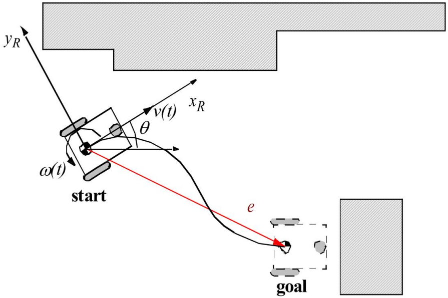
Mobile Robots

State-Space Models of Mobile Robots

Lecture 5

State-Space Models of Wheeled Mobile Robots

- The science of robotics is about synthesizing motion – in the environment, using a robot.
- To synthesise motion we need to:
  - understand what it is (i.e., *structure*),
  - and how to describe it – an overview of basic techniques for describing motion.
- The mobility analysis of the mobile robot can be reformulated into a state-space form.



State-Space Models of Wheeled Mobile Robots

Defining $\boldsymbol{q}$ as the vector of configuration coordinates

$$\boldsymbol{q} \triangleq [\boldsymbol{p}^{\mathsf{T}}, \boldsymbol{\beta}_s^{\mathsf{T}}, \boldsymbol{\beta}_{oc}^{\mathsf{T}}, \boldsymbol{\varphi}^{\mathsf{T}}]^{\mathsf{T}} \in \mathcal{C}, \quad \mathcal{C} \text{ – configuration space,}$$

where

$$\begin{aligned} \boldsymbol{p} = [x, y, \theta]^{\mathsf{T}} \quad & \text{– robot posture coordinates} \\ \boldsymbol{\beta}_s, \boldsymbol{\beta}_{oc}, \boldsymbol{\varphi} \quad & \text{– steering and spinning angles of the wheels} \end{aligned}$$

Four different state-space representations of WMRs:

1. The **posture kinematic model** – the simplest state-space model.
2. The **configuration kinematic model** – describes the kinematic behaviour of the whole robot, including all the configuration variables.
3. The **posture dynamic model** – feedback equivalent to the configuration dynamic model, makes a dynamical counterpart to the posture kinematic model.
4. The **configuration dynamic model** – the most general state-space model. It gives a complete description of the dynamics including the forces provided by the actuators.

Mobile Robot Kinematics

- Mobile robots can move unbound with respect to its environment:
  - there is no direct way to measure the robot's pose instantaneously,
  - robot motion must be integrated over time,
  - leads to inaccuracies of the motion estimation due to slippage.
- The goal of the kinematic modeling is to establish the robot velocity $\dot{\boldsymbol{p}} = [\dot{x}, \dot{y}, \dot{\theta}]^\mathsf{T}$ as a function of wheel speed $\dot{\varphi}_i$, steering angle β_i, steering speed $\dot{\beta}_i$ and the geometric parameters of the robot.
- Forward kinematic model in the global reference frame:

$$\dot{\boldsymbol{p}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f_F(\varphi_1, \ldots, \varphi_n, \dot{\varphi}_1, \ldots, \dot{\varphi}_n, \beta_1, \ldots, \beta_m, \dot{\beta}_1, \ldots, \dot{\beta}_m)$$

- Inverse kinematics:

$$[\varphi_1, \ldots, \varphi_n, \dot{\varphi}_1, \ldots, \dot{\varphi}_n, \beta_1, \ldots, \beta_m, \dot{\beta}_1, \ldots, \dot{\beta}_m)]^\mathsf{T} = f_I(\dot{x}, \dot{y}, \dot{\theta})$$

Posture Kinematic Model

- Whatever the type of robot, the velocity vector $\dot{\boldsymbol{p}}(t)$ is restricted to belong to a distribution Δ_c defined as

$$\dot{\boldsymbol{p}}(t) \in \Delta_c \triangleq \operatorname{span}\left\{\operatorname{col}[R(\theta)B(\beta_s)]\right\}, \tag{1}$$

  where the columns of the matrix $B(\beta_s)$ constitute a basis of the null space $\mathcal{N}(C_1^*(\beta_s))$.
- This is equivalent to the following statement:

$$\forall\, t \;\; \exists\, \boldsymbol{\eta}(t): \quad \dot{\boldsymbol{p}}(t) = R(\theta)B(\beta_s)\boldsymbol{\eta}(t) \tag{2}$$

- The dimension of the distribution Δ_c, and hence of the vector $\boldsymbol{\eta}$, is equal to the degree of mobility δ_m of the robot.
- In the case where the robot has no steering wheels, the matrix B is constant, otherwise the matrix B explicitly depends on the orientation angles β_s.

