11. The Laplace transform

1. The s-domain
2. Strategy of the Laplace Transform
3. Transfer function
4. Filter design in the s-domain

[Smith, ch. 32]

1. The s-domain

The **Laplace transform** changes a signal in the **continuous time** domain into a signal in the **s-domain**, also called the **s-plane**.

The Laplace transform allows the time domain to be ***complex***; however in nearly all practical applications, the time domain signal is completely **real**.

The s-domain is **a complex plane**, i.e., there are real numbers along the horizontal axis and imaginary numbers along the vertical axis:

- the **real** axis is expressed by the variable, σ,
- the **imaginary** axis uses the variable, ω, the **natural frequency**,
- each **location** is represented by the complex variable s, where:

$$s = \sigma + j\,\omega.$$

Each point in the s-domain has a ***value*** that is a complex number.

The **s-plane** is **continuous** and extends to **infinity** in all four directions.

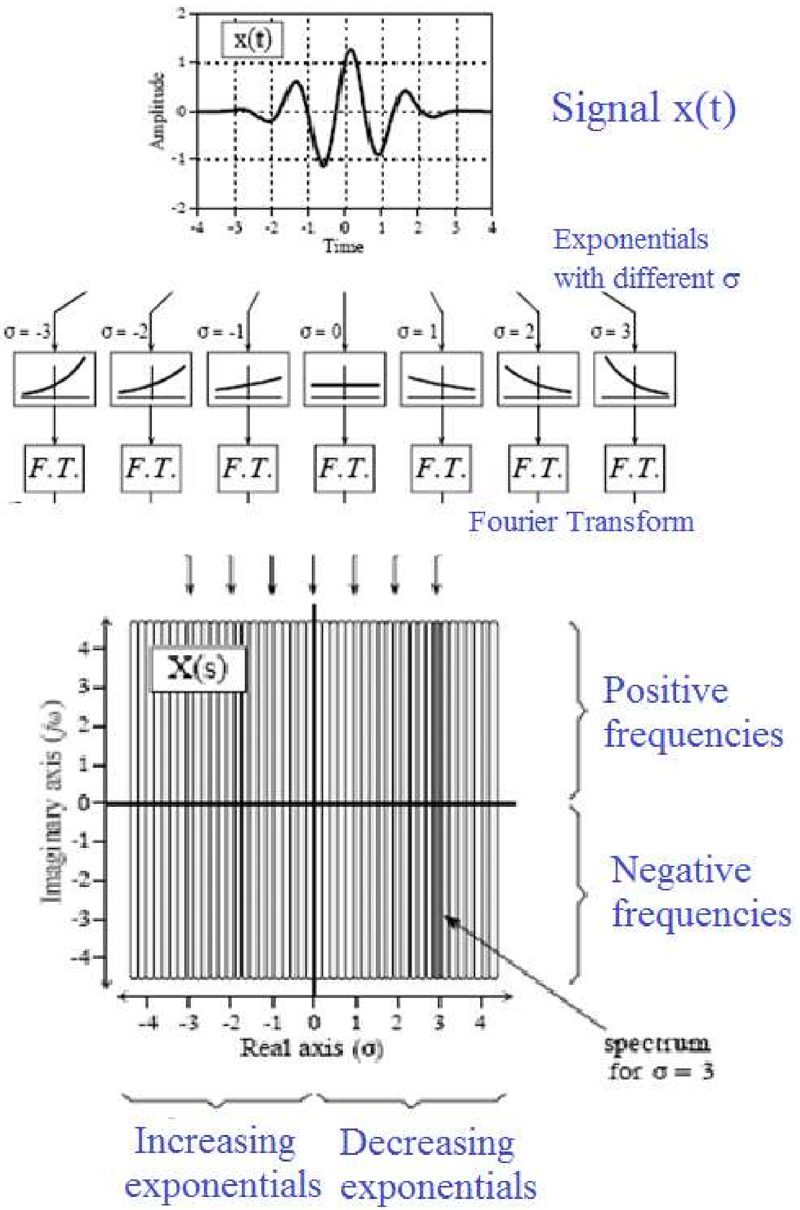


Fig. 1 The Laplace transform.

The Laplace transform converts a signal in the *time domain*, $x(t)$, into a signal in the *s-domain*, $X(s)$ or $X(\sigma , \omega)$.

1) Laplace- vs. Fourier transform

Notation

Signals in the s-domain are represented by capital letters. For example, $x(t)$ is transformed into an s-domain signal, $X(s)$, or alternatively, $X(\sigma , \omega)$.

The Laplace transform analyzes signals in terms of **sinusoids** *and* **exponentials**. This makes the Fourier transform a **subset** of the Laplace transform.

The **complex Fourier Transform** is given by:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\; dt$$

This can be expanded into the **Laplace transform** by first **multiplying** the time domain signal by the **exponential term**:

$$X(\sigma,\omega) = \int_{-\infty}^{\infty} [x(t)\, e^{-\sigma t}]\, e^{-j\omega t}\; dt$$

The **Laplace transform**:

$$X(\sigma,\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\; dt$$

Equivalent form:

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\; dt$$

where the term, e^{-st} , is a **complex exponential**.

Complex exponentials are a compact way of representing both **sinusoids** and **exponentials** in a single expression.

The values in the s-plane **along the y-axis** ($\sigma = 0$) are exactly **equal** to the Fourier transform.

