Warsaw University of Technology

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Department of Robotics

Kinematic Model of the ABB IRB-7600 Manipulator

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CONTENTS

DENAVIT-HARTENBERG PARAMETERS AND JOINT-LINK DESCRIPTION MATRICES

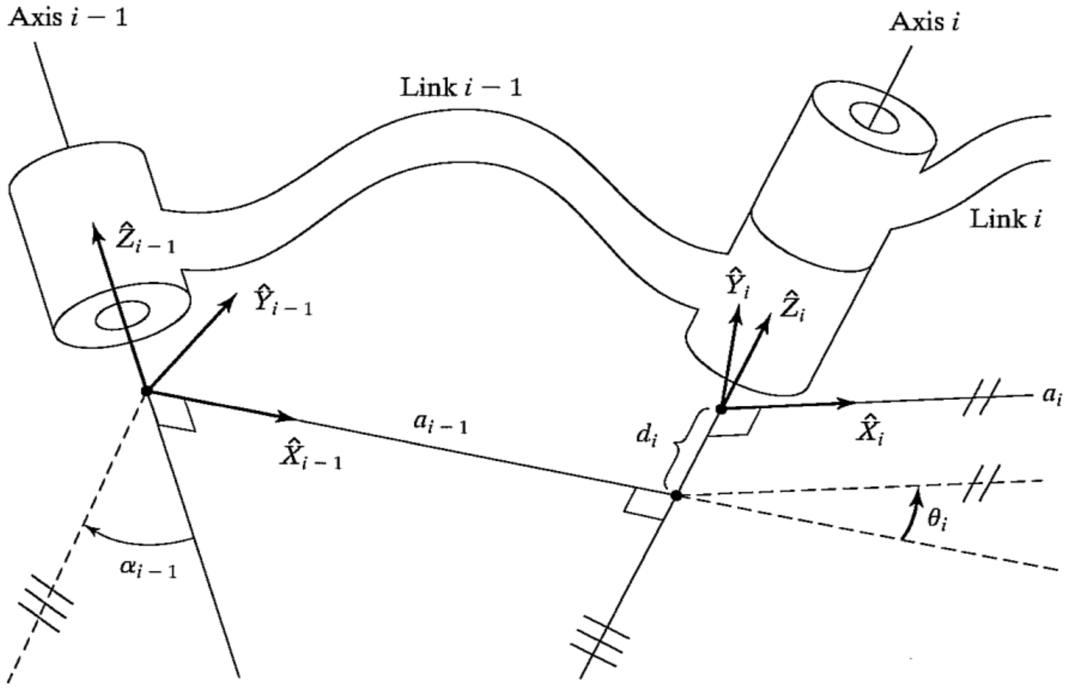


Figure 1. The description of frame {i} with respect to frame {i −1} (Craig, 2005)

These modified D-H parameters can be described as:

α_{i-1}: Twist angle between the joint axes Z_i and Z_{i-1} measured about X_{i-1}.

a_{i-1}: Distance between the two joint axes Z_i and Z_{i-1} measured along the common normal.

θ_i: Joint angle between the joint axes X_i and X_{i-1} measured about Z_i.

d_i: Link offset between the axes X_i and X_{i-1} measured along Z_i.

$${}^{i-1}_{i}T = Rot(X_{i-1}, \alpha_{i-1}) Trans(X_{i-1}, \alpha_{i-1}) Rot(Z_i\ , \theta_i) Trans(Z_i, d_i)$$

$${}^{i-1}_{i}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_{i-1} & -s\alpha_{i-1} & 0 \\ 0 & s\alpha_{i-1} & c\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -d_i s\alpha_{i-1} \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & d_i c\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{……………………………….Equation 1} $$

