EMARO

Modeling and Control of Manipulators

Part II:
Control of Manipulators, *by P. Tatjewski*

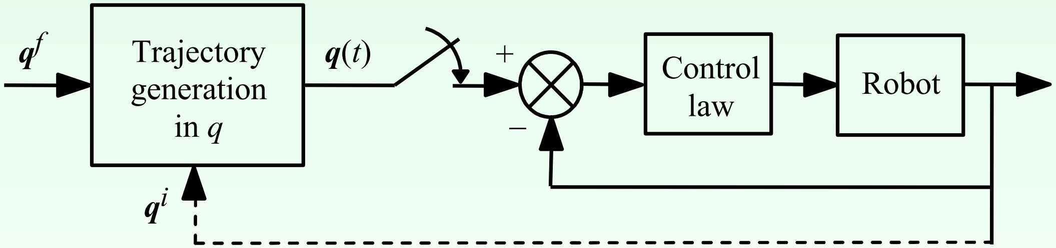
Trajectory generation

for point-to-point control

A path and a trajectory

- **A path:** a sequence of points along which a manipulator should move.
  - Defined in task space or joint space.
  - In simplest case only two points given: initial q^i and final q^f, leads to a point-to-point motion,
- **A trajectory** $q(t)$: a function of time passing through the path points.

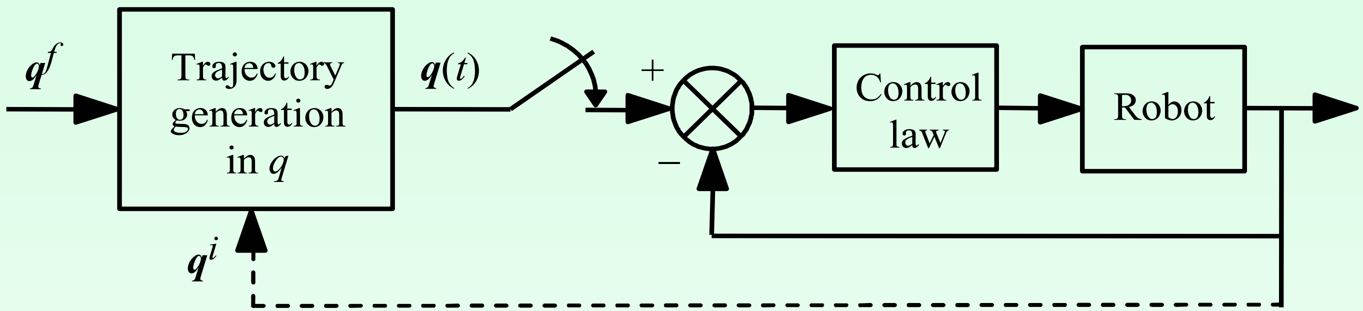
Computed as a **reference (set-point) trajectory** for the manipulator control system:



In the case of a **point-to-point motion**, there are two solutions:

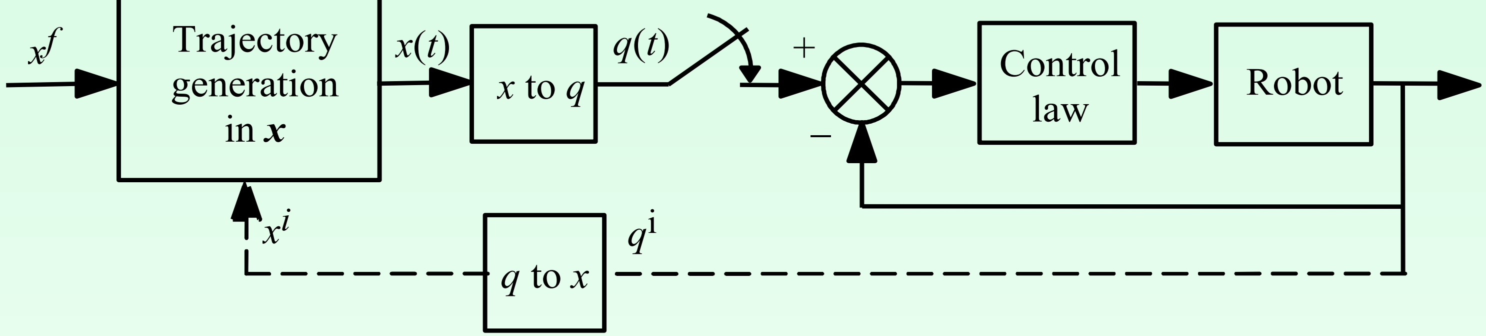
- A step change of controller set-point from q^i to q^f – the trajectory is generated dynamically by the controller according to the dynamics of the control system;
- A continuous trajectory $q(t)$ between q^i and q^f is generated, the controller performs **trajectory tracking** – *a better solution*.

Trajectory generation in the joint space



- Needs less on-line computations (computation of inverse model not needed).
- No risk of singular configurations.
- Actuator constraints easily available and preserved.
- End-effector path not accurately predictable, risk of collisions if possible.
- Best suited to fast motions in relatively free space.

Trajectory generation in the task space



- Needs more on-line computations (computation of inverse kinematics model needed).
- More difficult to preserve actuator constraints.
- May fail when crossing singular configurations.
- Task-space trajectory tracking, collision avoidance more accurate.