2) Relating the s-domain to the time domain signal

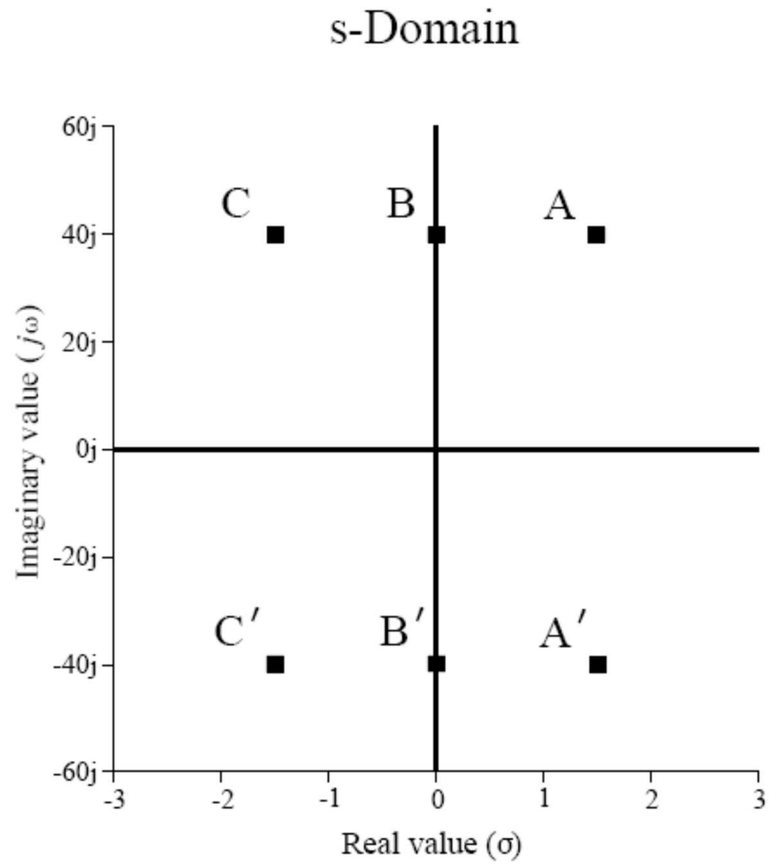
- In FT: each point in the ***frequency domain***, identified by a value of ω, corresponds to two sinusoids, cos(ω*t*) and sin(ω*t*).
- In LT: each point in the s-domain corresponds to σ and ω**.**

The **value** at each location in the **s-plane** is a **complex** number.

- The **real part** is found by multiplying the signal by the **exponentially** weighted **cosine** wave and then **integrated** from -∞ to ∞.
- The **imaginary part** is found in the same way, except the **exponentially** weighted **sine** wave is used instead.

Example (fig. 2) For **real-valued** signals, points at A, B, C (positive frequencies) are the complex **conjugates** of the points at A’, B’, C’ (negative frequencies). Treating these points **in pairs** allows us to operate in the time domain with only real numbers. For example:

$$\mathrm{Re}\, X(\sigma = 1.5,\, \omega = \pm 40) = \int_{=\infty}^{\infty} x(t)\cos(40t)\, e^{-1.5t}\; dt$$



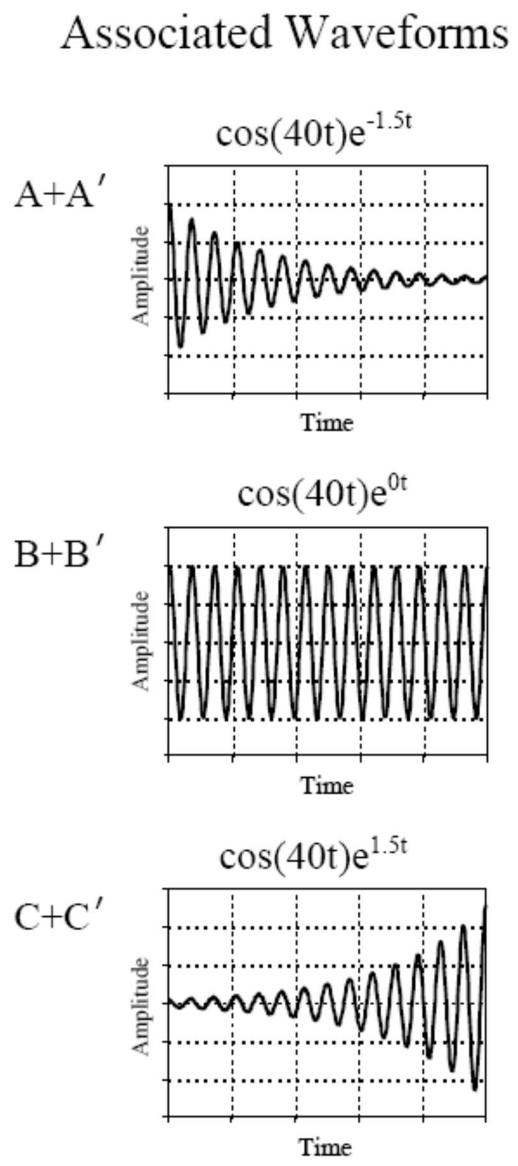


Fig. 2 Waveforms associated with real parts of locations in the s-domain. Real parts of location and corresponding locations with the same real parts.

Observations

The Laplace transform is analyzing a specific class of time domain signals:

- ***impulse responses*** that consist of **sinusoids** *and* **exponentials**.

Such systems are *very common* in science and engineering:

- **sinusoids** and **exponentials** are solutions to ***differential equations***, the mathematics that controls much of our physical world.

For other waveforms (such as the rectangular pulse) the resulting s-domain is meaningless.

Exercises

Task 11-1

Let the time domain signal is a **rectangular pulse** of width *two* and height *one*. The complex Fourier transform of this signal is a **sinc** function in the real part, and an entirely **zero** signal in the imaginary part. Show this by using the **Laplace transform** and simplifying the s-domain representation of this rectangular pulse.

2. Strategy of the Laplace Transform

System analysis in the s-domain

Let the **impulse response** be ***infinite* in length**, i.e. from $t = 0$ to $t = +\infty$.

- If the system is **stable**, the amplitude of the impulse response will become **smaller** as time increases, reaching a value of **zero** at $t = +\infty$.
- The system is **unstable** if the impulse response will **increase** in amplitude as time increases, becoming **infinitely large**.

Probing the time-domain signal

- The Laplace transform **probes** the time domain waveform to identify the ***frequencies*** of the sinusoids, and the ***decay constants*** of the exponentials.

- **Probing** means ***multiplying*** the signal with base waveforms, and then ***integrating*** the result from $t = -\infty$ to $+\infty$.

Cancel the signal and its s-transform

We want to find combinations of σ and ω that exactly **cancel** current **impulse response**.

This cancellation can occur in two forms - the **area under the curve** can be:

1. ***zero*** (i.e. the signal is cancelled and the **transform is zero**) or
2. just ***barely infinite*** (the signal is repeatedly of limited amplitude, but the transform is of **infinite value**).

All other results are not interesting and can be ignored.