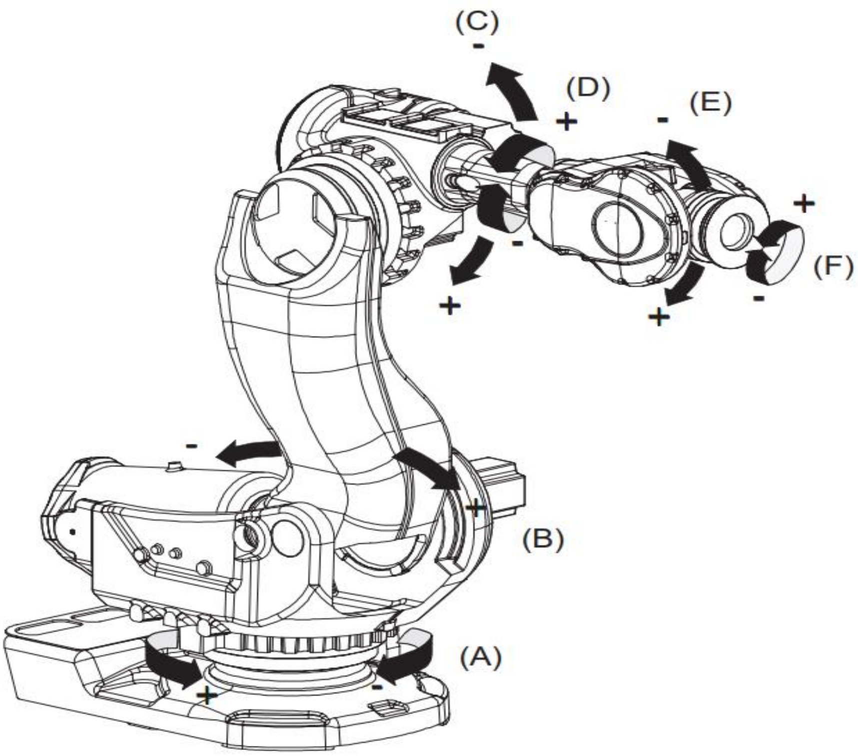
Algorithm producing the kinematic model of a manipulator:-

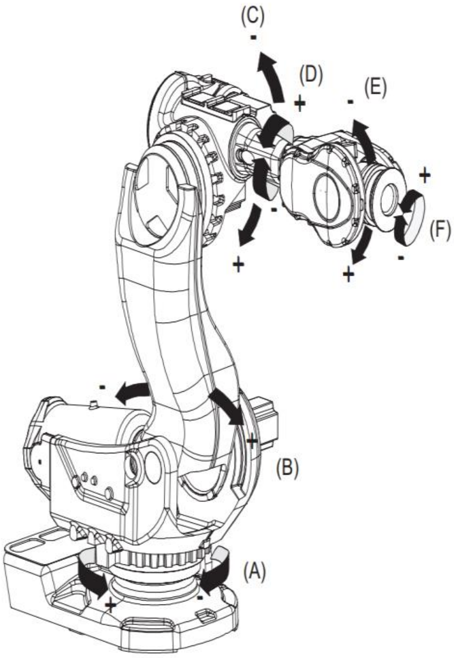
1. Determine the link and joint parameters
   - ✓ α_{i-1}: Twist angle between the joint axes Z_i and Z_{i-1} measured about X_{i-1}.
   - ✓ a_{i-1}: Distance between the two joint axes Z_i and Z_{i-1} measured along the common normal.
   - ✓ θ_i: Joint angle between the joint axes X_i and X_{i-1} measured about Z_i.
   - ✓ d_i: Link offset between the axes X_i and X_{i-1} measured along Z_i.

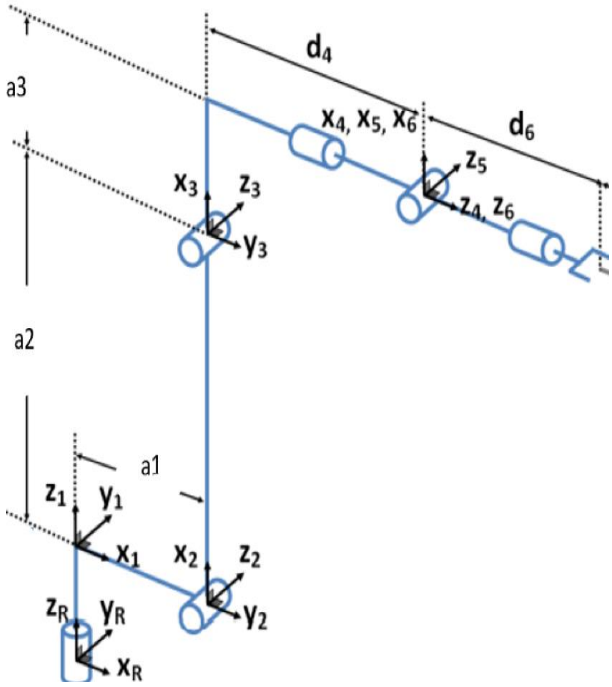
2. Determine the homogeneous matrices ${}^{i-1}_{i}T$ using equation 1.
3. Solve the direct kinematics problem

