Quickselect Routine
Time complexity analysis

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1 Common Elements

The quickselect routine involves choosing a pivot (which completes in fixed time) and partitioning (which completes in linear time). These two steps are the same as for a quicksort routine. To complete the routine, the desired element must be selected from one of three subsets. If the pivot is the desired element, the quickselect routine is finished; otherwise, the routine must recurse into the subset which contains the desired element.

2 Best Case

In the best case, the pivot happens to be the desired element on the first try, and the selection portion is performed in linear time.

$$\begin{aligned} T(n) &= 1 + n + 1 \\ &= n + 2 \\ &= O(n) \end{aligned}$$

3 Worst Case

In the worst case, the pivot is not the desired element until the list is whittled down to a single element, and the worst possible pivot is chosen each time. That is, the largest value or smallest value that is not the desired element is always chosen as the pivot. In this case, the recursion is performed on a list one item smaller than the previous list.

$$T(n) = 1 + n + T(n-1)$$

Dropping the constant term and following the recursion produces n equations.

$$\begin{aligned}
T(n) &= n + T(n-1) \\
T(n-1) &= n - 1 + T(n-2) \\
T(n-2) &= n - 2 + T(n-3) \\
&\vdots \\
T(2) &= 2 + T(1) \\
T(1) &= 1 + T(0)
\end{aligned}$$

Summing the equations and cross-cancelling like values produces a polynomial result.

$$\begin{aligned}
T(n) &= n + (n-1) + (n-2) + \cdots + 2 + 1 \\
&= \sum_{i=1}^{n} i \\
&= \frac{n(n-1)}{2} \\
&= O(n^2)
\end{aligned}$$

4 Average Case

In the average case, assume that the pivot is not the desired element until it is the only element, and the pivot is chosen somewhere in the middle. Therefore, assume that the recursion is performed on a list half the size of the previous list.

$$T(n) = 1 + n + T\left(\frac{n}{2}\right)$$

Dropping the constant term and following the recursion produces the following series of equations.

$$
\begin{aligned}
T(n) &= n + T\left(\frac{n}{2}\right) \\
T\left(\frac{n}{2}\right) &= \frac{n}{2} + T\left(\frac{n}{4}\right) \\
T\left(\frac{n}{4}\right) &= \frac{n}{4} + T\left(\frac{n}{8}\right) \\
&\vdots \\
T(2) &= 2 + T(1) \\
T(1) &= 1 + \ldots
\end{aligned}
$$

Summing the equations and cross-cancelling like values produces a linear result.

$$
\begin{aligned}
T(n) &= n + \frac{n}{2} + \frac{n}{4} + \cdots + 2 + 1 + \ldots \\
&= \sum_{i=0}^{\infty} \frac{n}{2^i} \\
&= n \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i \\
&= n \left(\frac{1}{1 - \frac{1}{2}}\right) \\
&= 2n \\
&= O(n)
\end{aligned}
$$