Kalman Filter: Sensor Calibration

Jay A. Farrell *

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Abstract

This document presents an example Kalman filter design to calibrate one sensor relative to another. The standard Kalman filter equations and their derivation can be found in many text books. The final equations are summarized in Appendix A. This document applies and simplifies those equations for the sensor calibration problem.

If you find errors in the document or implementation, please do let me know.

1 Notation

In this document, the equality symbol '$=$' will be used in its normal sense. The symbol '$\doteq$' will be used to indicate computations that will be implemented in the software.

This distinction is meant to clearly differentiate between calculations and analysis. The analysis is necessary to understand how and why the approach works.

Boldface symbols represent matrices or vectors. The variable k is the discrete-time of a measurement that occurs with sample period T. Using this notation $\boldsymbol{x}_k = \boldsymbol{x}(k) = \boldsymbol{x}(t)|_{t=kT}$.

2 Problem Statement

An application has two sensors that measure a variable ψ_k:

$$v_k = \psi_k + \eta_k \tag{1}$$
$$z_k = (1-a)\,\psi_k - b - \nu_k \tag{2}$$

where a represents an unknown scale factor error and b represents an unknown sensor bias. The symbols $\eta_k \sim N(0, \sigma_\eta^2)$ and $\nu_k \sim N(0, \sigma_\nu^2)$ represent independent, Gaussian random measurement noise with covariance σ_ν^2 and σ_η^2, satisfying $0 < \sigma_\nu \ll \sigma_\eta$. The symbol ψ_k represents the variable that the sensors are designed to measure.

3 Model Derivation

This section manipulates the equations and assumptions of the problem statement to derive a state-space model for the Kalman filter design.

3.1 Measurement Model

The approach will estimate a vector $\boldsymbol{x} = [\theta_1, \theta_2]^\top$ such that over the ensemble of data, $E\langle v_k\rangle = \theta_1 E\langle z_k\rangle + \theta_2$.

For each k, our goal is to estimate $\boldsymbol{x}$ to achieve:

$$v_k = \theta_1 z_k + \theta_2 \tag{3}$$
$$= \mathbf{h}_k\,\boldsymbol{x} \tag{4}$$

where $\mathbf{h}_k = [z_k \quad 1]$. In the approach that follows, we will define $y_k \doteq v_k$ and account for the measurement noise using $w_k = \nu_k - \eta_k$ with the model

$$y_k = \mathbf{h}_k\,\boldsymbol{x} + w_k \tag{5}$$

where $w_k \sim N(0, \sigma_w^2)$ and $\sigma_w^2 = \sigma_\eta^2 + \sigma_\nu^2$.

3.2 State Time-Transition Model

The state variable $\boldsymbol{x}$ is an unknown constant. Therefore, its discrete-time state-space model is $\boldsymbol{x}_k = \boldsymbol{x}_{k-1}$. This corresponds to the standard state-space model

$$\boldsymbol{x}_k = \mathbf{\Phi}\,\boldsymbol{x}_{k-1} + \boldsymbol{\omega}_{k-1} \tag{6}$$

where $\mathbf{\Phi} = \mathbf{I}$ and $\boldsymbol{\omega}_k \sim N(\mathbf{0}, \mathbf{Q})$. Because $\boldsymbol{x}$ is a constant, $\mathbf{Q} = \mathbf{0}$.

3.3 State-space Model

Eqns. (4) and (6) combine to give the state-space model:

$$\boldsymbol{x}_k = \mathbf{\Phi}\,\boldsymbol{x}_{k-1} + \boldsymbol{\omega}_{k-1} \tag{7}$$
$$y_k = \mathbf{h}_k\,\boldsymbol{x} + w_k. \tag{8}$$

Note that these equations are the state-space model. They will be used to design the state-estimator, but they are not themselves computed in the Kalman filter implementation.

4 Kalman Filter Implementation

The standard Kalman filter equations are presented in Appendix A. This section applies and simplifies those equations for the sensor calibration problem.