   ${}^{0}_{n}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T \dots\dots\dots\, {}^{n-1}_{n}T = \prod_{i=1}^{n} {}^{i-1}_{i}T$

   Where n – the end-effector frame
4. Solve the inverse kinematics problem







i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	a_1	$-\frac{\pi}{2}$	0	$\theta_2 - \frac{\pi}{2}$
3	a_2	0	0	θ_3
4	a_3	$-\frac{\pi}{2}$	d_4	θ_4
5	0	$\frac{\pi}{2}$	0	θ_5
6	0	$-\frac{\pi}{2}$	d_6	θ_6

$$ {}^{i-1}_{\ \ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -d_i s\alpha_{i-1} \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & d_i c\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{1}T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} c(\theta_2 - \frac{\pi}{2}) & -s(\theta_2 - \frac{\pi}{2}) & 0 & a_1 \\ 0 & 0 & 1 & 0 \\ -s(\theta_2 - \frac{\pi}{2}) & -c(\theta_2 - \frac{\pi}{2}) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{4}T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{4}_{5}T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{5}_{6}T = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ -s\theta_6 & -c\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

THE FORWARD KINEMATIC

The direct kinematic problem is to compute the orientation and location of the frame of reference of the last link knowing all the configuration variables of a manipulator (i.e. the θ values for revolute joints and d values for prismatic joints). In this case, all joints are revolute, thus all configuration variables are θ. knowing all of them, we can compute the matrix ${}^{0}_{n}T$ which is the solution of the direct kinematic problem. It is defined as:

$$ {}^{0}_{n}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T \ldots\ldots\ldots.. {}^{n-1}_{n}T = \prod_{i=1}^{n} {}^{i-1}_{i}T \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1 $$

Once the homogeneous transformation matrix of each link is obtained, forward kinematic chain can be applied to achieve the position and orientation of the robot end effector with respect to the global reference frame.

$$ {}_2^0T = {}_1^0T * {}_2^1T = \begin{bmatrix} c1*c2 & -c1*s2 & -s1 & a1*c1 \\ c2*s1 & -s1*s2 & c1 & a1*s1 \\ -s2 & -c2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ldots\ldots 2 $$

$$ {}_3^0T = {}_1^0T * {}_2^1T * {}_3^2T = \begin{bmatrix} c1*c2*c3 - c1*s2*s3 & -c1*c2*s3 - c1*c3*s2 & -s1 & a1*c1 + a2*c1*c2 \\ c2*c3*s1 - s1*s2*s3 & -c2*s1*s3 - c3*s1*s2 & c1 & a1*s1 + a2*c2*s1 \\ -c2*s3 - c3*s2 & s2*s3 - c2*c3 & 0 & -a2*s2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

……3

$$ {}_3^0T = {}_1^0T * {}_2^1T * {}_3^2T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix} $$

$r_{11} = c1*c2*c3 - c1*s2*s3$

$r_{12} = - c1*c2*s3 - c1*c3*s2$

$r_{13} = -s1$

$r_{14} = a1*c1 + a2*c1*c2$

$r_{21} = c2*c3*s1 - s1*s2*s3$

$r_{22} = - c2*s1*s3 - c3*s1*s2$

$r_{23} = c1$

$r_{24} = a1*s1 + a2*c2*s1$

$r_{31} = - c2*s3 - c3*s2$

$r_{32} = s2*s3 - c2*c3$

$r_{33} = 0$

$r_{34} = - a2*s2$

$r_{41} = 0$

$r_{42} = 0$

$r_{43} = 0$

$r_{44} = 1$

$$ {}^{0}_{4}T = {}^{0}_{3}T * {}^{3}_{4}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix} \text{...}4 $$

r_{11} = s1*s4 + c4*(c1*c2*c3 - c1*s2*s3)

