

# Allan Standard Deviation to Gauss-Markov State-Space Model: Cookbook with Example

Jay A. Farrell

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## Abstract

This article walks through the process for extracting first-order, Gauss-Markov, state-space models corresponding to an Allan Standard Deviation (ASD) plot for a range of delays. This article skips most derivations, many of which are presented or pointed to in [1].

The methods herein apply to both gyros and accelerometers. To be generic, the article will use the terminology of phase and frequency. The term *frequency* refers to either the IMU angular rate or specific force measurement. The term *phase* refers to the integral of frequency, which for an IMU refers to attitude or velocity.

Given values for the parameters:  $N$ ,  $B$ ,  $T_p$ , and  $F_s$ , the script `ASD_to_GaussMarkovFirstOrder.py` available at <https://github.com/jaffarrell/AV-Matlab-SW.git> implements all the computations that are presented herein in a blue font.

## 1 Introduction

Any real physical system is impossible to model perfectly. Often such systems can be modeled sufficiently for a well-defined purpose by a linear system.

The purpose of the IMU error model is to communicate to an optimal state estimation routine, the accuracy of the IMU measurements so that it can properly incorporate them into its data processing. One axis of the accelerometer (i.e., specific force) or gyro measurement in IMU frame can be modeled as

$$y(t) = s(t) + z(t) + d(t) \quad (1)$$

where  $s(t)$  represents the signal (i.e., specific force or angular rate) the instrument is designed to measure,  $z(t)$  represents the stochastic errors, and  $d(t)$  represents non-stochastic errors (e.g., scale factor, non-orthogonality, saturation). This document is only concerned with the stochastic errors:  $z(t)$ . The non-stochastic errors are addressed by other methods.

### 1.1 ASD Parameters

Manufacturers often follow the IEEE standard [1] and specify their IMU's typical stochastic errors through either an ASD plot or selected parameters extracted from it. Examples of important parameters are: phase random walk  $N$ , bias instability  $B$ , and sample frequency  $F_s$ . An example ASD plot is shown in Fig. 1. The method to read the ASD parameters from this plot is described in Section 3.

### 1.2 Stochastic Error: PSD and ASD

The Power Spectral Density (PSD) and ASD of the stochastic error  $z(t)$  are directly related to each other. The PSD can be represented as an infinite sum of terms:

$$S_z(f) = \cdots + N^2 + \frac{B^2}{2\pi f} + \frac{K^2}{(2\pi f)^2} + \cdots, \quad (2)$$

where  $f$  denotes frequency in Hz. Each even term in this series can be perfectly represented by a finite dimensional continuous-time linear system. For the odd terms, there is no finite dimensional linear system to represent any one of them. Due to (1) our inability to model the odd terms and (2) there being an infinite number of even term, there does not exist a perfect finite dimensional linear system model for the IMU stochastic error.

Fortunately, due to the nature of the problem and good design by the manufacturer, only a few terms in the series are important.

- The navigation engineer, understanding aliasing and Nyquist frequency ideas will ensure that the sample frequency  $F_s$  of the IMU is much larger than the bandwidth  $BW$  of the vehicle on which it will be mounted (i.e.,  $F_s \gg BW$ ). The ASD plot has its smallest delay at the sampling period ( $T_s = 1/F_s$ ). All important signal information is at much lower frequencies (much higher delays).
- The manufacturer will ensure that the measurement quantization error is much smaller than the instrument measurement noise. Therefore for high frequencies below  $F_s$  (i.e., small delays above  $T_s$ ) the phase white random walk noise ( $N$ ) will dominate.
- At mid-to-lower frequencies (higher delays) time-correlated errors become important. For intermediate frequency ranges, the bias instability ( $B$ ) and bias random walk ( $K$ ) parameters are important.
- The designer of the navigation system can include states in the system to model the time-correlated states. These are often referred to as time-varying bias estimates. Because the navigation system can accurately estimate the low-frequency (long-delay) portion to the stochastic error, this portion of the PSD (or ASD) curve is not important.

The  $N$ ,  $B$ , and  $K$  parameters can be selected (or approximated) directly from the ASD curve.

