in (0, 1)$, so that.

$$F\{\tau_p \leqslant T\} \geqslant \theta \tag{4}$$

Therefore, we can find $p_1 \in Z$ such that for all$p \geqslant p_1$,$F\{\tau_p \leqslant T\} \geqslant \theta$ holds true.

Define a function $U \in C^2$ such that$U : R^4_+ \to R_+$, satisfying.

$$\begin{aligned}
U(S, I, R, Z) &= S + I + R + Z\\
&\quad - (\log S + \log I + \log R + \log Z) - 4
\end{aligned} \tag{5}$$

In fact, $\log y \leqslant y - 1$ holds for all$y > 0$, so the function U is nonnegative. We assume that$p \geqslant p_0$,$T > 0$, and using the *Itô* formula for the above Equation (5), we obtain the following expressions:

$$\begin{aligned}
DU(S, I, R, Z) &= \left(1 - \tfrac{1}{S}\right)\left(\beta - \tfrac{\lambda SI}{1+Z} - (\alpha+\kappa)S - wmZS\right) + \tfrac{\xi_1^2}{2}\\
&\quad + \left(1 - \tfrac{1}{I}\right)\left(\tfrac{\lambda SI}{1+Z} - (\gamma+\varepsilon+\kappa)I\right) + \tfrac{\xi_2^2}{2}\\
&\quad + \left(1 - \tfrac{1}{R}\right)(\gamma I + \alpha S + \varepsilon I - \kappa R + wmZS) + \tfrac{\xi_3^2}{2}\\
&\quad + \left(1 - \tfrac{1}{Z}\right)\left(\tfrac{aI}{1+bI} - a_0 Z\right) + \tfrac{\xi_4^2}{2}\\
&= \beta + 3\kappa + \alpha + \gamma + \varepsilon + a_0 + \tfrac{\lambda I}{1+Z} + wmZ + \alpha S + \tfrac{aI}{1+bI}\\
&\quad + \tfrac{\xi_1^2}{2} + \tfrac{\xi_2^2}{2} + \tfrac{\xi_3^2}{2} + \tfrac{\xi_4^2}{2} - (\alpha+\kappa)S - \tfrac{\beta}{S} - \kappa I - \tfrac{\lambda S}{1+Z}\\
&\quad - \kappa R - \tfrac{(\gamma+\varepsilon)I + \alpha S + wmZS}{R} - a_0 Z - \tfrac{aI}{(1+bI)Z}
\end{aligned}$$

The above equation implies that.

$$\begin{aligned}
DU(S, I, R, Z) &\leqslant \beta + 3\kappa + \alpha + \gamma + \varepsilon + a_0 + (\lambda + wm + \alpha + a)B\\
&\quad + \tfrac{\xi_1^2}{2} + \tfrac{\xi_2^2}{2} + \tfrac{\xi_3^2}{2} + \tfrac{\xi_4^2}{2} := K
\end{aligned} \tag{6}$$

Hence,

$$\begin{aligned}
&\Psi\left[U(S(\tau_p \wedge T), I(\tau_p \wedge T), R(\tau_p \wedge T), Z(\tau_p \wedge T))\right]\\
&\quad \leqslant U(\xi(0)) + \Psi\left[\int_0^{\tau_p \wedge T} K dt\right] \leqslant U(\xi(0)) + KT
\end{aligned} \tag{7}$$

For all$p \geqslant p_1$, let$\tau_p \leqslant T = \Omega_p$, then$F(\Omega_p) \geqslant \theta$.We can find$S(\tau_p,\sigma)$,$I(\tau_p,\sigma)$,$R(\tau_p,\sigma)$, $Z(\tau_p,\sigma)$ whose value is either $\frac{1}{p}$ or$p$.Therefore,$U(S(\tau_p), I(\tau_p), R(\tau_p), Z(\tau_p))$ is not less than $\log k + \frac{1}{k} - 1$ or$k - 1 - \log k$.Therefore,

$$U(S(\tau_p), I(\tau_p), R(\tau_p), Z(\tau_p)) \geqslant \psi\left[\left(\log k + \frac{1}{k} - 1\right) \wedge (k - 1 - \log k)\right] \tag{8}$$

Using the above equations,i.e., Eq.(4) and Eq.(7), this ultimately leads to the following assertion:

$$\begin{aligned} U(\xi(0)) + KT &\geqslant \psi\left[1_{\Omega(\sigma)} U(S(\tau_p), I(\tau_p), R(\tau_p), Z(\tau_p))\right] \\ &\geqslant \theta\left[\left(\log k + \tfrac{1}{k} - 1\right) \wedge (k - 1 - \log k)\right] \end{aligned} \tag{9}$$

Using the notion $1_{\Omega(\sigma)}$ as an indicator function ofΩ, as p approaches∞, the contradiction arises, i.e.,$\infty > U(\xi(0)) + BT = \infty$, which proves the conclusion.

4. The equilibrium points and basic reproduction number

In this section, we solve the equilibrium points and find the basic reproduction number of the model according to the method mentioned in [26], and the relevant calculations are as follows:

Set all right-hand sides of system (2) to 0:

$$\begin{cases} 0 = \beta - \lambda \frac{S(t)I(t)}{1+Z(t)} - (\alpha+\kappa)S(t) - wmZ(t)S(t) \\ 0 = \lambda \frac{S(t)I(t)}{1+Z(t)} - (\gamma+\varepsilon+\kappa)I(t) \\ 0 = \gamma I(t) + \alpha S(t) + \varepsilon I(t) - \kappa R(t) + wmZ(t)S(t) \\ 0 = \frac{aI(t)}{1+bI(t)} - a_0 Z(t) \end{cases} \tag{10}$$

Two solutions are obtained by solving system (2). They are the two equilibrium points of the system: $E^1(S_1^*, I_1^*, R_1^*, Z_1^*)$ and$E^2(S_2^*, I_2^*, R_2^*, Z_2^*)$,where E^1 is the equilibrium point of rumour elimination and E^2 is the equilibrium point of rumour propagation. In addition, there are $S_1^* = \frac{\beta}{\alpha+\kappa}, I_1^* = 0, R_1^* = \frac{\alpha\beta}{(\alpha+\kappa)\kappa}$ and$Z_1^* = 0$.