r_{12} = c4*s1 - s4*(c1*c2*c3 - c1*s2*s3)

r_{13} =- c1*c2*s3 - c1*c3*s2

r_{14} = a1*c1 + a3*(c1*c2*c3 - c1*s2*s3) - d4*(c1*c2*s3 + c1*c3*s2) + a2*c1*c2

r_{21} =c4*(c2*c3*s1 - s1*s2*s3) - c1*s4

r_{22} =- c1*c4 - s4*(c2*c3*s1 - s1*s2*s3)

r_{23} =- c2*s1*s3 - c3*s1*s2

r_{24} =a1*s1 + a3*(c2*c3*s1 - s1*s2*s3) - d4*(c2*s1*s3 + c3*s1*s2) + a2*c2*s1

r_{31} = -c4*(c2*s3 + c3*s2)

r_{32} = s4*(c2*s3 + c3*s2)

r_{33} = s2*s3 - c2*c3

r_{34} = - a2*s2 - a3*(c2*s3 + c3*s2) - d4*(c2*c3 - s2*s3)

r_{41}=0

r_{42} =0

r_{43} =0

r_{44} =1

$$ {}^{0}_{5}T = {}^{0}_{4}T * {}^{4}_{5}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix} \text{...}5 $$

r_{11} = c5*(s1*s4 + c4*(c1*c2*c3 - c1*s2*s3)) - s5*(c1*c2*s3 + c1*c3*s2)

r_{12} = - c5*(c1*c2*s3 + c1*c3*s2) - s5*(s1*s4 + c4*(c1*c2*c3 - c1*s2*s3))

r_{13} =s4*(c1*c2*c3 - c1*s2*s3) - c4*s1

r_{14} = a1*c1 + a3*(c1*c2*c3 - c1*s2*s3) - d4*(c1*c2*s3 + c1*c3*s2) + a2*c1*c2

r_{21} =- s5*(c2*s1*s3 + c3*s1*s2) - c5*(c1*s4 - c4*(c2*c3*s1 - s1*s2*s3))

r_{22} =s5*(c1*s4 - c4*(c2*c3*s1 - s1*s2*s3)) - c5*(c2*s1*s3 + c3*s1*s2)

r_{23} =c1*c4 + s4*(c2*c3*s1 - s1*s2*s3)

r_{24} =a1*s1 + a3*(c2*c3*s1 - s1*s2*s3) - d4*(c2*s1*s3 + c3*s1*s2) + a2*c2*s1

r_{31} =- s5*(c2*c3 - s2*s3) - c4*c5*(c2*s3 + c3*s2)

r_{32} = c4*s5*(c2*s3 + c3*s2) - c5*(c2*c3 - s2*s3)

r_{33} = -s4*(c2*s3 + c3*s2)

r_{34} = - a2*s2 - a3*(c2*s3 + c3*s2) - d4*(c2*c3 - s2*s3)

r_{41}=0

r_{42} =0

r_{43} =0

r_{44} =1

$${}^{0}_{6}T = {}^{0}_{5}T * {}^{5}_{6}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix} \text{..6}$$

r_{11} = s6*(c4*s1 - s4*(c1*c2*c3 - c1*s2*s3)) - c6*(s5*(c1*c2*s3 + c1*c3*s2) - c5*(s1*s4 + c4*(c1*c2*c3 - c1*s2*s3)))

r_{12} = s6*(s5*(c1*c2*s3 + c1*c3*s2) - c5*(s1*s4 + c4*(c1*c2*c3 - c1*s2*s3))) + c6*(c4*s1 - s4*(c1*c2*c3 - c1*s2*s3))

r_{13} =- c5*(c1*c2*s3 + c1*c3*s2) - s5*(s1*s4 + c4*(c1*c2*c3 - c1*s2*s3))

r_{14} = a1*c1 - d6*(c5*(c1*c2*s3 + c1*c3*s2) + s5*(s1*s4 + c4*(c1*c2*c3 - c1*s2*s3))) + a3*(c1*c2*c3 - c1*s2*s3) - d4*(c1*c2*s3 + c1*c3*s2) + a2*c1*c2

r_{21} =- c6*(s5*(c2*s1*s3 + c3*s1*s2) + c5*(c1*s4 - c4*(c2*c3*s1 - s1*s2*s3))) - s6*(c1*c4 + s4*(c2*c3*s1 - s1*s2*s3))