The posture kinematic model can be expressed as follows:

$$\dot{\boldsymbol{p}}(t) = R(\theta)B(\beta_s)\boldsymbol{\eta} \tag{3}$$

$$\dot{\beta}_s = \mu \tag{4}$$

These equations can be rewritten in more compact form

$$\dot{\boldsymbol{q}} = G(\boldsymbol{q})\boldsymbol{u}, \tag{5}$$

where

$$\boldsymbol{q} \triangleq \boldsymbol{p}, \quad G(\boldsymbol{q}) \triangleq R(\theta)B, \quad \boldsymbol{u} \triangleq \boldsymbol{\eta} \quad \text{when } \delta_s = 0$$

or

$$\boldsymbol{q} \triangleq \begin{bmatrix} \boldsymbol{p} \\ \beta_s \end{bmatrix}, \; G(\boldsymbol{q}) \triangleq \begin{bmatrix} R(\theta)B(\beta_s) & 0 \\ 0 & I \end{bmatrix}, \; \boldsymbol{u} \triangleq \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\mu} \end{bmatrix} \quad \text{when } \delta_s \geq 1$$

Above equations describe the kinematics of the nonholonomic driftless robotic system.

The posture kinematic model:

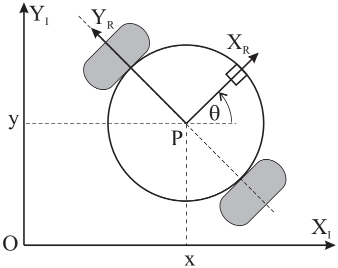
$$\dot{\boldsymbol{q}} = G(\boldsymbol{q})\boldsymbol{u},$$

$$\begin{aligned} \boldsymbol{q} &\in \mathcal{C} \quad \dim \mathcal{C} = n, \qquad & \mathcal{C} &\text{ – configuration space} \\ \boldsymbol{u} &\in \mathcal{U} \quad \dim \mathcal{U} = m, \qquad & \mathcal{U} &\text{ – command (input) space} \end{aligned}$$

with $m < n$.

- Describes the relation between the velocity input commands and the generalized velocities.
- Provides all feasible directions of instantaneous motion.
- Used for
  - studying the accessibility of $\mathcal{C}$ (system "controllability")
  - planning of feasible paths/trajectories
  - design of motion control algorithms
  - incremental (relative) localization of WMR – odometry

Differential Drive – Revisited



The robot's pose and velocity in the global frame:

$$\boldsymbol{p}_I = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}_I, \qquad \dot{\boldsymbol{p}}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}_I$$

The transformation R between the two frames:

$$\dot{\boldsymbol{p}}_R = R(\theta)\dot{\boldsymbol{p}}_I$$
$$\dot{\boldsymbol{p}}_I = R^{\mathsf{T}}(\theta)\dot{\boldsymbol{p}}_R$$

where

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

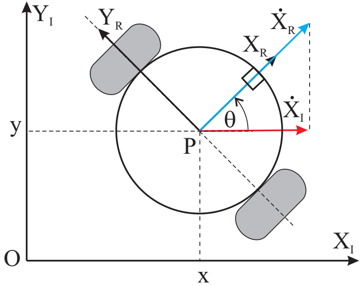
Differential Drive – Velocity Mapping I

The velocity in the local coordinate frame:

$$\dot{\boldsymbol{p}}_R = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}_R = \begin{bmatrix} v \\ 0 \\ \omega \end{bmatrix}_R$$

where v – linear velocity, and ω – angular velocity.