Next, we find the basic reproduction number using the equilibrium point of rumour elimination. We focus on the spread categories contained in spreaders, namely,

$$f = \lambda \frac{1}{1+Z(t)} S(t)I(t) - (\gamma+\varepsilon+\kappa)I(t) \tag{11}$$

Now we can obtain the following two matrices.

$$\begin{cases} G = \begin{pmatrix} \frac{\lambda\beta}{\alpha+\kappa} & 0 \\ 0 & 0 \end{pmatrix} \\ V = \begin{pmatrix} \gamma+\varepsilon+\kappa & 0 \\ \frac{\lambda\beta}{\alpha+\kappa} & \alpha+\kappa \end{pmatrix} \end{cases} \tag{12}$$

Thus,

$$GV^{-1} = \begin{pmatrix} \frac{\lambda\beta}{(\alpha+\kappa)(\gamma+\varepsilon+\kappa)} & 0 \\ 0 & 0 \end{pmatrix} \tag{13}$$

Then, the basic reproduction number of system (2) is given by Equation (14).

$$R_0 = \rho(GV^{-1}) = \frac{\lambda\beta}{(\alpha+\kappa)(r+\varepsilon+\kappa)} \tag{14}$$

For the equilibrium point of rumour propagation, there are:

$$\begin{cases} S_2^* = \frac{\gamma+\varepsilon+\kappa}{\lambda}\left(\frac{aI_2^*}{a_0(1+bI_2^*)} + 1\right) \\ R_2^* = \frac{1}{\kappa}\left(\begin{aligned} &(\gamma+\varepsilon)I_2^* + \frac{\alpha(\gamma+\varepsilon+\kappa)}{\lambda}\left(\frac{aI_2^*}{a_0(1+bI_2^*)} + 1\right) \\ &+ \frac{wma(\gamma+\varepsilon+\kappa)I_2^*}{\lambda a_0(1+bI_2^*)}\left(\frac{aI_2^*}{a_0(1+bI_2^*)} + 1\right) \end{aligned} \right) \\ Z_2^* = \frac{aI_2^*}{a_0(1+bI_2^*)} \end{cases} \tag{15}$$

where I_2^* is the nonnegative real root of Equation (16).

$$\begin{aligned} &\beta - (\gamma+\varepsilon+\kappa)I \\ &- \frac{\gamma+\varepsilon+\kappa}{\lambda}\left((\alpha+\kappa) + wm\frac{aI}{a_0(1+bI)}\right)\left(\frac{aI}{a_0(1+bI)} + 1\right) \\ &= 0 \end{aligned} \tag{16}$$

Eq.(16) can be reduced to the form,$AI^3 + BI^2 + CI + D = 0$, where:

$$\begin{cases} A = -a_0^2 b^3 \lambda(\gamma+\varepsilon+\kappa) \\ B = a_0^2 b^2 \lambda\beta - (2a_0^2 b\lambda + wma^2 + wmaa_0 b)(\gamma+\varepsilon+\kappa) \\ \quad - (aa_0 b + a_0^2 b^2)(\alpha+\kappa)(\gamma+\varepsilon+\kappa) \\ C = 2a_0^2 b\lambda\beta - (wmaa_0 + a_0^2\lambda)(\gamma+\varepsilon+\kappa) \\ \quad - (aa_0 + 2a_0^2 b)(\alpha+\kappa)(\gamma+\varepsilon+\kappa) \\ D = a_0^2\lambda\beta - a_0^2(\alpha+\kappa)(\gamma+\varepsilon+\kappa) \end{cases} \tag{17}$$

According to the Shengjin formula, the equation satisfies$A_1 = B^2 - 3AC$,$B_1 = BC - 9AD$, and$C_1 = C^2 - 3BD$. The total discriminant is$\Delta = B_1^2 - 4A_1C_1$.For$R_0 > 1$, according to the discriminant conditions, Equation (16) has at least one nonnegative real root.

In system (2), the following expressions are contained:

$$\begin{cases} \left.\frac{dS(t)}{dt}\right|_{S(t)=0} = \beta \geqslant 0 \\ \left.\frac{dI(t)}{dt}\right|_{I(t)=0} = 0 \\ \left.\frac{dR(t)}{dt}\right|_{R(t)=0} = \gamma I(t) + \alpha S(t) + \varepsilon I(t) + wmZ(t)S(t) \geqslant 0 \\ \left.\frac{dZ(t)}{dt}\right|_{Z(t)=0} = \frac{aI(t)}{1+bI(t)} \geqslant 0 \end{cases} \tag{18}$$

Recall that the total number of individuals at time t is$N(t) = S(t) + I(t) + R(t)$.Then,

$$\frac{dN(t)}{dt} = \beta - dN(t) \tag{19}$$

where$\frac{dN(t)}{dt} \leqslant \beta$, and namely,$\lim_{t\to\infty} \sup N \leqslant \beta$. Thus,$\beta$ is the upper bound ofS, I, R..

According to the boundary of $\frac{dZ(t)}{dt} = \frac{aI^2(t)}{1+bI^2(t)} - a_0 Z(t)$ and$I$, $\lim_{t\to\infty} \sup Z \leqslant \frac{a\beta^2}{a_0(1+b\beta^2)}$ are established.

Thus, the invariant set Γ is obtained.

$$\Gamma = \left\{(S,I,R,Z) \in R_+^4 : S+I+R \leqslant \beta, Z \leqslant \frac{a\beta^2}{a_0(1+b\beta^2)}, S,I,R,Z \geqslant 0\right\} \tag{20}$$

Namely, in the nonnegative setR_+^4, the solution of the model is always inΓ.

5. Stability analysis of the equilibrium points for system (2)

In this section, we analyse the local and global stability of the system equilibrium points and verify their rationality.

5.1. Local stability analysis of the equilibrium points for system (2)

In this section, we will focus on the local asymptotic stability of the equilibrium points E^1 and E^2 of system (2) through linearization technology and the Routh-Hurwitz criterion [27].

Theorem 5.1. *If$R_0 < 1$, the equilibrium point of rumour elimination E^1 for system (2) is locally asymptotically stable; if$R_0 > 1$, the equilibrium point of rumour elimination E^1 for system (2) is unstable.*

Proof. The Jacobian matrix of system (2) is as follows.

$$J = \begin{pmatrix} -\left(\lambda\frac{1}{1+Z}I+\alpha+\kappa+wmZ\right) & -\lambda\frac{1}{1+Z}S & 0 & \lambda SI\frac{1}{(1+Z)^2}-wmS \\ \lambda\frac{1}{1+Z}I & \lambda\frac{1}{1+Z}S-(\gamma+\varepsilon+\kappa) & 0 & -\lambda SI\frac{1}{(1+Z)^2} \\ \alpha+wmZ & \gamma+\varepsilon & -\kappa & wmS \\ 0 & \frac{a}{(1+bI)^2} & 0 & -a_0 \end{pmatrix} \tag{21}$$

Then, the Jacobian matrix of system (2) at the equilibrium point of rumour elimination E^1 is:

$$J(E^1) = \begin{pmatrix} -(\alpha+\kappa) & -\lambda S & 0 & -wmS \\ 0 & \lambda S-(\gamma+\varepsilon+\kappa) & 0 & 0 \\ \alpha & \gamma+\varepsilon & -\kappa & wmS \\ 0 & a & 0 & -a_0 \end{pmatrix} \tag{22}$$

Thus, the characteristic equation of system (2) at E^1 can be expressed by Equation (23).

$$\left|\delta U - J(E^1)\right| = 0 \tag{23}$$

where δ is the characteristic value and U is the identity matrix.