r_{22} =s6*(s5*(c2*s1*s3 + c3*s1*s2) + c5*(c1*s4 - c4*(c2*c3*s1 - s1*s2*s3))) - c6*(c1*c4 + s4*(c2*c3*s1 - s1*s2*s3))

r_{23} = s5*(c1*s4 - c4*(c2*c3*s1 - s1*s2*s3)) - c5*(c2*s1*s3 + c3*s1*s2)

r_{24} = a1*s1 - d6*(c5*(c2*s1*s3 + c3*s1*s2) - s5*(c1*s4 - c4*(c2*c3*s1 - s1*s2*s3))) + a3*(c2*c3*s1 - s1*s2*s3) - d4*(c2*s1*s3 + c3*s1*s2) + a2*c2*s1

r_{31} =s4*s6*(c2*s3 + c3*s2) - c6*(s5*(c2*c3 - s2*s3) + c4*c5*(c2*s3 + c3*s2))

r_{32} = s6*(s5*(c2*c3 - s2*s3) + c4*c5*(c2*s3 + c3*s2)) + c6*s4*(c2*s3 + c3*s2)

r_{33} = c4*s5*(c2*s3 + c3*s2) - c5*(c2*c3 - s2*s3)

r_{34} = - a2*s2 - a3*(c2*s3 + c3*s2) - d4*(c2*c3 - s2*s3) - d6*(c5*(c2*c3 - s2*s3) - c4*s5*(c2*s3 + c3*s2))

r_{41}=0

r_{42} =0

r_{43} =0

r_{44} =1

$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$ ……………………………Orientation of the end effector

r_{14} =P_X

r_{24} =P_Y ………………………………………Position of the end effector

r_{34} =P_Z

THE INVERSE KINEMATIC

The next task was to compute the inverse kinematic problem. Its definition is that knowing the homogeneous matrix ${}^{0}_{n}T$ determine the joint variables θi, i.e. to determine the configuration variables with which the last links frame of reference can be placed in a desired position. The general idea of solving the inverse kinematics problem is to compare the elements of a matrix obtained through solving direct kinematic problem and a known, constant matrix. For the inverse kinematic problem in this case, multiplication of the homogeneous matrices from right to left was much more useful. Therefore, multiplication of the matrices yielded:

$${}^{2}_{3}T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{3}_{4}T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{2}_{4}T = {}^{2}_{3}T * {}^{3}_{4}T = \begin{bmatrix} c3*c4 & -c3*s4 & -s3 & a2 + a3*c3 - d4*s3 \\ c4*s3 & -s3*s4 & c3 & c3*d4 + a3*s3 \\ -s4 & -c4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{1}_{4}T = {}^{1}_{2}T * {}^{2}_{4}T$$
$$= \begin{bmatrix} c2*c3*c4 - c4*s2*s3 & s2*s3*s4 - c2*c3*s4 & -c2*s3 - c3*s2 & a1 - s2*(c3*d4 + a3*s3) + c2*(a2 + a3*c3 - d4*s3) \\ -s4 & -c4 & 0 & 0 \\ -c2*c4*s3 - c3*c4*s2 & c2*s3*s4 + c3*s2*s4 & s2*s3 - c2*c3 & -c2*(c3*d4 + a3*s3) - s2*(a2 + a3*c3 - d4*s3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{4}_{6}T = {}^{4}_{5}T * {}^{5}_{6}T = \begin{bmatrix} c5*c6 & -c5*s6 & -s5 & -d6*s5 \\ s6 & c6 & 0 & 0 \\ c6*s5 & -s5*s6 & c5 & c5*d6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{1}_{6}T = {}^{1}_{4}T * {}^{4}_{6}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix} \quad \text{...}7$$

r_{11} = c5*c6*(c2*c3*c4 - c4*s2*s3) - s6*(c2*c3*s4 - s2*s3*s4) - c6*s5*(c2*s3 + c3*s2)

r_{12} = s5*s6*(c2*s3 + c3*s2) - c6*(c2*c3*s4 - s2*s3*s4) - c5*s6*(c2*c3*c4 - c4*s2*s3)