Point-to-point trajectories in joint space

The problem: to find a trajectory connecting initial and final points, satisfying specified constraints at these points (and possibly also at certain internal points) on velocities and/or accelerations.

Basic constraints: - on positions:

$$q(t_0) = q_0$$
$$q(t_f) = q_f$$

- on velocity:

$$\dot{q}(t_0) = v_0$$
$$\dot{q}(t_f) = v_f$$

- on acceleration:

$$\ddot{q}(t_0) = \alpha_0$$
$$\ddot{q}(t_f) = \alpha_f$$

Additional similar constraints possible on internal points of the trajectory.

Standard shapes of point-to-point trajectories:

- Cubic polynomial trajectory,
- Quintic polynomial trajectory,
- LSPB (trapezoidal) trajectory,
- Minimum time trajectory.

Cubic polynomial trajectory

Polynomial joint trajectory satisfying prescribed values of velocity at initial and final point:

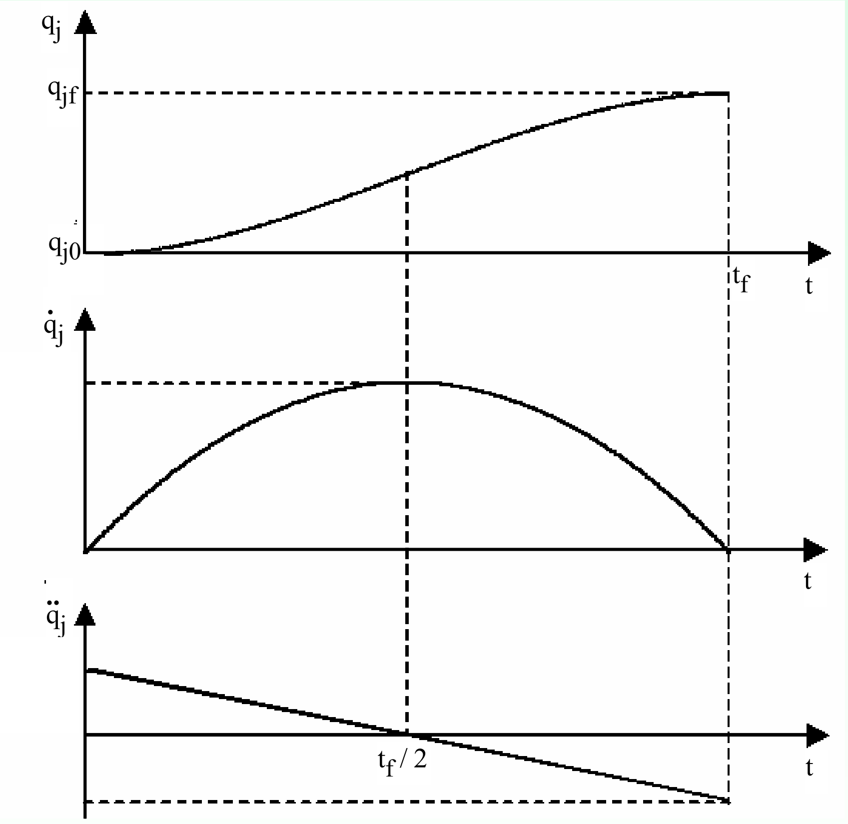
$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
$$(\dot{q}(t) = a_1 + 2a_2 t + 3a_3 t^2)$$

End-points constraints:

$$q(t_0) = q_0 \qquad \dot{q}(t_0) = v_0$$
$$q(t_f) = q_f \qquad \dot{q}(t_f) = v_f$$

Four constraints on four parameters (for each joint variable) – a system of 4 linear equations:

$$q_0 = a_0 + a_1 t_0 + a_2 t_0^{\,2} + a_3 t_0^{\,3},$$
$$v_0 = a_1 + 2a_2 t_0 + 3a_3 t_0^{\,2}$$
$$q_f = a_0 + a_1 t_f + a_2 t_f^{\,2} + a_3 t_f^{\,3}$$
$$v_f = a_1 + 2a_2 t_f + 3a_3 t_f^{\,2}$$



Cubic polynomial trajectory with zero velocities at initial and final points, for t_0=0

Cubic polynomial trajectory – explicit formula

Assuming zero initial time (temporarily) and zero velocity at initial and final point we have:

$$q_0 = a_0,$$
$$0 = a_1$$
$$q_f = a_0 + a_1 t_f + a_2 t_f^{\,2} + a_3 t_f^{\,3}$$
$$0 = a_1 + 2a_2 t_f + 3a_3 t_f^{\,2}$$

$$\Longrightarrow \quad q_f = q_0 + a_2 t_f^{\,2} + a_3 t_f^{\,3}, \qquad 0 = 2a_2 t_f + 3a_3 t_f^{\,2}$$

From the last equation:

$$a_2 = -\frac{3}{2} a_3 t_f \quad \Longrightarrow \quad q_f - q_0 = -\frac{3}{2} a_3 t_f t_f^{\,2} + a_3 t_f^{\,3} \quad \Longrightarrow \quad a_3 = 2(q_0 - q_f)\frac{1}{t_f^{\,3}}$$

Thus $a_2 = -3(q_0 - q_f)\dfrac{1}{t_f^{\,2}}$ and the trajectory equation for $t_0=0$:

$$q(t) = q_0 + \frac{3(q_f - q_0)}{t_f^{\,2}} t^2 - \frac{2(q_f - q_0)}{t_f^{\,3}} t^3, \qquad t_0 = 0.$$