### 1.3 Finite Dimension, Gauss-Markov Model

Section 1.2 explained why, typically, only three parameters ( $N$ ,  $B$ , and  $K$ ) are necessary to model the stochastic errors over the frequency (delay) range of interest. However, due to the  $B$  term, which appears as an odd-term in the PSD series of eqn. (1.2), these three terms theoretically still require an infinite order linear system to model them exactly.

More practically, the two  $N$  and  $K$  terms can be perfectly modeled by a first-order Gauss-Markov model. This  $N$  and  $K$  only model, may model the stochastic error well at short and long delays, but not adequately model it in the mid-range of delays corresponding to the flat portion of the ASD curve. That flat portion is often more important to performance than the long delay portion. The three  $N$ ,  $B$ , and  $K$  terms can also often be adequately modeled, over the frequency (delay) range of interest, by a first-order Gauss-Markov model.

### 1.4 Why First-Order?

The order of the stochastic error model is important because the computational load of the matrix operations, for example in a Kalman filter, are  $O(n^3)$  where  $n$  is the order of the state vector. That state vector includes the vehicle state ( $n_v = 9$ ), the number of states ( $6n_d$  used to model  $d(t)$ ), and the number of states ( $6n_z$ ) needed to implement the model of  $z(t)$ . The factor of 6 comes from the fact that the IMU has 6 measurements. Therefore the computational load is on the order of  $O((n_v + 6(n_d + n_z))^3)$ .

The purpose of the Gauss-Markov model is to communicate to the estimation algorithm the frequency distribution (i.e., time-correlation structure) of the stochastic errors. If the ASD plot over the important range of delays (or the PSD plot over the important range of frequencies) can be adequately represented by a first-order model, there is a high computational penalty for implementing a higher-order model.

## 2 First-Order, Gauss-Markov Models

This section summarizes a few facts about first-order Gauss-Markov state-space models.

### 2.1 Continuous-Time

The continuous-time model has the form

$$z(t) = b(t) + n(t), \quad \text{where } n(t) \sim \mathcal{N}(0, S_n) \quad (3)$$

$$\frac{db(t)}{dt} = -\mu b(t) + \zeta(t), \quad \text{where } \zeta(t) \sim \mathcal{N}(0, S_\zeta), \quad (4)$$

and  $\mu \geq 0$ . The symbols  $n(t)$  and  $\zeta(t)$  represent independent, continuous-time, Gaussian, white noise processes with power spectral densities of  $S_n$  and  $S_\zeta$ , respectively. For the discussion herein, these PSD's are assumed to be constant.

This model is discussed in Example 4.19 on p. 140 in [2], which shows that the time evolution of the  $P(t) = \text{cov}(b(t))$  is described by

$$\dot{P}(t) = -2\mu P(t) + S_\zeta. \quad (5)$$

For  $\mu > 0$ , as  $t$  increases towards infinity,  $\dot{P}(t)$  approaches 0, and  $P(t)$  approaches its steady-state value

$$P_\infty = \frac{S_\zeta}{2\mu}. \quad (6)$$

If  $\mu = 0$  (i.e., the bias is modeled as a pure random walk), the system has no steady-state covariance as  $t \rightarrow \infty$ ; instead,  $P(t)$  grows to infinity, which is not a realistic covariance for an IMU bias. The model parameters (i.e.,  $\mu$ ,  $S_n$ , and  $S_\zeta$ ) can be estimated from the ASD plot, as will be discussed in Section 3.

### 2.2 Discrete-time

The discrete-time model has the form

$$z_k = b_k + n_k, \quad \text{where } n_k \sim \mathcal{N}(0, \sigma_n^2) \quad (7)$$

$$b_{k+1} = \phi b_k + \zeta_k, \quad \text{where } \zeta_k \sim \mathcal{N}(0, \sigma_\zeta^2), \quad (8)$$

where  $|\phi| \leq 1$ . The subscript  $k$  on any variable denotes the value of the variable at the discrete instant of time  $t_k = kT_s$ , where  $T_s > 0$  is the IMU sampling rate. For example,  $z_k = z(kT_s)$ . The symbols  $n_k$  and  $\zeta_k$  represent independent, Gaussian, discrete-time, white noise processes with covariances of  $\sigma_n^2$  and  $\sigma_\zeta^2$ , respectively. For the discussion herein, these covariances are assumed to be constant.