r_{13} =- s5*(c2*c3*c4 - c4*s2*s3) - c5*(c2*s3 + c3*s2)

r_{14} = a1 - s2*(c3*d4 + a3*s3) + c2*(a2 + a3*c3 - d4*s3) - d6*s5*(c2*c3*c4 - c4*s2*s3) - c5*d6*(c2*s3 + c3*s2)

r_{21} =- c4*s6 - c5*c6*s4

r_{22} = c5*s4*s6 - c4*c6

r_{23} =s4*s5

r_{24} =d6*s4*s5

r_{31} =s6*(c2*s3*s4 + c3*s2*s4) - c5*c6*(c2*c4*s3 + c3*c4*s2) - c6*s5*(c2*c3 - s2*s3)

r_{32} = c6*(c2*s3*s4 + c3*s2*s4) + s5*s6*(c2*c3 - s2*s3) + c5*s6*(c2*c4*s3 + c3*c4*s2)

r_{33} = s5*(c2*c4*s3 + c3*c4*s2) - c5*(c2*c3 - s2*s3)

r_{34} = d6*s5*(c2*c4*s3 + c3*c4*s2) - s2*(a2 + a3*c3 - d4*s3) - c2*(c3*d4 + a3*s3) - c5*d6*(c2*c3 - s2*s3)

r_{41}=0

r_{42} =0

r_{43} =0

r_{44} =1

$${}^{0}_{1}T^{-1} * {}^{0}_{n}T_{d} = {}^{0}_{1}T^{-1} * {}^{0}_{n}T$$

$${}^{0}_{n}T_{d} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_X \\ r_{21} & r_{22} & r_{23} & P_Y \\ r_{31} & r_{32} & r_{33} & P_Z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}_{1}T^{-1} = \begin{bmatrix} c1 & s1 & 0 & 0 \\ -s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}_{1}T^{-1} * {}^{0}_{n}T_{d} = \begin{bmatrix} c1*r11 + r21*s1 & c1*r12 + r22*s1 & c1*r13 + r23*s1 & c1*px + py*s1 \\ c1*r21 - r11*s1 & c1*r22 - r12*s1 & c1*r23 - r13*s1 & c1*py - px*s1 \\ r31 & r32 & r33 & pz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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At first the elements ${}^{1}_{6}T24$ and ${}^{1}_{6}T23$ had been compared, which enabled to easily determine the angle θ1. The calculations are following:

r_{24} =d6*s4*s5

r_{23} =s4*s5

$\frac{r_{24}}{r_{23}}=\frac{d6*s4*s5}{s4*s5}=\frac{c1*py-px*s1}{c1*r23-r13*s1}$ Assume $s4$ and $s5$ are different from zero.

$d6(c1*r23-r13*s1)=c1*py-px*s1$

$\frac{d6(c1*r23-r13*s1)-c1*py}{c1}=\frac{-px*s1}{c1}$

$\frac{d6}{c1}(c1*r23-r13*s1)-py=-px*tg1$

$d6*r23-d6*r13*tg1-py=-px*tg1$

$d6*r23-py=d6*r13*tg1-px*tg1$

tg=$\left(\frac{d6*r23-py}{d6*r13-px}\right)$

$\theta_1=\text{arc tg}\left(\frac{d6*r23-py}{d6*r13-px}\right)$ ……………………………………………………………..9

Having calculated the value of θ1 it can be now treated as a known value. With this information available, a simple set of two equations from the comparison of elements ${}^{1}_{6}T14$ and ${}^{1}_{6}T34$ can be stated:

Let's take c2*c3 - s2*s3 as c23 and c2*s3 + c3*s2= s23

r_{34} = d6*s5*c4*(s23) - s2*(a2 + a3*c3 - d4*s3) - c2*(c3*d4 + a3*s3) - c5*d6*(c23)= $pz-d6*r33$

r_{14} = a1 - s2*(c3*d4 + a3*s3) + c2*(a2 + a3*c3 - d4*s3) - d6*s5*c4*(c23) - c5*d6*(s23)= $c1*px+py*s1$

$(-s2*(a2+a3*c3-d4*s3)-c2*(c3*d4+a3*s3)-c5*d6*(c23)+d6*s5*c4*(s23))^2=(pz-d6*r33)^2$