The characteristic value at the equilibrium point of rumour elimination E^1 can be obtained from solution (24).

$$\begin{cases} \delta_1 = -(\alpha+\kappa) \\ \delta_2 = -\kappa \\ \delta_3 = -a_0 \\ \delta_4 = \frac{\lambda\beta}{\alpha+\kappa} - (\gamma+\varepsilon+\kappa) \end{cases} \tag{24}$$

where the characteristic values ofδ_1, δ_2 and δ_3 are negative. According to the Routh-Hurwitz criterion, if$R_0 < 1$, δ_4 is negative at this time. The corresponding four characteristic values are negative, so the equilibrium point of rumour elimination E^1 for system (2)is locally asymptotically stable;if$R_0 > 1$, δ_4 is positive at this time. Therefore, the equilibrium point of rumour elimination E^1 for system (2) is unstable when$R_0 > 1$.

Theorem 5.2. *If$R_0 > 1$, system (2) has a unique equilibrium point of rumour propagationE^2. At this time, the characteristic equation of Jacobian matrix J at the equilibrium point of rumour propagation E^2 is$\delta^4 + A_2\delta^3 + B_2\delta^2 + C_2\delta + D_2 = 0$; if the characteristic equation satisfies $A_2B_2 > C_2$ and$A_2(B_2C_2 - A_2D_2) > C_2^2$, then the equilibrium point of rumour propagation E^2 for system (2) is locally asymptotically stable.*

Proof. The characteristic equation of the Jacobian matrix J of system (2) at the equilibrium point of rumour propagation is shown in Equation (25):

$$\delta^4 + A_2\delta^3 + B_2\delta^2 + C_2\delta + D_2 = 0 \tag{25}$$

Namely,

$$\begin{cases} A_2 = \frac{\lambda I_2^*}{1+Z_2^*} + 2\kappa + \alpha + a_0 + wmZ_2^* \\ B_2 = \frac{\lambda S_2^* I_2^* a}{\left(1+Z_2^*\right)^2\left(1+bI_2^*\right)^2} + \frac{\lambda^2 S_2^* I_2^*}{\left(1+Z_2^*\right)^2} + (a_0+\kappa)\left(\frac{\lambda I_2^*}{1+Z_2^*} + \alpha + \kappa + wmZ_2^*\right) + a_0\kappa \\ C_2 = \frac{\lambda S_2^* I_2^* a\left(\alpha+2\kappa+wmZ_2^*\right)}{\left(1+Z_2^*\right)^2\left(1+bI_2^*\right)^2} + \frac{(a_0+\kappa)\lambda^2 S_2^* I_2^*}{\left(1+Z_2^*\right)^2} + \frac{\lambda awmS_2^* I_2^*}{\left(1+Z_2^*\right)\left(1+bI_2^*\right)^2} \\ \quad + a_0\kappa\left(\frac{\lambda I_2^*}{1+Z_2^*} + \alpha + \kappa + wmZ_2^*\right) \\ D_2 = \frac{\lambda S_2^* I_2^* a\kappa\left(\alpha+\kappa+wmZ_2^*\right)}{\left(1+Z_2^*\right)^2\left(1+bI_2^*\right)^2} + \frac{\lambda^2 a_0\kappa S_2^* I_2^*}{\left(1+Z_2^*\right)^2} + \frac{\lambda a\kappa wmS_2^* I_2^*}{\left(1+Z_2^*\right)\left(1+bI_2^*\right)^2} \end{cases} \tag{26}$$

where $A_2 > 0$ and$D_2 > 0$. If $A_2B_2 > C_2$ and$A_2(B_2C_2 - A_2D_2) > C_2^2$, according to the Routh-Hurwitz criterion, the characteristic values of $J(E^2)$ are all negative values or negative real parts. Then, according to the Hartman-Grobman theorem [28], if the above conditions are met, the equilibrium point of rumour propagation E^2 is locally asymptotically stable.

5.2. Global stability analysis of the equilibrium points for system (2)

In this section, we will focus on the global asymptotic stability of the equilibrium points E^1 and E^2 of system (2) through the Lyapunov function approach [29] and LaSalle's invariance principle [30].

System (2) is rewritten as follows:

$$\frac{dX}{dt} = F(X,Y), \frac{dY}{dt} = H(X,Y), H(X,0) = 0 \tag{27}$$

where $X \in R^3$ refers to individuals who do not spread rumours, and $Y \in R$ refers to individuals who spread rumours. $M = (X_0, 0)$ is the equilibrium point of rumour elimination for system (2).

Lemma 5.3. [31]. *If$R_0 < 1$, the following two conditions are satisfied:*

c_1 : For$\frac{dX}{dt} = F(X,0)$, X_0 is globally asymptotically stable.

c_2 : For$\forall(X,Y) \in \Gamma$, there exists$H(X,Y) = D_YH(X_0,0)Y - \hat{H}(X,Y)$, where $\hat{H}(X,Y) \geqslant 0$,$D_YH(X_0,0)$ is an $m-$matrix, and Γ is the variable boundary defined in (13). Then, the equilibrium point of rumour elimination $M = (X_0, 0)$ for the system is globally asymptotically stable.

Theorem 5.4. *If$R_0 < 1$, the equilibrium point of rumour elimination E^1 for system (2) is globally a symptotically stable.*

Proof. According to Lemma 5.3, system (2) can be rewritten into this equation form$\frac{dX}{dt} = F(X,Y)$, $\frac{dY}{dt} = H(X,Y)$, where:

$$F(X,Y)=\left(\beta-\frac{\lambda SI}{1+Z}-(\alpha+\kappa)S-wmZS,\gamma I+\alpha S+\varepsilon I-\kappa R+wmZS,\frac{aI}{1+bI}-a_0Z\right)^T \tag{28}$$

$$H(X,Y)=\frac{\lambda SI}{1+Z}-(\gamma+\varepsilon+\kappa)I \tag{29}$$

and $H(X,0)=0, X=(S,R,Z)^T, Y=I$ are established.

Meanwhile,$M_0=E^1=(X_0,0)$. Here,$X_0=\left(\frac{\beta}{\alpha+\kappa},\frac{\alpha\beta}{(\alpha+\kappa)\kappa},0\right)$ is determined. When$t\to\infty$,$X\to X_0$ is established.

Thus, X_0 is globally a symptotically stable, and it satisfies c_1 in Lemma 1.