Locations in the s-plane that:

1. produce a **zero** cancellation are called **zeros** of the system.
2. produce the "just **barely infinite**" type of cancellation are called **poles**.

Example. Pole-zero example (fig. 3).

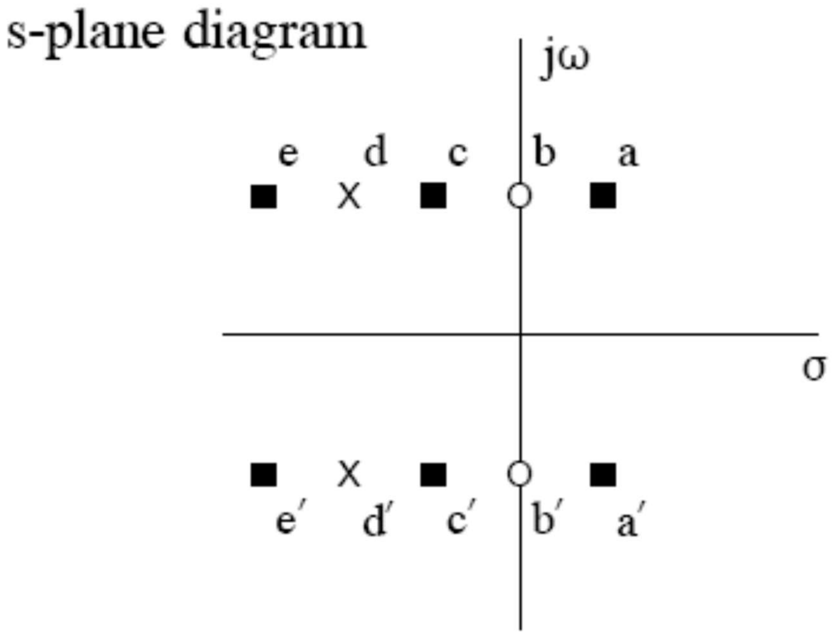
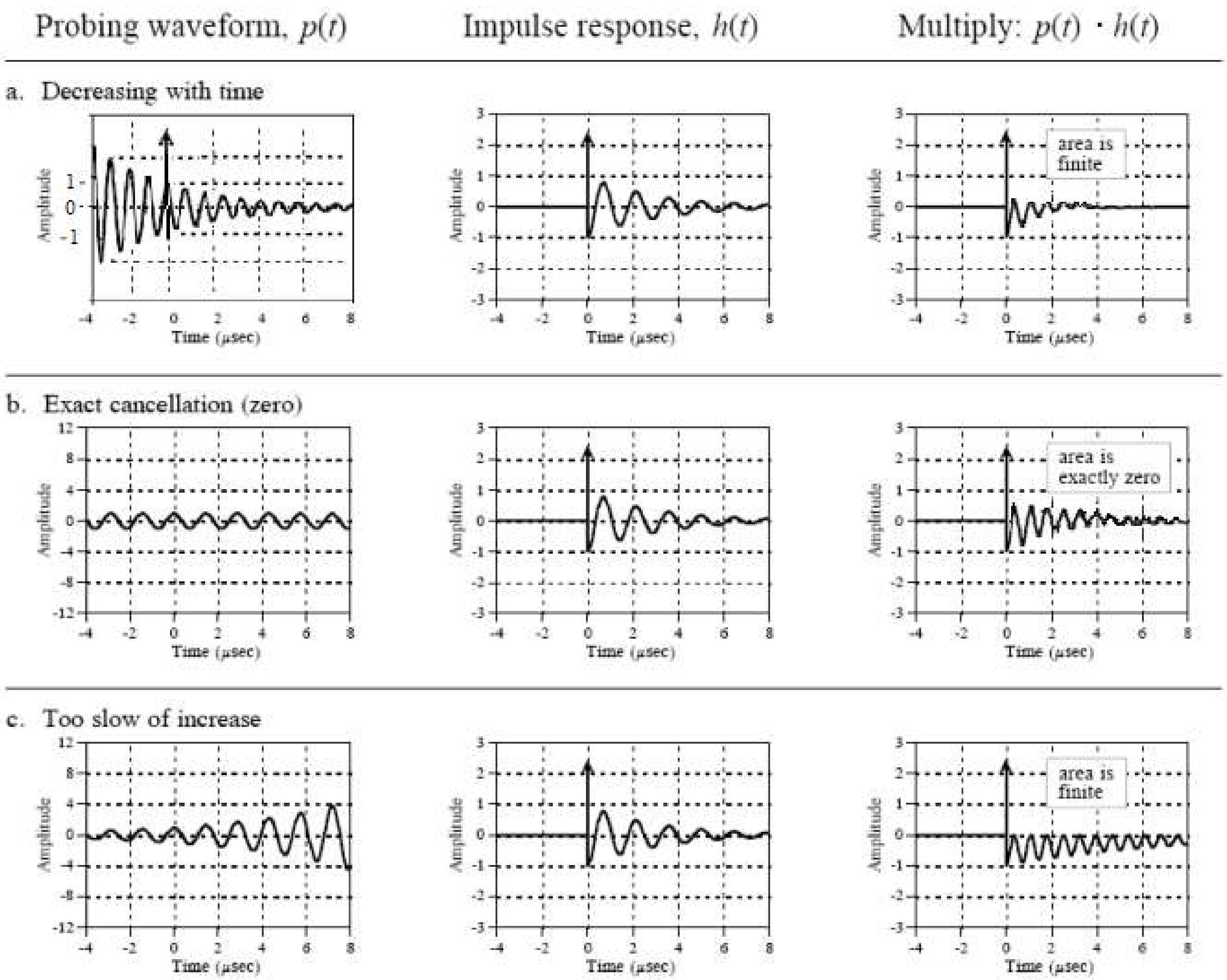


Fig. 3 We analyze a notch filter which has two poles (represented by ×) and two zeros (represented by **o**).