$(-s2*(c3*d4+a3*s3)+c2*(a2+a3*c3-d4*s3)-d6*s5*c4*(c23)-c5*d6*(s23))^2=(c1*px+py*s1-a1)^2$

$(-s2*(a2+a3*c3-d4*s3)-c2*(c3*d4+a3*s3)-c5*d6*(c23)+d6*s5*c4*(s23))^2+(-s2*(c3*d4+a3*s3)+c2*(a2+a3*c3-d4*s3)-d6*s5*c4*(c23)-c5*d6*(s23))^2=(pz-d6*r33)^2+(c1*px+py*s1-a1)^2$

$a1+c1*(-px+d6*r11)-py*s1+d6*r22*s1)$=G

$pz-d6*r33=F$

$-c3^2{d^2}_4-2a3*c3*d_4*s3-a3^2*s3^2=-F^2*c_2^2+2*F*G*c2*s2-s2^2*G^2$

$a2^2 + a3^2 + 2 * a2 * \ a3 * c3 + d4^2 = F^2 + G^2 + 2 * a2 * d_4 * s3$

$a3 * c3 + \frac{-F^2 - G^2 + a2^2 + a3^2 + d4^2}{2 * a2} = d_4 * s3$

$\frac{-F^2 - G^2 + a2^2 + a3^2 + d4^2}{2 * a2} = H$

$a3 * c3 + H = d_4 * s3$

$$\theta_3 = arctg\left(\frac{Hd4 + \sqrt{a_3^2(-H^2 + a_3^2 + d_4^2}}{(a_3^2 + d_4^2)\sqrt{1 - \frac{(Hd4 + \sqrt{a_3^2(-H^2 + a_3^2 + d_4^2)}^2}{(a_3^2 + d_4^2)}}}\right)$$

$$\theta_3 = arctg\left(\frac{Hd4 + \sqrt{a_3^2(-H^2 + a_3^2 + d_4^2}}{(a_3^2 + d_4^2)\sqrt{1 - \frac{(Hd4 + \sqrt{a_3^2(-H^2 + a_3^2 + d_4^2)}^2}{(a_3^2 + d_4^2)}}}\right) \text{..10}$$

Having calculated the value of θ1 and θ_3 it can be now treated as a known value.

$a2 + a3 * c3 - d_4 * s3 = J = -Gc2 - Fs2$

$$\theta_2 = arctg\left(\frac{-FJ + \sqrt{G^2(F^2 + G^2 + J^2}}{(F^2 + G^2)^2\sqrt{1 - \frac{(-FJ + \sqrt{G^2(F^2 + G^2 - J^2})^2}{(F^2 + G^2)^2}}}\right)$$

$$\theta_2 = arctg\left(\frac{-FJ - \sqrt{G^2(F^2 + G^2 + J^2}}{(F^2 + G^2)^2\sqrt{1 - \frac{(-FJ - \sqrt{G^2(F^2 + G^2 - J^2})^2}{(F^2 + G^2)^2}}}\right) \text{..11}$$

From the comparison of elements ${}_6^1T13$ and ${}_6^1T33$

r_{13} =- s5*c4 *(c23) - c5*(s23) = $c1 * r13 + r23 * s1$

r_{33} = s5*c4 (s23) - c5*(c23) = $r33$

$s5 * c4\ (s23)\ -\ c5 * (c23)\ =\ r33$

$-\ s5 * c4\ * (c23) =\ c1 * r13 + r23 * s1 + c5 * (s23)$

$s5 * c4 \; *= \frac{-1}{(c23)} \left(c1 * r13 + r23 * s1 + c5 * (s23) \right)$..........Substitute in the above equation

$$\frac{-(s23)}{(c23)} \left(c1 * r13 + r23 * s1 + c5 * (s23) \right) \; - \; c5 * (c23) \; = \; r33$$

$$r33 = \frac{-c1 * s23 * r13}{(c23)} - \frac{r23 * s1 * s23}{(c23)} - \frac{c5}{(c23)}$$

$$r33 * c23 = -c1 * s23 - r23 * s1 * s23 - c5$$

$$c5 = -c1 * s23 * r13 - r23 * s1 * s23 - c23 * r33 \,, \; s5 = \sqrt{1 - c5^\wedge 2}$$