For any initial time t_0 (and any t_f, q_0, q_f) we have:

$$\boxed{q(t - t_0) = q_0 + 3\frac{(q_f - q_0)}{(t_f - t_0)^2}(t - t_0)^2 - 2\frac{(q_f - q_0)}{(t_f - t_0)^3}(t - t_0)^3}$$

Quintic polynomial trajectory

Polynomial joint trajectory satisfying prescribed values of velocity and acceleration at initial and final point:

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

$$\dot{q}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$$

$$\ddot{q}(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3$$

End-points constraints:

$$q(t_0) = q_0 \qquad \dot{q}(t_0) = v_0 \qquad \ddot{q}(t_0) = \alpha_0$$

$$q(t_f) = q_f \qquad \dot{q}(t_f) = v_f \qquad \ddot{q}(t_f) = \alpha_f$$

Six constraints on six parameters (for each joint variable) – result in a system of 6 linear equations:

$$q_0 = a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 + a_4 t_0^4 + a_5 t_0^5$$

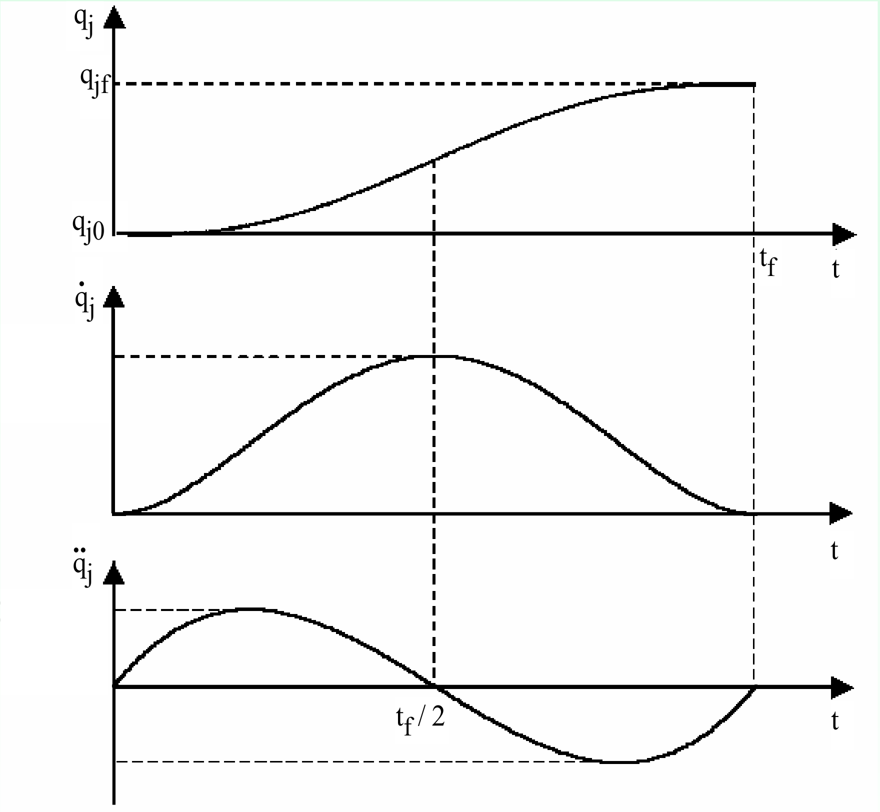
$$v_0 = a_1 + 2a_2 t_0 + 3a_3 t_0^2 + 4a_4 t_0^3 + 5a_5 t_0^4$$

$$\alpha_0 = 2a_2 + 6a_3 t_0 + 12a_4 t_0^2 + 20a_5 t_0^3$$

$$q_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5$$

$$v_f = a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_0^3 + 5a_5 t_0^4$$

$$\alpha_f = 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3$$



Quintic polynomial trajectory with zero velocities and accelerations at initial and final points, for $t_0=0$

Quintic polynomial trajectory – explicit formula

Assuming $t_0=0$ (temporarily), $v_0=0$, $v_f=0$, $\alpha_0=0$, $\alpha_f=0$, we have from first 3 equations:

$q_0 = a_0,\ 0 = a_1,\ 0 = 2a_2,$ the remaining 3 equations are then:

$$q_f = q_0 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5, \Longrightarrow a_3 = (q_0 - q_f)/t_f^3 - a_4 t_f - a_5 t_f^2,$$

$$0 = 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4,$$

$$0 = 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3,$$

$$\Downarrow$$

$$0 = 3(q_f - q_0)/t_f + a_4 t_f^3 + 2a_5 t_f^4, \Longrightarrow a_4 = -3(q_0 - q_f)/t_f^4 - 2a_5 t_f,$$

$$0 = 6(q_f - q_0)/t_f^2 + 6a_4 t_f^2 + 14a_5 t_f^3, \Longrightarrow a_5 = 6(q_0 - q_f)/t_f^5,$$

Taking the all above into account we obtain the trajectory equation for $t_0=0$:

$$q(t) = q_0 + 10\frac{(q_f - q_0)}{t_f^3}t^3 - 15\frac{(q_f - q_0)}{t_f^4}t^4 + 6\frac{(q_f - q_0)}{t_f^5}t^5$$

For any initial time t_0 (and any t_f, q_0, q_f) we have:

$$\boxed{q(t - t_0) = q_0 + 10\frac{(q_f - q_0)}{(t_f - t_0)^3}(t - t_0)^3 - 15\frac{(q_f - q_0)}{(t_f - t_0)^4}(t - t_0)^4 + 6\frac{(q_f - q_0)}{(t_f - t_0)^5}(t - t_0)^5}$$

LSPB (trapezoidal) trajectory

LSPB - Linear Segments with Parabolic Blends trajectory, a symmetric trajectory with trapezoidal shape of velocity profile, with constant velocity in central part of the path.