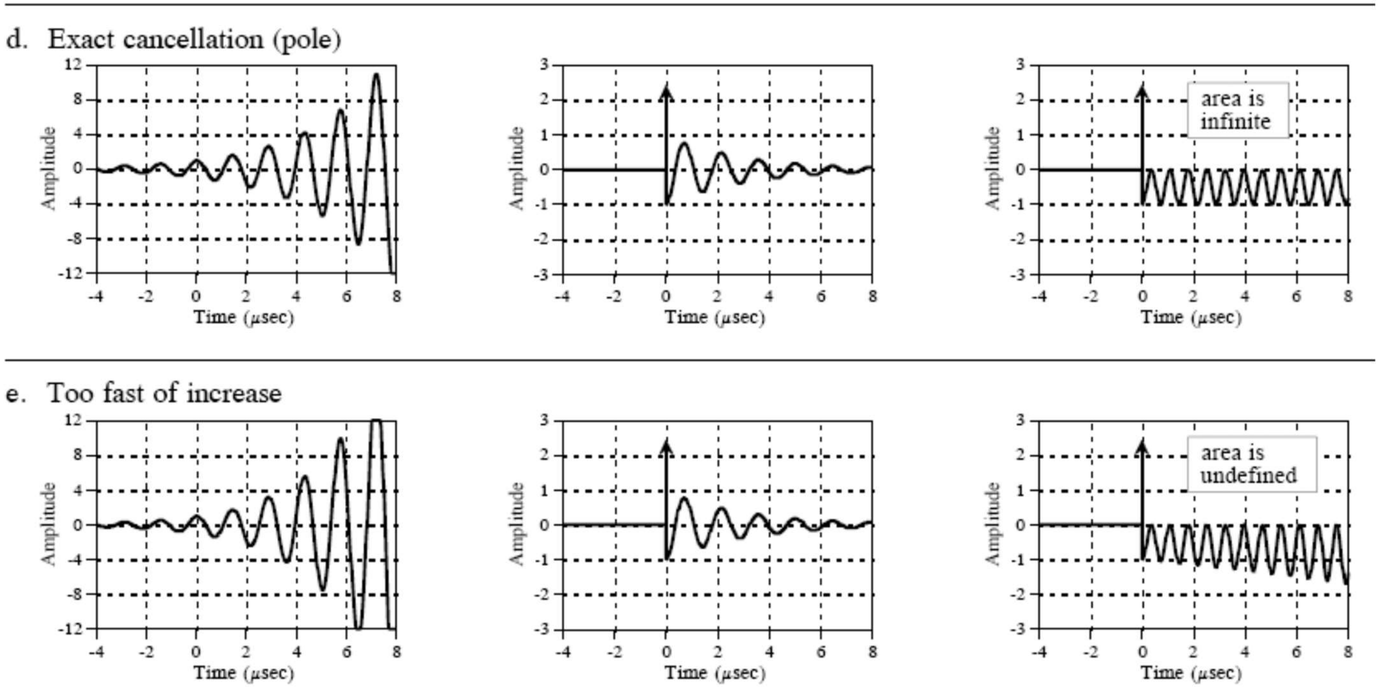
We can “probe” five location pairs (a,a’, b,b’, c,c’, d,d’, e,e’) to analyze this system (in figure 4), i.e. probing the impulse response $h(t)$ of a notch filter by 5 weighted cosine or sine waves $p(t)$, corresponding to real or imaginary parts of locations in the s-plane.



a) The product $p(t) \cdot h(t)$ decreases with time. The area (Laplace transform) is **finite**.

(b) Exact cancellation - a ***zero*** of the system.

(c) The weighted cosine is increasing too slow - the area is **finite**.



(d) The product signal is of limited amplitude, but the area is infinite – a **pole** of the system.

(e) The product is unlimited - the *integral* **does not converge**.

Fig. 4 The Laplace transform is like probing the system's impulse response with various exponentially decaying sinusoids. Probing waveforms that produce a **limitation** or **cancellation** of the time domain signal are called **poles** or **zeroes**, respectively.

Poles

In (d), the probing waveform **increases at exactly the same rate** that the impulse response decreases. This makes the product of the two waveforms to have **limited amplitude**. This point is on the borderline of the region of convergence. Values for σ and ω that produce this type of exact limitation are called **poles** of the system.

A **decreasing probe cannot limit** a **decreasing** impulse response but only an **increasing** one:

- **A stable system** will not have any poles with $\sigma > 0$ (for a **decreasing** probe). All of the poles in a **stable** system are in **the left half** of the s-plane.

- **Poles in the right half** of the s-place show that the system **is unstable** (i.e., an impulse response that ***increases*** with time).

3. Transfer function

Recall from *Fourier* analysis:

- the frequency spectrum of the output signal divided by the frequency spectrum of the input signal is equal to the system's **frequency response**, $H(\omega)$.

An extension into the s-domain.

- The signal $H(s)$ is called the system's **transfer function** and is equal to the s-domain representation of the output signal divided by the s-domain representation of the input signal.
- Further, $H(s)$ is equal to the **Laplace transform** of the impulse response.

1) Filter analysis in the s-domain

- Arrange *H*(*s*) to be *one **polynomial** over another*. For example:

$$H(s) = \frac{as^2 + bs + c}{As^2 + Bs + C}$$

Note: it is always possible to express the **transfer function** in this form *if* the system is controlled by **differential equations**.

- Make a **factoring** of *the **numerator** and **denominator** polynomials* (break these polynomials into components that each contain a single *s*), e.g.

$$H(s) = \frac{(s - z_1)(s - z_2)(s - z_3)\cdots}{(s - p_1)(s - p_2)(s - p_3)\cdots}$$

The roots of the numerator, z_i , are the **zeros** of the equation, while the roots of the denominator, p_i , are the **poles**.

Factoring an s-domain expression is straightforward if the numerator and denominator are ***second-order*** *polynomials*, or less.

- The **roots** of a second-order polynomial,

$$ax^2 + bx + c,$$

  can be found by using the quadratic equation:

$$x_{1,2} = b \pm \text{sqrt}(b^2 - 4\ a\ c)\ /\ 2a.$$

  The factored form is:

$$H(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)}$$

- A **second-order** system has a maximum of two zeros and two poles.
- The **number of zeros** will be equal to or less than the number of poles.

Polynomials **greater than second order** require more complicated numerical methods, for example a ***cascade*** of *second-order* stages.

2) The Importance of Poles and Zeros

- The **pole-zero diagram** is the common display of s-domain data.
- **Every pole** and **zero** is exactly the same shape and size as every other pole and zero. The only **unique characteristic** a pole or zero has is its **location**.