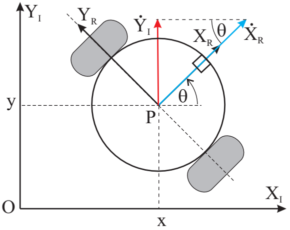
The transformation between the two frames:



The $\dot{x}_I$ velocity component

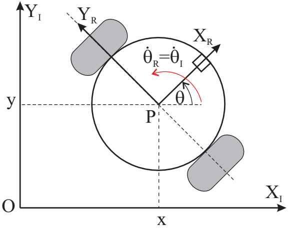
$$\dot{x}_I = \dot{x}_R \cos\theta$$

The transformation between the two frames:



The $\dot{y}_I$ velocity component

$$\dot{y}_I = \dot{x}_R \sin\theta$$



The $\dot{\theta}_I$ velocity component

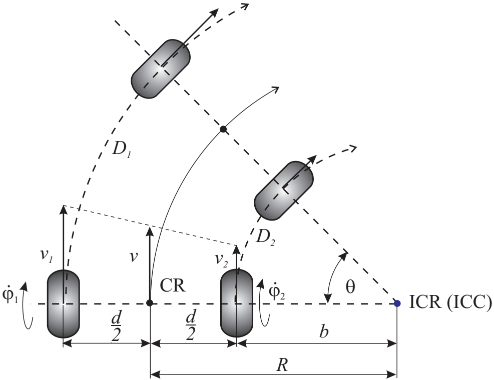
$$\dot{\theta}_I = \dot{\theta}_R$$

hence we have

$$\dot{\boldsymbol{p}}_I = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

Instantaneous Center of Rotation (ICR)

At each time instant both wheels follow a path that moves around the ICR (ICC) with the same angular rate $\omega = \frac{d\theta}{dt}$, and thus



$$\omega R = v$$

$$\omega(R + \frac{d}{2}) = v_1$$

$$\omega(R - \frac{d}{2}) = v_2$$

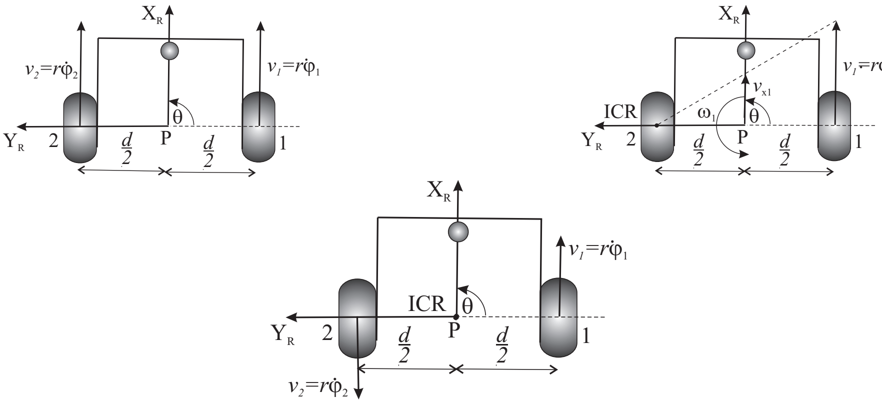
$$v = \frac{v_1 + v_2}{2}$$

$$\omega = \frac{v_1 - v_2}{d}$$

The radius R is the signed distance from ICR to CR:

$$R = \frac{v}{\omega} = \frac{d}{2}\frac{v_1 + v_2}{v_1 - v_2} = \frac{d}{2}\frac{\dot{\varphi}_1 + \dot{\varphi}_2}{\dot{\varphi}_1 - \dot{\varphi}_2}$$

ICR for Differential Drive



- For $\dot{\varphi}_1 = \dot{\varphi}_2$, robot moves forward or backward, $R = \infty$.
- For $\dot{\varphi}_1 \neq 0$, $\dot{\varphi}_2 = 0$, turning around one of the wheels, $R = l$.
- For $\dot{\varphi}_1 = -\dot{\varphi}_2$, robot rotates in place, $R = 0$.