The time evolution of the  $P_k = \text{cov}(b_k)$  is described by

$$P_{k+1} = \phi^2 P_k + \sigma_\zeta^2. \quad (9)$$

For  $|\phi| < 1$ , as  $k \rightarrow \infty$ , a steady-state is reached wherein  $P_{k+1} = P_k$ ; therefore, the steady-state covariance satisfies

$$P_\infty = \frac{\sigma_\zeta^2}{1 - \phi^2}. \quad (10)$$

If  $\phi = 1$ , then no steady-state exists and  $P_k \rightarrow \infty$  as  $k \rightarrow \infty$ . The parameters (i.e.,  $\phi$ ,  $\sigma_n^2$ , and  $\sigma_\zeta^2$ ) are discussed in Section 2.3.

### 2.3 Parameter Choice for Equivalent Models

The continuous and discrete-time models will represent equivalent models in the sense that their noise statistics are equal at the times  $t_k = kT_s$  if the discrete-time model parameters are selected as

$$\phi = \exp(-\mu T_s), \quad \sigma_n^2 = S_n F_s, \quad \text{and} \quad \sigma_\zeta^2 = \frac{1 - \phi^2}{2\mu} S_\zeta. \quad (11)$$

Appendix A presents a unit analysis to check the final equation.

#### 2.3.1 Summary

The continuous-time model parameters are easily ready from the ASD plot. Translation to an equivalent discrete-time model is useful for a few reasons:

- Discrete-time estimators are easier to implement and require discrete-time models.
- The discrete-time model is easily simulated. If it is correct, then the ASD plot made from its simulated data should match the ASD plot of the instrument of interest.

### 3 ASD Graph to State-Space Model

This goals of this section are to show how to extract the  $N$ ,  $B$ , and  $T_p$  parameters from an ASD plot, and how to use those parameters to define a first-order, continuous-time, state-space model whose discrete-time equivalent model will generate data to reproduce the ASD graph.

Samples along the ASD curve for one axes of an accelerometer are shown as blue dots in the figures herein.

1. Find the portion of the ASD graph that has a slope of  $-1/2$ . Draw a tangent line, and define phase random walk parameter  $N$  as the value of this ASD tangent line where it crosses the vertical line at  $\tau = 1$ . See the example orange lines in Fig. 1.
2. The continuous-time phase random walk power spectral density is  $S_n = N^2$ . See eqn. (22) in [1].
3. The discrete-time phase random walk covariance is  $\sigma_n^2 = S_n F_s$ . See eqn. (62) in [1] and its discussion.
4. The values of the bias instability  $B$  and  $T_p$  are selected together.
  - By definition  $B$  is the value of the ASD plot at which its slope is zero. For many inexpensive IMU's this flat section may not exist or may be difficult to select due to the noisiness of the ASD plot. The ASD plot in Fig. 1 is very clean and a reasonable value of  $B$  can be read off.
  - There is not finite-dimensional state space-model that can exactly reproduce the bias instability. See [1].
  - This example selects the first-order model of eqn. (4) to approximate the bias instability portion of the ASD plot. The shape of the ASD plot corresponding to this model is sketched as the green asymptotes and black curve in Fig. 2. It has a peak at  $T_p$  and its asymptotes have slope  $\pm 1/2$ ; therefore, for delays near, but smaller than  $T_p$ , the parameters  $B$  and  $T_p$  can be selected to produce a flat region with the desired height. Given values for  $B$  and  $T_p$ , according to eqns. (34) and (37) in [1], the script computes:

$$\begin{aligned}
 T_b &= \frac{T_p}{1.89}, & \text{a parameter for the ASD model} \\
 \mu &= \frac{1}{T_b} & \text{the decay rate in eqn. (4)} \\
 S_\zeta &= \frac{B^2}{0.4365^2 T_b} & \text{PSD of } \omega(t) \\
 \bar{P}_\infty &= \frac{S_b}{2\mu} & \text{steady state covariance of } b(t)
 \end{aligned}$$

The script also provides tools to convert the continuous-time model to its equivalent discrete-time model, to simulate the model of produce sample data, and to plot the Allan standard deviation plots. The red dots in Fig. 3 show the ASD for the data simulated by the equivalent discrete-time model.