On the other hand, according to X_0 and$R_0=\frac{\lambda\beta}{(\alpha+\kappa)(r+\varepsilon+\kappa)}$, the following formula holds:

$$H(X,Y)=-(r+\varepsilon+\kappa)(1-R_0)I-\lambda I\left(\frac{\beta}{\alpha+\kappa}-S\right) \tag{30}$$

where $\hat{H}(X,Y)=\lambda I\left(\frac{\beta}{\alpha+\kappa}-S\right)$ is established. Because $S\leqslant\frac{\beta}{\alpha+\kappa}$ is true, $\hat{H}(X,Y)\geqslant 0$ is established$\forall(X,Y)\in\Gamma$.This satisfies c_2 in Lemma 1. Thus, the equilibrium point of rumour elimination E^1 for system (2) is globally asymptotically stable.

Theorem 5.5. *If$R_0>1$, the equilibrium point of rumour propagation E^2 for system (2) is globally a symptotically stable.*

Proof. We define the Lyapunov function of the following formula (31).

$$V(t)=\frac{1}{3}\left[(S-S_2^*)+(I-I_2^*)+(R-R_2^*)\right]^2 \tag{31}$$

We then calculate the derivative of $V(t)$ with respect to time$t$:

$$\begin{aligned}\frac{dV(t)}{dt}&=\frac{2}{3}\left[(S-S_2^*)+(I-I_2^*)+(R-R_2^*)\right]\left[d(S_2^*+I_2^*+R_2^*)-d(S+I+R)\right]\\&=-\frac{2}{3}\left[(S-S_2^*)+(I-I_2^*)+(R-R_2^*)\right]\left[d(S-S_2^*)+d(I-I_2^*)+d(R-R_2^*)\right]\end{aligned} \tag{32}$$

Namely,$\frac{dV(t)}{dt}\leqslant 0$. If and only if$S=S_2^*$,$I=I_2^*$ and$R=R_2^*$, $\frac{dV(t)}{dt}=0$ holds. According to LaSalle's invariance principle, the equilibrium point of rumour propagation E^2 for system (2) is globally asymptotically stable.

6. Optimal control analysis for deterministic and stochastic systems

In our work, we will discuss the application of the Pontryagin maximum principle to the optimal control of rumour propagation in both deterministic and stochastic systems. Moreover, we study the control of rumour propagation systems (1) and (2) by applying the two control variables given as follows.

I. $u_1(t)$ represents the control variable used to reduce the number of spreaders by adjusting the intervention intensity of external information.
II. $u_2(t)$ represents the control variable used to reduce the number of spreaders through external control measures.

The goal of our work is to minimize the number of spreaders and maximize the number of stiflers by using $u_1(t)$ and $u_2(t)$ values that are as small as possible.

6.1. Deterministic optimal control

In this section, we explore how to effectively control the spread of rumours in a deterministic system and improve the model by adding control variables. The control variables are as follows: $u_1(t)$ represents the intervention intensity of external information, and $u_2(t)$ represents the inhibition rate of spreaders.

We define the allowable control set in Equation (33):

$$P=\left\{u_1(t),u_2(t)\in L^2(0,T):0\leqslant t\leqslant T;0\leqslant u_1(t)\leqslant 1,0\leqslant u_2(t)\leqslant 1\right\} \tag{33}$$

where T represents the final time when the control strategy ends. We use u_1 and u_2 to represent these two control variables to facilitate the expression below.

Next, we use $kI(t)$ to represent the weight of spreaders, and we use k_1 and k_2 to represent the respective weights of information intervention intensity and inhibition rate of rumour spreaders. The terms $k_1u_1^2$ and $k_2u_2^2$ represent the corresponding costs of information dissemination and restraining spreaders, respectively. By controlling our ultimate goal, on the one hand, we keep the number of users who receive, read and believe rumours on social networks to a minimum – that is, the number of spreaders. On the other hand, we minimize the cost of information dissemination and restrain spreaders. Thus, the control problem is as follows:

$$\min J(u_1,u_2)=\int_0^T\left[kI(t)+\frac{1}{2}\left(k_1u_1^2+k_2u_2^2\right)\right]dt \tag{34}$$

Subject to.

$$\begin{aligned}&\tfrac{dS}{dt}=\beta-\tfrac{\lambda SI}{1+Z}-(\alpha+\kappa)S-u_1mZS\\&\tfrac{dI}{dt}=\tfrac{\lambda SI}{1+Z}-(\gamma+\varepsilon+\kappa)I-u_2I\\&\tfrac{dR}{dt}=\gamma I+\alpha S+\varepsilon I-\kappa R+u_1mZS+u_2I\\&\tfrac{dZ}{dt}=\tfrac{aI}{1+bI}-a_0Z\end{aligned} \tag{35}$$

where$k$,k_1 and k_2 are normal numbers and J represents the total cost of controlling the spread of rumours. Therefore, we seek the optimal control$u^*=\left(u_1^*,u_2^*\right)$; then,$J(u^*)=\min J(u_1,u_2)$ can be obtained.

Theorem 6.1. *For the above optimal control problem, there must be an optimal control $u^*=\left(u_1^*,u_2^*\right)$ in$P$, such that$J(u^*)=\min J(u_1,u_2)$.*

Proof. Here, we use the method mentioned in [32] to prove the existence of optimal control. First, according to the theorem, the control set P is a convex closed set. Second, because the control variables and state variables are nonnegative, the necessary convexity of the objective function defined by the equation in u_1 and u_2 is satisfied in this minimum problem. Therefore, our optimal system is bounded, which ensures the compactness required for the existence of optimal control. Moreover, the integrand function of the objective is convex on the control set$P$.

Next, we use the Pontryagin maximum principle to find the optimal solution of the control problem and define the Hamilton function:

$$H(S,I,R,Z,u_1,u_2,\lambda_1,\lambda_2,\lambda_3,\lambda_4)$$
$$= kI(t) + \frac{1}{2}\left(k_1u_1^2 + k_2u_2^2\right)$$
$$+ \lambda_1\left[\beta - \frac{\lambda SI}{1+Z} - (\alpha+\kappa)S - u_1mZS\right]$$
$$+ \lambda_2\left[\frac{\lambda SI}{1+Z} - (\gamma+\varepsilon+\kappa)I - u_2I\right]$$
$$+ \lambda_3[\gamma I + \alpha S + \varepsilon I - \kappa R + u_1mZS + u_2I]$$
$$+ \lambda_4\left[\frac{aI}{1+bI} - a_0Z\right] \tag{36}$$

whereλ_1,λ_2, λ_3 and λ_4 are covariate variables.