For $t \in [t_0, t_b]$ and $t \in [t_f - t_b, t_f]$

where t_b is the **blend time,** the trajectory is described by a quadratic polynomial:

$$q(t) = a_0 + a_1 t + a_2 t^2 \qquad (\dot{q}(t) = a_1 + 2a_2 t)$$

For $t \in [t_b, t_f - t_b]$

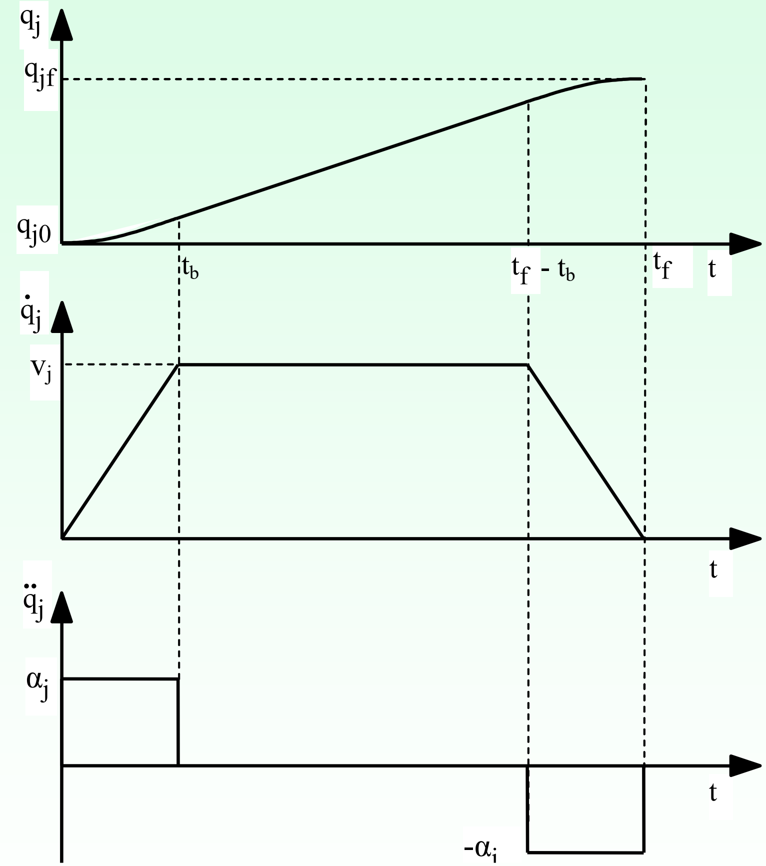
constant velocity, say v, is assumed.

The constraints (assuming t_0 = 0, temporarily):

$$q(0) = q_0,\ q(t_f) = q_f,$$
$$\dot{q}(t_b) = \dot{q}(t_f - t_b) = v,$$

Assume also that:

$$\dot{q}(0) = \dot{q}(t_f) = 0$$



LSPB (trapezoidal) trajectory with zero velocities at the end points, for $t_0=0$

LSPB (trapezoidal) trajectory

Calculation of the LSPB trajectory (for given velocity v and acceleration α):

For $t \in [t_0, t_b]$:

$$q(t) = a_0 + a_1 t + a_2 t^2, \qquad \dot{q}(t) = a_1 + 2a_2 t, \qquad \ddot{q}(t) = 2a_2 = \alpha$$

$$q(0) = q_0 \;\Rightarrow\; a_0 = q_0,$$

$$\dot{q}(0) = 0 \;\Rightarrow\; a_1 = 0 \;\Rightarrow\; \dot{q}(t_b) = 2a_2 t_b \;\Rightarrow\; a_2 = \frac{v}{2t_b} \;\Rightarrow\; \boxed{\alpha = \frac{v}{t_b}}$$

Thus: $$q(t) = q_0 + \frac{v}{2t_b} t^2 \;\Rightarrow\; \boxed{q(t) = q_0 + \frac{\alpha}{2} t^2}$$

$$q(t_b) = q_0 + \frac{\alpha}{2} t_b^2 = q_0 + \frac{v^2}{2\alpha} \quad \text{as} \quad \boxed{t_b = \frac{v}{\alpha}} \quad (*)$$

For $t \in [t_b, t_f - t_b]$:

$$q(t) = q(t_b) + v(t - t_b)$$

$$q(t) = q_0 + \frac{v^2}{2\alpha} + v(t - t_b) \;\Rightarrow\; \boxed{q(t) = q_0 + \frac{v^2}{2\alpha} + v(t - \frac{v}{\alpha})}$$