Posture Kinematic Model for Differential-Drive Robot – Method 1

- If one wheel spins while the other wheel contributes nothing and is stationary, since P is halfway between to wheels, then the robot will move instantaneously with half the speed

$$\dot{x}_{r1} = v_{x1} = \frac{r\dot{\varphi}_1}{2}, \quad v_{y1} = 0; \qquad \dot{x}_{r2} = v_{x2} = \frac{r\dot{\varphi}_2}{2}, \quad v_{y2} = 0$$

- These two components can be added to compute $\dot{x}_R$:

$$\dot{x}_R = \dot{x}_{r1} + \dot{x}_{r2} = v_{x1} + v_{x2}, \qquad \dot{y}_R = v_{y1} + v_{y2} = 0$$

- If the wheel 1 (right wheel) spins forward alone, then the robot pivots around the wheel 2 (left wheel) *counterclockwise*. Since the linear velocity of this wheel is $v_{x1} = r\dot{\varphi}_1$, the contributed angular velocity at point P is given by

$$\omega_1 = \frac{r\dot{\varphi}_1}{d}$$

- Similarly, forward spin of the wheel 2 results in *clockwise* rotation at P

$$\omega_2 = -\frac{r\dot{\varphi}_2}{d}$$

- Angular velocity is then

$$\dot{\theta} = \omega_1 + \omega_2 = \frac{r}{d}(\dot{\varphi}_1 - \dot{\varphi}_2)$$

- Thus, combining above two formulas yields a kinematic model for the differential-drive robot:

$$\dot{\boldsymbol{p}}_I = R^{\mathsf{T}}(\theta)\dot{\boldsymbol{p}}_R = R^{\mathsf{T}}(\theta)\begin{bmatrix} v_{x1}+v_{x2} \\ v_{y1}+v_{y2} \\ \omega_1+\omega_2 \end{bmatrix} = \tag{6}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{r(\dot{\varphi}_1+\dot{\varphi}_2)}{2} \\ 0 \\ \frac{r(\dot{\varphi}_1-\dot{\varphi}_2)}{d} \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{r}{2} & \frac{r}{2} \\ 0 & 0 \\ \frac{r}{d} & -\frac{r}{d} \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix}$$

Posture Kinematic Model for Differential-Drive Robot – Method 2

- Forward kinematics for differentially driven robot can also be expressed as

$$\dot{\boldsymbol{p}}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = R^{\mathsf{T}}(\theta)B(\beta_s)\boldsymbol{\eta} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix},$$

  where matrix B is a constant matrix

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

  and variables η_1, η_2 are linear v and angular ω velocities of the robot.
- These velocities are related to the angular velocity of the wheels via the one-to-one mappings:

$$\eta_1 = v = \frac{r}{2}(\dot{\varphi}_1 + \dot{\varphi}_2)$$

$$\eta_2 = \omega = \frac{r}{d}(\dot{\varphi}_1 - \dot{\varphi}_2)$$

- Posture kinematic equation can also be obtained from the constraint equations.
- The rolling constraints in this case are

$$J_1(\beta_s)R(\theta)\dot{\boldsymbol{p}}_I - J_2\dot{\boldsymbol{\varphi}} = 0 \tag{7}$$

- and sliding constraints

$$C_1(\beta_s)R(\theta)\dot{\boldsymbol{p}}_I = 0 \tag{8}$$

- Fusing these two equations yields:

$$\begin{bmatrix} J_1(\beta_s) \\ C_1(\beta_s) \end{bmatrix} R(\theta)\dot{\boldsymbol{p}}_I = \begin{bmatrix} J_2\dot{\boldsymbol{\varphi}} \\ 0 \end{bmatrix} \tag{9}$$

- Two driving wheels are not steerable, and therefore matrices $J_1(\beta_s)$ and $C_1(\beta_s)$ simplify to J_{1f} and C_{1f}, respectively.