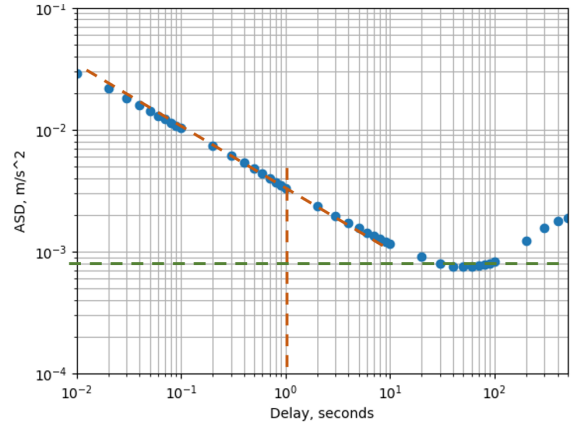


Figure 1: Allan Standard Deviation Plot marked for estimation of  $N$  and  $B$ . The slanted orange line with slope  $-1/2$  intersects the vertical line at  $\tau = 1$  for  $N = 3.3e-3$  m/s/s/rHz. The horizontal dashed green line defines  $B = 8.0e-3$  m/s/s.

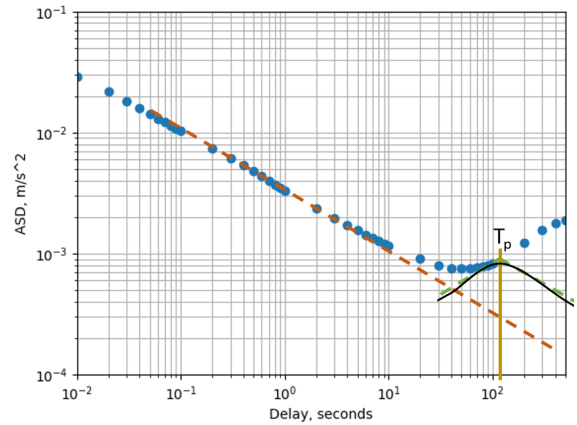


Figure 2: Allan Standard Deviation Plot marked with approximations for the first-order Gauss-Markov model.

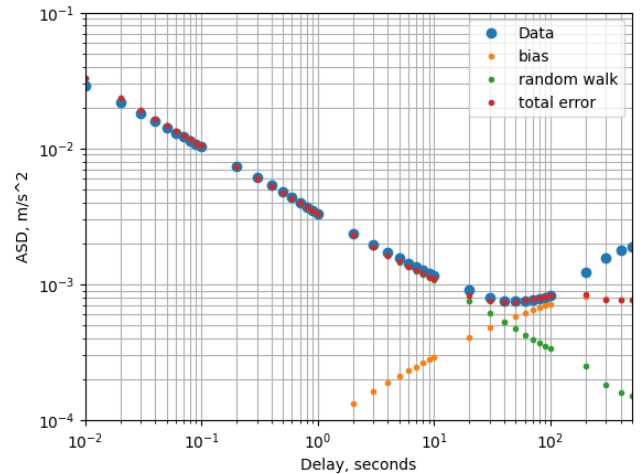


Figure 3: Allan Standard Deviation Plot.

## 4 Results

The Jupyter notebook outputs the following:

- Continuous-time model is (units depend on context):  
 $z(t) = b(t) + n(t)$ ,  
 $db(t)/dt = -6.3000e-03 * b(t) + w(t)$ ,  
 where  $n(t)$  is white noise with PSD  $S_n = 1.0890e-05$  and  $w(t)$  is the bias process noise with PSD  $S_w = 2.6783e-08$ . The steady-state covariance of the bias process is  $P_{b,ss,c} = 2.1256e-06$ .
- Discrete-time model is (units depend on context):  
 $z[k] = b[k] + n[k]$ ,  
 $b[k+1] = 0.9999 * b[k] + w[k]$ ,  
 where  $n[k]$  is white noise with covariance  $Q_n = 1.0890e-03$  (std  $3.3000e-02$ ) and  $w[k]$  is the bias process with covariance  $Q_w = 2.6781e-10$  (std  $1.6365e-05$ ). The steady-state covariance of the bias process is  $P_{b,ss,d} = 2.1256e-06$ .

Note that both models have the same steady-state covariance for the bias.

The discrete-time model is easily simulated to yield data sequences that can be used to check the validity of the error model.