Poles and **zeros** provide a full representation of the transfer function value at ***any point*** *in the s-plane*.
Thus we can **completely** describe the characteristics of the system using only a ***few parameters***.

Example

In the case of a ***RLC* notch filter**, we only need to specify four complex parameters to represent the system: z_1, z_2, p_1, p_2 (each consisting of a real and an imaginary part).

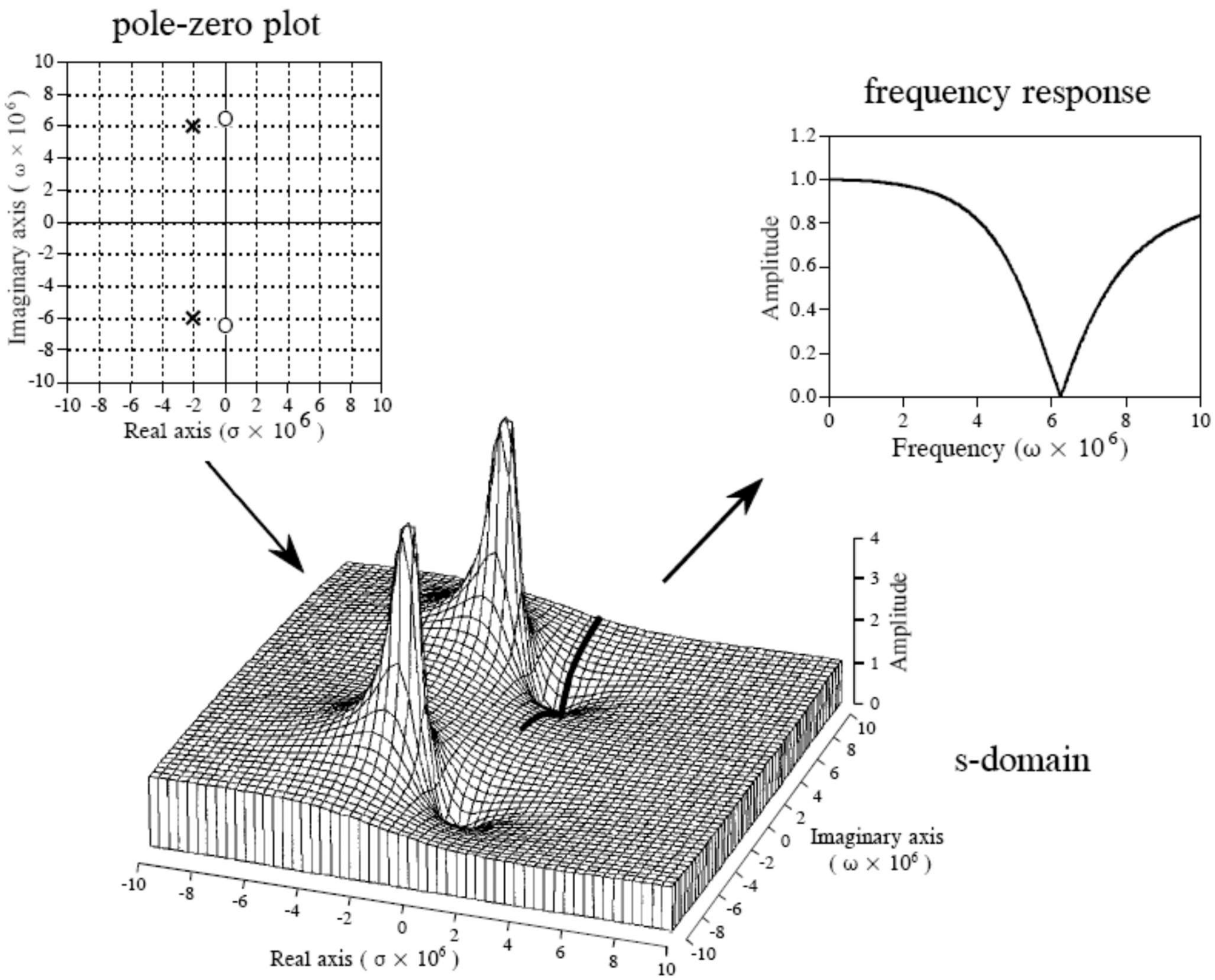


Fig. 5 The relationship between the pole-zero plot, the s-domain, and the frequency response. If we are near a pole, the value will be large; if we are near a zero, the value will be small.

Observe: $(s - z_0)$ is the **distance** between the arbitrary location, s, and the zero located at z_0.

The **value at each location**, s, is equal to:

the **distances** to all of the **zeros** ***multiplied***,

divided by

the **distances** to all of the poles ***multiplied*****.**

Conclusion

1. The Laplace transform calculates an s-domain representation from the **physical system**, usually displayed as a **pole-zero diagram**.
2. **Poles** and **zeros** allow to obtain the $H(s)$ value at ***any point*** s.
3. The **system's *frequency response*** is equal to the values of $H(s)$ along the imaginary axis.