- To employ the fixed wheel constraint formulas for pure rolling constraint

$$[\sin(\alpha+\beta) \;\; -\cos(\alpha+\beta) \;\; (-l)\cos(\beta)]\; R(\theta)\dot{\boldsymbol{p}}_I(t) - r\dot{\varphi}(t) = 0, \tag{10}$$

- and sliding constraint

$$[\cos(\alpha+\beta) \;\; \sin(\alpha+\beta) \;\; l\sin(\beta)]\; R(\theta)\dot{\boldsymbol{p}}_I(t) = 0, \tag{11}$$

- we must first identify each wheel's values for α and β.
- Suppose that robot moves forward along $+X_R$ axis. In this case:
  - for the right wheel: $\alpha = -\pi/2$, $\beta = \pi$
  - for the left wheel: $\alpha = \pi/2$, $\beta = 0$
  - the distance $l = \frac{d}{2}$
- Note the value of β for the right wheel is necessary to ensure that positive spin causes motion in the $+X_R$ direction.

- Compute the J_{1f} and C_{1f} matrix. Because the two fixed standard wheels are parallel, equation (11) results in only one independent equation, and equation (9) gives

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 & l \\ 1 & 0 & -l \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{bmatrix} R(\theta)\dot{\boldsymbol{p}}_I - \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix} = 0 \tag{12}$$

- Inverting this equation yields the kinematic model:

$$\dot{\boldsymbol{p}}_I = R^{\mathsf{T}}(\theta) \begin{bmatrix} 1 & 0 & l \\ 1 & 0 & -l \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} J_2\dot{\boldsymbol{\varphi}} \\ 0 \end{bmatrix} =$$

$$R^{\mathsf{T}}(\theta) \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2l} & -\frac{1}{2l} & 0 \end{bmatrix} \begin{bmatrix} J_2\dot{\boldsymbol{\varphi}} \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{r(\dot{\varphi}_1+\dot{\varphi}_2)}{2} \\ 0 \\ \frac{r(\dot{\varphi}_1-\dot{\varphi}_2)}{2l} \end{bmatrix}$$

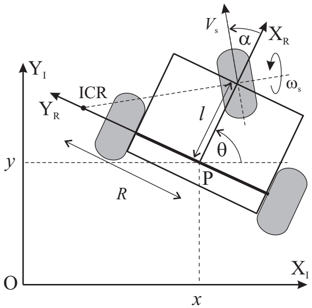
Differential-Drive Robot – Example

A differential-drive robot is positioned at a $\theta = \frac{\pi}{4}$ angle with respect to the global reference frame and has wheels with a radius of $r = 1$, and the distance between wheels is $d = 0.5$. The angular speeds of the wheels are the following: left wheel $\omega_l = \dot{\varphi}_l = 3$ and right wheel $\omega_r = \dot{\varphi}_r = 1$.
What is the robot velocity with respect to the global reference frame?

$$\dot{\boldsymbol{p}}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \theta \end{bmatrix}_I = R^\mathsf{T}(\theta)\dot{\boldsymbol{p}}_R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{r(\dot{\varphi}_l+\dot{\varphi}_r)}{2} \\ 0 \\ \frac{r(\dot{\varphi}_l-\dot{\varphi}_r)}{d} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1\cdot(3+1)}{2} \\ 0 \\ \frac{1\cdot(3-1)}{0.5} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 4 \end{bmatrix}$$

Posture Kinematic Model for Tricycle



$$v_s(t) = \omega_s(t) r$$

$$R = l \tan\left(\frac{\pi}{2} - \alpha(t)\right);$$

$$\omega(t) = \frac{v_s}{l} \sin \alpha(t);$$

$$v(t) = v_s \cos \alpha(t)$$

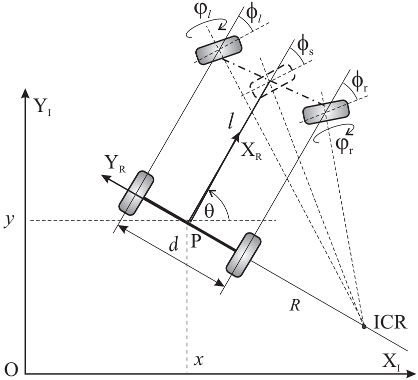
Forward kinematics for tricycle vehicle can be expressed as

$$\dot{\boldsymbol{p}}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix},$$

where

$$\eta_1 = v(t) = v_s(t) \cos \alpha(t); \qquad \eta_2 = \omega(t) = \frac{v_s}{l}(t) \sin \alpha(t)$$