LSPB (trapezoidal) trajectory

Relations between v, α, and t_b, t_f: Due to symmetry, we have for $t=t_f/2$:

$$q(\frac{t_f}{2}) = \frac{q_0+q_f}{2}, \quad \text{i.e.:} \quad q_0 + \frac{v^2}{2\alpha} + v(\frac{t_f}{2} - \frac{v}{\alpha}) = \frac{q_0+q_f}{2} \quad \Rightarrow \quad -\frac{v^2}{2\alpha} + v\frac{t_f}{2} = \frac{q_f-q_0}{2}$$

which gives upon solving for t_f:

$$\boxed{t_f = \frac{q_f-q_0}{v} + \frac{v}{\alpha}} \qquad (**)$$

and, as $t_b = \dfrac{v}{\alpha}$:

$$\boxed{t_b = t_f - \frac{q_f-q_0}{v}} \qquad (***)$$

Condition to have constant velocity phase: $t_b < \dfrac{t_f}{2}$, inserting t_b from (***) results in:

$$\frac{t_f}{2} > t_f - \frac{q_f-q_0}{v} \quad \Rightarrow \quad v < 2\frac{q_f-q_0}{t_f}$$

Inserting vt_f from (**) into the last inequality:

$$vt_f = q_f - q_0 + \frac{v^2}{\alpha} < 2(q_f-q_0) \quad \Rightarrow \quad \boxed{q_f - q_0 > \frac{v^2}{\alpha}}$$

LSPB (trapezoidal) trajectory

For $t \in [t_f - t_b, t_f]$ the (quadratic) trajectory can be found knowing that $\dot{q}(t_f) = 0$, $q(t_f) = q_f$ and the acceleration is equal to $-\alpha$.

From $\dot{q}(t_f) = 0$ and the acceleration $-\alpha$ we conclude that

$$\dot{q}(t) = \alpha(t_f - t)$$

which results in $$q(t) = -\frac{\alpha}{2} t^2 + \alpha t_f t + c$$

From $q(t_f) = q_f$ we have $$c = q_f - \alpha t_f t_f + \frac{\alpha}{2} t_f^2 = q_f - \frac{\alpha}{2} t_f^2$$

thus

$$q(t) = -\frac{\alpha}{2} t^2 + \alpha t_f t + q_f - \frac{\alpha}{2} t_f^2$$

Due to

$$t_f = \frac{q_f - q_0}{v} + \frac{v}{\alpha}$$

we get finally

$$\boxed{q(t) = q_f - \frac{\alpha}{2}\left(t - \frac{q_f - q_0}{v} - \frac{v}{\alpha}\right)^2}$$

LSPB (trapezoidal) trajectory

Complete description of the LSPB trajectory (given velocity v and acceleration α, for $t_0 = 0$):

$$q(t) = \begin{cases} q_0 + \dfrac{\alpha}{2} t^2 & \text{for} \quad 0 \le t \le t_b, \\ q_0 + \dfrac{v^2}{2\alpha} + v\left(t - \dfrac{v}{\alpha}\right) & \text{for} \quad t_b \le t \le t_f - t_b, \\ q_f - \dfrac{\alpha}{2}\left(t - \dfrac{q_f - q_0}{v} - \dfrac{v}{\alpha}\right)^2 & \text{for} \quad t_f - t_b \le t \le t_f, \end{cases}$$

where

$$t_b = \frac{v}{\alpha} \qquad (*)$$

$$t_f = \frac{q_f - q_0}{v} + \frac{v}{\alpha} \qquad (**)$$

Applying maximum velocity v and maximum acceleration α results in LSPB trajectory with **minimum time** $t_f = t_{fmin}$.

Alternatively, when assuming final time t_f and blend time t_b, the appropriate constant velocity v and acceleration α result from the formulae (*) and (**).

LSPB (trapezoidal) trajectory – for $t_0 > 0$

Complete description of the LSPB trajectory for $t_0 \neq 0$ (given velocity v and acceleration α), assuming:

t_m – time interval of motion, $t_f = t_0 + t_m$, (t_f - final time),

t_b – tength of single blend time interval (incremental blend time)

$$q(t-t_0)=\begin{cases} q_0+\dfrac{\alpha}{2}(t-t_0)^2 & \text{for} \quad t_0\le t\le t_0+t_b, \\ q_0+\dfrac{v^2}{2\alpha}+v(t-t_0-\dfrac{v}{\alpha}) & \text{for} \quad t_0+t_b\le t\le t_f-t_b, \\ q_f-\dfrac{\alpha}{2}\left(t-t_0-\dfrac{q_f-q_0}{v}-\dfrac{v}{\alpha}\right)^2 & \text{for} \quad t_f-t_b\le t\le t_f, \end{cases}$$

where

$$t_b=\frac{v}{\alpha} \qquad (*)$$

$$t_m=\frac{q_f-q_0}{v}+\frac{v}{\alpha} \qquad (**)$$

Bang-bang (minimum time) trajectory

When in the LSPB design $q_f - q_0 \le \dfrac{v^2}{\alpha}$

then the constant velocity phase vanishes, and LSPB trajectory reduces to the **bang-bang trajectory.**