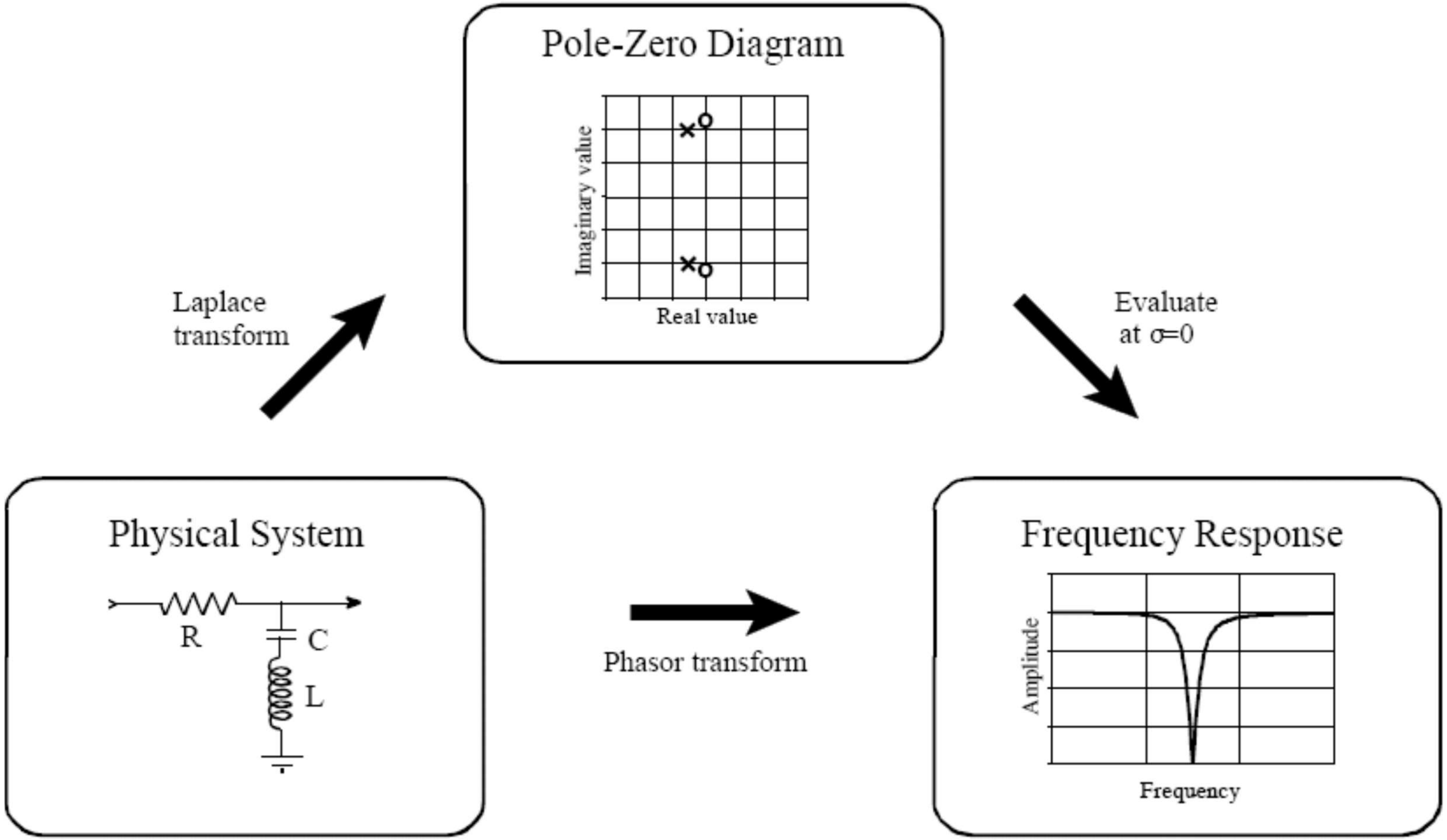


Fig. 6. The *phasor* transform allows the frequency response to be directly calculated from the parameters of the physical system. The Laplace transform calculates an s-domain representation from the physical system, usually expressed in the form of a pole-zero diagram.

4. Filter design in the s-domain

The **design of systems *directly*** in the s-domain involves two steps:

1. The **s-domain** is designed by specifying **the number and location** of the **poles** and **zeros**, with the goal of obtaining the **best frequency response** for given task.
2. An **electronic circuit** is derived that provides this s-domain representation.

1) Designing with *biquads*

Factoring of the s-domain expression is very difficult if the system contains more than two poles or two zeros. A common solution is to implement multiple poles and zeros in **successive stages**:

- For example, a 6 pole filter is implemented as three successive stages, with each stage containing up to two poles and two zeros.
- Each of these stages can be represented in the s-domain by a quadratic numerator divided by a quadratic denominator, a **biquad**.

A low-pass **Butterworth filter** design

Such **filter** is designed by placing a selected number of poles evenly around the **left-half of a circle**.

Each **two poles** in this configuration require **one biquad** stage.

The Butterworth filter is **maximally flat**:

- It has the sharpest transition between the passband and stopband *without peaking* in the frequency response.
- The more poles used, the faster the transition.
- Since all the poles in the Butterworth filter lie on the same circle, all the cascaded stages use the same values for *R* and *C*.
- The only thing different between the stages is the amplification.

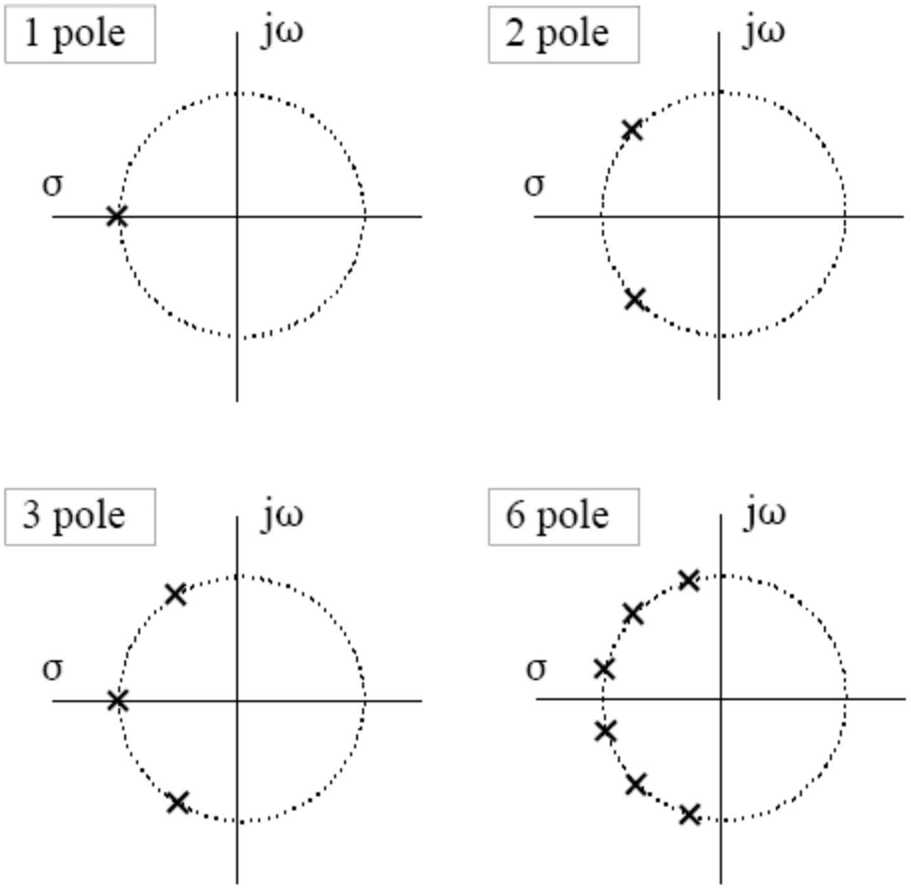


Fig. 7 The Butterworth s-plane. The low-pass Butterworth filter is created by placing **poles** equally around the left-half of a circle. The more poles used in the filter, the faster the roll-off.