Theorem 6.2. *If u_1^* and u_2^* are the optimal control variables of the optimal control problem andS^*,I^*, R^* and Z^* are the corresponding optimal state variables, thenλ_1,λ_2, λ_3 and λ_4 satisfy the following covariate equations:*

$$\frac{d\lambda_1}{dt} = \lambda_1\left(\frac{\lambda I^*}{1+Z} + \alpha + \kappa + u_1mZ^*\right) - \lambda_2\frac{\lambda I^*}{1+Z}$$
$$- \lambda_3(\alpha + u_1mZ^*)\frac{d\lambda_2}{dt}$$
$$= -k + \lambda_1\frac{\lambda S^*}{1+Z} + \lambda_2\left(u_2 + \gamma + \varepsilon + \kappa - \frac{\lambda S^*}{1+Z}\right)$$
$$- \lambda_3(\gamma + \varepsilon + u_2) - \lambda_4\frac{a}{(1+bI^*)^2}\frac{d\lambda_3}{dt}$$
$$= \lambda_3\kappa\frac{d\lambda_4}{dt}$$
$$= -\lambda_1\left(\frac{\lambda S^*I^*}{(1+Z^*)^2} - u_1mS^*\right) - \lambda_2\frac{\lambda S^*I^*}{(1+Z^*)^2}$$
$$- \lambda_3u_1mS^* + \lambda_4a_0 \tag{37}$$

The boundary conditions are$\lambda_1(T) = 0, \lambda_2(T) = 0, \lambda_3(T) = 0, \lambda_4(T) = 0$.

In addition, the corresponding optimal control variables u_1^* and u_2^* are:

$$\begin{aligned} u_1^* &= \min\left\{\max\left\{0, \tfrac{(\lambda_1-\lambda_3)mZ^*S^*}{k_1}\right\}, 1\right\} \\ u_2^* &= \min\left\{\max\left\{0, \tfrac{(\lambda_2-\lambda_3)I^*}{k_2}\right\}, 1\right\} \end{aligned} \tag{38}$$

Proof. If u_1^* and u_2^* are the optimal control variables of the optimal control problem, then S^*, I^*, R^*, Z^* are the corresponding optimal state variables. According to the Pontryagin maximum principle, there are covariate variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ that satisfy the following covariate equations:

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial S}, \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial I}, \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial R}, \frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial Z} \tag{39}$$

The boundary condition is$\lambda_1(T) = 0, \lambda_2(T) = 0, \lambda_3(T) = 0$, $\lambda_4(T) = 0$. According to the optimal condition, when $u_1 = u_1^*$ and$u_2 = u_2^*$, $\frac{\partial H}{\partial u_1} = 0$ and $\frac{\partial H}{\partial u_2} = 0$ hold. Therefore:

$$u_1^* = \frac{(\lambda_1 - \lambda_3)mZ^*S^*}{k_1}, u_2^* = \frac{(\lambda_2 - \lambda_3)I^*}{k_2} \tag{40}$$

According to the defined control set$P$, the following results are obtained:

$$u_1^* = \begin{cases} 0 & \frac{(\lambda_1-\lambda_3)\mathrm{mZ^*S^*}}{\mathrm{k_1}} < 0 \\ \frac{(\lambda_1-\lambda_3)mZ^*S^*}{k_1} & 0 \leqslant \frac{(\lambda_1-\lambda_3)\mathrm{mZ^*S^*}}{\mathrm{k_1}} \leqslant 1 \\ 1 & \frac{(\lambda_1-\lambda_3)\mathrm{mZ^*S^*}}{\mathrm{k_1}} > 1 \end{cases} \tag{41}$$

$$u_2^* = \begin{cases} 0 & \frac{(\lambda_2-\lambda_3)\mathrm{I^*}}{\mathrm{k_2}} < 0 \\ \frac{(\lambda_2-\lambda_3)I^*}{k_2} & 0 \leqslant \frac{(\lambda_2-\lambda_3)\mathrm{I^*}}{\mathrm{k_2}} \leqslant 1 \\ 1 & \frac{(\lambda_2-\lambda_3)\mathrm{I^*}}{\mathrm{k_2}} > 1 \end{cases} \tag{42}$$

Its correspondence can be rewritten into the above form.

6.2. Stochastic optimal control

In this section, according to relevant references [33–35], we study the stochastic optimal control of the improved rumour propagation model. Meanwhile, we consider system (1) with stochastic disturbance under the control parameters u_1 andu_2, and we explore how to effectively control the spread of rumours through control variables. We have the following stochastic control system:

$$dS(t) = \left[\beta - \frac{\lambda S(t)I(t)}{1+Z(t)} - (\alpha+\kappa)S(t) - u_1mZ(t)S(t)\right]dt$$
$$+ \Phi_1S(t)dW_1(t)dI(t)$$
$$= \left[\frac{\lambda S(t)I(t)}{1+Z(t)} - (\gamma+\varepsilon+\kappa)I(t) - u_2I(t)\right]dt$$
$$+ \Phi_2I(t)dW_2(t)dR(t)$$
$$= [\gamma I(t) + \alpha S(t) + \varepsilon I(t) - \kappa R(t) + u_1mZ(t)S(t) + u_2I(t)]dt$$
$$+ \Phi_3R(t)dW_3(t)dZ(t)$$
$$= \left[\frac{aI(t)}{1+bI(t)} - a_0Z(t)\right]dt + \Phi_4Z(t)dW_4(t) \tag{43}$$

We set the following two vectors:

$$u(t) = [u_2(t), u_1(t)]' \quad x(t) = [x_4(t), x_3(t), x_2(t), x_1(t)]' \tag{44}$$

Equation (43) can be rewritten into the form of the Ito process:

$$dx(t) = f(x(t), u(t))dt + g(x(t))dz(t) \tag{45}$$

where dz,f and g are the vectors defined as follows:

$$dz = \varepsilon_t\sqrt{dt} \tag{46}$$

where ε_t is a random number extracted from the normal distribution, with an average value of 0 and a standard deviation of 1.

$$f_1(x(t), u(t)) = \beta - \frac{\lambda S(t)I(t)}{1+Z(t)} - (\alpha+\kappa)S(t)$$
$$- u_1mZ(t)S(t)f_2(x(t), u(t))$$
$$= \frac{\lambda S(t)I(t)}{1+Z(t)} - (\gamma+\varepsilon+\kappa)I(t)$$
$$- u_2I(t)f_3(x(t), u(t))$$
$$= \gamma I(t) + \alpha S(t) + \varepsilon I(t) - \kappa R(t)$$
$$+ u_1mZ(t)S(t) + u_2I(t)f_4(x(t), u(t))$$
$$= \frac{aI(t)}{1+bI(t)} - a_0Z(t) \tag{47}$$

$$g_1 = \Phi_1 S(t), g_2 = \Phi_2 I(t), g_3 = \Phi_3 R(t), g_4 = \Phi_4 Z(t) \tag{48}$$

The following is the modified Hamilton equation [36]:

$$\frac{dp}{dt} = -f_x p - \frac{1}{2}\left(g^2\right)_x \omega, \mathrm{p(T) = c} \tag{49}$$

$$\frac{d\omega}{dt} = -2\omega f_x - pf_{xx} - \frac{1}{2}\left(g^2\right)_{xx}\omega, \omega(\mathrm{T}) = 0 \tag{50}$$

$$H = pf + \frac{1}{2}g^2\omega \tag{51}$$

Recall that the control target is as follows.