Posture Kinematic Model for Ackerman Steering (car-like drive)



$$\cot\phi_l - \cot\phi_r = \frac{d}{l}$$

$$\cot\phi_s = \frac{d}{2l} - \cot\phi_r$$

or

$$\cot\phi_s = \cot\phi_l - \frac{d}{2l}$$

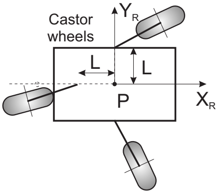
$$\dot{\boldsymbol{p}}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_s \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ \frac{1}{l}\tan\phi_s & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix},$$

singularity at $\phi_s = \pm\frac{\pi}{2}$

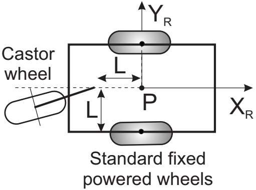
where

$$\eta_1 = \frac{r}{2}(\dot{\varphi}_l + \dot{\varphi}_r); \qquad \eta_2 = \frac{r}{2d}(\dot{\varphi}_l - \dot{\varphi}_r)$$

Kinematic Models of Three-Wheel Configurations



(3, 0)



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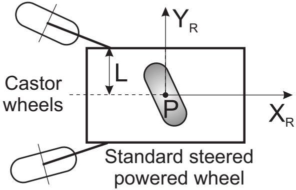
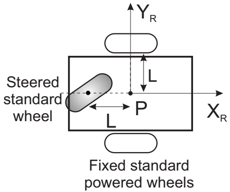


(2, 1)



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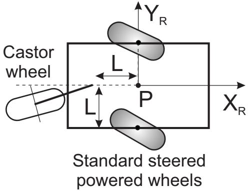


(1, 2)

Kinematic Models of Three-Wheel Configurations

$$\dot{\boldsymbol{q}} = P(\beta_s)\boldsymbol{u}$$

Type	$\boldsymbol{q}$	$P(\beta_s)$ or P	Kinematic Model
(3,0)	x y θ	$I_{3\times 3}$	$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\\ \eta_3\end{bmatrix}$
(2,0)	x y θ	$\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\cos\theta & 0\\ \sin\theta & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}$
(2,1)	x y θ β_{s1}	$\begin{bmatrix}-\sin\beta_{s1} & 0\\ \cos\beta_{s1} & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\beta}_{s1}\end{bmatrix} = \begin{bmatrix}-\sin(\theta+\beta_{s1}) & 0 & 0\\ \cos(\theta+\beta_{s1}) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\\ \mu_1\end{bmatrix}$
(1,1)	x y θ β_{s3}	$\begin{bmatrix}0\\ L\sin\beta_{s3}\\ \cos\beta_{s3}\end{bmatrix}$	$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\beta}_{s3}\end{bmatrix} = \begin{bmatrix}-L\sin\theta\sin\beta_{s3} & 0\\ L\cos\theta\sin\beta_{s3} & 0\\ \cos\beta_{s3} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\eta_1\\ \mu_1\end{bmatrix}$
(1,2)	x y θ β_{s1} β_{s2}	$\begin{bmatrix}-2L\sin\beta_{s1}\sin\beta_{s2}\\ L\sin(\beta_{s1}+\beta_{s2})\\ \sin(\beta_{s2}-\beta_{s1})\end{bmatrix}$	$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\beta}_{s1}\\ \dot{\beta}_{s2}\end{bmatrix} = \begin{bmatrix}-L[\sin\beta_{s1}\sin(\theta+\beta_{s2})+\sin\beta_{s2}\sin(\theta+\beta_{s1})] & 0 & 0\\ -L[\sin\beta_{s1}\cos(\theta+\beta_{s2})+\sin\beta_{s2}\cos(\theta+\beta_{s1})] & 0 & 0\\ \sin(\beta_{s2}-\beta_{s1}) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\eta_1\\ \mu_1\\ \mu_2\end{bmatrix}$