From the derivations above, the model parameters in eqns. (19 -20) are $\mathbf{\Phi} = \mathbf{I} \in \Re^{2\times 2}$, $\mathbf{B} = \mathbf{0} \in \Re^{2\times 1}$, $\mathbf{\Gamma} = \mathbf{I} \in \Re^{2\times 2}$, $\mathbf{h}_k = [z_k \quad 1]$, $\mathbf{Q} = \mathbf{0}$, and $\mathbf{R} = \sigma_w^2$.

*† Professor at the Dept. of Electrical and Computer Engineering, UC Riverside. {farrell}@ece.ucr.edu.

4.1 Time Propagation

This subsection discusses the implementation of Eqns. (21-22) for the sensor calibration problem.

4.1.1 Time Propagation: State

Using the definitions of the model parameters stated at the start of Section 4, eqn. (21) simplifies to

$$\hat{\boldsymbol{x}}_k^- \doteq \hat{\boldsymbol{x}}_{k-1}^+. \tag{9}$$

4.1.2 Time Propagation: Covariance

The uncertainty in the state estimate $\hat{\boldsymbol{x}}_k$ is characterized by the error-state covariance matrix:

$$cov(\hat{\boldsymbol{x}}_k) = \mathbf{P}_k. \tag{10}$$

Dropping the time subscript for this paragraph, to simplify the notation, the structure of $\mathbf{P}$ is

$$\mathbf{P} = \left[\begin{array}{cc} \sigma_{x_1}^2 & \rho\sigma_{x_1}\sigma_{x_2} \\ \rho\sigma_{x_1}\sigma_{x_2} & \sigma_{x_2}^2 \end{array} \right] \tag{11}$$

where σ_*^2 represents the covariance of variable $*$ and ρ is the cross-correlation between variable x_1 and x_2. Therefore, at any time $\mathbf{P}$ quantifies the error covariance of each element of the state vector. The only assumption for this to be valid is that the state-space model must be accurate.

For the definitions of the model parameters stated at the start of Section 4, eqn. (22) simplifies to

$$\mathbf{P}_k^- \doteq \mathbf{P}_{k-1}^+ + \mathbf{Q} \tag{12}$$
$$\mathbf{P}_k^- \doteq \mathbf{P}_{k-1}^+. \tag{13}$$

With every measurement update, eqn. (17) decreases $\mathbf{P}$. As $\mathbf{P}$ decreases, so does $\mathbf{K}$. When $\mathbf{Q}$ is zero, as is correct for the assumed model, both $\mathbf{P}$ and $\mathbf{K}$ approach $\mathbf{0}$ as k approaches ∞. This is typically not desirable, as later measurements are ignored relative to sensor calibration. Therefore, eqn. (12) is often used with $\mathbf{Q} > \mathbf{0}$, but small.

4.2 Measurement Update

The measurement residual is computed as

$$r_k \doteq v_k - \hat{y}_k^-, \tag{14}$$

where $\hat{y}_k^- = \mathbf{h}_k\,\hat{\boldsymbol{x}}_k^-$. The covariance of the residual is

$$S_k \doteq \mathbf{h}_k\,\mathbf{P}_k^-\,\mathbf{h}_k^\top + \sigma_w^2. \tag{15}$$

The Kalman measurement update is

$$\mathbf{K}_k \doteq \mathbf{P}_k^-\,\mathbf{h}_k^\top\,(S_k)^{-1} \tag{16}$$
$$\mathbf{P}_k^+ \doteq \mathbf{P}_k^- - \mathbf{K}_k\,(\mathbf{h}_k\,\mathbf{P}_k^-) \tag{17}$$
$$\boldsymbol{x}_k^+ \doteq \boldsymbol{x}_k^- + \mathbf{K}_k\,r_k. \tag{18}$$

The posterior state $\boldsymbol{x}_k^+$ and covariance $\mathbf{P}_k^-$ are now available as initial conditions for the next time update.