$$\min J(u_1, u_2) = \int_0^T \left[kI(t) + \frac{1}{2}\left(k_1u_1^2 + k_2u_2^2\right)\right]dt \tag{52}$$

Thus, the Hamilton function is:

$$\begin{aligned} H &= kI + \frac{1}{2}\left(k_1u_1^2 + k_2u_2^2\right) \\ &\quad + p_1\left[\beta - \frac{\lambda SI}{1+Z} - (\alpha+\kappa)S - u_1 mZS\right] \\ &\quad + p_2\left[\frac{\lambda SI}{1+Z} - (\gamma+\varepsilon+\kappa)I - u_2 I\right] \\ &\quad + p_3[\gamma I + \alpha S + \varepsilon I - \kappa R + u_1 mZS + u_2 I] \\ &\quad + p_4\left[\frac{aI}{1+bI} - a_0 Z\right] + \frac{1}{2}g^2\omega \end{aligned} \tag{53}$$

By solving the derivatives of the Hamilton function with respect to u_1 andu_2, the optimal control variables u_1^* and u_2^* are obtained. As shown in Equation (54), the stochastic optimal control problem is solved by using the stochastic maximum principle [37].

$$\begin{aligned} u_1^* &= \min\left\{\max\left\{0, \tfrac{(p_1-p_3)mZ^*S^*}{k_1}\right\}, 1\right\} \\ u_2^* &= \min\left\{\max\left\{0, \tfrac{(p_2-p_3)I^*}{k_2}\right\}, 1\right\} \end{aligned} \tag{54}$$

7. Numerical simulations

In this section, we conduct approximate simulations of deterministic and stochastic systems. The selection of parameters is feasible in the process of rumour propagation. We use the Runge-Kutta method and the Milstein method [38] to conduct the numerical simulations of deterministic and stochastic systems under optimal control, and we give the relevant calculation model as shown in the following Formula (55).

Table 2 Parameter values in the system.

Parameter	Value	Parameter	Value
β	0.70	k	10
λ	0.16/0.48	k_1	0.55
κ	0.05	k_2	0.35
γ	0.25	Φ_1	0.40
ε	0.05	Φ_2	0.40
α	0.84/0.52	Φ_3	0.40
m	0.017	Φ_4	0.40
w	0.20	$S(0)$	95
a	0.01	$I(0)$	5
b	1	$R(0)$	0
a_0	0.06	$Z(0)$	5

Meanwhile, referring to [14,39] and relevant real data, we obtain the model parameters and values as shown in Table 2.

$$\begin{aligned} S_{i+1} &= S_i + \left[\beta - \frac{\lambda S_i I_i}{1+Z_i} - (\alpha+\kappa)S_i - wmZ_iS_i\right]\Delta t \\ &\quad + \Phi_1 S_i\sqrt{\Delta t}\zeta_{1,i} + \frac{1}{2}\Phi_1^2 S_i\left(\zeta_{1,i}^2 - 1\right)\Delta t I_{i+1} \\ &= I_i + \left[\frac{\lambda S_i I_i}{1+Z_i} - (\gamma+\varepsilon+\kappa)I_i\right]\Delta t + \Phi_2 I_i\sqrt{\Delta t}\zeta_{2,i} \\ &\quad + \frac{1}{2}\Phi_2^2 I_i\left(\zeta_{2,i}^2 - 1\right)\Delta t R_{i+1} \\ &= R_i + [\gamma I_i + \alpha S_i + \varepsilon I_i - \kappa R_i + wmZ_iS_i]\Delta t + \Phi_3 R_i \\ &\quad \times \sqrt{\Delta t}\zeta_{3,i} + \frac{1}{2}\Phi_3^2 R_i\left(\zeta_{3,i}^2 - 1\right)\Delta t Z_{i+1} \\ &= Z_i + \left[\frac{aI_i}{1+bI_i} - a_0 Z_i\right]\Delta t + \Phi_4 Z_i\sqrt{\Delta t}\zeta_{4,i} \\ &\quad + \frac{1}{2}\Phi_4^2 Z_i\left(\zeta_{4,i}^2 - 1\right)\Delta t \end{aligned} \tag{55}$$

where $\zeta_{k,i} (i = 1, 2, 3, 4)$ are four independent Gaussian random variables with $N(0, 1)$ and time interval$\Delta t > 0$.

7.1. Stationary and extinction analysis

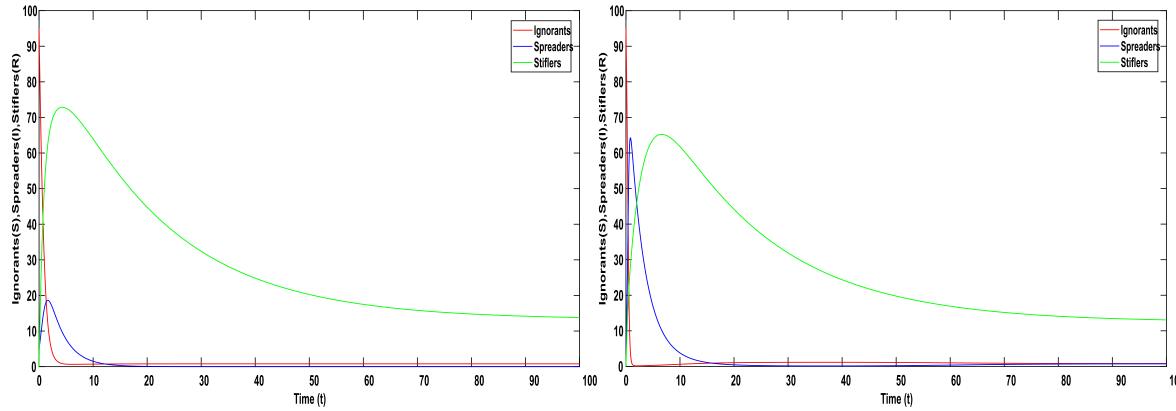
Next, we use numerical simulation combined with the parameters in Table 2 to simulate the stability and extinction of the system. The forward time interval we take is$[0, 100]$. The initial values given by the model are$S(0) = 95, I(0) = 5, R(0) = 0, Z(0) = 5$,and the parameters are$\beta = 0.7$,$\lambda = 0.16$,$\alpha = 0.84$,$\kappa = 0.05$,$w = 0.2$,$m = 0.017$,$\gamma = 0.25$,$\varepsilon = 0.05$,$a = 0.01$,$b = 1$, and $a_0 = 0.06$.We can obtain$R_0 = 0.3596 < 1$. Fig. 2 (a) verifies Theorem 3.Then, we use the same initial values and parameters:$\beta = 0.7$, $\lambda = 0.48$, $\alpha = 0.84$, $\kappa = 0.05$, $w = 0.2$, $m = 0.017$, $\gamma = 0.25$, $\varepsilon = 0.05$, $a = 0.01$, $b = 1$, $a_0 = 0.06$, and we can obtain $R_0 = 1.6842 > 1$. Fig. 2 (b) verifies Theorem 4.