The pure rolling condition for conventional wheels can be written in the compact form:

$$J_1(\beta_c, \beta_s)R(\theta)\dot{\boldsymbol{p}} + J_2\dot{\boldsymbol{\varphi}} = 0, \tag{13}$$

where $R(\theta)$ is a rotation matrix, and $J_1(\beta_s)$ denotes a matrix with projections for all wheels to their motions along their individual wheel planes:

$$J_1(\beta_c, \beta_s) = \begin{bmatrix} J_{1f} \\ J_{1s}(\beta_s) \\ J_{1c}(\beta_c) \end{bmatrix},$$

where

- J_{1f} a constant matrix of size $(N_f \times 3)$,
- matrix $J_{1c}(\beta_c)$ of size $(N_c \times 3)$,
- matrix $J_{1s}(\beta_s)$ of size $(N_s \times 3)$,
- matrix J_2 is a constant diagonal $(N \times N)$ matrix whose entries are radii r of all wheels.

The nonslip conditions for caster wheels can be summarized as

$$C_{1c}(\beta_c)R(\theta)\dot{\boldsymbol{p}} + C_{2c}\dot{\beta}_c = 0, \tag{14}$$

where

- matrix $C_{1c}(\beta_c)$ of size $(N_c \times 3)$, whose entries derived from the nonslip constraints for castor wheel,
- matrix $C_{2c} = \mathrm{diag}(d)$ is a constant diagonal matrix, whose entries are equal to d (assuming that castor offset d is the same for every castor wheel).

Now, we can derive the equations of evolution of the orientation (steering) velocities $\dot{\beta}_c$ and of the rotation velocities $\dot{\varphi}$ of the castor wheels:

$$\dot{\boldsymbol{\beta}}_c = -\underbrace{C_{2c}^{-1} C_1(\boldsymbol{\beta}_c) R(\theta)}_{D} \dot{\boldsymbol{p}} \tag{15}$$

$$\dot{\boldsymbol{\varphi}} = \underbrace{J_2^{-1} J_1(\boldsymbol{\beta}_s, \boldsymbol{\beta}_c) R(\theta)}_{E} \dot{\boldsymbol{p}} \tag{16}$$

By combining these equations with the posture kinematic model, the state equations for $\boldsymbol{\beta}_c$ and $\boldsymbol{\varphi}$ are the following

$$\dot{\boldsymbol{\beta}}_c = D(\boldsymbol{\beta}_c) B(\boldsymbol{\beta}_s) \boldsymbol{\eta} \tag{17}$$

$$\dot{\boldsymbol{\varphi}} = E(\boldsymbol{\beta}_s, \boldsymbol{\beta}_c) B(\boldsymbol{\beta}_s) \boldsymbol{\eta} \tag{18}$$

Defining $\boldsymbol{q}$ as the vector of configuration coordinates

$$\boldsymbol{q} \triangleq [\boldsymbol{p}^{\mathsf{T}}, \boldsymbol{\beta}_s^{\mathsf{T}}, \boldsymbol{\beta}_{oc}^{\mathsf{T}}, \boldsymbol{\varphi}^{\mathsf{T}}]^{\mathsf{T}}$$

the evolution of the configuration coordinates can be described in the compact form, called the **configuration kinematic model**

$$\dot{\boldsymbol{q}} = S(\boldsymbol{q})\boldsymbol{u}, \tag{19}$$

where

$$S(\boldsymbol{q}) \triangleq \begin{bmatrix} R(\theta)B(\boldsymbol{\beta}_s) & 0 \\ 0 & I \\ D(\boldsymbol{\beta}_c)B(\boldsymbol{\beta}_s) & 0 \\ E(\boldsymbol{\beta}_s, \boldsymbol{\beta}_c)B(\boldsymbol{\beta}_s) & 0 \end{bmatrix}, \tag{20}$$

and

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\mu} \end{bmatrix}$$

The vector $\boldsymbol{q}$ is the vector of generalized coordinates that fully describe the pose and the configuration of the mobile robot.