Denoting by v_s the maximum velocity,

attained at $t_s = t_f/2$,

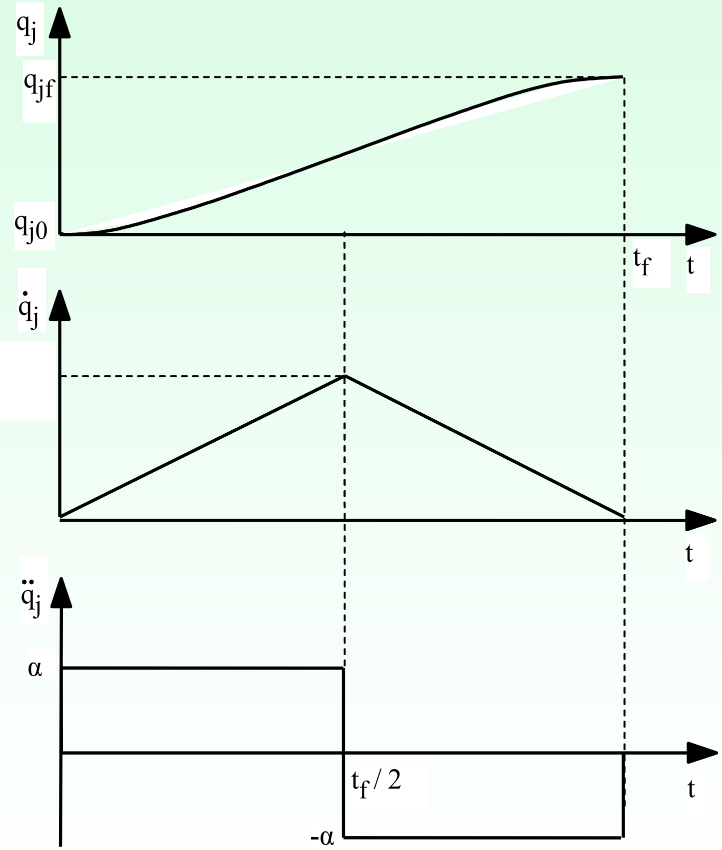
we have (assuming t_0=0):

$$v_s = \alpha t_s = \frac{\alpha t_f}{2}$$

$$\frac{q_f - q_0}{2} = \frac{\alpha t_s^2}{2}$$

Hence the switching time and final time:

$$t_s = \frac{t_f}{2} = \sqrt{\frac{q_f - q_0}{\alpha}}$$



Bang-bang trajectory with zero velocities at end points

Synchronization of trajectories

In decentralized point-to-point control of a n-joint manipulator, it is recommended that trajectory for each joint should reach the final point at the same final time t_f.

In the case of **cubic, quintic and bang-bang trajectories** the synchronization is simple:

- assuming the same initial time t_0,
- the same final time t_f. should be assumed, under the feasibility conditions: as the trajectories are symmetric, maximal velocity and acceleration occur at the middle points, at time $(t_0+t_f)/2$:
  - for the cubic and quintic trajectories:

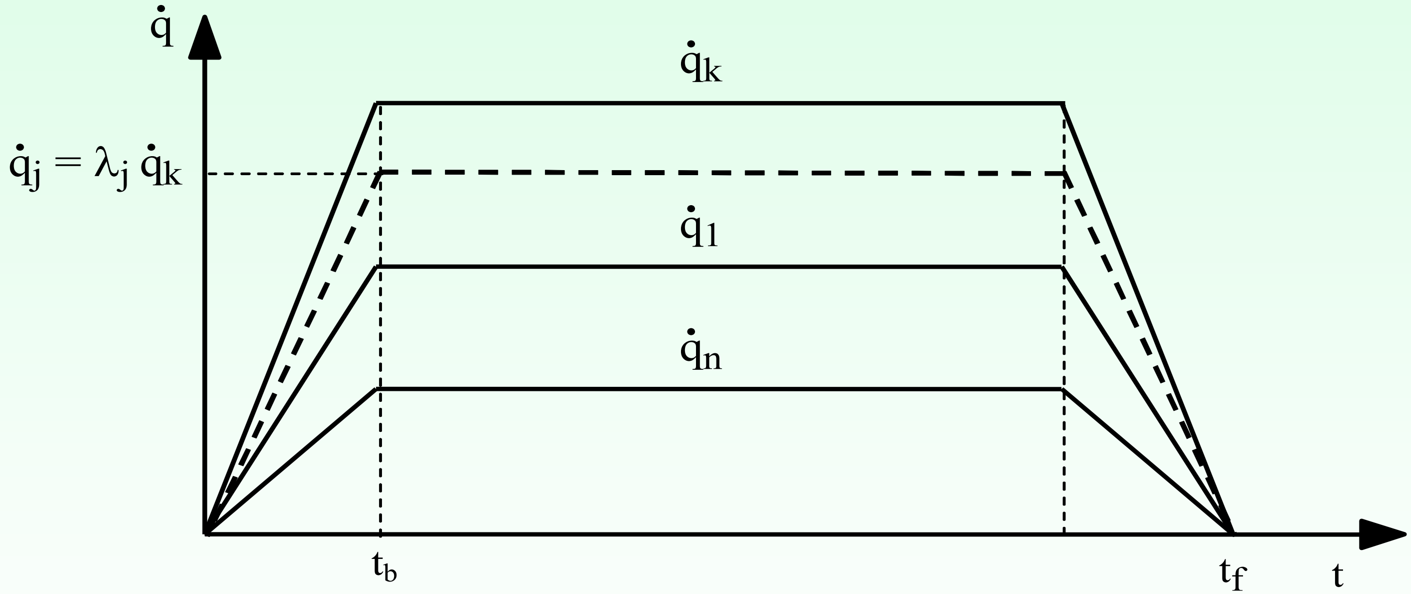
$$\dot{q}_i\left(\frac{t_0+t_f}{2}\right) \le v_{i,\max}, \quad \ddot{q}_i\left(\frac{t_0+t_f}{2}\right) \le \alpha_{i,\max}, \quad i=1,\dots,n$$