A The Discrete-Time Kalman Filter

The standard Kalman filter for a linear system modeled as:

$$\boldsymbol{x}_{k+1} = \boldsymbol{\Phi}\,\boldsymbol{x}_k + \mathbf{B}\,\mathbf{u}_k + \boldsymbol{\Gamma}\,\boldsymbol{\omega}_k \tag{19}$$
$$\mathbf{y}_k = \mathbf{h}_k\,\boldsymbol{x}_k + \mathbf{w}_k. \tag{20}$$

with $\boldsymbol{\Phi}$, $\mathbf{B}$, $\boldsymbol{\Gamma}$, $\mathbf{h}$ all known; a known prior distribution $\boldsymbol{x}_0 \sim N(\hat{\boldsymbol{x}}_0, \mathbf{P}_0)$; a known deterministic input sequence $\mathbf{u}_k$; and, two zero mean, white Gaussian random sequences $\boldsymbol{\omega}_k$ and $\mathbf{w}_k$ with known covariance matrix sequences $\mathbf{Q}$ and $\mathbf{R}$. In addition, it is assumed that $\boldsymbol{\omega}_k$ and $\boldsymbol{\eta}_k$ are uncorrelated with each other and with $\boldsymbol{x}_0$.

For a system with the model in the previous paragraph, the objective is to estimate the state sequence $\boldsymbol{x}_k$ optimally, using only the current and past measurements $\{\mathbf{y}_k\}_{j=1}^k$.

The Kalman filter [1,2] solves this problem, see also Chapter 5 in [3]. The Kalman filter has two types of updates.

Time Propagation: The time propagation step transports the state estimate between the times of two measurements. The equations are:

$$\hat{\boldsymbol{x}}_k^- \doteq \boldsymbol{\Phi}\,\hat{\boldsymbol{x}}_{k-1}^+ + \mathbf{B}\,\mathbf{u}_{k-1}. \tag{21}$$
$$\mathbf{P}_k^- \doteq \boldsymbol{\Phi}\,\mathbf{P}_{k-1}^+\,\boldsymbol{\Phi}^\top + \boldsymbol{\Gamma}\mathbf{Q}\boldsymbol{\Gamma}^\top \tag{22}$$

Measurement Update: The measurement update step corrects the state estimate to account for the new infomation in the measurement $\mathbf{y}_k$. The equations are:

$$\mathbf{K}_k \doteq \mathbf{P}_k^-\mathbf{h}_k^\top(\mathbf{h}_k\mathbf{P}_k^-\mathbf{h}_k^\top + \mathbf{R})^{-1} \tag{23}$$
$$\mathbf{P}_k^+ \doteq \mathbf{P}_k^- - \mathbf{K}_k\mathbf{h}_k\,\mathbf{P}_k^- \tag{24}$$
$$\mathbf{r}_k \doteq \mathbf{y}_k - \mathbf{h}_k\hat{\boldsymbol{x}}_k^- \tag{25}$$
$$\hat{\boldsymbol{x}}_k^+ \doteq \hat{\boldsymbol{x}}_k^- + \mathbf{K}_k\,\mathbf{r}_k. \tag{26}$$

The derivation of these equations can be found in the many books on optimal estimation. Those same books will present various forms of the Kalman filter. All forms yield the same state estimates in theory, but have different computational and numeric properties.

Note that both the state estimate and the covariance matrix have two values at each measurement time: $\hat{\mathbf{z}}_k^+$ and $\hat{\mathbf{z}}_k^-$ and $\mathbf{P}_k^-$ and $\mathbf{P}_k^+$. The superscript minus sign represents the values before including $\mathbf{y}_k$. The superscript plus sign represents the values after including $\mathbf{y}_k$.

References

[1] R. E. Kalman, “A New Approach to Linear Filtering and Prediction Problems,” *Transactions of the ASME–Journal of Basic Engineering*, vol. 82, no. D, pp. 35–45, 1960.

[2] R. E. Kalman and R. S. Bucy, “A new approach to linear filtering and prediction theory,” *ASME Journal of Basic Engineering, Series D*, vol. 83, pp. 95–108, 1961.

[3] J. A. Farrell, *Aided Navigation: GPS with High Rate Sensors.* McGraw Hill, 2008.