The constraints (13), and (14) can be summarized under the following compact form

$$J(\boldsymbol{q})\dot{\boldsymbol{q}} = \mathbf{0},$$

where the matrix $J(\boldsymbol{q})$ is the Jacobian of the constraints, i. e.,

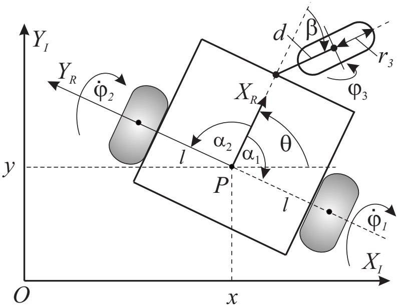
$$J(\boldsymbol{q}) = \begin{bmatrix} J_1(\beta_s, \beta_c)R(\theta) & 0 & 0 & J_2 \\ C_{1c}(\beta_c)R(\theta) & 0 & C_{2c} & 0 \\ C_1^*(\beta_s)R(\theta) & 0 & 0 & 0 \end{bmatrix}$$

The two matrices $S(\boldsymbol{q})$ and $J(\boldsymbol{q})$ satisfy the relation

$$S(\boldsymbol{q})J(\boldsymbol{q}) = \mathbf{0}.$$

Configuration Kinematic Model – Differential Drive

We consider the differential drive robot with two fixed actuated wheels and one passive castor wheel.



- For this robot $\delta_m = 2$, $\delta_s = 0$, and the vector of configuration coordinates $\boldsymbol{q} = [x, y, \theta, \beta, \varphi_1, \varphi_2, \varphi_3]^{\mathsf{T}}$, where $\beta = \beta_{c3}$.
- Wheel parameters:
  - for the right wheel: $\alpha_1 = -\frac{\pi}{2}$, $\beta_1 = \pi$
  - for the left wheel: $\alpha_2 = \frac{\pi}{2}$, $\beta_2 = 0$
  - for the castor wheel: $\alpha_3 = 0$, $\beta_3 = \beta(t)$

Configuration Kinematic Model – Differential Drive

- Pure rolling constraint for a standard wheel:

$$\left[-\sin(\alpha+\beta(t)) \;\; \cos(\alpha+\beta(t)) \;\; l\cos(\beta(t))\right] R^{\mathsf{T}}(\theta)\dot{\boldsymbol{p}}(t) + r\dot{\varphi}(t) = 0$$

- Nonslip constraint for a standard wheel:

$$\left[\cos(\alpha+\beta(t)) \;\; \sin(\alpha+\beta(t)) \;\; d + l\sin(\beta(t))\right] R^{\mathsf{T}}(\theta)\dot{\boldsymbol{p}}(t) + d\,\dot{\beta}(t) = 0$$

Motion constraints for this robot are the following:

- Pure rolling constraints

$$\begin{bmatrix} -1 & 0 & -l \\ -1 & 0 & l \\ -\sin\beta & \cos\beta & l\cos\beta \end{bmatrix} R^{\mathsf{T}}(\theta) \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r_3 \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\varphi}_3 \end{bmatrix} = \mathbf{0}$$

- Nonslip constraints

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ \cos\beta & \sin\beta & d + l\sin\beta \end{bmatrix} R^{\mathsf{T}}(\theta) \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix} \dot{\beta} = \mathbf{0}$$

Configuration Kinematic Model – Differential Drive

Configuration kinematic model for three-wheel differential drive robot is described by the matrix $S(\boldsymbol{q})$:

$$S(\boldsymbol{q}) = \begin{bmatrix} \sin\theta & 0 \\ \cos\theta & 0 \\ 0 & 1 \\ \frac{1}{d}\cos\beta & -(1+\frac{l}{d}\sin\beta) \\ -\frac{1}{r} & -\frac{l}{r} \\ \frac{1}{r} & -\frac{l}{r} \\ -\frac{1}{r_3}\sin\beta & -\frac{l}{r_3}\cos\beta \end{bmatrix}, \qquad \dot{\boldsymbol{q}} = S(\boldsymbol{q})\boldsymbol{u}$$

It results from the configuration model that

$$\dot{\varphi}_1 + \dot{\varphi}_2 = -\frac{2l}{r}\dot{\theta}$$

This means that the variable $\varphi_1 + \varphi_2 + \frac{2l}{r}\theta =$ const. (i.e. it has a constant value along any trajectory compatible with the constraints).