$$s5 = \sqrt{1 - (-c1 * r13 * s23 - r23 * s1 * s23 - c23 * r33)^\wedge 2}$$

tg=$\left(\frac{s5}{c5} \right)$

$$\theta_5 = \text{arc tg} \left(\frac{\sqrt{1 - (-c1 * r13 * s23 - r23 * s1 * s23 - c23 * r33)^\wedge 2}}{-c1 * r13 * s23 - r23 * s1 * s23 - c23 * r33} \right)$$

$$\theta_5 = \text{arc tg} \left(\frac{\sqrt{1 - (c1 * r13 + r23 * s1) * s23 - c23 * r33)^\wedge 2}}{(-c1 * r13 - r23 * s1) * s23 - c23 * r33} \right) \text{...12}$$

Knowing the value of θ5, it is easier to calculate the value of θ4 by using the comparison of elements ${}^{1}_{6}T23$, than by solving again the equations from the previous set of equations obtained by comparing elements ${}^{1}_{6}T13$ and ${}^{1}_{6}T33$. Thus the following equation was obtained.

s4*s5= $c1 * r23 - r13 * s1$

$$s4 = \frac{c1 * r23 - r13 * s1}{s5}$$

Assuming that $s5 \neq 0$, the value of θ4 can be computed using the following:

$$c4 = \sqrt{1 - (\frac{c1 * r23 - r13 * s1}{s5})^\wedge 2}$$

$$\theta_4 = \text{arc tg} \left(\frac{\frac{c1 * r23 - r13 * s1}{s5}}{\sqrt{1 - (\frac{c1 * r23 - r13 * s1}{s5})^\wedge 2}} \right) \text{...13}$$

The value of θ6 was calculated with the comparison of elements ${}^{1}_{6}T21$ and ${}^{1}_{6}T22$:

r_{21} =- c4*s6 - c5*c6*s4= $c1 * r21 - r11 * s1$

r_{22} = c5*s4*s6 - c4*c6 = $c1 * r22 - r12 * s1$

In order to simplify the notation, right sides of the above notations were substituted with:

E=- c4*s6 - c5*c6*s4= $c1 * r21 - r11 * s1$

F= c5*s4*s6 - c4*c6 = $c1 * r22 - r12 * s1$

E+c4*s6=- c5*c6*s4   square both sides

F= c5*s4*s6 - c4*c6

$(E + c4 * s6)^2 = (- c5 * c6 * s4)^2$

$(F + c4 * c6)^2 = (c5 * s4 * s6\ )^2$

$F^2 + 2Fc4 * c6 + (c4 * c6)^2 = c5^2 * s4^2 * s6^2$

$E^2 + 2Ec4 * s6 + (c4 * s6)^2 = c5^2 * s4^2 * c6^2$

$F^2 + 2Fc4 * c6 + (c4 * c6)^2 + E^2 + 2Ec4 * s6 + (c4 * s6)^2 = c5^2 * s4^2 * s6^2 + c5^2 * s4^2 * c6^2$

$F^2 + 2F * c4 * c6 + (c4)^2 + E^2 + 2E * c4 * s6 = c5^2 * s4^2$

For the purpose of simplifying the calculations, the polar coordinates were introduced. The left side of the equation was defined as I and the following substitutions were made:

$I = c5^2 * s4^2 - E^2 - F^2 - (c4)^2 = 2F * c4 * c6 + 2E * c4 * s6$

$G = 2F * c4 = xs\varphi \qquad x = \sqrt{G^2 + H^2}$

$H = 2E * c4 = xc\varphi \qquad \varphi = arctg(\frac{G}{H})$

$\sin(\varphi + \theta 6) = \frac{I}{x}$

$\cos(\varphi + \theta 6) = \pm\sqrt{1 - \left(\frac{H}{x}\right)^2}$

$\varphi + \theta 6 = arctg\left(\frac{\frac{I}{x}}{\pm\sqrt{1-\left(\frac{H}{x}\right)^2}}\right)$

$\theta 6 = arctg\left(\frac{\frac{I}{x}}{\pm\sqrt{1-\left(\frac{H}{x}\right)^2}}\right) - arctg(\frac{G}{H})$ 14