2) Classic pole-zero patterns

The **Chebyshev filter** achieves a sharper transition than the Butterworth:

- in the s-domain, this corresponds to the circle of poles being flattened into an **ellipse**,
- the more **flattened** the ellipse, the more ripple in the passband, and the sharper the transition.

When formed from a cascade of stages, this requires **different** values of resistors and capacitors in each stage.

The **elliptic filter** achieves the sharpest possible transition:

- by allowing **ripple** in **both** the pass-band and the stop-band,
- in the s-domain, this corresponds to placing **zeros** directly on the imaginary axis, with the first one near the cutoff frequency.
- The poles and zeros of the elliptic filter do not lie in a simple geometric pattern, but in an arrangement involving elliptic functions and integrals.

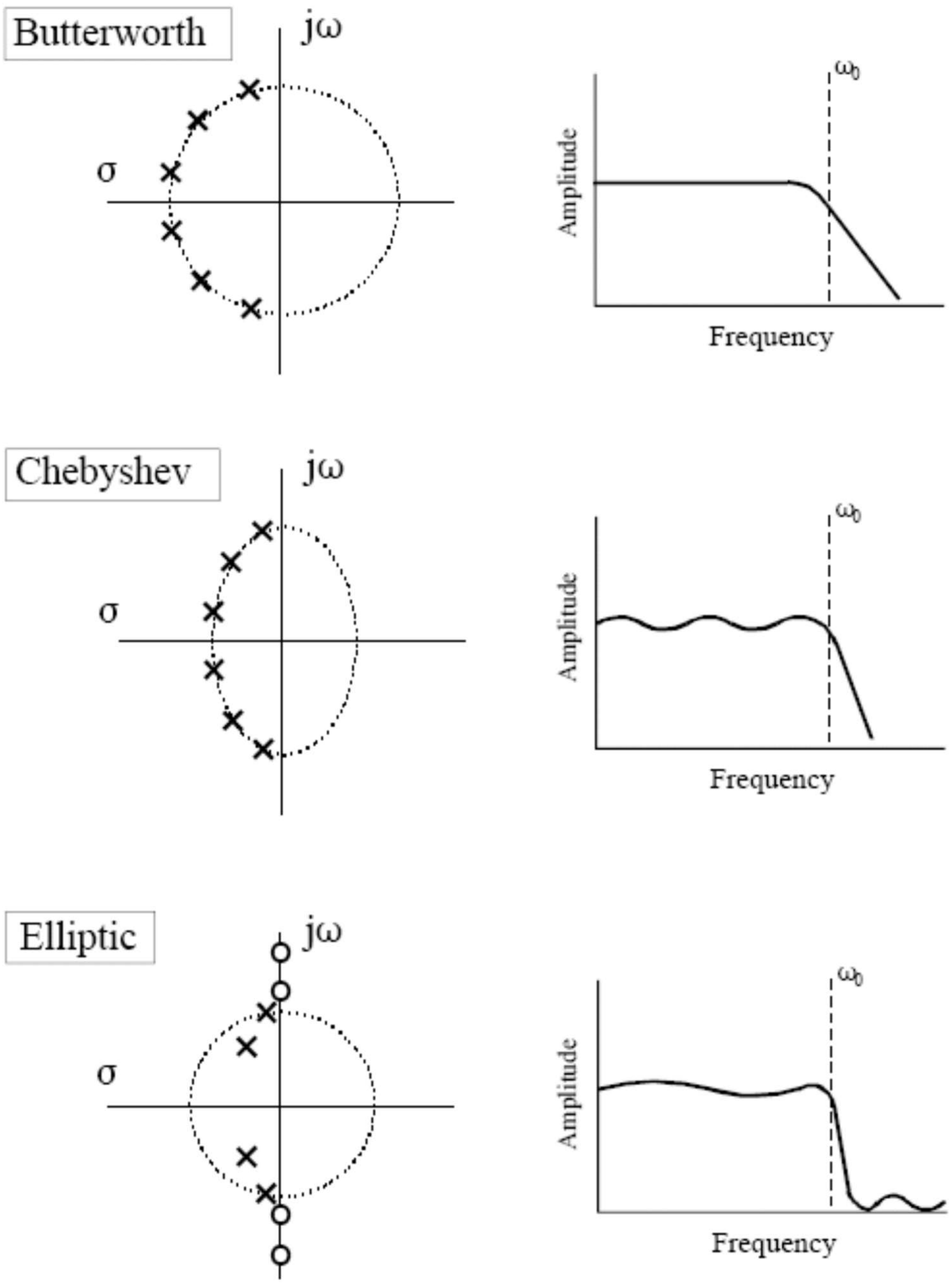


Fig. 8 Classic pole-zero patterns.

Butterworth filters have poles equally spaced around a circle, resulting in a maximally flat response.

Chebyshev filters have poles placed on an ellipse, providing a sharper transition, but at the cost of ripple in the passband.

Elliptic filters add zeros to the stopband. This results in a faster transition, but with ripple in the passband *and* stopband.

Each **biquad** produces two poles, hence **even order filters** (2 pole, 4 pole, 6 pole, etc.) can be constructed by **cascading** biquad stages.

Odd order filters (1 pole, 3 pole, 5 pole, etc.) require a **single pole** on the real axis. This is a simple RC circuit added to the cascade.
For example, a 9 pole filter can be constructed from 5 stages:

- 4 biquads,
- plus one stage consisting of a single capacitor and resistor.

3) High-pass filter design

Design a **low-pass filter** and perform a transformation into the s-domain:

1. Calculate the low-pass filter **pole** locations, and then write the **transfer function,** $H(s)$.
2. The transfer function of the corresponding high-pass filter is found by replacing each "s" with "$1/s$", and then rearranging the expression to again be in the pole-zero form. This defines **new pole** and **zero locations** that implement the high-pass filter.