  - For the bang-bang trajectories:

$$\alpha_{i,\max}\frac{t_f-t_0}{2} \le v_{i,\max}, \quad i=1,\dots,n$$

Synchronized LSPB minimum time trajectories

All three phases of motion: acceleration, constant velocity and deceleration should have the same duration t_b, $t_m - 2\,t_b$ and t_b :



Synchronized LSPB velocity profiles for n-link manipulator

Synchronized LSPB minimum time trajectories

For the case of 2-joint robot we have:

$$t_{b1} = \frac{v_1}{\alpha_1}, \qquad t_{f1} = \frac{|q_{f1} - q_{01}|}{v_1} + t_{b1} = \frac{d_1}{v_1} + \frac{v_1}{\alpha_1}, \qquad \text{where} \quad d_1 = |q_{f1} - q_{01}|$$

$$t_{b2} = \frac{v_2}{\alpha_2}, \qquad t_{f2} = \frac{|q_{f2} - q_{02}|}{v_2} + t_{b2} = \frac{d_2}{v_2} + \frac{v_2}{\alpha_2}, \qquad \text{where} \quad d_2 = |q_{f2} - q_{02}|$$

The synchronized trajectories should satisfy, denoting by λ_i and β_i factors scaling velocities and accelerations:

$$t_f = \frac{d_1}{\lambda_1 v_1} + \frac{\lambda_1 v_1}{\beta_1 \alpha_1} = \frac{d_2}{\lambda_2 v_2} + \frac{\lambda_2 v_2}{\beta_2 \alpha_2}, \qquad t_b = \frac{\lambda_1 v_1}{\beta_1 \alpha_1} = \frac{\lambda_2 v_2}{\beta_2 \alpha_2}$$

under the realizability condition $t_f \geq \max\{t_{f1}, t_{f2}\}$

Hence

$$\frac{d_1}{\lambda_1 v_1} = \frac{d_2}{\lambda_2 v_2} \quad \Rightarrow \quad \boxed{\lambda_2 = \lambda_1 \frac{v_1 d_2}{v_2 d_1}},$$

$$\frac{\lambda_1 v_1}{\beta_1 \alpha_1} = \frac{\lambda_2 v_2}{\beta_2 \alpha_2} \quad \Rightarrow \quad \frac{v_1}{\beta_1 \alpha_1} = \frac{\lambda_2}{\lambda_1} \frac{v_2}{\beta_2 \alpha_2} \quad \Rightarrow \quad \frac{v_1}{\beta_1 \alpha_1} = \frac{v_1 d_2}{v_2 d_1} \frac{v_2}{\beta_2 \alpha_2} \quad \Rightarrow \quad \boxed{\beta_2 = \beta_1 \frac{\alpha_1 d_2}{\alpha_2 d_1}}$$

Synchronized LSPB minimum time trajectories

Obviously

$$0 \le \lambda_1 \le 1, \quad 0 \le \lambda_2 \le 1$$

thus

$$\lambda_2 \le 1 \;\Rightarrow\; \lambda_1 \frac{v_1 d_2}{v_2 d_1} \le 1 \;\Rightarrow\; \lambda_1 \le \frac{v_2 d_1}{v_1 d_2} \;\Rightarrow\; \boxed{\lambda_1 = \min\{1, \frac{v_2 d_1}{v_1 d_2}\}}$$

Similarly

$$0 \le \beta_1 \le 1, \quad 0 \le \beta_2 \le 1$$

$$\beta_2 \le 1 \;\Rightarrow\; \beta_1 \frac{\alpha_1 d_2}{\alpha_2 d_1} \le 1 \;\Rightarrow\; \beta_1 \le \frac{\alpha_2 d_1}{\alpha_1 d_2} \;\Rightarrow\; \boxed{\beta_1 = \min\{1, \frac{\alpha_2 d_1}{\alpha_1 d_2}\}}$$

It is convenient to index the joint with the largest $\frac{d_j}{v_j}$ as the first, then $\lambda_1 = 1$.

After calculation of the factors for the first joint, we calculate:

$$\lambda_2 = \lambda_1 \frac{v_1 d_2}{v_2 d_1}, \quad \beta_2 = \beta_1 \frac{\alpha_1 d_2}{\alpha_2 d_1}$$

and

$$t_b = \frac{\lambda_1 v_1}{\beta_1 \alpha_1}, \qquad t_f = \frac{d_1}{\lambda_1 v_1} + \frac{\lambda_1 v_1}{\beta_1 \alpha_1} \qquad (t_f \ge \max\{t_{f1}, t_{f2}\})$$

The above formulae can be generalized for more joints.