### 4.1 Check: ASD Plot

Fig. 3 shows the ASD graph for the simulate output of the state-space error model as red dots. These closely match the ASD plot for the instrument data for delay  $\tau \in [T_s, 100]$  seconds.

The figure also shows the ASD plot for both components  $b$  (orange) and  $v$  (green) that are summed to create the overall output  $z$ . The value of  $B$  was selected to get the appropriate height of the the ASD for  $b$  and the value of  $T_p$  was adjusted so that the overall plot had the desired shape for  $\tau \in [10, 100]$  seconds.

### 4.2 Check: Covariance

The main purpose of the model is to communicate to the optimal estimation routine the magnitude and correlation structure of the IMU errors. The IMU outputs are integrated to produce angles or velocities.

Therefore, this section compares the growth of the integrated error in a few ways:

- The error model produces 1000 simulated integrated error trajectories. Each is plotted as a narrow opaque curve in the top portion of Fig. 4. At each time instant the robust standard deviation is computed and plotted as a blue line. The theoretical error covariance is computed using the state-space model and eqn. (9) (i.e., eqn. (4.125) in [2]). This is plotted as the black line. Note that the blue and black lines match closely at all times.
- After removing a constant prior estimate of the bias, the instrument data produces 1000 simulated integrated error trajectories. Each is plotted as a narrow opaque curve in the bottom portion of Fig. 4. At each time instant the robust standard deviation is computed and plotted as a blue line. The theoretical error covariance is computed using the state-space model and eqn. (9) (i.e., eqn. (4.125) in [2]). This is plotted as the black

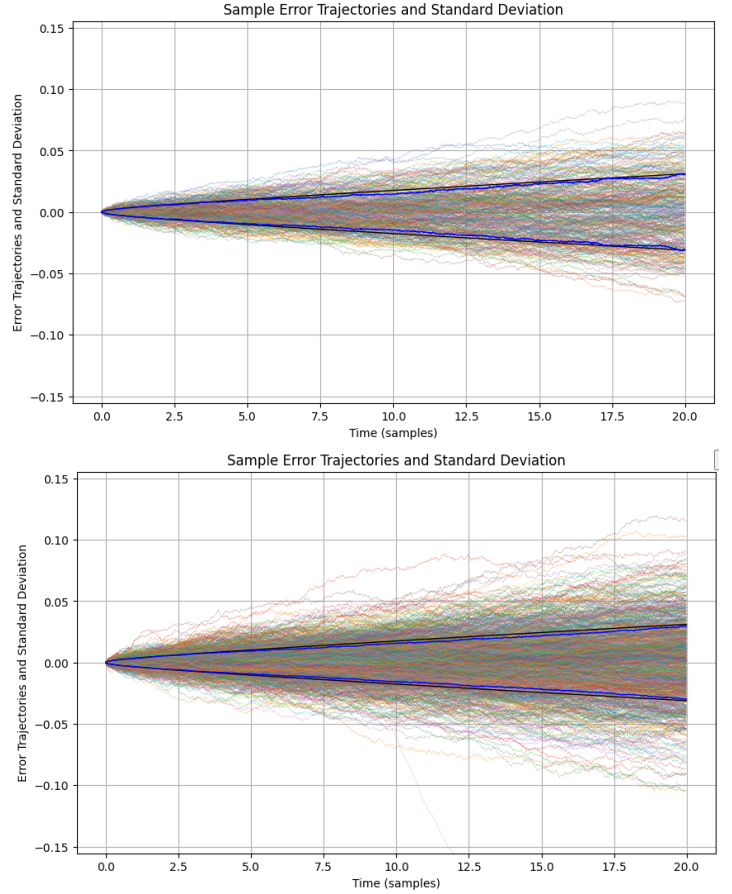


Figure 4: Covariance Check Graphs. Top: One thousand sample error trajectories computed by integrating the output of the state-space error model. Bottom: One thousand sample error trajectories computed by integrating the IMU data after subtracting a prior estimate of the bias. Black solid curve is the theoretical standard deviation from the state-space model. Blue solid curve is the sample standard deviation at each time instant over the sample trajectories.

line. Note that the blue and black lines match closely at all times.