The design of **high-pass filters** using analog circuits:

- the "$1/s$" for "s" replacement in the s-domain corresponds to swapping the resistors and capacitors in the circuit.
- In the s-plane, this swap places the poles at a new position, and adds two zeros directly at the origin. This results in the frequency response having a value of zero at DC (zero frequency).

Exercises

Task 11-2

Design a low-pass filter (as a Butterworth filter with two poles in the s-plane) and obtain (by a transformation in the s-domain) a corresponding high-pass filter. Obtain its frequency response.

Exercises

Task 11-1

Let the time domain signal is a **rectangular pulse** of width *two* and height *one*. The complex Fourier transform of this signal is a **sinc** function in the real part, and an entirely **zero** signal in the imaginary part. Show this by using the **Laplace transform** and simplifying the s-domain representation of this rectangular pulse.

Solution

The s-domain signal corresponding to this rectangular pulse is a two-dimensional complex-valued signal:

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\; dt = \int_{-1}^{1} 1 \cdot e^{-st}\; dt = \frac{e^{s} - e^{-s}}{s}$$

expressed in terms of the complex location s, and the complex value $X(s)$.

Now replace s with $\sigma + j\omega$, and then separate the real and imaginary parts:

$$\operatorname{Re} X(\sigma,\omega) = \frac{\sigma \cdot \cos(\omega) \cdot [e^{\sigma} - e^{-\sigma}] + \omega \cdot \sin(\omega) \cdot [e^{\sigma} + e^{-\sigma}]}{\sigma^2 + \omega^2}$$

$$\operatorname{Im} X(\sigma,\omega) = \frac{\sigma \cdot \sin(\omega) \cdot [e^{\sigma} + e^{-\sigma}] - \omega \cdot \cos(\omega) \cdot [e^{\sigma} - e^{-\sigma}]}{\sigma^2 + \omega^2}$$

These equations reduce to the **Fourier transform** along the y-axis (σ=0):

$$\operatorname{Re} X(\sigma,\omega)\,|_{\sigma=0} = \frac{2\sin(\omega)}{\omega}$$

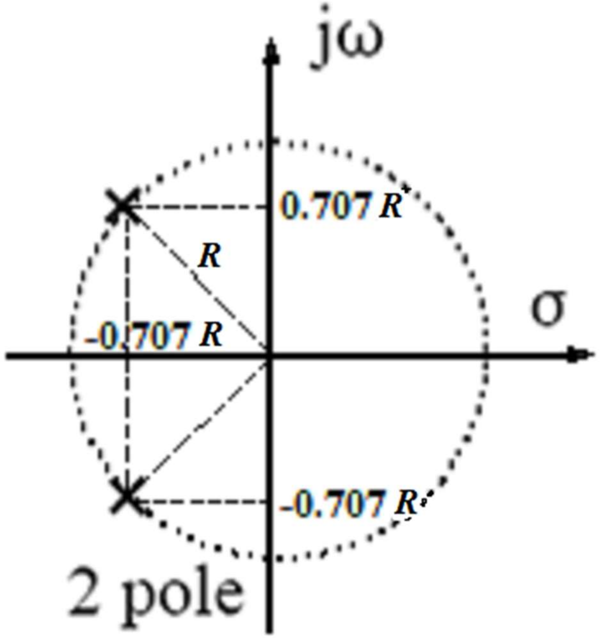
$$\operatorname{Im} X(\sigma,\omega)\,|_{\sigma=0} = 0$$

Task 11-2

Design a low-pass filter (as a Butterworth filter with two poles in the s-plane) and obtain (by a transformation in the s-domain) a corresponding high-pass filter. Obtain its frequency response.

Solution

1) Place two poles evenly around the **left-half of a circle**.



$$p_1 = R\,e^{j(3\pi/4)} = -0.7071 \cdot R + j \cdot 0.7071 \cdot R = \sigma + j\omega$$

$$p_2 = R\,e^{j(-3\pi/4)} = -0.7071 \cdot R - j \cdot 0.7071 \cdot R = \sigma - j\omega$$

No zeros.

Assume that **radius of the circle is *one*** (*R*=1) corresponding to the cutoff frequency of $\omega_C=1$ [rad/s],.

Transmittance function:

$$\boxed{H(s) = \frac{1}{(s-p_1)(s-p_2)}}$$

$$H(s) = \frac{1}{[s-(-0.7071 + j \cdot 0.7071)][s-(-0.7071 - j \cdot 0.7071)]}$$

low pass → high-pass

s → 1/s

$$\boxed{H_{high}(s) = \frac{1}{(1/s - p_1)(1/s - p_2)} = \frac{s^2}{(1 - p_1 s)(1 - p_2 s)} = \frac{s^2}{1 - p_1 s - p_2 s + p_1 p_2 s^2}}$$

$$H_{high}(s) = \frac{s^2}{1 - (-0.7071 + j0.7071 - 0.7071 - j0.7071)s + (-0.7071 + j0.7071)(-0.7071 - j0.7071)s^2}$$

$$\boxed{H_{high}(s) = \frac{s^2}{1 + 1.4142 s + s^2}}$$

Frequency response (H(s)$|_{\sigma=0}$):

$$H_{high}(\sigma + j\omega) = \frac{(\sigma + j\omega)^2}{1 + 1.4142(\sigma + j\omega) + (\sigma + j\omega)^2}$$

$$H_{high}(j\omega) = \frac{(j\omega)^2}{1 + 1.4142(j\omega) + (j\omega)^2} = \frac{-\omega^2}{1 - \omega^2 + 1.4142 j\omega} = \frac{\omega^2}{(\omega^2 - 1) - 1.4142 j\omega}$$

In particular:

$$Mag(H_{high}(\omega = 0)) = \frac{0}{1} = 0$$

$$Mag(H_{high}(\omega = 1)) = \frac{1}{1.4142} = 0.7071 = \frac{\sqrt{2}}{2}$$

$$Mag(H_{high}(\omega = 2)) = \frac{4}{\sqrt{9+8}} \cong 0.9701$$

$$Mag(H_{high}(\omega = 3)) = \frac{9}{\sqrt{64+18}} \cong 0.9939$$