7.2. Optimal control of deterministic and stochastic systems

Here, we use numerical simulations in conjunction with the parameters listed in Table 2 to simulate optimal control for deterministic and stochastic systems with and without information intervention. At the same time, to compare the results between the deterministic and stochastic models, we produced numerical simulation graphs of both models under the conditions of optimal control and external information intervention. In addition, to show the advantages of the random rumour propagation model in our paper, we complete the comparison with the existing SIR rumour propagation model.

The forward time interval we take is$[0, 20]$. For numerical simulation, we use the fourth-order Runge-Kutta iterative method and the Milstein method to analyse the deterministic and stochastic systems.

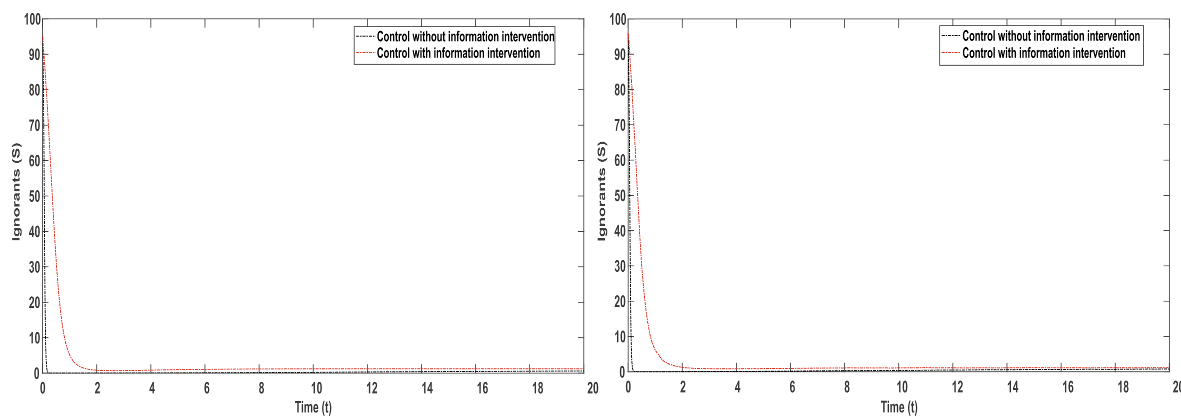
The initial values given by the model are $S(0) = 95$, $I(0) = 5, R(0) = 0, Z(0) = 5$, and the parameters are $\beta = 0.7$, $\lambda = 0.48$, $\alpha = 0.52$, $\kappa = 0.05$, $w = 0.2$,$m = 0.017$, $\gamma = 0.25$, $\varepsilon = 0.05$, $a = 0.01$, $b = 1$, $a_0 = 0.06$, $k = 10$, $k_1 = 0.55$, and $k_2 = 0.35$.We use the forward push back algorithm to solve the control problem and then consider the impact of information intervention on the number of ignorants, spreaders, and stiflers in the deterministic and stochastic systems. They are



(a) $S(t), I(t), R(t)$ in a stability state of changes

(b) $S(t), I(t), R(t)$ in a extinction state of changes

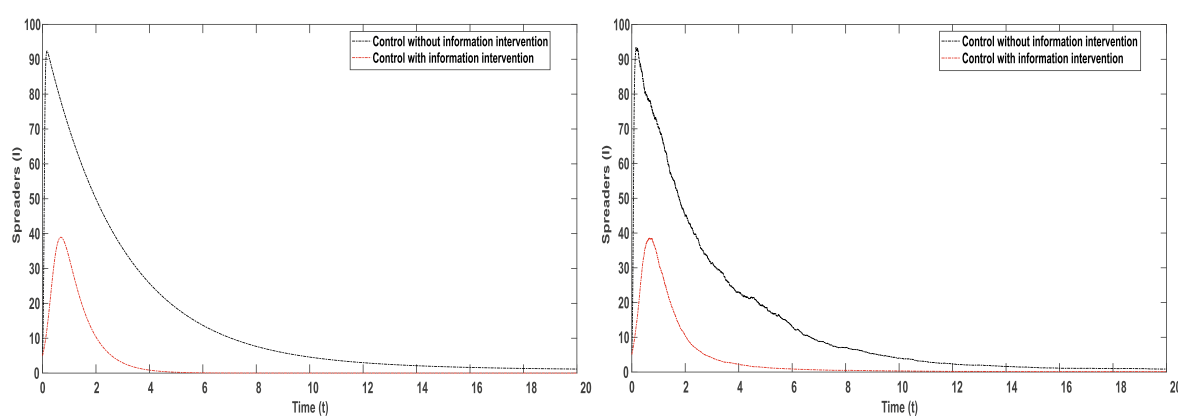
Fig. 2 Plots of changes in the number of $S(t), I(t)$ and $R(t)$ in a stability and extinction state.



(c) Under the optimal control, $S(t)$ with and without information intervention in the deterministic system.

(d) Under the optimal control, $S(t)$ with and without information intervention in the stochastic system.

Fig. 3 Under the optimal control, plots of changes in the number of $S(t)$ with and without information intervention in the deterministic and stochastic systems.



(e) Under the optimal control, $I(t)$ with and without information intervention in the deterministic system.

(f) Under the optimal control, $I(t)$ with and without information intervention in the stochastic system.

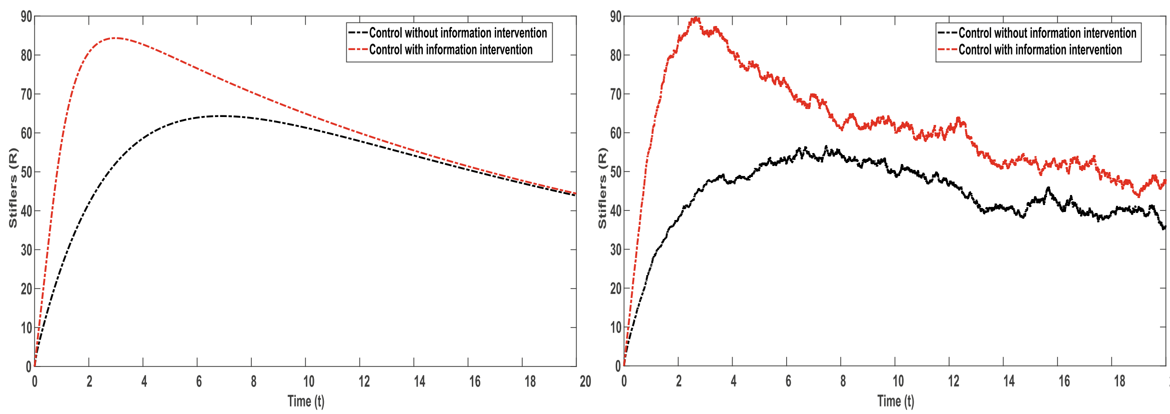
Fig. 4 Under the optimal control, plots of changes in the number of $I(t)$ with and without information intervention in the deterministic and stochastic systems.

shown in (Fig. 3), (Fig. 4) and (Fig. 5),respectively. Furthermore, Fig. 6 shows the different performances of $R(t)$ in deterministic and stochastic systems under the conditions of optimal control and external information intervention. At the same time, Fig. 6 also shows the comparison between the existing SIR rumour propagation model and the stochastic model in this paper. Fig. 7 shows the behaviour of the optimal control in the deterministic and stochastic systems.