## A Unit Analysis:Check

The equation

$$\sigma_{\zeta}^2 = \frac{1 - \phi^2}{2\mu} S_{\zeta} \quad (12)$$

from Section 2.3 is derived by setting eqn. (6) equal to eqn. (10).

To get comfortable with this equation and with the (sometimes confusing) units associated with PSD's, it is useful to perform an analysis of the units of each term in this equation.

**Gyro:** The units of the bias are  $rad/s$ . The covariance  $P$  has units  $(rad/s)^2$ . Therefore, from eqn. (9), the units of  $\sigma_{\zeta}^2$  are also  $(rad/sec)^2$ , because all summed items have the same units. The time derivative of the covariance  $\dot{P}$  has units  $\frac{(rad/s)^2}{s}$ . Therefore, each term being added in eqn. (5) including the PSD  $S_{\zeta}$  has these same units. The numerator of eqn. (12) (i.e.,  $(1 - \phi^2)$ ) has no units. The denominator  $(2\mu)$  has units of reciprocal seconds. The units of the product on the right-hand side are thus the units of  $S_{\zeta}$  (i.e.,  $\frac{(rad/s)^2}{s}$ ) divided by the units of  $\mu$  (i.e.,  $\frac{1}{s}$ ), which yields  $(rad/s)^2$ .

Note that the driving noise  $\zeta$  has units of  $\frac{rad}{s^2}$ . The units of its PSD are:

$$\frac{(rad/s)^2}{s} = \frac{rad^2}{s^3} = \frac{rad^2}{s^3} \frac{s}{s} = \frac{(rad/s^2)^2}{Hz}. \quad (13)$$

**Accelerometer:** The units of the bias are  $m/s^2$ . The covariance  $P$  has units  $(m/s^2)^2$ . Therefore, from eqn. (9), the units of  $\sigma_{\zeta}^2$  are also  $(m/s^2)^2$ , because all summed items have the same units. The time derivative of the covariance  $\dot{P}$  has units  $\frac{(m/s^2)^2}{s}$ . Therefore, each term being added in eqn. (5) including the PSD  $S_{\zeta}$  has these same units. The numerator of eqn. (12) (i.e.,  $(1 - \phi^2)$ ) has no units. The denominator  $(2\mu)$  has units of reciprocal seconds. The units of the product on the right-hand side are thus the units of  $S_{\zeta}$  (i.e.,  $\frac{(m/s^2)^2}{s}$ ) divided by the units of  $\mu$  (i.e.,  $\frac{1}{s}$ ), which yields  $(m/s^2)^2$ .

Note that the driving noise  $\zeta$  has units of  $\frac{m}{s^3}$ . The units of its PSD are:

$$\frac{(m/s^2)^2}{s} = \frac{m^2}{s^5} = \frac{m^2}{s^5} \frac{s}{s} = \frac{(m/s^3)^2}{Hz}. \quad (14)$$

In both cases, the unit analysis shows that both the left and right sides of equation 12 have the same units (as they should) and that

the units of the PSD are the units of the driving noise squared per Hz.

## B Approximate Discrete-Time Driving Covariance

Due to the fact that  $\phi = \exp(-\mu T_s)$ ,

$$\begin{aligned} 1 - \phi^2 &= 1 - (1 + (-2\mu T_s) + 0.5(-2\mu T_s)^2 + \dots) \\ &= 2\mu T_s - 0.5(2\mu T_s)^2 + \dots \end{aligned}$$

When  $\mu T_s \ll 1$ , which is typically the case by design, because the sample period  $T_s$  is much less than the correlation time of the IMU bias, the higher-order terms are small and may be dropped:

$$1 - \phi^2 \approx 2\mu T_s.$$

In this case, eqn. (12) simplifies to

$$\sigma_{\zeta}^2 \approx \frac{2\mu T_s}{2\mu} S_{\zeta} = T_s S_{\zeta}. \quad (15)$$

## References

- [1] J. A. Farrell, F. O. Silva, F. Rahman, and J. Wendel, "Inertial measurement unit error modeling tutorial: Inertial navigation system state estimation with real-time sensor calibration," *IEEE Control Systems Magazine*, vol. 42, no. 6, pp. 40–66, 2022.
- [2] J. A. Farrell, *Aided Navigation: GPS with High Rate Sensors*. McGraw-Hill, Inc., 2008.