Synchronized LSPB minimum time trajectories

Example 1: $v_1 = 2,\ \ \alpha_1 = 1,\quad q_{01} = 0,\ q_{f1} = 10 = d_1,\ \ (d_1 / v_1 = 5)$

$v_2 = 1,\ \ \alpha_2 = 2,\ q_{02} = 0,\ q_{f2} = 4 = d_2,\quad (d_2 / v_2 = 4)$

We have:

$$t_{bj} = \frac{v_j}{\alpha_j},\quad t_{fj} = \frac{d_j}{v_j} + \frac{v_j}{\alpha_j} \quad \Rightarrow \quad \begin{aligned} &t_{b1} = 2,\ t_{f1} = 5+2 = 7, \\ &t_{b2} = 0.5,\ t_{f1} = 4+0.5 = 4.5 \end{aligned}$$

Scaling:

$$\lambda_1 = \min\left\{1, \frac{v_2 d_1}{v_1 d_2}\right\} = \min\left\{1, \frac{5}{4}\right\} = 1,$$

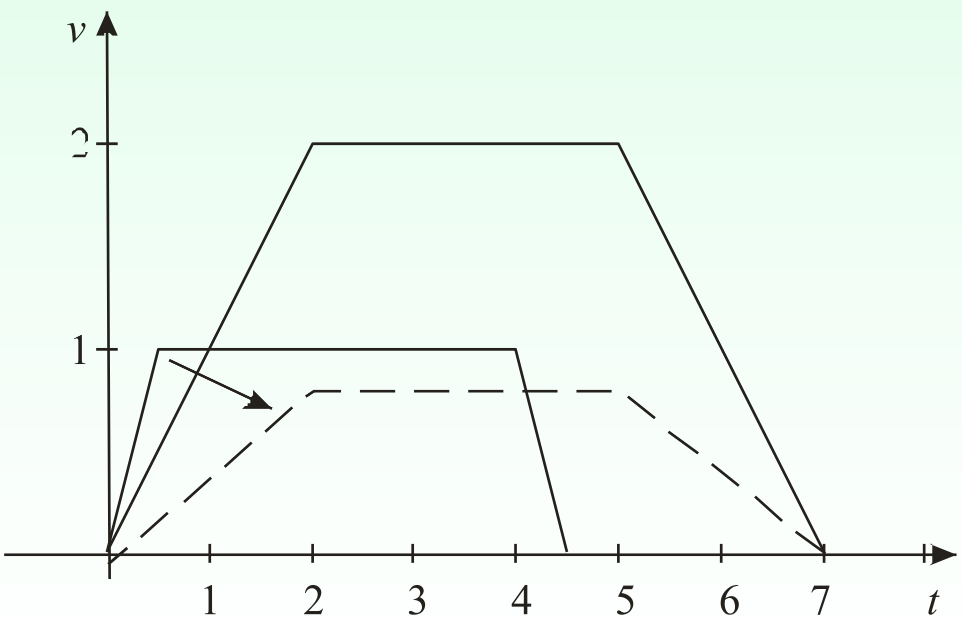
$$\beta_1 = \min\left\{1, \frac{\alpha_2 d_1}{\alpha_1 d_2}\right\} = \min\{1, 5\} = 1,$$

$$\lambda_2 = \lambda_1 \frac{v_1 d_2}{v_2 d_1} = \frac{4}{5} = 0.8,$$

$$\beta_2 = \beta_1 \frac{\alpha_1 d_2}{\alpha_2 d_1} = \frac{4}{20} = 0.2,$$

$$v_2^{sc} = 1 \cdot 0.8 = 0.8,\ \ \alpha_2^{sc} = 2 \cdot 0.2 = 0.4,$$

$$t_{b2}^{sc} = 2,\ \ t_{f2}^{sc} = \frac{4}{0.8} + 2 = 7$$



Synchronized LSPB minimum time trajectories

Example 2: $v_1 = 2,\ \ \alpha_1 = 2,\quad q_{01} = 0,\ \ q_{f1} = 10 = d_1,\quad (d_1 / v_1 = 5)$

$v_2 = 1,\ \ \alpha_2 = 0.4,\ \ q_{02} = 0,\ \ q_{f2} = 4 = d_2,\quad (d_2 / v_2 = 4)$

We have: $t_{bj} = \dfrac{v_j}{\alpha_j},\quad t_{fj} = \dfrac{d_j}{v_j} + \dfrac{v_j}{\alpha_j}\quad \Rightarrow$

$t_{b1} = 1,\ \ t_{f1} = 5 + 1 = 6,$

$t_{b2} = 2.5,\ \ t_{f1} = 4 + 2.5 = 6.5$

Scaling:

$$\lambda_1 = \min\left\{1, \frac{v_2 d_1}{v_1 d_2}\right\} = \min\left\{1, \frac{5}{4}\right\} = 1,$$

$$\beta_1 = \min\left\{1, \frac{\alpha_2 d_1}{\alpha_1 d_2}\right\} = \min\left\{1, \frac{4}{8}\right\} = 0.5,$$

$$\lambda_2 = \lambda_1 \frac{v_1 d_2}{v_2 d_1} = \frac{4}{5} = 0.8,$$

$$\beta_2 = \beta_1 \frac{\alpha_1 d_2}{\alpha_2 d_1} = 0.5\frac{8}{4} = 1,$$

$$v_1^{sc} = 1 \cdot 2 = 2,\ \ \alpha_1^{sc} = 2 \cdot 0.5 = 1,$$

$$v_2^{sc} = 1 \cdot 0.8 = 0.8,\ \ \alpha_2^{sc} = 1 \cdot 0.4 = 0.4,$$

$$t_{b1}^{sc} = 2,\ \ t_{f2}^{sc} = \frac{10}{2} + 2 = 7$$

$$t_{b2}^{sc} = 2,\ \ t_{f2}^{sc} = \frac{4}{0.8} + 2 = 7$$