The results of (Fig. 3), (Fig. 4) and (Fig. 5) show that the intervention of external information can effectively reduce the number of spreaders in the process of rumour propagation, expand the population of stiflers, make more people quickly stop believing a rumour, and accelerate the elimination of a rumour. This further demonstrates the effectiveness of the model in this paper.

Fig. 6 illustrates the comparison before the model through the perspective of stifler change. The result of (Fig. 6) (k) shows that under optimal control and the same external information intervention, the stochastic model performs better than the deterministic model. The result of (Fig. 6) (l) shows that the stochastic rumour propagation model proposed in this paper performs better than the existing SIR rumour propagation model.

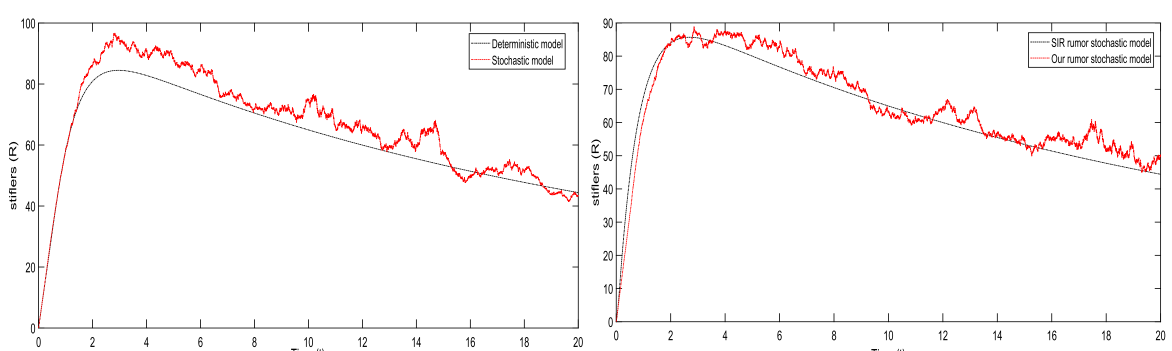
The results in Fig. 7 show that the maximum value of the control variable is reached at the beginning in both deterministic and stochastic systems, then decreases rapidly after a period of time, and finally decreases to 0. This shows that after the strict measures in the initial stage, with the extension of time, the control intensity can slowly reduce and tend to normalize. The decay rate of $u_2(t)$ is the slowest. Namely, in the actual process of blocking the spread of rumours, we need to invest most in the intensity of external information intervention and less so in controlling the spread of people.



(g) Under the optimal control, $R(t)$ with and without information intervention in the deterministic system.

(h) Under the optimal control, $R(t)$ with and without information intervention in the stochastic system.

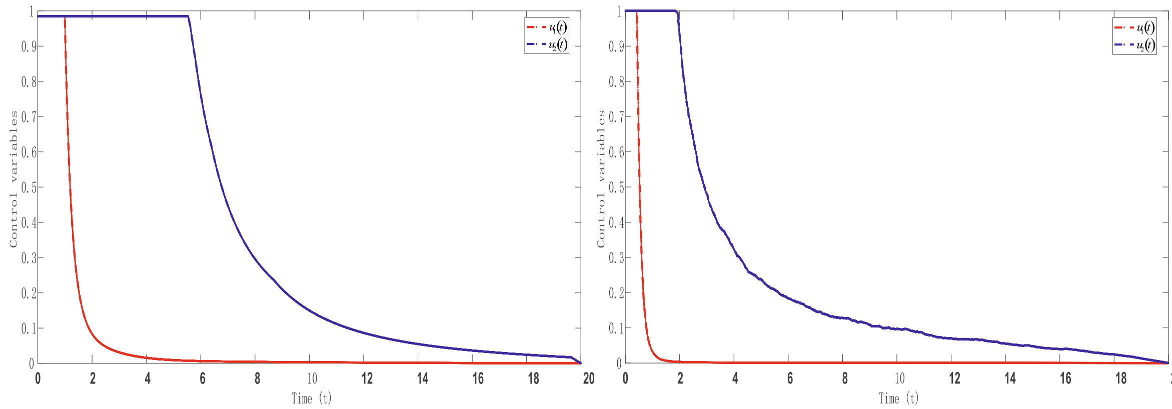
Fig. 5 Under the optimal control, plots of changes in the number of $R(t)$ with and without information intervention in the deterministic and stochastic systems.



(k) Under optimal control and external information intervention, $R(t)$ in the deterministic and stochastic systems.

(l) Under optimal control, $R(t)$ changes in comparison between SIR rumor stochastic system and our rumor stochastic systems.

Fig. 6 (k): Under the optimal control and external information intervention, plot of changes in the number of $R(t)$ in the deterministic and stochastic systems;(l): Under the optimal control, plot of changes in the number of $R(t)$ in the SIR rumour stochastic system and this paper's rumour stochastic system.



(i) $u_1(t), u_2(t)$ in case of deterministic system. (j) $u_1(t), u_2(t)$ in case of stochastic system.

Fig. 7 Plots of $u_1(t)$ and $u_2(t)$ in the case of deterministic and stochastic systems.

8. Conclusion

In this paper, the authors propose an improved SIR rumour propagation model under the intervention of external information. On the one hand, we use the characteristics of rumour dynamics and the next generation matrix method to calculate the basic reproduction number R_0 and obtain the threshold of rumour propagation. Moreover, we demonstrate the local and global stability of system equilibrium points E^1 andE^2. On the other hand, we comprehensively investigate the effects of information intervention and the control rate of spreaders on the dynamic properties of rumour propagation, and we analyse the optimal control strategy to block rumour propagation under deterministic and stochastic methods. For further understanding, we use the Runge–Kutta and Milstein methods to achieve numerical simulations of deterministic and stochastic systems under optimal control. We draw the following conclusions: 1) Under optimal control and the same external information intervention, the stochastic model shows better results than the deterministic model. 2) The stochastic rumour propagation model proposed in this paper performs better than the existing stochastic SIR rumour propagation model.

Authors' contributions

H.H.W. and Y.Z.Z. proposed and designed thisstudy, did all numerical simulations described in the paper, interpreted the results, and wrote the paper. J.W.Z. and C.Y.Z., H.H.W. performed an analytical treatment of a stochastic differential equation. Both authors contributed to the discussions.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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